Precalculus

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Part I

Archimedes

Chapter 1

Introduction

This book has been modified from a previous project, where it comprised the first several chapters of another project titled *Best of Calculus*. That book is here:

https://github.com/telliott99/calculus_book

I decided to make these chapters separate because the overall size of the big book makes it hard to focus on the Precalculus topics. These chapters have been the recipient of much recent attention on my part.

I wrote much of this originally as short explanations for my son Sean as he studied calculus in high school. It bothers me that so often the good stuff gets left out at that level.

(The image is a detail from a painting entitled "School of Athens", and it was used as the front cover of a wonderful book annotating the Heath translation of Euclid's

Elements).

It took a genius to figure it out the first time, but it is within anyone's grasp to appreciate what they found. I imagine myself looking over Archimedes' shoulder as he explains it to me.

There is a lot of geometry here. Most scientists I've met loved geometry in school. Proof is central to the enterprise. One of the most interesting features of this book is the natural use of proofs that I have tried to make as simple and easy to follow as possible.

My favorite authors on calculus including precalculus are Morris Kline, Richard Hamming, and Gil Strang. I highly recommend Simmons, if you can find a copy.

Finally, a saying attributed to Manaechmus (speaking to Alexander the Great), "there is no royal road to geometry". Which means, practically, learning mathematics requires that you follow the argument with pencil and paper and work out each step yourself, to your own satisfaction. That is the only way of really learning, and at heart, one of the reasons I wrote this book.

I express my sincere thanks to the authors of my favorite books, which are listed in the references and mentioned at various places in the text. Almost everything in here was appropriated from them, and styled to my taste. I offer my profound thanks also to Eugene Colosimo, S.J. He was, for me, the best of a bunch of very special teachers.

If I stole your figure off the internet, I'm sorry. I intended to redraw it but have not yet found the time.

We start with my favorite mathematician, Archimedes.

You can find the current version of this book on github here:

https://github.com/telliott99/precalculus

Chapter 2

Area of a circle

- Albert Einstein

> Any fool can know. The point is to understand.

In this first unit we will develop the most famous of Archimedes geometrical contributions, a theorem on the volume of the sphere.

Before we get there we need to talk about circles (a topic to which he also contributed) and look at the volume of cones and pyramids. These are topics in geometry that come before the volume of the sphere.

area

But even before that, we need a brief introduction to area and volume. In geometry there are lines and curves and each of these has length.

Figures in the plane have area: triangles, squares and rectangles, and straight-sided polygons, so-called rectilinear figures. But also, circles and ellipses and parabolas.

Then there are solid figures, like cubes and pyramids, and cones and spheres, that have volume.

For a rectangular figure, it is easy to see why the definition of area as length times width makes sense. For a cube, the volume is the length times width times height.

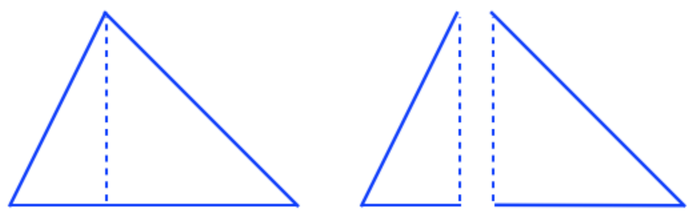
One of the miracles of calculus is that it can give us areas and volumes of curved figures. But some of those results were available from Greek geometry, even before calculus. We'll see a bit of that here. As we start reasoning about circles, we recognize that the area of a circle is going to be something *squared*, because it occupies space in the plane.

According to wikipedia

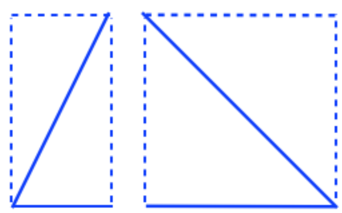
https://en.wikipedia.org/wiki/Area_of_a_circle

Eudoxus of Cnidus, born in the 5th century (408 BCE), proved that the area of a circle, like that of regular polygons, is proportional to both horizontal and vertical dimensions, and thus is proportional to the radius squared.

An idea that I bet you've run into, and we will use a lot, is that the area of a triangle is one-half the base times the height. This is easy to prove. Here we show only an acute triangle, but the theorem is true for all: any triangle can be divided into two right triangles.



The two pieces can then be pasted to their rotated equals to form two rectangles.

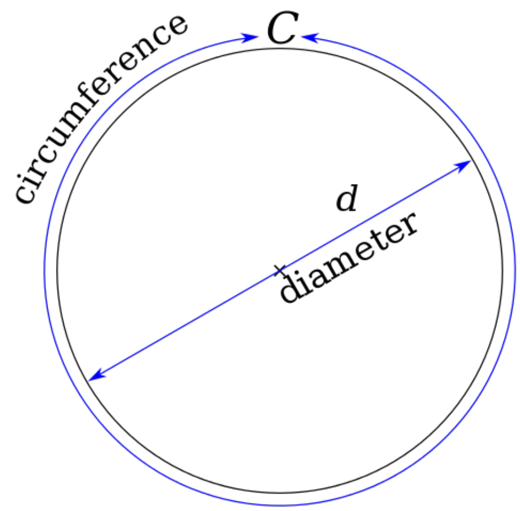


The total area of each rectangle is the total base times the height. The total area is the total base times the same height. This is twice the area of the original triangle.

□

circumference

We move on to the circle. A fundamental result about circles is that the ratio of the circumference of a circle to its diameter is independent of the size of the circle. All circles have the same shape.



The proportionality constant was named in the early 1700s and popularized by Euler a few decades later:

$$\pi = \frac{C}{d}$$

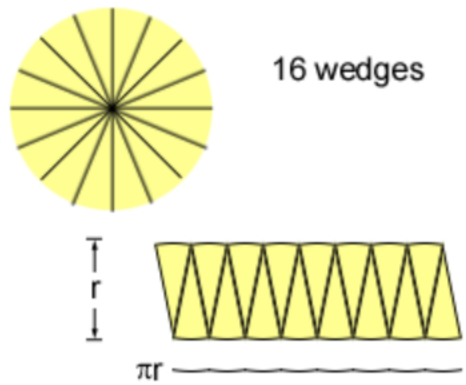
Since the radius is one-half the diameter, $2r = d$ and

$$2\pi r = C$$

This is usually stated as a self-evident fact, but it is actually a theorem to be proved. We defer that one for the moment.

area of a circle: pizza proof

Imagine dividing a circle into wedges, like you might do with a pizza. Here, the pie has been divided into 16 parts.



Since the pieces are triangular, it is easy to stack them next to each other with the bases and tips alternating, as shown. Of course the bases are not straight, but have the same curvature as the edge of the circle.

The length of the short side is the radius, r, although it is angled. The original perimeter or circumference is divided into the top and the bottom of the figure, so the length of the long side is approximately one-half the circumference and thus, with length times width, we obtain

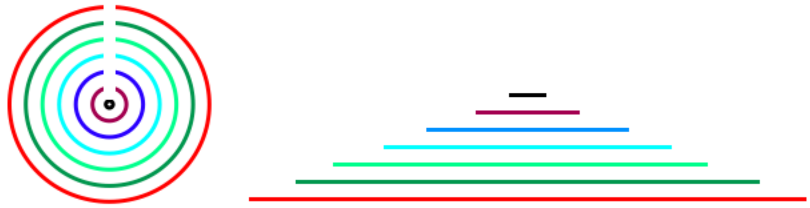
$$A = r \cdot \frac{1}{2} \cdot 2\pi r = \pi r^2$$

The trick is to imagine what happens if we subdivide the circle into many slices. The more slices, the more vertical the side, and the straighter the edges. If there are infinitely many slices, the edges will be perfectly straight and this calculation becomes exact.

The pizza proof is very much like one attributed to Leonardo da Vinci, among others.

concentric rings

Another idea is to remove concentric strips from the edge and stack them.



This is actually the same calculation as we did previously, just based on a different idea. Here, we imagine the concentric strips infinitely thin.

We obtain a triangle of height r and base $2\pi r$ so its area is

$$\frac{1}{2}\, 2\pi r \cdot r = \pi r^2$$

concentric rings

A formal proof that this triangle has the same area as the circle was given by Archimedes and is found in his *Measurement of a Circle*, proposition 1. However, many sources, including

http://www.math.tamu.edu/~dallen/masters/Greek/eudoxus.pdf

attribute the proof to Eudoxus, who was perhaps the second most famous mathematician of antiquity, and a colleague of Plato in Athens.

I love the proof, but some people have struggled with it, especially so early in the book. If you struggle, just move on. But please take a shot at it first.

We will sketch the idea and discuss it more formally **here**.

The method used is called proof by contradiction, *reductio ad absurdum*, which Hardy called

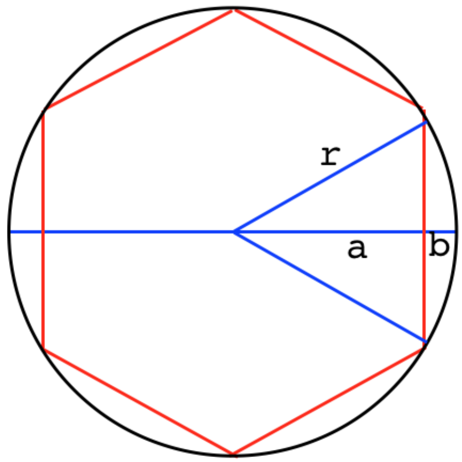
> one of the mathematicians finest weapons

One begins with an assumption. A slightly strained analogy might be, turning into a narrow street and, having missed the sign, assuming it goes your way. Later, when you meet a semi trailer head-on you know there is a problem somewhere in the logic.

inscribed polygon

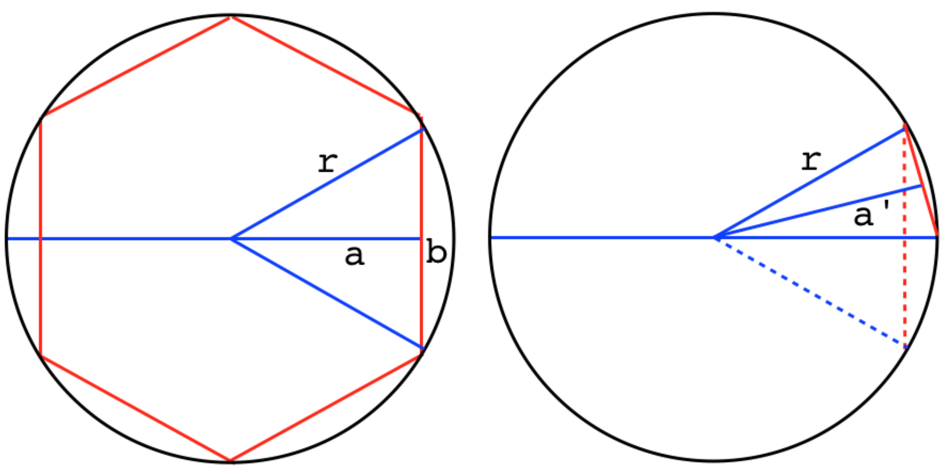
Draw a circle. Call the actual, correct, yet still unknown value for the area of the circle A.

The idea is to draw a regular polygon (all sides equal length) inside the circle.



Here we have drawn a hexagon (6-sides). If the lines from the center to the vertices have length r, then they divide the hexagon into six triangles, each with height a and base length b, where $a < r$ and b is less than the arc length. Clearly the area of the triangle is less than the area of that sector of the circle.

The base of the triangle is closest to the center (and farthest from the edge) at the center of the base, where the line segment from the center is called the apothem (labeled a). Using trigonometry, it isn't hard to calculate the area of the polygon, but we don't need to. Just call it P.



Archimedes now says, let us double the number of sides (right panel). What happens then?

The new 12-sided polygon and its component triangles will have a larger apothem a', and the total of all the bases all the way around will be larger and closer to the true circumference of the circle. There is obviously less white space between the triangle's base and the perimeter of the circle.

Thus the new P will be larger, and closer to A.

This doubling trick is easy to do. We don't even have to carry it out, just imagine doing so. We can make the difference between the area of the polygon P and the circle's true area A as as small as you please.

If your boss decides it's not close enough, just double the number of sides *again*.

That was all setup, here is the punchline.

proof

Archimedes says, let us suppose that the true area of the circle A is *not* actually equal to T (which is exactly πr^2) but is larger. Just suppose. In symbols, we are assuming that

$$T < A$$

We've already seen that $P < A$

We know we can make P as close to A as we please.

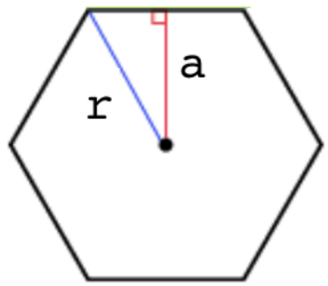
And therefore (the key point) we can make P *closer* to A than T is. The meaning of $T < A$ is that there must be some daylight between T and A. The side-doubling operation can get us into that window.

So now, by doubling, we have obtained a new P that is larger than T. We have established that

$$T < P < A$$

If $T < A$ there is really no other choice.

But look at the figure below. No matter how many sides our many-sided polygon has, $a < r$ and and the base must be less than the circumference of the circle so clearly $P < T$ for any polygon.



This is a contradiction. We have shown by two arguments which are both logically correct that $P < T$ and also that $P > T$. There is something wrong.

The resolution is that assumption we made above, that $A > T$, cannot be right. Therein lies our problem. A is not greater than T. It is either less than or equal to T.

But now try it the other way around. Circumscribe the circle with a hexagon that goes around the outside and run the argument again, and you will find that it cannot be true that $A < T$ either.

But if A is neither less than nor greater than T there is only one possibility, equality:

$$A = T = \pi r^2$$

□

Plutarch, talking about Archimedes:

> It is not possible to find in all geometry more difficult and intricate questions, or more simple and lucid explanations. Some ascribe this to his natural genius; while others think that incredible effort and toil produced these, to all appearances, easy and unlabored results. No amount of investigation of yours would succeed in attaining the proof, and yet, once seen, you immediately believe you would have discovered it; by so smooth and so rapid a path he leads you to the conclusion required.

There is one additional point. Archimedes actually provides a way of calculating the improved area of each successive polygon (or its perimeter, it is really the same problem) obtained by side-doubling.

Each cycle gives a smaller and smaller improvement, which means that there is a limiting value of this process.

That value is π, when talking about the perimeter for a unit diameter, or equivalently when talking about the area, for a unit radius. We will see how this works later, suffice it to say that the side-doubling trick gives us a way to calculate the value of π to any accuracy we have the patience to compute.

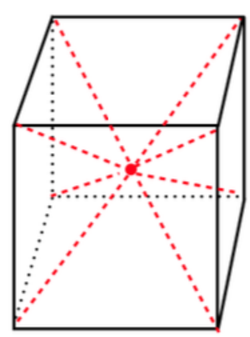
Chapter 3

Volume of a cone

We need a formula for the volume of a cone in order to find the volume of the sphere. And in order to find the cone's volume, we start with something simpler, a pyramid with a square base.

Consider a cube with all eight edges having length s. So each of the six faces is a square with sides of length s and area s^2.

Label the central point inside the solid as P. Draw lines connecting each of the 8 external vertices to P, something like this.

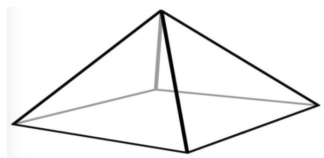


Now we imagine slicing on planes that connect adjacent pairs of lines.

You can't do this in real life by slicing up a single cube or rectangular solid, because the cuts to form one surface would ruin some of the other pieces. The cuts must enter the solid at a corner and then pivot on a line ending at the exact center.

Perhaps you could do it with a *light saber* since the beam comes to a point.

The result would be 6 identical pieces (square pyramids) looking something like this



The procedure described generates pyramids with height $s/2$. So they are a little squat, but just bear with me.

We started with a cube so that the six resulting solids would be identical.

Unfortunately you can either have six pieces come out exactly the same, as we've done, or make it so some of the pieces come out with equal base and height, but you can't do both at the same time by this construction.

Let the six identical pyramid volumes each be V, and their sum is equal to the volume that we started with. We have that

$$6V = s^3$$

$$V = \frac{1}{6}s^3$$

If we factor out the height $h = s/2$ we obtain

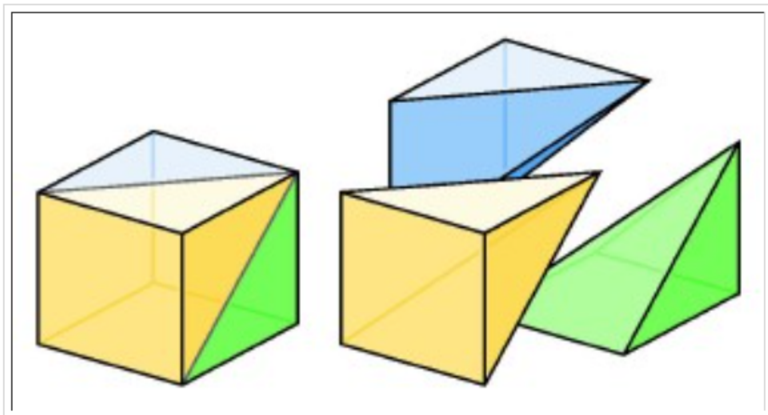
$$V = \frac{1}{3}\ hs^2$$

This is the volume for each pyramid with base area s^2 and height $s/2$.

The volume depends on both the area of the base and the height. You can show this by starting with solids that are longer in one-dimension. Since here $h = s/2$ it all works out.

better way

Here is an even better way to slice a cube

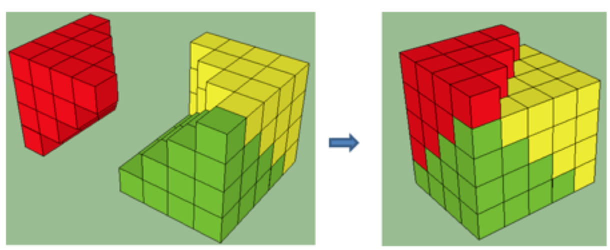


Three congruent pyramids meet along a diagonal of a cube.

When I first saw this, I thought it was a trick. But in fact, we have produced 3 identical square pyramids (they are called oblique because the apex is not in the center).

http://www.math.brown.edu/~banchoff/Beyond3d/chapter2/section02.html

Here is a version in blocks:



I know it sounds complicated but it's really not.

real world

I found a fun way to do the demonstration easily and safely. I was going to cut some wood on the table saw, but this is way better.

Get a thick piece of cheese and cut out a cube as large as you can make it and with everything squared up as accurately as you can.

Then cut straight down on a diagonal all the way through the cube, resulting in two identical pieces.

If you now take each of the pieces and orient them with the new angled surface resulting from the cut facing up, you can then make another diagonal cut straight down for each.

You will end up with a large piece and a small one. Do the same with the other half.

It seems remarkable, but the two smaller ones can be assembled into a single shape identical to each of the large pieces.

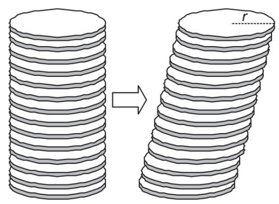
You have thus de-constructed the cube into three identical pyramids.

Good luck! It is OK to eat the demonstration afterwards.

cones

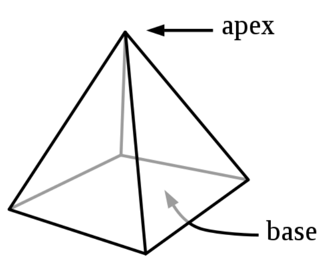
Of course, a pyramid is not a cone. But it will turn out that the volume is independent of the shape of the base. It just depends on the area. This is proved by what seem to be "hand-waving" arguments before calculus.

For example, there is Cavalieri's principle, also called the "method of indivisibles", or the "stack of quarters" argument.



If we slice a volume into small segments and then slide the slices around, the volume doesn't change. So we expect that a pyramid and a right pyramid of the same base and height have the same volume.

https://en.wikipedia.org/wiki/Cavalieri's_principle



And there doesn't seem any reason why the shape of the base should matter either, only its area. Hence we substitute a circular cross-section, i.e., a cone.

For a cone we finally obtain

$$V = \frac{1}{3}\pi r^2 h$$

Knowing basic calculus allows us to see easily where the factor of one-third comes from in the formula for the volume of a pyramid or a cone. It comes from integrating

$x^2\ dx$ and obtaining $x^3/3$. Get there we will, young padawan.

However, it is interesting to see how things might have been glimpsed in the age before calculus.

algebraic derivation of the constant 1/3

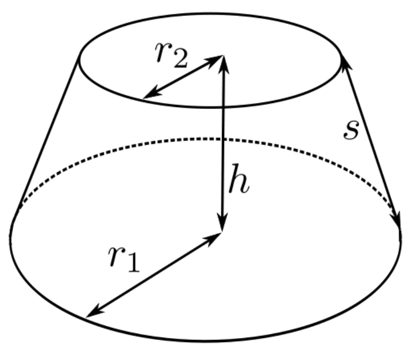
I found an algebraic argument on the web at

https://web.maths.unsw.edu.au/~mikeh/webpapers/paper47.pdf

Let us assume for this proof that the volume of a cone is proportional to both the area of the base and the height: $V = cAh$; our objective is to find the constant of proportionality.

It takes a bit of algebra to see, but gives the value for the proportionality constant as 1/3.

Consider a conical frustum, a cone with the top lopped off.



Suppose the area of the base is A and the height of the frustum is h.

Calculate the volume of the frustum as the difference between that of a larger cone with base A and height $h+e$ (e for extra), and that of a small cone (the part cut off to form the frustum) with base area a and height e.

$$V = cA(e+h) - cae$$

Now, the area of the base of a cone is π times the radius squared, and the radius is proportional to the height (depending on how sharply the side slants). Hence

$$a = ke^2$$

The area of the small base is proportional to the height squared with proportionality constant k, and the same for the large one:

$$A = k(e+h)^2$$

so we rearrange things

$$k = \frac{a}{e^2} = \frac{A}{(e+h)^2}$$

Hence

$$\frac{\sqrt{a}}{e} = \frac{\sqrt{A}}{e+h}$$

Let us manipulate this expression to find e in terms of h. It just requires a bit of facility with square roots:

$$\frac{\sqrt{A}}{\sqrt{a}} = \frac{e+h}{e} = 1 + \frac{h}{e}$$

$$\frac{h}{e} = \frac{\sqrt{A}}{\sqrt{a}} - 1 = \frac{\sqrt{A} - \sqrt{a}}{\sqrt{a}}$$

$$e = \frac{\sqrt{a}}{\sqrt{A} - \sqrt{a}} \cdot h$$

And then

$$e + h = \ [\ \frac{\sqrt{a}}{\sqrt{A} - \sqrt{a}} + 1\ ]\ h = \frac{\sqrt{A}}{\sqrt{A} - \sqrt{a}} \cdot h$$

Substituting into what we had above for the volume:

$$V = cA(e+h) - cae$$

$$= cA\ [\frac{\sqrt{A}}{\sqrt{A} - \sqrt{a}} \cdot h\ ] - ca\ [\frac{\sqrt{a}}{\sqrt{A} - \sqrt{a}} \cdot h\ ]$$

$$= c\ [\ \frac{A\sqrt{A} - a\sqrt{a}}{\sqrt{A} - \sqrt{a}}\ ]h$$

This really looks like a mess.

But suppose we let $m = \sqrt{A}$ and $n = \sqrt{a}$ and then

$$\sqrt{A} - \sqrt{a} = m - n$$

then the numerator above is really just $m^3 - n^3$.

We can factor that, we get $m^3 - n^3 = (m-n)(m^2+mn+n^2)$, which you can confirm by multiplying back out. So the first term $(m-n)$ cancels the denominator. We now have:

$$V = c(m^2 + mn + n^2)h$$
$$V = c(A + \sqrt{A}\sqrt{a} + a)h$$

And, the punchline. Consider what happens as a gets larger and closer to A.

We say: let $a \to A$. But just say $a = A$.

The expression in parentheses becomes $3A$. Hence:

$$V = c(3A)h$$

But if $a = A$, the frustum has become a cylinder, whose volume we know. It is equal to Ah.

$$V = c(3A)h = Ah$$

Therefore $c = 1/3$.

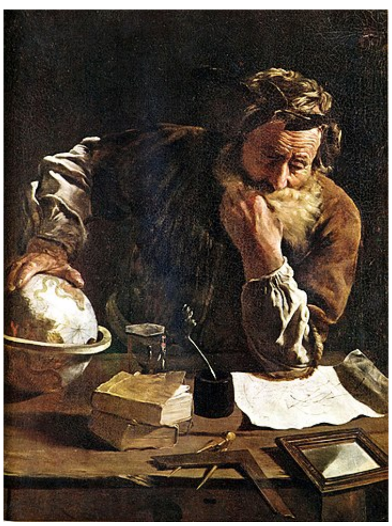
□

Chapter 4

Archimedes and the sphere

biography

Archimedes is often ranked as the greatest of the Greek mathematicians. He stands with Newton, Euler and Gauss, the best of the moderns.



Here are two images, the first is a painting by Fetti from 1620 that is in the wikipedia article on Archimedes:

https://en.wikipedia.org/wiki/Archimedes

The other is the image on the famous Fields medal, which is sometimes described as the "Nobel prize" for mathematics.

https://en.wikipedia.org/wiki/Fields_Medal

Archimedes lived and died (c.287-212 BC) in the beautiful city of Syracuse, found on the southeastern coast of modern Sicily. He is famous for many inventions, derivations and discoveries, but was evidently proud of the formula for the volume of the sphere.

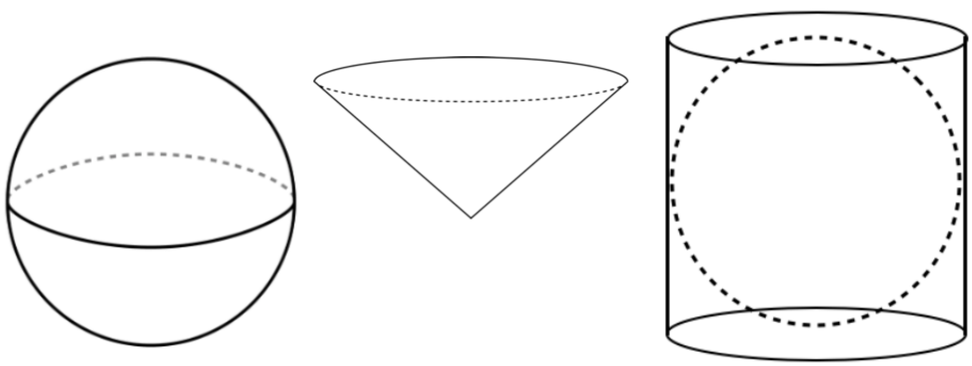
The very simple result is that the volume is two-thirds that of a cylinder that just encloses the sphere.

Because of his discovery, there was a sculpture of the sphere and cylinder carved on his gravestone, located near the Agrigentine gate of Syracuse. The grave was rediscovered by the Roman orator Cicero, covered by brush after 137 years of neglect. It is now lost again.

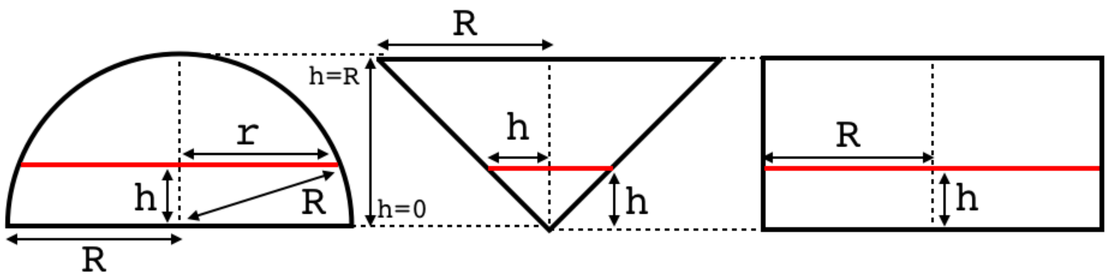
slices of solids

The following is Archimedes simple but subtle argument.

We compare a half-sphere and an inverted cone to a cylinder.



Below is a diagram showing a **vertical** cross-section through the center of each solid so we can visualize the geometry. The radius R is the same for all three. In addition, the cone and cylinder have overall height equal to R.



Now, imagine making a **horizontal** slice through each solid at an arbitrary but constant height h, shown by the red lines. I hope you can visualize each of these red slices, which are perpendicular to the page.

Each slice is a circle. Any cross-section of a sphere is a circle.

For the cylinder and cone, cross-sections perpendicular to the central axis are circles as well.

The question we ask is: what is the area for each slice?

To answer that, we need to determine the radius for each red circle.

Moving right-to-left, the radius of the cylinder is just R. For the cone, the radius at each height h is equal to h, since $R = H$. And for the sphere, we use the Pythagorean theorem to find that

$$r^2 + h^2 = R^2$$
$$r^2 = R^2 - h^2$$

For more on this theorem see **here**.

The first insight of the proof is to recognize that the radius squared for the sphere's slice (r^2), plus the radius squared for the cone (h^2) is equal to R^2, the radius squared for the cylinder.

Since the area of each circle is proportional to to the radius squared (namely $A = \pi r^2$ and so on) and

$$\pi r^2 + \pi h^2 = \pi R^2$$

so the areas add too. Our just famously and remarkable simple result: *sphere plus cone equals cylinder.*

invariance

The second crucial insight of the proof is to recognize that this property is invariant, it does not depend on which height we choose to make the slice. The three slices obtained at any height h add up like this. So if we imagine making a bunch of slices for each solid and adding them all up to find the volume, the volumes will add too.

This idea is now called Cavalieri's principle, though it was called the "method of indivisibles" before that.

The volume of the cylinder is simply πR^3. The volume of the cone is known to be one-third the area of the base times the height, or $1/3\ \pi R^3$.

Subtract to find that the area of the half-sphere is $2/3\ \pi R^3$, and therefore the volume of the whole sphere is

$$V_{\text{sphere}} = \frac{4}{3}\pi R^3$$

There is a bit of a trick here to hide the idea introduced in calculus, which makes this thinking rigorous. The sphere and cone have variable widths, which means that the radius will be different on the top of a slice compared to the bottom. Therefore, the slices have to be made very thin. In calculus they become infinitely thin, but we add up infinitely many of them.

Moreover, Archimedes was subject to certain limitations (discussed by Bressoud), which lead him to formulate the argument in terms of moments (masses of the solids and their centers). I have left that complication out of this discussion.

knowledge before proof

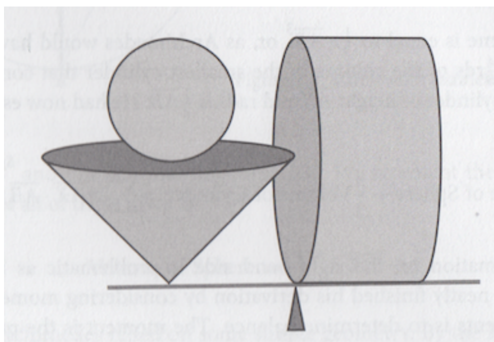
Archimedes said that he discovered the correct result by balancing the three objects on a fulcrum.

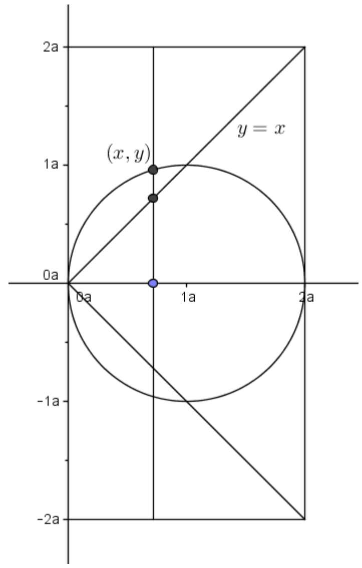
According to Archimedes (in the Method, translation by Heath)

> For certain things which first became clear to me by a mechanical method had afterward to be demonstrated by geometry...it is of course easier, when we have previously acquired by the method some knowledge of questions, to supply the proof than it is to find the proof without any previous knowledge. This is a reason why, in the case of the theorems the proof of which Eudoxus was the first to discover, namely, that the cone is a third part of the cylinder, and the pyramid a third part of the prism,

having the same base and equal height, we should give no small share of the credit to Democritus, who was the first to assert this truth...though he did not prove it.

From his description, what Archimedes actually balanced is a set-up like that shown below:





https://proofwiki.org/wiki/File:SphereVolume.png

We have a:

- sphere with radius R
- cone with radius $2R$ and height $2R$

Their combined volumes:

$$V = \frac{4}{3}\,\pi R^3 + \frac{1}{3}\,\pi(2R)^2 \cdot 2R$$

$$= \frac{12}{3}\,\pi R^3 = 4\pi R^3$$

balanced against

- a cylinder with radius $2R$ and height $2R$ with

$$V = \pi(2R)^2 \cdot 2R = 8\pi R^3$$

However, the moment of the cylinder is at one-half the distance from the fulcrum as that of the sphere-cone combination, so it balances.

Part II

Numbers and proof

Chapter 5

Integers

Integers

The *natural* or counting numbers which everyone learns very early in life are $1, 2, 3$ and so on.

One can get hung up on the question of whether the natural numbers would exist without the problem of counting a dozen sheep or all twenty of our fingers and toes. Leopold Kronecker famously said "God made the integers; all else is man's handiwork".

We will not worry about where they come from.

Mathematicians refer to the *set* of natural numbers and give that set a special symbol, $\mathbb{N}$. We write

$$\mathbb{N} = \{1, 2, 3 \dots\}$$

The brackets contain between them the elements or members of the set. The dots mean that this sequence continues forever.

How can we decide whether a particular n is in the set if we can't enumerate all of its members? We can tell by its form whether some n is a natural number or not.

If this seems problematic, you might call $\mathbb{N}$ a class instead (Hamming); we carry out *classification* to decide whether n is a natural number.

The notion of an unending sequence can be unnerving upon first encounter.

construction of N

To construct the set $\mathbb{N}$, start with the smallest element, 1. Then

$$1 + 1 = 2$$
$$2 + 1 = 3$$
$$3 + 1 = 4$$
$$\ldots$$

Add successive elements by forming $a_n + 1 = a_{n+1}$.

$\mathbb{N}$ is an infinite set.

We say there is no largest number in $\mathbb{N}$, no largest $n \in \mathbb{N}$. The symbol $\in$ means "in the set" or "is a member of the set".

Proof:

Suppose $\mathbb{N}$ did have a largest member, M.

Well, what about $M + 1$? By the definition we can construct it and it is clearly a member of the set, but $M + 1 > M$ so M is not the largest number in the set.

This is a proof by contradiction that $\mathbb{N}$ is infinite.

$\square$

set membership

Sometimes people say that

$$0 \in \mathbb{N}$$

(0 is a part of the set) but most do not, and we will follow the definition given above. If you wanted to be explicit about this you could write

$$0 \notin \mathbb{N}$$

What do we mean by infinity? We mean an upper bound on the natural numbers, and later, all rational and indeed all real numbers.

All numbers $n \in \mathbb{N}$ have the property that n is contained in the interval $[1..\infty)$. However, ∞ is *not* considered part of the interval, and that is the meaning of the the right parenthesis.

∞ is not a number so it probably doesn't even make sense to write $\infty \notin \mathbb{N}$.

least element

$\mathbb{N}$ does not have a greatest number, but it does have a smallest or least one. If pairwise comparisons are carried out, a single element, the number 1, has the property that $1 \leq n$ for all numbers $n \in \mathbb{N}$. As we go on, we will find that other types of numbers (rationals and real numbers), do not have a least positive number.

well-ordered property

Since we can also find the least member of the set excluding 1, written $\mathbb{N} \setminus 1$, we can order every number in $\mathbb{N}$.

This property is called the **well-ordered** property.

the Integers

The set $\mathbb{Z}$ contains all the members of $\mathbb{N}$ plus their negatives, as well as the special number 0, often called the additive identity since $0 + n = n$ for all $n \in \mathbb{N}$.

$$\mathbb{Z} = \{\cdots - 2, -1, 0, 1, 2, \dots\}$$

$\mathbb{Z}$ stands for the German word *Zahlen*, number. The set $\mathbb{Z}$ is usually referred to as the integers.

$\mathbb{Z}$ is also an infinite set and also has the well-ordered property. To show this simply order all numbers $n > 0$ with respect to zero using $<$, and all the numbers $n < 0$ using $>$.

Chapter 6

Primes

prime numbers

As you know, the positive integers larger than 1 are of two types:

∘ a prime number p has only two factors, p itself and 1

∘ a composite number has at least one additional factor. Either the number is a perfect square of a prime, or it has an even number of additional factors: $a_1 \dots a_k$.

The first ten primes are:

```
2 3 5 7 11 13 17 19 23 29 ...
```

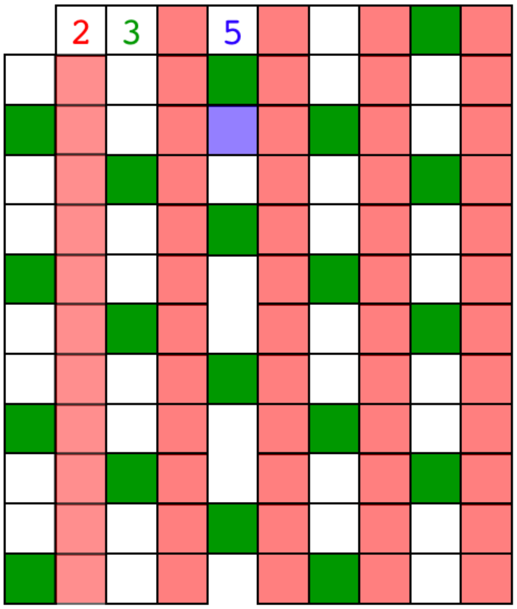
The sieve of Eratosthenes

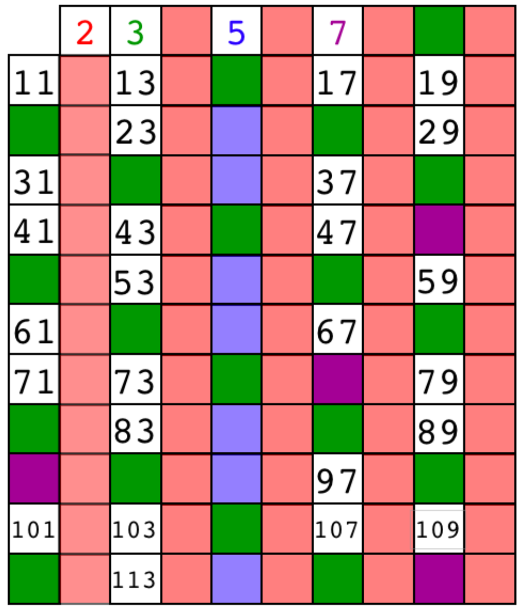
Eratosthenes is famous in mathematics for his "sieve" which allows one to determine which numbers are prime in an economical fashion.

We will take note of him again in talking about the circumference of the earth. He was a contemporary of Archimedes and became the chief librarian at the Library of Alexandria when he was only about 35 years old.

The sieve operates by first writing down all the integers to some upper limit (here 120). To carry out the process manually it is convenient to use rows with 10 values, so there are 12 rows in all here. Most of the boxes have not yet been numbered (below, left).

Starting with the first prime number, 2, eliminate all the numbers divisible by 2 (all the red numbers, or even numbers). Here this has been done by coloring red all squares with numbers ending in $2, 4, 6, 8, 0$.





Next, do the same thing with 3 (green). 6 was already eliminated previously, but odd multiples of 3 like 9, 15 and 21 go away at this step.

The next larger number that still has a white square is 5. All the squares eliminated at this step are white ones in the fifth row, starting with 25. Continue with 7, eliminating $49, 77, 91$ and 119.

The sieve now ends (for this upper bound of 120).

The rule is that if the number for the next round, the smallest number not yet eliminated, is larger than the square root of the upper limit ($\sqrt{120}$), we terminate.

So 7 is the last value used, because after that round the smallest remaining integer is 11, but we terminate since $11^2 = 121 > 120$.

The graphic shows all the numbers which have yet to be eliminated after the round of 7. All of these numbers, 11, 13, 17, and so on, as well as those used as divisors for each round of the sieve $(2, 3, 5, 7)$, are prime numbers.

By testing for division by $2, 3, 5$ and 7, we have found the first 30 prime numbers.

```
 2   3   5   7  11  13  17  19  23  29
31  37  41  43  47  53  59  61  67  71
73  79  83  89  97 101 103 107 109 113
```

From a performance standpoint, it is important that we do not need to carry out division. All that is really needed is repeated addition. Coding this algorithm in, say, Python is a good challenge.

A bigger challenge is to come up with a method to *grow* the list of primes on demand. This can be done by keeping track of the first value to be tested above the limit, for each prime in the current list.

infinite primes

Euclid has a theorem and a proof that the number of primes is infinite.

Proof:

By contradiction.

Suppose the set of primes is finite, and that p_1, $p_2 \dots p_k$ are all of the primes. Construct the following numbers:

$$P = (p_1 \cdot p_2 \cdot \dots p_k)$$

$$Q = P + 1$$

For a prime number p to evenly divide Q, it must divide the difference between Q and P. But that difference is 1 and so can't be divided evenly by any prime.

Therefore, none of the known primes divides Q and at least one of these is true:

- Q is a prime not in the set of known primes
- the set was originally incomplete

The assumption that the set of primes is finite leads to a contradiction.

□

Even for a relatively small number of primes, we may encounter the second situation. Start with the first prime: 2:

$$2 + 1 = 3 \text{ (prime)}$$

$$2 \cdot 3 + 1 = 7 \text{ (prime)}$$

$$2 \cdot 3 \cdot 7 + 1 = 43 \text{ (prime)}$$

$$2 \cdot 3 \cdot 7 \cdot 43 + 1 = 1807$$

1807 is *not* prime. ($1807 = 13 \cdot 139$).

testing primality

This is a pretty deep subject. However, a simple filter to apply first is to ask: is the last digit one of $\{0, 2, 4, 6, 8\}$, i.e. is the number even? Does the number end in 5? Or is the sum of the digits divisible by 3 or 9?

There is a trick for the last test. If a number is divisible by 3 its digits add to a multiple of three. Suppose the number is:

$$abcd = a \cdot 10^3 + b \cdot 10^2 + c \cdot 10^1 + d$$

$$= a(9 \cdot 10^2 + 1) + b(9 \cdot 10^1 + 1) + c(9 \cdot 10 + 1) + d$$

If $x|(y + z)$ and $x|y$ then it must be that $x|z$. Since 9 times anything is divisible by 3, it follows that 3 must divide $a + b + c + d$ for $abcd$ to be divisible by 3.

A similar thing is true of 9 except that the digits must add only to 9.

A more general observation is that all primes greater than 3 are of the form $4k+1$ or $4k + 3$, for integer k. That's because $4k$ and $4k + 2$ are even, and $4k + 4 = 4(k + 1)$.

Any composite number n has a unique prime factorization. Its smallest prime factor p has the property (easily proved):

$$p^2 \leq n$$

Therefore, it suffices to check whether the prime numbers less than or equal to the square root of n divide n. If the square root is not an integer, we need check only the next smallest integer, what is called the *floor* of the value. If no prime less than that divides n, then n is a prime.

This can be improved still more.

https://en.wikipedia.org/wiki/Primality_test

Chapter 7

Prime factorization

We will prove that every integer has a unique *prime factorization*.

Examples:

$$6006 = 2 \cdot 3 \cdot 7 \cdot 11 \cdot 13$$

$$144 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3$$

$$12 = 2 \cdot 2 \cdot 3$$

This is also called *the fundamental theorem of arithmetic*.

$$n = p_1 \cdot p_2 \dots p_k$$

In the list of the prime factors of n, a factor may be repeated. To compare two factorizations for uniqueness, we suppose they are sorted (say, from smallest to greatest).

Example:

Let's try 123456789.

At first it seems easy. I get two factors of 3, leaving 13717421.

Then my luck ran out. The smallest prime factor was too large for me to find by hand. I used Python:

```
def f(n):
    for i in range(2,int(n**0.5)):
        if n % i == 0:
```

```
            print i, n/i
```

Running this

```
>>> f(13717421)
3607 3803
```

These are the two prime factors of that number.

background

We will prove that the unique prime factorization theorem is true.

Before we start on that, remember that when we say that one integer evenly *divides* another one, written as $a|n$ or a is a factor of n, what we mean is that we can find another integer k such that

$$a \cdot k = n$$

a times k is exactly equal to n.

And if there is also a number m where a evenly divides m, we write $a|m$ and mean that

$$a \cdot j = m$$

So then addition or subtraction of $m + n$ gives

$$m + n = a \cdot j + a \cdot k = a(k + j)$$

$$m - n = a \cdot j - a \cdot k = a(k - j)$$

If $a|m$ and $a|n$, a divides their sum or difference.

abnormal numbers

Hardy and Wright (*Theory of Numbers*, sect. 2:11) have a proof of prime factorization, different than the standard one, which I find quite elegant.

Its greatest virtue is that it simplifies some other proofs including of Euclid's lemma, which is tedious.

Proof.

By contradiction.

Hardy:

> Let us call numbers which can be factored into primes in more than one way, *abnormal*, and let n be the smallest abnormal number.

Start by supposing that there are two different factorizations of n:

$$n = p_1 \cdot p_2 \dots p_k$$

and

$$n = q_1 \cdot q_2 \dots q_j$$

where the p's and q's are all primes.

Different factorizations

As a preliminary result, consider the possibility that some p is equal to a q.

Let us rearrange if necessary so that factor is first: $p_1 = q_1$ and

$$n = p_1 \cdot p_2 \dots p_k$$

$$n = p_1 \cdot q_2 \dots q_j$$

But now, n/p_1 is abnormal, because it has two different prime factorizations.

That is impossible, because n is the smallest abnormal number.

Thus, we have that no p is a q and no q is a p.

If there do exist abnormal numbers with two factorizations, those factorizations must be completely different.

inequality

In this part, we establish that $p_1 \cdot q_1 < n$.

We may take p_1 to be the least p and q_1 to be the least q.

Since n is composite, either $p_1^2 = n$ (and p_1 is the only factor of n),

Or p_1 times the largest $p_k = n$.

In the second case, $p_1 < p_k$, $p_1 \cdot p_k = n$ and so $p_1 \cdot p_1 < n$.

A similar result holds for q_1.

But, since $p_1 \neq q_1$, only one of p_1 or q_1 at most, can be squared to give n. Either $p_1 \cdot p_1 < n$ or $q_1 \cdot q_1 < n$ or if there is more than one p and more than one q, both statements are true.

From this it follows that $p_1 \cdot q_1 < n$.

the contradiction

Let $N = n - p_1 q_1$.

We have $0 < N < n$ and also that N is not abnormal.

We're given that $p_1|n$ and so, from the above equality $N = n - p_1 q_1$ and our preliminary reminder about what factorization means, it must be that $p_1|N$.

A similar result is true for q_1, namely $q_1|N$.

Hence both p_1 and q_1 appear in the unique factorizations of both N and $p_1 q_1$.

We have that $N = n - p_1 q_1$ and we've shown that $p_1 q_1|N$ and certainly $p_1 q_1|p_1 q_1$. It follows that $p_1 q_1|n$ and hence $q_1 = n/p_1$.

But n/p_1 is less than n and has the unique prime factorization $p_2 \cdot p_3 \ldots p_k$.

Since q_1 is not a p, this is impossible.

Hence there cannot be any abnormal numbers.

□

This is also called the *fundamental theorem of arithmetic*.

Chapter 8

Induction

the problem

Suppose we have some theorem that we think *might be true* for all numbers n, because we've tried it on a few different values of n and the theorem is true for all of them.

A classic example (Courant and Robbins) is this prime number generator:

$$p(n) = n^2 - n + 41$$

The remarkable function $p(n)$ produces a prime number for integer $0 < n < 41$.

```
  41   43   47   53   61   71   83   97
 113  131  151  173  197  223  251  281
 313  347  383  421  461  503  547  593
 641  691  743  797  853  911  971 1033
1097 1163 1231 1301 1373 1447 1523 1601
1681
```

But, for $n = 41$, the last two terms cancel in

$$p(n) = n^2 - n + 41$$

and then n^2 is divisible by n, thus the result cannot be prime.

By testing them all, I found that 41 is the largest prime smaller than 2000 with this property (I don't know of a proof that no more exist). The primes with this property are:

```
 2  3  5 11 17 41
```

Hamming has some other examples of theorems with many true candidates, but which are false. Here is one:

$$f(n) = n(n-1)(n-2)\dots(n-k)$$

$f(n) = 0$ for all $0 \leq n \leq k$, but will never be zero for any other $n > k$.

That is because there are only k zeroes of a kth degree polynomial. (As an aside, this is a consequence of the *fundamental theorem of algebra*).

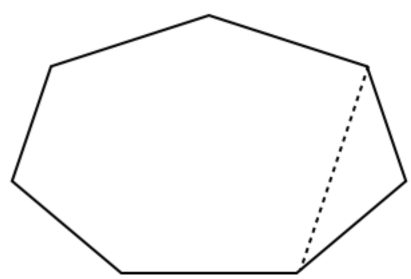
By choosing k large and expanding the definition, we can generate as many true cases as you have patience for.

Furthermore, for any function $g(n)$, $f(n) + g(n)$ will have the same property.

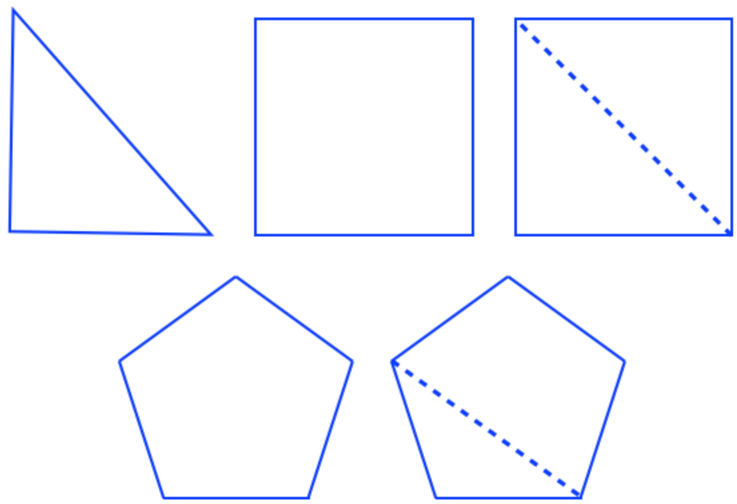
induction in geometry

In the figure below is a polygon—an irregular heptagon. Actually, there are three polygons altogether, there is the heptagon with $n + 1$ sides, the hexagon with only n sides that would result from cutting along the dotted line, and the triangle that is cut off.

We want to find a formula for the sum of the internal angles that depends only on the number of sides or vertices.



The first part of the answer is to guess.



We know that for a triangle ($n = 3$), the sum of the angles is 180°, and the sum does not depend on whether the triangle is acute, right or obtuse.

Continuing with the square ($n = 4$), we can draw the diagonal and observe that the sum of all the angles is twice 180° or 360°. The partition into two triangles can be carried out with any quadrilateral, it does not require any sides being equal.

From this we guess that the formula may be:

$$S_n = (n - 2) \cdot 180$$

And indeed, in going from $n = 4$ to $n = 5$ sides we can think of the pentagon as being a quadrilateral with an extra triangle.

And in the first figure, you can see that by adding the extra vertex to go to the $n + 1$-gon, we added a triangle, or perhaps you'd rather say than in going from $n + 1$ to n we lost a triangle.

In all cases, the difference between n and $n + 1$ is 180°.

The formula *seems* to work.

We can use induction to *prove* that it is correct.

The proof has two parts. We must verify the formula for a base case like the triangle, which we've done. You may wish to check that it works for the square as well, but that's not strictly necessary.

The second part of the proof is to verify that in going from n to $n+1$, we add another $180°$. The formula for n sides is $(n-2)180°$, adding another triangle gives:

$$(n-2)180° + 180°$$

That must be equal to what the formula gives for $n+1$ sides:

$$((n+1)-2)180°$$

Substituting x for $180°$ and equating the two, we have

$$(n-2)x + x = ((n+1)-2)x$$

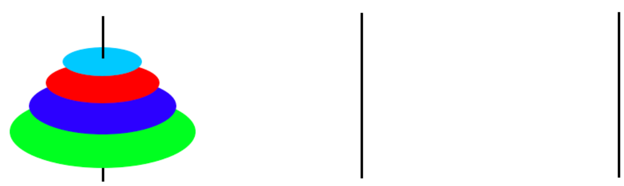
$$n - 2 + 1 = n + 1 - 2$$

$$n = n$$

which is certainly correct.

□

Towers of Hanoi



In this famous game the goal is to move a set of disks from one peg to another. Let us choose the one on the right as the target.

https://en.wikipedia.org/wiki/Tower_of_Hanoi

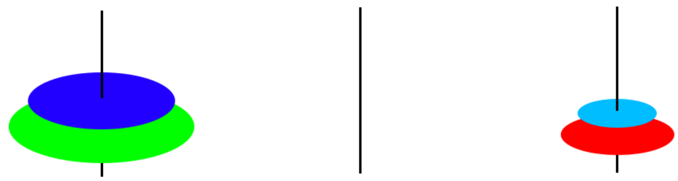
The rules are:

◦ Only one disk may be moved at a time.

◦ Each move consists of taking the upper disk from one of the pegs and sliding it onto another peg, on top of the other disks that may already be present on that peg.

◦ No disk may be placed on top of a smaller disk.

Here is an intermediate stage of the game:



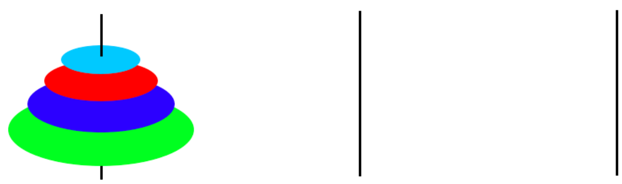
The next move is to place the blue disk on the middle peg. I think you can take it from there.

We can solve the puzzle for any number of disks n.

Proof:

By induction.

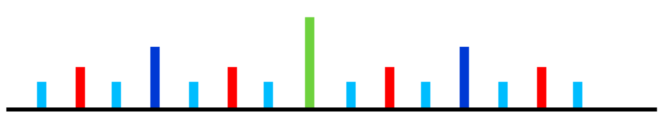
Start from the first position:



Suppose we know how to move $n-1$ disks from one peg to another. Move them to the middle peg, then move the nth disk to the right peg, then place all the $n-1$ disks on top. We have moved n disks.

The base case is to move the single light blue disk. That's trivial. The only thing to watch is if the number of disks is even or odd. If even, choose peg 2, otherwise peg 3.

Which peg is to be moved at each stage is shown in this graphic:



> The puzzle was invented by the French mathematician Édouard Lucas in 1883. There is a legend about a Vietnamese temple which contains a large room with three time-worn posts in it surrounded by 64 golden disks. The monks of Hanoi, acting out the command of an ancient prophecy, have been moving these disks, in accordance with the rules of the puzzle, since

> that time. The puzzle is therefore also known as the Tower of Brahma puzzle. According to the legend, when the last move of the puzzle is completed, the world will end.

summary

We can visualize an inductive proof as a kind of chain. We show that the base case is true, for some value of n. Then we show that if the formula works for n , it must work for $n + 1$.

> Mathematical induction proves that we can climb as high as we like on a ladder, by proving that we can climb onto the bottom rung (the basis) and that from each rung we can climb up to the next one (the step).

- Graham, Knuth and Patashnik

[There is a variant called *strong* induction where we know some statement is true for *all* $0 < k \leq n$ and use it to prove something about $n + 1$.]

A few more examples:

sum of digits and divisibility

It is very easy to check whether any number n is divisible by 9. Simply add up all the digits of say, 234783738:

$$2 + 3 + 4 + 7 + 8 + 3 + 7 + 3 + 8$$

$$= 5 + 1 + 1 + 1 + 1 + 1 + 0 + 8$$

$$= 1 + 8 = 9$$

Yes, 234783738 is a multiple of 9.

We propose that

$9|(10^n - 1)$ for all integers $n \geq 0$.

The statement $9|n$ means "9 divides n".

Suppose we know that $9|10^k - 1$ for some n. We mean that

$$10^k - 1 = 9x$$

for some x. Multiply by 10:

$$10 \cdot (10^k - 1) = 10 \cdot 9x$$

$$10^{k+1} - 10 = 9 \cdot 10x$$

$$10^{k+1} - 1 = 9 \cdot 10x + 9 = 9(10x + 1)$$

The right-hand side is clearly divisible by 9, and then so is the left-hand side.

The base case is $9|0$ which is true by definition but may be confusing. Try $n = 1$, then $9|(10 - 1)$ is certainly correct.

□

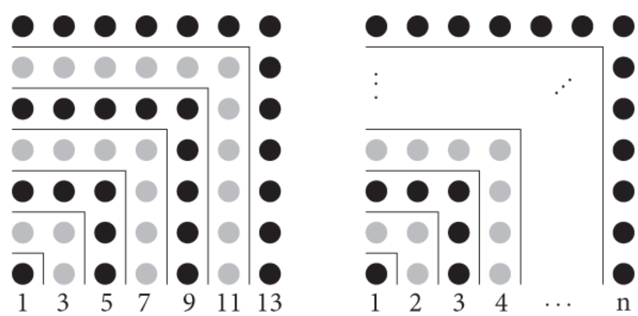
Given this, it is easy to show that the sum of digits method always works.

We demonstrated that this works for 3 previously.

Odd number theorem

Here is a simple but very useful inductive proof.

The *odd number theorem* says that the sum of the first n odd numbers is equal to n^2. Here is a "proof without words".



We prove this by induction.

$$(0 \cdot 2 + 1) + (1 \cdot 2 + 1) + \ldots \cdots + ((n - 1) \cdot 2 + 1) = n^2$$

Notice that the nth odd number is $2 \cdot (n - 1) + 1$.

Our formula says that

$$1 + 3 + 5 + \cdots + (2n-1) = n^2$$

If you like the summation style:

$$\sum_{k=0}^{n} 2k - 1 = n^2$$

As an example, the first five odd numbers are

$$1 + 3 + 5 + 7 + 9 = 25 = 5^2$$

So, if we consider the next odd number, n changes to $n+1$. The left-hand side gets another term: we add $2 \cdot (n+1) - 1$ to it. That is equal to $2n+1$.

To maintain the equality, add the same quantity to the right-hand side:

$$n^2 + 2n + 1 = (n+1)^2$$

Rearrange the result, and that's our formula back again. We have proved the inductive step.

To finish, note that the base case is simply

$$1 = 1^2$$

□

The binomial theorem gives the cofactors for a binomial expansion like:

$$(a+b)^5 = a^5 + 5ab^4 + 10a^2b3 + 10a^3b^2 + 5a^4b + b^5$$

We will prove this theorem using induction **here**.

proof of induction

According to Hamming, if you are not convinced by the ladder analogy, here is another proof that induction works:

Suppose the statement is not true for every positive (non-negative) integer. Then there are some false cases. Consider the set for which the statement is false. *If* this is a non-empty set, then it would have a least integer, which is m. Now consider the preceeding case, which is $m - 1$. This $(m - 1)$th case must be true by definition, and we know that there is such a case because as a basis for the induction we showed that there was at least one true case. We now apply the step forward, starting from this true case $m - 1$, and conclude that the next case, case m, must be true. But we assumed that it was *false*! A contradiction.

Therefore, there are no false cases.

□

Chapter 9

Sum of integers

In calculus, we will compute Riemann sums, and to do that we need to find formulas for the sum of squared integers, cubed integers, and so on. To keep it simple, let's start with the integers from 1 to n.

In a previous chapter we introduced the method called induction. Probably the most famous example of an inductive proof is that for the sum of integers.

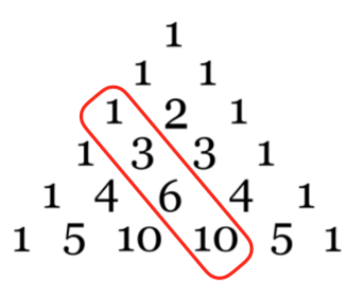
$$S_n = 1 + 2 + \cdots + n$$

proof

The numbers we seek are called the triangular numbers. These are

$$1, 3, 6, 10 \cdots$$

These are generated as the third diagonal of Pascal's triangle.:



Suppose someone has sent us, anonymously, a formula which they claim gives the sum of the first n integers, namely

$$S_n = \frac{n(n+1)}{2}$$

Assume the formula is correct for S_n. Add $n+1$ to both sides. The left-hand side becomes S_{n+1}, so we have:

$$S_{n+1} = \frac{(n)(n+1)}{2} + (n+1)$$

Rearranging:

$$\begin{aligned} &= \frac{n(n+1) + 2(n+1)}{2} \\ &= \frac{(n+1)(n+2)}{2} \end{aligned}$$

which is exactly what we'd get by substituting $n+1$ for n in the original formula.

Alternatively, sometimes it's clearer to assume the $n-1$ case and prove the formula is correct for n:

$$\begin{aligned} S_{n-1} &= \frac{n(n-1)}{2} \\ S_n &= \frac{n(n-1)}{2} + n \\ &= \frac{n(n-1) + 2n}{2} \\ &= \frac{n(n-1+2)}{2} = \frac{n(n+1)}{2} \end{aligned}$$

So we have proven that if the S_n formula is correct, then so is the one for S_{n+1}.

How do we know that S_n is correct?

Just check the *base case*:

$$S_1 = \frac{1(1+1)}{2} = 1$$

Since S_1 is clearly correct, S_2 must be also, and this continues all the way to S_n.

$$S_1 \Rightarrow S_2 \Rightarrow \ldots S_{n-1} \Rightarrow S_n \Rightarrow S_{n+1}$$

Therefore, it must be true for *every* integer n.

$\square$

There is a famous story about Gauss. As a schoolboy, he "saw" how to add the integers from 1 to 100 as two parallel sums.

Added together horizontally, these two series must equal twice the sum of 1 to 100.

But vertically, we notice that each sum is equal to $n+1$, and we have n of them.

$$
\begin{array}{ccccc|c}
1 & 2 & \ldots & 99 & 100 & S_n \\
100 & 99 & \ldots & 2 & 1 & S_n \\
\hline
101 & 101 & & 101 & 101 &
\end{array}
$$

So, again

$$2S_n = n(n+1)$$

$$S_n = \frac{1}{2}\, n(n+1)$$

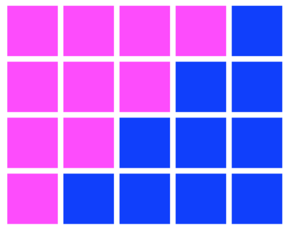
For $n = 100$ the value of the sum is 5050, which is what Gauss wrote on his slate and presented to the teacher immediately on being given the problem as a make-work exercise.

One way of looking at this result is that between 1 and 100 there are 100 representatives of the "average" value in the sequence, which (because of the monotonic steps) is $(100+1)/2 = 50.5$.

Or alternatively, view the sum as ranging from 0 to 100 (with the same answer). Now there are 101 examples of the average value $(100+0)/2 = 50$).

proof without words

Here is a striking *visual proof* of the formula to obtain T_n, the n^{th} such number. The total number of circles in the figure below is $n \times (n+1)$ and this is exactly two times the sum of the integers from 1 to n.



$$2S = n(n+1)$$

Derivation using sums

It seems a shame to spoil such a beautiful proof "without words" as the one above by saying anything more, but I can't resist. I'd like to derive the equation we have been using using algebra. The general method will help us later.

For any number, and in particular, any integer k it is true that

$$(k+1)^2 = k^2 + 2k + 1$$

So consider what happens if we sum the values from $k = 1 \to n$ for each of these terms

$$\sum_{k=1}^{n} (k+1)^2 = \sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} 2k + \sum_{k=1}^{n} 1$$

If the equation is valid for any individual k, then the sum is also valid, plugging in all k up to n.

Rearranging

$$\sum_{k=1}^{n} (k+1)^2 - \sum_{k=1}^{n} k^2 = \sum_{k=1}^{n} 2k + \sum_{k=1}^{n} 1$$

Now think about the left-hand side in our equation.

$$\sum_{k=1}^{n} (k+1)^2 - \sum_{k=1}^{n} k^2$$

We have a bunch of terms starting with 2^2:

$$2^2 + 3^2 + \cdots + n^2 + (n+1)^2$$

we also have a bunch of terms to subtract starting with 1^2:

$$1^2 + 2^2 + 3^2 + \cdots + n^2$$

Almost everything cancels. This is called a "collapsing" or "telescoping" sum. We have

$$(n+1)^2 - 1 = n^2 + 2n$$

Bringing back the right-hand side we obtain:

$$n^2 + 2n = \sum_{k=1}^{n} 2k + \sum_{k=1}^{n} 1$$

We can bring the constant factor 2 out of the sum, and also, we recognize that the sum of the value 1 a total of n times is just n.

$$n^2 + 2n = 2\sum_{k=1}^{n} k + n$$

Subtract n from both sides and divide by 2:

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

That's it!

Part III

Lines and triangles

Chapter 10

Lines and angles

Euclid and the postulates

Greek geometry starts hundreds of years before Euclid, who was a contemporary of Alexander the Great (356-323 BC).

We know that Euclid lived after Plato (d 347 BC), and before Archimedes (b 287 BC). Except that he worked in Alexandria, all other details of his life and death are shrouded in mystery.

After more than 2000 years, Euclid's book *Elements* is still an excellent place to begin surveying the foundations of geometry. It is a textbook, an organized collection of what was known about the subject at the time, or at least, everything that a well-educated student should know.

This book of consists of *propositions*, which include constructions (geometric figures) drawn with a pencil on a piece of paper, using a straight-edge or a compass or both.

Often proofs of propositions build on previous items in the book. Euclid does not prove everything. Bertrand Russell was famously disappointed about that.

Here are Euclid's first three postulates — statements that are assumed to be true:

∘ A straight line segment can be drawn joining any two points.

∘ Any straight line segment can be extended indefinitely in a straight line.

∘ Given any straight line segment, a circle can be drawn having the segment as the radius and one endpoint as the center.

Let us assume these as well. We will use them often.

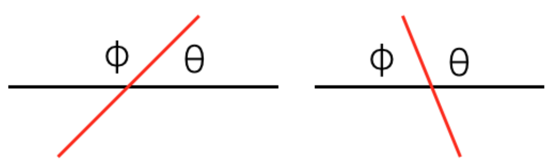
We finesse the difficulty in defining what is meant by *straight* in the real world. If you've ever done any carpentry, for example, you probably know that unknown edges are determined to be straight by comparison with a known straight edge. We use an imaginary perfect straight-edge and draw a straight line as "the shortest distance between two points".

The fourth postulate is:

∘ All right angles are congruent, that is, equal to each other.

This one prompts a different question: what is a *right angle*?

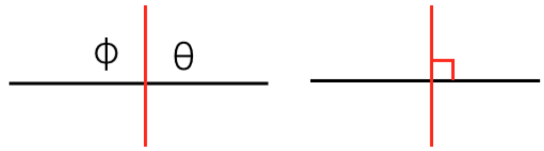
If one line segment (piece of a line) is drawn crossing a second one, forming their *intersection*, let us refer to the angles formed on the same side of a line as *supplementary* angles (also sometimes called adjacent angles).



On the left, one of the angles, ϕ, is larger than the other one, θ. On the right, we have $\phi < \theta$.

The third possibility is that $\theta = \phi$. The definition of a right angle is that

∘ if two supplementary angles are equal, they are both right angles.



A right angle is frequently designated by drawing a small square, as seen in the right panel above.

Regardless of whether $\theta < \phi$, $\theta > \phi$, or $\theta = \phi$, the sum of the two angles $\phi + \theta$ is equal to two right angles or 180 degrees.

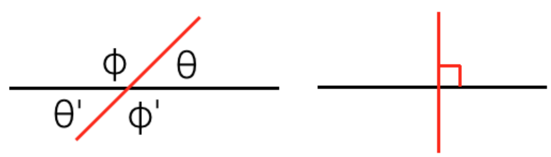
There is nothing particularly special about using 90 degrees as the measure of a right angle, 180 degrees for any two supplementary angles including right angles, or 360 degrees for one whole turn.

Well, there is one thing: there are *approximately* 360 days in a year, which marks the sun's track across the sky.

In his book, *Measurement*, Lockhart adopts the convention that a whole turn is equal to 1.

Later, we'll see that one whole turn can be defined using a different unit of measure as 2π *radians*, and that convention turns out to be quite important for calculus.

Now, consider those angles lying below the horizontal:



We said that the sum of the two angles $\phi + \theta$ is equal to two right angles, but so are the sums $\theta' + \phi$ and $\theta + \phi'$, for the same reason. As a result

$$\phi + \theta = \theta + \phi'$$

We conclude that $\phi = \phi'$ and $\theta = \theta'$.

This is called the *vertical angle theorem*.

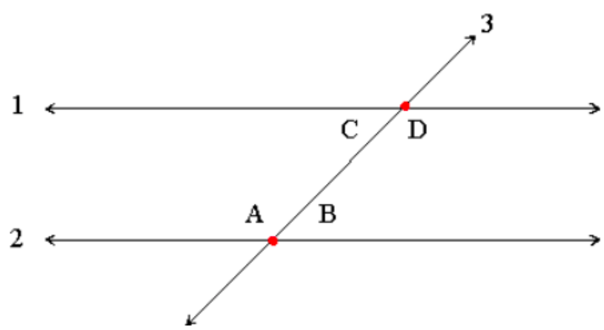
On the right, if any one of the angles where two lines cross is a right angle, then all four are right angles.

parallel postulate

So far, all this seems rather obvious. The fifth and final postulate is more subtle:

∘ If two lines are drawn which intersect a third in such a way that the sum of the inner angles on one side is less than two right angles, then the two lines inevitably must intersect each other on that side if extended far enough.

Line 1 and line 2 are parallel, if and only if, $A + C = B + D = 180 = 2$ right angles.



This postulate is equivalent to what is known as the parallel postulate.

http://mathworld.wolfram.com/EuclidsPostulates.html

But we also know from the properties of two lines given above that the supplementary angles $A + B = 180$ add up to 180 degrees. So

$$A + C = 180 = A + B$$

and then

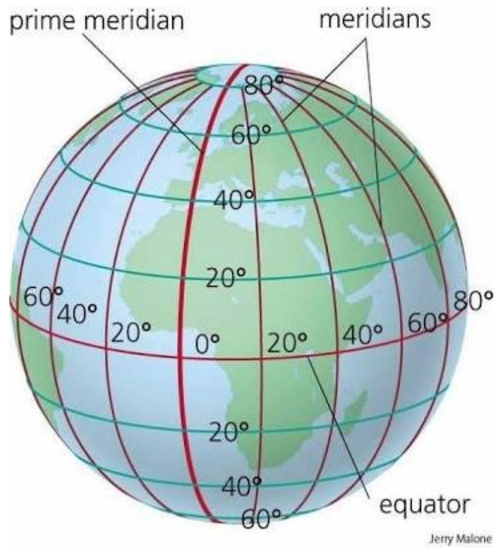
$$C = B$$

This is called the theorem on *alternate interior angles* between two parallel lines.

We note that adoption of this postulate is a choice. The definition works for geometry in the flat plane.

But, consider a familiar situation where this is not true. Suppose we are doing geometry on the surface of a sphere, such as the earth.

Then, two adjacent lines of longitude can be drawn so as to cross the equator at right angles, and the lines are parallel there, but they will meet (intersect) at the poles.



The parallel postulate only holds for geometry on a *flat* surface.

axioms

Euclid also lists five axioms, things which are assumed. Here are two examples:

∘ Things that are equal to the same thing are also equal to one another.

∘ If equals are added to equals, then the wholes are equal.

These seem quite reasonable.

We will see how to proceed from the postulates and axioms to various proofs. *Given these assumptions*, we can prove theorems that must be true.

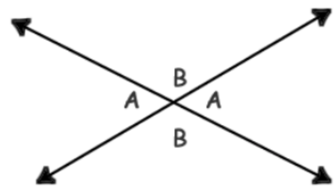
Thales

I'm a big fan of William Dunham's books — several of them are listed in the References.

Dunham has written a lot about the history of mathematics in Greece, starting with Thales (624-546 BC), who was from a Greek town called Miletus on the coast of Asia Minor (modern Turkey). He lived long before Euclid (about 300 years before, 600

BC). Although none of his writing survives, it is believed that Thales proved several early theorems including one we saw above.

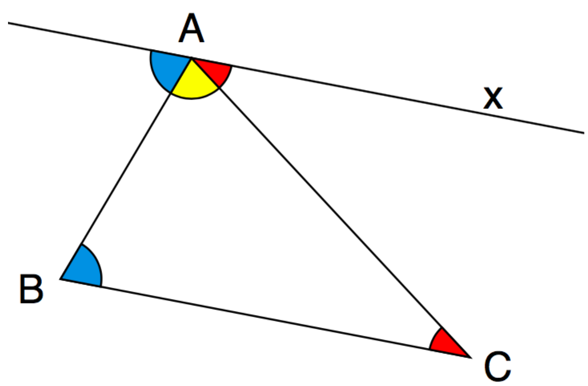
∘ The vertical angles formed by two straight lines crossing, are equal.



This theorem, which we already proved, depends on a property of straight lines. In the proof, we used the axiom "equals added to equals are equal", alternatively "equals subtracted from equals are equal."

A very important theorem attributed to Thales is the following:

∘ The angle sum of a triangle is equal to two right angles.



This theorem depends on the ideas we developed above. Draw a line segment through A parallel to BC. Now, use alternate interior angles and follow the colors to the result.

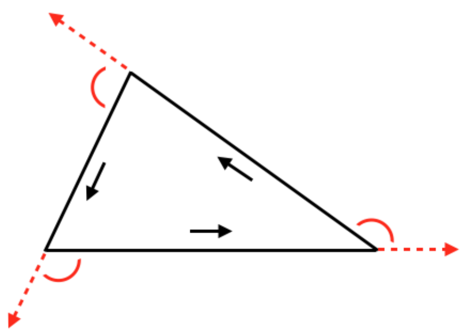
We will see one more theorem ascribed to Thales (it is actually called Thales' theorem), in the next chapter. It is about isosceles triangles (two sides equal).

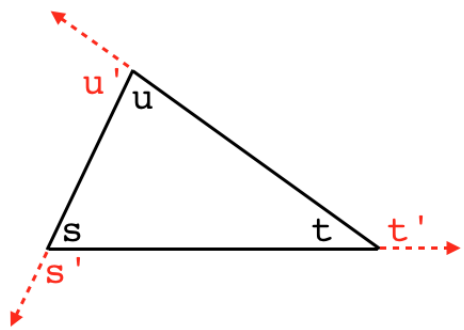
another proof

Here is a different proof of the theorem on the sum of angles in a triangle adding to 180 degrees. It never hurts to re-prove things by a different method. It serves as a check on both the result, and the methods.

Imagine walking around the perimeter of a triangle in the counter-clockwise direction. At each vertex we turn left by a certain number of degrees, θ, called the exterior angle. After passing through all three vertices, we must end up facing in the same direction as we started.

The sum of the exterior angles is 360°.





$$s' + t' + u' = 360$$

In addition, for each vertex, the interior angle plus the exterior angle add up to 180 degrees. If we add all three pairs, we obtain

$$(s + s') + (t + t') + (u + u') = 3 \cdot 180 = 540$$

By subtraction

$$s + t + u = 180$$

summary

Make sure you understand each of these theorems:

- supplementary angles
- vertical angles

- alternate interior angles
- triangle sum

Chapter 11

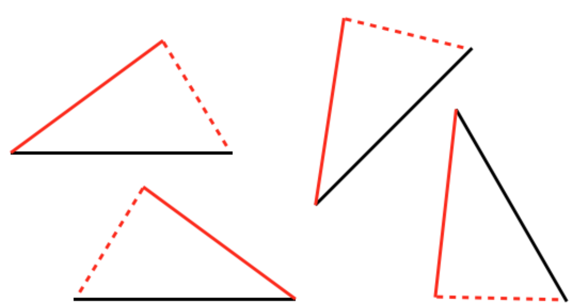
Congruent triangles

congruent triangles

∘ Two triangles are *congruent* if and only if they have the same three side lengths. This is often abbreviated SSS (side-side-side).

As we'll see, some other equalities are equivalent to (they imply) congruence and SSS equality.

By this definition, a triangle and its mirror image are congruent. The three triangles shown below are all congruent, even though two are flipped (they are mirror images of the other two).



Having the same three sides means that the shape is the same, and all three angles are the same — the shapes are superimposable, with the proviso that we allow the shape to be flipped over.

similar triangles

Some triangles are *similar* but not congruent.

Similarity means that the three angles are the same but the triangles are of different overall sizes. We might say that they are the same but *scaled* differently.

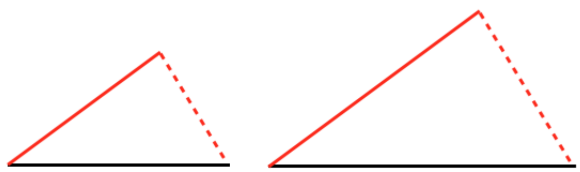
We can call this AAA (angle-angle-angle). For similar triangles, the three corresponding pairs of sides are in the same proportions, but re-scaled by a constant of proportion.

∘ Two triangles are similar if they have the same three angles.

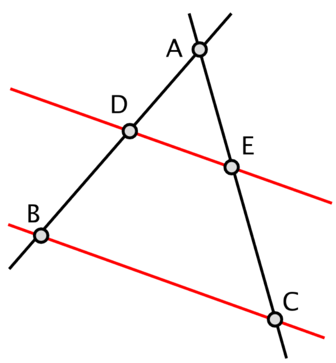
Because of the angle sum theorem, if any two angles of a pair of triangles are known to be equal, then the third one must be equal as well.

∘ Two triangles are similar if they have two angles known to be equal.

Similar triangles have their sides in the same proportion.



Given any triangle, draw a line parallel to one side, which also joins the other two sides. The new triangle with that side as its base is similar to the given triangle. Similarity means that all the angles are equal. This is easily proved using the theorem on alternate interior angles.



In this example, these ratios are all equal

$$\frac{AD}{AB} = \frac{AE}{AC} = \frac{DE}{BC}$$

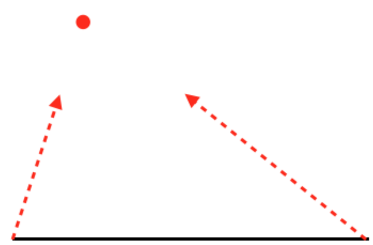
$$\frac{AD}{DB} = \frac{AE}{EC} = \frac{DE}{BC - DE}$$

In addition to SSS (side-side-side), there are three other conditions that lead to congruence of two triangles when they are satisfied, namely

- SAS (side-angle-side)
- ASA (angle-side-angle)
- AAS (angle-angle-side)

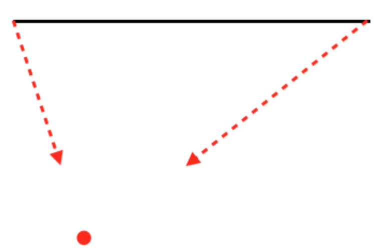
constructions

The way I think about these conditions is to imagine trying to construct a triangle from the given information, and ask whether it is uniquely determined. Suppose we know ASA. The situation is thus:



Draw the known side, then using the known angles, start two other sides from the ends of that side. They must cross at a unique point.

But... actually, if we start the two lines pointing below the horizontal, there is another solution, the mirror image. This triangle is also congruent to the one above.

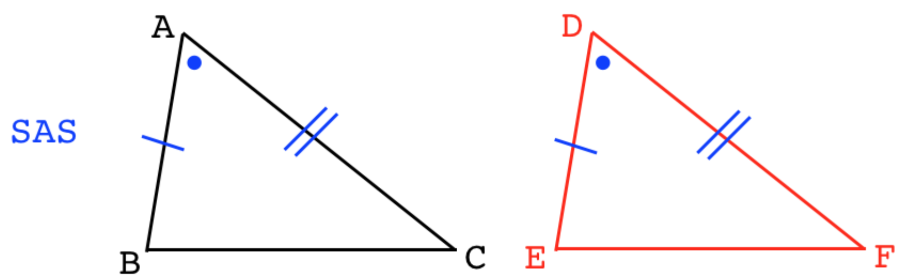


If we know two angles we also know the third, because they must add to 180 degrees. For this reason, ASA and AAS imply that we have exactly the same information, because we know all three angles and (this part is important) we also know *which* two angles flank the known side.

Alternatively, it is enough to know which angle faces the known side.

SAS, ASA, AAS but not ASS

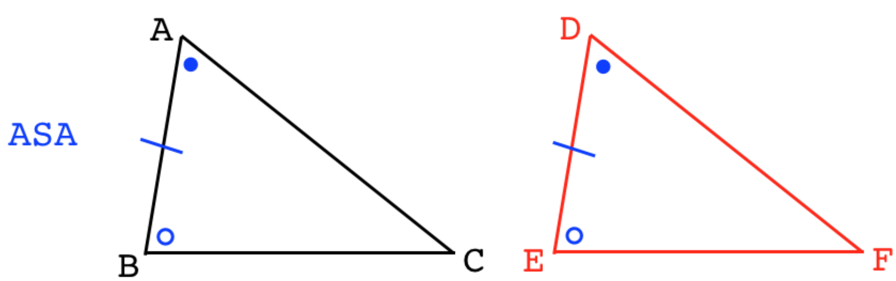
SAS is very commonly used to prove congruence.



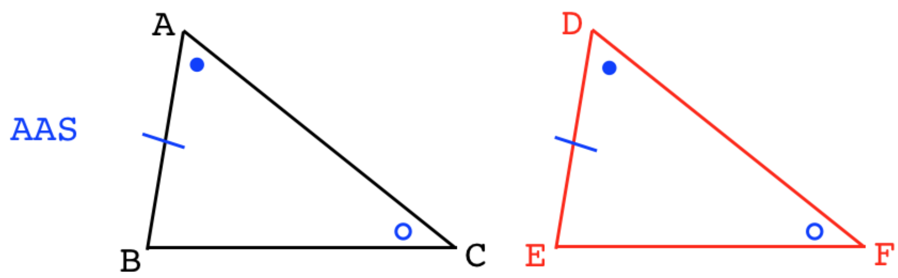
In this diagram, sides of equal length are indicated by one or more hash marks. Equal angles are indicated by dots (another common method is to draw an arc with a hash across it).

The other methods for proving congruence use two equal angles and a side. Two equal angles imply the third angle is also equal (since they add to a half-circle or 180 degrees), so the two triangles are similar. To prove they are congruent, It is important that the equal sides are flanked by the same angles, or equivalently, are opposite the same angle.

These methods using two angles are referred to as ASA

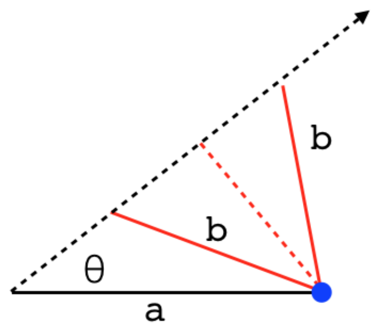


and AAS.



one that doesn't work

There is one set of three that doesn't work, that is ASS (angle-side-side).



Here we know sides a and b and the angle θ adjacent to a and facing opposite side b. Imagine b swinging on a hinge at the blue dot. If $b < a$, there are two points where b can intersect with the side projecting from angle θ. There is no unique solution, so the triangle is not determined.

If it had been the case that $b > a$, or alternatively that b formed a right angle with the third side, then the triangle *would* be determined.

I think Tony Randall said it best

https://www.youtube.com/watch?v=KEP1acj29-Y

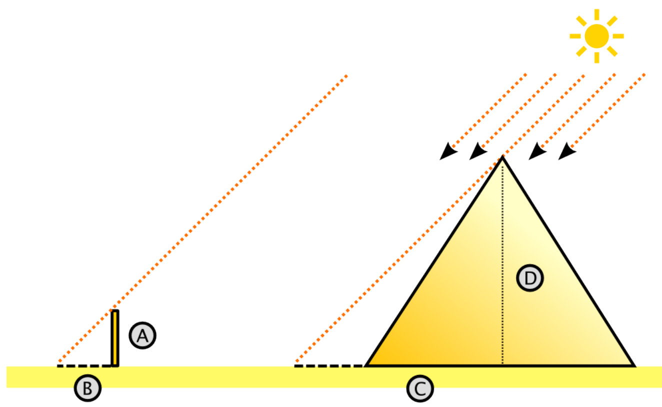
pyramid height

As we said, Thales was from Miletus and he lived around 600 BC. Thales is believed to have traveled extensively and was likely of Phoenician heritage. As you probably know, the Phoenicians were famous sailors who founded many settlements around the Mediterranean.

They competed with the mainland Greeks and later with the Romans for colonies, and their major city, Carthage, was destroyed much later by the Romans in the third Punic War.

During his travels, Thales went to Egypt, home to the great pyramids at Giza, which were already ancient then. They had been built just around around 2560 BC (dated by reference to Egyptian kings) and were already 2000 years old at that time!

The story is that Thales asked the Egyptian priests about the height of the Great Pyramid of Cheops, and they would not tell him. So he set about measuring it himself. The current height is 480 feet. He used similar triangles.



Chapter 12

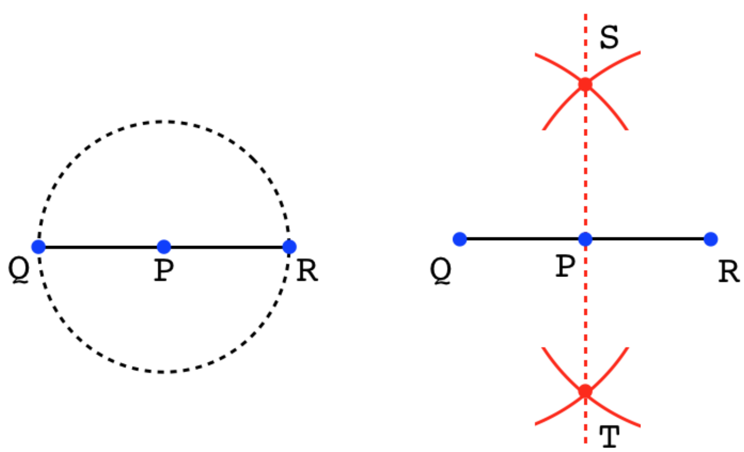
Perpendicular bisector

perpendicular at a point

We often will wish to construct a line or line segment perpendicular to another line segment. This may be specified to occur at a particular (given) point, either on the line, or not on the line.

For the first case, consider the horizontal line below (left panel) and suppose we know a point on the line P and wish to construct the vertical line through P. The procedure is to use the compass to mark off points Q and R on the line an equal distance from P.

This can be done by drawing a circle with center at P.



Using the compass again (right panel), find S equally distant from Q and R ($QS = RS$). This can be done by using the compass to draw slightly larger circles of the same radius on centers Q and R.

The line segment SP will be perpendicular to the line containing QPR.

There is a restriction in Euclid's Elements to a collapsible compass, one which loses its setting when lifted from the page.

That might make it impossible to find S as just stated. However, one can simply draw a circle of radius exactly equal to QR centered at Q and another one with the same radius, centered at R.

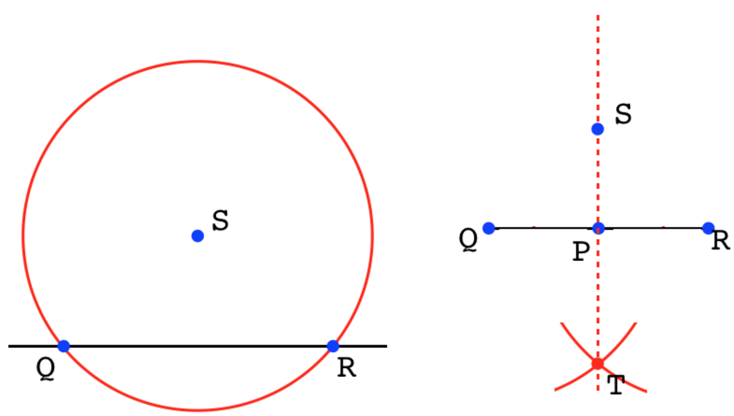
bisect a line segment

Suppose that we had not known the point P when we started the procedure above, but already had two points Q and R.

Then the line through S and T crosses QR at its midpoint, and we have found the point P that bisects QR.

perpendicular through a point

Alternatively, suppose we know the line and the point S but not P, and we wish to construct a vertical through the line that also passes through S. Find Q and R on the line an equal distance from S ($QS = RS$), as radii of a circle centered at S (left panel, below). Their exact position is unimportant.



Also find T such that $QT = RT$, using circles with radius QR centered at Q and R, as before.

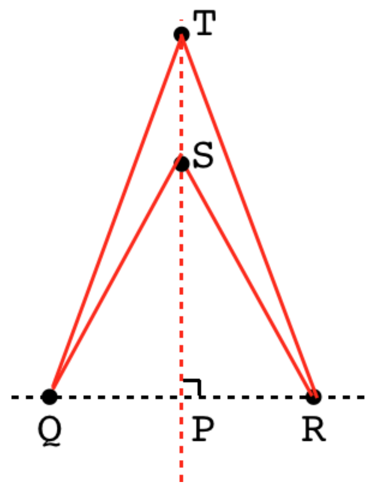
The line segment ST is perpendicular to the line segment containing QR, and passes through S, as required.

Also, see the video at the url:

https://www.mathopenref.com/constperpextpoint.html

bisector properties

Using what we've just learned, suppose we know two points Q and R. We find the point P equidistant between them and construct the perpendicular bisector PS. Then the two sides SQ and SR have equal length. Triangle $\triangle SQR$ is isosceles.



Proof.

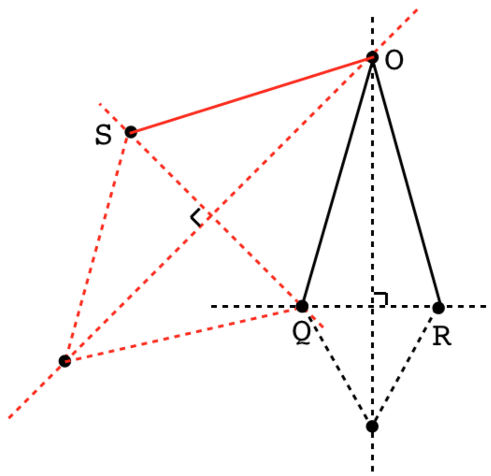
By construction, $PQ = PR$, $\angle SPR$ is a right angle, and side SP is shared. Hence the two triangles $\triangle SPQ$ and $\triangle SPR$ are congruent, by SAS.

$\square$

This is true for *any* point on the line drawn through S and P. For example, $TQ = TR$ in the figure above.

three points

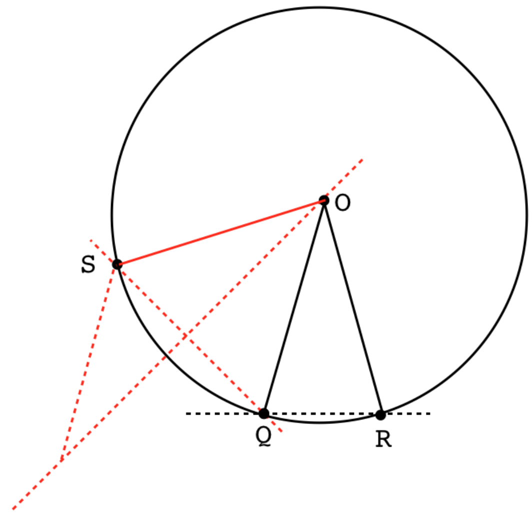
Now, suppose we have three points: Q, R and S. We find the perpendicular bisector of QR and also, the perpendicular bisector of QS. Extend them to where they meet, at point O.



What can we say about point O?

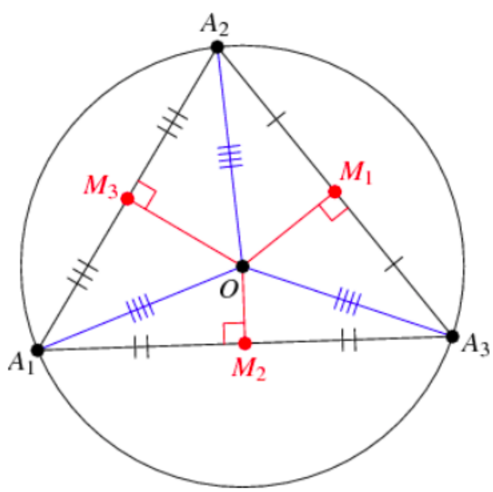
- O is equidistant from Q and R.
- O is also equidistant from Q and S.

Therefore, $OQ = OR = OS$. If we draw a circle on center O, it will pass through all three points.

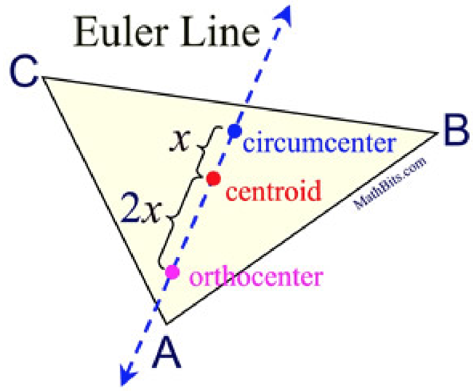


circumcenter

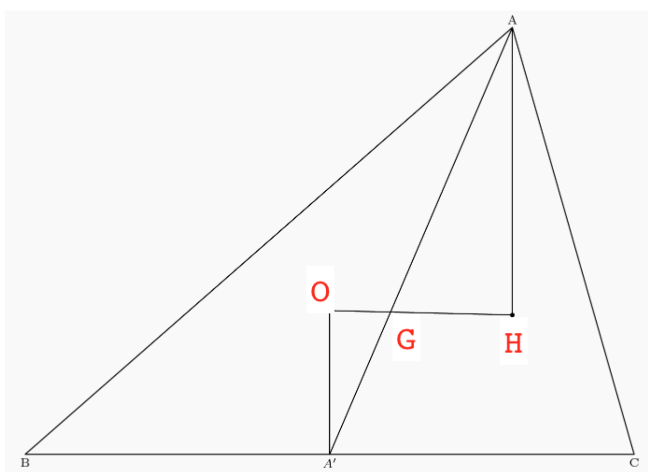
The point where the perpendicular bisectors cross has a special name, it is called the circumcenter.



There are other special points where interesting circles can be drawn.



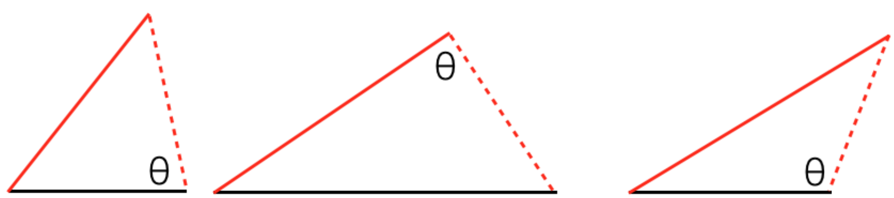
- orthocenter H: where altitudes cross
- centroid G: where angular bisectors cross
- circumcenter O: where perpendiculars to side bisectors cross



Chapter 13

Special triangles

There are several adjectives one can use to describe different types of triangles. For example: acute, right, and obtuse.



The acute triangle (left) has all three angles smaller than a right angle. The right triangle, naturally, has one right angle (it can't have two — why?)

We'll say a lot more about right triangles later.

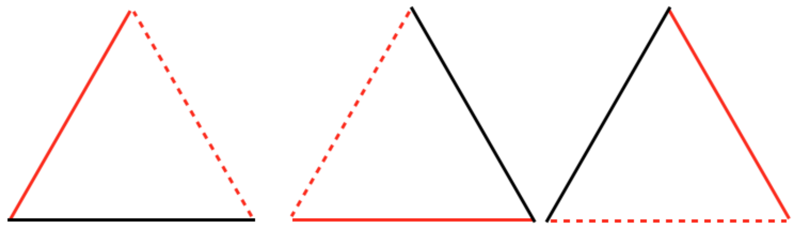
Finally, an obtuse triangle has one angles larger than a right angle (right, above).

symmetry

One can also talk about the situation where either two sides, or all three sides, have the same length.

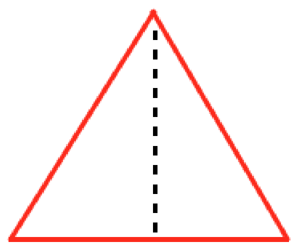
An equilateral triangle has all three sides the same, while an isosceles triangle has two sides the same length.

The most important consequence of three sides equal for an equilateral triangle, is rotational symmetry. Three turns of 120 degrees, and we're back where we started.



The implication of that is that the three angles are also equal. There is no reason to choose one larger than the other.

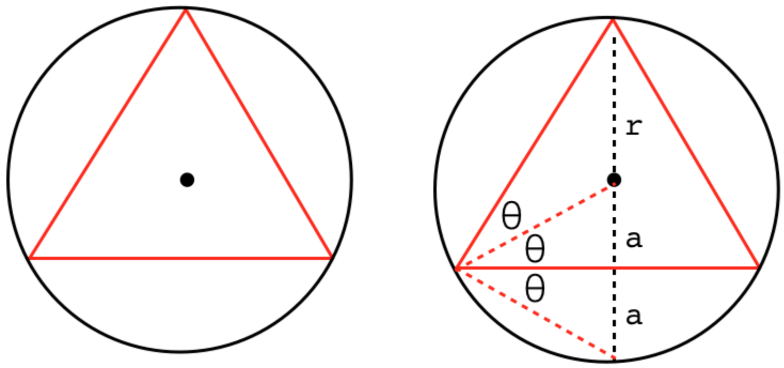
In the next figure the two smaller triangles obtained by dividing in half an equilateral triangle (all sides equal), are congruent.



By divide in half, we mean bisect the base and draw the line from the top vertex. We have SSS.

circumscribed circle

Here is a fun construction for an equilateral triangle. Any triangle fits into a unique circle. We will prove this elsewhere.

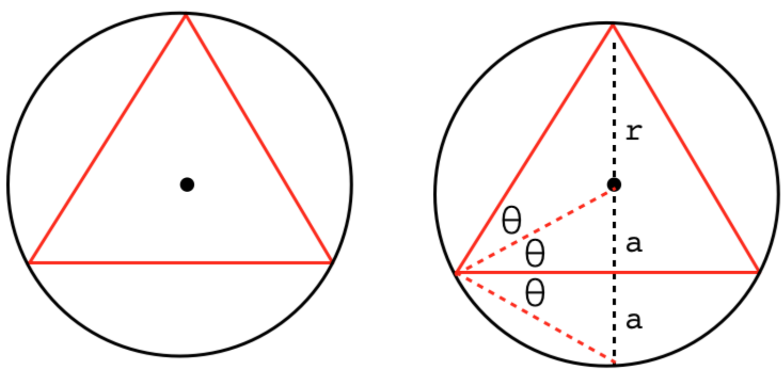


If we draw the radius to a vertex of the triangle, and then to the end of the diameter, it looks tantalizing.

Here is the proof.

First, the radius is an altitude, and it divides the vertex angle in half, as we've been saying. So that accounts for two of the angles labeled θ. The third comes about because any triangle with two points on the diagonal and a third anywhere on the circle is a right triangle. We'll prove that when we get to circles.

As a result, by measure θ is 30 degrees. Since a right triangle is 90, we assign the third θ.



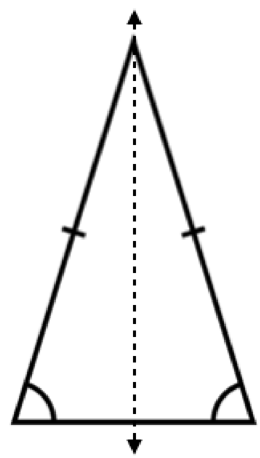
So now we have a smaller angle of a right triangle, and a shared side. The two triangles are congruent. That accounts for the duplicated a in the figure.

Thus, the altitude of the equilateral triangle is $3/4$ of the diameter of the circle that just encloses it. And the point where the altitudes meet in an equilateral triangle is $1/3$ of the way up from the base, since $r = 2a$.

We will see that such a point where the altitudes cross is unique and exists for any triangle, and it always has the same measure as a fraction of the altitude. This is called Ceva's theorem.

theorem from Thales

∘ The base angles of an isosceles triangle are equal. Also, if the two base angles are equal, the triangle is isosceles.



My favorite proof of this theorem is from reflective or mirror image symmetry (above).

Start with the two sides equal and draw a line to the midpoint of the base opposite. The figure has reflective symmetry, thus the angle is bisected.

We prove this more carefully now.

notation

The Greeks, including Euclid, adhere to certain conventions. For example, points are always labeled with letters, line segments are referred to by the endpoints, and angles by the line segments that determine them, as in $\angle ABC = \angle DEF$.

I don't know about you but I find myself tracing out angles from the three points, again and again.

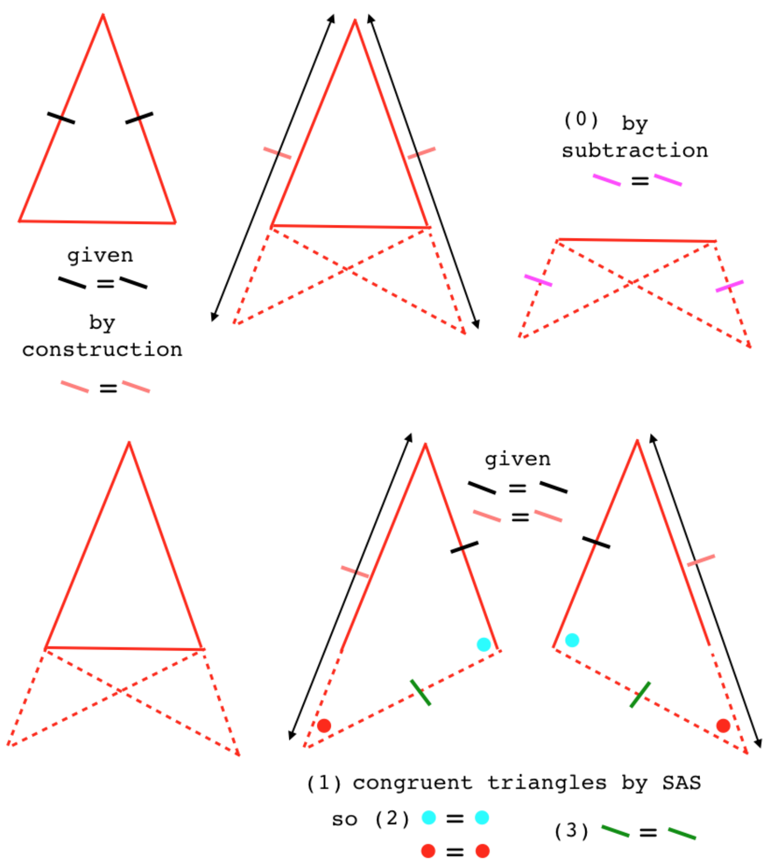
We could give labels to the angles like $\alpha, \beta \dots$, and to the sides opposite vertices as a opposite A and so on.

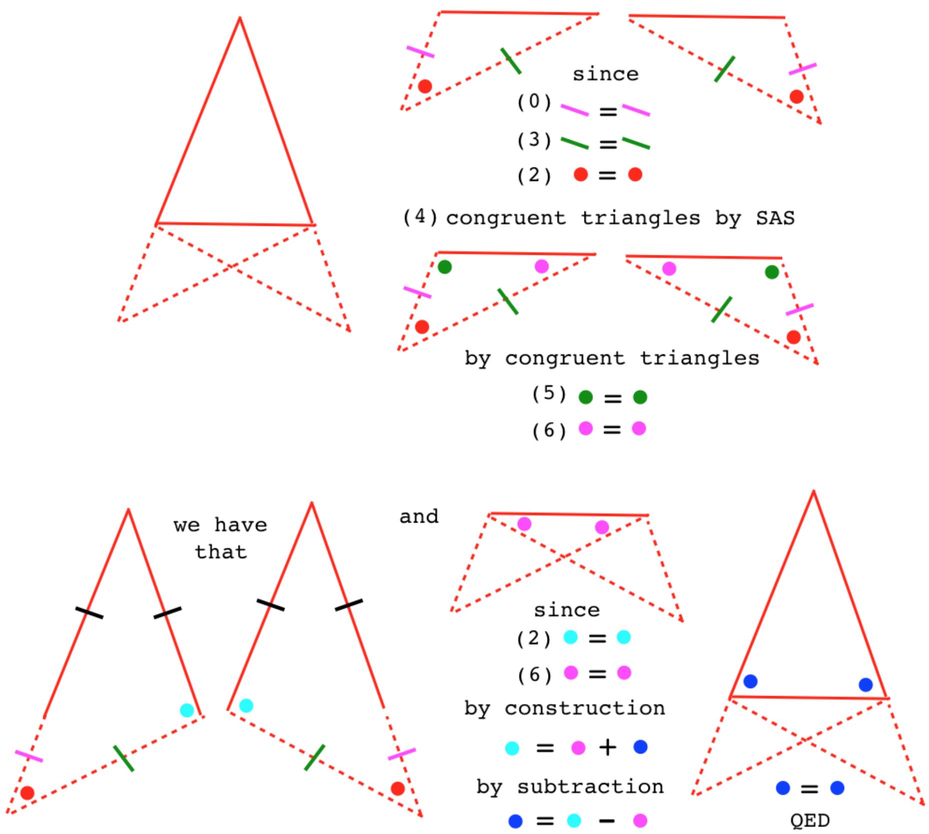
But let's go with something even more dramatic. Dispense with labels altogether and use colored dots for equal angles and colored bars for equal lengths. We give the famous proof of Thales' theorem from Euclid's *Elements*.

Prop. I.5

In isosceles triangles the angles at the base equal one another, and, if the equal straight lines are produced further, then the angles under the base equal one another.

In what follows, all the pieces are with reference to the initial construction, first figure, below.



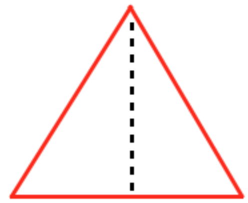


$\square$

The theorem says that the base angles are equal $\iff$ the two sides sides are equal (not the base).

The symbol $\iff$ means *if and only if*, so $A \iff B$ means that both $A \to B$ and $B \to A$.

Above we said that in this figure the two smaller triangles obtained by dividing an equilateral triangle in half, are congruent. The dotted line is called an *altitude* of the triangle.



An altitude meets the side opposite in a right triangle.

Because the left and right sides of the original triangle are equal, the base angles are equal, by the property of isosceles triangles which we just proved. The angles where the altitude meets the base are both right angles, by symmetry and by the definition of the altitude.

Therefore we have AAS, and the two halves are congruent.

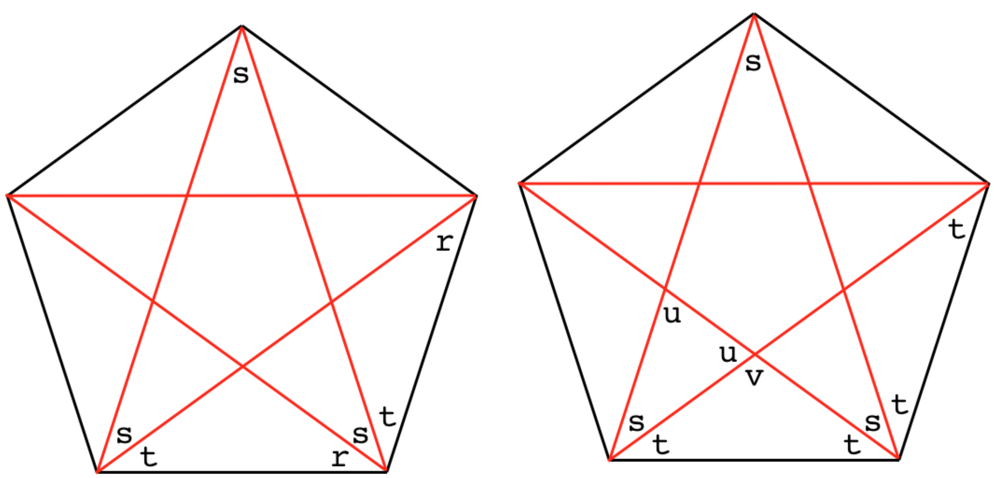
So, the two angles at the top where the altitude meets the sides are also equal (as the third angle with the other two angles determined).

Chapter 14

Pentagon

In this chapter we explore some properties of a regular pentagon. The pentagon has five-fold rotational symmetry. Draw all of the internal chords of the figure and label a few angles.

By rotational symmetry each of the five vertices of the pentagon has the same three components, the central one labeled s, and two flanking ones r and t.

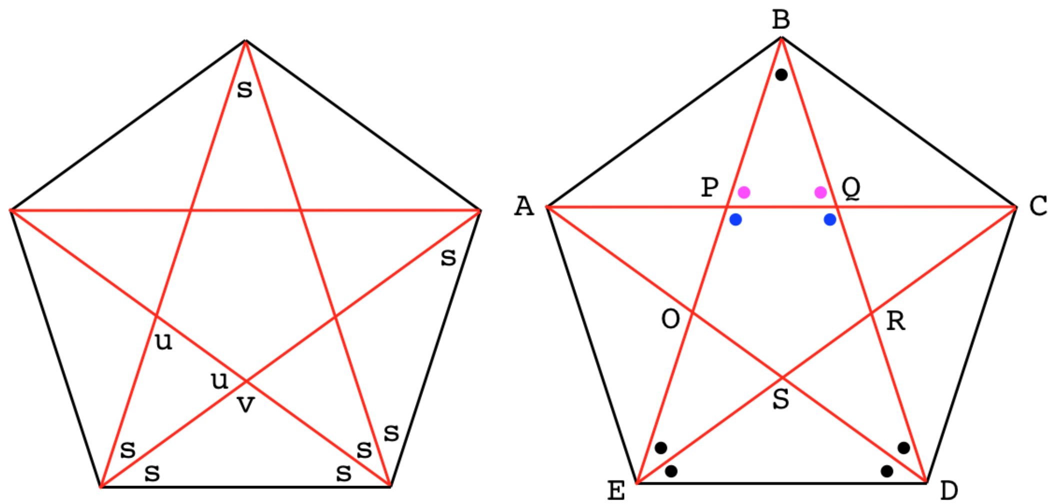


But $r = t$, by Thales' theorem, using two sides of the pentagon. Hence we relabel, immediately (right panel).

We compute two triangle sums:

$$3s + 2t = 4t + s, \qquad 2s = 2t$$

Hence, $s = t$. Relabel t as s (left panel, below):



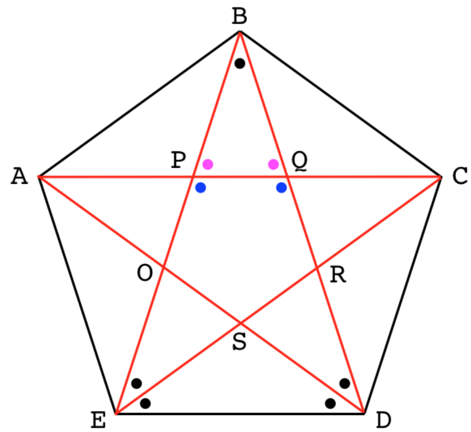
We observe that $5s = \pi$. We again compute two triangle sums:

$$5s = v + 2s, \qquad v = 3s$$

$$5s = 2u + s, \qquad u = 2s$$

Since $v = 3s$, its measure is the same as the vertex angle of the pentagon. Thus the inner figure is also a regular pentagon.

We do not need the angle labels any more, just the equalities.

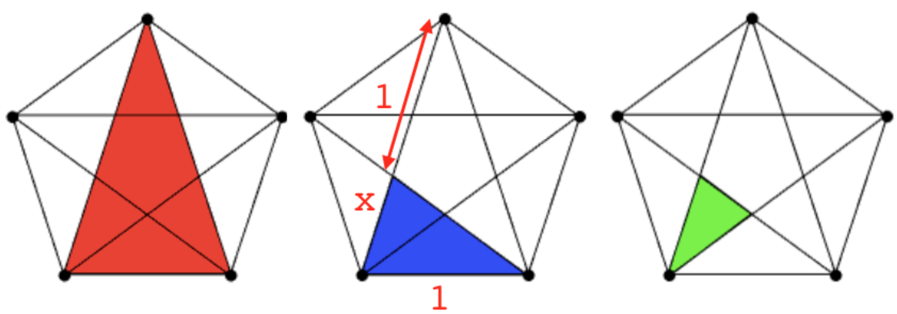


$\triangle BED$ is similar to $\triangle BPQ$. Hence the side of the inner pentagon is in the same measure to the side of the original pentagon as the ratio of BP to BE.

Now observe that AC is parallel to ED, because they have the equal alternate interior angles of two parallel lines. (Or simply add the included angles). Our drawing is filled with parallelograms.

One can draw two types of isosceles triangles using the chords and sides of the pentagon. One is tall and skinny, the other short and fat.

The tall, skinny type have base angles equal to s. Here are three examples:



If we take the side length of the original pentagon to be 1, then all the edges of regular parallelograms in the figure also have side length 1, so the long side length of the red triangle is 1 plus some other value, equal to the base of the blue triangle. Let's call that extra part, x.

We use the fact that red and blue are similar and form the ratio ϕ of the long side to the base (red on the left, blue on the right):

$$\frac{1+x}{1} = \frac{1}{x}$$

Rearrange:

$$x^2 + x - 1 = 0$$
$$x = \frac{-1 \pm \sqrt{1+4}}{2}$$

Of course, ϕ is the golden ratio where we have taken the positive branch of the square root:

$$\phi = 1 + x = \frac{1+\sqrt{5}}{2}$$

From the similarity of the green triangle, if the side of the inner pentagon is y:

$$\frac{x}{y} = \phi$$

$$y = \frac{x}{1+x}$$

Chapter 15

Euclid's Elements

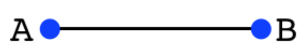
In this chapter we will study some nine or ten *Propositions* from the first volume Euclid's *Elements*. We also prove the *external angle theorem*.

The book was put together as a compendium of geometry for students. One thing we will see is how the propositions build on one another.

The first three propositions are *constructions*, e.g. the very first asks us to construct a triangle with all three sides equal, an equilateral triangle. The first statement below is Euclid's voice.

Prop. I.1

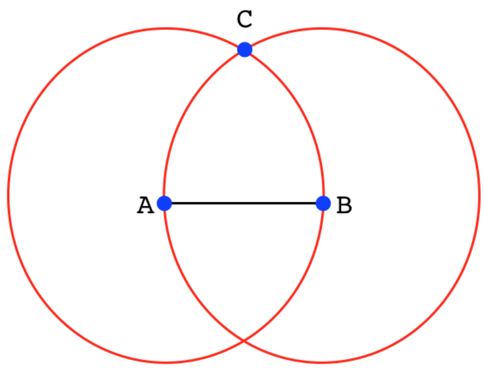
To construct an equilateral triangle on a given line segment.



The tools we have are a straight-edge and a compass. The compass is collapsible, meaning that it cannot be used to transfer distances since it loses its setting when lifted from the page. As we'll see in the next part, this is a problem with a solution.

Euclid was smart enough to know about compasses and how to set them. The idea he had was this: to make the fewest possible assumptions. A non-collapsible compass was a luxury he didn't need, since he could accomplish the same end without it, as we will see.

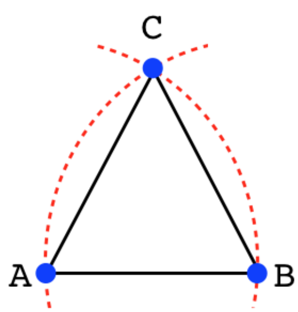
The first step is to draw two circles on centers A and B.



The circles are drawn with each radius equal to the line segment AB. It is a property of circles that all points on the circle are at the same distance from the center. Thus all points on the left-hand circle are equidistant from A, and all points on the second one are equidistant from B.

Therefore, the point C where the circles cross is equidistant from *both* A and B.

For this, we don't really need the entire circles, just the part where the arcs cross at C.



Now use the straight edge to draw $\triangle ABC$. Since $AC = AB$ and $BC = AB$, we know that $AC = BC$. The triangle is equilateral.

We put a little box to show that the proof is complete.

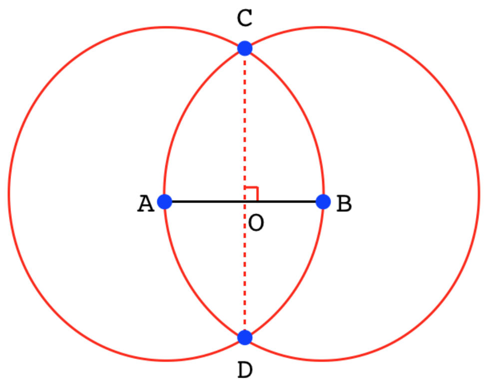
$\square$

The proof doesn't stand on its own. We used one definition (D) and a common notion (CN).

◦ D I.15 all radii of a circle are equal.

◦ CN I.1 things which equal the same thing also equal one another.

If we look again at the figure, and label the other point where the circles cross as D:



Note: CD is the perpendicular bisector of AB. Euclid doesn't have the tools to prove that yet, so he leaves it for now.

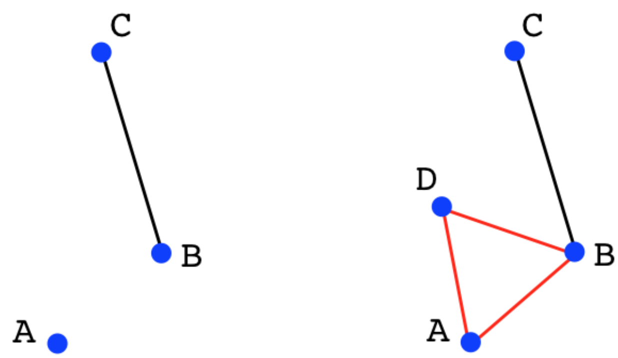
If desired, we could draw on tools from the last chapter. Clearly $\triangle ABC$ and $\triangle ABD$ are equilateral triangles with sides of the same length. From that, we deduce that $\triangle ACD$ and $\triangle CBD$ are isosceles. And from that, we can easily prove that all four small triangles with O as a vertex are equal and therefore right angles.

We leave this as an exercise.

Prop. I.2

To place a straight line equal to a given straight line with one end at a given point.

We will construct a line segment at A equal in length to BC (left panel). The first thing is to draw the line segment AB and construct an equilateral triangle on it (right panel).

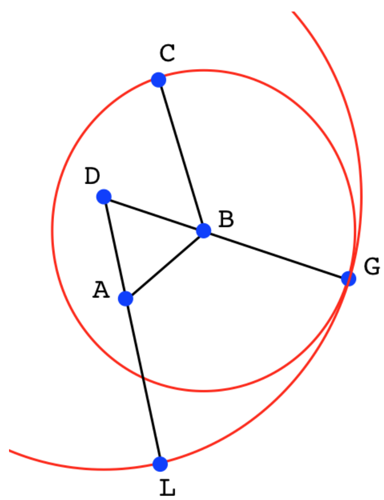


We know how to do this (from $PI.1$).

Next, construct a circle on center B with radius BC and extend the line segment DB to point G.

Then, construct a circle on center D with radius DG and extend DA to that circle at point L.

We have:



As common radii of the circle on center B, we have $BC = BG$.

As common radii of the circle on center D, we have $DL = DG$.

As sides of an equilateral triangle, we have $DA = DB$.

We use CN $I.3$: if equals are subtracted from equals, then the remainders are equal. Thus, $AL = BG$. But we had above that $BC = BG$. Therefore, $AL = BC$, by CN $I.1$.

Q.E.D. or "quod erat demonstrandum", in Latin

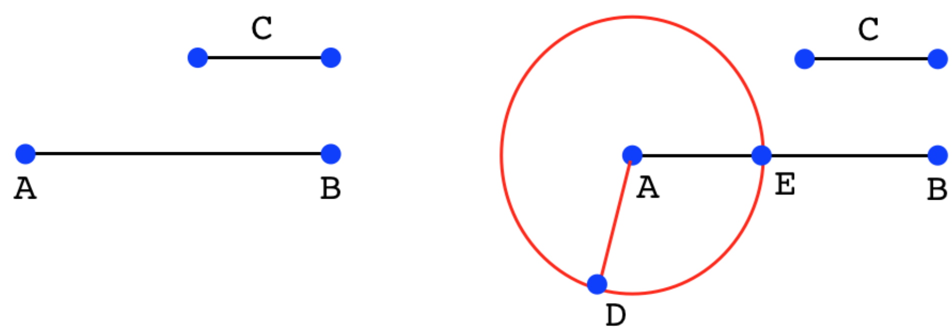
And in the original Greek *the very thing it was required to have shown.*

□

Note in passing, the orientation is determined by AB. We have not shown how to transfer the length with an arbitrary orientation. We will solve this next.

Prop. I.3

To cut off the lesser of two unequal straight lines from the greater.



In the left panel, we have the line segment AB and a smaller one just labeled C. To do the construction, use the method of P $I.2$ and transfer C to point A, forming AD.

Next, use AD as the radius of a circle on center A. Then, $AE = AD$, but $AD = C$. Hence

$$BE = AB - AE = AB - AD = AB - C$$

as required.

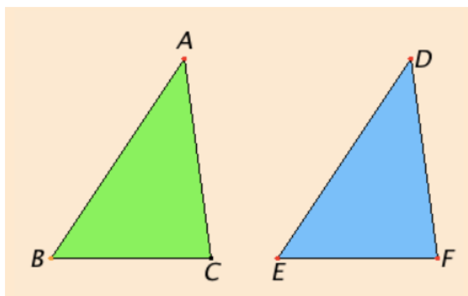
□

At this point, we have a method to mark off a given length from a larger length, even though all we have is a collapsing compass. Therefore, going forward, we can act as if we have a standard compass, that holds its setting after being lifted from the paper.

We also have the means to an important *trichotomy*. Comparing two line segments, one of three things must be true: either the first is smaller than the second, they are equal, or the second is smaller than the first.

Prop. I.4

If two triangles have two sides equal to two sides respectively, and have the angles contained by the equal straight lines equal, then they also have the base equal to the base, the triangle equals the triangle, and the remaining angles equal the remaining angles respectively, namely those opposite the equal sides.



This is not a construction, unlike the previous three propositions. It is a method for proving congruence (equality) of two triangles

$$\triangle ABC \cong \triangle DEF$$

Elsewhere in this book we would call the method SAS or *side angle side*. Given that $AB = DE$ and $AC = DF$ and that the angles between them at the vertices A and D are also equal, the two triangles are congruent: all three angles and all three sides are equal.

This (P I.4) is a proof that SAS is correct.

The proof is by superposition. The facts establish the positions of the points B and C, which determines AB and so the angles at vertices B and C.

Euclid says that if we lift up $\triangle ABC$ and lay it on top of $\triangle DEF$ then B coincides with E and C coincides with F so $BC = EF$.

□

This seems perhaps a little shaky logically, and it's not a method of proof that Euclid uses much.

But one might instead have taken this proposition as a postulate. The source, above, says that David Hilbert claims that under the hypotheses of the proposition it is true that the two base angles are equal, and then proves that the bases are equal.

We have used SAS to prove SSS, that all three sides are equal.

Prop. I.5

In isosceles triangles the angles at the base equal one another, and, if the equal straight lines are produced further, then the angles under the base equal one another.

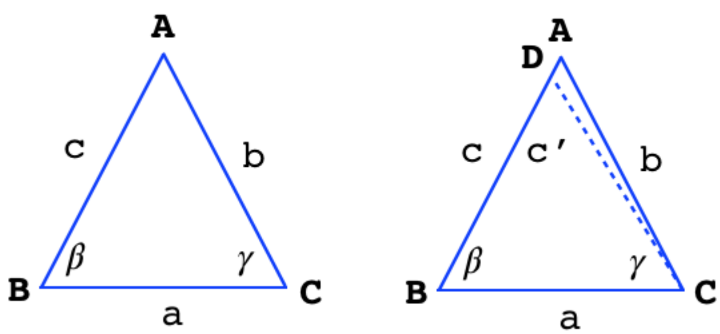
We proved this theorem in the previous chapter on isosceles triangles.

Euclid's proof of the converse is short and introduces the method of contradiction, or *reductio ad absurdum*. That is the next proposition.

Prop. I.6

If in a triangle two angles equal one another, then the sides opposite the equal angles also equal one another.

Suppose we have $\triangle ABC$ with equal angles $\beta = \gamma$ at the base (left panel).



We will assume that the two sides b and c are not equal. Then one of them is greater. Let c be greater, then cut off b from c at point D such that the new length $c' = b$.

The new triangle has sides c' and a, which flank angle β, while for the original we have side b and side a flanking γ. But we constructed $c' = b$, are given that $\beta = \gamma$, and the side a is common.

Therefore the $\triangle DBC \cong \triangle ACB$ by SAS.

But this means that the less equals the greater, which is absurd.

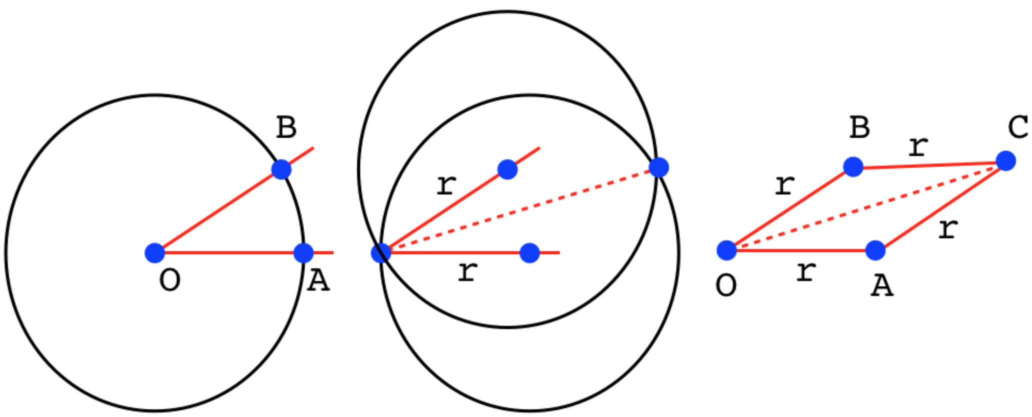
Therefore c cannot be unequal to b. It therefore equals it.

Our original assumption that b does not equal c must be false.

□

Prop. I.9

To bisect a given angle.



As radii of a circle on center O, we first find points A and B equidistant from O (left panel). Let that distance be r.

As radii of circles on the centers A and B that pass through O (so the radius is equal to r), we find C equidistant from A and B (middle panel), with radius also equal to r.

Thus, $OA = OB = AC = BC$ (right panel).

So $\triangle OAC \cong \triangle OBC$.

Therefore $\angle BOC$ is equal to $\angle AOC$ and the given angle is bisected.

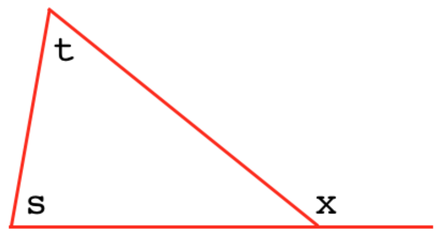
□

We will do three more. They are short, sweet and powerful.

In these examples, we use letters late in the alphabet (s, t, u, v) for angles, while a, b, c are labels for sides.

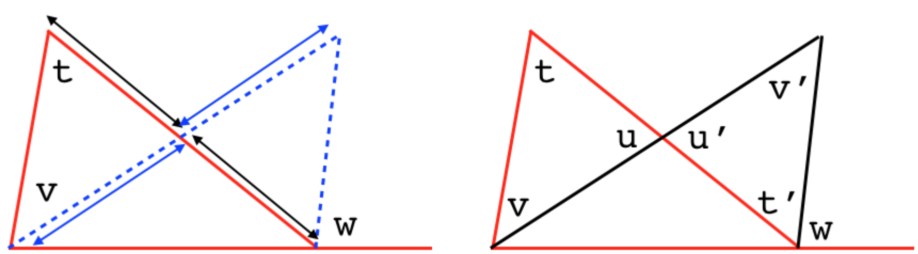
Prop. I.16

In any triangle, if one of the sides is produced (extended), then the exterior angle is greater than either of the interior and opposite angles.



The claim is that the exterior angle x is greater than either of the interior angles: s or t.

Find the midpoint of the side opposite s and draw the indicated line segment (right panel, below), so that the two segments marked with black arrows are equal, as well as the segments marked with blue arrows.



By SAS and the vertical angle theorem, the two smaller triangles to the left and right with dotted lines are congruent, as indicated in the right panel by the labels on the angles: $t = t', u = u', v = v'$.

The original external angle x is seen to be composed of $t' + w$, that is

$$x = t' + w$$

so clearly (the whole is greater than its parts):

$$x > t'$$

but since $t = t'$:

$$x > t$$

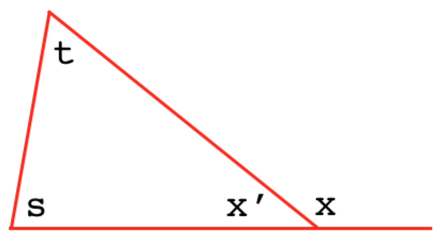
We can make a similar construction and proof for angle s.

The exterior angle is greater than either of the interior and opposite angles.

□

external angle theorem

The external angle theorem is extremely useful, so let's take a break from Euclid and prove it now. The proof is simple for us:



As supplementary angles, $x + x' = 180$ degrees. As the three angles of a triangle, $s + t + x' = 180$ degrees as well. Things equal to the same thing are equal to each other:

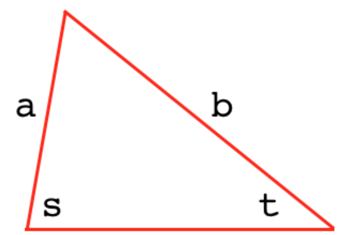
$$x + x' = s + t + x'$$

$$x = s + t$$

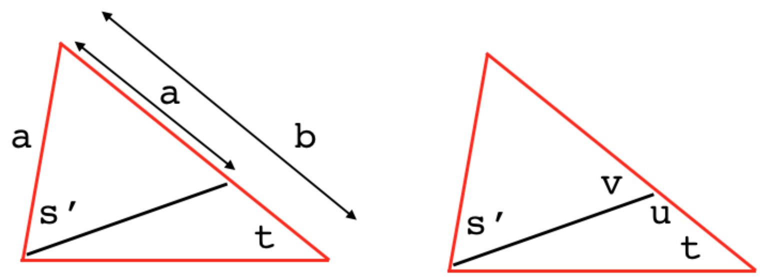
The question of why Euclid doesn't use supplementary angles here is complicated. For now it is enough to say that he just doesn't.

Prop. I.18

In any triangle, a greater side is opposite a greater angle.



Given $b > a$, mark off a on b.



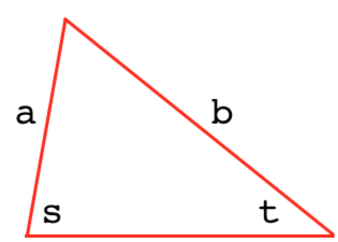
By the external angle theorem (I.16)

$$v > t$$

But $v = s'$ (by isosceles $\triangle$, $I.5$) so

$$s' > t$$

And since $s > s'$



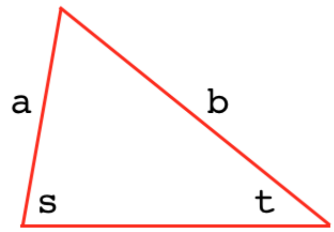
$$s > t$$

$\square$

We get the converse almost for free.

Prop. I.19

In any triangle, a greater angle is opposite a greater side.



We are given $s > t$ and want to prove $a < b$. We proceed by considering the other possibilities.

It cannot be that $a = b$ because then $s = t$ by isosceles $\triangle$ $(I.5)$, but we are given $s > t$.

So then suppose $a > b$. By the previous proposition $(I.18)$, we would have that $t > s$. But this is again contrary to what we were given. Hence $b > a$.

$\square$

We have made use of the trichotomy from before, that there are only three possibilities:

$$a < b, \qquad a > b, \qquad a = b$$

This applies to line segments and angles as well as many other things.

This is enough of the *Elements* to give us a good taste of the basics of Greek geometry of lines and triangles, and methods of proof. There is more to come: Pythagoras, and circles with their arcs and tangents.

Part IV

Right triangles

Chapter 16

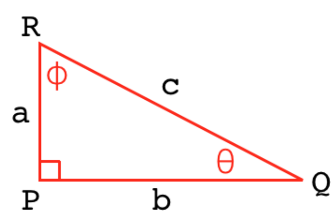
Right triangles

The main result we are headed for is the Pythagorean Theorem. Before we get there, however, it is worthwhile to continue our development of basic geometry with a discussion about right angles and right triangles.

A right triangle is a triangle containing one right angle. Right angles (and right triangles) are special. We saw previously that the definition of a right angle is that two of them add up to one straight line or 180 degrees.

Since we proved that the sum of the three angles in any triangle is equal to one straight line, by extension, the sum of angles in any triangle is also equal to two right angles.

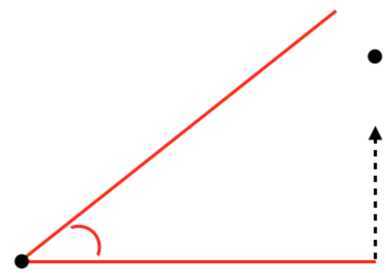
In the figure below, the angle at vertex P is a right angle. It is common to mark a right angle with a little square, as shown, but these are a pain to draw, so I will sometimes not do that. The side opposite P, namely c, is the hypotenuse.



Since the sum of angles in a triangle is equal to two right angles, the sum of the angles θ and ϕ above is also equal to a right angle, or 90 degrees. Angles θ and ϕ are said to be *complementary*. This fact is often exploited in proofs.

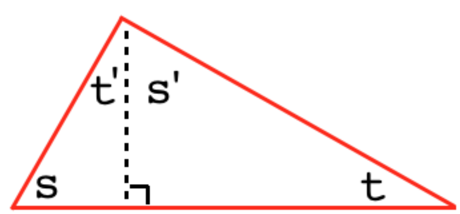
right triangles

For two right-triangles, if one hypotenuse is equal to the other, and also one set of legs equal, the two triangles are congruent.



In the figure, imagine the hypotenuse swinging on the hinge of its vertex with the horizontal base. There is only one angle where it will terminate on the vertical side with the correct length. This determines the angle between the known sides, or alternatively, the length of the third side.

altitudes



In the large right triangle above, we know that

$$s + t = 90$$

When we draw the perpendicular to the hypotenuse that goes through the upper vertex, that is an *altitude* of the triangle. Because of the right angle, we obtain two smaller right triangles. Thus

$$s + t' = 90$$

$$s' + t = 90$$

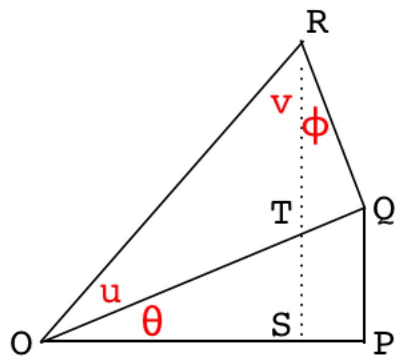
By subtracting appropriately we get

$$s - s' = 0$$

$$s = s'$$

and similarly for t and t'.

stacked triangles



Suppose we are given that $\angle OPQ$ and $\angle OQR$ are right angles. We draw the altitude RS and observe that the angle at vertex S is a right angle.

Therefore, in triangle ORS, the sum $\theta + u + v$ is equal to one right angle. At the same time, in triangle OQR, the sum $u + v + \phi$ is also equal to one right angle. Therefore

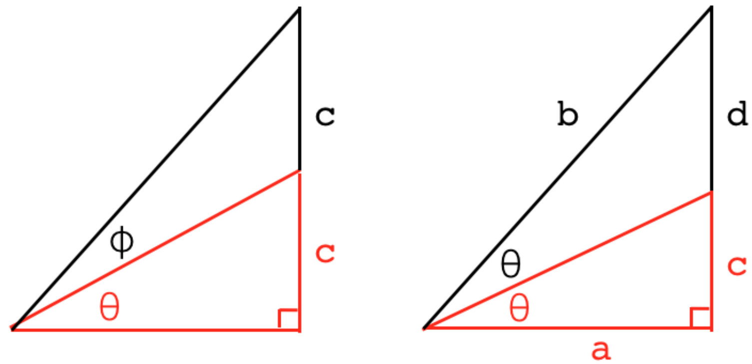
$$\theta = \phi$$

Further, $\triangle QRT$ and $\triangle OPQ$ are similar triangles.

angle bisector

With that background, we now consider a classic problem: angle bisectors. We saw previously how to construct one.

Now, consider the diagram below.



Suppose we are given that the large triangle is a right triangle.

We draw a line joining the vertex on the left with the side opposite.

This line could in general be drawn anywhere, however two interesting cases are when the side opposite is bisected (left panel), or when the angle at the left is bisected (right panel). These two cases are not the same. In the first $\phi \neq \theta$ and in the second, $c \neq d$.

Suppose we choose the second possibility, equal angles. We are in a position to prove an important theorem.

angle bisector theorem

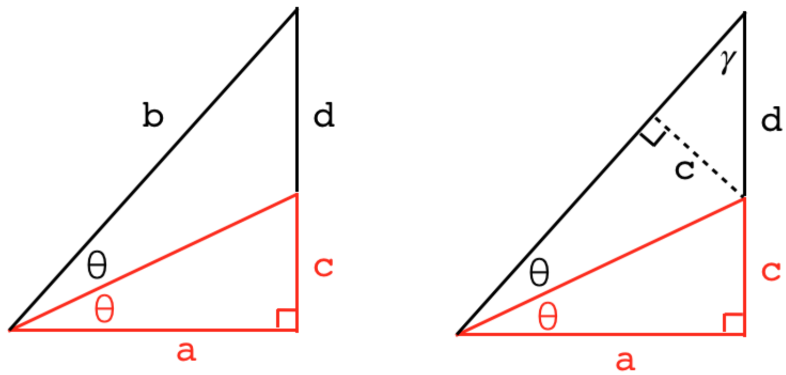
With reference to the two figures above, we are to prove that

$$\frac{d}{b} = \frac{c}{a}$$

The sides and bases are in proportion for a right triangle with bisected angle.

Proof.

Draw an altitude for the upper of the two small triangles, meeting the side of length b.



The red triangle and the one directly above it are congruent (right panel). They share a side (the hypotenuse of each), and they are right triangles with the same smaller angle θ. Therefore, the altitude we just drew has length c.

The small triangle with sides c and d (top) is similar to the original large triangle. The reason is that they are both right triangles containing the smaller angle γ.

By similar triangles, we form equal ratios of the angle opposite γ to the hypotenuse:

$$\frac{a}{b} = \frac{c}{d}$$

This is rearranged simply to give

$$\frac{d}{b} = \frac{c}{a}$$

which is what we were asked to prove.

□

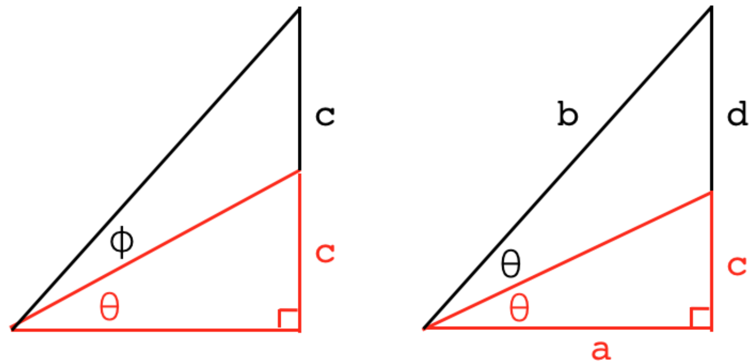
The result can be pushed a little further:

$$\frac{a}{b} = \frac{c}{d}$$

Here's the key point

$$\frac{a+b}{b} = \frac{c+d}{d}$$

$$\frac{a+b}{c+d} = \frac{b}{d} = \frac{a}{c}$$



which is a surprising result and becomes important in looking at Archimedes method for approximating the value of π.

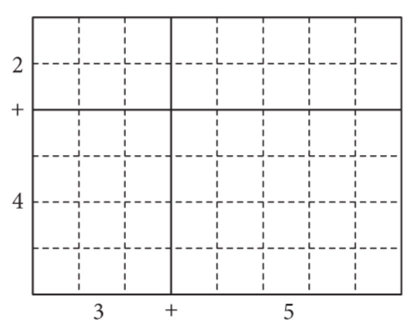
Chapter 17

Area

One aspect of calculus will be to determine the area of figures in the plane, particularly figures bounded by curves, as well as volumes in space. This is the magic of calculus, that we can make curves conform to rectilinear concepts of area and volume.

Since this introductory section is about Euclidean geometry, let's just say a few words about the area of a triangle. But we'll start with the rectangle.

To find the area of a rectangle, we must first fix a unit length. Then multiply the width by the height.



This particular figure (from Lockhart) shows the distri butive law in action:

$$(3 + 5) \cdot (4 + 2)$$

$$= 3 \cdot 4 + 3 \cdot 2 + 5 \cdot 4 + 5 \cdot 2$$

$$= 48$$

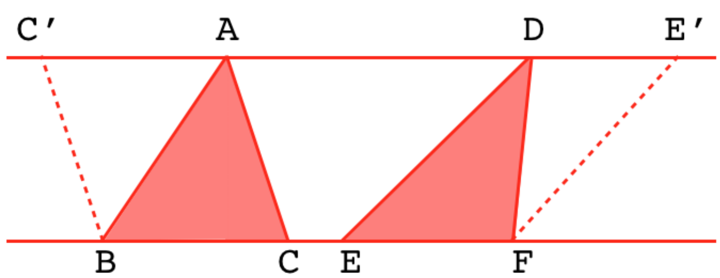
Any combination of numbers that add up to 8, times any combination of numbers that add up to 6, gives the same result.

The next figure is a parallelogram, a four-sided figure whose two pairs opposite sides are parallel (left panel). As a consequence of the theorems we saw previously, the opposing angles are equal, and the adjacent angles add up to 180 degrees.

To find the area, we cut off a right triangle from the left and re-attach it on the right. The angles add up to form a straight line along the base and a right triangle at the upper right. The area is clearly $h \times b$.

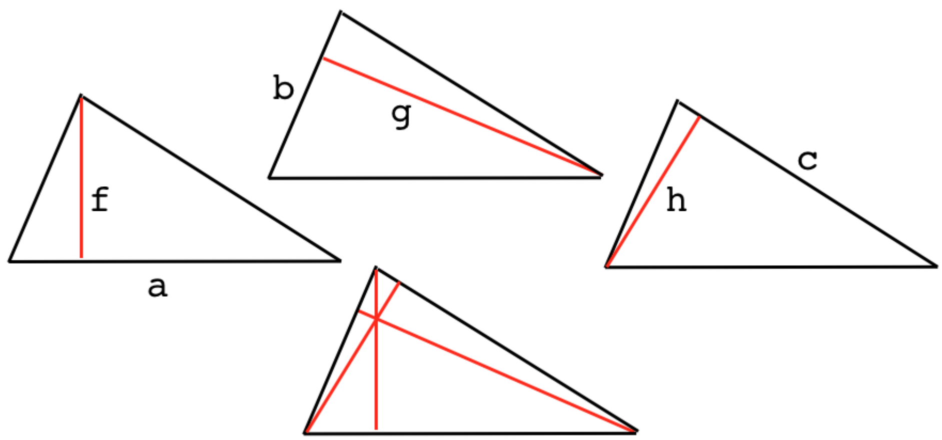
What about triangles? Previously, we talked about an approach where any triangle can be divided into two right triangles.

Alternatively, any triangle can be turned into a parallelogram, by attaching a rotated image of itself, like this:



It is easy to show that $\triangle ABC \cong \triangle ABC'$ and $\triangle DEF \cong \triangle DE'F$. For example, by construction AC is parallel to BC' so $\angle ACB = \angle AC'B$. We'll leave it to you to complete the proof.

An acute triangle is on the left and an obtuse triangle on the right. Since the area of each triangle is one-half that of its corresponding parallelogram (because we added the same area to make the parallelogram), the area of a triangle is one-half the base times the height.



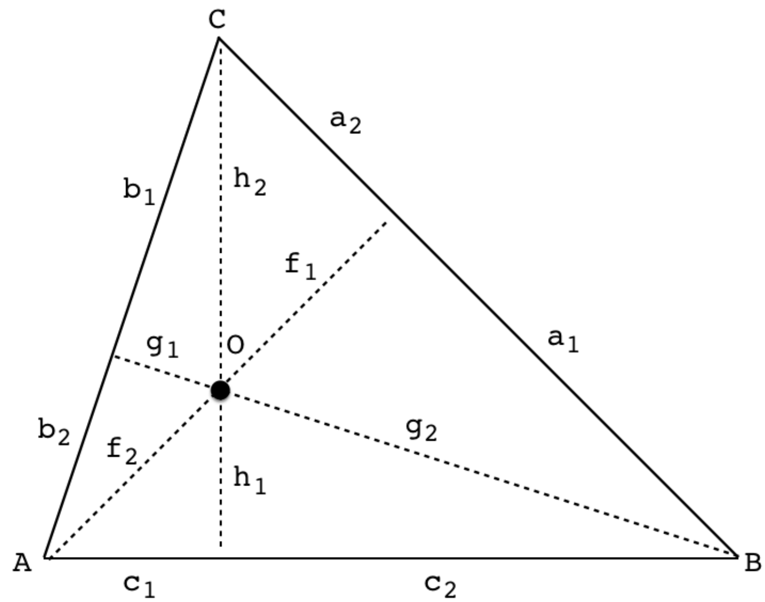
In the figure above, the area is

$$A = \frac{1}{2}\ af = \frac{1}{2}\ bg = \frac{1}{2}\ ch$$

We can choose any side of the triangle to be the base and then multiply $1/2\cdot$ base $\cdot$ height to get the area. We must always get the same answer!

If you accept the argument about the parallelogram above, it must be true, because the area of the triangle has to be the same no matter how you calculate it.

Here's a proof by counting up the area of smaller triangles:



In $\triangle ABC$ with sides a, b, c, drop the three altitudes from each of the three vertices to form right angles on the opposing sides. Ceva's theorem says that these altitudes cross at a single point (we will prove this later). Label the parts of the sides and the altitudes as shown in the diagram.

The area of the whole $\triangle ABC$ is equal to the sum

$$\triangle BOC + \triangle AOC + \triangle AOB$$

Using the rule, *twice* the area is

$$2A = af_1 + bg_1 + ch_1$$

But each of these smaller areas can be computed in different ways. In particular $\triangle BOC$ can be viewed as having base g_2 and height b_1, while $\triangle AOB$ can be viewed as having base b_2 and height g_2, so (twice) the total area is also

$$2A = b_1 g_2 + b_2 g_2 + bg_1$$

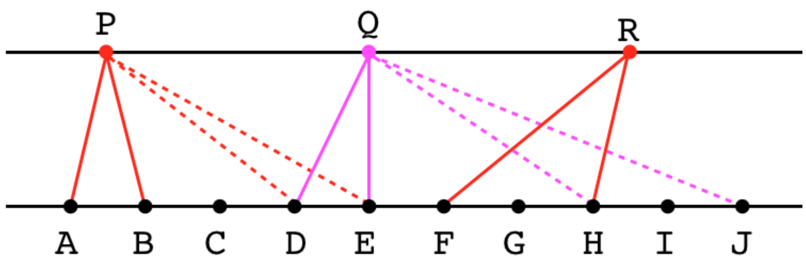
$$= bg_2 + bg_1 = bg$$

Similar calculations can be carried out for the other two sides. Hence the area is the same regardless of which side is chosen as the base.

□

A corollary is that all triangles with the same base and height have the same area.

Draw two parallel lines. Mark off equal distances between adjacent points A through J on the bottom. Now pick any point on the top and draw the triangle with two *equidistant* points on the bottom. Any other triangle drawn with an equal base has the same area.



In this figure the areas of $\triangle PAB$, $\triangle PDE$, and QDE are equal, as are $\triangle QHJ$ and $\triangle RFH$. Further, the latter two have twice the area of any of the first three.

platonic solids

https://en.wikipedia.org/wiki/Platonic_solid

> In three-dimensional space, a Platonic solid is a regular, convex polyhedron. It is constructed by congruent (identical in shape and size) regular (all angles equal and all sides equal) polygonal faces with the same number of faces meeting at each vertex. Five solids meet these criteria.

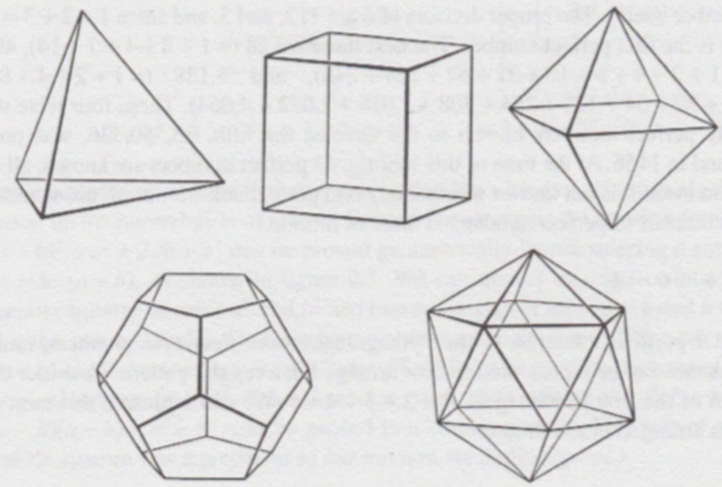


Figure 2.3. The five regular or Platonic solids

These are: (i) tetrahedron, (ii) cube, (iii) octagon, (iv) dodecagon, and (v) icosahedron.

There is a wonderful, simple proof that there are only five of them. Any solid requires at least three sides meeting at each vertex, otherwise the joint between two sides can just flap, like a hinge. Furthermore, the total of all the vertex angles added up must be less than 360 degrees, since otherwise the figure would be planar, not 3-dimensional.

So, three equilateral triangles total $60 \times 3 = 180$, four total $60 \times 4 = 240$ and five total $60 \times 5 = 300$. Six would be a hexagon lying in the plane. Three squares total $90 \times 3 = 270$, while four give a square array in the plane. Finally, three pentagons give $108 \times 3 = 324$. And that's it. Three hexagons would give $120 \times 3 = 360$, which gives an array in the plane.

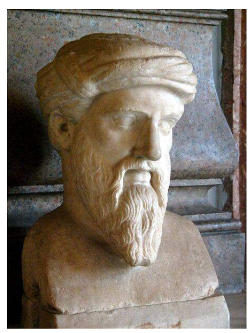
Proving that all the angles and side lengths come out correctly, so that the possible solids actually can be constructed is another matter, however. Euclid devotes book XIII of *The Elements* to this:

https://mathcs.clarku.edu/~djoyce/elements/bookXIII/bookXIII.html#props

Chapter 18

Pythagoras

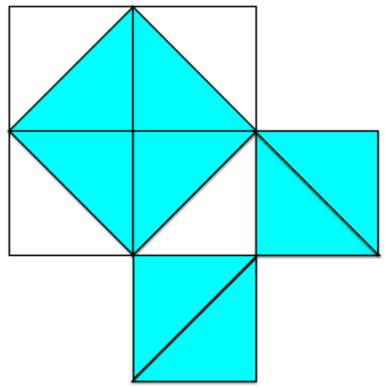
The most famous theorem of Greek geometry is also the most useful in Calculus.



Pythagoras (c.570-c.495 BC) was much younger than Thales but may have encountered him as a young man. Like many other Greek mathematicians, Pythagoras was not from the mainland, but from one of the islands, in his case, Samos, which is not far from Miletus, where Thales lived.

Pythagoras was famous as a philosopher as well as a mathematician. In fact, he founded a famous "school" and it is not sure now which of the theorems developed by this school are due to Pythagoras, and which to his disciples. It is not even clear whether the Pythagorean theorem, as we know it, was known to Pythagoras.

However, it's pretty certain that they knew something. The $3, 4, 5$ right triangle and many other Pythagorean triples (see below) had been known for a thousand years (since 1500 BC). Here is a special case, easily proved, for an isosceles right triangle.

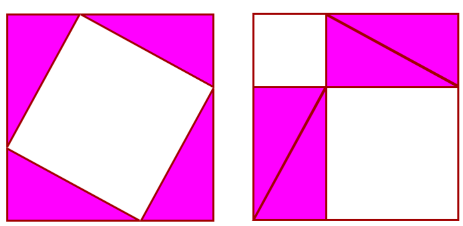


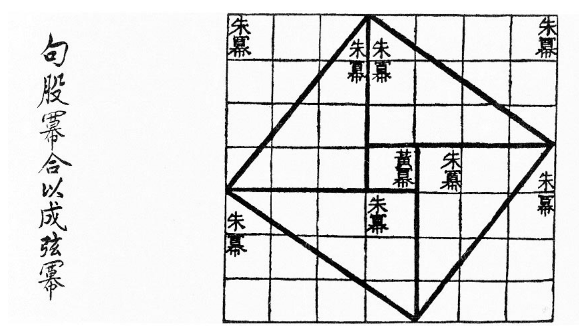
The area of the square on the hypotenuse is equal to twice the area on each side.

There are literally hundreds of proofs of the general theorem, that if a and b are the shorter sides of a right triangle and c is the hypotenuse, then

$$a^2 + b^2 = c^2$$

This one is sometimes called the "Chinese proof." I can easily imagine proceeding from the figure above to this one by simply rotating the inner square.





It really needs no explanation, but ..

In the left panel we have a large square box that contains within it a white square, whose side is also the hypotenuse of the four identical right triangles contained inside. Altogether the four triangles plus the white area add up to the total.

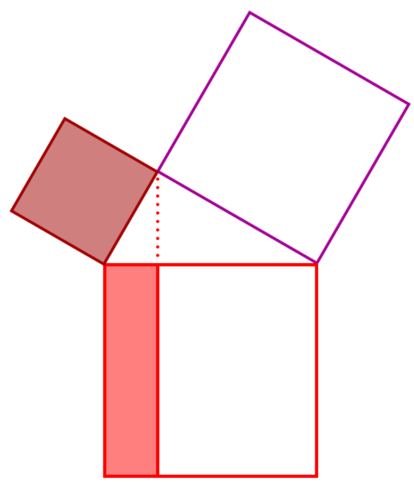
We simply rearrange the triangles. Now we evidently have the same area left over from the four triangles, because they still have the same area and the surrounding box has not changed.

But clearly, now the white area is the sum of the squares on the second and third sides of the triangles. Hence the two white squares on the right are equal in area to the large white square on the left. □

This proof is contained in the Chinese text Zhoubi Suanjing (right panel, above).

https://en.wikipedia.org/wiki/Zhoubi_Suanjing

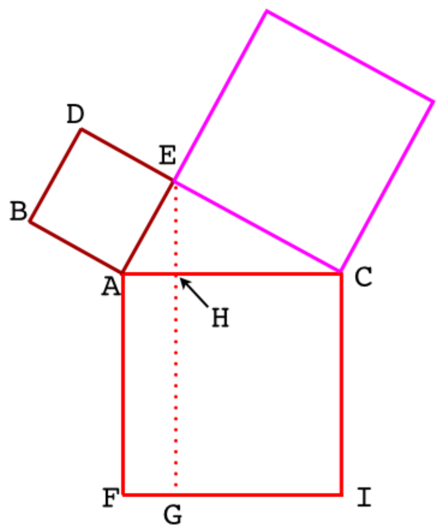
Euclid's proof



My favorite proof relies on the construction above (Euclid $I.47$, sometimes called the "bridal chair" or the "windmill"), where the central triangle is a right triangle, and the other constructions are squares. It is a bit more detailed, but it is a gem of a proof, from Euclid, which is a justification for including it.

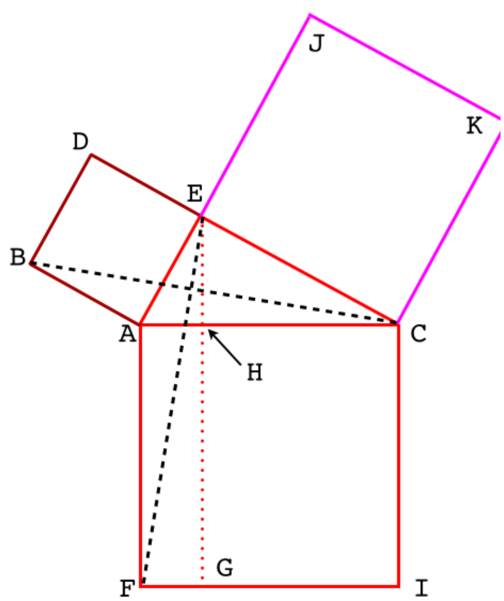
What we will show is that the part of the large square in red is equal in area to the entire small square, in maroon.

We label some points as shown:



First, drop a vertical line EHG, constructing the rectangle $AFGH$.

Finally, sketch dotted lines for the long sides of two triangles:



The crucial point is this: we will show that triangle ΔABC is congruent to triangle ΔAEF.

Use side-angle-side (SAS). The two sets of sides are evidently equal

$$AB = AE, \quad AC = AF$$

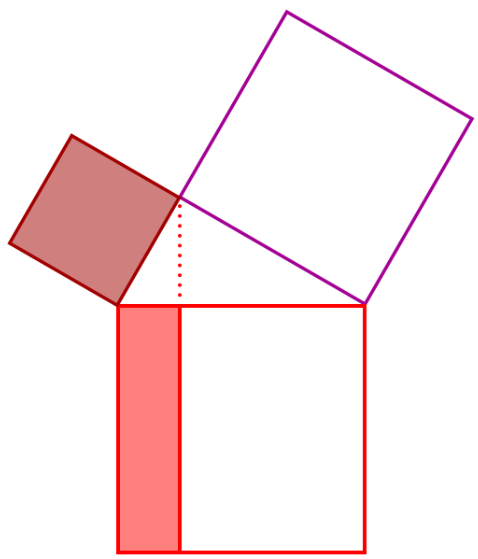
because these are given as sides of two squares.

What about the included angle? The angles $\angle BAC$ and $\angle EAF$ each contain a right angle plus the shared angle $\angle EAC$. So they are themselves equal, and thus we have proved SAS and thus the congruence relationship:

$$\Delta ABC = \Delta AEF$$

The next part of the proof is to tilt triangle ΔABC to the left and see that it has base AB and altitude AE so its area is one-half that of the small square $ABDE$. On the other hand triangle ΔAEF has base AF and altitude AH (as well as FG) so its area is one-half that of the rectangle $AFGH$.

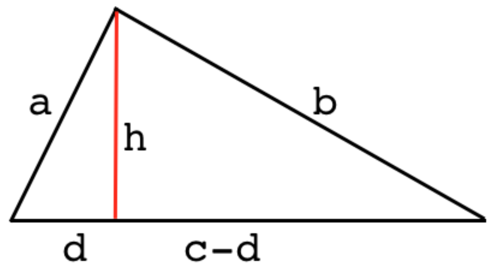
Hence we have proved that the two colored areas in this figure are equal:



Finally, we could proceed to do the same thing on the right side of the figure, but we just appeal to symmetry. All the equivalent relationships will hold.

□

There are several hundred proofs of the Pythagorean theorem. Many of them are algebraic. Here is a classic:



We know that when an altitude is drawn in a right triangle, the two resulting right triangles are similar (use complementary angles if you need to convince yourself again). So we have equal ratios of sides. Here are two sets:

hypotenuse to short side

$$\frac{a}{d} = \frac{b}{h} = \frac{c}{a}$$

hypotenuse to long side

$$\frac{a}{h} = \frac{b}{c-d} = \frac{c}{b}$$

From the first

$$a^2 = cd$$

From the second

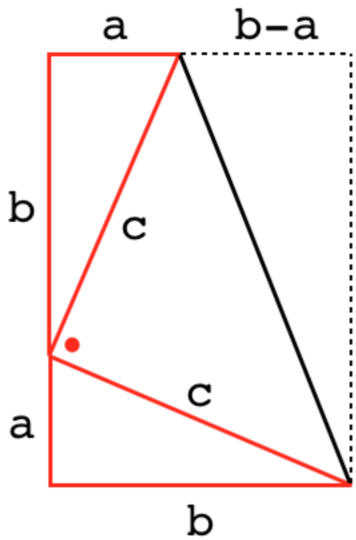
$$b^2 = c(c-d)$$

Just add

$$a^2 + b^2 = cd + c^2 - cd = c^2$$

Garfield

There is one by a future President of the United States, James A. Garfield. (He was a congressman at the time).



Draw a right triangle and a rotated copy as shown. The angles opposite sides a and b are complementary angles. So the angle marked with a dot is a right angle, and the triangle with sides labeled c is a right triangle.

The area of the quadrilateral is the product of the side $(a+b)$ and the *average* of a and b (top and bottom). This can be seen intuitively. The halfway point of the solid red line has horizontal dimension $(a+b)/2$. Hence

$$A = (a+b) \cdot \frac{1}{2}(a+b)$$

If you're worried about that argument, just subtract the area of the triangle with two dotted sides from the quadrilateral that includes it:

$$A = (a+b)b - \frac{(a+b)(b-a)}{2}$$

$$= (a+b)(b - \frac{b}{2} + \frac{a}{2})$$

$$= (a+b) \cdot \frac{1}{2}(a+b)$$

which is just what we said. So now:

$$= \frac{a^2}{2} + ab + \frac{b^2}{2}$$

But we can also calculate the area of the quadrilateral as the sum of the three triangles:

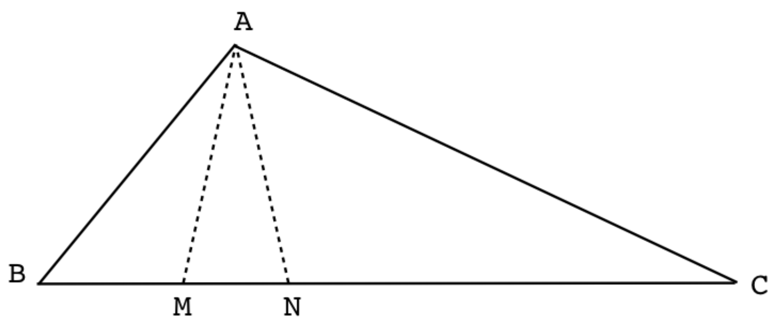
$$A = \frac{ab}{2} + \frac{ab}{2} + \frac{c^2}{2}$$

Equate the two and the result follows almost immediately.

□

Corollary

There are several important corollaries of the Pythagorean theorem. We'll derive one later called the law of cosines. Here is another from the Islamic geometer Ibn Quorra, who brought algebraic techniques, shunned by the Greeks, to geometry.



Let $\triangle ABC$ be *any* triangle (here it is obtuse). Draw AM and AN so that the new angles $\angle AMB$ and $\angle ANC$ are equal to $\angle A$. The corresponding triangles are similar to the original, because they share the angle of measure A plus one other from the original triangle.

Then

$$BM : AB = AB : BC$$

Thus, $AB^2 = BM \times BC$. Similarly

$$NC : AC = AC : BC$$

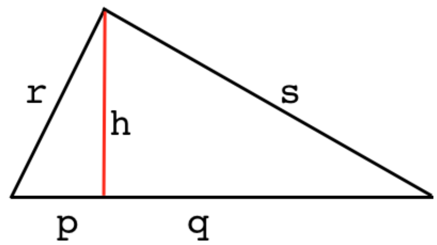
So $AC^2 = NC \times BC$ Therefore

$$\begin{aligned} AB^2 + AC^2 &= BM \times BC + NC \times BC \\ &= (BM + NC) \times BC \end{aligned}$$

In the case where the angle at vertex A is a right angle, then M coincides with N, and $BM + NC = AC$, and this reduces to the Pythagorean theorem.

geometric mean

As a slight detour from calculus, but on the topic of this chapter



We will show that

$$h^2 = pq$$

$$h = \sqrt{pq}$$

That is, h is the geometric mean of these two values p and q.

Proof.

Using the Pythagorean theorem with the two small triangles (also right triangles), we obtain:

$$h^2 + p^2 = r^2$$

$$h^2 + q^2 = s^2$$

Summing

$$2h^2 + p^2 + q^2 = r^2 + s^2$$

Using the theorem with the big triangle:

$$r^2 + s^2 = (p+q)^2$$

$$= p^2 + 2pq + q^2$$

Equating the two expressions for $r^2 + s^2$ we get:

$$2h^2 + p^2 + q^2 = p^2 + 2pq + q^2$$

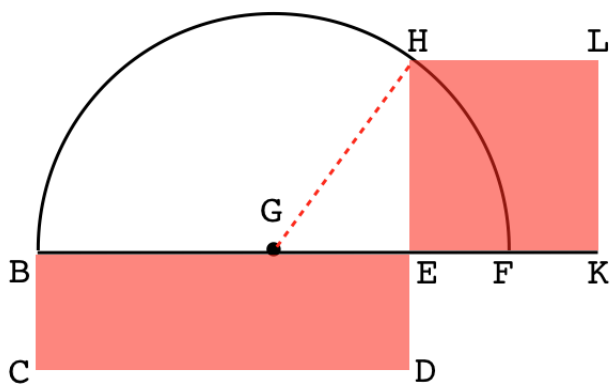
$$h^2 = pq$$

$$h = \sqrt{pq}$$

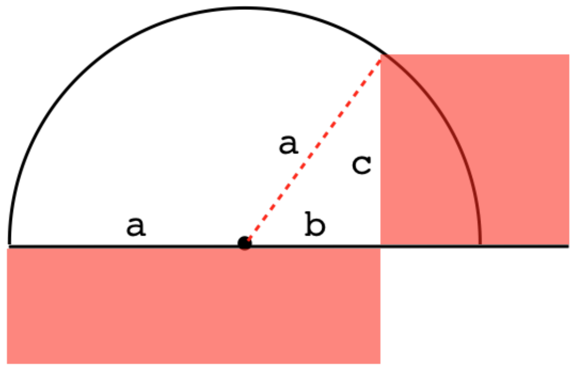
Chapter 19

Hippocrates

Hippocrates of Chios (470-410 BC) was a major figure in Greek geometry. (Not to be confused with the physician of the same name, from Kos). Hippocrates focused on quadrature, the process of constructing (with straight-edge and compass) a square with area equal to a given geometric figure, particularly curved figures, called lunes. Here is one of the first of these—construction of the square equivalent to a given rectangle.



The construction says to: (i) extend BE horizontally, (ii) mark off the same distance as DE to construct EF, (iii) find the midpoint G of BF, (iv) draw the half-circle of radius BG, (v) extend DE up to meet the circle at H, construct the square of side the same length as EH.



As suggested by the dotted line in the figure, the proof invokes the Pythagorean theorem. The long side of the rectangle is $a + b$, while its short side is $a - b$, so the area is

$$A = (a + b)(a - b) = a^2 - b^2$$

but Pythagoras says that is equal to c^2.

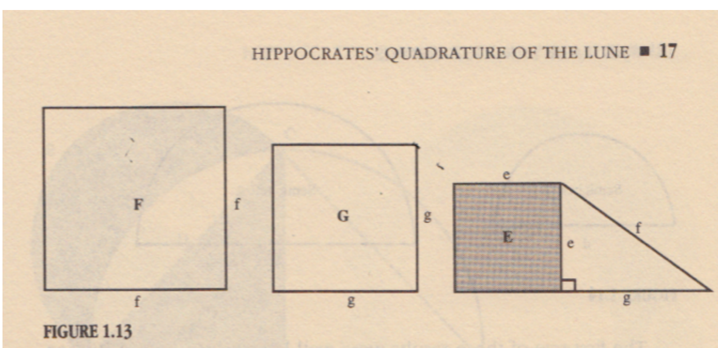
□

This is a slight restatement of our proof about the geometric mean.

The side of the square, c is the geometric mean of the sides of the rectangle.

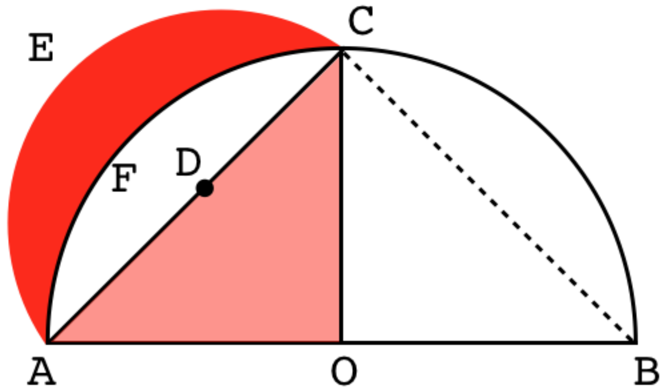
$$c = \sqrt{(a + b)(a - b)}$$

Hippocrates "squared" rectangles, triangles and polygons. A lot of his constructions depended on Pythagoras as suggested by this figure:



where two squares resulting from manipulation of part of a polygon need to be subtracted to obtain the final result.

Hippocrates moved on to curves, trying to find squares with area equal to that under or between two curves. That turns out to be a class of problems where few have solutions (in fact, only five, according to Dunham). Famously, it is impossible to square the circle. However, here is one that is possible, it is an example of (the) quadrature of the lune.



We will prove that the two shaded regions are equal in area.

Consider the smaller semicircle with base ADC, which is also the hypotenuse of the right triangle. Let radius AD be equal to r. Let the large semicircle have radius AO equal to R. Pythagoras tells us that

$$R^2 + R^2 = (2r)^2$$

$$R^2 = 2r^2$$

Let the area of the triangle be T. (Its value is $R^2/2$ but that's not needed).

The segment of the larger semicircle (white) is the area of the quadrant minus the area of the triangle

$$\pi \frac{R^2}{4} - T$$

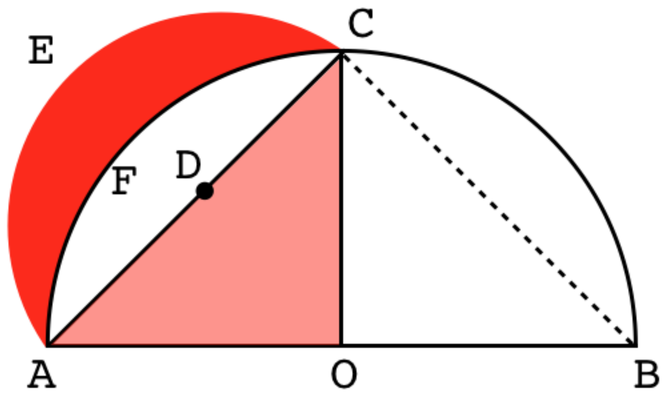
The area of the red lune is the area of the small semicircle minus the white area

$$\pi \frac{r^2}{2} - \ [\ \pi \frac{R^2}{4} - T\ ]$$

$$= \pi \frac{r^2}{2} - \pi \frac{2r^2}{4} + T$$

$$= T$$

Which is just the area of the triangle.



Part V

Circles

Chapter 20

Circle and triangle

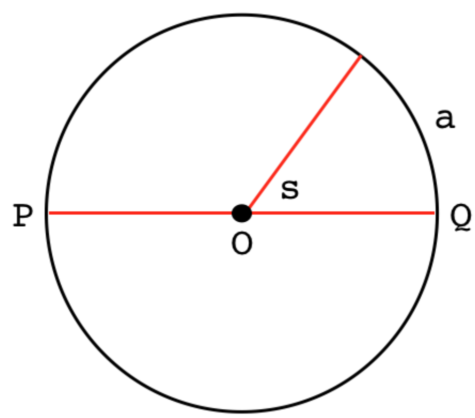
From a previous chapter, Euclid's third postulate was:

◦ Given any straight line segment, a circle can be drawn having the segment as radius and one endpoint as center. The tool to do this is a compass.

If the radius is extended so that it cuts the circle at two points, it is called a diameter.

We saw previously that one can construct a line perpendicular to any given line. If that line is constructed perpendicular to the diameter at the point where it meets the circle, the new line is called a tangent line. By definition, the tangent line touches the circle at a single point.

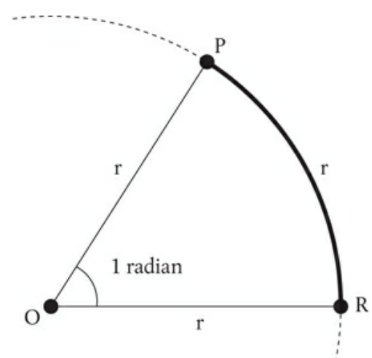
arcs of a circle



In calculus and analytical geometry angles are defined in terms of radians of arc. In the figure above, the angle s is *defined to be equal to* the length of arc a that it sweeps out, or subtends in a unit circle.

For a unit circle with radius = 1, the total circumference is 2π, so the arc swept out by the angle θ, measured in radians, is in the same ratio to 2π as the ratio of the angle's measure in degrees to 360°.

It seems natural then to adopt the arc length as a measure of the angle, where 360° is equal to 2π *radians*, and an angle of 90°, for example, a right angle, is equal to $\pi/2$ radians.



72. Definition of a radian.

One can easily divide 360 by 2π to find that one radian is approximately 57°.

To convert some more measures of angles in degrees to radians:

$$180^\circ = \pi,\ 90^\circ = \frac{\pi}{2}$$

$$60^\circ = \frac{\pi}{3},\ 45^\circ = \frac{\pi}{4},\ 30^\circ = \frac{\pi}{6}$$

Central angle and subtended arc are numerically equal, but remember that they are dimensionally different. Arc is a length, angle is just an angle.

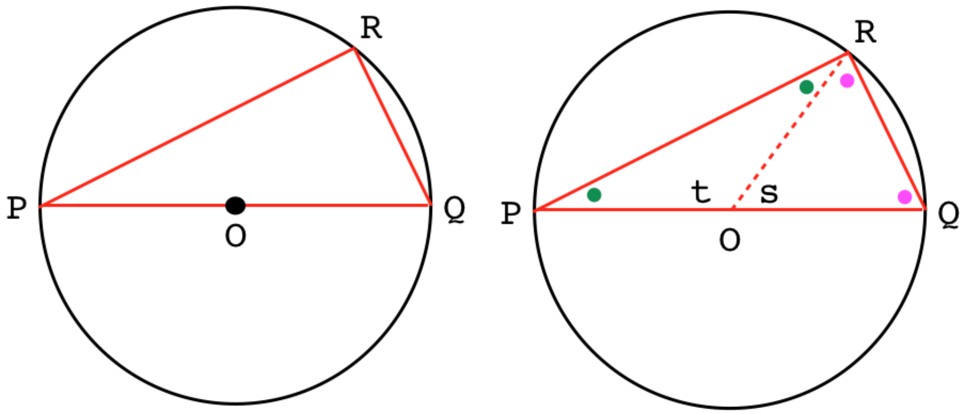
Thales' theorem

In this chapter, we introduce a few more theorems concerning circles, starting with the last of Thales' theorems:

- Any angle inscribed in a semicircle is a right angle.

Think of three points on the circumference of the circle as forming a triangle. If two points are on a diameter of the circle, the angle formed at any arbitrary but distinct third point is always a right angle.

To prove: $\angle PRQ$ is a right angle.

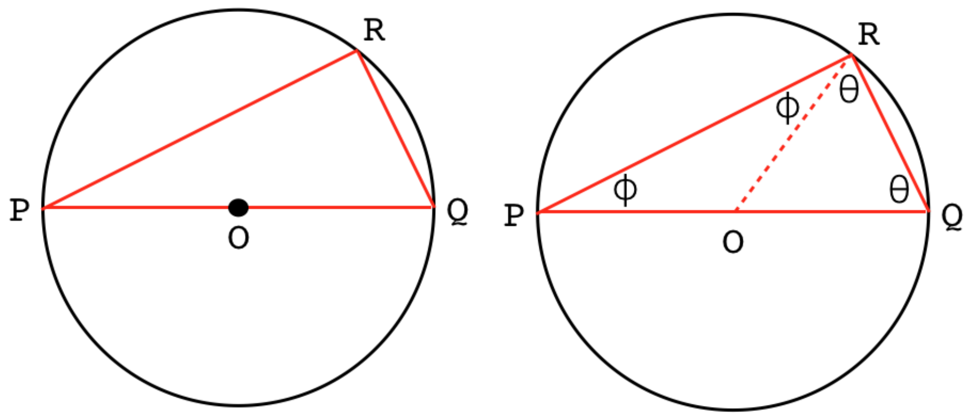


Solution:

Draw the radius OR. Notice that the two smaller triangles produced ($\triangle OPR$ and $\triangle OQR$) are both isosceles, since two of their sides are radii of the circle.

Therefore, in each triangle the two angles marked with dots of the same color are equal (by P I.5).

Since $\angle PRQ$ contains one angle of each type, it is equal to one-half the angle sum for the triangle, i.e., $\angle PRQ$ is a right angle.



To restate this: in the figure above, $\angle PRQ = \phi + \theta$. Since the full measure of the triangle is $180° = \pi$ radians, and

$$\phi + \phi + \theta + \theta = \pi$$

it follows that

$$\phi + \theta = \pi/2$$

$\square$

angles on the perimeter

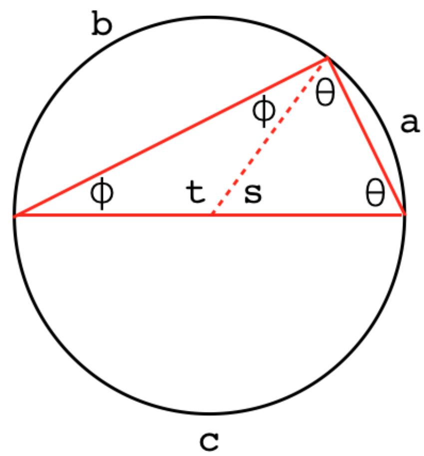
The arc swept out by the right angle $\angle PRQ$ is clearly equal to π, because that arc is one-half of a circle.

But the angles on the perimeter of the circle subtending the very same arc add up to

$$\angle PRQ = \phi + \theta = \pi/2$$

What's going on?

To clarify, let us label the arcs on the circle.



a is the arc swept out by angle s, and a and s have the same measure by definition, although one is a length and the other an angle.

$$s = a$$

By the external angle theorem, we know that

$$s = 2\phi$$

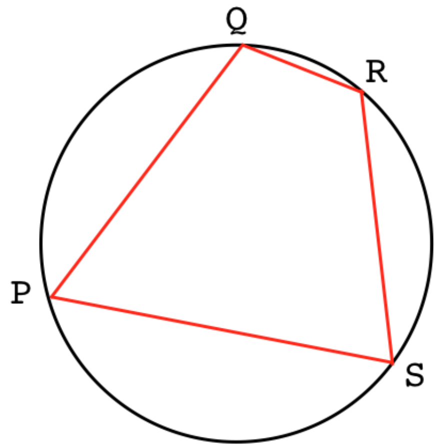
and so conclude that

$$\phi = \frac{s}{2}$$

The angle ϕ lying on the perimeter sweeps out the same arc as s, even though ϕ is one-half of s.

ϕ is farther away from the arc, so we get a bigger arc for the same angular measure. The arc length swept out by an angle on the perimeter is twice the angle's measure in radians.

This leads to a wonderful simple theorem about quadrilaterals.

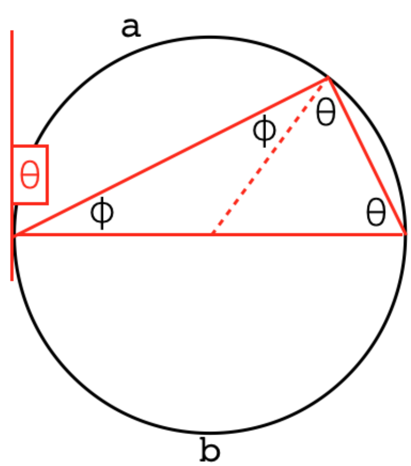


For *any* quadrilateral whose four vertices lie on a circle, the opposing angles are supplementary (they sum to 180°).

Proof: together, opposing angles exactly subtend the whole arc of the circle.

tangent

Consider the chord PR and draw the tangent at P.



The angle between the tangent and the chord equals θ because $\theta + \phi$ is a right angle

Take a chord of the circle, draw the diameter and the tangent. The same rule applies to both angles: one between the chord and the diameter, and the second between the chord and the tangent. The arc is twice the measure of the angle.

geometric mean

We showed in the chapter on the Pythagorean theorem that the altitude of a right triangle is the geometric mean of the two components of the base.

$$h^2 = pq$$
$$h = \sqrt{pq}$$

According to wikipedia:

https://en.wikipedia.org/wiki/Geometric_mean

The fundamental property of the geometric mean is that (letting m be the *geometric mean* here):

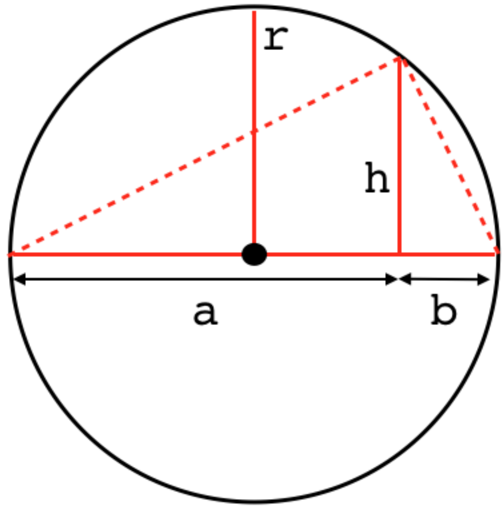
$$m \ [\ \frac{x_i}{y_i}\ ] \ = \frac{m(x_i)}{m(y_i)}$$

and one consequence is that

> This makes the geometric mean the only correct mean when averaging normalized results; that is, results that are presented as ratios to reference values.

A number of examples are given in the article.

This section is here because originally, there was a proof-without-words that the geometric mean is always less than or equal to the arithmetic mean.



I decided to add some words. A right triangle is inscribed in a semicircle. As just mentioned, the altitude h squared is equal to the product of chord segments (we will prove this geometrically in the next chapter as well).

$$h^2 = ab$$

$$h = \sqrt{ab}$$

But we also have that $a + b = 2r$ and hence

$$r = \frac{a+b}{2}$$

Do you recognize these? The second expression is the arithmetic mean of a and b, while the first is the geometric mean.

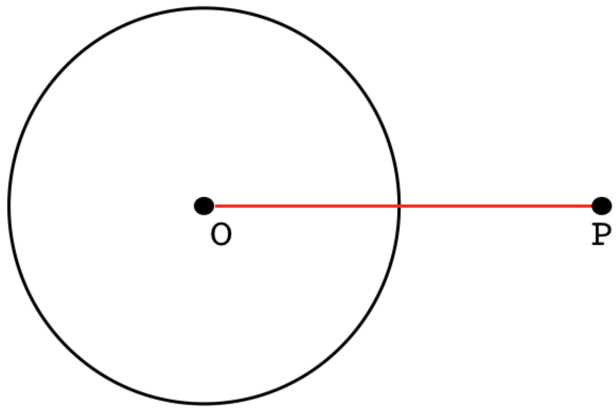
The geometry shows that $h \leq r$ so:

$$\sqrt{ab} \leq \frac{a+b}{2}$$

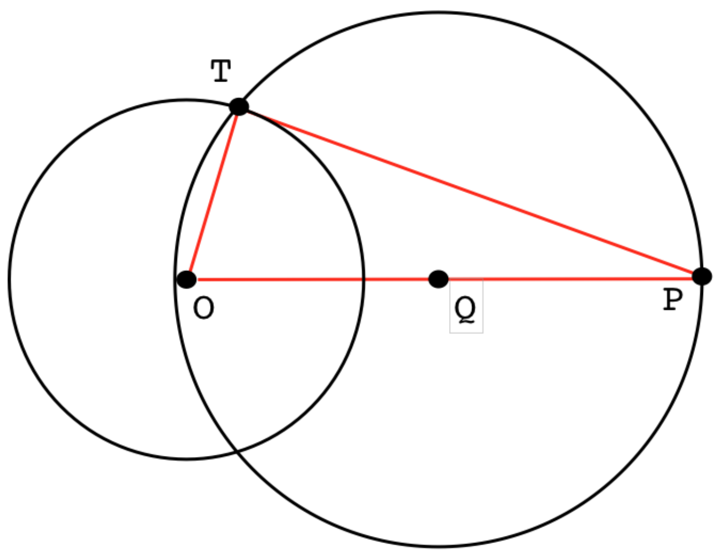
The geometric mean is always less than the arithmetic mean, except when $a = b$, when they are equal (or all of n values are equal).

tangents

Thales theorem provides a way to construct the tangent to a circle passing through any exterior point P.

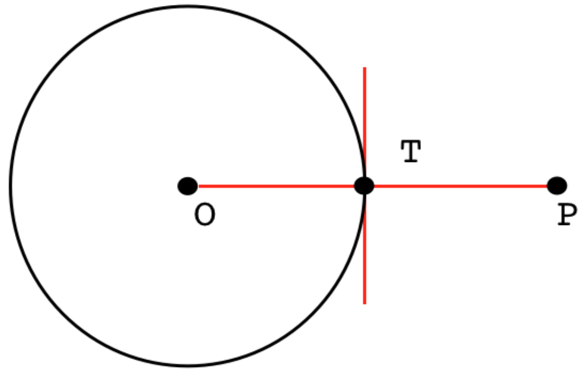


Use OP as the diameter of a circle. Draw the line segment OP and divide it in half by erecting the perpendicular bisector at Q. Use that Q as the center of a new circle. The point T is the intersection of the two circles.



By Thales theorem, $\angle OTP$ is a right angle, and since OT is a radius of the original circle, TP is the tangent at the point T.

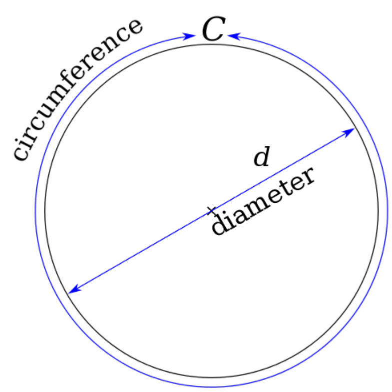
To construct a tangent on a circle at a given point T



Extend OT to P. Construct the perpendicular bisector at T. That is the tangent of the circle.

Chapter 21

Pi is a constant



We began the book with a bold claim: the ratio of the circumference of a circle to its diameter is a constant, independent of the length of the diameter:

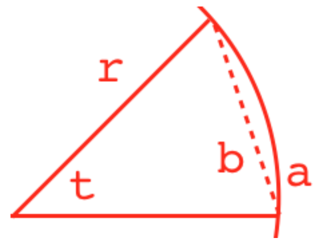
$$\pi = \frac{C}{d} = \frac{C}{2r}$$

We did not prove this theorem at the time but will do so now.

We need the idea of limits, which was introduced previously, and a property of similar triangles. The theorem is: if two triangles are similar, then their sides are proportional to each other.

Consider an arc a of a circle and two radii.

The triangle corresponding to that arc has base b. We can restate Archimedes argument about inscribed polygons by saying that, in the limit, as the inscribed polygon gets very close to being the same as the circle, $b \to a$.

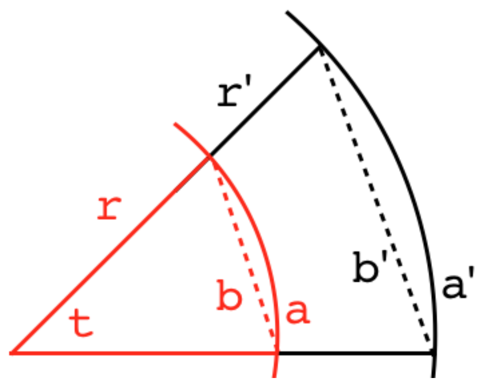


So if there are n pieces ($t = 2\pi/n$), the ratio of the circumference to the arc is just n and we have

$$n = \frac{na}{a} = \frac{C}{a} = \frac{C}{b}$$

The last step is "in the limit."

Now draw a larger arc. In the same way $b' \to a'$.



and

$$n = \frac{C'}{b'}$$

since t and n haven't changed:

$$n = \frac{C'}{b'} = \frac{C}{b}$$

But by similar triangles the ratios are equal:

$$\frac{r}{b} = \frac{r'}{b'}$$

so

$$\frac{C'}{C} = \frac{b'}{b} = \frac{r'}{r}$$
$$\frac{C'}{r'} = \frac{C}{r}$$

Suppose for a moment that $C = 2\pi r$ and $C' = 2\pi' r'$ and we don't know how π compares to π':

$$\frac{2\pi' r'}{r'} = \frac{2\pi r}{r}$$

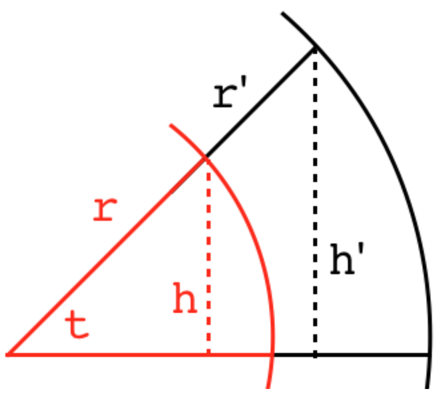
We have that $\pi = \pi'$.

□

Second proof:

Here is a simple variant which assumes something we will prove in the section on sine and cosine. If this is confusing, it can easily be skipped.

Drop the altitude h in each of the two similar triangles. The ratio h/r is equal to $\sin t$, but the arc length is equal to t, measured in radians.



In the limit that $n \to \infty$, the ratio between t and $\sin t = h/r$ is equal to our "special limit":

$$\lim_{n\to\infty} \frac{t}{\sin t} = 1$$

If the ratio to the sine is equal to 1, so is the ratio to its inverse and thus the ratio s/r is constant, which is what we wanted to prove.

□

Chapter 22

Arcs of a circle

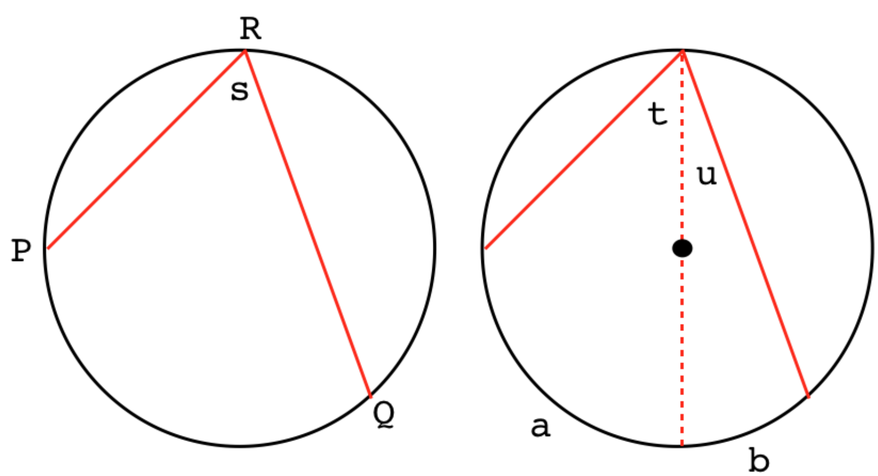
We have previously established some central facts about circles including Thales' theorem about the angle intercepting a half-arc of the circle being a right angle. We will need some of these results later.

Now, we do more. We can generalize the results for all arcs.

arcs encompassing diameter

The examples so far directly contain the diameter in some way.

Consider the arc swept out by the angle s in this figure (left panel).



We can prove that the measure of the angle s is equal to one-half the arc swept out between P and Q.

Draw the diameter (right panel): By our previous work:

$$2t = a, \qquad 2u = b$$

and

$$s = t + u$$

$$2s = 2t + 2u = a + b$$

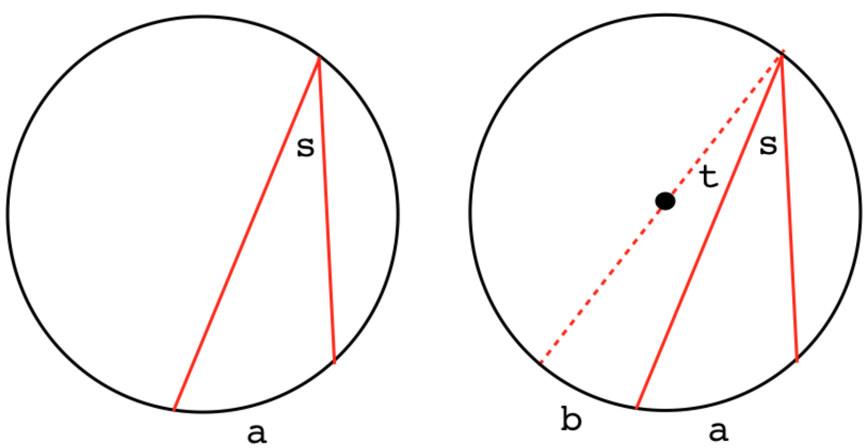
$$s = \frac{1}{2}\,(a + b)$$

Thus, we have proved the theorem for

- the special case where the arc is a diameter of the circle
- the general case where the diameter is one chord flanking the arc
- another general case where the arc includes the diameter.

arc without diameter

However, the theorem is true even if the angle does not include the diameter. We use subtraction.



On the right, draw the diameter. There are two arcs which include the diameter: one with angle t and one with angle $s + t$. We obtain the generalized arc with angle s by subtracting the result for t from that for $s + t$.

$$2t = b$$

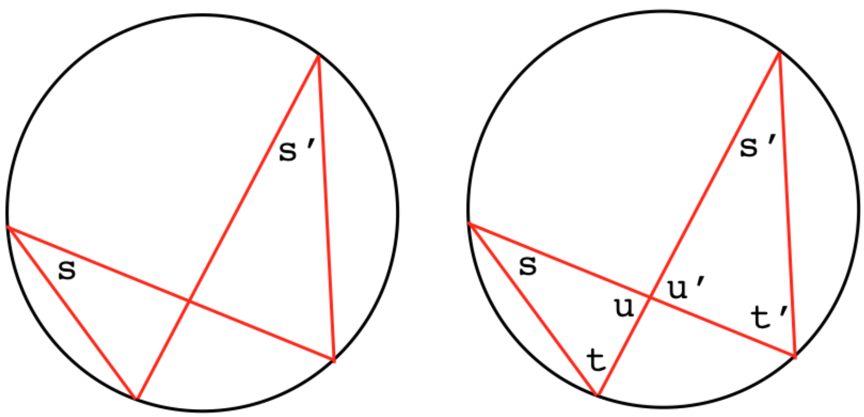
$$2(s + t) = a + b$$

Subtract:

$$2s = a$$

$$s = \frac{1}{2}\ a$$

As a corollary, any two angles with vertexes (vertices) on the circle that cut off the same arc are equal. In the figure below, $s = s'$ (left panel)



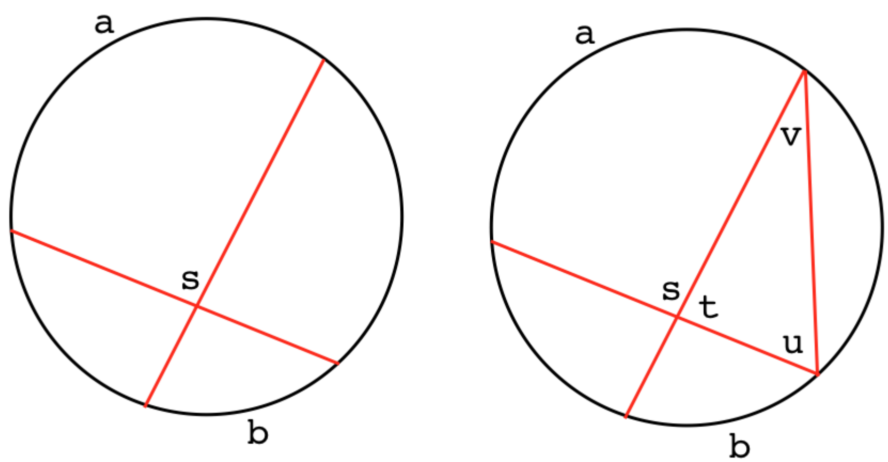
and $t = t'$ (right panel). So $u = u'$ by both the vertical angle theorem and by of the triangle sum theorem. Therefore, the two triangles are similar. We will come back to this.

Intersecting chords

Given two chords, to prove:

$$s = \frac{1}{2}(a + b)$$

s is the average of the two arc lengths.



Solution: Draw a triangle (right panel, above).

$$2v = b$$

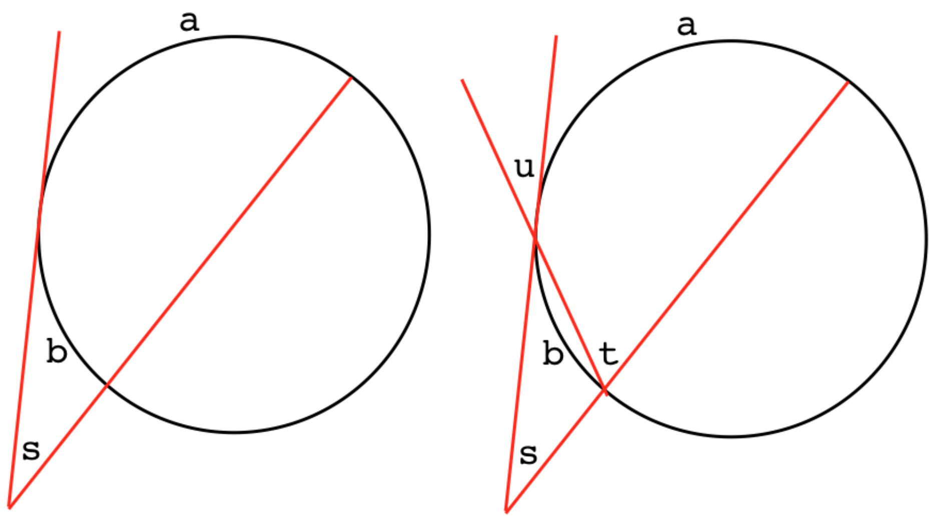
$$2u = a$$

The external angle is the sum of the two opposing interior angles.

$$s = u + v = \frac{1}{2}\ (a + b)$$

Tangent and secant

Rather than having all three points on the circle, one point is now outside. We have the same arc swept out by the endpoints (a), but the included angle is smaller, and there is a new small piece of arc length b.



To prove:

$$s = \frac{1}{2}(a - b)$$

Solution: Draw the triangle. By our previous work:

$$2t = a$$

By the vertical angle theorem the unlabeled angle inside the triangle is equal to u and it subtends arc b so

$$2u = b$$

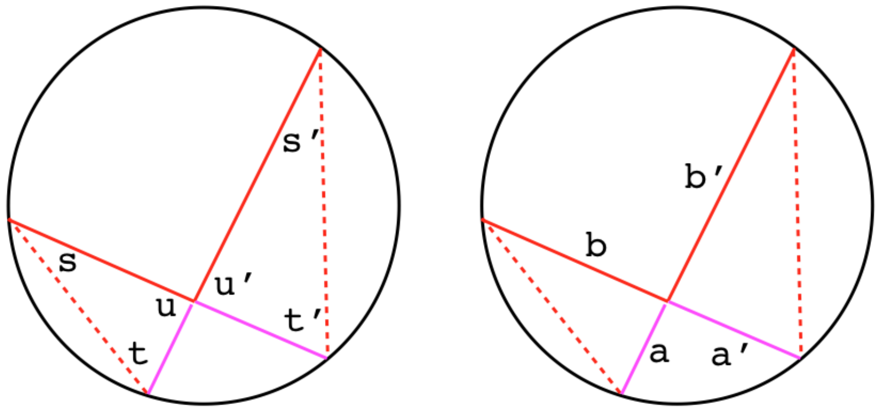
By the exterior angle theorem

$$t = s + u$$

$$s = t - u = \frac{1}{2}\ (a - b)$$

Chord segments

Finally, there is a simple algebraic relationship between chord segments. Draw two chords of the circle.



Draw the two triangles. The angles are equal to their primed counterparts ($s = s'$ and $t = t'$ because they subtend equal arcs, while $u = u'$ by the vertical angle theorem).

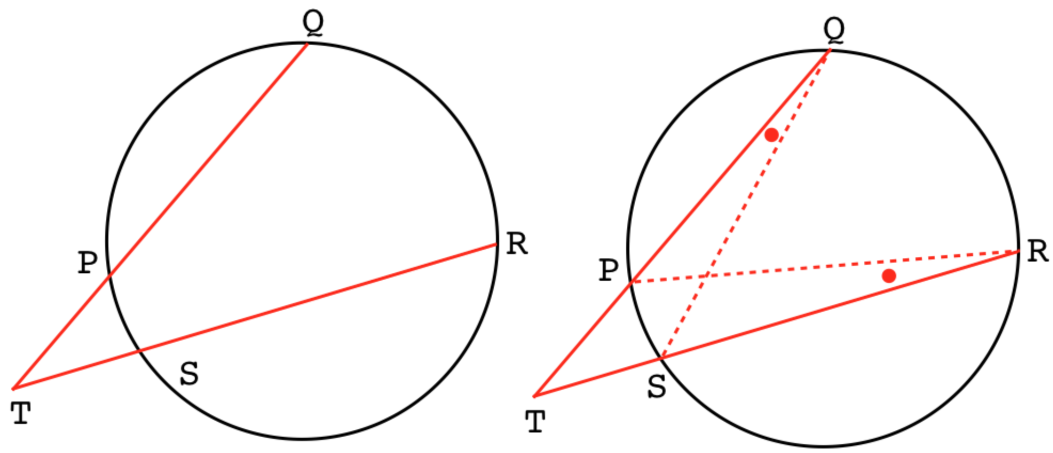
Therefore, the triangles are similar. Similar sides are indicated by color or style, and labeled with primes in the right panel, where the letters refer to the sides.

$$a/b = a'/b'$$

$$ab' = a'b$$

For any two chords that cross in a circle, the products of the chord segments are equal. We use this later in looking at the spherical cap.

We can prove a similar theorem about chords extended from a circle.



The two angles marked with red dots subtend the same arc, so they are equal. The angle at vertex T is shared, therefore $\triangle QST \cong \triangle PRT$.

By similar triangles we have that

$$\frac{TP}{TS} = \frac{TQ}{TR}$$

$$TP \cdot TR = TS \cdot TQ$$

Chapter 23

Eratosthenes

This part of the book is focused on geometry, and we take a look at Eratosthenes in this chapter as an important Greek scholar.

The widely held theory, that the ancient world believed the earth to be flat, is just wrong. People with any level of sophistication not only knew the earth is roughly spherical but also knew its size within a few percent of the true value.

One likely basis is the false story that Columbus had trouble getting financing for his proposed trip to China because everyone thought he would fall off the edge of the earth. This was a tall tale invented by Washington Irving, who also made up several remarkable fables about George Washington.

The real reason the Italians and the Portuguese thought Columbus would fail is that they had a pretty good idea of the size of the spherical earth and thus of the distance to China, while the over-optimistic Columbus believed it was about half the true value. The prospective financiers knew that he was not able to carry the supplies necessary for a trip of this length.

Morris Kline (*Mathematics and the Physical World*) says that the error is due to geographers after Eratosthenes, who reduced the estimated circumference from 24,000 to 17,000 miles.

Eratosthenes

Views of the Greek philosophers on the earth and its sphericity are detailed here https://www.iep.utm.edu/thales/#SH8d

Here is a partial quotation:

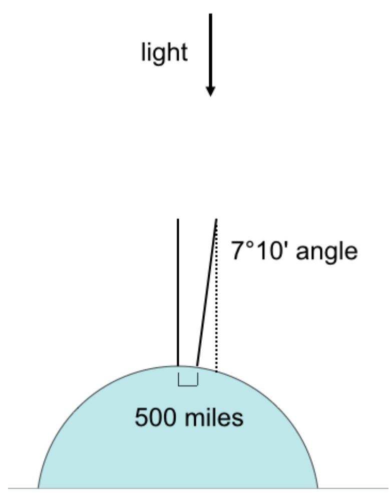
> There are several good reasons to accept that Thales envisaged the earth as spherical. Aristotle used these arguments to support his own view [...] . First is the fact that during a solar eclipse, the shadow caused by the interposition of the earth between the sun and the moon is always convex; therefore the earth must be spherical. In other words, if the earth were a flat disk, the shadow cast during an eclipse would be elliptical. Second, Thales, who is acknowledged as an observer of the heavens, would have observed that stars which are visible in a certain locality may not be visible further to the north or south, a phenomen[on] which could be explained within the understanding of a spherical earth.

https://en.wikipedia.org/wiki/Eratosthenes

Eratosthenes (ca. 276 - 195 BCE) measured the circumference of the earth from this observation: at high noon on June 21st there was no shadow seen at Syene, e.g., allegedly from a stick in the ground. Some people say it was a deep well, where the bottom was illuminated at midday.

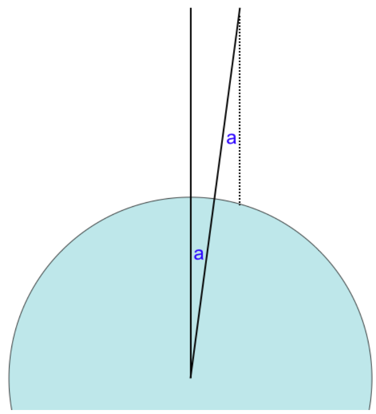
Syene is presently known as Aswan. It is on the Nile about 150 miles upstream of Luxor, which includes the famous site called the Valley of the Kings. At 24.1 degrees north latitude, Aswan or Syene is close enough to having the sun directly overhead on June 21. (The "Tropic of Cancer" is at 23 degrees, 26 minutes north).

This news about the lack of a shadow at Syene reached Alexandria, a famous center of learning of the ancient world. Alexandria lies on the Mediterranean some 500 miles north of Syene, and anyone there who was looking could observe that at high noon on June 21st there *was a shadow*. This shadow Eratosthenes measured to be some 7 degrees and a bit (7 degrees and 10 minutes).



A full 360 degrees divided by 7 degrees and a bit is approximately 50. So we can calculate on this basis that the circumference of the earth is about $50 \times 500 = 25000$ miles. That's pretty close to the correct value.

For this calculation, we assume that the sun's rays are effectively parallel (not a bad assumption given a distance of 93 million miles). Then we just use this:



an application of the alternate-interior-angles theorem.

It is curious how the distance from Alexandria to Syene was calculated.

Kline:

> Camel trains, which usually traveled 100 stadia a day, took 50 days to reach Syene. Hence the distance was 5000 stadia...It is believed that a stadium was 157 meters.

We obtain

$$157 \times 5000 \times 50 = 39,250 \text{ km}$$

That's a much better estimate than a method that relies on camels really deserves.

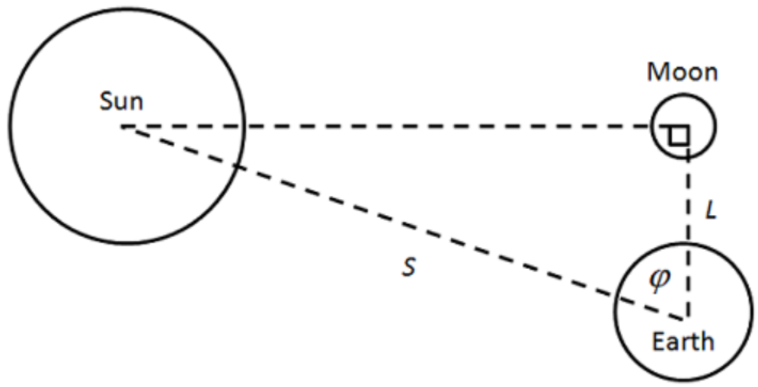
Aristarchus

Aristarchus of Samos (310-230 BCE) wrote a famous book in which he calculated the relative sizes of the sun and the moon and their distances from earth.

One straightforward observation is that the apparent size of the sun and moon in the sky is about the same. This can be seen during a solar eclipse, or observed at any other time by holding a disk up at a fixed distance from the eye, (while taking care to block most of the sun's rays). The value is approximately one-half degree.

Since the distance to the sun is much greater than that to the moon (see below), we can infer that the sun is much larger than the moon.

The central idea of Aristarchus is that, at half moon, the geometry of the three orbs is like this:



In other words, when the phase is half moon and that moon is exactly overhead, the sun has not yet set, but is a bit above the horizon.

If S is the distance to the sun and L is that to the moon, he estimated that

$$18 < \frac{S}{L} < 20$$

with the same ratio for their sizes. Unfortunately, this is not a particularly good estimate. The true value is about 390. Aristarchus obtained a value of 20 for the Earth-Moon distance in Earth radii. The correct value is about 60. Much better estimates were obtained later, by Hipparchus and Ptolemy.

However, Aristarchus made up for this by being the first person to propose a heliocentric theory of the solar system: that the earth and planets rotate around the sun.

https://en.wikipedia.org/wiki/On_the_Sizes_and_Distances_(Aristarchus)

quick estimate

Here is an estimate for the earth-moon distance based on a lunar eclipse.

One measures the time it takes for a complete, total eclipse. From the first shadow of the earth on the moon to the last, that time is about 3 hr. The moon has moved approximately 1 earth diameter in its orbit in that time.

However, we must correct for the fact that the first and last shadows occur on opposite edges of the moon. It was noted that the shape of the eclipse suggests the earth's diameter (at that distance) is about 2.5 moon diameters. So the moon has actually moved $(2.5 + 1.0)/2.5 = 1.4$ earth diameters in the given time. The relevant time becomes 2.14 hr.

Any correction for the true size of the earth's diameter is minimal because the earth-moon system is so far from the source of illumination.

The other piece of information we need is the time for a full revolution, one lunar cycle. This part is tricky. Naively, you'd look for the moon to be in the same place against the fixed stars (27 days, c. 8 hr). This is off because the earth has moved in the meantime — there is a parallax error. As a rough correction, mutliply by 360/330 degrees. The result in hours is 715.

The circumference of the orbit is then

$$715/2.143 = 333$$

earth diameters.

This gives a radius of 53 earth diameters, which is not too far from 60.

Chapter 24

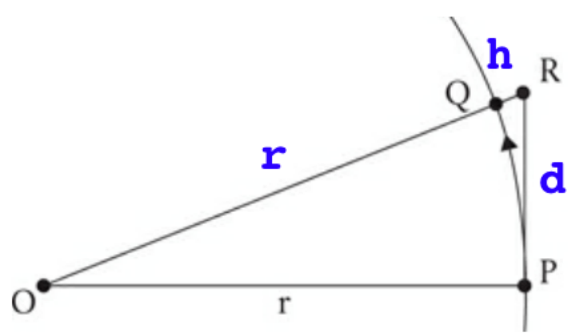
Circular orbits

Pythagoras and Newton

A previous chapter looked in detail at Pythagoras' Theorem, which is used incessantly from here on out. Here, we explore one use of the Pythagorean theorem and provide a taste of orbital mechanics, which is a particular focus of calculus. Newton made early calculations similar to these, which increased confidence about his famous inverse-square law and inspired the mathematics that led to the explanation of elliptical orbits.

Although the orbits of the planets around the sun are ellipses, they are very nearly circular and we will make that approximation for what follows here.

We use the Pythagorean Theorem to make another approximation. Using r for the (fixed) radius of the orbit for the moment, because the construction has capital letters for the points, including the symbol R:



$$r^2 + d^2 = (r+h)^2 = r^2 + 2rh + h^2$$
$$d^2 = 2rh + h^2$$

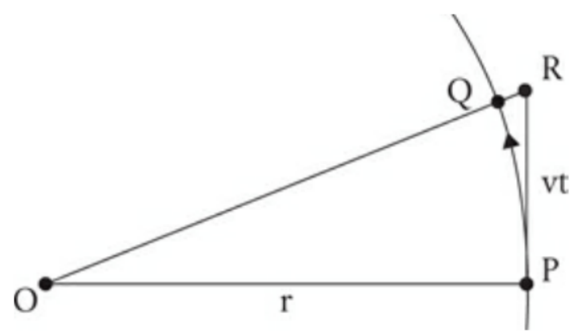
If $h << r$ then we can ignore the very small quantity h^2 and obtain

$$d^2 = 2rh$$
$$r = \frac{d^2}{2h}, \quad h = \frac{d^2}{2r}$$

If the planet were not accelerated, then it would move from P to R, a distance d, and this is equal to the velocity $\times$ time:

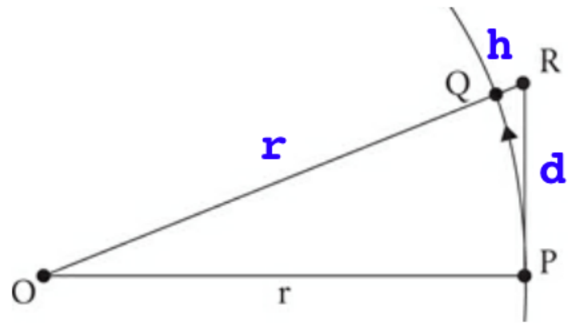
$$d = vt$$

At this point, we use an idea from calculus. *For a small enough segment of the orbit*, this distance PR is the same as the arc length PQ.



So we substitute for $d^2 = (vt)^2$ into the equation from above

$$h = \frac{d^2}{2r} \approx \frac{(vt)^2}{2r}$$



Also, for a small enough part of the orbit (again), h and d are perpendicular to each other as well.

At this point we use the additional assumption that the force is directed toward the sun. We might say that the distance *fallen* by the planet in this short time is h.

By the standard equation of motion, under gravitational acceleration g is related to h and the time t by this equation:

$$h = \frac{1}{2}gt^2$$

We combine the two different expressions for h

$$h = \frac{1}{2}gt^2 \approx \frac{(vt)^2}{2r}$$

$$g \approx \frac{v^2}{r}$$

Note: we have not covered this yet. If this idea (dependence on t^2) is completely new to you, you may want to come back to this part after going through the first chapter on calculus.

The equation $a = v^2/r$ comes even more easily with a little bit of calculus and the use of vectors. See **here**.

Kepler's Third Law

The famous mathematician Johannes Kepler (of whom much more later also), working with observational data from Tycho Brahe, had the following values for the radius

R of the (assumed circular) orbit and the period T (time for completion of one orbit), for five planets.

Orbital data for the six planets known in Kepler's time

	$\overline{r}$ (units of $\overline{r}_{\text{Earth}}$)	T (years)
Mercury	0.387	0.241
Venus	0.723	0.615
Earth	1.000	1.000
Mars	1.524	1.881
Jupiter	5.203	11.862

On the basis of this data, Kepler published his **third law** (in 1619, about 10 years after the first two). K3 states that

$$T^2 = kR^3$$

The square of the period is proportional to the cube of the radius of the orbit. The data in the table has been scaled so that $k = 1$.

For a circular orbit, the orbital speed, the magnitude of the velocity $v = |\mathbf{v}|$, is constant.

The period times the speed is equal to the circumference.

$$vT = C = 2\pi R$$
$$T = \frac{2\pi R}{v}$$

K3 above says that

$$R^3 = T^2$$
$$= \frac{(2\pi)^2 R^2}{v^2}$$

Hence

$$v^2 \approx \frac{1}{R}$$

We showed above that the acceleration for a circular orbit is

$$a = \frac{v^2}{R} = v^2 \cdot \frac{1}{R}$$

so we conclude that that

$$g = a \approx \frac{1}{R} \cdot \frac{1}{R} = \frac{1}{R^2}$$

if the acceleration of gravity g is directed toward the sun, with a magnitude that is inversely proportional to the square of the distance, then we can explain Kepler's third law by running this chain of reasoning in reverse.

comparing the moon to an apple

Earlier we worked out that the acceleration is

$$a = \frac{v^2}{R}$$

Let's figure out the acceleration of the moon. We make a decision to work in English units for this one.

The moon averages about 237 thousand miles from earth (221.5 - 252.7 thousand miles). The earth's circumference is about 24.9 thousand miles so its radius is about 3.96 miles. Thus, the ratio of the moon's distance to the center of the earth, compared to my distance to the center of the earth, is about 60 : 1 (ranging between 56-64).

What is the moon's velocity? The distance it travels in one complete orbit (in feet) is:

$$2\pi \cdot 2.4 \times 10^5 \cdot 5280$$

The time that takes in seconds is

$$28 \cdot 24 \cdot 3600$$

$$v = \frac{2\pi \cdot 2.4 \times 10^5 \cdot 5280}{28 \cdot 24 \cdot 3600}$$

The acceleration is v^2/R so we square everything except the radius.

$$a = \frac{(2\pi)^2 \cdot 2.4 \times 10^5 \cdot 5280}{(28 \cdot 24 \cdot 3600)^2} = 0.0085$$

That's in feet per second.

We compare this value to the acceleration measured at the surface of the earth, which is 32.2 in the same units. The ratio is 3788, which is just over $(61.5)^2$.

Newton:

> I began to think of gravity extending to the orb of the Moon . . . and computed the force requisite to keep the Moon in her Orb with the force of gravity at the surface of the earth . . . & found them answer pretty nearly. All this was in the two plague years of 1665-1666. For in those days I was in the prime of my age for invention & minded mathematicks and Philosophy more than at any time since.

Part VI

More number sets

Chapter 25

Rationals

The integers are great, they give us an infinite supply of numbers.

However, there is a problem with division. For

$$p \in \mathbb{N}, \quad q \in \mathbb{Z}$$

very often the result of $p \div q$ is not contained in $\mathbb{N}$ or even in $\mathbb{Z}$ — the result is not an integer. We say these sets are not *closed* under division.

For example $3 \div 2 = ?$

So, we just leave the result as

$$\frac{p}{q} = \frac{3}{2}$$

where p/q is in "lowest terms", i.e. they have no common factor other than 1. Of course if p and q have a common divisor, then we can divide both top and bottom by the largest common divisor.

q must not be zero because division by zero is not defined.

theorem

• Between *any* two rational numbers it is always possible to find another rational number.

Consider two rational numbers, not equal. Let

$$s = \frac{p_1}{q_1} \quad t = \frac{p_2}{q_2}$$

Suppose $s < t$.

The *average* of these two numbers is:

$$r = \frac{1}{2} \; [\; s + t \;]$$

Then

$$2r = s + t$$

$$2r - 2s = t - s$$

We have that $s < t$, so $t - s > 0$ and then

$$r - s > 0$$

$$r > s$$

A similar argument will show that

$$r < t$$

so

$$s < r < t$$

□

Thus, one can always find a new rational number that lies between two known rational numbers. In particular, there is no *smallest* positive rational number.

decimal representation

Every rational number can be represented as a decimal, using the method called long division.

Consider 1/2

$$2\overline{)1.000}$$

We say that 2 does not *go into* 1, since $2 > 1$, so we have the first part of our result as 0, followed by a decimal point. But 2 does go into 10 exactly 5 times, giving 0.5. The remainder is zero and so the division process terminates.

Consider 1/8.

$$8\overline{)1.000}$$

∘ 8 goes into 10 once, leaving 2 as remainder

∘ 8 goes into 20 twice, leaving 4.

∘ 8 goes into 40 exactly 5 times with no remainder.

The result is 0.125.

The other possibility is that in going through the process a remainder comes up that has been seen previously. If this happens then the sequence will repeat forever.

If we don't terminate with zero, then this must eventually happen, because there are only as many as q possible remainders.

Thus, for example

$$1/7 = 0.142857142857\ldots$$

which contains 142857, repeating.

decimals to fractions

Conversely, every repeating decimal can be represented as a rational number. For example

```
      1 x r =      0.142857142857...
1000000 x r = 142857.142857...
 999999 x r = 142857
```

$$r = \frac{142857}{999999} = \frac{1}{7}$$

since 7×142857 equals 999999 exactly.

You can do this trick with

```
     r = 0.333
10 x r = 3.33
 9 x r = 3
```

$$r = \frac{3}{9} = \frac{1}{3}$$

or even

```
    r = 0.4999
10 x r = 4.999
 9 x r = 4.5
```

$$r = \frac{4.5}{9} = \frac{1}{2}$$

and

```
    r = 0.9999
10 x r = 9.999
 9 x r = 9
```

$$r = \frac{9}{9} = 1$$

This is one of the subtleties of numbers. In what sense can we say that

$$0.5 = 0.4999\ldots$$

$$1 = 0.9999\ldots$$

Most everyone is OK with the example $1/3 = 0.3333\ldots$ but some may be uneasy with the other two.

Ultimately, we justify the result as defined by evaluation of a limit.

Consider 0.9999. If n is the number of places in the result, then as $n \to \infty$ the number being shown approaches 1 as its limit. We'll come back to this after considering the real numbers.

ordering

For two rational numbers a and b there are only three cases:

$$r_1 = r_2, \qquad r_1 < r_2, \qquad r_1 > r_2$$

It is a property of the integers, that if

$$a < b$$

then for $c > 0$

$$ca < cb$$

using that property

$$\frac{p}{q} < \frac{s}{t} \iff pt < qs$$

p/q is less than s/t if and only if $pt < qs$. (If only one of p and q or s and t is negative, associate the minus sign with the numerator).

Ordering of the integers guarantees ordering of the rational numbers. For any rational numbers, if

$$\frac{p}{q} < \frac{s}{t}$$

then for $c > 0$

$$c \cdot \frac{p}{q} < c \cdot \frac{s}{t}$$

For that matter, it is generally true for real numbers (which include integers and rationals) that if

$$a < b$$

for $c > 0$

$$ca < cb$$

intervals

We denote the numbers greater than u and less than v as lying in the interval (u, v). With parentheses, the interval described is *open*, it does not include the boundary values.

To describe a *closed* interval, write $[u, v]$. This interval includes all the values in the first one, plus it also includes u and v.

Because of the density property described below, any interval such as

$$I = [0, 1]$$

contains an *infinite* quantity of rational numbers.

density

Consider the set of all points

$$x = \frac{p}{10^n}$$

for all natural numbers n and integers p.

It is clear that simply by increasing the value of n, we can construct a set of equally spaced rational numbers as tightly clustered as we wish.

The rational numbers are said to be *dense* on the number line.

Chapter 26

Infinity

infinity

The symbol for infinity is ∞.

In the old days, they used to write things like

$$\frac{1}{0} = \infty, \qquad \frac{1}{\infty} = 0$$

John Wallis wrote $24/0 = \infty$, in 1656, which is when the ∞ symbol was introduced with its current definition. Even Euler argued that $n/0 = \infty$ when it suited him,

It is claimed that the symbol derives from the Roman symbol for 100 million. That's interesting. I never knew any symbols larger than M, for one thousand. And I'm not sure I believe it, but that's what some people say.

According to

https://notevenpast.org/dividing-nothing/

> On 21 September 1997, the USS Yorktown battleship was testing "Smart Ship" technologies on the coast of Cape Charles, Virginia. At one point, a crew member entered a set of data that mistakenly included a zero in one field, causing a Windows NT computer program to divide by zero. This generated an error that crashed the computer network, causing failure of the ship's propulsion system, paralyzing the cruiser for more than a day.

no division by zero

There is a fundamental problem when we set up a division problem and 0 is in the denominator. What goes wrong when we attempt to divide by zero?

$$\frac{a}{0} = ?$$

Well, what do we mean by an expression such as

$$\frac{a}{b} = c$$

By *definition*, we mean that we will try to find c such that

$$c \cdot b = a$$

For the integers, of course, there is the problem of a possible remainder. Let us leave that aside for a minute.

Suppose we have $c \cdot b = a$ but then take b to be very small though not 0. In that case, the number c may get very large. That's OK.

We can make b as small as we wish by making c large enough or vice versa. And we can say that as $b \to 0$, then $c \to \infty$.

But we can't say $a/0 =$ some number.

If there were such a number (say $a/0 = \infty$, infinity), then what about

$$\frac{b}{0} = ??, \qquad \frac{c}{0} = ??$$

It would mean that whatever the expression $b/0$ is equal to, when multiplied by zero, we would obtain any number whatsoever. This makes no sense.

Here is another, perhaps silly, example.

$$0 \cdot 1 = 0$$

$$0 \cdot 2 = 0$$

so

$$0 \cdot 1 = 0 \cdot 2$$

but then

$$1 = 2$$

By definition, we do not allow division by zero.

infinity is not a number

And we can't answer the question what is $2 \cdot \infty$? If we allowed multiplication by ∞ then the only reasonable answer would be

$$2 \cdot \infty = \infty$$

so then also

$$n \cdot \infty = \infty$$

where n is any number. But then say

$$2 \cdot \infty = 3 \cdot \infty$$

so, cancelling

$$2 = 3$$

This would be a mess.

By definition, *infinity is not a number* and division by 0 is *undefined.*

limits

Often people say that calculus is all about limits, and they are certainly where you start in proving the theoretical basis of the field.

We will keep the discussion of limits and ϵ-δ formalism to a minimum for the reasons explained in the Introduction. But let us try to establish an intuitive idea about what we mean when we say "in the limit as $N \rightarrow \infty$".

Above we had that there is no greatest integer.

A corollary of that is the limit

$$\lim_{n \to \infty} \frac{(n+1)-n}{n} = 0$$

Why? As n increases without bound, the difference between successive numbers, as a fraction of n, tends to zero.

To get an idea about this, first simplify by multiplying by $1/n$ on top and bottom. Then we have

$$\lim_{n \to \infty} \frac{(1+1/n-1)}{1} = \frac{1}{n}$$

We say that $1/n$ *tends* to zero as $n \rightarrow \infty$, and so does $[(n+1)-n]/n$.

Chapter 27

Euclid's algorithm

Consider two natural numbers a and b. Usually a is allowed to be an integer (i.e., it can be negative), but to keep things simple here we will say that $a, b \in \mathbb{N}$, a and b are positive integers.

We can find their *greatest common divisor*, written (a, b). First we write the unique prime factorization of a and b:

```
180 =           2 x 2 x 3 x 3 x 5
140 =           2 x 2 x         5 x 7
gcd(140,180) = 2 x 2 x         5 = 20
```

Pick out the common factors and the $\gcd(a, b)$ will be their product. It is important that we do not need to actually factor a and b.

(We will develop a theorem on unique prime factorization in another chapter).

The algorithm works like this. Find integers $r \geq 0$ and $q > 0$ such that

$$a = b \cdot q + r$$

∘ If $r = 0$ we are done: b divides a equally. Otherwise

∘ switch $a = b$ and $b = r$ and repeat.

Then b is the gcd of the original a and b.

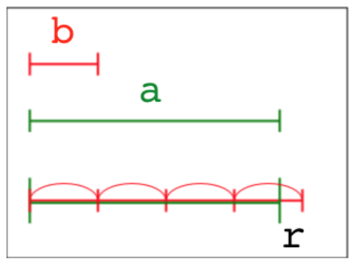
In our example

```
180 = 140 x 1 + 40
140 =  40 x 3 + 20
 40 =  20 x 2 + 0
gcd =  20
```

Here is the reason this works. First, we can always find q and r such that

$$a = b \cdot q + r$$

This is a version of the Archimedean property for positive integers.



It may be paraphrased by saying

> given a bathtub full of water and a teaspoon, it is possible to empty the bathtub.

Either $a = b \cdot q$ and we are done or:

$$b \cdot q < a < b \cdot q + b$$

So then

$$a - bq > 0$$

$$a - bq < b$$

With $r = a - bq$, we obtain $0 < r < b$.

Let u be the largest integer that divides both a and b (the greatest common divisor)

$$a = su$$

$$b = tu$$

Then

$$su = q \cdot tu + r$$

$$r = su - q \cdot tu$$
$$r = u(s - q \cdot t)$$

So u divides r.

Hence every common divisor of a and b is also a divisor of b and r.

recursive program

Here are two examples of programs in different styles that implement the algorithm (with no error checking):

```
def gcd(a,b):
    r = a % b
    if r == 0:
        return b
    return gcd(b,r)
```

```
def gcd(a,b):
    r = a % b
    while r != 0:
        a,b = b,r
        r = a % b
    return b
```

The first version is *recursive*, it may call itself. The second uses a **while** loop to accomplish the same thing.

Chapter 28

Real numbers

There is a big problem with rational numbers which you probably know: some numbers cannot be expressed as the ratio of two integers, as a first example, the number which when multiplied by itself is equal to 2, written $\sqrt{2}$.

The discovery that one cannot find integer p and q such that

$$(\frac{p}{q})^2 = 2$$

is due to the Pythagorean school and was most unwelcome since it screwed up their cherished theory of the universe.

Some say that they drowned the guy who discovered it by throwing him overboard, and that his name was Hippasus. Like most stories about Greek mathematicians, the truth is unknown.

We will see that there is a similar problem (called irrationality) with $\sqrt{3}$, $\sqrt{5}$, $\sqrt{7}$, etc., as well as with $3^{1/3}$ and so on.

Proof.

For $\sqrt{2}$:

We assume that there does exist a rational number p/q such that

$$\frac{p}{q} = \sqrt{2}$$

We will show that this assumption leads to a contradiction.

A crucial part of the proof is that we suppose p/q to be in lowest terms and in particular, that p and q are not both even. It would be easy to recognize the case if they were both even, for then each would have their terminal digit in the set $\{0, 2, 4, 6, 8\}$.

Another fact we will need is that every odd number, when squared, gives an odd result. Proof: every odd number can be written as $2k + 1$ (for non-negative integer k) and then

$$(2k + 1)^2 = 4k^2 + 4k + 1$$

which is an odd number. Therefore, if n^2 is even, n is also even.

So go back to

$$\frac{p}{q} = \sqrt{2}$$

Move the q term to the right-hand side and square both sides:

$$p^2 = 2q^2$$

This implies that p^2 and p are even, using the result from above. So we can write that $p = 2m$. But now

$$(2m)^2 = 2q^2$$
$$2m^2 = q^2$$

which implies that q is *also* even.

We started with the assumption that p and q are not both even, but now we've reached a contradiction. We conclude that there do not exist two integers p and q such that $p/q = \sqrt{2}$.

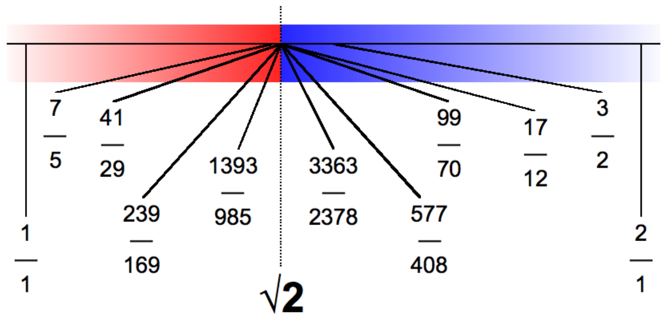
discussion

To quote Hardy (*A Mathematician's Apology*):

> The proof is by reductio ad absurdum, and reductio ad absurdum, which Euclid loved so much, is one of a mathematician's finest weapons. It is a far finer gambit than any chess gambit: a chess player may offer the sacrifice of a pawn or even a piece, but a mathematician offers *the game*.

The numbers like $\sqrt{2}$ are said to be *irrational* numbers and the set of these, plus all the other numbers is called the set of real numbers $\mathbb{R}$.

This led Dedekind to formulate the famous Dedekind cut. Visualize the standard number line as an infinite line on (an infinite) piece of paper.



Each real number corresponds to a cut, a knife-edge coming down somewhere on this number line. Every other number that is not equal to this one, is either $>$ or $<$ the number specified by the cut.

One position is $\sqrt{2}$, another is $3/2$ and so on.

proof using prime factors

The fundamental theorem of arithmetic says that any positive integer greater than 1 can be expressed as a product of its prime factors

$$n = p_1 \cdot p_2 \dots p_k$$

where this factorization is unique (if the factors are sorted first), and multiple copies allowed. For example

$$60 = 2 \cdot 2 \cdot 3 \cdot 5$$

A corollary says that the square of any integer (a perfect square) has an even number of prime factors since

$$n^2 = p_1^2 \cdot p_2^2 \dots p_k^2$$

In the expression from above

$$p^2 = 2q^2$$

the number of prime factors on the left is therefore even, but the number on the right is odd. This is a contradiction. Therefore p and q cannot both be integers.

continued fractions

Square roots can be represented as continued fractions. Some smart person figured out that we can write this:

$$(\sqrt{2}-1)(\sqrt{2}+1) = 2 - 1 = 1$$

Now, rearrange to get a substitution we will use repeatedly

$$\sqrt{2} - 1 = \frac{1}{\sqrt{2}+1}$$

Add one and subtract one on the bottom right:

$$\sqrt{2} - 1 = \frac{1}{2 + \sqrt{2} - 1}$$

And substitute for $\sqrt{2} - 1$:

$$= \frac{1}{2 + \frac{1}{\sqrt{2}+1}}$$

Lather, rinse, and repeat:

$$= \frac{1}{2 + \frac{1}{2+\sqrt{2}-1}} = \frac{1}{2 + \frac{1}{2+\frac{1}{\sqrt{2}+1}}}$$

Clearly, this goes on forever.

$$\sqrt{2} - 1 = \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \dots}}}$$

Add 1 to the value of the *continued fraction* to get an expression for the square root of 2.

The numerators are all 1, so this is called a simple continued fraction. The continued fraction representation of $\sqrt{2}$ is usually written as [1 : 2], meaning that there is an initial 1 followed by repeated 2's.

This fraction goes on forever (since $\sqrt{2}$ is irrational). One can view the existence of the infinite continued fraction as a proof of irrationality.

We can turn the above into an approximate decimal representation of $\sqrt{2}$, by truncating the infinite expansion at the .... Then the last fraction is 5/2. Invert and add, repeatedly:

```
2 + 1/2 = 5/2
2 + 2/5 = 12/5
2 + 5/12 = 29/12
2 + 12/29 = 71/29
2 + 29/71 = 171/71
2 + 71/171 = 413/171
```

To terminate we need to use that initial 1:

```
1 + 171/413 = 584/413 = 1.414043
```

To six places, $\sqrt{2} = 1.414213$. We have only three places, but can get more (convergence is relatively slow, however).

geometric proof

There are many other proofs of the irrationality of the square root of 2.

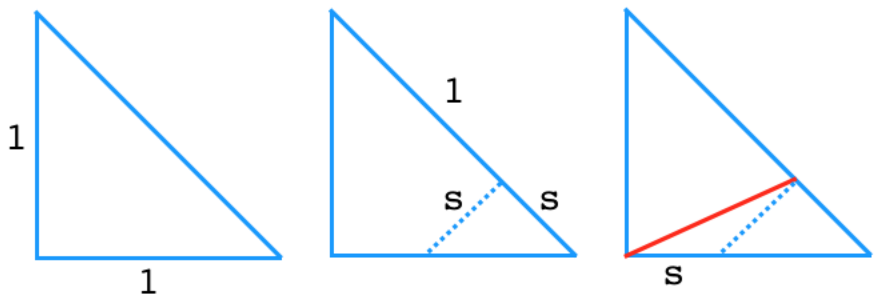
https://www.cut-the-knot.org/proofs/sq_root.shtml

Here we will look at one more, before considering a more general proof for all non-perfect squares. This one is from Tom Apostol (see the link). A more elaborate exposition is:

https://jeremykun.com/2011/08/14/the-square-root-of-2-is-irrational-geometric-proof/

Draw an isosceles triangle with side length 1, then Pythagoras tells us that the hypotenuse is equal in length to $\sqrt{2}$ (left panel).

Our hypothesis is that the length of the hypotenuse is a rational number, and that its ratio to the side is in "lowest terms".



Now mark off the length of the side on the hypotenuse and erect a perpendicular (middle panel). The new small triangle that is formed is also isosceles (it is a right triangle and it also contains one of the complementary angles of the original right triangle). By hypothesis, its side length s is the difference of two rational numbers, so it is a rational number.

Furthermore, the triangle with the red base and blue sides of length 1 is isosceles (right panel), so by complementary angles the triangle with the red base and one side a dotted line has equal angles at its base and so is isosceles. All the lengths marked s are equal.

Therefore, the hypotenuse of the new, small right triangle is a rational number, since it is equal to $1 - s$.

We are back where we started, with an isosceles triangle that has all rational sides.

It is clear that this process can continue forever. The sides will never be in "lowest terms" because we can always form a new similar but smaller right triangle, which amounts to evenly dividing both the sides and the hypotenuse by a rational number.

general proof

I found a long algebraic proof of the general irrationality of roots and it is discussed **here**. What follows is a much simpler proof based on the fundamental theorem of arithmetic.

We suppose that there exist two integers a and b such that

$$(\frac{a}{b})^2 = n$$

According to the fundamental theorem of arithmetic, both a and b have a unique prime factorization. Suppose that gives $a = a_1 \cdot a_2 \ldots a_i$ and likewise for b so:

$$(\frac{a_1 \cdot a_2 \ldots a_i}{b_1 \cdot b_2 \ldots b_j})^2 = n$$

If every factor b were some a_i, then we could cancel all of them and so a/b would be an integer.

If a/b is to be rational but not an integer, there must be at least one prime factor of b that cannot be cancelled. Call that (those) q, so in lowest terms we have

$$(\frac{a_1 \cdot a_2 \ldots}{q_1 \ldots})^2 = n$$

But then, after squaring, we will have q_1^2 in the denominator and no corresponding factor of either q_1 or q_1^2 in the numerator. Thus, they cannot be canceled and the result cannot be an integer.

This proves that the only n with rational square roots are perfect squares with integer roots.

The proof also applies generally to other powers like cube and the fourth and fifth power and so on.

other irrational numbers

There are many other irrational numbers besides these square roots. The proof that e is irrational is easy, but since we haven't introduced the exponential yet we need to wait.

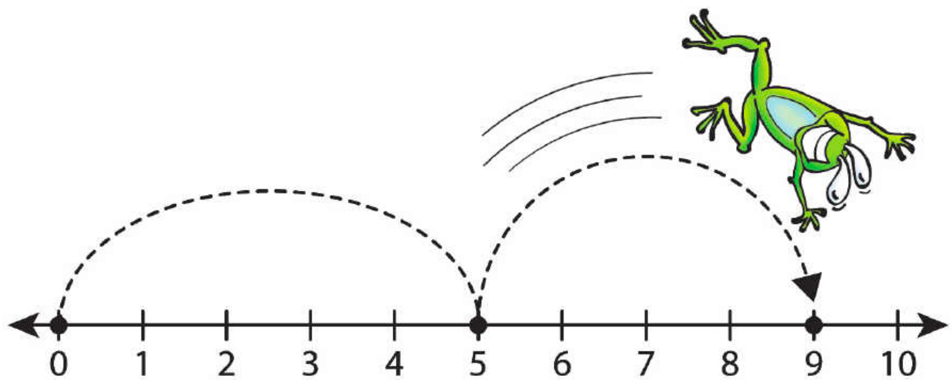
The proof that π is irrational is a bit harder, so we defer that as well.

density

number line

A simple tool to visualize all of the real numbers is the familiar number line. Here is the number line with numbers marked from $\mathbb{N}$, but obviously we could also draw one for $\mathbb{Z}$ or $\mathbb{Q}$.

We explore the application of the number line to $\mathbb{R}$ as we proceed.



We might simply assume that to every point on the number line there corresponds a rational or irrational number, and that this total collection obeys the same laws of arithmetic as the rational numbers do.

As mentioned above, the need for the real numbers is indicated by empty "holes" in the number line corresponding to the irrational numbers like $\sqrt{2}$.

A problem that arises is how to specify an irrational number non-geometrically and other than as the solution to an equation such as $r^2 = 2$. We saw above a method involving continued fractions.

approximations

In all cases we write particular real numbers as *approximations*. For example, the square root of 2 lies between 1 and 2 because

$$1^2 = 1 < 2$$

$$2^2 = 4 > 2$$

Implying that $\sqrt{2} < 2$. At the second place:

$$1.4^2 = 1.96 < 2$$

$$1.5^2 = 2.25 > 2$$

Implying that $\sqrt{2} < 1.5$. At the third:

$$1.41^2 = 1.9881 < 2$$

$$1.42^2 = 2.0164 > 2$$

Implying that $\sqrt{2} < 1.42$.

This process may be continued for as long as desired.

We can never write down the decimal value of $\sqrt{2}$ exactly, but only approximate it to greater and greater precision. It goes on forever.

In carrying out this recursive process, suppose we know 1.41 and we seek the next digit. Rather than try all the digits in order starting with 1, there is a better way.

Try to estimate the error from the previous round.

For example $1.41^2 = 1.9881$ so we are short of 2.0000 by 0.0119.

$1.42^2 = 2.0164$ so the difference is 0.0283 and the fraction of the difference that we're under is $119/283 = 0.4205$. In fact, the next two digits of the approximation to $\sqrt{2}$ are 42.

However, we will see a much better method for obtaining this value later, called Newton's method.

At the seventh place

$$1.414213^2 = 1.9999984093689998.. < 2$$

$$1.414214^2 = 2.0000012377960004 > 2$$

Because any repeating decimal can be written as a fraction, we know that the sequence cannot repeat (any apparent repeat will be illusory).

It is a curious fact that all the digits of π, *to whatever accuracy you desire*, can be found in the correct order, somewhere within the digital expansion of e or ϕ or indeed, any irrational number. The converse is also true.

Another way to say the same thing is that *any* finite sequence can be found within *any* infinite sequence, and in as many copies as you have the patience to discover. The sequence 271828 is found starting around digit 33,790 of π, but 2718281 (adding the next digit of e) is not found within the first million digits of π. You just need more.

limit of a sequence

The real number $\sqrt{2}$ is defined to be the limit of the sequence

$1.4, 1.41, 1.414, \ldots 1.414214 \ldots$

as the number of terms $n \to \infty$.

In a similar way, the number e can be viewed as

$$\lim_{n\to\infty} (1 + \frac{1}{n})^n$$

And the number π can be viewed as the limit of the method of exhaustion applied to the area of a unit circle.

density of numbers

We showed previously that between any two rational numbers, including 0 and the *smallest* positive number, one can find another rational number which lies between them.

Three related statements are also true.

∘ for any two rational numbers one can find a real number which lies between them

∘ for any two real numbers one can find a rational number which lies between them

∘ for any two real numbers one can find a real number which lies between them

Proofs of these are readily accessible but will be given separately **here**.

This property of the real (and even the rational) numbers, that there is no closest number to any given number, accounts for virtually all of the theoretical difficulties in calculus which are solved by the use of limits and the apparatus of δ and ϵ or alternatively, neighborhoods. We will get to that in a bit.

Part VII

Algebra

Chapter 29

Basic algebra

There is very little algebra that needs to be memorized for where we're going. We just finished a chapter about the sum of integer squares. If you can follow that, you're in good shape. If not, go back and work through it again, carefully.

Here is a bit more:

inequality

You have surely seen and used the symbols $>$ (greater than), and $<$ (less than) before we used them a second ago.

Among the axioms of the number systems is the collection of *order axioms*. A few definitions:

- $x < y$ means that $y - x$ is positive
- $y > x$ means that $x < y$

For arbitrary numbers a and b only one of three statements is true:

- $a < b$
- $a = b$
- $a > b$

There is no attempt or need to be systematic here. Let us just mention that these properties (and their kin) are true not just for natural numbers, but also for the rational numbers and the real numbers, as we will see in due course.

Here are a few important theorems about order which we will use often:

- If $a < b$, and c is any number, then $a + c < b + c$
- If $a < b$, then $-b < -a$
- If $a < b$ and $c > 0$, then $ac < bc$

The first one above implies the second and the third.

algebraic operations

- addition: $a + b$
- subtraction: $a - b = a + (-b)$

The negative integers and 0 solve the problem of how to evaluate $a - b$ when $b \geq a$.

- multiplication: $a \cdot b$, also often written ab (but not $a \times b$, at this level).

And then:

- division a/b, equivalent to finding a number c such that $c \cdot b = a$.

algebra

As you know, the basic axioms of algebra include the following:

- Commutativity for addition and multiplication:

$$a + b = b + a, \qquad a \cdot b = b \cdot a$$

- Associativity for addition and multiplication:

$$(a + b) + c = a + (b + c), \qquad (a \cdot b) \cdot c = a \cdot (b \cdot c)$$

- Distributivity of addition over multiplication:

$$a \cdot (b + c) = a \cdot b + a \cdot c$$

- Additive identity: $0 + a = a$.
- Multiplicative identity: $1 \cdot a = a$.

binomial theorem

One basic idea from algebra is the binomial theorem:

$$(a+b)^2 = (a+b)(a+b) = a^2 + 2ab + b^2$$

so then

$$\begin{aligned}(a+b)^3 &= (a+b)(a^2 + 2ab + b^2) \\ &= a^3 + 3a^2b + 3ab^2 + b^3\end{aligned}$$

And in general, to get the next $n+1$ power, go through the expansion for the n power and multiply each term separately by a and b.

You will find that the cofactors are given by Pascal's triangle.

$$\begin{array}{ccccccccccc} &&&&&1&&&&& \\ &&&&1&&1&&&& \\ &&&1&&2&&1&&& \\ &&1&&3&&3&&1&& \\ &1&&4&&6&&4&&1& \\ 1&&5&&10&&10&&5&&1 \end{array}$$

$$\begin{aligned}(a+b)^3 &= a^3 + 3a^2b + 3ab^2 + b^3 \\ (a+b)^4 &= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4 \\ (a+b)^5 &= a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5 \\ &\dots\end{aligned}$$

If you substitute $-b$ for b you will find that everything is exactly the same, except those terms with b raised to an odd power have acquired a minus sign.

$$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$$

Hence

$$(a-b)(a^2 - 2ab + b^2) = a^3 - 3a^2b + 3ab^2 - b^3$$

The binomial theorem is usually stated and worked with in terms of positive integers.

But it actually works for negative integers and fractional powers as well. The big difference is that the series of terms is *infinite*.

Newton discovered that, though he didn't prove it. He just used it as a tool. He found that

$$\frac{1}{1+x} = (1+x)^{-1}$$
$$= 1 - x + x^2 - x^3 + x^4 + \cdots$$

Newton checked this by multiplying:

$$(1+x)(1 - x + x^2 - x^3 + x^4 + \cdots)$$

$$= 1 - x + x^2 - x^3 + x^4 + \cdots$$
$$+x - x^2 + x^3 - x^4 + \cdots = 1$$

But be careful! What happens if $x = -1$?

factoring

The classic quadratic equation is often written

$$y = ax^2 + bx + c$$

This is a parabola that opens up ($a > 0$).

Depending on the values of the *coefficients* a, b, c, this equation may or may not have solutions when $y = 0$

$$ax^2 + bx + c = 0$$
$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

Suppose that r and s are such solutions, then

$$(x-r)(x-s) = 0$$

and so

$$x^2 - (r+s)x + rs = 0$$

A fair amount of effort in algebra goes into guessing values of r and s that work, that have the appropriate sum and difference to match the equation we are given.

However, the answers frequently are not integers, and in that case, this approach is doomed.

A formula that always works is the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

I said "always works". To be more precise, it works if there are any solutions. If the discriminant $D = b^2 - 4ac$ is negative, then we're trying to take a square root of a negative number, and we don't know how to do that in the real numbers.

We will develop this more in the chapters on analytic geometry. That's really all you will need.

Chapter 30

Exponential and logarithm

Principal and interest

Suppose I put 100 dollars in the bank, and the people at the bank say that after one year, they will give me an additional \$10 at that time. They will pay 10% interest for the year on the principal P of \$100.

However, suppose I bargain with them. I get them to promise to pay me half the interest (5%) at the six-month mark, and the rest after one year. My account will hold \$105 after six months, and the interest due for the second half will be 5% of \$105, which is \$5.25 for a total of \$10.25.

The equation to describe this situation is that if the rate of interest for the year is r and the year is broken up into n periods when interest will be paid, the total amount at the end will be:

$$A = P(1 + \frac{r}{n})^n$$

In the example, we have $r = 0.10$ and $n = 2$ so

$$A = 100(1 + 0.05)^2 = 110.25$$

This is compound interest. If there are additional years t, the exponent will be nt rather than n.

And now we start wondering what happens if the bank pays every month so that $n = 12$ or every day so $n = 365$ or even every second. What happens if the interest is compounded *continuously*?

$$A = \lim_{n \to \infty} P \ [\ (1 + \frac{r}{n})^n\ ]$$

Now it turns out that in the limit as n approaches ∞ these two expressions are equal

$$(1 + \frac{r}{n})^n = \ [\ (1 + \frac{1}{n})^n\ ]^r$$

The same factor r can be either in the numerator of the second term inside or up in the exponent outside.

A quick proof is:

$$\lim_{n \to \infty} (1 + \frac{r}{n})^n$$

$$= \lim_{n \to \infty} (1 + \frac{r}{n})^{(n/r)r}$$

Define $m = n/r$ and so as $n \to \infty$, so does $m \to \infty$ and then we have

$$\lim_{m \to \infty} (1 + \frac{1}{m})^{(m)r}$$

and the r is outside. m is just a dummy variable so we write:

$$\lim_{n \to \infty} \ [\ (1 + \frac{1}{n})^n\ ]^r$$

□

Therefore, going back to what we were working on, let us bring out the factor r and obtain

$$A = P(1 + \frac{1}{n})^{nr}$$

$$A = P \ [\ (1 + \frac{1}{n})^n\ ]^r$$

Thus, the important question is, what is the value of this expression?

$$A = \lim_{n \to \infty} (1 + \frac{1}{n})^n$$

It does not depend on r. It will turn out that this limit is equal to the number e.

$$e = 2.71828\ 18284\ 59045 \ldots$$

That's really all we need to worry about with respect to e, for now.

As far as general exponents go, I'm sure you know that:

$$x^a \ x^b = x^{a+b}$$

$$(x^a)^e = x^{ae}$$

$$x^{-a} = \frac{1}{x^a}$$

$$x^{1/2} = \sqrt{x}$$

working with logarithms

The logarithm and exponential functions are inverses. If we have that

$$y = b^x$$

for some $b > 0, b \neq 1$, then we say that

$$x = \log_b \ y$$

Putting them together

$$y = b^{\ \log_b y}$$

The usual bases are 10 (common logarithm, $\log_{10}$, or just log), e (natural logarithm or ln), and 2 (binary logarithm or $\log_2$).

The rules for exponents are simple, if p and q are two numbers and we know the logarithms of p and q to base b

$$p = b^u; \quad q = b^v$$

then their product can be computed as:

$$pq = b^u \cdot b^v = b^{u+v}$$

It helps if we can actually compute b^{u+v}. In the old days there were tables of logarithms, so you just looked up the answer in the table.

The second rule is that:

$$(b^u)^v = b^{uv}$$

And in terms of logarithms we write

$$\log_b (b^u)^v = \log b^{uv} = v \log_b (b^u)$$

For example

$$2^2 = 2 \times 2 = 4$$

$$2^3 = 2 \times 2 \times 2 = 8$$

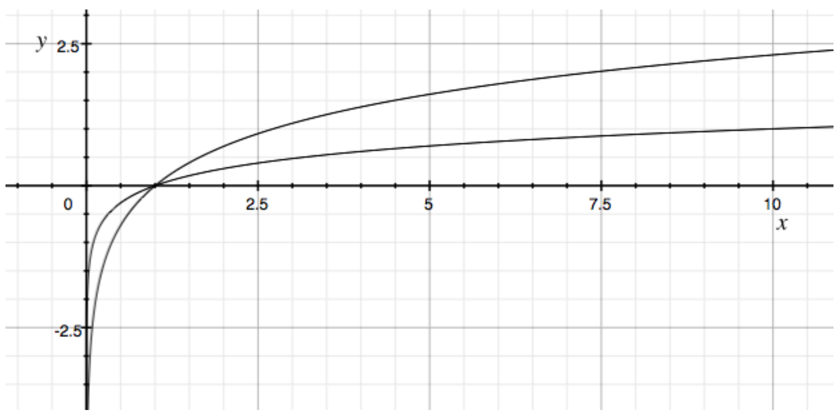
$$4 \times 8 = 2^2 \times 2^3 = 2^{2+3} = 2^5$$

$$= 2 \times 2 \times 2 \times 2 \times 2 = 32$$

and

$$(2^2)^3 = 4^3 = 64 = 2^6 = 2^{2 \times 3}$$

Here is a plot of $\log_{10}(x)$ and $\ln x$:



The first function reaches the value 1 when $x = 10$ and the second reaches the value 1 when $x = e$. Both have the value 0 at $x = 1$ because $b^0 = 1$ for any base, so the logarithm to any base of 1 is equal to 0.

It turns out that if we take the logarithm of x (where x is any number > 1) to two *different* bases, the ratio of the logarithms is a constant, independent of the value of x.

change of bases

This is nicely shown by the change of bases formula.

$$\log_b x = \frac{\log_a x}{\log_a b}$$

Start with an expression with b as the base:

$$y = b^x$$

and by the definition of the logarithm

$$x = \log_b y$$

To derive the formula, take the logarithm to the base a on both sides of the first expression:

$$\log_a y = \log_a (b^x)$$

Now, just invoke the second rule on the right-hand side

$$= x \log_a b$$

and substitute for x from the second expression above

$$= \log_b y \log_a b$$

We're basically done.

y can be any value, so replace it by x

$$\log_a x = \log_b x \log_a b$$

Rearranging:

$$\log_b x = \frac{\log_a x}{\log_a b}$$

One way I remember this is that first the logarithms to different bases are connected by some constant k

$$\log_b x = k \log_a x$$

and we substitute for k the inverse of the log to the *same* base as we have in the numerator:

$$\log_b x = \frac{\log_a x}{\log_a b}$$

that is, I remember that we want $\log_a$ something *over* $\log_a$ something on the right.

Alternatively, you might look at the other formula

$$\log_a x = \log_a b \log_b x$$

and imagine the b's canceling in some way.

One other thing we can do is to set $x = a$ in the above formula. We start from

$$\log_b x = \frac{\log_a x}{\log_a b}$$

then with $x = a$

$$\log_b a = \frac{\log_a a}{\log_a b}$$

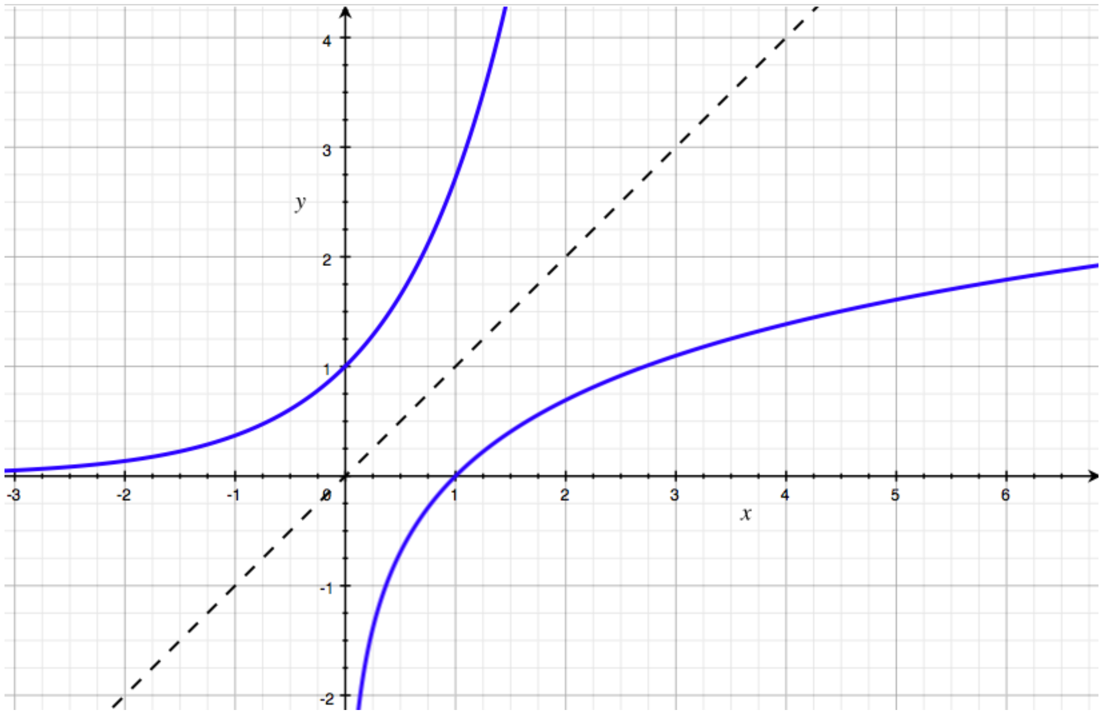
but $\log_a a = 1$ so

$$\log_b a = \frac{1}{\log_a b}$$

And that makes perfect sense. If we multiply by some factor k to convert from the logarithm in base a to base b, we must multiply by the inverse of the same factor to convert back again.

For the figure above of the common log (base 10) and the natural logarithm, $\ln 10 = 2.303$, and that looks about right, when $x = 10$ the first function is 1.0 and the second one is about 2.3.

The logarithm and the exponential are inverse functions, we can see that if we plot them together:



The upper curve is $y = e^x$ and the lower one is $y = \ln x$. As inverse functions, they are symmetric about the line $y = x$.

fractional exponents

The introduction above dealt mainly with integer exponents, but of course you know that the practical use of logarithms depends on fractional values. The simplest way to see how this works is to consider the square root.

$$\sqrt{2} \times \sqrt{2} = 2$$

If we think about what the exponent u to the base 2 would be such that

$$2^u = \sqrt{2}$$

We observe that by the rules for exponents

$$\sqrt{2} \times \sqrt{2} = 2^u \times 2^u = 2^{u+u} = 2^1$$

That is

$$u + u = 1$$

so $u = 1/2$. By the same logic the n^{th} root of b is $b^{1/n}$. And of course

$$(b^2)^{1/2} = b^{2\times 1/2} = b^1$$

Feynman has a nice description of how logarithms were calculated (see Lectures, volume 1, Chapter 22, Algebra;

http://www.feynmanlectures.caltech.edu/I_22.html)

The basic idea is to take repeated square roots of the base (10), and then combine those to form the required value.

Less than 1

Fractional exponents leads to consideration of $0 < x < 1$. Write

$$x\ \frac{1}{x} = 1$$

Take the logarithm of both sides

$$\log(x\ \frac{1}{x}) = \log 1 = 0$$

$$= \log x + \log \frac{1}{x}$$

Thus

$$\log \frac{1}{x} = -\log x$$

Chapter 31

Fibonacci sequence

Continuing with the topic of induction, let's introduce the Fibonacci numbers and Binet's formula.

These are numbers in a series formed by adding together the two previous numbers in the series:

$$F_n = F_{n-2} + F_{n-1}$$

or

$$F_{n+1} = F_{n-1} + F_n$$

It remains to choose the first two numbers, which are 1 and 1. Thus the first ten Fibonacci numbers are

$$1 \quad 1 \quad 2 \quad 3 \quad 5 \quad 8 \quad 13 \quad 21 \quad 34 \quad 55$$

Fibonacci example

We will use induction to prove that the sum of the first n Fibonacci numbers is

$$1 + 1 + 2 + \cdots + F_n = F_{n+2} - 1$$

We assume that the formula is correct for F_{n-1}:

$$1 + 1 + 2 + \cdots + F_{n-1} = F_{n+1} - 1$$

Add F_n to both sides

$$1 + 1 + 2 + \cdots + F_n = F_n + F_{n+1} - 1$$

$$= F_{n+2} - 1$$

This completes the induction.

The base case is

$$1 = 2 - 1$$

□

Another way to check this is to write the sum as

```
1 + 1 + 2 + 3 + 5 + ..       .. + F_{n}   + F_{n+1}
    1 + 1 + 2 + 3 + 5 + ..   .. + F_{n-1} + F_{n}
--------------------------------------------------
1 + 0 + 1 + 1 + 2 + ..       .. + F_{n-2} + F_{n-1}
```

Subtracting the second sum from the first, we obtain the third:

$$\sum F_{n+1} - \sum F_n = \sum F_{n-1} + 1$$

$$F_{n+1} = \sum F_{n-1} + 1$$

which rearranges to give the formula.

Binet

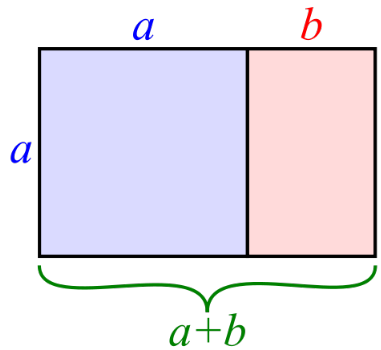
Binet's formula is an explicit formula for F_n which saves us from calculating all the intermediate numbers:

$$F_n = \frac{\phi^n - \psi^n}{\phi - \psi}$$

where ϕ is the Golden Ratio $(1 + \sqrt{5})/2$ and ψ is its conjugate $(1 - \sqrt{5})/2$.

Golden ratio

The basic definition involves the following construction:

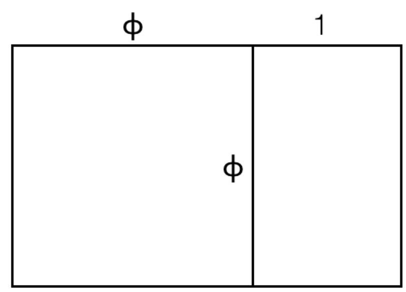


We start with a square of side length a and then extend one side by length b, forming two rectangles. When the ratios of side lengths for these two rectangles are the same, then that ratio is the golden ratio:

$$\phi = \frac{a}{b} = \frac{a+b}{a}$$

(ϕ is often written Φ).

Rescaling of the figure in both dimensions doesn't change the ratios, so let $b = 1$. Then (changing to the symbol ϕ):



$$\frac{\phi}{1} = \frac{\phi+1}{\phi}$$

$$\phi^2 = 1 + \phi$$

We will need only this last result below.

The equation can be solved numerically using the quadratic formula. Put everything on the left-hand side

$$\phi^2 - \phi - 1 = 0$$

Recall that the solutions are

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

We obtain

$$\phi = \frac{1 + \sqrt{5}}{2} \qquad \psi = \frac{1 - \sqrt{5}}{2}$$

This gives values of $\phi \approx 1.61803$ and $\psi \approx -0.61803$.

It seems like $\phi = 1 - \psi$. Proof:

$$2\phi = 1 + \sqrt{5}$$

$$2\psi = 1 - \sqrt{5}$$

Adding them together

$$2(\phi + \psi) = 2$$

What a nice symmetry:

$$\phi + \psi = 1$$

$$\phi = 1 - \psi$$

$$\psi = 1 - \phi$$

Note that since ψ is a solution it is also true that

$$\psi^2 = 1 + \psi$$

An alternative proof of that is:

$$\psi^2 = (1 - \phi)^2$$

$$= 1 - 2\phi + \phi^2$$

$$= 1 - 2\phi + 1 + \phi$$

$$= 2 - \phi = 2 - (1 - \psi) = 1 + \psi$$

We can also do the arithmetic

$$\phi^2 = \frac{1 + \sqrt{5}}{2} \cdot \frac{1 + \sqrt{5}}{2} = \frac{6 + 2\sqrt{5}}{4} = 1 + \phi$$

$$\psi^2 = \frac{1 - \sqrt{5}}{2} \cdot \frac{1 - \sqrt{5}}{2} = \frac{6 - 2\sqrt{5}}{4} = 1 + \psi$$

back to the proof

Wikipedia says that you can prove Binet's formula using induction.

$$F_n = \frac{\phi^n - \psi^n}{\phi - \psi}$$

https://en.wikipedia.org/wiki/Mathematical_induction#Example:_Fibonacci_numbers

That is an entertaining challenge. For induction we assume that the formula is correct for F_{n-1} and F_n and must prove that:

$$F_{n+1} = F_n + F_{n-1}$$

$$= \frac{\phi^n - \psi^n}{\phi - \psi} + \frac{\phi^{n-1} - \psi^{n-1}}{\phi - \psi}$$

$$= \frac{(\phi^n + \phi^{n-1}) - (\psi^n + \psi^{n-1})}{\phi - \psi}$$

Although one could imagine it is more complicated, the simple idea is to try to show that

$$\phi^{n+1} = \phi^n + \phi^{n-1}$$

and the same for ψ, and that will complete the proof of the inductive step.

Write

$$\phi^{n+1} = \phi^2 \ \phi^{n-1} = (1 + \phi) \ \phi^{n-1} = \phi^{n-1} + \phi^n$$

similarly

$$\psi^{n+1} = \psi^2 \ \psi^{n-1} = (1 + \psi)\psi^{n-1} = \psi^{n-1} + \psi^n$$

This shows that the inductive step is valid.

Now we just need to verify the base cases. We should check at least the first two of them, because there are two values in the recursion formula $F_n + F_{n-1}$.

It's a matter of convenience whether we consider the series to start with $n = 1$ or $n = 0$. If the latter, then the zeroth Fibonacci number is 0, and the first is 1 and we obtain the same series.

For $n = 0$ we have $\phi^0 - \psi^0$ which is just zero.

For $n = 1$, we have

$$\frac{\phi^1 - \psi^1}{\phi - \psi} = 1$$

We decide to continue with $n = 2$

$$\phi^2 = \phi + 1$$

$$\psi^2 = \psi + 1$$

so

$$\frac{\phi^2 - \psi^2}{\phi - \psi} = \frac{\phi + 1 - \psi - 1}{\phi - \psi} = 1$$

This completes the proof.

□

At this point we note the curious pattern

$$\phi^2 = \phi + 1$$

$$\phi^3 = \phi^2 + \phi = 2\phi + 1$$

$$\phi^4 = \phi^2 + 2\phi + 1 = 3\phi + 2$$

$$\phi^5 = 2\phi^2 + 3\phi + 1 = 5\phi + 3$$

$$\phi^6 = 4\phi^2 + 4\phi + 1 = 8\phi + 5$$

so it looks like

$$\phi^n = F_n \ \phi + F_{n-1}$$

The coefficients *are* the Fibonacci numbers.

The same is true for ψ since $\psi^2 = \psi + 1$.

We could prove this by induction, and it would be a proof of Binet's formula as well because the coefficient of ϕ^n or ψ^n is equal to F_n so

$$\phi^n = F_n \ \phi + F_{n-1}$$

$$\psi^n = F_n \ \psi + F_{n-1}$$

and then

$$\frac{\phi^n - \psi^n}{\phi - \psi} = \frac{(F_n \ \phi + F_{n-1}) - (F_n \ \psi + F_{n-1})}{\phi - \psi}$$

$$= F_n \ \frac{\phi - \psi}{\phi - \psi} = F_n$$

Part VIII

Analytic geometry

Chapter 32

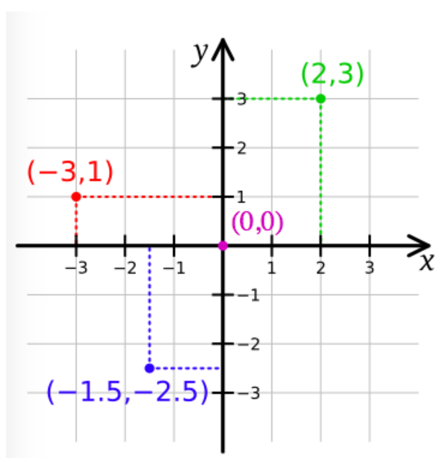
Lines and slopes

It is difficult today to put ourselves in the place of those who tried to reason about mathematics through the ages.

The Greeks lacked algebra, and although the Romans worked with numbers they did not have decimal notation. The concept of 0 came much later (from India), and even in the Middle Ages there was as yet no such thing as the equals sign =, which dates from 1557.

https://en.wikipedia.org/wiki/Table_of_mathematical_symbols_by_introduction_date

The invention of analytic geometry is often ascribed solely to Descartes, but Fermat also had his own version. There are two fundamental ideas.



The first is to orient two number lines on a piece of paper, at right angles, and then consider pairs of numbers (x, y) in the 2D plane. Such pairs or tuples are called points.

Descartes published this idea in 1637. The presentation would be difficult to recognize as our current system, but the germ is there: axes where the position of a variable could be marked. Only the positive numbers would be shown, and the axes not necessarily perpendicular. As to the proofs, here is wikipedia on the subject:

> His exposition style was far from clear, the material was not arranged in a systematic manner and he generally only gave indications of proofs, leaving many of the details to the reader. His attitude toward writing is indicated by statements such as "I did not undertake to say everything," or "It already wearies me to write so much about it," that occur frequently. In conclusion, Descartes justifies his omissions and obscurities with the remark that much was deliberately omitted "in order to give others the pleasure of discovering [it] for themselves."

The second idea of analytic geometry is to plot all the points that satisfy some mathematic relationship between x and y, for example the parabola $y = x^2$.

To do this, pick a few values of x and calculate the corresponding values of y. For example: $(0, 0), (\pm 1, 1), (\pm 2, 4), \dots$. Plot these points, and then finally, sketch the graph of the curve, without actually trying to plot *all* of the individual points (of which there is an infinite number). We make the assumption here that the function being plotted is continuous, so that the sketch of a curve between two points that are close enough together will be fairly smooth and if the x-values are close to the plotted x, the corresponding y-values will not be not too different from the plotted y.

point

A point is simply an ordered pair (x, y) such as $(1, 3)$. Often points have integer components, but they don't have to be.

distance formula

The x- and y-axes are perpendicular to one another (a fancy word for that is *orthogonal*).

Suppose we pick two particular points (s, t) and (u, v), plot them on a graph, and then draw the line that connects them. Recall Euclid's first two postulates:

∘ A straight line segment can be drawn joining any two points.

∘ Any straight line segment can be extended indefinitely in a straight line.

The distance between the two points is given by the Pythagorean formula, where Δx is the change in x and Δy is the change in y:

$$d = \sqrt{\Delta x^2 + \Delta y^2}$$

It is often easier to use the squared distance and avoid the square root:

$$d^2 = \Delta x^2 + \Delta y^2$$

$$= (s - u)^2 + (t - v)^2$$

Switching the order of (s, t) and (u, v) doesn't change the result.

formulas for a line

Now we want to derive an equation that describes (is valid for) all the points or pairs of values (x, y) on this line. A general approach is to say that the line has some slope m, which is defined as Δy, divided Δx:

$$m = \frac{\Delta y}{\Delta x} = \frac{y - y'}{x - x'}$$

This is called the *point-slope equation.* For any two particular points (s, t) and (u, v) one can plot a line between them. The slope is

$$m = \frac{s - u}{t - v}$$

One can write the two points in either order, with the same result since:

$$\frac{s - u}{t - v} = \frac{u - s}{v - t}$$

Depending on the details, the value of m might be zero, for a horizontal line, where all the values of y are the same (which happens when $s = u$). Or it might be undefined, for a vertical line, where all the values of x are identical ($t = v$).

In most cases, however, $m \neq 0$ and $m \in (-\infty, \infty)$. That is, m is usually non-zero and not infinite.

Except in the case of the vertical line, we can write

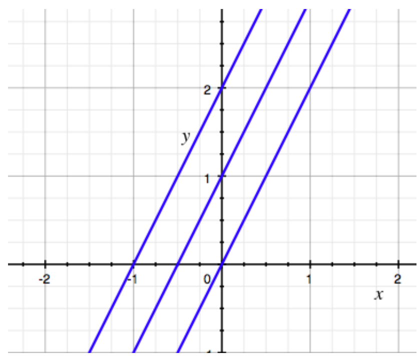
$$y = mx + y_0$$

for any point (x, y) on a given line, where y_0 is the y-intercept, the value of y obtained when $x = 0$.

[The choice of b for the y-intercept is the usual notation, but it conflicts with another b that we will see in a minute.]

$y = mx + y_0$ is the *slope-intercept equation* of the line.

The equation of a line is determined by both the slope and one point on the line, for example the y-intercept. One can draw a whole family of parallel lines with the same slope and different y-intercepts. Here are three lines $y = 2x + y_0$ for $y_0 = \{0, 1, 2\}$.



The value of x corresponding to $y = 0$ is the x intercept

$$x_0 = -\frac{y_0}{m}$$

The point-slope equation is easily derived from the second one. Suppose we have $y = mx + y_0$:

Plugging in for specific points (s, t) and (u, v) we have

$$t = ms + y_0$$

$$v = mu + y_0$$

Subtracting:

$$v - t = m(u - s)$$

which rearranges to give the desired result.

intersections

Often one has two lines (or curves) and we want to find the point(s) that lie on both. We might have

$$y = 2x - 1$$

$$y = -x + 8$$

Substitute from the second into the first:

$$2x - 1 = -x + 8$$

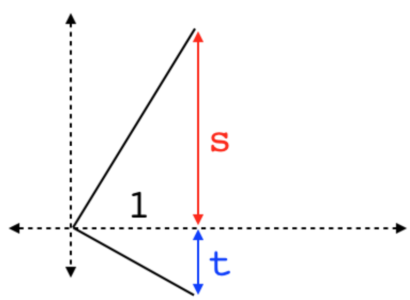
$$3x = 9$$

$$x = 3$$

From the first equation, $y = 5$, and we check that $x = 3, y = 5$ solves the second equation as well.

orthogonality

If two lines cross each other at right angles we say they are *orthogonal*. In that case the slopes have a special relationship. Their product is equal to -1.



Here is a simple proof. Draw the two lines going through the origin, forming a right angle there. The first has slope s, so it goes through the point $(1, s)$, the second goes through $(1, t)$.

Recall from the chapter on the Pythagorean theorem that the altitude squared is equal to the product of the two pieces of the base. Here:

$$1^2 = 1 = st$$

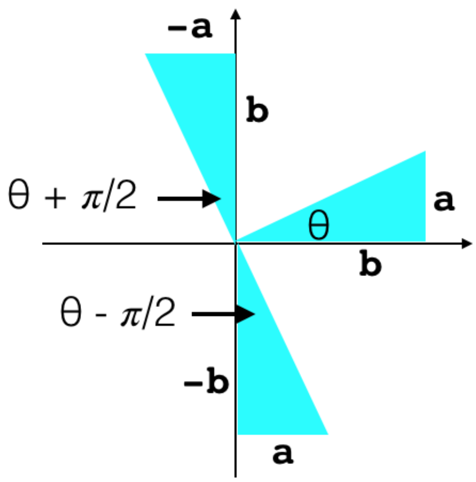
These are the lengths, i.e. the absolute values of the slopes. Thus $|s| = 1/|t|$. But

Clearly the sign of t is negative. So we arrive at

$$s \cdot (-t) = 1$$

$$m_1 = -\frac{1}{m_2}$$

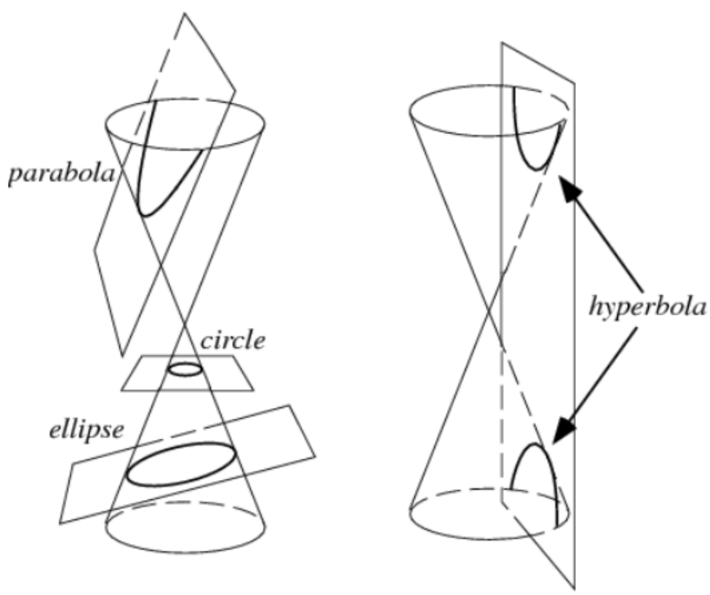
We'll see a natural easy proof of this once we look at trigonometry. Here is a hint:



Chapter 33

Circles again

We now consider what are called quadratic forms, as distinguished from linear equations (i.e., for lines). The quadratics contain a squared term (or a term that mixes x and y).



The simplest example is the equation for a unit circle centered at the origin:

$$x^2 + y^2 = 1$$

Pythagoras tells us that for a point (x, y), the square of the distance from the origin is $x^2 + y^2$. This equation describes all the points whose distance from the origin is equal to $\sqrt{1} = 1$. But all the points equi-distant to a point form a circle. We generalize

$$x^2 + y^2 = r^2$$

It is clear that when $y = 0$, $x = \pm\, r$. r is the radius of the circle.

Now, what happens if we displace the unit circle from the origin so its center is at $(1, 0)$? What this amounts to is adding 1 to the x value of every point. If we solve for x

$$x = \sqrt{1 - y^2}$$

and then add 1

$$x = \sqrt{1 - y^2} + 1$$

$$(x - 1)^2 = 1 - y^2$$

$$(x - 1)^2 + y^2 = 1$$

Or, more generally

$$(x - h)^2 + (y - k)^2 = r^2$$

where the origin of the circle is at (h, k).

Multiplying out:

$$x^2 - 2hx + h^2 + y^2 - 2ky + k^2 = r^2$$

$$x^2 + y^2 - 2hx - 2ky + (h^2 + k^2 - r^2)$$

Comparing to the most general form for a quadratic

$$Ax^2 + By^2 + Cxy + Dx + Ey + F = 0$$

We see that

$$A = 1, \quad B = 1, \quad C = 0$$

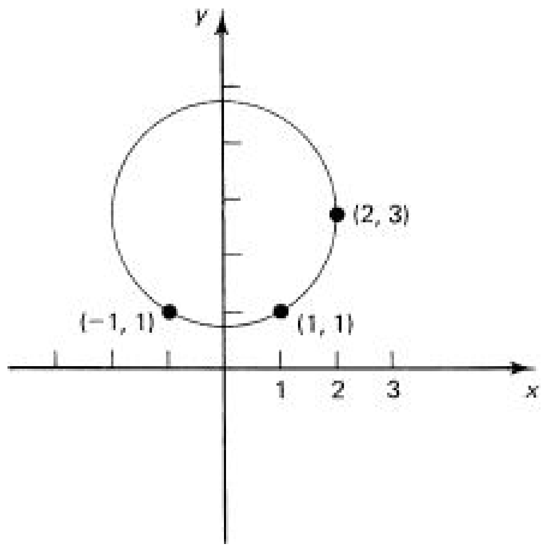
and in fact, this is true for all circles. (If $A = B \neq 1$, just divide all the terms by A).

Moreover

$$D = -2h, \quad E = -2k, \quad F = h^2 + k^2 - r^2$$

This equation can help us solve the following problem from Hamming: find the equation of the circle that passes through the following three points:

$$(-1, 1), (1, 1), (2, 3)$$



We write

$$x^2 + y^2 + Dx + Ey + F = 0$$

From the values of x and y at each of the three points we get

$$1 + 1 - D + E + F = 0$$

$$1 + 1 + D + E + F = 0$$

$$4 + 9 + 2D + 3E + F = 0$$

Three equations in three unknowns. We can do that.

Adding the first two equations together:

$$4 + 2(E + F) = 0$$

so $E + F = -2$.

Subtracting the first two equations (or substituting the result for $E+F$) tells us that $D = 0$.

Adding (-3) times the second equation to the third gives:

$$1 + 6 - D - 2F = 0$$

$$7 - 2F = 0$$

$F = 7/2$, and since $E + F = -2$, $E = -11/2$.

So the solution is

$$x^2 + y^2 - \frac{11}{2}y + \frac{7}{2} = 0$$

You can check that it works for all three points:

$$(-1, 1), (1, 1), (2, 3)$$

The first two are easy, while the third gives

$$4 + 9 - \frac{11}{2}3 + \frac{7}{2} = 0$$

$$8 + 18 - 33 + 7 = 0$$

which looks correct.

completing the square

We can improve this by completing the square. We see that

$$y^2 - \frac{11}{2}y + (\frac{11}{4})^2 = (y - \frac{11}{4})^2$$

We must add that back to the right-hand side of the original to obtain:

$$x^2 + (y - \frac{11}{4})^2 = (\frac{11}{4})^2 - \frac{7}{2}$$

The center is at $(0, 11/4)$. The radius doesn't come out cleanly but r^2 is

$$\frac{121}{16} - \frac{56}{16} = \frac{65}{16}$$

so r is slightly more than 2.

Or recall that we had:

$$D = -2h, \quad E = -2k, \quad F = h^2 + k^2 - r^2$$

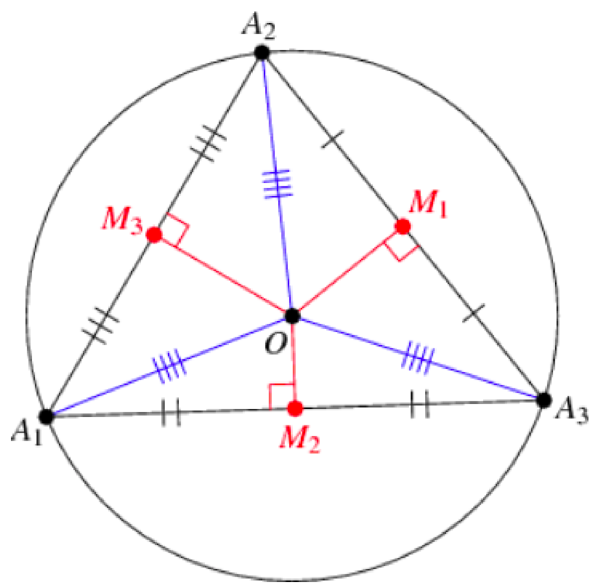
From this, we have that $h = 0$ and $k = -E/2 = 11/4$, and the radius is more complicated, as we said.

plane geometry

We can check our work by solving the problem using a technique from plane geometry. Again, we want the circle passing through three points:

$$(-1,1),(1,1),(2,3)$$

Take two of the points to be placed on a circle and construct the line segment joining them (a chord of the circle). Find the midpoint of the chord and erect a perpendicular bisector through the midpoint. Now, every point lying on the bisector is equidistant from the two starting points. Proof: draw the two triangles including that point, the two starting points and the midpoint of the bisector. The two triangles are congruent. Here is the general picture.

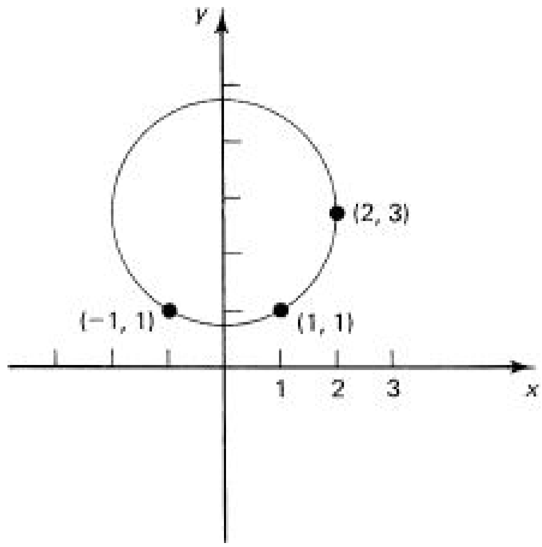


It's a bit trickier to prove that *every* point that is equidistant from the two points lies on the bisector. We assume that.

Since every point that is equidistant from the two points lies on the bisector, the radius of the circle lies on the bisector.

Then, erect a perpendicular bisector of a chord joining another pair chosen from the three points. This new bisector and the first one meet at the center of the circle.

In our case two points $(-1,1),(1,1)$ are symmetric about the y-axis. Therefore it is clear that the perpendicular bisector for these two points is the y-axis.



For the second bisector, form the vector between $(1,1)$ and $(2,3)$ as $\mathbf{v} = \langle 1,2 \rangle$. The midpoint is at $(1,1) + \mathbf{v}/2 = (3/2, 2)$.

The slope of the bisector is the negative inverse of the slope for the chord which is $-1/2$ so the equation of the bisector is

$$y - y_0 = -\frac{1}{2}(x - x_0)$$

Plugging in the point that we know, we obtain

$$y - 2 = -\frac{1}{2}(x - 3/2)$$

We want to solve for y when $x = 0$, crossing the first bisector, the y-axis

$$y - 2 = -\frac{1}{2}(-3/2)$$

$$y = \frac{11}{4}$$

So the center is at $(0, 11/4)$, which matches what we had before. We compute the distance to one of the points $(1,1)$ as

$$d = \sqrt{1^2 + (11/4 - 1)^2} = \sqrt{1 + 49/16}$$

which also matches our previous result.

quadratics

The technique of completing the square comes from the standard equation

$$(x+p)^2 = x^2 + 2px + p^2$$

We run into problems where we have the $2px$ but not the p^2. For example

$$x^2 + y^2 + Dx + F = 0$$

Focus on

$$x^2 + Dx$$

we want to turn this into

$$(x + \text{something})^2$$

if D is like $2p$ we need to add something like p^2:

$$x + Dx + \frac{D^2}{4}$$

$$= (x + \frac{D}{2})^2$$

Since we added $D^2/4$ on the left, we must also add it on the right. We obtain

$$(x + \frac{D}{2})^2 + y^2 + F = \frac{D^2}{4}$$

You don't believe me? Multiply it out

$$x^2 + Dx + \frac{D^2}{4} + y^2 + F = \frac{D^2}{4}$$

To form $(x + D/2)^2$ on the left-hand side, we added $D^2/4$ (what we needed) to both sides.

Again, the general equation for a quadratic is

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

In starting to work with one of these, the first thing to do is to see if there is a term which "mixes" x and y, that is, whether there is some term like Bxy. If there is, we might think about rotating the curve so that it is in a standard orientation.

We'll talk about how to do that **here**, in the context of the ellipse. However, the approach is general.

Let us assume we've done that, we relabel the new A, C etc. and assume here that $B = 0$.

Once in standard orientation, the next thing we might do is to translate the quadratic so that it is centered on the origin. We do that by completing the square for both x and y. We did some of that in this chapter, and we'll talk more about it **here** in the context of the parabola. Once again, however, the approach is general.

Cases:

- Both A and C present, and $F < 0$. If
  - A and C are both > 0: it's an ellipse.
  - A and C are of opposite signs: it's a hyperbola.
- A, C and F are all negative: it's imaginary.
- Only one squared term is present, but we still have the other variable

$Ax^2 + Ey + F = 0$: it's a hyperbola.

Not every quadratic equation gives a conic. Some are "degenerate". For example, having done all the right manipulations, we might end up with something like

$$A'(x-h)^2 + B'(y-k)^2 = 0$$

which has only $x = h$ and $y = k$ as a solution. It's a point.

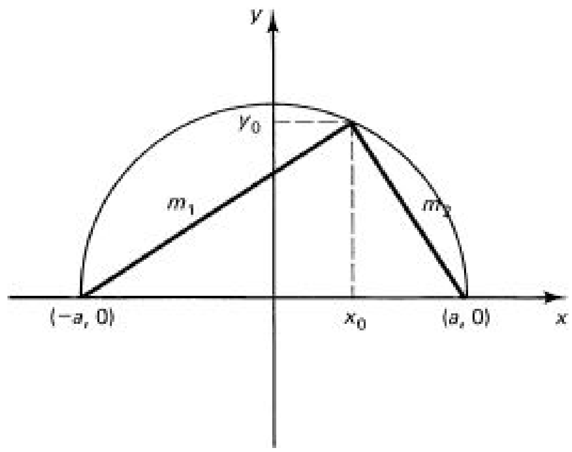


Figure 6.2-3 Angle in a semicircle

Here is another problem from Hamming. We need to prove that the angle above is a right angle. Suppose the equation of the circle is

$$x^2 + y^2 = a^2$$

The point on the circle is (x_0, y_0).

Our first solution uses slopes and points. The line from $(-a, 0)$ to (x_0, y_0) has slope

$$m_1 = \frac{y_0}{x_0 + a}$$

The line from $(a, 0)$ to (x_0, y_0) has slope

$$m_2 = \frac{y_0}{a - x_0}$$

Two lines meet at a right angle if the product of their slopes is equal to -1.

$$m_1 m_2 = \frac{y_0}{x_0 + a} \frac{y_0}{a - x_0}$$

$$= \frac{y_0^2}{a^2 - x_0^2} = \frac{y_0^2}{x_0^2 + y_0^2 - x_0^2} = -1$$

This was not pretty, it's just good exercise.

And here is a proof using vectors and the dot product. Consider the semicircle centered on the origin with radius a, so the ends of the diameter are at $(x = \pm\, a, 0)$.

Form the vectors from those ends to an arbitrary point (x, y) on the perimeter:

$$\mathbf{u} = \langle x + a, y \rangle$$

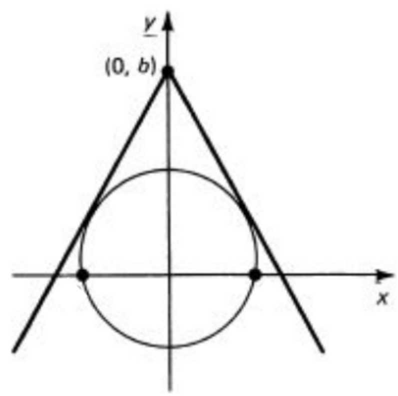
$$\mathbf{v} = \langle x - a, y \rangle$$

Notice that

$$\begin{aligned} \mathbf{u} \cdot \mathbf{v} &= (x + a)(x - a) + y^2 \\ &= x^2 - a^2 + y^2 = 0 \end{aligned}$$

because $x^2 + y^2 = a^2$ for any point on the circle.

As our last example, consider the problem of finding the equation of a line tangent to a circle that goes through some arbitrary point b.



We take the circle to have radius a and be centered at the origin. We take the point b to be on the y-axis. The equation of the line on the right side is

$$\frac{y - y_0}{x - x_0} = m = \frac{y - b}{x}$$

$$y = mx + b$$

(well, of course).

For the point or points where the line intersects the circle we also have

$$y = \sqrt{a^2 - x^2}$$

$$\sqrt{a^2 - x^2} = mx + b$$
$$a^2 - x^2 = m^2x^2 + 2bmx + b^2$$
$$(m^2 + 1)x^2 + 2bmx + b^2 - a^2 = 0$$

From the quadratic equation:

$$x = \frac{-2bm \pm \sqrt{4b^2m^2 - 4(m^2 + 1)(b^2 - a^2)}}{2(m^2 + 1)}$$

We are looking for the case where there is a single solution so the discriminant under the square root must be equal to zero:

$$4b^2m^2 = 4(m^2 + 1)(b^2 - a^2)$$
$$m^2b^2 = m^2b^2 - m^2a^2 + b^2 - a^2$$
$$0 = -m^2a^2 + b^2 - a^2$$
$$m = \pm\frac{\sqrt{b^2 - a^2}}{a}$$

This makes sense since if $a = b$ the single tangent should be horizontal with zero slope. Notice that if $a^2 > b^2$ there is no real solution. This corresponds to having b inside the circle.

Part IX

Trigonometry

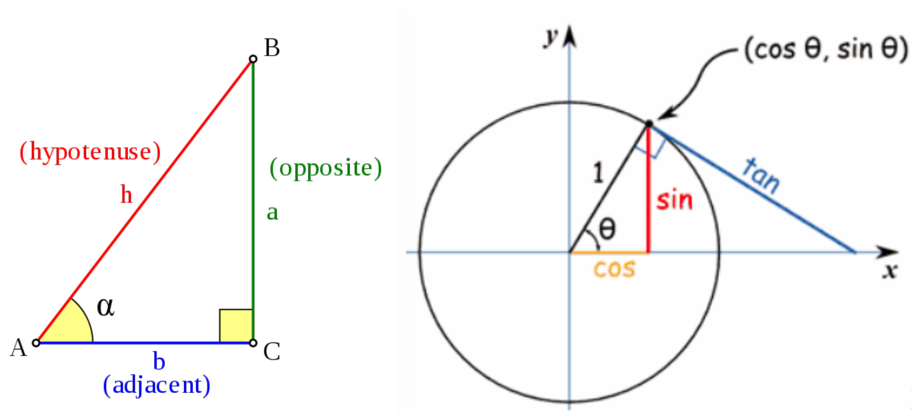
Chapter 34

Six functions

basic definitions

The most elementary trigonometric functions are the sine and cosine. These are defined in geometry as ratios of the lengths of the sides of a right triangle.

Looking at the left panel, we say that the sine of the angle α is the ratio *opposite-over-hypotenuse*, while the cosine of α is the ratio *adjacent-over-hypotenuse*. Tangent is the ratio *opposite-over-adjacent*. The names are abbreviated to three letters in formulas.



Using the notation for the sides from the figure:

$$\sin\alpha = \frac{a}{h}, \qquad \cos\alpha = \frac{b}{h}, \qquad \tan\alpha = \frac{a}{b} = \frac{\sin\alpha}{\cos\alpha}$$

The "unit circle" is a circle of radius 1 with its center positioned at the origin of coordinates, the place where the x and y axes cross. From the right panel of the diagram you can see that any point (x, y) on the unit circle can be described in radial coordinates as

$$x = \cos\theta \quad y = \sin\theta$$

In the diagram, all three right triangles are similar because the red line is an altitude of the largest right triangle. Thus, by similar triangles, the blue side has this relationship

$$\frac{\text{blue side}}{1} = \frac{\sin\theta}{\cos\theta}$$

which explains why it is labeled as tan (α).

If the vertex labeled B is denoted angle β (the complementary angle of α), then the notions of opposite and adjacent switch so that:

$$\sin\alpha = \cos\beta, \qquad \cos\alpha = \sin\beta$$

If the circle has radius r then

$$x = r\cos\theta \quad y = r\sin\theta$$

Stewart:

> The mathematicians of ancient India built on the Greek work to make major advances in trigonometry. They [used] the sine (sin) and cosine (cos) functions, which we still do today. Sines first appeared in the Surya Siddhanta, a series of Hindu astronomy texts from about the year 400, and were developed by Aryabhata in Aryabhatiya around 500. Similar ideas evolved independently in China.

The other functions are the inverses of sine, cosine and tangent, namely: cosecant, secant and cotangent. The secant (inverse cosine) comes up somtimes, but the other two are not especially important in calculus.

However, there is one context that we will look at, namely, Archimedes determination of the value of π. The crucial step in that approach will turn out to be the calculation of the cotangent of the half-angle $\theta/2$ given the values of cotangent and cosecant for angle θ.

The main relationship or identity is derived from the Pythagorean theorem. We had above that for a unit circle

$$x = r \cos \theta \qquad y = r \sin \theta$$

Since x and y are the sides of a right triangle whose hypotenuse is r

$$x^2 + y^2 = r^2$$

and for a unit circle

$$\cos^2 \theta + \sin^2 \theta = 1$$

which is usually written

$$\sin^2 \theta + \cos^2 \theta = 1$$

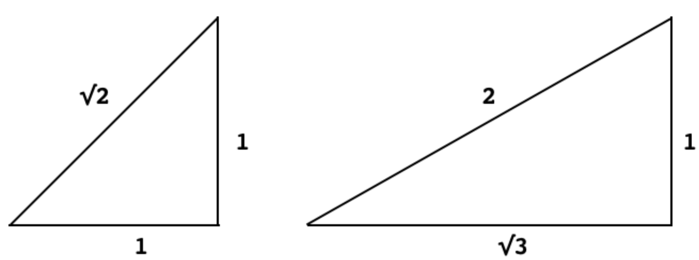
and transformed to

$$1 + \tan^2 \theta = \sec^2 \theta$$

particular values

We can easily determine the values for these functions for three special cases.

The first is the angle 45 degrees or $\pi/4$. Draw an isosceles right triangle with sides of length 1 (left panel).



Then the hypotenuse has length $\sqrt{2}$ (from Pythagoras) and the values are

$$\sin \frac{\pi}{4} = \frac{1}{\sqrt{2}} = \cos \frac{\pi}{4}$$

$$\tan \frac{\pi}{4} = 1$$

For the other two, bisect an equilateral triangle and erase one half (right panel). The smaller angle is 30 degrees or $\pi/6$ and its complement is 60 degrees or $\pi/3$.

The values are

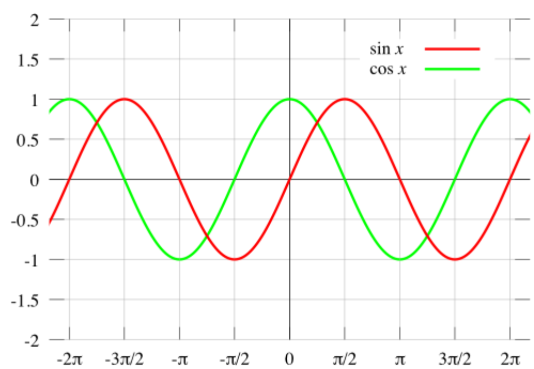
$$\sin\frac{\pi}{6} = \frac{1}{2} = \cos\frac{\pi}{3}, \qquad \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2} = \sin\frac{\pi}{3}$$

$$\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$$

We can easily verify that

$$(\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2 = 1, \qquad (\frac{1}{\sqrt{2}})^2 + (\frac{1}{\sqrt{2}})^2 = 1$$

graph



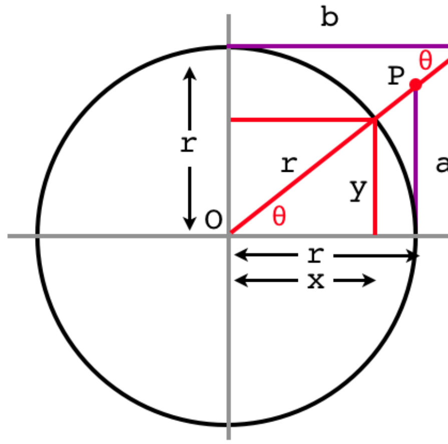
Savov:

> The sine function represents a fundamental unit of vibration. The graph of $\sin(x)$ oscillates up and down and crosses the x-axis multiple times. The shape of the graph of $\sin(x)$ corresponds to the shape of a vibrating string.

Imagine a circle placed to the left of a graph. I think of the sine function as the "shadow" of the point (x, y) as it travels around the circle at the same constant speed as the point on the graph "moves" to the right.

visualization of all six functions

Consider a unit circle. Extend the radius with the angle θ and then draw the vertical and horizontal tangents to the circle a and b.



The original triangle with sides x, y, r is similar to the triangle with sides r, a, OP, and both are similar to the triangle with sides b, r, OQ.

$$x, y, r \sim r, a, OP, \sim b, r, OQ$$

By similar $\triangle$

$$\frac{a}{r} = \frac{y}{x} = \tan\theta$$

But $r = 1$ so

$$a = \tan\theta$$

If you imagine a point moving around the circle a will get very large as $\theta \to \pi/2$, and in fact, approaches ∞ there (becomes undefined).

The segment OP is (by similar $\triangle$) to r as

$$\frac{OP}{r} = \frac{r}{x}$$

$$OP = \frac{1}{\cos\theta} = \sec\theta$$

The horizontal from the y-axis to Q is b. Consider θ near the top of the figure. By similar $\triangle$, the relations we had were

$$r/b = y/x = \tan\theta$$

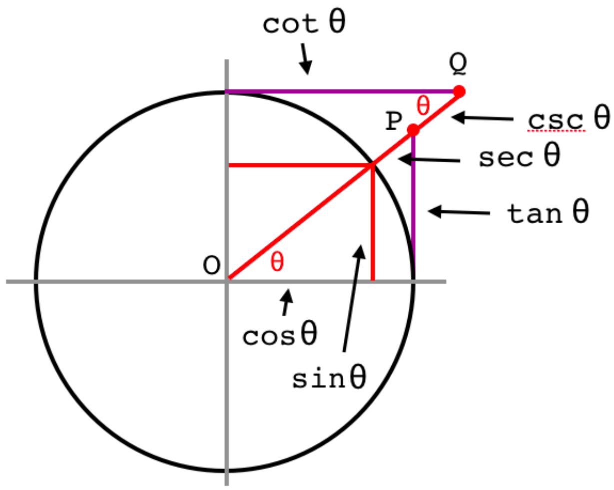
since $r = 1$

$$b = \frac{r}{\tan\theta} = \frac{1}{\tan\theta} = \cot\theta$$

Finally

$$r/OQ = 1/OQ = \sin\theta$$

$$OQ = \frac{1}{\sin\theta} = \csc\theta$$



Chapter 35

Sum of angles

cosine of a sum

The sum of angle formulas (i.e. formulas for the sine and cosine of the sum or difference of two angles) are used often in calculus, not only for working problems, but even in finding an expression for the "derivative" of sine and cosine.

You really must know them. I think it's so important that we will show three ways of finding these formulas — not all in this chapter. The easiest way to remember them uses Euler's equation, and we won't be ready for that until later. See **here**.

There are four equations: $\sin s \pm t$ and $\cos s \pm t$.

I've memorized only this one:

$$\cos s - t = \cos s \cos t + \sin s \sin t$$

By $\cos s - t$ we mean $\cos(s - t)$, but have left off the parentheses.

Say "cos cos" and then recall the difference in sign.

check

I like this version because it can be checked easily. Set $s = t$:

$$\cos s - t = \cos 0 = 1 = \cos^2 s + \sin^2 s$$

which is our favorite trigonometric identity and obviously correct.

change signs

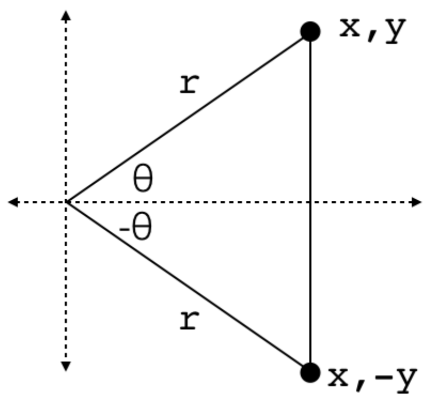
For $\cos s + t$ flip the sign on the second term.

$$\cos s + t = \cos s \cos t - \sin s \sin t$$

This is simply a result of the fact that

$$\cos -\theta = \cos \theta$$

$$\sin -\theta = -\sin \theta$$



The diagram shows the reason: $\cos \theta = \cos -\theta = x/r$ while $\sin \theta = y/r = -(\sin -\theta) = -(-y/r)$.

Proof:

$$\cos(s - (-u)) = \cos s \cos(-u) + \sin s \sin(-u)$$

Since $\cos -x = \cos x$ and $\sin -x = -\sin x$:

$$\cos(s + u) = \cos s \cos u - \sin s \sin u$$

But u is just a dummy variable (it could be any symbol), so

$$\cos(s + t) = \cos s \cos t - \sin s \sin t$$

sine of a sum

We will look at the proof for the sine formula later, for now just write it:

$$\sin s + t = \sin s \cos t + \sin t \cos s$$

Say "sin cos" and then, that here + goes with +. Like most things having to do with sine and cosine, there is a change of sign when moving from one to the other.

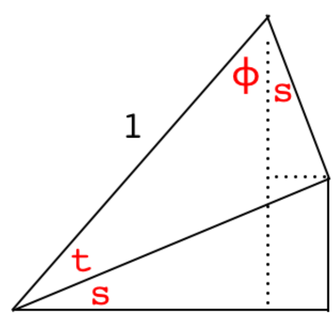
For $\sin s - t$, flip the sign on the second term, as before.

proof

Here is a geometric proof of both of the sum of angles formulas, using similar triangles. The key is to draw an inspired diagram.

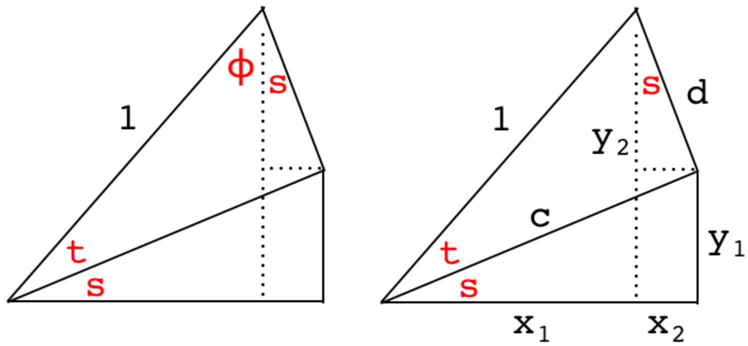
Consider a right triangle, with one of the angles labeled s. Construct another right triangle containing angle t, and scale it so that the base adjacent to angle t is just as long as the hypotenuse of the triangle containing angle s, and draw them one on top of the other as shown:

Scale the joined triangles so that the hypotenuse of the second triangle has unit length. Our crucial insight is to draw vertical and horizontal dotted lines as shown below.



The angle s is part of a right triangle with angle t adjacent, where the third acute angle is ϕ. But ϕ is also part of a second right triangle containing t plus the angle adjacent to ϕ. Therefore, that adjacent angle is also equal to angle s.

We add some labels to the sides of the triangles and calculate the sine and cosine of s, t and $s + t$:



Since I already know the result I am looking for, I write what we had before

$$\cos s \cos t - \sin s \sin t$$

From the figure

$$\cos s = \frac{x_1 + x_2}{c}; \quad \cos t = \frac{c}{1}; \quad \cos s \cos t = x_1 + x_2$$

The sine of s is a little trickier, look at the small right triangle at the top of the figure

$$\sin s = \frac{x_2}{d}; \quad \sin t = \frac{d}{1}; \quad \sin s \sin t = x_2$$

The difference is

$$\cos s \cos t - \sin s \sin t = x_1$$

but from the diagram it's clear that

$$\cos s + t = x_1$$

□

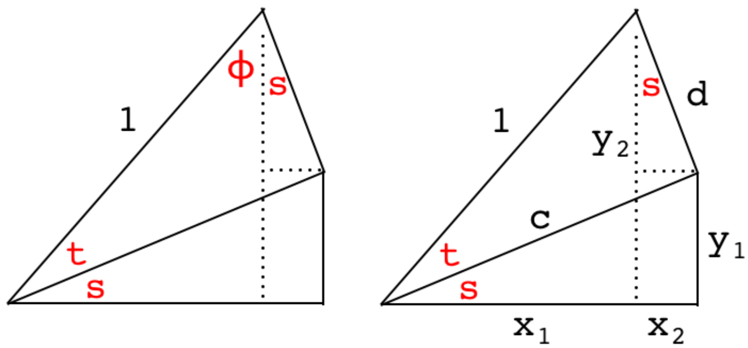
As a quick check we can ask what happens to the formula

$$\cos s + t = \cos s \cos t - \sin s \sin t$$

when $t = 0$. Then the first term is the cosine of s, and the second term is equal to 0. The formula is symmetrical with respect to s and t.

extension to sine

Referring back to the diagram (and again, with our goal clearly in mind)



$$\sin s = \frac{y_1}{c}; \quad \cos t = \frac{c}{1}; \quad \sin s \cos t = y_1$$

$$\sin t = \frac{d}{1}; \quad \cos s = \frac{y_2}{d}; \quad \sin t \cos s = y_2$$

But

$$\sin s + t = y_1 + y_2 = \sin s \cos t + \sin t \cos s$$

Using the even/odd function rules, we get

$$\sin s - t = c + d = \sin s \cos t - \sin t \cos s$$

And that's all four of them.

another calculation

We found previously that

$$\sin \frac{\pi}{4} = \cos \frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

$$\sin \frac{\pi}{6} = \cos \frac{\pi}{3} = \frac{1}{2}; \quad \sin \frac{\pi}{3} = \cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$$

These angles correspond to 30, 45 and 60 degrees. It might be nice to have sine and cosine of 15 and 75 degrees as well. That would make even divisions of the first 90 degrees. We can get them as the sum and difference of $\pi/4$ and $\pi/6$.

Let $s = \pi/4$ and $t = \pi/6$. Then

$$\begin{aligned}
\sin\frac{\pi}{12} &= \sin s - t = \sin s\cos t - \sin t\cos s \\
&= \frac{1}{\sqrt{2}}\cdot\frac{\sqrt{3}}{2} - \frac{1}{2}\cdot\frac{1}{\sqrt{2}} = \frac{\sqrt{3}-1}{2\sqrt{2}} \\
\cos\frac{\pi}{12} &= \cos s - t = \cos s\cos t + \sin s\sin t \\
&= \frac{\sqrt{3}}{2}\cdot\frac{1}{\sqrt{2}} - \frac{1}{2}\cdot\frac{1}{\sqrt{2}} = \frac{\sqrt{3}+1}{2\sqrt{2}}
\end{aligned}$$

We just check that $\sin^2\theta + \cos^2\theta = 1$:

$$\frac{(\sqrt{3}-1)^2 + (\sqrt{3}+1)^2}{(2\sqrt{2})^2}$$

$$= \frac{3 - 2\sqrt{3} + 1 + 3 + 2\sqrt{3} + 1}{8} = 1$$

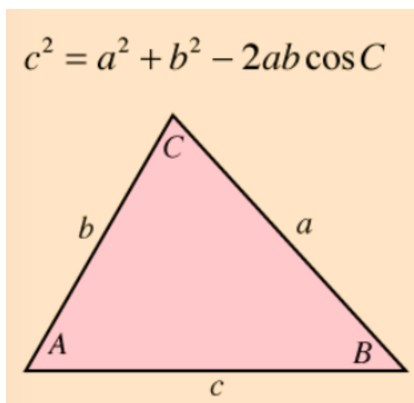
We can calculate similarly for $s+t = 5\pi/12$ or just switch sine and cosine from $\pi/12$.

Chapter 36

Law of cosines

Law of cosines

Designate the lengths of a triangle's sides as a, b, c and the angle between sides a and b as C (because it is opposite side c). The law of cosines says that



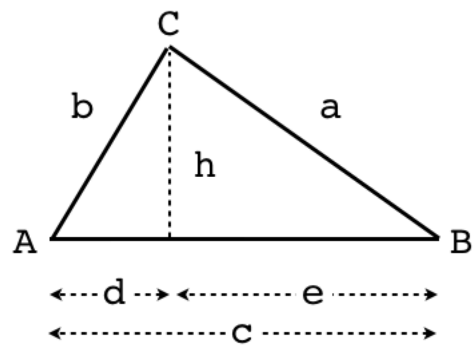
$$c^2 = a^2 + b^2 - 2ab\cos C$$

Lockhart calls this the "generalized" Pythagorean theorem. We can view the term $-2ab\cos C$ as a correction term which disappears in the case where $\angle C$ is 90 degrees.

derivation

The result follows from the Pythagorean Theorem. (In fact, we can reuse the same diagram that was shown for the algebraic proof of the theorem).

For a triangle with sides a, b and c and angles opposite those sides A, B and C, divide the third side into two lengths $c = d + e$ using the vertical altitude from vertex C.



$$a^2 - e^2 = h^2 = b^2 - d^2$$

So

$$a^2 = e^2 + b^2 - d^2$$

Since $d = c - e$ and thus $d^2 = c^2 - 2ce + e^2$:

$$a^2 = e^2 + b^2 - (c^2 - 2ce + e^2)$$

$$= b^2 - c^2 + 2ce$$

but $e = a \cos B$ so

$$a^2 = b^2 - c^2 + 2ac \cos B$$

rearrange to give a more familiar form (this is the law of cosines)

$$b^2 = a^2 + c^2 - 2ac \cos B$$

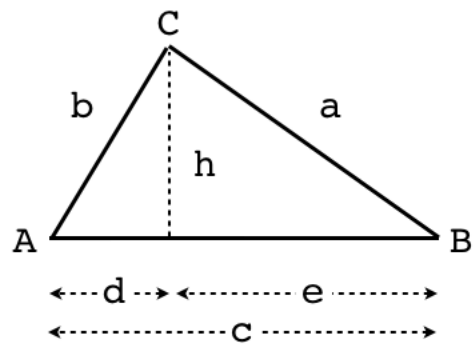
Any side of a triangle can be expressed in terms of the other two and the cosine of the angle between them. Thus, for example

$$c^2 = a^2 + b^2 - 2ab \cos C$$

$$a^2 = b^2 + c^2 - 2bc \cos A$$

Law of sines

I'll just mention that there is another law called the law of sines. In contrast to the law of cosines, it is fairly trivial.



$$\frac{h}{b} = \sin A \qquad \frac{h}{a} = \sin B$$

Therefore

$$h = b \sin A = a \sin B$$
$$\frac{\sin A}{a} = \frac{\sin B}{b}$$

We could do the same construction and argument with A and C or B and C. Therefore

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Part X

Two basic operations in calculus

Chapter 37

Simple slopes

To introduce the two fundamental ideas in calculus, consider two measuring devices used while driving a car. Most good drivers look fairly often at the speedometer, which measures speed or velocity, or how fast you're going.

On the other hand, if someone gives you directions like "go three and a half miles and then turn left (where the old gas station used to be)" you will be watching your odometer.

Velocity times time = distance. We can think of speed and velocity as the same for now. Distance divided by time is velocity.

Velocity is the *rate of change* of distance with time, it has units of distance divided by time (say, miles per hour).

In calculus we say that the velocity is the **derivative** of the distance with respect to time, and the distance is the **integral** of the velocity with respect to time.

We can speak of velocity at a particular time t, as in "our current velocity is 60 miles per hour." But the distance, the integral, must be evaluated between appropriate starting and stopping points for the time. In our example, you must first look at your odometer *before* you start on that 3.5 mile drive.

time-dependence

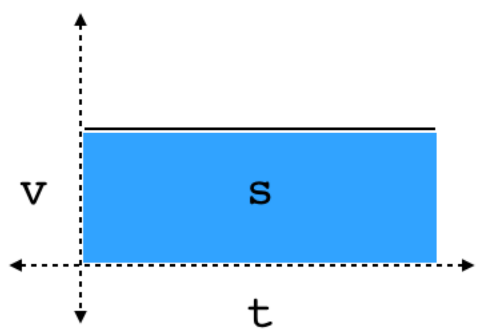
Distance equals velocity times time.

This is easy if the velocity is constant. Travel west on the interstate at exactly 60 miles per hour for 2 hours and your distance will be 120 miles from where you started (provided you don't start in Los Angeles). It is standard to use s to refer to the distance traveled and v for velocity. If the velocity is constant then:

$$s = vt$$

According to the internet, s is from the Latin "spatium", for "space, room, or distance."

Suppose we plot velocity as a *function of time* with v on the y-axis and t on the x-axis.



Since the velocity is constant, the result is a straight horizontal line. Furthermore, the distance traveled is the *area under the curve* (and above the x-axis) which is the area of a rectangle with sides v and t and as we said

$$s = vt$$

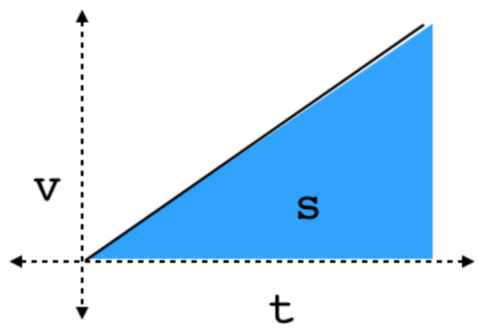
However, for most interesting problems velocity is not constant.

Imagine maintaining pressure on the gas pedal in the car steadily so that, starting from a stop at zero time, after 1 second your velocity is 10 mph, after 2 seconds it is 20 mph, after 3 seconds, 30 mph. If we continue at the same rate of acceleration, we'll go from 0 to 60 mph in 6 seconds, which is quite a respectable time.

This example has constant acceleration. Here, we say that v is a constant function of time, and write

$$v = at$$

where a is the acceleration.



What about the distance?

If a is not zero then v changes with time. If a is non-zero and constant, then v changes at a constant rate. Starting from 0, the final velocity will be $v = at$, but the distance traveled is no longer the product

$$s = v \times t \stackrel{?}{=}$$

because this v is the final velocity and that is not the correct v to use. For variable velocity, the distance traveled is the *average* velocity times the time. For smooth (constant) acceleration from zero to v, the average velocity is the average of the initial and final velocities:

$$v_{\text{avg}} = \frac{1}{2}\ (v_i + v_f) = \frac{1}{2}\ v$$

So the correct equation is:

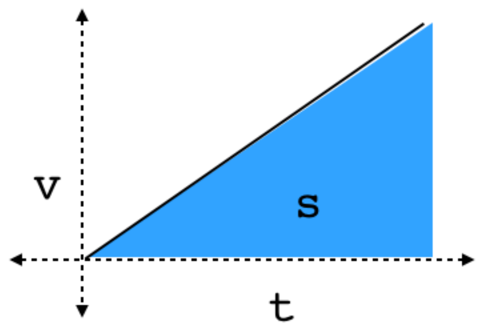
$$s = v_{\text{avg}}\ t = \frac{1}{2}\ v \cdot t$$

and since $v = at$

$$s = \frac{1}{2}at^2$$

In this case, if we plot velocity as a function of time, we obtain a straight line that extends diagonally up with respect to the x-axis. The distance traveled is the area under the curve, below the line and above the x-axis.

The shape whose area is needed is a triangle. This also accounts for the factor of 1/2.



You probably know that if a mass m is dropped from a tall building like the Tower of Pisa, then the distance it has fallen goes like the square of the time. The equation is:

$$s = \frac{1}{2}gt^2$$

where g is the acceleration due to gravity.

Notice that this is the same equation as we had earlier. The reason is that g is approximately constant near the surface of the earth. g is about 10 in units of m/s^2. A fall of four seconds is about 80 meters.

Galileo knew this formula (at least, he knew the t^2 part of it), which he obtained not from experiments at the Tower of Pisa, but by timing the descent of balls down an inclined plane.

initial position and velocity

If you want to be more complete and say that the starting point is not necessarily the origin of the coordinate system, add a constant s_0 to describe the initial distance from the origin and obtain:

$$s = vt + s_0$$

and similarly, a constant v_0 to describe the initial velocity as shown above.

The full equation of motion is

$$s = \frac{1}{2}at^2 + v_0 t + s_0$$

We'll say much more about this later.

power rule

We will introduce the theory of calculus more formally in the next section of the book. For now, we just talk about a simple rule called the power rule.

Switching notation to y and x, suppose that y is a *function* of x and write $y = f(x)$.

Here are three types of dependency (with c as a constant), with three corresponding types of graph.

$$y = c$$

$$y = cx$$

$$y = cx^2$$

These are (respectively) the equations of: (i) a horizontal line, since y is constant, (ii) any other non-vertical line (y is proportional to x), and (iii), a parabola.

We ask "what happens if we change x a little bit" and use the notation dx to refer to this little bit of x.

What happens to y? y will usually change by a small amount. Call that amount dy.

However, in the first case, $y = c$, y does not actually depend on x at all. The result (dy, the change in y for a change in x, dx) is zero.

$$y = c, \quad dy = 0 \cdot dx$$

The ratio dy/dx is the slope of the curve formed by plotting y against x. We call that slope the *derivative* of the function $f(x)$.

Divide both sides by dx and rewrite the above as:

$$\frac{dy}{dx} = 0$$

The plot is a horizontal line with slope 0.

In the second case, y is a linear function of x, the change in y, dy is the change dx multiplied by c:

$$y = cx, \quad dy = c \cdot dx$$

rearranging.

$$\frac{dy}{dx} = c$$

In analytical geometry, we calculate the slope of a line as $\Delta y/\Delta x$.

For a line, the slope is constant and so it doesn't matter which two points with coordinates $(x_1, y_1), (x_2, y_2)$ we choose for the calculation. The following is true for *any* two pairs (x, y) on the line:

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

Above we had the example where $v = at$ with constant a. Then $dv/dt = a$.

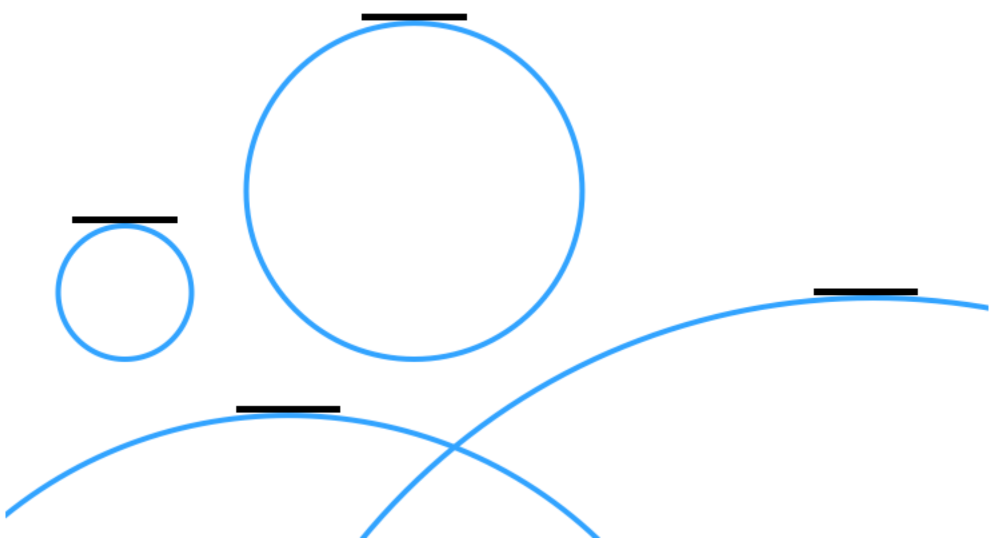
The third case is different.

$$y = cx^2$$

For a parabola, the slope of the curve at a point (the slope of the tangent to the curve $y = cx^2$) depends on the choice of x. The slope is steeper the further out you go in a positive direction on the x-axis.

It seems impossible to compute the slope of this curve in the standard way, by picking a second point near (x, y) and then calculating $\Delta y/\Delta x$, because the slope changes as we go out along the curve.

The key insight is that if x_1 is sufficiently close to x_2 the slope is constant. It's like saying that the earth is flat *locally*. If you detect any curvature, just zoom in a bit. In the figure



the line has constant length, but the distance to the circle from the end of the line decreases as we increase the size of the circle.

In calculus, we keep the curve the same size and decrease the length of the line, and then magnify the whole picture, until we get something like the figure above.

Just zoom in until the line is a good enough approximation to the shape of the circle, if the curve doesn't look flat enough, zoom in some more.

As we are accelerating in the car, with constantly changing velocity, we can still have a unique velocity at a particular instant in time.

In other words, for a very small change Δx in either direction from x, we get the same slope, *if* Δx is small enough.

If it's not, we can always make it smaller. That's the beauty of the real numbers.

Or, put still another way, when they built your house they didn't worry about the curvature of the earth. If r is the radius of the earth in feet, and the house is $a = 50$ feet long, the drop due to curvature is $r - b$

```
r = 21120000
a =       50
b = 21119999.9999408
```

That is 0.00006 feet over the length of a 50 foot house, much much less than 1/16 of an inch. Roughly 6 parts in 10000 compared to 1 part in 192.

Since the changes in x and y are so small, we use the new nomenclature: dy and dx.

power rule

To actually calculate slopes for curves (and straight lines), use the power rule.

For a horizontal line with zero slope:

$$y = c$$

$$\frac{dy}{dx} = 0$$

For a line with a slope c:

$$y = cx$$

$$\frac{dy}{dx} = c$$

For the parabola, the rule says that if $y = cx^2$, the slope or derivative is

$$\frac{dy}{dx} = 2cx$$

We've been writing c as the constant, so as not to confuse it with a, the acceleration. In analytic geometry, a parabola is usually written with a constant a, called the shape factor:

$$y = ax^2$$

Then, the slope is $2ax$.

If we had

$$y = ax^2 + bx + c$$

with a, b, c all constant, then the slope would be $2ax + b$.

The above uses our three rules from above, plus one more, that when taking the derivative of a polynomial, the derivative of the whole is simply the summed derivatives for each term.

For the equation of motion under gravity

$$s = \frac{1}{2}at^2 + v_0 t + s_0$$

$$v = \frac{ds}{dt} = at + v_0$$

$$\frac{dv}{dt} = a$$

Notice how the 1/2 and the 2 cancel in the second equation.

Continuing to the cubic, if y depends on x^3 like

$$y = cx^3$$

then

$$\frac{dy}{dx} = 3cx^2$$

The general form of the power rule is that if

$$y = x^n$$

then

$$\frac{dy}{dx} = nx^{n-1}$$

The exponent has been reduced by 1 power, and the value of that exponent applied as a factor in front of the expression.

This rule had already been discovered before Newton. It's a toss-up whether Fermat or Cavalieri was first. We will prove this later, but for now we just want to introduce the idea and practice using it.

note

If you already know some calculus you're probably jumping out of your chair while reading this chapter because you've had it pounded into you that dy/dx is not a quotient and believe that you can't simply multiply both sides of the equation by dx.

Well, you can. And I'll explain why as we go along.

Chapter 38

Easy pieces

Integration

Differentiation breaks things up into small pieces dx or dr. Integration adds up many little pieces. The symbol for integration is a relaxed S that stands for summation: $\int$.

As Thompson says

> The word "integral" simply means "the whole." If you think of the duration of time for one hour, you may (if you like) think of it as cut up into 3600 little bits called seconds. The whole of the 3600 little bits added up together make one hour.

We boldly claim that from the point of view of problem-solving, integration is simply the inverse of differentiation.

Mathematicians hate this kind of talk, because it trivializes a profound statement, the fundamental theorem of calculus.

But for practical problem-solving our counter-claim is that this profundity *doesn't matter*. It is also likely to confuse the beginning student, another reason to put it aside for the time being. We'll return to this issue later, when we cover the theory of the subject very lightly.

The sum of a bunch of small pieces dy is equal to the sum of a bunch of small pieces dx times cx, when $dy/dx = cx$ describes how y changes with small changes in x at any particular point.

The key idea is *at any point.* The relationship between dy and dx depends on where you are on the curve. That's why we need integration.

Write

$$dy = f(x)\ dx$$

We want to solve

$$\int dy = \int f(x)\ dx$$

The sum of all the little pieces dy is just y

$$y = \int f(x)\ dx$$

Now, this surely sounds a little vague. But it will turn out that

$$F(x) = \int f(x)\ dx = y$$

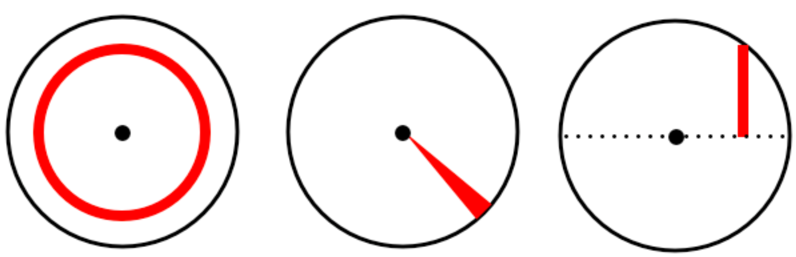
exactly when the derivative of $F(x)$ is $f(x)$:

$$\frac{dF}{dx} = F'(x) = f(x)$$

This is the first of two bright ideas we need to solve an equation like $\int f(x)\ dx$. Just find $F(x)$ such that the derivative of $F(x)$ is $f(x)$.

Area of the circle

Let's spend some time analyzing the area of a circle. This provides crucial insight into what integral calculus can do.



Integration is used to compute areas and volumes, and other sums, by adding up many little pieces.

To calculate the area of a circle, we find the pieces we will use with one of three basic strategies: rings, slices of pie, or rectangles of area underneath the function obtained by solving $x^2+y^2 = R^2$ (using the positive square root). These three approaches are illustrated in the figure above.

rings

In the first approach (left panel), we imagine the area being computed by adding up the individual areas of a series of very thin, concentric rings.

The total area to be computed is that of a circle of a definite, fixed size, and we denote the radius of this circle by capital R, a constant. On the other hand, the series of rings ranges from the origin of the circle to the circumference of the outmost ring. Each one of this progression of rings has a radius, so we use the lowercase r to describe them, with r being a variable—r varies from 0 at the origin to R at the outside of the circle.

Think about an individual ring, for example the outermost ring, which is similar to the circular peel or rind surrounding a thin slice of lemon. We are working with areas here, in two dimensions, so the slice we imagine to be infinitely thin, and we are working with it as a cross-section or ring.

The area of the ring is the length times the width. The length is the circumference, $2\pi R$ for the outermost ring, but in general, for any of the inner rings it is $2\pi r$. The length is multiplied by the width of the slice, which is a small element of radius, dr. The small element of area contributed by an individual ring is dA:

$$dA = 2\pi r \ dr$$

Another way to explain this equation is to ask the question:

how does area change with increasing radius?

If we take a circle and increase its radius by a little bit, how does the area change? The answer is, it changes in proportion to the circumference, $2\pi r$.

Another way to say the same thing is that the derivative is

$$\frac{dA}{dr} = 2\pi r$$

Proceeding from the first equation, the total area is the sum of the areas for the series of rings.

$$A = \int dA = \int_0^R 2\pi r \ dr$$

It's worth emphasizing how this view is different than the examples of integration one usually sees first in a calculus book: these pieces of area are not rectangles but circles. But it poses most clearly the question we are trying to answer, "how does area change as r changes"?

In order to actually determine a value for the area we need two principles. The first is, as we mentioned before, that the solution to

$$\int f(x) \ dx$$

is $F(x)$ if and only if the derivative of $F(x)$ is equal to $f(x)$.

Continuing with our problem

$$\int 2\pi r \ dr = 2\pi \int r \ dr$$

In this step we used a fundamental rule that a constant can come "out from under" the integral sign. That's not surprising. We already know that (at least in the power rule) the derivative of a constant times some function is that constant times the derivative of the function. We will show that is a general rule later.

Now, we need to find a function whose derivative is r.

$$2\pi \int r \ dr$$

We know that function, it is r^2, with an extra factor of 1/2.

$$= 2\pi \ [\ \frac{1}{2} \ r^2 \ ] \ = \pi r^2$$

Combining all the coefficients we have $\int 2\pi r \ dr = \pi r^2$ precisely because the derivative of πr^2 is just $2\pi r$.

The second principle we need comes from the Fundamental Theorem of Calculus, which takes account of the bounds on the integral (in this case 0 and R). The bounds are written attached to the integral as

$$\int_0^R$$

and on the expression to be evaluated attached to a vertical bar

$$\Big|_{r=0}^{r=R}$$

like this

$$2\pi \int_{r=0}^{r=R} r\ dr = \pi\ r^2\ \Big|_{r=0}^{r=R}$$

We say that the answer is this function, "evaluated between the bounds 0 and R."

The value of such a definite integral is $F(x)$ evaluated at the upper limit minus the value of $F(x)$ evaluated at the lower limit:

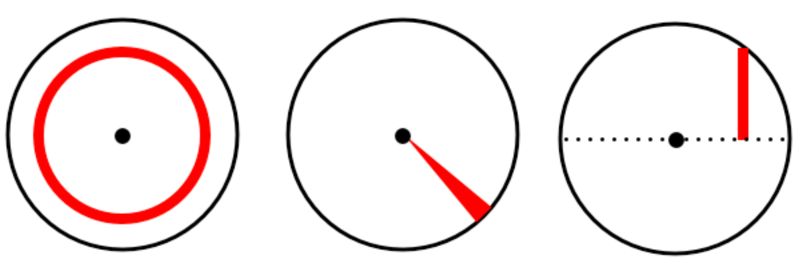
$$= \pi R^2 - \pi(0)^2 = \pi R^2$$

which appears to be correct.

Note in passing that the lower bound doesn't have to be 0, it could be some $\rho < R$. Then we'd have the area of a ring rather than a circle. And another thing, it's not uncommon to leave out the variable from the bounds, and write it like this:

$$2\pi \int_0^R r\ dr$$

wedges



In the second method (middle panel), we need to first find the area of a wedge. For a thin enough slice, this is a triangle, with a familiar formula: one-half the base times the height. The height is R, the radius of the circle.

For the base we need the length of a piece of arc of a circle. Recall that by definition, if we have a unit circle, then the angle of a wedge is equal to the arc it cuts out, and vice-versa, the arc is equal to the angle. (Thus, the total length if we go all the way around the unit circle is 2π).

For a circle with radius R, the length going all the way around is $2\pi R$, and the length of arc for any angle θ is θ times R.

The area we want is built up of a series of wedges that are almost infinitely slender, with angle $d\theta$, so these wedges have bases measuring $R\ d\theta$. The area of each triangular wedge is one-half the height times the base or

$$dA = \frac{1}{2} R\ R\ d\theta$$

For the total area

$$A = \int dA = \int \frac{1}{2} R\ R\ d\theta$$

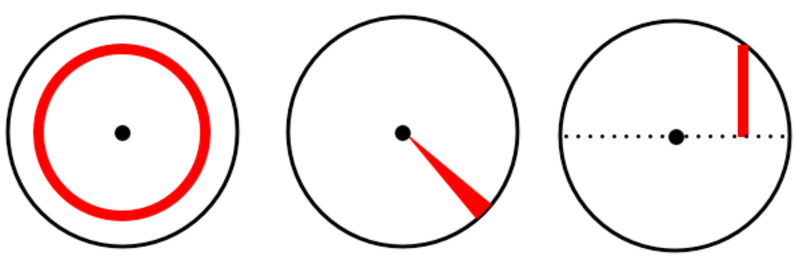
again we see that constants can come outside the integral

$$= \frac{1}{2} R^2 \int_{\theta=0}^{\theta=2\pi} d\theta$$

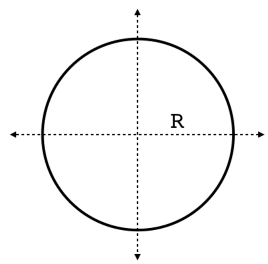
$$= \frac{1}{2} R^2 \theta \ \Big|_{\theta=0}^{\theta=2\pi}$$

$$= \pi R^2$$

area under the curve



The third view (right panel) is the most familiar, but has a somewhat harder calculation. We calculate the area under the positive square root in the equation for a circle (right panel), lying above the x-axis, and then multiply by two to get the whole thing.



$$x^2 + y^2 = R^2$$
$$y = f(x) = \sqrt{R^2 - x^2}$$

To get the area, we need to integrate:

$$\int y\ dx = \int_{-R}^{R} \sqrt{R^2 - x^2}\ dx$$

We will work through this problem **later**, after we review a few more techniques that are useful in doing integration problems.

Of course, the answer will turn out to be just what you'd expect. In fact, this must be so. If we solve the same problem by correctly using two different techniques and get different answers, then at least one of the techniques is wrong.

The area beneath the circle $y = \sqrt{R^2 - x^2}$ and above the x-axis is

$$\frac{1}{2}\pi R^2$$

which is multiplied by 2 to get the area of the whole circle.

Volume of the sphere

We think about how the volume of the sphere depends on r ($r = 0 \to R$). An incremental change dr changes the volume by adding a thin shell of volume equal to the surface area of the sphere ($4\pi r^2$) times dr. That is

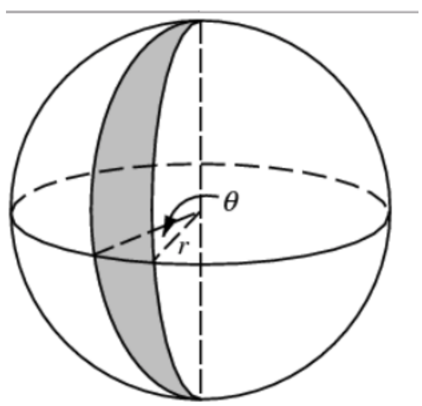
$$dV = 4\pi r^2 \ dr$$

$$V = \int dV = \int_0^R 4\pi r^2 \ dr$$

$$= 4\pi \ \ \frac{1}{3} r^3 \ \Big|_0^R = \frac{4}{3}\pi R^3$$

It's really as simple as that. Of course, you need to know the formula for the surface area to do it that way. Alternatively, if you know the volume of the sphere, taking the derivative is an easy way to get a formula for the surface area.

The image shows a "spherical lune", or segment of the surface of the sphere, as an aid to visualizing the whole surface.



We'll say a lot more about the volume of the sphere **later**.

technical note

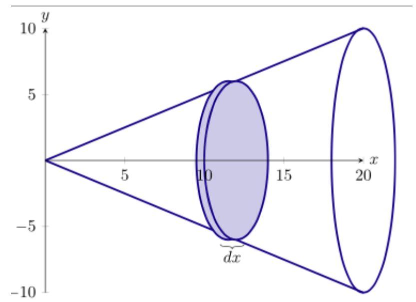
We should point out that this connection between volume and surface area is not true for *every* solid.

As an example, the surface area of a cube of side s is $6s^2$, which would have volume $2s^3$ if the relationship were always correct. In fact, there is something special about the *radial symmetry* of circles and spheres, and their lack of sharp corners and edges.

Here is one more example, to calculate the volume of a cone.

volume of a cone

We lay a cone along the x-axis with its vertex at the origin, opening to the right.



The cone is three-dimensional with the third axis (z) coming up out of the page. The intersection with the xy-plane is a triangle.

Can you see that in the xy-plane y is a linear function of x, i.e. $y = kx$ where k is a constant. The constant k is actually the ratio of the radius R to the height H. That is equal to $\Delta y/\Delta x$.

$$y = \frac{R}{H}x$$

If we slice the cone into thin sections perpendicular to the x-axis, each little piece is a circle with radius y and area πy^2. For a thin enough slice, the volume is that area times the width of the slice:

$$dV = \pi y^2 \ dx$$

Finding the volume of an individual piece is the important part of the calculus argument.

Now we just substitute the value of y in terms of x

$$dV = \pi \ [\frac{R}{H}]^2 x^2 \ dx$$

add up all the little volumes by setting up the integral

$$V = \int dV = \int \pi \ [\frac{R}{H}]^2 x^2 \ dx$$

We apply the basic rule that constant terms can move "out from under" the integral sign:

$$= \pi \ [\frac{R}{H}]^2 \int x^2 \ dx$$

This is a corollary of the result that constants are just carried through in taking the derivative.

We recognize that the value x lies in the interval between 0 and H, $[0, H]$, so these are the "bounds" on the integral, which we write as $\int_0^H$:

$$= \pi \, [\frac{R}{H}]^2 \int_0^H x^2 \, dx$$

and then just follow the rule for doing a problem like this: $\int x^2 = x^3/3$. So

$$= \pi \, [\frac{R}{H}]^2 \, [\frac{x^3}{3}] \, \Big|_0^H$$

$$= \frac{1}{3} \pi R^2 H$$

This is the answer precisely because the derivative of the result $(x^3/3)$ is equal to the integrand we started with (x^2).

Once again, we obtain the formula of one-third times the area of the base times the height. No matter what the shape of the base is, the area of each slice will be proportional to x^2 and we will end up with a formula involving one-third at the end.

We will see several other methods for obtaining this result.

Note in passing that we can obtain the volume of a frustum (a cone whose top has been cut off) as

$$= \pi \, [\frac{R}{H}]^2 \, [\frac{x^3}{3}] \, \Big|_{h_1}^{h_2}$$

$$= \pi \, [\frac{R}{H}]^2 \, [\frac{{h_2}^3}{3} - \frac{{h_1}^3}{3} \,]$$

The geometers have given us an even more elegant formula (**here**).

Part XI

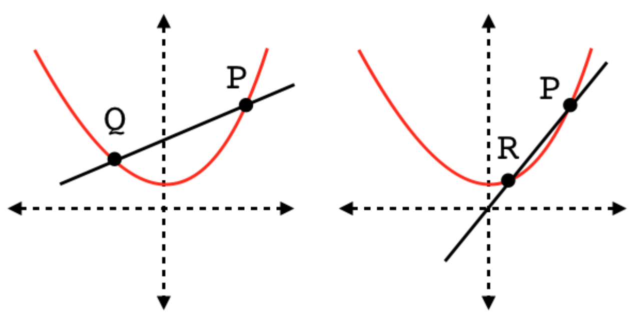
Basic calculus

Chapter 39

Difference quotients

In this chapter we look at the geometric interpretation of the derivative — which is the traditional way to begin calculus. The general approach was developed by Fermat.

Think for a minute about a curve such as the one shown in the figure, corresponding to some unspecified function $f(x)$, which looks like it's probably a parabola.



At an arbitrary point P on the curve, for some value of x, we plot $y = f(x)$. This is Descartes' genius idea. The point on the graph of $f(x)$ at x has coordinates $P = (x, f(x))$.

Now consider a point Q near P but also on the curve, For the x-coordinate of Q, a small change is made to x.

We might call that small amount Δx, but many authors use h, a simpler notation, and we will do so as well. The value of the function at $x + h$ is $f(x + h)$ and so Q

has coordinates $Q = (x + h, f(x + h))$.

In this example, h is negative, but that makes no difference. We drew it that way so it's easier to see how the approximation to the slope gets better as we go along.

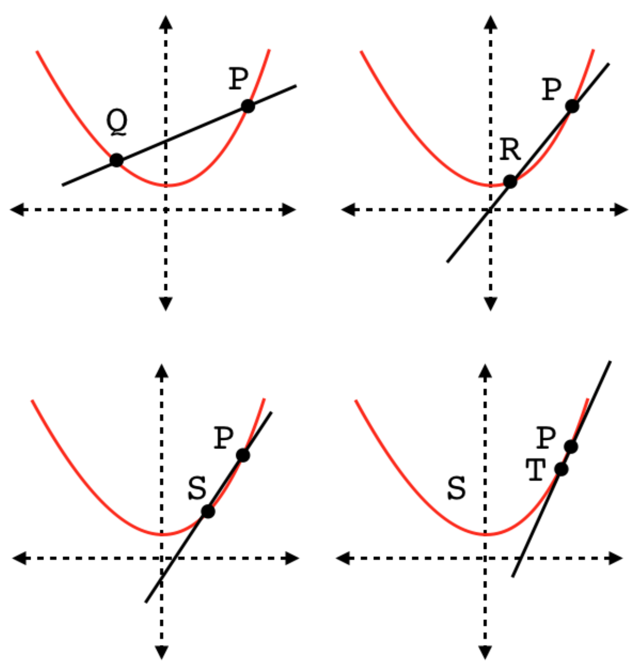
The slope of the (secant) line connecting Q and P is

$$\frac{\Delta y}{\Delta x} = \frac{f(x+h) - f(x)}{x + h - x} = \frac{f(x+h) - f(x)}{h}$$

This is a famous quantity, it's called the **difference quotient**.

The goal of differential of calculus is to find the slope of the *tangent* to the curve at the point P. What we have is an expression for the slope of the secant line PQ, which is close but not quite the same thing.

To go from the secant to the tangent, we ask "what happens to this expression as h gets smaller and smaller and approaches zero." The second point where the secant meets the curve comes closer and closer to the first one.



In mathematical language, we say the slope of the tangent is equal to the limit of the difference quotient as h tends to 0:

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

We'll say a bit more about limits in the next chapter, but for the moment you can think about

$$\lim_{h \to 0}$$

as meaning, "substitute $h = 0$ and see what happens to the expression of interest."

x squared

Let's try a couple of examples and look for a pattern.

$$f(x) = x^2$$

For this function, we write that the difference quotient is

$$\frac{(x+h)^2 - x^2}{h}$$

$$= \frac{x^2 + 2xh + h^2 - x^2}{h}$$

$$= \frac{2xh + h^2}{h}$$

Now divide by the denominator h

$$= 2x + h$$

Finally, to get the slope of the tangent, we evaluate the limit

$$\lim_{h \to 0} 2x + h = 2x$$

In evaluating the limit, we ask: what happens to this expression as h approaches 0. In this case, it cannot actually reach zero, because then our previous step of dividing by h would not be allowed. But we let h become really really small, and take advantage of the property of the limit which says that an expression can have a limit at c even if it can't be evaluated at c itself.

At every point on the curve $y = x^2$, the slope of the tangent line to the curve is $2x$. So the slope at $x = 0$ is 0, and the slope at $x = 2$ is 4, and so on.

This process of computing the difference quotient and then finding the limit as $h \to 0$ is called "taking the derivative." It produces an expression which is called the derivative of y with respect to x, in this case

$$\frac{dy}{dx} = 2x$$

and we can interpret this as the slope of the tangent to the curve of $f(x)$ at the point x.

Another useful shorthand uses the f from $f(x)$. We adopt the convention that the derivative of $f(x)$ can be written $f'(x)$.

$$f'(x) = 2x$$

To be even more succinct we might write y' for $f'(x)$.

If we repeat this exercise with a leading constant a (that is, for $f(x) = ax^2$), we find that every term in the numerator of the difference quotient will contain a, and the final result will be $2ax$. Constants just get carried through.

square root

Now look at the square root:

$$f(x) = \sqrt{x}, \quad (x \geq 0)$$

The difference quotient for this function is

$$\frac{\sqrt{x+h} - \sqrt{x}}{h}$$

Clean up the numerator by multiplying by the conjugate

$$\frac{\sqrt{x+h} - \sqrt{x}}{h} \; \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$$

$$= \frac{x + h - x}{h \; (\sqrt{x+h} + \sqrt{x})}$$

$$= \frac{h}{h\ (\sqrt{x+h} + \sqrt{x})}$$

$$= \frac{1}{\sqrt{x+h} + \sqrt{x}}$$

We evaluate the limit

$$\frac{dy}{dx} = \lim_{h \to 0} \ \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

inverse

Consider the inverse function

$$f(x) = 1/x, \quad (x \neq 0)$$

$$\frac{\frac{1}{x+h} - \frac{1}{x}}{h}$$

Clean up the numerator

$$\frac{\frac{1}{x+h} - \frac{1}{x}}{h} \ \frac{(x)(x+h)}{(x)(x+h)}$$

$$= \frac{x - (x+h)}{h\ (x)\ (x+h)}$$

$$= \frac{-h}{h\ (x)\ (x+h)}$$

$$= -\frac{1}{(x)\ (x+h)}$$

We evaluate the limit:

$$\lim_{h \to 0} \ -\frac{1}{(x)\ (x+h)}$$

$$\frac{dy}{dx} = -\frac{1}{x^2}$$

There's a pattern here. We will use the notation $f'(x)$ to indicate the slope of the curve $f(x)$ at x

$$f(x) = x^2 \quad \Rightarrow \quad f'(x) = 2x$$

$$f(x) = \sqrt{x} = x^{1/2} \quad \Rightarrow \quad f'(x) = \frac{1}{2}x^{-1/2}$$

$$f(x) = \frac{1}{x} = x^{-1} \quad \Rightarrow \quad f'(x) = -\frac{1}{x^2} = -x^{-2}$$

The general formula is

$$f(x) = x^n \quad \Rightarrow \quad f'(x) = nx^{n-1}$$

This is easily proved (for integer n) using the binomial expansion for $(x+h)^n$ for integral n ($n \in 1, 2, \dots$). We need only the first three terms:

$$(x+h)^n = x^n + nx^{n-1}h + n\frac{(n-1)}{2}x^{n-2}h^2 + \dots$$

The key point is that the last term shown and all subsequent terms contain powers of h^2 or higher.

After division by h, for each of these terms there will remain one or more terms of h, and in the limit $\lim_{h \to 0}$ these become zero.

$$\begin{aligned}
&\lim_{h \to 0} \ \frac{(x+h)^n - x^n}{h} \\
&= \lim_{h \to 0} \ \frac{x^n + nx^{n-1}h + n\frac{(n-1)}{2} \ x^{n-2}h^2 + \dots - x^n}{h} \\
&= \lim_{h \to 0} \ \frac{nx^{n-1}h + n\frac{(n-1)}{2} \ x^{n-2}h^2 + \dots}{h} \\
&= \lim_{h \to 0} \ nx^{n-1} + n\frac{(n-1)}{2} \ x^{n-2}h + \dots \\
&= nx^{n-1}
\end{aligned}$$

Another question is what to do with a sum or difference of polynomials, such as

$$f(x) + g(x)$$

If you write out the difference quotient

$$\frac{f(x+h) - f(x) + g(x+h) - g(x)}{h}$$

everything can be exactly as before, just grouping all terms with $f(x)$ and those with $g(x)$ separately.

$$[f(x) + g(x)]' = f'(x) + g'(x)$$

We showed above by computing the difference quotient directly that

$$f(x) = \sqrt{x}$$

$$f'(x) = \frac{1}{2\sqrt{x}}$$

Here is another approach to the same problem. Consider

$$y = x^2$$

$$\frac{dy}{dx} = 2x$$

Solve for x as a function of y:

$$x = \sqrt{y}$$

We can do algebra with *differentials* (with some constraints):

$$\frac{dy}{dx}\,\frac{dx}{dy} = 1$$

$$2x\,\frac{dx}{dy} = 1$$

$$\frac{dx}{dy} = \frac{1}{2x} = \frac{1}{2\sqrt{y}}$$

In observing the inverse relationship, remember that x and y are related by the equation $y = x^2$. For example, when $x = 2$, $dy/dx = 2x = 4$.

Using the relationship $f(x)$, when $x = 2$, $y = 4$, and $dx/dy = 1/2\sqrt{y} = 1/2\sqrt{4} = 1/4$, which is indeed the inverse of 4.

In this last section, after solving for x as a function of y, y is the *independent* variable. We can switch back to our usual notation:

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$$

problem

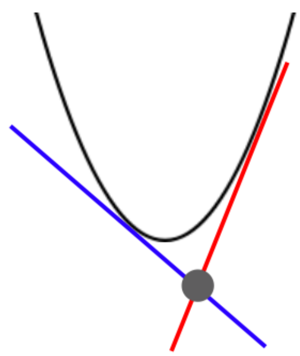
I found the following problems on the web. They are great practice and show what kinds of problems this approach of differentiation can solve. To prove:

Let $(a, f(a))$ and $(b, f(b))$ be two distinct points on the graph of a differentiable function f. Suppose that the tangent lines of f at these two points intersect, and call the point of intersection (c, d). Verifying the following facts is elementary.

1. If $f(x) = x^2$, then $c = (a+b)/2$, the arithmetic mean of a and b.
2. If $f(x) = \sqrt{x}$, then $c = \sqrt{ab}$, the geometric mean of a and b.
3. If $f(x) = 1/x$, then $c = 2ab/(a+b)$, the harmonic mean of a and b.

1

Here is a diagram for the first one:



The claim is that the x-coordinate of the point will be half-way between the x-coordinates for the two points on the parabola. We have:

$$y = f(x) = x^2$$

$$y' = f'(x) = 2x$$

At $x = a$, the slope is $2a$ and the equation of a line through the point (a, a^2) is

$$y - a^2 = 2a(x - a)$$

At $x = b$, the equation is

$$y - b^2 = 2b(x - a)$$

To see where the lines cross, we set the y's to be equal, and solve for x:

$$2a(x-a)+a^2 = 2b(x-b)+b^2$$
$$2ax-a^2 = 2bx-b^2$$
$$2x(a-b) = a^2-b^2$$
$$= (a+b)(a-b)$$
$$x = \frac{1}{2}(a+b)$$

2

We have:

$$y = f(x) = \sqrt{x}$$
$$y' = f'(x) = \frac{1}{2\sqrt{x}}$$

At $x = a$, the slope is $1/2\sqrt{a}$ and the equation of a line through the point $(a, \sqrt{a})$ is

$$y - \sqrt{a} = \frac{1}{2\sqrt{a}}\ (x-a)$$

At $x = b$, the equation is

$$y - \sqrt{b} = \frac{1}{2\sqrt{b}}\ (x-b)$$

We set the y's to be equal

$$\frac{1}{2\sqrt{a}}\ (x-a)+\sqrt{a} = \frac{1}{2\sqrt{b}}\ (x-b)+\sqrt{b}$$

and solve for x. Multiply by $2\sqrt{a}\sqrt{b}$

$$(x-a)\sqrt{b}+2a\sqrt{b} = (x-b)\sqrt{a}+2b\sqrt{a}$$

Multiply through and cancel

$$x\sqrt{b}+a\sqrt{b} = x\sqrt{a}+b\sqrt{a}$$
$$x(\sqrt{b}-\sqrt{a}) = b\sqrt{a}-a\sqrt{b}$$
$$= \sqrt{a}\sqrt{b}(\sqrt{b}-\sqrt{a})$$

$$x = \sqrt{ab}$$

3

We have:

$$y = f(x) = \frac{1}{x}$$

$$y' = f'(x) = -\frac{1}{x^2}$$

At $x = a$, the slope is $-1/a^2$ and the equation of a line through the point $(a, 1/a)$ is

$$y - 1/a = -\frac{1}{a^2}\ (x - a)$$

At $x = b$, the equation is

$$y - 1/b = -\frac{1}{b^2}\ (x - b)$$

We set the y's to be equal

$$-\frac{1}{a^2}\ (x - a) + 1/a = -\frac{1}{b^2}\ (x - b) + 1/b$$

and solve for x:

$$(\frac{1}{b^2} - \frac{1}{a^2})x = 2(\frac{1}{b} - \frac{1}{a})$$

$$(\frac{1}{b} + \frac{1}{a})x = 2$$

$$(a + b)x = 2ab$$

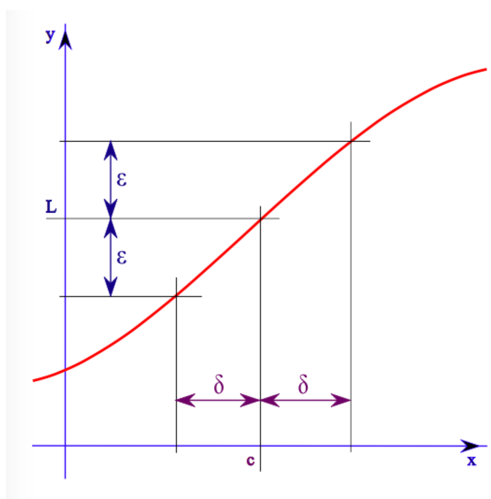
$$x = \frac{2ab}{a + b}$$

Chapter 40

Limit concept

Limit concept

Consider the graph of a function $f(x)$. We might choose a power of x similar to $y = x^2$ or $y = x^3 - x$, which affirmatively has two properties that are of interest here: continuity and differentiability (we'll get to those ideas in a bit). Let's just say $y = f(x)$ is a "good" function. The functions we deal with in this book are all "good."



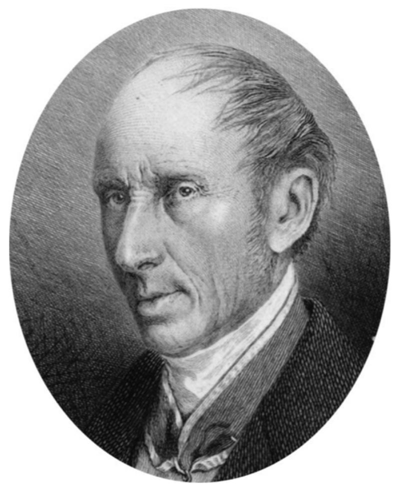
Focus on the neighborhood of a point on the x-axis, $x = c$.

By inspection of the graph, for points near c, the value of f at those points is not too different from L.

(It is also true here that the value of $f(x)$ at c is equal to L. This matters for continuity but not for limits).

We would like to say that the *limit* of $f(x)$ as x *approaches* c is equal to L. The idea is that we can make $f(x)$ as close to L as we please, provided we choose x sufficiently close to c.

> When the values successively attributed to a variable approach indefinitely to a fixed value, in a manner so as to end by differing from it by as little as one wishes, this last is called the limit of all the others. —Cauchy



Modern mathematicians don't like that word "approach", which conjures up movement and the involvement of time.

They also don't like reasoning from what they see in a graph, in part because no graph can show the whole function for the general case. To free ourselves from graphs and pictures, we will use an algebraic method from the formal apparatus of calculus.

There are two equivalent approaches, neighborhoods, and epsilon-delta formalism. Let's look at neighborhoods briefly.

neighborhoods

First, an *interval* between two real numbers a and b ($a < b$) contains every real number $a < x < b$.

$$(a, b) = x \mid a < x < b$$

The "$|$" means x "such that" the condition $a < x < b$ holds.

A *closed* interval $[a, b]$ includes the endpoints, $a \leq x \leq b$, while an *open* interval (a, b) excludes them. Half-open intervals like $[a, b)$ may be defined, and an interval with $\pm\infty$ as an endpoint is always open on that end, for example: $[a, \infty)$, because infinity *is not a number*.

Any open interval with a point p as its midpoint is called a *neighborhood* of p. Let r be the distance from p to the boundary of a particular neighborhood; r may be large or very very small. We denote a neighborhood of p as $N(p)$. $N(p)$ consists of all those values of x such that

$$|x - p| < r$$

.

which we would write more formally as

$$N(p) = x \mid |x - p| < r$$

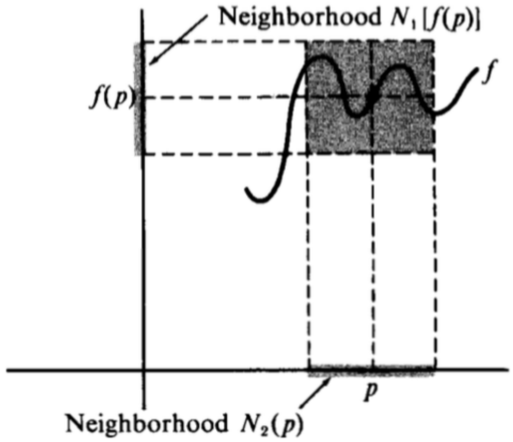
.

To say that the limit $f(x) \to L$ exists, we mean that for every neighborhood $N_1(L)$, no matter how small, there exists some neighborhood $N_2(p)$ such that $f(x)$ is contained within $N_1(L)$, written as

$$f(x) \in N_1(L)$$

whenever $x \in N_2(p)$.

If $N_1(L)$ is very small, then $N_2(p)$ may need to be very small as well, to guarantee that $f(x)$ is contained within N_1. Here is an example where this condition is satisfied.



The idea of a neighborhood is a nice abstraction to hide the apparatus of modern calculus, which we save for the Addendum.

An important fact about limits has to with the case where $x = p$. It is *not* necessary that $f(p) = L$. This relaxed condition is in fact crucial for calculus.

example 1

Limits can be easy or hard, depending on the problem. Here is one found in the previous chapter on difference quotients:

$$\lim_{h \to 0} \frac{2xh + h^2}{h}$$

When you see something like this, what you are supposed to do is reason about what happens as the variable h approaches 0 (gets smaller and smaller). The first step in that is to figure out what would happen if h actually would become zero.

Here, each term has a limit of 0 when h is zero, so we will have $0/0$. The zero on the bottom is trouble, it means that the expression becomes undefined.

However, suppose we first cancel h on top and bottom to obtain

$$\lim_{h \to 0} \frac{2x + h}{1} = \lim_{h \to 0}\ 2x + h$$

Now, the answer is just $2x$. This is valid as long as h approaches zero but is never actually equal to it.

Recall that we can have a limit for $f(x)$ as x approaches c, even if $f(c)$ does not exist.

example 2

Here is another important expression. What is the value of $f(x)$ as h approaches zero?

$$\cos h < f(x) < \frac{1}{\cos h}$$

Since $\cos 0 = 1$, the two outside terms both approach 1 in the limit as h approaches zero. Since $f(x)$ lies between them, it must also approach 1.

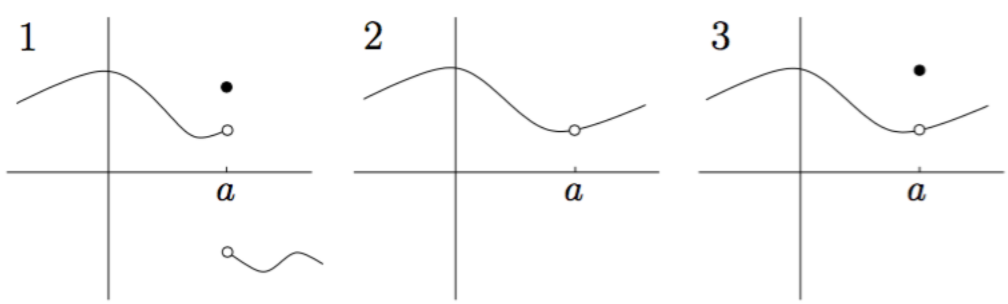
This is called the *squeeze theorem*.

The magical thing is that this is true even if, when $h = 0$, $x = 0/0$. We'll see this when we look at calculus of sine and cosine.

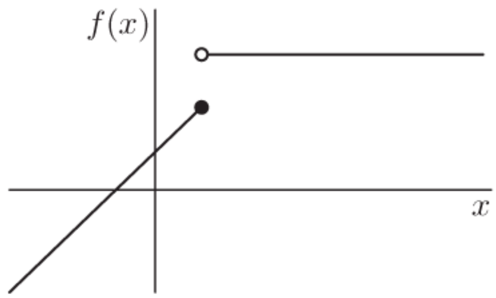
Continuity

Continuity has an intuitive definition: as Euler said, if we can graph a function *without lifting our pencil from the paper*, then the function is continuous.

Here are some graphs showing examples of how continuity can fail.



A filled circle means that the function yields that y-value for the corresponding x-value of the point, while an open circle means it does not. The function may yield some other value, or simply be undefined.



$$f(x) = \begin{cases} x+1 & \text{if } x \leq 1 \\ 3 & \text{if } x > 1 \end{cases}$$

For a function to be continuous at a point $x = c$, we imagine that if we vary x in neighborhood of c, then $f(x)$ should not change in value by too much.

Again, we will call that value L, the limit of $f(x)$ as $x \to c$. For L to exist we require that the two one-sided limits be equal. If we approach c from the high side ($x > c$) or the low side ($x < c$), the limit must be the same.

Very important: continuity requires, in addition, that $f(c)$ be equal to L.

Differentiability

For a function to be differentiable, we require that the limit

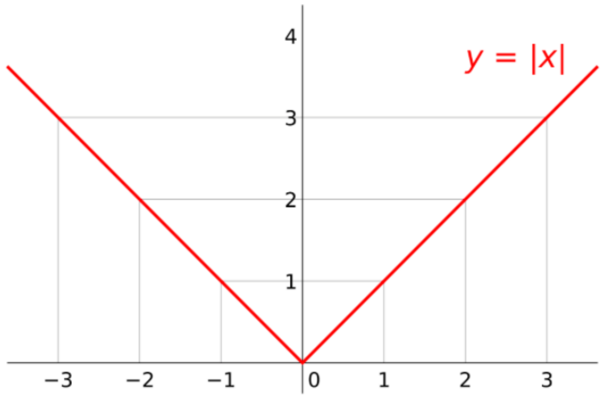
$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

exists. An example of a function that is continuous but not differentiable at a particular point is the absolute value function.

example: absolute value

An algebraic definition of the absolute value function is piecewise:

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$



The function $f(x) = |x|$ is continuous at $x = 0$ because the two one-sided limits exist and are equal to each other. They are also equal to $f(0) = 0$.

However, there is no defined slope at $x = 0$. The difference quotient gives different results for positive Δx (positive slope) than for negative Δx (negative slope).

Without getting too technical

> Note that the graph of the absolute value function is "all in one piece", but has a "sharp point" at the origin. We will not attempt to make these descriptions precise, other than to say that the fact that the graph comes "all in one piece" is a feature of continuity, and that graphs of differentiable functions are "smooth" in that they do not have "sharp points." The unambiguous and demonstrably true statement here is that the absolute value function is continuous at 0 but is not differentiable at 0.

https://oregonstate.edu/instruct/mth251/cq/Stage5/Lesson/diffVsCont.html

Chapter 41

Higher derivatives

We have defined the derivative of a function $f(x)$ as a limit

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

It is the limit of the difference quotient as $h \to 0$.

We introduced the power rule to obtain the derivative of integer powers of x. In addition, we said that the derivative of a sum of two or more functions is the sum of the derivatives.

The derivative is just a function itself. Consider a quadratic like

$$y = ax^2 + bx + c$$

The derivative is

$$y' = 2ax + b$$

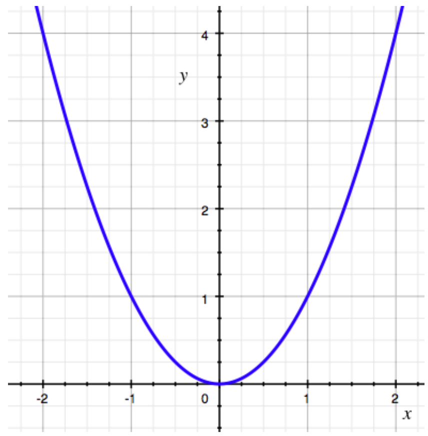
Since the derivative is just a function, why not take the derivative of the derivative, which is called

$$\frac{d^2y}{dx^2}$$

or more compactly, y double prime:

$$y'' = 2a$$

What is the meaning of the second derivative? It gives the slope of the slope, or how the slope is changing with a change in x. For a parabola that opens up, like this one



the second derivative is positive. This means that the slope continues to increase as x increases.

There is nothing to stop us from taking more derivatives. For a quadratic $y''' = 0$, which is not very interesting, but consider the cubic:

$$y = x^3$$

$$y' = 3x^2$$

$$y'' = 6x$$

$$y''' = 6$$

extrema

A very important use of the derivative is to find a maximum or minimum of a function. At such a point the slope is zero because the curve is headed sideways, just for a moment. For the quadratic, the slope is zero at the vertex:

$$y' = 0 = 2ax + b$$

$$x = -\frac{b}{2a}$$

You should recognize this equation from geometry. Without getting into details, it is obtained there by completing the square. Or alternatively, write the equation of a parabola whose vertex is (h, k)

$$(y - k) = a(x - h)^2$$

multiply out

$$y = ax^2 - 2ahx + h^2 + k$$

By comparison with the standard form

$$y = ax^2 + bx + c$$

it's clear that

$$-2ahx = bx$$

so the x-coordinate of the vertex is

$$h = -\frac{b}{2a}$$

The vertex is the maximum or minimum of a quadratic, depending on the sign of a. We can tell the difference by looking at the second derivative again:

$$y'' = 2a$$

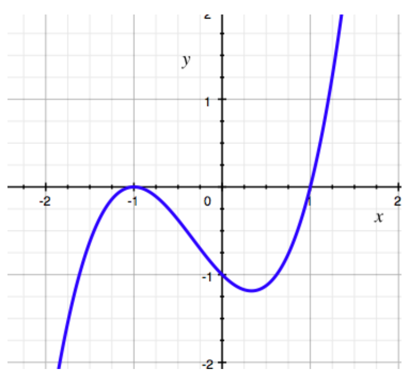
If $a > 0$ (a parabola that opens up), then the second derivative is positive and we have found a minimum value. If $y'' < 0$, we have a maximum.

Consider the cubic

$$\begin{aligned} y &= (x+1)(x+1)(x-1) \\ &= (x^2 + 2x + 1)(x-1) \\ &= x^3 + x^2 - x - 1 \end{aligned}$$

From the first form, we can easily see that the roots are $x = \pm\ 1$. These are the values where the function crosses the y-axis, that is, where its value is zero.

The graph looks like this:



The first derivative is

$$
\begin{aligned}
y' &= 3x^2 + 2x - 1 \\
&= (3x - 1)(x + 1)
\end{aligned}
$$

This expression is zero when $x = -1$ or $x = 1/3$. That does match the places where the curve is horizontal, as we can see.

The second derivative is

$$y'' = 6x + 2$$

For the first value $x = -1$, the second derivative is negative, and this corresponds to a local maximum for the function. For $x = 1/3$, the second derivative is positive, and this is a minimum. We say "local" because there may be more extreme values, as is the case here.

A maximum corresponds to a negative value for the slope of the slope because the slope is first positive, then zero, then negative. Its change with increasing x is negative.

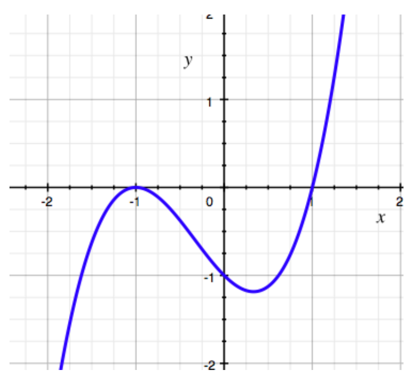
Again, the second derivative of our cubic is

$$y'' = 6x + 2$$

Setting this equal to zero, we obtain

$$x = -\frac{1}{3}$$

This point on the curve is an *inflection point*. It is a point (actually the only point for this curve) where the rate of change of the slope, which is negative to the left $x = -1/3$, changes to positive, and there is an instant where it is zero.



We will see this in connection with the Gaussian or normal curve. It's an interesting fact that the first standard deviation corresponds to the inflection point of the curve. At that point the second derivative of the function is equal to zero.

Rectangular area

Here is a classic problem.

We wish to construct a rectangle with the maximum area *for a fixed perimeter* (without the second statement the area would be infinite). Let's call the sides x and y, and the semi-perimeter S (constant) and so our constraint is that

$$S = x + y$$

The area is then

$$A = xy$$

substitute

$$\begin{aligned} A &= x(S - x) \\ &= Sx - x^2 \end{aligned}$$

Take the first derivative and set it equal to zero:

$$A' = S - 2x = 0$$

$$x = \frac{S}{2} = y$$

A square has the maximum area for a given perimeter constructed with right angles, as expected. We'll see many challenging problems of this type later on.

Chapter 42

Differentials

infinitesimals

We say that the derivative dy/dx is the slope of the tangent to the curve $y = f(x)$ at some particular point (x, y); it is the slope of a line that just touches the curve.

And it is frequently called the slope of the curve at the point (x, y).

We saw a formal definition for the derivative in terms of an expression for Δy as a function of Δx, which is then divided by Δx. We determine what is called the "limit" as Δx approaches zero.

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

There appears to be a contradiction here. On one hand, we're saying that Δx must approach zero. On the other hand, it cannot really be zero, because division by zero is not defined. So how close does it have to be to zero?

Are dy and dx small, really small, really really small, or almost zero?

The official answer requires a cumbersome apparatus of limits, and it would say that dy/dx is not a quotient at all, but rather a single entity, the limit of a quotient, as we just said:

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

However, our simple answer is that dx and dy are separable entities and they are just as small as they need to be. Take the example we used previously:

$$\frac{dy}{dx} = 3cx^2$$

Put the dx on the right-hand side:

$$dy = 3cx^2 \ dx$$

What this expression says is that for a small change in x which we call dx, we will obtain a small change in y called dy with the given relationship. $3cx^2$ gives the proportionality between dx and dy.

Suppose we evaluate $3cx^2$ at some particular x_0. Then it is a number, it has a fixed value depending on where we are on the curve. So write it as $k = 3cx_0^2$ and then:

$$dy = k \ dx$$

When we write this, we are making a *linear approximation* to the quadratic function. dy is not exactly equal to $k \ dx$, for most situations we are ignoring quadratic and higher terms.

Here, we treat dy and dx as very small but non-zero quantities. If there should ever be a problem because we've chosen Δx too large, just reduce it by some factor (1/10, 10^{-6}, 1/googol), whatever is needed,

https://en.wikipedia.org/wiki/Googol

and try again until the problem disappears (it will). Make dx and dy really really small. If that's not small enough, try making them smaller still.

By this trick, we free ourselves from limits. If you want to multiply by dx on both sides of an equality

$$\frac{dy}{dx} = nx^{n-1}$$

$$dx \cdot \frac{dy}{dx} = dy = nx^{n-1} \ dx$$

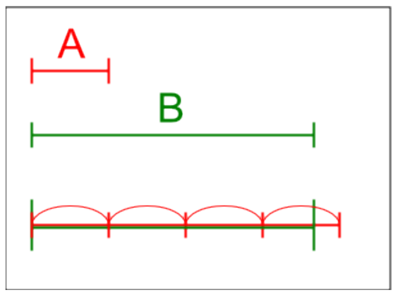
Feel free, go ahead and do it.

fancy

If you want to read more about the derivative as a limit, why dy/dx is not a quotient, and so on, you can look at any standard calculus book. Or start here

https://math.stackexchange.com/questions/21199/is-frac-textrmdy-textrmdx-not-a-ratio

The bottom line is that we want dy and dx to be very small compared to y and x, but one of the properties of the real numbers is that no matter how small small we choose A (or dx), there exists a positive integer n such that $n \cdot A > B$ (or $n \cdot dx > x$). This is called the Archimedean property of the real numbers.



Effectively what limits and neighborhoods do is to say, OK smart guy, you go first. Pick n. Then once you've picked n very large, we can always find dx very very small so that $n \cdot dx$ is still small compared with x. That's the whole trick.

However, in practice none of this is a problem because we view dy and dx as very small. Although often we only care about their ratio, some times we will need to separate them. This is legal, trust me.

Chapter 43

Fundamental theorem of calculus

> Calculus has a long history. Although Newton and Leibniz are credited with the invention of calculus in the late 1600s, almost all the basic results predate them. One of the most important is what is now called the Fundamental Theorem of Calculus (ftc), which relates derivatives to integrals.

https://mathcs.clarku.edu/~ma120/FTC.pdf

The usual way to begin the study of calculus is to think about the slope of the tangent to a curve at a point. If the point is some particular value of x, say $x = a$, then this is

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

For any x in the domain of $y = f(x)$, we say that this slope is the limit

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

We may call this construct by various names such as the derivative of y with respect to x, dy/dx, or $f'(x)$.

The notation dy/dx is due to Liebnitz, and $f'(x)$ is due to Lagrange. For situations where x is a function of time t, Newton would write $\dot{x}$, $\ddot{x}$ and so on. Newton called his method "fluxions", concentrating on derivatives with respect to time.

$$y = f(x)$$
$$\frac{dy}{dx} = f'(x)$$

Evaluation of this limit for $f(x) = x^n$, $f(x) = e^x$, and $f(x) = \sin(x)$ then follows.

Some rules (product rule, chain rule and so on) will be introduced that allow us to calculate derivatives of more complicated functions. We also learn to keep note of various functions and their derivatives because it is essential to be able to "go backwards."

The inverse of differentiation is integration. By definition

$$y = f(x) = \int dy$$

Now

$$\frac{dy}{dx} = f'(x)$$
$$dy = f'(x)\ dx$$
$$y = f(x) = \int dy = \int f'(x)\ dx$$

There is an idea in integration which is really profound. We already introduced it in previous chapters by considering a ball or solid sphere in 3D space and the outer surface of the ball (technically, that *is* the sphere, but no matter). As Archimedes showed 2200 years ago, the volume of the sphere is this function of the cube of the radius.

$$V = f(r) = \frac{4}{3}\pi r^3$$

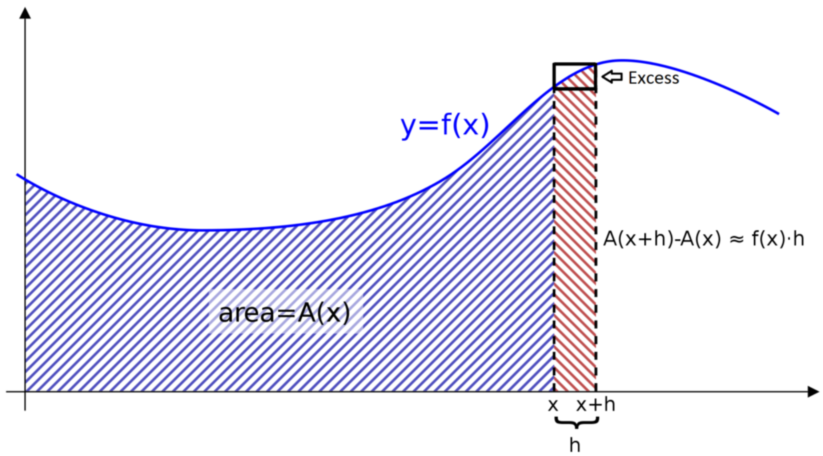
The question was: how does the volume change when the radius changes by a little bit? The big idea is to realize that the answer is exactly the same as the limit we gave above, it is the derivative of V with respect to r.

$$\frac{d}{dr}V = \frac{dV}{dr} = \frac{d}{dr}\frac{4}{3}\pi r^3 = 4\pi r^2$$

It is no accident that this result is the same as the formula for the surface area of the sphere. Increasing the radius r by a little bit dr, the new volume that is added is approximately the surface area $4\pi r^2$ times dr, the volume of the shell at the radius r.

$$dV = 4\pi r^2 \; dr$$

This is a completely general idea. The way it is usually introduced is to consider the graph of a function $f(x)$ in the plane.



We think not just about $f(x)$ itself, but the total area underneath the curve. Area is a function. Its value depends on the bounds. Let us fix the left-hand boundary (say, at $x = a$), but leave the right-hand bound as a variable, call it x.

The big idea is that the total area under the curve (the region in blue) is some as yet unknown function of x, $A = F(x)$.

The way we find $F(x)$ is to ask the question: how does the area $F(x)$ change when x changes by a little bit, say h? If you look at the figure it is clear that the answer is the area of the red rectangle, the added area is just $f(x)$ times h. If h is small enough, the answer is exact.

Recast in mathematical terms

$$\lim_{h \to 0} \; f(x) \; h = \lim_{h \to 0} \; F(x+h) - F(x)$$

We just divide by h

$$\lim_{h \to 0} f(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h}$$

and recognize that the term on the left does not depend any longer on h so

$$f(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h}$$

$f(x)$ is the derivative of $F(x)$:

$$f(x) = F'(x)$$

To find the area function F, we just need to find a function that, when we differentiate it, gives us $f(x)$.

The fundamental theorem of calculus states this principle. Usually, a new variable is introduced to remove any confusion that might arise with respect to x, which in the discussion above, we actually used in two different ways. Write

$$F(x) = \int_a^x f(t)\ dt$$

In the equation above, the real variable is x. t is what's called a dummy variable, since it might replaced with any other symbol without changing anything.

We have two different functions F and f. The value for each will vary with the value of x (since we evaluate at the right-hand bound $t = x$). The value of F depends also on the left-hand bound a. Anyway, having written

$$F(x) = \int_a^x f(t)\ dt$$

The Fundamental Theorem of Calculus (FTC) states:

$$F'(x) = f(x)$$

which is what we figured out before.

The FTC has a second part, which is

$$\int_a^b f(x)\ dx = F(b) - F(a)$$

This gives us the way in which areas (and volumes and so on) are actually calculated. Start with the function $f(x)$. We find $F(x)$, and then just evaluate it at the endpoints a and b. The difference is the area under the curve $f(x)$ between the two bounds a and b.

The proof of this second part is usually done with what are called Riemann sums. We look at those a bit later in the book.

Chapter 44

Sum of squares

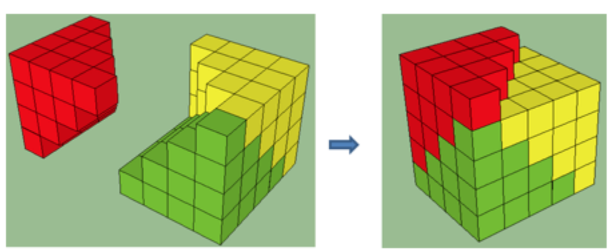
We want to find a formula for the sum of the squares of the first n integers, which is written variously as

$$\frac{n\ (n+1)(2n+1)}{6}, \qquad \frac{n(n+1)}{2} \cdot \frac{(2n+1)}{3}$$
$$\frac{1}{6}\ (2n^3+3n^2+2n), \qquad \frac{n^3}{3}+\frac{n^2}{2}+\frac{n}{3}$$

and my favorite:

$$\frac{1}{3}\ n \cdot (n+1)(n+\frac{1}{2})$$

which explains the proof without words:



sum of squares

We us use exactly the same method as for the sum of integers to determine the formula for

$$S_n = 1^2 + 2^2 + \ldots n^2$$

Since the formula for the sum of integers has a square, we expect that this will be a cubic. Write

$$(k+1)^3 = k^3 + 3k^2 + 3k + 1$$

$$\sum_{k=1}^{n}(k+1)^3 = \sum_{k=1}^{n} k^3 + \sum_{k=1}^{n} 3k^2 + \sum_{k=1}^{n} 3k + \sum_{k=1}^{n} 1$$

Moving the first term from the right-hand side to the left, we obtain a telescoping sum as the difference, just like before:

$$(n+1)^3 - 1 = \sum_{k=1}^{n} 3k^2 + \sum_{k=1}^{n} 3k + \sum_{k=1}^{n} 1$$

Now it's just some messy algebra. The second term on the right-hand side includes our previous result:

$$n^3 + 3n^2 + 3n = 3\sum_{k=1}^{n} k^2 + 3\ \frac{n(n+1)}{2} + n$$

$$6\sum_{k=1}^{n} k^2 = 2(n^3 + 3n^2 + 3n) - 3n(n+1) - 2n$$

$$6\sum_{k=1}^{n} k^2 = n\ [\ 2(n^2 + 3n + 3) - 3(n+1) - 2\ ]$$

$$6\sum_{k=1}^{n} k^2 = n\ [\ 2n^2 + 3n + 1\ ]$$

$$6\sum_{k=1}^{n} k^2 = n\ (n+1)(2n+1)$$

$$\sum_{k=1}^{n} k^2 = \frac{n\ (n+1)(2n+1)}{6}$$

which can be re-written in various ways including:

This formula is also written variously as

$$\frac{n(n+1)}{2} \cdot \frac{(2n+1)}{3}$$

$$\frac{1}{6}\ (2n^3 + 3n^2 + 2n)$$

$$\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{3}$$

But I find it easiest to remember the first version.

We can check it by induction. The base case is easy

$$\frac{1(2)(3)}{6} = 1$$

Now for the induction step:

$$\frac{n(n+1)(2n+1)}{6} + (n+1)^2$$

$$= \frac{n+1}{6}\ [\ (n)(2n+1) + 6(n+1)\ ]$$

Look at what's in the brackets

$$(n)(2n+1) + 6(n+1)$$

$$= 2n^2 + 7n + 6$$

$$= (n+2)(2n+3)$$

$$= (n+1+1)(2(n+1)+1)$$

So altogether we have

$$= \frac{(n+1)(n+1+1)(2(n+1)+1)}{6}$$

which indeed, is the formula we had above, substituting $n+1$ for n.

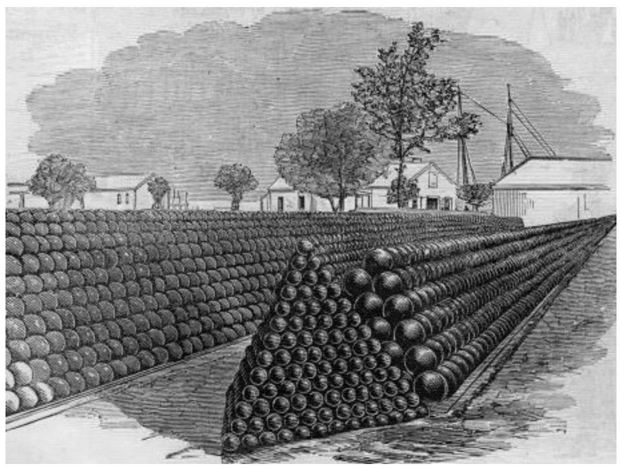
$\square$

Strang's proof

Here is another approach, from Strang's *Calculus*. He says "the best place to start is a good guess". So again, our goal is to find a formula for:

$$S = \sum_{k=1}^{n} k^2$$

Perhaps we visualize a pile of cannonballs



Each layer contains a square number of cannonballs (1, then 4, then 9, etc.). The shape is a pyramid with dimensions $n \times n \times n$. We know the formula for the volume of a pyramid, and guess

$$S_n = \frac{1}{3}n^3$$

To test it, check whether this difference is n^2 (as it should be):

$$S_n - S_{n-1} = \frac{1}{3}n^3 - \frac{1}{3}(n-1)^3$$

Now

$$(n-1)^2 = n^2 - 2n + 1$$
$$(n-1)^3 = (n-1)(n^2 - 2n + 1)$$

$$= n^3 - 3n^2 + 3n - 1$$

So

$$S_n - S_{n-1} = \frac{1}{3}(n^3 - n^3 + 3n^2 - 3n + 1)$$

We see that our guess is off by the residual terms

$$\frac{1}{3}(3n^2 - 3n + 1)$$
$$= n^2 - n + \frac{1}{3}$$

Strang says: the guess needs *correction terms*. To cancel 1/3 in the difference, subtract $n/3$ from the sum. And to add back n in the difference, add back $1 + 2 + \cdots + n(n+1)/2$ to the sum. Our new guess is

$$S_n = \frac{1}{3}n^3 + \frac{n(n+1)}{2} - \frac{n}{3}$$
$$= \frac{n}{6}(2n^2 + 3(n+1) - 2)$$
$$= \frac{n}{6}(2n+1)(n+1)$$
$$= \frac{n(n+1)(2n+1)}{6}$$

which may be easier to remember as

$$S_n = \frac{n(n+1)}{2} \cdot \frac{2n+1}{3}$$

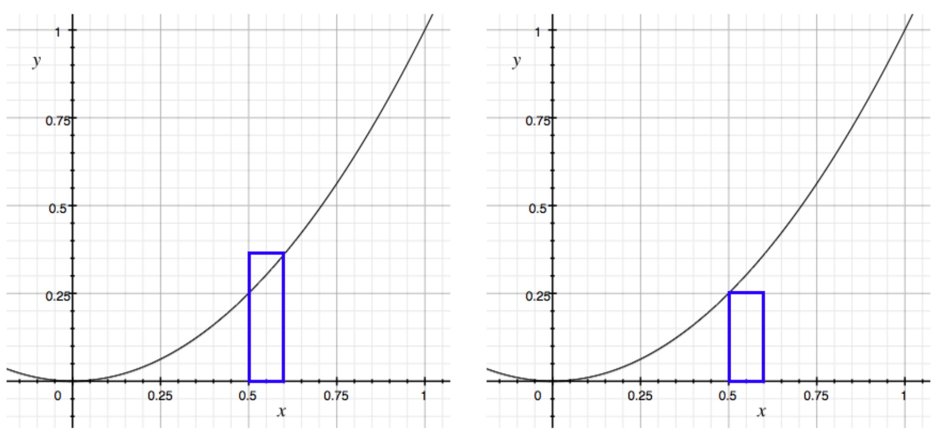
Chapter 45

Riemann sums

Riemann sums

We introduced integration as simply the reverse of differentiation. The fundamental theorem of calculus gives us a means of evaluating integrals between two bounds.

Starting with Courant, however, calculus courses have sought a more formal approach. The first (though not the only) method is to compute what are called Riemann sums for the the area bounded under a curve.

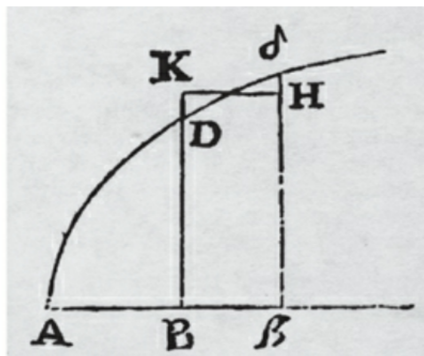


Our first example is to calculate the area under the curve $f(x) = x^2$. The areas of many small rectangles are added to form the sum.

The key is to set up a calculation or expression for the area in terms of a variable number of rectangles, N. Although any rectangle can only be an approximation to a curved surface, if we use many skinny rectangles the approximation will be very good, and *in the limit* as $N \to \infty$, as we use an infinite number of rectangles, it will be exactly right.

The proof that it *is* right is to show that the sum of a set of rectangles bounding the curve below, and the sum for a second set bounding the curve above, *converge to the same limit* as $N \to \infty$.

Another way to see this is to look at this diagram from Newton's book *De Analysi* (from Acheson's book *The Calculus Story*).



Newton's argument is that if we draw a box as illustrated, there must be one height, one point along the top of the rectangle where the area under the curve but not in the box, to the right, is exactly equal to the area over the curve and in the box, to the left. At that point, the errors exactly cancel.

If this point is always bounded by the y-values of the left and right sides of the box, then as the boxes get smaller and smaller, the balance point will always be included somewhere in the rectangle, so the answer will come out exactly right. (This latter assumption breaks down at maxima and minima, but only for finite boxes. Let's admire the diagram and move on to the actual method).

example

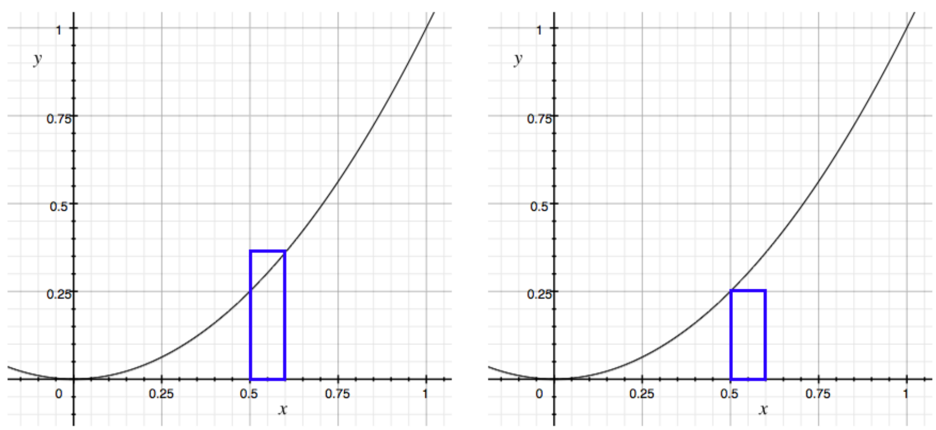
We start with x^2. It is the simplest power curve, and we actually know the answer due to work by Archimedes, called the **quadrature**.

Consider the region bounded by the x-axis, the y-axis, the line $x = 1$, and the curve $y = x^2$.

Partition the region on the x-axis between $x = 0$ and $x = 1$ into N segments. Each segment will contain a tall, thin rectangle that extends from the x-axis up to the curve.

Here is a figure that illustrates the basic idea. The region between 0 and 1 is divided into 10 segments, so the width of each segment is 0.1.

In the left panel, the blue rectangle shown is the sixth segment; its left and right bounds are $x = 0.5$ and $x = 0.6$. The height is $x^2 = 0.6^2 = 0.36$. The right panel is the same, except the height corresponds to the value at the left-hand bound $x^2 = 0.5^2$.



We compute the area (using the first set of rectangles) as

$$\begin{aligned} A &= 0.1(0.1^2 + 0.2^2 + \cdots + 1.0^2) \\ &= 0.1(0.01 + 0.04 + \cdots + 1.0) \\ &= 0.1(3.85) = 0.385 \end{aligned}$$

This is obviously an over-estimate of the area (as it will be for any function that increases over the interval), but the trick is that as the number of rectangles becomes very large, the result will converge to the exact area we want.

area as a limit

We divide the region into N intervals. Each interval has width $1/N$. The x-values of the right hand side of these boxes are

$$\frac{1}{N}, \frac{2}{N} \cdots \frac{N}{N}$$

We can rewrite this as

$$\sum_{k=1}^{N} \frac{k}{N}$$

We will first compute the sums for the set of boxes that is an overestimate, it uses the x-value of the right hand side of each box times the width of each box:

$$A = \sum_{k=1}^{N} f(\frac{k}{N}) \cdot \frac{1}{N}$$

Since the function is x^2 this is

$$A = \sum_{k=1}^{N} (\frac{k}{N})^2 \cdot \frac{1}{N}$$

And since N is a constant, it can be pulled out from the summation:

$$A = \frac{1}{N^3} \sum_{k=1}^{N} k^2$$

Now we need an expression for the sum of the squares of the first N integers. We will show below that the formula is $N \cdot (N+1)/2 \cdot (2N+1)/3$. Distributing the factor of $1/N^3$ over each term, we obtain

$$A = \frac{N}{N} \cdot \frac{N+1}{2N} \cdot \frac{2N+1}{3N}$$

If $N \to \infty$ and we take the limit, the result is the value of the integral

$$\begin{aligned} I &= \lim_{N \to \infty} \frac{N}{N} \cdot \frac{N+1}{2N} \cdot \frac{2N+1}{3N} \\ &= 1 \cdot \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3} \end{aligned}$$

The reason is that as $N \to \infty$ the difference between N and $N+1$ (or $N-1$ and N becomes negligible compared to the size of N, therefore the ratio $(N+1)/2N$, for example is equal to $1/2$ in the limit. In other words, for, say

$$\lim_{N \to \infty} \frac{1}{2N} \ (N+1) = \lim_{N \to \infty} \frac{1}{2}(1 + \frac{1}{N}) = \frac{1}{2}$$

We must still compute the sums for the set of boxes that is an underestimate (for finite N), with the x-value used for the function taken from the left-hand side. This is the series

$$\frac{0}{N}, \frac{1}{N} \cdots \frac{N-1}{N}$$

If the function is $f(x) = x^2$ the sum is

$$A = \sum_{k=0}^{N-1} (\frac{k}{N})^2 \cdot \frac{1}{N}$$

Since the first term is zero, just remove it and start the index from 1

$$A = \sum_{k=1}^{N-1} (\frac{k}{N})^2 \cdot \frac{1}{N}$$

For the integral, we obtain

$$I = \lim_{N \to \infty} \frac{N-1}{N} \cdot \frac{N}{2N} \cdot \frac{2N-1}{3}$$

We obtain exactly the same result as before. This is really the proof that the method works. In the limit of infinite N, the methods which over-estimate and under-estimate the area converge to the same value, therefore the result is exactly correct.

integer sums

We have previously seen that

$$\sum_{k=0}^{k=n} k^2 = \frac{1}{3}\ n \cdot (n + \frac{1}{2})(n+1)$$

which can be written in various ways including

$$\frac{n(n+1)}{2}\ \frac{2n+1}{3}$$

We plug that expression into the Riemann Sum:

$$= \frac{1}{N^3} \frac{N(N+1)}{2} \, \frac{2N+1}{3}$$

Each of the terms in $N(N+1)(2N+1)$ is grouped with one of the N's in the denominator at the left

$$= \frac{1}{6} \, \frac{N}{N} \, \frac{(N+1)}{N} \, \frac{(2N+1)}{N}$$

In the limit as N gets very large.

$$\lim_{N \to \infty} \frac{N}{N} = 1$$

$$\lim_{N \to \infty} \frac{N+1}{N} = \lim_{N \to \infty} 1 + \frac{1}{N} = 1$$

$$\lim_{N \to \infty} \frac{2N+1}{N} = \lim_{N \to \infty} 2 + \frac{1}{N} = 2$$

Thus, the final answer is 1/3, which agrees with Archimedes.

n cubed

The height of the first interval is $(1/N)^3$ and that of the kth interval is $(k/N)^3$. The total area is:

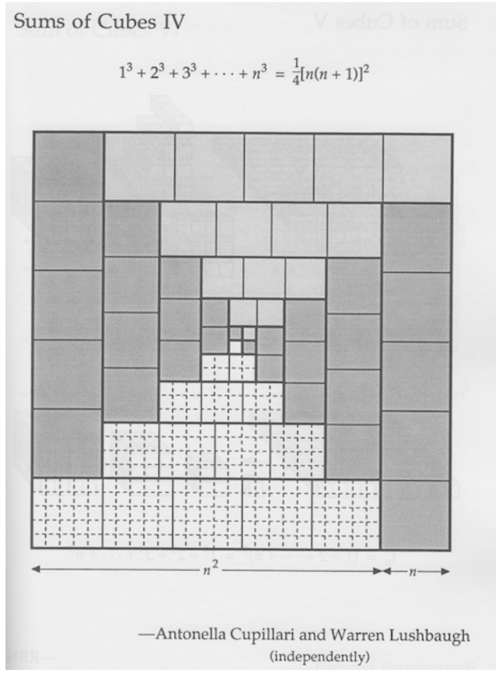
$$\sum_{k=1}^{N} (\frac{k}{N})^3 \times \frac{1}{N}$$

Since N is a constant, it can be pulled out from the summation:

$$\frac{1}{N^4} \sum_{k=1}^{N} k^3$$

So now we need an expression for the sum of the cubes of the first N integers.

Yet another ingenious "proof without words."



$$\sum_{k=1}^{N} k^3 = \frac{1}{4}\ [\ n(n+1)\ ]^2$$

$$= \frac{1}{4}\ N(N+1)N(N+1)$$

As before, each of the four factors of N in the denominator cancels an N or $N+1$ on top and we're left with just $1/4$.

It turns out that if you let the interval be $[0, b]$ or even $[a, b]$, we obtain the expressions you will be used to from integral calculus, namely

$$\int_a^b n^2 = \frac{n^3}{3}\ \Big|_a^b$$

and

$$\int_a^b n^3 = \frac{n^4}{4}\ \Big|_a^b$$

Part XII

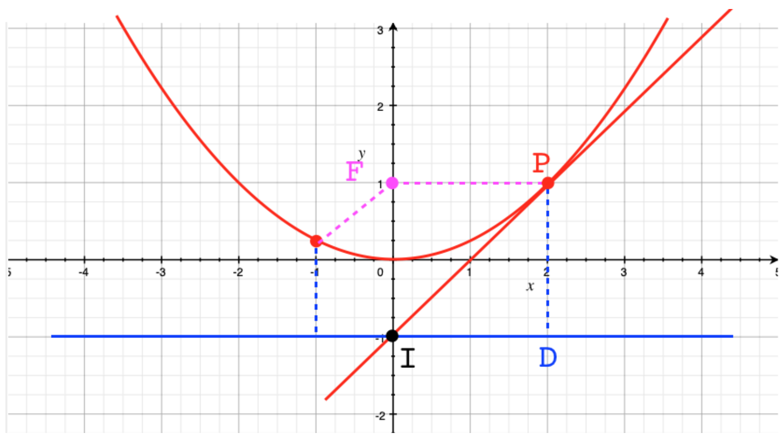
Conic sections

Chapter 46

Parabola

The parabola is one of a larger class of geometric figures called the conic sections.

It is pretty complicated to look at parabolas in the way that the Greeks did, so we will fudge a little and use analytic geometry to get a general formula for the curve in Cartesian coordinates, so we can draw the curve.



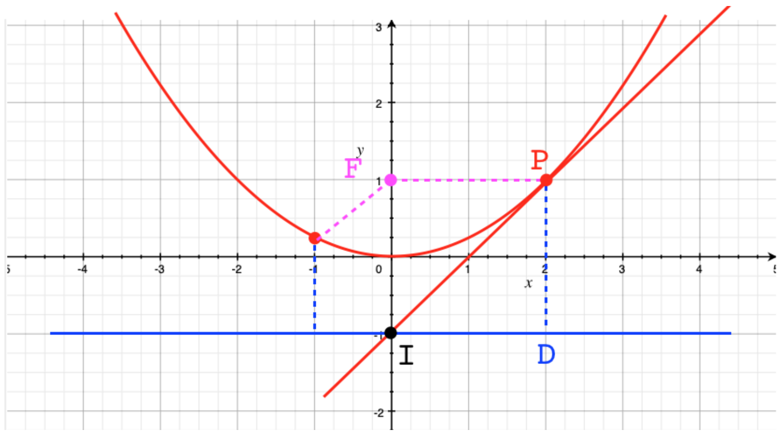
The equation of a general parabola is simply $y = ax^2$, where a is called the shape factor of the parabola.

The one in the figure is a little flatter than we usually draw, we'll see the reason for that choice in a minute. At $x = 1$, $y = 1/4$, so $a = 1/4$. Consistent with that, at $x = 2$ we have $y = ax^2 = 1$.

focus and directrix

We need one more idea to start our geometric look at the parabola.

The geometric definition is this. Pick a point on the y-axis a distance p up from the origin, colored magenta in the figure. This point is called the focus (F).



Then draw a line parallel to the x-axis which intersects the y-axis the same distance p below the origin. This line is called the directrix. It is colored blue and its equation is $y = -p$.

The parabola consists of all those points whose *distance to the focus is equal to the vertical distance to the directrix.*

It is another fact that we will establish later that the distance p is related to a by the equation:

$$4ap = 1$$

which explains our choice of a. We need the parabola flat enough to see F and the line $y = -p$ clearly.

If we consider the point $P = (2, 1)$ we can compute the distance to the focus as simply $\Delta x = 2$ and to the directrix as $\Delta y = 1 + 1 = 2$.

slope of the tangent

One last fact we will import from the future (to be justified in a bit). At any point (x, y) on the parabola, the slope is $2ax$. Therefore, the slope of the tangent to the curve at $x = 2$ is

$$m = 2 \cdot 1/4 \cdot 2 = 1$$

By inspection of the graph we see that the tangent line goes through $I = (0, -1)$ which gives a point slope formula of $y = x - 1$. It is easy to verify that $(2, 1)$ and $(0, -1)$ are both on the line, and of course the slope is 1, as advertised.

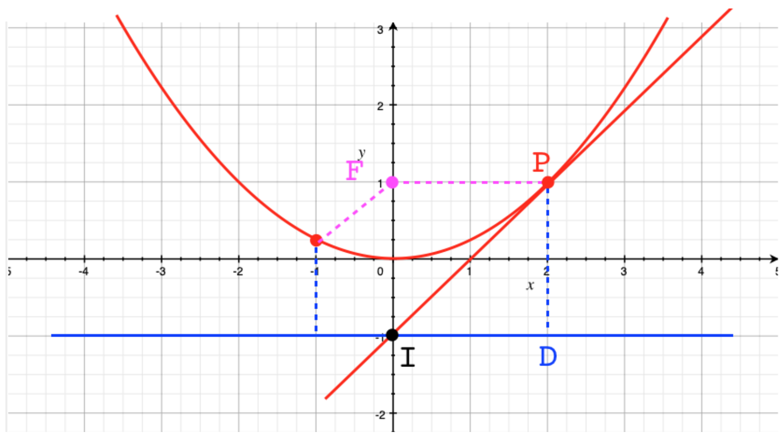
Notice that the x-intercept, x_0 is

$$0 = x - 1, \qquad x = 1$$

This is exactly halfway on the x-axis between P and I.

another point

From the equation of the curve we can get that $(x = \pm 1, y = 1/4)$ is on the curve, for either value of x. The distance from each point to the directrix is just 1 1/4.



The distance to the focus is

$$\sqrt{1^2 + (\frac{3}{4})^2} = \sqrt{\frac{25}{16}} = \frac{5}{4}$$

which checks.

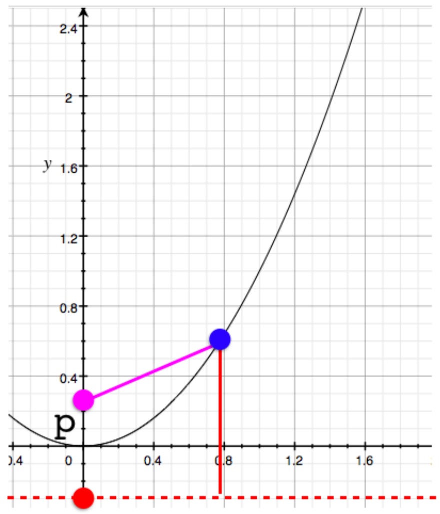
And this should not be a surprise. A look at the figure will show that in units of 1/4, we have a 3-4-5 right triangle.

computing p

Pick an arbitrary point on a parabola (in blue), with coordinates (x, ax^2). The squared distance to the focus (magenta point) is

$$\Delta x^2 + \Delta y^2 = x^2 + (ax^2 - p)^2$$

while the squared distance to the directrix (red line) is $(ax^2+p)^2$ because Δx is zero.



For the correct choice of p these distances are to be equal:

$$(ax^2 - p)^2 + x^2 = (ax^2 + p)^2$$

$$a^2x^4 - 2apx^2 + p^2 + x^2 = a^2x^4 + 2apx^2 + p^2$$

Canceling two terms on each side

$$-2apx^2 + x^2 = +2apx^2$$

Divide by x^2

$$-2ap + 1 = 2ap$$

$$4ap = 1$$

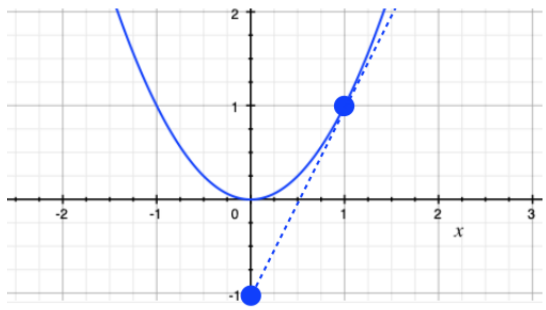
$$ap = \frac{1}{4}$$

The shape factor a determines the distance of the focus from the origin, which is p, and from the directrix, which is $2p$.

slope of the tangent

It will turn out that the slope of the tangent to $y = ax^2$ at any fixed point x is equal to $2ax$.

This is literally the first result from differential calculus, but we will also see a way to find it using analytical geometry, as well as a vector approach later on.



Thus, the equation of a line passing through the point (x, ax^2) with the given slope is

$$y' - ax^2 = 2ax(x' - x)$$

where (x', y') is any other point on the line.

What *that* means is that the x-intercept of the tangent line ($y' = 0$, $x' = x_0$) is:

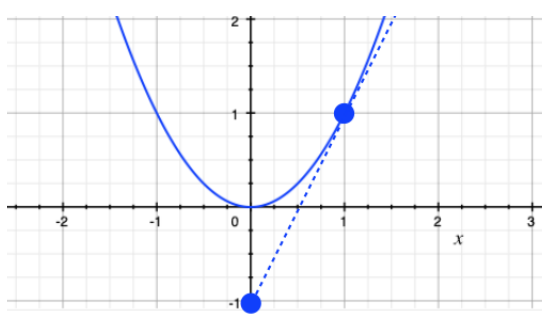
$$-ax^2 = 2axx_0 - 2ax^2$$

$$ax^2 = 2axx_0$$

$$x = 2x_0$$

$$x_0 = \frac{1}{2}x$$

The tangent line passes through the x axis halfway back toward the origin.

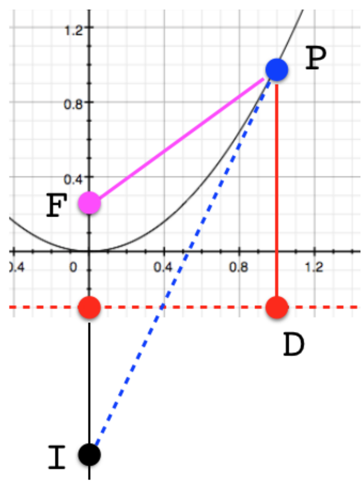


And what *that* means is that the y-intercept is symmetrical with the original point (as far below the x-axis as the point is above it). Here's the algebra:

$$y_0 - ax^2 = 2ax(0 - x)$$

$$y_0 = -ax^2$$

And then finally, if the point on the parabola is P, the focus F, the intersection with the directrix D, and the y-intercept I



the quadrilateral $FPDI$ is a regular parallelogram with all four equal sides, and its long diagonal (the tangent line) makes equal angles with FP and PD.

If PD is extended vertically, the angle it makes with the tangent line is equal to the angle between FP and the tangent line, so that for example, all vertical light rays entering a parabola will reflect and then come together at the focus.

analytic geometry

A general formula for a parabola with its vertex at the point (h, k) is

$$y - k = a(x - h)^2$$

where a is called the *shape factor*. It governs how steeply the curve rises (and by its sign, in which direction it opens). Multiplying out:

$$y - k = a(x^2 - 2xh + h^2)$$

$$y = ax^2 - 2ahx + ah^2 + k$$

In this form the cofactors are usually simplified as

$$y = ax^2 + bx + c$$

where

$$b = -2ah; \quad c = ah^2 + k$$

This means that any parabola's shape is solely governed by the value of a.

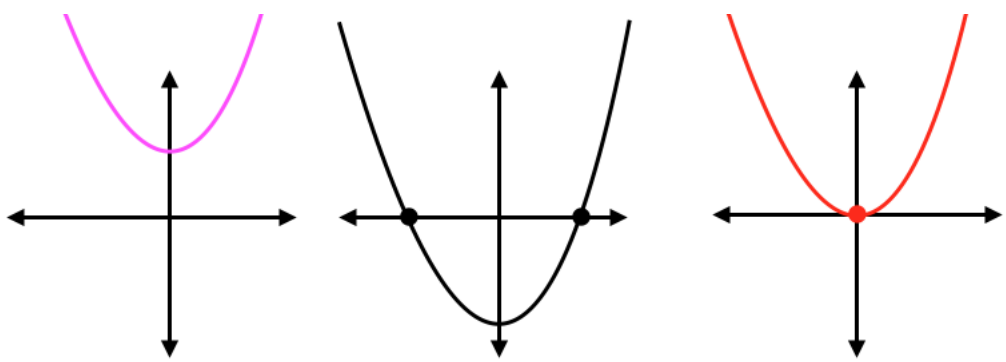
If the equation is given in the second form then we can find:

$$h = -\frac{b}{2a}$$

$$\begin{aligned} k &= c - ah^2 \\ &= c - \frac{b^2}{4a} \end{aligned}$$

Probably the most common thing we're asked to do with a quadratic equation like this is to find the roots, the values of x for which $y = 0$ is a solution. These are the points where the graph of the curve crosses the x-axis.

It is possible to have 0, 1 or 2 roots. The black curve has two roots, the red curve has one. The latter's equation is $y = x^2$, the former is $y = x^2 - 1$ and then we can see that $x^2 = 1$ has two real solutions $x = \pm 1$.



On the left, the magenta curve does not cross the y-axis. Its equation is $y = x^2 + 1$, and there are no (real) solutions, no values of x that solve the equation when $y = 0$.

$$0 = x^2 + 1$$

$$x^2 = -1$$

To find the roots of

$$ax^2 + bx + c = 0$$

We can guess solutions by trying to factor into a form like:

$$(x-s)(x-t) = 0$$

The case of a single root occurs when $s = t$ so we have $a(x-s)^2 = 0$. A common example of that is a parabola with its vertex at the origin, so $s = 0$ and $y = ax^2$ (right panel, above).

Roots do not have to be integers (or even rational). An arguably more productive and certainly more general approach to finding them is the process of *completing the square*.

First, multiply through by $1/a$ and rearrange:

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

The key insight is to recognize that if we add $(b/2a)^2$ to both sides, the left-hand side will become a perfect square:

$$x^2 + \frac{b}{a}x + (\frac{b}{2a})^2 = -\frac{c}{a} + (\frac{b}{2a})^2$$

$$(x + \frac{b}{2a})^2 = -\frac{c}{a} + (\frac{b}{2a})^2$$

$$x + \frac{b}{2a} = \pm\sqrt{-\frac{c}{a} + (\frac{b}{2a})^2}$$

Multiplying top and bottom of the first term under the square root gives a common factor:

$$x + \frac{b}{2a} = \pm\sqrt{-\frac{4ac}{4a^2} + (\frac{b}{2a})^2}$$

which can come out of the square root and then matches what's in the second term on the left-hand side:

$$x + \frac{b}{2a} = \pm\frac{\sqrt{-4ac + b^2}}{2a}$$

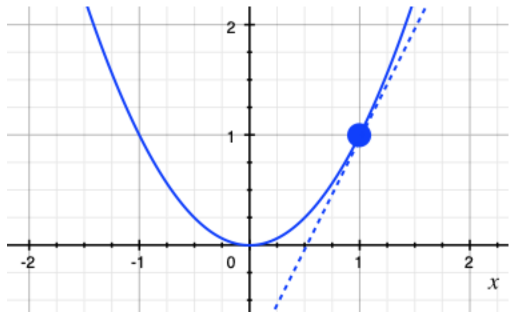
which we rearrange slightly to give the standard *quadratic formula*:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

part 1

Consider the simplest parabola: $y = x^2$.

The point $(1,1)$ is on the curve, because $(x = 1, y = 1)$ satisfies the equation $y = x^2$.



Suppose we know that the slope of the tangent to the curve at the point $(1,1)$ is equal to 2.

(Using calculus to find this result is trivial, we'll also show a non-calculus method in part three, below).

The equation of the tangent line is

$$y' - y = m(x' - x)$$

Plugging in for $(x', y') = (1, 1)$:

$$y - 1 = 2(x - 1)$$

$$y = 2x - 1$$

Now suppose that we knew only the parabola and this slope, but we did not know the point where the tangent meets the curve, and so do not know the y-intercept.

We have the equation of a line:

$$y = 2x + y_0$$

We seek points which are simultaneously on the line and the curve. They must satisfy both equations.

Since this is a tangent line, we seek the value for which this expression has only a single solution. The tangent "touches" the curve at a single point.

So, substitute for y from the equation for the curve:

$$x^2 = 2x + y_0$$

$$x^2 - 2x - y_0 = 0$$

Now look at the quadratic formula we would use to solve this equation for x:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

There is a single solution when the part under the square root (called the discriminant) is equal to zero.

$$b^2 - 4ac = 0$$

$$b^2 = 4ac$$

$$(-2)^2 = 4(-y_0)$$

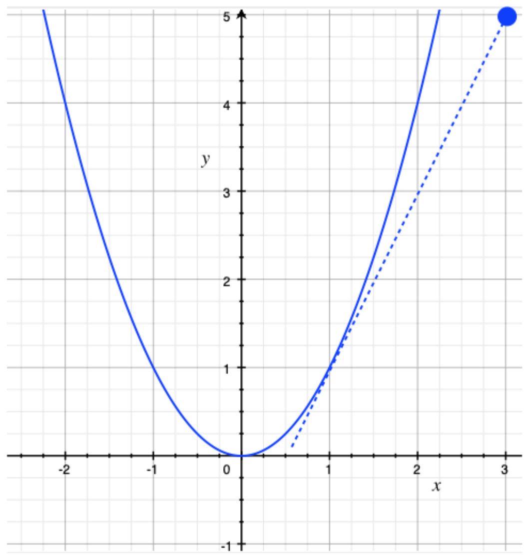
$$y_0 = -1$$

Therefore, the equation of the tangent line is $y = 2x - 1$, which matches what we had before.

In general, $y = 2x + y_0$ is a *family* of lines. For $y_0 = -1$, there is a single solution for x to be both on the line and the parabola. For $y_0 < -1$, there are no solutions, while for $y_0 > -1$ there are two solutions, because the line actually traces out a secant of the parabola, passing through the curve at two points.

part 2

Now suppose we have the same parabola and a point not on the parabola, but in the plane and outside of the "cup" of the parabola, such as $(3, 5)$. We seek the equations of tangent lines to the parabola that go through this point.



There will be two of them. We show just one in the figure.

The equations of lines passing through this point, with different slopes m are given by:

$$(y' - y) = m(x' - x)$$

Here, let (x', y') be $(3, 5)$ and then multiply by -1:

$$5 - y = m(3 - x)$$

$$y - 5 = m(x - 3)$$

Since values of (x, y) are both on the line and the parabola $y = x^2$, we can plug in for y:

$$x^2 - 5 = mx - 3m$$

$$x^2 - mx + (3m - 5) = 0$$

As before, solutions are given by the quadratic equation. The value of the slope m giving a single solution (zero discriminant) is:

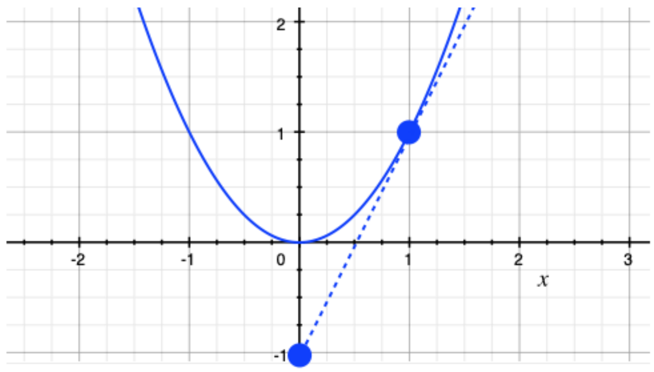
$$(-m)^2 - 4(3m - 5) = 0$$

$$m^2 - 12m + 20 = 0$$

$$(m-2)(m-10) = 0$$

$$m = 2, \quad m = 10$$

We knew the first one already, because the point $(3, 5)$ is on the line $y = 2x - 1$. This is the tangent to the curve at $(1, 1)$, which has slope $m = 2$.



Actually, there is always another solution which we haven't found explicitly and isn't shown on the graph either. Any vertical line (with infinite slope) passes through only a single point on the parabola.

Basically what this amounts to is that in the equation

$$x = \frac{m \pm \sqrt{(-m)^2 - 4(3m-5)}}{2}$$

as m gets very large, only the term $(-m)^2$ matters under the square root, so we have

$$x = \frac{m \pm \sqrt{(-m)^2}}{2}$$

if we choose the negative root, then as $m \to \infty$, $m - \sqrt{m^2} \to 0$.

part 3

Now suppose we are given the same parabola again and also a point on it such as (x_1, y_1).

Any line through that point has the equation:

$$y - y' = m(x - x')$$

To find the equation of a tangent line through that point we need the slope m.

If there is a point (x, y) that is on the line and *also* on the parabola, it must satisfy $y = ax^2$ as well, so:

$$ax^2 - ax'^2 = m(x - x')$$

$$ax^2 - mx - ax'^2 + mx' = 0$$

Certainly $x = x'$ is a solution.

The value of m must be such that there are *no other solutions*.

Write the quadratic equation to solve for x:

$$x = \frac{m \pm \sqrt{m^2 - 4a(mx' - ax'^2)}}{2a}$$

There is a single solution when the discriminant is zero, that is, when

$$x = \frac{m}{2a}$$

$$m = 2ax$$

Since $x = x'$ for the tangent line

$$m = 2ax'$$

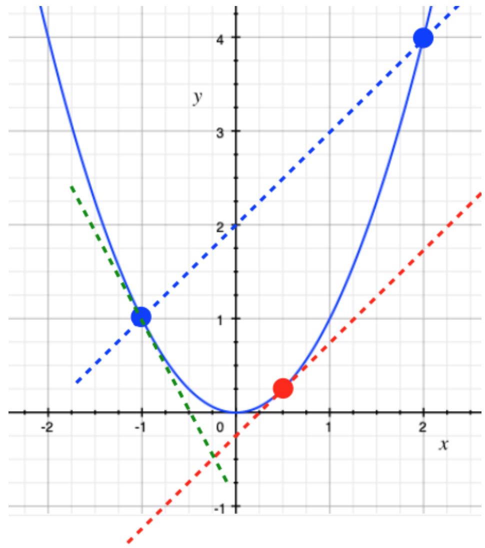
as expected.

The slope of the tangent line is $2ax'$ and in particular, at the point $(1, 1)$, the slope is equal to 2.

further comment

The slope of the parabola has some simple interesting properties. For example, pick any two points (x, y) and (x', y') on our standard parabola.

The slope of the line that connects those two points is equal to the slope of the parabola at the point whose x-value is halfway in between.



For the first part:

$$m = \frac{y' - y}{x' - x}$$

$$= \frac{ax'^2 - ax^2}{x' - x}$$

$$= a \; [\; \frac{x'^2 - x^2}{x' - x} \;]$$

$$= a(x' + x)$$

For the midpoint

$$x_m = \frac{1}{2}(x' + x)$$

and the slope is

$$2a \cdot \frac{1}{2}(x' + x)$$

$$= a(x' + x)$$

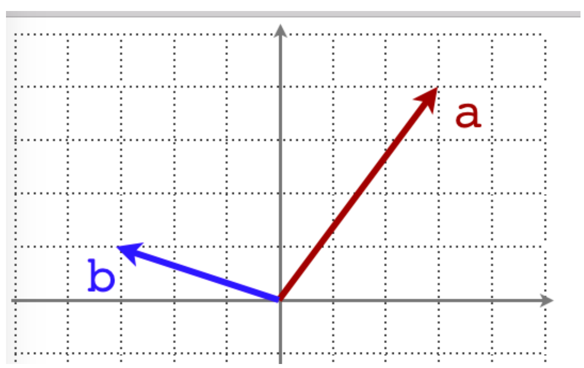
A similar result is that if we pick any two points (x, y) and (x', y'), and draw their slopes, the point where the two slope lines meet has its x-value exactly halfway in between x and x'.

Chapter 47

Vector dot product

In this chapter, we look at a few useful properties and operations of vectors in two- and three-dimensional space. I assume that you have already encountered vectors before, so this is not totally new.

From a geometrical point of view, a vector is a mathematical object that has both magnitude and direction. For example, in the standard 2D-coordinate system, the (maroon) vector $\langle\ 3, 4\ \rangle$ goes out from the origin three units in the x-direction and four units in the y-direction.



Vectors are written in bold type:

$$\mathbf{a} = \langle\ 3, 4\ \rangle$$

$$\mathbf{b} = \langle\ -3, 1\ \rangle$$

A vector has one property of a line, slope, but the fixed magnitude means that a vector does not extend to infinity as a line does. The squared length of a vector can be computed as the sum of the squares of its components, according to Pythagoras.

$$(\text{length } \mathbf{a})^2 = |\mathbf{a}|^2 = 3^2 + 4^2$$

By convention, we allow vectors to move about in space. We mean that two vectors of the same length, and pointing in the same direction are considered to be the same object, regardless of where they are located in space. (Some physics problems don't allow this, but in math it's the usual case).

So if we have the vector $\mathbf{v} = \langle\ 1, 1\ \rangle$ starting at the origin $(0, 0)$ and ending at the point $(1, 1)$, and compare it to a second vector $\mathbf{u}$ that starts from $(2, 0)$ and ends at $(3, 1)$, those are considered to be the same vector.

As you might guess, the vector that connects two points (x_1, y_1) and (x_2, y_2) is

$$\mathbf{p} = \ \langle\ x_2 - x_1, y_2 - y_1 \rangle$$

If we do the subtraction in reverse we have

$$\mathbf{q} = \ \langle\ x_1 - x_2, y_1 - y_2 \rangle$$

$$\mathbf{p} = -\mathbf{q}$$

Vectors add by adding their components:

$$\mathbf{a} = \langle\ 3, 4\ \rangle$$

$$\mathbf{b} = \langle\ -3, 1\ \rangle$$

$$\mathbf{a} + \mathbf{b} = \langle\ 0, 5\ \rangle$$

Subtraction works the same way.

From a linear algebra point of view, a vector is simply an ordered collection of numbers

$$\mathbf{u} = \ \langle\ u_1, u_2, \cdots\ u_n \rangle$$

where n could be very large, even infinite.

However, a lot of work is done in two or three dimensions (officially $\mathbb{R}^2$ and $\mathbb{R}^3$), and the principles developed there carry over nicely into n-dimensional space. So let's start by thinking about a two-dimensional vector

$$\mathbf{u} = \langle\, u_1, u_2 \,\rangle$$

As I've said, the vector **u** can be thought of as an arrow that goes from the origin to the point (u_1, u_2). It has both length and direction, with the length given by

$$|\mathbf{u}| = \sqrt{u_1^2 + u_2^2}$$

and its direction is

$$\frac{u_2}{u_1} = \tan\ \theta, \qquad \theta = \tan^{-1}\ \frac{u_2}{u_1}$$

where θ is the angle the vector makes (rotating counter-clockwise) from the positive x-axis.

Any vector can be converted into a *unit vector*, a vector of length one, by dividing by its length. For example if $\mathbf{u} = \langle\, 1, 2 \,\rangle$ then

$$\hat{\mathbf{u}} = \frac{1}{|\mathbf{u}|}\ \mathbf{u} = \langle\ \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \rangle$$

$\hat{\mathbf{u}}$ is a unit vector pointing in the same direction as **u**.

The line through the origin with slope $m = u_2/u_1$ and equation

$$y = mx$$

can be thought of as the extension of vector **u** obtained by multiplying some t times **u** for all $t \in \mathbb{R}$. We have stretched the vector to infinity, and beyond!

The standard unit vectors point in the direction of the x, y and z axes.

$$\hat{\mathbf{i}} = \langle\, 1, 0, 0 \,\rangle$$

$$\hat{\mathbf{j}} = \langle\, 0, 1, 0 \,\rangle$$

$$\hat{\mathbf{k}} = \langle\, 0, 0, 1 \,\rangle$$

We can write the vector using these unit vectors as

$$\mathbf{a} = \langle\, 3, 4 \,\rangle = 3 \cdot \hat{\mathbf{i}} + 4 \cdot \hat{\mathbf{j}}$$

Dot product

We now introduce a procedure for multiplying two vectors, the *dot product*, and derive the relationship between the dot product of two vectors and the angle between them. Suppose we have two vectors

$$\mathbf{a} = \langle a_1, a_2 \rangle$$

$$\mathbf{b} = \langle b_1, b_2 \rangle$$

Geometrically, we might think of these as being one vector extending from the origin in the x, y-plane to the point (a_1, a_2), and the other vector extending from the origin to (b_1, b_2). The dot product is defined as

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2$$

We can extend this to a pair of vectors in n-dimensional space

$$\mathbf{a} = \langle a_1, a_2, \ldots a_n \rangle$$

$$\mathbf{b} = \langle b_1, b_2, \ldots b_n \rangle$$

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n = \sum_{i=0}^{n} a_i b_i$$

The two vectors being multiplied (whose dot product is computed) must have the same dimension, the same n. Also, the result of the multiplication—the dot product—is a number. This is in contrast to another form of vector multiplication (the cross-product) which yields a vector as the result.

notation

The dot $(\cdot)$ in the dot product may also used to set apart two multiplicands in scalar multiplication, to increase clarity. So, you ask, how can we tell what is meant? Well, consider

$$v \cdot \frac{1}{v}$$

$$\mathbf{a} \cdot \mathbf{b}$$

It's a dot product if the two objects are vectors, otherwise it's multiplication.

Some properties

The dot product obeys the usual rules: it is associative, commutative and distributive.

The commutative property of the dot product:

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$$

follows from the same property for multiplication of real numbers, since

$$\mathbf{a} \cdot \mathbf{b} = \sum_n a_n b_n$$

$$= \sum_n b_n a_n = \mathbf{b} \cdot \mathbf{a}$$

For the distributive property, suppose

$$\mathbf{b} = \mathbf{c} + \mathbf{d}$$

Then

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot (\mathbf{c} + \mathbf{d}) = \mathbf{a} \cdot \mathbf{c} + \mathbf{a} \cdot \mathbf{d}$$

You can easily verify this by computing each term of the respective products.

$$\mathbf{b} = \ \langle b_1, b_2 \rangle \ = \mathbf{c} + \mathbf{d} = \ \langle c_1 + d_1, c_2 + d_2 \rangle$$

$$\mathbf{a} \cdot \mathbf{b} = a_1(c_1 + d_1) + a_2(c_2 + d_2)$$

$$= a_1 c_1 + a_1 d_1 + a_2 c_2 + a_2 d_2$$

$$= a_1 c_1 + a_2 c_2 + a_1 d_1 + a_2 d_2$$

$$= \mathbf{a} \cdot \mathbf{c} + \mathbf{a} \cdot \mathbf{d}$$

Another example that we will need below is

$$(\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) = \mathbf{a} \cdot \mathbf{a} - \mathbf{a} \cdot \mathbf{b} - \mathbf{b} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b}$$

by the commutative property

$$= \mathbf{a} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b} - 2\ \mathbf{a} \cdot \mathbf{b}$$

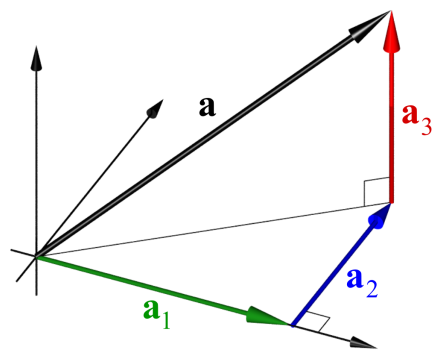
Length of a vector

As we said, the length of a vector $\mathbf{a} = \langle a_1, a_2 \rangle$, designated $|\mathbf{a}|$, is computed by a straightforward application of the Pythagorean Theorem:

$$|\mathbf{a}|^2 = a_1^2 + a_2^2$$

We leave the result as the square for simplicity.

This is easily extended to more dimensions by sequential application of the same method.



In $\mathbb{R}^3$:

$$|\mathbf{a}|^2 = a_1^2 + a_2^2 + a_3^2$$

In $\mathbb{R}^n$:

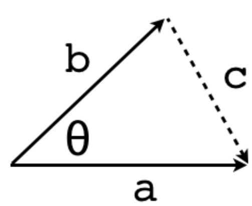
$$|\mathbf{a}|^2 = a_1^2 + a_2^2 + \cdots + a_n^2$$

Notice that

$$|\mathbf{a}|^2 = \mathbf{a} \cdot \mathbf{a}$$

Relation to θ

Now we are ready for the main idea. Suppose we draw two vectors $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{R}^2$ with their tails at the same point. Designate the angle between them as θ and the vector representing the side opposite as $\mathbf{c}$.



The orientation of $\mathbf{c}$ doesn't matter for the argument that follows. As shown

$$\mathbf{b} + \mathbf{c} = \mathbf{a}$$

$$\mathbf{c} = \mathbf{a} - \mathbf{b}$$

Compute the dot product of $\mathbf{c}$ with itself

$$\mathbf{c} \cdot \mathbf{c} = (\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b})$$

Recalling the result from above, this is

$$\mathbf{c} \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b} - 2\ \mathbf{a} \cdot \mathbf{b}$$

Since

$$|\mathbf{a}|^2 = \mathbf{a} \cdot \mathbf{a}$$

and so on, we have that

$$\mathbf{c} \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b} - 2\ \mathbf{a} \cdot \mathbf{b}$$

$$|\mathbf{c}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2\ \mathbf{a} \cdot \mathbf{b}$$

Does this remind you of the **law of cosines**?

$$c^2 = a^2 + b^2 - 2ab \cos\theta$$

Comparing the two equations, we see that

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\ |\mathbf{b}|\ \cos\theta$$

This relationship is extremely useful because it allows us to compute the cosine of the included angle via the dot product.

Even more important, two vectors which are perpendicular will have $\cos\theta = 0$, so their dot product is zero. Two vectors in pointed in the same direction have $\cos\theta = 1$ so it's just the product of the magnitudes.

This result extends to vectors in $\mathbb{R}^n$. Proof: choose a coordinate system where the two vectors lie in the same plane. Then apply the standard method.

For example, suppose I have the vector

$$\mathbf{u} = \langle p, q \rangle$$

Find a vector $\mathbf{v}$ perpendicular to $\mathbf{u}$.

$$\mathbf{v} = \langle q, -p \rangle$$

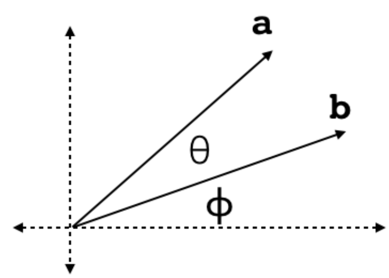
$\mathbf{v}$ is perpendicular to $\mathbf{u}$ because

$$\mathbf{u} \cdot \mathbf{v} = pq + q(-p) = 0$$

How to find a vector in $\mathbb{R}^5$ perpendicular to $\langle 1, 1, 1, 1, 0 \rangle$? Any vector of the form $\langle 0, 0, 0, 0, k \rangle$ will do, where k is some real number.

Alternate derivation

Here is another approach which doesn't depend on knowing the law of cosines, but uses the addition rule for cosine instead.



Vector $\mathbf{a}$ forms an angle θ with vector $\mathbf{b}$. $\mathbf{b}$ forms an angle ϕ with the x-axis, so the angle between $\mathbf{a}$ and the x-axis is $\theta + \phi$.

Find the dot product using components. If $a = |\mathbf{a}|$ and $b = |\mathbf{b}|$ then

$$a_x = a \cos(\theta + \phi)$$

$$b_x = b \cos \phi$$

$$a_y = a \sin(\theta + \phi)$$

$$b_y = b \sin \phi$$

So

$$\mathbf{a} \cdot \mathbf{b} = a_x b_x + a_y b_y$$

$$= ab \; [\; \cos(\theta + \phi) \cos \phi + \sin(\theta + \phi) \sin \phi \;]$$

Using the rule

$$\cos s - t = \cos s \cos t + \sin s \sin t$$

the part in parentheses is

$$\cos(\theta + \phi) \cos \phi + \sin(\theta + \phi) \sin \phi$$

$$= \cos(\theta + \phi - \phi) = \cos \theta$$

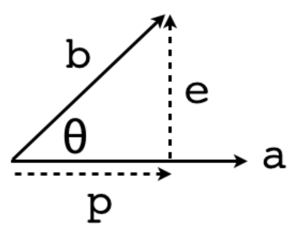
Another important property is that the value of the dot product is *independent* of the coordinate system chosen, because rotation or translation cannot change the lengths of the vectors nor the angle between them.

Projection

If $|\mathbf{a}| = 1$ we say that $\mathbf{a}$ is a *unit vector*. In that case

$$\mathbf{b} \cdot \mathbf{a} = |\mathbf{b}| \cos \theta$$

Looking at the figure, $|\mathbf{b}| \cos \theta$ is the length of the *projection* of $\mathbf{b}$ on $\mathbf{a}$. (Recall that the dot product is a scalar—a number—and not a vector).



The result, $\mathbf{b} \cdot \mathbf{a} = |\mathbf{b}| \cos \theta$, is the length of the part of $\mathbf{b}$ that extends in the same direction as $\mathbf{a}$. The corresponding vector is

$$\mathbf{p} = (\mathbf{b} \cdot \mathbf{a})\, \mathbf{a}$$

The other component of $\mathbf{b}$ is the part that is perpendicular to $\mathbf{p}$

$$\mathbf{p} + \mathbf{e} = \mathbf{b}$$

We compute $\mathbf{e}$ as the difference $\mathbf{b} - \mathbf{p}$. $\mathbf{e}$ is the part of $\mathbf{b}$ that is perpendicular to the projection. As a final note, the formula given here is a simplification for the situation in which $\mathbf{a}$ is a unit vector. If not, the complete formula is:

$$\mathbf{p} = \frac{\mathbf{b} \cdot \mathbf{a}}{\mathbf{a} \cdot \mathbf{a}}\, \mathbf{a}$$

Vectors allow simple proofs for some geometric theorems such as Ceva's theorem and the law of cosines.

example

Here is a problem from Nahin:

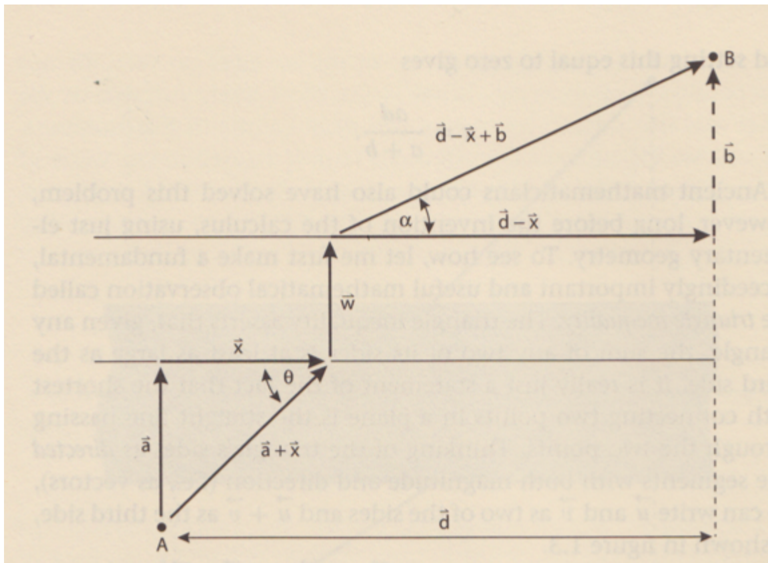


FIGURE 1.4. Bridge geometry in vector notation.

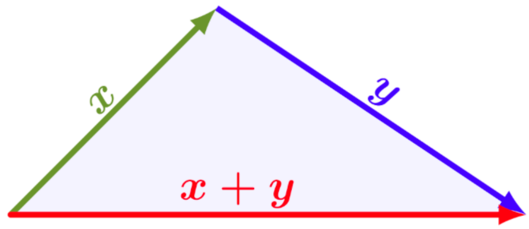
Two towns are on opposite sides of a river at points A and B. It is desired to choose the site of a bridge so as to minimize the distance between the two towns when traveling over the bridge. The problem can be set up algebraically and solved by differential calculus. However, the vector approach is more fun, and allows us to introduce the important *triangle inequality*.

Vectors are shown in the figure: $\mathbf{a}$ is the perpendicular distance from A to the river, and similarly for $\mathbf{b}$. $\mathbf{x}$ determines the placement of the bridge. If the horizontal distance between A and B is $\mathbf{d}$, then $\mathbf{d} - \mathbf{x}$ is the horizontal distance between B and the bridge. The distance across the bridge is $\mathbf{w}$, which cannot be changed. Its length will just be added onto our shortest path.

We want to choose $\mathbf{x}$ so that the path from A to B is the shortest. The path from A to the bridge is $\mathbf{a} + \mathbf{x}$, that from the bridge to B is $\mathbf{b} + \mathbf{d} - \mathbf{x}$ so all together we have (taking the lengths of the vectors)

$$L = |\mathbf{a} + \mathbf{x}| + |\mathbf{b} + \mathbf{d} - \mathbf{x}|$$

The triangle inequality says that the lengths of two sides of a triangle add to be larger than or equal to the length of the third side.



$$|\mathbf{x}| + |\mathbf{y}| \geq |\mathbf{x} + \mathbf{y}|$$

The rule is that the minimal value for the sum $|\mathbf{x}| + |\mathbf{y}|$ occurs when they point in the same direction.

In our problem, the minimum length occurs when $\mathbf{a} + \mathbf{x}$ and $\mathbf{b} + \mathbf{d} - \mathbf{x}$ point in the same direction. In other words, when $\theta = \alpha$.

Then, by similar triangles,

$$\frac{x}{a} = \frac{d - x}{b}$$

$$bx = ad - ax$$

$$x = \frac{ad}{a + b}$$

example

Here is one from Strang.

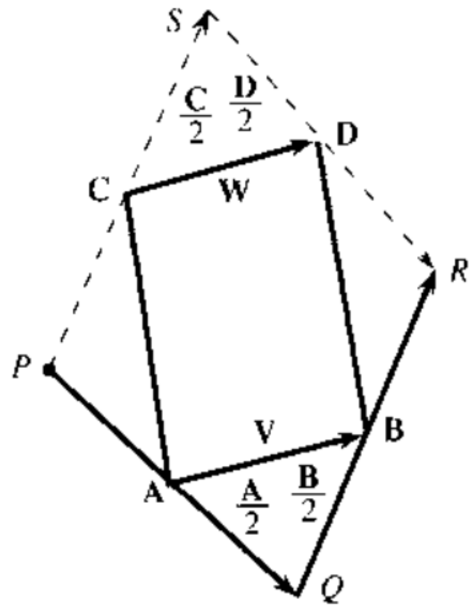


Fig. 11.4 Four midpoints

Consider *any* four-sided figure in space, such as $PQRS$ in the figure. (Note: $|\mathbf{A}| \neq |\mathbf{B}|$, and so on, and S is not co-planar with P, Q, R. I claim that the midpoints of the sides form a parallelogram $ABCD$.

We will prove that $\mathbf{V} = \mathbf{W}$.

The figure makes it almost obvious.

$$\mathbf{V} = \frac{\mathbf{A}}{2} + \frac{\mathbf{B}}{2}$$

$$\mathbf{W} = \frac{\mathbf{C}}{2} + \frac{\mathbf{D}}{2}$$

The segment from P to R can be covered in two ways

$$\mathbf{A} + \mathbf{B} = \mathbf{C} + \mathbf{D}$$

Divide both sides by 2 and obtain

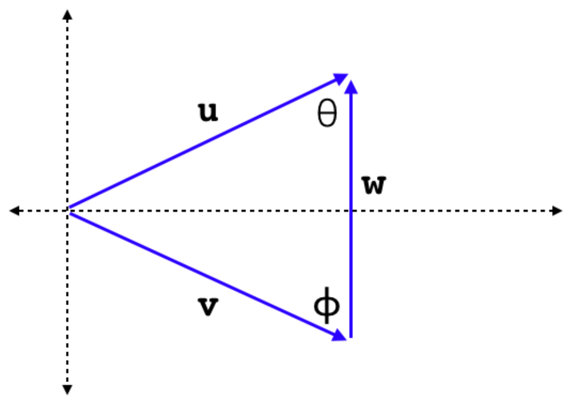
$$\frac{\mathbf{A}}{2} + \frac{\mathbf{B}}{2} = \frac{\mathbf{C}}{2} + \frac{\mathbf{D}}{2}$$

$$\mathbf{V} = \mathbf{W}$$

□

example

And here is one from Euclid:



We are given a triangle with two sides the same length (isosceles). Without loss of generality, draw the triangle with its vertex at the origin and the midpoint of the third side on the x-axis.

To prove: $\theta = \phi$.

Let

$$\mathbf{u} = \langle a, b \rangle$$
$$\mathbf{v} = \langle a, -b \rangle$$
$$\mathbf{w} = \langle 0, 2b \rangle$$

We compute the dot products so that the angle between the vectors is acute and the dot product is > 0.

$$\mathbf{u} \cdot \mathbf{w} = 2b^2$$
$$= |\mathbf{u}||\mathbf{w}| \cos\theta = \sqrt{a^2 + b^2}\ 2b \cos\theta$$
$$\cos\theta = \frac{b}{\sqrt{a^2 + b^2}}$$

which is also obvious from the figure. We didn't need vectors for this.

$$(-\mathbf{w}) \cdot \mathbf{v} = 2b^2$$
$$= \sqrt{a^2 + b^2}\ 2b \cos\phi$$

$$\cos\phi = \frac{b}{\sqrt{a^2+b^2}}$$

We obtain the same result for $\cos\phi$ as for $\cos\theta$ and then finally

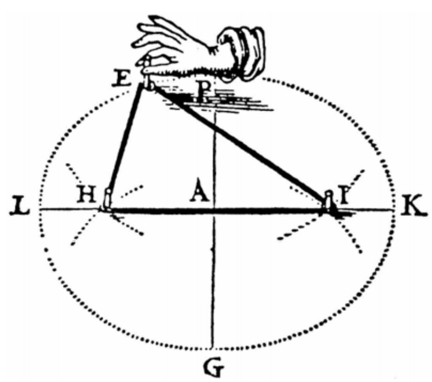
$$\theta = \phi$$

Chapter 48

Ellipse

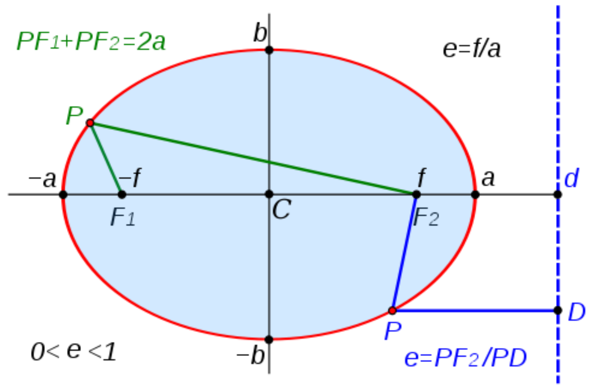
This chapter on the ellipse uses a small amount of trigonometry in the middle section, and that's why it is placed a bit later than the other material on analytic geometry.

construction



Learning how to draw an ellipse using two pins and a circular piece of string holding a pencil is an early adventure in mathematics. The ellipse is the set of all points whose combined distance to the two pins (foci) is the same.

The drawing is reproduced from a 17th century book in Acheson (see the References).



The pin positions with respect to the origin or center are called the foci, lying at the points shown in the figure as $(\pm f, 0)$.

We will use the notation c: the focus in the first quadrant is at the point $(c, 0)$.

The lengths of the axes (called semi-major and semi-minor) are usually labeled a and b.

Consider the situation when the pencil is at the point $P = (0, a)$. The distance to the left focus is $c + a$, so the length L of the string is twice that

$$L = 2(c + a)$$

The combined distance from each point on the ellipse to the two foci is the length of the string minus the distance between the two foci

$$L - 2c = 2(c + a) - 2c = 2a$$

standard equation

We learn in algebra that the equation for an ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

We will derive this equation below.

The relation of a and c to b can be seen from the point $Q = (0, b)$ (see previous figure) where the combined distance to the two foci is just

$$QF_1 + QF_2$$

From what we said above the distance is $2a$, but Pythagoras also gives us

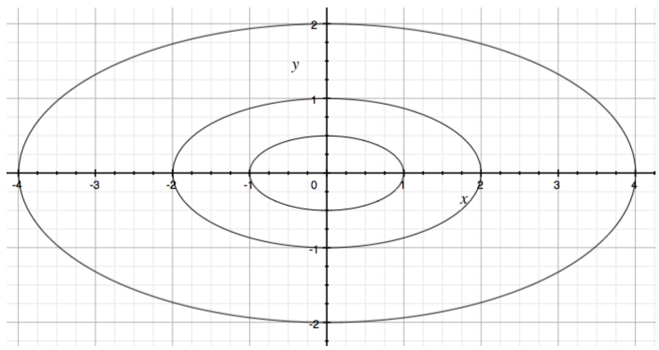
$$QF_1 + QF_2 = 2a = 2\sqrt{b^2 + c^2}$$

so

$$b^2 + c^2 = a^2$$
$$c^2 = a^2 - b^2$$

Given a and b one can then find c easily.

Here are three ellipses drawn with the same center.



The difference is an adjustment in the value on the right-hand side of the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = r^2$$

where $r = \{1/2, 1, 2\}$. This is equivalent to scaling both a and b by the same factor of r

$$\frac{x^2}{(ra)^2} + \frac{y^2}{(rb)^2} = 1 = (\frac{x/a}{r})^2 + (\frac{y/b}{r})^2$$

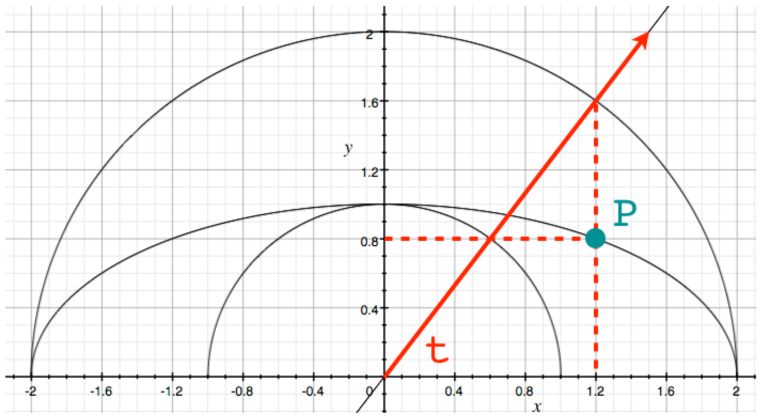
When $r = 2$ we need to make the string a bit less than twice as long, because the length c is also involved:

$$\frac{L_2}{L_1} = \frac{ra + c}{a + c}$$

parametrization

An alternative view is the one below, which shows (black curves) the upper half of two circles of radius $r = 1$ and $r = 2$ and an ellipse whose equation is

$$\frac{x^2}{2^2} + \frac{y^2}{1} = 1$$



Here $a = 2$ and $b = 1$.

The standard parametrization of the ellipse is

$$x = a\cos t$$

$$y = b\sin t$$

which I had trouble visualizing, until I drew the picture. The thing is that the parameter t is *not* the angle that a ray to P makes with the x-axis, as it is for the circle. Instead, to find the x value of P corresponding to t, we extend the ray with angle t to the larger circle, with radius a, where we read off the x-value as

$$x = a\cos t$$

We go back to find the intersection of the same ray with the small circle to get

$$y = b\sin t$$

The algebraic way to do this is to show that the parametrization is equivalent to the original formulation

$$x^2 = a^2\cos^2 t$$

$$y^2 = b^2\sin^2 t$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \cos^2 t + \sin^2 t = 1$$

as expected.

Derivation of the equation of the ellipse

Although it is a bit tedious, it's a reasonable exercise to derive the equation of the ellipse from the geometric constraint. Recall that a is the length of the semi-major axis and c the distance to each of the foci from the origin.

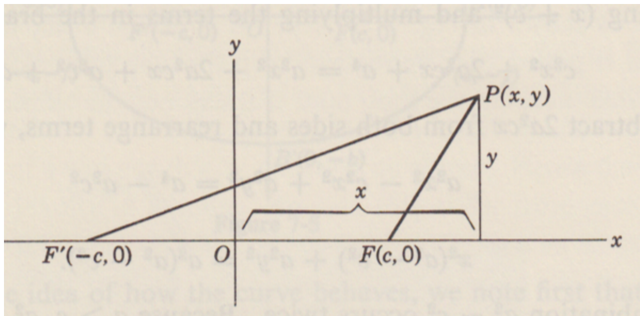


Figure 7-4

For any point x, y on the ellipse, the distance to the focus in the first quadrant is

$$\sqrt{(x-c)^2+y^2}$$

and combined distances to both foci are equal to $2a$ so

$$2a = \sqrt{(x+c)^2+y^2} + \sqrt{(x-c)^2+y^2}$$

Now we just do some algebra. Pick one square root and rearrange

$$\sqrt{(x-c)^2+y^2} = 2a - \sqrt{(x+c)^2+y^2}$$

Square both sides

$$(x-c)^2+y^2 = 4a^2 - 4a\sqrt{(x+c)^2+y^2} + (x+c)^2+y^2$$

Cancel y^2

$$(x-c)^2 = 4a^2 - 4a\sqrt{(x+c)^2+y^2} + (x+c)^2$$

But

$$(x+c)^2 - (x-c)^2 = 4xc$$

so

$$0 = 4a^2 - 4a\sqrt{(x+c)^2+y^2} + 4xc$$
$$a^2 + xc = a\sqrt{(x+c)^2+y^2}$$

Square again

$$a^4 + 2a^2xc + x^2c^2 = a^2(x^2 + 2xc + c^2 + y^2)$$

$$a^4 + 2a^2xc + x^2c^2 = a^2x^2 + 2a^2xc + a^2c^2 + a^2y^2$$

Cancel $2a^2xc$

$$a^4 + x^2c^2 = a^2x^2 + a^2c^2 + a^2y^2$$

Gather terms

$$a^4 - a^2c^2 = a^2x^2 - x^2c^2 + a^2y^2$$

$$a^2(a^2 - c^2) = x^2(a^2 - c^2) + a^2y^2$$

Recall that $b^2 = a^2 - c^2$

$$b^2a^2 = b^2x^2 + a^2y^2$$

Divide by a^2b^2

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

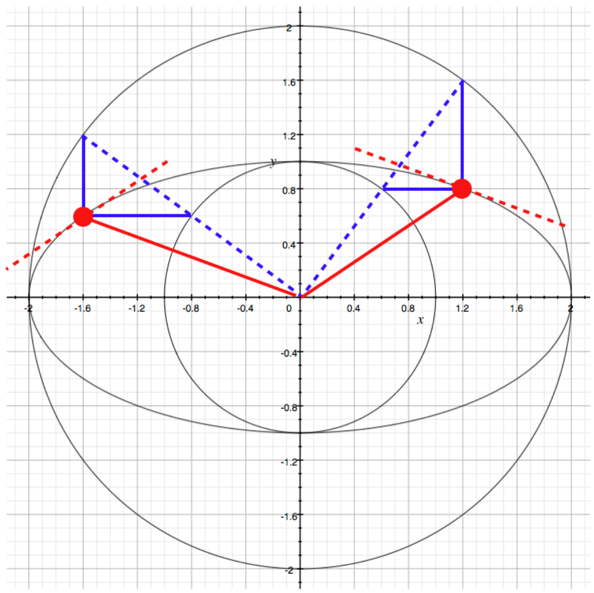
rotation

Let's return to the diagram of the ellipse with two bounding circles of radius a and radius b. There is a new diagram below. Consider the coordinates of the point $P = (x, y)$ (the red dot in the first quadrant) as functions of the angle t. As we said, t is *not* the angle of a ray from the origin to P.

Draw a ray (blue dotted line) from the origin makes an angle t with the x-axis. As before, extend the ray to the outer circle. The radius is a, the angle is t, and

$$a \cos t = x$$

This is the parametrization of the ellipse introduced previously.



The ray drawn with angle t has the same x-intercept with the outer circle as our point P on the ellipse. Similarly, the intercept of the ray with the inner circle has the same y-value as the point P on the ellipse.

We estimate the point $P = (1.2, 0.8) = (6/5, 4/5)$. Using our algebraic equation:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Recall that $a = 2$ and $b = 1$ so

$$x^2 + 4y^2 = 4$$

Plugging in for x^2 and y^2 we get

$$\frac{36}{25} + 4\,(\frac{16}{25}) = \frac{100}{25} = 4$$

as expected. Reading off the intercepts for the ray with angle t (dotted blue line) with the outer circle, we have the point $(1.2, 1.6)$ at a distance 2 from the origin. Thus,

$$\frac{1.2}{2} = 0.6 = \cos t$$

$$t \approx 0.927 \text{ rad} \approx 53^\circ$$

Looking again at the figure, we want to consider what happens for the angle $u = t + \pi/2$. This is the dotted blue ray in the second quadrant.

We might calculate the values of sine and cosine for u, but notice that if we view u as a vector, its *dot product* with t must be equal to zero. The coordinates of the intercept of the rotated vector with the outer circle are $(-1.6, 1.2)$, so the cosine of the angle u is

$$\cos u = -0.8$$

$$u \approx 2.498 = t + \frac{\pi}{2} \text{ rad} \approx 143°$$

We confirm that

$$2.498 - 0.927 = 1.57 = \frac{\pi}{2}$$

The coordinates of the point on the ellipse are $(-1.6, 0.6)$, which we check against the formula

$$x^2 + 4y^2 = 4$$

$$(1.6)^2 + 4(0.6)^2 = 2.56 + 4(0.36) = 4$$

(no clean fractions this for this one).

tangent

Finally, and this is really the crucial result:

the vector to the point, call it Q, on the ellipse (red dot in the second quadrant) is the *tangent to the ellipse* for the point P in the first quadrant.

How did this happen? Recall what we did. We had

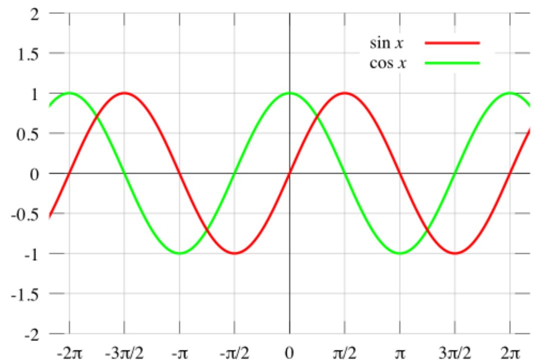
$$x = a \cos t$$

$$y = b \sin t$$

The rotated point $Q = (x', y')$ is

$$x' = a \cos(t + \frac{\pi}{2})$$

$$y' = b \sin(t + \frac{\pi}{2})$$



Sine is like cosine, but shifted to the right by $\pi/2$

$$\cos\theta = \sin(\theta + \frac{\pi}{2})$$

$$\sin\theta = -\cos(\theta + \frac{\pi}{2})$$

So

$$x' = a\cos(t + \frac{\pi}{2}) = -a\sin t$$

$$y' = b\sin(t + \frac{\pi}{2}) = b\cos t$$

Let's look at the position vector, which can be written $\mathbf{r}(t)$, since it's a function of the angle t or the time, but we will just use $\mathbf{r}$. It has components x and y.

$$\mathbf{r} = \langle x, y \rangle = \langle a\cos t, b\sin t \rangle$$

Now, the tangent to the ellipse is precisely the direction in which a particle at (x, y) is currently moving on the ellipse. The tangent vector points in the same direction as the velocity vector, but $\mathbf{v}$ is just the time-derivative of the position vector.

$$\begin{aligned}\mathbf{v} &= \frac{d\mathbf{r}}{dt} \\ &= \langle \frac{dx}{dt}, \frac{dy}{dt} \rangle \\ &= \langle -a\sin t, b\cos t \rangle \\ &= \langle x', y' \rangle\end{aligned}$$

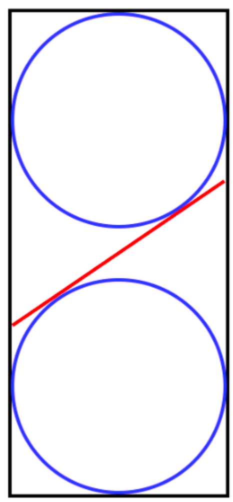
These two methods — using the time-derivative as the tangent, and rotation of t by $\pi/2$ — generate the same vector. And that's the point. :)

Starbird

Here is a neat approach to the ellipse that I saw in one of Michael Starbird's lectures.

Imagine a glass cylinder, shown here in cross-section and colored black. The cylinder has been sliced through at an angle by a plane, and we suppose a flat piece of glass in the shape of an ellipse is glued between the two halves.

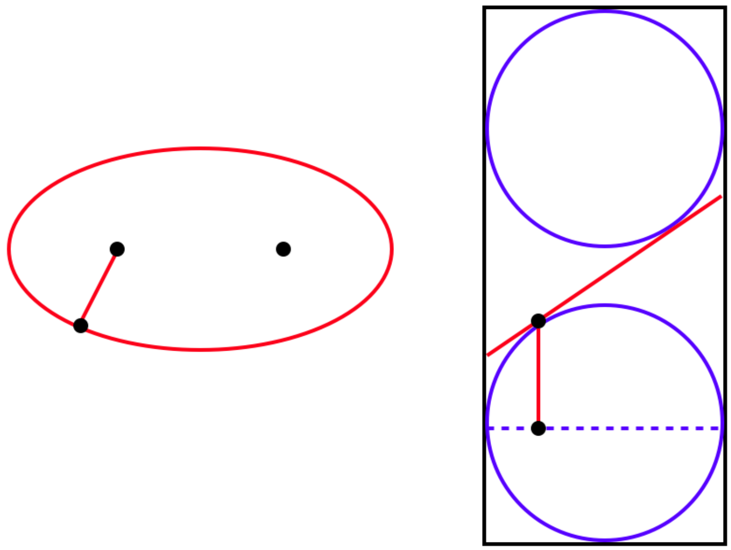
The elongated region in red (formed at the plane of the cut) is the ellipse, and the cylinder is oriented so that at each horizontal position going across the page, the two points on the ellipse are at the same vertical position. We see the plane of the cut edge-on.



Two spheres that fit snugly inside the cylinder lie above and below the ellipse, just touching it. The planar surface of the ellipse is tangent to the spheres, touching each one at a single point.

We claim that the points where the spheres touch the ellipse are the foci of the ellipse.

By the nature of the construction, the two spheres just fit inside the cylinder. That means the intersection where the spheres touch the cylinder is a circle, the lower one is shown with a dotted blue line in the next figure.

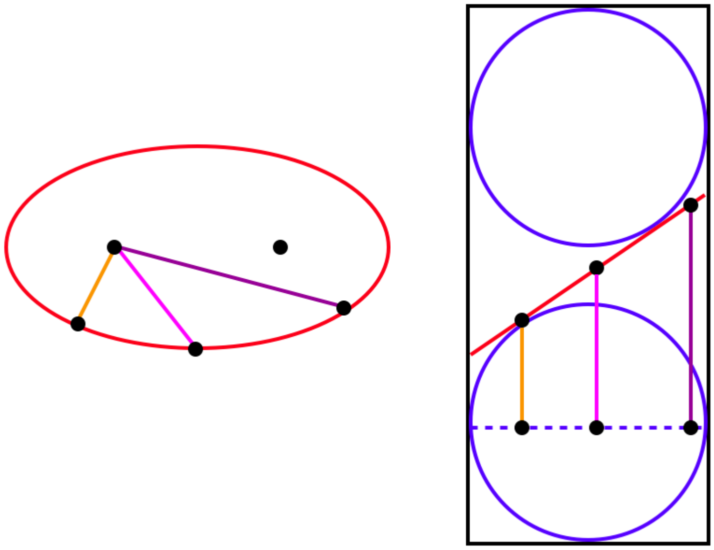


Now consider any point on the ellipse. On the left, we see one point on the ellipse together with two interior points we claim are foci, with a line drawn from our point to one of the foci.

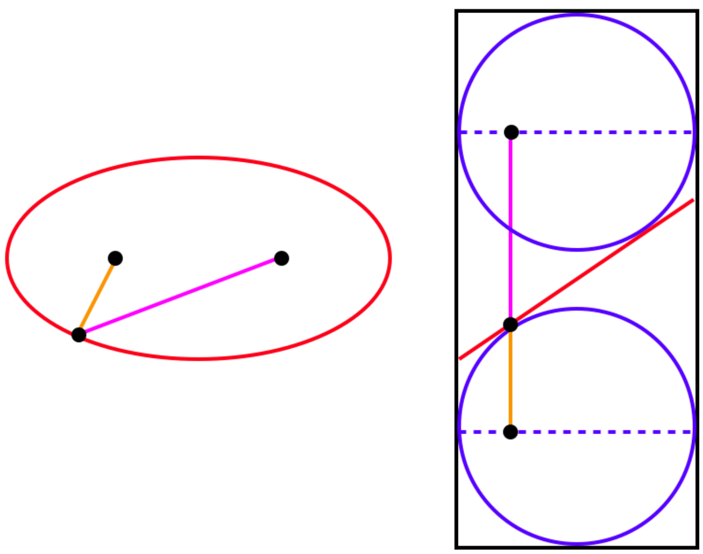
We said that this point is the point where the ellipse touches the lower sphere. We conclude that the line we've drawn from the edge of the ellipse to the focus is a tangent to the sphere.

A second tangent of interest is the perpendicular dropped vertically down the surface of the cylinder, shown in the right panel. Since they are both tangents, this line is the same length as line to the focus.

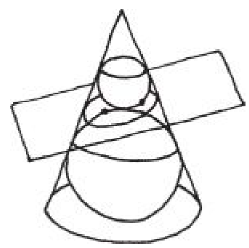
But the construction, and this equality, holds for any point on the ellipse, as shown in the next figure.



Finally, this is true for both spheres (below). The sum of the perpendicular tangents for any point is a constant.



Thus, the points where the spheres touch the ellipse are its foci, because the sum of the distances to any point on the ellipse, which is equal to the sum of the vertical tangents, is a constant.



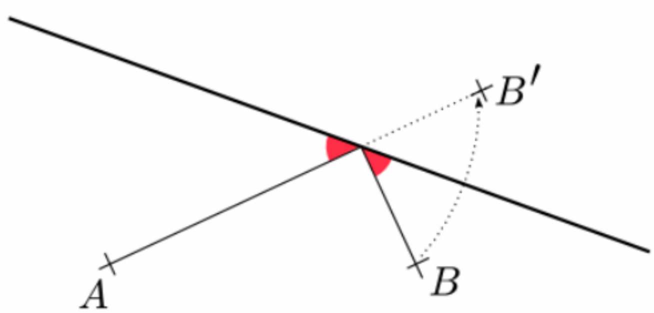
According to Lockhart, the same argument can be used to prove that the cross sections of a cone are ellipses (which seems strange at first since we've been demonstrating that the cross-sections of cylinders are also ellipses).

reflected rays

In any ellipse, the segments from the foci to any point on the ellipse make equal angles with the tangent. This means that light rays emitted from one focus and striking anywhere on the ellipse will pass through the other focus upon reflection. It is the principle behind "whispering galleries."

Here is a simple geometric proof.

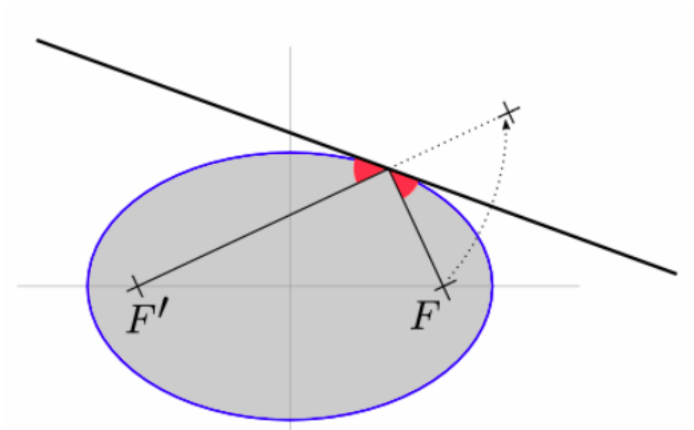
In another **chapter**, we considered the problem of the "shortest path."



The problem is to go from A to the line and then back to B by the shortest path. The clever solution is to place B' on the other side of the line at the same distance away. By definition (see Euclid) the shortest path A to B' is a straight line.

We can use vertical angles (or supplementary angles twice) and then similar triangles to prove that the two angles colored red are equal.

Now consider an enhanced diagram of the same situation:



We draw the tangent to the ellipse. By definition, the tangent has only a single point on the curve. This point lies at a distance $2a$ from the combined foci. All other points on the line are farther away from the two foci than the point of intersection. (You would have to make the string bigger to draw the ellipse that goes through any of those points).

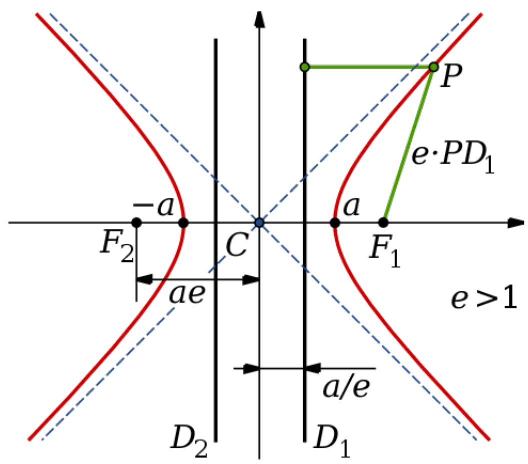
Therefore, the path shown is the shortest path from F' to the tangent and then to F. But we know that for the shortest path the angles colored red are equal.

http://math.stackexchange.com/questions/1063977/how-to-geometrically-prove-the-focal-property-of-ellipse

Chapter 49

Hyperbola

Here is a hyperbola as shown in the wikipedia article on the subject.



Hyperbolas of this type (that open ”east-west”) have equations of the form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Rearranging

$$\frac{x^2}{a^2} = 1 + \frac{y^2}{b^2}$$

so the minimum value of x occurs when $y = 0$ and $x = a$.

The *conjugate* hyperbola of this one is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$$

or equivalently

$$-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

opens "north-south."

And, although I will wait to deal with this complication, we have to mention another very common hyperbola

$$xy = c$$

where it must be true that $x \neq 0$ and $y \neq 0$.

Another feature of hyperbolas is the asymptote, the straight line which is approached when $x, y >> a, b$. In the case of the first example

$$\frac{y^2}{b^2} = \frac{x^2}{a^2} - 1$$

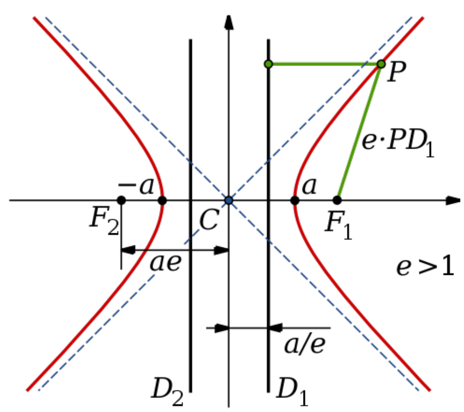
$$y^2 = \frac{b^2}{a^2}x^2 - \frac{1}{a^2}$$

but for large x and y this approaches

$$y^2 = \frac{b^2}{a^2}x^2$$

$$y = \pm\frac{b}{a}x$$

As the diagram suggests:



The following diagram gives geometric meaning to the b coefficient which really derives from the slope of the asymptotic line. We go vertically up from $x = a$ to the asymptote and then go left to the $y-$axis, that intercept is b.

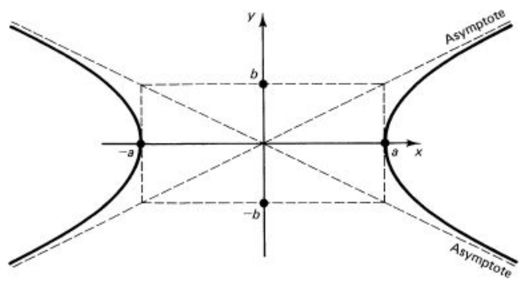


Figure 6.6-1 Hyperbola

geometry

Kline gives the following string and pencil construction for the hyperbola.

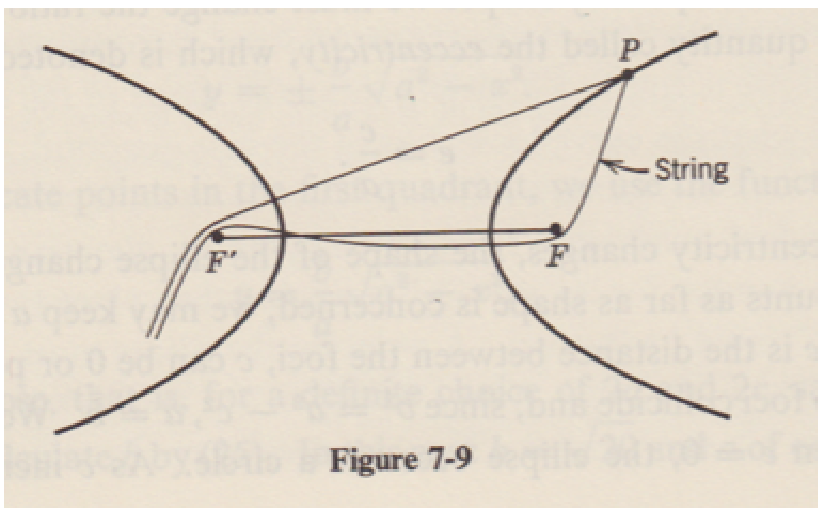


Figure 7-9

Pick two foci F and F' and loop a long piece of string around them, holding it tight. Then place the pencil at some point P on a line between the two foci, at a fixed position in the upper loop.

Now let the string slowly slip up past F' in both directions, increasing the length of PF and PF' by the same amount for each small slip. What this amounts to is that the difference $PF - PF'$ is constant.

If we place the origin halfway between F and F' then

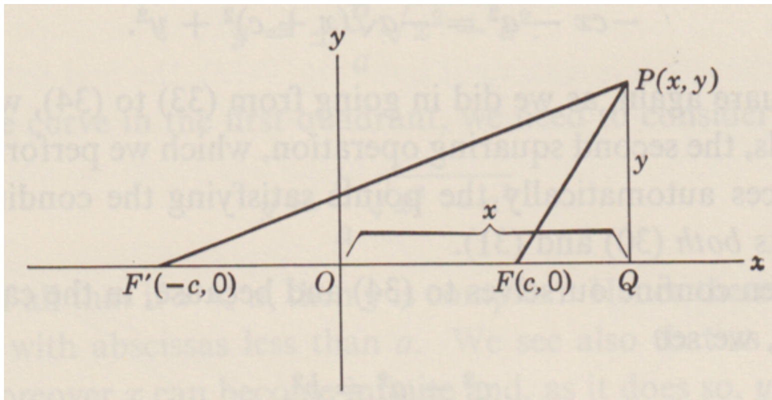


Figure 7-10

$$PF = \sqrt{(x-c)^2 + y^2}$$
$$PF' = \sqrt{(x+c)^2 + y^2}$$

and the difference $PF' - PF$ is

$$\sqrt{(x+c)^2 + y^2} - \sqrt{(x-c)^2 + y^2}$$

and if the constant distance

$$PF' - PF = 2a$$

then

$$\sqrt{((x+c)^2 + y^2} - \sqrt{(x-c)^2 + y^2} = 2a$$

Now we repeat the approach we took for the ellipse:

$$\sqrt{((x+c)^2 + y^2} = 2a + \sqrt{(x-c)^2 + y^2}$$

Square

$$(x+c)^2 + y^2 = 4a^2 + 4a\sqrt{(x+c)^2 + y^2} + (x-c)^2 + y^2$$

Cancel y^2

$$(x+c)^2 = 4a^2 + 4a\sqrt{(x+c)^2 + y^2} + (x-c)^2$$

Since

$$(x+c)^2 - (x-c)^2 = 4cx$$

we have

$$4cx = 4a^2 + 4a\sqrt{(x+c)^2 + y^2}$$
$$cx - a^2 = a\sqrt{(x+c)^2 + y^2}$$
$$c^2x^2 - 2ca^2x + a^4 = a^2(x+c)^2 + a^2y^2$$
$$c^2x^2 - 2ca^2x + a^4 = a^2x^2 + 2a^2cx + a^2c^2 + a^2y^2$$
$$(c^2 - a^2)x^2 - a^2y^2 = (c^2 - a^2)a^2$$

Define b^2 slightly differently here

$$b^2 = c^2 - a^2$$

so

$$b^2x^2 - a^2y^2 = b^2a^2$$
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

which looks familiar.

Chapter 50

Headlight problem

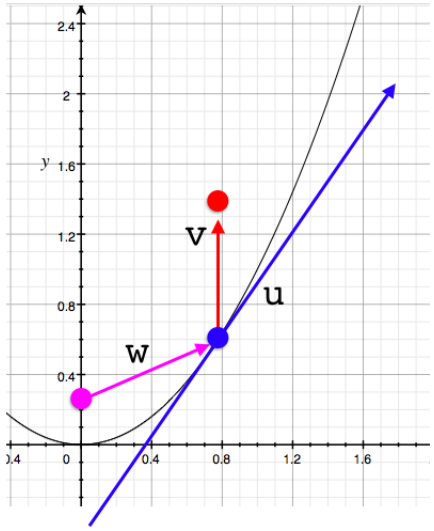
The reflective property of the parabola asserts that if a light ray emitted from the focus bounces off any point of the parabola, it then travels off in the vertical direction.

Snell's law for reflection says that the angle of incidence and reflection to the inside surface of the parabola must be equal. It is curious that this law has Snell's name on it, since the fact was known to Euclid, and Heron had a proof of it. The proof depends on the assumption that light travels the shortest path between two points.

We're going to make use of vectors in the first proof. We require simply the idea of what a vector is, and the dot product. That's the motivation for the introduction to vectors in the previous chapter. There is an alternative geometric derivation at the end.

derivation

In any case, applying that law to this problem, the angle of incidence is the angle of the magenta vector $\mathbf{w}$ with the tangent vector $\mathbf{u}$. This is equal to the angle of reflection, the angle of the tangent $\mathbf{u}$ with the vertical vector $\mathbf{v}$.



We assert that there exists a point on the y-axis (the focus, colored magenta), with the property that when we draw a vector to any point on the parabola, the angle that this vector makes with the tangent to the parabola is equal to the angle the tangent makes with the vertical.

Let the distance of this point from the origin be p. Then

$$\mathbf{w} = \langle x, ax^2 - p \rangle$$

The tangent has slope $2ax$ so

$$\mathbf{u} = \langle 1, 2ax \rangle$$

Scale the vertical to be a unit vector

$$\mathbf{v} = \langle 0, 1 \rangle$$

By the definition of the dot product, the cosine of the angle between $\mathbf{w}$ and $\mathbf{u}$ is

$$\frac{\mathbf{w} \cdot \mathbf{u}}{u\ w}$$

By the equal angle constraint, this is equal to the cosine of the angle between $\mathbf{u}$ and $\mathbf{v}$

$$\frac{\mathbf{u} \cdot \mathbf{v}}{u\ v} = \frac{\mathbf{w} \cdot \mathbf{u}}{u\ w}$$

Since $v = 1$ we have

$$w\ (\mathbf{u} \cdot \mathbf{v}) = \mathbf{w} \cdot \mathbf{u}$$

That's the important logic of the solution.

Now it's just algebra: The length of $\mathbf{w}$ is

$$w = \sqrt{x^2 + (ax^2 - p)^2}$$

while

$$\mathbf{u} \cdot \mathbf{v} = 2ax$$

$$\mathbf{w} \cdot \mathbf{u} = x + 2ax(ax^2 - p)$$

So

$$w\ (\mathbf{u} \cdot \mathbf{v}) = \mathbf{w} \cdot \mathbf{u}$$

$$\sqrt{x^2 + (ax^2 - p)^2}\ (2ax) = x + 2ax(ax^2 - p)$$

Divide by $2ax$:

$$\sqrt{x^2 + (ax^2 - p)^2} = \frac{1}{2a} + (ax^2 - p)$$

Square both sides

$$x^2 + (ax^2 - p)^2 = \frac{1}{(2a)^2} + \frac{1}{a}(ax^2 - p) + (ax^2 - p)^2$$

A nice cancelation:

$$x^2 = \frac{1}{(2a)^2} + \frac{1}{a}(ax^2 - p)$$

We can also cancel the x^2:

$$0 = \frac{1}{(2a)^2} + \frac{1}{a}(-p)$$

and finally cancel an a:

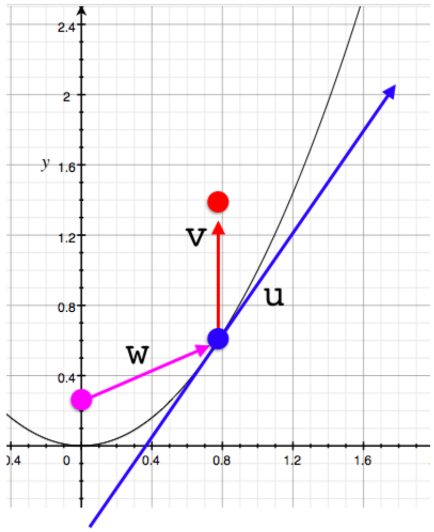
$$0 = \frac{1}{4a} - p$$

$$p = \frac{1}{4a}$$

The point $(0, 1/4a)$ is, as we saw before, the focus of the parabola.

Since p is independent of x, this property holds for every point on the parabola.

geometric proof



An alternative, more geometric approach is to note that the angle the vector **u** makes with the vertical at (x, ax^2) is equal to the angle **u** makes with the y-axis (just off the image to the bottom).

This angle is equal to the angle between **w** and **u** if and only if the triangle is isosceles, that is, if length of the vector **w** is equal to the distance between $(0, p)$ and the intersection of **u** with the y-axis.

We start by exploring the properties of a line through the point (x, ax^2) with slope equal to $2ax$.

From this point on, the point on the parabola is *fixed*. We want to write an equation for a line with the same slope as the parabola at this point, the same slope as the vector **u**.

We will be re-using x as a variable. To reduce confusion, label the fixed value at the point as $\hat{x}$, so then $\hat{y} = a\hat{x}^2$, and the slope is $2a\hat{x}$.

The point-slope formula for the line is

$$2a\hat{x} = \frac{\Delta y}{\Delta x} = \frac{y - \hat{y}}{x - \hat{x}} = \frac{y - a\hat{x}^2}{x - \hat{x}}$$

The intersection with the y-axis occurs at $y = 0$ so there

$$2a\hat{x} = \frac{-a\hat{x}^2}{x - \hat{x}}$$

$$2 = \frac{-\hat{x}}{x - \hat{x}}$$

$$2x - 2\hat{x} = -\hat{x}$$

$$x = \frac{\hat{x}}{2}$$

The intersection of $\mathbf{u}$ with the x-axis is at $\hat{x}/2$.

For the intersection with the y-axis, $x = 0$ and then

$$2a\hat{x} = \frac{y - a\hat{x}^2}{-\hat{x}}$$

$$-2a\hat{x}^2 = y - a\hat{x}^2$$

$$y = -a\hat{x}^2$$

What we've discovered is that the point of intersection is the same distance below the x-axis as our point on the parabola $(\hat{x}, a\hat{x}^2)$ is above it. We could have used congruent triangles proceeding from the discovery above that the intersection of with the x-axis is at $\hat{x}/2$.

Our goal is to show that the triangle is isosceles:

$$a\hat{x}^2 + p = w$$

$$a\hat{x}^2 + p = \sqrt{\hat{x}^2 + (a\hat{x}^2 - p)^2}$$

$$(a\hat{x}^2 + p)^2 = \hat{x}^2 + (a\hat{x}^2 - p)^2$$

Continuing

$$a^2\hat{x}^4 + 2ap\hat{x}^2 + p^2 = \hat{x}^2 + a^2\hat{x}^4 - 2ap\hat{x}^2 + p^2$$

Does this look familiar?

Cancel two terms

$$2ap\hat{x}^2 = \hat{x}^2 - 2ap\hat{x}^2$$

$$4ap\hat{x}^2 = \hat{x}^2$$

$$4ap = 1$$

$$p = \frac{1}{4a}$$

And we already proved this is true, if the magenta point we start from is the focus.

Hence the lengths are equal, the triangle is isosceles, and the corresponding angles are equal. The point we've been using is just the focus.

Chapter 51

Rotation

Yet another thing that can happen to make life complicated is rotation.

Consider the rotated parabola. You are probably used to seeing examples where it opens to the right or left. These are obtained by having an equation like

$$a(y-k)^2 = (x-h)$$

with $x = g(y)$

Then, the easiest thing to do is to switch x for y, solve the problem, and switch back at the end.

But it is also possible to rotate through a different angle, like 45°. What happens then? Well, basically we replace x and y by u and v with

$$x = u\cos\theta - v\sin\theta$$

$$y = u\sin\theta + v\cos\theta$$

(I derived these **here**). For 45°, $\sin\theta = \cos\theta = 1/\sqrt{2}$. Let

$$k = \sin\theta = \cos\theta = 1/\sqrt{2}$$

Substitute for x and y as given above

$$y = ax^2 + bx + c$$

$$ku + kv = a(ku - kv)^2 + b(ku - kv) + c$$

$$u + v = ak(u^2 - 2uv + v^2) + b(u - v) + \frac{c}{k}$$

Now, we might attempt to solve this for v in terms of u, but there is a new term $-2uv$ which mixes up u and v. That is what gives a parabola that is not symmetric with respect to either the x-axis or the y-axis.

general problem

The most general equation for a parabola, ellipse or hyperbola is

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

This includes rotated versions of all three.

Kline says (Chapter 7) to consider a rotation through an angle θ. I will use t for θ We wrote above

$$x = u\ cost - v\ sint$$

$$y = u\ sint + v\ cost$$

First compute the products:

- $x^2 = u^2 \cos^2 t - 2uv \sin t \cos t + v^2 \sin^2 t$
- $xy = u^2 \sin t \cos t + uv \cos^2 t - uv \sin^2 t - v^2 \sin t \cos t$
- $y^2 = u^2 \sin^2 t + 2uv \sin t \cos t + v^2 \cos^2 t$

Now try substituting into the general equation (I know, it's a mess). We collect the coefficients for all the terms u^2, uv, v^2, etc., separately:

- $[A \cos^2 t + B \sin t \cos t + C \sin^2 t\ ]\ u^2$
- $[-2A \sin t \cos t + B \cos^2 t - B \sin^2 t + 2C \sin t \cos t\ ]\ uv$
- $[A \sin^2 t - B \sin t \cos t + C \cos^2 t\ ]\ v^2$
- $[D \cos t + E \sin t\ ]\ u$
- $[-D \sin t + E \cos t\ ]\ v$

The insight is this: we must choose t so as to eliminate the coefficient of the term that mixes u and v: namely uv.

$$-2A \sin t \cos t + B \cos^2 t - B \sin^2 t + 2C \sin t \cos t = 0$$

Recall those sum of angles formulas!

$$\cos^2 t - \sin^2 t = \cos 2t$$

$$2 \sin t \cos t = \sin 2t$$

So

$$-A \sin 2t + B \cos 2t + C \sin 2t = 0$$

giving

$$\tan 2t = \frac{B}{A - C}$$

example

Consider

$$xy = 1$$

Here A and C are zero, while $B = 1$. What angle's tangent is not defined? $\pi/2$. As $2t$ approaches $\pi/2$, its tangent approaches ∞. So the value of t we seek is $t = \pi/4$.

We go back and compute the coefficients for all the other terms. Since only $B \neq 0$ and since $\sin \pi/4 = \cos \pi/4 = 1/\sqrt{2}$, we get

$$[\ \frac{A}{2} + \frac{B}{2} + \frac{C}{2}\ ]\ u^2 +\ [\ \frac{A}{2} - \frac{B}{2} + \frac{C}{2}\ ]v^2 = 1$$

$$= \frac{u^2}{2} - \frac{v^2}{2} = 1$$

which is the equation of a rectangular hyperbola opening left and right.

test

Suppose you run into a general conic equation with some version of

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

Ask these questions to decide what you have:

- **Are both variables squared?**

  No: It's a parabola.
  Yes: Go to the next test....

  - **Do the squared terms have opposite signs?**

    Yes: It's an hyperbola.
    No: Go to the next test....

    - **Are the squared terms multiplied by the same number?**

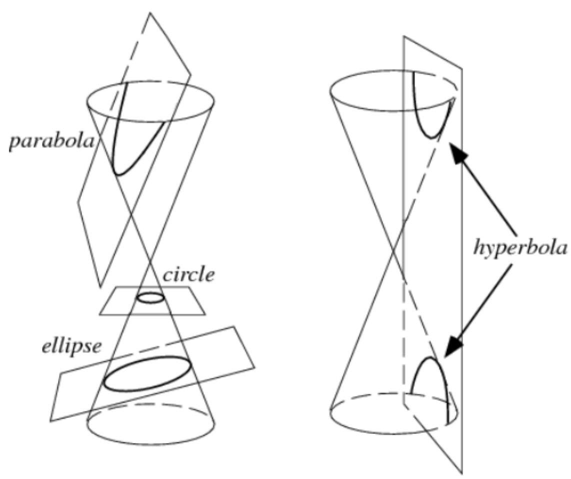
      Yes: It's a circle.
      No: It's an ellipse.

Kline goes through the effort of showing that, after rotating to a standard orientation, *every* equation of the general form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

can be translated to the origin to give a standard parabola, circle, ellipse or hyperbola.

conic sections



Everyone learns in high school that the conic sections can be obtained by slicing a double cone with a plane and taking the points that belong to both. Here is a simple example.

The level curves of a cone are

$$x^2 + y^2 = r^2$$

and the equation of the cone is $z = kr$ where $k = H/R$ is a constant. Suppose $k = 1$.

Suppose we also have a plane like

$$z = y + 1$$

This plane has normal vector

$$\mathbf{N} = \langle 0, -1, 1 \rangle$$

so the x-axis lies in the plane because $\mathbf{N} \cdot \hat{\mathbf{i}} = 0$. Another vector orthogonal to both and also in the plane is $\langle 0, 1, 1 \rangle$.

The normal vector to the cone depends on where you are, but if you are at $x = 0, y = 1$ then it would be

$$\mathbf{N} = \langle 0, 1, -1 \rangle$$

In the yz-plane it points down at a 45 degree angle. The two normal vectors are the same (within sign), so if there is a solution it should be a parabola.

We can see that there should be a solution, because the plane intersects the y-axis at $y = -1$ ($z = 0$) and the z-axis at $z = 1$. If you draw a sketch, one point is outside the cone and the other inside it, so the plane must cut the cone.

Every point on the intersection of the plane and the cone satisfies both equations:

$$\sqrt{x^2 + y^2} = 1 + y$$

$$x^2 + y^2 = 1 + 2y + y^2$$

$$x^2/2 = y + 1/2$$

This is a parabola but it is *not* the parabola formed by the intersection. It is the projection of that intersection onto the xy-plane.

Such projections are linear transformations, which simply amount to rescaling of the variables x to x' and y' to y' (in this case only the latter) without changing the nature of the curve—a parabola is still a parabola.

However, an ellipse may become a circle, and vice-versa.

In this case, the normal vector forms an angle of 45 degrees with the vertical z-axis since

$$\cos\theta = \frac{\langle 0,-1,1\rangle \cdot \langle 0,0,1\rangle}{\sqrt{1+1}\,\sqrt{1}} = \frac{1}{\sqrt{2}}$$

This is the factor by which the actual curve is stretched compared to the projection in the plane.

For the general problem, we would need to rotate all the points on the curve using angles obtained from the normal vector. We want to tilt $\mathbf{N}$ so that it points straight up and has its magnitude unchanged.

https://en.wikipedia.org/wiki/Rotation_matrix

In 3D we we could rotate points (or the coordinate system) first with respect to the xy-plane (ignoring z) using the standard transformation with this matrix

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

This is the same rotation that we had before — it leaves the z-coordinate unchanged. The relevant t is calculated from the x and y components of $\mathbf{N}$ using $t = \tan^{-1} y/x$. Then use the given matrix, or perhaps switch the signs on the sine.

After $\mathbf{N}$ has been rotated so that it lies along either the x- or y-axis, then rotate in the xz-plane or yz-plane until N is vertical.

Having done this, I believe there should be no mixed terms containing xy, so we won't need to rotate to remove those, as done before.

Part XIII

Archimedes

Chapter 52

Value of pi

Archimedes and π

Since Archimedes is a strong presence in this book, we will discuss his method for approximating the value of π, the ratio of the circumference of a circle to its diameter. The commonly cited result is

> The ratio of the circumference of any circle to its diameter is less than 3 1/7 but greater than 3 10/71.

In decimal that is $3.140845.. < \pi < 3.1428571$.

However, to some extent this misses the main idea, that Archimedes described an iterative procedure which can be used to calculate the value of π *to any desired accuracy*.

Although the idea is beautiful, his argument is somewhat unwieldy in detail, so instead we will use modern trigonometry to achieve the same result more economically.

For a discussion of Archimedes actual method (based on a translation by Heath), see this web page

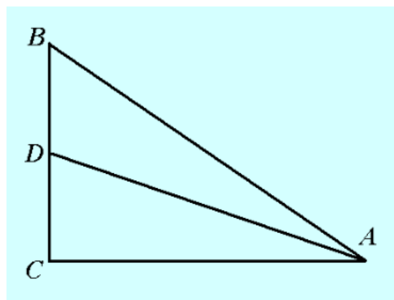
`https://itech.fgcu.edu/faculty/clindsey/mhf4404/archimedes/archimedes.html`

and I have worked out the same proof in detail in this **chapter**.

In addition, we will connect the trigonometry to easy formulas for the perimeter and area of inscribed and circumscribed polygons. The first part is in this chapter, and the second part has been split out into another **chapter**, which is in the Addendum.

If this material is too esoteric, it can be skipped without loss of continuity in the rest of the book.

I should also point out that although we don't follow Archimedes exactly, a key element which he relies upon is the proof that, for an angle bisector in a right triangle, the adjacent sides are in the same proportion as the two segments formed where the bisector meets the other side.



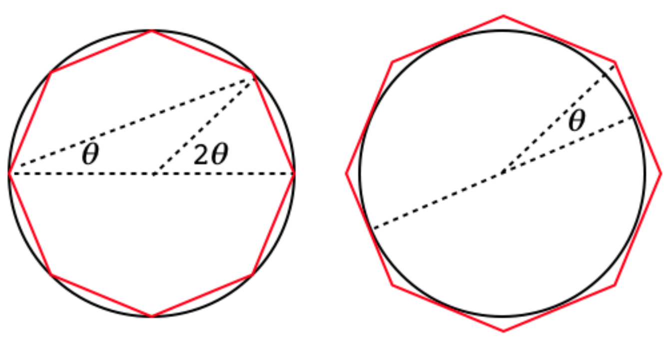
Here:

$$\frac{AB}{AC} = \frac{BD}{DC}$$

We showed a proof of this earlier (**here**).

the method

We will approximate the value of π by squeezing it between the perimeter of an inscribed polygon, which is less than the circumference of the circle, and the perimeter of a circumscribed polygon, which is greater than the circumference of the circle.



We use a circle of *diameter* equal to 1 (rather than the radius, which is more usual). The circumference of the circle is then equal to π, the value which gets squeezed between the two perimeters.

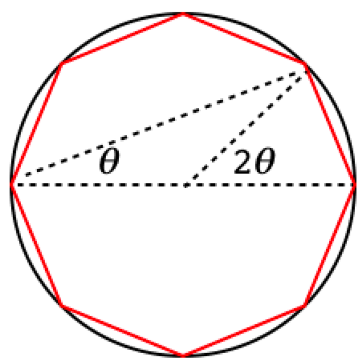
The figure shows a sketch of the polygons when $n = 8$. We will be increasing the number of sides by a factor of 2 at each step, so these are really 2^n-gons with $n = 3$ here.

Finding perimeters in terms of angle θ

For the left panel, we have 8 sides, so the central angle (marked 2θ) is equal to

$$\frac{2\pi}{8} = \frac{\pi}{4} = 45^\circ$$

and θ is one-half that.



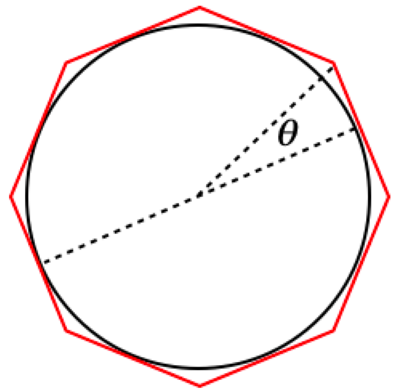
By a standard theorem (from Thales), the triangle above containing angle θ, with the diameter as one side, and two other vertices also on the circle, is a right triangle. The inscribed n-gon side of length S (shown in red) is equal to $\sin\theta$, since the hypotenuse of the triangle is the diameter of the circle, which is equal to 1.

The total perimeter is $8 \cdot S$.

[Alternatively, use half the angle at the center of the circle (i.e. θ). Then half the length of the red line $S/2$, divided by the radius ($r = 1/2$) gives $S = \sin\theta$, the same result.]

For the right panel, we have the same circle (now showing the outside polygon, circumscribing the circle), it is just rotated slightly.. One dashed line extends a bit further to the vertex of the n-gon outside. The angle marked θ is one-half the angle we marked as 2θ previously since now the diameter comes down to the middle of the side.

We compute the whole length of the side T as follows. The half-side is $T/2$ and the hypotenuse of the triangle is one-half the unit diameter, which is $1/2$, so $T = \tan\theta$. The total perimeter is $8 \cdot T$.



All of this gives us two simple equations for the two perimeters. At each stage there are 2^n sides, the length of each short side S on the inside equals $\sin\theta$ and the length of each short side on the outside T is equal to $\tan\theta$, where $\theta = 2\pi/2^n$.

The total length of the inside perimeter is $nS = n\sin\theta$ and that of the outside is $nT = n\tan\theta$. When we go from θ to $\theta/2$ and n to $2n$, we must compute the new values S' and T' from S and T using the half-angle formulas, and then also multiply by 2 to take account of the change from n to $2n$ for the total circumference.

The base case

If we go back to the square ($n = 2, 2^n = 4$), then the angle θ is $\pi/4$.

The tangent is $T = \tan\pi/4 = 1$ and the sine is $S = \sin\pi/4 = 1/\sqrt{2}$.

Our formulas say that on the inside, the perimeter is $4S = 4/\sqrt{2} = 2\sqrt{2}$ and on the outside, the perimeter is $4T = 4$.

From simple geometry, we can calculate that the circumscribing square has a side length which is twice the radius of the circle, that is, 1 for our circle with unit diameter, so its perimeter is 4, which checks.

Similarly, an inscribed square can be decomposed into four isosceles right triangles with sides of length $1/2$ and hypotenuse $1/\sqrt{2}$, so the total perimeter is $4/\sqrt{2}$, which also checks.

Now, what we are going to do is to increase n in steps of 1, that increases 2^n by a factor of $2^1 = 2$ each time. Doubling n halves the angle. So all we need is a way to compute trigonometric functions of $\theta/2$, knowing the values for θ, so we can calculate what happens to the perimeter. We already know how to do that.

Half angle formulas

We have derived these elsewhere. Refer to this **chapter**.

The unprimed values refer to angle θ, while the primed ones have angle $\theta/2$.

$$C' = \sqrt{\frac{1}{2}(1+C)}$$

This can be rearranged (e.g.) to give $2[C']^2 = 1 + C$, which we'll use in a second.

$$S' = \frac{S}{2C'}$$

$$\begin{aligned} T' = \frac{S'}{C'} &= \frac{S}{2[C']^2} \\ &= \frac{S}{1+C} \end{aligned}$$

So, given S, C and T, first calculate C' and T' and then S'. To get the perimeters, remember that factor of two from doubling n, the number of sides.

another approach

This web page originally got me started with this derivation

`http://personal.bgsu.edu/~carother/pi/Pi3d.html`

(Unfortunately, the link is dead now, probably because the University took Dr. Carother's pages down when he died, idiots). It has been preserved by the way-back machine:

https://web.archive.org/web/20171024182015/http://personal.bgsu.edu/~carother/pi/Pi3d.html

On that page, there was given an arguably simpler pair of formulas listed, namely, for an inside perimeter p and an outside perimeter P

$$P' = \frac{2pP}{p+P}$$

$$p' = \sqrt{pP'}$$

The first equation can be rearranged to give

$$\frac{1}{P'} = \frac{1}{2}\,[\frac{1}{P} + \frac{1}{p}\,]$$

which is the definition of the harmonic mean of p and P, while the second equation is the geometric mean.

Since in our derivation p and P are the same multiple of S and T, it seems like the same relationships should hold for the sine and tangent, but we must remember the extra factor of 2.

From the half-angle formulas, we said that

$$T' = \frac{S}{1+C}$$

Multiply top and bottom on the right by T:

$$T' = \frac{ST}{T+S}$$

Recall that S is the same as p, within a factor of n, and that T is the same as P, within the same factor.

$$p = nS$$

$$P = nT$$

while

$$P' = 2nT'$$

Going back to

$$T' = \frac{ST}{T+S}$$

$$2nT' = \frac{2 \cdot nS \cdot nT}{nT + nS}$$
$$P' = \frac{2pP}{p + P}$$

This is what was given.

For the second one

$$S' = \frac{S}{2C'}$$
$$= \frac{S}{2}\frac{T'}{S'}$$

Then

$$4[S']^2 = S \cdot 2T'$$
$$[2nS']^2 = nS \cdot 2nT'$$

Changing variables, $p' = 2nS'$

$$[p']^2 = pP'$$

Finally

$$p' = \sqrt{pP'}$$

which matches what was given.

Calculation

Let's run a simulation to see what kind of numbers we get. Start with the square ($n = 2$, $2^n = 4$) Previously we found that $S = 1/\sqrt{2}$ and $T = 1$ so

$$p = 2^n S = \frac{4}{\sqrt{2}} = 2.8284$$
$$P = 2^n T = 4$$

Let's try a script to calculate this to larger n.

https://gist.github.com/telliott99/19f521c807210171a4847b319104b3df

Output:

```
> python pi.py
 2 2.8284271247  4.0000000000
```

```
 3 3.0614674589  3.3137084990
 4 3.1214451523  3.1825978781
 5 3.1365484905  3.1517249074
 6 3.1403311570  3.1441183852
 7 3.1412772509  3.1422236299
 8 3.1415138011  3.1417503692
 9 3.1415729404  3.1416320807
10 3.1415877253  3.1416025103
11 3.1415914215  3.1415951177
12 3.1415923456  3.1415932696
13 3.1415925766  3.1415928076
14 3.1415926343  3.1415926921
15 3.1415926488  3.1415926632
16 3.1415926524  3.1415926560
17 3.1415926533  3.1415926542
18 3.1415926535  3.1415926537
19 3.1415926536  3.1415926536
>
```

That looks pretty good to me, although it's a bit slow to converge.

This is really quite amazing. Archimedes has not only calculated π to 3 significant figures. More important, he has provided us with an iterative procedure that can be used to calculate the value to *any precision we desire*. As an engineer, Archimedes knew that 3.1416 is precise enough, so he stopped.

After all, no one wants to be William Shanks, or one of these guys:

Quote:

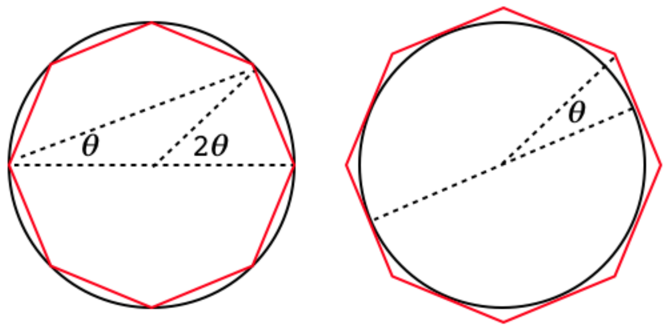
> [He] calculated pi to [n] digits, but *not all were correct.*

There is an additional **chapter** which substantially extends the above discussion, showing a geometric derivation of the basic relationships and developing new formulas involving the areas as well as the perimeters of the sectors of inscribed and circumscribed polygons.

area

I became aware later that there is yet another way to apply the method, and that is to calculate the *areas* of inscribed and circumscribed polygons. We'll go through this briefly.

For this approach we use a unit circle (radius 1) rather than a diameter of 1, as we did above. As before, we define θ to be the central angle of the half-sector (i.e. $\theta = 2\pi/2n$).



Rather than draw an entirely new figure, just imagine in the left panel that we draw the angle bisector of angle 2θ. The area of each new triangle is then $\sin\theta\cos\theta/2$ and the total area of the inner polygon is

$$a = n\sin\theta\cos\theta = nSC$$

in the notation we adopted previously in this chapter. And, as before, to progress to a' we have a factor of 2 as well as the new values S' and C':

$$a' = 2nS'C'$$

For the circumscribed or outer polygon, we just have what we had before, that the side length of the triangle in the right panel is $\tan\theta$ so the total area is

$$A = nT$$

Bring in the half-angle formulas as follows:

$$a' = 2nS'C' = 2n \cdot \frac{S}{2C'} \cdot C' = nS$$

That is slick, but we need an expression for nS:

$$aA = nSC \cdot n\frac{S}{C} = [nS]^2$$

$$aA = [a']^2$$
$$a' = \sqrt{aA}$$

This is like, and yet subtly different than what we had when calculating the perimeter. Since

$$A = nT$$

and

$$A' = 2nT'$$
$$= 2n\frac{ST}{S+T} = 2\frac{nS \cdot nT}{nS + nT}$$
$$A' = 2\frac{a'A}{a' + A}$$

Compare

$$a' = \sqrt{aA} \qquad A' = 2\frac{a'A}{a' + A}$$
$$p' = \sqrt{pP'} \qquad P' = 2\frac{pP}{p + P}$$

However, it turns out that when you take account of the differing size of the circle for perimeter and area methods, and thus the initial values of p, P, a and A, the different order of operations results in precisely the same calculation.

Chapter 53

Archimedes circle proof

∘ Let A be the area of the circle

∘ Let T be the area of the triangle formed with base $2\pi r$ and height r (i.e. the area of T is πr^2).

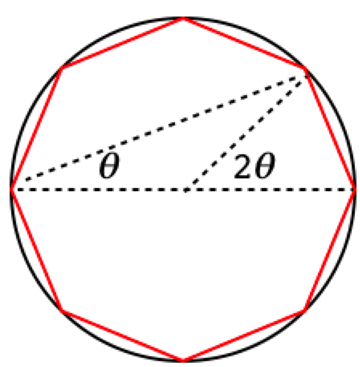
The method of proof is by finding a contradiction. We will assume something next, and then prove that a contradiction results, so the assumption must be incorrect

∘ Assume $A > T$.

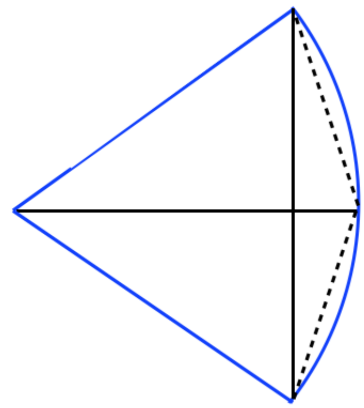
That is, the difference $A - T$ is non-zero and positive:

$$A - T > 0$$

Using the methods described **here**, we know that it is possible to construct an inscribed polygon whose area differs from A by *as little as we please*



Here is a single sector:



In the figure, suppose that this is a sector of a blue circle, and the black vertical is one side of an inscribed polygon of n sides,.

In the next step, we find a way to double the number of sides (dotted lines). This is a simple construction, just divide the secant between two adjacent vertices, and draw the radius through that point to the edge of the circle. By this means we can obtain a series, like $6, 12, 24, 48, 96 \dots$ sides, as Archimedes did.

Clearly, the polygon with $2n$ sides is still contained within the circle, but its area more closely approximates that of the circle. This can be repeated forever.

Call the area of the inscribed polygon P.

So what we meant by as little as we please is that P can be made closer to A than T is, simply on the assumption that $T < A$. Since this is an *inscribed* polygon, we have

$$A - P < A - T$$

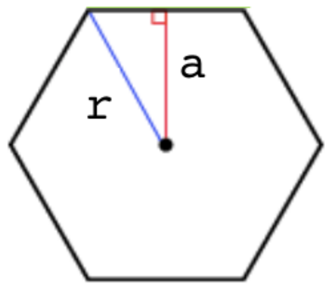
Add $-A$ to both sides:

$$-P < -T$$

Now, add $P + T$ to both sides:

$$T < P$$

However, for an inscribed polygon, the area is the number of sides n times the length of the base of each side, which is the perimeter, times the apothem (the vertical to the sides of the polygon), labeled a, times $1/2$.



In the figure, it must be that $a < r$.

But the perimeter is certainly less than the circumference of the circle ($2\pi r$) so:

$$P < \frac{1}{2} \cdot 2\pi r \cdot r$$

By this second argument, we have shown that the area of the regular polygon, $P < T$. However, we first showed that $T < P$. We have reached a contradiction.

Therefore, our assumption that $A > T$, is incorrect. A is *not* greater than T:

$$A \not> T$$

A similar argument assuming $A < T$ also leads to a contradiction.

Since A is neither greater than nor smaller than T it must be equal to T.

$$A = T = \frac{1}{2} \cdot 2\pi r \cdot r = \pi r^2$$

The analysis is taken from Dunham's *Journey Through Genius*.

Chapter 54

Value of pi revisited

As discussed in a previous **chapter**, Archimedes used paired inscribed and circumscribed polygons to develop an iterative procedure that can be used to calculate the value of π *to any desired accuracy.* Although the method is beautiful, his argument is unwieldy in detail, so we used modern trigonometry to achieve the same result more economically.

There are, in addition, two other sets of formulas that also reach this end, one based on perimeters, and the other on areas. These formulas are intriguing because they are simple, and it is not surprising that they are connected.

For example, consider a circle of unit *diameter*, so that π is equal to the perimeter. If p and P are the inside and outside perimeters for polygons whose sectors have central angle θ, and the same symbols are used with primes for angle $\theta/2$, then:

$$P' = 2\frac{pP}{p+P}$$

$$p' = \sqrt{pP'}$$

The corresponding formulas for inside (a) and outside (A) areas are (for a circle of unit radius)

$$A' = 2\frac{a'A}{a'+A}$$

$$a' = \sqrt{aA}$$

Notice that these two similar sets of formulas are subtly different. For example, to go from p and P to the primed version, we start with the first formula, while for area we must start with the square root. Part of our purpose in this chapter is to show that this works. (I must confess, I still do not have a simple explanation for *why* it is true).

inspiration

Originally, I was thinking about trying to implement Archimedes actual method for calculating π. However, the details of the approach are pretty painful. Instead, I worked through the problem using trigonometry.

It's striking that the formulas for the inside and outside perimeters are so simple, namely $n \sin \theta$ and $n \tan \theta$. The rest just follows from the half-angle formulas.

The web page which originally got me started with the harmonic and geometric mean formulas has been preserved by the wayback machine:

https://web.archive.org/web/20171024182015/http://personal.bgsu.edu/~carother/pi/Pi3d.html

On the very same day that I was revising the previous chapter to better integrate these two approaches, I came across another page which gives a "proof without words" of Gregory's Theorem (that is our subject).

https://divisbyzero.com/2018/09/28/proof-without-word-gregorys-theorem/

It gives these two formulas:

$$I_{2n} = \sqrt{I_n C_n}$$

$$C_{2n} = \frac{2}{1/I_{2n} + 1/C_n}$$

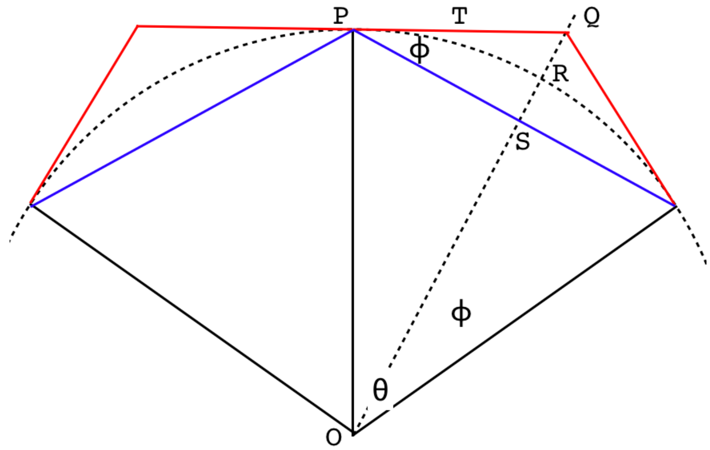
I found this notation a bit awkward, so I substituted the versions given above:

$$a' = \sqrt{aA}$$

$$A' = 2\frac{a'A}{a' + A}$$

Here, we mainly follow the development from that page and its "proof without words". One difference is that we will start with the geometry and work backward to the formulas. Let's deal with the perimeter first and then do the area.

basic setup



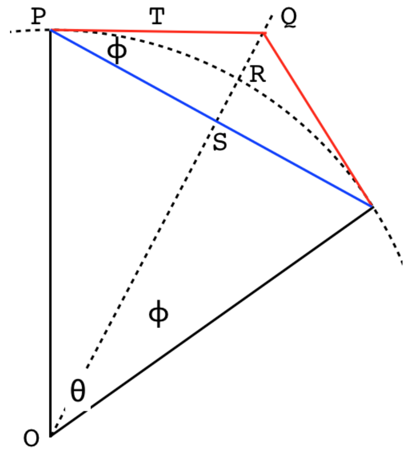
Draw a circle centered at O (only an arc of the circle is shown).

Points on the circle are chosen such that the arc length is an integral fraction of the whole. Equivalently, set $n\theta = 2\pi$.

Two adjacent sectors are shown in the figure above. The two polygons might be drawn so that the vertices of the internal and external figures are on the same ray, with parallel sides. However, the construction shown is more convenient.

The precise scale does not matter to the argument (nor the value of n). If it should turn out that the arc length as drawn is not exactly right, increase or decrease the radius of the circle and then fit it to the figure, keeping two points on the perimeter, and adjust O to be at the center of the adjusted circle.

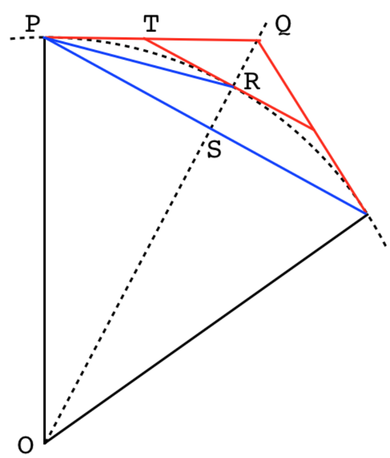
Two red lines comprise this sector's external perimeter P, while a single blue line is the inscribed perimeter p. The lines of the external perimeter are both tangent to the circle, and the whole figure is symmetric in each sector, with one blue and two red lines.



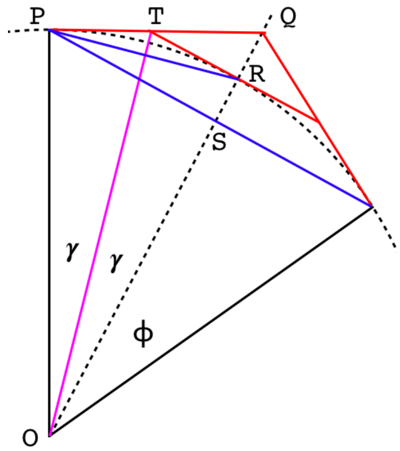
$\angle PSR$ is a right angle. Proof: we simply appeal to symmetry, or point out the congruent triangles. Since $\phi = \theta/2$, we have SAS.

Next, draw the perimeters p' and P' for the polygon with $2n$ sides and sector angle $\phi = \theta/2$.

It is convenient to rotate the internal perimeter by $\theta/2$ with respect to the external one, a bit to the left when we draw p' and a bit to the right for P'. Both p' and P' touch the circle at R.



A central relationship we use below is that $\triangle PRT$ is isosceles. For a proof, draw OT and appeal to symmetry.

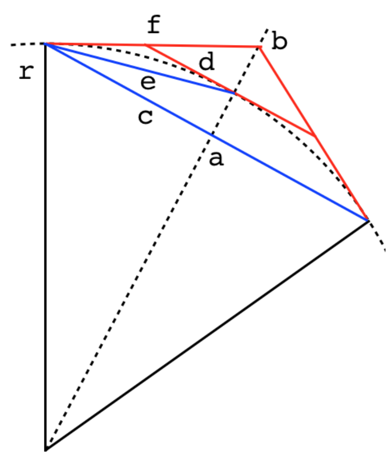


Or note that OT bisects $\angle POQ$ so $\triangle POT \cong \triangle ROT$ by SAS.

A consequence is that PR bisects $\angle QPS$. This can also be proved by an argument based on the sum of internal angles for an n-gon,

It looks as if the segment of the vertical that extends beyond the radius might be equal to that part below down to what looks like the "strut" of a kite. However, this is not true. We will show what this ratio is equal to in just a bit.

Rather than use the vertices as points of reference, we will label the line segments.

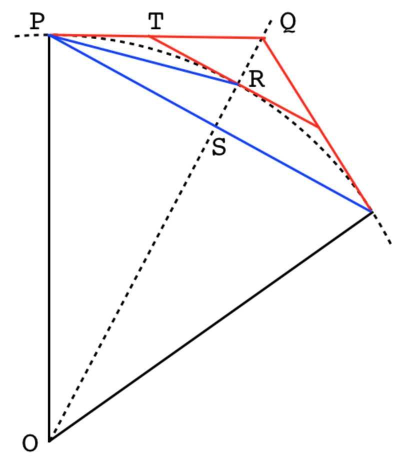


Just to be clear: a is the part of the radius extended to point S above, while b extends to Q. c and d are the lengths of the indicated lines *in the half-sector*, not all the way across, and f is the entire length of PQ.

We're ready to proceed.

basic geometry: perimeters

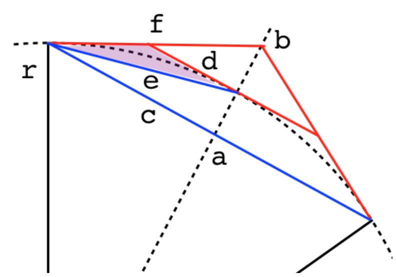
As we said, the key observation is that $\triangle PRT$ is isosceles.



Because of that, and since $\angle SPR = \angle PRT$ by the alternate interior angles theorem, $\angle SPR = \angle TPR$.

Therefore the cosines are also equal, namely:

$$\frac{c}{e} = \frac{e/2}{d}$$



(To see the midpoint of e, drop an altitude in the isosceles triangle, shown in purple).

Therefore:

$$2dc = e^2$$

Now, c is the entirety of p in this half-sector. But d is only one-half of P'.

Hence $2d \cdot c$ is equal to pP', and since $e = p'$, we have that

$$pP' = [p']^2$$

which was our second rule for the perimeters.

The first rule was

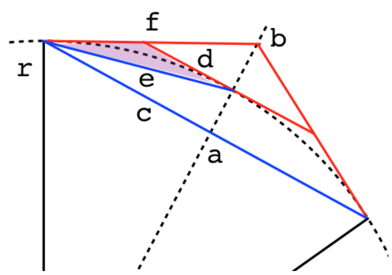
$$P' = 2\frac{pP}{p + P}$$

In geometric terms, we must show that

$$2d = 2\frac{cf}{c + f}$$

$$cd + df = cf$$

Taking another look at the diagram:



The small triangle with base d ($\triangle QRT$ above) has slanted side $f - d$ (subtracting d because, again, $\triangle PRT$ is isosceles). By similar triangles, we have

$$\frac{d}{f - d} = \frac{c}{f}$$

$$df = cf - cd$$

$$cd + df = cf$$

But this is what we needed to prove.

□

basic geometry: areas

The area formulas for inside (a) and outside (A) polygons are those for a circle of unit radius (so that π is the area):

$$A' = 2\frac{a'A}{a' + A}$$

$$a' = \sqrt{aA}$$

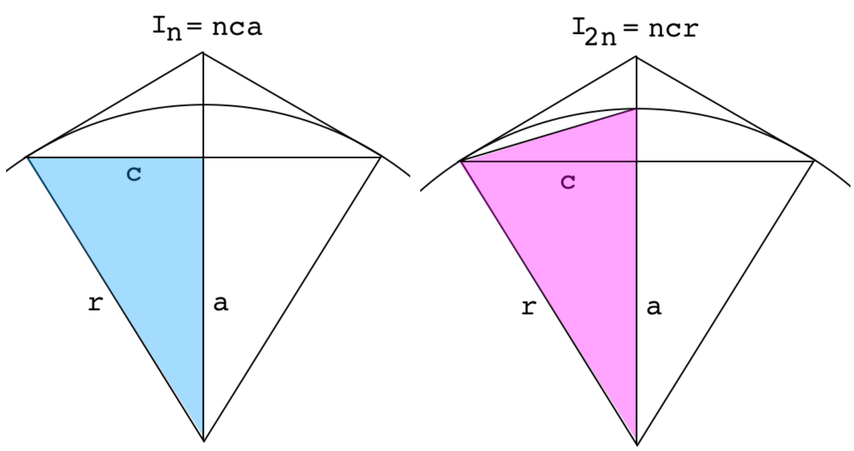
However, having reached this point, we need another symbol for area, because a is currently the line segment corresponding to p/n. Let's use I and C for the inside and outside areas, to match the source.

We will also adopt their n and $2n$ notation, It's a bit clumsy but that will make it easier to match things up.

$$C_{2n} = 2 \cdot \frac{I_{2n}C_n}{I_{2n} + C_n}$$

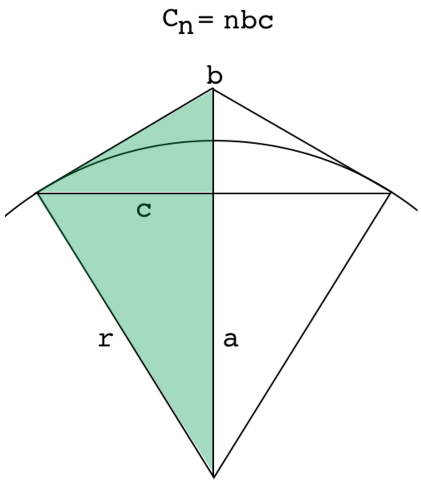
$$I_{2n} = \sqrt{I_n C_n}$$

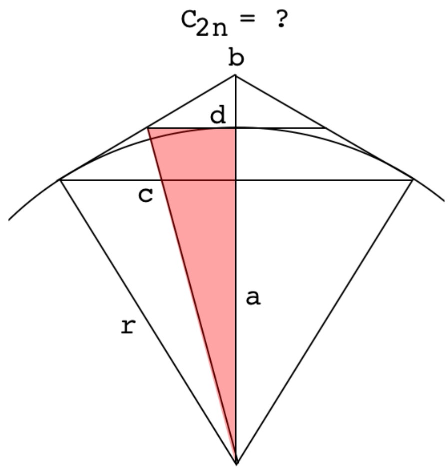
The first two areas are I_n and I_{2n}



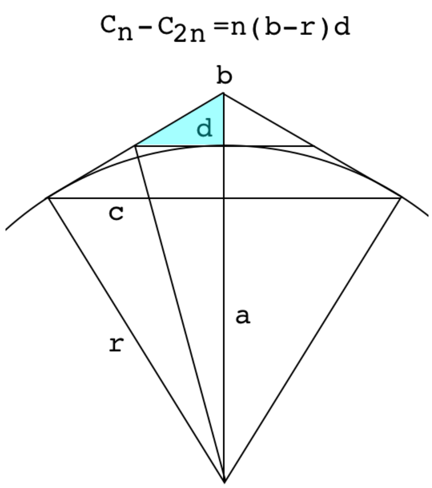
We compute these areas for the whole sector of angle θ, so there are two congruent triangles with base a (or base r) and height c. Multiply by n if you like to get the entire polygon, but every expression will have a factor of n, and we'll be looking at ratios, so we can just not worry about it.

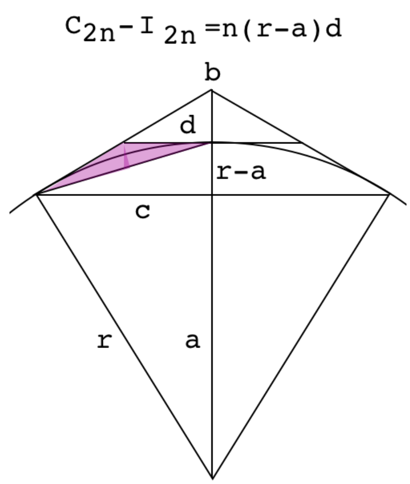
The third easy one is C_n:





We write the last one (C_{2n}) as two different differences.





Let's gather all these expressions in one place, forming ratios:

$$\frac{I_{2n}}{I_n} = \frac{ncr}{nca} = \frac{r}{a}$$

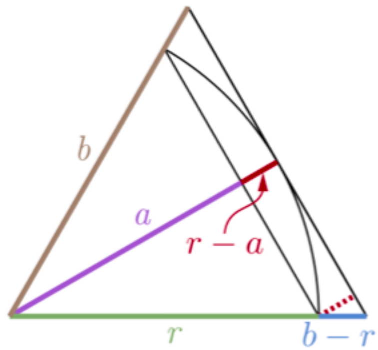
$$\frac{C_n}{I_{2n}} = \frac{ncb}{ncr} = \frac{b}{r}$$

$$\frac{C_n - C_{2n}}{C_{2n} - I_{2n}} = \frac{n(b-r)d}{n(r-a)d} = \frac{b-r}{r-a}$$

We will prove that these three ratios are all equal to each other.

We will have used the geometry to prove what the source calls their Lemmas, and those can be used in turn to prove the original Gregory formulas.

But the proof is easy:



It's just a matter of similar triangles:

$$\frac{r}{a} = \frac{b}{r} = \frac{b-r}{r-a}$$

That's the "without words" part.

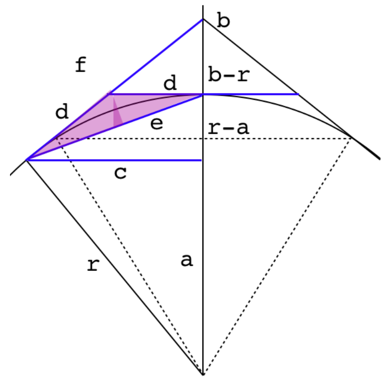
For that very last part, you can work out the dimensions of the tiny similar triangle, or you can say:

$$\frac{r}{a} = \frac{b}{r}$$
$$\frac{r}{a} - \frac{a}{a} = \frac{b}{r} - \frac{r}{r}$$
$$\frac{r-a}{a} = \frac{b-r}{r}$$

which is easily rearranged to give the desired result.

□

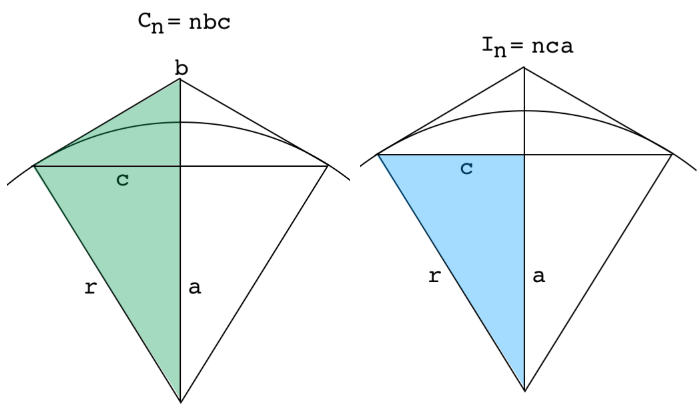
This can also be proved using the **angle bisector theorem**.



The side labeled e bisects the angle formed by the two sides labeled c and f. Therefore

$$\frac{b-r}{f} = \frac{r-a}{c} \quad \Rightarrow \quad \frac{b-r}{r-a} = \frac{f}{c}$$

But f and c are two sides of a triangle which is similar to the colored portions below:



Therefore

$$\frac{b}{r} = \frac{r}{a} = \frac{f}{c} = \frac{b-r}{r-a}$$

As we said.

algebra

Moving on to the geometric mean formula is not hard. From above we have that

$$\frac{I_{2n}}{I_n} = \frac{C_n}{I_{2n}}$$

$$[I_{2n}]^2 = I_n C_n$$

Translated back into the A, a area notation

$$a' = \sqrt{aA}$$

This is just what we wanted to show.

For the other formula, what we have is:

$$\frac{C_n - C_{2n}}{C_{2n} - I_{2n}} = \frac{C_n}{I_{2n}}$$

$$I_{2n}(C_n - C_{2n}) = C_n(C_{2n} - I_{2n})$$

$$2I_{2n}C_n = C_nC_{2n} + I_{2n}C_{2n}$$

$$= C_{2n}(C_n + I_{2n})$$

So

$$C_{2n} = 2 \cdot \frac{I_{2n}C_n}{C_n + I_{2n}}$$

$$C_{2n} = 2 \cdot \frac{1}{1/I_{2n} + 1/C_n}$$

And we're done. In our preferred notation

$$A' = 2 \cdot \frac{1}{1/a' + 1/A}$$

historical note

The area-based formulas given above are due to James Gregory.

https://divisbyzero.com/2018/09/28/proof-without-word-gregorys-theorem/

As an aside, the Fundamental Theorem of Calculus (FTC) is usually thought about (taught and learned) using the language of functions, and ascribed mainly to Leibnitz, with some credit to the two Isaacs, Newton and his university lecturer, Barrow.

https://arxiv.org/abs/1111.6145

Amazingly enough, Gregory published a geometric (Euclidean) proof of the FTC in 1668! That predates Liebnitz (1693) by more than 25 years. This is motivation to give considerable credit to individuals other than Newton and Liebnitz (e.g. Fermat, Pascal, Wallis, Gregory, etc.) in the invention of the calculus.

test

I wrote a simple test of the area formulas using Python.

The script is here:

https://gist.github.com/telliott99/5269b48672cdaeca95c9c9c9d163321d

It gives this output:

```
> python script.py
    4 2.0000000000 4.0000000000
    8 2.8284271247 3.3137084990
   16 3.0614674589 3.1825978781
   32 3.1214451523 3.1517249074
   64 3.1365484905 3.1441183852
  128 3.1403311570 3.1422236299
  256 3.1412772509 3.1417503692
  512 3.1415138011 3.1416320807
 1024 3.1415729404 3.1416025103
 2048 3.1415877253 3.1415951177
 4096 3.1415914215 3.1415932696
 8192 3.1415923456 3.1415928076
16384 3.1415925766 3.1415926921
32768 3.1415926343 3.1415926632
65536 3.1415926488 3.1415926560
>
```

The digits of the output appear to be identical or nearly so. The only difference is that in this script I computed 2^n to give the number of sides. In the previous chapter, we just print n.

details

That's very curious. The first four lines of output from the perimeter version:

```
  2 2.8284271247  4.0000000000
 3 3.0614674589  3.3137084990
 4 3.1214451523  3.1825978781
 5 3.1365484905  3.1517249074
```

and the first five from the area version:

```
 4 2.0000000000 4.0000000000
 8 2.8284271247 3.3137084990
16 3.0614674589 3.1825978781
32 3.1214451523 3.1517249074
64 3.1365484905 3.1441183852
```

It's pretty clear that we are doing the same calculation. It's just that the first column is shifted up by one row.

To confirm that, the perimeter calculation is:

initialization:

$$p = 2\sqrt{2} \qquad P = 4$$

recurrence:

$$P' = \frac{2pP}{p+P} \qquad p' = \sqrt{pP'}$$

The area version is:

initialization:

$$a = 2 \qquad A = 4$$

recurrence:

$$a' = \sqrt{aA} \qquad A' = \frac{2a'A}{a'+A}$$

They give identical results: $A = P$, at each round, but a matches p', or to put it the other way around, p' is retarded by one cycle compared to a'.

Let's try one round of calculation by hand:

$$p = 2\sqrt{2} \qquad P = 4$$

$$P' = \frac{2pP}{p+P} = \frac{2 \cdot 2\sqrt{2} \cdot 4}{2\sqrt{2}+4} = \frac{2 \cdot 2\sqrt{2} \cdot 4}{2\sqrt{2}(1+\sqrt{2})} = \frac{8}{1+\sqrt{2}} = 3.31371$$

$$p' = \sqrt{pP'} = \sqrt{2\sqrt{2} \cdot \frac{8}{1+\sqrt{2}}} = 4\sqrt{\frac{1}{1+1/\sqrt{2}}} = 3.06147$$

The area calculation:

$$a' = \sqrt{aA} = \sqrt{2 \cdot 4} = \sqrt{8} = 2.828427$$

$$A' = \frac{2a'A}{a'+A} = \frac{2\cdot\sqrt{8}\cdot 4}{\sqrt{8}+4} = \frac{8}{1+\sqrt{2}}$$

A' is the same as P'.

The next round for a' is

$$a' = \sqrt{aA} = \sqrt{\sqrt{8}\cdot\frac{8}{1+\sqrt{2}}} = 4\sqrt{\frac{1}{1+1/\sqrt{2}}}$$

Perhaps someday I'll have a deeper understanding, Undoubtedly, there is a series here.

Chapter 55

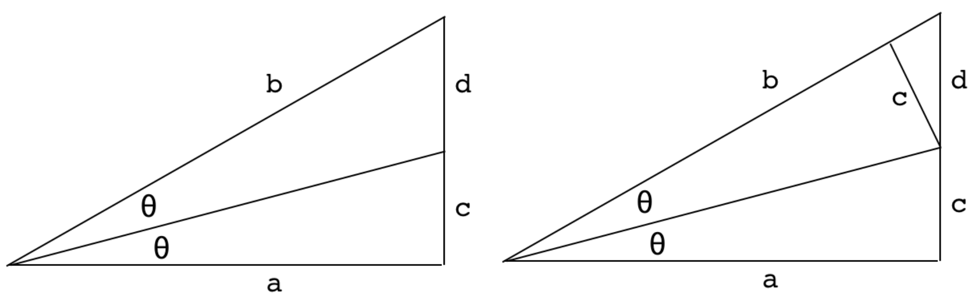
Archimedes and pi

We're going to follow a page that in turn follows Archimedes argument for the approximation of π.

https://itech.fgcu.edu/faculty/clindsey/mhf4404/archimedes/archimedes.html

Before we start, let's review some ideas related to angle bisection. Recall that if we have an angle bisector in a right triangle (left panel), the theorem says that

$$\frac{a}{c} = \frac{b}{d}$$



The proof (which we showed in an earlier chapter) involves drawing the altitude of the top triangle, forming two congruent triangles and a smaller one which is easily shown to be similar to the original triangle (right panel). By similar triangles, we have

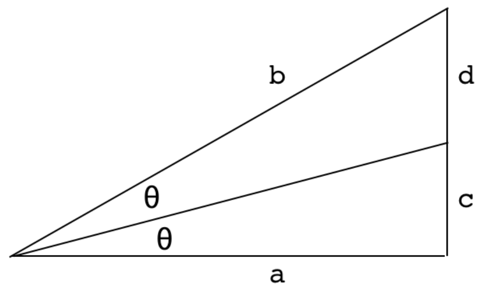
$$\frac{d}{c} = \frac{b}{a}$$

which can be rearranged to give the desired statement. A corollary follows:

$$\frac{a}{b} = \frac{c}{d}$$

$$\frac{a+b}{b} = \frac{c+d}{d}$$

$$\frac{a+b}{c+d} = \frac{b}{d} = \frac{a}{c}$$

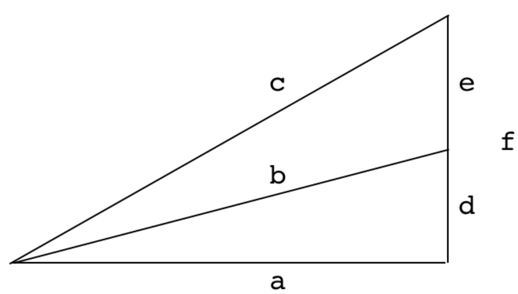


This doesn't seem obvious to me, in fact it seems counter-intuitive. Nonetheless, we will use it extensively for what follows.

overview

There are three steps which we will repeat (eventually, four times).

Let's relabel the figure now:



• Obtain ratios for the cosecant and cotangent (c/f and a/f). In the first round, these are just $\sqrt{3}$ and 2.

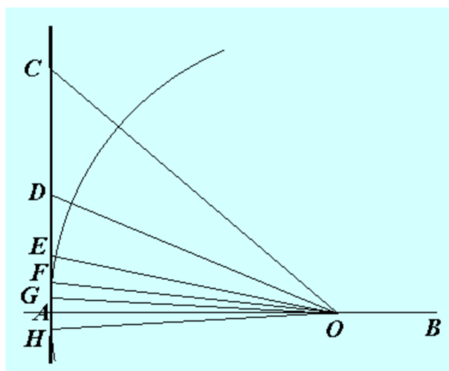
• Add them together, obtaining $(c+a)/f$ and observe that this ratio is also equal to a/d, by the corollary of the angle bisector theorem above. This gives us the cotangent of the half-angle.

• Obtain the cosecant of the half-angle, b/d, by using the Pythagorean theorem. Namely:

$$a^2 + d^2 = b^2$$
$$\sqrt{\frac{a^2}{d^2} + 1} = \frac{b}{d}$$

Part A, round 1

Draw a circle with radius OA and tangent AC, and let $\angle AOC$ be one-third of a right angle.



Note: the figure appears to have been compressed in width. The angle bisectors don't look right and the original angle looks more like 45 than 30. We'll use it anyway.

In what follows we list the claim first:

• $OA : AC > 265 : 153$

Followed by the proof:

Since the triangle is a 30-60-90 triangle, $OA = \sqrt{3}$ and $AC = 1$, so the ratio is just $\sqrt{3}$. 265/153 is a (very good) approximation, just slightly smaller than the true value.

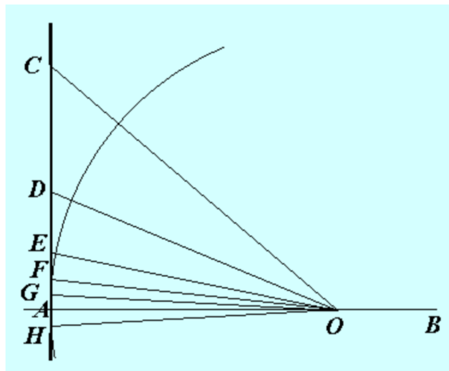
• $OC : AC = 306 : 153$

The cosecant = 2. The denominator has been chosen to match the previous ratio.

Now draw the angle bisector OD.

• $CO : OA = CD : DA$

This is just the angle bisector theorem.



• $(CO + OA) : CA = OA : AD$

This ratio is equal to that from the bisector theorem by the corollary. Start with

$$CO : OA = CD : DA$$
$$\frac{CO + OA}{OA} = \frac{CD + DA}{DA} = \frac{CA}{DA}$$

This crucial step gives us the cotangent of the half-angle formed by the angle bisector OD.

• $OA : AD > 571 : 153$

We just add numerators for the first two ratios above, leaving the result over the common denominator.

• $OD : AD > 591\ 1/8 : 153$

Finally, we want $OD : AD$. By the Pythagorean Theorem

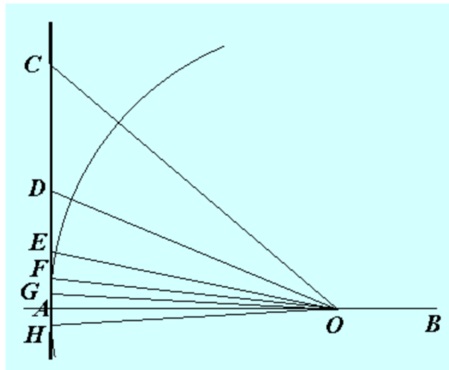
$$OD^2 = OA^2 + AD^2$$
$$(OD : AD)^2 = (OA : AD)^2 + 1$$

Now do $571^2 = 326041; 153^2 = 23409$ so the sum of numerators is 349450 and the square root is 591 1/6.

Archimedes approximates the result as 591 1/8 : 153. Remember that we are looking for a lower bound, so smaller is OK.

Part A, round 2

Now draw the angle bisector OE.



- From above, we have that $OA : AD > 571 : 153$ and $OD : AD > 591\ 1/8 : 153$.
- $OA : AE > 1162\ 1/8 : 153$

This calculation invokes the angle bisector corollary again. Rather than repeat the derivation, just add the inputs:

$$591\ 1/8 : 153 + 571 : 153$$

which adds to give the result above, 1162 1/8 : 153.

- $OE : AE > 1172\ 1/8 : 153$

Use the Pythagorean theorem to write:

$$OE^2 = AE^2 + OA^2$$

$$\frac{OE^2}{AE^2} = \frac{OA^2}{AE^2} + 1$$

We have $(1162\ 1/8)^2$ and $153^2 = 23409$.

Write

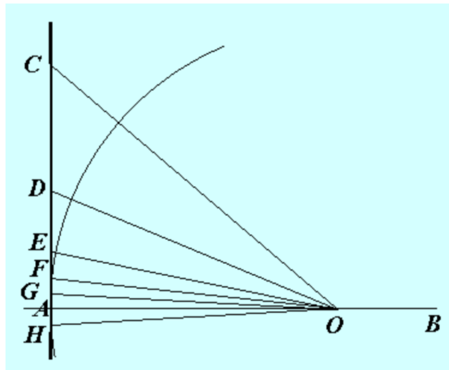
$$1162^2 + 1162/4 + 153^2 = 1350244 + 290\ 1/2 + 23409$$

$$= 1373943\ 1/2 + 1/64 = 1373943\ 33/64$$

The square root is 1172 1/8.

Part A, round 3

Now draw the angle bisector OF.



• From above, we have that $OA : AE > 1162\ 1/8 : 153$ and $OE : AE > 1172\ 1/8 : 153$.

• $OA : AF > 2334\ 1/4 : 153$

This calculation invokes the angle bisector corollary again.

$$\frac{OA}{FA} = \frac{OE}{EA} + \frac{OA}{EA}$$
$$1162\ 1/8 : 153 + 1172\ 1/8 : 153$$

which adds to give the result above.

• $OF : FA > 2339\ 1/4 : 153$

Use the Pythagorean theorem to write:

$$\frac{OF^2}{FA^2} = \frac{OA^2}{FA^2} + 1$$

We have $(2334\ 1/4)^2$ and $153^2 = 23409$.

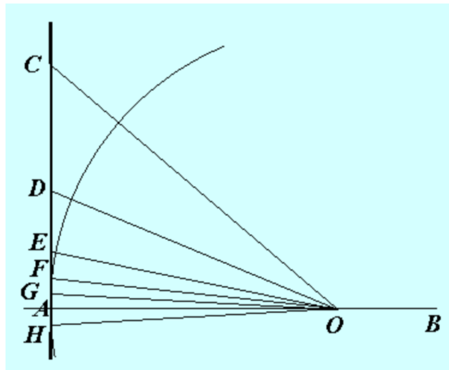
Write

$$\begin{aligned} 2334^2 + 2334/2 + 153^2 &= 5447556 + 1167 + 23409 \\ &= 5472132 + 1/16 \end{aligned}$$

The square root is 2339 1/4.

Part A, round 4

Now draw the angle bisector OG.



• From above, we have that $OA : FA > 2334\ 1/4 : 153$ and $OF : FA > 23391/4 : 153$

Add

• $OA : AG > 4673\ 1/2 : 153$

We're almost done. The original distance AC was 1/12 the perimeter of a circumscribed polygon, so we would multiply by 12 to get the ratio to the radius, but we want the ratio to the diameter so that gives a factor of 2 on the bottom for a total factor of 6.

There is an additional factor for the four "halvings" of $2^4 = 16$. Hence we obtain

$$153 \times 96 = 14688$$

and then invert to get the ratio of the circumference to the diameter:

$$\frac{14688}{4673\ 1/2} = 3 + \frac{668\ 1/2}{4673\ 1/2}$$

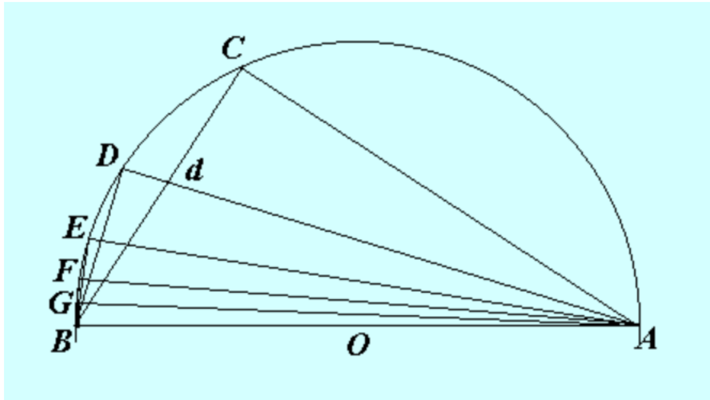
The fraction is just less than 1/7.

$1/7 = 0.142857$, while $668\ 1/2/4673\ 1/2 = 0.14304$.

We conclude that $\pi < 3\ 1/7$.

Part B

For Part B we use this diagram for an inscribed polygon.



As before $\triangle ABC$ is a $30-60-90$ right triangle.

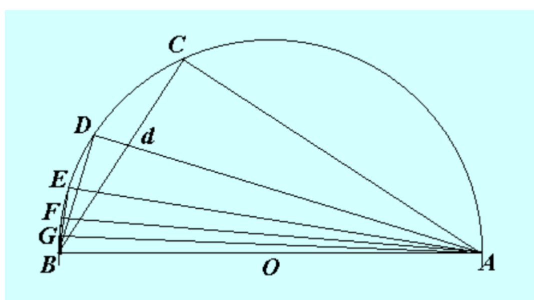
- $AC : BC < 1351 : 780$.

This ratio is an approximation for $\sqrt{3}$. It is an even better approximation than the previous one, and also, crucially, it is just slightly *more* than the true value, whereas 265/153 was slightly less.

Part B, round 1

Let AD bisect the angle, and then join BD.

- $\angle BAD = \angle dAC = \angle dBD$.



The first statement just restates the construction as an angle bisector. The second follows from the fact that the two angles have vertices on the circle and cut off the same arc.

As a consequence, $\triangle dBD \sim \triangle dAC$.

- $AD : DB < 2911 : 780$

Start with the similar triangles above and write three ratios of long side (not hypotenuse) to short side

$$AD : BD = BD : Dd = AC : Cd$$

Note: the source has $AB : Bd$ but this seems to be an error. That is a ratio of two hypotenuses and so is not equal to the others. As a result, I was unable to follow this part of the proof:

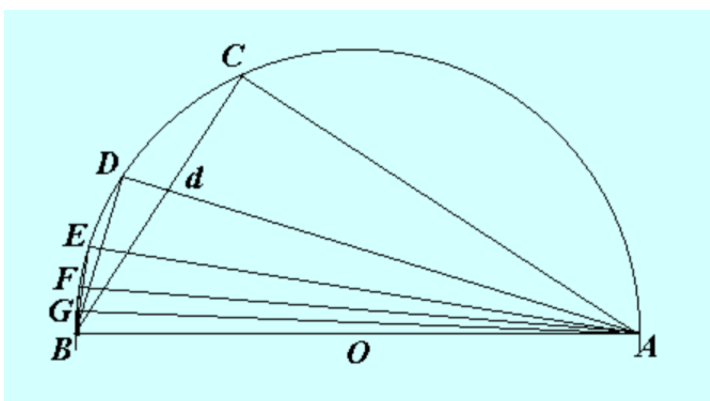
$$\begin{aligned} \boldsymbol{AD : BD = BD : Dd = AB : Bd} \\ \boldsymbol{= (AB + AC) : (Bd + Cd)} \\ \boldsymbol{= (AB + AC) : BC} \end{aligned}$$

or $\boldsymbol{(BA + AC) : BC = AD : DB.}$

However, I was able to prove the last statement

$$(AB + AC) : BC = AD : DB$$

The proof is as follows.



We have that $\triangle ABC$ is a right triangle and that AD and thus Ad is the angle bisector for $\angle BAC$. Therefore, we have by our favorite theorem that

$$(AB + AC) : BC = AC : Cd$$

We also have that $\triangle ABD$ is a right triangle and by virtue of the angle bisector construction, $\triangle ABD$ is similar to $\triangle ACd$. Therefore:

$$AC : Cd = AD : DB$$

These two lines combine to give the desired result.

The ratio $AD : DB$ is what we need going forward, and we get that from the other part: $(AB + AC) : BC$. We have that the cotangent $AC : BC < 1351 : 780$, and $AB : BC$ is the cosecant whose value is 2 so we multiply $780 \times 2 = 1560$, and then add 1351 to get the numerator of the result listed above.

- $AB : BD < 3013\ 3/4 : 780$

From the Pythagorean theorem: $AD^2 + BD^2 = AB^2$ so

$$AB^2 : BD^2 = AD^2 : BD^2 + 1$$

$AD : DB < 2911 : 780$ So we obtain $2911^2 = 8473921$ and $BD^2 = 608400$ so we have that

$$AB^2 : BD^2 = 9082321 : 608400$$

$$AB : BD < 3013\ 3/4 : 780$$

Part B, round 1 summary

Let's summarize what we did in round 1. We started with the cosecant and the cotangent for $\triangle ABC$, namely $AB : BC$ and $: AC : BC$.

We used the relationship $(AB + AC) : BC = AD : DB$ to obtain the cotangent of the bisected angle, and then we used the Pythagorean theorem in this form

$$AB^2 : BD^2 = AD^2 : BD^2 + 1$$

to get the cosecant from the cotangent. Thus we have

- $AD : DB < 2911 : 780$, the cotangent.
- $AB : BD < 3013\ 3/4 : 780$, the cosecant.

Part B, round 2

Now, let AE bisect the angle, and then join BE.

Rather than go through the geometry again, let's just substitute letters. First the cotangent

$$(AB + AD) : BD = AE : EB$$

Then the cosecant.

$$AB^2 : BE^2 = AE^2 : BE^2 + 1$$

For the first part we have $2911 : 780 + 3013\ 3/4 : 780 = 5924\ 3/4 : 780$.

We reduce the denominator to 240. This amounts to dividing by 3 1/4. 5924 3/4 divided by 3 1/4 is exactly equal to 1823.

- $AE : EB = 1823 : 240$, the cotangent.

For the second part we have the previous number squared and added to 1 and then take the square root. $1823^2 = 3323329; 240^2 = 57600$; so we have 3380929 and the square root is $< 1838\ 3/4$, but the source gives the fraction as a bit larger 1838 9/11.

- $AB : BE = 1838\ 9/11 : 240$, the cosecant.

Part B, round 3

Now, let AF bisect the angle, and then join BF. Substitute letters (carefully, looking at the diagram). First the cotangent

$$(AB + AE) : BE = AF : FB$$

Then the cosecant.

$$AB^2 : BF^2 = AF^2 : BF^2 + 1$$

For the first part we have $1838\ 9/11 : 240 + AE : EB = 1823 : 240 = 3661\ 9/11 : 240$.

We reduce the denominator, this time to 66. This amounts to multiplication by 11/40. So the numerator is multiplied by the same factor giving

- $AF : FB = 1007 : 66$, the cotangent.

For the second part we have the previous number squared and added to 1 and then take the square root.$1007^2 = 1014049; 66^2 = 4356$, so we have 1018405 and the square root is 1009 1/6.

- $AB : FB = 1009\ 1/6 : 66$, the cosecant.

Part B, round 4

Finally, let AG bisect the angle, and then join BG. Substitute letters. First the cotangent

$$(AB + AF) : BF = AG : GB$$

Then the cosecant.

$$AB^2 : BG^2 = AG^2 : BG^2 + 1$$

For the first part we have

- $AG : GB = 2016\ 1/6 : 66$, the cotangent.

Now do $2016^2 = 4064256; 66^2 = 4356$ so that's 4068612 and the square root is 2017 1/12 while the source gives

- $AB : GB < 2017\ 1/4 : 66$, the cosecant.

Almost done. The side BG is a side of an inscribed regular polygon of 96 sides. We multiply $66 \times 96 = 6336$ and compute the ratio of the inverse.

I am not sure how Archimedes came up with it, but it is easy to verify that the ratio which is less than π is greater than:

$$\frac{6336}{2017\ 1/4} > 3\ 10/71$$

We combine parts A and B to make our final statement that

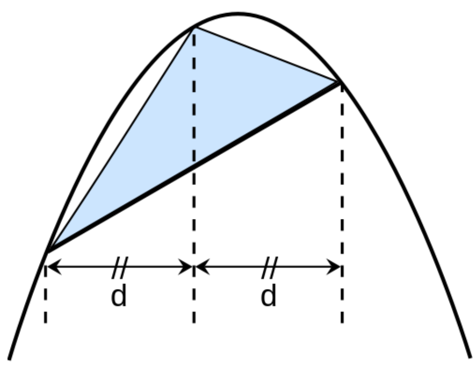
$$3\ 10/71 < \pi < 3\ 1/7$$

Chapter 56

Archimedes and quadrature

Let's talk about Archimedes, and parabolas.

Here is a figure from wikipedia, showing a parabola and a chord of the parabola, which might be drawn between any two points. A triangle is constructed from the chord in the following way: the point dividing the horizontal distance in half is found and that is used for the x-value of the third point.



The Greek genius Archimedes showed that the total area underneath the curve, between the two outside vertices of the triangle, is 4/3 times the area of the triangle shown in blue. The method he used is called the "quadrature of the parabola" and it is (from our modern perspective) a relatively simple though still revolutionary idea.

One very interesting consequence is that the slope of the tangent to the parabola at this midway point is equal to the slope of the chord.

The general equation of a parabola is

$$y = ax^2 + bx + c$$

But for any given parabola, we can translate it to the origin and the parabola at the origin with the same shape is

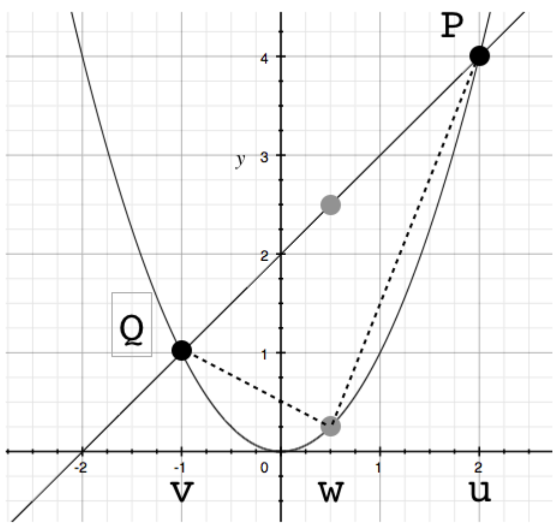
$$y = ax^2$$

This can be demonstrated by completing the square.

If we pick two points on the parabola at $x = u$ and $x = v$, then the corresponding coordinates are

$$P = (u,\ au^2)$$

$$Q = (v,\ av^2)$$



P is the right-hand point in the figure. Let us say that $au^2 > av^2$ and and the slope m of the chord that connects them is

$$m = \frac{au^2 - av^2}{u - v} = \frac{a(u^2 - v^2)}{u - v} = \frac{a(u - v)(u + v)}{u - v}$$

so

$$m = a(u + v)$$

We can see that this formula gives the correct answer for $u = -v$, since the slope at the vertex is 0. Now label the midpoint $x = w$

$$w = \frac{1}{2}(u + v)$$

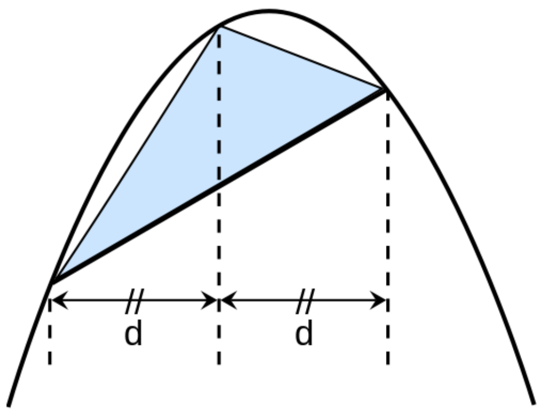
And the slope at w (from calculus) is

$$f'(w) = 2aw == 2a\ \frac{1}{2}(u+v) = a(u+v)$$

So the proposition is correct.

Quadrature

Another interesting thing about this figure is that the area of the triangle can be found from the length of the vertical coming down from the top.



If we simply turn the graph sideways in our mind, then the two small triangles share the part of this line within the blue region, which is their "base", b. And they both have "height" d, since w was chosen as half way between u and v, so their areas are equal, and the total area of the two together is

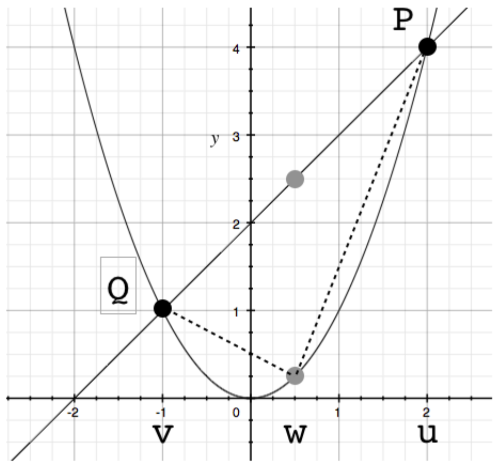
$$A = bd$$

We want to find an expression for the area only in terms of u and v (no b or d). Let's look at the second version of the figure again below.

To repeat, what we found above is that the slope at the point on the parabola corresponding to $x = w$ is equal to the slope of the line that connects v and u, and more important to us now, that the area of the combined triangle (vertices u, v, w) is

$$A = (u-w)\ b = \frac{1}{2}(u-v)\ b$$

where b is the distance parallel to the y-axis between the two points marked in gray.



The length of the "base" b is the average of the y-values for $x = u$ and $x = v$, minus aw^2.

$$b = \frac{1}{2}(au^2 + av^2) - aw^2$$

and from before

$$w = \frac{1}{2}(u + v)$$

so we have

$$b = \frac{1}{2}(au^2 + av^2) - a\ [\ \frac{1}{2}(u + v)\ ]^2$$

Factor out $a/4$

$$= \frac{1}{4}a\ [\ 2u^2 + 2v^2 - (u + v)^2\ ]$$

$$= \frac{1}{4}a\ [\ 2u^2 + 2v^2 - u^2 - 2uv - v^2\ ]$$

$$= \frac{1}{4}a\ [\ u^2 - 2uv + v^2\ ]$$

$$b = \frac{1}{4}a\ (u - v)^2$$

The area is

$$\boxed{A = bd = \frac{1}{8}a\ (u - v)^3} \qquad (56.1)$$

check

We'll check three cases to see if this makes sense. First if

$$u = v$$

then the area is zero and $w = u = v$, so that's good. Second, if

$$u = -v$$

then

$$A = \frac{1}{8}a\ (u - v)^3 = \frac{1}{8}a\ (2u)^3 = au^3$$

We compare this result with a direct computation by geometry. In the figure we have two symmetric triangles with individual area

$$\frac{1}{2}u\ au^2$$

The total area is twice that, so it checks. Finally, suppose we have $v = 0$

$$A = \frac{1}{8}a\ (u - v)^3$$

This one is harder to see, but we have that

$$d = \frac{1}{2}(u - v) = \frac{1}{2}u$$

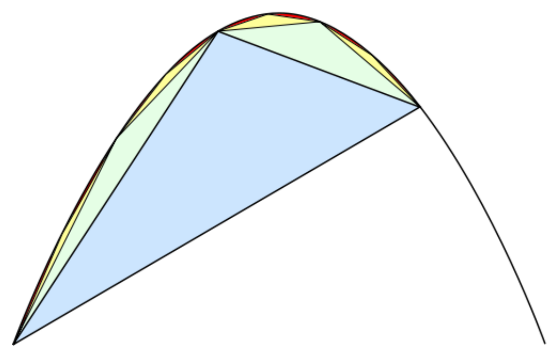
b is the distance between the average y-value which is $(1/2)au^2$ and $aw^2 = a(u/2)^2$

$$b = a\ (\frac{1}{2}u)^2 - \frac{1}{2}\ [\ au^2 - 0\ ]\ =\ \frac{1}{4}a\ u^2$$

$$A = bd = \frac{1}{8}au^3$$

so they all check.

Quadrature of the parabola



The reason for the whole preceding argument is this. The area formula is

$$A = bd = \frac{1}{8}a\ (u - v)^3 = k(u - v)^3, \quad k = const$$

It is solely a function of $u - v$. Suppose we draw two new triangles (in light green, above). For each of these triangles the distance between the new vertices is one-half what we had before. So everything that we have for the big blue triangle is also true for these two new ones, just adjusted by a factor of $u' - v' = (1/2)(u - v)$.

What this means is that the area of each light green triangle is in the ratio to the blue one of $(1/2)^3 = 1/8$. But there are two of these new triangles, so the new area we added is in the ratio $1/4$.

Suppose we do it again, constructing the yellow triangles. The new area of each is in the ratio $(1/4)^3 = 1/64$ but there are now 4 of these yellow triangles so the total area is in the ratio $1/16 = (1/4)^2$

If we call the area of the original triangle T, that of the blue plus the light green is

$$A = T + \frac{1}{4}T$$

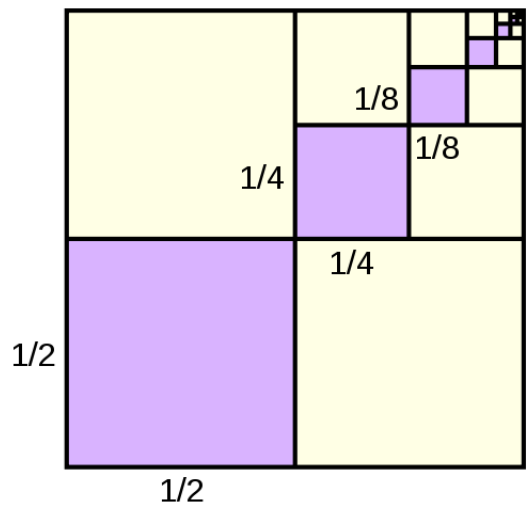
and with the addition of the yellow it is

$$A = T + \frac{1}{4}T + \frac{1}{16}T$$

so, as an infinite series it is

$$A = T(1 + \frac{1}{4} + \frac{1}{16} + \cdots)$$

Here is Archimedes' proof that the sum of this series (not counting the first term) is 1/3.



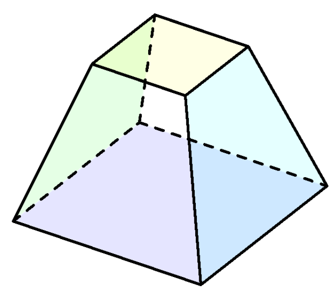
So the total is 4/3, and the complete area under the parabola is 4/3 the area of the triangle drawn as we described!

This called the "method of exhaustion", and not just because it's a lot of work.

Chapter 57

Frustum

A frustum is the (bottom) part of a larger pyramid or cone that remains after the original solid is cut by a horizontal plane and the upper, small pyramid or cone removed.



If we call the dimensions of the larger pyramid H (height) and B (base), then its volume is

$$V = \frac{1}{3}HB^2$$

Similarly, if we call the dimensions of the pyramid that has been removed h and b its volume is

$$V = \frac{1}{3}hb^2$$

The volume of the frustum is just the difference

$$V = \frac{1}{3}(HB^2 - hb^2)$$

If we're dealing with a cone rather than a pyramid, then replace B^2 with R^2 and b^2 with r^2 and multiply the whole thing by π.

alternative formula

However, there is another formula for the volume of the frustum, which is perhaps more interesting.

If we call the altitude or height of the frustum a, where $a = H - h$, this formula is

$$V = \frac{1}{3}a(B^2 + Bb + b^2)$$

$$V = \frac{1}{3}(H - h)(B^2 + Bb + b^2)$$

We'd like to derive this. The key insight here is that by similar triangles

$$\frac{b}{h} = \frac{B}{H}, \qquad h = \frac{b}{B}H$$

while

$$\begin{aligned} a &= H - h \\ &= H - \frac{b}{B}H \\ &= H(1 - \frac{b}{B}) \\ &= H(\frac{B - b}{B}) \end{aligned}$$

reverse proof

The proof proceeds easily in the reverse direction. Start with the answer:

$$V = \frac{1}{3}a(B^2 + Bb + b^2)$$

Substitute for a

$$V = \frac{1}{3}\ H(1 - \frac{b}{B})(B^2 + Bb + b^2)$$

Part of this simplifies dramatically

$$(1 - b/B)(B^2 + Bb + b^2)$$

$$= B^2 + Bb + b^2 - bB - b^2 - \frac{b^3}{B}$$

$$= B^2 - \frac{b^3}{B}$$

Hence we have that

$$V = \frac{1}{3}\ H(B^2 - -\frac{b^3}{B})$$

Multiplying out, the first term is $1/3\ HB^2$, as desired.

The second is

$$-\frac{1}{3}\ \frac{H}{B}b^3$$

Recall that $h = bH/B$ so this is just $-1/3\ hb^2$, and we're done.

□

forward proof

How would you proceed if you didn't know the answer? The formula we can easily work out is for the frustrum as the difference in volumes of two pyramids:

$$V = \frac{1}{3}\ (HB^2 - hb^2)$$

and it's not much of a stretch to substitute for $h = Hb/B$

$$V = \frac{1}{3}\ (HB^2 - \frac{H}{B}b^3)$$

$$= \frac{H}{3}\ (B^2 - \frac{b}{B}b^2)$$

and then since

$$a = H(1 - \frac{b}{B})$$

you notice the connection to the previous expression and imagine trying to factor out

$$(1 - \frac{b}{B})(p + q + r) = (B^2 - \frac{b}{B}b^2)$$

where p, q and r will need to be determined. Obviously, we will need two sets of cancellations.

We need a term of B^2

$$(1 - \frac{b}{B})(B^2 + q + r) = (B^2 - \frac{b}{B}b^2)$$

but from $(b/B) \cdot B^2$ we get a term $-bB$ that needs a corresponding term bB from somewhere.

Similarly we must have a term b^2 to generate $b/B \cdot b^2$ so

$$(1 - \frac{b}{B})(B^2 + q + b^2) = (B^2 - \frac{b}{B}b^2)$$

but then from the 1 we get a term b^2 that needs a corresponding $-b^2$.

Then the inspiration: bB gives us both of these things.

$$(1 - \frac{b}{B})(B^2 + bB + b^2) = (B^2 - \frac{b}{B}b^2)$$

I'm not saying it was easy!

Now we just pick up the $(1/3)H$

$$V = \frac{1}{3}H(1 - \frac{b}{B})(B^2 + bB + b^2)$$

and recall that

$$a = H(1 - \frac{b}{B})$$

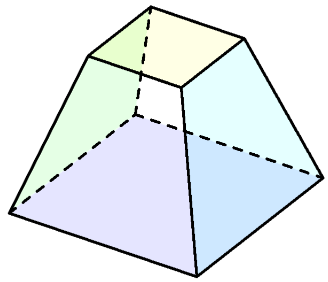
so

$$V = \frac{1}{3}a(B^2 + bB + b^2)$$

slant height

The slant height is the length of a cone or frustum along its outside edge. In the case of a cone, we can obtain it from the height and 1/2 the length of the base using the Pythagorean theorem.

For a frustum



consider the triangle containing an altitude down from the outside edge on the top.

The height of the triangle is just a, and the base has length $(B-b)/2$.

If the slant height is c then Pythagoras says that

$$c^2 = a^2 + (\frac{B-b}{2})^2$$

$$a = \sqrt{c^2 - (\frac{B-b}{2})^2}$$

Part XIV

More geometry

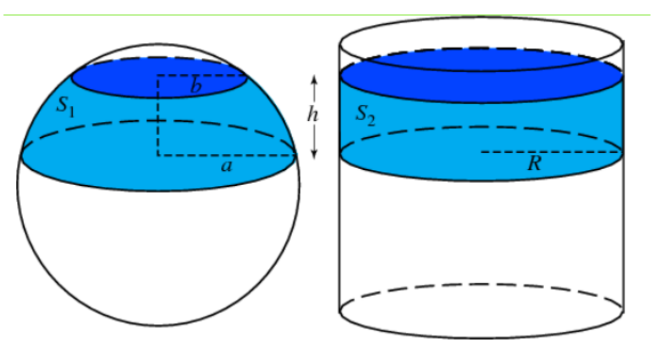
Chapter 58

Hatbox Theorem

A famous result of Archimedes, among many others, is called the hat-box theorem, which states that the surface area of a sphere is equal to the lateral surface area of a cylinder which just encloses it, as we had above. (Lateral area does not count the end pieces).

For a sphere and cylinder of radius R, the cylinder has surface area of the circumference $2\pi R$ times the height $2R$ for a total of $4\pi R^2$.

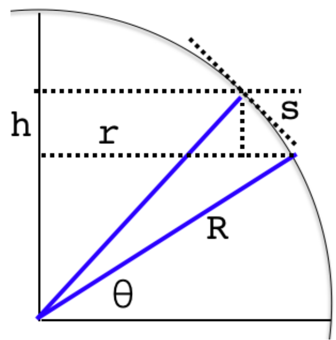
Archimedes showed that this is true not just for the whole, but for any slice or section through the sphere. That's pretty amazing.



Let's sketch a geometric proof briefly.

First, consider a thin strip of surface area extending around the sphere on a great circle (such as the equator). The surface area will be the circumference times the width of the belt, or $2\pi R \times h$.

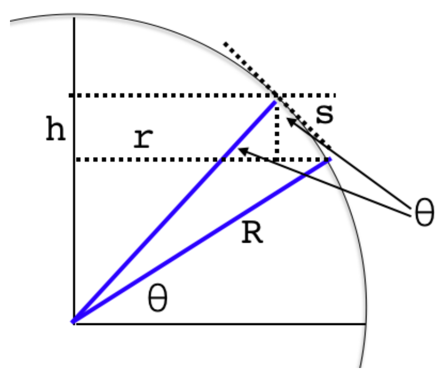
Second, this is true even for a belt that is not at the equator. Consider this figure:



What is the surface area contained between two horizontal cuts of the sphere along the dotted lines? Such a strip is called a "spherical belt". If the slice is very thin, then the circumference at the top and bottom of the slice will be approximately the same, with radius $r = R\cos\theta$.

To get the surface area, we must multiply the circumference $2\pi r$ by the width of the belt. The width is not the height h but s (called the slant height), because of the tilt of the surface.

For a very thin slice, the angle θ won't change much in going from the first blue radius R to the second one. Seeing this, it is then not hard to work out that the angle between s and h in the right triangle containing them both is equal to θ



so

$$\cos\theta = \frac{h}{s}$$

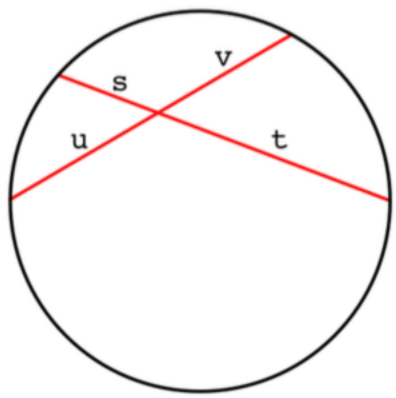
The area is

$$a = 2\pi r s = 2\pi R \cos\theta \frac{h}{\cos\theta} = 2\pi R h$$

The same as before.

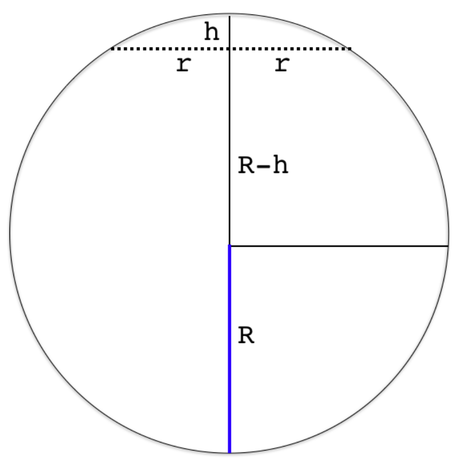
Third, consider the "belt" at the very top of the sphere. This region is more of a "spherical cap", like a contact lens or one of the poles of earth.

We recall a famous result concerning chords of a circle.



$$st = uv$$

This relationship holds for any two chords of the circle. In particular, it holds for the chord consisting of $r + r$ and the chord consisting of $h + (2R - h)$.



By the above theorem

$$r^2 = h(2R - h) = 2Rh - h^2 \approx 2Rh$$

Since h is small, we can neglect a factor of h^2.

The area of the circle is

$$a = \pi r^2 = 2\pi Rh$$

so we have the same rule as above.

Therefore, for *every* belt of height h, the area is $2\pi Rh$.

In summing up the contributions from each belt of width h_i

$$A = \sum a = \sum 2\pi Rh_i = 2\pi R \sum h_i$$

But $\sum h_i$ is just equal to R so we have

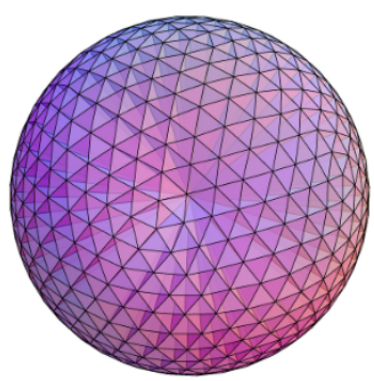
$$A = 2\pi R^2$$

The surface area of a hemisphere of radius R is twice the area of its great circle. Thus, the total area of the sphere is $4\pi R^2$.

geometric proof

Here is a geometric proof that assumes we know the volume of the sphere is

$$V = \frac{4}{3}\,\pi R^3$$



Divide the whole sphere up into triangular prisms. Each one has volume $dV = 1/3\ R\ dA$. So for the whole thing the volume =

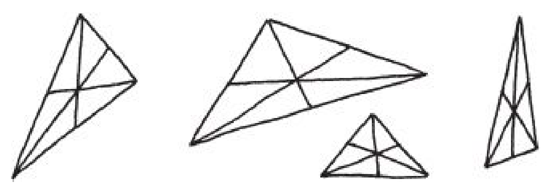
$$\frac{4}{3}\ \pi R^3 = \frac{1}{3} R\ A$$

$$A = 4\pi R^2$$

Chapter 59

Ceva's theorem

Ceva's Theorem says that if we start with a triangle and draw line segments connecting each vertex with the midpoint of the opposite side, then the three line segments cross at a single, unique point. The question is: how do we know this? We can obviously draw a line to the midpoint of the opposing side from two vertices, and these will cross at some point.

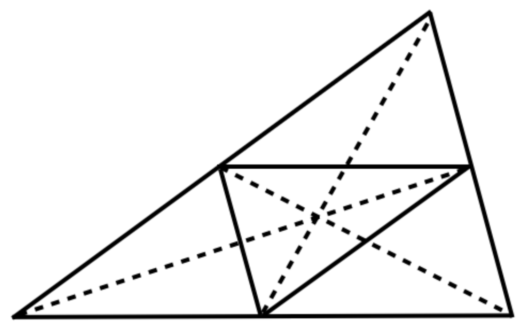


Then, we can either draw to the midpoint of the opposing side from the third vertex, and ask whether that line goes through the central point, or we can draw a line to the point and extend it to the opposing side, and then ask whether it bisects the side. That's the question. We will show that the answer is yes.

Furthermore, it is possible to show that for any of these line segments, this point (called the *centroid*) lies one-third of the length from the side, and two-thirds of the length from the vertex. We will use calculus to compute the centroid later in the book (and get the same answer).

In other write-ups I've shown one proof of this using similar triangles. The first approach shown here is one outlined in Lockhart's book *Measurement*. And the second one uses vectors.

Lockhart's proof



The idea is to connect the midpoints of the sides. If we do that, the construction results in four smaller triangles. It is easy to show that these triangles are congruent, and are similar to the large one we started with. (By similar triangles, a short side opposite a midpoint/vertex is parallel to the side containing the midpoint. I'm sure you can finish the proof).

Because of the congruent triangles, we also have three congruent parallelograms, and these have rotational symmetry around the centroid. Therefore the centroid is a single point.

If you don't like that argument, notice that the dotted lines play the same role for each of the small triangles, extending from a vertex to the midpoint of the opposite side.

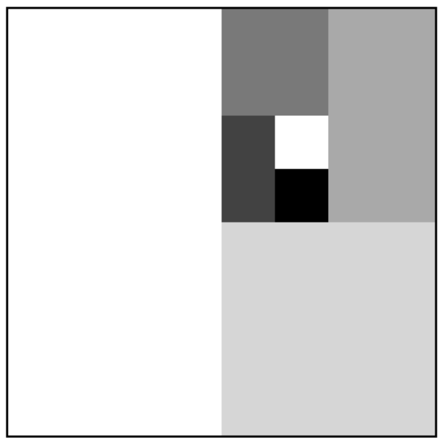
What this means is that *if* the centroid is a single point, then centroids of the larger triangle and the small central triangle are the same point. But we can just continue in the same way, inscribing a new, even smaller, triangle, using the midpoints of the small central triangle, and this process can be extended *ad infinitum*. Hence, in the limit, we will reach a single point.

We can locate the centroid by imagining that we find successive midpoints of a length from opposite ends left and right. The first point is at $1/2$ of the length (from the left), the second comes back from 1 by $1/4$ so is at 0.75 (at the right), the third is at $0.5 + 1/8$ (from the left), so every second round we get closer to the centroid by advancing from the left by

$$S = \frac{1}{2} + \frac{1}{8} + \frac{1}{32} + \dots$$

Now, we can either assume this sum is finite (for now) or recognize that it is certainly smaller than

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots = 1$$



So if

$$S = \frac{1}{2} + \frac{1}{8} + \frac{1}{32} + \dots$$

then

$$2S = 1 + \frac{1}{4} + \frac{1}{16} + \dots$$

and

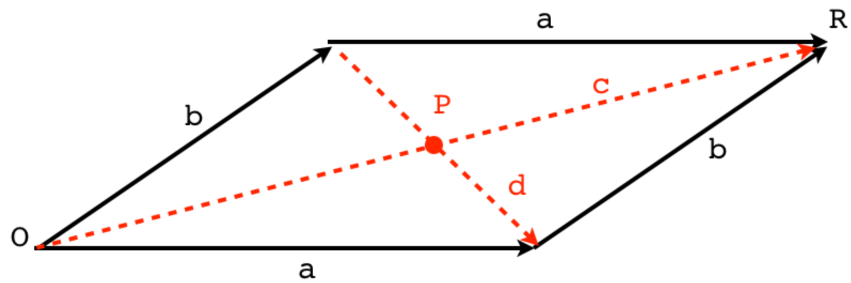
$$3S = 1S + 2S = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$$

That is, $3S = 1 + 1$, so $S = 2/3$.

using vectors

Ceva's Theorem says that if we start with a triangle and draw the line segments connecting each vertex with the midpoint of the opposite side, the three line segments cross at a single, unique point. Furthermore, it is possible to show that for any of these line segments, the *centroid* lies one-third of the length from the side, and two-thirds of the length from the vertex.

Using vectors makes everything particularly simple. As a warmup, let's start by looking at the midpoint of the diagonals for a parallelogram including the triangle of interest. We will prove that the two diagonals cross at their mid-points (at P).



by construction:

$$\mathbf{c} = \mathbf{a} + \mathbf{b}$$

$$\mathbf{b} + \mathbf{d} = \mathbf{a} \;\Rightarrow\; \mathbf{d} = \mathbf{a} - \mathbf{b}$$

Let's define P as the point we reach by going halfway along $\mathbf{c}$

$$\mathbf{c}/2 = (\mathbf{a} + \mathbf{b})/2$$

What we need to show is that if we do

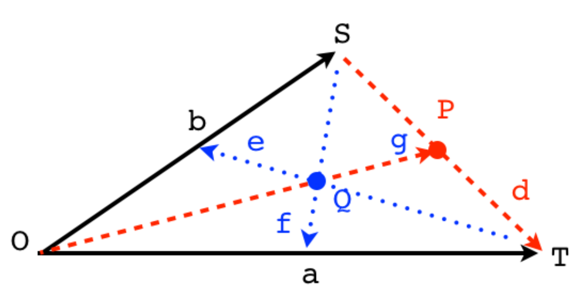
$$\mathbf{b} + \mathbf{d}/2$$

we arrive at P. Since $\mathbf{d} = \mathbf{a} - \mathbf{b}$

$$\mathbf{b} + \mathbf{d}/2 = \mathbf{b} + (\mathbf{a} - \mathbf{b})/2 = (\mathbf{a} + \mathbf{b})/2$$

□

Vectors make that pretty easy. We will use this result below. Now, here is the triangle.



Construct a line to the midpoint of the opposing side

$$\mathbf{b} + \mathbf{f} = \mathbf{a}/2 \;\Rightarrow\; \mathbf{f} = \mathbf{a}/2 - \mathbf{b}$$

$$\mathbf{a} + \mathbf{e} = \mathbf{b}/2 \;\Rightarrow\; \mathbf{e} = \mathbf{b}/2 - \mathbf{a}$$

Refer to the first diagram for $\mathbf{c} = (\mathbf{a} + \mathbf{b})$. The halfway point is:

$$\mathbf{a}/2 + \mathbf{b}/2 = \mathbf{g}$$

By the property proved in the first part, we know that this bisects the side $\mathbf{d}$.

It makes it a little easier when we know that Q is two-thirds of the way along the line segment. We have three paths to move to Q. The first two are constructed using the property that the vector bisects the opposing side.

from S:

$$\mathbf{b} + \frac{2}{3}\mathbf{f} = \mathbf{b} + \frac{2}{3}(\mathbf{a}/2 - \mathbf{b}) = (\mathbf{a} + \mathbf{b})/3$$

from T:

$$\mathbf{a} + \frac{2}{3}\ \mathbf{e} = \mathbf{a} + \frac{2}{3}\ (\mathbf{b}/2 - \mathbf{a}) = (\mathbf{a} + \mathbf{b})/3$$

from O:

$$\frac{2}{3}\mathbf{g} = \frac{2}{3}\ \frac{1}{2}\mathbf{c} = (\mathbf{a} + \mathbf{b})/3$$

□

By moving two-thirds of the way along $\mathbf{g}$, one-third of the way along $\mathbf{c}$, we arrive at the same point defined to be two-thirds of the way along the bisectors of opposing sides drawn from S and T.

factor

How would we find the factor of 2/3 if we didn't already know? Here is one way. Call that unknown factor r. By symmetry it is the same for all three lines.

$$r\ (\mathbf{a} + \mathbf{b})/2 + (1 - r)\ \mathbf{e} = \mathbf{b}/2$$

$$r\ (\mathbf{a} + \mathbf{b})/2 + (1 - r)\ (\mathbf{f} = \mathbf{a}/2$$

Substitute for $\mathbf{e}$ and $\mathbf{f}$:

$$r\ (\mathbf{a} + \mathbf{b})/2 + (1 - r)\ (\mathbf{b}/2 - \mathbf{a}) = \mathbf{b}/2$$

$$r\ (\mathbf{a}+\mathbf{b})/2+(1-r)\ (\mathbf{a}/2-\mathbf{b})=\mathbf{a}/2$$

add

$$r\ (\mathbf{a}+\mathbf{b})+(1-r)\ (-\mathbf{b}/2-\mathbf{a}/2)=\mathbf{b}/2+\mathbf{b}/2$$

$$\frac{3}{2}\ r\ (\mathbf{a}+\mathbf{b})=\mathbf{a}+\mathbf{b}$$

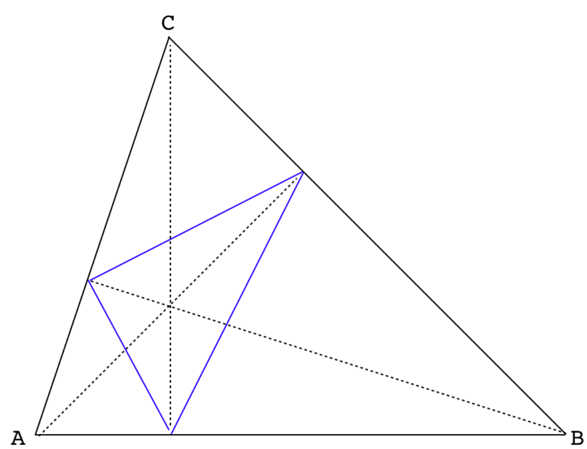
$$r=2/3$$

Chapter 60

Orthocenter

An altitude of a triangle is a line extended from a vertex so as to form a right angle with the opposing side. The orthocenter is the point where the three altitudes of a triangle meet.

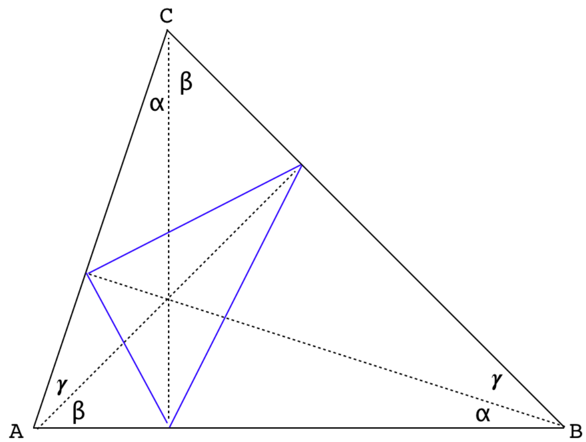
Assume for now that the three altitudes *do* meet at a single point, we will come back to this question later.



The altitudes are drawn as dotted lines in the figure above. The points where these lines meet their opposite sides have also been connected, forming another triangle outlined in blue.

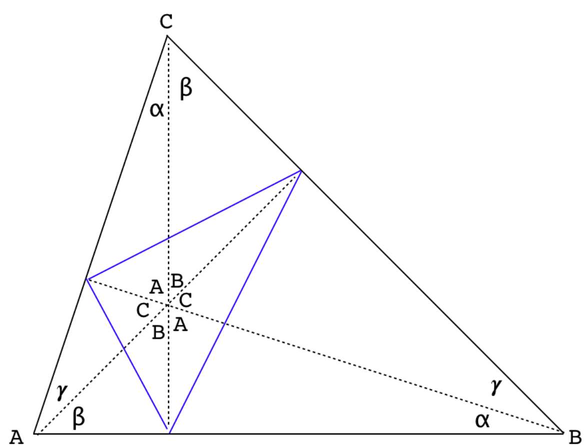
The same construction for the centroid, formed from lines that bisect the opposing sides, gives four small triangles which are all congruent. In the case of the orthocenter, we will show that the three outer triangles are similar (t hough obviously not congruent), while the one in the center is different.

The first observation is that the angles A, B, and C are divided by the altitudes into two parts with the measures repeated as shown below.



Proof: there are two right triangles formed that include A as a base angle. The corresponding complementary angles must be equal, and these are labeled α. A similar argument gives β and γ.

Switching our attention to the central angles, we can show that these have measures equal to the vertices.

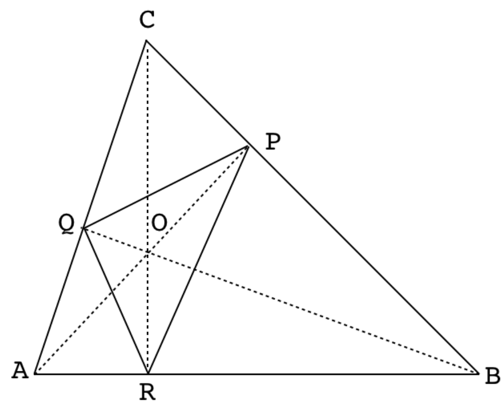


The argument builds on the one above, each central angle is part of a right triangle with one of α, β, or γ as the complementary angle.

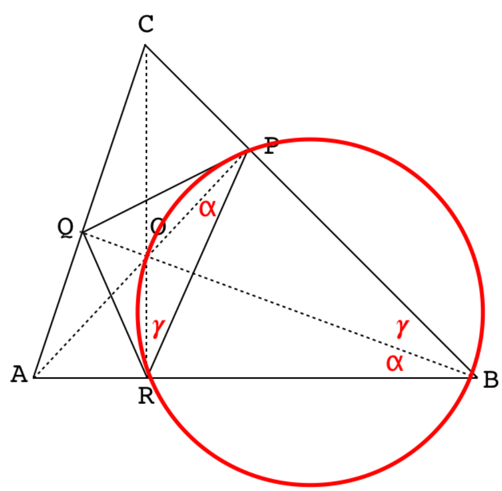
angle bisectors

The next fact we need is that the altitudes are angle bisectors for the inscribed triangle.

Here is a neat and quick proof from Courant and Robbins.

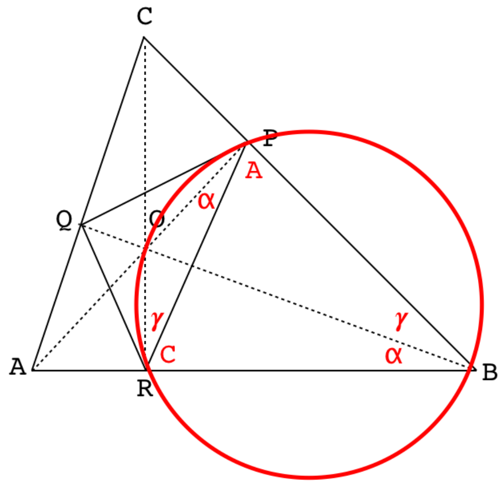


Label the vertices of the altitudes as P, Q, R and the orthocenter as O. Since $\angle OPB$ is a right angle and so is $\angle ORB$, the quadrilateral $OPBR$ containing both can be inscribed into a circle with OB as the diameter.

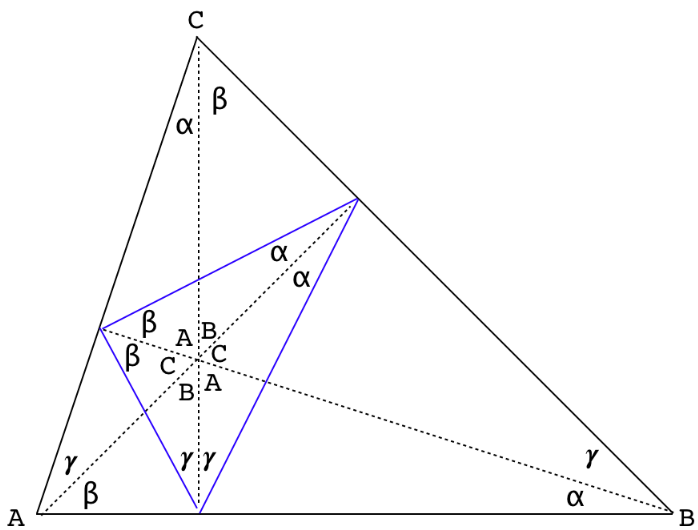


Now, we use the theorem that if two angles on the circumference of a circle sweep out the same arc, they are equal to each other This allows labeling of α and γ as shown.

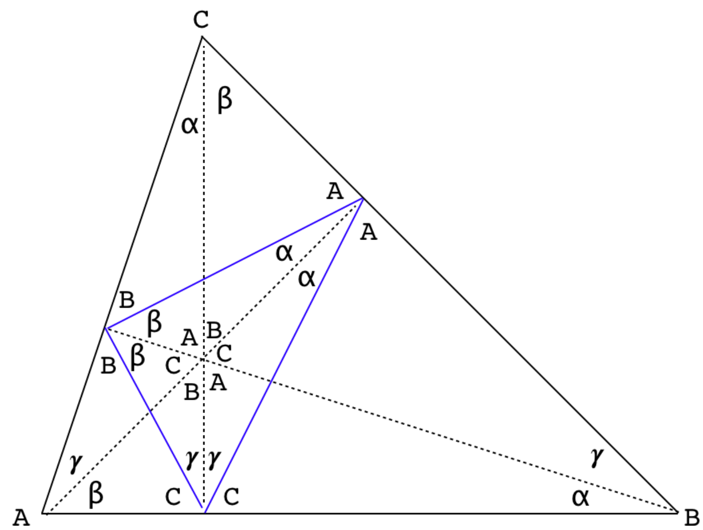
Then, since A is the complement of α, and C is the complement of γ, we obtain



We could draw two more circles, but instead just invoke symmetry:



So finally, we have



Thus, we have established that the altitudes are angle bisectors for the included triangle, and that the three small outer triangles are congruent, by AAA.

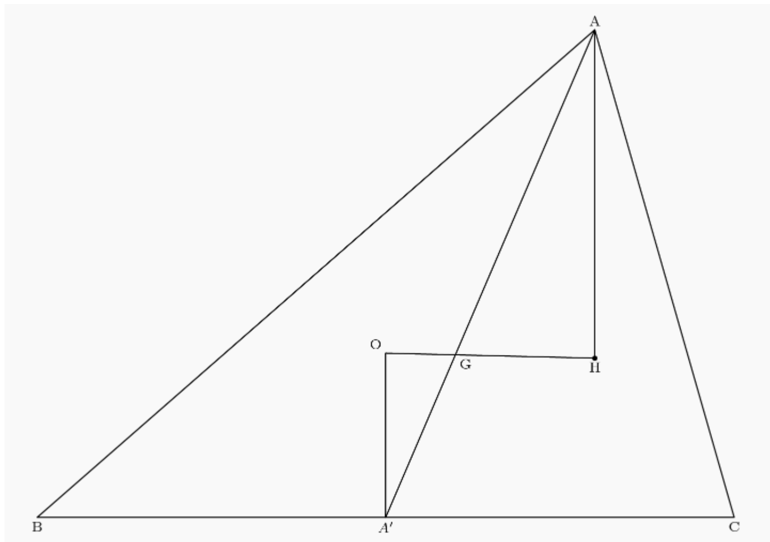
Orthocenter exists

For the above derivation, we assumed that the three altitudes do indeed meet at a single point. This is a direct consequence of Ceva's Theorem, which we've seen before.

Below we give an alternative proof, due to Euler, which is stunning, following

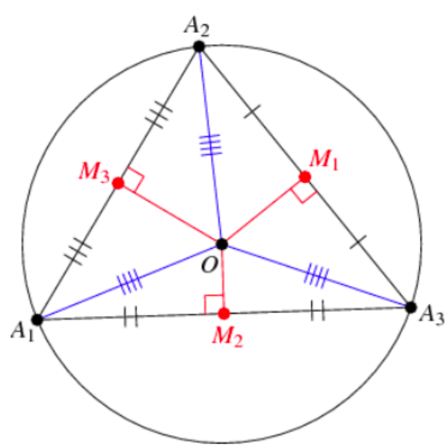
`https://artofproblemsolving.com/wiki/index.php/Orthocenter`

Borrowing their figure:



The orientation is reversed from what we had above. First, the point O is the circumcenter of the triangle: the center of the circle which contains all three vertices of the triangle.

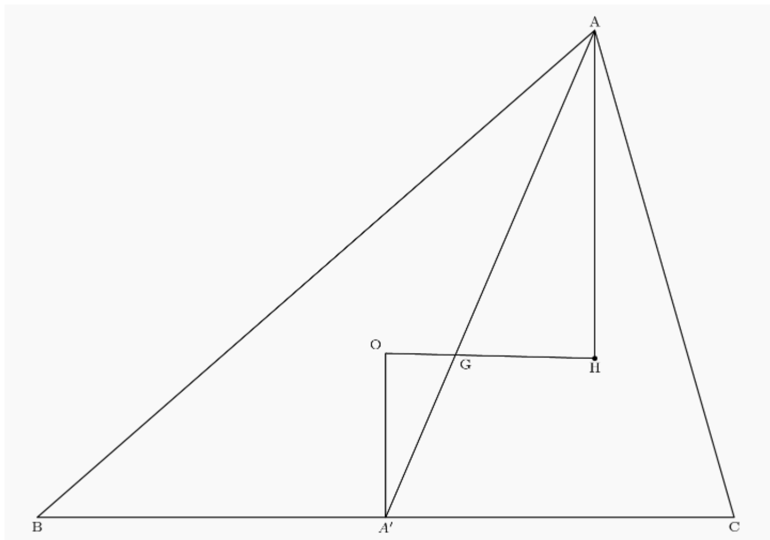
Clearly, this circle has a center. The classic construction is to bisect each side (here BC is bisected at A'), and erect a perpendicular. The point where the three perpendiculars cross is the circumcenter, which is the center of the circle.



So, assume we have done this and that point is O.

The next point, G, is the centroid. One way to find this point is to draw all three lines connecting vertices with the midpoints of the opposite side (AA'). However, if you recall, the distance from the vertex A to G is twice the distance from the midpoint A' to G. Hence we draw point G using arithmetic.

Now, extend OG by twice its length, to H. ($2OG = GH$).



Because AG is twice $A'G$ and GH is twice OG and the two triangles share both angle $\angle OGA'$ (equal to $\angle AGH$), they are similar triangles.

Since $\angle A'OG$ is a right angle, therefore so is $\angle AHG$. This means that AH is perpendicular to BC. Thus, AH is a part of the altitude from A to BC (the whole altitude is not shown).

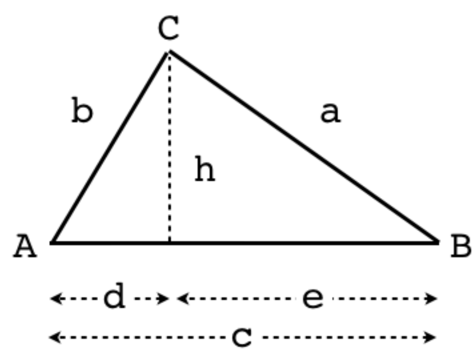
The same construction could be done for the other two vertices, each time ending at H. This shows that H is unique, and that H is on all three altitudes.

This proof also demonstrates that the orthocenter, centroid and circumcenter lie on a single line, and that the distance from centroid to orthocenter is twice that from centroid to circumcenter.

Chapter 61

Heron's formula

Heron's Formula can be used to compute the area of a triangle from the lengths of its sides. It is a simple formula that does not explicitly include the altitude h or the parts of side c.



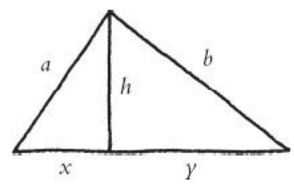
If s is the semi-perimeter

$$s = \frac{1}{2}(a + b + c)$$

then

$$A = \sqrt{s + (s - a) + (s - b) + (s - c)}$$

Lockhart's version



Here the triangle is labeled slightly differently than the one above. The bottom side c is split into x and y. We can write three equations:

$$x^2 + h^2 = a^2$$
$$y^2 + h^2 = b^2$$
$$x + y = c$$

Our ultimate objective is an equation that contains only a, b and c. Lockhart gives us a target for the first part of the derivation:

$$2xc = c^2 + a^2 - b^2$$

Let's just start manipulating equations to get there. Subtract the second from the first:

$$x^2 - y^2 = a^2 - b^2$$

Square the third

$$x^2 + 2xy + y^2 = c^2$$

Add the two new equations

$$2x^2 + 2xy = c^2 + a^2 - b^2$$

Substitute for y

$$2x^2 + 2x(c - x) = c^2 + a^2 - b^2$$
$$2xc = c^2 + a^2 - b^2$$

Finally a slight rearrangement:

$$x = \frac{c^2 + a^2 - b^2}{2c} = \frac{c}{2} + \frac{a^2 - b^2}{2c}$$

This says that to find the point where c is divided into x and y, we move from the center $c/2$ a distance of $(a^2 - b^2)/2c$.

The corresponding equation for y is

$$y = \frac{c}{2} - \frac{a^2 - b^2}{2c}$$

which is easily checked by adding together the final two equations, obtaining $x+y = c$.

For the area, we will need h somehow. It is easier to use h^2.

$$\begin{aligned} h^2 &= a^2 - x^2 \\ &= a^2 - \frac{(c^2 + a^2 - b^2)^2}{(2c)^2} \end{aligned}$$

The area squared is

$$\begin{aligned} A^2 &= \frac{1}{4}c^2h^2 \\ &= \frac{1}{4}c^2a^2 - \frac{1}{4}c^2\frac{(c^2 + a^2 - b^2)^2}{(2c)^2} \end{aligned}$$

Lockhart:

> the algebraic form of this measurement is aesthetically unacceptable. First of all, it is not symmetrical; second, it's hideous. I simply refuse to believe that something as natural as the area of a triangle should depend on the sides in such an absurd way. It must be possible to rewrite this ridiculous expression...

Here's a start:

$$16A^2 = (2ac)^2 - (c^2 + a^2 - b^2)^2$$

This is much better, notice that we have only a, b and c.

We will now go through two difference of squares manipulations. First

$$\begin{aligned} 16A^2 &= [\ 2ac + (c^2 + a^2 - b^2)\ ]\ [\ 2ac - (c^2 + a^2 - b^2)\ ] \\ &= [\ (a + c)^2 - b^2)\ ]\ [\ b^2 - (a - c)^2\ ] \\ &= (a + c + b)(a + c - b)(b + a - c)(b - a + c) \end{aligned}$$

So

$$A = \sqrt{\frac{a+b+c}{2} \cdot \frac{a+c-b}{2} \cdot \frac{a+b-c}{2} \cdot \frac{-a+b+c}{2}}$$

At this point, we recognize the semi-perimeter $s = (a+b+c)/2$ and then we see that each of the other terms is s minus one of the sides

$$A = \sqrt{s \cdot (s-a) \cdot (s-b) \cdot (s-c)}$$

check

As a simple example, if we have a right triangle with sides 3,4,5, then the area is one-half of 3 times 4 = 6. The semi-perimeter is s

$$s = \frac{(3+4+5)}{2} = \frac{12}{2} = 6$$

We have

$$A = \sqrt{6(6-5)(6-4)(6-3)} = \sqrt{6(1)(2)(3)} = 6$$

Chapter 62

Euclid's formula

Pythagorean triples

The simplest right triangle with integer sides is $3, 4, 5$:

$$3^2 + 4^2 = 5^2$$

any multiple n will work

$$(3n)^2 + (4n)^2 = (5n)^2$$

but that's not so interesting. The triples which are not multiples of another triple are called *primitive*.

There is a small table of triples in this discussion of Euclid X:29 by Joyce:

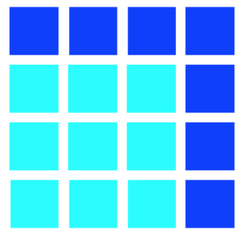
https://mathcs.clarku.edu/~djoyce/elements/bookX/propX29.html

	1	3	5	7	9	11	13
3	3 : 4 : 5						
5	5 : 12 : 13	15 : 8 : 17					
7	7 : 24 : 25	21 : 20 : 29	35 : 12 : 37				
9	9 : 40 : 41	27 : 36 : 45	45 : 28 : 53	63 : 16 : 65			
11	11 : 60 : 61	33 : 56 : 65	55 : 48 : 73	77 : 36 : 85	99 : 20 : 101		
13	13 : 84 : 85	39 : 80 : 89	65 : 72 : 97	91 : 60 : 109	117 : 44 : 125	143 : 24 : 145	
15	15 : 112 : 113	45 : 108 : 117	75 : 100 : 125	105 : 88 : 137	135 : 72 : 153	165 : 52 : 173	195 : 28 : 197

We can explain the first column from Joyce

$$(3, 4, 5) \quad (5, 12, 13) \quad (7, 24, 25) \quad (9, 40, 41)$$

using this graphic



$$n^2 + (2n+1) = (n+1)^2$$

where $2n+1$ is the count of dark blue squares in the top column plus the rightmost row.

Of course, this is just basic algebra. Now, if that odd number is also a perfect square then we have that

$$2n+1 = a^2$$

so

$$(n+1)^2 = n^2 + a^2$$

Every odd number squared gives an odd perfect square:

$$3^2 = 9$$
$$5^2 = 25$$
$$7^2 = 49$$

So every odd number (≥ 3) is the basis for one of the entries, and the other two values can be computed as

$$b = \frac{a^2 - 1}{2}, \qquad c = b + 1$$

We can also explain the first diagonal

$$(8, 15, 17) \quad (12, 35, 37) \quad (16, 63, 65) \quad (20, 99, 101)$$

The first value is $4n$ for $n = 2, 3, 4 \ldots$.

The other two values are $4n^2 \pm 1$. This works because

$$(4n^2+1)^2 = (4n^2-1)^2 + (4n)^2$$

The fourth powers cancel and the ones cancel and we have

$$8n^2 = -8n^2 + 16n^2$$

which is correct.

Here is one set of primes I generated by brute-force search in Python with the constraints that either $a \leq 50$ or $b \leq 50$, no common factor, and none of the squares larger than 500.

```
  3   4   5      5  12  13      7  24  25      8  15  17
  9  40  41     11  60  61     12  35  37     13  84  85
 15 112 113     16  63  65     17 144 145     19 180 181
 20  21  29     20  99 101     21 220 221     23 264 265
 24 143 145     25 312 313     27 364 365     28  45  53
 28 195 197     29 420 421     31 480 481     32 255 257
 33  56  65     36  77  85     36 323 325     39  80  89
 40 399 401     44 117 125     44 483 485     48  55  73
```

opposite parity

Euclid gave a formula for Pythagorean triples, which we will explain since he's so important in this book.

First of all, a and b are of opposite parity (one odd and one even) since if they were both even, then c would be even so the triple would not be primitive. They cannot both be odd either because then c would be even and

$$(2x+1)^2 + (2y+1)^2 = (2z)^2$$

$$4x^2 + 4x + 1 + 4y^2 + 4y + 1 = 4z^2$$

The right-hand side is divisible by 4 but the left is not. This is impossible.

Let a be the odd value and b the even one.

$$a^2 + b^2 = c^2$$

$$a^2 = c^2 - b^2 = (c+b)(c-b)$$

perfect squares

If we look at examples from above, say

$$15^2 = (17+8)(17-8) = 25 \cdot 9$$

$$33^2 = (65+56)(65-56) = 121 \cdot 9$$

$$55^2 = (73+48)(73-48) = 121 \cdot 25$$

The two terms on the right are perfect squares. This appears to always be true.

co-prime

Next, $(c+b)$ and $(c-b)$ must be co-prime.

If d were to divide both $(c+b)$ and $(c-b)$, then d^2 must divide a^2 and so d divides a, but this doesn't help because we don't know about b and c individually ($3|9$ but not either 2 or 7).

Silverman argues that if d were to divide both $(c+b)$ and $(c-b)$ then since

$$(c+b) + (c-b) = 2c$$

$$(c+b) - (c-b) = 2b$$

d must divide both $2b$ and $2c$. And since b and c are co-prime, d must divide 2 or 1. But d also divides a^2 and a^2 is odd, so $d \neq 2$. Hence $d = 1$.

conclusion

The crucial observation is that if $(c+b)$ and $(c-b)$ are co-prime and their product is square, they must themselves be perfect squares.

$$s^2 = c+b, \qquad t^2 = c-b$$

$$\frac{s^2+t^2}{2} = c, \qquad \frac{s^2-t^2}{2} = b$$

$$a^2 = s^2t^2$$

$$a = st$$

Using the formula

What we have derived above is often written slightly differently. Let us rewrite this, but remembering that b is even and a is odd:

For every integer m, n, with $m > n$, a Pythagorean triple is given by

$$a = m^2 - n^2 \qquad b = 2mn \qquad c = m^2 + n^2$$

This works because

$$(m^2 - n^2)^2 + (2mn)^2 = (m^2 + n^2)$$

Canceling the fourth powers, we have

$$-2m^2n^2 + 4m^2n^2 = 2m^2n^2$$

So c is a sum of squares. For the first column c is:

$$2^2 + 3^2, \quad 3^2 + 4^2, \quad 4^2 + 5^2$$

a the difference, which is 1. For the second c is

$$1^2 + 4^2, \quad 1^2 + 6^2, \quad 1^2 + 8^2, \quad 1^2 + 10^2$$

Clearly a number of patterns will be found.

This is discussed further in a separate chapter **here**.

https://en.wikipedia.org/wiki/Pythagorean_triple#Enumeration_of_primitive_Pythagorean_triples

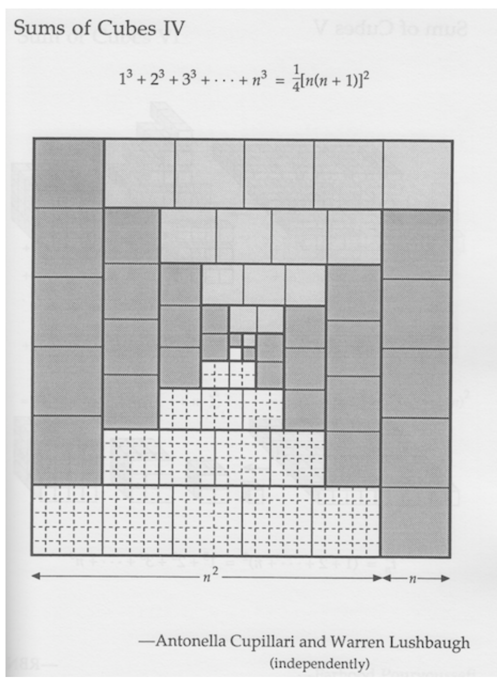
A thousand years before Pythagoras, the Babylonians knew the triple $4601, 4800, 6649$. It seems unlikely that they found this by random search.

Chapter 63

Sum of cubes

The formula is

$$\sum_{k=1}^{n} k^3 = [\ \sum_{k=1}^{n} k\ ]^2$$



Let's prove this using induction.

The "base case" is pretty simple. For $n = 2$

$$1^3 + 2^3 = 1 + 8 = 9$$

and

$$\frac{n^2(n+1)^2}{2^2} = \frac{2^2(3^2)}{2^2} = 3^2 = 9$$

Now for the induction step what we need to show is that what we get assuming the formula for n is correct and then adding the term $(n+1)^3$

$$\frac{n^2(n+1)^2}{2^2} + (n+1)^3$$

is equal to what we get by plugging $n+1$ into the formula.

$$\frac{(n+1)^2(n+2)^2}{2^2}$$

We need to show that eqn 2 is equal to eqn 3.

$$\frac{n^2(n+1)^2}{2^2} + (n+1)^3 = \frac{(n+1)^2(n+2)^2}{2^2}$$

First, we can factor out and cancel $(n+1)^2$ from both sides. So then we have

$$\frac{n^2}{2^2} + (n+1) \stackrel{?}{=} \frac{(n+2)^2}{2^2}$$

$$n^2 + 4(n+1) \stackrel{?}{=} (n+2)^2$$

That looks correct!

$\square$

derivation by collapsing sum

We proceed exactly as before

$$(k+1)^4 = k^4 + 4k^3 + 6k^2 + 4k + 1$$

Sum each term from $k = 1 \to k = n$

$$\sum_{k=1}^{n}(k+1)^4 = \sum_{k=1}^{n} k^4 + \sum_{k=1}^{n} 4k^3 + \sum_{k=1}^{n} 6k^2 + \sum_{k=1}^{n} 4k + \sum_{k=1}^{n} 1$$

Rearrange and compute the collapsing sum.

$$\sum_{k=1}^{n}(k+1)^4 - \sum_{k=1}^{n} k^4 = \sum_{k=1}^{n} 4k^3 + \sum_{k=1}^{n} 6k^2 + \sum_{k=1}^{n} 4k + \sum_{k=1}^{n} 1$$

$$(n+1)^4 - 1 = \sum_{k=1}^{n} 4k^3 + \sum_{k=1}^{n} 6k^2 + \sum_{k=1}^{n} 4k + \sum_{k=1}^{n} 1$$

Substitute for the right-hand sum

$$(n+1)^4 - 1 = \sum_{k=1}^{n} 4k^3 + \sum_{k=1}^{n} 6k^2 + \sum_{k=1}^{n} 4k + n$$

Rearrange some more

$$\sum_{k=1}^{n} 4k^3 = (n+1)^4 - 1 - \sum_{k=1}^{n} 6k^2 - \sum_{k=1}^{n} 4k - n$$

Expand the term $(n+1)^4$ and pick up the $-1-n$:

$$\begin{gathered}(n+1)^4 - 1 - n \\ = n^4 + 4n^3 + 6n^2 + 4n + 1 - 1 - n \\ = n^4 + 4n^3 + 6n^2 + 3n\end{gathered}$$

Factor out an n

$$= (n)(n^3 + 4n^2 + 6n + 3)$$

And another $n+1$

$$= (n)(n+1)(n^2+3n+3)$$

Recall our previous results:

$$\sum_{k=1}^{n} 6k^2 = 6\sum_{k=1}^{n} k^2$$
$$= 6\ \frac{n(n+1)(2n+1)}{6}$$
$$= n(n+1)(2n+1)$$

Similarly

$$\sum_{k=1}^{n} 4k = 4\sum_{k=1}^{n} k$$
$$= 4\ \frac{n(n+1)}{2}$$
$$= 2n(n+1)$$

Substitute all three of these results (and pull out the factor of 4 from the sum):

$$4\sum_{k=1}^{n} k^3 = (n)(n+1)(n^2+3n+3) - n(n+1)(2n+1) - 2n(n+1)$$

Just a bit more algebra. See that we have $n(n+1)$ in each term. We have

$$= n(n+1)\ [\ (n^2+3n+3) - (2n+1) - 2\ ]$$
$$= n(n+1)\ [\ n^2+3n+3-2n-1-2\ ]$$
$$= n(n+1)\ [\ n^2+n\ ]$$
$$= n(n+1)\cdot n(n+1)$$

So all together we have

$$4\sum_{k=1}^{n} k^3 = n(n+1)\cdot n(n+1)$$

$$\sum_{k=1}^{n} k^3 = \frac{n(n+1)}{2}\cdot\frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} k^3 = \ [\ \frac{n(n+1)}{2}\ ]^2$$

A remarkable simplification!

Looking deeper

$$\sum_{k=1}^{n} k^3 = [\ \sum_{k=1}^{n} k\ ]^2$$

We want to try to understand something more about why this is true.

A web search revealed the answer. Here's an interesting pattern for the cubes of integers

$$1^3 = 1$$

$$2^3 = 8 = 3 + 5$$

$$3^3 = 27 = 7 + 9 + 11$$

$$4^3 = 64 = 13 + 15 + 17 + 19$$

$$5^3 = 125 = 21 + 23 + 25 + 27 + 29$$

If you want a formula for n^3, notice that the first term is $n^2 - n + 1$ and the last term is $n^2 - n + 2n - 1$, and the number of terms for each sum equals n. (There are n odd numbers between 1 and $2n - 1$).

In other words, the sum of all the cubes of integers from 1^3 to n^3 is equal to the sum of all the odd numbers up to $n^2 - n + 2n - 1 = n^2 + n - 1$.

How many of these numbers are there? A little thought should convince you that the answer is $(n^2+n)/2$. For example, with $n = 5$, our last odd number is $5^2+5-1 = 29$, and we have $(25 + 5)/2 = 15$ terms.

We want the sum of the first $(n^2 + n)/2$ odd numbers.

Let's look at another pattern

$$1 = 1$$

$$2^2 = 4 = 1 + 3$$

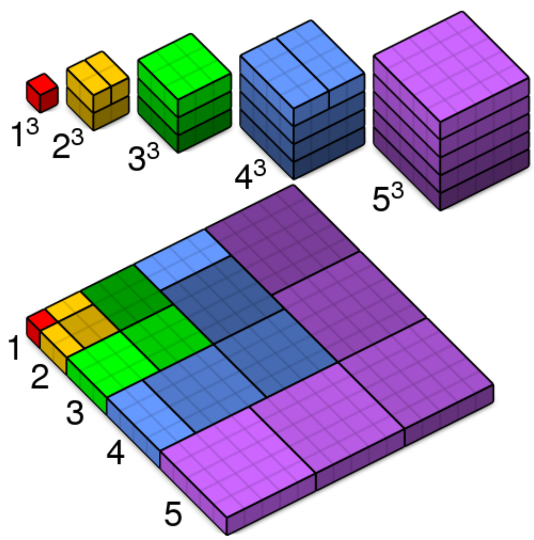
$$3^2 = 9 = 1 + 3 + 5$$

$$4^2 = 16 = 1 + 3 + 5 + 7$$

$$5^2 = 25 = 1 + 3 + 5 + 7 + 9$$

The *odd number theorem* says that the sum of the first n odd numbers is equal to n^2. We want the sum of the first $(n^2 + n)/2$ odd numbers, so that's $((n^2 + n)/2)^2$. And that's how we get our formula.

Here is another beautiful proof without words:



The length of the bottom pattern is a triangular number, which is itself a sum of squares. When squared it equals the sum of cubes.

another method

Here is another approach that I found very early in Hamming (Chapter 2) and have not seen in other books. It is called the method of *undetermined coefficients*.

We observe that the sum of integers formula has order n^2, while the sum of squares has order n^3, so we expect the sum of cubes would have n^4.

$$\sum_{k=0}^{k=n} k^3 = an^4 + bn^3 + cn^2 + dn + e$$

and if $n = 0$ the sum is zero so $e = 0$.

The right-hand side is

$$an^4 + bn^3 + cn^2 + dn$$

The inductive step is to write the formula for $m - 1$, and then add m^3 to it.

The right-hand side is just the formula, writing m for n

$$am^4 + bm^3 + cm^2 + dm$$

The left-hand side is the formula for $(m - 1)$, plus m^3 from the induction step:

$$a(m-1)^4 + b(m-1)^3 + c(m-1)^2 + d(m-1) + m^3$$

We work with the left-hand side. Expand each term using the binomial theorem:

$$a\ [\ m^4 - 4m^3 + 6m^2 - 4m + 1\ ]$$

$$b\ [\ m^3 - 3m^2 + 3m - 1\ ]$$

$$c\ [\ m^2 - 2m + 1\ ]$$

$$d\ [\ m - 1\ ]$$

Next, group the cofactors by the corresponding powers:

$$[\ a\ ]\ m^4$$

$$[\ -4a + b + 1\ ]\ m^3$$

$$[\ 6a - 3b + c\ ]\ m^2$$

$$[\ -4a + 3b - 2c + d\ ]\ m$$

$$a - b + c - d$$

Now to the point. The cofactors for *each power* of m must cancel exactly.

am^4 cancels on left and right, likewise bm^3, cm^2 and dm. That leaves four equations.

$$-4a + 1 = 0$$

$$6a - 3b = 0$$

$$-4a + 3b - 2c = 0$$

$$a - b + c - d = 0$$

We find that $a = 1/4$, $b = 1/2$, $c = 1/4$, $d = 0$. So then finally the formula is

$$an^4 + bn^3 + cn^2 + dn$$

$$= \frac{n^4 + 2n^3 + n^2}{4}$$

$$\frac{(n^2 + n)^2}{2^2} = [\ \frac{n(n+1)}{2}\ ]^2$$

which is exactly what we will have from other approaches.

Hamming uses this method to get a general formula, but we will not need that, because we will show how to use the binomial theorem to get what is necessary.

Part XV

Addendum

Chapter 64

References

- Acheson. *The Calculus Story.*
- Alcock. *Mathematics Rebooted.*
- Corral. *Trigonometry.*
- Courant, Robbins, Stewart. *What is Mathematics?.*
- Densmore. *Euclid's Elements* (annotated Heath translation).
- Dunham. *Euler: The Master of Us All.*
- Dunham. *Journey Through Genius.*
- Dunham. *The Mathematical Universe.*
- Hamming. *Methods of Mathematics Applied to Calculus, Probability, and Statistics.*
- Kline. *Calculus.*
- Lockhart. *Measurement.*
- Maor. *e, the Story of a Number.*
- Maor. *To Infinity and Beyond.*
- Nahin. *An Imaginary Tale. The Story of* $\sqrt{-1}$.
- Silverman. *A Friendly Introduction to Number Theory.*
- Simmons. *Precalculus mathematics in a nutshell.*

- Spivak. *The Hitchhiker's Guide to Calculus.*
- Stewart. *Significant Figures.*
- Strang. *Calculus.*
- Strogatz. *The Joy of x.*
- Thompson. *Calculus Made Easy.*