Chapter 10

Lines and angles

Euclid and the postulates

Greek geometry starts hundreds of years before Euclid, who was a contemporary of Alexander the Great (356-323 BC).

We know that Euclid lived after Plato (died 347 BC), and before Archimedes (born 287 BC). Except that he worked in Alexandria, all other details of his life and death are shrouded in mystery.

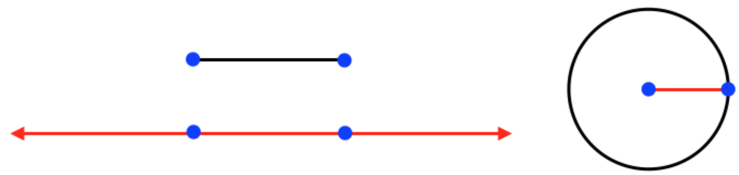
After more than 2000 years, Euclid's book *Elements* is still an excellent place to begin surveying the foundations of geometry. It is a textbook, an organized collection of everything that a well-educated student was expected to know about the subject at the time.

This book of consists of *propositions*, which include constructions (geometric figures) drawn with a pencil on a piece of paper, using a straight-edge or a compass or both.

Often proofs of propositions build on previous items in the book. Euclid does not prove everything. Bertrand Russell was famously disappointed about that.

Here are Euclid's first three postulates — statements that he assumed to be true:

• A straight line segment can be drawn joining any two points.



• Any straight line segment can be extended indefinitely in a straight line.

• Given any straight line segment, a circle can be drawn having the segment as the radius and one endpoint as the center.

Let us assume these as well. We will use them often.

We finesse the difficulty in defining what is meant by *straight* in the real world. If you've ever done any carpentry, you probably know that unknown edges are determined to be straight by comparison with another edge known to be straight.

In geometry, we use an imaginary perfect straight-edge to draw a straight line as "the shortest distance between two points".

The fourth postulate is:

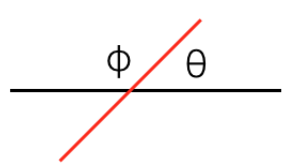
• All right angles are congruent, that is, equal to each other.

This one prompts two different questions. The first one is:

◦ how do we measure an angle?

measure of an angle

Consider the diagram below. One line segment is drawn crossing a second one, forming their *intersection*



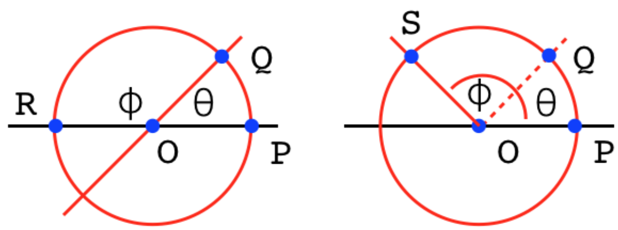
Two angles are labeled. One, ϕ, is larger than the other one, θ.

$$\phi > \theta$$

Writing the statement $\phi > \theta$ is easy, but the implication is that we have some way of talking about the *measure* of an angle.

Our answer is to construct a circle around the central point, and call the distance along the circumference between the points where the lines cross the circle, the measure of the angle.

If that distance along the edge is larger for $\angle\phi$, then $\phi > \theta$. In the left panel, the arc between Q and R (call it arc QR) is larger than arc PQ.



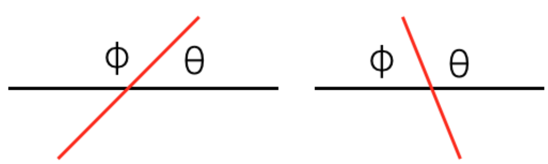
We don't need to actually measure the arc.

Instead we can use a standard compass to lay off the linear distance from Q to R starting at P (right panel). Since S is further around the circle, $\phi > \theta$.

As for our second question:

- what is a *right angle*?

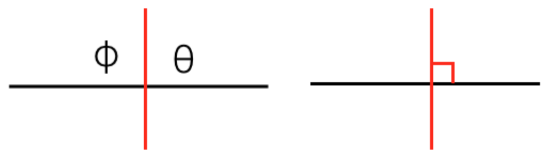
Let us call angles formed on the same side of a line, *supplementary* angles (also sometimes called adjacent angles).



On the left, one of the angles, ϕ, is larger than the other one, θ. On the right, we have $\phi < \theta$.

There is of course a third possibility, namely that $\theta = \phi$. The definition of a right angle is that

- if two supplementary angles are equal, then they are both right angles.



A right angle is frequently designated by drawing a small square, as seen in the right panel above.

Regardless of whether $\theta < \phi$, $\theta > \phi$, or $\theta = \phi$, the sum of the two angles $\phi + \theta$ is equal to two right angles or 180 degrees.

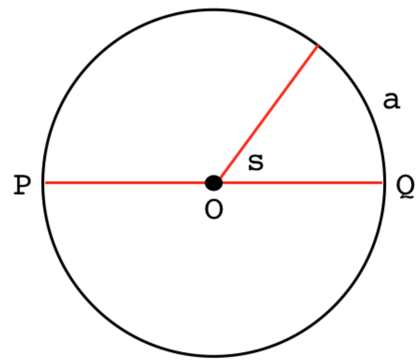
The convention that there are 360 degrees total in a circle dates to the time of the Babylonians (c. 2400 BC).

There is nothing particularly special about using 90 degrees as the measure of a right angle, 180 degrees for any two supplementary angles including right angles, or 360 degrees for one whole turn.

Well, there is one thing: there are *approximately* 360 days in a year, which marks the sun's track across the sky.

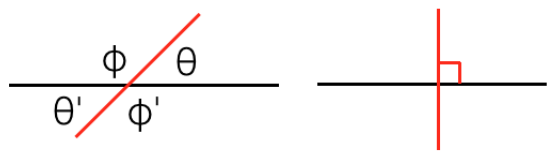
In his book, *Measurement*, Lockhart adopts the convention that a whole turn is equal to 1.

We'll just mention here that one whole turn can be defined using a different unit of measure as 2π *radians*, and that convention turns out to be quite important for calculus.



In the figure above, we *define* the measure of the angle s to be equal to the arc it sweeps out or subtends, a, in a circle of radius 1, a *unit* circle. Angles are not lengths, but numerically, the measure of the angle is the measure of the arc.

Now, consider those angles lying below the horizontal:



We said that the sum of the two angles $\phi + \theta$ is equal to two right angles, but so are the sums $\theta' + \phi$ and $\theta + \phi'$, for the same reason. As a result

$$\phi + \theta = \theta + \phi'$$

We conclude that

$$\phi = \phi'$$

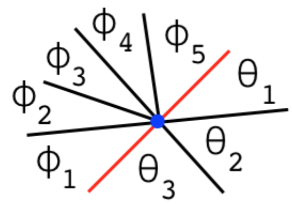
and

$$\theta = \theta'$$

This is called the *vertical angle theorem.*

On the right, if any one of the angles where two lines cross is a right angle, then all four are right angles.

Finally, the idea is generalizable to more than just two angles. In the figure below



$$s = \phi_1 + \phi_2 + \phi_3 + \phi_4 + \phi_5$$
$$t = \theta_1 + \theta_2 + \theta_3$$
$$s = t = \pi$$
$$s + t = 2\pi$$

Both sums are equal to two right angles.

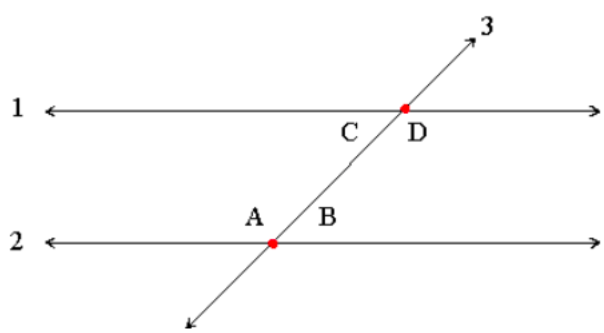
- when one or more lines cross a given line at the same point, the angles formed on one side of the given line have a total measure equal to two right angles.
- the total of all the angles formed at that point is equal to four right angles.

parallel postulate

So far, all this seems rather obvious. The fifth and final postulate is more subtle.

In the figure, line 1 and line 2 are parallel, *if and only if*

$$A + C = B + D = 180 = 2 \text{ right angles}$$



• If two lines are drawn which intersect a third in such a way that the sum of the inner angles on one side is less than two right angles, then the two lines inevitably must intersect each other on that side if extended far enough.

This postulate is equivalent to what is known as the parallel postulate.

http://mathworld.wolfram.com/EuclidsPostulates.html

alternate interior angles

We come to a very important theorem.

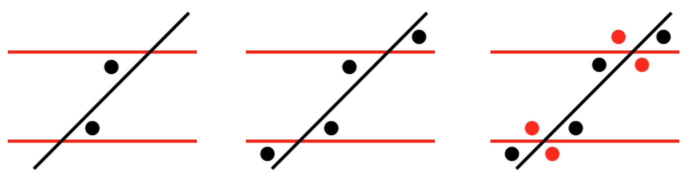
In the figure above, the supplementary angles $A + B = 180$ also add up to 180 degrees. So

$$A + B = 180 = A + C$$

and then

$$B = C$$

This is called the theorem on *alternate interior angles* between two parallel lines.



In the figure above (left panel), we're given that the two horizontal lines are parallel.

The indicated angles are equal because they are alternate interior angles of two parallel lines (parallel postulate).

In the middle panel, two additional equalities are established by the vertical angle theorem. Then on the right, we use the supplementary angle theorem.

Note that the conclusions for the angles marked with a red dot are also themselves consistent vertical angles and alternate interior angles.

summary

Make sure you learn and understand each of these theorems:

• supplementary angles

- vertical angles
- alternate interior angles
- triangle sum

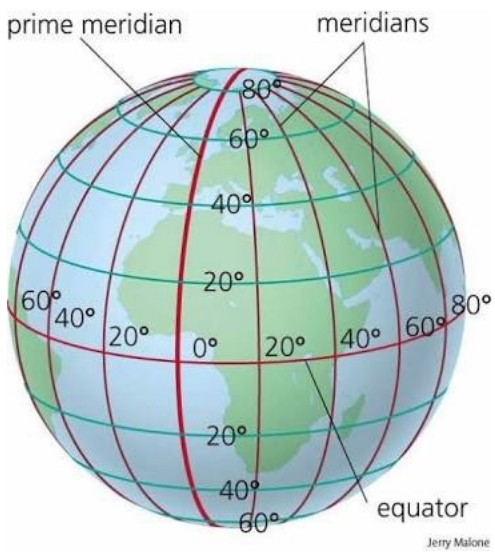
Another point to remember: these are two-way, *if and only if* theorems. So for example, if the two alternate interior angles of a traversal are equal, the two lines are parallel.

flat geometry

Adoption of the parallel postulate is a choice. The definition works for geometry in the flat plane.

But consider a familiar situation where this is not true. Suppose we are doing geometry on the surface of a sphere, such as the earth.

Then, two adjacent lines of longitude can be drawn so as to cross the equator at right angles, and the lines are parallel there, but they will meet (intersect) at the poles.



The parallel postulate only holds for geometry on a *flat* surface.

axioms

Euclid also lists five axioms, things which are assumed. Here are two examples:

• Things that are equal to the same thing are also equal to one another.

• If equals are added to equals, then the wholes are equal.

These seem quite reasonable.

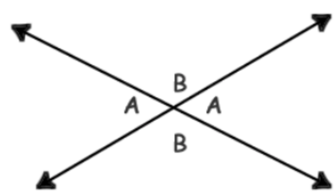
We will see how to proceed from the postulates and axioms to various proofs. Given these *assumptions*, we can prove theorems that must be true.

Thales

I'm a big fan of William Dunham's books — several of them are listed in the References.

Dunham has written a lot about the history of mathematics in Greece, starting with Thales (624-546 BC), who was from a Greek town called Miletus on the coast of Asia Minor (modern Turkey). He lived long before Euclid (about 300 years before, 600 BC). Although none of his writing survives, it is believed that Thales proved several early theorems including one we saw above.

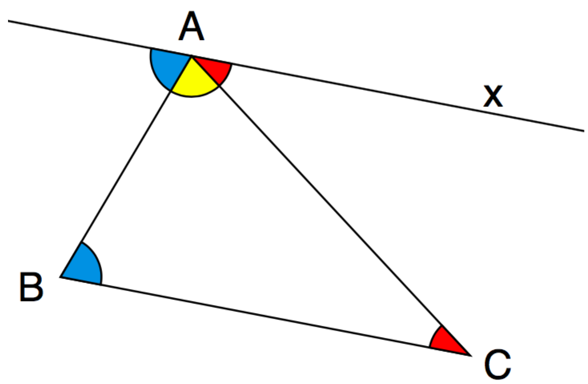
• The vertical angles formed by two straight lines crossing, are equal.



This theorem, which we already proved, depends on a property of straight lines. In the proof, we used the axiom "equals added to equals are equal", alternatively "equals subtracted from equals are equal."

A very important theorem attributed to Thales is the following:

• The angle sum of a triangle is equal to two right angles.



This theorem depends on the ideas we developed above. Draw a line segment through A parallel to BC. Now, use alternate interior angles and follow the colors to the result.

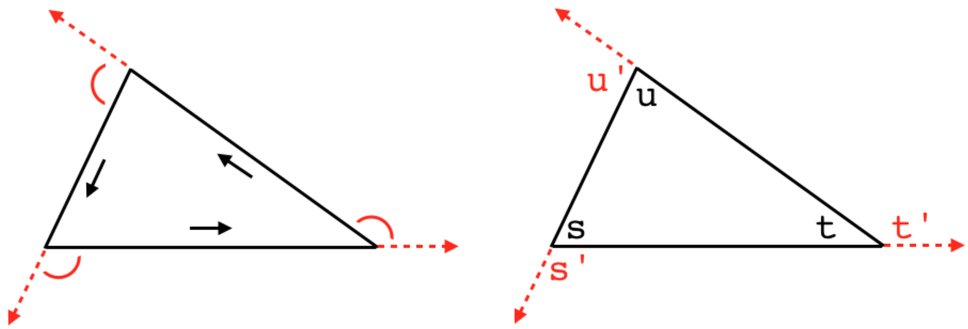
We will see one more theorem ascribed to Thales (it is actually called Thales' theorem), in the next chapter. It is about isosceles triangles (two sides equal).

another proof

Here is a different proof of the theorem on the sum of angles in a triangle adding to 180 degrees. It never hurts to re-prove things by a different method. It serves as a check on both the result, and the methods.

Imagine walking around the perimeter of a triangle in the counter-clockwise direction. At each vertex we turn left by a certain number of degrees, θ, called the exterior angle. After passing through all three vertices, we must end up facing in the same direction as we started.

The sum of the exterior angles is 360°.



$$s' + t' + u' = 360$$

In addition, for each vertex, the interior angle plus the exterior angle add up to 180 degrees. If we add all three pairs, we obtain

$$(s + s') + (t + t') + (u + u') = 3 \cdot 180 = 540$$

By subtraction

$$s + t + u = 180$$

Chapter 11

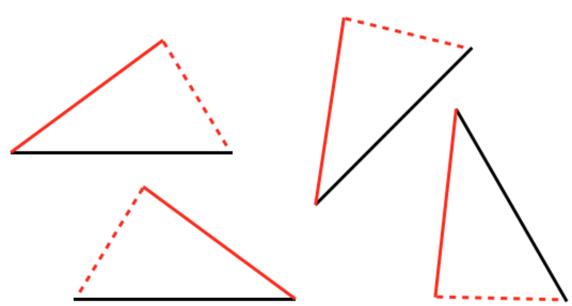
Congruent triangles

congruence

• Two triangles are *congruent* if and only if they have the same three side lengths.

Congruence is a fancy word for equal. The condition we use is often abbreviated SSS (side-side-side).

By this definition, a triangle and its mirror image are congruent. The three triangles shown below are all congruent, even though two are flipped (they are mirror images).



Having three sides equal means that the shape is the same. The three angles are also equal — and the shapes are superimposable, with the proviso that we allow the shape to be flipped over.

In addition to SSS (side-side-side), there are three other conditions that lead to congruence of two triangles when they are satisfied, namely

• SAS (side-angle-side)

• ASA (angle-side-angle)

• AAS (angle-angle-side)

similarity

Some triangles are *similar* but not congruent.

Similarity means that the three angles are the same but the triangles are of different overall sizes. We might say that they are the same but *scaled* differently.

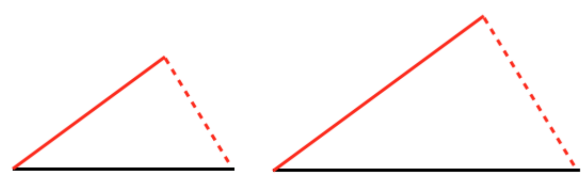
We can call this AAA (angle-angle-angle). For similar triangles, the three corresponding pairs of sides are in the same proportions, but re-scaled by a constant of proportion.

• Two triangles are similar if they have the same three angles.

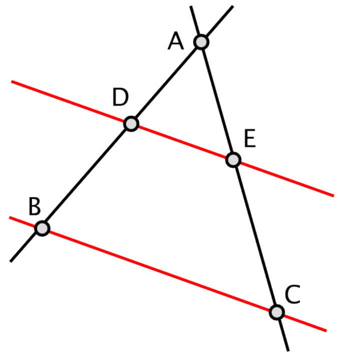
Because of the angle sum theorem, if any two angles of a pair of triangles are known to be equal, then the third one must be equal as well.

• Two triangles are similar if they have two angles known to be equal.

Similar triangles have their sides in the same proportion. This is known as the AAA similarity theorem.



Given any triangle, draw a line parallel to one side, which also joins the other two sides. The new triangle with that side as its base is similar to the given triangle.



In this example, these ratios are equal

$$\frac{AD}{AB} = \frac{AE}{AC} = \frac{DE}{BC}$$

and you can find others.

Statements about similarity:

∘ similar triangles have all three angles equal

∘ if two similar triangles are superimposed, the two sides that do not coincide with each other, are parallel

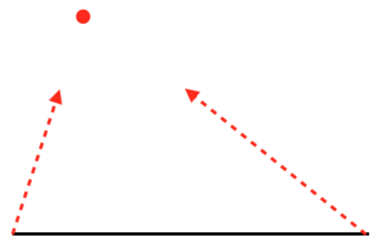
∘ similar triangles have their similar sides in the same ratio

It is easy to see why the first two statements are equivalent. Just use the alternate interior angles theorem.

The third is harder. For now we will assume the theorem: that AAA and sides in proportion are both the same as similarity.

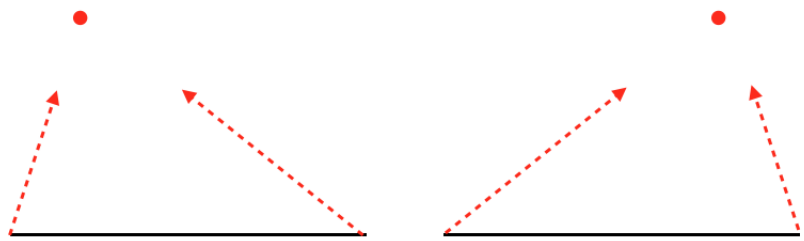
constructions

The way I think about the congruence conditions is to imagine trying to construct a triangle from the given information, and ask whether it is uniquely determined. Suppose we know ASA. The situation is thus:



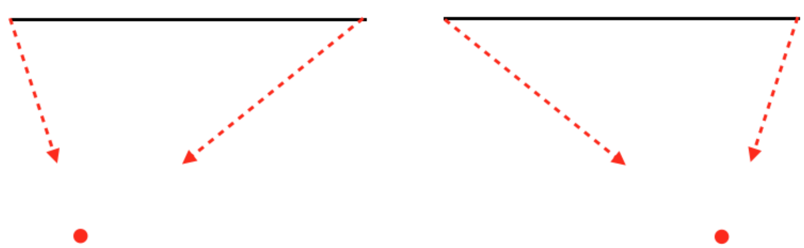
Draw the known side, then using the known angles, start two other sides from the ends of that side. They must cross at a unique point.

But... actually, if we start the two lines from opposite ends of the horizontal



there is another solution, the mirror image. These two triangles are congruent to the one above.

I'm tempted to add still another:



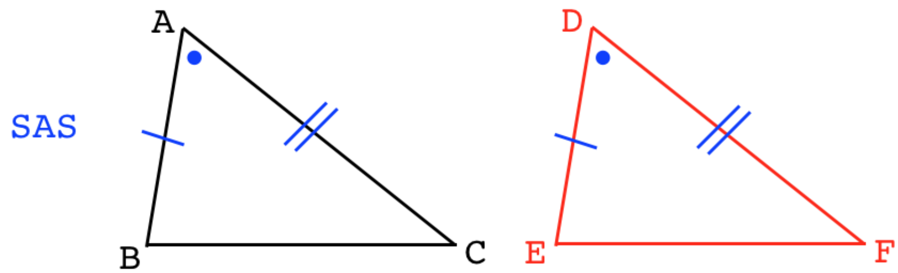
But this doesn't give anything new. These are merely rotated versions of the ones above. Congruent triangles include the two mirror images and that's it.

If we know two angles we also know the third, because they must add to 180 degrees. For this reason, ASA and AAS imply that we have exactly the same information, because we know all three angles and we know one side.

This restriction is crucial: we must also know *which* two angles flank the known side. Alternatively, it is enough to know which angle is opposite to the known side.

SAS, ASA, AAS but not ASS

SAS is very commonly used to prove congruence.

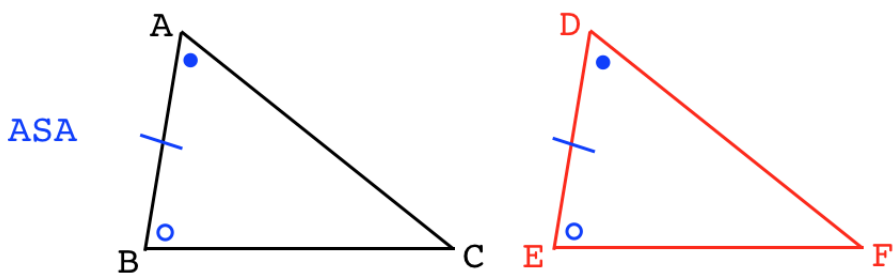


In this diagram, sides of equal length are indicated by one or more hash marks.

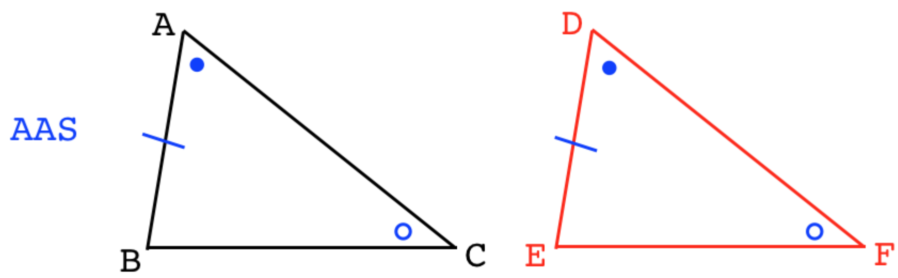
Equal angles are usually indicated by dots in this book. (Dots are easier to place on the figures, and lend themselves to color-coding; the common method for pencil and paper is to draw an arc with a hash across it).

The other methods for proving congruence use two equal angles and a side. Two equal angles imply the third angle is also equal (since they add to a half-circle or 180 degrees), so the two triangles are similar. To prove they are congruent, we need one side.

These methods using two angles are referred to as ASA



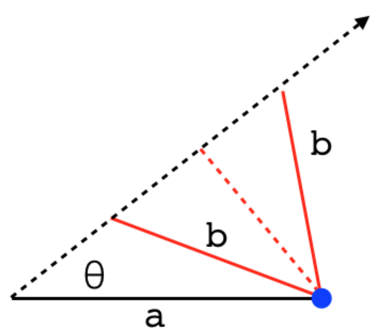
and AAS.



Because AA implies AAA, these tell unambiguously which angle is opposite the known side.

one that doesn't work

There is one set of three that doesn't work in the general case, that is ASS (angle-side-side).



Here we know sides a and b and the angle θ adjacent to a and facing opposite side b. Imagine b swinging on a hinge at the blue dot. If $b < a$, there are two points where b can intersect with the side projecting from angle θ. There is no unique solution, so the triangle is not determined.

If it had been the case that $b > a$, or alternatively that b formed a right angle with the third side, then the triangle *would* be determined.

I think Tony Randall said it best

https://www.youtube.com/watch?v=KEP1acj29-Y

Chapter 12

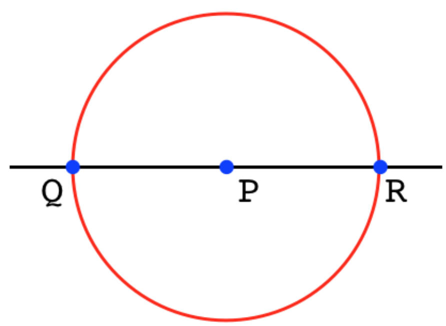
Perpendicular bisector

perpendicular at a point

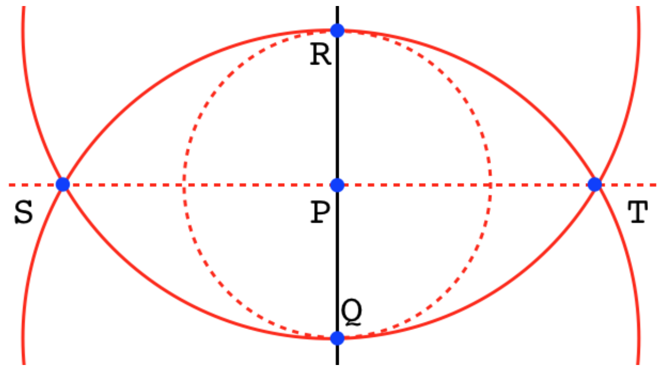
It's common to want to construct a line segment perpendicular to another line segment. The perpendicular might be specified to occur at a particular given point, either on the line, or alternatively through some other point not on the line.

For the first case, consider the horizontal line segment below. Suppose we know a point P on the line and wish to construct the vertical line through P.

Use the compass to mark off points Q and R on both sides, an equal distance from P. This can be done by drawing a circle with center at P. The radius is PR (also equal to QP).



Find S equally distant from Q and R. This can be done by using the compass to draw larger circles of the same radius on centers Q and R. Since we have the room, I've drawn much larger circles of radius equal to QR.



The figure is rotated by 90 degrees to conserve vertical space on the page.

$QS = RS$ because they are both radii of circles of equal radius.

The line segment SP will be perpendicular to the line containing QPR, because of theorems about triangles with two equal sides (isosceles) which we will prove later.

collapsible compass

Note briefly that is a restriction in Euclid's *Elements* to a *collapsible* compass, which is a compass that loses its setting when lifted from the page. That means that generally, you wouldn't be able to draw two circles of the same radius on different centers.

We got around that restriction by drawing the circles on Q and R with the same radius QR.

We will call a compass that is able to hold its setting, a *standard* compass, and explain why the distinction is important to Euclid in the chapter devoted to his book. But we also note that within the first few pages of that book, it is shown how to use a collapsible compass to carry out the very construction we said we couldn't do, namely, construct two circles on Q and R with equal radius and that radius not equal to QR or QP.

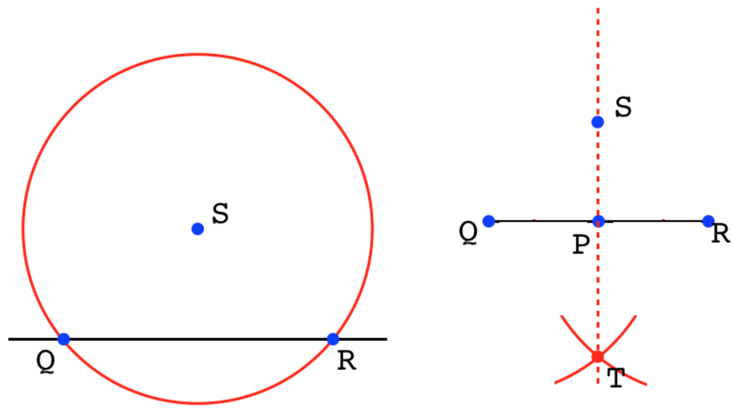
bisect a line segment

Suppose that we had not known the point P when we started the procedure above, but already had two points Q and R.

Then the line through S and T crosses QR as a perpendicular at its midpoint, and we have found the point P that bisects QR.

perpendicular through a point

Alternatively, suppose we know the line and the point S but not P, and we wish to construct a vertical through the line that also passes through S. Find Q and R on the line an equal distance from S ($QS = RS$), as radii of a circle centered at S (left panel, below). Their exact position is unimportant.



Now repeat the previous construction, using Q and R.

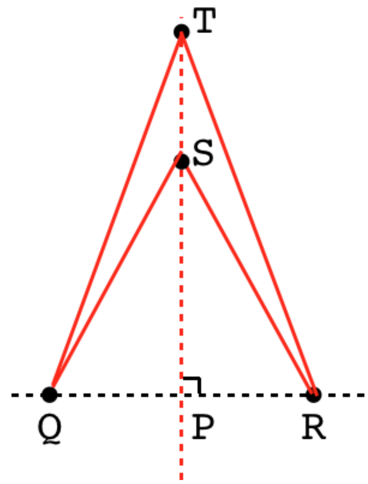
The line segment ST is perpendicular to the line segment containing QR, and passes through S, as required.

Also, see the video at the url:

https://www.mathopenref.com/constperpextpoint.html

bisector properties

Using what we've just learned, suppose we know two points Q and R. We find the point P equidistant between them and construct the perpendicular bisector PS. Then the two sides SQ and SR have equal length. Triangle $\triangle SQR$ is isosceles.



Proof.

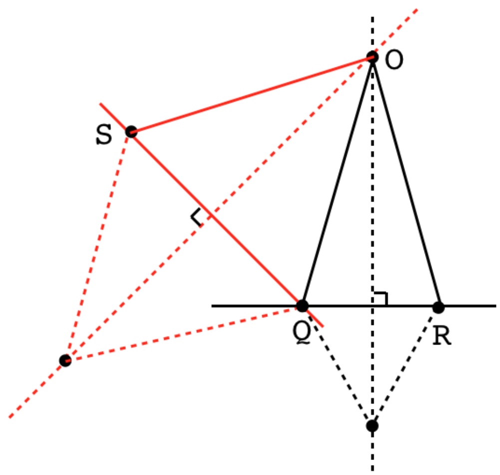
By construction, $PQ = PR$, $\angle SPR$ is a right angle, and side SP is shared. Hence the two triangles $\triangle SPQ$ and $\triangle SPR$ are congruent, by SAS.

□

This is true for *any* point on the line drawn through S and P. For example, $TQ = TR$ in the figure above.

three points

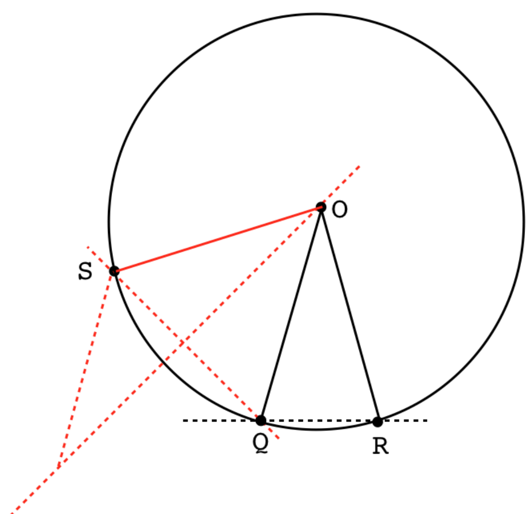
Now, suppose we have three points: Q, R and S. We find the perpendicular bisector of QR and also, the perpendicular bisector of QS. Extend them to where they meet, at point O.



What can we say about point O?

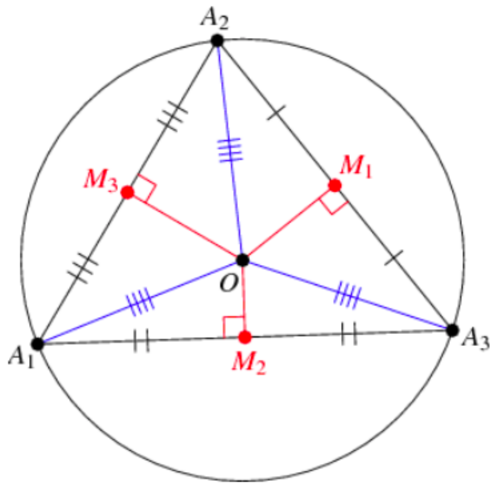
- O is equidistant from Q and R.
- O is also equidistant from Q and S.

Therefore, $OQ = OR = OS$. If we draw a circle on center O with radius OR, it will pass through all three points.



circumcenter

The point where the perpendicular bisectors cross has a special name, it is called the circumcenter.

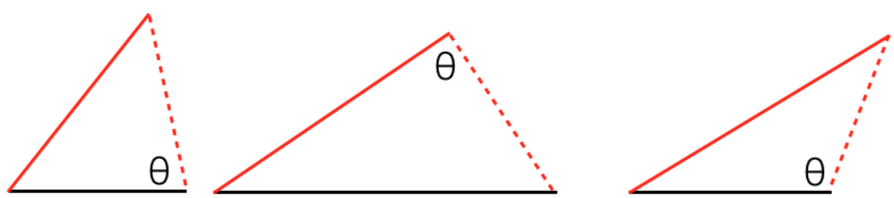


There are other special points where interesting circles can be drawn. We'll talk about them in a bit.

Chapter 13

Special triangles

There are several adjectives one can use to describe different types of triangles. For example: acute, right, and obtuse.



The acute triangle (left) has all three angles smaller than a right angle. The right triangle, naturally, has one right angle.

We'll say a lot more about right triangles later.

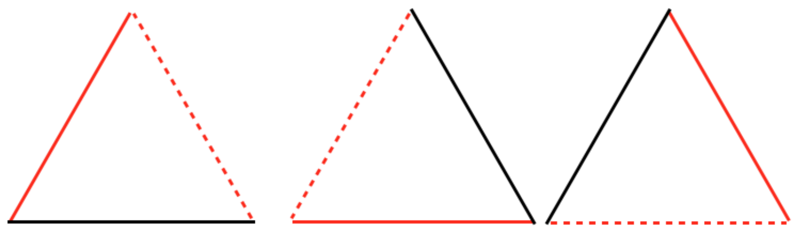
Finally, an obtuse triangle has one angle larger than a right angle (right, above).

symmetry

One can also talk about the situation where either two sides, or all three sides, have the same length.

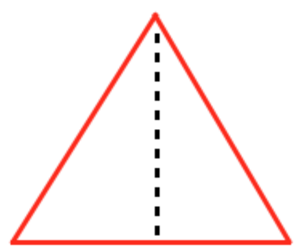
An equilateral triangle has all three sides the same, while an isosceles triangle has two sides the same length.

The most important consequence of three sides equal for an equilateral triangle, is rotational symmetry. Three turns of 120 degrees, and we're back where we started.



The implication of that is that the three angles are also equal. There is no reason to choose one larger than any other.

In the next figure the two smaller triangles obtained by dividing in half an equilateral triangle (all sides equal), are congruent.



By divide in half, we mean bisect the base and draw the line from the top vertex. We have SSS.

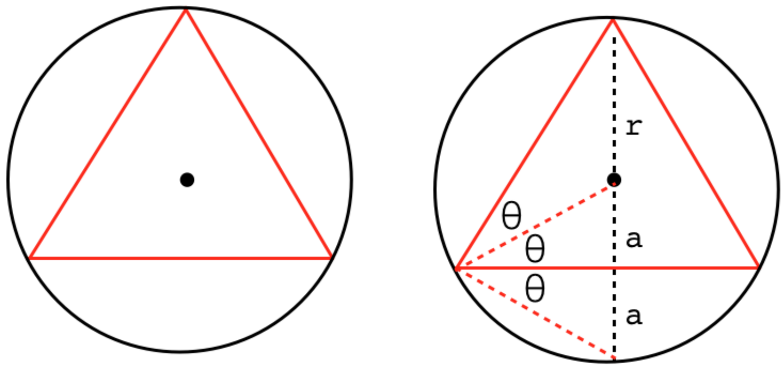
We also have that the two angles at the base of the bisector are equal and supplementary angles. Therefore, they are both right angles.

□

circumscribed circle

Here is a fun construction based on an equilateral triangle.

Any triangle fits into a unique circle. We will prove this elsewhere.



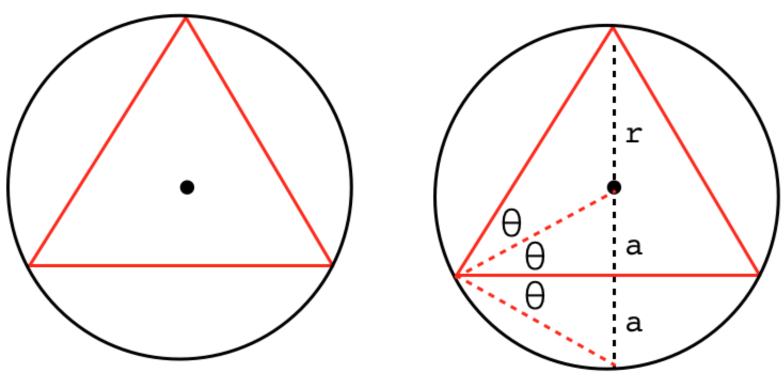
If we draw the radius to a vertex of the triangle, and then to the end of the diameter, it makes you wonder whether the three angles marked θ are equal.

Proof.

First, the radius is an altitude, and it divides the vertex angle in half, as we've been saying. So that accounts for two of the angles labeled θ.

The third comes about because any triangle with two points at the ends of the diagonal and a third anywhere on the circle is a right triangle. We'll prove that when we get to circles.

As a result, θ measures 30 degrees. Since a right triangle is 90, we assign the third θ.



So now we have a smaller angle of a right triangle, and a shared side. The two triangles are congruent. That accounts for the duplicated a in the figure.

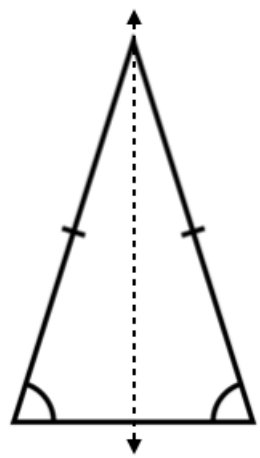
Thus, the altitude of the equilateral triangle is 3/4 of the diameter of the circle that just encloses it. And the point where the altitudes meet in an equilateral triangle is

1/3 of the way up from the base, since $r = 2a$.

We will see that such a point where the altitudes cross is unique and exists for any triangle, and it always has the same measure as a fraction of the altitude. This is called Ceva's theorem.

theorem from Thales

• The base angles of an isosceles triangle are equal. Also, if the two base angles are equal, the triangle is isosceles.



My favorite proof of this theorem is from reflective or mirror image symmetry (above).

Start with the two sides equal and draw a line to the midpoint of the base opposite. The figure has reflective symmetry, thus the angle is bisected.

We prove this more carefully now.

notation

The Greeks, including Euclid, adhere to certain conventions. For example, points are always labeled with letters, line segments are referred to by the endpoints, and angles by the line segments that determine them, as in $\angle ABC = \angle DEF$.

I don't know about you but I find myself tracing out angles from the three points, again and again.

We could give labels to the angles like $\alpha, \beta \ldots$ and so on, to the sides opposite vertices as a opposite A and so on.

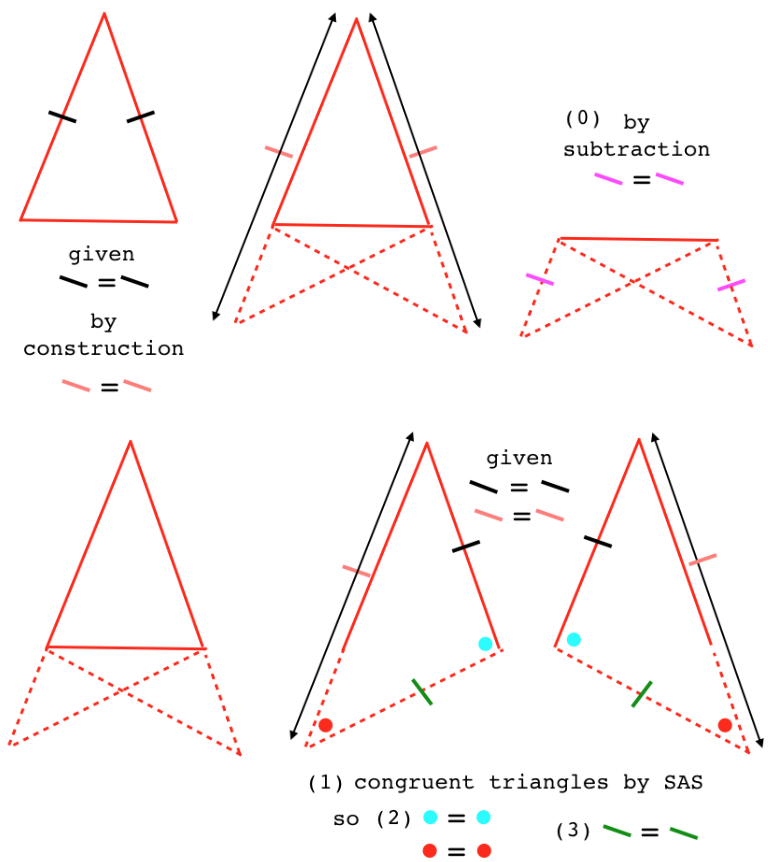
Boldly, we choose to be even more dramatic. Dispense with labels altogether and use colored dots for equal angles and colored bars for equal lengths.

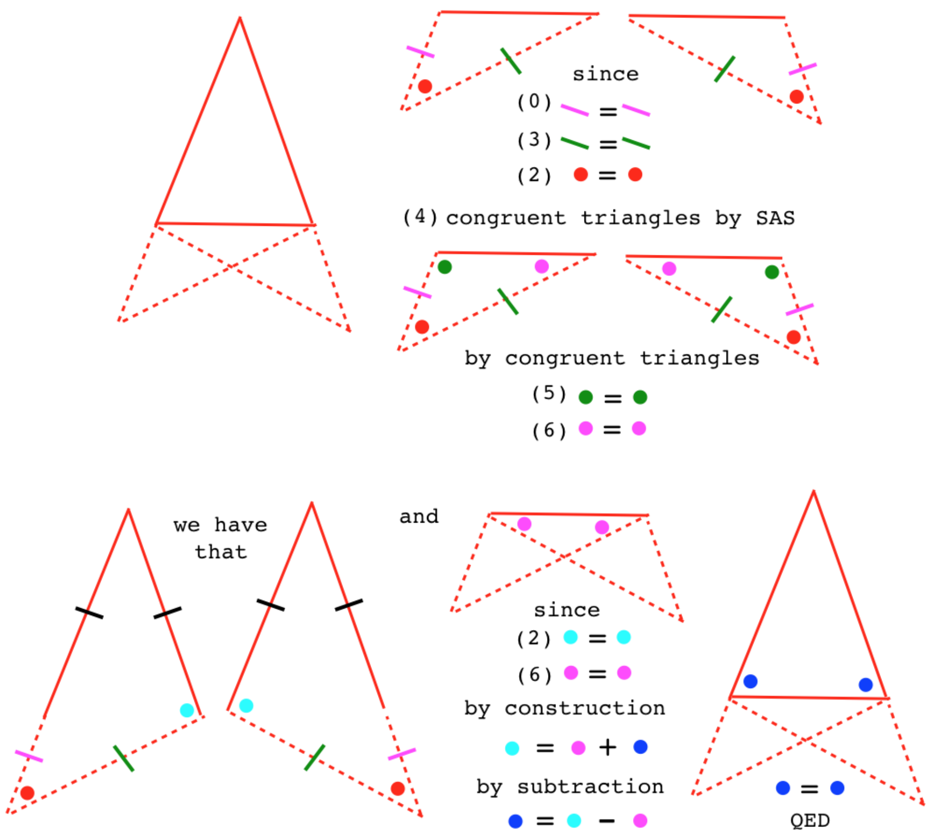
Here is the famous proof of Thales' theorem from Euclid's *Elements*.

Prop. I.5

In isosceles triangles the angles at the base equal one another, and, if the equal straight lines are produced further, then the angles under the base equal one another.

In what follows, all the pieces are with reference to the initial construction, first figure, below.





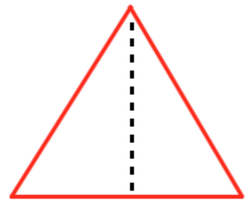
□

The theorem says that the base angles are equal $\iff$ the two sides sides are equal (not the base).

The symbol $\iff$ means *if and only if*, so $A \iff B$ means that both $A \rightarrow B$ and $B \rightarrow A$.

So far, our proof runs only in the forward direction. We will do the converse later.

Above we said that in this figure the two smaller triangles obtained by dividing an equilateral triangle in half, are congruent. The dotted line is called an *altitude* of the triangle.



An altitude meets the side opposite in a right triangle.

Because the left and right sides of the original triangle are equal, the base angles are equal, by the property of isosceles triangles which we just proved. The angles where the altitude meets the base are both right angles, by symmetry and by the definition of the altitude.

Therefore we have AAS, and the two halves are congruent.

So, the two angles at the top where the altitude meets the sides are also equal (as the third angle with the other two angles determined).

Chapter 14

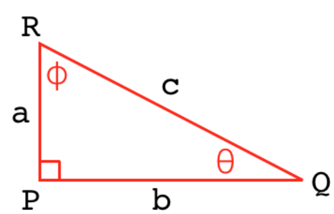
Right triangles

The main result we are headed for is the Pythagorean Theorem. Before we get there, however, it is worthwhile to continue our development of basic geometry with a discussion about right angles and right triangles.

A right triangle is a triangle containing one right angle. Right angles (and right triangles) are special. We saw previously that the definition of a right angle is that two of them add up to one straight line or 180 degrees.

Since we proved that the sum of the three angles in any triangle is equal to one straight line, by extension, the sum of angles in any triangle is also equal to two right angles.

In the figure below, the angle at vertex P is a right angle. It is common to mark a right angle with a little square, as shown, but these are a pain to draw, so I will often not do that. The side opposite P, namely c, is the *hypotenuse*.



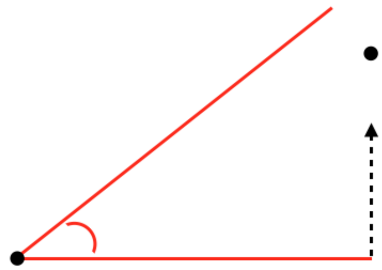
Since the sum of angles in a triangle is equal to two right angles, the sum of the angles θ and ϕ above is also equal to one right angle, or 90 degrees.

Angles θ and ϕ are said to be *complementary.* This fact is often exploited in proofs.

- the two smaller angles in a right triangle are complementary and add to 90 degrees.

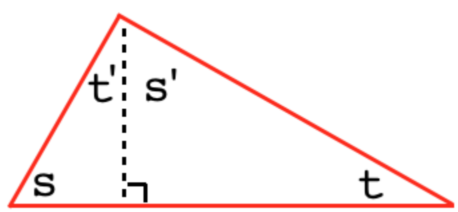
right triangles

For two right-triangles, if one hypotenuse is equal to the other, and also one set of legs equal, the two triangles are congruent.



In the figure, imagine the hypotenuse swinging on the hinge of its vertex with the horizontal base. There is only one angle where it will terminate on the vertical side with the correct length. This determines the angle between the known sides, or alternatively, the length of the third side.

altitudes



In the large right triangle above, we know that

$$s + t = 90$$

When we draw the perpendicular to the hypotenuse that goes through the upper vertex, that is an *altitude* of the triangle. Because of the right angle, we obtain two smaller right triangles. Thus

$$s + t' = 90$$

$$s' + t = 90$$

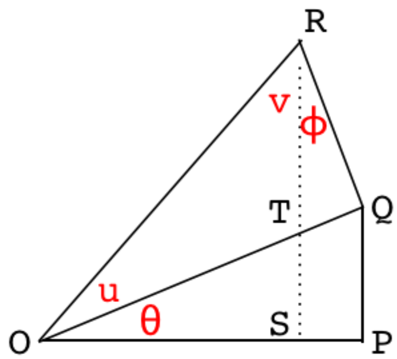
Hence

$$s + t = s + t'$$

$$t = t'$$

and similarly for s and s'.

stacked triangles



Suppose we are given that $\angle OPQ$ and $\angle OQR$ are right angles. We draw the altitude RS and observe that the angle at vertex S is a right angle.

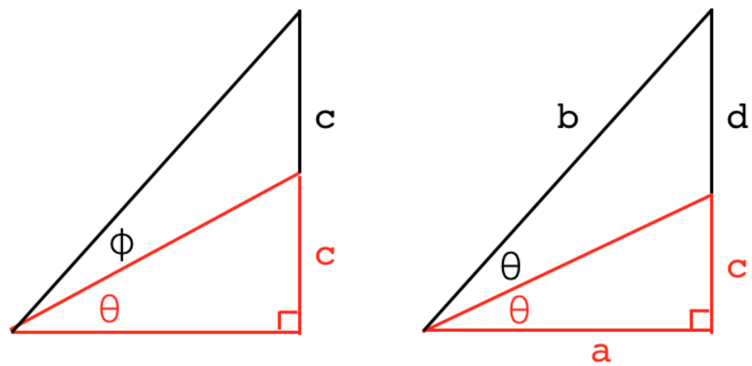
Therefore, in triangle ORS, the sum $\theta + u + v$ is equal to one right angle. At the same time, in triangle OQR, the sum $u + v + \phi$ is also equal to one right angle. Therefore

$$\theta = \phi$$

Further, $\triangle QRT$ and $\triangle OPQ$ are similar triangles.

angle bisector

With that background, we now consider a classic problem: angle bisectors.



Suppose we are given that the large triangle is a right triangle.

We draw a line joining the vertex on the left with the side opposite.

This line could in general be drawn anywhere, however two interesting cases are when the side opposite is bisected (left panel), or when the angle at the left is bisected (right panel). These two cases are not the same. In the first $\phi \neq \theta$ and in the second, $c \neq d$.

Suppose we choose the second possibility, equal angles. We are in a position to prove an important theorem.

angle bisector theorem

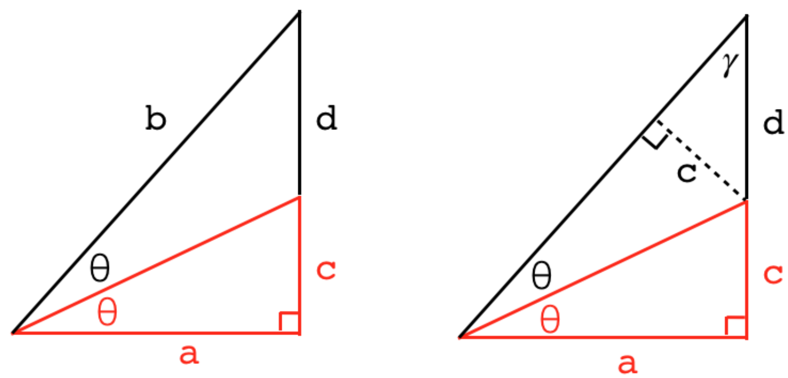
With reference to the two figures above, we are to prove that

$$\frac{d}{b} = \frac{c}{a}$$

The sides and bases are in proportion for a right triangle with bisected angle.

Proof.

Draw an altitude for the upper of the two small triangles, meeting the side of length b.



The red triangle and the one directly above it are congruent (right panel). They share a side (the hypotenuse of each), and they are right triangles with the same smaller angle γ. Therefore, the altitude we just drew has length c.

The small triangle with sides c and d (at the top) is similar to the original large triangle. The reason is that they are both right triangles containing the smaller angle γ.

By similar triangles, we form equal ratios of the angle opposite γ to the hypotenuse:

$$\frac{a}{b} = \frac{c}{d}$$

This is rearranged simply to give

$$\frac{d}{b} = \frac{c}{a}$$

which is what we were asked to prove.

□

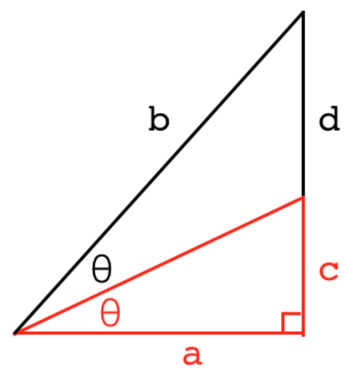
The result can be pushed a little further:

$$\frac{a}{b} = \frac{c}{d}$$

Here's the key point

$$\frac{a+b}{b} = \frac{c+d}{d}$$

$$\frac{a+b}{c+d} = \frac{b}{d} = \frac{a}{c}$$



which is a surprising result and becomes important later in looking at Archimedes method for approximating the value of π.

Chapter 15

Euclid's Elements

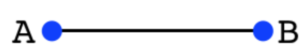
In this chapter we will study some nine or ten *Propositions* from the first volume Euclid's *Elements*. We also prove the *external angle theorem*.

The book was put together as a compendium of geometry for students. One thing we will see is how the propositions build on one another.

The first three propositions are *constructions*, e.g. the very first asks us to construct a triangle with all three sides equal, an equilateral triangle. The first statement below is Euclid's voice.

Prop. I.1

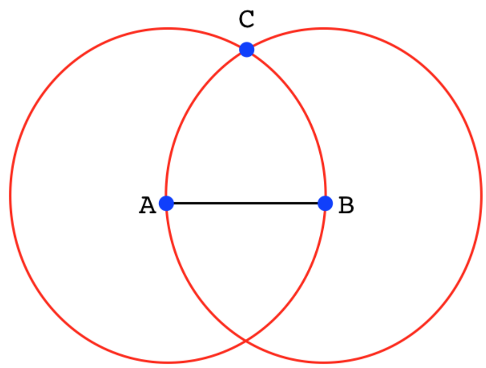
To construct an equilateral triangle on a given line segment.



The tools we have are a straight-edge and a compass. The compass is collapsible, meaning that it cannot be used to transfer distances since it loses its setting when lifted from the page. As we'll see in the next part, this is a problem with a solution.

Euclid was smart enough to know about compasses and how to set them. The idea he had was this: to make the fewest possible assumptions. A non-collapsible compass was a luxury he didn't need, since he could accomplish the same end without it, as we will see.

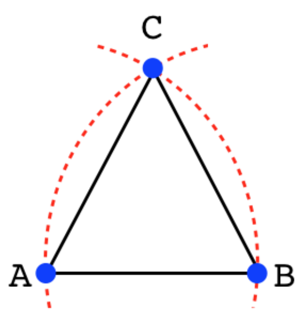
The first step is to draw two circles on centers A and B.



The circles are drawn with each radius equal to the line segment AB. It is a property of circles that all points on the circle are at the same distance from the center. Thus all points on the left-hand circle are equidistant from A, and all points on the second one are equidistant from B.

Therefore, the point C where the circles cross is equidistant from *both* A and B.

For this, we don't really need the entire circles, just the part where the arcs cross at C.



Now use the straight edge to draw $\triangle ABC$. Since $AC = AB$ and $BC = AB$, we know that $AC = BC$. The triangle is equilateral.

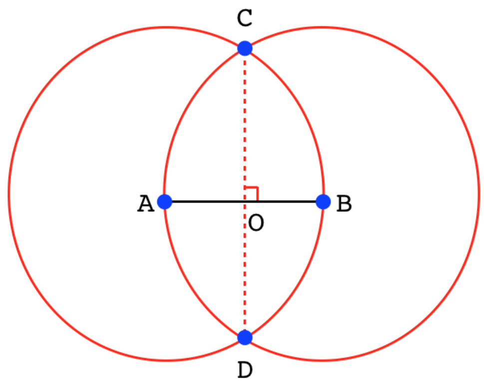
We put a little box to show that the proof is complete.

□

The proof doesn't stand on its own. We used one definition (D) and a common notion (CN).

- D I.15 all radii of a circle are equal.
- CN I.1 things which equal the same thing also equal one another.

If we look again at the figure, and label the other point where the circles cross as D:



Note: CD is the perpendicular bisector of AB. Euclid doesn't have the tools to prove that yet, so he leaves it for now.

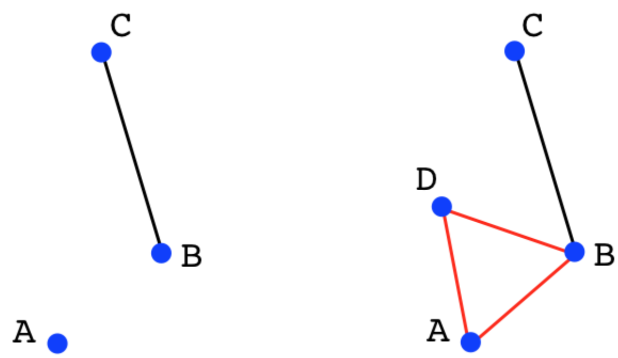
If desired, we could draw on tools from the last chapter. Clearly $\triangle ABC$ and $\triangle ABD$ are equilateral triangles with sides of the same length. From that, we deduce that $\triangle ACD$ and $\triangle CBD$ are isosceles. And from that, we can easily prove that all four small triangles with O as a vertex are equal and therefore right angles.

We leave this as an exercise.

Prop. I.2

To place a straight line equal to a given straight line with one end at a given point.

We will construct a line segment at A equal in length to BC (left panel). The first thing is to draw the line segment AB and construct an equilateral triangle on it (right panel).

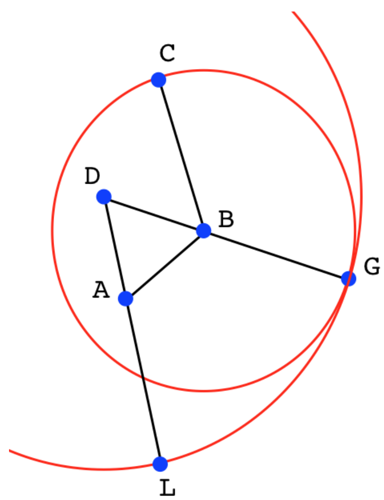


We know how to do this (from $PI.1$).

Next, construct a circle on center B with radius BC and extend the line segment DB to point G.

Then, construct a circle on center D with radius DG and extend DA to that circle at point L.

We have:



As common radii of the circle on center B, we have $BC = BG$.

As common radii of the circle on center D, we have $DL = DG$.

As sides of an equilateral triangle, we have $DA = DB$.

We use CN $I.3$: if equals are subtracted from equals, then the remainders are equal. Thus, $AL = BG$. But we had above that $BC = BG$. Therefore, $AL = BC$, by CN $I.1$.

Q.E.D. or "quod erat demonstrandum", in Latin

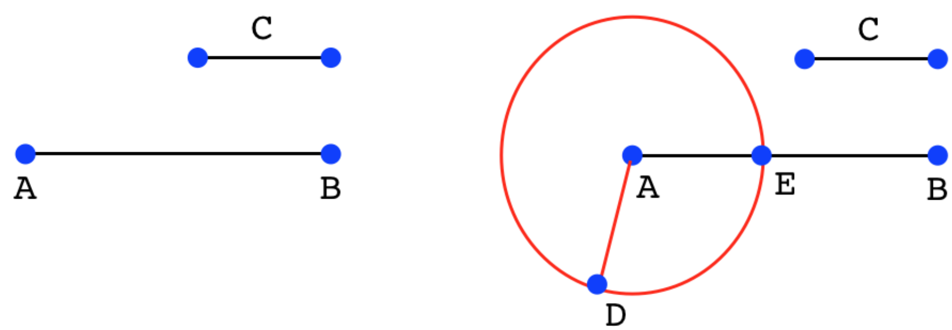
And in the original Greek *the very thing it was required to have shown.*

□

Note in passing, the orientation is determined by AB. We have not shown how to transfer the length with an arbitrary orientation. We will solve this next.

Prop. I.3

To cut off the lesser of two unequal straight lines from the greater.



In the left panel, we have the line segment AB and a smaller one just labeled C. To do the construction, use the method of P $I.2$ and transfer C to point A, forming AD.

Next, use AD as the radius of a circle on center A. Then, $AE = AD$, but $AD = C$. Hence

$$BE = AB - AE = AB - AD = AB - C$$

as required.

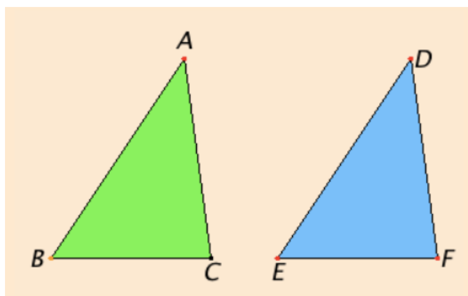
□

At this point, we have a method to mark off a given length from a larger length, even though all we have is a collapsing compass. Therefore, going forward, we can act as if we have a standard compass, that holds its setting after being lifted from the paper.

We also have the means to an important *trichotomy*. Comparing two line segments, one of three things must be true: either the first is smaller than the second, they are equal, or the second is smaller than the first.

Prop. I.4

If two triangles have two sides equal to two sides respectively, and have the angles contained by the equal straight lines equal, then they also have the base equal to the base, the triangle equals the triangle, and the remaining angles equal the remaining angles respectively, namely those opposite the equal sides.



This is not a construction, unlike the previous three propositions. It is a method for proving congruence (equality) of two triangles

$$\triangle ABC \cong \triangle DEF$$

Elsewhere in this book we would call the method SAS or *side angle side*. Given that $AB = DE$ and $AC = DF$ and that the angles between them at the vertices A and D are also equal, the two triangles are congruent: all three angles and all three sides are equal.

This (P I.4) is a proof that SAS is correct.

The proof is by superposition. The facts establish the positions of the points B and C, which determines AB and so the angles at vertices B and C.

Euclid says that if we lift up $\triangle ABC$ and lay it on top of $\triangle DEF$ then B coincides with E and C coincides with F so $BC = EF$.

□

This seems perhaps a little shaky logically, and it's not a method of proof that Euclid uses much.

But one might instead have taken this proposition as a postulate. The source, above, says that David Hilbert claims that under the hypotheses of the proposition it is true that the two base angles are equal, and then proves that the bases are equal.

We have used SAS to prove SSS, that all three sides are equal.

Prop. I.5

In isosceles triangles the angles at the base equal one another, and, if the equal straight lines are produced further, then the angles under the base equal one another.

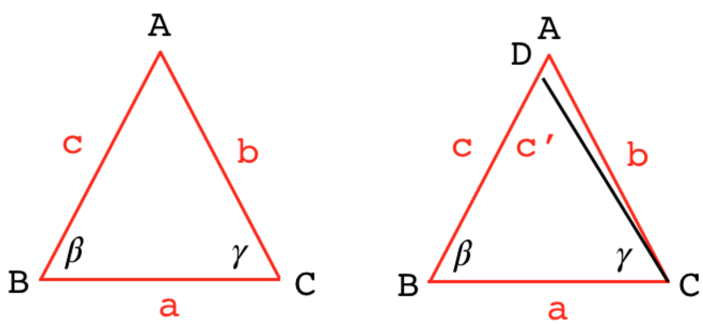
We proved this theorem in the previous chapter on isosceles triangles.

Euclid's proof of the converse is short and introduces the method of contradiction, or *reductio ad absurdum*. That is the next proposition.

Prop. I.6

If in a triangle two angles equal one another, then the sides opposite the equal angles also equal one another.

Suppose we have $\triangle ABC$ with equal angles $\beta = \gamma$ at the base (left panel).



We will assume that the two sides b and c are not equal. Then one of them is greater. Let c be greater, then cut off b from c at point D such that the new length $c' = b$.

The new triangle has sides c' and a, which flank angle β, while for the original we have side b and side a flanking γ. But we constructed $c' = b$, are given that $\beta = \gamma$, and the side a is common.

Therefore the $\triangle DBC \cong \triangle ACB$ by SAS.

But this means that the less equals the greater, which is absurd.

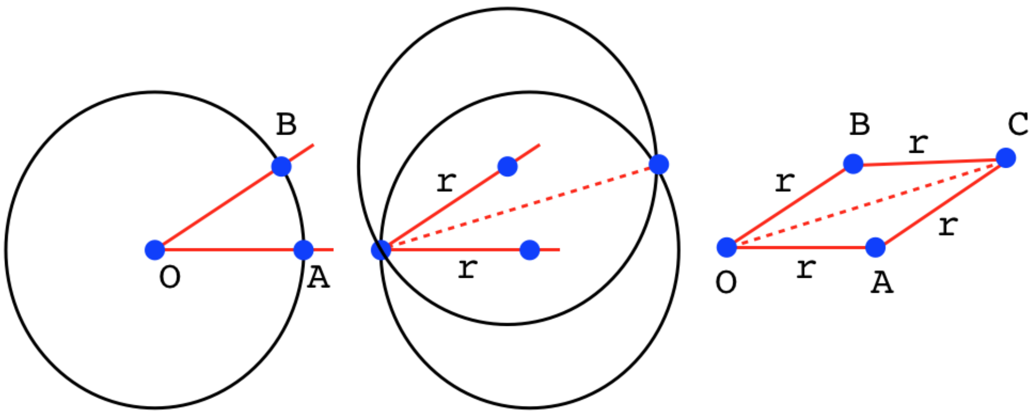
Therefore c cannot be unequal to b. It therefore equals it.

Our original assumption that b does not equal c must be false.

$\square$

Prop. I.9

To bisect a given angle.



As radii of a circle on center O, we first find points A and B equidistant from O (left panel). Let that distance be r.

As radii of circles on the centers A and B that pass through O (so the radius is equal to r), we find C equidistant from A and B (middle panel), with radius also equal to r.

Thus, $OA = OB = AC = BC$ (right panel).

So $\triangle OAC \cong \triangle OBC$.

Therefore $\angle BOC$ is equal to $\angle AOC$ and the given angle is bisected.

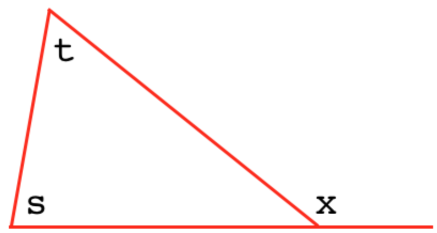
$\square$

We will do three more. They are short, sweet and powerful.

In these examples, we use letters late in the alphabet (s, t, u, v) for angles, while a, b, c are labels for sides.

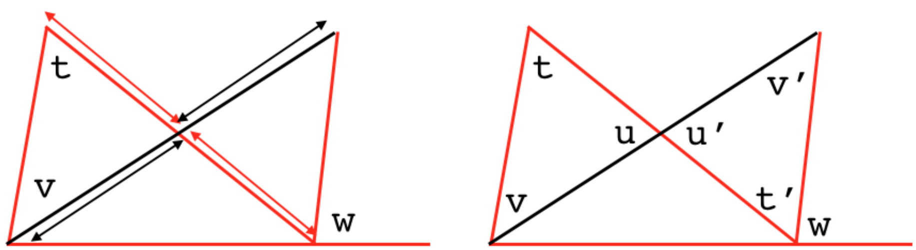
Prop. I.16

In any triangle, if one of the sides is produced (extended), then the exterior angle is greater than either of the interior and opposite angles.



The claim is that the exterior angle x is greater than either of the interior angles: s or t.

Find the midpoint of the side opposite s and draw the indicated line segment (below), so that the two segments marked with black arrows are equal, as well as the segments marked with red arrows.



By SAS and the vertical angle theorem, the two smaller triangles to the left and right are congruent, as indicated in the right panel by the labels on the angles: $t = t', u = u', v = v'$.

The original external angle x is seen to be composed of $t' + w$, that is

$$x = t' + w$$

so clearly (the whole is greater than its parts):

$$x > t'$$

but since $t = t'$:

$$x > t$$

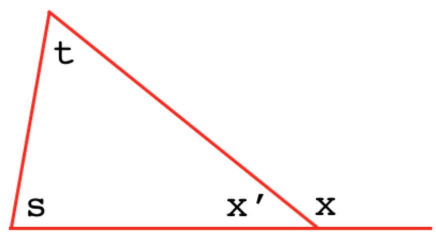
We can make a similar construction and proof for angle s.

The exterior angle is greater than either of the interior and opposite angles.

□

external angle theorem

The external angle theorem is extremely useful, so let's take a break from Euclid and prove it now. The proof is simple for us:



As supplementary angles, $x + x' = 180$ degrees. As the three angles of a triangle, $s + t + x' = 180$ degrees as well. Things equal to the same thing are equal to each other:

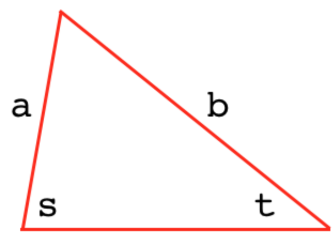
$$x + x' = s + t + x'$$

$$x = s + t$$

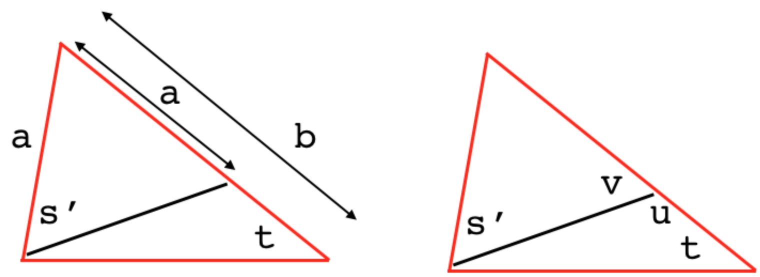
The question of why Euclid doesn't use supplementary angles here is complicated. For now it is enough to say that he just doesn't.

Prop. I.18

In any triangle, a greater side is opposite a greater angle.



Given $b > a$, mark off a on b.



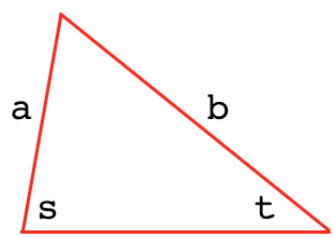
By the external angle theorem (I.16)

$$v > t$$

But $v = s'$ (by isosceles $\triangle$, $I.5$) so

$$s' > t$$

And since $s > s'$



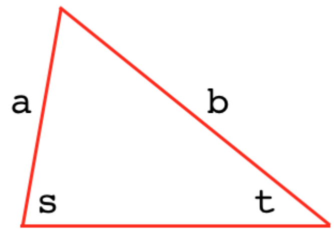
$$s > t$$

$\square$

We get the converse almost for free.

Prop. I.19

In any triangle, a greater angle is opposite a greater side.



We are given $s > t$ and want to prove $a < b$. We proceed by considering the other possibilities.

It cannot be that $a = b$ because then $s = t$ by isosceles $\triangle$ ($I.5$), but we are given $s > t$.

So then suppose $a > b$. By the previous proposition ($I.18$), we would have that $t > s$. But this is again contrary to what we were given. Hence $b > a$.

$\square$

We have made use of the trichotomy from before, that there are only three possibilities:

$$a < b, \qquad a > b, \qquad a = b$$

This applies to line segments and angles as well as many other things.

This is enough of the *Elements* to give us a good taste of the basics of Greek geometry of lines and triangles, and methods of proof. There is more to come: Pythagoras, and circles with their arcs and tangents.

Part IV

Pythagoras

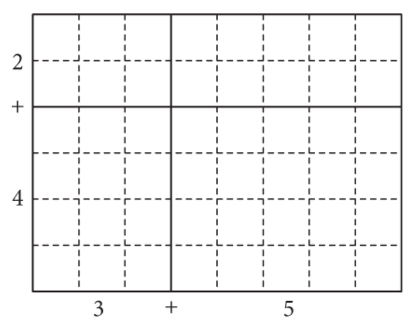
Chapter 16

Area

One aspect of calculus will be to determine the area of figures in the plane, particularly figures bounded by curves, as well as volumes in space. This is the magic of calculus, that we can make curves conform to rectilinear concepts of area and volume.

Since this introductory section is about Euclidean geometry, let's just say a few words about the area of a triangle. But we'll start with the rectangle.

To find the area of a rectangle, we must first fix a unit length. Then multiply the width by the height.



This particular figure (from Lockhart) shows the distributive law in action:

$$(3+5)\cdot(4+2)$$
$$= 3\cdot 4 + 3\cdot 2 + 5\cdot 4 + 5\cdot 2$$
$$= 48$$

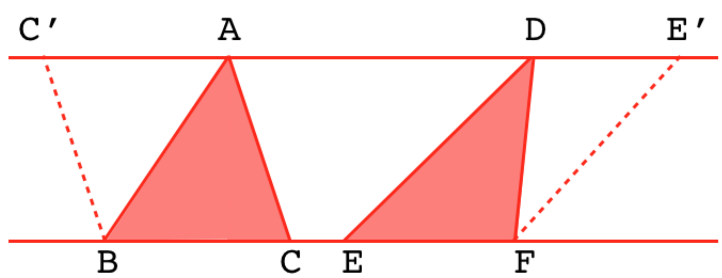
Any combination of numbers that add up to 8, times any combination of numbers that add up to 6, gives the same result.

The next figure is a parallelogram, a four-sided figure whose two pairs opposite sides are parallel (left panel). As a consequence of the theorems we saw previously, the opposing angles are equal, and the adjacent angles add up to 180 degrees.

To find the area, we cut off a right triangle from the left and re-attach it on the right. The angles add up to form a straight line along the base and a right triangle at the upper right. The area is clearly $h \times b$.

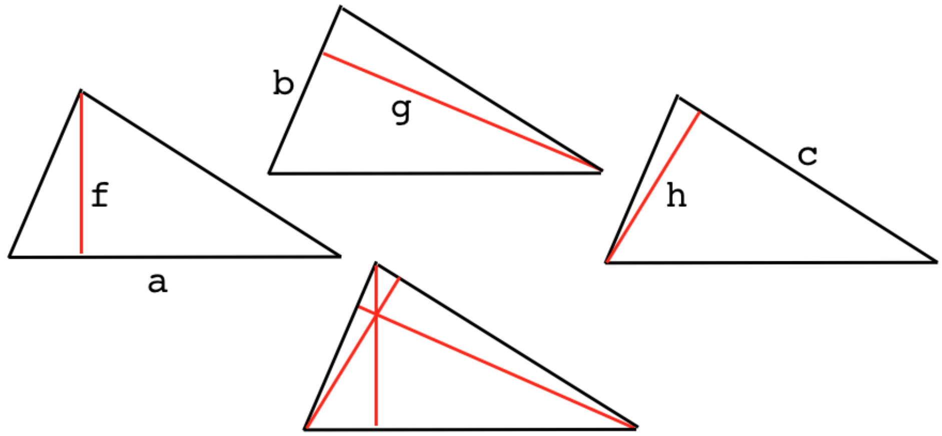
What about triangles? Previously, we talked about an approach where any triangle can be divided into two right triangles.

Alternatively, any triangle can be turned into a parallelogram, by attaching a rotated image of itself, like this:



It is easy to show that $\triangle ABC \cong \triangle ABC'$ and $\triangle DEF \cong \triangle DE'F$. For example, by construction AC is parallel to BC' so $\angle ACB = \angle AC'B$. We'll leave it to you to complete the proof.

An acute triangle is on the left and an obtuse triangle on the right. Since the area of each triangle is one-half that of its corresponding parallelogram (because we added the same area to make the parallelogram), the area of a triangle is one-half the base times the height.



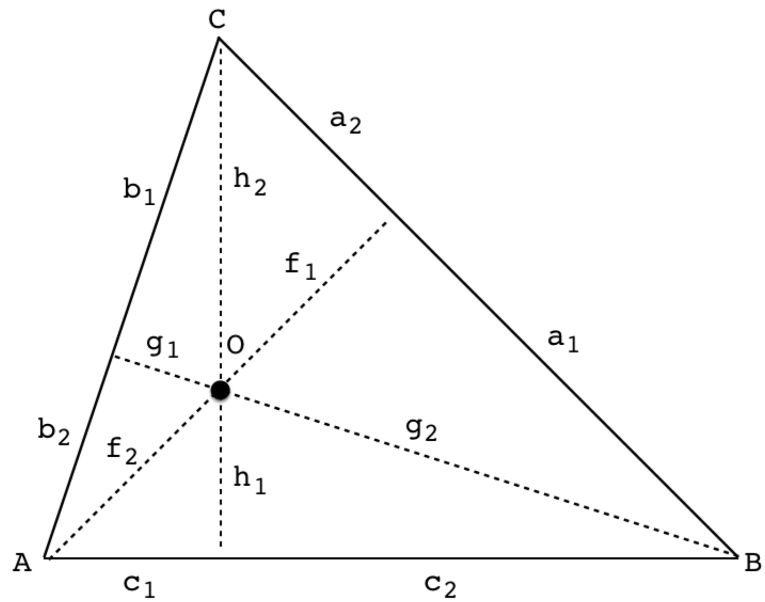
In the figure above, the area is

$$A = \frac{1}{2}\ af = \frac{1}{2}\ bg = \frac{1}{2}\ ch$$

We can choose any side of the triangle to be the base and then multiply $1/2\cdot$ base $\cdot$ height to get the area. We must always get the same answer!

If you accept the argument about the parallelogram above, it must be true, because the area of the triangle has to be the same no matter how you calculate it.

Here's a proof by counting up the area of smaller triangles:



In $\triangle ABC$ with sides a, b, c, drop the three altitudes from each of the three vertices to form right angles on the opposing sides. Ceva's theorem says that these altitudes cross at a single point (we will prove this later). Label the parts of the sides and the altitudes as shown in the diagram.

The area of the whole $\triangle ABC$ is equal to the sum

$$\triangle BOC + \triangle AOC + \triangle AOB$$

Using the rule, *twice* the area is

$$2A = af_1 + bg_1 + ch_1$$

But each of these smaller areas can be computed in different ways. In particular $\triangle BOC$ can be viewed as having base g_2 and height b_1, while $\triangle AOB$ can be viewed as having base b_2 and height g_2, so (twice) the total area is also

$$2A = b_1 g_2 + b_2 g_2 + bg_1$$

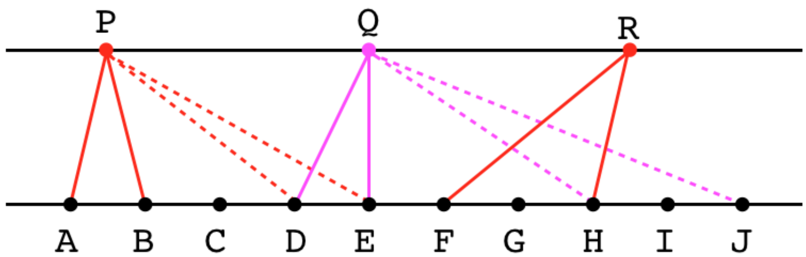
$$= bg_2 + bg_1 = bg$$

Similar calculations can be carried out for the other two sides. Hence the area is the same regardless of which side is chosen as the base.

□

A corollary is that all triangles with the same base and height have the same area.

Draw two parallel lines. Mark off equal distances between adjacent points A through J on the bottom. Now pick any point on the top and draw the triangle with two *equidistant* points on the bottom. Any other triangle drawn with an equal base has the same area.



In this figure the areas of $\triangle PAB$, $\triangle PDE$, and QDE are equal, as are $\triangle QHJ$ and $\triangle RFH$. Further, the latter two have twice the area of any of the first three.

platonic solids

https://en.wikipedia.org/wiki/Platonic_solid

> In three-dimensional space, a Platonic solid is a regular, convex polyhedron. It is constructed by congruent (identical in shape and size) regular (all angles equal and all sides equal) polygonal faces with the same number of faces meeting at each vertex. Five solids meet these criteria.

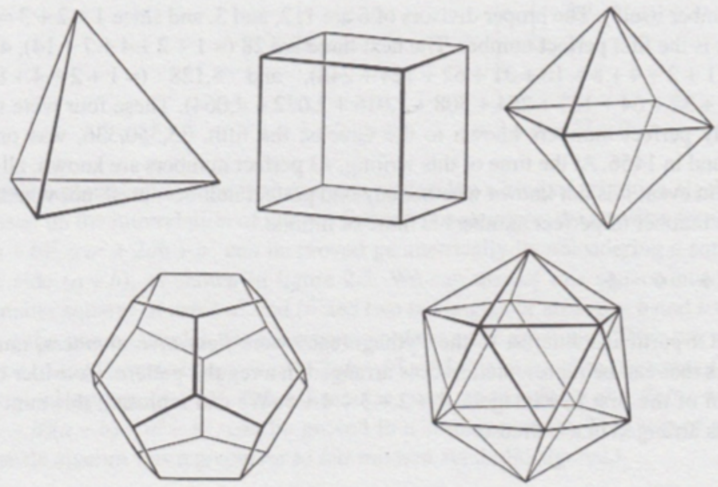


Figure 2.3. The five regular or Platonic solids

These are: (i) tetrahedron, (ii) cube, (iii) octagon, (iv) dodecagon, and (v) icosahedron.

There is a wonderful, simple proof that there are only five of them. Any solid requires at least three sides meeting at each vertex, otherwise the joint between two sides can just flap, like a hinge. Furthermore, the total of all the vertex angles added up must be less than 360 degrees, since otherwise the figure would be planar, not 3-dimensional.

So, three equilateral triangles total $60 \times 3 = 180$, four total $60 \times 4 = 240$ and five total $60 \times 5 = 300$. Six would be a hexagon lying in the plane. Three squares total $90 \times 3 = 270$, while four give a square array in the plane. Finally, three pentagons give $108 \times 3 = 324$. And that's it. Three hexagons would give $120 \times 3 = 360$, which gives an array in the plane.

Proving that all the angles and side lengths come out correctly, so that the possible solids actually can be constructed is another matter, however. Euclid devotes book XIII of *The Elements* to this:

https://mathcs.clarku.edu/~djoyce/elements/bookXIII/bookXIII.html#props

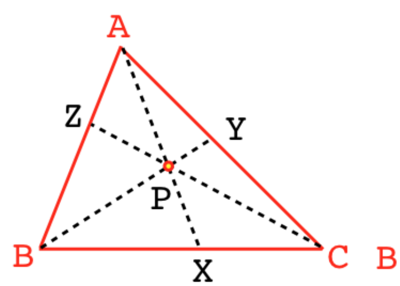
Chapter 17

Ceva's theorem

There are some special points in triangles including orthocenters, which we've already mentioned, but also circumcenters, incenters and centroids.

They have the common feature that three lines are drawn crossing at a single point. We need to establish the conditions under which this assertion is true.

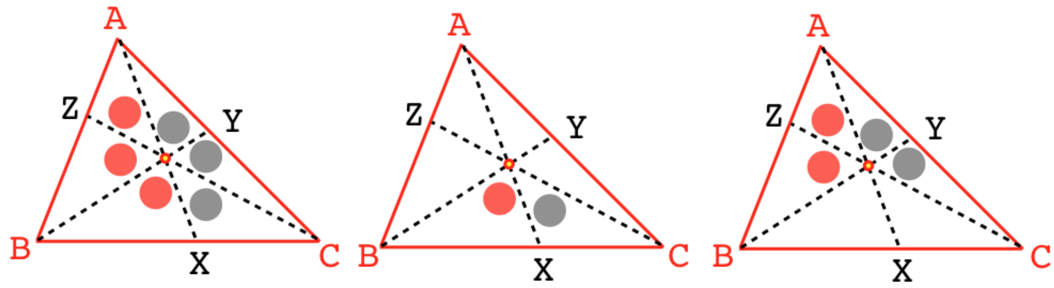
We begin with the triangle shown below, picking a point P to be *any point* inside the triangle. Now draw line segments from each vertex through P and extend them to the opposing side.



Since P can be anywhere, the ratio can be anything. Let's call it x.

$$\frac{BX}{XC} = x$$

Line AX divides the whole triangle into two parts.

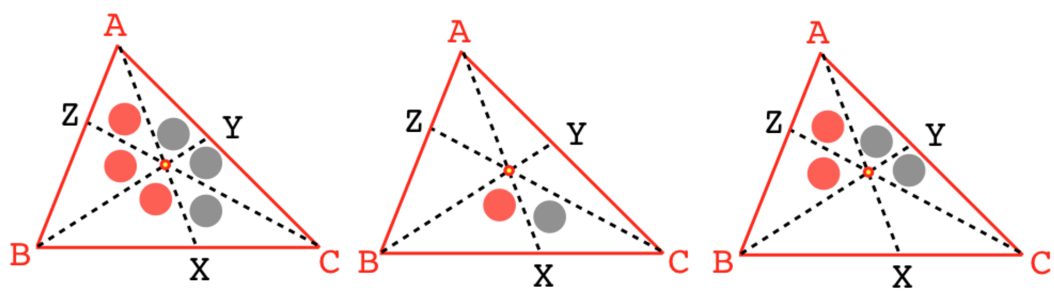


We know that the area of $\triangle ABX$ is in the same proportion to the area of $\triangle AXC$ as x, because they share the same height, while x is the ratio of their bases.

$$BX = x \cdot XC$$
$$A_{ABX} = \frac{1}{2}h \cdot BX = \frac{1}{2}hx \cdot XC = xA_{AXC}$$

Now consider the lower pair of triangles $\triangle BPX$ and $\triangle CPX$

These two also have their areas in the ratio x, for the same reason.



By subtraction, $\triangle ABP$ and $\triangle ACP$ also have ratio x.

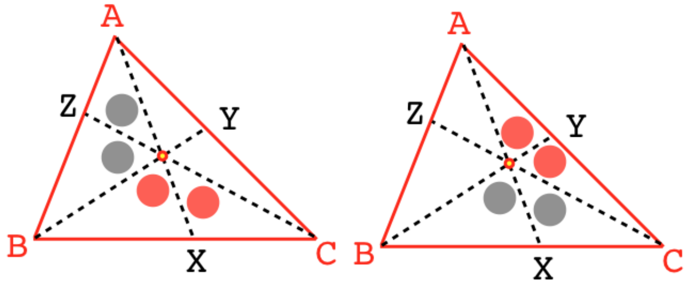
So, altogether, we have that

$$\frac{BX}{XC} = \frac{|ABX|}{|ACX|} = \frac{|BPX|}{|CPX|} = \frac{|ABP|}{|ACP|} = x$$

more sides

By the same reasoning, if $y = CY/YA$

$$\frac{|BCP|}{|ABP|} = y$$



and if $z = AZ/ZB$

$$\frac{|ACP|}{|BCP|} = z$$

Then

$$xyz = \frac{|ABP|}{|ACP|}\ \frac{|BCP|}{|ABP|}\ \frac{|ACP|}{|BCP|}$$

But all terms cancel, so

$$xyz = 1$$

And this is of course true not just for the areas but for the original line segments

$$xyz = \frac{BX}{XC}\ \frac{CY}{YA}\ \frac{AZ}{ZB} = 1$$

$\square$

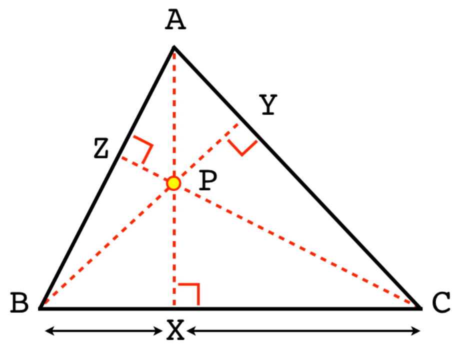
This proof also works in reverse,

$$xyz = 1 \iff \text{3 lines cross at point P}$$

We will just assume that part.

orthocenter

So now, for this triangle



if α is the angle at vertex A and so on, then for example,

$$BX = AB\ cos\beta$$

and

$$\frac{BX}{XC} = \frac{AB\ cos\beta}{AC\ cos\gamma}$$
$$\frac{CY}{YA} = \frac{BC\ cos\gamma}{AB\ cos\alpha}$$
$$\frac{AZ}{ZB} = \frac{AC\ cos\alpha}{BC\ cos\beta}$$

When we construct this ratio, all the terms cancel.

$$\frac{ABcos\beta}{AC\ cos\gamma}\ \frac{BC\ cos\gamma}{AB\ cos\alpha}\ \frac{AC\ cos\alpha}{BC\ cos\beta} = 1$$

which means that

$$\frac{BX}{XC}\ \frac{CY}{YA}\ \frac{AZ}{ZB} = 1$$

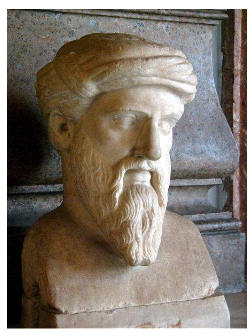
Therefore, the 3 altitudes all cross at a single point. That point is the orthocenter, and this is a proof that it exists.

□

Chapter 18

Pythagoras

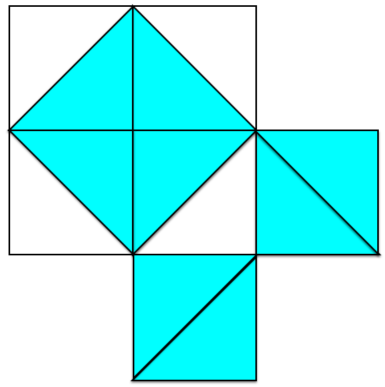
The most famous theorem of Greek geometry is also the most useful in Calculus.



Pythagoras (c.570-c.495 BC) was much younger than Thales but may have encountered him as a young man. Like many other Greek mathematicians, Pythagoras was not from the mainland, but from one of the islands, in his case, Samos, which is not far from Miletus, where Thales lived.

Pythagoras was famous as a philosopher as well as a mathematician. In fact, he founded a famous "school" and it is not sure now which of the theorems developed by this school are due to Pythagoras, and which to his disciples. It is not even clear whether the Pythagorean theorem, as we know it, was known to Pythagoras.

However, it's pretty certain that they knew something. The $3, 4, 5$ right triangle and many other Pythagorean triples (see below) had been known for a thousand years (since 1500 BC). Here is a special case, easily proved, for an isosceles right triangle.

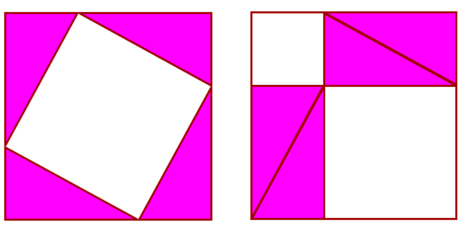


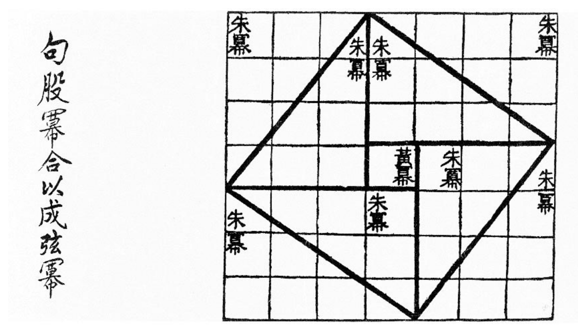
The area of the square on the hypotenuse is equal to twice the area on each side.

There are literally hundreds of proofs of the general theorem, that if a and b are the shorter sides of a right triangle and c is the hypotenuse, then

$$a^2 + b^2 = c^2$$

This one is sometimes called the "Chinese proof." I can easily imagine proceeding from the figure above to this one by simply rotating the inner square.





It really needs no explanation, but ..

In the left panel we have a large square box that contains within it a white square, whose side is also the hypotenuse of the four identical right triangles contained inside. Altogether the four triangles plus the white area add up to the total.

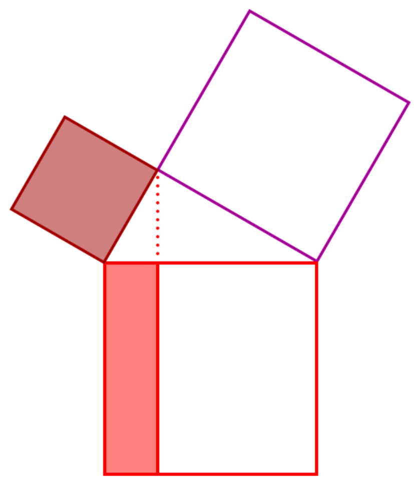
We simply rearrange the triangles. Now we evidently have the same area left over from the four triangles, because they still have the same area and the surrounding box has not changed.

But clearly, now the white area is the sum of the squares on the second and third sides of the triangles. Hence the two white squares on the right are equal in area to the large white square on the left. □

This proof is contained in the Chinese text Zhoubi Suanjing (right panel, above).

https://en.wikipedia.org/wiki/Zhoubi_Suanjing

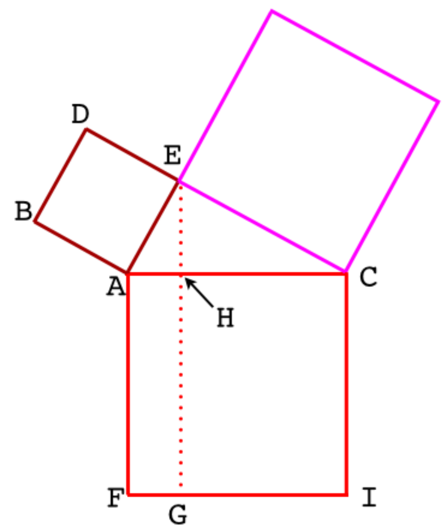
Euclid's proof



My favorite proof relies on the construction above (Euclid $I.47$, sometimes called the "bridal chair" or the "windmill"), where the central triangle is a right triangle, and the other constructions are squares. It is a bit more detailed, but it is a gem of a proof, from Euclid, which is a justification for including it.

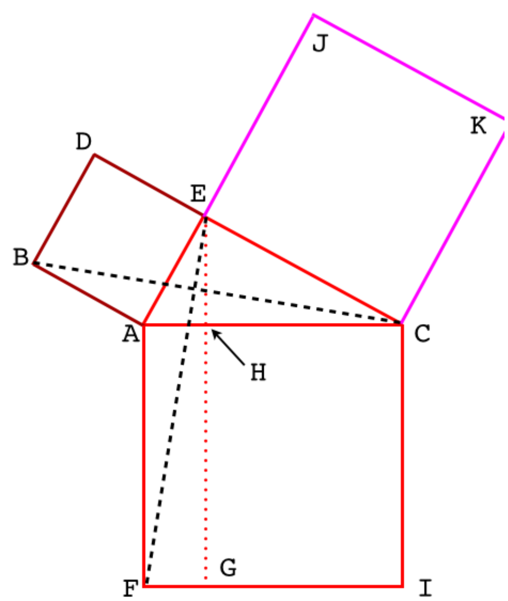
What we will show is that the part of the large square in red is equal in area to the entire small square, in maroon.

We label some points as shown:



First, drop a vertical line EHG, constructing the rectangle $AFGH$.

Finally, sketch dotted lines for the long sides of two triangles:



The crucial point is this: we will show that triangle ΔABC is congruent to triangle ΔAEF.

Use side-angle-side (SAS). The two sets of sides are evidently equal

$$AB = AE, \quad AC = AF$$

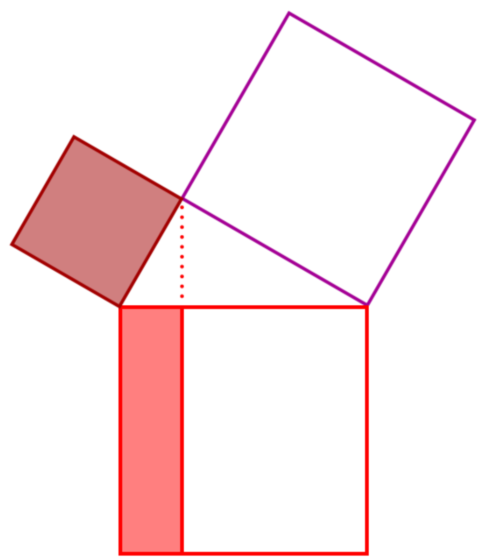
because these are given as sides of two squares.

What about the included angle? The angles $\angle BAC$ and $\angle EAF$ each contain a right angle plus the shared angle $\angle EAC$. So they are themselves equal, and thus we have proved SAS and thus the congruence relationship:

$$\Delta ABC = \Delta AEF$$

The next part of the proof is to tilt triangle ΔABC to the left and see that it has base AB and altitude AE so its area is one-half that of the small square $ABDE$. On the other hand triangle ΔAEF has base AF and altitude AH (as well as FG) so its area is one-half that of the rectangle $AFGH$.

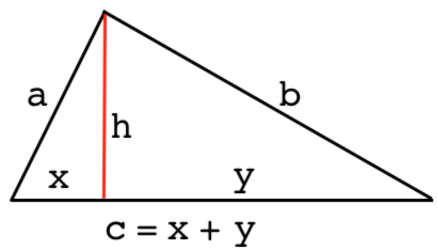
Hence we have proved that the two colored areas in this figure are equal:



Finally, we could proceed to do the same thing on the right side of the figure, but we just appeal to symmetry. All the equivalent relationships will hold.

□

There are several hundred proofs of the Pythagorean theorem. Many of them are algebraic. Here is a classic:



We know that when an altitude is drawn in a right triangle, the two resulting right triangles are similar (use complementary angles if you need to convince yourself again). So we have equal ratios of sides. Here are two sets:

hypotenuse to short side

$$\frac{a}{x} = \frac{b}{h} = \frac{c}{a}$$

hypotenuse to long side

$$\frac{a}{h} = \frac{b}{y} = \frac{c}{b}$$

From the first

$$a^2 = cx$$

From the second

$$b^2 = cy$$

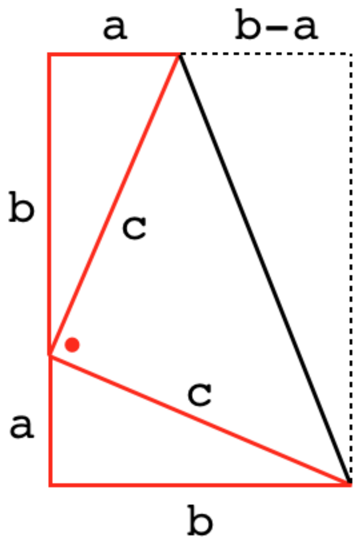
Just add

$$a^2 + b^2 = cx + cy$$

$$= c(x + y) = c^2$$

Garfield

There is one by a future President of the United States, James A. Garfield. (He was a congressman at the time).



Draw a right triangle and a rotated copy as shown. The angles opposite sides a and b are complementary angles. So the angle marked with a dot is a right angle, and the triangle with sides labeled c is a right triangle.

The area of the quadrilateral is the product of the side $(a + b)$ and the *average* of a and b (top and bottom). This can be seen intuitively. The halfway point of the solid red line has horizontal dimension $(a + b)/2$. Hence

$$A = (a + b) \cdot \frac{1}{2}(a + b)$$

If you're worried about that argument, just subtract the area of the triangle with two dotted sides from the quadrilateral that includes it:

$$A = (a + b)b - \frac{(a + b)(b - a)}{2}$$

$$= (a + b)(b - \frac{b}{2} + \frac{a}{2})$$

$$= (a + b) \cdot \frac{1}{2}(a + b)$$

which is just what we said. So now:

$$= \frac{a^2}{2} + ab + \frac{b^2}{2}$$

But we can also calculate the area of the quadrilateral as the sum of the three triangles:

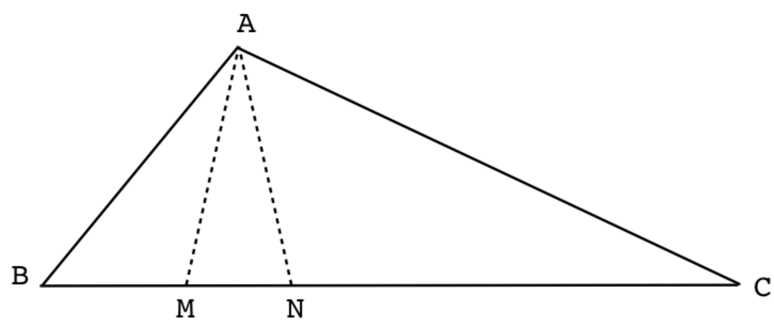
$$A = \frac{ab}{2} + \frac{ab}{2} + \frac{c^2}{2}$$

Equate the two and the result follows almost immediately.

□

Corollary

There are several important corollaries of the Pythagorean theorem. We'll derive one later called the law of cosines. Here is another from the Islamic geometer Ibn Quorra, who brought algebraic techniques, shunned by the Greeks, to geometry.



Let $\triangle ABC$ be *any* triangle (here it is obtuse). Draw AM and AN so that the new angles $\angle AMB$ and $\angle ANC$ are equal to $\angle A$. The corresponding triangles are similar to the original, because they share the angle of measure A plus one other from the original triangle.

Then

$$BM : AB = AB : BC$$

Thus, $AB^2 = BM \times BC$. Similarly

$$NC : AC = AC : BC$$

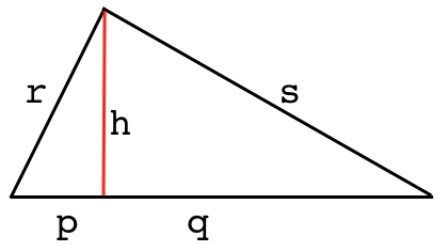
So $AC^2 = NC \times BC$ Therefore

$$\begin{aligned} AB^2 + AC^2 &= BM \times BC + NC \times BC \\ &= (BM + NC) \times BC \end{aligned}$$

In the case where the angle at vertex A is a right angle, then M coincides with N, and $BM + NC = AC$, and this reduces to the Pythagorean theorem.

geometric mean

As a slight detour from calculus, but on the topic of this chapter



We will show that

$$h^2 = pq$$

$$h = \sqrt{pq}$$

That is, h is the geometric mean of these two values p and q.

Proof.

Using the Pythagorean theorem with the two small triangles (also right triangles), we obtain:

$$h^2 + p^2 = r^2$$

$$h^2 + q^2 = s^2$$

Summing

$$2h^2 + p^2 + q^2 = r^2 + s^2$$

Using the theorem with the big triangle:

$$r^2 + s^2 = (p+q)^2$$

$$= p^2 + 2pq + q^2$$

Equating the two expressions for $r^2 + s^2$ we get:

$$2h^2 + p^2 + q^2 = p^2 + 2pq + q^2$$

$$h^2 = pq$$

$$h = \sqrt{pq}$$