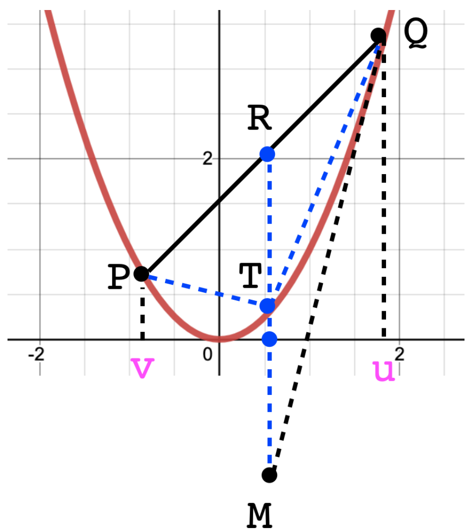
# Archimedes' Quadrature (Lemma)

We verify one of Archimedes lemmas (propositions) very quickly.

Let the parabola be $y = x^2$ with points $P = (v, v^2)$ and $Q = (u, u^2)$. The secant $PQ$ has equation $y = (u + v)x - uv$ (check both points). The tangent $QM$ has equation $y = 2ux - u^2$ (ditto).




So then let $w$ lie somewhere between $v$ and $u$, and then $T_y = w^2$. We can get $R$ and $M$ from the equations above as $R_y = (u+v)w - uv$ and $M_y = 2uw - u^2$.

The first ratio is *

$$\frac{PR}{PQ} = \frac{w - v}{u - v}$$

by similar triangles (bases and hypotenuses in same proportion).

Then the other ratio that we want to match it is $RT/RM$.

$$|RT| = R_y - T_y = (u+v)w - uv - w^2$$

$$|RM| = R_y - M_y = (u+v)w - uv - 2uw + u^2$$
$$= vw - uv - uw + u^2$$

We can be guided by the first ratio. We want $(u-v)$ on the bottom.

$$|RM| = u(u-v) + w(v-u) = (u-v)(u-w)$$

So then let's try to find $(u-w)$ on top

$$|RT| = (u+v)w - uv - w^2$$

$$= uw - uv + wv - w^2$$

$$= u(w-v) + w(v-w) = (u-w)(w-v)$$

Thus

$$\frac{RT}{RM} = \frac{(u-w)(w-v)}{(u-v)(u-w)} = \frac{w-v}{u-v} = \frac{PR}{QR}$$

□

* *Proof* (additional).

The secant has equation $y = kx + y_0$ where $k = (u+v)$. So $|RP|^2$

$$= \Delta y^2 + \Delta x^2 = \ [\ kw - y_0 - (kv - y_0)\ ]^2 + (w-v)^2$$

$$(k^2+1)(w-v)^2$$

$|PQ|^2$ is exactly the same, with $u$ substituted for $w$. So in the ratio of the square roots we have just $(w-v)/(u-v)$.