Brahmagupta without trig

Brahmagupta without trigonometry

I found an old geometry book online (Johnson, *Advanced Euclidean Geometry* (1929)).

`https://www.isinj.com/mt-usamo/Advanced%20Euclidean%20Geometry%20-%20Roger%20Johnson%20(Dover,%201960).pdf`

Theorem 109 is Brahmagupta's theorem, though he isn't mentioned. The proof is really interesting because there are no trig functions. We rely on a clever construction, ratios from similar triangles, and a deep concept about ratios of areas in similar triangles.

However, it also involves some wonky algebra, which we'll go through in a separate section.

As we said before Brahmagupta's theorem is that the area of a cyclic quadrilateral with sides a, b, c and d is

$$K = \sqrt{(s-a)(s-b)(s-c)(s-d)}$$

where s is the semi-perimeter, namely $2s = a + b + c + d$.

construction

As with most proofs, there is a construction that makes everything possible.

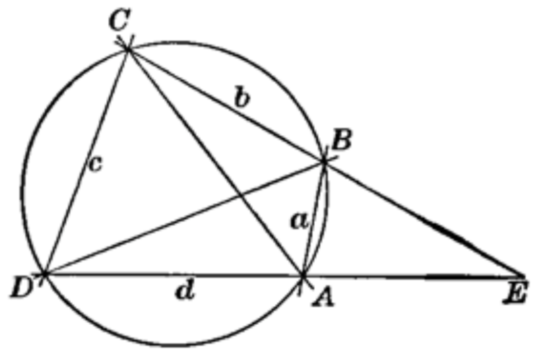

Fig. 28

The cyclic quadrilateral $ABCD$ is extended on two sides that converge to form a triangle.

Johnson says that "if $ABCD$ were a rectangle, the proof would follow trivially." Which is a good thing, since in that case we can't draw the triangle.

Proof.

$$K = \sqrt{(s-a)(s-b)(s-c)(s-d)}$$

Since $a = c$ and $b = d$

$$K^2 = (s-a)^2\ (s-b)^2$$

$$K = (s-a)(s-b)$$

Since $2s = a + b + c + d = 2a + 2b$:

$$K = \ [\frac{2a+2b}{2} - a\ ]\ [\ \frac{2a+2b}{2} - a\ ] = ba$$

□

Let x and y be the extended sides CE and DE in the triangle, with side c as the third side of $\triangle CDE$.

Recalling our previous work, we notice that $\angle BAE = \angle C$, since $\angle BAD$ is supplementary to both. So $\triangle ABE \sim \triangle CDE$.

From similar triangles we have

$$\frac{x}{c} = \frac{y-d}{a}, \qquad \frac{y}{c} = \frac{x-b}{a}$$

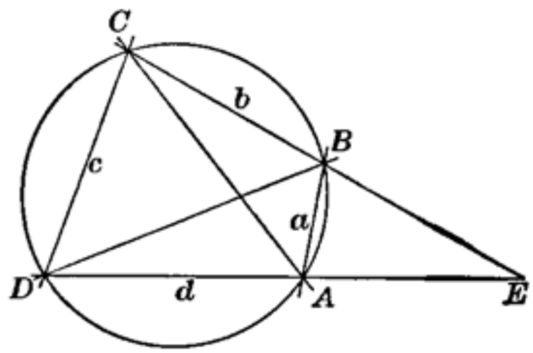

Fig. 28

Johnson simply says that adding these and solving for $x+y$ we may obtain

$$x + y + c = \frac{c}{c-a}(a + b - c + d)$$

and similarly find the other three expressions that we need (each term appears singly negative in one of the four). Let's see.

fancy algebra

First rewrite the ratios as

$$ax = cy - cd, \qquad ay = cx - bc$$

In the first case we want $x + y$ so let's add

$$ax + ay = cx + cy - bc - cd$$

$$(a-c)(x+y) = -c(b+d)$$

$$x + y = \frac{c}{c-a}(b+d)$$

and then a trick

$$\begin{aligned} x + y + c &= \frac{c}{c-a}(b+d) + \frac{c(c-a)}{c-a} \\ &= \frac{c}{c-a}\ (-a+b+c+d) \end{aligned}$$

The second one is also $x + y$ but has minus c

$$\begin{aligned} x + y - c &= \frac{c}{c-a}(b+d) - \frac{c(c-a)}{c-a} \\ &= \frac{c}{c-a}\ (a+b-c+d) \end{aligned}$$

The third one needs to be $x - y$ which we get as

$$ax = cy - cd, \qquad -ay = -cx + bc$$

$$\begin{aligned} ax - ay &= -cx + cy + bc - cd \\ &= -c(x-y) + c(b-d) \end{aligned}$$

so

$$\begin{aligned} (x-y)(a+c) &= c(b-d) \\ x - y &= \frac{c}{a+c}(b-d) \end{aligned}$$

and then

$$\begin{aligned} x - y + c &= \frac{c}{a+c}(b-d) + \frac{c(a+c)}{a+c} \\ &= \frac{c}{a+c}(a+b+c-d) \end{aligned}$$

The last one is $y - x$ so

$$y - x = \frac{c}{a+c}(d-b)$$

and then

$$-x+y+c = \frac{c}{a+c}(d-b) + \frac{c(a+c)}{a+c}$$
$$= \frac{c}{a+c}(a-b+c+d)$$

Assembling everything, on the left-hand side we have

$$(x+y+c)(x+y-c)(x-y+c)(-x+y+c)$$

and on the right hand side we have

$$[\ \frac{c^2}{c^2-a^2}\ ]^2\ (-a+b+c+d)(a-b+c+d)(a+b-c+d)(a+b+c-d)$$

I think that's quite a lot of algebra to just skip over.

converting to the semi-perimeter

By Heron's formula we have for the area of $\triangle CDE$:

$$K = \frac{1}{4}\ \sqrt{(x+y+c)(-x+y+c)(x-y+c)(x+y-c)}$$

But by our algebraic manipulations that is equal to

$$K = \frac{1}{4}\ \frac{c^2}{c^2-a^2}\ \sqrt{(-a+b+c+d)(a-b+c+d)(a+b-c+d)(a+b+c-d)}$$

Then, each of the terms under the square root can be connected to the semi-perimeter s because, for example

$$2s = a+b+c+d$$

so

$$2s-2a = -a+b+c+d$$
$$2(s-a) = (-a+b+c+d)$$

We accumulate a factor of 16 under the square root, which just cancels the 4 leaving

$$K = \frac{c^2}{c^2 - a^2} \sqrt{(s-a)(s-b)(s-c)(s-d)}$$

This is for the area of the triangle $\triangle CDE$. It is bigger than what's under the square root by the factor of $c^2/(c^2 - a^2)$, which is greater than one.

It remains to connect this area to that of the quadrilateral. Naturally, it will turn out that they are connected by the same factor.

connecting the areas

The last idea is that

$$\Delta_{CDE} = (ABCE) + \Delta_{ABE}$$

where these terms are all areas.

But $\triangle CDE \sim \triangle ABE$ and the ratio between the areas is

$$\frac{\Delta_{ABE}}{\Delta_{CDE}} = \frac{a^2}{c^2}$$

This goes back to the Pythagorean theorem. The ratio of areas of two similar triangles is proportional to the squares on corresponding sides.

Proof. Let the two similar triangles have sides abc and ABC. If ϕ is the angle between a and b, and also between A and B, then the ratio of areas is

$$\frac{ab \sin\phi \cdot 1/2}{AB \sin\phi \cdot 1/2} = \frac{ab}{AB} = \frac{a^2}{A^2}$$

□

So dividing both sides by Δ_{CDE} we obtain:

$$\frac{(ABCD)}{\Delta_{CDE}} = \frac{\Delta_{CDE}}{\Delta_{CDE}} - \frac{\Delta_{ABE}}{\Delta_{CDE}}$$

$$= 1 - \frac{a^2}{c^2} = \frac{c^2 - a^2}{c^2}$$

so

$$(ABCD) = \frac{c^2 - a^2}{c^2}\ \Delta_{CDE}$$

But this is just the factor that we are looking for. Multiply what we had before by this value and obtain the area of $ABCD$ as

$$K = \sqrt{(s-a)(s-b)(s-c)(s-d)}$$

$\square$