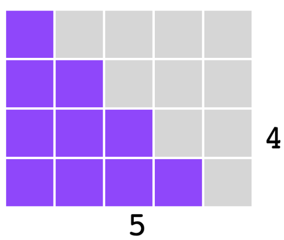
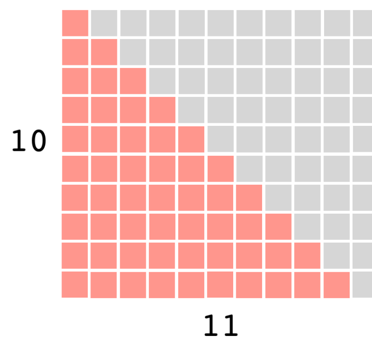
Sum of integers

In calculus, we will need formulas for the sum of the squares of the first n integers, the cubed integers, and so on. To keep it simple, let's just start with plain integers, $1 + 2 + 3 + \cdots + n$.

It seems likely that someone noticed the following pattern pretty early:





The number of colored squares is one-half the total in each case. If we call the number of squares in each column n, then the total is $n(n+1)$ and one-half that is

$$S_n = \frac{n(n+1)}{2}$$

which is also, evidently the sum of the integers from 1 to n.

This is representative of a class of demonstrations that are often called "proofs without words." Except that we have added some words.

The partial sums are called the *triangular* numbers.

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1   2   3   4   5   6
1   3   6  10  15  21
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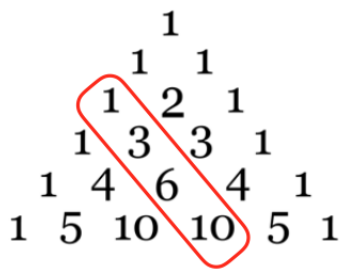
Each number in the second row is the sum of the number to the left plus the one above. The triangular numbers are

$$1, 3, 6, 10 \cdots$$

They can be generated as the third diagonal of Pascal's triangle.:



There is a famous story about Gauss. As a schoolboy, he "saw" how to add the integers from 1 to 100 as two parallel sums.

$$\begin{array}{cccc|l} 1 & 2 & \ldots & 99 & 100 \;\; S_n \\ 100 & 99 & \ldots & 2 & 1 \;\; S_n \\ \hline 101 & 101 & & 101 & 101 \end{array}$$

Added together horizontally, these two series must equal twice the sum of 1 to 100.

But vertically, we notice that each sum is equal to $n+1$, and we have n of them.

So, again

$$2S_n = n(n+1)$$
$$S_n = \frac{1}{2}\, n(n+1)$$

For $n = 100$ the value of the sum is 5050, which is what Gauss is said to have written on his slate and presented to the teacher immediately on being given the problem as a make-work exercise.

One way of looking at this result is that between 1 and 100 there are 100 representatives of the "average" value in the sequence, which (because of the monotonic steps) is $(100+1)/2 = 50.5$.

Or alternatively, view the sum as ranging from 0 to 100 (with the same answer). Now there are 101 examples of the average value $(100{+}0)/2 = 50)$.

induction

Perhaps you have seen the method called induction. Probably the most famous example of an inductive proof is that for the formula we've been working with, the sum of integers.

$$S_n = 1 + 2 + \cdots + n$$

Proof.

Suppose someone has sent us, anonymously, a formula which they claim gives the sum of the first n integers, namely

$$S_n = \frac{n(n+1)}{2}$$

Assume the formula is correct for S_n. Add $n+1$ to both sides. The left-hand side becomes S_{n+1}, so we have:

$$\begin{aligned} S_{n+1} &= \frac{(n)(n+1)}{2} + (n+1) \\ &= (n+1)(\frac{n}{2} + 1) \\ &= \frac{(n+1)(n+2)}{2} \end{aligned}$$

which is exactly what we'd get by substituting $n+1$ for n in the original formula.

Alternatively, sometimes it's clearer to assume the $n-1$ case and prove the formula is correct for n:

$$\begin{aligned} S_{n-1} &= \frac{n(n-1)}{2} + n \\ S_n &= n\ [\ \frac{(n-1)}{2} + 1\ ] \\ &= \frac{n(n+1)}{2} \end{aligned}$$

So we have proven that if the S_n formula is correct, then so is the one for S_{n+1}.

How do we know that S_n is correct?

Just check the *base case*:

$$S_1 = \frac{1(1+1)}{2} = 1$$

Since S_1 is clearly correct, S_2 must be also, and this continues all the way to S_n.

$$S_1 \Rightarrow S_2 \Rightarrow \ldots S_{n-1} \Rightarrow S_n \Rightarrow S_{n+1}$$

Therefore, it must be true for *every* integer n.

□

Derivation using sums

I'm going to derive the equation we have been using using algebra. The general method will help us later.

For any number, and in particular, any integer k it is true that

$$(k+1)^2 = k^2 + 2k + 1$$

So consider what happens if we sum the values from $k = 1 \rightarrow n$ for each of these terms

$$\sum_{k=1}^{n}(k+1)^2 = \sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} 2k + \sum_{k=1}^{n} 1$$

If the equation is valid for any individual k, then the sum is also valid, plugging in all k up to n.

Rearranging

$$\sum_{k=1}^{n}(k+1)^2 - \sum_{k=1}^{n} k^2 = \sum_{k=1}^{n} 2k + \sum_{k=1}^{n} 1$$

Now think about the left-hand side in our equation.

$$\sum_{k=1}^{n}(k+1)^2 - \sum_{k=1}^{n} k^2$$

We have a bunch of terms starting with 2^2:

$$2^2 + 3^2 + \cdots + n^2 + (n+1)^2$$

we also have a bunch of terms to subtract starting with 1^2:

$$1^2 + 2^2 + 3^2 + \cdots + n^2$$

Almost everything cancels. This is called a "collapsing" or "telescoping" sum. We have

$$(n+1)^2 - 1 = n^2 + 2n$$

Bringing back the right-hand side we obtain:

$$n^2 + 2n = \sum_{k=1}^{n} 2k + \sum_{k=1}^{n} 1$$

We can bring the constant factor 2 out of the sum, and also, we recognize that the sum of the value 1 a total of n times is just n.

$$n^2 + 2n = 2\sum_{k=1}^{n} k + n$$

Subtract n from both sides and divide by 2:

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

That's it!

another method

Here is another approach that I found very early in Hamming (Chapter 2) and have not seen in other books. He calls it the method of *undetermined coefficients.*

We suspect that the sum of integers formula has order n^2, because if we represent the integers as collections of squares or circles or something, and then stack them up next to each other it forms a triangle.

Another way to think about it is to suppose that the highest term is of degree 1, just n:

$$S_n = kn$$

but that would mean that

$$S_{n+1} = k(n+1) \neq kn + (n+1)$$

That's a contradiction.

And still another way is to look at the growth of the sum

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 0 + 1 =  1
 1 + 2 =  3
 3 + 3 =  6
 6 + 4 = 10
10 + 5 = 15
15 + 6 = 21
21 + 7 = 28
..
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When adding an odd number, the sum grows from 1 times the number to twice, three and then four times the number. It goes up by another factor of n every two steps.

It's obviously growing faster than n.

Given a term of n^2 the most general formula would be

$$\sum_{k=0}^{k=n} k = an^2 + bn + c$$

and if $n = 0$ the sum is zero so $c = 0$. The right-hand side is now $an^2 + bn$.

We can actually solve for a and b using S_1 and S_2.

$$S_1 = 1 = an^2 + bn = a + b$$

$$S_2 = 3 = 4a + 2b$$

Multiply the first equation by -4 and add to the second one:

$$-1 = -2b$$

So $b = 1/2$ and then $1 = a + 1/2$ so $a = 1/2$ as well.

general solution

The inductive step is to write the formula for $m-1$, and then add m to it.

The right-hand side is just the formula, writing m for n

$$am^2 + bm$$

The left-hand side is the formula for $(m-1)$, plus m from the induction step:

$$a(m-1)^2 + b(m-1) + m$$
$$= am^2 - 2am + a + bm - b + m$$

Both terms on the right-hand side cancel and we are left with

$$-2am + a - b + m = 0$$

What makes this method great is that the terms containing each power of m (here just m^1 and m^0), must individually cancel. That is

$$-2am + m = 0$$

$$-2a + 1 = 0$$

which gives $a = 1/2$. And the other part is

$$a - b = 0$$

So $b = a$ and then

$$S_n = an^2 + bn = \frac{1}{2}(n^2 + n) = \frac{n(n+1)}{2}$$

series proof

This proof does little more than the proof without words given above, and in fact depends on another one, but it is interesting because it uses series directly.

We see that the sum of the first n odd numbers is equal to n^2.

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2$$

This formula is easily proven by induction, since the next term is $2n+1$, which when added to n^2 gives $(n + 1)^2$.

Count up the same number of 1's

$$1 + 1 + 1 + \cdots + 1 = n$$

Add term by term to the first series

$$2 + 4 + 6 + \cdots + 2n = n^2 + n$$

But that result is just twice what we want:

$$1 + 2 + 3 + \cdots + n = (n^2 + n)/2$$