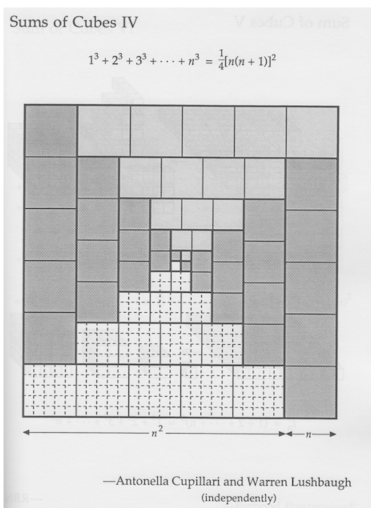
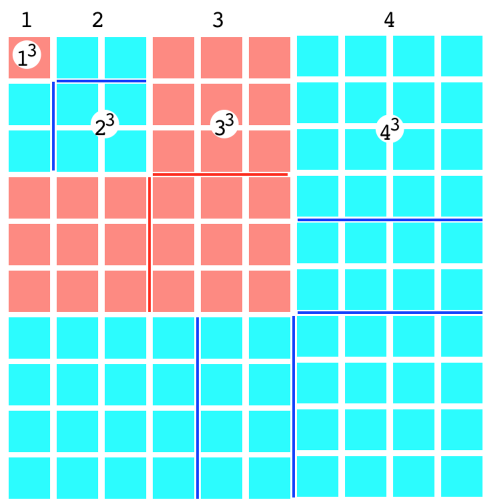
Sum of cubes

The formula is

$$\sum_{k=1}^{n} k^3 = [\ \sum_{k=1}^{n} k\ ]^2$$

which we can understand using these diagrams





or one later on in this write-up

Proof.

By induction.

The "base case" is pretty simple. For $n = 1$:

$$1^3 = 1$$

and

$$[\ \frac{n(n+1)}{2}\ ]^2 = 1^2$$

It's not necessary, but for $n = 2$

$$1^3 + 2^3 = 1 + 8 = 9$$

and

$$[\ \frac{n(n+1)}{2}\ ]^2 = \ [\ \frac{2(3)}{2}\ ]^2 = 9$$

For the induction step what we need to show is by assuming the formula for n is correct and then adding the term $(n+1)^3$

$$[\ \frac{n(n+1)}{2}\ ]^2 + (n+1)^3$$

is equal to what we get by plugging $n+1$ into the formula.

$$[\ \frac{(n+1)(n+2)}{2}\ ]^2$$

Set the two equations equal and factor out $(n+1)^2$ from both sides. That leaves

$$(\frac{n}{2})^2 + (n+1) = (\frac{n+2}{2})^2$$

$$n^2 + 4(n+1) = (n+2)^2$$

which looks correct.

□

derivation by collapsing sum

We proceed exactly as we did for integers and squares of integers:

$$(k+1)^4 = k^4 + 4k^3 + 6k^2 + 4k + 1$$

Sum each term from $k = 1 \to k = n$ (and suppress the indices as before)

$$\sum (k+1)^4 = \sum k^4 + \sum 4k^3 + \sum 6k^2 + \sum 4k + \sum 1$$

Rearrange and simplify the collapsing sum.

$$(n+1)^4 - 1 = \sum 4k^3 + \sum 6k^2 + \sum 4k + \sum 1$$

Place factors before the sums and substitute our known formulas.

$$n^4 + 4n^3 + 6n^2 + 4n = 4\sum k^3 + 6\frac{n(n+1)(2n+1)}{6} + 4\frac{n(n+1)}{2} + n$$

$$n^4 + 4n^3 + 6n^2 + 3n = 4\sum k^3 + n(n+1)(2n+1) + 2n(n+1)$$

Work with the left-hand side:

$$n^4 + 4n^3 + 6n^2 + 3n = n(n^3 + 4n^2 + 6n + 3)$$

We have to believe there will be an $(n+1)$ in there. Find it by long division and then the left-hand side is

$$= n(n+1)(n^2 + 3n + 3)$$

So factor $n(n+1)$ from each term on the right-hand side after the sum, and bring it all over to the left-hand side as a subtraction:

$$n(n+1)(n^2 + 3n + 3 - (2n+1) - 2) = 4\sum k^3$$

$$n(n+1)(n^2 + n) = 4\sum k^3$$

$$\sum k^3 = \ [\ \frac{n(n+1)}{2}\ ]^2$$

A remarkable simplification!

Looking deeper

$$\sum_{k=1}^{n} k^3 = [\ \sum_{k=1}^{n} k\ ]^2$$

We want to try to understand something more about why this is true.

A web search revealed the answer. Here's an interesting pattern for the cubes of integers

$$1^3 = 1$$

$$2^3 = 8 = 3 + 5$$

$$3^3 = 27 = 7 + 9 + 11$$

$$4^3 = 64 = 13 + 15 + 17 + 19$$

$$5^3 = 125 = 21 + 23 + 25 + 27 + 29$$

If you want a formula for n^3, notice that the first term is $n^2 - n + 1$ and the last term is $n^2 - n + 2n - 1$, and the number of terms for each sum equals n. (There are n odd numbers between 1 and $2n - 1$).

In other words, the sum of all the cubes of integers from 1^3 to n^3 is equal to the sum of all the odd numbers up to $n^2 - n + 2n - 1 = n^2 + n - 1$.

How many of these numbers are there? A little thought should convince you that the answer is $(n^2 + n)/2$. For example, with $n = 5$, our last odd number is $5^2 + 5 - 1 = 29$, and we have $(25 + 5)/2 = 15$ terms.

We want the sum of the first $(n^2 + n)/2$ odd numbers.

Let's look at another pattern

$$1 = 1$$

$$2^2 = 4 = 1 + 3$$

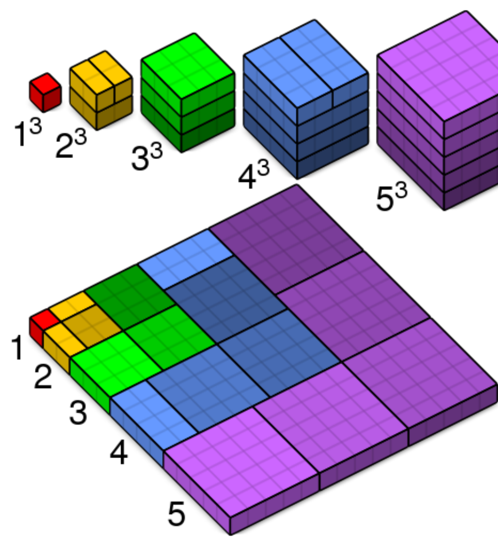
$$3^2 = 9 = 1 + 3 + 5$$

$$4^2 = 16 = 1 + 3 + 5 + 7$$

$$5^2 = 25 = 1 + 3 + 5 + 7 + 9$$

The *odd number theorem* says that the sum of the first n odd numbers is equal to n^2. We want the sum of the first $(n^2 + n)/2$ odd numbers, so that's $((n^2 + n)/2)^2$. And that's how we get our formula.

Here is another beautiful proof without words:



The length of the bottom pattern is a triangular number, which is itself a sum of squares. When squared it equals the sum of cubes.

another method

Here is another approach that I found very early in Hamming (Chapter 2) and have not seen in other books. It is called the method of *undetermined coefficients*.

We observe that the sum of integers formula has order n^2, while the sum of squares has order n^3, so we expect the sum of cubes would have n^4.

$$\sum_{k=0}^{k=n} k^3 = an^4 + bn^3 + cn^2 + dn + e$$

and if $n = 0$ the sum is zero so $e = 0$.

The right-hand side is

$$an^4 + bn^3 + cn^2 + dn$$

The inductive step is to write the formula for $m - 1$, and then add m^3 to it.

The right-hand side is just the formula, writing m for n

$$am^4 + bm^3 + cm^2 + dm$$

The left-hand side is the formula for $(m-1)$, plus m^3 from the induction step:

$$a(m-1)^4 + b(m-1)^3 + c(m-1)^2 + d(m-1) + m^3$$

We work with the left-hand side. Expand each term using the binomial theorem:

$$a\ [\ m^4 - 4m^3 + 6m^2 - 4m + 1\ ]$$

$$b\ [\ m^3 - 3m^2 + 3m - 1\ ]$$

$$c\ [\ m^2 - 2m + 1\ ]$$

$$d\ [\ m - 1\ ]$$

Next, group the cofactors by the corresponding powers:

$$[\ a\ ]\ m^4$$

$$[\ -4a + b + 1\ ]\ m^3$$

$$[\ 6a - 3b + c\ ]\ m^2$$

$$[\ -4a + 3b - 2c + d\ ]\ m$$

$$a - b + c - d$$

Now to the point. The cofactors for *each power* of m must cancel exactly.

am^4 cancels on left and right, likewise bm^3, cm^2 and dm. That leaves four equations.

$$-4a + 1 = 0$$

$$6a - 3b = 0$$

$$-4a + 3b - 2c = 0$$

$$a - b + c - d = 0$$

We find that $a = 1/4$, $b = 1/2$, $c = 1/4$, $d = 0$. So then finally the formula is

$$an^4 + bn^3 + cn^2 + dn$$

$$= \frac{n^4 + 2n^3 + n^2}{4}$$

$$\frac{(n^2 + n)^2}{2^2} = \ [\ \frac{n(n+1)}{2}\ ]^2$$

which is exactly what we will have from other approaches.

Hamming uses this method to get a general formula, but we will not need that, because we will show how to use the binomial theorem to get what is necessary.