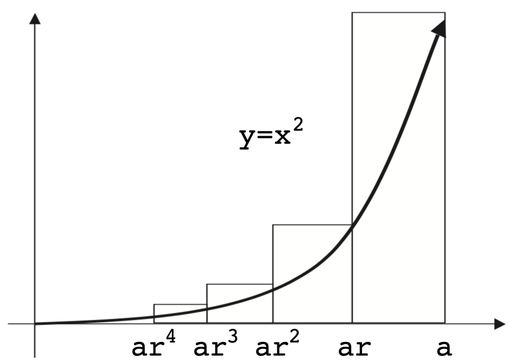
Fermat area

Fermat had the idea of approximating the area under a curve by adding up a series of rectangles. To start with, we will look at the curve $y = x^2$. And for simplicity, we will start from $x = 0$ and go up to $x = a$.

But the really great idea is to divide the number line from $0 \to a$ into segments using $r < 1$. So going backward from a we have $ar < a$ and so on: i.e. $\dots ar^3 < ar^2 < ar < a$.



The number of rectangles is infinite, and the approximation to the area will become exact as we let $r \to 1$.

The last rectangle has base $a - ar$ and height a^2 so

$$S = (a - ar) \cdot (a)^2 + \dots$$

The next term is for the rectangle with base $a = ar - ar^2$ and height $(ar)^2$:

$$S = (a - ar) \cdot (a)^2 + (ar - ar^2) \cdot (ar)^2 + \dots$$

The third term is

$$(ar^2 - ar^3) \cdot (ar^2)^2$$

which gives

$$S = (a - ar) \cdot (a)^2 + (ar - ar^2) \cdot (ar)^2 + (ar^2 - ar^3) \cdot (ar^2)^2 + \dots$$
$$= a^3(1 - r) \; [\; 1 + r^3 + r^6 + \dots \;]$$

This is a geometric series! Since $r < 1$ it converges and the value of the sum is

$$S = \frac{a^3(1 - r)}{1 - r^3}$$

But $1 - r^3$ can be factored into

$$1 - r^3 = (1 - r)(1 + r + r^2)$$

which means we have

$$S = \frac{a^3}{1 + r + r^2}$$

And then as $r \to 1$ this becomes just

$$S = \frac{a^3}{3}$$

which matches Archimedes.

general case

Then

$$S = (a - ar) \cdot (a)^n + (ar - ar^2) \cdot (ar)^n + (ar^2 - ar^3) \cdot (ar^2)^n + \dots$$

We can factor out a^{n+1} and $1 - r$ just like before

$$S = (a^{n+1})(1 - r)(1 + r^{n+1} + (r^{n+1})^2 + \dots \;) \;]$$

This is also a geometric series with

$$S = \frac{a^{n+1}(1-r)}{1-r^{n+1}} = \frac{a^{n+1}}{1+r+r^2+\cdots+r^n}$$

$$= \frac{a^{n+1}}{n+1}$$