non-perfect squares have irrational square roots

`https://twitter.com/wtgowers/status/1542096796880625664`

one

The numbers $a, b, m, , A, B$ below are all integers, while d and n are *positive* integers.

As a ratio of integers, a/b is a rational number. Every integer multiple of a/b is also a multiple of $1/b$. So the multiple is either an integer or is *at least* $1/b$ away from an integer.

For example, with $b = 10$, the first positive integer multiple would be $10/b = 1$, bounded by $9/b$ and $11/b$, which are $1/10$ away from 10.

two

Let d be any positive integer that is not a perfect square. Choose m such that $m < \sqrt{d} < m + 1$. For example, with $d = 5$, $m = 2$.

Consider a positive integer n and

$$(\sqrt{d} - m)^n$$

I claim that for some A and B

$$(\sqrt{d} - m)^n = A\sqrt{d} + B$$

The binomial expansion of the left-hand side has terms that are integer powers of $\sqrt{d}$. The even powers are integers and the odd ones are multiples of $\sqrt{d}$. Collecting like terms, we obtain the result.

three

But the left-hand side $(\sqrt{d}-m)^n$ tends to zero as n gets large, without ever equalling zero. So the right-hand side $A\sqrt{d} + B$ can be made arbitrarily small but non-zero.

Equivalently, $A\sqrt{d}$ can be made arbitrarily close to an integer (namely, $-B$) without actually being an integer.

By step one, $\sqrt{d}$ is not rational.

$\square$