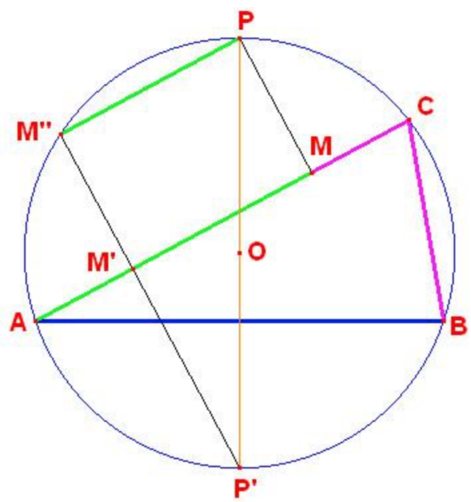
Broken Chord

Here is a somewhat more complicated proof of the broken chord theorem which I also found on the web, attributed to Bùi Quang Tuản. It has slightly different notation than I've used before. In the figure below, P is the point on the circle midway between A and B, and the vertical drops to M.

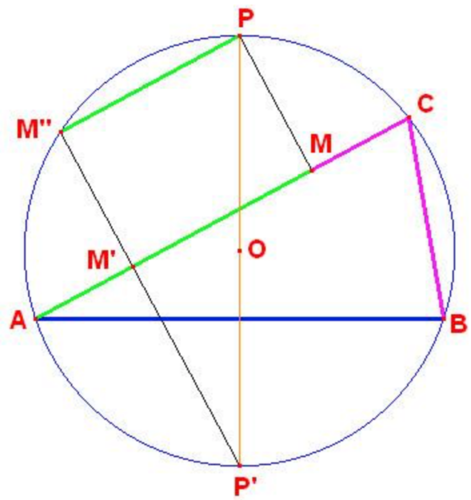
The claim is that $AM = MC + BC$.



A key step in the proof will be to show that $MM'M''P$ is a rectangle, and a consequence is that $AM' = MC$. This conclusion relies on a lemma, namely, that given such a rectangle the extensions on the longer chord, AM' and MC are equal.

This is a basic result. There is a unique perpendicular bisector for any two parallel chords in a circle. As a result, if a rectangle is drawn with two points lying on the circle, and if the opposing side is extended to meet the circle, the extensions are equal. We showed this earlier.

Proof.



Draw the diameter from P through the center of the circle at O to P'. Since P is midway between A and B, this diameter is also the perpendicular bisector of AB (relying on the same chapter cited above).

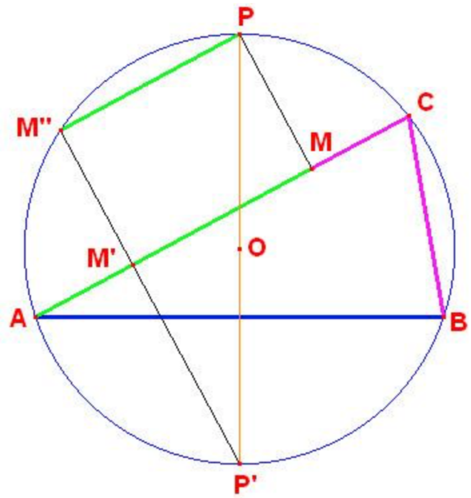
Lay off a chord the same length as CB from P to M''. Complete the second triangle by drawing $M''P'$. We have that the angle at vertex P' is equal to the angle at vertex A, since they are inscribed angles of equal chords.

(However, the other two angles are not equal and the two large triangles are not similar, which is perhaps obvious since AB is not a diameter of the circle).

Since the angle at A is equal to the angle at P', and the smallest triangles with those vertices also contain two vertical angles, they are similar triangles. And since PP' is perpendicular to AB, the third angle in both triangles is a right angle. Therefore, the angles at M'

are right angles.

The vertex at M'' is also a right angle, by Thales' theorem.



Since $\angle PMM'$ is a right angle, and we have established that both $\angle PM''M'$ and $MM'M''$ are right angles, the fourth angle MPM'' is also right, and $MM'M''P$ is a rectangle.

As opposing sides in a rectangle, $M'M = M''P = CB$.

Because $MM'M''P$ is a rectangle, we reference the preliminary discussion to conclude that

$$AM' = MC$$

Adding equals to equals

$$AM' + M'M = MC + CB$$

$$AM = MC + CB$$

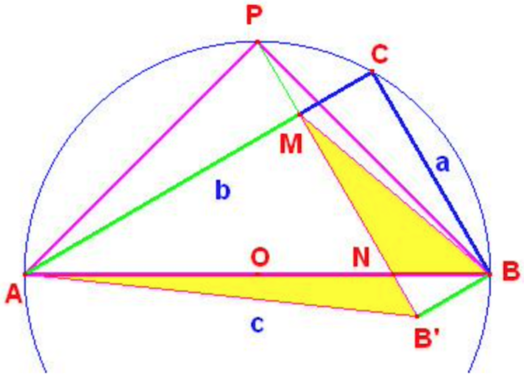
$\square$

The key to the construction was to make PM'' equal to CB. An alternative approach is to construct $\angle P'$ on the diagonal PP' such that $\angle P' = \angle A$. From this it follows that $M''P = CB$. We also have the two small similar triangles, which leads to right angles at M'. This implies that $MM'M''P$ is a rectangle.

`https://www.cut-the-knot.org/triangle/BrokenChordBQT.shtml`

Pythagorean theorem

Tuần also extended the broken chord theorem to a proof of the Pythagorean theorem. Here is a diagram from the web.



We start with two right triangles (AB is a diameter of the circle). One of the triangles, $\triangle APB$, is isosceles.

The sides of $\triangle ABC$ are labeled as a, b and c, opposite the corresponding vertices. Side BC has length a.

PM is drawn perpendicular to AC. By the broken chord theorem,

$$AM = MC + BC$$

Twice that is

$$AM + MC + BC = AC + BC = b + a$$

so

$$AM = \frac{b+a}{2}$$

while

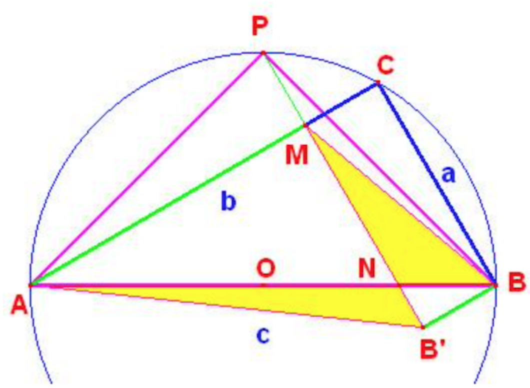
$$\begin{aligned} MC &= AM - BC \\ &= \frac{b+a}{2} - a = \frac{b-a}{2} \end{aligned}$$

PM is extended to meet the diagonal at N and past it to B'. B' is chosen so that $B'BCM$ is a rectangle. Thus side MB' is equal to BC and so to a.

We make two preliminary claims. The first is that $\triangle PMC$ is a right *isosceles* triangle. Let us accept that provisionally.

$$PM = MC = \frac{b-a}{2}$$

The second is that the areas of the two triangles shaded yellow are equal.



We reason as follows. Add $\triangle NB'B$ to both. The area of $\triangle AB'B$ is equal to that of $\triangle MB'B$ because they have the same base BB' and the same altitude, a.

By subtraction

$$(\triangle AB'N) = (\triangle MNB)$$

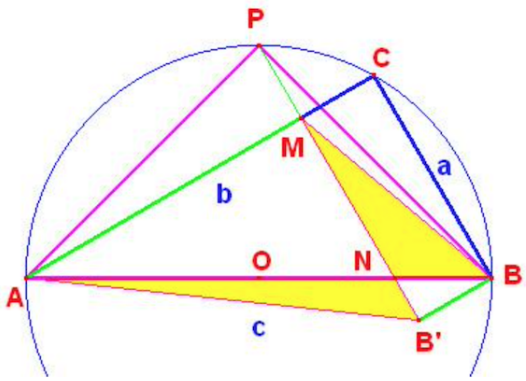
We find the area of $\triangle APB$ in two different ways.

The first is $c^2/4$ (since it is one-quarter of a square with sides c).

The second way is as the sum of several smaller triangles:

$$(\triangle APM) + (\triangle PMB) + (\triangle AMN) + (\triangle MNB)$$
$$(\triangle APM) + (\triangle PMB) + (\triangle AMN) + (\triangle AB'N)$$
$$(\triangle APM) + (\triangle PMB) + (\triangle AMB')$$

The second line follows from the previous claim about the yellow triangles, and the third is by simple addition of areas.



So then the areas are

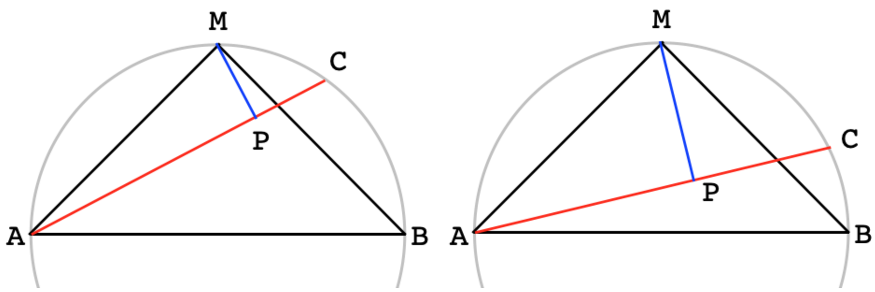
$$(\triangle APM) = \frac{1}{2} \cdot AM \cdot MC = \frac{1}{2} \cdot \frac{b+a}{2} \cdot \frac{b-a}{2}$$
$$(\triangle PMB) = \frac{1}{2} \cdot PM \cdot MC = \frac{1}{2} \cdot \frac{(b-a)}{2} \cdot \frac{(b-a)}{2}$$
$$(\triangle AMB') = \frac{1}{2} \cdot PM \cdot MC = \frac{1}{2} \cdot a \cdot \frac{b+a}{2}$$

We compute 8 times the sum, so as not to deal with fractions:

$$(b^2 - a^2) + (b^2 - 2ab + a^2) + (2ab + 2a^2) = 2b^2 + 2a^2$$

Previously, we calculated the area as $c^2/4$, and 8 times that is $2c^2$. The result follows immediately.

The proof is not yet complete. We must show that $\triangle PMC$ is isosceles. Simplifying the figure:

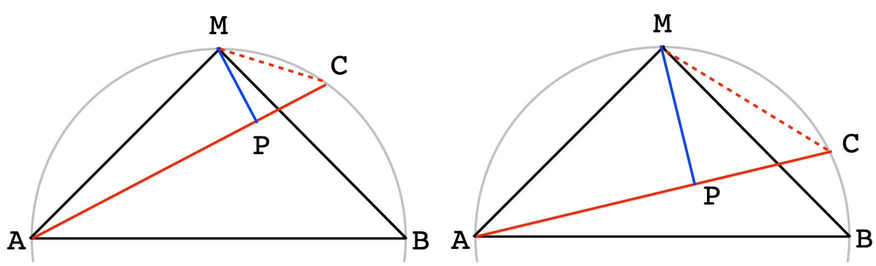


Let AB be a diameter of the circle and $\triangle AMB$ isosceles. Let C be any point on the perimeter, with $MP \perp AC$. Then, we claim that $MP = PC$.

It seems reasonable. If $C \to B$, the P becomes the origin and the statement is true, while if $C \to M$, both vanish.

Proof.

Connect the two vertices by drawing MC.



Clearly $\angle MCA$ is one-half of a right angle since it intercepts the same arc as $\angle ABM$, by the peripheral angle theorem. Since $\angle MPC$ is right, it follows that $\triangle PMC$ is isosceles (by complementary angles) and so $MP = PC$ (by the converse of the isosceles triangle theorem).

□

Notice how we use the information that $\triangle AMB$ is isosceles.