Quadratics, summary

Summary

The most general equation for a quadratic is

$$y = ax^2 + bx + c$$

where a, b and c are all constants. These are the letters everyone uses for them.

The graph of any particular quadratic equation traces out a parabola, which is shaped like the nose cone of a rocket but is open inside, like a cup.

○ a determines how steeply the sides curve and its sign tells in what direction the "cup opens", up or down.

○ b and a together determine where the vertex lies. The vertex is the very bottom or top of the cup.

○ c shifts the parabola's vertical position and is also the y-value when $x = 0$.

If a is positive, the cup opens up, and y has its minimum value at the vertex. If a is negative, then the cup opens down, and y has its maximum value at the vertex.

Two basic questions to ask about any quadratic are

○ Where is the vertex located? What is the x-value at the vertex?

◦ Where are the roots? These are the x-values where the graph crosses the x-axis, where $y = 0$.

We will call the vertex's x-value m for minimum/maximum. The formula is

$$m = -\frac{b}{2a}$$

When $x = m$, then y attains its min or max value.

If you need to find that y value, plug $x = m = -b/2a$ into the equation and crank away.

To "solve" a quadratic means to find its zeros, or roots, which means $y = 0$.

$$ax^2 + bx + c = 0$$

It is very useful to play in Desmos with the form of the equation produced by factoring out a:

$$y = a \; [\; x^2 + \frac{b}{a}x + \frac{c}{a} \;]$$

That leading a still determines shape, but it no longer changes where the vertex or the zeros are. It is the cofactor of x, namely b/a, which does that.

As a result, we can make an equation easier to work with by factoring out a from x^2 in this way.

In looking for the roots, because of that 0 on the right-hand side, dividing by a does not change the roots. The same values of x satisfy this equation

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

Sometimes we can guess the zeros or roots. For s and t to be zeros, it must be that

$$(x - s)(x - t) = 0$$

$$= x^2 - (s+t)x + st$$

By comparison with the other form (see the previous equation), it should be apparent that

$$\frac{b}{a} = -(s+t) \qquad \frac{c}{a} = st$$

See if you can find two factors that add to give negative $-b/a$ and multiply to give c/a. As an example

$$x^2 + 2x - 63 = y$$

Find integers two units apart that multiply to give 63, realizing that one of them has a minus sign

$$(x-7)(x+9) = x^2 + 2x - 63 = 0$$

This equation equals 0 when $x = 7$ or $x = -9$.

The vast majority of equations do not have integer roots, but this exercise seems valuable because it focuses attention on what it means to be a root.

Another situation in which the zeros are easy is if $c = 0$ because then

$$ax^2 + bx + 0 = x(ax + b) = 0$$

One zero is $x = 0$ and the other is when $ax + b = 0$ so $x = -b/a$.

The x position of the vertex is always the average of the two zeros. That works for the last example and if you look back you see that we had s and t as the roots and

$$\frac{b}{a} = -(s+t)$$

But

$$m = -\frac{b}{2a} = \frac{s+t}{2}$$

m is the average of the roots and we can also write

$$x^2 - 2mx + \frac{c}{a} = 0$$

The zeros can always be found using the quadratic formula. Here is my favorite, simple version of that:

$$x = m \pm \sqrt{m^2 - c/a}$$

You must memorize both the formula for m and this one.

The easiest derivation of the quadratic formula is by "completing the square". Start with the previous equation and rearrange to give

$$x^2 - 2mx = -c/a$$

The bright idea is to turn the left-hand side into $(x - m)^2$ which we do by adding m^2 to both sides:

$$x^2 - 2mx + m^2 = m^2 - c/a$$

$$(x - m)^2 = m^2 - c/a$$

The rest is just algebra.