Tangent to conic section

https://maa.org/sites/default/files/kung11010356273.pdf

The equation below is a general form for any conic section

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

If the point $P = (x_0, y_0)$ lies on the conic section, then it satisfies the above equation.

Next, Kung says, let

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F =$$
$$A(x - x_0)^2 + B(x - x_0)(y - y_0) + C(y - y_0)^2$$

We've simply taken the higher powers of the conic and plugged in $(x - x_0)$ and $(y - y_0)$.

One thing to note is that if $x = x_0$ and $y = y_0$ (we're at P), then the right-hand side is zero. So P is on the conic since now we have

$$Ax_0^2 + Bx_0y_0 + Cy_0^2 + Dx_0 + Ey_0 + F = 0$$

If the right-hand side is not zero (not both $x = x_0$ and $y = y_0$), there are some consequences. No other point on the conic can satisfy the equation, since any point on the conic makes the left-hand side equal to zero.

Also, we lose all the higher power terms when multiplying out. Canceling the common terms:

$$\begin{gathered}Dx + Ey + F = \\ A(-2x_0x + x_0^2) + B(-y_0x - x_0y + x_0y_0) + C(-2y_0y + y_0^2)\end{gathered}$$

Gathering terms we have

$$\begin{aligned}(2Ax_0 + By_0 + D)x + (Bx_0 + 2Cy_0 + E)y \\ -Ax_0^2 - Bx_0y_0 - Cy_0^2 + F = 0\end{aligned}$$

This equation is linear in x and y. It is the equation of a line.

So we have the equation of a line passing through (x_0, y_0) where no other point on the line is on the conic section. In other words, it is the equation of the tangent line to the curve.

A more sophisticated treatment, given in the article, is that the equation of any line can be parametrized.

Suppose $x = x_0 + \lambda_1 t$ and $y = y_0 + \lambda_2 t$ then

$$(x - x_0) = \lambda_1 t$$

$$(y - y_0) = \lambda_2 t$$

Substituting into Kung's equation

$$\begin{gathered}Ax^2 + Bxy + Cy^2 + Dx + Ey + F = \\ A(x - x_0)^2 + B(x - x_0)(y - y_0) + C(y - y_0)^2\end{gathered}$$

we have

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F =$$

$$(A\lambda_1^2 + B\lambda_1\lambda_2 + C\lambda_2^2)t^2 = Kt^2$$

If $t = 0$, we're at (x_0, y_0).

If $t \geq 0$, and $K = 0$, then each point on the line satisfies the conic section. (i.e. it is degenerate).

If both $K \neq 0$ and $t \neq 0$ then the left-hand side equals Kt^2 and is not zero.

So no point on the conic

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

other than (x_0, y_0) satisfies the condition

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = Kt^2$$

No other point on the conic is also on the line, and no other point on the line also belongs to the conic.

The example given is

$$x^2 - xy + y^2 - 7 = 0$$

The point $(-1, 2)$ is on the curve.

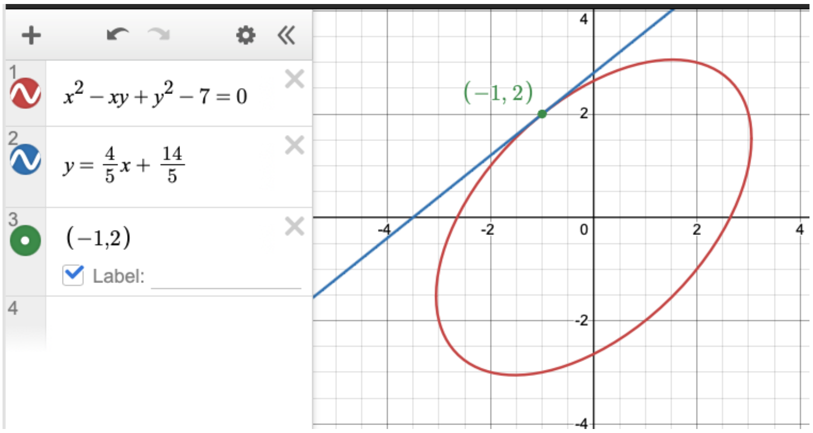
Plugging into what we had above, the tangent line through $(-1, 2)$ is

$$x^2 - xy + y^2 - 7 = (x+1)^2 - (x+1)(y-2) + (y-2)^2$$

$$-7 = 2x + 1 + 2x - y + 2 - 4y + 4$$

$$5y = 4x + 14$$

$$y = \frac{4}{5}x + \frac{14}{5}$$



A second example is the unit circle. We have

$$x^2 + y^2 - 1 = 0 = (x - x_0)^2 + (y - y_0)^2$$

$$-1 = -2xx_0 + x_0^2 - 2yy_0 + y_0^2$$

$$2yy_0 = -2xx_0 + x_0^2 + y_0^2 + 1$$

$$y = -\frac{x_0}{y_0}\, x + \left[\ \frac{x_0^2}{2y_0} + \frac{y_0}{2} + 1\ \right]$$

The *slope* of the tangent to a unit circle at any point (x, y) is $-x/y$.

The easiest calculus derivation uses implicit differentiation:

$$2x\ dx + 2ydy = 0$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

For the general parabola in standard orientation:

$$ax^2 + bx + c - y = 0$$

write

$$ax^2 + bx + c - y = a(x - x_0)^2$$

$$bx + c - y = -2ax_0x + x_0^2$$

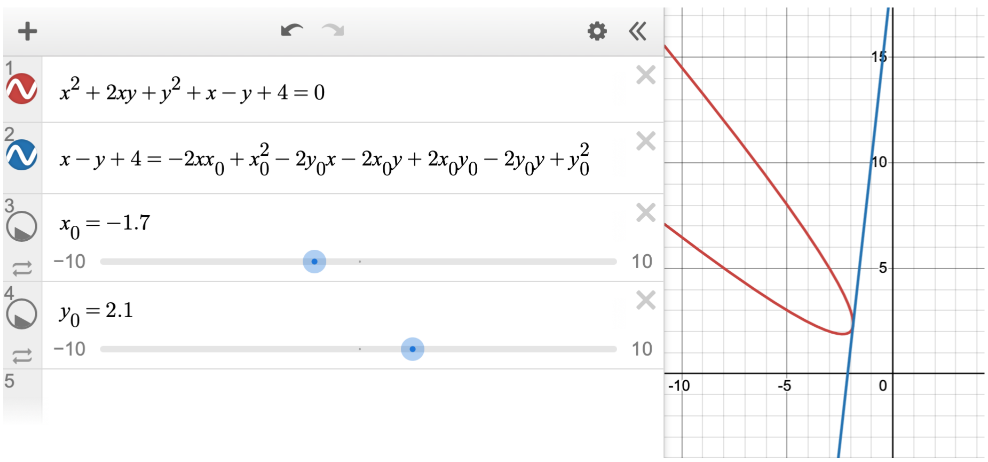
$$(2ax_0 + b)x + c - x_0^2 = y$$

And indeed $2ax_0 + b$ is the slope of the tangent to the general parabola at (x_0, y_0).

As a final example

$$x^2 + 2xy + y^2 + x - y + 4 = (x - x_0)^2 + 2(x - x_0)(y - y_0) + (y - y_0)^2$$

$$x - y + 4 = -2xx_0 + x_0^2 - 2y_0x - 2x_0y + 2x_0y_0 - 2y_0y + y_0^2$$



Sliding the sliders, the line stays on the parabola. It works!.