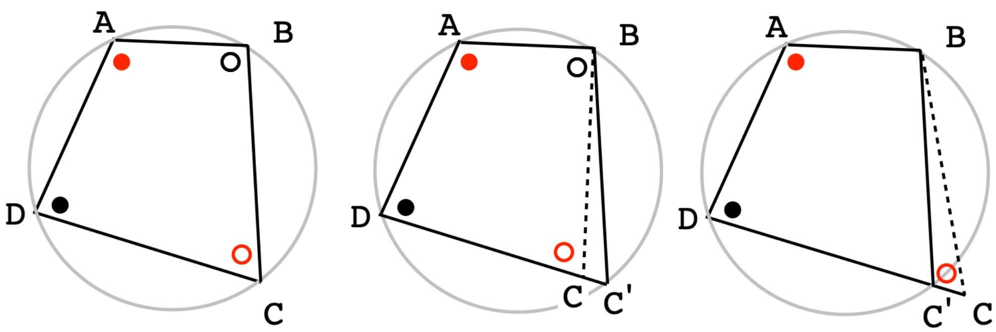
Parallelogram proof of Ptolemy's theorem

preliminary

The cyclic quadrilateral theorem says that *if* four points lie on a circle, then opposing angles are supplementary. We will derive the converse: if opposing angles are supplementary, the four points lie on a circle.

Suppose we are given that the angles B and D are supplementary, and that points A, B, and D lie on the circle as drawn. *Claim.* Then C also lies on the circle.



Proof. Since the sum of $B + D = \pi$ and the sum of all four angles equals 2π, it follows that the angles A and C are also supplementary.

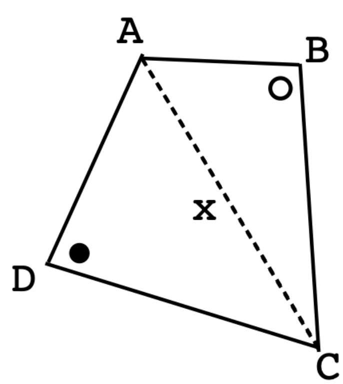
Suppose that C does *not* lie on the circle, but is internal. Extend DC to the circle at C' and draw BC' (middle panel). A and C' are supplementary, by the forward theorem, so $\angle C = C'$ (i.e. $\angle BCD = \angle BC'D$. But as the external angle of $\triangle BCC'$, $\angle BCD > \angle BC'D$. This is a contradiction.

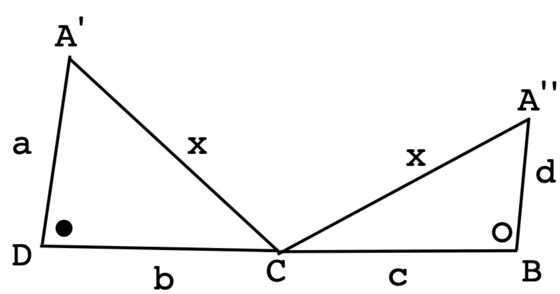
Or suppose that C is external (right panel). Find the point where DC meets the circle at C'. As before, we must have that $C = C'$. But as the external angle of $\triangle BC'C$, $\angle BC'D > \angle BCD$. This is a contradiction. Since C is neither outside nor inside the circle, it must lie on the circle.

$\square$.

main

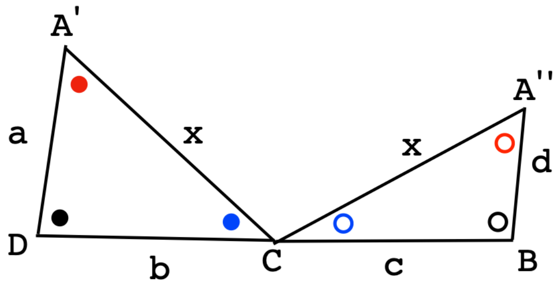
We consider a quadrilateral $ABCD$ with diagonal $AC = x$ and we're given that B and D are supplementary. It follows then that $ABCD$ is a *cyclic* quadrilateral — all four vertices lie in a circle, by our lemma.



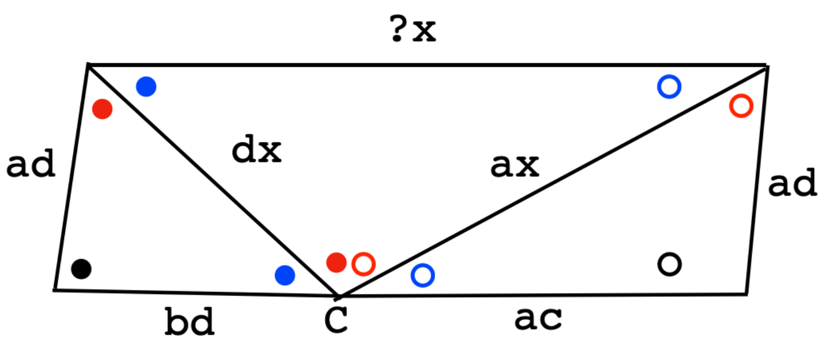


We cut the quadrilateral into two triangles and lay them along the horizontal line DCB. Because of the supplementary angles, $A'D \parallel A''B$.

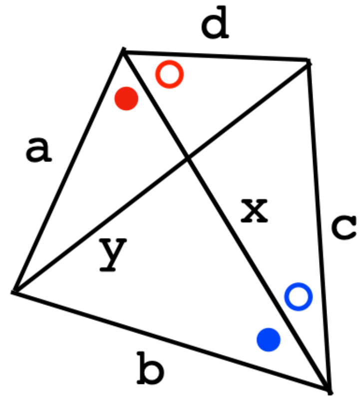
We relabel the sides for ease of reference. We can also go back to the original quadrilateral and get some more angles.



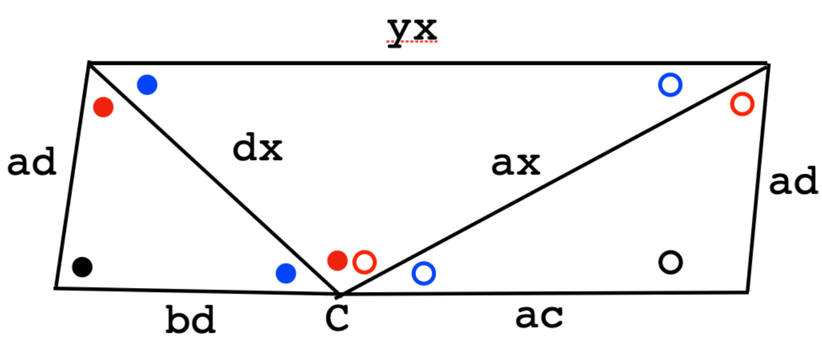
The trick then is to rescale the two triangles in the last figure so that the parallel sides are also equal and we have a parallelogram.



An easy way to do that is to scale the left one by d and the right one by a.



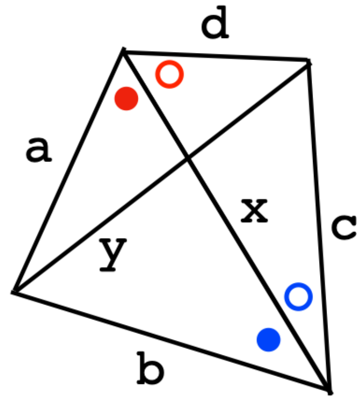
The new triangle that we generate is similar to one in the original figure. It has a central angle composed of the open and filled red dots and flanking sides proportional to a and d with ratio x. The third side is then similar to y but scaled by x to give xy.



Opposing equal sides of the parallelogram are

$$bd + ac = xy$$

which is Ptolemy's theorem.


d
a
x
c
y
b

□