Chapter 49

Pi is irrational

We will show that there do not exist two integers a and b such that $a/b = \pi$. The proof that π is irrrational is from

https://mindyourdecisions.com/blog/2013/11/08/proving-pi-is-irrational-a-step-by-step-guide-to-a-simple-proof/

https://projecteuclid.org/download/pdf_1/euclid.bams/1183510788

Proof.

The proof is by contradiction. After the assumption (that π is rational), there are three major parts:

- Create a function $f(x)$ that depends on a and b, as well as a constant integer n.

- Prove that if π is rational, the integral of $f(x) \sin x$ over a certain interval is an integer, for all values of n.

- Prove that the same integral tends to 0 as $n \to \infty$.

The contradiction is a proof that there do not exist such integers a and b.

part 1

In this part we derive some properties of the derivative of a special function $f(x)$.

Assume $\pi = a/b$, for integer a and b. Define (for integer n):

$$f(x) = \frac{x^n (a - bx)^n}{n!}$$

We can verify some properties of the function.

$$f(x) = \frac{x^n (a - bx)^n}{n!}$$

- 1a. $f(0) = 0$. Since $x^n = 0$ for $x = 0$, the result follows.
- 1b. $f(x) = f(\pi - x)$.

Verify this as follows:

$$\begin{aligned} f(\pi - x) &= \frac{(a/b - x)^n \ [\ a - b(a/b - x)\ ]^n}{n!} \\ &= \frac{(a/b - x)^n \ (bx)^n}{n!} \\ &= \frac{(a - bx)^n \ x^n}{n!} = f(x) \end{aligned}$$

- 1c. Any derivative $f^{(k)}(0)$ is an integer.

We use the binomial theorem to expand

$$(a - bx)^n = a^n + a^{n-1}(-bx) + a^{n-2}(-bx)^2 \cdots + (-bx)^n$$

Multiplied by x^n we obtain a series of terms

$$x^n \ldots (-b)^n x^{2n}$$

That is

$$f(x) = \frac{1}{n!} \; [\; c_0 x^n + c_1 x^{n+1} + \cdots + c_n x^{2n} \;]$$

Since a and b are integers, the coefficients c_k which are products of a and b are also integers. Furthermore, their sign alternates from plus to minus.

Suppose we take the derivative k times

If $k < n$, then all the derivatives contain x^{n-k} or a greater power, and will be zero when $x = 0$. Differentiating more than $2n$ times, all the terms vanish.

When k is intermediate between n and $2n$ there will be exactly one constant term (not a power of x) which is therefore non-zero at $x = 0$.

Besides the factor of c_k, another cofactor of the non-zero term of the $f^{(k)}$ derivative is k! Recall that we have $n!$ in the denominator and $n \le k \le 2n$. Since $k \ge n$, $k!/n!$ is an integer.

- 1d. $f^{(k)}(x) = (-1)^k f^{(k)}(\pi - x)$.

By the chain rule, the first derivative

$$\frac{d}{dx} f(\pi - x) = (-1) f'(\pi - x) = -f'(x)$$

Subsequent derivatives alternate in sign: $f^{(2)}(x) = f^{(2)}(\pi - x)$ and $f^{(k)}(x) = (-1)^k f^{(k)}(\pi - x)$

- 1e. From (1d)

$$f^{(k)}(\pi) = \pm f^{(k)}(\pi - x) = \pm f^{(k)}(0)$$

so it follows from (1c) that $f^{(k)}(\pi)$ is an integer.

part 2

We have

$$f(x) = \frac{x^n(a-bx)^n}{n!}$$

and construct an even weirder function:

$$g(x) = (-1)^0 f(x) + (-1)^1 f^{(2)}(x) + (-1)^2 f^{(4)}(x) \cdots + (-1)^n f^{(2n)}(x)$$

The terms simply alternate in sign.

$$g(x) = f(x) - f^{(2)}(x) + f^{(4)}(x) \cdots + (-1)^n f^{(2n)}(x)$$

∘ 2a. $g(x)$ evaluated at either 0 or π is an integer.

Since $f^{(k)}(0)$ and $f^{(k)}(\pi)$ are integers for all k, so is $g(x)$.

Here's where it gets a little tricky.

∘ 2b. The second derivative $g^{(2)}(x)$ is

$$g^{(2)}(x) = f^{(2)}(x) - f^{(4)}(x) \cdots + (-1)^n f^{(2n+2)}(x)$$

where the last term is equal to zero, by 1c.

So

$$g^{(2)}(x) + g(x) = f(x)$$

∘ 2c. We form the composite function $f(x)\sin x$ and obtain the anti-derivative by guessing the answer in the first step:

$$\frac{d}{dx}\ [\ g'(x)\sin x - g(x)\cos x\ ]$$

$$= g^{(2)}(x)\sin x + g'(x)\cos x - g'(x)\cos x + g(x)\sin x$$

$$= g^{(2)}(x)\sin x + g(x)\sin x$$

$$= f(x)\sin x$$

Yep, that's it.

Thus, by the Fundamental Theorem of Calculus:

$$\int_0^\pi f(x)\sin x \; dx = g'(x)\sin x - g(x)\cos x\Big|_0^\pi$$
$$= g'(\pi)\sin\pi - g(\pi)\cos\pi - g'(0)\sin 0 + g(0)\cos 0$$
$$= g(\pi) + g(0)$$

We have shown previously (2a) that $g(\pi) + g(0)$ is an integer.

In summary, we have established that $\int_0^\pi f(x)\sin x \; dx$ is an integer for all integer n.

Of course, this is all based on the premise that π rational.

part 3

We now expose the contradiction.

Consider the open range $0 < x < \pi$. We have

$$f(x) = \frac{x^n(a-bx)^n}{n!}$$

What about $f(x)\sin x$?

Well, $\sin x > 0$ over this range, only becoming zero at the limits.

Since $x < \pi = a/b$, the term $a - bx$ is also positive over this range. It only becomes equal to 0 when $x = \pi = a/b$. So $f(x)\sin x > 0$ always over this range.

How about the upper bound? $0 < \sin x < 1$ except at the bounds so

$$0 < f(x) \sin x < f(x)$$

This is what the source says. It isn't exactly correct because at $x = \pi/2$, the sin term is equal to one and we have equality.

Go back to the definition

$$f(x) = \frac{x^n(a - bx)^n}{n!}$$

The largest value of x^n is π^n. The largest value of $a - bx$ is a. So

$$0 < f(x) \sin x < \frac{a^n \pi^n}{n!}$$

Integrate each term. The integrand on the right is constant, hence we get just an additional factor of π.

$$0 < \int_0^\pi f(x) \sin x \ dx < \frac{a^n \pi^{n+1}}{n!}$$

Rewrite it slightly as

$$\pi \frac{a^n \pi^n}{n!}$$

Finally, evaluate what happens in the limit as $n \to \infty$.

We know that the exponential series e^x converges. From that we know that the terms $x^n/n!$ tend to zero. Therefore the above ratio (with $x = a\pi$) does too.

More carefully, form the ratio

$$\frac{c_{n+1}}{c_n} = \frac{n!}{a^n \pi^n} \cdot \frac{a^{n+1} \pi^{n+1}}{(n+1)!}$$

$$= \frac{a\pi}{n+1}$$

Therefore, for sufficiently large n, the upper bound tends to zero.

We have reached a contradiction.

There do not exist two integers a and b such that $a/b = \pi$.

$\square$