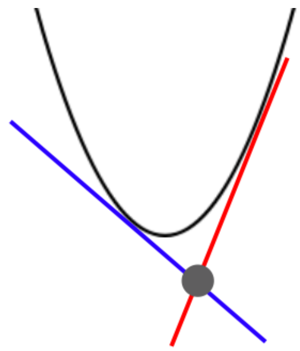
# Tangents

Consider the graph of the curve $y = f(x)$. Suppose we draw the tangent to the curve at two points, $x_1, y_1$ and $x_2, y_2$. What are the coordinates of the point where they meet, if they do so?



If we draw the tangent line at a point $(x_0, y_0)$ on the curve, the slope is the derivative at that point, and the equation of the tangent line is

$$\frac{y - y_0}{x - x_0} = m = f'(x_0)$$

$$y = f'(x_0)(x - x_0) + y_0$$

for every point $(x, y)$ on the line.

So for these two points and a function $f(x)$, we have two lines

$$y = f'(x_1)(x - x_1) + y_1$$

$$y = f'(x_2)(x - x_2) + y_2$$

The point where the two lines cross has the same coordinates $(x, y)$. So

$$f'(x_1)(x - x_1) + y_1 = f'(x_2)(x - x_2) + y_2$$

Solving for $x$

$$[\ f'(x_1) - f'(x_2)\ ]\ x = y_2 - y_1 + f'(x_1)x_1 - f'(x_2)x_2$$

**parabola**

Suppose the function is $y = f(x) = ax^2$ so $f'(x) = 2ax$. Performing the substitutions we obtain

$$2a(x_1 - x_2)x = a(x_2^2 - x_1^2) + 2ax_1^2 - 2ax_2^2$$

We can cancel the $a$, divide by 2 and factor the difference of squares:

$$(x_1 - x_2)x = \frac{(x_2 - x_1)(x_2 + x_1)}{2} + (x_1 - x_2)(x_1 + x_2)$$

Factor out the common term $(x_2 - x_1)$ (one has a minus sign):

$$\begin{aligned} x &= -\frac{(x_2 + x_1)}{2} + (x_1 + x_2) \\ &= \frac{x_1 + x_2}{2} \end{aligned}$$

A remarkably simple answer!

**square root**

The function is $y = \sqrt{x}$ so $f'(x) = 1/2\sqrt{x}$. We have

$$[\ \frac{1}{2\sqrt{x_1}} - \frac{1}{2\sqrt{x_2}}\ ]\ x = \sqrt{x_2} - \sqrt{x_1} + \frac{x_1}{2\sqrt{x_1}} - \frac{x_2}{2\sqrt{x_2}}$$

Multiply by 2 and simplify the last two terms

$$[\ \frac{\sqrt{x_2} - \sqrt{x_1}}{\sqrt{x_1}\sqrt{x_2}}\ ]\ x = 2\sqrt{x_2} - 2\sqrt{x_1} + \sqrt{x_1} - \sqrt{x_2}$$

$$[\ \frac{\sqrt{x_2}-\sqrt{x_1}}{\sqrt{x_1x_2}}\ ]\ x = \sqrt{x_2}-\sqrt{x_1}$$

$$x = \sqrt{x_1x_2}$$

The first one was the arithmetic mean, this is the geometric mean!

Restating the general result

$$[\ f'(x_1) - f'(x_2)\ ]\ x = y_2 - y_1 + f'(x_1)x_1 - f'(x_2)x_2$$

**inverse**

The function is $y = f(x) = 1/x$ so $f'(x) = -1/x^2$. We have

$$[\ \frac{1}{x_2^2} - \frac{1}{x_1^2}\ ]\ x = \frac{1}{x_2} - \frac{1}{x_1} + \frac{1}{x_2} - \frac{1}{x_1}$$

$$(\frac{1}{x_2} - \frac{1}{x_1})(\frac{1}{x_2} + \frac{1}{x_1})\ x = 2(\frac{1}{x_2} - \frac{1}{x_1})$$

$$(\frac{1}{x_2} + \frac{1}{x_1})\ x = 2$$

$$\frac{1}{x} = \frac{1}{2}\cdot(\frac{1}{x_1} + \frac{1}{x_2})$$