Circular cap

I found a nice problem on Twitter. The points on the circle are "evenly spaced." The secant has the length shown, and we're asked for the shadedd area.

I always start by recognizing that any problem like this can be re-scaled to the unit circle. It will make the arithmetic less complicated, decreasing the chances of a mistake.

Let us consider the areas of an angular sector, followed by a circular cap.

For a sector of the circle swept out by a central angle θ, the sector area or slice of the pie is the same fraction of the total area of the circle

(π), as the ratio of θ to the whole circumference, 2π.

$$\frac{A_{\text{sector}}}{\pi} = \frac{\theta}{2\pi}$$
$$A_{\text{sector}} = \frac{\theta}{2}$$

Next, for what we're calling a circular cap (the area between a secant of the circle and the circle itself), we have to subtract a triangle.

That triangle is composed of two right triangles, each with a central angle of $\phi = \theta/2$ and sides $\sin\theta/2$, and $\cos\theta/2$. So the area of the entire triangle for a sector of angle $\theta = 2\phi$ is simply the product of the sine and cosine of ϕ.

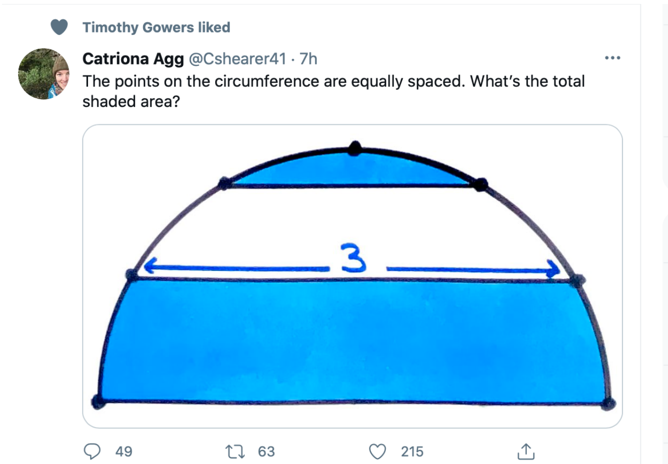
The area of the cap is the difference

$$A_{\text{cap}} = \frac{\theta}{2} - \sin\theta/2\cos\theta/2$$

We will rewrite this in terms of the half-angle $\phi = \theta/2$. The area of the cap for angle 2ϕ is

$$A = \phi - \sin\phi\cos\phi$$

In the problem we're given, one of the areas in question (the larger blue region) is not the area of a sector or a cap alone, but the difference between the area of the semi-circle and the cap.



The points are equally spaced, with 12 of them for the whole circle, so the angle corresponding to two adjacent points is $2\pi/12 = \pi/6$. The blue cap at the top has $\theta = \pi/3$, so $\phi = \pi/6$ and then

$$A = \frac{\pi}{6} - \sin\frac{\pi}{6} \cdot \cos\frac{\pi}{6}$$

We recognize that $\sin\pi/6 \cdot \cos\pi/6 = \sqrt{3}/4$, but stay with what we have for a minute.

The cap consisting of the white portion plus the blue above it has twice the angle, so that cap is

$$A = \frac{\pi}{3} - \sin\frac{\pi}{3} \cdot \cos\frac{\pi}{3}$$

The blue region on the bottom is the difference between the area of a semicircle and the previous result

$$A = \frac{\pi}{2} - \frac{\pi}{3} + \sin\frac{\pi}{3} \cdot \cos\frac{\pi}{3}$$

and the total blue area is

$$A = \frac{\pi}{2} - \frac{\pi}{3} + \sin\frac{\pi}{3} \cdot \cos\frac{\pi}{3} + \frac{\pi}{6} - \sin\frac{\pi}{6} \cdot \cos\frac{\pi}{6}$$

In this problem, the angles are complementary, which means that $\sin \pi/3 = \cos \pi/6$ and the reverse, hence those terms cancel. We are left with simply

$$A = \frac{\pi}{2} - \frac{\pi}{3} + \frac{\pi}{6} = \frac{\pi}{3}$$

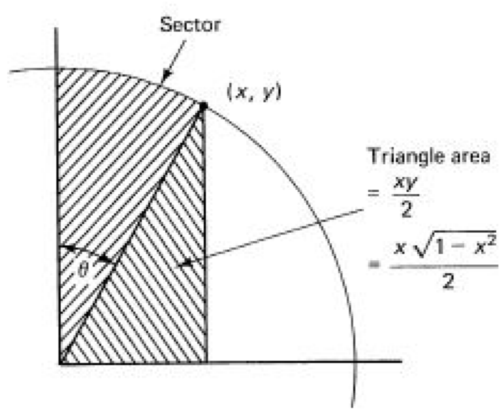
The area is 1/3 the area of the complete circle.

calculus

It's interesting that the arithmetic above mirrors what is seen in the calculus solution. This latter method provides us with a way to determine the area bounded by horizontal secants between any two angles. The unshaded area would be

$$A = \int_{\pi/6}^{\pi/3} \cos^2\theta \; d\theta$$

We justify this integral with the following picture. Turn the previous view sideways by a quarter-turn to the right.



The area under the curve is $\int y\ dx$. For this parametrization of the circle (notice which angle is labeled in the figure):

$$x = \sin\theta$$

$$y = \cos\theta$$

So $dx = \cos\theta\ d\theta$ and then

$$\int y\ dx = \int \cos^2\theta\ d\theta$$

This integral is not one we can solve immediately. But it turns out, when playing with different expressions, we may notice that the derivative of the product $\sin x \cos x$ is, by the product rule:

$$[\ \sin x \cos x\ ]' \ = \sin^2 x - \cos^2 x$$

and applying our favorite trigonometric identity we get

$$\sin^2 x - \cos^2 x = 1 - 2\cos^2 x$$
$$\cos^2 x = \frac{1}{2} - \frac{[\sin x \cos x]'}{2}$$

Integrating

$$\int \cos^2 x\ dx = \frac{x}{2} - \frac{\sin x \cos x}{2}$$

We have discovered that the integral of $\cos^2 x$ involves both x and the product of the sine and cosine of x. This is what we saw before in the geometrical approach.

The factor of 2 comes in because this integral is actually the area between the curve of the circle and the axis, which is half the result we need.

Since ϕ_1 and ϕ_2 are complementary angles, the terms with $\sin x \cos x$ cancel, just as before, and we have

$$A = x \Big|_{\phi_2}^{\phi_1} = \phi_2 - \phi_1 = \frac{\pi}{6}$$

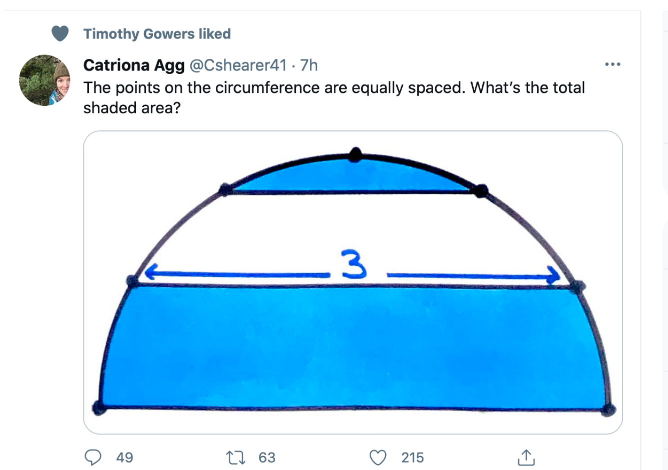
I am missing a factor of 2. The reason is that we have found the area that is unshaded and what we need is the area in blue.

That area is

$$A = \frac{\pi}{2} - \frac{\pi}{6} = \frac{\pi}{3}$$

re-scaling

And now I notice that the problem does not contain a radius of 1 but rather a particular secant of length 3.



We calculate the radius of this circle using the Pythagorean theorem. Draw a 30-60-90 right triangle with one side equal to $3/2$ and the hypotenuse equal to r. The other side is $3/2$ divided by $\sqrt{3}$, which means that

$$r^2 = (3/2)^2 + (\sqrt{3}/2)^2 = \frac{12}{4} = 3$$

The total area of the circle is π times this and the shaded part is $1/3$ of it. The final answer is just π.