Tangents

Let $T = (x, y)$ be the point where the tangent line meets a unit circle centered at the origin. Write the equation of the tangent line in terms of its slope $-x/y$ as

$$-\frac{x}{y} = \frac{y - (-1)}{x - 1/2}$$

$$y^2 + y = -x^2 + \frac{1}{2}x$$

Since $x^2 + y^2 = 1$ everywhere on the circle,

$$y + 1 = \frac{1}{2}x$$

This is a relationship between the x- and y- values at both tangent points, but it is clearly not the equation of the tangent line, confirmed since $(1/2, -1)$ is not a solution. The trick here is to use the information from the circle *again*:

$$\sqrt{1 - x^2} = \frac{1}{2}x - 1$$

$$1 - x^2 = \frac{1}{4}x^2 - x + 1$$

$$\frac{5}{4}x^2 - x = 0$$

$$(\frac{5}{4}x - 1)x = 0$$

This has two solutions, $x = 0$ and $x = 4/5$, which are indeed the $x-$coordinates of the points of tangency as a more sophisticated analysis will confirm. Then $y = \sqrt{1 - x^2} = -3/5$ (the minus sign because this is the fourth quadrant). So the slope is

$$\frac{-x}{y} = \frac{-4/5}{-3/5} = \frac{4}{3}$$

The height of the triangle is then $2/3$ and the area is easily found to be $1/6$.