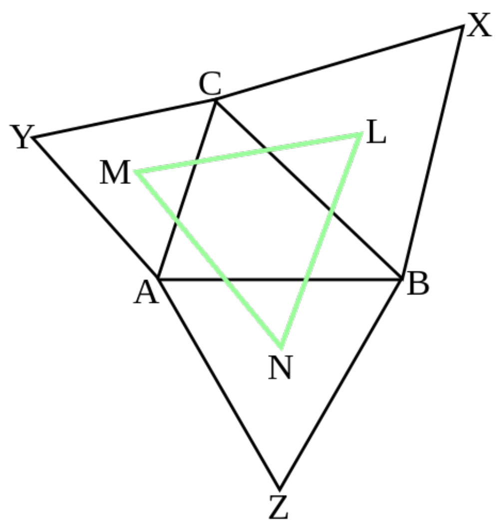
Napoleon's Theorem

This is a famous problem where equilateral triangles are constructed on the sides of an arbitrary triangle. The new triangles may lie outside the original triangle, or lie inside and overlap extensively with the original triangle.

Here is the first case:



It is claimed that in both cases, the incenters of the equilateral triangles form a fourth equilateral triangle.

Vectors can easily solve this problem.

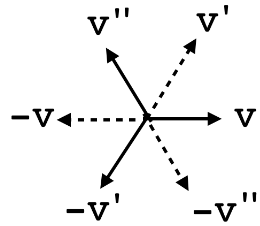
Proof.

We use the general notation $\mathbf{v}'$ to denote a vector $\mathbf{v}$ after rotation.

For this problem involving equilateral triangles, let the angle of rotation be 60°. With this definition it follows that

$$\mathbf{v}''' = -\mathbf{v}$$

Specify the direction of rotation as counter-clockwise.

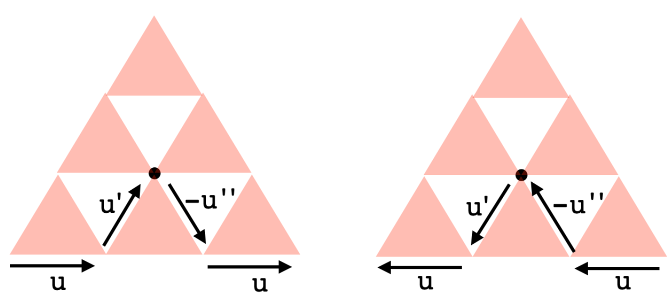


Another result we will need is that a clockwise rotation by one step gives $-\mathbf{v}''$. It is easy to see why.

The incenters of an equilateral triangle are also the centroids, so the distance up from the base is one-third of the median/altitude.

Here's a tiling proof that two vectors of length 1/3 the base, with appropriate rotation by 60°, can reach the incenter and then go to the adjacent vertex.

We specify this first path as counterclockwise, so the relevant figure is the one on the right.



Refer back to the first figuree to see that the side a is opposite vertex A and the incenter of the triangle formed with sides a is L.

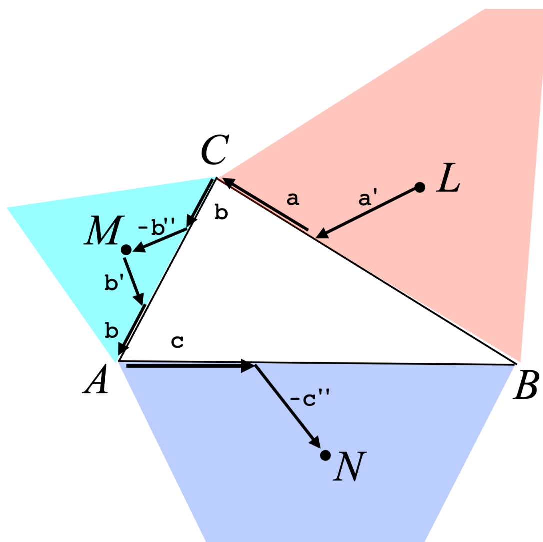
In vector notation the path along sides a and b between incenters L and M by a *counter-clockwise* path is

$$\mathbf{p} = \mathbf{a}' + \mathbf{a} + \mathbf{b} + (-\mathbf{b}'')$$

and between M and N it's

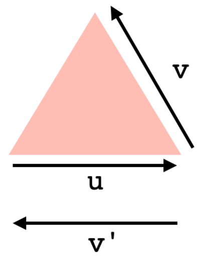
$$\mathbf{q} = \mathbf{b}' + \mathbf{b} + \mathbf{c} + (-\mathbf{c}'')$$

Here's the whole thing



The important idea is that in a counter-clockwise traversal of any equilateral triangle, two sides in order $\mathbf{u}, \mathbf{v}$ have the property that $\mathbf{u} + \mathbf{v}' = \mathbf{0}$, where $\mathbf{0}$ is the zero vector.

It's easy to see why:



We have the paths $\mathbf{p}$ and $\mathbf{q}$. Rotate $\mathbf{q}$:

$$\mathbf{q}' = \mathbf{b}'' + \mathbf{b}' + \mathbf{c}' + (-\mathbf{c}''')$$

Substitute $-\mathbf{c}''' = \mathbf{c}$:

$$\mathbf{q}' = \mathbf{b}'' + \mathbf{b}' + \mathbf{c}' + \mathbf{c}$$

Add them together.

$$\mathbf{p} + \mathbf{q}' = \mathbf{a}' + \mathbf{a} + \mathbf{b} + (-\mathbf{b}'') + \mathbf{b}'' + \mathbf{b}' + \mathbf{c}' + \mathbf{c}$$

Cancel two terms

$$\mathbf{p} + \mathbf{q}' = \mathbf{a}' + \mathbf{a} + \mathbf{b} + \mathbf{b}' + \mathbf{c}' + \mathbf{c}$$

The vector sum for a closed path is zero ($\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$), so

$$\mathbf{p} + \mathbf{q}' = \mathbf{a}' + \mathbf{b}' + \mathbf{c}'$$

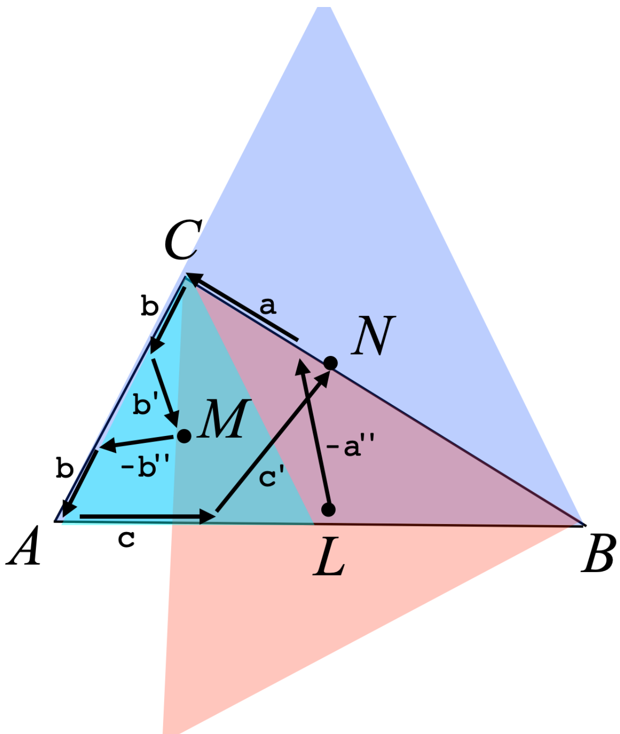
The sum is zero even if the triangle is rotated:

$$\mathbf{p} + \mathbf{q}' = \mathbf{0}$$

$\square$

second problem

The really nice thing about this is that the second problem becomes trivial once we know how. Just to make it interesting, we keep the rotation counter-clockwise but make a clockwise traversal.



Although it may not seem like it, this is a clockwise tour around LMN.

Therefore, we start with $-\mathbf{a}''$, and our test will be to rotate the *first* vector before addition.

$$\mathbf{p}' + \mathbf{q} = -\mathbf{a}''' + \mathbf{a}' + \mathbf{b}' + \mathbf{b}'' + (-\mathbf{b}'') + \mathbf{b} + \mathbf{c} + \mathbf{c}'$$

$$\mathbf{p}' + \mathbf{q} = \mathbf{a} + \mathbf{a}' + \mathbf{b}' + \mathbf{b} + \mathbf{c} + \mathbf{c}' = \mathbf{0}$$

You can see how this turns out.

□