# Rational root theorem

Consider this polynomial with integer coefficients:

$$a_0 + a_1 x + \cdots + a_n x^n = 0$$

The theorem says that *if* there is a rational root $p/q$ of this equation (a solution), then it must be true that $p|a_0$ ($p$ is a divisor of $a_0$) and $q|a_n$.

As usual, we require that $p/q$ be in "lowest" terms, so that there is no remaining common factor. $p$ and $q$ are coprime.

*Proof.*

The proposed solution can be written as

$$a_0 + a_1(p/q) + \cdots + a_n(p/q)^n = 0$$

Multiply by $q^n$

$$a_0 q^n + a_1 p q^{n-1} + \cdots + a_n p^n = 0$$

Now simply isolate the first term

$$a_1 p q^{n-1} + \cdots + a_n p^n = -a_0 q^n$$

Factor out $p$ on the left-hand side:

$$p(a_1 q^{n-1} + \cdots + a_n p^{n-1}) = -a_0 q^n$$

Since $p$ divides the product on the left-hand side, it must divide either $a_0$ or $q$. But $p$ is co-prime to $q$. Therefore $p|a_0$.

An analogous argument shows that $q|a_n$.

**caution**

I have written the polynomial starting with $a_0$ so that the index will increase from left-to-right.

However, we usually write simple polynomials with the higher powers first:

$$a_n x^n + \cdots + a_1 x + a_0 = 0$$

Our result says that the numerator of any rational solution must evenly divide $a_0$, the constant at the end, and further that the denominator of any rational solution must evenly divide the cofactor of the highest power of $x$, i.e. $a_n$

**applications**

Consider

$$x^2 - 2 = 0$$

The solution is $\sqrt{2}$.

For a rational solution $p/q$, it must be that $p$ divides $-2$ and $q$ divides 1. This means that $p \in \{1, -1, 2, -2\}$ and $q = 1$.

None of the four forms $p/q$ solves the equation. Thus, there is no solution in the rational numbers.

The condition $q = 1$ means that, *if* there is a solution, it must be an integer. And this is true for any square root. Therefore, the only rational square roots are those of the perfect squares.

Consider the cube root:

$$x^3 - 2 = 0$$

The same analysis holds. None of the possible solutions is an actual one. The cube root of 2 is also irrational.

Now consider

$$7x^3 - 3x^2 + 14x - 6 = 0$$

We have that $q$ is either 1 or 7 and $p$ is either $1, 2$ or 3 or one of their negatives, so the possibilities are

$$1/1, 2/1, 3/1, 6/1, 1/7, 2/7, 3/7, 6/7$$

together with their negatives.

By trial and error, we discover that $x = 3/7$ is a solution since

$$7(3/7)^3 - 3(3/7)^2 + 14(3/7) - 6 = 0$$

$$\frac{27}{49} - \frac{27}{49} + \frac{42}{7} - 6 = 0 \quad \checkmark$$

Which means that $7x - 3$ must factor out of the equation, so that $x = 3/7$ gives zero for that term.

By "long division" we factor to obtain:

$$= (7x - 3)(x^2 + 2)$$

and recognize that the term $x^2 + 2$ does not have any rational roots (it cannot be equal to zero for any value of $x$).

https://www.quora.com/Whats-your-favorite-mathematical-theorem

https://en.wikipedia.org/wiki/Rational_root_theorem