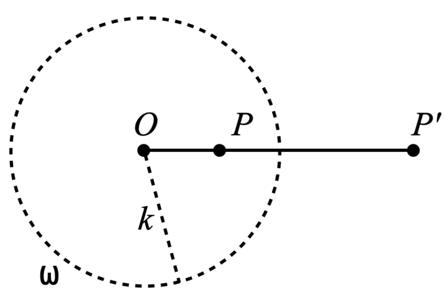
Inversive Geometry

Inversion in a circle maps a point in the plane to (typically) a different point in the plane. The mapping is continuous, so if points lie close together before inversion they are close together afterwards. A locus of points is transformed into another locus, e.g. a circle is in most cases transformed into another circle.

To carry out inversion, one starts by choosing a reference circle. We will refer to ω, the *inversion circle*, with its *inversion center* at O having radius k. We show ω as a dotted circle.

Let P be an arbitrary point, distinct from O. The inverse of P is called P'. P' lies on the ray through OP according to the following rule:



$$OP \cdot OP' = k^2 \qquad \frac{OP}{k} = \frac{k}{OP'}$$

We can write $I_\omega(P) = P'$.

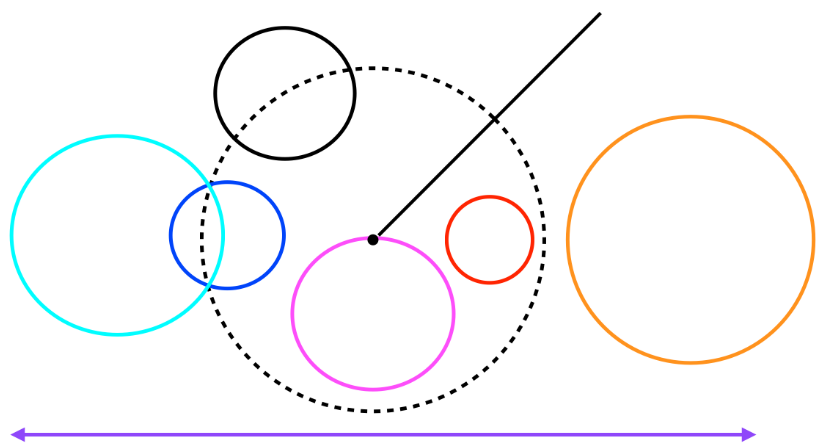
Rather than the ray, we will just draw a line segment, with definite endpoints, and also as usual just call it line segment a line, for brevity.

A locus of transformed points is referred to as the image of the input locus under inversion. In some cases the image of a point is the same as the original. Points lying on ω invert into themselves, because then $OP = k = OP'$.

You can see in the example above that OP is about one-half of k, so then the length of OP' is twice k. The image of a point close to O may be very far away.

O is a special point in more than one way. As P approaches O, the inverse P' moves farther and farther away.

Here's an overview, with the *inversion circle* represented as a dotted circle. The blue circle maps to the one in cyan and the red circle maps to the one in tan. The converse is also true.

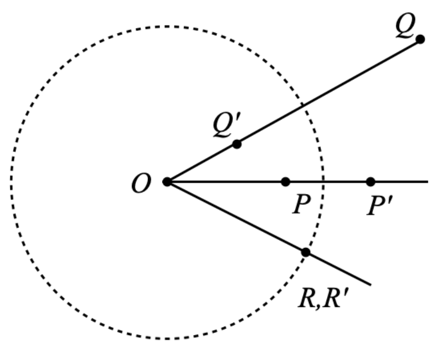


Points outside the inversion circle get mapped to points inside, and vice-versa. A line through the center maps to itself.

A locus through the center of the inversion circle (magenta circle) is a special case — it maps to the purple line. The black circle is another special case, it maps to a circle that looks just like original, although all of the points have been swapped (except the two lying on the inversion circle).

The complete theory invents a "point at infinity" which is the inverse of O. We won't worry about that. We'll just think of the set of possible inputs to I as the "punctured plane", with O missing.

When we say something like "a circle through O", we are not worrying about the point O itself.



We call points that are invariant under inversion, fixed. Let R be a point on ω, so the length of OR is k. Then we have that

$$I_\omega(R) = R' = R$$

Much of our work with inversions involves similar triangles. Coxeter introduces it this way:

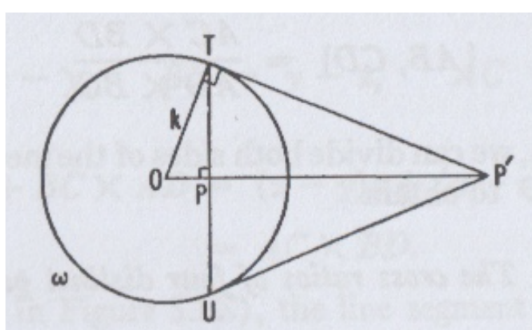


Figure 5.3A

Draw the chord $TU \perp OP$. Then P' is the point where the tangents from T and U meet. From similar triangles $\triangle OTP' \sim \triangle OPT$, so we have:

$$\frac{OP'}{k} = \frac{k}{OP}$$

An analogous transformation in three dimensions uses a sphere, in space.

inversion is an involution

If P' is the inverse of P, then the inverse of P' is P.

Proof. Let P'' be the inverse of P', with $OP' \cdot OP'' = k^2$. But $k^2 = OP \cdot OP'$. It follows that $OP'' = OP$. $\square$

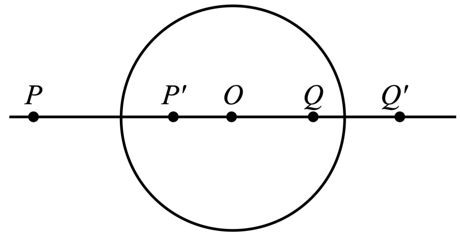
If P lies inside ω, then P' lies outside, and vice-versa. There is a famous essay which analogizes how one would hunt a lion in the desert by various mathematical or physical techniques. Here is the one using inversion:

> We place a *spherical* cage in the desert, enter it and lock it. We perform an inversion with respect to the cage. The lion is then inside the cage, and we are outside. *H. Petard*

inversion of a line through O

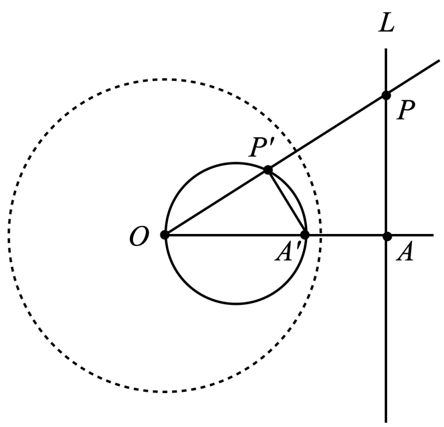
We now consider the results of inverting either a line or a circle. There are two cases for each, depending on whether the line or circle goes through, O, the center of ω. They are all related, as we will see.

The inverse of a line through O is the same line.



Let P and Q be any two points on a line through O. The inverse points P' and Q' lie on the same line.

inversion of a line not through O



The inverse of a line not through O is a circle through O.

This is the first result that is a little surprising.

Proof.

Let L be a line, not passing through O and external to ω, and let P and A be two points on the line. Pick A such that $OA'A \perp PA$.

We have that

$$OP \cdot OP' = k^2 = OA \cdot OA'$$
$$\frac{OP}{OA} = \frac{OP'}{OA'}$$

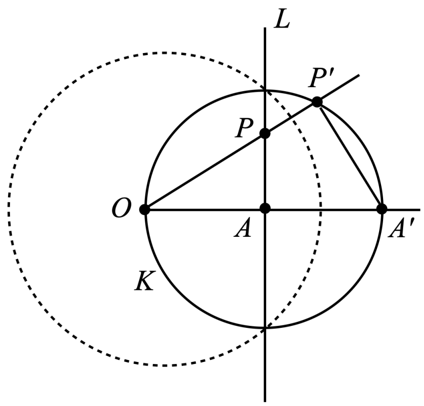
It follows that $\triangle PAO \sim \triangle A'P'O$ and in particular, $\angle OP'A'$ is a right angle because $\angle OAP$ is a right angle.

But this is true for *every* point on L other than A.

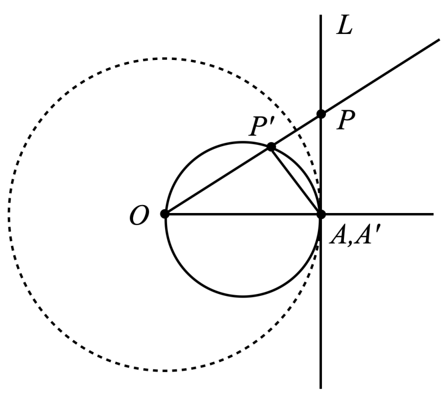
By the inverse of Thales' circle theorem, it follows that A' and P' lie on a circle. This circle passes through O and in fact, has diameter OA'.

□

The diameter of this circle is perpendicular to the line L, so L is parallel to the tangent to the circle at O (not shown).



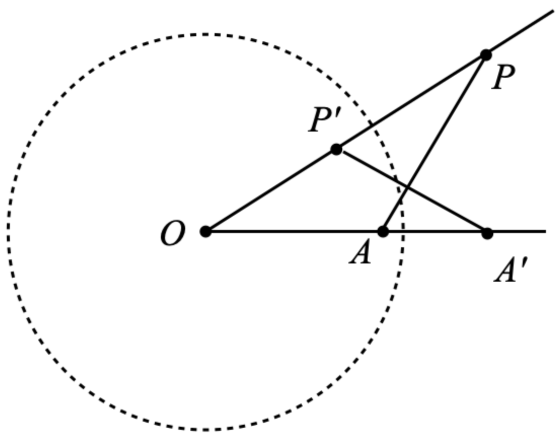
The same proof still works for the case where L passes through ω, or even when it is tangent (below). This is because, in both cases, $\triangle OAP \sim \triangle OP'A'$.



note on similar triangles

For any points A and P, we can form similar triangles by using the fundamental equations

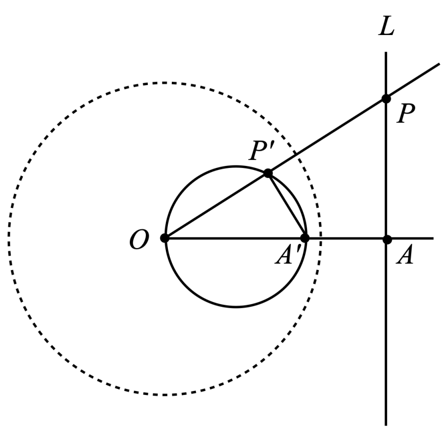
$$OP \cdot OP' = k^2 = OA \cdot OA'$$
$$\frac{OP}{OA} = \frac{OP'}{OA'}$$



This works even when one of A or P lies inside, and one outside ω.

inversion of a circle through O

Let K be a circle, passing through O.



The inverse of a circle through O is a line not through O.

This follows directly from the previous theorem, since the inverse of the inverse of any point is the same point.

As we saw from the previous result, it does not matter whether the circle is entirely internal to ω, is internally tangent, or extends outside.

Ptolemy's Theorem

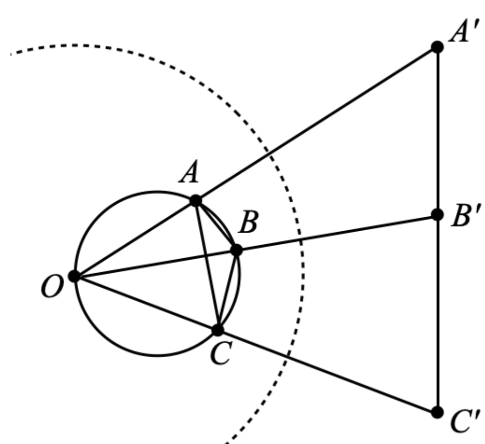
With only what we have established so far, we can prove an important theorem.

Let A, B and C lie on a circle through O. Then, as we showed above, the inverse of this circle is a line not through O and here, entirely external to the inversion circle.

Ptolemy says that

$$AB \cdot OC + BC \cdot OA = AC \cdot OB$$

Proof.



$A'B' + B'C' = A'C'$. We will leverage this identity to find the result.

We have three pairs of similar triangles. To help stay organized, we write out the ratio boxes.

AB	$A'B'$
OB	OA'
OA	OB'

AC	$A'C'$
OC	OA'
OA	OC'

BC	$B'C'$
OC	OB'
OB	OC'

The symmetry is pretty clear. We have that

$$\frac{A'B'}{AB} = \frac{OA'}{OB} \Rightarrow A'B' = \frac{AB \cdot OA'}{OB}$$

Also

$$\frac{B'C'}{BC} = \frac{OC'}{OB} \Rightarrow B'C' = \frac{BC \cdot OC'}{OB}$$

Then

$$\frac{A'C'}{AC} = \frac{OC'}{OA} \Rightarrow A'C' = \frac{AC \cdot OC'}{OA}$$

Form the sum and clear the denominators:

$$AB \cdot OA' \cdot OA + BC \cdot OC' \cdot OA = AC \cdot OC' \cdot OB$$

Divide by OC'

$$AB \cdot \frac{OA'}{OC'} \cdot OA + BC \cdot OA = AC \cdot OB$$

Can you find the appropriate entry for the last step? I obtain

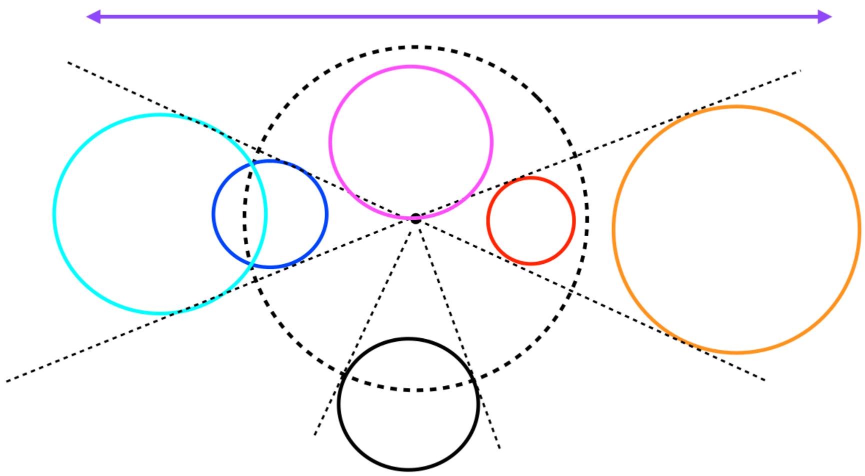
$$AB \cdot OC + BC \cdot OA = AC \cdot OB$$

$\square$

This is Ptolemy's theorem.

We pause to reflect on one of the most useful and important theorems in all of geometry.

We now consider circles not through O. This is the point where the theory gets a little more sophisticated.



inversion of a circle not through O

The inverse of a circle not through O is another circle not through O.

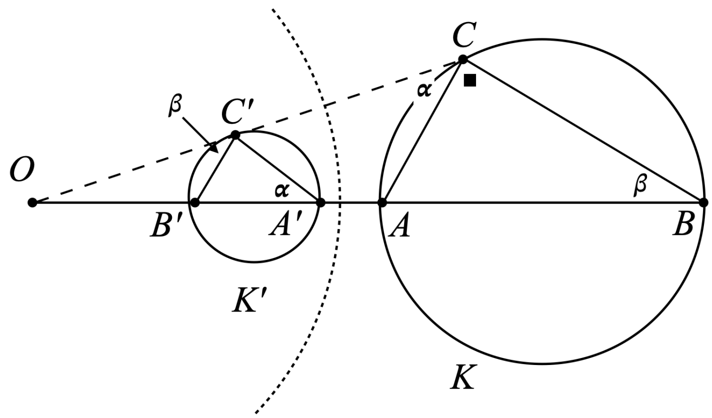
We can again draw different diagrams depending on where the circles lie with respect to ω. We need to make sure that the proof or proofs work for all of them.

The cases we've already seen include a circle through the inversion center, which inverts into a line extending infinitely out (magenta and purple).

Consider first the case where K is entirely external to ω.

Proof.

Let K be a circle, not passing through O, and not even intersecting with ω. Draw the diameter AB of K, such that the extension passes through O.



Let C be an otherwise arbitrary point on K, not on AB. Find the inverses of A, B and C. Draw $\triangle A'B'C'$.

As before

$$\frac{OA'}{OC'} = \frac{OA}{OC}$$

so $\triangle OB'C' \sim \triangle OCB$ and $\triangle OA'C' \sim \triangle OCA$.

From similar triangles, we have that the angles marked α and β are equal.

By Thales' circle theorem, $\angle ACB$ is a right angle. So from $\triangle BOC$:

$$\angle BOC + \alpha + \beta + \angle ACB = 180$$

But from $\triangle B'OC'$ (with $\angle B'OC' = \angle BOC$):

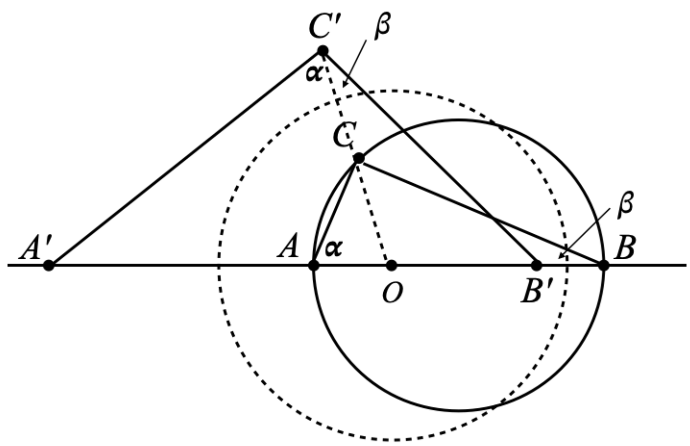
$$\angle B'OC' + \alpha + \beta + \angle AC'B' = 180$$

It follows that $\angle A'C'B' = \angle ACB$, so $\angle A'C'B'$ is also a right angle, and thus it lies on the circle whose diameter is $A'B'$. This is true for every point C on circle K, other than A and B.

$\square$

Second case: let the circle K intersect ω, with the inversion center O inside K.

Draw the diameter AB of circle K which goes through O. Let C be any other point on circle K.



Proof.

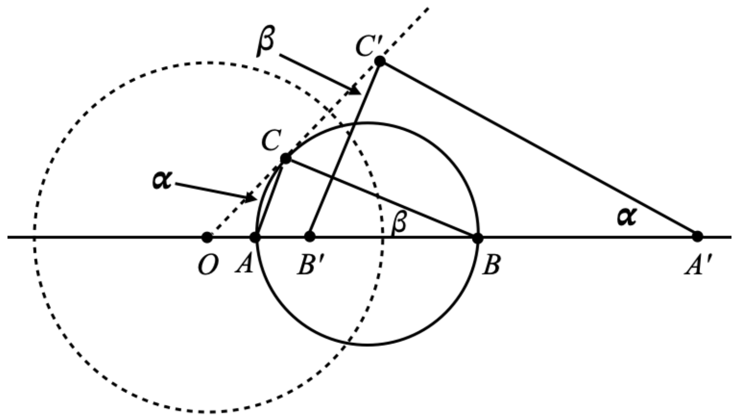
We have $\triangle OAC \sim \triangle OC'A'$ and $\triangle OBC \sim \triangle OC'B'$. From similar triangles, the angles marked as α are equal, as are those marked as β.

$\angle ACB$ is right, but $\alpha + \beta$ is supplementary to $\angle ACB$ to $\alpha + \beta$ so the sum is 90.

It follows that $\angle A'C'B' = \alpha + \beta$ is also right. Therefore C' lies on a circle whose diameter is $A'B'$.

$\square$.

Third case: the circle K intersects ω but the inversion center O is external to K.



Draw the diameter AB of circle K which, extended, goes through O. Let C be any other point on circle K.

Proof.

The angles marked α and β are equal (in pairs) by similar triangles.

$\angle ACB$ is right. So $\angle BOC + \alpha + \beta$ is equal to a right angle.

But $\angle BOC + \alpha + \beta + \angle A'B'C'$ is equal to two right angles.

Thus, $\angle A'C'B'$ is also right.

□

The fourth and last case is where the circle K and ω are concentric. But then the inverse of K is also a concentric circle.

Invariants

Either a given point is on a circle (or line) or it is not. If it is on the locus, then it must also lie on the image of the locus under inversion.

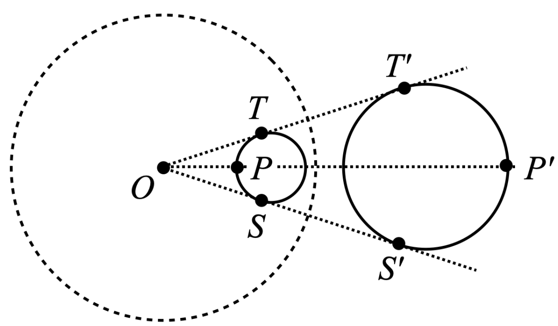
This seems quite obvious, yet subtle. We will not prove it here.

Then if a circle or a line cuts a circle at two points, after inversion the resulting figures will still have two points of intersection.

And if a line is tangent to a circle or if two circles are tangent, or a line cuts a line, after inversion the resulting figures will still be tangent (in the first two cases) and in all cases, share one point of intersection.

The next figure shows a circle K not through O and its inverse K'. We notice that lines tangent to the original circle K are also tangent to the inverse circle K'

However, although the center of K and the center of K' are on the same line through O, they are *not* inverses.



We'll see why in just a bit.

Inversion preserves angles

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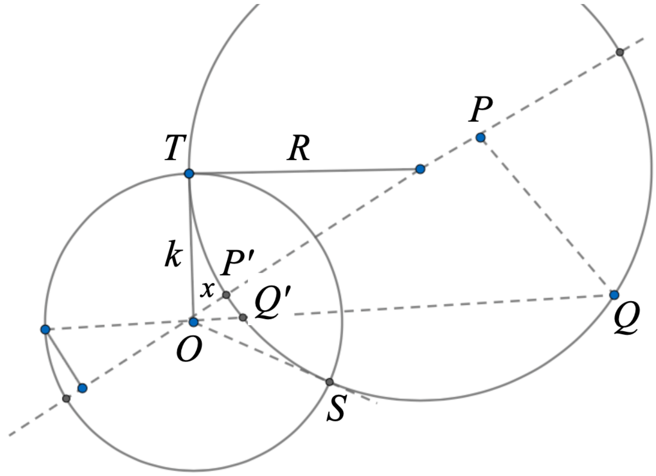
As a special case (and the only one we will use for the Feuerbach theorem, below), if two circles are orthogonal, then their inverses are also orthogonal.

Two circles are orthogonal if, at the points where they cross, the two tangents are perpendicular. Equivalently, the two radii to that point must be perpendicular.

Then, considering a circle orthogonal to ω, the claim is that inversion in ω leaves such a circle invariant, or fixed.



Proof.

Let the inversion circle ω have its center at O and have radius of length k.

Another circle K with radius of length R is positioned so that at the points where the two circles cross, the tangents (and radii) are perpendicular.

The first thing we notice is that the points S and T lie on ω, so they are fixed, their inverses are the same as the original point.

Next consider a pair of points lying P and P' on a diameter of K, P' on a distance x from the inversion center, while P lies a distance $2R + x$ from the inversion center. The product of the distances is

$$(2R + x)x = 2Rx + x^2$$

But we also have from the geometry that

$$k^2 + R^2 = (R + x)^2$$
$$k^2 = 2Rx + x^2$$

The product of the two distances is just k^2, so the inverse of P is P'

and vice-versa. We now know that four points of the inverted circle are also on the original circle. The result follows.

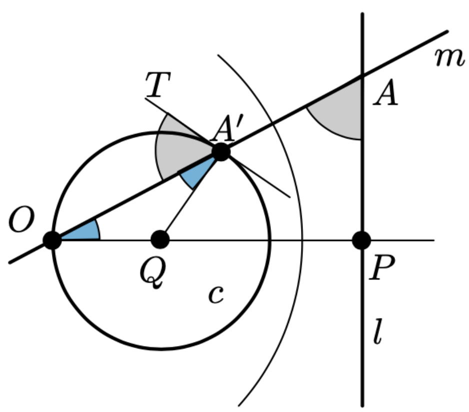
Even more generally, we may consider an arbitrary secant QQ'. Recall from the secant tangent theorem that $OQ \cdot OQ' = OT^2 = k^2$.

Any pair of points on any secant of circle K are their own inverses, so they both lie on the inverted circle. Thus, the circle is fixed under inversion.

□

A relatively simple proof for circles that are not necessarily orthogonal comes from the web. I have lost the author's name.

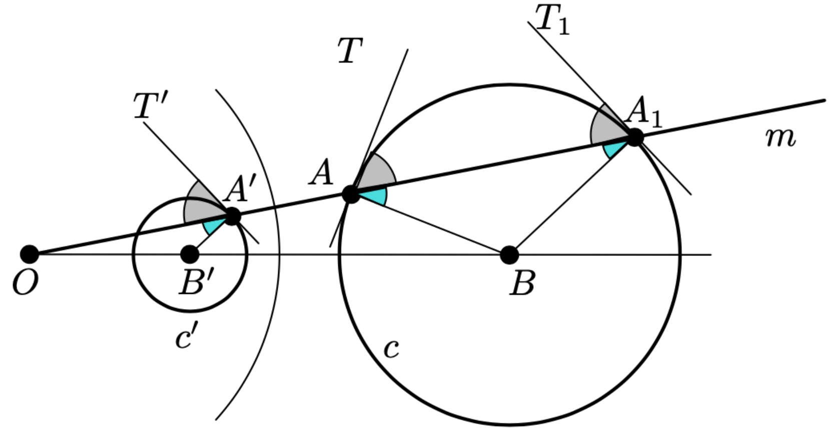
We start with the special case where one circle goes through O:



The line m through O transforms into itself. The circle is also through O, and is the inverse of the line l. The angle between l and m is easily shown to be equal to the angle between m and the image of the line l, namely the circle, at A'.

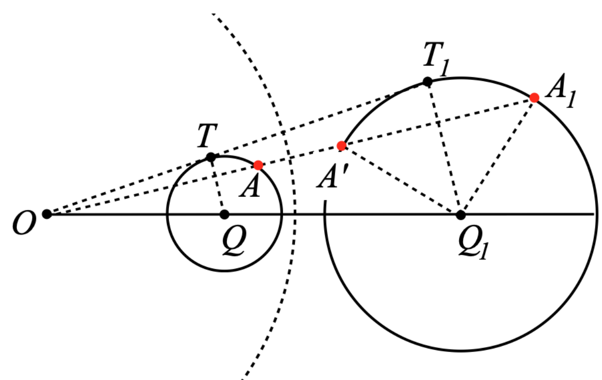
The shaded gray angle at A is complementary to $\angle AOP$. But $\angle AOP = \angle QOA' = \angle QA'O$, and $\angle QA'O$ is complementary to the gray shaded angle at A'. □

In the general case, suppose the circles on centers B and B' are inverses through I_ω. Let A and A' be inverse points as usual.



This does *not* mean that $I_\omega(B') = B$!

However, the green angles at A' and A_1 *are* equal by similar triangles. This requires a bit of work. We simplify the diagram and change the notation as far as the center of the circle, and also introduce tangent points T.

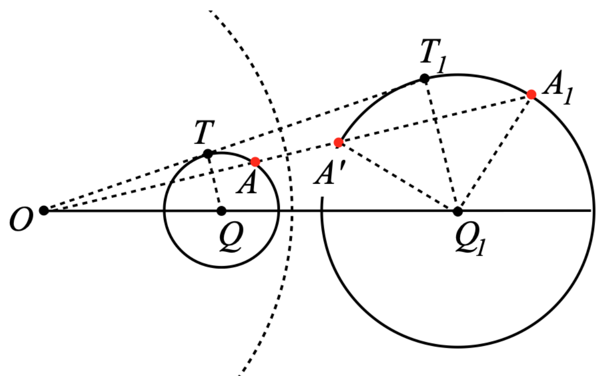


By the secant-tangent theorem, $OA_1 \cdot OA' = OT_1^2$ and this is true wherever A' and A_1 lie.

By the definition of inversion $OA \cdot OA' = k^2$. So

$$\frac{OA_1}{OA} = \frac{OT_1^2}{k^2}$$

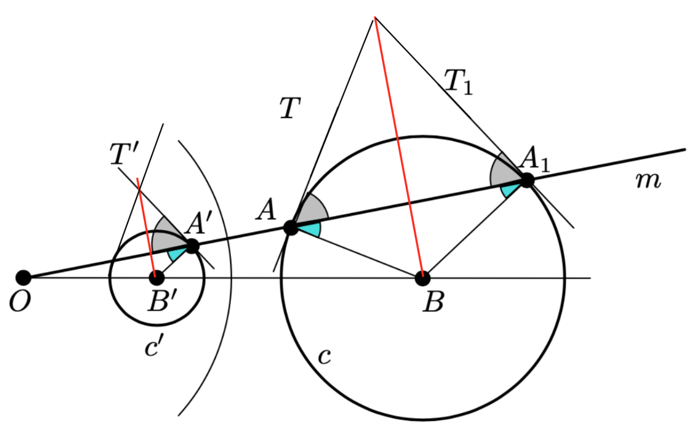
which is a constant. Call it c. So then $OA_1 = c \cdot OA$ for every pair of points A_1 and A (*at the second point of intersection*) lying on the same line through circles Q and Q_1.



Since $\triangle OQT \sim \triangle OQ_1T_1$, that includes Q and Q_1 so $OQ_1 = c \cdot OQ$.

This relationship is called a homothety. The entire circle on Q_1 is a bigger version of the one on Q.

It follows that $\triangle OAQ \sim \triangle OA_1Q_1$. We now switch back to the original notation: $\triangle OA'B' \sim \triangle OA_1B$, which gives two of the equal angles colored green in the figure below. The third one (at A) is equal since $\triangle BAA_1$ is isosceles.



The gray angles are both complementary to the green angles. The second one in circle B (at A) is equal by reflection across a line of symmetry through B and $\perp m$.

$\square$

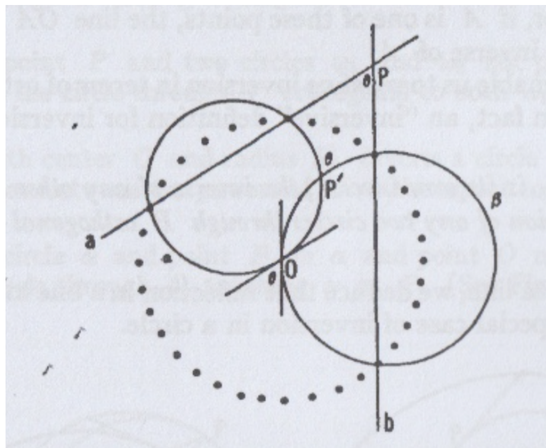


Figure 5.5A

Proof. (Alternate).

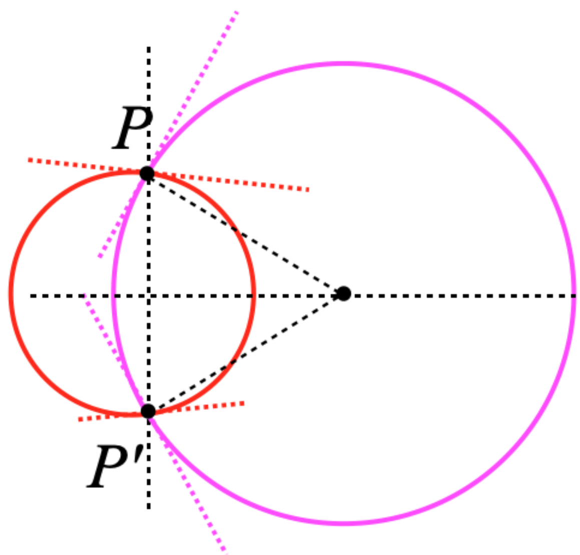
This is section 5.5 of Coxeter.

Consider two circles intersecting at P (not drawn). The supplementary angles formed between the circles are naturally defined in terms of the tangents to the two circles at P. Suppose the angles are θ and ϕ. Now consider just the two tangent lines a and b, which form $\angle\theta$ in the figure above.

We perform an inversion with respect to circle ω on center O. The inverse of a line not through O is a circle through O, where the tangent to the new circle at O is parallel to the original line. So the inverse of line a is circle α with its tangent parallel to a, and similarly for b. We can see that the angle between these two tangents at O is also equal to θ, since they are parallel to the original lines a and b.

As we said above, if P lies on both circles before inversion, it lies on both circles (or lines) after inversion. Thus the tangents at P' are equal to those at P and form the same angle.

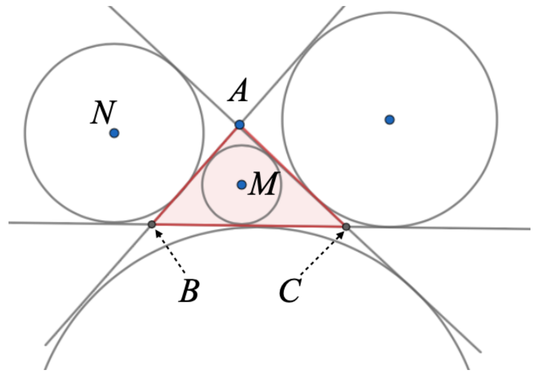
We can also see that for intersecting circles, the angles at the two points of intersection are the same although the tangents have a mirror image symmetry.



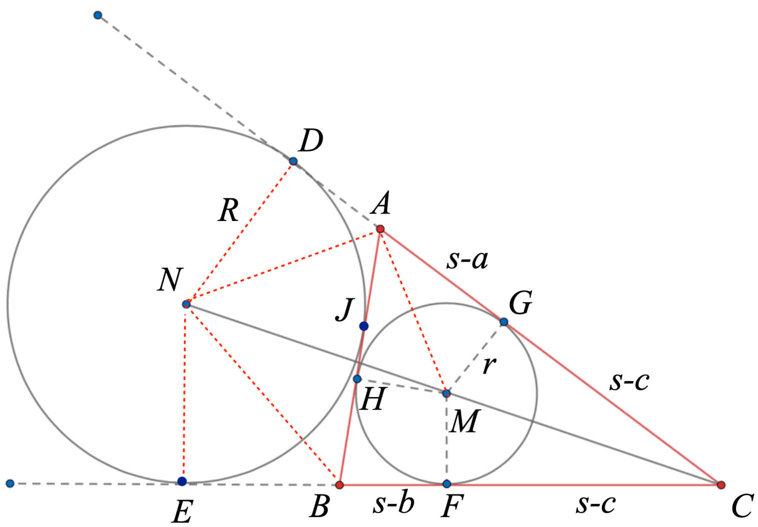
excircles

To review briefly, *excircles* are so named by reference to *incircles*. The incircle to a triangle is tangent to all three sides and lies inside the triangle. An excircle is tangent to one side and also tangent to extensions of the other two sides and it is external to the triangle.

In $\triangle ABC$, with opposite sides a, b and c, the incircle on center M is that circle to which all three sides are tangent.



Any triangle has a single incircle and three excircles, one for each side of the triangle. The excircle on side c (opposite $\angle C$) is formed by extending sides a and b (i.e. BC and AC above) and then finding the circle that is tangent to those two extensions and also tangent to side c.



Let N be the center of the excircle on side c. N lies on the bisector of the original $\angle C$. This follows from the definition that CD and CE are tangent to this excircle.

Since AD and AJ are also tangents to the excircle from point A, if the angle supplementary to $\angle A$ (i.e. $\angle BAD$) is bisected, then the bisector also runs through N, so N can be constructed by finding the intersection of the bisectors of $\angle DAB$ and $\angle C$. The bisector of $\angle ABE$ goes through N as well.

s is the *semi-perimeter* of the triangle, one-half the sum of the side lengths. $AG = u, BF = v, FC = w$. As tangents from a point, the perimeter of the triangle has two copies of each, so the semi-perimeter has one:

$$u + v + w = s$$

But $v + w = a$, so $AG = u = s - a$ and so on.

It is also easy to show that the tangents from A to the excircle on side c are

$$AD = AJ = s - b$$

In other words, J and H each divide side c into lengths $s-a$ and $s-b$.
Proof. Let $AD = p$ and $BE = q$. Then

$$p + (s - a) + (s - c) = q + (s - b) + (s - c)$$

$$p - a = q - b$$

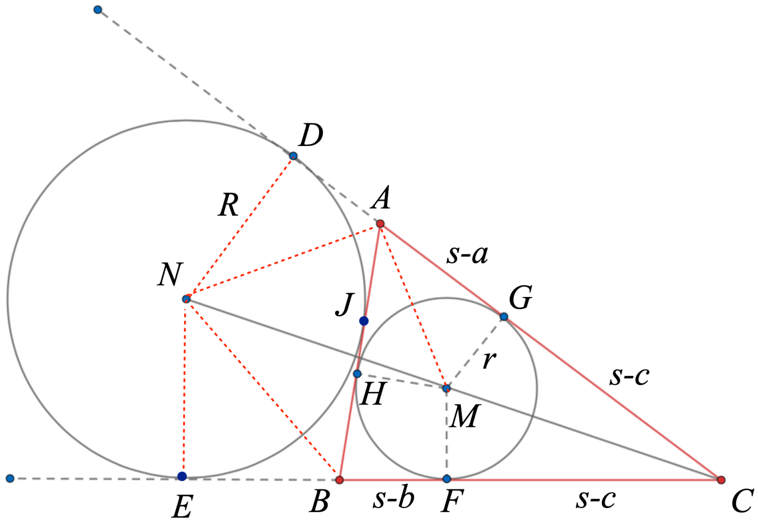
But we also know that $p + q = c$ so

$$c - q - a = q - b$$

$$2q = -a + b + c = 2(s - a)$$

$$q = BE = BJ = s - a$$

□

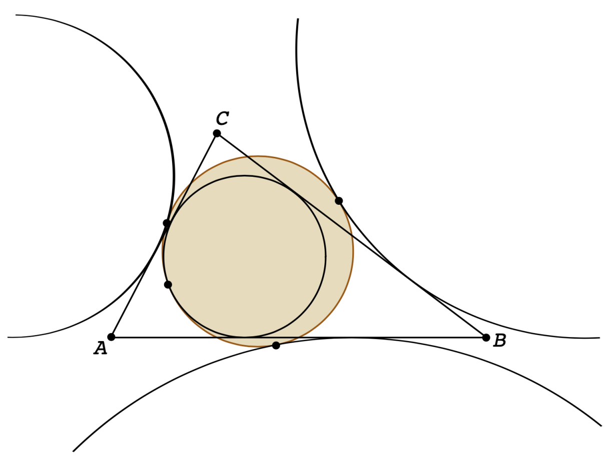


The length of the long tangents $CD = CE$ is easily shown to be just s.

$$(s-a)+(s-b)+(s-c) = 3s-(a+b+c) = s$$

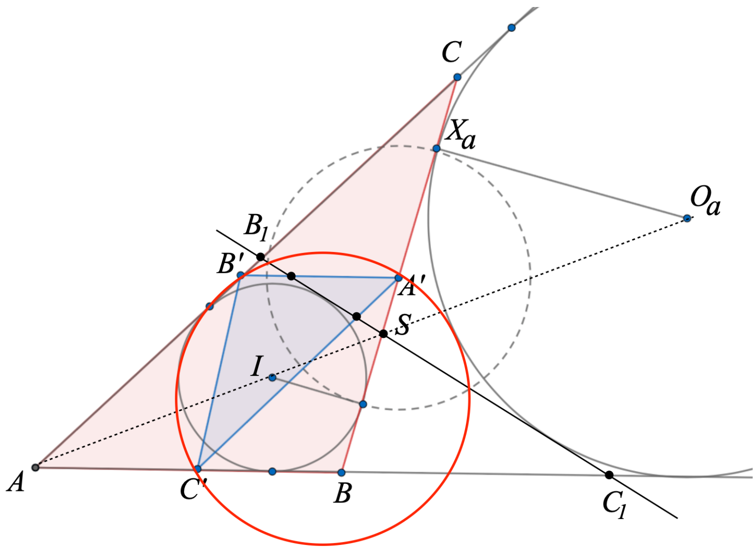
Feuerbach's Theorem

Finally, we reach this famous theorem.



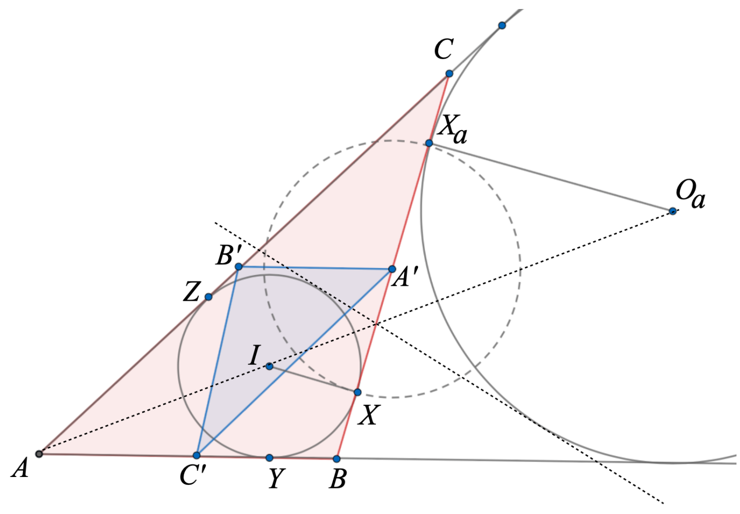
Recall that the nine point circle, passing through the midpoints of the sides, is the circumcircle of the medial triangle, i.e., it is formed with these points as the vertices.

Feuerbach's theorem says that this circle (in red below) is tangent (internally) to the incircle and (externally) to each of the excircles of a triangle.



We previously looked at the incircle and three excircles of a triangle. Recall that the nine point circle also goes through the feet of the altitudes and the bisectors of that part of each altitude between the orthocenter and each vertex.

Let $\triangle ABC$ have the sides bisected at A', B' and C'. The incircle is drawn on center I, and one excircle is drawn tangent to side $a = BC$, on center O_a. $\triangle A'B'C'$ is also drawn.



I

We know from previous work that $BX = BY = s{-}b$, $AY = AZ = s{-}a$ and $CX = CZ = s - c$. As well, the entire tangent from A to the excircle has length s, the semi-perimeter.

Radii are drawn from the incenter I and the excenter O_a, perpendicular to side BC. BC is tangent to the two circles at these points. It follows that

$$IX \parallel O_aX_a$$

We also know from previous work with excircles that $BX = X_aC = s - b$. So

$$XX_a = a - 2(s-b) = a - (a-b+c) = b - c$$

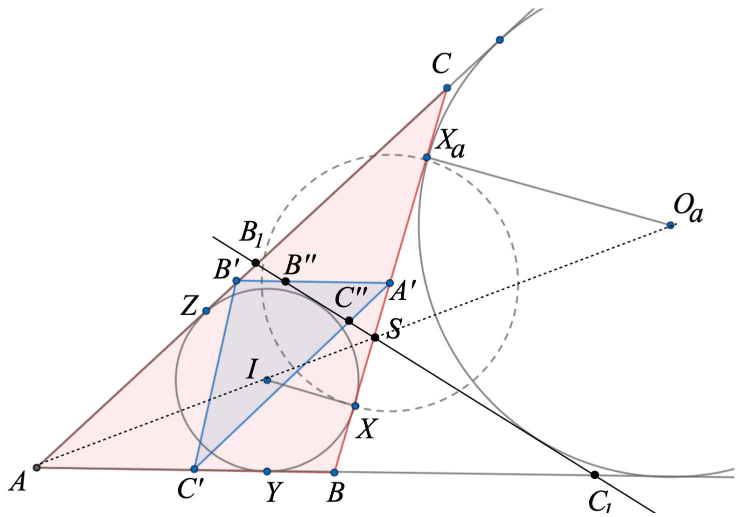
We also have that $XA' = X_aA'$. That's because A' is the midpoint of side $BC = a$, and we've subtracted equal lengths from both ends with $BX = X_aC$.

Therefore, we can draw the circle as shown, with A' as the center and radius $A'X = A'X_a = (b-c)/2$.

We will refer to that circle as ω. It will be the inversion center for the transformation.

We also draw another line tangent to both incircle and excircle. Label the endpoints as B_1 and C_1.

Also mark three other points: where $A'B'$ crosses B_1C_1 at B'', where $A'C'$ crosses B_1C_1 at C'', and finally, the point where BC crosses B_1C_1, at S.



II

The center of the inversion circle ω is at A', which is also on the nine-point circle. The inverse of a circle through the inversion center is a line.

The claim is that B'' is the inverse of B', and C'' is the inverse of C' so the straight line B_1C_1 which goes through those two points is the inverse of the nine point circle containing points A', B' and C'.

To begin with we obtain some more lengths. Use the angle bisector theorem to get an expression for the length of BS and CS, the components of side a produced by the bisector. The claim is that

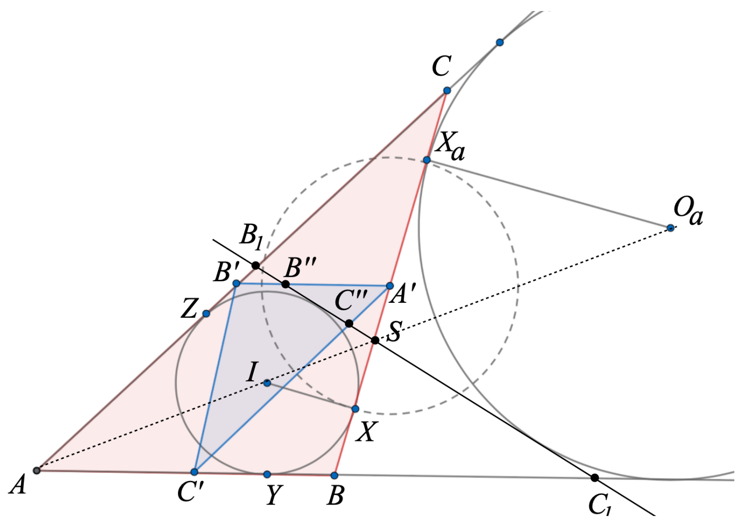
$$BS = \frac{ac}{b+c} \qquad CS = \frac{ab}{b+c}$$

This looks plausible because the sum is just $a = BC$, which checks.

Coxeter explain it thus:

> Since S (like I and I_a) lies on the bisector of the angle A, S divides the segment BC, of length a, in the ratio $b : c$.

Thus the two parts of a are the fractions $b/(b+c)$ and $c/(b+c)$.



Working through the algebra just to confirm this, let $BS = u$ and $CS = v$. Then $u/c = v/b$ so $u/v = c/b$. Add one to both sides

$$\frac{u+v}{v} = \frac{b+c}{b}$$

but $u+v = a$ so we have $v = ab/(b+c)$. Alternatively add one to both sides of the inverse of what's above. Then

$$\frac{u+v}{u} = \frac{b+c}{c}$$

and the second result follows easily as well.

Another claim is that

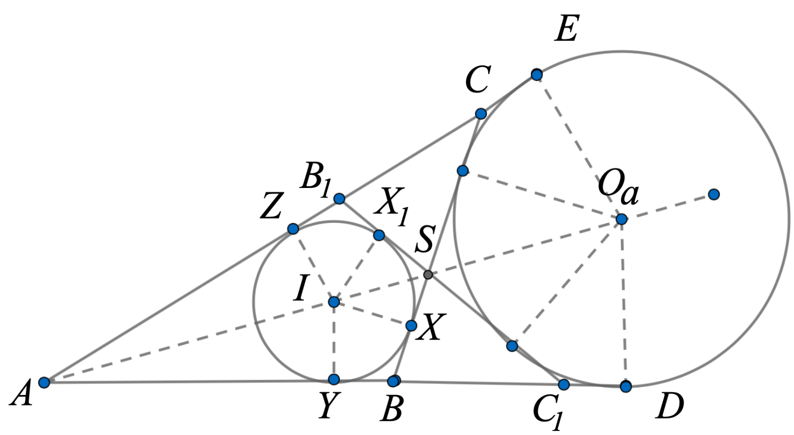
$$SA' = \frac{a(b-c)}{2(b+c)}$$

We get this from $BC = a$, so $BA' = a/2$ and then

$$SA' = \frac{a}{2} - \frac{ac}{b+c}$$

Placed over a common denominator and subtracting, that checks.

The last claim, which I missed in a youtube video of this proof but is in the source (Coxeter), is that $BC_1 = B_1C = b - c$. This is easier to see for B_1C so let us start there. The whole of side AC has length b. so somehow it must be the case that AB_1 has length c and then B_1C is the difference.



We must show that $B_1C = BC_1 = b - c$. There is a lot of symmetry in the figure. Both circles have their centers on the bisector of $\angle A$.

We have that $\triangle SIX$ and $\triangle SIX_1$ have three sides equal (shared side, radius, and two tangents from a point), so they are congruent, which gives $\angle ASX = \angle ASX_1$.

It follows that $\triangle ASB$ and $\triangle ASB_1$ are equiangular and share side AS, so they are congruent. Thus $AB = AB_1 = c$.

So $B_1C = b - c$, by subtraction, and

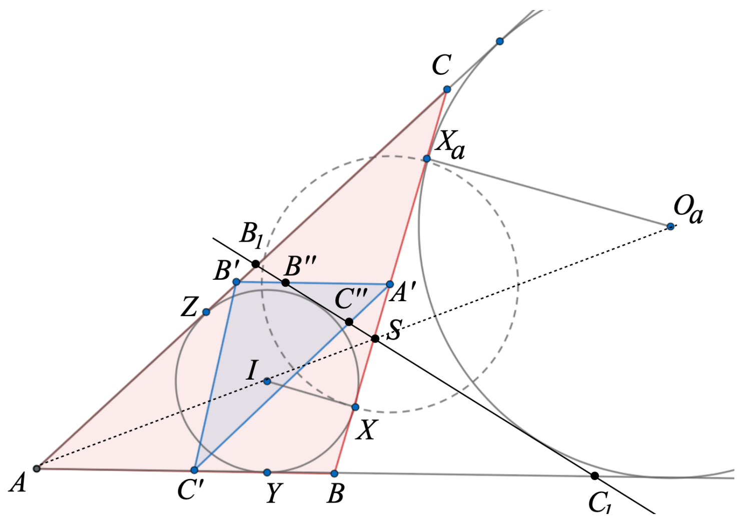
$$B_1Z = b - (s - a) - (b - c) = a + c - s = s - b$$

One can also get this from $SB = SB_1$ (congruent triangles, above) and $SX = SX_1$, so by subtraction, $BX = B_1X_1 = B_1Z$.

The whole $\angle ASC = \angle ASC_1$ because the parts are equal (from congruent triangles and then vertical angles). It follows that $\triangle ASC \cong \triangle ASC_1$. So $AC = AC_1$ and $AB = AB_1$, and then finally $B_1C = BC_1$, also by subtraction.

We do not need it but we can also show that $BC = B_1C_1$. $\triangle ABC \cong \triangle AB_1C_1$.

III



Next, we must find two pairs of similar triangles. These are

$$\triangle SA'B'' \sim \triangle SBC_1 \qquad \triangle SA'C'' \sim \triangle SCB_1$$

These are easy because we have parallel lines and either vertical angles in the first case, or shared $\angle CSB_1$ in the second.

We have

$$\frac{SA'}{A'B''} = \frac{SB}{BC_1}$$

$$A'B'' = SA' \cdot BC_1 \cdot \frac{1}{SB}$$

Plugging in the results from above

$$= \frac{a(b-c)}{2(b+c)} \cdot (b-c) \cdot \frac{b+c}{ac}$$

and since $A'B' = c/2$:

$$A'B' \cdot A'B'' = (\frac{b-c}{2})^2$$

For the second triangle we obtain:

$$\frac{SA'}{A'C''} = \frac{SC}{B_1C}$$

$$A'C'' = SA' \cdot B_1C \cdot \frac{1}{SC}$$

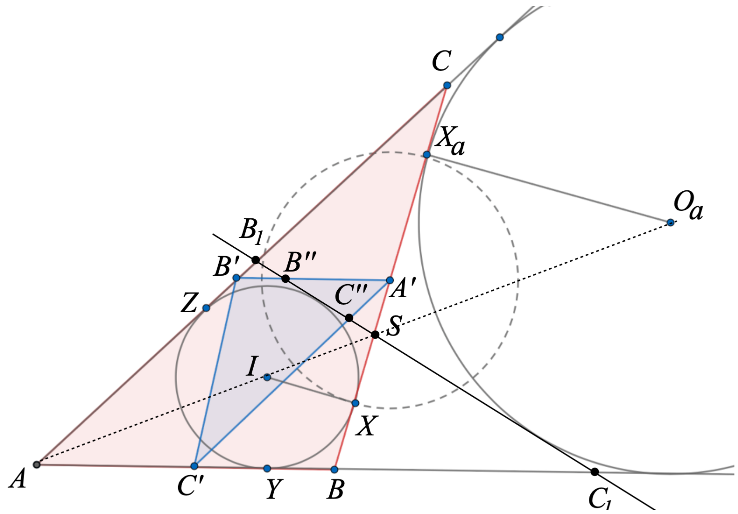
And again, plugging in:

$$= \frac{a(b-c)}{2(b+c)} \cdot (b-c) \cdot \frac{b+c}{ab}$$

and since $A'C' = b/2$:

$$A'C' \cdot A'C'' = (\frac{b-c}{2})^2$$

But this is (the same for both), namely r^2.



So B'' and C'' are the inverses of B' and C' respectively, with respect to ω.

IV

We have an inversion circle centered on A' with diameter XX_a that we have labeled ω.

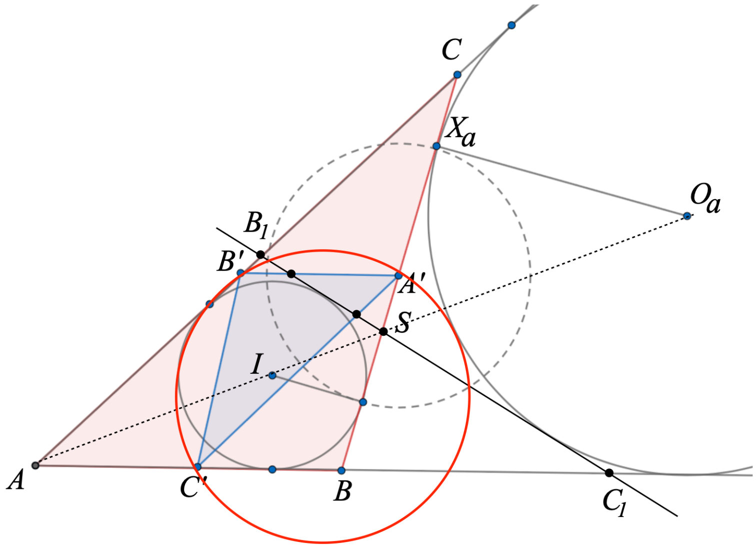
We have shown that the inverse of the nine-point circle is the line B_1C_1, which passes through BC at S and is tangent to both the incircle and to the excircle (these points are not labeled).

Since the inverses of the incircle and excircle are orthogonal with respect to ω, they are fixed (they invert into themselves).

So then, finally, in the inverted system we have line B_1C_1 and the incircle and excircle, with the line tangent to both of these circles.

Points where two curves cross or touch are still points where the inverted figures cross or touch, after inversion.

Since B_1C_1 is tangent to both circles, its inverse is still tangent to both circles. The inverse of B_1C_1 is the nine point circle, so that circle is tangent to both the other circles, i.e. the incircle and the excircle, and to the other two excircles, by extension.



other ideas

Finally, a note about other approaches. One can show that two circles are tangent externally (such as the excircle and nine-point circle), by showing that the distance between the two centers is equal to the sum of the radii.

For internal tangency, the distance between centers must be the difference of the two radii.

Thus, if one can calculate *where* the centers lie, then this requirement can be checked. A judicious choice of coordinate system will help, and you can find such proofs on the internet. However, I think the inversion proof presented in detail here is a lot more fun, and quite simple, once we understand what inversion is all about.