The square root of 3

Archimedes uses two approximations for $\sqrt{3}$: 265/153 and 1350/780. Recall that $\sqrt{3}$ is irrational so it doesn't have an exact decimal representation.

I wrote a script to search for these and other close approximations.

```
> python approx_sqrt3.py
     0      0      0      1      1
     1      1      2      2      1
     3      5      2      6      9
     4      6     12      7      1
    11     19      2     20     37
    15     25     50     26      1
    41     71      2     72    141
    56     96    192     97      1
   153    265      2    266    529
   209    361    722    362      1
   571    989      2    990   1977
   780   1350   2700   1351      1
  2131   3691      2   3692   7381
  2911   5041  10082   5042      1
  7953  13775      2  13776  27549
 10864  18816  37632  18817      1
>
```

The first column has the denominator i (we search from 1 to 50000).

The algorithm is really brute force. For each possible i, we search all the integers larger than i until we find one j such that $3 \cdot i^2 < j^2$. In other words, j is the smallest integer such that j/i is larger than $\sqrt{3}$.

Having the closest j (and $j-1$) for each i, we test whether

$$j^2 - 3 \cdot i^2 < 5$$

$$3 \cdot i^2 - (j-1)^2 < 5$$

If either is true, we print all the values e.g.

```
   153    265       2    266    529
```

In the third column we find repeatedly, 2.

What this means is that the square of the value in column 2, plus 2, is exactly three times the square of the value in column 1. For example:

```
153^2 = 23409
265^2 = 70225
3 x 23409 = 70227
```

Since $265^2/(153^2 + 2) = 3$, $265/153$ is just barely less than $\sqrt{3}$.

The error is $2/23409 \approx 8 \times 10^{-5}$.

In column 5 we see the number 1 repeated.

This is the difference between 3 times the square of the value in column 4 and the square of the value in column 1. For example:

```
780 1350  2700 1351     1
```

```
780^2 = 608400
1351^2 = 1825201
3 x  608400 = 1825200
```

So $1351/780$ is just barely greater than $\sqrt{3}$, the error is $1/608400 \approx 1.6 \times 10^{-6}$.

Continued fractions

There is another way to find such numbers. A continued fraction is an expression like:

$$1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \dots}}}$$

This particular continued fraction is equal to the famous number ϕ.

$$\phi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \dots}}}$$

But notice, the second term on the right-hand side is $1/\phi$ so we can write

$$\phi = 1 + \frac{1}{\phi}$$

$$\phi^2 = \phi + 1$$

For more on ϕ see **here**.

Square roots can be represented as continued fractions. We look first at the slightly easier case of $\sqrt{2}$, before tackling $\sqrt{3}$.

$$(\sqrt{2} - 1)(\sqrt{2} + 1) = 1$$

Rearrange to get a substitution we will use again

$$\sqrt{2} - 1 = \frac{1}{\sqrt{2} + 1}$$

At the same time, add one and subtract one on the bottom right:

$$\sqrt{2} - 1 = \frac{1}{2 + \sqrt{2} - 1}$$

substitute

$$= \frac{1}{2 + \frac{1}{\sqrt{2}+1}}$$

Add one and subtract one again and then substitute again

$$= \frac{1}{2 + \frac{1}{2+\sqrt{2}-1}} = \frac{1}{2 + \frac{1}{2+\frac{1}{\sqrt{2}+1}}}$$

Clearly, this goes on forever.

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \dots}}}$$

The numerators are all 1, so this is a *simple* continued fraction for $\sqrt{2}$.

The continued fraction representation of $\sqrt{2}$ is $[1 : 2]$, meaning that there is an initial 1 followed by repeated 2's.

This fraction goes on forever (since $\sqrt{2}$ is irrational). To turn this into an approximate decimal representation of $\sqrt{2}$, ignore the .... Then the last fraction is 5/2. Invert and add, repeatedly:

```
2 + 1/2 = 5/2
2 + 2/5 = 12/5
```

```
2 + 5/12 = 29/12
2 + 12/29 = 71/29
2 + 29/71 = 171/71
2 + 71/171 = 413/171
```

To terminate we need to use that initial 1:

```
1 + 171/413 = 584/413 = 1.414043
```

To six places, $\sqrt{2} = 1.414213$. We have only three places, but can easily get more.

square root of 3

The continued fraction representation of $\sqrt{3}$ is $[1, 1, 2, 1, 2, \dots]$, which can be shortened to $[1 : (1, 2)]$.

Here is a derivation:

$$(\sqrt{3} - 1)(\sqrt{3} + 1) = 2$$
$$\sqrt{3} - 1 = \frac{2}{\sqrt{3} + 1}$$
$$\frac{\sqrt{3} - 1}{2} = \frac{1}{\sqrt{3} + 1}$$

both of which we will use again. However, going further, add and subtract on the bottom right

$$\sqrt{3} - 1 = \frac{2}{\sqrt{3} + 1} = \frac{2}{2 + \sqrt{3} - 1}$$

Divide top and bottom by 2

$$= \frac{1}{1 + \frac{\sqrt{3}-1}{2}}$$

and substitute giving

$$= \frac{1}{1 + \frac{1}{\sqrt{3}+1}}$$

That's the end of step 1.

Now, for the second step, we focus on that last fraction

$$\frac{1}{\sqrt{3}+1} = \frac{1}{2+\sqrt{3}-1} = \frac{1}{2+\frac{2}{\sqrt{3}+1}}$$

Then for step three, we focus again on the last fraction, which is what we worked with in the first part.

$$\frac{2}{\sqrt{3}+1} = \frac{1}{1+\frac{1}{\sqrt{3}+1}}$$

So now both terms repeat:

$$\sqrt{3} - 1 = \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \dots}}}}$$

which is $[1 : (1, 2)]$, as we said.

We can get approximations for $\sqrt{3}$ similar to what we did for $\sqrt{2}$. Unlike previously, here there are two possibilities. We start with either one of

$$1 + \frac{1}{2 + \dots}$$

$$2 + \frac{1}{1 + \dots}$$

and proceed by ignoring the dots.

The first gives

```
1 + 1/2 = 3/2
2 + 2/3 = 8/3
1 + 3/8 = 11/8
2 + 8/11 = 30/11
1 + 11/30 = 41/30
2 + 30/41 = 112/41
1 + 41/112 = 153/112

1 + 112/153 = 265/153 = 1.732026
```

The actual value is $\sqrt{3} = 1.732051$, to six places. We have four.

The second gives

```
2 + 1 = 3
1 + 1/3 = 4/3
2 + 3/4 = 11/4
1 + 4/11 = 15/11
2 + 11/15 = 41/15
1 + 15/41 = 56/41
2 + 41/56 = 153/56
1 + 56/153 = 209/153
2 + 153/209 = 571/209
1 + 209/571 = 780/571

1 + 571/780 = 1351/780 = 1.732051
```

The actual value is $\sqrt{3} = 1.732051$, to six places. We have all six.

It is believed that this is how Archimedes came up with those approximations. (He doesn't say).