Sum of squares

In this write-up we're going to look at a method to derive the formula for the sum of the squares of integers from $1 \dots n$.

The key idea is to first write

$$(k+1)^3 = k^3 + 3k^2 + 3k + 1$$

(which is true for any particular k), and then sum all n equations for $1 \dots n$.

$$\sum_{k=1}^{n} (k+1)^3 = \sum_{k=1}^{n} k^3 + \sum_{k=1}^{n} 3k^2 + \sum_{k=1}^{n} 3k + \sum_{k=1}^{n} 1$$

This looks a little frightening. So the next thing is to suppress the indices in the sums. They will always be $1 \dots n$. Move one term to the left-hand side.

$$\sum (k+1)^3 - \sum k^3 = 3 \sum k^2 + 3 \sum k + n$$

As before, the difference of sums on the left-hand side simplifies dramatically. On the right, substitute the previous result.

$$(n+1)^3 - 1 = 3 \sum k^2 + 3 \cdot \frac{n(n+1)}{2} + n$$

$$n^3 + 3n^2 + 2n = 3 \sum k^2 + 3 \cdot \frac{n(n+1)}{2}$$

The best thing to do here is to factor the left-hand side. We see there is an n, there will also be a factor of $(n+1)$:

$$n^3 + 3n^2 + 2n = n(n^2 + 3n + 2)$$

$$= n(n+1)(n+2)$$

So now, in one step, we will move the last term on the right-hand side to the left, and also factor out the $n(n+1)$ giving

$$n(n+1)(n+2-\frac{3}{2}) = 3\sum k^2$$

$$n(n+1)(n+\frac{1}{2}) = 3\sum k^2$$

$$\sum k^2 = \frac{n(n+1)}{2} \cdot \frac{(2n+1)}{3}$$

This can be re-written in various ways, for example

$$S_{n^2} = \frac{1}{3} \cdot n(n+1)(n+\frac{1}{2})$$

proof by induction

We can check it by induction. The base case is easy

$$\frac{1(2)(3)}{6} = 1$$

Now for the induction step:

$$\frac{n(n+1)(2n+1)}{6} + (n+1)^2$$

$$= \frac{n+1}{6}\ [\ (n)(2n+1) + 6(n+1)\ ]$$

Look at what's in the brackets

$$\begin{aligned}(n)(2n+1)+6(n+1) \\ = 2n^2+7n+6 \\ = (n+2)(2n+3) \\ = ((n+1)+1)(2(n+1)+1)\end{aligned}$$

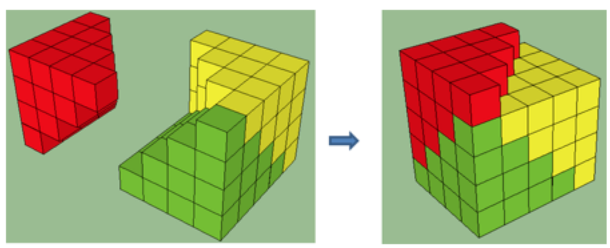
So altogether we have

$$= \frac{(n+1)((n+1)+1)(2(n+1)+1)}{6}$$

which indeed, is the formula we had above, substituting $n+1$ for n.

$\square$

Here are a proof without words:



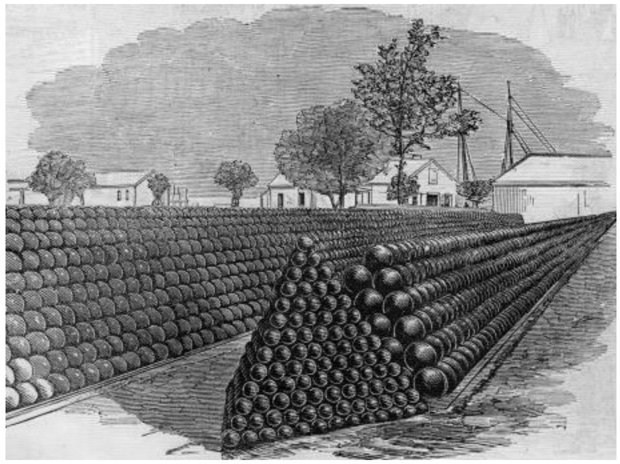
The pieces have a base of $n(n+1)$ and there are three of them rising to a height of $n+\frac{1}{2}$.

Strang's proof

Here is another approach, from Strang's *Calculus*. He says "the best place to start is a good guess". So again, our goal is to find a formula for:

$$S = \sum_{k=1}^{n} k^2$$

Perhaps we visualize a pile of cannonballs



Each layer contains a square number of cannonballs (1, then 4, then 9, etc.). The shape is a pyramid with dimensions $n \times n \times n$. We know the formula for the volume of a pyramid, and guess

$$S_n = \frac{1}{3}n^3$$

To test it, check whether this difference is n^2 (as it should be):

$$S_n - S_{n-1} = \frac{1}{3}n^3 - \frac{1}{3}(n-1)^3$$

Since

$$(n-1)^3 = n^3 - 3n^2 + 3n - 1$$

then

$$S_n - S_{n-1} = \frac{1}{3}(n^3 - n^3 + 3n^2 - 3n + 1)$$

We see that our guess is off by the residual terms

$$\frac{1}{3}(3n^2 - 3n + 1)$$

$$= n^2 - n + \frac{1}{3}$$

Strang says: the guess needs *correction terms*. To cancel $1/3$ in the difference, subtract $n/3$ from the sum. And to add back n in the difference, add back $1+2+\cdots+n(n+1)/2$ to the sum. Our new guess is

$$\begin{aligned} S_n &= \frac{1}{3}n^3 + \frac{n(n+1)}{2} - \frac{n}{3} \\ &= \frac{n}{6}(2n^2 + 3(n+1) - 2) \\ &= \frac{n}{6}(2n+1)(n+1) \\ &= \frac{n(n+1)(2n+1)}{6} \end{aligned}$$

which may be easier to remember as

$$S_n = \frac{n(n+1)}{2} \cdot \frac{2n+1}{3}$$

undetermined coefficients

We guess that the formula for sum of squares will have terms of degree 3: powers of n^3.

$$S_n = an^3 + bn^2 + cn + d$$

Since $S_0 = 0$ we conclude that $d = 0$. The sum for $n-1$ would be

$$S_{n-1} = a(n-1)^3 + b(n-1)^2 + c(n-1)$$

If we add n^2 to this, we should have the previous expression:

$$a(n-1)^3 + b(n-1)^2 + c(n-1) + n^2 = S_n = an^3 + bn^2 + cn$$

When we expand the parentheses, each term on the right is canceled, leaving

$$a(-3n^2 + 3n - 1) + b(-2n + 1) - c + n^2 = 0$$

And, as before, the coefficients of different powers of n must all separately balance.

$$-3a + 1 = 0$$
$$a = \frac{1}{3}$$

Then

$$3a - 2b = 0$$
$$b = \frac{3}{2}a = \frac{1}{2}$$

and $-a + b - c = 0$, which gives $c = b - a = 1/6$ so

$$S_n = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}$$

We have a factor of $n/6$ and the rest is

$$2n^2 + 3n + 1 = (2n+1)(n+1)$$

so

$$S_n = \frac{n(n+1)(2n+1)}{6}$$