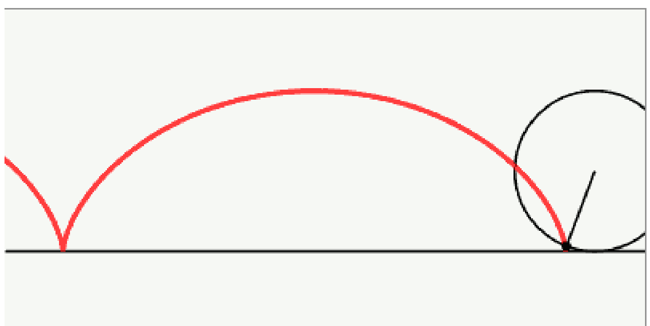
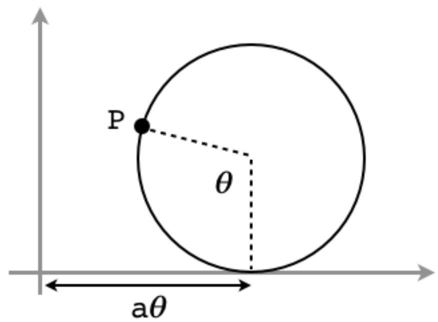
Cycloid

Imagine a bicycle with one tire marked at a particular point on the rim, say with fluorescent paint or a small light. Time starts at $t = 0$ with that point P in contact with the x axis at $(0, 0)$. Then the bike rolls to our right. As the tire rotates the fixed point P on the rim traces a curve



We want to find equations that give the point P as a function of time. We will parametrize the curve, yielding parametric equations $x(t)$, $y(t)$.

The second diagram shows the angle through which the wheel has turned as θ, but we will use t for θ here.



The x displacement of the vertical straight down from the center of the tire is just at, where a is the radius of the wheel, it is equal to the arc on the circumference of the wheel from the point which is currently in contact with the ground, going around up to P.

It is reasonably easy to derive the desired parametric equations, using vectors, especially once you know the answer. For x, we have the vector that goes from $(0, 0)$ to the contact point with the ground. As indicated in the figure, that is at.

We need to subtract the distance $a\ sin\ t$ from that. Basically the rationale is that the motion is a standard parametric circle which has been rotated by 90 degrees clockwise and then inverted. The rotation changes cosine to sine, and the inversion brings the subtraction.

It's easier to see for $t < \pi/2$, but it is true always. Check some other values of t like π or $3\pi/2$ to confirm. This is the usual circular motion.

For y, we have a constant factor of a above the x axis, then the additional displacement is $-a\ cos\ t$. So for $t = 0$ we have the additional displacement is -a (we were on the ground), for $t = \pi/2$ it is zero, and for $t = \pi$ it is plus a for a total of $2a$.

The parametric equations are then

$$x(t) = at - a \sin t$$

$$y(t) = a - a\cos t$$

Taking derivatives:

$$x'(t) = a - a\cos t$$

$$y'(t) = a\sin t$$

We can get dy/dx, what we usually call y', by simple division:

$$y' = \frac{dy}{dx} = \frac{a\sin t}{a - a\cos t}$$

$$= \frac{\sin t}{1 - \cos t}$$

Simmons uses half-angle formula like so:

$$\sin t = 2\sin t/2 \cos t/2$$

$$\cos t = \cos^2 t/2 - \sin^2 t/2$$

$$= 1 - 2\sin^2 t/2$$

So the ratio is

$$y' = \frac{2\sin t/2 \cos t/2}{2\sin^2 t/2} = \cot t/2$$

aside about Archimedes

It struck me that $\cot t/2$ is one of the terms in the relationship Archimedes used in approximating π. There we had

$$\cot 2\theta + \csc 2\theta = \cot\theta$$

$$\cot t + \csc t = \cot t/2$$

So somehow, if what we have above is correct

$$\frac{\sin t}{1 - \cos t} = \cot t/2 = \cot t + \csc t$$

Factor the right-hand side

$$= \frac{1}{\sin t}(\cos t + 1)$$

Then

$$\sin^2 t = (1 - \cos t)(\cos t + 1)$$
$$= 1 - \cos^2 t$$

and one more step gives our favorite identity.

Area under the arc

We want

$$A = \int_{t=0}^{t=2\pi} y\ dx$$
$$= \int_{t=0}^{t=2\pi} (a - a\cos\ t)(a - a\cos\ t)\ dt$$
$$a^2 \int_{t=0}^{t=2\pi} (1 - \cos\ t)(1 - \cos\ t)\ dt$$
$$a^2 \int_{t=0}^{t=2\pi} (1 - 2\cos\ t + \cos^2\ t)\ dt$$

If you don't remember the result for $\int \cos^2 t\ dt$, we will derive the double angle formula later and convert from $\sin^2$ to $\cos^2$. Write:

$$A = a^2(t - 2\sin\ t + \frac{1}{2}t + \frac{1}{4}\sin 2t)\ \Big|_0^{2\pi}$$

$$a^2(2\pi - 0 + \pi + 0 - 0 + 0 - 0 - 0) = 3\pi a^2$$

A very simple answer.

Roberval

The cycloid curve was not known to the ancients, as far as our sources have anything to say. The first mention is around 1500, and by the time of Galileo a century later, it was something he thought about quite a bit. Galileo is said to have used Archimedes method of cutting out shapes and weighing them) to show that the area was approximately 3 times that of the generating circle.

The cycloid became very popular in the decades before Newton. Roberval came up with a clever application of Cavlieri's principle, to find the area under the curve.

The idea is to draw a second curve, called the *companion curve*, by adding a half-circle to the x-values on the cycloid curve, as shown.

The areas marked by horizontal lines are equal, by Cavalieri's principle.

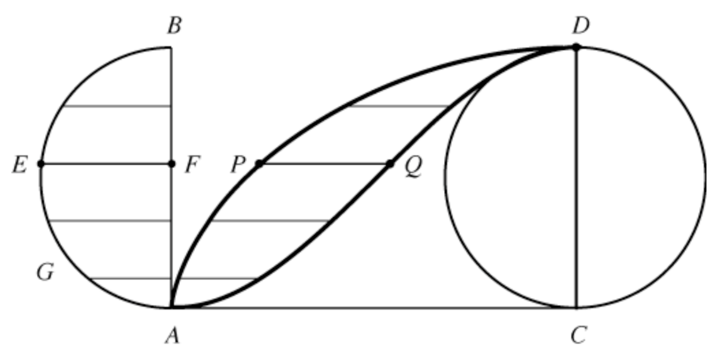


Figure 4. The companion curve to the cycloid.

https://maa.org/sites/default/files/pdf/cmj_ftp/CMJ/January%202010/3%20Articles/3%20Martin/08-170.pdf

But the curve AQD apparently divides the rectangular area $ACDB$ in half, by symmetry. (It is actually the sine curve from $-\pi/2$ to $\pi/2$, just scaled up by a factor of a in both directions). Since the rectangle has width πa and height $2a$, its area is $2\pi a^2$, so one-half that is πa^2.

The area under the cycloid curve is then three-quarters of the total, which (between $0 \to \pi$) is $(3/2)\pi a^2$, and the area under one complete lobe is twice that or $3\pi a^2$.

Let's see if we can figure out a parametric equation for the companion curve. $x(t)$ for the cycloid was $x(t) = at - a\sin t$. The companion curve gets an additional length in the x-direction of $a\sin t$. So $x(t) = at$. That was easy!

$y(t)$ is unchanged. Our integral is just $\int y\ dx$ and $dx = a\ dt$ so

$$A = \int (a - a\cos t)\ a\ dt$$

The bounds are $0 \to \pi$ so finally

$$A = a^2 \int_0^\pi 1 - \cos t\ dt$$

$$= a^2\ [\ t - \sin t\ ]\Big|_0^\pi$$

$$= \pi a^2$$

which matches what we had before.

slope

Descartes gave the slope of the tangent line to the cycloid by the following construction:

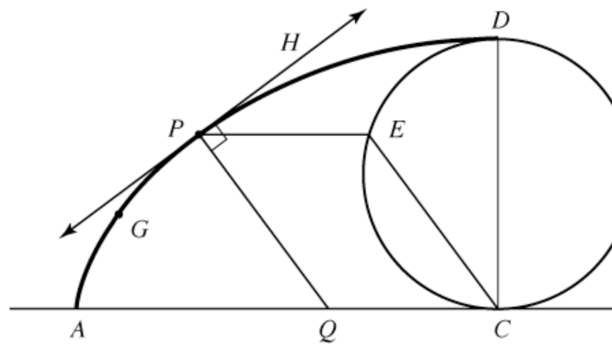


Figure 6. Descartes' tangent construction.

Given P on the cycloid, draw PE horizontally to find E on the circle, then draw PQ parallel to EC. The tangent is perpendicular to PQ.

I like this construction a lot because it reminds me of what Roberval did with the parabola. For any t, the distance moved horizontally is equal to the distance along the rim of the generating circle. The tangent should add to or subtract from both values at the same rate.

In other words, I expect that the tangent should be the average of the tangent to the circle at E and the horizontal, namely one-half of the former.

If C is taken to be $(0,0)$, the coordinates of E are (x', y), where

$$x' = -a \sin t$$

and the slope of $PQ \parallel EC$ will be

$$\frac{0 - (a - a\cos t)}{0 - (-a \sin t)} = \frac{\cos t - 1}{\sin t} = \cot t - \csc t$$

The slope of the tangent is the negative inverse

$$\frac{\sin t}{1 - \cos t}$$

which matches.

Recall that we had before that

$$y' = \cot t/2$$

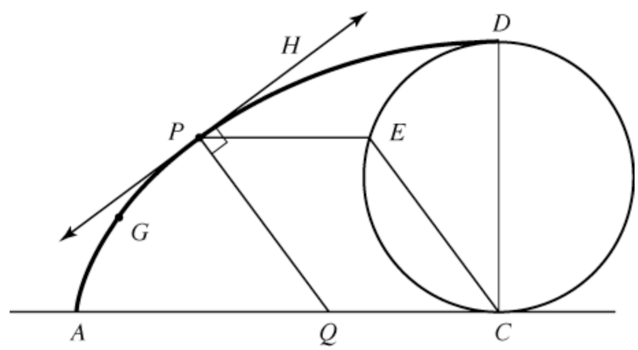


Figure 6. Descartes' tangent construction.

which seems like a problem at first but $t \neq \angle ECQ$.

If we had drawn the center of the circle as O, then t would be $\angle EOC$, and since $\triangle EOC$ is isosceles

$$t + 2 \cdot \angle OCE = 180$$

$$t/2 + \angle OCE = 90$$

but $\angle OCE$ ($\angle DCE$) and $\angle ECQ$ are also complementary so

$$t/2 = \angle ECQ = \angle EPQ$$

The angle made by the tangent ($\angle HPE$) is complementary, so its tangent is the cotangent of $t/2$.

Note also that $\angle EDC = \angle ECQ = \angle EPQ$, so it is also complementary to the tangent angle.

Arc length

We wish to determine the arc length and area under the curve for one complete revolution of the wheel.

We want to use a slightly different version of the usual formula for arc length

$$ds^2 = dx^2 + dy^2$$

$$(\frac{ds}{dt})^2 = (\frac{dx}{dt})^2 + (\frac{dy}{dt})^2$$

$$ds = \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}\ dt$$

$$= \sqrt{(a - a\cos\ t)^2 + (a\sin\ t)^2}\ dt$$

This expands to

$$a\sqrt{1 - 2\cos\ t + \cos^2 t + \sin^2 t}\ dt$$

$$= a\sqrt{2 - 2\cos\ t}\ dt$$

The length is

$$L = \int_0^{2\pi} a\sqrt{2 - 2\cos\ t}\ dt$$

$$= a\sqrt{2}\ \int_0^{2\pi} \sqrt{1 - \cos\ t}\ dt$$

double angle

$$\cos(s - t) = \cos s \cos t + \sin s \sin\ t$$

(check: if $s = t$ then $\cos 0 = 1$, which is correct).

So

$$\cos(s + t) = \cos s \cos t - \sin s \sin\ t$$

Let $s = t$ and $u = 2s$, then

$$\cos 2s = \cos u = \cos^2\ (\frac{u}{2}) - \sin^2\ (\frac{u}{2})$$

$$\cos u = 1 - \sin^2\ (\frac{u}{2}) - \sin^2\ (\frac{u}{2})$$

$$2\sin^2\ (\frac{u}{2}) = 1 - \cos u$$

u is just a dummy variable, so we can switch back to t

$$2\sin^2\ (\frac{t}{2}) = 1 - \cos t$$

finishing up

We have that

$$L = a\sqrt{2}\ \int_0^{2\pi} \sqrt{1 - \cos\ t}\ dt$$

And

$$1 - \cos t = 2\sin^2(\frac{t}{2})$$

$$\sqrt{1 - \cos t} = \sqrt{2}\sin(\frac{t}{2})$$

So

$$L = a\sqrt{2}\ \int_0^{2\pi} \sqrt{2}\sin\ (\frac{t}{2})\ dt$$

$$= 2a\ \int_0^{2\pi} \sin\ (\frac{t}{2})\ dt$$

$$= 2a\ (-2)\cos\ (\frac{t}{2})\ \Big|_0^{2\pi}$$

$$= -4a\ (\cos\ \pi - \cos\ 0)$$

$$= -4a\ (-1 - 1) = 8a$$

Also a very simple answer to the problem.

tautochrone

`https://en.wikipedia.org/wiki/Tautochrone_curve`

We can parametrize using the angle of the circle of radius a:

$$x(\theta) = a\theta - a\sin\theta \qquad y(\theta) = a - a\cos\theta$$

$$\frac{dx}{d\theta} = a - a\cos\theta \qquad \frac{dy}{d\theta} = a\sin\theta$$

To make things very clear, we will keep track of the difference between the parameters t and θ. Let's say that

$$\theta = \omega t$$

but $\omega = 1$ so

$$\theta = t \qquad d\theta = dt$$

and then

$$x(t) = at - a\sin t \qquad y(t) = a - a\cos t$$

$$x' = \frac{dx}{dt} = a - a\cos t \qquad y' = \frac{dy}{dt} = a\sin t$$

$$\frac{dy}{dx} = \frac{\sin t}{1-\cos t} = \cot\frac{t}{2}$$

The last identity can be obtained in various ways, as we saw before.

Now, we solve the tautochrone system, using Simmons (*Calculus Gems*), who follows Abel.

Inverting the coordinate system, and the curve, doesn't change the parametrization. (Try it). An increase in y converts potential energy to kinetic energy such that

$$\frac{1}{2}mv^2 = mgy$$

$$v = \sqrt{2gy}$$

Since $v = ds/dt$

$$dt = \frac{ds}{v} = \frac{ds}{\sqrt{2gy}}$$

The arc length differential is

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + y'^2}\ dx$$

then

$$dt = \frac{\sqrt{1 + y'^2}}{\sqrt{2gy}}\ dx$$

This says that the time T_1 to arrive at $x = x_1$ is

$$T_1 = \int_0^{t_1} dt = \int_0^{x_1} \frac{\sqrt{1 + y'^2}}{\sqrt{2gy}}\ dx$$

We want to rewrite this in terms of θ.

$$ds^2 = dx^2 + dy^2$$

$$(\frac{ds}{d\theta})^2 = (\frac{dx}{d\theta})^2 + (\frac{dy}{d\theta})^2$$

$$\frac{ds}{d\theta} = \sqrt{a^2(1 - \cos\theta)^2 + a^2 \sin^2\theta}$$

$$ds = \sqrt{a^2(1 - \cos\theta)^2 + a^2 \sin^2\theta}\ d\theta$$

Then

$$dt = \frac{ds}{\sqrt{2gy}}$$

$$= \frac{\sqrt{a^2(1 - \cos\theta)^2 + a^2 \sin^2\theta}}{\sqrt{2gy}}\ d\theta$$

$$= \frac{\sqrt{2a^2(1 - \cos\theta)}}{\sqrt{2ga(1 - \cos\theta)}}\ d\theta$$

$$\int dt = \sqrt{\frac{a}{g}} \int d\theta \qquad\qquad t = \sqrt{\frac{a}{g}}\ \theta$$

This says that the time to arrive at angle θ is $\sqrt{a/g} \cdot \theta$. In particular, if we go all the way to the bottom, $\theta = \pi$ and the time is $\pi \cdot \sqrt{a/g}$.

It isn't clear to me why you can't plug in different bounds on the angle and get a different time.

tautochrone property

The tautochrone property is that it should take the same time to descend to the bottom of the curve regardless of where we start.

Suppose we start at y_0, then

$$v = \frac{ds}{dt} = \sqrt{2g(y - y_0)}$$

ds is unchanged, so substituting

$$dt = \frac{ds}{v} = \frac{\sqrt{2a^2(1 - \cos\theta)}}{\sqrt{2g(y - y_0)}}\ d\theta$$

Now,

$$\begin{aligned} y - y_0 &= (a - a\cos\theta) - (a - a\cos\theta_0) \\ &= \cos\theta_0 - \cos\theta \end{aligned}$$

so our integrand is

$$dt = \sqrt{\frac{a}{g}}\ \frac{\sqrt{1 - \cos\theta}}{\sqrt{\cos\theta_0 - \cos\theta}}\ d\theta$$

substitution

We need to manipulate $1 - \cos\theta$. As we saw before

$$\cos\theta = \cos^2\theta/2 - \sin^2\theta/2$$

$$= 1 - 2\sin^2\theta/2$$

but also

$$\cos\theta = 2\cos^2\theta/2 - 1$$

Use the first one to substitute on top

$$dt = \sqrt{\frac{a}{g}}\ \frac{\sqrt{2}\ \sin\theta/2}{\sqrt{\cos\theta_0 - \cos\theta}}\ d\theta$$

and the second one on the bottom:

$$\cos\theta_0 - \cos\theta = 2\cos^2\theta_0/2 - 1 - (2\cos^2\theta/2 - 1)$$

$$= 2(\cos^2\theta_0/2 - \cos^2\theta/2)$$

so

$$dt = \sqrt{\frac{a}{g}}\ \frac{\sin\theta/2}{\sqrt{\cos^2\theta_0/2 - \cos^2\theta/2}}\ d\theta$$

final part

We want to get the constant $\cos^2\theta_0/2$ out from under the square root. Simmons gives the following substitution (I know it looks a little strange):

$$u = \frac{\cos\theta/2}{\cos\theta_0/2}$$

$$\cos^2\theta_0/2 \cdot u^2 = \cos^2\theta/2$$

so

$$dt = \sqrt{\frac{a}{g}}\ \frac{\sin\theta/2}{\cos\theta_0/2 \cdot \sqrt{1-u^2}}\ d\theta$$

And then also

$$du = -\frac{\sin\theta/2}{2\cdot\cos\theta_0/2}\ d\theta$$

so

$$dt = \sqrt{\frac{a}{g}}\ \frac{1}{1-u^2}\ (-2\ du)$$

Now

$$\int \frac{1}{1-u^2}\ du = \sin^{-1} u$$

What are the limits on u? The original limits on θ were $\theta_0 \to \pi$.

When $\theta = \theta_0$, $u = 1$, and when $\theta = \pi$, $u = k\cos\theta/2 = 0$. So finally

$$t = \int dt = \sqrt{\frac{a}{g}}(-2)\sin^{-1} u \ \Big|_1^0$$

Note the flipped limits. I get

$$t = \sqrt{\frac{a}{g}}(-2)(-\frac{\pi}{2})$$

which gets rid of that pesky minus sign and gives finally $t = \pi \cdot \sqrt{a/g}$. In other words, the time is the same, even if we start at y_0 rather than $y = 0$.