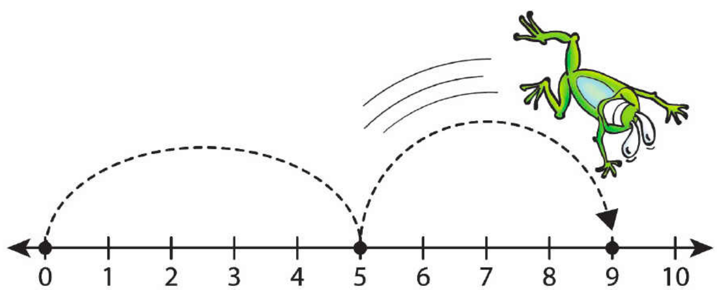
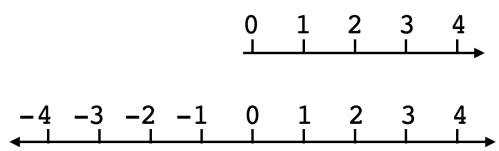
Minus one

You're familiar with the number line.



Here we see the result of $5 + 4$. Start on 0, facing right. First take 5 steps forward, then 4 more. The result is 9.

Negative numbers extend to the left of 0 on the number line (original line on top and the extended line below).



The concept of negative numbers seems a bit strange at first. How can you have -3 sheep?

But if you first owe someone \$3 and then you earn \$5, after returning your friend's money you'll have $5 - 3 = 2$ dollars. You could think of this as you had -\$3 (you had no money but also owed money) and then got paid \$5 so you have $-3 + 5 = 2$ dollars.

For multiplying with negative numbers we first have the rule that

$$-1 \cdot n = -n$$

for every positive number n. For example $-1 \cdot 2 = -2$.

And since multiplication is *commutative* (you can reverse the order of the terms without changing the result)

$$n \cdot -1 = -n$$

for every positive number n. For example $2 \cdot -1 = -2$.

The big question is what to do with

$$-1 \cdot -1 \stackrel{?}{=}$$

We're going to define this to be equal to something, and there are really only two choices we might think about for a possible answer: either 1 (that is, $+1$) or -1. Let's think about the second possibility and see why that causes trouble.

what's wrong with this

We're going to find that if we defined the above result as equal to -1, we run into a conflict with the fundamental properties of arithmetic.

One property is that 1 is the multiplicative identity, meaning that for any number n

$$1 \cdot n = n$$

Also, when multiplying any equation by the same number on both sides, equality is maintained. Multiplying the above by $1/n$ we get

$$1 = \frac{n}{n}$$

This second rule says that any number divided by itself is equal to 1.

So let's look at

$$-1 \cdot -1 = -1$$

Using 1 as the multiplicative identity, we have that $1 \cdot -1 = -1$ and equating that with our definition we have that

$$-1 \cdot -1 = 1 \cdot -1$$

Now multiply by $1/-1$, clearing one copy of -1 on each side. (Here we are using another fundamental property, associativity).

This leaves us with

$$-1 = 1$$

That's a real problem, what mathematicians call a *contradiction*. Choosing -1 as the answer in the original equation messes up our mathematical universe. Therefore, we are forced to define

$$-1 \cdot -1 = 1$$

[Note: the rule above was that any number divided by itself is equal to 1. The exception is that $0/0$ is not defined. It is not *permitted*. We'll explain why in a separate writeup.]

Strogatz (in *Joy of x*) shows this pattern

$$2 \cdot -1 = -2$$

$$1 \cdot -1 = -1$$

$$0 \cdot -1 = 0$$

$$-1 \cdot -1 =?$$

The pattern is clear. The answer is +1.

Here is a table

```
      -2 -1  0  1  2
      --------------
-2 |         0 -2 -4
-1 |      1  0 -1 -2
 0 | 0  0  0  0  0
 1 |-2 -1  0  1  2
 2 |-4 -2  0  2  4
```

Another explanation I like comes from Stewart (*Letters to a Young Mathematician*). Suppose we look at

$$(-1) \cdot (1 - 1) = 0$$

Clearly, this is zero, because the sum inside the parentheses is 0.

But suppose we expand it using the distributive law. The first term is $-1 \cdot 1 = -1$. If we put that -1 on the right-hand side it becomes $+1$ and then we have:

$$-1 \cdot -1 = 1$$