Zero

Consider the equation $xy = 1$.

Now suppose $y = 0$. What is x? The answer is that since 0 times *anything* is zero, there is no x that satisfies the equation when $y = 0$.

If we multiply both sides of the equation by $1/y$

$$x = \frac{1}{y}$$

It's less obvious now, but still true that if $y = 0$, then there is no real number x to satisfy the equation.

Thus, we learn in algebra not to do this

$$\frac{1}{0} \stackrel{?}{=}$$

We say that the result of division by zero is *undefined*.

Let's suppose the denominator is not 0 but instead is small and then ask what happens if it becomes smaller. So for example $1/0.1 = 10$, and using exponents

$$\frac{1}{10^{-1}} = 10^1$$

Then

$$\frac{1}{10^{-1000}} = 10^{1000}$$

The inverse of a very very very small number is a very very very large number. What's the largest number?

Infinity, symbolized ∞, is a candidate. Suppose we try treating ∞ as a number. Maybe

$$\frac{1}{0} \stackrel{?}{=} \infty$$

However, infinity is a strange beast. Consider

$$\infty + 1 \stackrel{?}{=}$$

Certainly some number one more than infinity is also infinite? ∞ is supposed to be the largest number. But then if

$$\infty + 1 = \infty$$

Subtracting ∞ on both sides (assuming that ∞ is just a number) we obtain

$$1 = 0$$

That's problematic, to say the least.

Continuing our exploration, suppose we try

$$\frac{1}{0} = \infty$$

What do we mean by the division operation? We mean that result of a/b is a number c such that

$$a = c \cdot b$$

So, applying the same logic to our problem

$$1 = \infty \cdot 0$$

But 0 times any number is defined to be equal to zero. Oops. We must also ask about 2/0. It must be that

$$\frac{2}{0} = 2 \cdot \frac{1}{0} = 2 \cdot \infty = \infty$$

So, working with the left-hand and very right-hand sides:

$$2 = \infty \cdot 0$$

and since $1 = \infty \cdot 0$ it follows that

$$2 = \infty \cdot 0 = 1$$

These are huge contradictions. We conclude that

- ∞ is not a number
- Division by 0 is not defined

another demonstration

Suppose we accept for a moment that $\frac{1}{0}$ is *something*, without specifying exactly what it is. Then consider

$$0 \cdot \frac{1}{0}$$

On one hand, if we use the rule that $0 \cdot a = 0$, where a is any number, then

$$0 \cdot \frac{1}{0} = 0$$

On the other hand, if we use the rule that any number divided by itself is 1

$$0 \cdot \frac{1}{0} = 1 \cdot \frac{0}{0} = 1 \cdot 1 = 1$$

We have to break at least one rule, and that's not desireable.