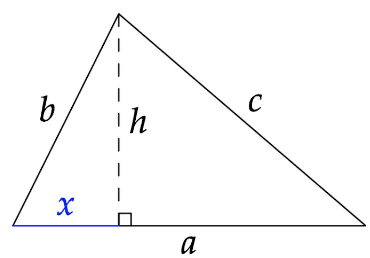
Heron's formula

In this short write-up we will derive Heron's formula for the area of a triangle in terms of the *semi-perimeter*. Start by drawing the altitude in any triangle (if obtuse, use the vertex with the obtuse angle).



$$x^2 + h^2 = b^2$$
$$(a-x)^2 + h^2 = c^2$$

Subtract the first from the second

$$(a-x)^2 - x^2 = c^2 - b^2$$
$$a^2 - 2ax = c^2 - b^2$$
$$c^2 = a^2 + b^2 - 2ax$$

Straightforward, this is the Pythagorean theorem with a correction term.

Let $\angle C$ be the angle opposite side c. Since $x = b\cos C$:

$$c^2 = a^2 + b^2 - 2ab\cos C$$

The law of cosines.

However, we will leave our result as $2ax$ and move on to write an expression for the area A. Going back to the right triangle:

$$h^2 = b^2 - x^2$$

$$h = \sqrt{b^2 - x^2}$$

$$= \sqrt{b^2 - (\frac{a^2 + b^2 - c^2}{2a})^2}$$

Now find the area (or twice that):

$$2A = ah$$

So

$$2A = a\sqrt{b^2 - (\frac{a^2 + b^2 - c^2}{2a})^2}$$

This seems pretty complicated, but notice, we have a^2 on the bottom under a square root, so we can cancel the leading factor of a.

Put the b^2 term on top of a common denominator:

$$2A = a\sqrt{\frac{4a^2b^2}{4a^2} - (\frac{a^2 + b^2 - c^2}{2a})^2}$$

$$4A = \sqrt{4a^2b^2 - (a^2 + b^2 - c^2)^2}$$

$$16A^2 = 4a^2b^2 - (a^2 + b^2 - c^2)^2$$

Now comes the part that makes this derivation beautiful. We will use the difference of squares:

$$16A^2 = \ [\ 2ab + (a^2 + b^2 - c^2)\ ]\ [\ 2ab - (a^2 + b^2 - c^2)\ ]$$

$$= \ [\ (a + b)^2 - c^2\ ]\ [\ c^2 - (a - b)^2\ ]$$

and then again

$$
\begin{aligned}
16A^2 &= (a+b+c)(a+b-c)(c-(a-b))(c+a-b) \\
&= (a+b+c)(a+b-c)(c-a+b)(c+a-b)
\end{aligned}
$$

We're basically done. The semi-perimeter s is

$$
\begin{aligned}
s &= \frac{a+b+c}{2} \\
2s &= a+b+c
\end{aligned}
$$

Thus

$$2(s-c) = a+b-c$$

and so on.

We had

$$16A^2 = (a+b+c)(a+b-c)(c-a+b)(c+a-b)$$

So

$$
\begin{aligned}
A^2 &= s(s-a)(s-b)(s-c) \\
A &= \sqrt{s(s-a)(s-b)(s-c)}
\end{aligned}
$$

□

This formula for the area of a triangle in terms of the three sides is ascribed to Hero of Alexandria.