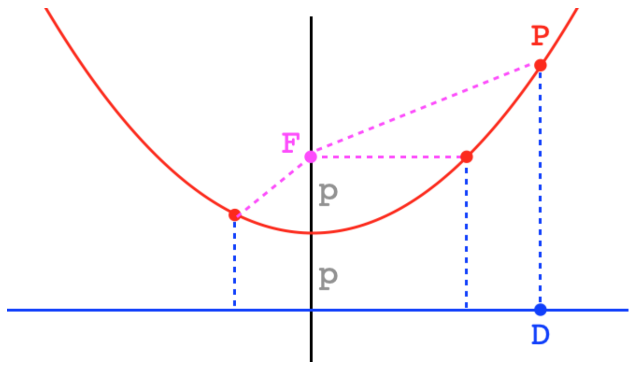
Tangent to the parabola by geometry

Here is the simple geometric definition of a parabola: choose a point F (colored magenta) and a line called the directrix (colored blue). Let the distance between F and the directrix be equal to $2p$.

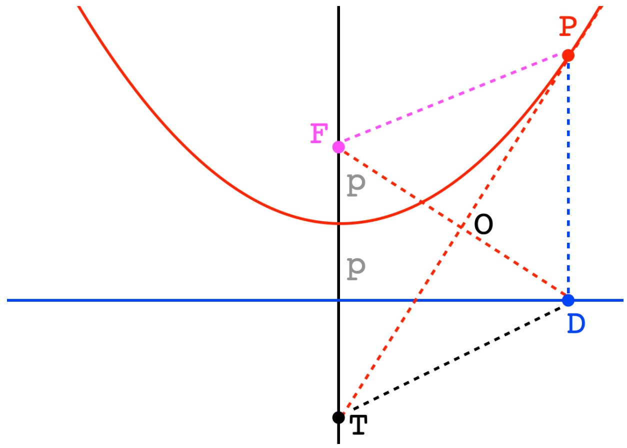


The parabola is the set of all points whose vertical distance to the directrix is equal to the distance to F (colored red). $FP = PD$.

Draw the perpendicular from F down to the directrix (colored black). The point where the parabola crosses the vertical black line is the point of closest approach both to F and to the directrix. This point is called the vertex of the parabola.

We wish to discover something about the tangent to the parabola at a general point P. The tangent is defined to be the line that just touches the curve at a single point.

Connect F to D and then draw the vertical bisector of FD, PT.



Because $\triangle FPD$ is isosceles, every point on PT is equidistant to both F and D by the standard properties of the perpendicular bisector of the base of an isosceles triangle.

Therefore, the point P where the vertical bisector PT meets PD is on the parabola, for this choice of D.

For any given value of D, there can be only one such point P, because there is only one point where PT and PD cross. (They must cross, since FD cannot be parallel to the directrix, hence PT cannot be vertical). PD and PT are never parallel, for any choice of D.

PT is the tangent

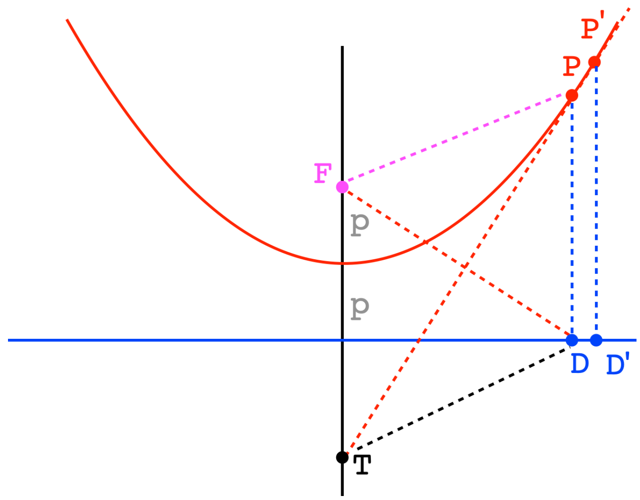
We will show that the perpendicular bisector PT is the tangent at P.

Proof.

Suppose PT is not the tangent.

Then, perhaps it would not cut the parabola at all. But clearly PT and PD must intersect, since they are not parallel. Therefore PT does touch the parabola.

Suppose instead that the perpendicular bisector is a secant and cuts through two points on the parabola.



Since the second point P' lies on the perpendicular bisector of FD, it follows that $FP' = P'D$. (Note: the original D, not D').

At the same time, as a point on the parabola, P' must satisfy the invariant: $FP' = P'D'$, where $P'D'$ is perpendicular to the directrix.

Then $FP' = P'D = P'D'$.

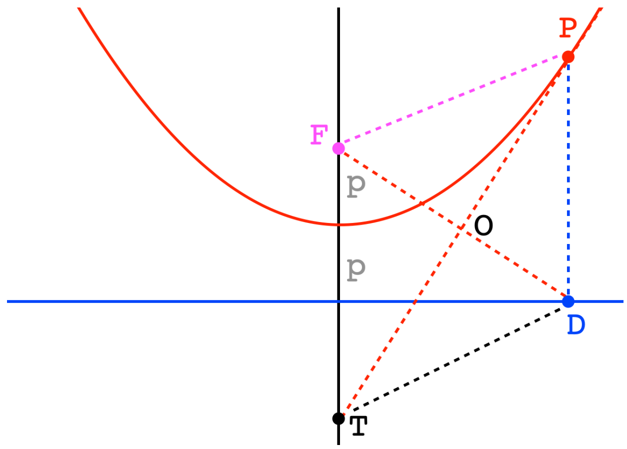
But $P'D$ cannot be equal to $P'D'$, since $P'D'$ is the shortest line segment connecting P' and the directrix, which means that $P'D$ is the hypotenuse in a right triangle and as such $P'D > P'D'$.

This is a contradiction.

Therefore the perpendicular bisector is not a secant, and since it does touch the parabola, it must be tangent.

□

a parallelogram



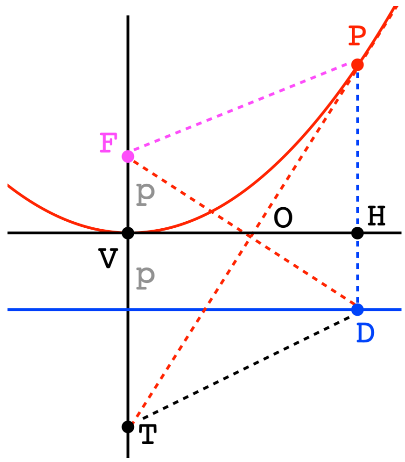
We are given that $FP = PD$ and all the angles at O are right angles. Since $OF = OD$ (perpendicular bisector) and OP is shared we have SSS so $\triangle FOP \cong \triangle POD$.

$FT \parallel PD$ (both are defined to be vertical). So $\angle TFO = \angle ODP$ (alternate interior angles), while the angles at O are all right angles and $OF = OD$. Therefore $\triangle POD \cong \triangle FOT$ by ASA.

As corresponding sides of congruent triangles, $FT = PD$, and given that $FT \parallel PD$, $FPDT$ is a parallelogram. Furthermore, it is a regular parallelogram with four sides equal.

horizontal axis

Draw the horizontal line through the vertex. We will show that this line also goes through the midpoint of the diagonals of the parallelogram.



Proof.

Consider the small triangle with one vertex at V, the vertex of the parabola, plus O and F, with vertical side length p. That triangle is congruent to another triangle through the vertical angle at O, $\triangle OHD$, because they are both right triangles with sides $FO = OD$. So they are congruent by AAS.

Alternatively, just note that OV and OH are altitudes in congruent right triangles.

Therefore $VO = OH$. Then it is clear that $\triangle VOT \cong \triangle POH$ by ASA.

The point P lies the same distance above the horizontal axis as the intersection with the vertical, T, lies below it, and point O lies equidistant between V and H.

$\square$

This completes the derivation of the slope of the tangent to the parabola at a general point P, using only geometry. The slope is rise over run, twice PH divided by VH or alternatively, PH divided by OH.

It is easier to deal with this relationship using the invention of Descartes and Fermat.

analytical geometry

Let y be the vertical distance from P down to the horizontal axis, PH. Let x be the horizontal distance of P from the vertical axis, VH.

In the language of analytical geometry the slope of PT, which we will call $\Delta y/\Delta x$, is equal to *twice* y/x.

We obtain an equation for the parabola as follows.

The total distance $PD = y + p$. Also, FP is the hypotenuse of a right triangle with sides x and $y - p$.

Since $FP = PD$, the squared lengths are also equal and we have:

$$x^2 + (y-p)^2 = (y+p)^2$$

$$x^2 - 2yp = 2yp$$

$$y = \frac{1}{4p}x^2$$

The coefficient of x^2 is usually denoted by a, as in $y = ax^2$. We see that a is related to p by $4ap = 1$.

The slope is

$$\frac{2y}{x} = 2 \cdot \frac{x^2}{4p} \cdot \frac{1}{x} = \frac{1}{2p}x = 2ax$$

This is, literally, the first result from calculus, which shows its power.