Equations for conics

Here we derive the equations for parabola, ellipse and hyperbola in standard orientation.

parabola

For a parabola in standard orientation and centered at the origin, the focus lies on the y-axis a distance c above the origin, while the directrix is a line parallel to the x-axis the same distance c below it.

Any parabola can be rotated if necessary, then translated to the origin in standard orientation with shape factor a and equation $y = ax^2$.

For any point on the parabola, the distance to the focus and the vertical distance down to the directrix are equal.

$$\sqrt{x^2 + (y-c)^2} = y + c$$

We can substitute for either x or y. Normally we choose x, so why not try y.

$$\sqrt{\frac{y}{a} + (y-c)^2} = y + c$$

$$\frac{y}{a} + (y-c)^2 = (y+c)^2$$

$$\frac{y}{a} = 4cy$$

$$4ac = 1$$

The focus $c = 1/(4a)$.

ellipse

For any point on the ellipse, the sum of the distances to the foci is constant. Let us call that constant $2a$, we will find that a is an important value in the standard graph, later.

$$L_1 = \sqrt{(x-c)^2 + y^2}$$

$$L_2 = \sqrt{(x+c)^2 + y^2}$$

So

$$\sqrt{(x-c)^2 + y^2} + \sqrt{(x+c)^2 + y^2} = 2a$$

Rearrange

$$\sqrt{(x-c)^2 + y^2} = 2a - \sqrt{(x+c)^2 + y^2}$$

$$(x-c)^2 + y^2 = 4a^2 - 4a\sqrt{(x+c)^2 + y^2} + (x+c)^2 + y^2$$

$$-4xc = 4a^2 - 4a\sqrt{(x+c)^2 + y^2}$$

$$a\sqrt{(x+c)^2 + y^2} = a^2 + xc$$

$$a^2(x^2 + 2xc + c^2 + y^2) = a^4 + 2a^2xc + x^2c^2$$

The terms with a factor of 2 cancel.

$$a^2x^2 + a^2c^2 + a^2y^2 = a^4 + x^2c^2$$

$$(a^2 - c^2)x^2 + a^2y^2 = (a^2 - c^2)a^2$$

Define $a^2 - c^2 = b^2$

$$b^2x^2 + a^2y^2 = b^2a^2$$

Divide by a^2b^2 and we're done.

We can see that if $y = 0$, then $x = \pm\, a$.

Similarly, if $x = 0$ then $y = \pm\, b$.

hyperbola

For the hyperbola, the math is very similar to the ellipse. The lengths are the same, but it is the absolute value of the difference that is constant.

We have

$$L_1 - L_2 = \pm\ 2a$$

$$\sqrt{(x+c)^2+y^2} - \sqrt{(x-c)^2+y^2} = \pm\ 2a$$

Rearrange

$$\sqrt{(x+c)^2+y^2} = \pm\ 2a + \sqrt{(x-c)^2+y^2}$$

$$(x+c)^2+y^2 = 4a^2 \pm\ 4c\sqrt{(x+c)^2+y^2} + (x-c)^2+y^2$$

$$4xc = 4a^2 \pm\ 4a\sqrt{(x-c)^2+y^2}$$

$$xc - a^2 = \pm\ a\sqrt{(x-c)^2+y^2}$$

$$x^2c^2 - 2xca^2 + a^4 = a^2(x^2 - 2xc + c^2 + y^2)$$

The terms with a factor of 2 cancel.

$$x^2c^2 + a^4 = a^2x^2 + a^2c^2 + a^2y^2$$

$$x^2(c^2-a^2) - a^2y^2 = a^2(c^2-a^2)$$

Define $b^2 = c^2 - a^2$

$$x^2b^2 - a^2y^2 = a^2b^2$$

Divide by a^2b^2 and we're done.

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

And we see that again, when $y = 0$, $x = \pm\ a$. Rearranging

$$\frac{x^2}{a^2} = 1 + \frac{y^2}{b^2}$$

When x is large

$$\frac{x^2}{a^2} \approx \frac{y^2}{b^2}$$

$$y \approx \pm \frac{b}{a} x$$

This is the equation of a straight line with slope b/a through the origin.

For the ellipse, we define $a^2 - c^2 = b^2$, and this makes perfect sense geometrically when considering a point at say $(0, b)$. There is a right triangle with sides b and c and hypotenuse a.

Presumably, there is a geometrical explanation for $a^2 + b^2 = c^2$ in the case of the hyperbola.