gcd example

A challenging example of factorization is to convert a speed in miles per hour into feet per second. It turns out that

$$15 \ \frac{\text{miles}}{\text{hour}} = 22 \ \frac{\text{feet}}{\text{second}}$$

which is usually abbreviated

$$15 \ \frac{\text{mi}}{\text{hr}} = 22 \ \frac{\text{ft}}{\text{sec}}$$

To start with we need to remember or look up how many feet there are in a mile (5280 feet per mile), and perhaps, calculate 60 seconds per minute x 60 minutes per hour = 3600 seconds per hour.

So then write the computation:

$$15 \ \frac{\text{mi}}{\text{hr}} \cdot 5280 \ \frac{\text{ft}}{\text{mi}} \cdot \frac{1}{3600} \ \frac{\text{hr}}{\text{sec}} = ? \ \frac{\text{ft}}{\text{sec}}$$

We note in passing how important it is to see that the units are correct: miles cancel top and bottom, as do hours, leaving the answer in units of feet per second, as desired. This makes it easy to write the fractions with the correct orientation.

So then, what about calculating

$$15 \cdot \frac{5280}{3600}$$

We see immediately that we can cancel a factor of 10

$$15 \cdot \frac{528}{360}$$

There are two ways to do this. The first is to write out all the prime factors.

$$528 = 2 \cdot 264 = 2^2 \cdot 132 = 2^3 \cdot 66 = 2^4 \cdot 3 \cdot 11$$
$$360 = 2 \cdot 36 \cdot 5 = 2^3 \cdot 3^2 \cdot 5$$

which means that

$$\frac{528}{360} = \frac{2 \cdot 11}{3 \cdot 5}$$

but that 15 in the denominator cancels the leading factor of 15, leaving our answer as

$$2 \cdot 11 \ \frac{\text{ft}}{\text{sec}} = 22 \ \frac{\text{ft}}{\text{sec}}$$

as promised.

There is another way to do this problem, which is arguably better (easier). To see why, imagine that our problem had been 5290/3600. Then we would proceed to 529/360.

In checking the possible prime factors of 529, we must check at least to 19, since $20^2 < 529$, so $19^2 < 529$ as well, but possibly not to 23. Compute the square of 23 to be sure. Oops. $529 = 23^2$.

Okay, what if our problem had been 5090? $2, 3, 5, 7, 11, 13, 17, 19$ all must be checked first, and 23 squared, before we realize that 509 is prime.

Euclid's algorithm

Here's another way, it is called Euclid's algorithm for the gcd (greatest common divisor). Its big advantage is avoiding factorization, which is a hard problem because of possible large prime factors.

To make it more challenging, we will pretend we didn't notice that both numbers are multiples of 10.

Find the largest integer which multiplies 3600 and the result is still smaller than or equal to 5280. Then subtract to obtain the remainder and write:

$$5280 = 3600 \cdot 1 + 1680$$

Now repeat the same process, but use 3600 and 1680.

$$3600 = 1680 \cdot 2 + 240$$

and again

$$1680 = 240 \cdot 7 + 0$$

We stop when the remainder is zero. Then, 240 is the gcd of the two numbers we started with. We almost can read off

$$3600 = 15 \cdot 240$$

since $2 \cdot 7 + 1 = 15$. For the other, when I see 5280/240 I know right away it's at least 20 because $20 \cdot 240$ is 4800 and then what's left is twice 240 so that gives 22. The calculation was

$$15 \cdot \frac{5280}{3600}$$

and it becomes

$$15 \cdot \frac{22 \cdot 240}{15 \cdot 240} = 22$$

In the second part, it helps that we *know* both 3600 and 5280 must be even multiples of 240. I think that's at least as easy as finding all the prime factors.