Square roots

Theorem:

If a positive integer has a square root that is a rational number, it must be a perfect square.

Alternative formulations:

◦ Every positive integer that is not a perfect square has an irrational square root.

◦ Every rational square root is also a positive integer.

We will use the *fundamental theorem of arithmetic*: every positive integer has a unique prime factorization. Thus

$$a = p_1 \cdot p_2 \cdot p_3 \ldots$$

The factors p_1 etc. are not necessarily all different. Example:

$$36 = 2 \cdot 2 \cdot 3 \cdot 3$$

Proof.

Let n be a positive integer whose square root is a rational number, i.e. the ratio of two positive integers a and b:

$$\sqrt{n} = \frac{a}{b}$$

Square both sides and rearrange:

$$n \cdot b^2 = a^2$$

By the preliminary result, a has a unique prime factorization:

$$a = p_1 \cdot p_2 \cdot p_3 \ldots$$

So

$$a^2 = p_1^2 \cdot p_2^2 \cdot p_3^2 \ldots$$

The number of copies of each distinct factor of a^2 is even, and the total number of factors is also even.

Because of the equality $nb^2 = a^2$, we write:

$$n \cdot b^2 = p_1^2 \cdot p_2^2 \cdot p_3^2 \ldots$$

But b is a positive integer, so

$$b^2 = q_1^2 \cdot q_2^2 \cdot q_3^2 \ldots$$

thus

$$n \cdot q_1^2 \cdot q_2^2 \cdot q_3^2 \cdots = p_1^2 \cdot p_2^2 \cdot p_3^2 \ldots$$

This means that every q on the left-hand side must be one of the p's on the right-hand side.

Let us factor out all the q's from both sides, leaving

$$n = p_x^2 \cdot p_y^2 \cdot p_z^2 \ldots$$

Every prime factor of n occurs an even number of times on the right-hand side. But this means that n must be a perfect square.

We cannot have

$$3 = p_x^2 \cdot p_y^2 \cdot p_z^2 \ldots$$

because 3 is not a perfect square. Furthermore, we can't have

$$6 = p_x^2 \cdot p_y^2 \cdot p_z^2 \dots$$

$$2 \cdot 3 = p_x^2 \cdot p_y^2 \cdot p_z^2 \dots$$

The prime factorization of any integer is unique. But 2 and 3 are present only once on the left-hand side, and each p is present an even number of times on the right-hand side. This is a contradiction.

n must be of the form

$$n = p_x^2 \cdot p_y^2 \cdots = (p_x \cdot p_y \dots)^2$$

That is, a perfect square.

□