Rotation

introduction

The *conic sections* are so named because they can be formed as the intersection of a plane and a right circular cone[1]. These curves — circle, ellipse, parabola and hyperbola — are usually presented in *standard* orientation.

For example, $y = ax^2$ (with $a > 0$) is a parabola with its center at the origin, opening up. The curve $x = ay^2$ is not materially different, since we obtain the previous curve merely by exchanging symbols.

$y = ax^2 + bx + c$ has its center displaced but still opens up. In fact, any conic not centered at the origin will have terms $(x - h)$ and/or $(y - k)$ which can be replaced by x and y to move the center to the origin. In the example, this can be done by *completing the square.*

$x^2 + y^2 = r^2$ is a circle with its center at the origin.

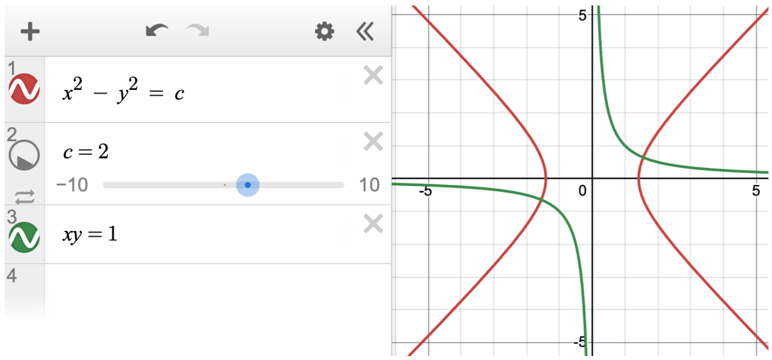
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

are respectively, the equations of an ellipse, and a hyperbola. Each is centered at the origin.

If $a > b$ the ellipse has its long dimension along the x-axis.

[1] A right circular cone has for its base a circle, each point of which is the same distance from the vertex of the cone. Thus, a line from the vertex to the center of the circle is perpendicular to the plane of the circle.

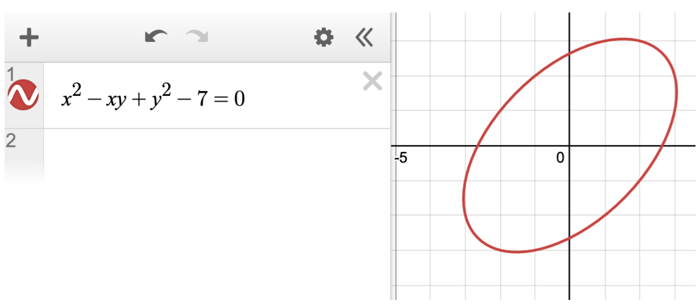
The hyperbola (with $a = b$, and rearranged as $x^2 - y^2 = c$, is shown in the graph below as the red curve.



All of these are in standard orientation. They have either the x- or the y-axis as an axis of symmetry.

But that hyperbola can be *rotated*. In fact, the familiar equation $xy = 1$ is a hyperbola, rotated by $45°$ (green curve in the figure above above). The point of closest approach to the center is the same for each, $\sqrt{2} \approx 1.4$, and the arms are perpendicular — a consequence of having $a = b$.

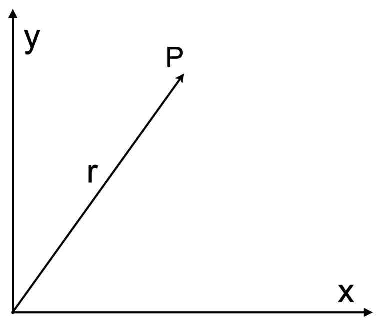
Any conic can be rotated. You could even rotate a circle, but the graph wouldn't look any different. :) Here is a rotated ellipse.

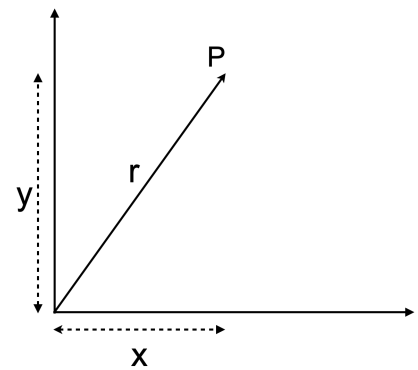


We want to understand how to transform equations of graphs from their rotated versions to standard form, and back again. One clue is to notice the presence of terms that mix x and y, such as $-xy$ in the equation for the rotated ellipse, above. Not all rotated forms have xy terms, but many do.

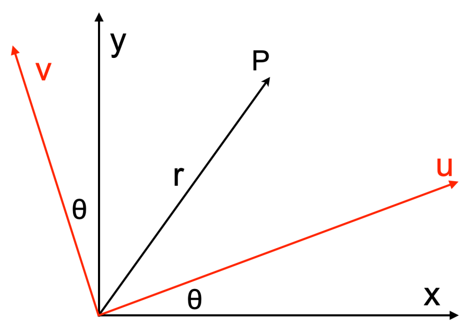
basic idea

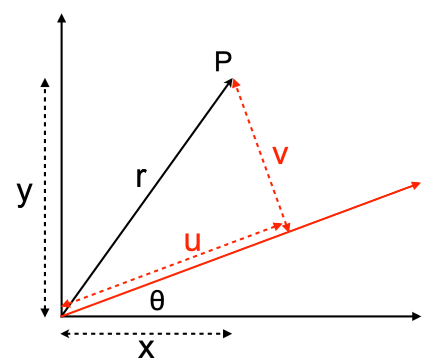
Start with the standard picture from analytical geometry, courtesy of Fermat and Descartes. Pick a point in the *Cartesian plane* for the origin. Draw perpendicular x- and y-axes (the fancy word is orthogonal).





The point P lies a distance r from the origin. In the xy-coordinate system, $P = (x, y)$. Drop the vertical from P to the x-axis, then the intersection with the x-axis lies x units from the origin. Similarly, P is y units from the x-axis.





Now consider a second coordinate system with u- and v-axes, rotated counter-clockwise (CCW) by an angle θ. In the new system, $P = (u, v)$. Of course the length of the vector to P is still r.

We will find a formula to give u as a function of x and y (and also θ), abbreviated $u(x, y, \theta)$. Similarly, we will get $v(x, y, \theta)$.

There are also equations that go in the opposite direction, namely $x(u, v, \theta)$ and $y(u, v, \theta)$. This can be understood as rotation of the coordinate system CW, or as rotation CCW by an angle $-\theta$.

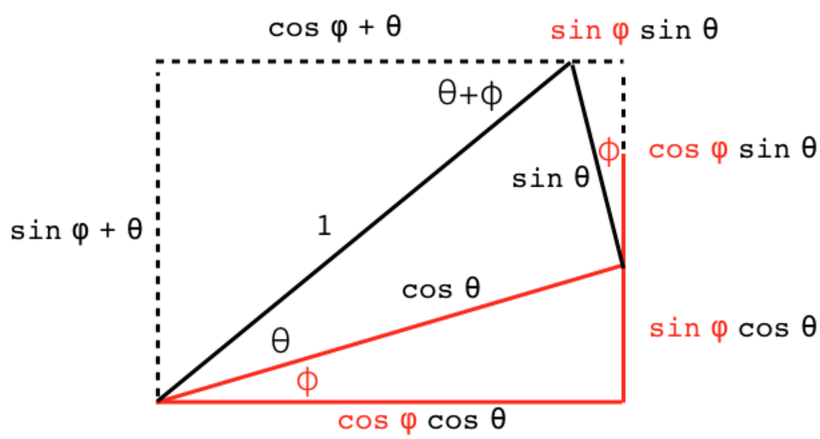
Before getting into specifics, note that a rotation of the coordinate system CCW, from x, y-axes to u, v-axes, can equally be viewed as the rotation of the point P or a vector from the origin to P, in the opposite direction, i.e. CW. P gets closer to the "horizontal" u-axis than it was to the x-axis.

Although we end up rotating points and vectors and the graphs of conic sections, the equations are typically derived by thinking about rotations of the coordinate system.

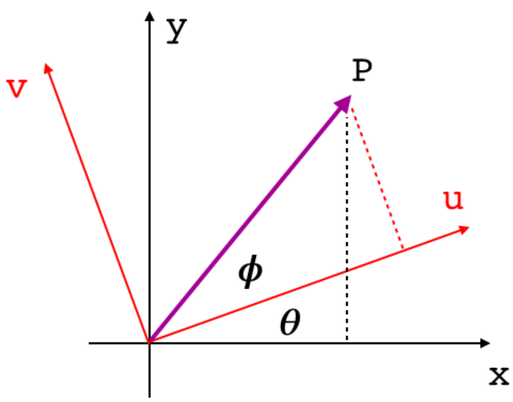
visual derivations

Stewart

Probably the easiest method I have found, in Stewart, depends on knowing the sum of angles formulas.



It goes like this. Label the angle between the vector to P and the u-axis as ϕ.



Let the point P lie on the unit circle, with $r = 1$. The coordinates of point P in the u, v system are naturally expressed in terms of ϕ:

$$u = \cos\phi \qquad v = \sin\phi$$

while x and y are naturally expressed in terms of the combined angle $\theta + \phi$.

$$x = \cos(\theta + \phi) \qquad y = \sin(\theta + \phi)$$

Now, use the sum formula for cosine:

$$x = \cos\theta\cos\phi - \sin\theta\sin\phi$$

Plug in from $u = \cos\phi$ and $v = \sin\phi$:

$$x = u\cos\theta - v\sin\theta$$

In the same way:

$$y = \sin(\theta + \phi)$$

$$= \sin\theta\cos\phi + \cos\theta\sin\phi$$

$$= u\sin\theta + v\cos\theta$$

These are equations that start from (u,v) coordinates and yield (x,y) coordinates. Re-stating them

$$x = u\cos\theta - v\sin\theta$$

$$y = u\sin\theta + v\cos\theta$$

We will call this transformation the action of the *operator* T

$$T\langle u,v\rangle = \langle x,y\rangle$$

We choose T to remember that the minus sign is in the top equation: It is more usual to call the operator R with a subscript like R_{cw}, but there are two problems. One must remember where the minus sign goes, and there is always confusion between rotation of vectors and rotation of the coordinate system, which are opposites.

We compute the square of the length to the origin:

$$x^2 + y^2 = (u\cos\theta - v\sin\theta)^2 + (u\sin\theta + v\cos\theta)^2$$

$$= u^2\cos^2\theta - 2uv\sin\theta\cos\theta + v^2\sin^2\theta + u^2\sin^2\theta + 2uv\sin\theta\cos\theta + v^2\cos^2\theta$$

$$= u^2\cos^2\theta + v^2\sin^2\theta + u^2\sin^2\theta + v^2\cos^2\theta$$

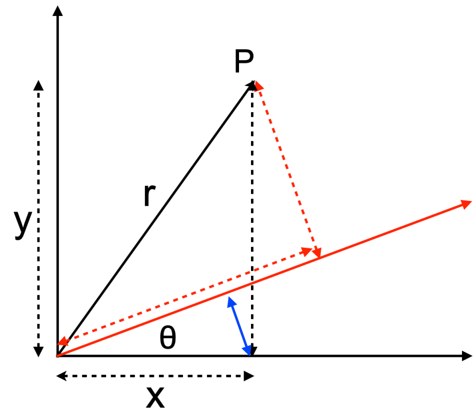
we have u^2 and v^2 separately multiplying $\sin^2\theta + \cos^2\theta = 1$. Thus

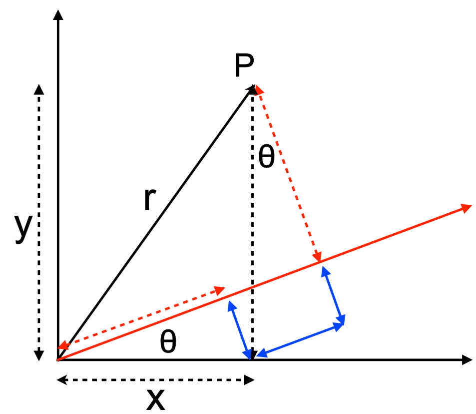
$$x^2 + y^2 = u^2 + v^2 = r^2$$

drawing rectangles

Next we look at two methods that work by drawing rectangles to help us see the pieces.

The key step of the first approach is to draw a line that *makes x the hypotenuse* of a right triangle. That's the blue line in the figure below.



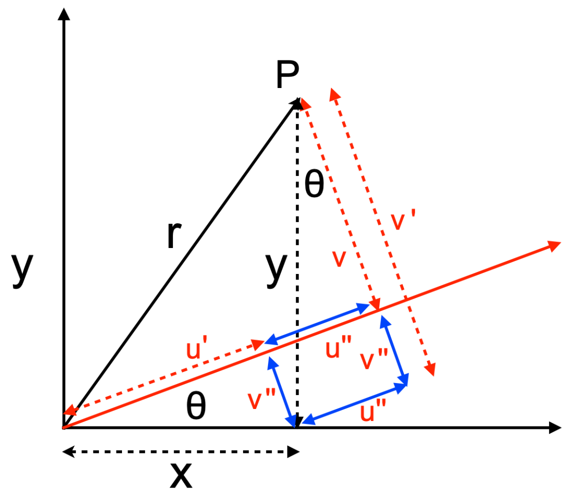


Then, draw the adjacent side of the rectangle to make y the hypotenuse of a (different) right triangle.

θ, the angle between x and u axes, is also the angle between y and v axes. Note the angle near P in the diagram. This comes easily from the properties of right triangles and the vertical angles theorem. On the other hand, it is also a trivial consequence of the fact that the axes are orthogonal: $x \perp y$ and $u \perp v$.

Basic trigonometry gives:

$$\sin\theta = \frac{v''}{x} = \frac{u''}{y} \qquad \cos\theta = \frac{u'}{x} = \frac{v'}{y}$$



$$u' = x\cos\theta \qquad u'' = y\sin\theta$$
$$v' = y\cos\theta \qquad v'' = x\sin\theta$$

We can therefore write u as the sum of two segments:

$$u = u' + u'' = x\cos\theta + y\sin\theta$$

And write v as the difference of two segments:

$$v = v' - v'' = y\cos\theta - x\sin\theta$$

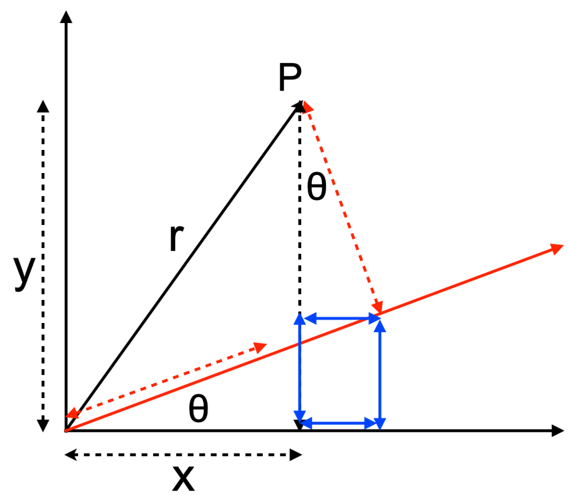
I will call this operator B because the minus sign is in the second (bottom) equation:

$$B\langle x, y\rangle = \langle u, v\rangle$$

another rectangle picture

There is a different rectangle picture which gives $x(u, v, \theta)$ and $y(u, v, \theta)$. Above, we drew a rectangle whose sides made x and y the hypotenuse of two different right triangles.

Instead, we now draw lines (and a rectangle) to make u and v the hypotenuse of two different right triangles.



We leave it as an exercise to derive $x(u, v, \theta)$ and $y(u, v, \theta)$. These equations are the same as given in Stewart's method.

algebra

We can manipulate $u(x, y, \theta$ and $v(x, y, \theta)$ to find $x(u, v, \theta)$ and $v(x, y, \theta)$ algebraically. We can also do the reverse.

$$u = x\cos\theta + y\sin\theta$$
$$v = -x\sin\theta + y\cos\theta$$

To solve for x we need to lose y.

Multiply the first equation by $\cos\theta$ and the second by $-\sin\theta$ and add. The terms with y in them cancel.

$$u\cos\theta - v\sin\theta = x(\cos^2\theta + \sin^2\theta) = x$$

Also, multiply the first equation by $\sin\theta$ and the second by $\cos\theta$ and add:

$$u\sin\theta + v\cos\theta = y(\cos^2\theta + \sin^2\theta) = y$$

CW rotation is CCW rotation by minus θ

Above we've given the equations to interconvert $x(u,v,\theta)$ plus $y(u,v,\theta)$ and $u = (x,y,\theta)$ plus $v(x,y,\theta)$.

Another way to derive the second from the first is to switch symbols x,y for u,v, and at the same time, substitute $-\theta$ for θ.

What this amounts to is to think of x,y as being the rotated axes, but rotated in the opposite direction (CW rather than CCW).

$$x = u\cos\theta - v\sin\theta$$

switch

$$u = x\cos -\theta - y\sin -\theta$$

$$= x\cos\theta + y\sin\theta$$

(Recall that $\cos -x = \cos x$ and $\sin -x = -\sin x$).

For y

$$y = u\sin\theta + v\cos\theta$$

switch

$$v = x\sin -\theta + y\cos -\theta$$

$$v = -x\sin\theta + y\cos\theta$$

When we substitute $-\theta$ for θ, the $\sin\theta$ term switches signs, but the cosine term does not. The reason is that sine is an odd function, $\sin -t = -\sin t$, while cosine is even, $cos - t = \cos t$.

examples

parabola

Consider $y = x^2$ rotated through an angle of $45°$.

We need expressions for x and y in terms of u and v. Let us just try T and see where we get.

$$x = u\cos\theta - v\sin\theta$$
$$v = u\sin\theta + v\cos\theta$$

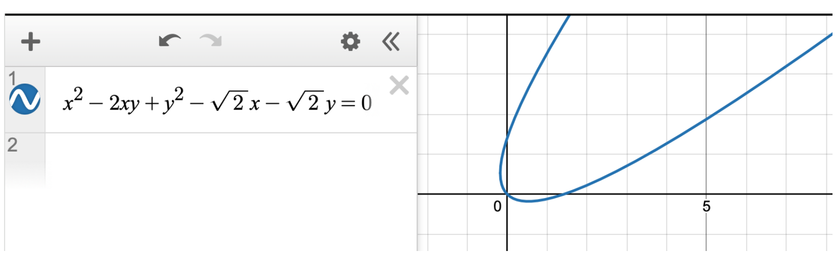
Furthermore, $\cos\theta = \sin\theta = 1/\sqrt{2}$. Let $k = \sqrt{2}$.

$$x = \frac{u-v}{k}$$
$$y = \frac{u+v}{k}$$

so

$$y = x^2$$
$$\frac{u+v}{k} = \frac{1}{k}^{2} \cdot (u-v)^2$$
$$k(u+v) = (u-v)^2 = u^2 - 2uv + v^2$$
$$u^2 - 2uv + v^2 - k(u+v) = 0$$

Desmos doesn't like u and v so just for the graph, we must change symbols to x and y:



We've got a parabola rotated 45° CW. Its center is still at the origin.

hyperbola

The standard equation of a hyperbola is usually given as

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Let $a = b = 2$

$$x^2 - y^2 = 2$$

Going back to our first result:

$$x = u\cos\theta - v\sin\theta$$

$$x^2 = u^2\cos^2\theta - 2uv\sin\theta\cos\theta + v^2\sin^2\theta$$

$$y = u\sin\theta + v\cos\theta$$

$$y^2 = u^2\cos^2\theta + 2uv\sin\theta\cos\theta + v^2\sin^2\theta$$

So

$$x^2 - y^2 = -4uv\sin\theta\cos\theta = 2$$

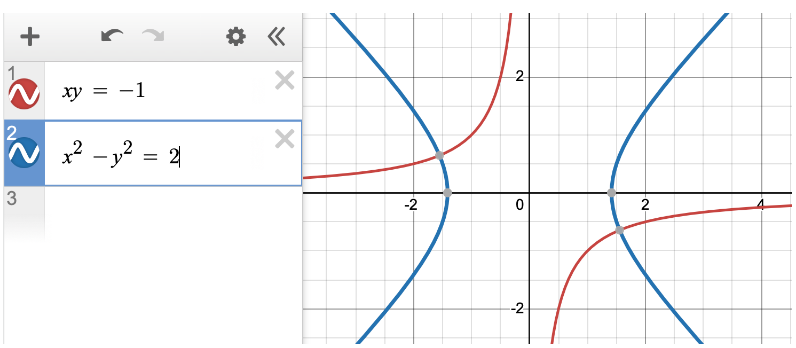
If $\theta = 45°$

$$-2uv = 2$$

So

$$uv = -1$$

Again, switch to x, y for Desmos:



It can be confusing to understand which direction we've gone. We used the T operator which takes points in terms of uv as input and gives xy points. However, the way we used it, we started with x, y points and ended up with u, v, which is CW. We started with $x^2 - y^2 = 2$ and rotated to give $uv = -1$.

finding the angle of rotation

Suppose we are given a conic section like:

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

and we wish to rotate it to the standard orientation.

Take the equations for T (CCW rotation of coordinates):

$$x = u\cos\theta - v\sin\theta$$

$$y = u\sin\theta + v\cos\theta$$

and substitute. As a preliminary

$$x^2 = u^2\cos^2\theta - 2uv\sin\theta\cos\theta + v^2\sin^2\theta$$

$$xy = u^2\sin\theta\cos\theta + uv\cos^2\theta - uv\sin^2\theta - v^2\sin\theta\cos\theta$$

$$y^2 = u^2\sin^2\theta + 2uv\sin\theta\cos\theta + v^2\cos^2\theta$$

We only need to consider the terms that have uv in them. Remembering to pick up A, B, C we have for the cofactors of uv:

$$-2A\sin\theta\cos\theta + B\cos^2\theta - B\sin^2\theta + 2C\sin\theta\cos\theta$$

What is needed is to make this term zero.

$$(C - A)(2\sin\theta\cos\theta) + B(\cos^2\theta - \sin^2\theta) = 0$$

The sum of angles formulas come in handy now:

$$(C - A)(\sin 2\theta) + B(\cos 2\theta) = 0$$

$$\cot 2\theta = \frac{A - C}{B}$$

The hyperbola $xy = 1$ has both A and C equal to zero. What angle has a cotangent of zero? The one whose cosine is zero, namely $\pi/2$ or 90°. That is 2θ, so our final answer is 45°.

For a second example

$$x^2 - 2xy + y^2 - \sqrt{2}x - \sqrt{2}v = 0$$

First note that $B^2 = 4AC$ so this passes the parabola test. Then, again we have $A - C = 1 - 1 = 0$ so that makes $\cot 2\theta = 0$, with the same angle as the final answer.

Finally, in the introduction we had a rotated ellipse:

$$x^2 - xy + y^2 - 7 = 0$$

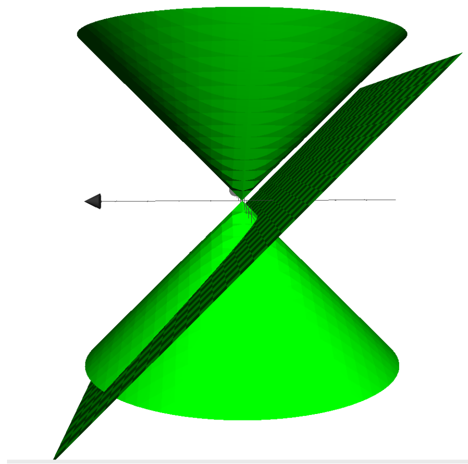
Once again $A - C = 0$ so the angle is 45°.

slicing a cone

plane with normal along x-axis

The conic sections can be formed as the intersection of a plane and two identical but inverted copies of a right circular cone.

If the orientation of the plane is precisely right — the plane must be inclined at the same angle as the cone — then the result is a parabola.



Otherwise it's either a closed curve, that is, an ellipse or even a circle, or the plane cuts both nappes of a double cone to give the two separate parts of a hyperbola. In certain cases, called *degenerate*, the intersection can be a line or pair of lines or even just a single point. To give one example: $x^2 - y^2 = 0$.

`https://en.wikipedia.org/wiki/Degenerate_conic`

In three dimensions, the equation of a cone whose axis of symmetry is the z-axis is:

$$z = cr$$

where c is a constant and

$$r = \sqrt{x^2 + y^2}$$

For simplicity choose $c = 1$ and then

$$x^2 + y^2 = z^2$$

Let us choose a plane oriented along the x-axis. That would be a plane with no x in its normal vector and an equation like

$$y + kz + 1 = 0$$

We can adjust k later to obtain a parabola.

$$z = \frac{1}{k}(-y - 1)$$

$$z^2 = \frac{1}{k^2}(y + 1)^2$$

$$k^2(x^2 + y^2) = y^2 + 2y + 1$$

$$k^2x^2 + (k^2 - 1)y^2 - 2y - 1 = 0$$

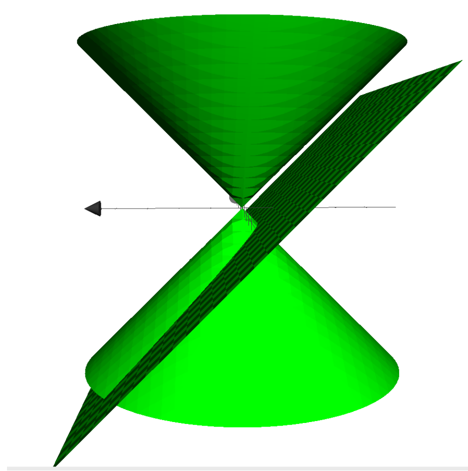
Since there is no xy, $B = 0$, and we must have $4AC = 0$ so $k^2 = 1$ and then

$$x^2 - 2y - 1 = 0$$

$$y = \frac{1}{2}x^2 - \frac{1}{2}$$

A parabola in standard orientation.

Try to visualize what we just did. We picked a plane with normal vector $\langle 0, 1, k\rangle$, so that vector points out along the y-axis with a variable amount of z, depending on k. In the end, we had $k = 1$ so the vector was $\langle 0, 1, 1\rangle$, which makes a $45°$ angle between the y- and z-axes. This matches the cone, whose surface is also inclined at that angle.



The result is a parabola in the x, y-plane. This is the projection of the intersection. To obtain the actual parabola, we must stretch it by $1/\cos\phi$, where ϕ is the angle the plane we chose makes with the x, y-plane.

Going back to

$$k^2x^2 + (k^2 - 1)y^2 - 2y - 1 = 0$$

Let $k^2 = 2$, then

$$2x^2 + y^2 - 2y - 1 = 0$$

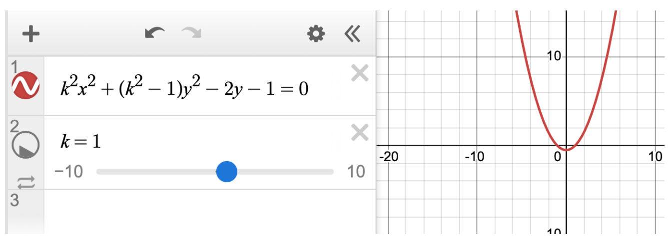
Complete the square on y:

$$2x^2 + y^2 - 2y + 1 - 2 = 0$$

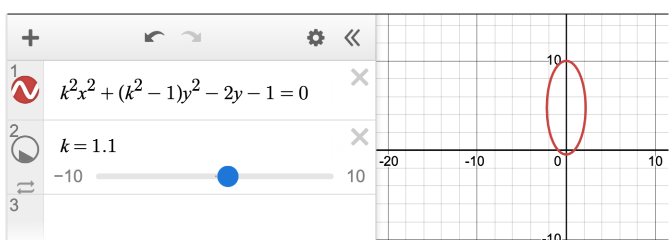
$$2x^2 + (y - 1)^2 = 2$$

An ellipse centered at $(0, 1)$.

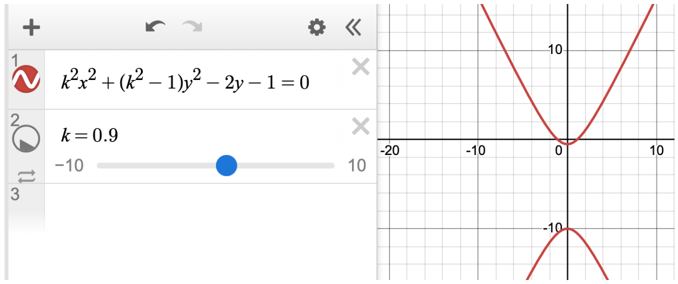
Actually, with Desmos we don't even have to solve the equation. If $k^2 = 1$ it's a parabola:



If $k^2 > 1$ then the sign on y^2 is positive and we have a ellipse as we just saw above.



If $k^2 < 1$ then the sign on y^2 is negative and we have a hyperbola.

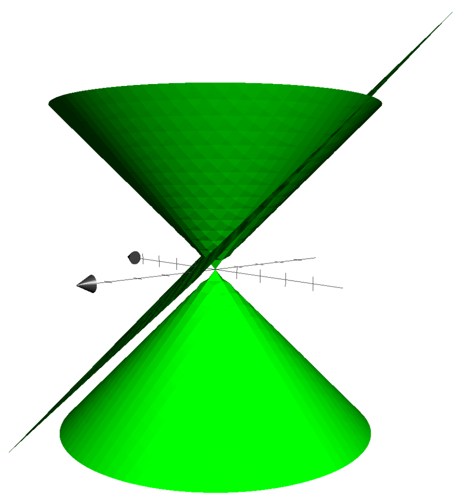


plane at $45°$

Let's do a more complicated example. For the plane, we will choose:

$$x + y + kz = 1$$

where (again) k is a constant we can vary later to obtain a parabola. It is noteworthy that the normal vector to this plane is $\langle 1, 1, k\rangle$, which means that it slices on a angle $45°$ to both x- and y-axes.



Then

$$z = \frac{1}{k}(1 - x - y)$$

$$x^2 + y^2 = z^2 = \frac{1}{k^2}(1 - x - y)^2$$

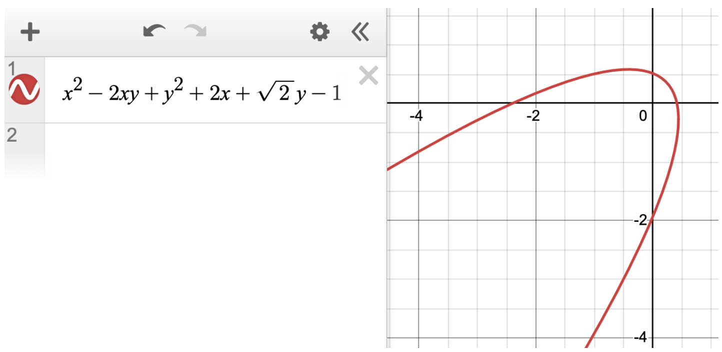
$$x^2 + y^2 = \frac{1}{k^2}(1 - 2x - 2y + x^2 + 2xy + y^2)$$

And

$$(k^2 - 1)x^2 - 2xy + (k^2 - 1)y^2 + 2x + 2y - 1 = 0$$

We obtain a parabola with $k^2 = 2$ since then $4AC = 4 = B^2$. Simplifying:

$$x^2 - 2xy + y^2 + 2x + 2y - 1 = 0$$



We wish to rotate this parabola to standard orientation. The angle is, once again, 45°.

The points should be rotated CCW by the angle to get the form that opens down. Then it's just a question of the sign of the cofactor of x^2.

Let $c = \sin\theta = \cos\theta = 1/\sqrt{2}$. We substitute $cx + cy$ for x and $-cx + cy$ for y.

Factoring out the c we have $c(x + y)$ and $c(-x + y)$. In the squared and xy terms we have $c^2 = \frac{1}{2}$.

For $2x + 2y$ we substitute to get

$$2c(x + y) + 2c(-x + y) = 4cy = 2\sqrt{2}y$$

So that gives

$$\frac{1}{2}(x + y)^2 - (x + y)(x - y) + \frac{1}{2}(-x + y)^2 + 2\sqrt{2}y - 1 = 0$$

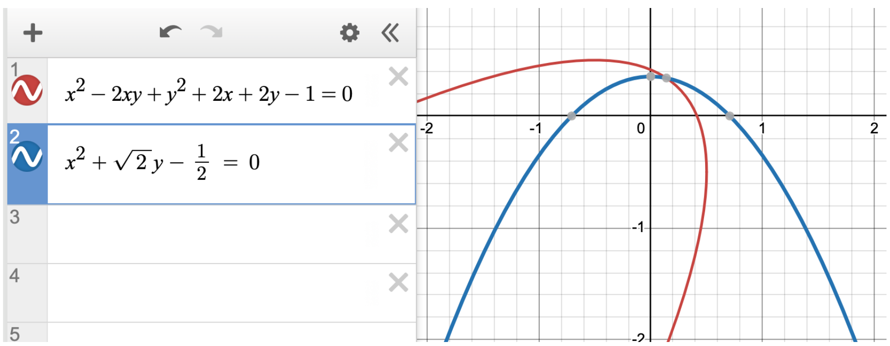
$$(x+y)^2 - 2(x+y)(-x+y) + (-x+y)^2 + 4\sqrt{2}y - 2 = 0$$
$$x^2 + 2xy + y^2 + 2x^2 - 2y^2 + x^2 - 2xy + y^2 + 4\sqrt{2}y - 2 = 0$$

The xy terms drop out.

$$x^2 + y^2 + 2x^2 - 2y^2 + x^2 + y^2 + 4\sqrt{2}y - 2 = 0$$

So do the y^2 terms:

$$x^2 + 2x^2 + x^2 + 4\sqrt{2}y - 2 = 0$$
$$4x^2 + 4\sqrt{2}y - 2 = 0$$
$$x^2 + \sqrt{2}y - \frac{1}{2} = 0$$



The intersection of the two curves has an awkward angle and solution. It is not the vertex of the unrotated curve,

Any point, when rotated, will still lie at the same distance from the origin as it did before rotation. That includes the vertex of the parabola, and it's easy to check.

Let's see. When $x = 0$ we have

$$\sqrt{2}y = \frac{1}{2}$$
$$y = \frac{1}{2\sqrt{2}}$$

So that's the distance from the vertex to the origin. Since the angle is 45°, if x is at the vertex of the unrotated curve then we have an isosceles right triangle and Pythagoras tells us that

$$2x^2 = (\frac{1}{2\sqrt{2}})^2$$

$$x = \frac{1}{4}$$

Plugging that into

$$x^2 - 2xy + y^2 + 2x + 2y - 1 = 0$$

$$\frac{1}{16} - \frac{y}{2} + y^2 + \frac{1}{2} + 2y - 1 = 0$$

$$y^2 + \frac{3}{2}y - \frac{7}{16} = 0$$

$$(y + \frac{7}{4})(y - \frac{1}{4}) = 0$$

The positive root is $y = 1/4$ as expected. The unrotated vertex is at $(1/4, 1/4)$ and the rotated one is at $(0, 1/(2\sqrt{2}))$. We rotate in a circle around the origin, and the distance to the origin for any rotated point is unchanged.

slope

One other thing we can do is to apply Kung's method to find the slope at any point (x_0, y_0).

`https://maa.org/sites/default/files/kung11010356273.pdf`

We have the parabola

$$x^2 - 2xy + y^2 + 2x + 2y - 1 = 0$$

We set that equal to the higher powers of $(x - x_0)$ and $(y - y_0)$

$$(x - x_0)^2 - 2(x - x_0)(y - y_0) + (y - y_0)^2$$

which gives

$$x^2 - 2xy + y^2 + 2x + 2y - 1 = (x - x_0)^2 - 2(x - x_0)(y - y_0) + (y - y_0)^2$$

$$2x + 2y - 1 = -2x_0x + x_0^2 + 2y_0x + 2x_0y - 2x_0y_0 - 2y_0y + y_0^2$$

Gathering like terms

$$2y - 2x_0y + 2y_0y = -2x_0x - 2x + 2y_0x + x_0^2 - 2x_0y_0 + y_0^2 + 1$$

$$y - x_0y + y_0y = -x_0x - x + y_0x + \frac{(x_0 - y_0)^2}{2} + \frac{1}{2}$$

This is the equation of a line with slope

$$\frac{-x_0 - 1 + y_0}{1 - x_0 + y_0}$$

and y-intercept

$$\frac{\frac{(x_0-y_0)^2}{2} + \frac{1}{2}}{1 - x_0 + y_0}$$

If we let the vertex be $(x_0, y_0) = (1/4, 1/4)$ the x_0 terms cancel y_0 terms so the slope there is -1.

The y-intercept is

$$\frac{1/2}{1} = \frac{1}{2}$$

In standard orientation, the slope is zero at the vertex but here, the graph is angled 45° CW. .

general equation

Let us try to write a somewhat general equation for a conic section in terms of the cone and the plane.

For the cone, let

$$cr = z$$

$$c^2r^2 = z^2 = c^2x^2 + c^2y^2$$

The larger c is, the steeper the cone.

For the normal vector to the plane, we keep things simple by making the vector perpendicular to the x-axis. Let the y-component be 1 and then adjust the tilt of the plane, with the z component as k. We have

$$y + kz + d = 0$$

where d is a scalar that adjusts the position of the plane but not its orientation.

The bigger k is, the shallower the angle of the plane. There will be an inverse relationship between c and k when we have a parabola for the intersection.

Solve for the plane equation for z:

$$z = \frac{1}{k} \cdot (-1)(y + d)$$

$$z^2 = \frac{1}{k^2} \cdot (y^2 + 2dy + d^2)$$
$$= c^2x^2 + c^2y^2$$

This becomes

$$c^2k^2x^2 + (c^2k^2 - 1)y^2 - 2dy - d^2 = 0$$

There is no term that mixes x and y (we're oriented due north). To have a parabola, we must make y^2 go away so

$$c^2k^2 - 1 = 0$$

which means that $k = 1/c$, as suspected. The steeper the cone, the shallower the normal vector for the plane, in order to have a parabola.

Finally

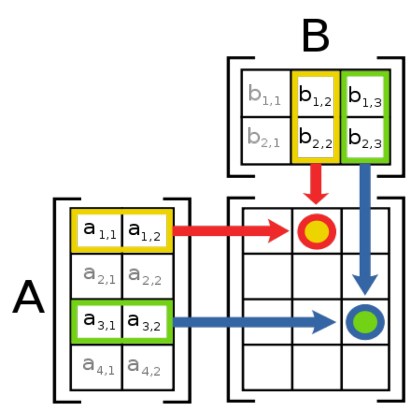
$$y = \frac{x^2}{2d} - \frac{d}{2} = 0$$

matrix form

Rotation is a great topic to use for an introduction to matrix multiplication.

matrix multiplication

Matrix multiplication works like this



where the entry in row i column j of the result is computed from the dot product of row i from the matrix times column j from the second one. In the case of vectors, there is only a single column to worry about.

The way I visualize matrix multiplication is to write it like this:

$$\begin{array}{cc} & \begin{bmatrix} x \\ y \end{bmatrix} \\ \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} & \begin{bmatrix} \cdot \\ \cdot \end{bmatrix} \end{array}$$

To find the first value in the result (what we're going to call u) multiply the row to the left by the column above:

$$\langle \cos\theta, \sin\theta \rangle \cdot \langle x, y \rangle = x\cos\theta + y\sin\theta = u$$

This is called the *dot product.* Multiply term by term and then add to obtain the result. Do the second one:

$$\langle -\sin\theta, \cos\theta \rangle \cdot \langle x, y \rangle = -x\sin\theta + y\cos\theta = v$$

A convenient notation is to write the equations as a matrix multiplying a vector. We have two sets:

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix}$$

We adopt the unconventional strategy of naming the matrices after the position of the minus sign, T when it is on top, and B when it is on the bottom. The advantage is there is never any question about where the sign is in the matrix.

We had:

$$T\langle u, v \rangle = \langle x, y \rangle$$

$$B\langle x,y\rangle = \langle u,v\rangle$$

The two ways of writing a vector, $\langle x,y\rangle$, and

$$\begin{bmatrix} x \\ y \end{bmatrix}$$

are equivalent for our purposes (a purist might complain).

T changes (u,v) into (x,y). It rotates the coordinate system CW and vectors CCW. One way to test is

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

The unit vector $\hat{\mathbf{i}}$ (one unit straight out along the x-axis) has been rotated into the first quadrant.

It is even simpler to test with the particular case of $\theta = \pi/2$ because then

$$T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The unit vector $\hat{\mathbf{i}}$ has been rotated into $\hat{\mathbf{j}}$.

The difference between T and B is just a matter of moving a minus sign.

In vector terms, this is called finding the transpose. It is that simple because the two terms have the same absolute value and it's a 2 x 2 matrix.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

If you know about matrices, you will realize that the two matrices must be inverses, since rotating first one direction and then the other, by the same angle, leaves the point unchanged.

$$T \cdot B = I$$

To invert a 2×2 matrix, transpose it and divide by the determinant (which is 1 here).

The transpose of

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^T = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

rotation by $+\theta$ then $-\theta$

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

We get the identity matrix, as required.

$$I\langle x, y\rangle = \langle x, y\rangle$$

rotation by θ then ϕ

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} =$$

$$= \begin{bmatrix} \cos\theta\cos\phi - \sin\theta\sin\phi & -\sin\theta\cos\phi - \cos\theta\sin\phi \\ -\sin\theta\cos\phi - \cos\theta\sin\phi & cos\theta\cos\phi - \sin\theta\sin\phi \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta+\phi & -\sin\theta+\phi \\ \sin\theta+\phi & \cos\theta+\phi \end{bmatrix}$$

Rotation by θ followed by rotation by ϕ gives the same result as rotation by $\theta+\phi$ in one step.

Incidentally, we have recovered the sum of angles formulas. Sum of angles is inherent in the rotation of axes formulas.

linear combinations

We can look at this in one final way. Write

$$\begin{bmatrix} x \\ y \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

In this representation, the vector $\langle x, y \rangle$ is a linear combination of the unit vectors $\hat{\mathbf{i}} = \langle 1, 0 \rangle$ and $\hat{\mathbf{j}} = \langle 0, 1 \rangle$.

Then suppose we decide to use a different set of unit vectors. The new ones (for the rotated axes) are $\langle \cos\theta, \sin\theta \rangle$ and $\langle -\sin\theta, \cos\theta \rangle$.

If you compute their lengths, it is clear that they are, in fact, unit vectors. If you compute the dot products, it's clear that they are orthogonal. (Two vectors are orthogonal $\iff$ their dot product is zero).

In the new basis:

$$\begin{bmatrix} u \\ v \end{bmatrix} = x \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} + y \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

Written as a matrix multiplication, this is

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} u \\ v \end{bmatrix} = T \begin{bmatrix} x \\ y \end{bmatrix}$$