Pythagorean triples

The simplest right triangle with integer sides is a $3, 4, 5$ right triangle::

$$3^2 + 4^2 = 5^2$$

but of course any multiple k will work

$$(3k)^2 + (4k)^2 = (5k)^2$$

However, that's not so interesting. The triples which are not multiples of another triple are called *primitive*. There is a small table of triples in this discussion of Euclid X:29 by Joyce:

`https://mathcs.clarku.edu/~djoyce/elements/bookX/propX29.html`

	1	3	5	7	9	11	13
3	3 : 4 : 5						
5	5 : 12 : 13	15 : 8 : 17					
7	7 : 24 : 25	21 : 20 : 29	35 : 12 : 37				
9	9 : 40 : 41	27 : 36 : 45	45 : 28 : 53	63 : 16 : 65			
11	11 : 60 : 61	33 : 56 : 65	55 : 48 : 73	77 : 36 : 85	99 : 20 : 101		
13	13 : 84 : 85	39 : 80 : 89	65 : 72 : 97	91 : 60 : 109	117 : 44 : 125	143 : 24 : 145	
15	15 : 112 : 113	45 : 108 : 117	75 : 100 : 125	105 : 88 : 137	135 : 72 : 153	165 : 52 : 173	195 : 28 : 197

We can see that the entries in each column are similar. For example, in the first column

$$(3, 4, 5) \quad (5, 12, 13) \quad (7, 24, 25) \quad (9, 40, 41)$$

The values a_n differ by a constant:

In column 1, the Δ is 2:

$$a_{n+1} = a_n + 2$$

In the second and third columns, Δ is 6 and 10, respectively for the a_n.

For b and c the Δ each step is the same but the scale ratchets upward (as it must since we don't want shared factors). For the first column, the rule is

$$\Delta = 4 + 4k$$

The first difference is 8, then 12, then 16 and so on.

It is conventional to write a as odd. We observe that, in the table, b is even and c is odd.

factors

Which brings us to a first, elementary rule about squares: if n is even then so is n^2, while if n is odd then so is n^2. To see this, write $n = 2k$ for $k \in 1, 2, 3 \ldots$ as the definition of an even number. Then $n^2 = 4k^2$, which is even.

On the other hand, if n is odd, write $n = 2k + 1$ with $k \in 0, 1, 2 \ldots$, so $n^2 = 4k^2 + 4k + 1$, which is odd. Since there are only these two cases, we can conclude that the converse is also true: an even square comes from an even number, etc.

As a result, we find that for the triples we care about, a and b are not both even, because a^2 and b^2 would be even, as would c^2, so then c would be even, and the triple would not be primitive.

more about the table

In column 2, the Δ for a_n is 6, in column 3 it is 10, and in column 4 it is 14.

In column 2 we have differences of 12, 16 and so on for b and c.

$$\Delta = 8 + 4k$$

The table is set up so that the rows have the same Δ for b and c.

There are other rules.

The difference in the first b or c from each column to the next is 4 for b and $12, 20, 28$ etc. for c; and the difference for the first a is 12 also $12, 20, 28$.

There is some underlying formula to explain all this regularity, and we aim to find it.

even and odd

Let us go back to

$$a^2 + b^2 = c^2$$

We said that a and b cannot both be even, because then c would be even. Or rather, they can, but in that case we are not interested.

The other possible cases are, either a and b both odd, or one is even and one odd. In the first case we have that c is even because odd plus odd is even. So then

$$(2i+1)^2 + (2j+1)^2 = (2k)^2$$
$$4i^2 + 4i + 4j^2 + 4j + 2 = 4k^2$$

The left-hand side is not evenly divisible by 4, but the right-hand side is. This is impossible. Hence one of a and b is even and one odd. Let a be odd, as we saw in the table above.

more about factoring

Rearrange the equation:

$$b^2 = c^2 - a^2 = (c+a)(c-a)$$

Since b is even, we can write $b = 2t$

$$4t^2 = (c+a)(c-a)$$

Now we come to an argument about common factors. There are some basic facts we can deduce. Let

$$p + q = r$$

Suppose that p and q share a common factor, f. So then

$$fj + fk = f(j+k) = r$$

By the fundamental theorem of arithmetic, if f is a factor of the left-hand side, it is also a factor of r. In a similar way, suppose that p and r share a common factor, f. Then

$$r - p = fk - fj = f(k-j) = q$$

and again, all three must have the common factor. But we have agreed that these cases do not interest us.

The same argument applies to squares, since if there is a common factor, it will be present as f^2.

We conclude that a, b and c are all relatively prime. No two of them can share a common factor.

Let us go back to

$$4t^2 = (c+a)(c-a)$$
$$t^2 = \frac{(c+a)}{2} \cdot \frac{(c-a)}{2}$$

Recall that a and c are both odd, so their sum and difference are both even. Therefore the two factors on the right-hand side are integers, while t^2 is a perfect square, namely, that of t.

Furthermore, those two factors have no common factor, by the argument we just made.

The crucial inference is that both factors are themselves perfect squares.

That is, there exist integers u and v such that

$$m^2 = \frac{(c-a)}{2}$$

$$n^2 = \frac{(c+a)}{2}$$

with $n > m$.

Adding

$$m^2 + n^2 = c$$

Subtracting

$$n^2 - m^2 = a$$

Go back again to

$$4t^2 = (c+a)(c-a)$$
$$= m^2 n^2$$
$$2t = mn = b$$

We have not limited m and n in any way except to say that they are not equal so one is larger than the other $> m$. Every primitive triple must have an integer m and n with these properties:

$$c = m^2 + n^2, \qquad a = n^2 - m^2, \qquad b^2 = 2mn$$

So finally not only do m and n exist with these properties, but any integer m and n will satisfy the Pythagorean condition:

$$\begin{aligned} a^2 + b^2 &= (n^2 - m^2)^2 + (2mn)^2 \\ &= n^4 - 2n^2m^2 + m^4 + 4n^2m^2 \\ &= n^4 + 2n^2m^2 + m^4 \\ &= (n^2 + m^2)^2 = c^2 \end{aligned}$$

So any integer m, n, with $n > m$ will work.

For 3-4-5, $n = 2$, $m = 1$.

This is a proof that this formula gives all Pythagorean triples.

another derivation

Start with our favorite:

$$\sin^2 x + \cos^2 x = 1$$

$$\tan^2 x + 1 = \frac{1}{\cos^2 x}$$

$$\cos^2 x = \frac{1}{1 + \tan^2 x}$$

And then, the double-angle formula for sine:

$$\sin 2s = 2 \sin s \cos s$$

$$= 2\frac{\sin s}{\cos s}\cos^2 s$$

$$= 2\tan s \;\; \frac{1}{1+\tan^2 s}$$

Let $a = \tan s$, then

$$\sin 2s = \frac{2a}{1+a^2}$$

cosine

$$\cos 2s = \cos^2 s - \sin^2 s$$

$$= \;\left[\; \frac{\cos^2 s}{\cos^2 s} - \frac{\sin^2 s}{\cos^2 s} \;\right]\; \cos^2 s$$

$$= \;\left[\; \frac{1-\tan^2 s}{1+\tan^2 s} \;\right]$$

so

$$\cos 2s = \frac{1-a^2}{1+a^2}$$

In general, a can be anything. But if a is a rational number, then we can obtain the corresponding sides of a right triangle with rational lengths as well.

The sides are: $2a, 1-a^2$ with the hypotenuse:

$$\sqrt{4a^2 + (1 - 2a^2 + a^4)}$$

$$\sqrt{1 + 2a^2 + a^4)}$$

$$= 1 + a^2$$

Suppose $a = \frac{2}{3}$. Then, we have side lengths: $\frac{4}{3} = \frac{12}{9}, \frac{5}{9}$, and $\frac{13}{9}$, which can be converted to integers: $12, 5, 13$.

In general, if $a = \tan s = p/q$ then the sides are

$$\frac{2p}{q}, \qquad 1 - \frac{p^2}{q^2}, \qquad 1 + \frac{p^2}{q^2}$$

which as integers will be

$$2pq, \qquad q^2 - p^2, \qquad q^2 + p^2$$

This formula was found by Euclid.

https://en.wikipedia.org/wiki/Pythagorean_triple

If p and q are two odd integers the sum and difference of squares is even so we can write

$$pq, \qquad \frac{q^2 - p^2}{2}, \qquad \frac{q^2 + p^2}{2}$$

Courant

The fundamental equation can be rewritten in terms of two rational numbers as

$$(\frac{a}{c})^2 + (\frac{b}{c})^2 = 1$$

Let $x = a/c$ and $y = b/c$ and then

$$x^2 + y^2 = 1$$

In other words, if x and y are rational numbers the point (x, y) lies on the unit circle.

Now for some algebra

$$y^2 = 1 - x^2 = (1 + x)(1 - x)$$

$$\frac{y}{1 + x} = \frac{1 - x}{y}$$

Courant says, define these two equivalent expressions as equal to t. And before going further note that since t is the ratio of two rational numbers, it is also rational. Let that $t = u/v$.

Then we can write

$$x + ty = 1$$
$$y = t(1 + x)$$

And then "we find immediately" some expressions for x and y. I get there, but more slowly.

If we take the first equation and substitute for y

$$x + t^2(1 + x) = 1$$
$$x + t^2x = 1 - t^2$$
$$x = \frac{1 - t^2}{1 + t^2}$$

and then

$$\begin{aligned} y &= t(1 + x) \\ &= t(1 + \frac{1 - t^2}{1 + t^2}) \\ &= t(\frac{1 + t^2 + 1 - t^2}{1 + t^2}) \\ &= \frac{2t}{1 + t^2} \end{aligned}$$

Going back to u and v:

$$\begin{aligned} x = \frac{1 - t^2}{1 + t^2} &= \frac{1 - (u/v)^2}{1 + (u/v)^2} \\ &= \frac{v^2 - u^2}{v^2 + u^2} \end{aligned}$$

and

$$y = \frac{2(u/v)}{1+(u/v)^2} = \frac{2uv}{v^2+u^2}$$

Going back to a, b and c

$$\frac{a}{c} = \frac{v^2-u^2}{v^2+u^2}$$

$$\frac{b}{c} = \frac{2uv}{v^2+u^2}$$

the result

Being careful, we recognize there could be a common factor of r top and bottom, but if we insist on lowest terms then

$$a = v^2 - u^2$$

$$b = 2uv$$

$$c = v^2 + u^2$$

This formula for triples is in Euclid's *Elements.* These are often written in terms of m and n but we've followed Courant's derivation.

more

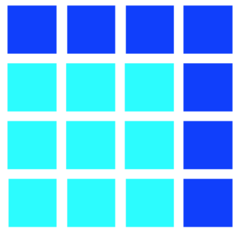
Going back to Joyce's table of triples:

	1	**3**	**5**	**7**	**9**	**11**	**13**
3	3 : 4 : 5						
5	5 : 12 : 13	15 : 8 : 17					
7	7 : 24 : 25	21 : 20 : 29	35 : 12 : 37				
9	9 : 40 : 41	27 : 36 : 45	45 : 28 : 53	63 : 16 : 65			
11	11 : 60 : 61	33 : 56 : 65	55 : 48 : 73	77 : 36 : 85	99 : 20 : 101		
13	13 : 84 : 85	39 : 80 : 89	65 : 72 : 97	91 : 60 : 109	117 : 44 : 125	143 : 24 : 145	
15	15 : 112 : 113	45 : 108 : 117	75 : 100 : 125	105 : 88 : 137	135 : 72 : 153	165 : 52 : 173	195 : 28 : 197

We can explain the first column

$$(3, 4, 5) \quad (5, 12, 13) \quad (7, 24, 25) \quad (9, 40, 41)$$

using this graphic



$$n^2 + (2n + 1) = (n + 1)^2$$

where $2n + 1$ is the count of dark blue squares in the top column plus the rightmost row.

Of course, this is just basic algebra. However, if that odd number is also a perfect square we have that

$$2n + 1 = a^2$$

so

$$(n + 1)^2 = n^2 + a^2$$

Every odd number, when squared, gives an odd perfect square:

$$3^2 = 9$$
$$5^2 = 25$$
$$7^2 = 49$$

So every odd number (≥ 3) is the basis for one of the entries. Its two paired values in the triple can be computed as

$$b = \frac{a^2 - 1}{2}, \qquad c = b + 1$$

We can also explain the first diagonal

$$(8, 15, 17) \quad (12, 35, 37) \quad (16, 63, 65) \quad (20, 99, 101)$$

The first value is $4n$ for $n = 2, 3, 4 \dots$.

The other two values are $4n^2 \pm 1$. This works because

$$(4n^2 + 1)^2 = (4n^2 - 1)^2 + (4n)^2$$

The fourth powers cancel and the ones cancel and we have

$$8n^2 = -8n^2 + 16n^2$$

which is correct.

code

Here is a Python script to generate triples by exhaustive search:

`https://gist.github.com/telliott99/b543f41d84155bc9171df68b6350e256`

And here is one that implements Euclid's formula:

`https://gist.github.com/telliott99/144c1a7e90740eb1614ca8ceb5bdeed9`

Here is some output (m, n, a, b, c) from the second script, sorted on m and n:

```
> python triples2.py
  1   2   3   4   5
  1   4   8  15  17
  1   6  12  35  37
```

1	8	16	63	65
1	10	20	99	101
1	12	24	143	145
1	14	28	195	197
2	3	5	12	13
2	5	20	21	29
2	7	28	45	53
2	9	36	77	85
2	11	44	117	125
2	13	52	165	173
3	4	7	24	25
3	8	48	55	73
3	10	60	91	109
3	14	84	187	205
4	5	9	40	41
4	7	33	56	65
4	9	65	72	97
4	11	88	105	137

There are some interesting patterns in lists of triples. Here is one:

3	4	5
5	12	13
7	24	25
9	40	41
11	60	61
13	84	85
15	112	113
17	144	145
19	180	181
21	220	221

23	264	265
25	312	313
27	364	365

For every step $\Delta a = 2$, we get Δb increasing in steps of 4, with $c = b+1$. If we think of the step size as $4 + 4k$, then the first entry matches as well.

In terms of m and n, we have $b = 2mn$ so mn goes like $2, 6, 12, 20 \ldots$, which looks like $n = m + 1$, starting with $m = 1$. Each step of 1 in m gives a step of 4 in $2mn = b$.

$$a = n^2 - m^2 = (m+1)^2 - m^2 = 2m + 1$$
$$c = n^2 + m^2 = (m+1)^2 + m^2 = 2m^2 + 2m + 1$$

This explains the Δ of 2 for a. We can also explain the steps for c.