# Bertrand's paradox

Grinstead and Snell's wonderful *Introduction to Probability* has this problem (example 2.6). It's called Bertrand's paradox. We are asked to draw a chord of a unit circle randomly.

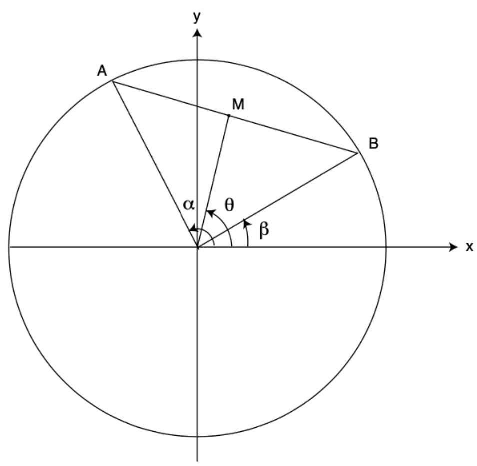



Figure 2.9: Random chord.

Here we might say, let's choose each of three angles $\alpha$, $\beta$ and $\theta$ randomly (uniform density) from $[0, 2\pi]$.

But there is no reason why the radius to $B$ cannot lie along the $x$-axis, so there are really only two choices.

The question is posed: what is the probability that the length of this random chord is $> \sqrt{3}$.

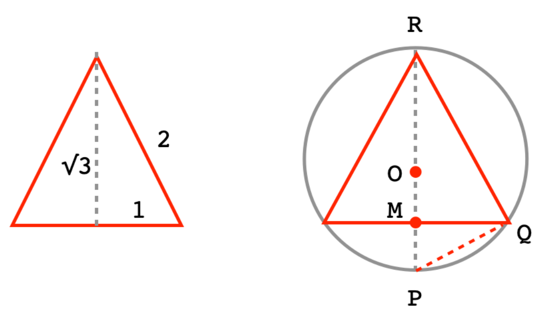
However, there are several different approaches to parametrize the problem, and randomizing the different parameters leads to different results.

**equilateral triangles**

Let's review briefly some properties of equilateral triangles. We drop an altitude and observe the ratio of side lengths. It is convenient to start with a side length of 2 for the original triangle, then in the bisected copies the sides are in the ratio 1-2-$\sqrt{3}$ (the 1 by bisection, and $\sqrt{3}$ by the Pythagorean theorem).

The angle at each vertex of the original equilateral triangle is $\pi/3$, so the new triangles have angles of $\pi/6$, both by bisection or because the altitude forms an angle of $\pi/2$ at the base, so the sum of angles theorem gives us the last angle.

In the right panel, the equilateral triangle is inscribed in a unit circle, so $OR = OP = OQ = 1$. We claim that the line segment $OM$ has a length of $1/2$.




*Proof.*

$\angle PQR$ is a right angle, by Thales' circle theorem, and $\angle MRQ$ is shared, so $\triangle PQR$ is similar to $\triangle RMQ$. Therefore, $\angle RPQ = \angle MQR = \pi/3$.

Therefore the sides of $\triangle PQR$ are also in the ratio 1-2-$\sqrt{3}$, with $PQ/PR = 1/2$ and so $PQ = OP = OQ$. Thus, $\triangle OPQ$ is isosceles.

$QM \perp OP$ so $MQ$ is the bisector of both $\angle PQO$ and the base $OP$.

Therefore, $OM$ is one-half of $OP$ and has a length of $1/2$.

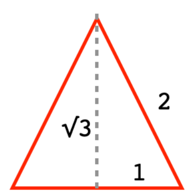
□

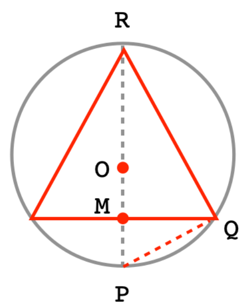
**first parametrization**

We have just shown that the altitude of the inscribed equilateral triangle in a unit circle has length $3/2$. This means that the ratio of the inscribed triangle to the standard one is $\sqrt{3}/2$.

And that means the side length of the inscribed equilateral triangle is $\sqrt{3}$.

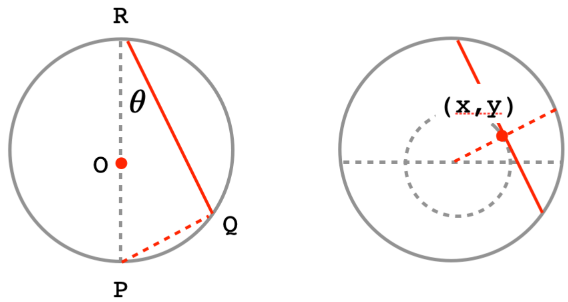
That explains the length chosen for the chord in this problem. We see that if $M$ is chosen at random anywhere along $OP$, one-half of the time the chord formed will be larger than $\sqrt{3}$.




So the probability we are asked to give is just $1/2$.

**second parametrization**




The second parametrization has the same triangle we just saw, $\triangle PQR$. The angle at vertex $R$ is $\theta$.

$\theta$ can lie in the interval $[0, \pi/2]$, and in the event that $\theta < \pi/6$, the chord length $RQ > \sqrt{3}$.

The probability that the chord is greater than $\sqrt{3}$ in length is $1/3$, since $\pi/6$ is one-third of $\pi/2$.

**third parametrization**

Finally, we imagine picking two coordinates $(x, y)$ at random from the interior of the circle. We place the midpoint of the chord $M$ at $(x, y)$.

If $M$ is such that $r = \sqrt{x^2 + y^2} < 1/2$, then $M$ will be closer to the center of the circle than $1/2$ and so the chord length will be $> \sqrt{3}$.

The number of points that have this property is proportional to the relative areas of the inside small circle, and the larger circle around is.

$$\frac{\pi(1/2)^2}{\pi} = \frac{1}{4}$$

We see that, depending on which parameter is randomized, we obtain a probability of $1/2$, $1/3$ or $1/4$.

In Jaynes' words, the problem is not well-formed.