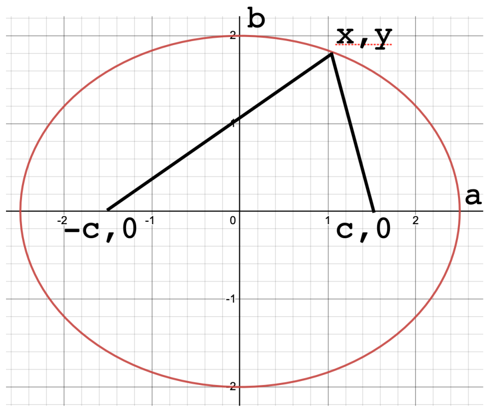
Algebra of the ellipse

Below is an ellipse which crosses the x-axis at $(\pm a, 0)$ and the y-axis at $(0, \pm b)$. By definition, the distances from every point on the ellipse to the two foci, $(\pm c, 0)$, combined, is a constant, L.



Note that if $(x, y) = (a, 0)$ then the two distances are $a - c$ and $a + c$ which add up to $L = 2a$.

If the point is $(x, y) = (0, b)$ then by the Pythagorean theorem we have:

$$2\sqrt{b^2 + c^2} = L = 2a$$

$$a^2 = b^2 + c^2$$

$$b^2 = c^2 - a^2$$

We'll come back to this.

This example has $a = 2.5$ and $b = 2$.

$2.5^2 - 2^2 = 6.25 - 4 = 2.25 = 1.5^2$. So even though the ratio is pretty moderate $a/b = 5/4$, the foci are still at $3/5 \cdot a$.

For any general point we can write another equation using Pythagoras twice:

$$L = \sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = 2a$$

Rearrange

$$\sqrt{(x+c)^2 + y^2} = 2a - \sqrt{(x-c)^2 + y^2}$$

Square

$$(x+c)^2 + y^2 = 4a^2 - 4a\sqrt{(x-c)^2 + y^2} + (x-c)^2 + y^2$$

Cancel y^2, x^2 and c^2

$$2xc = 4a^2 - 4a\sqrt{(x-c)^2 + y^2} - 2xc$$

Rearrange and cancel 4

$$a\sqrt{(x-c)^2 + y^2} = a^2 - xc$$

Square again

$$a^2(x-c)^2 + a^2y^2 = a^4 - 2a^2xc + x^2c^2$$

Expand

$$a^2x^2 - 2a^2xc + a^2c^2 + a^2y^2 = a^4 - 2a^2xc + x^2c^2$$

Cancel $-2a^2xc$

$$a^2x^2 + a^2c^2 + a^2y^2 = a^4 + x^2c^2$$

Rearrange and factor

$$(a^2 - c^2)x^2 + a^2y^2 = a^2(a^2 - c^2)$$

Substitute (from way back):

$$b^2x^2 + a^2y^2 = a^2b^2$$

Finally

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$