Square Roots and Irrational Numbers

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Date: 10/08/97 at 17:00:17
From: Terry Dobbins
Subject: Irrational numbers

My question is: Will all square roots of positive numbers that
are not perfect squares be irrational numbers?

I am a new teacher and this was asked of me by another teacher.
I think that it is a true statement but I can't prove it.

Thanks for the help.
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Date: 10/08/97 at 18:18:08
From: Doctor Tom
Subject: Re: Irrational numbers

Yes. They are all irrational. The proof is similar to the proof that
sqrt(2) is irrational.

In case you haven't seen that, here's how it goes:

Suppose sqrt(2) is rational. Then you can write sqrt(2) as a/b, where
a and b are integers, and the fraction is reduced to lowest terms.

So a^2/b^2 = 2 so a^2 = 2*b^2. So a is even. Since it's even, write
a = 2*c.  (2c)^2 = 2*b^2 or 4c^2 = 2b^2 or 2c^2 = b^2, so b is also
even. But then you didn't reduce a/b to lowest terms since they both
have a factor of 2.

To show that sqrt(p) is irrational where p is a prime number, the same
approach works, except instead of saying "a is even," you'll be saying
"a is a multiple of p."  The proof goes the same way, except that you
find that a and b are both multiples of p, and hence your original
fraction wasn't reduced as you said it was.
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For an arbitrary number n that's not a perfect square, you can factor it as follows:

n = p1^n1*p2^n2*p3^n3*... for a finite number of terms. At least one of the n1, n2, n3, ... must be odd, or n is a perfect square. Suppose n1 is the one that's odd. If n1 is 1, just go through the same proof above and show that the a and b in your a/b are multiples of p1. If n is odd and bigger than one, write your a/b as a*p1^((n1-1)/2)/b. That'll get rid of the part of the product of primes that's a perfect factor of p1.

To make this concrete, suppose I want to show that 216 does not have a rational square root.

216 = 2^3*3^3.

If 216 has a rational square root, it will be 2*sqrt(216/2^2) = 2*sqrt(54), so let sqrt(216) = 2*a/b, reduced to lowest terms. Then 4a^2/b^2 = 54, so 2a^2/b^2 = 27, or 2a^2 = 27b^2, so b must be even. There's already a contradiction.

-Doctor Tom, The Math Forum
Check out our web site! http://mathforum.org/dr.math/

Date: 10/08/97 at 18:31:06
From: Doctor Wallace
Subject: Re: Irrational numbers

Dear Terry,

The answer is yes, all non-perfect square square roots are irrational. Remember that a rational number is one that can be expressed as the ratio of 2 integers. If you look in our archives, you'll find a proof for the fact that the square root of 2 is irrational. Search on the terms irrational and square root of 2. I won't repeat the details here, except to say that the proof involves assuming that the square root of 2 IS rational, and working to a contradiction. The proof is simple and elegant. If I remember correctly, there is also a proof in the archives for the square root of 3.

As to a general proof that ALL non-perfect square square roots are irrational, I'm not sure. I know that one exists, though. Perhaps it is accomplished through extension of the two proofs I mentioned.

I hope this helps. Don't hesitate to write back if you have more questions.

-Doctor Wallace, The Math Forum
Check out our web site! http://mathforum.org/dr.math/

Date: 09/17/2001 at 17:41:50
From: Kerry
Subject: Irrational roots vs. roots of perfect squares

You answered the question about perfect squares by saying that all roots of numbers that are not perfect squares are irrational. Did you mean the roots of all WHOLE numbers that are not perfect squares are irrational?

Date: 09/17/2001 at 22:39:38
From: Doctor Peterson
Subject: Re: Irrational roots vs. roots of perfect squares

Hi, Kerry.

When we talk about "perfect squares," it is generally assumed that the context is whole numbers, and that is true here. There are certainly non-whole numbers that are not the squares of integers, but whose square roots are rational; 4/9 is an example. And if we were to extend the meaning of "perfect square" to mean "any number that is not the square of a rational number," then we wouldn't be saying much.

But when we look at whole numbers, whenever the square root is not an integer, it will be irrational.

- Doctor Peterson, The Math Forum
  http://mathforum.org/dr.math/

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