Vertex and roots

min/max value

We start with the standard form for a quadratic

$$y = ax^2 + bx + c$$

We have been told that the vertex of the graph lies at the point with

$$x = -\frac{b}{2a}$$

We might call that point x_m for min/max but I prefer to just call it m:

$$x = m = -\frac{b}{2a}$$

It's the x-value at the vertex. If you like, you can also figure out the value of y at the vertex by plugging in to the equation, but we won't do that.

Instead, we want to focus on the zeros or roots of the equation, those values of x that give $y = 0$. (We ignore the complication that for some equations, some graphs, they may not cross the x-axis so there are no such values in the normal way of thinking. We'll explain later).

$$0 = ax^2 + bx + c$$

And having that 0 immediately suggests

$$0 = x^2 + \frac{b}{a}x + \frac{c}{a}$$

The same x-values that give zero in the first equation also give zero in the second one, but the second is simpler to solve.

average of s and t

Here is the equation we wrote previously using the zeros s and t

$$0 = (x - s)(x - t)$$

$$0 = x^2 - (s + t)x + st$$

Compare with

$$0 = x^2 + \frac{b}{a}x + \frac{c}{a}$$

If these are two forms of the *same* equation, then the cofactors of x must match:

$$-(s + t) = \frac{b}{a}$$

$$s + t = -\frac{b}{a}$$

And if, as we said, the vertex $x = m$ is the average of s and t

$$m = \frac{1}{2}(s + t)$$

$$= \frac{1}{2}(-\frac{b}{a}) = -\frac{b}{2a}$$

So that explains where the formula for the vertex m comes from.

The other part c/a must also match. That is

$$\frac{c}{a} = st$$

we'll come back to that.

completing the square

We have

$$m = -\frac{b}{2a}$$

Re-arranging:

$$-2m = \frac{b}{a}$$

Let us plug that into the standard form

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

$$x - 2mx = -\frac{c}{a}$$

We want the left-hand side be a perfect square.

$$(x + \text{ something })^2$$

The trick is to see that if we add m^2 it will work with m as the something

$$x - 2mx + m^2 = (x - m)^2$$

Of course, we must add m^2 on both sides of the original equation so from

$$x - 2mx = -\frac{c}{a}$$

we get

$$x - 2mx + m^2 = m^2 - \frac{c}{a}$$

$$(x - m)^2 = m^2 - \frac{c}{a}$$

Now it's just a little algebra. We take the square root, which means we can have either the positive or the negative branch.

$$x - m = \pm \sqrt{m^2 - c/a}$$

$$x = m \pm \ \sqrt{m^2 - c/a}$$

This is a second formula to memorize. If you plug in the value of m (i.e. $m = -b/2a$) and also c/a, then it will give us the roots of the equation that we started with.

problems

In the previous write-up we had this problem

$$y = -16t^2 + 64t + 144$$

and we want to know the time t when $y = 0$ so

$$0 = -16t^2 + 64t + 144$$

Divide both sides by 16

$$0 = -t^2 + 4t + 9$$

$m = -b/2a = 2$ and the roots are

$$x = 2 \pm \ \sqrt{4 - 4(-1)(9)} = 2 \pm \sqrt{40}$$

It's not a round number but $2 + \sqrt{40}$ is about 8.3 seconds. Notice that we pick the branch with a positive square root because we are interested only in positive values of t.

The sum of two numbers is 18 and their product is 56. What are they?

$$u + v = 18$$

$$uv = 56$$

$$u(18 - u) = 56$$

$$u^2 - 18u + 56 = 0$$

The factors of 56 are $2 \cdot 28$, $4 \cdot 14$, and $7 \cdot 8$.

$$4 + 14 = 18$$

so

$$(u-4)(u-14) = 0$$

The numbers are 4 and 14.

The product of two consecutive positive odd numbers is 255. Find the numbers.

Odd numbers can be described as $2n+1$ for some n, since that is one more than the even number $2n$. So we have

$$(2n+1)(2n+3) = 255$$

$$4n^2 + 8n + 3 = 255$$

$$4n^2 + 8n - 252 = 0$$

We are looking for zeros. So we can divide by 4:

$$n^2 + 2n - 63 = 0$$

Two numbers multiply to give -63 and add to give 2:

$$(n+9)(n-7) = 0$$

The roots are $-9, 7$.

But we specified positive numbers which means $n = 7$. The numbers are $2 \cdot 7 + 1 = 15$ and 17, easily confirmed.

Alternatively we can write the numbers as $2n+1$ and $2n-1$ and then

$$(2n+1)(2n-1) = 255$$

This gives a simpler solution.

$$4n^2 - 1 = 255$$

$$n^2 = 64$$

$$n = \pm 8$$

Again $n > 0$ so $n = 8$ and the numbers are $2n-1 = 15$ and $2n+1 = 17$.

quadratic formula

I use the formula

$$x = m \pm \sqrt{m^2 - c/a}$$

because it's relatively simple, but the one you will find in your book doesn't use m, it uses a, b and c.

Let us plug in $m = -b/2a$ and see what we get.

$$x = -\frac{b}{2a} \pm \sqrt{(\frac{-b}{2a})^2 - \frac{c}{a}}$$

The trick is to put$c$ over the same denominator as everything else

$$x = -\frac{b}{2a} \pm \sqrt{(\frac{-b}{2a})^2 - \frac{4ac}{4a^2}}$$

We multiplied the last term by $4a$ on top and bottom.

So then

$$x = -\frac{b}{2a} \pm \sqrt{\frac{b^2}{(2a)^2} - \frac{4ac}{(2a)^2}}$$

Factor out the square root of $(2a)^2$

$$x = -\frac{b}{2a} \pm \frac{1}{2a}\sqrt{(-b)^2 - 4ac}$$

And then combine everything over the common denominator so finally

$$x = \frac{-b \ \pm \ \sqrt{b^2 - 4ac}}{2a}$$

This is the quadratic formula which has been memorized by generations of algebra students. I really prefer

$$x = m \pm \ \sqrt{m^2 - c/a}$$

but it's the same thing.

alternative approach

There is a different way to get the quadratic formula. We said that m lies exactly halfway between s and t. Let the distance from m to s (or to t) be d. Then

$$m - d = s \qquad m + d = t$$

$$(m - d)(m + d) = st$$

$$m^2 - d^2 = st$$

So we have a formula for d^2 in terms of m, s and t.

$$d^2 = m^2 - st$$

But remember we know that

$$\frac{c}{a} = st$$

and said we'd come back? So

$$d^2 = m^2 - c/a$$

$$d = \pm \ \sqrt{m^2 - c/a}$$

and then the roots are

$$s = m - d = m - \sqrt{m^2 - c/a}$$

$$t = m + d = m + \sqrt{m^2 - c/a}$$

And that, I hope, shows the meaning of the quadratic formula.

The roots lie an equal distance d on either side of the vertex at m, and that distance is given by what's under the square root.

complex numbers

Let us take a look at what is under the square root. Here, it is probably simpler to use a, b and c.

The expression we have is called the *discriminant*

$$D = b^2 - 4ac$$

Depending on the particular example, D may be positive, negative, or even zero. And if it is the case that $D < 0$, then we'll have the square root of a negative number.

There are no real numbers with that property. And this corresponds to the case where the graph does not cross the x-axis. Then, there are no roots.

Notice that if $a > 0$, so the graph opens up, then by making c more and more positive we can eventually make $4ac > b^2$. We can always shift the graph up above the x-axis by adding more to c.

The way we explain these weird roots is to define i as a special number with the property $i = \sqrt{-1}$, or equivalently $i^2 = -1$, so then if $D < 0$, let's say

$$D = -p^2$$

where p^2 is a positive real number, then

$$\sqrt{D} = ip$$

So we have that the roots are

$$s = m - ip$$

$$t = m + ip$$

Take a look at s times t

$$\begin{aligned} st &= (m + ip)(m - ip) \\ &= m^2 - i^2p^2 \end{aligned}$$

but remember $i^2 = -1$ so

$$st = m^2 + p^2$$

and we get the standard form as

$$\begin{aligned} 0 &= (x - s)(x - t) \\ &= x^2 - (s + t)x + st \end{aligned}$$

m is the average of s and t, meaning that $2m = s+t$, and we computed just now $st = m^2 + p^2$ so

$$\begin{aligned} 0 &= x^2 - 2m + m^2 + p^2 \\ &= (x - m)^2 + p^2 \end{aligned}$$

The i has gone away.

This is a perfectly valid equation for y

$$y = (x - m)^2 + p^2$$

but it only has solutions as long as $y >= 0$! It is possible to have m and p both zero, so then $y = x^2$. But there is no way to have $y < 0$

because it is the sum of two squares, which are both either 0 or positive but never negative.

Which is another way of saying that the graph does not cross the x-axis. There is no x such that $y < 0$.

shifting the vertex

We won't prove it, but just say that any quadratic (in fact, any formula) can be re-written in a form like:

$$(y - k) = a(x - h)^2$$

where a is the shape factor, as usual, and (h, k) is the point where the vertex lies. (We abandon our favorite symbol m for this part).

Multiplying out

$$y - k = ax^2 - 2ahx + ah^2$$
$$y = ax^2 - 2ahx + ah^2 + k$$

by comparison with the standard form

$$y = ax^2 + bx + c$$

we have that

$$b = -2ah$$
$$h = -\frac{b}{2a}$$

Which matches what we said before. The x-value at the vertex is

$$x = h = -\frac{b}{2a}$$

intersection

You have a parabola and a line and want to know where (or if) they intersect. Write the equation of the line as

$$y = kx + y_0$$

To make life easier, we will consider a parabola that is translated to have its vertex at the origin so

$$y = ax^2$$

We are interested in the x-values that give equal y. So

$$kx + y_0 = ax^2$$

Gather like terms

$$0 = ax^2 + -kx - y_0$$

We have a quadratic. So

$$m = -\frac{-k}{2a} = \frac{k}{2a}$$

And the roots are

$$\begin{aligned} x &= \frac{k}{2a} \ \pm \ \sqrt{(\frac{k}{2a})^2} \\ &= \frac{k}{2a} \pm \frac{k}{2a} \\ &= 0, \frac{k}{a} \end{aligned}$$

The interesting thing is when there is only a single root, a single intersection. That happens when the line is the tangent to the parabola at the point of intersection.

There

$$x = \frac{k}{2a}$$
$$k = 2ax$$

The slope of the tangent line is $2ax$, which matches the result given by calculus.