Lagrange Trig identities

According to wikipedia

https://en.wikipedia.org/wiki/List_of_trigonometric_identities#Lagrange's_trigonometric_identities

the following two trigonometric identities are due to Lagrange:

$$\sum_{k=0}^{n} \sin k\theta = \frac{\cos \frac{1}{2}\theta - \cos(n + \frac{1}{2})\theta}{2 \sin \frac{1}{2}\theta}$$

$$\sum_{k=0}^{n} \cos k\theta = \frac{\sin \frac{1}{2}\theta + \sin(n + \frac{1}{2})\theta}{2 \sin \frac{1}{2}\theta}$$

We focus on the first one because we need it for the problem of the area under the cycloid curve. Rearrange:

$$2 \sin \frac{1}{2}\theta \cdot \sum_{k=0}^{n} \sin k\theta = \cos \frac{1}{2}\theta - \cos(n + \frac{1}{2})\theta$$

Looking for a derivation, the product of sines and the difference of cosines suggests we look at the sum formulas:

$$\cos A + B = \cos A \cos B - \sin A \sin B$$

$$\cos A - B = \cos A \cos -B - \sin A \sin(-B)$$
$$= \cos A \cos B + \sin A \sin B$$

Subtracting the first from the second gives

$$\cos(A-B)-\cos(A+B)=2\sin A\sin B$$

Let $B=\frac{\theta}{2}$ and $A=k\theta$:

$$\cos(k-\frac{1}{2})\theta-\cos(k+\frac{1}{2})\theta=2\sin k\theta\cdot\sin\frac{\theta}{2}$$

Now rearrange and sum over k. We'll start the sum from $k=1$ for the moment.

$$2\sin\frac{1}{2}\theta\cdot\sum_{k=1}^{n}\sin k\theta=\sum_{k=1}^{n}\cos(k-\frac{1}{2})\theta-\cos(k+\frac{1}{2})\theta$$

On the right-hand side, we notice that adjacent terms cancel. For example in

$$[\cos(k-\frac{1}{2})\theta-\cos(k+\frac{1}{2})\theta\ ]\ +\ [\ \cos(k+1-\frac{1}{2})\theta-\cos(k+1+\frac{1}{2})\theta\ ]$$

the two middle terms cancel. This is a telescoping sum.

So the result is

$$2\sin\frac{1}{2}\theta\cdot\sum_{k=1}^{n}\sin k\theta=\cos\frac{1}{2}\theta-\cos(n+\frac{1}{2})\theta$$

which rearranges to give the result shown at the beginning except that we need to add one term

$$\sin 0=0$$

and then

$$2\sin\frac{1}{2}\theta\cdot\sum_{k=0}^{n}\sin k\theta=\cos\frac{1}{2}\theta-\cos(n+\frac{1}{2})\theta$$

second derivation

A second derivation is to write the sum using Euler's formula

$$e^{ik\theta} = \cos k\theta + i \sin k\theta$$

and then

$$\sum_{k=0}^{n} \sin k\theta = \Im \sum_{k=0}^{n} e^{ik\theta} = \Im \sum_{k=0}^{n} (e^{i\theta})^k$$

We will, at the end, need only the imaginary part, $\Im$, of the right-hand side.

This is a geometric series with ratio $e^{i\theta}$. Leaving off the $\Im$ part:

$$= \frac{1 - r^{n+1}}{1 - r} = \frac{e^{i(n+1)\theta} - 1}{e^{i\theta} - 1}$$

This can be factored

$$= \frac{\left(e^{i(n+1)\theta/2}\right)\left(e^{i(n+1)\theta/2} - e^{-i(n+1)\theta/2}\right)}{e^{i\theta/2}\left(e^{i\theta/2} - e^{-i\theta/2}\right)}$$

and simplified

$$= e^{in\theta/2} \cdot \frac{\left(e^{i(n+1)\theta/2} - e^{-i(n+1)\theta/2}\right)}{\left(e^{i\theta/2} - e^{-i\theta/2}\right)}$$

Using Euler's formula again

$$= e^{in\theta/2} \cdot \frac{2i \sin(n+1)\theta/2}{2i \sin \theta/2}$$

$$= e^{in\theta/2} \cdot \frac{\sin(n+1)\theta/2}{\sin \theta/2}$$

$$= (\cos \frac{n\theta}{2} + i \sin \frac{n\theta}{2}) \cdot \frac{\sin(n+1)\theta/2}{\sin \theta/2}$$

Recall that we want only the imaginary part, so finally

$$\sum_{k=0}^{n} \sin k\theta = \sin n\theta/2 \cdot \frac{\sin(n+1)\theta/2}{\sin\theta/2}$$

$$\sin\frac{\theta}{2} \cdot \sum_{k=0}^{n} \sin k\theta = \sin\frac{n\theta}{2} \cdot \sin\frac{(n+1)\theta}{2}$$

comparison

If all this is true, it must be that somehow

$$\cos\frac{1}{2}\theta - \cos(n+\frac{1}{2})\theta = 2\sin\frac{n\theta}{2} \cdot \sin\frac{(n+1)\theta}{2}$$

In other words:

$$\cos\frac{1}{2}\theta - \cos(n\theta + \frac{\theta}{2}) = 2\sin(n\cdot\frac{\theta}{2}) \cdot \sin(\frac{n\theta}{2} + \frac{\theta}{2})$$

$$\cos B - \cos(2A+B) = 2\sin A \cdot \sin(A+B)$$

$$\cos(A+B-A) - \cos(A+B+A) = 2\sin A \cdot \sin(A+B)$$

So if $\alpha = n\theta/2 + \theta/2$ and $\beta = n\theta/2$ this is

$$\cos(\alpha-\beta) - \cos(\alpha+\beta) = 2\sin\beta \cdot \sin\alpha$$

which is where we started! So the two formulas are equivalent.

evaluation

The reason we are doing this is that this expression came up in the context of the area under the arch of the cycloid. We need to evaluate

$$\sum_{k=0}^{n} \sin k\theta$$

for $\theta = \frac{\pi}{n}$, as $n \to \infty$.

For the first formula

$$2 \sin \frac{1}{2}\theta \cdot \sum_{k=0}^{n} \sin k\theta = \cos \frac{1}{2}\theta - \cos(n + \frac{1}{2})\theta$$

we have that the right hand side is

$$\cos \frac{1}{2}\frac{\pi}{n} - \cos(n + \frac{1}{2})\frac{\pi}{n}$$

In the limit, the first term is $\cos 0 = 1$ and the second is $\cos \pi + 0 = -1$ so the difference is 2, which cancels the 2 on the left-hand side.

As a result, the sum is

$$\lim_{n \to \infty} \frac{1}{\sin \frac{\pi}{2n}}$$

which we approximate by the small angle formula as $2n/\pi$.

For the second formula

$$\sin \frac{\theta}{2} \cdot \sum_{k=0}^{n} \sin k\theta = \sin \frac{n\theta}{2} \cdot \sin \frac{(n+1)\theta}{2}$$

we have that the right-hand side is

$$\sin \frac{\pi}{2} \cdot \sin \frac{(\pi + \frac{\pi}{n})}{2}$$

which in the limit is $1 \cdot 1$.

Therefore, the sum is (as before)

$$\lim_{n \to \infty} \frac{1}{\sin \frac{\pi}{2n}}$$

which we approximate by the small angle formula as $2n/\pi$.