Derivatives of sine and cosine

Think about a ball traveling around in a circle.

Strang:

> We make the ball travel with constant speed, by requiring that the angle is equal to the time t... A full circle is completed at $t = 2\pi$.

Since the distance around is also 2π,

> the speed equals 1.

Here is a diagram of the situation when $t \approx 1.2$ radian.

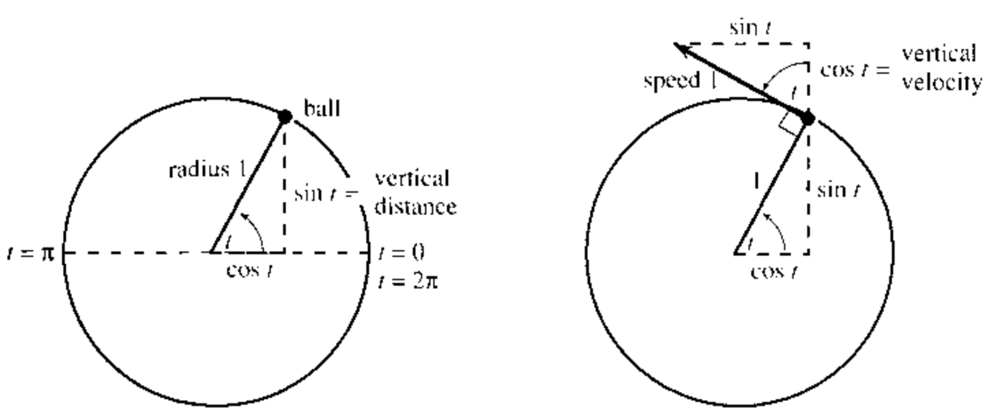


Fig. 1.16 Circular motion with speed 1, angle t, height $\sin t$, upward velocity $\cos t$.

The position of the ball is given by the radial vector out from the origin whose components are $x = \cos t, y = \sin t$.

The direction that the ball is going is given by the tangent to the curve. It can be seen from the diagram that the components of the tangent are $x = -\sin t, y = \cos t$. We know that this scaling of the vector is correct, because the magnitude of the vector is the speed, which is equal to 1, and by the Pythagorean theorem we have that $\sin^2 t + \cos^2 t = 1$.

What this means then, is that the derivative

$$\frac{d}{dt} \langle \sin t, \cos t \rangle = \langle \cos t, -\sin t \rangle$$

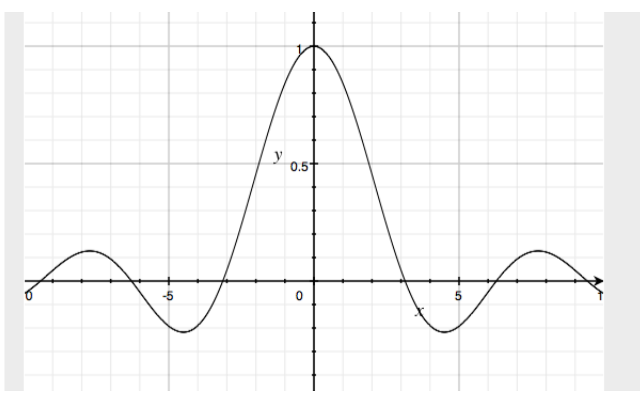
And in terms of the individual components, the derivative of the sine is the cosine and the derivative of the cosine is minus the sine.

limit

In applying the tools of calculus to this problem, we have a preliminary requirement, which is to look at a famous limit. We need the value of this limit

$$\lim_{x \to 0} \frac{x}{\sin x}$$

The limit of the ratio of the angle to its sine as the angle gets very small is equal to 1. One way to explore this is to use a plotting application:

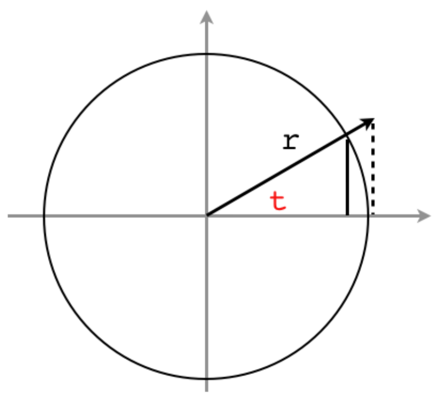


or a calculator such as that embedded in Python

```
>>> for i in range(1,100):
...     f = 1.0/i
...     print i, sin(f)/f
...
1 0.841470984808
..
97 0.999982286557
98 0.99998264621
99 0.999982995019
>>>
```

but (to be honest) these are cheating because when they calculate the sine of the angle they use a shortcut based on calculus.

Here is an actual proof that the ratio is equal to 1.



We'll be looking at the area of the right triangle whose hypotenuse is the radius, the area of the sector of the circle with the same angle t, and the area of the right triangle whose long side is equal to the radius.

Consider first the right triangle with radius r which lies entirely inside the circle. Its base is $r\cos t$ and its height is $r\sin t$, so its area is

$$A = \frac{1}{2} \cdot r\cos t \cdot r\sin t$$

$$= \frac{1}{2} r^2 \sin t \cos t$$

Consider next the sector of the circle (piece shaped like a slice of pie) containing the same angle, t. Recall that t is the length of the portion of the circumference along this sector (if t is measured in radians). If the circle is not a unit circle, then multiply by the radius.

t is some fraction of the total angular measure of the circle, namely $t/2\pi$, and we multiply by the total area of the circle to get the area of the sector:

$$A = \frac{t}{2\pi} \pi r^2 = \frac{1}{2} r^2 t$$

Finally, consider the right triangle containing the dotted line, whose base has length r. Because it is a similar triangle with the first one, its height (that dotted line) is in the same ratio to r, the base of the triangle, as $\sin t$ is to $\cos t$. Thus, its length is $r \tan t$.

The area of this triangle is

$$A = \frac{1}{2} \cdot r \cdot r \tan t$$

$$= \frac{1}{2} r^2 \ \frac{\sin t}{\cos t}$$

It is clear that the first triangle is smaller than the sector, and the sector is smaller than the second triangle *no matter how small t becomes.*

We can write

$$\frac{1}{2} r^2 \sin t \cos t < \frac{1}{2} r^2 t < \frac{1}{2} r^2 \ \frac{\sin t}{\cos t}$$

Now cancel $r^2/2$

$$\sin t \cos t < t < \frac{\sin t}{\cos t}$$

and divide by $\sin t$

$$\cos t < \frac{t}{\sin t} < \frac{1}{\cos t}$$

As $t \to 0$, both $\cos t$ and $1/\cos t$ approach the same limit, which is just the value of the cosine at $t = 0$, namely, 1. Therefore the ratio gets squeezed, and it approaches the same limit as well.

$$\lim_{x \to 0} \frac{x}{\sin x} = 1$$

Since the limit is 1, the inverse approaches the same limit. We have proved that:

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

algebra

We will need still another limit. We require

$$\lim_{h \to 0} \frac{(\cos h - 1)}{h}$$

Now

$$\frac{\cos h - 1}{h} = \frac{\cos h - 1}{h} \cdot \frac{\cos h + 1}{\cos h + 1}$$

The numerator on the right is

$$(\cos h - 1)(\cos h + 1) = \cos^2 h - 1$$

$$= -\sin^2 h$$

so we can assign one copy of $\sin h$ to each of the terms in the denominator:

$$-\frac{\sin h}{h} \cdot \frac{\sin h}{\cos h + 1}$$

The limit as $h \to 0$ of the first factor is equal to 1 (with a factor of -1) as we saw before, but the second one is $0/2 = 0$, so the whole thing is zero.

$$\lim_{h \to 0} \frac{\cos h - 1}{h} = 0$$

Now it's easy.

Difference quotient for sine

The limit just obtained allows us to find the derivatives of sine and cosine.

Set up the difference quotient for sine:

$$\frac{\sin(x+h) - \sin x}{h}$$

(Note that we are not yet equating this with the derivative, it is just the difference quotient itself. Therefore, we do not have to write the business about $\lim_{h \to 0}$).

Using the addition of angles formula:

$$= \frac{\sin x \cos h + \sin h \cos x - \sin x}{h}$$

Group the terms containing $\sin x$ and $\cos x$ separately

$$= \sin x \ \frac{(\cos h - 1)}{h} + \cos x \ \frac{\sin h}{h}$$

Evaluating the limit as $h \to 0$, the second term is

$$\cos x \ \lim_{h \to 0} \frac{\sin h}{h}$$

We are allowed to pull $\cos x$ out of the limit, because it does not depend on h. By the main result above (our "famous" limit), the limit part is equal to 1, so the whole expression is just equal to $\cos x$.

We just showed that the first term is zero, which means that we have in the end:

$$\frac{d}{dx} \sin x = \cos x$$

The derivative of the sine is the cosine.

Derivative of the cosine

Set up the difference quotient for cosine:

$$\frac{\cos(x+h) - \cos x}{h}$$

Using addition of angles

$$\frac{\cos x \cos h - \sin x \sin h - \cos x}{h}$$

Grouping like terms

$$= \cos x \ \frac{(\cos h - 1)}{h} - \sin x \ \frac{\sin h}{h}$$

But we as we just said:

$$\lim_{h \to 0} \frac{\cos h - 1}{h} = 0$$

so the first term is zero. By the original limit derived above, the second term is

$$\lim_{h \to 0} \left[-\sin x \ \frac{\sin h}{h} \right] = -\sin x$$

The derivative of the cosine is minus the sine.

$$\frac{d}{dx} \ \cos x = -\sin x$$