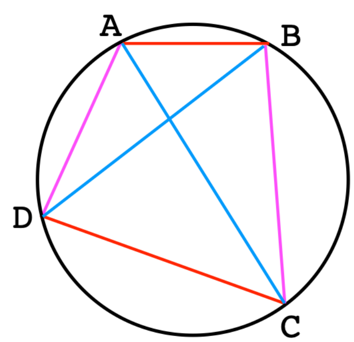
Ptolemy's theorem

Ptolemy was a Greek astronomer and geographer who lived at Alexandria in the 2nd century AD (died c. 168 AD). That is nearly 500 years after Euclid. Ptolemy was a popular name for Egyptian pharaohs in earlier centuries.

Our Ptolemy is known for many works including his book the *Almagest*, and important to us, for a theorem in plane geometry concerning cyclic quadrilaterals. These are 4-sided polygons all of whose vertices lie on a circle. Recall that any triangle lies on a circle, so this is a restriction on the fourth vertex of the polygon.



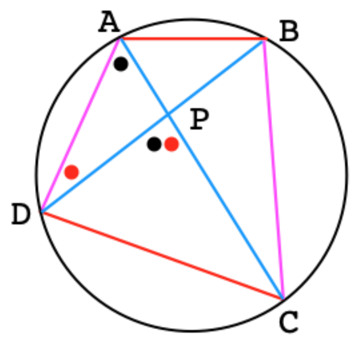
AB CD + BC AD = AC BD

Consider a cyclic quadrilateral $ABCD$. Draw the diagonals AC and BD. Ptolemy's theorem says that if we take the products of the two pairs of opposing sides and add them, the result is equal to the product of the diagonals.

$$AB \cdot CD + BC \cdot AD = AC \cdot BD$$

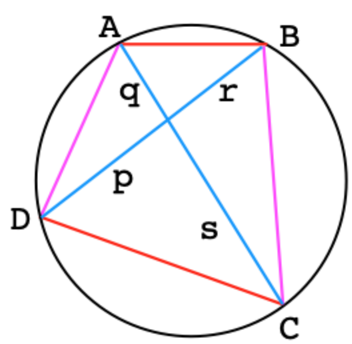
Proof. (adapted from wikipedia).

`https://en.wikipedia.org/wiki/Ptolemy%27s_theorem`



Let the angle s (red dot) subtend arc AB and the angle t (black dot) subtend arc CD. Then the central $\angle DPC = s + t$ and it has $\sin s + t$. The other central $\angle APD$ has the same sine, as it is supplementary to $s + t$.

Let the components of the diagonals be $AC = q + s$ and $BD = p + r$.



Twice the areas of the four small triangles will then be equal to

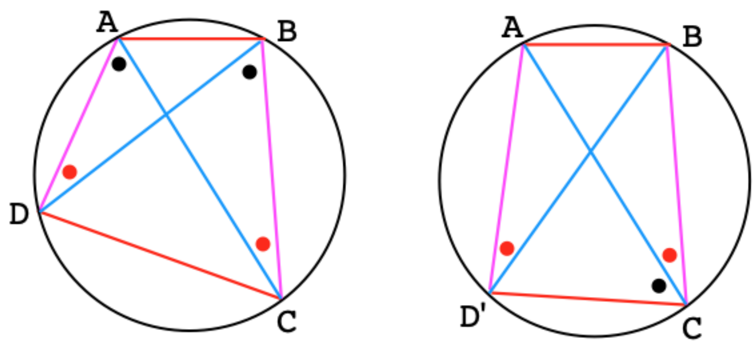
$$2A = (pq + qr + rs + sp) \sin s + t$$

Simple algebra will show that

$$(pq + qr + rs + sp) = (p + r)(q + s) = AC \cdot BD$$

The product of the diagonals times the sine of either central angle is equal to twice the area of the quadrilateral.

We're on to something. Now, the great idea.



Move D to D', such that $AD' = CD$ and $CD' = AD$.

$\triangle ACD \cong \triangle ACD$ by SSS, so they have the same area. Therefore the area of $ABCD$ is equal to the area of $ABCD'$.

Some of the angles switch with the arcs. In particular, angle t (black dot) now subtends arc AD'. As a result $s+t$ is the measure of the whole angle at vertex C. The whole angle at vertex A is supplementary, and the sine of the whole angle at vertex A is equal to that at C.

So twice the area of $\triangle ABD'$ is $AB \cdot AD' \cdot \sin s + t$, and twice that of $\triangle BCD'$ is $BC \cdot CD' \cdot \sin s + t$. Add these two areas, equate them with the previous result, and factor out the common term $\sin s + t$:

$$AC \cdot BD = AB \cdot AD' + BC \cdot CD'$$

But $AD' = CD$ and $CD' = AD$ so

$$AC \cdot BD = AB \cdot CD + BC \cdot AD$$

This is Ptolemy's theorem.

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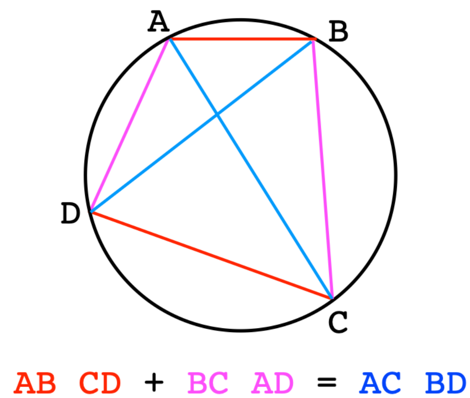
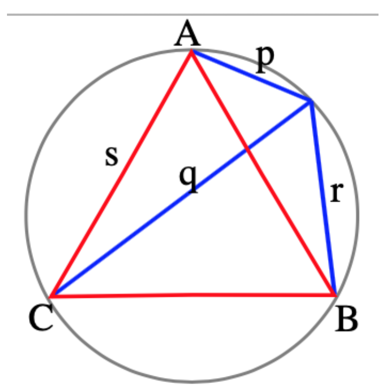


corollaries

Here are just a few of the results that follow from this remarkable theorem.

equilateral triangle



Inscribe an equilateral triangle in a circle and pick any point on the circle.

$$qs = ps + rs$$

$$q = p + r$$

Pythagorean theorem

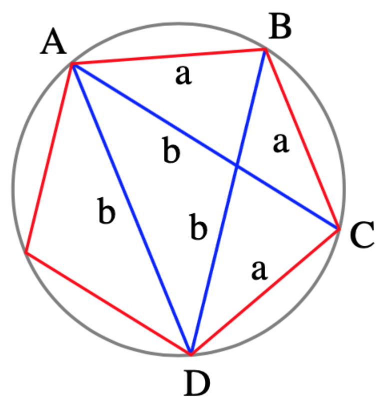
Let the quadrilateral be a rectangle. The the sum of squares of opposing sides is

$$a^2 + b^2$$

Triangles made by opposing diagonals are congruent, so the diagonals are equal in length. The diagonal is the hypotenuse, hence

$$a^2 + b^2 = h^2$$

golden mean in the pentagon



Take four vertices of the regular pentagon and draw two diagonals. From the theorem, we have

$$b \cdot b = a \cdot a + a \cdot b$$

$$\frac{b^2}{a^2} = 1 + \frac{b}{a}$$

Rather than use the quadratic equation, rearrange and add 1/4 to both sides to "complete the square":

$$\frac{b^2}{a^2} - \frac{b}{a} + \frac{1}{2^2} = 1 + \frac{1}{2^2}$$

So

$$(\frac{b}{a} - \frac{1}{2})^2 = \frac{5}{4}$$

$$\frac{b}{a} - \frac{1}{2} = \pm\,\frac{\sqrt{5}}{2}$$

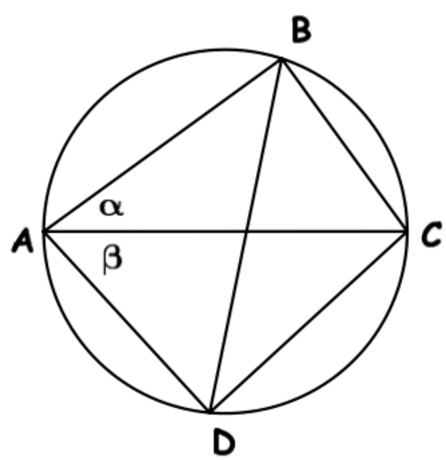
$$\frac{b}{a} = \frac{1 \pm \sqrt{5}}{2}$$

This ratio b/a is known as ϕ, the golden mean.

sum of angles

We will use Ptolemy's theorem to derive the formula for the sine of the sum of angles α and β.

Surowski gives this as a theorem but only gives hints for the proof. One is this: in the figure below AC is a diameter of the circle.



Proof.

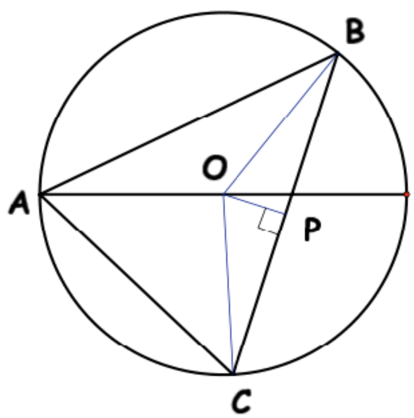
Ptolemy's theorem is that the product of opposing sides, summed, is equal to the product of the diagonals.

$$AD \cdot BC + AB \cdot DC = AC \cdot BD$$

As we said, in this special example, AC is a diameter. Therefore, the entire angles at vertices B and D are right angles, by Thales' theorem. Then, by complementary angles we have that $\sin \angle ACB = \cos \alpha$ and $\sin \angle ACD = \cos \beta$.

We begin by finding an expression for $\sin \alpha + \beta$.

We recall this proof:



The peripheral $\angle BAC$ is one-half the central angle subtending the same arc, $\angle BOC$. This is sometimes called the **inscribed angle theorem**.

Since OB and OC are radii of the circle, BOC is isosceles, and since OP is the altitude of an isosceles triangle, $\angle POC$ is one-half the central angle and thus equal to $\angle BAC$.

We find the sine of $\angle POC$ as

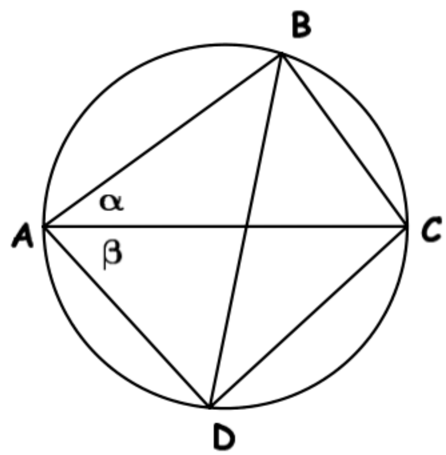
$$\sin\angle POC = \frac{PC}{OC} = \frac{BC}{2OC} = \frac{BC}{d}$$

where d is the diameter of the circle.

We saw this important result previously (**here**), and we used it in the **eyeball theorem**.

The sine of a peripheral angle is equal to the chord it cuts off, divided by the diameter.

Going back to the problem:



Ptolemy's theorem says that:

$$AD \cdot BC + AB \cdot DC = AC \cdot BD$$

by the work above:

$$\sin\alpha + \beta = \frac{BD}{d} = \frac{BD}{AC}$$

Take the formula from Ptolemy, and let $AC = d$.

Now, divide by d^2. The right-hand side is what we seek, $\sin\alpha+\beta$. The left-hand side is

$$= \frac{AD}{d} \cdot \frac{BC}{d} + \frac{AB}{d} \cdot \frac{DC}{d}$$

By the definitions of elementary trigonometry:

$$= \cos\beta \sin\alpha + \cos\alpha \cdot \sin\beta$$

This is indeed the formula for the sine of the sum of angles.

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