Long division

$$\frac{1+i}{2+i} = \ ?$$

method 1

Multiply by the fraction formed from the complex conjugate of the denominator:

$$\frac{1+i}{2+i} \cdot \frac{2-i}{2-i} = \frac{3+i}{5}$$

The length of the resulting vector is $\sqrt{10}/5 = \sqrt{2/5}$.

A slightly different approach is to use the inverse. z^{-1} is the inverse of z if and only if

$$z \cdot z^{-1} = 1, \qquad z^{-1} = \frac{1}{z}$$

so

$$\frac{w}{z} = w \cdot z^{-1}$$

What is the inverse of $2+i$? It is almost $2-i$ since

$$(2+i)(2-i) = 5$$

The inverse of $2+i$ is $(2-i)/5$. And you can see that in the calculation that we did on the first line.

method 2

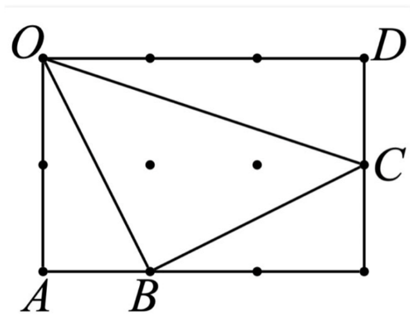
Write the numbers in r, θ notation as

$$\sqrt{2}\ e^{\pi/4}, \qquad \sqrt{5}\ e^{\theta},\ \text{with } \theta = \tan^{-1} 1/2$$

Dividing r-values gives us the same length as before. Subtracting the angles

$$\tan^{-1} 1 - \tan^{-1} 1/2 \stackrel{?}{=} \tan^{-1} 1/3$$

This is one of Gardner's problems.



Focusing on point O, it is clear that the central angle is 45° (because $\triangle OBC$ is isosceles and $\angle OBC$ is a right angle), so the sum of the other two angles is also 45°.

method 3

The last approach is long division. I'm not going to try to typeset this.

`https://twitter.com/MrHonner/status/1587593196606980097`

But let's just think about it. We're trying to divide $2 + i$ *into* $1 + i$. So the first factor is $1/2$ and we would write below $1 + i/2$ as what we need to subtract. The remainder is $i - i/2 = i/2$.

The next step is to divide $2 + i$ into $i/2$. The factor is $i/4$ and what we're subtracting is $-1/4$.

Next, divide $2+i$ into $1/4$. The factor is $1/8$ and what we're subtracting is $i/8$.

If you continue this, you will obtain two infinite series (see the web page)

$$S_r = \frac{1}{2} + \frac{1}{8} - \frac{1}{32} + \frac{1}{128} + \dots$$

$$S_i = i(\frac{1}{4} - \frac{1}{16} + \frac{1}{64} - \frac{1}{256} + \dots)$$

The sign of the first term of S_r is not quite right, but after that we have geometric series with the same common ratio $r = -1/4$.

Since $-1 < r < 1$, the series converge.

There are various ways to remember the formula for the sum. What I think of is to make the first term of the series equal to 1.

Here we must adjust the sign of the first term before doing the rest.

$$S_r = \frac{1}{2} + \frac{1}{8} - \frac{1}{32} + \frac{1}{128} + \dots$$

Add -1 on both sides:

$$S_r - 1 = -\frac{1}{2} + \frac{1}{8} - \frac{1}{32} + \frac{1}{128} + \dots$$

Make the first term 1

$$(-2) \cdot (S_r - 1) = 1 - \frac{1}{4} + \frac{1}{16} - \frac{1}{64} \dots$$

Now, the right-hand side is just

$$\frac{1}{1-r} = \frac{1}{5/4} = \frac{4}{5}$$

so, putting things together

$$(-2) \cdot (S_r - 1) = \frac{4}{5}$$

$$S_r - 1 = -\frac{2}{5}$$

I get $S_r = 3/5$, as expected.

For the second series we start with

$$S_i = i(\frac{1}{4} - \frac{1}{16} + \frac{1}{64} - \frac{1}{256} + \dots)$$

Remembering the equality $-i \cdot i = 1$

$$\begin{aligned}(-4i) \cdot S_i &= 1 - \frac{1}{4} + \frac{1}{16} - \frac{1}{64} \dots \\ &= \frac{4}{5}\end{aligned}$$

Applying the equality again, we obtain:

$$S_i = \frac{i}{5}$$

Putting the two results together we have that the result of the long division is $(3+i)/5$, which matches the other two methods.