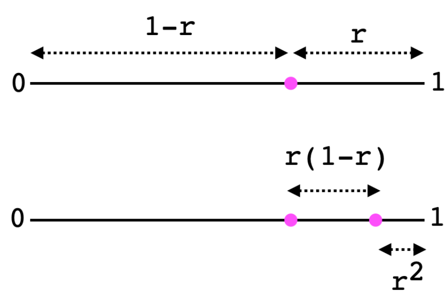
Infinite series

The most important elementary series is

$$1 + x + x^2 + \cdots = \sum_{n=0}^{\infty} x^n$$

which can also be written as shown.

I saw a post by Steven Strogatz on Twitter that shows a clever way of looking at it. Suppose we consider the distance from 0 to 1 and imagine that we will take multiple steps to move from $x = 0$ to $x = 1$.



We will use the rule that we always move a certain fraction of the distance remaining (like one of Zeno's paradoxes). This means that in the first step, we leave a certain fraction of the distance yet to be covered, r.

In the second step, we have that the distance remaining is r, the fraction that we will actually move is $1 - r$ and that distance corresponds to $r(1-r)$. So the total distance moved so far is

$$d = (1-r) + r(1-r)$$

If we continue forever we *will* get to 1

$$1 = (1-r) + r(1-r) + r^2(1-r) + \ldots$$

But then

$$\frac{1}{1-r} = 1 + r + r^2 + \ldots$$

And from our construction, clearly $0 < r < 1$, which is the standard restriction on r so that the sum of the series is meaningful.

We might think about writing this another way, namely that at each stage we move a fraction of the distance remaining f (with $f = 1-r$).

$$1 = f + f(1-f) + f(1-f)^2 + \ldots$$

Let $r = 1-f$ and then

$$r = fr + fr^2 + \ldots$$
$$\frac{1}{f} = 1 + r + r^2 + \ldots$$
$$\frac{1}{1-r} = 1 + r + r^2 + \ldots$$