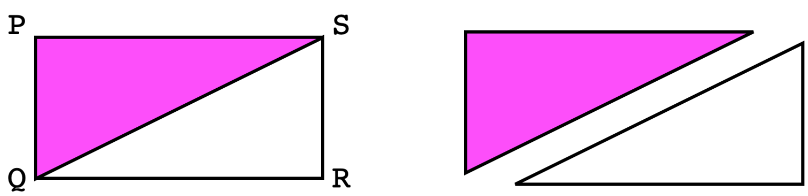
Similarity

We're going to do a couple short, simple proofs about, and then using, similarity. The last one is very famous, but we can get there in easy steps.

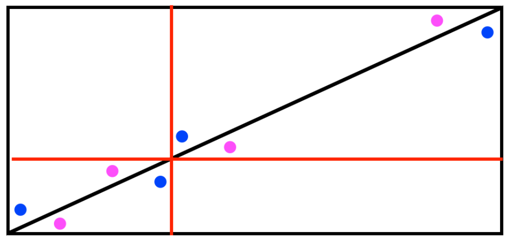
We start with the close connection between rectangles and right triangles. On the left we have the rectangle $PQRS$, where we have drawn one of the diagonals, QS. This divides the rectangle into two right triangles.



Any right triangle (right side) can be joined to a second, rotated, copy of itself to form a rectangle.

The *area* of a rectangle is the base times the height. Since the two triangles are congruent, the area of each is one-half of this. The important idea for us now is that the areas of the two congruent right triangles formed by drawing a diagonal are equal.

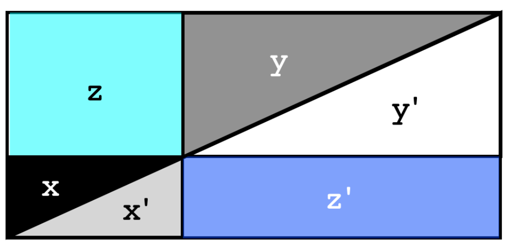
Next, we take a rectangle and draw the diagonal, and then pick any point on the diagonal and draw vertical and horizontal (red) lines through that point.



Notice that since the horizontal lines are parallel, the angles marked with a magenta dot are equal. The same is true of the angles marked with a blue dot, since the vertical lines are all parallel.

This means that all the right triangles in this figure are similar, since they have all 3 angles the same. Counting them all, there are six similar triangles.

In the figure below, x and x', as well as y and y' are pretty obvious, but there is also the pair of congruent triangles formed from the whole rectangle: that is $x + y + z$ and $x' + y' + z'$, where z and z' are the leftover rectangles in cyan and blue.



For the big congruent triangles, we have equality for the total of the pieces:

$$x + y + z = x' + y' + z'$$

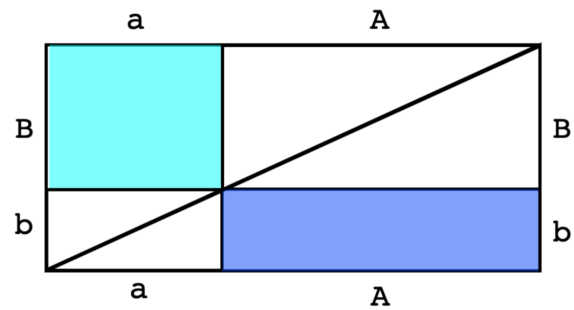
We also know that

$$x = x'$$

$$y = y'$$

It follows (after subtraction) that $z = z'$.

Now we just re-label the figure in terms of the sides of the similar triangles. The smallest triangle has sides a and b and the medium-sized one is A and B.



What are the areas of the two rectangles in cyan and blue? We have that they are equal, and in terms of the side lengths we have that

$$aB = Ab$$

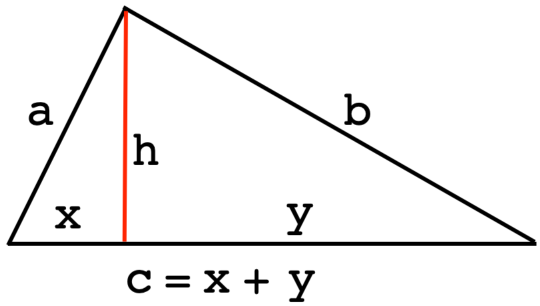
so

$$\frac{A}{a} = \frac{B}{b}, \qquad \frac{a}{b} = \frac{A}{B}$$

We have proved that for *any* two similar right triangles, the side lengths are in the same ratio. As we've seen before, the ratio can be formed either *within* or *between* triangles. The two ratios are different, but related. This is the forward version of the similar $\triangle$ theorem: equal angles $\Rightarrow$ equal ratios of sides.

hypotenuse

Consider the large right triangle in the figure below. If we draw the altitude or vertical line to the base, the result is three right triangles. You will recognize that all three right triangles are similar, because they have all three angles the same. The reason is complementary angles. I'm hoping that you've seen this part before.



The next step can be written simply but can also be somewhat confusing. I think it helps to say the words out loud and point to the corresponding sides at the same time.

Let's agree to call the three sides of a right triangle the *short* side, the *long* side, and the *hypotenuse*.

Using the theorem that we just proved, we can write the ratio of the long to the short side of each triangle and know that they are equal because these are similar triangles:

$$\frac{h}{x} = \frac{y}{h} = \frac{b}{a}$$

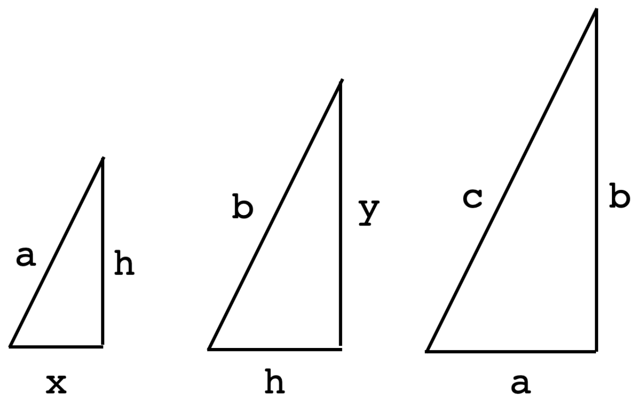
We have the small $\triangle$ on the left and the original, large $\triangle$ on the right.

But the same equation can be viewed as applying to the hypotenuse. It says that b, the hypotenuse of the medium $\triangle$, is to a, the hypotenuse of the small $\triangle$, as h, the short side of the medium $\triangle$, is to x, the short side of the small $\triangle$.

The ratio between each hypotenuse for two similar right triangles is the same as that between corresponding sides. It follows that our theorem about ratios applies to *all three sides* of the right triangles, including the hypotenuse.

Pythagorean theorem

We dissect the figure from above into the small and medium triangles plus the original, large one. Compare with the previous figure to be sure there's no mistake. (The scale is not exactly correct, we just want to be sure we can keep the corresponding sides straight).



Write the ratio of hypotenuse to short side for the small and large triangles:

$$\frac{a}{x} = \frac{c}{a}$$

and the ratio of hypotenuse to long side for the medium and large triangles:

$$\frac{b}{y} = \frac{c}{b}$$

Rearrange both equations slightly

$$a^2 = cx$$

$$b^2 = cy$$

Now just add them:

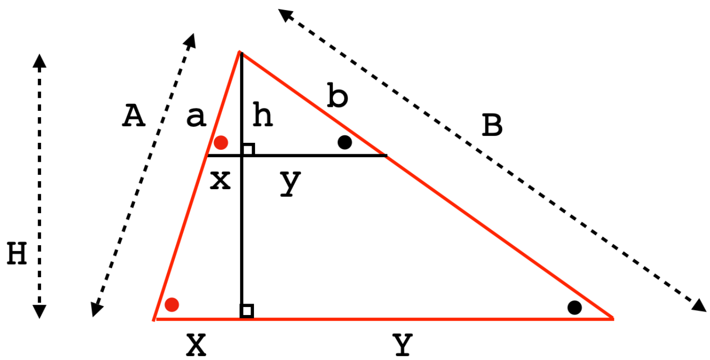
$$a^2 + b^2 = cx + cy$$

$$= c(x + y) = c^2$$

□

extension

We extend similarity to general triangles, as follows.



Any triangle can be dissected into two right triangles. From the properties of similar right triangles, we have that

$$\frac{a}{A} = \frac{b}{B} = \frac{h}{H} = \frac{x}{X} = \frac{y}{Y}$$

But then

$$\frac{x}{y} = \frac{X}{Y}$$

Add 1 to both sides

$$\frac{x+y}{y} = \frac{X+Y}{Y}$$

so, finally

$$\frac{x+y}{X+Y} = \frac{y}{Y} = \frac{h}{H}$$

All sides of two similar triangles are in the same ratio.

$\square$.