Acheson G163

Acheson presents this problem (though he does not show the solution). The algebra defeated me for quite a long time, but I finally got it.

We are given only the radii of the small circles, a, b and c, and are asked to find r, which I have written as R.

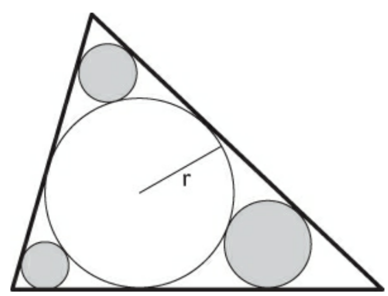


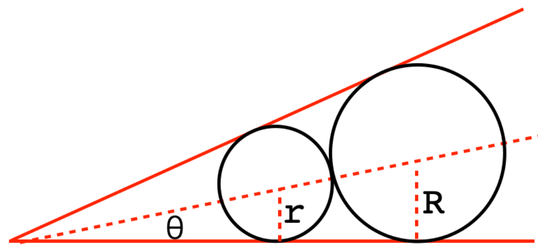
Fig. 163 A tricky problem from *The Ladies Diary* (1730).

Solution.

Let the relevant half-angles be α, β, and γ. These are half angles, so they sum to be one right angle. Therefore, any one angle is complementary to the sum of the other two.

$$\sin \alpha = \cos \beta + \gamma = \cos \beta \cos \gamma - \sin \beta \sin \gamma$$

Our previous work with this problem (in the geometry book)



showed that

$$\sin\theta = \frac{R-r}{R+r}, \qquad \cos\theta = \frac{2\sqrt{Rr}}{R+r}$$

Draw similar triangles involving the centers of the two circles to get this result.

We can apply this to the current problem, giving similar formulas for α, β and γ in terms of the radii a, b and c as well as R.

So the expression for $\sin\alpha$ above can be written as

$$\frac{R-a}{R+a} = \frac{2\sqrt{Rb}}{R+b} \cdot \frac{2\sqrt{Rc}}{R+c} - \frac{R-b}{R+b} \cdot \frac{R-c}{R+c}$$

Rearranging:

$$(R-a)(R+b)(R+c) + (R+a)(R-b)(R-c) = 4R\sqrt{bc} \cdot (R+a)$$

The first term is

$$\begin{aligned} &(R-a)(R+b)(R+c) \\ &= (R-a)(R^2 + Rb + Rc + bc) \\ &= R^3 + R^2 b + R^2 c + Rbc - R^2 a - Rab - Rac - abc \end{aligned}$$

The second term is

$$\begin{aligned} &(R+a)(R-b)(R-c) \\ &= (R+a)(R^2 - Rb - Rc + bc) \end{aligned}$$

$$= R^3 - R^2 b - R^2 c + Rbc + R^2 a - Rab - Rac + abc$$

The sum has four cancelations and four terms remaining:

$$2R^3 + 2Rbc - 2Rab - 2Rac = 4R\sqrt{bc} \cdot (R + a)$$

We can factor out $2R$

$$R^2 + bc - ab - ac = 2(R + a)\sqrt{bc}$$

$$R^2 + bc - ab - ac = 2R\ \sqrt{bc} + 2a\sqrt{bc}$$

This is a quadratic in R:

$$R^2 - 2\sqrt{bc}\ R + bc - ab - ac - 2a\sqrt{bc}$$

The discriminant is

$$4bc - 4(bc - ab - ac - 2a\sqrt{bc})$$

with another cancelation

$$4ab + 4ac + 8a\sqrt{bc}$$

The 4 comes out from the square root as 2 and we have

$$R = \frac{2\sqrt{bc} \pm 2\sqrt{ab + ac + 2a\sqrt{bc}}}{2}$$

$$R = \sqrt{bc} \pm \sqrt{ab + ac + 2a\sqrt{bc}}$$

But what is under the square root is a perfect square, namely

$$ab + ac + 2a\sqrt{bc} = (\sqrt{ab} + \sqrt{ac})^2$$

We have

$$R = \sqrt{bc} \pm (\sqrt{ab} + \sqrt{ac})$$

We will disregard the negative sign and obtain finally

$$R = \sqrt{bc} + \sqrt{ab} + \sqrt{ac}$$

which is the answer we were given. As expected, it is symmetric in a, b and c.

$\square$