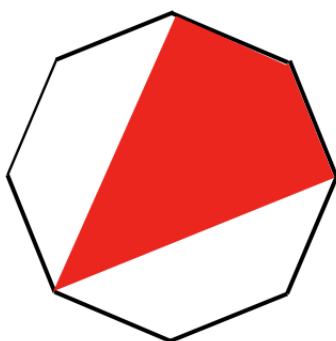


Introduction

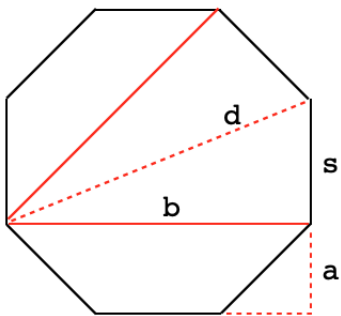
Here's a problem from the web. Two chords are drawn in an octagon, and we're asked what fraction of the area is contained between them.

Vincent Pantaloni @panlepan · Feb 25
The octagon that thinks it's David Bowie.
What fraction of this regular octagon is coloured in red ?
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inscribed in a square

The simplest way to look at this problem is to imagine the octagon inscribed in a square. Tilt the square so that the bottom edge of the octagon is horizontal.



We can calculate the total area of the octagon as the area of the square minus the small triangles at the corners. Focusing on them for a minute, we see that each is an isosceles right triangle with sides of length a and hypotenuse s .

$$a^2 + a^2 = 2a^2 = s^2$$

The area of each triangle is $a^2/2$ and there are four of them for $2a^2$.

Each side of the square has s plus two copies of a , so $b = s + 2a$ and the total area is b^2 or

$$A = (s + 2a)^2 = s^2 + 4as + 4a^2$$

Subtract $2a^2$

$$A_{\text{octagon}} = s^2 + 4as + 2a^2$$

This form obscures an important result. Substitute $s^2 = 2a^2$ from above:

$$A_{\text{octagon}} = 2a^2 + 4as + 2a^2 = 4(a^2 + as)$$

We have three lines that divide the octagon into four regions. The total area is four times something. Is this a coincidence?

The area of two triangles is

$$sb = s(s + 2a) = s^2 + 2as = 2(a^2 + as)$$

The triangles are one-half the total area of the octagon. The two trapezoids must be the other half.

The area of one trapezoid is (subtracting two triangles):

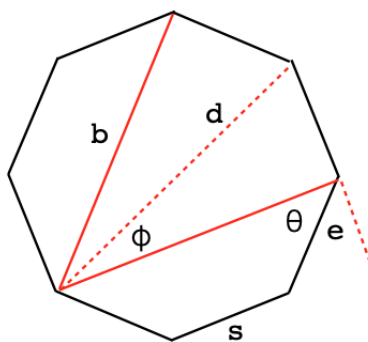
$$\begin{aligned} A_{\text{trapezoid}} &= ab - a^2 \\ &= a(s + 2a) - a^2 = as + a^2 \end{aligned}$$

So two triangles are equal in area to two trapezoids.

inscribed in a circle

Another way to look at this problem is to imagine the octagon inscribed in a circle. All the points of the octagon lie on the circle, which is of radius r , or diameter $d = 2r$ (below, left panel).

We reason that the two triangles are congruent (by symmetry) and they are both right triangles with a shared hypotenuse that is a diameter of the circle.



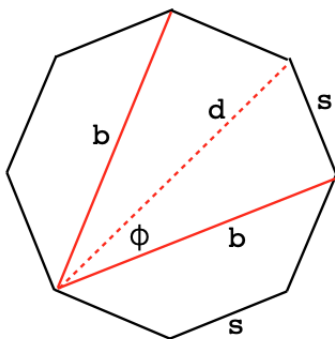
How do we know they are right triangles?

The external angle of the octagon is $e = \pi/4$ (we must turn by a full 2π in 8 turns so $8e = 2\pi$).

Two successive turns forms one right angle. Since $\theta + e = \pi/2$ and $e = \pi/4$, we have that $e = \theta = \pi/4$. The supplementary angle to $e + \theta$ is also a right angle.

Back to the triangles. The whole angle subtended is 2ϕ (by symmetry) and since 2ϕ is a peripheral angle of the circle, its arc is $1/4$ of the total, or $\pi/2$.

As a peripheral angle the measure of 2ϕ is one-half that. Thus, $2\phi = \pi/4$ and $\phi = \pi/8$.



trigonometry

The sides of the triangle can be found in terms of the diameter as $d \sin \phi$ and $d \cos \phi$. The area is

$$A_{\text{triangle}} = \frac{1}{2} d \sin \phi \cdot d \cos \phi$$

and the whole area for the region in red (two triangles) is

$$A_{\text{region}} = d^2 \sin \phi \cos \phi$$

The question asked what fraction of the octagon's area is shaded red.

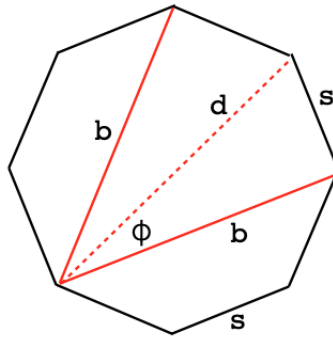
In the geometry book we derived a simple formula for the area of a regular polygon:

$$A_{\text{polygon}} = nr^2 \sin \frac{\pi}{n} \cos \frac{\pi}{n}$$

This is an octagon so $n = 8$. The angle is $\phi = \pi/8$ and since $r = d/2$

$$\begin{aligned}
 A_{\text{octagon}} &= 8 \cdot \frac{d^2}{2^2} \sin \phi \cos \phi \\
 &= 2 \cdot d^2 \sin \phi \cos \phi
 \end{aligned}$$

We have again that the area of the entire octagon is twice that of the two triangles. The octagon is divided by three lines into four equal parts.



calculating the area

If we want to actually calculate the area of the octagon, going back to the formula

$$A_{\text{octagon}} = 2 \cdot d^2 \sin \frac{\pi}{8} \cos \frac{\pi}{8}$$

The individual terms with sine and cosine of $\pi/8$ will take some figuring, but we know from the double angle formula that

$$\sin 2s = 2 \sin s \cos s$$

The sine of $\pi/4$ is $1/\sqrt{2}$, so

$$\sin \phi \cos \phi = \frac{1}{2\sqrt{2}}$$

and the area is then

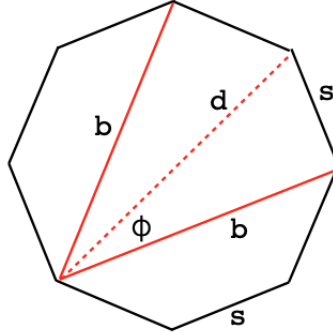
$$A_{\text{octagon}} = \frac{2d^2}{2\sqrt{2}} = \frac{d^2}{\sqrt{2}}$$

Previously, we calculated the area in terms of the length of the side.

$$\begin{aligned} A_{\text{octagon}} &= 4(a^2 + as) = 4\left(\frac{s^2}{2} + \frac{s^2}{\sqrt{2}}\right) \\ &= 2s^2(1 + \sqrt{2}) \end{aligned}$$

If these are truly equal then the relationship between d and s is

$$\begin{aligned} \frac{d^2}{\sqrt{2}} &= 2s^2(1 + \sqrt{2}) \\ d^2 &= s^2(2\sqrt{2} + 4) \end{aligned}$$



For one triangle, from Pythagoras we have that

$$\begin{aligned} d^2 &= s^2 + (s + 2a)^2 \\ &= s^2 + s^2 + 4as + 4a^2 \end{aligned}$$

with $a = s/\sqrt{2}$ and $2a^2 = s^2$

$$d^2 = 2s^2 + 2\sqrt{2}s^2 + 2s^2$$

$$= s^2(4 + 2\sqrt{2})$$

which matches what we had above.

As another check, write

$$\sin \frac{\pi}{8} = \frac{s}{d} = \frac{1}{\sqrt{(4 + 2\sqrt{2})}}$$

I obtained 0.382683.. for both sides.

half-angle formulas

Finally, another way to find the relationship between s and d is to use the half-angle formulas. We take the double angle formula for cosine and rewrite it as follows

$$\begin{aligned}\cos 2s &= \cos^2 s - \sin^2 s \\ &= 2 \cos^2 s - 1\end{aligned}$$

So

$$\cos^2 s = \frac{1 + \cos 2s}{2}$$

Using our favorite trigonometric identity again

$$\begin{aligned}\sin^2 s &= 1 - \cos^2 s \\ &= 1 - \frac{1 + \cos 2s}{2} = \frac{1 - \cos 2s}{2}\end{aligned}$$

The final formulas are obtained by taking square roots.

Leaving this as the square we have that

$$\sin^2 \frac{\pi}{8} = \frac{1 - \cos \frac{\pi}{4}}{2}$$

$$= \frac{1}{2} \left(1 - \frac{1}{\sqrt{2}} \right)$$

Somehow, it must be true that

$$\frac{1}{4 + 2\sqrt{2}} = \frac{1}{2} \left(1 - \frac{1}{\sqrt{2}} \right)$$

Factor out 1/2

$$\frac{1}{2 + \sqrt{2}} = 1 - \frac{1}{\sqrt{2}}$$

Multiply both sides by $\sqrt{2}$

$$\frac{1}{\sqrt{2} + 1} = \sqrt{2} - 1$$

Aha! A difference of squares:

$$(\sqrt{2} + 1)(\sqrt{2} - 1) = 2 - 1 = 1$$

□