TECHNISCHE UNIVERSITÄT WIEN

184.754 Seminar on Algorithms

Paper: Coloring the Vertices of 9-pt and 27-pt Stencils with Intervals

Camilo Tello Fachin

December 5, 2023

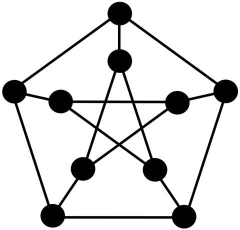
Outline

Intro Slide

Still to do..

VCP

- Given G(V,E)



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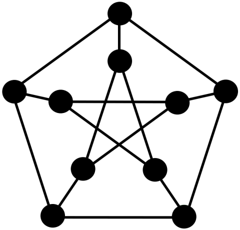

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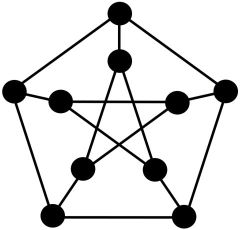

Formal Definition of VCP given G(V,E)

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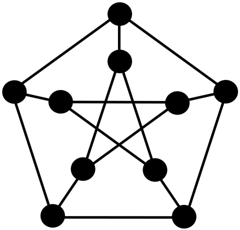
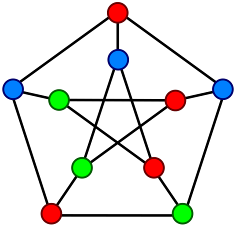


Colored G(V,E)



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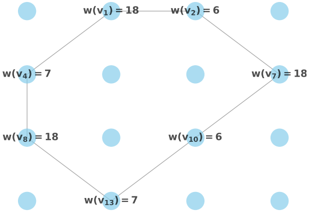

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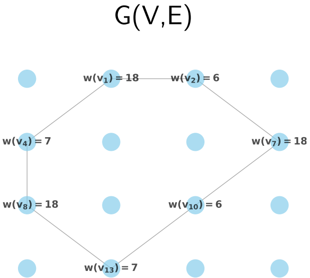

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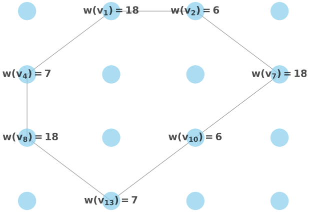

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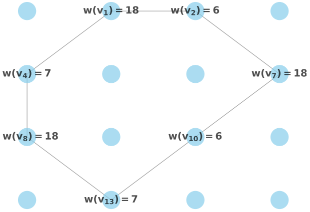

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- Vertex v is colored with open interval: $[\text{start}(v), \text{start}(v) + w(v))$
- Neighboring Vertices must have disjoint color intervals: $\forall (a, b) \in E :$ $[\text{start}(a), \text{start}(a) + w(a)) \cap [\text{start}(b), \text{start}(b) + w(b)) = \emptyset$.

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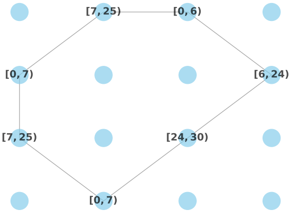

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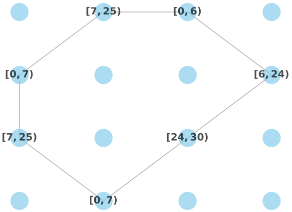

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Optimization Problem Instance:

Find a coloring start : $V \mapsto \mathbb{Z}^+$ that minimizes maxcolor.

- A maxcolor that is indeed minimal, is denoted with maxcolor* .

Definition: 2D Stencil Interval Vertex Coloring (2DS-IVC)

A problem where G is a 9-pt $2D$ stencil, composed of $X \times Y$ vertices on a $2D$ grid such that two vertices (i, j) and (i', j') are connected iff $|i - i'| \leq 1$ and $|j - j'| \leq 1$.

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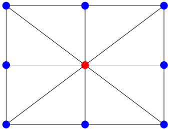

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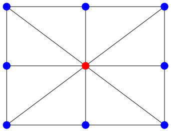
Definition: 3D Stencil Interval Vertex Coloring (3DS-IVC)

A problem where G is a 27-point $3D$ stencil, composed of $X \times Y \times Z$ vertices on a 3D grid such that two vertices (i, j, k) and (i', j', k') are connected iff $|i - i'| \leq 1$, $|j - j'| \leq 1$, and $|k - k'| \leq 1$.



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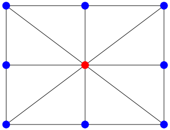

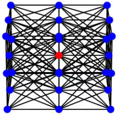

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Theorem

Let $G(V, E)$ be a clique K_n with n vertices, then

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- Particularly important result since the 2D-IVC's and 3D-IVC's are composed of K_4's and K_8's respectively.
- Therefore any identified K_4 in a 2D-IVC Problem and any K_8 in a 3D-IVC Problem are immediately (lose) lower bounds on maxcolor.

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A graph G(V,E) is considered Bipartite iff its vertices can be partitioned into two sets A and B, such that all edges have one endpoint in A and the other in B.

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- Another important result, since the 9-pt stencil contains a Bipartite 5-pt stencil and the 27-pt stencil constains at least a Bipartite 7-pt stencil, and all paths are Bipartite.
- Authors use this to construct approximation algorithms.

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- The clique with the largest weight from the cycle shown earlier would be of interval size 25, while the optimal coloring of the entire graph (the odd cycle) has maxcolor* $= 30$.
- Understanding of coloring of odd cycles yields new lower bounds.

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Theorem

Let G be an odd cycle, then there holds:

$$\mathit{maxcolor^*} = \mathit{max}(\mathit{maxpair},\ \mathit{minchain3})$$

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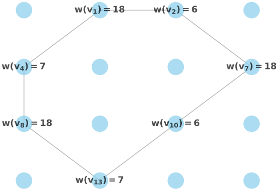
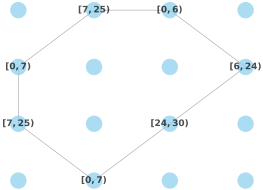
- One can verify this for the previous example





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Verify that no adjacent edges have overlapping intervals.
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Since $|E| \leq \frac{n(n-1)}{2}$ and checking if two intervals overlap is in $\mathcal{O}(1)$, the verification can be done in $\mathcal{O}(n^2)$ $\square$.

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Idea of Proof

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Since the 3DS-IVC problem is in NP and is harder than NAE-3SAT which is an NP-Complete problem, 3DS-IVC is NP-Complete $\square$.

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The proof yields:

Theorem

Deciding whether a 27-pt stencil can be colored with less than k colors is NP-Complete. [1]

Lemma

In the application section I will probably cite [2] and also [3].

References

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