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184.754 Seminar on Algorithms

Paper: Coloring the Vertices of 9-pt and 27-pt Stencils with Intervals

Camilo Tello Fachin
December 5, 2023

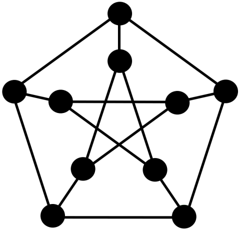
Outline

Intro Slide

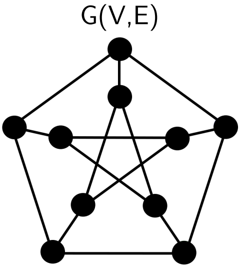
Still to do..

VCP

- Given G(V,E)



- Given G(V,E)
- Find a vertex coloring s.t. colors on adjacent vertices differ



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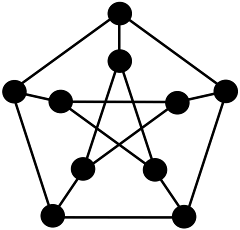

Formal Definition of VCP given G(V,E)

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$\forall v \in V : \forall w \, in \, \Gamma(v) : f(v) \neq f(w).$

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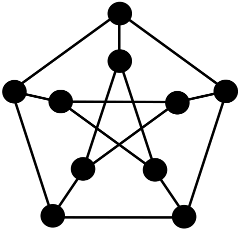
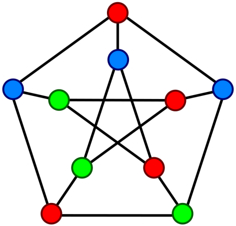


Colored G(V,E)



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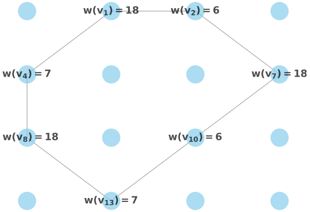

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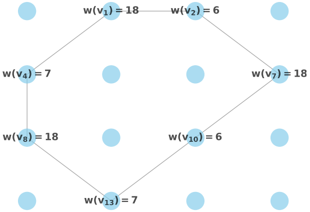

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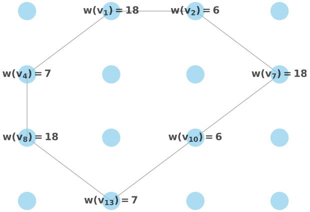

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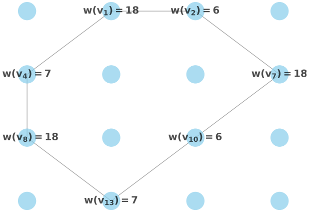

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- Vertex v is colored with open interval: $[\text{start}(v), \text{start}(v) + w(v))$
- Neighboring Vertices must have disjoint color intervals: $\forall (a, b) \in E :$ $[\text{start}(a), \text{start}(a) + w(a)) \cap$ $[\text{start}(b), \text{start}(b) + w(b)) = \emptyset$.

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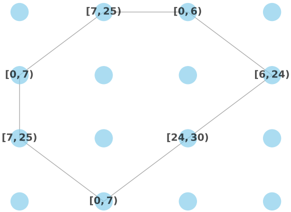
  $\text{maxcolor} = \max_{v \in V} start(v) + w(v)$ colors.

Formal IVC Problem Definition

Interval Colored G(V,E) with maxcolor = 30



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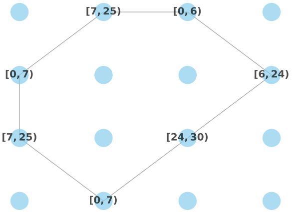

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Optimization Problem Instance:

Find a coloring start : $V \mapsto \mathbb{Z}^+$ that minimizes maxcolor.

- A maxcolor that is indeed minimal, is denoted with maxcolor* .

Definition: 2D Stencil Interval Vertex Coloring (2DS-IVC)

A problem where G is a 9-pt $2D$ stencil, composed of $X \times Y$ vertices on a $2D$ grid such that two vertices (i, j) and (i', j') are connected iff $|i - i'| \leq 1$ and $|j - j'| \leq 1$.

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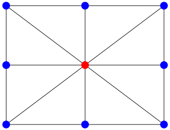

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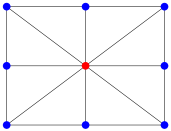
Definition: 3D Stencil Interval Vertex Coloring (3DS-IVC)

A problem where G is a 27-point $3D$ stencil, composed of $X \times Y \times Z$ vertices on a 3D grid such that two vertices (i, j, k) and (i', j', k') are connected iff $|i - i'| \leq 1$, $|j - j'| \leq 1$, and $|k - k'| \leq 1$.



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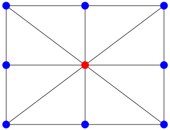

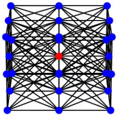

Theorem

Let $G(V, E)$ be a clique K_n with n vertices, then

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- Particularly important result since the 2D-IVC's and 3D-IVC's are composed of K_4's and K_8's respectively.
- Therefore any identified K_4 in a 2D-IVC Problem and any K_8 in a 3D-IVC Problem are immediately (lose) lower bounds on maxcolor.

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- Another important result, since the 9-pt stencil contains a Bipartite 5-pt stencil and the 27-pt stencil constains at least a Bipartite 7-pt stencil, and all paths are Bipartite.
- Authors use this to construct approximation algorithms.

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- The clique with the largest weight from the cycle shown earlier would be of interval size 25, while the optimal coloring of the entire graph (the odd cycle) has maxcolor* $= 30$.
- Understanding of coloring of odd cycles yields new lower bounds.

Definition

Let maxpair be the maximal sum of any two consecutive vertices. [1]

$$\text{maxpair} = \max_i w(i, i+1)$$

[1] Here G is cycle, neighbors of vertex x are $x+1$ and $x-1$.

Odd Cycles 2

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Let minchain3 be the minimum sum of any 3 consecutive vertices:

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Theorem

Let G be an odd cycle, then there holds:

$$\textit{maxcolor*} = \textit{max}(\textit{maxpair, minchain3})$$

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Odd Cycles 3

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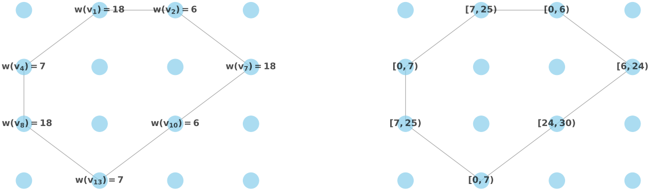
- Coinciding bounds conclude proof [1] $\square$.
- One can verify this for the previous example



Heuristics

Still to do..

In the application section I will probably cite [2] and also [3].

References

D. Durrman and E. Saule, "Coloring the vertices of 9-pt and 27-pt stencils with intervals," in 2022 IEEE International Parallel and Distributed Processing Symposium (IPDPS), 2022, pp. 963–973. DOI: 10.1109/IPDPS53621.2022.00098.

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