TECHNISCHE
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184.754 Seminar on Algorithms

Paper: Coloring the Vertices of 9-pt and 27-pt Stencils with Intervals

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December 6, 2023

Outline

Connection between VCP and Parallel Algorithms

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- Processors could be colors while the vertices could be tasks.

Introduction

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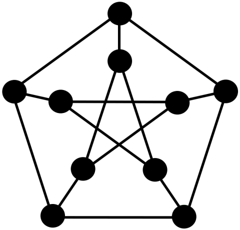
- optimal (minimal) colorings in such an application, can lead to runtime improvement $\propto$ degree of parallelism.
- Unfortunately problem is NP-Complete, heuristics are required for large instances to get useful solutions.

Vertex Color Problem Definition

- Given G(V,E)

G(V,E)



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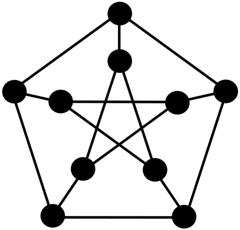

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find $f(v)$:
$\forall v \in V : \forall w \, in \, \Gamma(v) : f(v) \neq f(w).$

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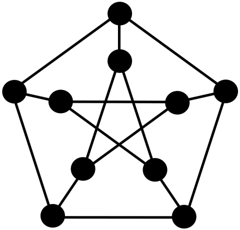

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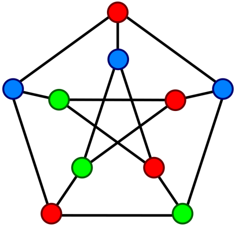
Colored



Interval Vertex Coloring Definitions

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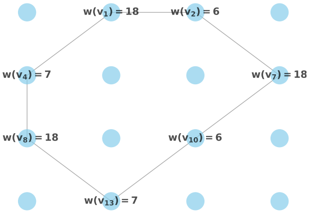

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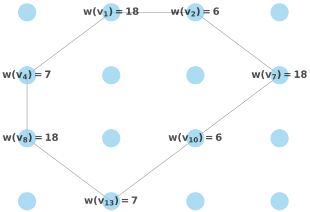

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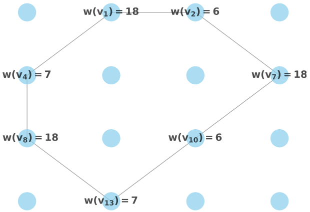

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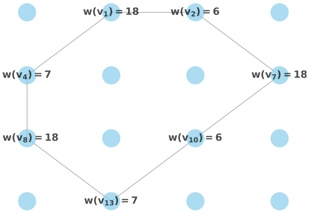

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- Vertex v is colored with open interval: $[\text{start}(v), \text{start}(v) + w(v))$
- Neighboring Vertices must have disjoint color intervals: $\forall (a, b) \in E$: $[\text{start}(a), \text{start}(a) + w(a)) \cap [\text{start}(b), \text{start}(b) + w(b)) = \emptyset$.

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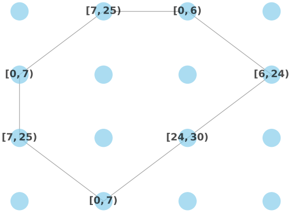
Formal IVC Problem Definition

Interval Colored G(V,E)
with maxcolor = 30



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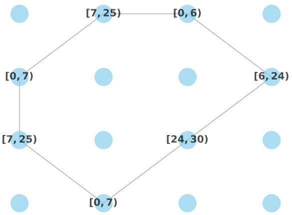

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Optimization Problem Instance:

Find a coloring start : $V \mapsto \mathbb{Z}^+$ that minimizes maxcolor.

- A maxcolor that is indeed minimal, is denoted with maxcolor* .

Definitions for Analysis

Definition: 2D Stencil Interval Vertex Coloring (2DS-IVC)

A problem where G is a 9-pt $2D$ stencil, composed of $X \times Y$ vertices on a $2D$ grid such that two vertices (i, j) and (i', j') are connected iff $|i - i'| \leq 1$ and $|j - j'| \leq 1$.

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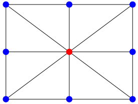

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Definition: 3D Stencil Interval Vertex Coloring (3DS-IVC)

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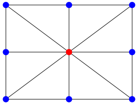

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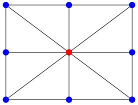

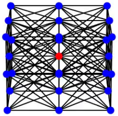

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Let $G(V,E)$ be a clique K_n with n vertices, then

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- Particularly important result since the 2D-IVC's and 3D-IVC's are composed of K_4's and K_8's respectively.
- Therefore any identified K_4 in a 2D-IVC Problem and any K_8 in a 3D-IVC Problem are immediately (lose) lower bounds on maxcolor.

Bipartite Graphs

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A graph G(V,E) is Bipartite iff it contains no odd cycles. In other words: its vertices can be partitioned into two sets A and B, such that all edges have one endpoint in A and the other in B.

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- Another important result, since the 9-pt stencil contains a Bipartite 5-pt stencil and the 27-pt stencil constains at least a Bipartite 7-pt stencil, and all paths are Bipartite.
- Authors use this to construct approximation algorithms.

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- The clique with the largest weight from the cycle shown earlier would be of interval size 25, while the optimal coloring of the entire graph (the odd cycle) has maxcolor* $= 30$.
- Understanding of coloring of odd cycles yields new lower bounds.

Odd Cycles 2

Definition

Let maxpair be the maximal sum of any two consecutive vertices. [1]

$$\text{maxpair} = \max_i w(i, i+1)$$

[1] Here G is cycle, neighbors of vertex x are $x+1$ and $x-1$.

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Theorem

Let G be an odd cycle, then it holds:

$$\mathit{maxcolor}^* = \mathit{max}(\mathit{maxpair},\ \mathit{minchain3})$$

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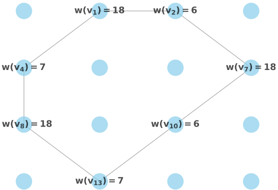
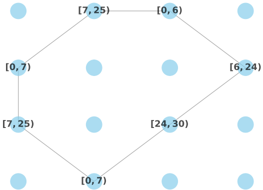
- One can verify this for the previous example





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Given a solution for 3DS-IVC, encoded with pairs of integers which are bounded between 0 and $\sum_{v \in V} w(v_i) \mapsto$ are solutions in polynomial space.
Verify that no adjacent edges have overlapping intervals.
$\forall (u, v) \in E$:

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Since $|E| \leq \frac{n(n-1)}{2}$ and checking if two intervals overlap is in $\mathcal{O}(1)$, the verification can be done in $\mathcal{O}(n^2)$ $\square$.

NAE-3SAT $\propto$ 3DS-IVC ?

Idea of Proof

- Construct an instance 3DS-IVC from an instance of NAE-3SAT in polynomial time.

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The proof yields:

Theorem

Deciding whether a 27-pt stencil can be colored with less than k colors is NP-Complete. [1]

Algorithms: Bipartite Decomposition

Recall: Theorem

For bipartite graphs there exists a coloring
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Definition

Let $c(x, y)$ be the lower end of the color interval associated with vertex (x, y) in that coloring. And let $\mathrm{RC} = \max c(x, y) + w(x, y)$ be the maximum color used by any of the rows.

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Corollary

$RC \leq maxcolor^*$ is a lower bound of the optimal number of colors of the instance since it is the optimal coloring of a subgraph of the original instance.

Algorithms: Bipartite Decomposition

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Algorithm: Bipartite Decomposition
  Data: 2DS-IVC instance with Y rows
  Result: Approximate coloring with at most 2 · maxcolor*
for each row r in the 2DS-IVC instance do
    if r is even then
        Color the chain of vertices in row r optimally using colors from [0, RC);
    else
        Color the chain of vertices in row r optimally using colors from [RC, 2RC);
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the Bipartite Decomposition is a 4-approximation algorithm for 3DS-IVC. (same approach) [1]

Space-Time Kernel Density Estimation

$$f(x,t) = \frac{1}{nh} \sum_{i=1}^{n} K\left(\frac{x - x_i}{h}, \frac{t - t_i}{h_t}\right)$$

- Parallelized summation leads to race condition.
- Implementation of space-time decomposition by Hohl et. al. [2].
- Parallelization by exploiting symmetry by Saule et. al. [3].

Application and Experiments

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Authors Approach

Application and Experiments

Space-Time Kernel Density Estimation

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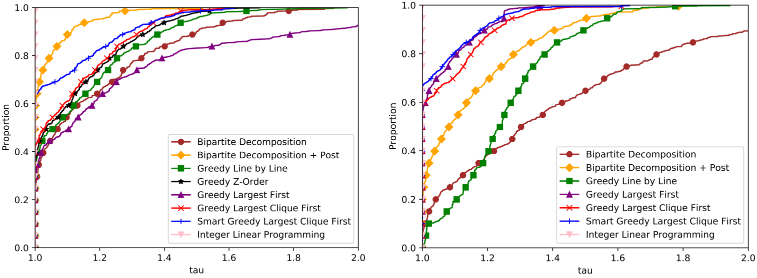
- number of datapoints within one voxel is set as weight.[2]
- Weight $\propto$ computational costs.

[2] Voxel is the 3D equivalent of a Pixel

Results



(a) 2D Results

(b) 3D Results

Figure: Performance Profiles with ILP. [1]

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