PHY 374 ORDINARY DIFFERENTIAL EQUATIONS Fall 2018

I. The Simple Euler Method

Consider the differential equation

$$y'(x) = f(x,y) \qquad \text{Eq.(1)}$$

with initial conditions $y(x_0) = y_0$. The Euler method is a simple numerical method for solving such equations. It's based on the idea that if we know $y(x)$ and $y'(x)$, we can estimate $y(x+\Delta x)$ by expanding y as a Taylor series about x,

$$y(x+\Delta x) = y(x) + y'(x)\Delta x + O(\Delta x^2) \qquad \text{Eq.(2)}$$

The leading term in the error is due to the truncation of the series and is $\propto y''\Delta x^2$. This error will be smaller the smaller Δx is. See Fig.1.

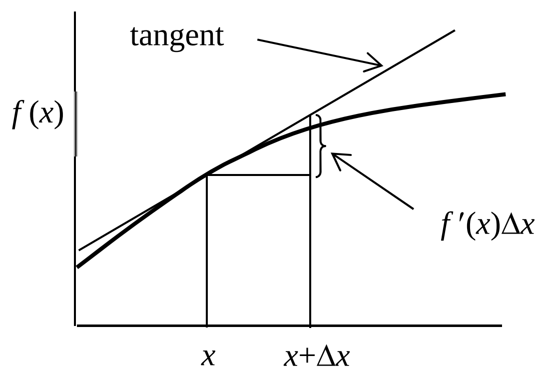

Fig. 1 The simple Euler method.

Once $y(x+\Delta x)$ is known, we can calculate $y(x+2\Delta x)$ in a similar fashion. Thus, we solve the differential equation by *discretizing* the solution interval with the x points given by

$$x_i = x_0 + i\Delta x \qquad \text{Eq.(3)}$$

and the solution at these points given by

$$y(x_{i+1}) = y(x_i) + y'(x_i)\Delta x \qquad \text{Eq.(4)}$$

This approach is also called the finite-differencing method, for obvious reasons. It can be extended to 2nd and higher order equations, and can also be applied to systems of 1st order equations.

As an example, suppose we want to solve the equation

$$y' = -y^2 + x \qquad \text{Eq.(5)}$$

over the interval $x \in [a,b]$ with $y(a) = y_0$. First we break our solution interval into N intervals, with $x_i = a+i\Delta x$ and $\Delta x = (b-a)/N$. Our finite differencing equation would read

$$y(x_{i+1}) = y(x_i) + (-y^2(x_i) + x_i)\Delta x \qquad \text{Eq.(6)}$$

Although this method is not the most accurate, it is extremely easy to program. To estimate the error, we note that the error for each step is $\approx y''\Delta x^2$, and there are N such steps so, the accumulated error is $\approx \bar{y}''\Delta x^2 N = \bar{y}''\Delta x^2 L/\Delta x = \bar{y}''L\Delta x$, where L is the solution interval length and $\bar{y}''$ is the average value of the second-derivative over the interval. Thus, the total error is $O(\Delta x)$ and to add another digit of accuracy to our solution we must make Δx ten times smaller. By carrying out our Taylor expansion out another term we get a more accurate method whose error is $O(\Delta x^2)$. We shall investigate this idea later.

We have so far discussed 1st-order equations. However, many times we need to simulate equations with higher derivatives. An example is Newton's second Law which is a 2nd order differential equation. However we can always write it as a system of 1st-order equations, as follows:

$$\begin{aligned} \frac{dv}{dt} &= F/m \\ \frac{dx}{dt} &= v \end{aligned} \qquad \text{Eq.(7)}$$