CT6 – P XS – 15

Series X Solutions

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Subject CT6

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Assignment X1 Solutions

Solution X1.1

> **Comment**
>
> *This question applies the Core Reading in Chapter 3 on the method of moments.*

Equating the sample and population moments we get:

$$e^{\hat{\mu}+½\hat{\sigma}^2} = 2,000 \qquad e^{2\hat{\mu}+\hat{\sigma}^2}(e^{\hat{\sigma}^2}-1) = 500^2 \tag*{[1]}$$

Substituting the square of the first equation into the second equation gives:

$$2,000^2(e^{\hat{\sigma}^2}-1) = 500^2 \quad \Rightarrow \quad \hat{\sigma}^2 = \ln\left(1+\frac{500^2}{2,000^2}\right) = 0.0606246 \tag*{[1]}$$

Substituting this back into the first equation gives:

$$\hat{\mu} = \ln 2,000 - ½\hat{\sigma}^2 = 7.57059 \tag*{[1]}$$

[Total 3]

Alternatively students could adopt an approach based on the first two non-central moments:

$$e^{\hat{\mu}+½\hat{\sigma}^2} = 2,000 \qquad e^{2\hat{\mu}+2\hat{\sigma}^2} = 500^2 + 2,000^2 \tag*{[1]}$$

Substituting the square of the first equation into the second equation gives:

$$2,000^2 e^{\hat{\sigma}^2} = 500^2 + 2,000^2 \quad \Rightarrow \quad \hat{\sigma}^2 = \ln\left(1.0625\right) = 0.0606246 \tag*{[1]}$$

Substituting this back into the first equation gives:

$$\hat{\mu} = \ln 2,000 - ½\hat{\sigma}^2 = 7.57059 \tag*{[1]}$$

[Total 3]

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Solution X1.2

Comment

This question applies the Core Reading in Chapter 4 on proportional reinsurance. Note there is also a recap on MGFs in Chapter 0.

If the claim amount, X, has a $gamma(\alpha,\lambda)$ distribution then:

$$\frac{\hat{\alpha}}{\hat{\lambda}}=2{,}000 \quad \text{and} \quad \frac{\hat{\alpha}}{\hat{\lambda}^2}=100^2$$

Solving these gives:

$$\hat{\alpha}=400 \quad \text{and} \quad \hat{\lambda}=0.2$$ [1]

Note these values are not actually required to answer Parts (a) and (b).

(a) ***Mean amount paid by the insurer***

If Y is the amount paid by the insurer then $Y=0.85X$. Hence:

$$E(Y)=0.85E(X)=0.85\times 2{,}000=\pounds 1{,}700$$ [1]

(b) ***Variance of the amount paid by the reinsurer***

If Z is the amount paid by the insurer then $Z=0.15X$. Hence:

$$\text{var}(Z)=0.15^2\,\text{var}(X)=0.15^2\times 100^2=\pounds^2 225$$ [1]

(c) ***MGF of the amount paid by the insurer***

We have:

$$M_Y(t)=E(e^{tY})=E(e^{t0.85X})=E(e^{(0.85t)X})=M_X(0.85t)$$ [½]

From Page 12 of the *Tables*, we have:

$$M_X(t)=\left(1-\frac{t}{0.2}\right)^{-400} \quad \text{or} \quad (1-5t)^{-400}$$

Hence:

$$M_Y(t) = \left(1 - \frac{0.85t}{0.2}\right)^{-400} \quad \text{or} \quad (1-4.25t)^{-400} \qquad [½]$$

[Total 4]

Solution X1.3

> **Comment**
>
> *This question applies the Core Reading in Chapter 1 on zero-sum games. Part (i) tests the minimax strategy and Part (ii) tests randomised strategies.*

(i) ***Minimax (pure) strategy for A and B***

The best worst case scenario (min max loss) for Player A is the 1 under strategy III.
The best worst case scenario (max min gain) for Player B is the 1 under strategy 3.

The strategies are in equilibrium (*ie* 1 is a saddle point – it is the highest in the column and the lowest in the row) as the 1 is the 'best worst case scenario' for *both* players. So pure strategy is used. The solution is (III, 3). [1]

The value of the game is 1. [1]

[Total 2]

(ii) ***Minimax (randomised) strategy for A***

The best worst case scenario (min max loss) for Player A is the 3 under strategy I.
The best worst case scenario (max min gain) for Player B is the 2 under strategy 2.

The strategies are *not* in equilibrium (*ie* there is no saddle point). So a randomised strategy is used.

Let p denote the probability that A chooses strategy I.

If B chooses Strategy 1 then A's expected loss would be:

$$E\left[\text{loss} \mid \text{B chooses 1}\right] = -3 \times p + 4 \times (1-p) = 4 - 7p \qquad [½]$$

If B chooses Strategy 2 then A's expected loss would be:

$$E\left[\text{loss} \mid \text{B chooses } 2\right] = 3\times p + 2\times(1-p) = 2 + p \qquad [½]$$

Assuming A wants to minimise her maximum expected loss she will choose p where the two straight lines corresponding to these expected losses cross. Solving gives:

$$4 - 7p = 2 + p \quad \Rightarrow \quad 8p = 2 \quad \Rightarrow \quad p = ¼ \qquad [½]$$

So A will choose strategy I will probability ¼ and strategy II with probability ¾. [½]

The value of the game is $4 - 7\times ¼ = 2¼$. [1]

[Total 3]

Solution X1.4

> ***Comment***
>
> *This question applies the Core Reading in Chapter 2. Part (i) tests conjugate priors and the derivation of the posterior distribution. Part (ii) tests the calculation of the Bayesian estimate under quadratic loss.*

(i) ***Conjugate prior for p***

The likelihood of observing $x_1, \dots x_n$ is given by:

$$L(p) = \frac{\Gamma(k+x_1)}{\Gamma(x_1+1)\Gamma(k)} p^k (1-p)^{x_1} \times \cdots \times \frac{\Gamma(k+x_n)}{\Gamma(x_n+1)\Gamma(k)} p^k (1-p)^{x_n}$$

$$\propto p^{nk}(1-p)^{\sum x_i} \qquad [1]$$

If the prior distribution for p is a $beta(\alpha,\beta)$, then its PDF is:

$$f(p) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} p^{\alpha-1}(1-p)^{\beta-1} \quad \propto \quad p^{\alpha-1}(1-p)^{\beta-1} \qquad [½]$$

So the PDF of the posterior distribution is given by:

$$f(p \mid \underline{x}) \propto \text{prior} \times \text{likelihood} = p^{\alpha-1}(1-p)^{\beta-1} \times p^{nk}(1-p)^{\sum x_i}$$

$$\propto p^{nk+\alpha-1}(1-p)^{\sum x_i + \beta - 1} \qquad [½]$$

This has the form of a $beta(\alpha^*, \beta^*)$ distribution, where $\alpha^* = nk + \alpha$ and $\beta^* = \sum x_i + \beta$. [½]

Both the posterior and the prior have beta distributions. Therefore the conjugate prior is indeed a beta distribution. [½]

[Total 3]

Alternatively students could adopt the following approach, where it is not necessary to do quite so much in terms of mathematical workings:

The likelihood of observing $x_1, \dots x_n$ is given by:

$$L(p) = \frac{\Gamma(k+x_1)}{\Gamma(x_1+1)\Gamma(k)} p^k (1-p)^{x_1} \times \cdots \times \frac{\Gamma(k+x_n)}{\Gamma(x_n+1)\Gamma(k)} p^k (1-p)^{x_n}$$

$$\propto p^{nk}(1-p)^{\sum x_i} \qquad [1]$$

The likelihood is of the form of a $beta\left(nk+1, \sum x_i + 1\right)$. [1]

The posterior PDF is proportional to the prior PDF multiplied by the likelihood. If the prior and the likelihood have beta forms, then the posterior must also have a beta distribution. [½]

Therefore the conjugate prior is indeed a beta distribution. [½]

[Total 3]

(ii) ***Bayesian estimate for p under quadratic loss***

Substituting in the given values we have a posterior distribution of $beta(\alpha^*, \beta^*)$ where:

$$\alpha^* = nk + \alpha = 12 \times 2 + 3 = 27 \quad \beta^* = \sum x_i + \beta = 8 + 4 = 12 \qquad [½]$$

The Bayesian estimate for p under quadratic loss is the mean of the posterior:

$$\frac{\alpha^*}{\alpha^* + \beta^*} = \frac{27}{39} = 0.692 \qquad [½]$$

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Solution X1.5

Comment

This question applies the Core Reading in Chapter 1 on statistical games and decision criteria. Part (i) tests the Bayes criterion solution and Part (ii) tests the minimax solution. Note that this question is Subject 106, April 2000, Question 1.

(i) ***Bayes criterion solution***

The total profit (in £000s) corresponding to each level of client-days is:

	θ_1	θ_2	θ_3
d_1	1,360	1,520	1,760
d_2	1,407	1,541	1,742
d_3	1,250	1,350	1,500

[1]

Notice that strategy d_3 is dominated by the two other strategies so we can discard it.

The expected total profit (in £000s) for each level of client-days is:

$$E(\text{profit under } d_1) = (1,360 \times 0.1) + (1,520 \times 0.6) + (1,760 \times 0.3) = 1,576$$ [½]

$$E(\text{profit under } d_2) = (1,407 \times 0.1) + (1,541 \times 0.6) + (1,742 \times 0.3) = 1,587.9$$ [½]

The Bayes criterion solution selects the client-days with the highest expected profit. Thus the solution is d_2, with an expected profit of £1,587,900. [1]

[Total 3]

(ii) ***Minimax and maximax solutions***

Remember that strategy d_3 is dominated by the two other strategies so we can discard it. The worst possible outcomes associated with d_1 and d_2 are:

1,360 1,407

The minimax solution is the best worst case scenario (*ie* maximises the minimum gain) which is 1,407, so the minimax solution is d_2. [1]

The best possible outcomes associated with d_1 and d_2 are:

1,760 1,742

The maximax solution is the best best case scenario (*ie* maximises the maximum gain) which is 1,760, so the maximax solution is d_1. [1]

[Total 2]

Solution X1.6

> **Comment**
>
> *This question applies the Core Reading in Chapter 2, on the derivation of the posterior distribution where the prior distribution is uninformative.*

The likelihood function for μ is given by:

$$L(\mu)=\frac{1}{x_1\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{\ln x_n-\mu}{\sigma}\right)^2}\times\cdots\times\frac{1}{x_n\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{\ln x_n-\mu}{\sigma}\right)^2}\propto e^{-\frac{1}{2}\sum\left(\frac{\ln x_i-\mu}{\sigma}\right)^2} \quad [1]$$

Since the prior distribution for μ is uninformative, we take the prior distribution to be constant. *Alternatively, we could say* $f(\mu)=1/k$ *where* $k\to\infty$. [½]

So the PDF of the posterior distribution for μ is given by:

$$f(\mu\,|\,\underline{x})\propto \text{prior}\times\text{likelihood}=\text{constant}\times e^{-\frac{1}{2}\sum\left(\frac{\ln x_i-\mu}{\sigma}\right)^2}\propto e^{-\frac{1}{2}\sum\left(\frac{\ln x_i-\mu}{\sigma}\right)^2} \quad [½]$$

We now want to write this as the PDF of a normal distribution, with μ as the variable. So we want it to look like:

$$f(\mu\,|\,\underline{x})\propto e^{-\frac{1}{2}\left(\frac{\mu-\mu_*}{\sigma_*^2}\right)^2}=e^{-\frac{1}{2\sigma_*^2}(\mu^2-2\mu_*\mu+\mu_*^2)} \quad (*)$$

So, expanding the bracket in our posterior and summing the terms (ignoring any expressions that do not involve μ, since these can be absorbed into the constant term):

$$f(\mu\,|\,\underline{x}) \propto e^{-\frac{1}{2\sigma^2}\sum(\ln x_i-\mu)^2} = e^{-\frac{1}{2\sigma^2}\sum\left((\ln x_i)^2-2\mu\ln x_i+\mu^2\right)}$$

$$= e^{-\frac{1}{2\sigma^2}\left(\sum(\ln x_i)^2-2\mu\sum\ln x_i+n\mu^2\right)} \qquad [½]$$

$$\propto e^{-\frac{1}{2\sigma^2}\left(-2\mu\sum\ln x_i+n\mu^2\right)}$$

$$= e^{-\frac{n}{2\sigma^2}\left(\mu^2-2\mu\frac{\sum\ln x_i}{n}\right)} \qquad [1]$$

Completing the square (the missing term is absorbed into the constant):

$$f(\mu\,|\,\underline{x}) \propto e^{-\frac{1}{2(\sigma^2/n)}\left(\mu-\frac{\sum\ln x_i}{n}\right)^2} \qquad [1]$$

Comparing this with equation $(*)$, we see that we have a normal distribution with parameters:

$$\mu_* = \frac{\sum\ln x_i}{n} \quad \text{and} \quad \sigma_*^2 = \frac{\sigma^2}{n} \qquad [½]$$

[Total 5]

Solution X1.7

> ***Comment***
>
> *This question applies the Core Reading in Chapter 3 on mixture distributions. The mixture distribution is also known as the marginal distribution. Marginal distributions were introduced in Subject CT3.*

The PDF of the marginal loss distribution is given by:

$$f_X(x) = \int f_{X,c}(x,c)\,dc = \int f_c(c) f_{X|c}(x,c)\,dc \qquad [½]$$

Here this gives us:

$$f_X(x) = \int_0^\infty \frac{\lambda^\alpha}{\Gamma(\alpha)} c^{\alpha-1} e^{-\lambda c} c\gamma x^{\gamma-1} e^{-cx^\gamma}\ dc \qquad [½]$$

We can move terms that do not depend on c outside the integral, so that:

$$f_X(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} \gamma x^{\gamma-1} \int_0^\infty c^\alpha e^{-c(\lambda+x^\gamma)}\, dc \qquad \text{equation (1)} \qquad [½]$$

We note that the integrand has the *form* of the PDF of a gamma distribution. We can transform the integrand into the exact PDF of a $Ga\left(\alpha+1, \lambda+x^\gamma\right)$ distribution as follows:

$$f_X(x) = \frac{\lambda^\alpha}{\left(\lambda+x^\gamma\right)^{\alpha+1}} \frac{\Gamma\left(\alpha+1\right)}{\Gamma\left(\alpha\right)} \gamma x^{\gamma-1} \int_0^\infty \frac{\left(\lambda+x^\gamma\right)^{\alpha+1}}{\Gamma\left(\alpha+1\right)} c^\alpha e^{-c\left(\lambda+x^\gamma\right)}\, dc \qquad [1]$$

The integral part of the expression is equal to 1 so that:

$$f_X(x) = \frac{\lambda^\alpha}{\left(\lambda+x^\gamma\right)^{\alpha+1}} \frac{\Gamma\left(\alpha+1\right)}{\Gamma\left(\alpha\right)} \gamma x^{\gamma-1} \qquad [1]$$

Now $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$, so the PDF of X is:

$$f_X(x) = \frac{\alpha\gamma\,\lambda^\alpha x^{\gamma-1}}{\left(\lambda+x^\gamma\right)^{\alpha+1}} \qquad [½]$$

The range of values for x is $x \geq 0$ as before. So this is the PDF of the Burr distribution (as given on Page 15 of the *Tables*) and we have shown the required result.

[Total 4]

Solution X1.8

Comment

This question applies the Core Reading in Chapter 2. Part (i) tests the derivation of the posterior distribution. Part (ii) tests the calculation of the Bayesian estimate under squared error loss, zero-one error loss and absolute error loss.

(i) ***Posterior distribution for λ***

The likelihood function for λ is:

$$f(\underline{x} \mid \lambda) = \frac{e^{-\lambda}\lambda^{x_1}}{x_1!} \times \cdots \times \frac{e^{-\lambda}\lambda^{x_n}}{x_n!} \quad \propto \quad e^{-n\lambda}\lambda^{\sum x_i} \qquad [1]$$

The prior distribution for λ is:

$$f(\lambda) = \frac{2^5}{\Gamma(5)}\lambda^4 e^{-2\lambda} \quad \propto \quad \lambda^4 e^{-2\lambda} \qquad [½]$$

So the PDF of the posterior distribution for λ is given by:

$$\begin{aligned} f(\lambda \mid \underline{x}) &\propto \text{prior} \times \text{likelihood} = \lambda^4 e^{-2\lambda} \times e^{-n\lambda}\lambda^{\sum x_i} \\ &\propto \lambda^{4+\sum x_i} e^{-(n+2)\lambda} \end{aligned} \qquad [½]$$

This has the form of a $Gamma(\alpha^*, \lambda^*)$ distribution, where $\alpha^* = 5 + \sum x_i$ and $\lambda^* = n+2$. From the data, we have $n = 10$ and $\sum x_i = 12$, so $\alpha^* = 17$ and $\lambda^* = 12$. [1]

[Total 3]

(ii)(a) ***Bayesian estimate for λ under squared error loss***

Using a squared error loss function, the Bayesian estimate for λ is the mean of the posterior distribution. So the Bayesian estimate is given by:

$$\frac{\alpha^*}{\lambda^*} = \frac{17}{12} = 1.4167 \qquad [1]$$

(ii)(b) ***Bayesian estimate for λ under zero-one loss***

Using a zero-one loss function, the Bayesian estimate for λ is the mode of the posterior distribution. The PDF of our $Gamma(17,12)$ posterior is given by:

$$f(\lambda \mid \underline{x}) = \frac{12^{17}}{\Gamma(17)} \lambda^{16} e^{-12\lambda}$$

Taking logs (to make it easier to differentiate):

$$\log f(\lambda \mid \underline{x}) = 17 \log 12 + 16 \log \lambda - 12\lambda - \log\left[\Gamma(17)\right] \qquad [½]$$

Differentiating with respect to λ and setting the result to zero gives us the mode of the posterior and the Bayesian estimate for λ:

$$\frac{d}{d\lambda}[\log f(\lambda \mid \underline{x})] = \frac{16}{\lambda} - 12 = 0 \quad \Rightarrow \quad \lambda = \frac{16}{12} = 1.333 \qquad [1]$$

Checking that we do indeed obtain a maximum:

$$\frac{d^2}{dx^2}[\log f(x)] = -\frac{(\alpha - 1)}{x^2} = -\frac{16}{\lambda^2} < 0 \quad \Rightarrow \quad \max \qquad [½]$$

(ii)(c) ***Bayesian estimate for λ under absolute loss***

Using an absolute error loss function, the Bayesian estimate for λ is the median of the posterior distribution. For $X \sim Gamma(17,12)$, we require the value of m such that:

$$P(X < m) = 0.5 \qquad [½]$$

Multiplying through by $2\lambda^*$ and using $2\lambda^* X \sim \chi^2_{2\alpha^*}$ from Page 12 of the *Tables*:

$$P(2\lambda^* X < 2\lambda^* m) = 0.5 \qquad [½]$$

$$P(\chi^2_{34} < 24m) = 0.5 \qquad [½]$$

From Page 169 of the *Tables* we see that $P(\chi^2_{34} < 33.34) = 0.5$. Hence:

$$24m = 33.34 \quad \Rightarrow \quad m = 1.389$$

So the Bayesian estimate for λ is 1.389. [½]

[Total 5]

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Solution X1.9

Comment

This question applies the Core Reading in Chapter 4 on excess of loss reinsurance. The result relating to the lognormal distribution can be found on Page 18 of the Tables. Its derivation is covered in Chapter 4.

The reinsurer's expected payment amount is given by $E(Z)$, where:

$$E(Z) = \int_{1,000}^{2,000} (x-1,000)f(x)\,dx \quad + \int_{2,000}^{\infty} 1,000f(x)\,dx \qquad [1]$$

$$= \int_{1,000}^{2,000} xf(x)\,dx \quad - \int_{1,000}^{2,000} 1,000f(x)\,dx \quad + \int_{2,000}^{\infty} 1,000f(x)\,dx \qquad [1]$$

$$= \int_{1,000}^{2,000} xf(x)\,dx - 1,000P(1,000 < X < 2,000) + 1,000P(X > 2,000)$$

Now, using the standard results for the lognormal distribution (see Page 18 of the *Tables*), we have:

$$E(Z) = e^{\mu+½\sigma^2}\left[\Phi\left(\frac{\ln 2,000-\mu}{\sigma}-\sigma\right)-\Phi\left(\frac{\ln 1,000-\mu}{\sigma}-\sigma\right)\right]$$
$$-1,000\left[\Phi\left(\frac{\ln 2,000-\mu}{\sigma}\right)-\Phi\left(\frac{\ln 1,000-\mu}{\sigma}\right)\right]$$
$$+1,000\left[1-\Phi\left(\frac{\ln 2,000-\mu}{\sigma}\right)\right] \qquad [2]$$

$$= e^7[\Phi(-0.69955)-\Phi(-1.04612)]-1,000[\Phi(1.30045)-\Phi(0.95388)]$$
$$+1,000[1-\Phi(1.30045)] \qquad [1]$$

$$= e^7(0.24210-0.14776)-1000(0.90328-0.82992)+1000\times(1-0.90328)$$

[½ mark for each of the 4 unique probabilities = 2]

$$= 103.456 - 73.36 + 96.72$$
$$= 126.82 \qquad [1]$$

[Total 8]

Solution X1.10

> ### *Comment*
>
> *This question applies the Core Reading in Chapter 1 on statistical games and decision criteria. Parts (i) and (ii) test the material on decision functions and the risk function. Part (iii) tests the material on the minimax and Bayes criterion solutions.*

(i) ***Four decision functions***

The observation from the $Bin(1, p)$ distribution could be 0 or 1. For each of these observations we can assign either the decision $\hat{p} = 0.2$ or $\hat{p} = 0.4$. This gives us the following 4 decision functions.

		decision functions			
		d_1	d_2	d_3	d_4
observation	0	0.2	0.2	0.4	0.4
x	1	0.2	0.4	0.2	0.4

[1]

(ii) ***Risk function***

The risk function, $R(d_i, \theta_j)$, is the *expected loss* for decision function d_i and state of nature θ_j.

Suppose nature was $p = 0.2$ then the probability of each observed value is:

$$P(X = 0) = 0.8 \text{ and } P(X = 1) = 0.2$$

So the expected loss for d_1 which always selects 0.2 is:

$$\begin{aligned} R(d_1, p = 0.2) &= P(X = 0)L(\hat{p} = 0.2, p = 0.2) + P(X = 1)L(\hat{p} = 0.2, p = 0.2) \\ &= (0.8 \times 0) + (0.2 \times 0) = 0 \end{aligned}$$ [½]

The expected loss for d_2 which selects 0.2 when $x = 0$ and 0.4 when $x = 1$ is:

$$\begin{aligned} R(d_2, p = 0.2) &= P(X = 0)L(\hat{p} = 0.2, p = 0.2) + P(X = 1)L(\hat{p} = 0.4, p = 0.2) \\ &= (0.8 \times 0) + (0.2 \times 200) = 40 \end{aligned}$$ [½]

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The expected loss for d_3 which selects 0.4 when $x=0$ and 0.2 when $x=1$ is:

$$\begin{aligned} R(d_3, p=0.2) &= P(X=0)L(\hat{p}=0.4, p=0.2) + P(X=1)L(\hat{p}=0.2, p=0.2) \\ &= (0.8\times 200) + (0.2\times 0) = 160 \end{aligned}$$ [½]

The expected loss for d_4 which always selects 0.4 is:

$$\begin{aligned} R(d_4, p=0.2) &= P(X=0)L(\hat{p}=0.4, p=0.2) + P(X=1)L(\hat{p}=0.4, p=0.2) \\ &= (0.8\times 200) + (0.2\times 200) = 200 \end{aligned}$$ [½]

Now suppose nature was $p=0.4$ then the probability of each observed value is:

$P(X=0)=0.6$ and $P(X=1)=0.4$

The expected loss for d_1 which always selects 0.2 is now:

$$\begin{aligned} R(d_1, p=0.4) &= P(X=0)L(\hat{p}=0.2, p=0.4) + P(X=1)L(\hat{p}=0.2, p=0.4) \\ &= (0.6\times 100) + (0.4\times 100) = 100 \end{aligned}$$ [½]

The expected loss for d_2 which selects 0.2 when $x=0$ and 0.4 when $x=1$ is now:

$$\begin{aligned} R(d_2, p=0.4) &= P(X=0)L(\hat{p}=0.2, p=0.4) + P(X=1)L(\hat{p}=0.4, p=0.4) \\ &= (0.6\times 100) + (0.4\times -50) = 40 \end{aligned}$$ [½]

The expected loss for d_3 which selects 0.4 when $x=0$ and 0.2 when $x=1$ is now:

$$\begin{aligned} R(d_3, p=0.4) &= P(X=0)L(\hat{p}=0.4, p=0.4) + P(X=1)L(\hat{p}=0.2, p=0.4) \\ &= (0.6\times -50) + (0.4\times 100) = 10 \end{aligned}$$ [½]

The expected loss for d_4 which always selects 0.4 is now:

$$\begin{aligned} R(d_4, p=0.4) &= P(X=0)L(\hat{p}=0.4, p=0.4) + P(X=1)L(\hat{p}=0.4, p=0.4) \\ &= (0.6\times -50) + (0.4\times -50) = -50 \end{aligned}$$ [½]

In tabular form, the expected losses are:

		decision			
		d_1	d_2	d_3	d_4
nature	$p=0.2$	0	40	160	200
	$p=0.4$	100	40	10	–50

[Total 4]

(iii) ***Minimax solution***

The minimax criterion will select the risk function with the smallest maximum loss. This is d_2. [1]

(iv) ***Bayes criterion solution***

We have $P(p=0.2)=\frac{2}{3}$ and $P(p=0.4)=\frac{1}{3}$, so the expected losses are:

$$E(\text{loss under } d_1) = \left(\tfrac{2}{3}\times 0\right)+\left(\tfrac{1}{3}\times 100\right)=33\tfrac{1}{3}$$
$$E(\text{loss under } d_2) = \left(\tfrac{2}{3}\times 40\right)+\left(\tfrac{1}{3}\times 40\right)=40$$
$$E(\text{loss under } d_3) = \left(\tfrac{2}{3}\times 160\right)+\left(\tfrac{1}{3}\times 10\right)=110$$
$$E(\text{loss under } d_4) = \left(\tfrac{2}{3}\times 200\right)+\left(\tfrac{1}{3}\times -50\right)=116\tfrac{2}{3}$$ [½ mark each]

The Bayes criterion solution is the decision function with the lowest expected loss. This is d_1. [1]

[Total 3]

Solution X1.11

Comment

This question applies the Core Reading in Chapter 4 on excess of loss reinsurance. Part (i) tests the material relating to the insurer's claims distribution. Part (ii) tests excess of loss reinsurance with inflation and the reinsurer's conditional claims distribution. There are several ways of completing Parts (i)(b) and (ii)(b). It is worth looking through our alternative solutions as you may find a more efficient approach. Markers, please give credit for any other sensible alternatives.

(i)(a) ***Probability claim involves the reinsurer***

Using the CDF from Page 14 of the *Tables*, we have:

$$P(X > 3,000) = 1 - F(3,000) = \left(\frac{7,500}{7,500 + 3,000}\right)^4 = 0.260308 \qquad [1]$$

(i)(b) ***Mean amount paid by the insurer***

The insurer pays Y where:

$$Y = \begin{cases} X & X < 3,000 \\ 3,000 & X > 3,000 \end{cases} \qquad [½]$$

So the expectation of Y is given by:

$$E(Y) = \int_0^{3,000} xf(x)\,dx + \int_{3,000}^{\infty} 3,000 f(x)\,dx \qquad [½]$$

$$= \int_0^{3,000} x\frac{4 \times 7,500^4}{(7,500 + x)^5}\,dx + 3,000P(X > 3,000)$$

We can evaluate the first integral using integration by parts. We can also use $P(X > 3,000)$ from Part (i)(a).

$$E(Y)=\left[-x\frac{7,500^4}{(7,500+x)^4}\right]_0^{3,000}+\int_0^{3,000}\frac{7,500^4}{(7,500+x)^4}\,dx+(3,000\times 0.260308) \qquad [1]$$

$$=-3,000\frac{7,500^4}{10,500^4}+\left[-\frac{7,500^4}{3(7,500+x)^3}\right]_0^{3,000}+780.925$$

$$=\cancel{-780.925}+\left[-\frac{7,500^4}{3\times 10,500^3}+\frac{7,500^4}{3\times 7,500^3}\right]+\cancel{780.925} \qquad [1]$$

$$=-911.079+2,500$$

$$=£1,588.92 \qquad [1]$$

[Total 5]

Alternatively, students could do integration by substitution:

The insurer pays Y where:

$$Y=\begin{cases} X & X<3,000 \\ 3,000 & X>3,000 \end{cases} \qquad [½]$$

So the expectation of Y is given by:

$$E(Y)=\int_0^{3,000} x\frac{4\times 7,500^4}{(7,500+x)^5}\,dx+3,000P(X>3,000) \qquad [½]$$

We use the substitution $u=7,500+x$ to obtain:

$$E(Y)=\int_{7,500}^{10,500}(u-7,500)\frac{4\times 7,500^4}{u^5}\,du+(3,000\times 0.260308) \qquad [1]$$

$$=\int_{7,500}^{10,500}\frac{4\times 7,500^4}{u^4}-\frac{4\times 7,500^5}{u^5}\,du+780.925$$

$$=\left[-\frac{4\times 7,500^4}{3u^3}+\frac{7,500^5}{u^4}\right]_{7,500}^{10,500}+780.925 \qquad [1]$$

$$=\left[-1,692.003-(-2,500)\right]+780.925=£1,588.92 \qquad [1]$$

[Total 5]

Alternatively, students could proceed by noting that the insurer's mean claim amount is the original claim amount less the reinsurer's mean claim amount:

The reinsurer Z where:

$$Z = \begin{cases} 0 & X < 3{,}000 \\ X - 3{,}000 & X > 3{,}000 \end{cases}$$ [½]

Now $E(Y) = E(X) - E(Z)$.

Since $X \sim Pa(4, 7500)$ then $E(X) = \dfrac{7{,}500}{4-1} = 2{,}500$ [½]

The reinsurer's mean claim amount on all claims is:

$$E(Z) = \int_{3{,}000}^{\infty} (x - 3{,}000) \frac{4 \times 7{,}500^4}{(7{,}500 + x)^5} \, dx$$ [½]

Using the substitution $u = x - 3{,}000$ gives:

$$E(Z) = \int_0^{\infty} u \frac{4 \times 7{,}500^4}{(10{,}500 + u)^5} \, du$$ [½]

We can transform the integrand into u multiplied by the PDF of a $Pa(4, 10500)$ distribution as follows:

$$E(Z) = \frac{7{,}500^4}{10{,}500^4} \int_0^{\infty} u \frac{4 \times (10{,}500)^4}{(10{,}500 + u)^5} \, du$$ [1]

The integral part of the expression above is the mean of a $Pa(4, 10500)$ distribution, which is $\dfrac{10{,}500}{4-1} = 3{,}500$ so that:

$$E(Z) = \frac{7{,}500^4}{10{,}500^4} \times 3{,}500 = 911.08$$ [½]

Finally $E(Y) = E(X) - E(Z) = £2{,}500 - £911.08 = £1{,}588.92$. [½]

[Total 5]

(ii)(a) ***Probability claim involves the reinsurer (with inflation)***

Claims are now $1.1X$, so using the CDF from Page 14 of the *Tables*, we have:

$$P(1.1X > 3{,}000) = P\left(X > \tfrac{3{,}000}{1.1}\right) = 1 - F\left(\tfrac{3{,}000}{1.1}\right)$$

$$= \left(\frac{7{,}500}{7{,}500 + \frac{3{,}000}{1.1}}\right)^4 = 0.28920 \qquad [1]$$

(ii)(b) ***Reinsurer's conditional mean with inflation***

Before inflation, the reinsurer pays Z where:

$$Z = \begin{cases} 0 & X < 3{,}000 \\ X - 3{,}000 & X > 3{,}000 \end{cases}$$

With 10% inflation, the reinsurer pays:

$$Z' = \begin{cases} 0 & 1.1X < 3{,}000 \quad \left(ie\ X < \frac{3{,}000}{1.1}\right) \\ 1.1X - 3{,}000 & 1.1X > 3{,}000 \quad \left(ie\ X > \frac{3{,}000}{1.1}\right) \end{cases} \qquad [1]$$

So the expectation of Z' is given by:

$$E(Z') = \int_{\frac{3{,}000}{1.1}}^{\infty} (1.1x - 3{,}000) f(x)\, dx = \int_{\frac{3{,}000}{1.1}}^{\infty} (1.1x - 3{,}000) \frac{4 \times 7{,}500^4}{(7{,}500 + x)^5}\, dx \qquad [1]$$

Using integration by parts, we get:

$$E(Z') = \left[-(1.1x - 3{,}000)\frac{7{,}500^4}{(7{,}500 + x)^4}\right]_{\frac{3{,}000}{1.1}}^{\infty} + \int_{\frac{3{,}000}{1.1}}^{\infty} 1.1 \frac{7{,}500^4}{(7{,}500 + x)^4}\, dx \qquad [1]$$

$$= 0 + \left[-1.1\frac{7{,}500^4}{3(7{,}500 + x)^3}\right]_{\frac{3{,}000}{1.1}}^{\infty}$$

$$= 0 - \left(-1.1\frac{7{,}500^4}{3\left(7{,}500 + \frac{3{,}000}{1.1}\right)^3}\right) = £1{,}084.52 \qquad [1]$$

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This is the mean amount paid by the reinsurer on *all* claims. To calculate the mean amount paid by the reinsurer on claims which involve the reinsurer we require the reinsurer's *conditional* mean:

$$E\left(Z' \mid X > \tfrac{3,000}{1.1}\right) = \frac{E(Z')}{P\left(X > \tfrac{3,000}{1.1}\right)} = \frac{1,084.52}{0.28920} = 3,750 \qquad [1]$$

[Total 6]

Alternatively, students could do integration by substitution – method (1):

With 10% inflation, the reinsurer pays:

$$Z' = \begin{cases} 0 & 1.1X < 3,000 \quad \left(ie\ X < \tfrac{3,000}{1.1}\right) \\ 1.1X - 3,000 & 1.1X > 3,000 \quad \left(ie\ X > \tfrac{3,000}{1.1}\right) \end{cases} \qquad [1]$$

And $$E(Z') = \int_{\frac{3,000}{1.1}}^{\infty} (1.1x - 3,000) f(x)\, dx = \int_{\frac{3,000}{1.1}}^{\infty} (1.1x - 3,000) \frac{4 \times 7,500^4}{(7,500 + x)^5}\, dx \qquad [1]$$

We use the substitution $u = 7,500 + x$ to obtain:

$$E(Z') = \int_{\frac{3,000}{1.1}+7,500}^{\infty} (1.1u - 11,250) \frac{4 \times 7,500^4}{u^5}\, du$$

$$= \int_{\frac{3,000}{1.1}+7,500}^{\infty} \frac{1.1 \times 4 \times 7,500^4}{u^4} - 11,250 \frac{4 \times 7,500^4}{u^5}\, du \qquad [1]$$

$$= \left[-\frac{1.1 \times 4 \times 7,500^4}{3u^3} + 11,250 \frac{7,500^4}{u^4} \right]_{\frac{3,000}{1.1}+7,500}^{\infty}$$

$$= \left[0 - (-1,084.52)\right] = £1,084.52 \qquad [1]$$

The reinsurer's *conditional* mean is:

$$E\left(Z' \mid X > \tfrac{3,000}{1.1}\right) = \frac{E(Z')}{P\left(X > \tfrac{3,000}{1.1}\right)} = \frac{1,084.52}{0.28920} = 3,750 \qquad [1]$$

[Total 6]

Alternatively, students could do integration by substitution – method (2):

With 10% inflation, the reinsurer pays:

$$Z' = \begin{cases} 0 & 1.1X < 3{,}000 \quad \left(ie\ X < \frac{3{,}000}{1.1}\right) \\ 1.1X - 3{,}000 & 1.1X > 3{,}000 \quad \left(ie\ X > \frac{3{,}000}{1.1}\right) \end{cases}$$ [1]

And $$E(Z') = \int_{\frac{3{,}000}{1.1}}^{\infty} (1.1x - 3{,}000) f(x)\,dx = \int_{\frac{3{,}000}{1.1}}^{\infty} (1.1x - 3{,}000) \frac{4 \times 7{,}500^4}{(7{,}500 + x)^5}\,dx$$ [1]

We use the substitution $u = 1.1x - 3{,}000$ *to obtain:*

$$E(Z') = \int_0^{\infty} \frac{u}{1.1} \frac{4 \times 7{,}500^4}{\left(7{,}500 + \frac{u + 3{,}000}{1.1}\right)^5}\,du$$ [½]

Multiplying the top and bottom of the integrand by 1.1^5 gives:

$$E(Z') = \int_0^{\infty} u \frac{4 \times (8{,}250)^4}{(11{,}250 + u)^5}\,du$$ [½]

We can transform the integrand into u multiplied by the PDF of a $Pa(4, 11250)$ distribution as follows:

$$E(Z') = \frac{8{,}250^4}{11{,}250^4} \int_0^{\infty} u \frac{4 \times (11{,}250)^4}{(11{,}250 + u)^5}\,du$$ [½]

The integral part of the expression above is now the mean of a $Pa(4, 11250)$, which is $\dfrac{11{,}250}{4-1} = 3{,}750$ so that:

$$E(Z') = \frac{8{,}250^4}{11{,}250^4} \times 3{,}750 = 1{,}084.52$$ [½]

The reinsurer's *conditional* mean is:

$$E\left(Z' \mid X > \tfrac{3,000}{1.1}\right) = \frac{E(Z')}{P\left(X > \tfrac{3,000}{1.1}\right)} = \frac{1,084.52}{0.28920} = 3,750 \qquad [1]$$

[Total 6]

Alternatively, students may remember and use the results from Chapter 4 relating to a Pareto distribution:

If $X \sim Pa(\alpha, \lambda)$ then $X' = kX \sim Pa(\alpha, k\lambda)$. [1]

With $\alpha = 4$, $\lambda = 7,500$ and $k = 1.1$, we have $X' \sim Pa(4, 8250)$. [1]

Also, if $X \sim Pa(\alpha, \lambda)$ and there is a retention level of M, then the reinsurer's conditional distribution is $Z \mid Z > 0 \sim Pa(\alpha, \lambda + M)$. [1]

With $X' \sim Pa(4, 8250)$ and a retention level of 3,000, we have $Z' \mid Z' > 0 \sim Pa(4, 11250)$. [1]

Hence the mean of the reinsurer's conditional distribution is given by:

$$E(Z' \mid Z' > 0) = \frac{11,250}{3} = 3,750 \qquad [1]$$

[Total 6]

Solution X1.12

Comment

This question applies various Core Reading ideas from Chapters 3 and 4. Part (i)(a) tests the material on maximum likelihood estimation where the sample is censored. Part (i)(b) tests the method of percentiles. Part (ii)(a) uses the reinsurer's conditional claims distribution. Part (ii)(b) tests maximum likelihood estimation, this time on non-censored data.

(i)(a) ***MLE***

The likelihood of observing the 7 known claims $(x_1, \dots, x_7)$ and the 3 unknown claims greater than £40,000 is:

$$L(\lambda) = f(x_1) \times \cdots \times f(x_7) \times P(X > 40{,}000)^3 \qquad [½]$$

$$= \lambda e^{-\lambda x_1} \times \cdots \times \lambda e^{-\lambda x_7} \times [1 - F(40{,}000)]^3$$

$$= \lambda^7 e^{-\lambda \sum x_i} \left[e^{-40{,}000\lambda} \right]^3 \qquad [½]$$

$$= \lambda^7 e^{-152{,}749\lambda} e^{-120{,}000\lambda} \qquad [½]$$

$$= \lambda^7 e^{-272{,}749\lambda} \qquad [½]$$

The log-likelihood is:

$$\ln L(\lambda) = 7 \ln \lambda - 272{,}749\lambda \qquad [½]$$

Differentiating with respect to λ:

$$\frac{d}{d\lambda} \ln L(\lambda) = \frac{7}{\lambda} - 272{,}749 \qquad [½]$$

Setting the derivative equal to zero and rearranging gives:

$$\frac{7}{\hat{\lambda}} - 272{,}749 = 0 \quad \Rightarrow \quad \hat{\lambda} = \frac{7}{272{,}749} = 0.0000257 \qquad [½]$$

Checking that we have a maximum:

$$\frac{d^2}{d\lambda^2} \ln L(\lambda) = -\frac{7}{\lambda^2} < 0 \quad \Rightarrow \quad \max \qquad [½]$$

(i)(b) ***Method of percentiles***

The sample median of our 10 claims is the $\tfrac{1}{2}(10+1)=5\tfrac{1}{2}$th value which by interpolation is:

$$\tfrac{1}{2}(28,506+36,834)=32,670 \qquad [1]$$

The population median satisfies:

$$F(m)=P(X<m)=1-e^{-\lambda m}=\tfrac{1}{2} \qquad [1]$$

Substituting the sample median into the above equation gives:

$$1-e^{-32,670\lambda}=\tfrac{1}{2} \quad \Rightarrow \quad \lambda=-\frac{\ln \tfrac{1}{2}}{32,670}=0.0000212 \qquad [1]$$

[Total 7]

(ii)(a) ***Conditional distribution***

Let Y be the amount paid by the insurer on a claim. Since the insurer only makes a payment if a claim is greater than the excess, Y has the conditional distribution:

$$Y=X-50,000 \mid X>50,000 \qquad [\tfrac{1}{2}]$$

The PDF of Y is given by:

$$g(y)=\frac{f(x)}{P(X>50,000)} \qquad x>50,000 \qquad [\tfrac{1}{2}]$$

$$\Rightarrow \quad g(y)=\frac{f(y+50,000)}{P(X>50,000)} \qquad y>0 \qquad [\tfrac{1}{2}]$$

Now $\quad P(X>50,000)=1-F(50,000)=\left(\dfrac{200,000}{200,000+50,000}\right)^{\theta} \qquad [\tfrac{1}{2}]$

$$f(y+50,000)=\frac{\theta\times 200,000^{\theta}}{(200,000+y+50,000)^{\theta+1}}=\frac{\theta\times 200,000^{\theta}}{(250,000+y)^{\theta+1}} \qquad [\tfrac{1}{2}]$$

$$\Rightarrow \quad g(y)=\frac{\theta\times 250,000^{\theta}}{(250,000+y)^{\theta+1}} \qquad y>0 \qquad [\tfrac{1}{2}]$$

This is the PDF of a Pareto distribution with parameters θ and 250,000.

(ii)(b) ***MLE***

The likelihood function of observing the sample data is given by:

$$L(\theta) = g(y_1) \times \cdots \times g(y_5) = \frac{\theta \times 250{,}000^{\theta}}{(250{,}000 + y_1)^{\theta+1}} \times \cdots \times \frac{\theta \times 250{,}000^{\theta}}{(250{,}000 + y_5)^{\theta+1}}$$

$$= \frac{\theta^5 \times 250{,}000^{5\theta}}{\prod (250{,}000 + y_i)^{(\theta+1)}} \qquad [1]$$

The log-likelihood is given by:

$$\ln L(\theta) = 5\ln\theta + 5\theta \ln 250{,}000 - (\theta+1)\ln\left[\prod (250{,}000 + y_i)\right]$$

$$= 5\ln\theta + 5\theta \ln 250{,}000 - (\theta+1)\sum \ln(250{,}000 + y_i) \qquad [½]$$

Differentiating with respect to θ :

$$\frac{d}{d\theta}\ln(\theta) = \frac{5}{\theta} + 5\ln 250{,}000 - \sum \ln(250{,}000 + y_i) \qquad [½]$$

For the MLE, set the derivative equal to zero and solve the equation. So:

$$\frac{5}{\theta} + 5\ln 250{,}000 = \sum \ln(250{,}000 + y_i)$$

$$\hat{\theta} = \frac{5}{\sum \ln(250{,}000 + y_i) - 5\ln 250{,}000} \qquad [½]$$

By calculation, $\sum \ln(250{,}000 + y_i) = 64.370$, so that [½]

$$\hat{\theta} = \frac{5}{64.370 - 5 \times 12.4292} = 2.25 \ \ (3 \text{ SF}) \qquad [½]$$

Now check that this does in fact give a maximum:

$$\frac{d^2}{d\theta^2}\ln L(\theta) = -\frac{5}{\theta^2} < 0 \quad \Rightarrow \quad \max \qquad [½]$$

[Total 7]

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Assignment X2 Solutions

Solution X2.1

Comment

This question applies the Core Reading in Chapter 8 on the collective risk model with excess of loss reinsurance. Note that this question is Subject 106, September 2001, Question 5. There are several ways of completing the integration in this question. It is worth looking through our alternative solution as you may find it more efficient. Markers, please give credit for any other correct alternatives.

If X_i is the ith individual claim amount before reinsurance, then the amount paid by the insurer on this individual claim is:

$$Y_i = \begin{cases} X_i & X_i < 200 \\ M & X_i \geq 200 \end{cases}$$

And $S_I = Y_1 + Y_2 + \cdots + Y_N$.

where $N \sim Bin(1000, 0.01)$ and $X_i \sim Exp(0.01)$.

We have:

$$E(N) = 1{,}000 \times 0.01 = 10 \qquad [½]$$

$$E(Y) = \int_0^{200} xf(x)\,dx + \int_{200}^{\infty} 200 f(x)\,dx \qquad [1]$$

$$= \int_0^{200} x0.01e^{-0.01x}\,dx + 200\left[1 - F(200)\right]$$

$$= \left[-xe^{-0.01x}\right]_0^{200} + \int_0^{200} e^{-0.01x}\,dx + 200e^{-2} \qquad [½]$$

$$= \cancel{-200e^{-2}} + \left[-100e^{-0.01x}\right]_0^{200} + \cancel{200e^{-2}} \qquad [½]$$

$$= 100(1 - e^{-2})$$

$$= 86.466 \qquad [½]$$

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Hence, using the formula on Page 16 of the *Tables*, we have:

$$E(S_I) = E(N)E(Y) = 10 \times 86.466 = £864.66$$ [1]

[Total 4]

Alternatively, students could proceed by noting that the insurer's mean individual claim amount is the original individual claim amount less the reinsurer's individual mean claim amount:

We have:

$$E(N) = 1{,}000 \times 0.01 = 10$$ [½]

The amount paid by the reinsurer on an individual claim is:

$$Z_i = \begin{cases} 0 & X_i < 200 \\ X_i - M & X_i \geq 200 \end{cases}$$

Now $E(Y) = E(X) - E(Z)$.

Since $X \sim Exp(0.01)$ then $E(X) = 100$. [½]

The reinsurer's individual mean claim amount is:

$$\begin{aligned} E(Z) &= \int_{200}^{\infty} (x-200)\, f(x)\, dx \\ &= \int_{200}^{\infty} (x-200)\, 0.01e^{-0.01x}\, dx \end{aligned}$$ [½]

Using the substitution $u = x - 200$ gives:

$$\begin{aligned} E(Z) &= \int_0^{\infty} u\, 0.01e^{-0.01(u+200)}\, du \\ &= e^{-2} \int_0^{\infty} u\, 0.01e^{-0.01u}\, du \end{aligned}$$ [½]

The integral part of the expression above is the mean of an $Exp(0.01)$ distribution, which is 100 so that $E(Z) = 100e^{-2}$. [½]

Finally $E(Y) = E(X) - E(Z) = 100 - 100e^{-2} = 86.466$. [½]

Hence, using the formula on Page 16 of the *Tables*, we have:

$$E(S_I) = E(N)E(Y) = 10 \times 86.466 = £864.66$$ [1]

[Total 4]

Solution X2.2

> **Comment**
>
> *This question applies assumed knowledge from Subject CT3 on conditional means and variances. In recent Subject CT6 exam papers, there have been many questions involving conditioning. Note that this question is Subject 106, September 2001, Question 2. The key results needed to answer this question can be found on Page 16 of the Tables.*

Using $E(Y) = E\left[E(Y \mid X)\right]$ from Page 16 of the *Tables*, we get:

$$E(Y) = E\left[E(Y \mid X)\right] = E(2X + 400) = 2E(X) + 400$$ [1]

But $E(X) = 50$, so:

$$E(Y) = 2 \times 50 + 400 = 500$$ [1]

Using $\text{var}(Y) = E\left[\text{var}(Y \mid X)\right] + \text{var}\left[E(Y \mid X)\right]$ from Page 16 of the *Tables*, we get:

$$\text{var}(Y) = E\left(\frac{X^2}{2}\right) + \text{var}(2X + 400)$$

$$= ½\left[E\left(X^2\right)\right] + 4\,\text{var}(X)$$ [1]

$$= ½\left(14^2 + 50^2\right) + 4 \times 14^2 = 2{,}132$$ [1]

So the standard deviation of Y is 46.17. [1]

[Total 5]

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Alternatively, students may use $\text{var}(Y) = E\left(Y^2\right) - \left[E(Y)\right]^2$

Using $E(Y) = E\left[E(Y \mid X)\right]$ from Page 16 of the *Tables*, we get:

$$E(Y) = E\left[E(Y \mid X)\right] = E(2X + 400) = 2E(X) + 400 \qquad [1]$$

But $E(X) = 50$, so:

$$E(Y) = 2 \times 50 + 400 = 500 \qquad [1]$$

Similarly:

$$\begin{aligned} E\left(Y^2\right) &= E\left[E\left(Y^2 \mid X\right)\right] \\ &= E\left[\text{var}(Y \mid X) + \left[E(Y \mid X)\right]^2\right] \end{aligned} \qquad [½]$$

Substituting in the results for $\text{var}(Y \mid X)$ and $E(Y \mid X)$ gives:

$$\begin{aligned} E\left(Y^2\right) &= E\left[\frac{X^2}{2} + (2X + 400)^2\right] \\ &= \frac{9}{2}E\left(X^2\right) + 1{,}600E(X) + 160{,}000 \\ &= \frac{9}{2}\left(14^2 + 50^2\right) + 1{,}600 \times 50 + 160{,}000 = 252{,}132 \end{aligned} \qquad [1]$$

Finally:

$$\begin{aligned} \text{var}(Y) &= E\left(Y^2\right) - \left[E(Y)\right]^2 \\ &= 252{,}132 - 500^2 = 2{,}132 \end{aligned} \qquad [½]$$

So the standard deviation of Y is 46.17. [1]

[Total 5]

Solution X2.3

> ### *Comment*
>
> *This question tests and applies the proof from Chapter 7 that the sum of compound Poisson random variables is also a compound Poisson random variable. This proof has been tested on a few occasions in the Subject CT6 exam.*

(i) ***Proof***

Using the formula from Page 16 of the *Tables*, we have:

$$M_S(t) = e^{\lambda[M_X(t)-1]} \quad [½]$$

$$M_T(t) = e^{\mu[M_Y(t)-1]} \quad [½]$$

The MGF of U is:

$$\begin{aligned} M_U(t) &= E(e^{tU}) = E(e^{t(S+T)}) \\ &= E(e^{tS})E(e^{tT}) \qquad \text{since } S \text{ and } T \text{ are independent} \\ &= M_S(t)M_T(t) \qquad [½] \\ &= e^{\lambda[M_X(t)-1]}e^{\mu[M_Y(t)-1]} \\ &= e^{(\lambda+\mu)\left[\frac{\lambda}{\lambda+\mu}M_X(t)+\frac{\mu}{\lambda+\mu}M_Y(t)-1\right]} \qquad [½] \end{aligned}$$

This is in the same format as the MGF of a compound Poisson distribution, hence by the uniqueness property of MGFs, U has a compound Poisson distribution with Poisson parameter $\lambda + \mu$ and individual claim size MGF of $M_Z(t) = \frac{\lambda}{\lambda+\mu}M_X(t) + \frac{\mu}{\lambda+\mu}M_Y(t)$. [1]

[Total 3]

(ii) ***Claim size distribution***

From Part (i), we see that the MGF of the individual claim size is a weighted average of the MGFs of X and Y, $M_Z(t) = \frac{3}{5}M_X(t) + \frac{2}{5}M_Y(t)$. Similarly, the individual claim size distribution will also be weighted average of the two individual probabilities:

$$P(Z = x) = \tfrac{3}{5}P(S = x) + \tfrac{2}{5}P(T = x)$$

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So the individual claim size distribution of U is given by:

$$P(Z=1)=\left(\tfrac{3}{5}\times\tfrac{1}{3}\right)+\left(\tfrac{2}{5}\times\tfrac{1}{2}\right)=\tfrac{2}{5} \qquad [1]$$

$$P(Z=2)=\left(\tfrac{3}{5}\times\tfrac{1}{3}\right)+\left(\tfrac{2}{5}\times\tfrac{1}{2}\right)=\tfrac{2}{5} \qquad [1]$$

$$P(Z=3)=\left(\tfrac{3}{5}\times\tfrac{1}{3}\right)+\left(\tfrac{2}{5}\times 0\right)=\tfrac{1}{5} \qquad [1]$$

[Total 3]

Solution X2.4

> ***Comment***
>
> *This question applies the Core Reading from Chapter 7 on MGFs of compound distributions. The question also requires knowledge of MGFs, PGFs, conditional means and mixture (or marginal) distributions. This is assumed knowledge from Subject CT3. However, there is a recap on MGFs and PGFs in Chapter 0, the conditional mean result is given on Page 16 of the Tables and mixture distributions are covered in Chapter 3. Note that this question is Subject 106, April 2001, Question 4.*

(i) ***MGF of S***

The moment generating function of S is given by:

$$M_S(t)=E(e^{tS})=E\left[E(e^{tS}\mid N)\right] \qquad [½]$$

But:

$$\begin{aligned}E(e^{tS}\mid N=n)&=E(e^{t(X_1+\ldots+X_n)})\\&=E(e^{tX_1})E(e^{tX_2})\ldots E(e^{tX_n}) && \text{since } X_1,\ldots,X_n \text{ independent}\\&=\left[M_X(t)\right]^n && \text{since } X_1,\ldots,X_n \text{ identical}\end{aligned}$$

It follows therefore that:

$$E(e^{tS}\mid N)=\left[M_X(t)\right]^N \qquad [½]$$

and so $M_S(t)=E\left[E(e^{tS}\mid N)\right]=E\left\{\left[M_X(t)\right]^N\right\}=G_N\left[M_X(t)\right]$ [1]

[Total 2]

(ii)(a) ***Probability***

We are given that the conditional distribution of $N \mid \lambda$ is $Poisson(\lambda)$. We want to find the unconditional (or mixture, or marginal) distribution of N. The probability function of this marginal distribution is given by:

$$P(N=n)=\int_{\lambda} P(N=n \mid \lambda) f(\lambda)\, d\lambda \qquad [½]$$

Here we have:

$$P(N=n)=\int_0^\infty \frac{e^{-\lambda}\lambda^n}{n!}\times 2e^{-2\lambda}\, d\lambda$$

$$=\frac{2}{n!}\int_0^\infty \lambda^n e^{-3\lambda}\, d\lambda \qquad [½]$$

We note that the integrand has the *form* of the PDF of a gamma distribution. We can transform the integrand into the exact PDF of a $Ga\left(n+1, 3\right)$ distribution as follows:

$$P(N=n)=\frac{2}{3^{n+1}}\int_0^\infty \frac{3^{n+1}}{n!}\lambda^n e^{-3\lambda}\, d\lambda \qquad [1]$$

The integral part of the expression is equal to 1 so that:

$$P(N=n)=\frac{2}{3^{n+1}}, \qquad n=0, 1, 2, \ldots \qquad [1]$$

(ii)(b) ***MGF***

We will use the result from Part (i). We have a compound distribution $S=X_1+\ldots+X_N$, where X is gamma with mean 2 and variance 2, and N has the distribution we derived in Part (ii)(a).

First we find the parameters of the gamma distribution. Using the formulae for the mean and variance of the gamma distribution:

$$\frac{\alpha}{\lambda}=2 \quad \text{and} \quad \frac{\alpha}{\lambda^2}=2$$

Solving these equations we get $\alpha=2$ and $\lambda=1$. [1]

From first principles:

$$G_N(t)=\sum_{n=0}^{\infty} t^n \frac{2}{3^{n+1}}=\tfrac{2}{3}+\tfrac{2}{3^2}t+\tfrac{2}{3^3}t^2+\cdots$$

This is a geometric series with $a=\frac{2}{3}$ and $r=\frac{1}{3}t$, so using $S_\infty=\frac{a}{1-r}$, $|r|<1$ we get:

$$G_N(t)=\frac{\frac{2}{3}}{1-\frac{1}{3}t}=\frac{2}{3-t} \qquad |t|<3$$ [1]

The moment generating function of the $Gamma(2,1)$ distribution is:

$$M_X(t)=(1-t)^{-2}, \qquad t<1$$

So using the formula we derived in Part (i), we get:

$$M_S(t)=G_N\left[M_X(t)\right]=\frac{2}{3-(1-t)^{-2}}$$ [1]

Alternatively, we could write this as:

$$M_S(t)=\frac{2(1-t)^2}{3(1-t)^2-1}=\frac{2-4t+2t^2}{2-6t+3t^2}$$

[Total 6]

Alternatively, students may identify that the probability function of N is a Type 2 negative binomial distribution.

We solve for the parameters α and λ of the gamma distribution for claim amounts:

$$\frac{\alpha}{\lambda}=2 \quad \text{and} \quad \frac{\alpha}{\lambda^2}=2 \quad \Rightarrow \quad \alpha=2 \text{ and } \lambda=1$$ [1]

The probability function for the number of claims from Part (ii)(a) is:

$$P(N=n)=\frac{2}{3^{n+1}}, \qquad n=0,1,2,\ldots$$

This is the probability function of a "Type 2" negative binomial distribution with parameters $p=2/3$, $q=1/3$ and $k=1$. [½]

Using the PGF formula for this distribution from Page 9 of the Tables, we get:

$$G_N(t) = \frac{(2/3)}{1-(1/3)t} = \frac{2}{3-t} \qquad |t|<3$$ [½]

The moment generating function of the $Gamma(2,1)$ distribution is:

$$M_X(t) = (1-t)^{-2}, \qquad t<1$$

So using the formula we derived in Part (i), we get:

$$M_S(t) = G_N\left[M_X(t)\right] = \frac{2}{3-(1-t)^{-2}}$$ [1]

[Total 6]

Alternatively, students may use $M_S(t) = M_N(\ln M_X(t))$ *from Page 16 of the Tables.*

We solve for the parameters α and λ of the gamma distribution for claim amounts:

$$\frac{\alpha}{\lambda} = 2 \quad \text{and} \quad \frac{\alpha}{\lambda^2} = 2 \quad \Rightarrow \quad \alpha = 2 \text{ and } \lambda = 1$$ [1]

The probability function for the number of claims from Part (ii)(a) is:

$$P(N=n) = \frac{2}{3^{n+1}}, \qquad n = 0, 1, 2, \ldots$$

This is the probability function of a "Type 2" negative binomial distribution with parameters $p = 2/3$, $q = 1/3$ and $k = 1$. [½]

Using the MGF formula for this distribution from Page 9 of the Tables, we get:

$$M_N(t) = \frac{(2/3)}{1-(1/3)e^t} = \frac{2}{3-e^t} \qquad |t|<3$$ [½]

The moment generating function of the $Gamma(2,1)$ distribution is:

$$M_X(t) = (1-t)^{-2}, \qquad t<1$$

Using the result $M_S(t) = M_N\left(\ln M_X(t)\right)$ from Page 16 of the *Tables*, we have:

$$M_S(t) = \frac{2}{3 - e^{\ln M_X(t)}}$$

$$= \frac{2}{3-(1-t)^{-2}} \qquad [1]$$

[Total 6]

Solution X2.5

Comment

This question applies the Core Reading from Chapter 6 on EBCT Model 1.

(i) ***Effect on Z***

The credibility factor Z is calculated as:

$$\text{credibility premium} = Z \times \bar{X}_i + (1-Z) \times \bar{X}$$

where n is the number of years of past data. As n increases, Z increases so more emphasis is put on the data from a particular risk (direct data). This makes sense, since the more information we have from the relevant risk, the less emphasis we will wish to place on the collateral data. [1]

$E[s^2(\theta)]$ is the average of the variability *within* each of the different risks in the group. As $E[s^2(\theta)]$ increases, Z decreases so more emphasis is put on the collateral data. This makes sense, since the more variable each individual risk's experience is, the less reliable it is. So we would expect to rely more on the collateral data. [1]

$\text{var}[m(\theta)]$ is a measure of the variability *between* the means of the different risks in the group. As $\text{var}[m(\theta)]$ increases, Z increases so more emphasis is put on the particular risk. This makes sense, since the larger the variability between the different risks, the less relevant the other risks are in assessing the premium of our particular risk. So we want to rely more on the direct data. [1]

[Total 3]

(ii) ***Credibility premium***

The means for each of the risks are:

$$\bar{X}_1 = 728 \quad \bar{X}_2 = 907 \quad \bar{X}_3 = 1{,}150 \quad \bar{X}_4 = 1{,}033$$

The estimators for the parameters are:

$$E[m(\theta)] \approx \bar{X} = \tfrac{1}{4}(728 + 907 + 1{,}150 + 1{,}033) = 954.5 \qquad [1]$$

$$E[s^2(\theta)] \approx \tfrac{1}{4}\sum_{i=1}^{4}\tfrac{1}{2}\sum_{j=1}^{3}(X_{ij} - \bar{X}_i)^2$$

$$= \tfrac{1}{4}(11{,}172 + 10{,}147 + 10{,}900 + 11{,}997) = 11{,}054 \qquad [1]$$

$$\text{var}[m(\theta)] \approx \tfrac{1}{3}\sum_{i=1}^{4}(\bar{X}_i - \bar{X})^2 - \tfrac{1}{3}\left[\tfrac{1}{4}\sum_{i=1}^{4}\tfrac{1}{2}\sum_{j=1}^{3}(X_{ij} - \bar{X}_i)^2\right]$$

$$= \tfrac{1}{3}\Big[(728 - 954.5)^2 + (907 - 954.5)^2 + (1{,}150 - 954.5)^2$$

$$+ (1{,}033 - 954.5)^2\Big] - \tfrac{1}{3} \times 11{,}054$$

$$= \tfrac{1}{3} \times 97{,}941 - \tfrac{1}{3} \times 11{,}054$$

$$= 28{,}962.3 \qquad [1]$$

So, the credibility factor is:

$$Z = \frac{3}{3 + \frac{11{,}054}{28{,}962.3}} = 0.88714$$

Thus the credibility premium for risk 3 is:

$$0.88714 \times 1{,}150 + 0.11286 \times 954.5 = 1{,}128 \qquad [1]$$

[Total 4]

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(iii) ***Addition of a fifth risk***

This risk has a much higher mean, so the variance of the means, $\text{var}[m(\theta)]$, will increase. It has a similar variance to the other risks, so the mean variance of each risk, $E[s^2(\theta)]$, will remain similar. [1]

Hence the proportionately larger $\text{var}[m(\theta)]$ will lead to a larger Z, because more emphasis will be put on the direct data. [1]

[Total 2]

Solution X2.6

> **Comment**
>
> *This question is very similar to one of the Core Reading Examples in Chapter 8. It is an application of the collective risk model but involves expenses as well as claims. Part (i) tests the calculation of the mean and variance and Part (ii) tests the application of a normal approximation to work out a probability.*

(i) ***Mean and variance of S***

Let N denote the number of claims in a year, X denote the amount of an individual claim, and Y denote the expense associated with the claim. Then:

$$S = T_1 + \cdots + T_N \text{ where } T_i = X_i + Y_i$$

S has a compound Poisson distribution with $N \sim Poi(0.25n)$, so using the appropriate formulae from Page 16 of the *Tables*:

$$E(S) = \lambda E(T) = 0.25n\,E(X+Y) = 0.25n\left[E(X)+E(Y)\right] \quad [1]$$

$$= 0.25n\left[\tfrac{1,800}{3} + \tfrac{35+85}{2}\right] = 165n \quad [1]$$

$$\text{var}(S) = \lambda E(T^2) = 0.25n\,E\left[(X+Y)^2\right] = 0.25n\,E\left[X^2 + 2XY + Y^2\right]$$

$$= 0.25n\left[E(X^2) + 2E(X)E(Y) + E(Y^2)\right] \quad [1]$$

Now using the formulae on Pages 13 and 14 of the *Tables*, we get:

$$E(X^2) = \tfrac{\Gamma(4-2)\Gamma(1+2)}{\Gamma(4)}1,800^2 = \tfrac{1!2!}{3!}1,800^2 = 1,080,000 \quad [½]$$

Alternatively, $E(X^2) = \text{var}(X) + E^2(X) = \frac{4(1{,}800)^2}{3^2 \times 2} + 600^2 = 1{,}080{,}000$.

$$E(Y^2) = \frac{1}{85-35}\frac{1}{2+1}(85^3 - 35^3) = 3{,}808\tfrac{1}{3}$$ [½]

Alternatively, $E(Y^2) = \text{var}(Y) + E^2(Y) = \frac{(85-35)^2}{12} + 60^2 = 3{,}808\frac{1}{3}$.

$$\Rightarrow \quad \text{var}(S) = 0.25n\left[1{,}080{,}000 + (2 \times 600 \times 60) + 3{,}808\tfrac{1}{3}\right] = 288{,}952\tfrac{1}{12}n$$ [1]

[Total 5]

Alternatively, students might determine the aggregate claim and expense amount at the policy level and then sum over n policies.

Let N_i denote the number of claims in a year on Policy i, X denote the amount of an individual claim, and Y denote the expense associated with the claim. Then:

$$S_i = T_1 + \cdots + T_{N_i} \text{ where } T_i = X_i + Y_i$$

and $S = S_1 + \cdots + S_n$.

S_i has a compound Poisson distribution with $N \sim Poi(0.25)$, so using the appropriate formulae from Page 16 of the *Tables*:

$$E(S_i) = \lambda E(T) = 0.25\,E(X+Y) = 0.25\left[E(X) + E(Y)\right]$$ [1]

$$= 0.25\left[\tfrac{1{,}800}{3} + \tfrac{35+85}{2}\right] = 165$$ [½]

$$\text{var}(S_i) = \lambda E(T^2) = 0.25\,E\left[(X+Y)^2\right] = 0.25\,E\left[X^2 + 2XY + Y^2\right]$$

$$= 0.25\left[E(X^2) + 2E(X)E(Y) + E(Y^2)\right]$$ [1]

Now using the formulae on Pages 13 and 14 of the *Tables*, we get:

$$E(X^2) = \frac{\Gamma(4-2)\Gamma(1+2)}{\Gamma(4)}1{,}800^2 = \frac{1!2!}{3!}1{,}800^2 = 1{,}080{,}000$$ [½]

Alternatively, $E(X^2) = \text{var}(X) + E^2(X) = \frac{4(1{,}800)^2}{3^2 \times 2} + 600^2 = 1{,}080{,}000$.

$$E(Y^2) = \tfrac{1}{85-35}\tfrac{1}{2+1}(85^3 - 35^3) = 3{,}808\tfrac{1}{3}$$ [½]

Alternatively, $E(Y^2) = \text{var}(Y) + E^2(Y) = \frac{(85-35)^2}{12} + 60^2 = 3{,}808\tfrac{1}{3}$.

$$\Rightarrow \quad \text{var}(S_i) = 0.25\left[1{,}080{,}000 + (2 \times 600 \times 60) + 3{,}808\tfrac{1}{3}\right] = 288{,}952\tfrac{1}{12}$$ [½]

Finally:

$$E(S) = E(S_1) + \cdots + E(S_n) = 165n$$ [½]

$$\text{var}(S) = \text{var}(S_1) + \cdots + \text{var}(S_n) = 288{,}952\tfrac{1}{12}n$$ [½]

The variance result follows if we assume that claims (and hence expenses) occur independently from policy to policy.

[Total 5]

Markers please also give credit if students use $\text{var}(S) = E(N)\,\text{var}(T) + \text{var}(N)E^2(T)$, *where* $\text{var}(T) = \text{var}(X) + \text{var}(Y)$ *as* X *and* Y *are independent (but please point out that using the shortcut compound Poisson formulae is quicker).*

Note that using $S = S_C + S_E$ *where* $S_C = X_1 + \cdots + X_N$ *and* $S_E = Y_1 + \cdots + Y_N$ *will work for* $E(S) = E(S_C) + E(S_E)$ *but since* S_E *and* S_C *are NOT independent (as an expense only occurs if a claim occurs) we can't use* $\text{var}(S) = \text{var}(S_C) + \text{var}(S_E)$.

(ii) ***Number of policies to make a profit***

Using a normal approximation for S, we have:

$$S \dot{\sim} N(165n, 288952n)$$

To make a profit we must have total outgo less than total premium income, *ie* $S < 190n$. So we require:

$$P(S < 190n) = P\left(Z < \frac{190n - 165n}{\sqrt{288{,}952n}}\right) = P\left(Z < 0.04651\sqrt{n}\right)$$ [1]

This probability will be at least 99% as long as:

$$0.04651\sqrt{n} \geq 2.3263$$ [½]

ie $n \geq 2{,}501.9$ [½]

Thus, the smallest value of n is 2,502. [1]

[Total 3]

Solution X2.7

> ### *Comment*
>
> *This question applies the Core Reading from Chapter 6 on EBCT Model 2.*

(i) ***EBCT Model 2 assumptions***

The assumptions are:

- The distribution of each X_j, $j = 1, \ldots, n$, depends on a parameter, θ, whose value is fixed (and the same for all the X_j's) but is unknown
- Given θ, the X_j's are independent (but not necessarily identically distributed)
- $E(X_j \mid \theta)$ does not depend on j
- $P_j \operatorname{var}(X_j \mid \theta)$ does not depend on j.

[½ mark each, total 2]

(ii)(a) ***Proof of unconditional mean***

Using the conditional expectation formula from Page 16 of the *Tables*:

$$E(X_j) = E[E(X_j \mid \theta)] = E[m(\theta)] \quad [1]$$

(ii)(b) ***Proof of unconditional variance***

Using the conditional variance formula from Page 16 of the *Tables*:

$$\begin{aligned}\operatorname{var}(X_j) &= E[\operatorname{var}(X_j \mid \theta)] + \operatorname{var}[E(X_j \mid \theta)] \\ &= E\left[\frac{s^2(\theta)}{P_j}\right] + \operatorname{var}[m(\theta)] \\ &= \frac{1}{P_j} E[s^2(\theta)] + \operatorname{var}[m(\theta)] \quad [1]\end{aligned}$$

[Total 2]

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(iii) ***EBCT Model 2 credibility premium***

Using the formula from Page 30 of the *Tables*, the credibility factor for Risk 1 is:

$$Z_1 = \frac{\sum_{j=1}^{n} P_{1j}}{\sum_{j=1}^{n} P_{1j} + \frac{E[s^2(\theta)]}{\text{var}[m(\theta)]}} = \frac{460}{460 + \frac{2.870}{0.1172}} = 0.9495 \qquad [1]$$

We have $\bar{x}_1 = \frac{980}{460} = 2.130$ and $\bar{x} = 2.275$. Hence the empirical Bayes credibility estimate for Risk 1 *per policy sold* is:

$$\begin{aligned}\text{cred estimate} &= Z_1\bar{x}_1 + (1 - Z_1)\bar{x} \\ &= 0.9495 \times 2.130 + (1 - 0.9495) \times 2.275 \\ &= 2.138 \qquad [1]\end{aligned}$$

Hence, the credibility premium for year 5 is:

$140 \times 2.138 = 299.3$ *ie* £299,300 [1]

[Total 3]

Solution X2.8

> ### *Comment*
>
> *This question applies the Core Reading from Chapter 5 on the binomial/beta model and credibility estimates. It also draws on material from Chapter 2 on the derivation of the posterior distribution and Bayesian estimates and material from Chapter 3 on MLE. Note that this question is Subject 106, April 2001, Question 8.*

(i)(a) ***Posterior distribution***

The prior distribution for θ has the form:

$$\text{Prior}(\theta) \propto \theta^{\beta-1}(1-\theta)^{\beta-1}$$

The random variable X has a binomial distribution with parameters m and θ. So the likelihood function based on a random sample $x_1,\ldots,x_n$ is:

$$\begin{aligned}L(\theta) &= \binom{m}{x_1}\theta^{x_1}(1-\theta)^{m-x_1} \times \binom{m}{x_2}\theta^{x_2}(1-\theta)^{m-x_2} \times \cdots \times \binom{m}{x_n}\theta^{x_n}(1-\theta)^{m-x_n} \\ &\propto \theta^{\sum x_i}(1-\theta)^{mn-\sum x_i}\end{aligned}$$ [1]

The posterior distribution for θ is proportional to the product of the likelihood function and the prior distribution:

$$\begin{aligned}\text{Posterior}(\theta) &\propto \theta^{\beta-1}(1-\theta)^{\beta-1} \times \theta^{\sum x_i}(1-\theta)^{mn-\sum x_i} \\ &= \theta^{\beta-1+\sum x_i}(1-\theta)^{\beta-1+mn-\sum x_i}\end{aligned}$$ [1]

We see that this has the form of another beta distribution, this time with parameters $\beta+\sum x_i$ and $\beta+mn-\sum x_i$. [1]

Note that we have here another conjugate prior distribution. The beta distribution is the conjugate prior in this particular case.

(i)(b) ***Maximum likelihood estimate***

We have already worked out the likelihood function based on the sample data:

$$L(\theta) = C\theta^{\sum x_i}(1-\theta)^{mn-\sum x_i}$$

Taking logs:

$$\log L = K + \sum x_i \log\theta + (mn - \sum x_i)\log(1-\theta)$$ [½]

Differentiating with respect to θ:

$$\frac{d}{d\theta}\log L = \frac{\sum x_i}{\theta} - \frac{mn-\sum x_i}{1-\theta}$$ [1]

Setting this expression equal to zero and rearranging, we get the MLE for θ:

$$\hat{\theta} = \frac{\sum x_i}{mn}$$ [1]

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We can check that this is a maximum by differentiating the log likelihood a second time:

$$\frac{d^2}{d\theta^2}\log L = -\frac{\sum x_i}{\theta^2} - \frac{mn - \sum x_i}{(1-\theta)^2} \qquad [½]$$

Since $mn - \sum x_i$ must be positive, this expression is negative, and the estimate we have found is indeed a maximum.

(i)(c) ***Bayesian estimate***

The Bayesian estimate under quadratic loss is the mean of the posterior distribution. We know that the posterior distribution is another beta distribution with parameters $\beta + \sum x_i$ and $\beta + mn - \sum x_i$. So the Bayesian estimate for θ is (using the formula for the mean of the beta distribution given in the *Tables*):

$$\hat{\theta} = \frac{\beta + \sum x_i}{\beta + \sum x_i + \beta + mn - \sum x_i} = \frac{\beta + \sum x_i}{2\beta + mn} \qquad [1]$$

We need to rewrite this as a credibility estimate:

$$Z\, g(\underset{\sim}{x}) + (1-Z)\mu$$

where μ is the mean of our prior distribution, so $\mu = ½$. [1]

We can split this up into a as follows:

$$\begin{aligned}\frac{\beta + \sum x_i}{2\beta + mn} &= \frac{\sum x_i}{2\beta + mn} + \frac{\beta}{2\beta + mn} \\ &= \frac{mn}{2\beta + mn} \times \frac{\sum x_i}{mn} + \frac{2\beta}{2\beta + mn} \times \frac{1}{2} \\ &= Z \times \frac{\sum x_i}{mn} + (1-Z) \times \mu \qquad [1]\end{aligned}$$

where $Z = \dfrac{mn}{2\beta + mn}$. [1]

(i)(d) ***Increase in data***

As the number of data points increases, both m and n increase (assuming that by data points we mean both the number of policies in the portfolio and also the number of observations in the sample). This means that for a fixed value of β, Z will increase as the amount of data increases, tending ultimately to one. This is not surprising, since we would expect to put more emphasis on the sample data (*ie* give it greater credibility) when the amount of sample data is large. [1]

[Total 11]

(ii)(a) $\boldsymbol{\beta} = 1$

In the first case we have:

$$\hat{\theta} = \frac{1+12}{2+60} = \frac{13}{62} = 0.20968 \qquad [1]$$

and $$Z = \frac{60}{2+60} = \frac{60}{62} = 0.96774 \qquad [1]$$

(ii)(b) $\boldsymbol{\beta} = 4$

In the second case we have:

$$\hat{\theta} = \frac{4+12}{8+60} = \frac{16}{68} = 0.23529 \qquad [1]$$

and $$Z = \frac{60}{8+60} = \frac{60}{68} = 0.88235 \qquad [1]$$

When $\beta = 1$, the prior variance is (using the formula for the variance of the beta distribution given in the *Tables*):

$$\frac{1}{2^2 \times 3} = 1/12 = 0.08333$$

When $\beta = 4$ the prior variance is:

$$\frac{16}{8^2 \times 9} = 0.02778$$

So when β is larger we have a smaller prior variance. This corresponds to the situation where we are more certain about the value of θ. Both prior distributions have a mean of ½, so we think that the value of θ might be somewhere near ½. However, if we choose a larger value for β we are saying that we are more certain that the value of θ is close to ½. [1]

This means that we want to put more emphasis on the prior distribution in our analysis, and less emphasis on the data available from the sample (*ie* the MLE). This means that we need a smaller value for Z, and this is in fact the case. Z is smaller when $\beta = 4$ than when $\beta = 1$. [1]

[Total 6]

Solution X2.9

Comment

This question applies the Core Reading from Chapter 8 on the individual risk model. Note that in this model, we assume a maximum of one claim per policy. Although this question refers explicitly to this model, exam questions are often more implicit. A characteristic of the individual risk model is that there is a maximum of one claim per risk. This is often the case with life insurance so any mention of life insurance, term assurance or whole life assurance in the question is a hint that this model is required.

(i) ***Formula and assumptions***

The formula for the total claims from the portfolio is:

$$S = X_1 + X_2 + \cdots + X_n$$

where X_i is the claim amount from the i th member (which may be zero). [½]

The assumptions underlying this model are:

- there are a fixed number of risks (*ie* members), n [½]
- claims occur independently for each member [½]
- the number of claims for each member is either 0 or 1. [½]

[Total 2]

(ii) ***Mean and variance***

Let $X = bI$, where I is an indicator random variable denoting whether or not a claim is paid, *ie* $P(I=0)=1-q$, $P(I=1)=q$ and b is the fixed benefit amount.

Therefore $E(I)=q$, $\text{var}(I)=q(1-q)$. [1]

So:

$$E(X)=bE(I)=bq$$ [1]

and:

$$\text{var}(X)=b^2\text{var}(I)=b^2q(1-q)$$ [1]

[Total 3]

Alternatively, students may adopt a similar approach but use the formulae for conditional mean and variance.

Let I be an indicator random variable denoting whether or not a claim is paid, *ie* $P(I=0)=1-q$, $P(I=1)=q$.

We have:

$$E(X\mid I)=bP(I=1)+0P(I=0)=bq$$ [½]

$$E(X^2\mid I)=b^2P(I=1)+0^2P(I=0)=b^2q$$ [½]

$$\text{var}(X\mid I)=E(X^2\mid I)-\left[E(X\mid I)\right]^2=b^2q-(bq)^2=b^2q(1-q)$$ [½]

Using the formulae on Page 16 of the *Tables*:

$$E(X)=E\left[E(X\mid I)\right]=E(bq)=bq$$ [½]

Note that the mean of a constant is a constant.

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$$\text{var}(X) = E\left[\text{var}(X \mid I)\right] + \text{var}\left[E(X \mid I)\right]$$

$$= E\left[b^2 q(1-q)\right] + \text{var}(bq)$$

$$= b^2 q(1-q)$$ [1]

Note that the variance of a constant is 0.

[Total 3]

Alternatively, students may consider each X as a compound binomial distribution.

X has a compound binomial distribution where the number of claims is given by $N \sim Bin(1,q)$ and the claim amount (conditional on there being a claim), Y, is given by $Y = b$. [1]

We can then use the compound mean and variance formulae from Page 16 of the Tables:

$$E(X) = E(N)E(Y) = bq$$ [1]

$$\begin{aligned}\text{var}(X) &= E(N)\,\text{var}(Y) + \text{var}(N)E^2(Y) \\ &= q \times 0 + q(1-q) \times b^2 \\ &= b^2 q(1-q)\end{aligned}$$ [1]

[Total 3]

(iii) ***Skewness***

We have:

$$\text{skew}(I) = E[(I - E(I))^3] = E\left[I^3 - 3I^2 E(I) + 2E^3(I)\right]$$

$$= E(I^3) - 3E(I)E(I^2) + 2E^3(I) = q - 3q^2 + 2q^3$$

$$= q(1-q)(1-2q)$$ [1]

So $\text{skew}(X) = \text{skew}(bI) = b^3 \text{skew}(I) = b^3 q(1-q)(1-2q)$. [1]

[Total 2]

Alternatively, students may consider each X as a compound binomial distribution and use a method based on MGFs/CGFs. This is a rather long-winded approach! Markers – please point out the quicker approach used previously if the student goes down the route below.

X has a compound binomial distribution where the number of claims is given by $N \sim Bin(1,q)$ and the claim amount (conditional on there being a claim), Y, is given by $Y=b$.

Using the compound MGF formula from Page 16 of the Tables:

$$M_X(t) = M_N(\ln M_Y(t)) = [(1-q)+qe^{bt}] \qquad [½]$$

We take natural logs to give the CGF:

$$C_X(t) = \ln M_X(t) = \ln[(1-q)+qe^{bt}]$$

The skewness can be obtained by differentiating the CGF three times with respect to t and then setting $t=0$:

$$C'_X(t) = \frac{qbe^{bt}}{[(1-q)+qe^{bt}]}$$

$$C''_X(t) = \frac{[(1-q)+qe^{bt}]qb^2e^{bt} - qbe^{bt}qbe^{bt}}{[(1-q)+qe^{bt}]^2}$$
$$= \frac{q(1-q)b^2e^{bt}}{[(1-q)+qe^{bt}]^2}$$

$$C'''_X(t) = \frac{[(1-q)+qe^{bt}]^2 q(1-q)b^3e^{bt} - q(1-q)b^2e^{bt}2[(1-q)+qe^{bt}]qbe^{bt}}{[(1-q)+qe^{bt}]^4}$$
$$= \frac{q(1-q)(1-2q)b^3e^{bt}}{[(1-q)+qe^{bt}]^4} \qquad [1]$$

Finally, setting $t=0$ gives:

$$Skew(X) = C'''_X(0) = q(1-q)(1-2q)b^3 \qquad [½]$$

[Total 2]

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(iv) ***Mean, variance and skewness of total***

If S is the total claim amount, then:

$$E(S) = 1,250 \times 50,000 \times 0.008 + 250 \times 20,000 \times 0.012 = 560,000 \qquad [1]$$

$$\begin{aligned} \text{var}(S) &= 1,250 \times 50,000^2 \times 0.008 \times 0.992 + 250 \times 20,000^2 \times 0.012 \times 0.988 \\ &= 2.59856 \times 10^{10} \end{aligned} \qquad [1]$$

$$\begin{aligned} \text{skew}(S) &= 1,250 \times 50,000^3 \times 0.008 \times 0.992 \times 0.984 \\ &\quad + 250 \times 20,000^3 \times 0.012 \times 0.988 \times 0.976 \\ &= 1.2433029 \times 10^{15} \end{aligned} \qquad [1]$$

Hence, the coefficient of skewness is:

$$\frac{\text{skew}(S)}{(\text{var}(S))^{3/2}} = \frac{1.2433029 \times 10^{15}}{\left(2.59856 \times 10^{10}\right)^{3/2}} = 0.297 \qquad [1]$$

Award follow-through marks for the coefficient of skewness if an incorrect skewness is derived in Part (iii).

[Total 4]

(v) ***Probability***

Using a normal approximation for S, we have:

$$S \doteq N(560\,000,\ 2.59856 \times 10^{10}) \qquad [½]$$

So the probability is given by:

$$\begin{aligned} P(S > 1,000,000) &\approx P\left(Z > \frac{1,000,000 - 560,000}{\sqrt{2.59856 \times 10^{10}}}\right) && [1] \\ &= 1 - P(Z < 2.72952) && [½] \\ &= 0.00317 && [1] \end{aligned}$$

[Total 3]

Markers – please award follow-through marks if an incorrect mean and/or variance are derived in Part (iii).

(vi) ***Comment***

A normal distribution gives the most accurate answers if the distribution is symmetrical. The coefficient of skewness in Part (iv) shows that the distribution is positively skewed, but not by very much. So the value is probably not that inaccurate. [1]

On the other hand, we are looking at a probability relating to the distribution of values in the upper tail, where a normal distribution is likely to approximate less well than at the centre of the distribution. [1]

[Total 2]

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Assignment X3 Solutions

Solution X3.1

> ### *Comment*
>
> *This question applies the Core Reading in Chapter 9 on the surplus process. Note that the claims process is discrete and so we need to use a logical approach to list through the combinations of claim amounts and claim numbers that will lead to ruin at time 2.*

The premium is:

$$c = (1+\theta)E(S) = 1.2 \times \lambda E(X) = 1.2 \times ¼ \times 60 = 18 \qquad [½]$$

Hence:

$$U(2) = 100 + 2 \times 18 - S(2) = 136 - S(2) \qquad [½]$$

So the probability of ruin is:

$$P[U(2) < 0] = P[136 - S(2) < 0] = P[S(2) > 136] \qquad [½]$$

Considering $P[S(2) < 136]$, remembering that $N(2) \sim Poi(2 \times ¼)$, we have:

number of claims	amount of claim(s)	probability
0 claims	0	$(e^{-0.5}) = 0.60653$
1 claim	50	$(0.5e^{-0.5}) \times 0.8 = 0.24261$
	100	$(0.5e^{-0.5}) \times 0.2 = 0.06065$
2 claims	50, 50	$(\frac{0.5^2}{2}e^{-0.5}) \times 0.8^2 = 0.04852$

[½ each probability = 2]

Hence:

$$P[S(2) < 136] = 0.9583 \quad \Rightarrow \quad P[U(2) < 0] = 0.0417 \qquad [½]$$

[Total 4]

Solution X3.2

> **Comment**
>
> *This question tests the bookwork in Chapter 11 on the statistical model for run-off triangles. This is Subject CT6, April 2006, Question 9(i). Markers, please point out the importance of learning this bit of Core Reading as well as the assumptions behind each of the run-off triangle models.*

Each incremental entry, C_{ij}, in the run-off triangle can be expressed in general terms as:

$$C_{ij} = r_j \, s_i \, x_{i+j} + e_{ij}$$ [1]

where:

- r_j is the development factor for Development Year j, representing the proportion of claim payments in year j. Each r_j is independent of the Accident Year i. [1]
- s_i is a parameter for Accident Year i representing exposure (*eg* number of claims or claim amount in respect of Accident Year *i*). [1]
- x_{i+j} is a parameter varying by calendar year (*eg* a measure of inflation). [½]
- e_{ij} is an error term. [½]

[Total 4]

Solution X3.3

Comment

This question tests and then applies the bookwork in Chapter 9 on the definition of the surplus process and the probability of ultimate ruin. Note that this is Subject 106, April 2003, Question 5(i) and (ii).

(i)(a) ***Surplus***

The surplus process $U(t)$ is given by:

$$U(t) = u + ct - S(t)$$

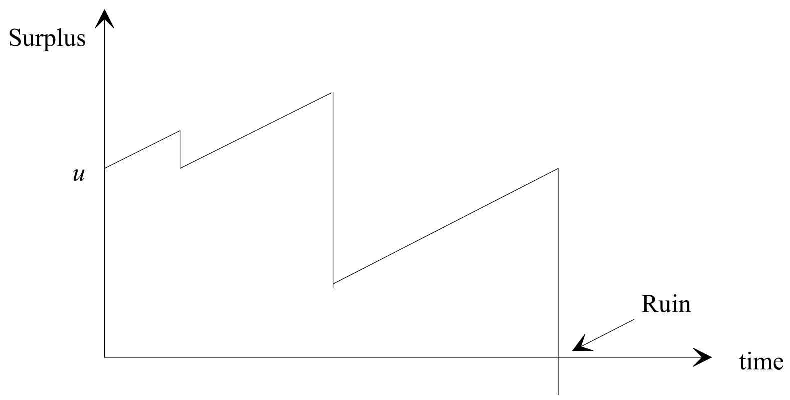
where c, the rate of premium income per unit time, is $c = (1+\theta)\lambda E[X]$ and $S(t)$, the aggregate claim amount, is $S(t) = X_1 + X_2 + \cdots + X_{N(t)}$. [1]

(i)(b) ***Probability of ruin***

The probability of ruin is given by:

$$\Psi(u) = P\left[U(t) < 0, t > 0\right] \quad [1]$$

The diagram showing a ruin event is as follows:



[1]

(i)(c) ***State probability of ultimate ruin***

$\Psi(u) = 1$ [1]

[Total 4]

(ii) ***Change of currency***

Let the surplus process under the new currency be $\tilde{U}(t)$. A surplus of 100 in the old currency is the same as a surplus of 250 in the new currency. So $\tilde{U}(t) = 2.5U(t)$. The probability of ruin, $\Psi(u)$, will be the same under the new currency since:

$$P\left(\tilde{U}(t) < 0\right) = P\left(2.5U(t) < 0\right) = P\left(U(t) < 0\right) \qquad [2]$$

Solution X3.4

> ### *Comment*
>
> *This question applies the bookwork in Chapter 9 on determining the adjustment coefficient and Lundberg's inequality for a Poisson process.*

(i)(a) ***Adjustment coefficient equation***

The adjustment coefficient is the unique positive root, R, of the equation:

$$\lambda + cr = \lambda M_X(r) \qquad \text{equation (1)} \qquad [½]$$

We have $P(X = 100) = 1$, hence:

$$E(X) = 100 \qquad \text{and} \qquad M_X(t) = e^{100t} \qquad [½]$$

$$\Rightarrow \quad c = (1+\theta)\lambda E(X) = 120\lambda \qquad [½]$$

Substituting these into equation (1) gives:

$$\lambda + 120\lambda R = \lambda e^{100R} \quad \Rightarrow \quad 1 + 120R = e^{100R} \qquad [½]$$

(i)(b) ***Adjustment coefficient***

Using the series expansion from Page 2 of the *Tables*:

$$1 + 120R \simeq \left(1 + 100R + \tfrac{(100R)^2}{2!} + \tfrac{(100R)^3}{3!}\right) \qquad [½]$$

$$0 = R\left(-20 + 5{,}000R + \tfrac{500{,}000}{3}R^2\right) \qquad [½]$$

Solving this gives $R = 0, 0.00357, -0.0336$. [½]

Since the adjustment coefficient is the positive root, $R = 0.00357$. [½]

[Total 4]

(ii) ***Minimum initial capital***

Using Lundberg's inequality:

$$e^{-RU} < 0.05$$ [1]

Using the value of R from Part (i)(b):

$$-RU < \ln 0.05 \quad \Rightarrow \quad RU > -\ln 0.05 \quad \Rightarrow \quad U > -\frac{\ln 0.05}{0.00357} = 838$$ [1]

[Total 2]

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Solution X3.5

Comment

This question applies the bookwork in Chapter 10 on showing whether or not a distribution is a member of the exponential family. It is really important that you can show that each of the exponential, gamma, Poisson, normal and binomial distributions belong to this family by the time of the exam.

(i)(a) ***Exponential family***

Rewriting the PDF we get:

$$f(y) = \frac{\alpha^\alpha}{\mu^\alpha \Gamma(\alpha)} y^{\alpha-1} e^{-\frac{\alpha}{\mu}y}$$

$$= \exp\left\{\alpha \ln \alpha - \alpha \ln \mu - \ln \Gamma(\alpha) + (\alpha - 1)\ln y - \frac{\alpha y}{\mu}\right\} \qquad [½]$$

$$= \exp\left\{\alpha\left(-\frac{y}{\mu} - \ln \mu\right) + \alpha \ln \alpha - \ln \Gamma(\alpha) + (\alpha - 1)\ln y\right\}$$

$$= \exp\left\{\frac{-\frac{1}{\mu}y - \ln \mu}{1/\alpha} + \alpha \ln \alpha - \ln \Gamma(\alpha) + (\alpha - 1)\ln y\right\} \qquad [½]$$

This is in the form of a member of the exponential family. Comparing with the formula on Page 27 of the *Tables*, we see that:

The natural parameter is $\theta = -\frac{1}{\mu}$. [½]

The dispersion (or scale) parameter is $\phi = \alpha$. [½]

Note that where a distribution has two parameters (eg normal, gamma, binomial), we can determine ϕ by taking it to be the "other" parameter (ie the parameter other than μ) in the pdf formulation. Here the two parameters are α and μ so ϕ is taken to be α. For a one parameter distribution (eg Poisson, exponential) we take ϕ to be 1.

Note also though that other possibilities for the dispersion parameter exist, eg $\phi = 1/\alpha$. Full marks should be awarded for these provided the other functions are expressed correctly, eg $a(\phi) = \phi$.

Hence:

$$a(\phi) = \tfrac{1}{\alpha} = \tfrac{1}{\phi} \qquad [½]$$

$$b(\theta) = \ln \mu = \ln\left(-\tfrac{1}{\theta}\right) = -\ln(-\theta) \qquad [1]$$

$$c(y,\phi) = \phi \ln \phi - \ln \Gamma(\phi) + (\phi - 1)\ln y \qquad [½]$$

Note: These functions need to be expressed in terms of θ and ϕ to obtain the marks.

(i)(b) ***Mean and variance***

Using the formulae on Page 27 of the *Tables*, we get:

$$E(Y) = b'(\theta) = -\tfrac{1}{\theta} = \mu \qquad \left(\text{or } \tfrac{\alpha}{\lambda} \text{ since } \mu = \tfrac{\alpha}{\lambda}\right) \qquad [1]$$

$$\text{var}(Y) = a(\phi)b''(\theta) = \tfrac{1}{\phi} \times \tfrac{1}{\theta^2} = \tfrac{1}{\phi\theta^2} = \tfrac{\mu^2}{\alpha} \quad \left(\text{or } \tfrac{\alpha}{\lambda^2} \text{ since } \mu = \tfrac{\alpha}{\lambda}\right) \qquad [1]$$

Working is essential to ensure students have used these formulae and not just copied the mean and variance from Page 12 of the Tables.

(i)(c) ***Variance function***

The variance function is given by:

$$V(\mu) = b''(\theta) = \frac{1}{\theta^2} = \mu^2 \qquad [1]$$

Students do need to express this in terms of the mean, μ, in order to obtain the mark.

Alternatively, students could calculate the variance function directly using the mean and variance of a gamma distribution given on Page 12 of the Tables.

$$\text{var}(Y) = a(\phi)V(\mu) \quad \Rightarrow \quad V(\mu) = \frac{\text{var}(Y)}{a(\phi)} = \frac{\alpha/\lambda^2}{1/\alpha} = \left(\frac{\alpha}{\lambda}\right)^2 = \mu^2$$

[Total 7]

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(ii) ***Negative binomial distribution***

We have:

$$P(X = x) = \frac{\Gamma(k+x)}{\Gamma(x+1)\Gamma(k)} p^k (1-p)^x$$

We need the PF to contain the mean, μ. Since:

$$\mu = \frac{k(1-p)}{p} \quad \Rightarrow \quad p = \frac{k}{k+\mu}$$

We get:

$$P(X = x) = \frac{\Gamma(k+x)}{\Gamma(x+1)\Gamma(k)} \left(\frac{k}{k+\mu}\right)^k \left(\frac{\mu}{k+\mu}\right)^x$$ [½]

Rearranging gives:

$$P(X = x) = \exp\left\{ x \ln\left(\frac{\mu}{k+\mu}\right) - k\ln(k+\mu) + \ln\frac{\Gamma(k+x)}{\Gamma(x+1)\Gamma(k)} + k\ln k \right\}$$ [½]

The first two terms correspond to $\dfrac{x\theta - b(\theta)}{a(\phi)}$, but since the natural parameter θ is a function of μ only, we need to separate out the scale parameter, which will be $\phi = k$, from these two terms. [½]

However there is no way to separate k from either the $x\ln\left(\frac{\mu}{k+\mu}\right)$ term or the $\ln(k+\mu)$ term. So it cannot be expressed in the exponential family form. [½]

[Total 2]

Solution X3.6

> **Comment**
>
> *This question applies the bookwork in Chapter 11 on the inflation-adjusted chain ladder method.*

Adjusting the incremental data for past inflation (*ie* change the figures to mid-2012 money terms):

Claim payments in mid-2012 money terms (£'000)		Development year		
		0	1	2
Accident year	2010	$830 \times 1.02 \times 1.025$ **867.765**	940×1.025 **963.5**	150
	2011	850×1.025 **871.25**	920	
	2012	1,120		

[1 mark for top left, ½ mark for other two bold entries = total 2]

Next, we accumulate, find the ratios and use the basic chain ladder to project the values:

×2.083084 ×1.081911

Claim payments in mid-2012 money terms (£'000)		Development year		
		0	1	2
Accident year	2010	867.765	1,831.265	1,981.265
	2011	871.25	1,791.25	**1,937.97**
	2012	1,120	**2,333.05**	**2,524.16**

[1 mark for 1st ratio, ½ mark for 2nd ratio, ½ mark for each bold entry = total 3]

Finally, we need incremental data again, so we can adjust for future inflation (*ie* calculate the actual money to be paid in each future year):

Actual money paid (£'000)		Development year		
		0	1	2
Accident year	2010			
	2011			146.72×1.03 **151.12**
	2012		$1,213.05 \times 1.03$ **1,249.44**	191.11×1.03^2 **202.75**

[1 mark for bottom right, ½ mark for other two bold entries = total 2]

So the estimated total future amount for outstanding claims is:

$$151.12 + 1{,}249.44 + 202.75 = 1{,}603.31 \simeq £1{,}603{,}000$$ [1]

[Total 8]

Rounding will affect the accuracy of students' answers. Our general guidelines are to round the final figure to the same degree of accuracy as the figures given in the question but to keep several more significant figures throughout the workings. In this case the final answer should be correct to at least 3 SF and the workings correct to at least 4 SF. For reference, our workings are based on non-rounded figures, ie storing non-rounded figures in the calculator memory.

Solution X3.7

> **_Comment_**
>
> *This question applies the bookwork in Chapter 11 on the Bornhuetter-Ferguson method. Note that this is a modified version of Subject CT6, September 2003, Question 6. Note the two different ways of getting to the revised estimate of the ultimate loss for 2012. Both are valid in the exam so work out which you are most comfortable with.*

The development factors are:

$$\text{DY } 1 \rightarrow \text{DY } 2 \qquad \frac{2{,}101}{1{,}923} = 1.09256$$ [1]

$$\text{DY } 0 \rightarrow \text{DY } 1 \qquad \frac{1{,}923 + 2{,}140}{1{,}417 + 1{,}701} = 1.30308$$ [1]

The initial estimate of the ultimate loss incurred for Policy Year 2012 is given by:

$$(\text{ultimate loss ratio}) \times (\text{earned premium}) = 0.92 \times £3{,}073 = £2{,}827.16$$ [1]

Next we calculate the emerging liability for Policy Year 2012:

$$(\text{intial ultimate loss}) \times (1 - \tfrac{1}{f}) = £2{,}827.16 \times \left(1 - \frac{1}{1.09256 \times 1.30308}\right)$$

$$= £841.37$$ [1]

In the formula above, f is the cumulative development factor (*ie* the product of all the development factors from the last known payment to the end).

So the revised estimate of the ultimate loss incurred for policies written in 2012 is:

$$£1,582 + £841.37 = £2,423.37 \qquad [1]$$

Alternatively, we can calculate the expected claims incurred to date as $£2,827.16 \times \dfrac{1}{1.09256 \times 1.30308} = £1,985.79$. *We have actually incurred £1,582. The actual is less than the expected figure by £403.79. Adjusting the initial estimate of the ultimate loss incurred by this gives a revised estimate of the ultimate loss incurred of* $£2,827.16 - £403.79 = £2,423.37$.

Since all our figures are in 000's, we have £2,423,370. Now the figures given in the table are claims incurred (rather than claims paid). So we need to use the fact that for policy year 2012 we have paid £441,000. Therefore, the amount left to pay is:

$$£2,423,370 - £441,000 \simeq £1,982,000 \text{ (4 SF)} \qquad [1]$$

[Total 6]

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Solution X3.8

Comment

This question applies the bookwork in Chapter 11 on the average cost per claim method. Markers, please don't award marks where development factors have been used instead of grossing up factors as the question specifically asks for the latter not the former.

(i) ***ACPC***

First divide each number in the first table by the corresponding entry in the second table to obtain the average cost incurred per claim reported:

Average cost per claim (£'000)		Development year		
		0	1	2
Accident year	2010	252 ÷ 56 = **4.5**	375 ÷ 74 = **5.0676**	438 ÷ 87 = **5.0345**
	2011	230 ÷ 49 = **4.6939**	343 ÷ 65 = **5.2769**	
	2012	208 ÷ 44 = **4.7273**		

[1]

Then calculate the grossing-up factors for the average cost incurred per claim reported:

Average cost per claim (£'000)		Development year		
		0	1	2
Accident year	2010	4.5 **89.384%**	5.0676 **100.657%**	5.0345 **100%**
	2011	4.6939 **89.536%**	5.2769 **100.657%**	5.2769 ÷ 1.00657 **5.2425**
	2012	4.7273 **89.460%**		4.7273 ÷ 0.89460 **5.2843**

[½ mark for each of the shaded figures = total 2]

Next calculate the grossing-up factors for the number of reported claims:

<table>
<tr><td colspan="2" rowspan="2">Number of reported claims</td><td colspan="3">Development year</td></tr>
<tr><td>0</td><td>1</td><td>2</td></tr>
<tr><td rowspan="3">Accident year</td><td>2010</td><td>56
64.368%</td><td>74
85.057%</td><td>87
100%</td></tr>
<tr><td>2011</td><td>49
64.120%</td><td>65
85.057%</td><td>65 ÷ 0.85057
76.419</td></tr>
<tr><td>2012</td><td>44
64.244%</td><td></td><td>44 ÷ 0.64244
68.489</td></tr>
</table>

[½ mark for each of the shaded figures = total 2]

The projected ultimate incurred loss for each accident year is then obtained by multiplying the ultimate figures for the average cost per claim and the number of reported claims.

Accident Year	ACPC (£'000)	Number of reported claims	Projected incurred claims (£'000)
2010	5.0345	87	438
2011	5.2425	76.419	400.6
2012	5.2843	68.489	361.9
		Total	**1,201** (4 SF)

[1]

The total claims paid is £950,000. So the estimated outstanding claims reserve is approximately:

$$£1,201,000 - £950,000 = £251,000$$ [1]

[Total 7]

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(ii) ***Assumptions***

- The 2010 accident year is fully run-off.
- For each origin year, the numbers of claims reported in each development year are constant proportions of the total number of claims reported from that accident year.
- For each origin year, the average claim amounts incurred in each development year are constant proportions of the total average claim amount incurred from that accident year.

[Total 2: subtract 1 mark per error or omission]

Solution X3.9

> ### *Comment*
>
> *This question applies the bookwork in Chapter 10 on GLMs. Part (i) tests maximum likelihood estimation and the scaled deviance of a particular model. Part (ii) tests the deviance and Pearson residuals and Part (iii) tests the significance of an extra parameter.*

(i)(a) ***Log-likelihood***

The PDF is:

$$f(y_i) = \frac{1}{\mu_i} e^{-\frac{1}{\mu_i} y_i} \qquad y_i > 0$$

which can be written as:

$$f(y_i) = \exp\left\{-\ln \mu_i - \frac{1}{\mu_i} y_i\right\}$$

Note that by putting all the terms inside the exponential, the maths is easier later on when we take natural logs.

So the likelihood is:

$$L(\mu_i) = \prod_{i=1}^{15} f(y_i) = \exp \sum_{i=1}^{15} \left\{-\ln \mu_i - \frac{1}{\mu_i} y_i\right\}$$

And hence the log-likelihood is:

$$\ln L(\mu_i) = \sum_{i=1}^{15} \left\{ -\ln \mu_i - \frac{1}{\mu_i} y_i \right\} \qquad \text{eqn (1)} \qquad [½]$$

Now substituting for the $\dfrac{1}{\mu_i}$ we get:

$$\ln L(\alpha,\beta) = \sum_{i=1}^{10} \{ \ln \alpha - \alpha y_i \} + \sum_{i=11}^{15} \{ \ln(\alpha+\beta) - (\alpha+\beta) y_i \} \qquad \text{eqn (2)} \qquad [½]$$

(i)(b) ***MLEs***

Differentiating with respect to α:

$$\frac{\partial}{\partial\alpha} \ln L(\alpha,\beta) = \sum_{i=1}^{10} \left\{ \frac{1}{\alpha} - y_i \right\} + \sum_{i=11}^{15} \left\{ \frac{1}{\alpha+\beta} - y_i \right\} \; = \; \frac{10}{\alpha} + \frac{5}{\alpha+\beta} - \sum_{i=1}^{15} y_i \qquad [1]$$

Differentiating with respect to β:

$$\frac{\partial}{\partial\beta} \ln L(\alpha,\beta) = \sum_{i=11}^{15} \left\{ \frac{1}{\alpha+\beta} - y_i \right\} \; = \; \frac{5}{\alpha+\beta} - \sum_{i=11}^{15} y_i \qquad [1]$$

Setting these two derivatives equal to zero, we obtain the equations:

$$\frac{10}{\hat\alpha} + \frac{5}{\hat\alpha+\hat\beta} - \sum_{i=1}^{15} y_i = 0 \qquad \text{eqn (3)}$$

$$\frac{5}{\hat\alpha+\hat\beta} - \sum_{i=11}^{15} y_i = 0 \qquad \text{eqn (4)}$$

From (4), we have $\dfrac{5}{\hat\alpha+\hat\beta} = \displaystyle\sum_{i=11}^{15} y_i$. Substituting this into (3) gives:

$$\frac{10}{\hat\alpha} - \sum_{i=1}^{10} y_i = 0 \quad \Rightarrow \quad \hat\alpha = \frac{10}{\sum_{i=1}^{10} y_i} \qquad [1]$$

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We can rearrange (4) to get:

$$\hat{\alpha} + \hat{\beta} = \frac{5}{\sum_{i=11}^{15} y_i}$$

Now using $\hat{\alpha} = \frac{10}{\sum_{i=1}^{10} y_i}$ we find that:

$$\hat{\beta} = \frac{5}{\sum_{i=11}^{15} y_i} - \frac{10}{\sum_{i=1}^{10} y_i} \qquad [1]$$

Markers please note that to show these are MLEs, we should really do some second order differentiation. However, this is complicated in a two-parameter case and would not be expected in the exam. For reference, this is covered in Subject CT3 Chapter 10, which explains how this would be done in practice.

(i)(c) ***Scaled deviance***

The scaled deviance is given by $2\{\ln L_S - \ln L_M\}$ where $\ln L_S$ is the log-likelihood of the saturated model and $\ln L_M$ is the log-likelihood of the current model.

In the saturated model the expected values, μ_i, are equal to the actual observed values, y_i. Hence replacing μ_i's with y_i's in equation (1) we get the log-likelihood for the saturated model:

$$\ln L_S = \sum_{i=1}^{15} \left\{ -\ln y_i - \frac{1}{y_i} y_i \right\} = \sum_{i=1}^{15} \{-\ln y_i - 1\} \qquad [1]$$

The log-likelihood of our current model is equation (2) with our estimates for α and β:

$$\ln L_M = \sum_{i=1}^{10} \{\ln \hat{\alpha} - \hat{\alpha} y_i\} + \sum_{i=11}^{15} \left\{ \ln(\hat{\alpha} + \hat{\beta}) - (\hat{\alpha} + \hat{\beta}) y_i \right\} \qquad [½]$$

Hence the scaled deviance is:

$$2\left\{\sum_{i=1}^{15}\left(-\ln y_i - 1\right) - \sum_{i=1}^{10}\left(\ln\hat{\alpha} - \hat{\alpha} y_i\right) - \sum_{i=11}^{15}\left(\ln(\hat{\alpha}+\hat{\beta}) - (\hat{\alpha}+\hat{\beta})y_i\right)\right\}$$

$$= 2\left\{\sum_{i=1}^{10}\left(-\ln y_i - 1 - \ln\hat{\alpha} + \hat{\alpha} y_i\right) + \sum_{i=11}^{15}\left(-\ln y_i - 1 - \ln(\hat{\alpha}+\hat{\beta}) + (\hat{\alpha}+\hat{\beta})y_i\right)\right\} \quad [½]$$

Using the given notation we see that $\hat{\alpha} = \frac{1}{\bar{y}_1}$ and $\hat{\beta} = \frac{1}{\bar{y}_2} - \frac{1}{\bar{y}_1}$. Hence $\hat{\alpha}+\hat{\beta} = \frac{1}{\bar{y}_2}$. Substituting these gives:

$$= 2\left\{\sum_{i=1}^{10}\left(-\ln y_i - 1 + \ln\bar{y}_1 + \frac{y_i}{\bar{y}_1}\right) + \sum_{i=11}^{15}\left(-\ln y_i - 1 + \ln\bar{y}_2 + \frac{y_i}{\bar{y}_2}\right)\right\} \quad [1]$$

as required.

[Total 8]

(ii)(a) ***Deviance residual***

The deviance residual for y_1 is given by:

$$\text{sign}(y_1 - \hat{\mu}_1)d_1 \quad [½]$$

where $\sum d_i^2$ gives the scaled deviance.

We have $\hat{\mu}_1 = \dfrac{1}{\hat{\alpha}} = \bar{y}_1 = 430$. [½]

So $\text{sign}(y_1 - \hat{\mu}_1) = \text{sign}(-5)$, which is negative. [½]

From Part (i)(c) we have:

$$d_1^2 = 2\left\{-\ln y_1 - 1 + \ln\bar{y}_1 + \frac{y_1}{\bar{y}_1}\right\} = 2\left\{-\ln 425 - 1 + \ln 430 + \frac{425}{430}\right\} = 0.000136 \quad [½]$$

Hence, the deviance residual is:

$$-\sqrt{0.000136} = -0.0117 \quad [½]$$

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(ii)(b) ***Pearson residual***

The Pearson residual for y_1 is given by:

$$\frac{y_1 - \hat{\mu}_1}{\sqrt{\text{var}(\hat{\mu}_1)}}$$ [½]

where $\text{var}(\hat{\mu}_1)$ is the variance of Y_1 with any μ_1's in the formula replaced by their estimate, $\hat{\mu}_1$.

Since the variance of $Y_1 \sim Exp\left(\frac{1}{\mu_1}\right)$ is $\text{var}(Y_1) = \mu_1^2$, we have $\text{var}(\hat{\mu}_1) = \hat{\mu}_1^2$. [½]

From (ii)(a), we had $\hat{\mu}_1 = 430$, hence the Pearson residual is:

$$\frac{425 - 430}{\sqrt{430^2}} = -0.0116$$ [½]

(ii)(c) ***Which residual is appropriate?***

The distribution of Pearson residuals is skewed for non-normal data, whereas the distribution of deviance residuals is symmetrical and approximately normal. Therefore the deviance residuals are appropriate for the exponential distribution. [1]

A histogram of the appropriate residuals should be symmetrical and approximately normally distributed if the model is a good fit to the data. [½]

Also a plot of the appropriate residuals against the variables and factors (our α's and β's in this question) should be patternless. [½]

[Total 6]

(iii)(a) ***Significant improvement?***

Using the fact that when subtracting scaled deviances of nested models we get a χ^2 distribution with degrees of freedom equal to the difference in the number of parameters, our test statistic is:

$$0.135 - 0.0120 = 0.123$$ [½]

Since the original model has 2 parameters (α and β) and the simplified model has 1 parameter (α), this is a realisation of a χ_1^2 random variable. [½]

The upper 5% critical value for a χ_1^2 is 3.841. Since $0.123 < 3.841$ the original model with β is **not** a significant improvement over the simplified model (*ie* adding the extra β does not significantly reduce the scaled deviance and hence improve the fit). [1]

Alternatively, we could use the approximation $\Delta(\text{deviance}) > 2 \times \Delta(\text{parameters})$*, since* $0.123 \not> 2 \times 1$ *the model with* β *is not a significant improvement.*

(iii)(b) ***Significance of the*** $\boldsymbol{\beta}$ ***parameter***

Using our formula for $\hat{\beta}$ from Part (i)(b) we get:

$$\hat{\beta} = \frac{5}{\sum_{i=11}^{15} y_i} - \frac{10}{\sum_{i=1}^{10} y_i} = \frac{5}{2{,}600} - \frac{10}{4{,}300} = -0.000403\,.$$

Since $|\hat{\beta}| = 0.000403 \not> 2 \times s.e.(\beta) = 2 \times 0.000769$ the β parameter is **not** considered significantly different from zero, and so should not be included in the model. [1]

This ties up with the result in Part (iii)(a) that the original model with β's is not a significant improvement over the simplified model and so the β parameter should not be included. [1]

[Total 4]

Solution X3.10

Comment

This question applies the bookwork in Chapter 10 on formulating linear predictors and determining the number of parameters for various models. It goes on to use the difference in scaled deviance as a test statistic to find out whether a particular model is a significant improvement over another.

(i) ***Completed table***

Model	Parameterised form of the linear predictor	Number of parameters	Scaled deviance
SA	$\alpha + \beta x$	2	238.4
SA + PT	$\alpha_i' + \beta x$	11	206.7
SA + PT + SA•PT	$\alpha_i' + \beta_i' x$	20	178.3
SA∗PT + NB	$\alpha_i' + \beta_i' x + B_j$	25	166.2
SA∗PT∗NB	$\alpha_{ij}'' + \beta_{ij}'' x$	120	58.9

[½ mark per entry = total 3]

Any other dummy variable is acceptable, eg $\gamma_i + \beta x$ *or* $T_i' + \beta x$ *for SA + PT, as long as the subscripts match up.*

Working:

SA + PT $\quad (\alpha + \beta x) + T_i = \alpha_i' + \beta x \quad 2 + (10 - 1) = 11$ parameters

SA + PT + SA•PT = SA∗PT

$= (\alpha + \beta x) * T_i = \alpha_i' + \beta_i' x \quad 2 \times 10 = 20$ parameters

SA∗PT + NB $\quad \alpha_i' + \beta_i' x + B_j \quad 20 + (6 - 1) = 25$ parameters

Note the rules for determining the number of parameters are given in Chapter 10 in the Example in Section 3.1.

(ii) ***Comparing models***

Comparing SA+PT with SA

The difference in the scaled deviances is $238.4 - 206.7 = 31.7$.

This is greater than 16.92, the upper 5% critical value of a $\chi^2_{11-2} = \chi^2_9$ distribution.

So SA + PT **is** a significant improvement over the SA model. [1]

Alternatively, using the $\Delta(\text{deviance}) > 2 \times \Delta(\text{parameters})$ *approximation, we get* $31.7 > 2 \times 9$ *so the SA+PT model **is** a significant improvement over the SA model.*

Comparing SA∗PT with SA+PT

The difference in the scaled deviances is $206.7 - 178.3 = 28.4$.

This is greater than 16.92, the upper 5% critical value of a $\chi^2_{20-11} = \chi^2_9$ distribution.

So SA ∗ PT **is** a significant improvement over the SA + PT model. [1]

Alternatively, using the $\Delta(\text{deviance}) > 2 \times \Delta(\text{parameters})$ *approximation, we get* $28.4 > 2 \times 9$ *so the SA ∗ PT model **is** a significant improvement over the SA+PT model.*

Comparing SA∗PT+NB with SA∗PT

The difference in the scaled deviances is $178.3 - 166.2 = 12.1$.

This is greater than 11.07, the upper 5% critical value of a $\chi^2_{25-20} = \chi^2_5$ distribution.

So SA ∗ PT + NB **is** a significant improvement over the SA ∗ PT model. [1]

Alternatively, using the $\Delta(\text{deviance}) > 2 \times \Delta(\text{parameters})$ *approximation, we get* $12.1 > 2 \times 5$ *so the SA ∗ PT model **is** a significant improvement over the SA ∗ PT+NB model.*

Comparing SA∗PT∗NB with SA∗PT+NB

The difference in the scaled deviances is $166.2 - 58.9 = 107.3$.

This is less than 118.7, the upper 5% critical value of a $\chi^2_{120-25} = \chi^2_{95}$ distribution.

We have to interpolate in the tables between χ^2_{90} *and* χ^2_{100} *to get the figure of 118.7.*

So SA ∗ PT ∗ NB is **not** a significant improvement over the SA ∗ PT+NB model. [1]

Alternatively, using the $\Delta(\text{deviance}) > 2 \times \Delta(\text{parameters})$ *approximation, we get* $107.3 \not> 2 \times 95$ *so the SA ∗ PT ∗ NB model is **not** a significant improvement over the SA ∗ PT+NB model.*

So the analyst should choose the SA ∗ PT+NB model. [1]

[Total 5]

(iii) ***Further information***

The analyst should also check:

- that the SA∗PT+NB model is a significant improvement when the order is different, *eg* add the NB factor *before* the PT factor [½]
- other models involving these rating factors, *eg* SA∗NB+PT [½]
- the residuals of the proposed model (to ensure that it is a good fit to the data) [½]
- the significance of the parameters of the proposed model (to ensure that all the estimated parameters are significantly different from zero). [½]

[Total 2]

Solution X3.11

Comment

This question applies the bookwork in Chapter 9 on determining the adjustment coefficient for a Poisson process in the presence of reinsurance. Part (i) considers proportional reinsurance and Part (ii) considers non-proportional reinsurance.

(i)(a) ***Minimum value of α***

The net premium is given by:

$$\begin{aligned} c_{net} &= (1+\theta)E(S)-(1+\xi)E(S_R) \\ &= (1+\theta)\lambda E(X)-(1+\xi)\lambda E(Z) \qquad [½] \\ &= 1.2\lambda\times 100-1.3\lambda\times 100(1-\alpha) \\ &= 130\alpha\lambda-10\lambda \qquad [½] \end{aligned}$$

The expected net claims received by the insurer are given by:

$$E(S_I)=\lambda E(Y)=\lambda\alpha E(X)=100\alpha\lambda \qquad [½]$$

Hence:

$$130\alpha\lambda-10\lambda>100\alpha\lambda \quad\Rightarrow\quad 30\alpha>10 \quad\Rightarrow\quad \alpha>\tfrac{1}{3} \qquad [½]$$

(i)(b) ***Adjustment coefficient***

The adjustment coefficient is the unique positive root, R, of the equation:

$$\lambda + c_{net}R = \lambda M_Y(R) \qquad \text{equation (1)} \qquad [½]$$

From Part (i)(a) we have $c_{net} = 130\alpha\lambda - 10\lambda$. Also:

$$M_Y(R) = E[e^{RY}] = E[e^{R\alpha X}] = M_X(\alpha R) = (1-100\alpha R)^{-1} \qquad [1]$$

Substituting into equation (1) gives:

$$\lambda + (130\alpha\lambda - 10\lambda)R = \lambda(1-100\alpha R)^{-1} \qquad [1]$$

Solving this:

$$\begin{aligned}
&1+(130\alpha-10)R = (1-100\alpha R)^{-1}\\
\Rightarrow\quad & 1 = (1+130\alpha R-10R)(1-100\alpha R)\\
\Rightarrow\quad & 1 = 1-100\alpha R+130\alpha R-13{,}000\alpha^2R^2-10R+1{,}000\alpha R^2\\
\Rightarrow\quad & 0 = 30\alpha R-13{,}000\alpha^2R^2-10R+1{,}000\alpha R^2\\
\Rightarrow\quad & 0 = 10R(3\alpha-1{,}300\alpha^2R-1+100\alpha R)\\
\Rightarrow\quad & R = 0 \text{ or } \frac{1-3\alpha}{100\alpha-1{,}300\alpha^2} \qquad [1]
\end{aligned}$$

Since R is the unique positive root, we have $R = \frac{1-3\alpha}{100\alpha-1{,}300\alpha^2}$. [½]

(i)(c) ***Maximum value of the adjustment coefficient***

In order to maximise R we need to differentiate the above equation with respect to α. Using the quotient rule, we get:

$$\begin{aligned}
\frac{dR}{d\alpha} &= \frac{(100\alpha-1{,}300\alpha^2)(-3)-(1-3\alpha)(100-2{,}600\alpha)}{(100\alpha-1{,}300\alpha^2)^2} \qquad [1]\\
&= \frac{-300\alpha+3{,}900\alpha^2-100+2{,}600\alpha+300\alpha-7{,}800\alpha^2}{(100\alpha-1{,}300\alpha^2)^2}\\
&= \frac{2{,}600\alpha-3{,}900\alpha^2-100}{(100\alpha-1{,}300\alpha^2)^2} \qquad [1]
\end{aligned}$$

Setting this equal to zero, we get:

$$39\alpha^2 - 26\alpha + 1 = 0 \quad \Rightarrow \quad \alpha = \frac{26 \pm \sqrt{26^2 - 4 \times 39}}{2 \times 39} = 0.6257 \text{ or } 0.0410 \qquad [1]$$

But from (i)(a) we know that $\alpha > \frac{1}{3}$, so we have $\alpha = 0.6257$. [½]

Markers please note that students can also substitute the given value of α in to show that the derivative is zero.

Substituting this value back into the equation in Part (i)(b), we get $R = 0.0019649$. [½]
[Total 10]

(ii)(a) ***Minimum value of R***

The minimum value of R is given by:

$$\frac{1}{b}\ln(1+\theta)$$

where b is the maximum value of an individual claim amount (which here is 200). Therefore, the minimum value of R is:

$$\frac{1}{200}\ln 1.2 = 0.00091161 \qquad [1]$$

This value is useful because the minimum value of R gives the maximum probability of ultimate ruin. [1]

(ii)(b) ***Relationship***

The reinsurer's premium is $(1+\xi)\lambda E(Z)$. Since λ and the security loading, ξ, are the same for both reinsurance arrangements, then $E[Z]$ must be the same under both. [½]

Under proportional reinsurance:

$$E[Z] = E[(1-\alpha)X] = (1-\alpha)E[X] = 100(1-\alpha) \qquad [1]$$

Under excess of loss reinsurance:

$$E[Z]=\int_M^{200}(x-M)f(x)\,dx=\int_M^{200}\frac{(x-M)}{200}\,dx=\left[\frac{(x-M)^2}{400}\right]=\frac{(200-M)^2}{400} \qquad [1]$$

Therefore:

$$100(1-\alpha)=\frac{(200-M)^2}{400} \qquad [½]$$

(ii)(c) ***Adjustment coefficient under excess of loss reinsurance***

Using $\alpha=0.6257$ in the relationship above, we get $M=77.64$. [1]

The equation for the adjustment coefficient is:

$$\lambda+c_{net}R=\lambda M_Y(R) \qquad \text{equation (2)}$$

For the net premium, it is important to remember that $E[Z]$ is the same under both reinsurance arrangements, so we can use the proportional reinsurance expression:

$$\begin{aligned}c_{net}&=(1+\theta)\lambda E(X)-(1+\xi)\lambda E(Z)\\&=1.2\lambda\times100-1.3\lambda\times100(1-\alpha)\\&=130\alpha\lambda-10\lambda=71.34\lambda\end{aligned} \qquad [½]$$

The moment generating function can be found as follows:

$$\begin{aligned}M_Y(R)=E[e^{RY}]&=\int_0^M e^{Rx}f(x)\,dx+\int_M^{200}e^{RM}f(x)\,dx=\int_0^M\frac{e^{Rx}}{200}\,dx+\int_M^{200}\frac{e^{RM}}{200}\,dx \qquad [1]\\&=\left[\frac{1}{200R}e^{Rx}\right]_0^M+\left[\frac{e^{RM}}{200}x\right]_0^M\\&=\frac{1}{200R}(e^{RM}-1)+\frac{e^{RM}(200-M)}{200} \qquad [1]\end{aligned}$$

Substituting these expressions into equation (2) and cancelling λ:

$$\frac{1}{200R}(e^{RM}-1)+\frac{e^{RM}(200-M)}{200}=1+71.34R \qquad [½]$$

Substituting $M = 77.64$ and $R = 0.003926$, the left hand side of the equation is 1.283 and the right hand side is 1.280, so 0.003926 is an approximate value for the insurer's adjustment coefficient. [1]

[Total 10]

Assignment X4 Solutions

Solution X4.1

Comment

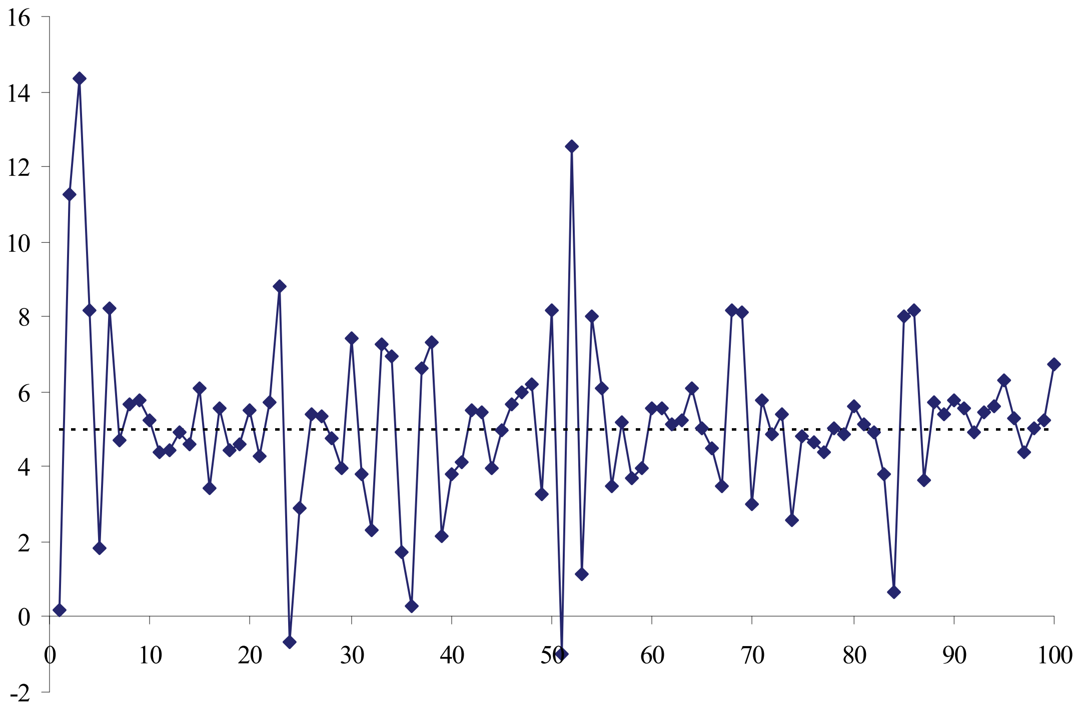
This question applies the Core Reading in Chapter 13 on ARCH models. Note that the model given in the question is an example of an ARCH(1) model.

The further X_{t-1} is from 5 the bigger the coefficient of Z_t and hence the bigger the fluctuation will be. [1]

The closer X_{t-1} is to 5, the smaller the coefficient, until at $X_{t-1}=5$ the process becomes $X_t = 5 + Z_t$, *ie* white noise with mean 5 and variance 1. [½]

The process has a mean of 5. [1]
[Maximum 2]

For information, a simulation of X_t against t is shown below:



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Solution X4.2

> ***Comment***
>
> *This question applies the Core Reading in Chapter 14 on the number of simulations required to ensure that the estimate of the mean of a random variable is within a certain distance from its true value with a given probability.*

We require:

$$P\left(\left|\bar{X}-\mu\right|<10\right)\geq 0.99$$ [½]

Now by the Central Limit Theorem, we know that:

$$\frac{\bar{X}-\mu}{200/\sqrt{n}} \mathrel{\dot{\sim}} N(0,1)$$ [½]

Using the appropriate value from Page 162 of the *Tables*, we get:

$$P\left(\left|\frac{\bar{X}-\mu}{200/\sqrt{n}}\right|\leq 2.5758\right)=0.99$$ [½]

Now:

$$\left|\frac{\bar{X}-\mu}{200/\sqrt{n}}\right|=2.5758 \quad\Rightarrow\quad \left|\bar{X}-\mu\right|=2.5758\frac{200}{\sqrt{n}}$$ [½]

We require $\left|\bar{X}-\mu\right|<10$. Hence:

$$2.5758\frac{200}{\sqrt{n}}<10 \quad\Rightarrow\quad \sqrt{n}>51.516 \quad\Rightarrow\quad n>2,653.9$$ [½]

So we need at least 2,654 simulations. [½]

[Total 3]

Alternatively, students may recall the formula given in the chapter on Monte Carlo simulation for working out the number of simulations required so that the absolute error falls is below a certain quantity with a given probability.

The number of simulations, n, required such that the absolute error $\left|\bar{X}-\mu\right|$ is less than ε with probability of at least $1-\alpha$ is given by:

$$n > \frac{z_{\alpha/2}^2 \hat{\tau}^2}{\varepsilon^2}$$ where $\hat{\tau}$ is an estimate of the sample standard deviation. [1]

In this question we have: $\varepsilon = 10$, $\alpha = 0.01$ and $\hat{\tau} = 200$ giving: [½]

$$n > \frac{z_{0.005}^2 200^2}{10^2} = \frac{2.5758^2 200^2}{10^2} = 2,653.9$$ [1]

So we need at least 2,654 simulations. [½]

[Total 3]

Solution X4.3

Comment

This question applies the Core Reading in Chapter 12 on the mean, variance and autocorrelation of a moving average time series.

(i) ***Mean and variance***

$$\begin{aligned} E(X_t) &= E(3.1 + \varepsilon_t + 0.25\varepsilon_{t-1} + 0.5\varepsilon_{t-2} + 0.25\varepsilon_{t-3}) \\ &= 3.1 + E(\varepsilon_t) + 0.25E(\varepsilon_{t-1}) + 0.5E(\varepsilon_{t-2}) + 0.25E(\varepsilon_{t-3}) = 3.1 \end{aligned}$$ [1]

$$\begin{aligned} \text{var}(X_t) &= \text{var}(3.1 + \varepsilon_t + 0.25\varepsilon_{t-1} + 0.5\varepsilon_{t-2} + 0.25\varepsilon_{t-3}) \\ &= \text{var}(\varepsilon_t) + 0.25^2\,\text{var}(\varepsilon_{t-1}) + 0.5^2\,\text{var}(\varepsilon_{t-2}) + 0.25^2\,\text{var}(\varepsilon_{t-3}) \\ &= 1.375\sigma^2 \end{aligned}$$ [1]

Alternatively, students could calculate the variance from the covariance so that:

$$\begin{aligned}
\text{var}(X_t) &= \text{cov}(X_t, X_t) \\
&= \text{cov}(3.1 + \varepsilon_t + 0.25\varepsilon_{t-1} + 0.5\varepsilon_{t-2} + 0.25\varepsilon_{t-3}, \\
&\qquad\qquad 3.1 + \varepsilon_t + 0.25\varepsilon_{t-1} + 0.5\varepsilon_{t-2} + 0.25\varepsilon_{t-3}) \\
&= \sigma^2 + 0.25^2\sigma^2 + 0.5^2\sigma^2 + 0.25^2\sigma^2 \\
&= 1.375\sigma^2
\end{aligned}$$

Students may omit the 3.1 as it will not affect the variance or covariance.

[Total 2]

(ii) ***Autocorrelation function***

The process is stationary as it the sum of stationary white noise terms, so we can calculate the autocovariance function (ignoring the 3.1's as they will not affect the results and noting that $\gamma_k = \gamma_{-k}$):

$$\gamma_0 = \text{cov}(X_t, X_t) = \text{var}(X_t) = 1.375\sigma^2 \text{ from Part (i)}$$

$$\begin{aligned}
\gamma_{\pm 1} &= \text{cov}(X_t, X_{t-1}) \\
&= \text{cov}(\varepsilon_t + 0.25\varepsilon_{t-1} + 0.5\varepsilon_{t-2} + 0.25\varepsilon_{t-3}, \varepsilon_{t-1} + 0.25\varepsilon_{t-2} + 0.5\varepsilon_{t-3} + 0.25\varepsilon_{t-4}) \\
&= 0.25\sigma^2 + (0.5)(0.25)\sigma^2 + (0.25)(0.5)\sigma^2 = 0.5\sigma^2
\end{aligned}$$

[½]

$$\begin{aligned}
\gamma_{\pm 2} &= \text{cov}(X_t, X_{t-2}) \\
&= \text{cov}(\varepsilon_t + 0.25\varepsilon_{t-1} + 0.5\varepsilon_{t-2} + 0.25\varepsilon_{t-3}, \varepsilon_{t-2} + 0.25\varepsilon_{t-3} + 0.5\varepsilon_{t-4} + 0.25\varepsilon_{t-5}) \\
&= 0.5\sigma^2 + 0.25^2\sigma^2 = 0.5625\sigma^2
\end{aligned}$$

[½]

$$\begin{aligned}
\gamma_{\pm 3} &= \text{cov}(X_t, X_{t-3}) \\
&= \text{cov}(\varepsilon_t + 0.25\varepsilon_{t-1} + 0.5\varepsilon_{t-2} + 0.25\varepsilon_{t-3}, \varepsilon_{t-3} + 0.25\varepsilon_{t-4} + 0.5\varepsilon_{t-5} + 0.25\varepsilon_{t-6}) \\
&= 0.25\sigma^2
\end{aligned}$$

[½]

$\gamma_{\pm k} = 0$ for $|k| > 3$ [½]

Since $\rho_k = \gamma_k / \gamma_0$, the autocorrelation function is:

$$\rho_0 = 1$$ [¼]

$$\rho_{\pm 1} = \frac{0.5\sigma^2}{1.375\sigma^2} = \frac{4}{11} \quad (= 0.364)$$ [½]

$$\rho_{\pm 2} = \frac{0.5625\sigma^2}{1.375\sigma^2} = \frac{9}{22} \quad (= 0.409)$$ [½]

$$\rho_{\pm 3} = \frac{0.25\sigma^2}{1.375\sigma^2} = \frac{2}{11} \quad (= 0.182)$$ [½]

$$\rho_{\pm k} = 0 \text{ for } |k| > 3.$$ [¼]

[Total 4]

Deduct 1 mark for students who only obtain the ACF for positive integer values as the question clearly states for all lags.

Solution X4.4

> **Comment**
>
> *Part (i) of this question tests the bookwork in Chapter 14 on pseudo-random numbers versus truly random numbers. Part (ii) applies the Core Reading in Chapter 14 on the inverse transform method, looking at both a discrete and a continuous distribution.*

(i) ***Advantages of pseudo-random numbers***

We can generate the same sequence of pseudo-random numbers more than once (whereas for truly random numbers the values would have to be recorded in a lengthy table). [1]

This reproducibility removes the variability of using different sets of random numbers, which is helpful for comparing different models. [1]

We only require a single routine to generate pseudo-random numbers (whereas for truly random numbers we need a lengthy table or a special piece of hardware). [1]

[Maximum 2]

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(ii)(a) ***Random variate from DRV***

The cumulative distribution function for X is:

x	0	1	2	3
$F(x) = P(X \le x)$	0.15	0.35	0.7	1

[½]

Since $F(2) < 0.764 \le F(3)$, the random variate is $x = 3$. [½]

(ii)(b) ***Random variate from CRV***

The distribution function for Y is:

$$F(y) = \int_0^y \tfrac{4}{3}\left(1 - \tfrac{1}{2}t\right) dt = \left[\tfrac{4}{3}\left(t - \tfrac{1}{4}t^2\right)\right]_0^y = \tfrac{4}{3}\left(y - \tfrac{1}{4}y^2\right) \qquad [1]$$

The random variate is obtained from:

$$0.764 = \tfrac{4}{3}\left(y - \tfrac{1}{4}y^2\right) \quad \Rightarrow \quad \tfrac{1}{3}y^2 - \tfrac{4}{3}y + 0.764 = 0 \qquad [1]$$

Hence:

$$y = \frac{\tfrac{4}{3} \pm \sqrt{\left(\tfrac{4}{3}\right)^2 - 4 \times \tfrac{1}{3} \times 0.764}}{2 \times \tfrac{1}{3}} = \cancel{3.307}, 0.693 \qquad [1]$$

The first solution is invalid as $0 < y < 1$.

[Total 4]

Solution X4.5

Comment

Part (i) of this question tests the bookwork in Chapter 12 on integrated time series of order d. Part (ii) applies the Core Reading in Chapter 12 on ARIMA time series.

(i) ***I(d)***

A process, *X,* is said to be $I(d)$ ("integrated of order d") if the d th difference, $\nabla^d X$, is a stationary process. [1]

Note that unless $d = 0$ then X is not stationary.

(ii)(a) ***Classify the time series $X_t = 0.6\varepsilon_{t\text{-}1} + \varepsilon_t$***

This is $MA(1)$ process and hence it is stationary (as it is the sum of stationary white noise terms). Therefore we can classify it as $ARIMA(0,0,1)$. [1]

(ii)(b) ***Classify the time series $X_t = 1.4X_{t\text{-}2} + \varepsilon_t + 0.5\varepsilon_{t\text{-}3}$***

This is an $ARMA(2,3)$ process. [½]

This process cannot be differenced, so to be able to classify it as an $ARIMA(2,0,3)$, we must check that it is $I(0)$, *ie* stationary.

Since $(1-1.4B^2)X_t = \varepsilon_t + 0.5\varepsilon_{t-3}$, the characteristic equation of the AR terms is:

$$\phi(\lambda) = 1 - 1.4\lambda^2 = 0 \quad \Rightarrow \quad \lambda = \pm 0.8452$$ [½]

Since both of the roots are less than one in magnitude the process is ***not*** stationary and so we cannot classify it as an $ARIMA(2,0,3)$. [½]

It is a non-stationary $ARMA(2,3)$ process. [½]

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(ii)(c) ***Classify the time series*** $\boldsymbol{X_t = 1.4X_{t\text{-}1} - 0.4X_{t\text{-}2} + \varepsilon_t + \varepsilon_{t\text{-}1}}$

This is an $ARMA(2,1)$ process. [½]

This process can be differenced as follows:

$$\begin{aligned} X_t - 1.4X_{t-1} + 0.4X_{t-2} &= \varepsilon_t + \varepsilon_{t-1} \\ (X_t - X_{t-1}) - 0.4(X_{t-1} - X_{t-2}) &= \varepsilon_t + \varepsilon_{t-1} \\ \nabla X_t - 0.4\nabla X_{t-1} &= \varepsilon_t + \varepsilon_{t-1} \end{aligned}$$ [½]

This process cannot be differenced again without a remainder term. We take $d = 1$.

Note that to find p and q we look at the subscripts of the differenced (not the original) time series.

To be able to classify it as an $ARIMA(1,1,1)$, we must check whether this differenced process is stationary (*ie* the original process is $I(1)$).

Since $(1 - 0.4B)\nabla X_t = \varepsilon_t + \varepsilon_{t-1}$, the characteristic equation of the differenced AR terms is:

$$\phi(\lambda) = 1 - 0.4\lambda = 0 \quad \Rightarrow \quad \lambda = 2.5$$ [½]

Alternatively, students can state that it is stationary as the coefficient of $\nabla X_t - 0.4\nabla X_{t-1}$ is less than one in magnitude.

Since the root is greater than one in magnitude the differenced process is stationary (*ie* the original process is $I(1)$). Therefore we can classify it as an $ARIMA(1,1,1)$. [½]

[Total 5]

Solution X4.6

> ### *Comment*
>
> *This question applies the Core Reading in Chapter 13 on the Box-Jenkins method of fitting a time series. Part (i) tests the bookwork on removing trends and Part (ii) looks at using graphs of the sample auto-correlation and sample partial auto-correlation functions to fit a time series.*

(i)(a) ***Removing seasonal variation***

Any ***one*** *of the following three methods for* ***two marks in total****.*

1. Seasonal differencing [½]

   Quarterly variation means that the period is four quarters (*ie* $\theta_t = \theta_{t+4}$). [½]

   So we subtract the value from 4 quarters ago:

   $$\nabla_4 q_t = q_t - q_{t-4}$$ [1]

2. Method of moving averages [½]

   Quarterly variation means that the period is four quarters (*ie* $\theta_t = \theta_{t+4}$). [½]

   So we find a symmetrical average of four terms about q_t:

   $$\tfrac{1}{4}\left(\tfrac{1}{2}q_{t-2} + q_{t-1} + q_t + q_{t+1} + \tfrac{1}{2}q_{t+2}\right)$$ [1]

3. Method of seasonal means [½]

   Quarterly variation means that the period is four quarters (*ie* $\theta_t = \theta_{t+4}$). [½]

   We first calculate estimates of the seasonal means from the data $\{q_1, \dots, q_{20}\}$:

   $$\hat{\theta}_1 = \tfrac{1}{5}(q_1 + q_5 + q_9 + q_{13} + q_{17}) - \hat{\mu}$$
   ... [½]
   $$\hat{\theta}_4 = \tfrac{1}{5}(q_4 + q_8 + q_{12} + q_{16} + q_{20}) - \hat{\mu}$$

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where $\hat{\mu} = \frac{1}{20}\sum q_i$ is the sample mean of the data.

Then we subtract the appropriate seasonal mean from the data:

$$z_t = \begin{cases} q_t - \hat{\theta}_1 & t = 1, 5, \ldots, 17 \\ \ldots & \\ q_t - \hat{\theta}_4 & t = 4, 8, \ldots, 20 \end{cases}$$ [½]

(i)(b) ***Linear trend***

*Any **one** of the following two methods for **one mark in total**.*

1. <u>Least squares trend removal</u>

Estimate a and b using least squares regression from CT3 [½]
(*ie* determine a and b that minimise $\sum y_t^2 = \sum (x_t - a - bt)^2$).

Then subtract the regression line from the observed values, $q_t - \hat{a} - \hat{b}t$. [½]

2. <u>Differencing</u>

Subtract the previous observed value:

$$\nabla q_t = q_t - q_{t-1}$$ [1]

[Total 3]

(ii) ***Fitted time series***

An appropriate time series is an $AR(2)$, $z_t = \alpha_1 z_{t-1} + \alpha_2 z_{t-2} + \varepsilon_t$. [1]

This is because the SPACF cuts of after lag 2 which indicates an $AR(2)$ process. [½]

And the SACF does not cut off but just decays. This is also consistent with an AR process. [½]

[Total 2]

Solution X4.7

Comment

Part (i) of the question applies the Core Reading in Chapter 14 on the Box-Muller and Polar algorithms for generating standard normal random variates. It goes on to transform these into variates from a lognormal distribution. Part (ii) looks at the reproducibility advantage of pseudo-random numbers.

(i)(a) ***Box-Muller formulae***

Using the formulae given on Page 39 of the *Tables* with the calculator set to radians, we obtain the following standard normal variates:

$$z_1 = \sqrt{-2\ln 0.846} \times \cos(2\pi \times 0.360) = -0.3686 \qquad [½]$$
$$z_2 = \sqrt{-2\ln 0.846} \times \sin(2\pi \times 0.360) = 0.4456 \qquad [½]$$

Note that if $Z \sim N(0,1)$ and $Y \sim N(\mu,\sigma^2)$ then we can apply the "standardising" relationship: $z = \dfrac{y-\mu}{\sigma}$ (or equivalently $y = \mu + \sigma z$) to flip between variates from the two distributions.

Recall also that if $Y = \log X \sim N(\mu,\sigma^2)$ then $X = e^Y \sim \log N(\mu,\sigma^2)$ and vice versa.

We can obtain values for $X \sim \log N(\mu,\sigma^2)$ using the transformation $X = e^{\mu+\sigma Z}$.

$$x_1 = e^{0.012-0.3686\sqrt{0.01}} = 0.9754 \qquad [½]$$
$$x_2 = e^{0.012+0.4456\sqrt{0.01}} = 1.058 \qquad [½]$$

(i)(b) ***Polar method***

First we need to obtain random variates from $V \sim U(-1,1)$. Since the distribution function is $F(v) = \frac{1}{2}(v+1)$, we have $v = 2u-1$ using the inverse transform method.

Hence, $v_1 = 0.692$, $v_2 = -0.280$ and $s = v_1^2 + v_2^2 = 0.557264$. [1]

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Using the Page 39 formulae, we obtain the following standard normal variates:

$$z_1 = 0.692\sqrt{\frac{-2\ln 0.557264}{0.557264}} = 1.002 \qquad [½]$$

$$z_2 = -0.280\sqrt{\frac{-2\ln 0.557264}{0.557264}} = -0.4056 \qquad [½]$$

We can obtain values for $X \sim \log N(\mu, \sigma^2)$ using the transformation $X = e^{\mu+\sigma Z}$:

$$x_1 = e^{0.012+1.002\sqrt{0.01}} = 1.118 \qquad [½]$$

$$x_2 = e^{0.012-0.4056\sqrt{0.01}} = 0.9718 \qquad [½]$$

[Total 5]

(ii) ***Different set of pseudo-random values?***

The company should use the same set of pseudo-random values so that any superior performance is not due to a more favourable combination of pseudo-random values. [1]

Solution X4.8

> ### *Comment*
>
> *This question applies the Core Reading in Chapter 13 on co-integration.*

(i) ***CPI and NAEI cointegrated***

CPI is price inflation, which drives wage inflation (the NAEI). So we would expect them to "move together". [1]

Both processes are not stationary but will have a trend (as prices and wages increase over time), so they may both be $I(1)$. [1]

[Total 2]

(ii) ***Cointegrated***

We need to show that $X \sim I(1)$, $Y \sim I(1)$ and $0.6X - Y \sim I(0)$. [½]

The first process can be differenced once as follows:

$$(X_n - X_{n-1}) - 0.2(X_{n-1} - X_{n-2}) = \varepsilon_n^x$$

$$\nabla X_n - 0.2\nabla X_{n-1} = \varepsilon_n^x \qquad [½]$$

Since the first difference is a stationary $AR(1)$ (as the coefficient is smaller than 1 in magnitude), this means X_n is $I(1)$. [½]

Alternatively, students could formally show the difference is stationary by showing that the root of the characteristic equation is greater than 1 in magnitude.

Differencing the second process gives:

$$\begin{aligned}(Y_n - Y_{n-1}) &= (0.6X_{n-1} + \varepsilon_n^y) - (0.6X_{n-2} + \varepsilon_{n-1}^y) \\ \nabla Y_n &= 0.6\nabla X_{n-1} + \nabla \varepsilon_n^y\end{aligned}$$ [½]

Since ∇X_n is stationary (and the white noise terms are stationary), then ∇Y_n is the sum of stationary terms and so is also stationary. Hence, Y_n is $I(1)$. [½]

If $(0.6, -1)$ is the cointegrating vector then $0.6X_n - Y_n$ should be stationary:

$$\begin{aligned}0.6X_n - Y_n &= 0.6(1.2X_{n-1} - 0.2X_{n-2} + \varepsilon_n^x) - (0.6X_{n-1} + \varepsilon_n^y) && [½] \\ &= 0.12X_{n-1} - 0.12X_{n-2} + 0.6\varepsilon_n^x - \varepsilon_n^y \\ &= 0.12\nabla X_{n-1} + 0.6\varepsilon_n^x - \varepsilon_n^y && [½]\end{aligned}$$

Since ∇X_{n-1} is stationary (and the white noise terms are stationary), then $0.6X_n - Y_n$ is the sum of stationary terms and so is also stationary (*ie* $0.6X - Y \sim I(0)$). [½]

[Total 4]

Solution X4.9

> ### *Comment*
>
> *Part (i) of this question tests the bookwork in Chapter 12 that shows how a stationary AR(1) series can be written in terms of an infinite MA series as well as finding its mean and variance. Part (ii) applies this bookwork to look at the distributional properties of this infinite MA series.*

(i)(a) ***Expressing X_t as a summation of white noise***

Substituting for X_{t-1} gives:

$$\begin{aligned} X_t &= \alpha X_{t-1} + Z_t \\ &= \alpha(\alpha X_{t-2} + Z_{t-1}) + Z_t \\ &= \alpha^2 X_{t-2} + Z_t + \alpha Z_{t-1} \end{aligned} \qquad [½]$$

Now substituting for X_{t-2} gives:

$$\begin{aligned} X_t &= \alpha^2 X_{t-2} + Z_t + \alpha Z_{t-1} \\ &= \alpha^2(\alpha X_{t-3} + Z_{t-2}) + Z_t + \alpha Z_{t-1} \\ &= \alpha^3 X_{t-3} + Z_t + \alpha Z_{t-1} + \alpha^2 Z_{t-2} \end{aligned} \qquad [½]$$

Continuing this pattern gives:

$$\begin{aligned} X_t &= (Z_t + \alpha Z_{t-1} + \alpha^2 Z_{t-2} + \alpha^3 Z_{t-3} + \cdots) \\ &= \sum_{j=0}^{\infty} \alpha^j Z_{t-j} \end{aligned} \qquad [1]$$

Note to markers: the full 2 marks should be given for obtaining the expression even if the students have not shown all of the repeated substitutions used.

(i)(b) ***Mean and variance***

Using the expression developed in Part (i)(a), we have:

$$E(X_t) = E\left(\sum_{j=0}^{\infty} \alpha^j Z_{t-j}\right) = \sum_{j=0}^{\infty} \alpha^j E(Z_{t-j}) = 0$$ [½]

$$\text{var}(X_t) = \text{var}\left(\sum_{j=0}^{\infty} \alpha^j Z_{t-j}\right) = \sum_{j=0}^{\infty} \alpha^{2j} \text{ var}(Z_{t-j}) = \sum_{j=0}^{\infty} \alpha^{2j}\sigma^2$$ [½]

Summing the infinite geometric series using $S_\infty = \frac{a}{1-r}$ with $a = \sigma^2$ and $r = \alpha^2$ (valid since $|r| = |\alpha^2| < 1$) gives: [½]

$$\text{var}(X_t) = \frac{\sigma^2}{1-\alpha^2}$$ [½]

[Total 4]

(ii)(a) ***Explanation of normal distribution and probability***

X_t is the sum of white noise terms which are normal random variables. Since the sum of independent normal random variables is itself a normal random variable, we have:[½]

$$X_t \sim N\left(0, \tfrac{\sigma^2}{1-\alpha^2}\right)$$ [½]

When $t = 10$, $\alpha = 0.6$ and $\sigma^2 = 50$, we have:

$$X_{10} \sim N(0, 78.125)$$

Hence:

$$P(X_{10} > 10) = P\left(Z > \frac{10-0}{\sqrt{78.125}}\right) = P(Z > 1.13137)$$ [1]

$$= 1 - 0.87105 = 0.12895$$ [1]

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(ii)(b) ***Comment if $\alpha > 1$***

For $\alpha > 1$ the mean would be unchanged, but the variance would be huge (as the infinite sum would no longer be convergent). Therefore the probability of being greater than 10 would be $P(Z > 0) = ½$. [1]

[Total 4]

Solution X4.10

> ### *Comment*
>
> *This question applies the Core Reading in Chapter 14 on the acceptance-rejection method.*

(i)(a) ***Generate uniform observations***

If $Y \sim U(-1,1)$ then $F(y) = \frac{y-(-1)}{1-(-1)} = \frac{y+1}{2}$

To generate observations, y, using the inverse transform method on a pseudo-random number u:

$$u = F(y) = \tfrac{y+1}{2} \quad \Rightarrow \quad y = 2u - 1$$ [½]

This gives:

–0.078, 0.932, 0.848 [½]

(i)(b) ***Acceptance-rejection method***

The PDF of the $U(-1,1)$ is $h(x) = \frac{1}{2}$ where $-1 < x < 1$.

The probability of accepting a value x generated from $U(-1,1)$ is:

$$g(x) = \frac{f(x)}{Ch(x)}$$ [½]

where $C = \max\limits_{-1<x<1} \left\{ \frac{f(x)}{h(x)} \right\} = \max\limits_{-1<x<1} \left\{ 1.2 - 0.6x^2 \right\}$. [½]

Differentiating to find the maximum and setting the derivative equal to zero gives:

$$\tfrac{d}{dx}\left\{1.2-0.6x^2\right\}=-1.2x=0 \quad\Rightarrow\quad x=0$$
$$\tfrac{d^2}{dx^2}\left\{1.2-0.6x^2\right\}=-1.2<0 \quad\Rightarrow\quad \text{max}$$

Alternatively, students may spot that $x=0$ *will give a maximum just by observation.*

Hence by substituting $x=0$, we get $C=1.2-0.6\times 0^2=1.2$. [½]

Therefore the probability of accepting a value x generated from $U(-1,1)$ is:

$$g(x)=\frac{f(x)}{Ch(x)}=\frac{0.6-0.3x^2}{1.2\times\frac{1}{2}}=1-\tfrac{1}{2}x^2 \qquad [½]$$

For $x=-0.078$ the probability of accepting this value is:

$$g(-0.078)=1-\tfrac{1}{2}(-0.078)^2=0.9970 \qquad [½]$$

Since $0.843<g(-0.078)=0.9970 \quad\Rightarrow\quad$ accept -0.078. [½]

Similarly for $x=0.932$ we have $g(0.932)=0.5657$. [½]

Since $0.427<g(0.932)=0.5657 \quad\Rightarrow\quad$ accept 0.932. [½]

Finally for $x=0.848$ we have $g(0.848)=0.6404$. [½]

Since $0.683>g(0.848)=0.6404 \quad\Rightarrow\quad$ reject 0.848. [½]

So the simulated values of X are:

–0.078, 0.932

[Total 6]

(ii) ***Proportion rejected***

The proportion of simulations accepted in the long run is $\frac{1}{C}=\frac{1}{1.2}=\frac{5}{6}$. Hence the proportion of simulations rejected in the long run is $\frac{1}{6}$. [1]

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Alternatively, we could calculate this from first principles: the probability that a value x will be rejected is $0.5x^2$. Since the values of x come from a $U(-1,1)$ distribution, the overall probability of rejection will be:

$$\int_{-1}^{1} 0.5x^2 \times \tfrac{1}{2}\,dx = \left[0.25 \times \tfrac{1}{3}x^3\right]_{-1}^{1} = \tfrac{1}{6}$$

(iii) ***Comparison of methods***

Integrating the density function would give a cubic distribution function, $F(x)$. Inverting a cubic function to get $x = F^{-1}(u)$ is not easy to do. [1]

Calculation of the distribution function is not required to obtain the mark:

$$F(x) = \int_{-1}^{x} (0.6 - 0.3t^2)\,dx = \left[0.6t - 0.1t^3\right]_{-1}^{x} = 0.6x - 0.1x^3 + 0.5$$

Solution X4.11

Comment

Part (i) of this question applies the Core Reading in Chapter 12 on the invertible property of time series. Part (ii) applies the Core Reading in Chapter 13 comparing different ways (method of moments, least squares regression and MLE) of fitting parameter values for a time series. Part (iii) focuses on two different ways of forecasting: k-step ahead and exponential smoothing.

(i) ***Invertible***

When $|\beta| < 1$ the process is invertible. [1]

Students can prove it as follows but it is not required to obtain the mark.

Rewriting the equation in terms of the backwards shift operator: $(1-\alpha B)X_n = (1+\beta B)e_n$

Solving the white noise characteristic equation gives: $1 + \beta\lambda = 0 \;\Rightarrow\; \lambda = -\frac{1}{\beta}$

The process is invertible if $|\lambda| > 1$, *hence if* $|\beta| < 1$.

(ii)(a) ***Method of moments***

Equating the given formula (with $k=1$ and $k=2$) to the observed values of the autocorrelation coefficients, gives:

$$\frac{(\alpha+\beta)(1+\alpha\beta)}{1+2\alpha\beta+\beta^2}=0.440 \qquad \text{and} \qquad \frac{(\alpha+\beta)(1+\alpha\beta)}{1+2\alpha\beta+\beta^2}\alpha=0.264$$

We can find α straight away by dividing these, which gives:

$$\alpha=\frac{0.264}{0.440}=0.6$$ [½]

Substituting $\alpha=0.6$ into the first equation gives:

$$\frac{(0.6+\beta)(1+0.6\beta)}{1+1.2\beta+\beta^2}=0.440$$ [½]

$$\Rightarrow \quad (0.6+\beta)(1+0.6\beta)=0.440(1+1.2\beta+\beta^2)$$

$$\Rightarrow \quad 0.16+0.832\beta+0.16\beta^2=0$$ [½]

The roots of this quadratic are:

$$\beta=\frac{-0.832\pm\sqrt{0.832^2-4\times0.16^2}}{2\times0.16}=-0.2,-5$$ [1]

Hence the required value is $\beta=-0.2$ (as we are told that the process is invertible). [½]

(ii)(b) ***Least squares estimation***

(1) Assume $\varepsilon_0=0$, and obtain (iteratively) expressions for ε_1, ε_2, …

(2) Obtain $\sum\varepsilon_i^2$ in terms of α and β.

(3) Calculate values of α and β which minimise this expression.

(4) Use these values of α and β to work backwards to time zero from the most recent known values of the time series, to determine an updated value for ε_0.

(5) Repeat the process to find improved estimates for α and β.

[½ each, maximum 2]

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(ii)(c) ***Maximum likelihood estimation***

They are equivalent if we assume that $\varepsilon_t \sim N(0,\sigma^2)$. [1]

[Total 6]

(iii)(a) ***1 and 2 step ahead estimates***

The fitted model is:

$$x_n = 0.6x_{n-1} + \varepsilon_n - 0.2\varepsilon_{n-1}$$

Hence the 1 and 2 step-ahead forecasts are:

$$\hat{x}_{80}(1) = 0.6x_{80} + \hat{\varepsilon}_{81} - 0.2\hat{\varepsilon}_{80} = (0.6\times 1.087) + 0 - (0.2\times 1.181) = 0.416$$ [1]

$$\hat{x}_{80}(2) = 0.6\hat{x}_{80}(1) + \hat{\varepsilon}_{82} - 0.2\hat{\varepsilon}_{81} = (0.6\times 0.416) + 0 - (0.2\times 0) = 0.2496$$ [1]

(iii)(b) ***Exponential smoothing***

Exponential smoothing uses the formula:

$$\hat{x}_n(1) = (1-\alpha)\hat{x}_{n-1}(1) + \alpha x_n$$

We are told that the estimate at time 79 for x_{80} was $\hat{x}_{79}(1) = 0.625$ and the smoothing parameter is $\alpha = 0.2$. Hence, our estimate of x_{81} is:

$$\hat{x}_{80}(1) = (1-\alpha)\hat{x}_{79}(1) + \alpha x_{80}$$ [½]

$$= (0.8\times 0.625) + (0.2\times 1.087)$$

$$= 0.7174$$ [½]

[Total 3]

Solution X4.12

Comment

Part (i) of this question tests the bookwork in Chapter 12 relating to the definition of an ARIMA time series. Part (ii) applies the Core Reading from Chapter 13 looking at how many times to difference time series data until it can be considered stationary. Part (iii) applies the Core Reading in Chapter 13 on testing the goodness of fit of a fitted model by looking at the residuals.

(i) ***General equation for an ARIMA(p, d, q) model***

Any one of the following expressions is a general equation for an $ARIMA(p,d,q)$:

$$\left(\nabla^d X_t - \mu\right) = \sum_{i=1}^{p} \alpha_i \left(\nabla^d X_{t-i} - \mu\right) + \varepsilon_t + \sum_{j=1}^{q} \beta_j \varepsilon_{t-j}$$

$$\nabla^d \left(X_t - \mu\right) = \sum_{i=1}^{p} \alpha_i \nabla^d \left(X_{t-i} - \mu\right) + \varepsilon_t + \sum_{j=1}^{q} \beta_j \varepsilon_{t-j}$$

$$(1-B)^d \phi(B)(X_t - \mu) = \theta(B)\varepsilon_t \text{ where } \phi(B) = 1 - \sum_{i=1}^{p} B^i \alpha_i \text{ and } \theta(B) = \sum_{j=1}^{q} B^j \beta_j$$

[1]

Note that the μ in the first formula is the mean of the differenced time series, whereas the μ in the other formulae is the (non-constant) mean of the original time series X_t. Markers please give credit for any of these expressions, providing the student appears to know what they're doing.

(ii) ***An appropriate value of d***

If the sample autocorrelation coefficients decay slowly from 1 then this indicates that differencing is required. Since this is not the case for $d = 0$ this means that differencing of the original series is not required. [1]

We would also expect the "correct" value of d to minimise the sample variance. This also indicates that $d = 0$. [1]

[Total 2]

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(iii)(a) ***Ljung-Box ("portmanteau") test***

We are testing whether the ACF of the residuals is zero (there is no correlation between them). This would indicate that the residuals are white noise, *ie* the $ARMA(1,1)$ model is a good fit. This is our null hypothesis.

From Page 42 of the *Tables*:

$$n(n+2)\sum_{k=1}^{m}\frac{r_k^2}{n-k} \sim \chi^2_{m-(p+q)}$$

We have $n=100$, $m=5$, $p=1$ and $q=1$. Hence the statistic is:

$$100(102)\left\{\frac{0.14^2}{99}+\frac{(-0.05)^2}{98}+\frac{0.1^2}{97}+\frac{0.12^2}{96}+\frac{(-0.02)^2}{95}\right\}=4.904 \qquad [1]$$

Under the null hypothesis this statistic has χ^2 with $5-(1+1)=3$ degrees of freedom. *We lose two degrees of freedom, one for each parameter that has been estimated.* [½]

The 5% critical value for the χ^2_3 distribution is 7.815. [½]

Since $4.904 < 7.815$ we have insufficient evidence at the 5% level to reject the null hypothesis. Hence we conclude that the residuals do not show any signs of correlations and the $ARMA(1,1)$ model is a good fit to the observed series. [1]

(iii)(b) ***Turning point test***

We are testing whether the residuals are patternless. This would indicate that the residuals are white noise, *ie* the $ARMA(1,1)$ model is a good fit. This is our null hypothesis.

From Page 42 of the *Tables*, the turning points of 100 uncorrelated values has mean and variance of:

$$E(T)=\tfrac{2}{3}\times 98 = 65\tfrac{1}{3} \qquad \text{and} \qquad \text{var}(T)=\tfrac{16\times 100-29}{90}=\tfrac{1{,}571}{90} \quad (=17.456) \qquad [½]$$

Using a normal approximation with continuity correction (since T is discrete), the observed value of 74 turning points corresponds to a standard normal value of:

$$z=\frac{73\tfrac{1}{2}-65\tfrac{1}{3}}{\sqrt{\tfrac{1,571}{90}}}=1.955$$ [1]

The 5% critical values are ± 1.96 (as it is a two-sided test). [½]

Alternatively, students may calculate the p-value:

$$P(T>74)=P\left(Z>\frac{73\tfrac{1}{2}-65\tfrac{1}{3}}{\sqrt{\tfrac{1,571}{90}}}\right)=1-\Phi(1.9547)=1-0.9747=0.0253$$

Since $1.955<1.960$ *(or alternatively since the p-value is greater than 2.5%)*, we have insufficient evidence at the 5% level to reject the null hypothesis. Hence we conclude that the residuals are patternless and the $ARMA(1,1)$ model is a good fit to the observed series. [1]

*Students who don't use a continuity correction will obtain a statistic of 2.074 (or a p-value of 1.90%). They will then reject the null hypothesis and conclude the $ARMA(1,1)$ model is **not** a good fit. Award students only 2 marks in total.*

(iii)(c) ***Sample correlations***

The ACF of the residuals should be zero for all lags except 0. A 95% confidence interval for the sample ACF is $\pm 2/\sqrt{n}=\pm 0.2$ (or accurately $\pm 1.96/\sqrt{n}=\pm 0.196$). [1]

Since all the values lie within this confidence interval, there is no evidence to suggest the model is not satisfactory. [1]

[Total 8]

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Solution X4.13

Comment

This question applies the Core Reading in Chapters 12 and 13 relating to various aspects of auto-regressive (AR) time series. Parts (i) and (ii) test the properties and the calculation of the auto-correlation and partial auto-correlation functions from Chapter 12. Parts (iii) and (iv) test the material in Chapter 13 on re-expressing a univariate time series as a multivariate time series and why this might be useful to do.

(i)(a) ***Stationary***

Since $(1-0.7B+0.1B^2)X_n = \varepsilon_n$, the characteristic equation of the AR terms is:

$$\phi(\lambda) = 1-0.7\lambda+0.1\lambda^2 = 0 \quad \Rightarrow \quad \lambda = 2,5$$ [½]

Since both of the roots are greater than one in magnitude the process ***is*** stationary. [½]

(i)(b) ***Invertible***

We can express the white noise term as follows:

$$\varepsilon_n = X_n - 0.7X_{n-1} + 0.1X_{n-2}$$ [½]

Since we will be able to calculate the residuals from the observed values the process ***is*** invertible. [½]

Use of the characteristic polynomial of the MA terms to identify invertibility will not be helpful here.

(i)(c) ***Purely indeterministic***

A random white noise term, ε_n, is added to each X_n. [½]

So the further in the future we get, the more random terms are added and so the less correlation there will be (hence x_n will be less useful in predicting x_{n+k} as k gets bigger).

So the process ***is*** purely indeterministic. [½] (i)(d) ***Markov***

X_n depends on X_{n-1} and X_{n-2}. [½]

Hence the conditional distribution of $X_n \mid X_{n-1} = x_{n-1}, X_{n-2} = x_{n-2}, \ldots, X_1 = x_1$ will not be the same as the conditional distribution of $X_n \mid X_{n-1} = x_{n-1}$. [1]

The process is ***not*** Markov. [½]
[Total 5]

(ii)(a) ***ACF***

Since the process is stationary, the Yule-Walker equations are:

$$\begin{aligned}\gamma_1 &= \text{cov}(X_n, X_{n-1}) = \text{cov}(0.7X_{n-1} - 0.1X_{n-2} + \varepsilon_n, X_{n-1}) \\ &= 0.7\gamma_0 - 0.1\gamma_1\end{aligned}$$ [½]

$$\begin{aligned}\gamma_2 &= \text{cov}(X_n, X_{n-2}) = \text{cov}(0.7X_{n-1} - 0.1X_{n-2} + \varepsilon_n, X_{n-2}) \\ &= 0.7\gamma_1 - 0.1\gamma_0\end{aligned}$$ [½]

And in general for $k > 2$, we have:

$$\begin{aligned}\gamma_k &= \text{cov}(X_n, X_{n-k}) = \text{cov}(0.7X_{n-1} - 0.1X_{n-2} + \varepsilon_n, X_{n-k}) \\ &= 0.7\gamma_{k-1} - 0.1\gamma_{k-2}\end{aligned}$$ [½]

Award students the generalisation half mark if they infer it from the pattern $\gamma_3 = 0.7\gamma_2 - 0.1\gamma_1$, $\gamma_4 = 0.7\gamma_3 - 0.1\gamma_2$, *... instead.*

So the autocorrelations are:

$$\rho_1 = \frac{\gamma_1}{\gamma_0} = \frac{0.7\gamma_0 - 0.1\gamma_1}{\gamma_0} = 0.7 - 0.1\rho_1$$ [½]

$$\Rightarrow \quad 1.1\rho_1 = 0.7 \quad \Rightarrow \quad \rho_1 = \frac{7}{11}$$ [½]

$$\rho_2 = \frac{\gamma_2}{\gamma_0} = \frac{0.7\gamma_1 - 0.1\gamma_0}{\gamma_0} = 0.7\rho_1 - 0.1 = \frac{19}{55}$$ [1]

And in general for $k > 2$, we have:

$$\rho_k = \frac{\gamma_k}{\gamma_0} = \frac{0.7\gamma_{k-1} - 0.1\gamma_{k-2}}{\gamma_0} = 0.7\rho_{k-1} - 0.1\rho_{k-2}$$ [½]

Note that the expression $\gamma_0 = 0.7\gamma_1 - 0.1\gamma_2 + \sigma^2$ is not required nor are calculation of the autocovariances $\gamma_0 = \frac{275}{162}\sigma^2$, $\gamma_1 = \frac{175}{162}\sigma^2$ and $\gamma_2 = \frac{95}{162}\sigma^2$.

We can use this shortcut when we only need the ACF of an $AR(p)$.

(ii)(b) ***General solution to the difference equation***

If $\rho_k = \dfrac{A}{2^k} + \dfrac{B}{5^k}$ is a solution of $\rho_k = 0.7\rho_{k-1} - 0.1\rho_{k-2}$ then it will satisfy the equation:

$$\frac{A}{2^k} + \frac{B}{5^k} = 0.7\left(\frac{A}{2^{k-1}} + \frac{B}{5^{k-1}}\right) - 0.1\left(\frac{A}{2^{k-2}} + \frac{B}{5^{k-2}}\right)$$ [½]

$$= A\left(\frac{0.7}{2^{k-1}} - \frac{0.1}{2^{k-2}}\right) + B\left(\frac{0.7}{5^{k-1}} - \frac{0.1}{5^{k-2}}\right)$$ [½]

$$= \frac{A}{2^k}(0.7 \times 2 - 0.1 \times 2^2) + \frac{B}{5^k}(0.7 \times 5 - 0.1 \times 5^2)$$ [½]

$$= \frac{A}{2^k} + \frac{B}{5^k}$$ [½]

We have:

$$\rho_3 = 0.7\rho_2 - 0.1\rho_1 = 0.7 \times \frac{19}{55} - 0.1 \times \frac{7}{11} = \frac{49}{275} = 0.17\dot{8}\dot{1}$$ [½]

$$\rho_4 = 0.7\rho_3 - 0.1\rho_2 = 0.7 \times \frac{49}{275} - 0.1 \times \frac{19}{55} = \frac{124}{1,375} = 0.090\dot{1}\dot{8}$$ [½]

Hence:

$$\rho_3 = \frac{A}{2^3} + \frac{B}{5^3} = \frac{49}{275} \qquad (1)$$ [½]

$$\rho_4 = \frac{A}{2^4} + \frac{B}{5^4} = \frac{124}{1,375} \qquad (2)$$ [½]

Dividing the equation (1) by 2 and subtracting equation (2) gives:

$$B = -\frac{5}{11}$$ [1]

Substituting this back into either equation gives:

$$A = \frac{16}{11}$$ [1]

An alternative solution involves using the information on Page 4 of the Tables to show the required result and/or a different method to solve for A and B.

We can write the difference equation $\rho_k = 0.7\rho_{k-1} - 0.1\rho_{k-2}$ in the form: $a\rho_k + b\rho_{k-1} + c\rho_{k-2} = 0$ with $a = 1$, $b = -0.7$ and $c = 0.1$. [½]

Now $b^2 - 4ac = 0.7^2 - 4 \times 1 \times 0.1 = 0.09$. [½]

From Page 4 of the Tables, the general solution to this difference equation has the form:

$$\rho_k = A\lambda_1^k + B\lambda_2^k$$ [½]

where λ_1 and λ_2 are the solutions to the quadratic equation $\lambda^2 - 0.7\lambda + 0.1 = 0$.

We solve the quadratic to find that $\lambda_1 = 0.5$ and $\lambda_2 = 0.2$, and: [½]

$$\rho_k = A\frac{1}{2^k} + B\frac{1}{5^k}$$ as required.

Note that the general autocorrelation function $\rho_k = 0.7\rho_{k-1} - 0.1\rho_{k-2}$ also holds for $k = 1$ and $k = 2$.

For example, using the results from Part (ii)(a) above:

$$\begin{aligned}\rho_1 &= 0.7 - 0.1\rho_1 \\ &= 0.7\rho_0 - 0.1\rho_{-1}\end{aligned}$$ [½]

And

$$\begin{aligned}\rho_2 &= 0.7\rho_1 - 0.1 \\ &= 0.7\rho_1 - 0.1\rho_0\end{aligned}$$ [½]

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Similarly, we can see that the formula $\dfrac{A}{2^k}+\dfrac{B}{5^k}=0.7\left(\dfrac{A}{2^{k-1}}+\dfrac{B}{5^{k-1}}\right)-0.1\left(\dfrac{A}{2^{k-2}}+\dfrac{B}{5^{k-2}}\right)$ holds for $k=1$ and $k=2$.

Hence:

$$\rho_1=\frac{A}{2^1}+\frac{B}{5^1}=\frac{7}{11} \qquad [½]$$

$$\rho_2=\frac{A}{2^2}+\frac{B}{5^2}=\frac{19}{55} \qquad [½]$$

Solving these simultaneously gives $B=-\dfrac{5}{11}$ and $A=\dfrac{16}{11}$. [2]

(ii)(c) ***PACF***

Using the formulae given on Page 40 with $\rho_1=\frac{7}{11}$ and $\rho_2=\frac{19}{55}$ from Part (ii)(a) gives:

$$\phi_1=\rho_1=\frac{7}{11} \qquad [½]$$

$$\phi_2=\frac{\rho_2-\rho_1^2}{1-\rho_1^2}=-\frac{1}{10} \qquad [½]$$

Since we have an $AR(2)$, the PACF, $\phi_k=0$ for $k>2$. [1]

[Total 12]

(iii) ***Univariate to multivariate***

We can express $X_n=0.7X_{n-1}-0.1X_{n-2}+\varepsilon_n$ as a $VAR(1)$ as follows:

$$\begin{pmatrix} X_n \\ X_{n-1}\end{pmatrix}=\begin{pmatrix}0.7 & -0.1\\ 1 & 0\end{pmatrix}\begin{pmatrix} X_{n-1} \\ X_{n-2}\end{pmatrix}+\begin{pmatrix}\varepsilon_n \\ 0\end{pmatrix}$$

[½ for each of the 4 vectors/matrices, total 2]

This corresponds to $\mathbf{X}_n=\mathbf{A}\mathbf{X}_{n-1}+\boldsymbol{\varepsilon}_n$ where:

$$\mathbf{X}_n=\begin{pmatrix} X_n \\ X_{n-1}\end{pmatrix} \qquad \mathbf{A}=\begin{pmatrix}0.7 & -0.1\\ 1 & 0\end{pmatrix} \qquad \boldsymbol{\varepsilon}_n=\begin{pmatrix}\varepsilon_n \\ 0\end{pmatrix}$$

(iv)(a) ***Stationary***

The $VAR(1)$ process in Part (iii) is stationary if the eigenvalues of matrix $\mathbf{A}$ are less than 1 in magnitude. [1]

The eigenvalues, λ, satisfy:

$$\det(\mathbf{A}-\lambda\mathbf{I}) = \det\left\{\begin{pmatrix} 0.7 & -0.1 \\ 1 & 0 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right\} = \det\begin{pmatrix} 0.7-\lambda & -0.1 \\ 1 & -\lambda \end{pmatrix} = 0 \qquad [½]$$

This gives:

$$(0.7-\lambda)(-\lambda)-(-0.1)(1) = \lambda^2 - 0.7\lambda + 0.1 = 0 \quad \Rightarrow \quad \lambda = 0.2, 0.5 \qquad [1]$$

Since both of the eigenvalues are less than one in magnitude the process ***is*** stationary. [½]

Note that this process is covered as an Appendix to Chapter 13.

(iv)(b) ***Markov***

$\mathbf{X}_n$ depends on $\mathbf{X}_{n-1}$ only. [½]

Alternatively, the conditional distribution of $\mathbf{X}_n \mid \mathbf{X}_{n-1} = \mathbf{x}_{n-1}, \mathbf{X}_{n-2} = \mathbf{x}_{n-2}, \ldots, \mathbf{X}_1 = \mathbf{x}_1$ *will the same as the conditional distribution of* $\mathbf{X}_n \mid \mathbf{X}_{n-1} = \mathbf{x}_{n-1}$.

So the process ***is*** Markov. [½]
[Total 4]

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