Topological Quantum Computing
An Introduction

Theodore Mefford
University of Maryland

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Abstract

Topological quantum computation offers an elegant solution to developing a fault-tolerant quantum computer by storing information non-locally in the states of quasiparticles. Unitary gate operations are achieved by braiding the quasiparticles and measurement is achieved through their fusion. This paper describes the underlying principles of topological quantum computation, exemplifies the process with a simple model, presents current topological phases of matter being explored, and discusses several issues it faces.

1 Introduction

The concept of a quantum computer has been known since Feynman introduced it in 1982 but the benefit of such a computer over a classical computer was unknown. The greatest challenge was to find problems that could be solved by exploiting the quantum nature of such a device with efficient speed-up over classical approaches. Shor found such a problem and solution when he introduced his factoring algorithm in 1994. With this theory firmly in place, the benefit of a quantum computer was obvious and the race to build such a computer began. However, the realization of a quantum computer comes with extensive technical difficulties and in particular the issue of decoherence, the loss of information due to interactions with the environment.

A tidy solution to decoherence was found by Kitaev when he realized that the quantum states of the computer could be encoded and operated on in a non-local way. Rather than fight decoherence, it might be possible to side-step the issue completely if certain topological phases of matter could be experimentally realized and controlled. A topological state is fundamentally different from physical states of matter and does not interact directly. This makes it immune to the type of external perturbations that decohere states in physical systems such as spin qubits.

Coinciding with the developments in quantum information theory were developments in condensed matter physics that opened the door to exploring topological phases of matter that might achieve topological quantum computation. The most important of these are the discovery of the fractional quantum Hall effect and the ability to have high temperature superconductivity. With the corresponding theoretical advances in quantum computation and the experimental discovery of topological phases of matter, the journey towards a quantum computer has found a new route.

2 Primary Principles

In order to understand topological quantum computation, it is useful to examine several of the key underlying concepts at the heart of its power. Firstly, the concept of uniqueness must be abandoned. We are classically accustomed to viewing the world deterministically through position and time. A single objects identity is its own and can be traced back through time. In quantum mechanics, the position of a particle is spread out in space as a wavefunction. When two particles become close their wave functions will overlap and it is not possible to discuss them as independent entities. A single wavefunction describing both is necessary. Topological quantum computation stretches this bound further. Anyons are not actual particles but quasiparticles, emergent excitations in solids that behave as if they were.

Secondly, elementary particles of the same type are indistinguishable from each other. They share the exact same mass, charge, spin and any other quantum number. Feynman famously tells of a call from John Wheeler:

"Feynman, I know why all electrons have the same charge and the same mass" "Why?" "Because, they are all the same electron!"

The idea being that all electrons worldliness are actually a single electron's weaving backwards and forwards through space-time. Indistinguishably allows universal theories that describe particles to be constructed on simple statistical rules.

Thirdly, exchange statistics provides a rich structure underpinning elementary quantum physics. Symmetries deal with changes in a system that leave the underlying properties unaffected and exchange statistics are a form of this symmetry. If two identical particles are exchanged and then exchanged back to their original position, by statistical symmetry it is intuitively clear that the physics will remain unchanged from if they had not been exchanged. Also, the particular path of the exchange is irrelevant so exchange statistics are topological by nature. Exchange statistics dictate the existence of bosons and fermion and, as we'll see, anyons.

3 Quantum Statistics and the Model of Nonabelian Anyons

The principles of the last section underpin the existence and behavior of elementary particles. In fact, they are enough to predict the existence of bosons and fermions. In three dimensions, quantum statistics offers two possibilities in the case that two particles are adiabatically exhanged since two such exchanges should the net effect on the wavefunction is multiplication by 1. Thus, there are two possible effects a single exchange could have on the wavefunction. It is either multiplied by a phase of +1 or -1. If the overall wave function change is by a minus sign then the particles are termed fermions. Whereas a state that is left the same is a boson.

To understand how anyons arise we employ a simple thought experiment. Consider the statistics of circulating one particle about another as shown in Figure 1. The path $\Psi(C_1)$ traverses around the other particle while the $\Psi(C_2)$ passes in front. Either path can be continuously deformed around the second particle to match the other path. As a consequence, the wavefunction of either path has to be exactly the same as the original.

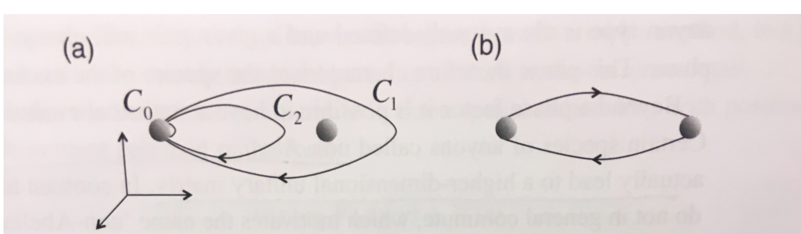


Figure 1: particle trajectories in 3-D (Panczos 2012)

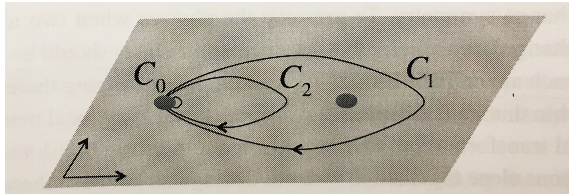


Figure 2: Particle trajectories in 2-D (Panczos 2012)

$$\Psi(C_0) = \Psi(C_1) = \Psi(C_2)$$

This is statistically necessary and is equivalent to the exchange statistics giving rise to fermions and bosons.

In two dimensions, we obtain different statistics due to the inability to deform a topological trajectory of one particle around another without cutting through that particle. In fact, this allows a wavefunction to pick up any arbitrary phase when two particles are exchanged. These exotic particles with phases other than 0 and π were duly named "anyons".

Now consider the same scenario of revolving particles around each other but with time displayed as the third axis. These topological exchanges take place in 2+1 dimensions and can be viewed as worldlines originating from some initial position to their final position. These world lines braid around each other and can be viewed in two dimensions if care is taken in representing the way in which the particles exchange since a counter-clockwise exchange is non-Abelian with a clockwise one. This braiding can be represented using generators σ as seen in Figure 3.

Worldline depictions provide a convenient means to keep track of an array of anyons histories. Exchanges are represented as braiding that has a clear depiction of which anyon passed under the other, as different rotation direction does not necessarily lead to the same statistics. Combining two braids into one easily portrays the process fusion. It is unnecessary to worry about the shape of the braids, as long as they do not cross, since the information is topological.

4 Computation with Anyons

Anyons are quasiparticles that arise in topological systems and when braided around each other they acquire non-trivial phases. In the last section we gained a useful visual represent with worldlines for depicting how such interactions occur. In this section we discuss the general features of a model of topological quantum computation before more rigorously defining it for non-Abelian anyons.

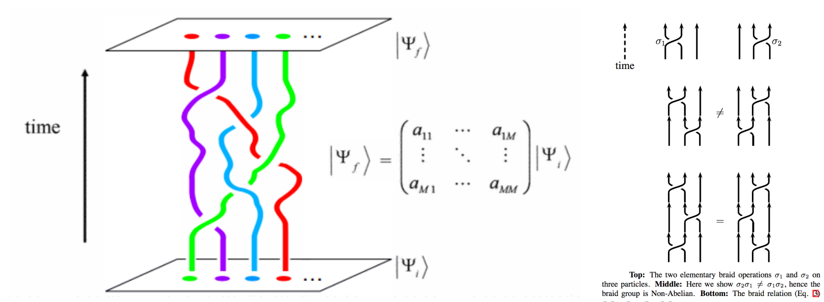


Figure 3: Worldlines in 2+1 dimensions (Nayak 2008)

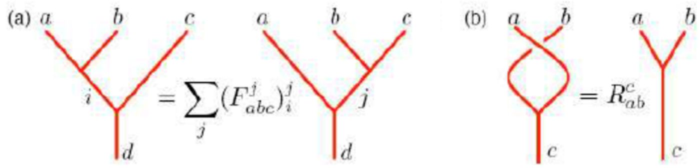


Figure 4: Potrayal of F-Matrix realations (Researchgate)

Fusion is the process in which two particle fuse to form a third. The fusion of two anyons can result in a variety of outcomes and fusion rules indicate what these possible outcomes are. In contrast to the single fusion channel for Abelian anyons, non-Abelian anyons have multiple, giving rise to higher a dimensional fusion space. This leads to much richer statistical dynamics for non-Abelian anyons that corresponds to a non-trivial evolution between the different possible outcomes.

The fusion of several anyons into a single anyon can occur several ways depending upon the order in which the anyons individually fused. To relate these differing routes to the same result, the F-matrix is introduced. One possible pathway can be translated into another by a rotation in F-space. As an example, consider the fusion of three particles a,b, and c into a single particle d. This can occur by a fusing particle a with b into a new particle i which then fuses with c into d. Alternatively, b could fuse with c into j which then fuses with a into j. The F-matrix translates these paths into one another.

The Hilbert space of anyons corresponds to the fusion process. The evolution of two anyons that have fused is assigned a distinct state. This leads to an inherent orthogonality between states since different anyons can be distinguished.

Analogous to how various routes of fusion can lead to the same anyon, various exchanges before a fusion can lead to the same anyon. In the simplest case, consider particles a and b that fuse to c.(Fig 5) They could simply fuse, but they could also exchange before fusing into c. R, the exchange matrix, relates these differing braiding patterns that lead to the same end result.

With fusion and braiding rules in place, a model of anyonic quantum computation can be explored. Begin with a set of anyons that are prepared in a well-defined fusion state. Braiding anyons with the R matrix and fusing them with the F matrix perform logical gate operations that are non-trivial. It has been proven that a universal set of gate can be realized if the F and R matrices span a dense set of unitaries action on the qubits. After the desired gate operations have been performed by braiding and fusion, the anyons can be fused in a series and a final outcome is obtained. Thus, topological systems that can be initialized, braided, and measured will be able to perform universal quantum computation.

5 A Formal Model of Non-Abelian Anyons

Non-Abelian braiding statistics occurs in higher dimensional representations where there is a degenerate set of g particles at fixed locations. An exchange of particles, σ_i can be represented by a g x g unitary matrix $\rho(\sigma_i)$ acting on these states. The exchange of particles *1* and *2* is represented by $\rho(\sigma_1)$ and of particles *2* and *3* by $\rho(\sigma_2)$. If $\rho(\sigma_1)$ and $\rho(\sigma_2)$ do not commute then the particles obey non-Abelian statistics. To create such statistics it is generally the case that multiple types of anyons are present. Further, by bringing two anyons close together they may be approximated as a single particle whose total quantum number is the combination of their individual quantum numbers. This concept is fusion.

A theory of a 2-D medium with a mass gap and where the particles carry locally conserved charges is said to be model of non-Abelian anyons with the following four defining features.

1)Finite label set ($a,b,c,...$)

This finite list of particle labels indicate possible values of the conserved charge that a particle can carry. If isolated, a particle's label never changes. There exists a special label "*0*" indicating the trivial charge and a special label "*C*" for the charge conjugation operator a - a.

2)Fusion rules

These specify the possible values of the charge that can result when two charged particles are combined. The fusion of particle a and b is represented as a x b and obey the following rules:

$$a \text{ x } b = \sum_c N_{ab}^c$$
$$\sigma_i \text{ x } \sigma_j = \sum_k N_{ij}^k \sigma_k$$
$$N_{ij}^k \sigma_k = N_{ji}^k \sigma_k$$
$$\sum_l N_{jk}^l \; N_{il}^{\,m} = \sum_l N_{ij}^l N_{lk}^m$$

The first rule refers to what are called fusion channels which denote the different possibilities that exist for combining topological quantum numbers and where N_{ab}^c may be matrices. When a and b fuse, the resulting particle may be particle c if $N_{ab}^c \neq 0$. Fusion channels are a way of accounting for different degenerate multiple particle states.

3)F-Matrix

There are two natural ways to decompose the topological Hilbert space V_{abc} of three anyons in terms of fusion spaces of pairs of particles. The F-matrix relates these two orthonormal bases: $\Psi_\alpha = F_{ab}\Psi_b$ where $a,b = 1,\psi$. The F-matrices are analogous to the corresponding quantities for SU(2) representation where there are multiple states in which 3 spins can form a total spin J. As SU(2) describes how these form, the F-matrix describes how three charges, i,j and k, fuse to a topological charge l. The various ways to fuse form bases and the F-matrix is the unitary transformation between these bases.

4)R-Matrix

The braid group has four distinguishing features. Any braid can be obtained by multiplying elementary braids. The inverse of any braid exists. There exists a vacuum, and associativity for disjoint σ_i. The braid group's σ_i generator representation satisfy:

$$\sigma_i \sigma_j = \sigma_j \sigma_i \text{ for } |\text{i-j}| \geq 2$$
$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$$

The R-Matrix carries these features and relates the braiding of two particles. When two neighboring particles are exchanged counter-clockwise, their total charge c is unaltered. Since the particles swap positions, the fusion space V_{ab} changes to the isomorphic space V_{ba}. The unitary R-matrix represents this isomorphism. The R-matrix also determines the topological spin of the label a i.e. the phase acquired when the particles is rotated by 2π. The resulting phase for a counter-clockwise exchange of a and b which then fuse to c is given by R_c^{ab}.

An anyon model is non-Abelian if for some a and b,

$$\dim(\oplus V_{ab}) \cong \sum_c N_{ba} \geq 2$$

where

$$\dim(V_{ab}) \cong N_{ba}$$

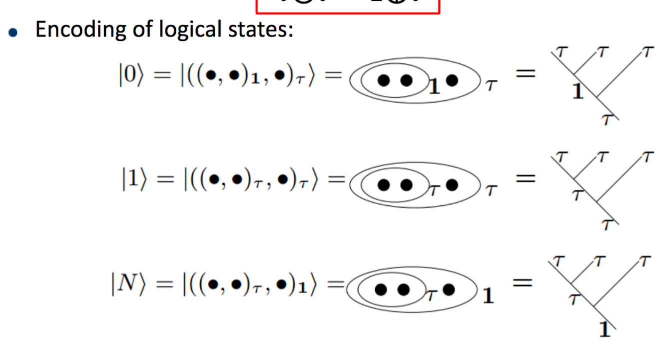
Then there exists a topological Hilbert space that encodes non-trivial information. This encoding is non-local and the information is a collective property of two anyons. When the particle with labels a and b are far apart, different states in the topological Hilbert space look identical to local observers. In particular, the quantum states are invulnerable to decoherence due to local interactions with the environment.

When the quantum state is hidden from the environment, it is hidden from us as well. To read out the quantum state, the information can be made locally visible again by bringing the two particles together and fusing them into a single object. By observing whether the label is equal to zero or not, the information is measured.

6 Yang-Lee Model (Fibonacci Model)

This is the simplest of all nonabelian anyon models and thus provides a good example for how a topological quantum computer would function. The charge of a particle can either be *1* (trivial) or τ (nontrivial and self-conjugate) Anyons have charge τ and two anyons can fuse in either of two ways and the quantum states can be encoded as:

$$\boxed{\tau \otimes \tau = 1 \oplus \tau}$$



The fusion rule determines F-matrix and R-matrix uniquely and the resulting nontrivial braiding properties are adequate for universal quantum computing.

From fusion rules and pentagon equation for Fibonacci model:

$$F^{\tau\tau\tau}{}_1 = F^{1\tau\tau}{}_\tau = F^{\tau 1\tau}{}_\tau = F^{\tau\tau 1}{}_\tau = 1$$

$$F^{\tau\tau\tau}{}_\tau = \begin{pmatrix} \frac{1}{\varphi} & \frac{1}{\sqrt{\varphi}} \\ \frac{1}{\sqrt{\varphi}} & -\frac{1}{\varphi} \end{pmatrix}$$

where ϕ is the golden ratio $\varphi = \frac{(1+\sqrt{5})}{2}$

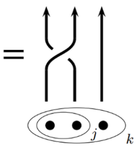
From hexagon equation and Yang-baxter relation:

$$R^{\tau 1}{}_\tau = R^{1\tau}{}_\tau = 1$$

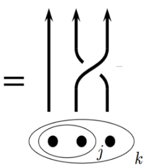
$$R^{\tau\tau} = \begin{pmatrix} e^{i4\pi/5} & 0 \\ 0 & -e^{i2\pi/5} \end{pmatrix}$$

Braiding matrices obtained from F and R:

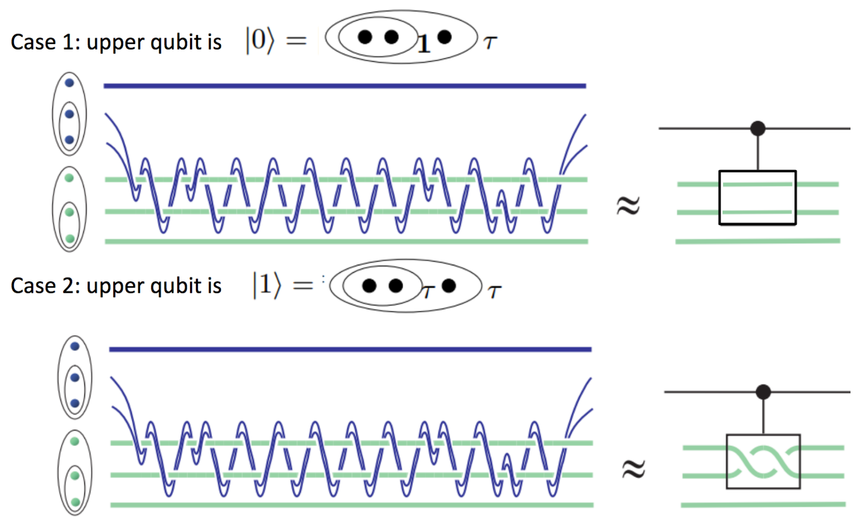
$$\begin{pmatrix} |0\rangle \\ |1\rangle \\ |N\rangle \end{pmatrix} \rightarrow \underbrace{\left(\begin{array}{cc|c} e^{-4\pi i/5} & 0 & 0 \\ 0 & -e^{-2\pi i/5} & 0 \\ \hline 0 & 0 & -e^{-2\pi i/5} \end{array}\right)}_{\rho(\sigma_1)} \begin{pmatrix} |0\rangle \\ |1\rangle \\ |N\rangle \end{pmatrix} \qquad \sigma_1 =$$



$$\begin{pmatrix} |0\rangle \\ |1\rangle \\ |N\rangle \end{pmatrix} \rightarrow \underbrace{\left(\begin{array}{cc|c} -e^{-\pi i/5}/\phi & -ie^{-i\pi/10}/\sqrt{\phi} & 0 \\ -ie^{-i\pi/10}/\sqrt{\phi} & -1/\phi & 0 \\ \hline 0 & 0 & -e^{-2\pi i/5} \end{array}\right)}_{\rho(\sigma_2)} \begin{pmatrix} |0\rangle \\ |1\rangle \\ |N\rangle \end{pmatrix} \qquad \sigma_2 =$$



By combining short braids, long braids are obtained with arbitrary accuracy that can simulate a desired single qubit unitary operation. The gate operations represented by braided are:



The braids $\rho(\sigma_1)$ and $\rho(\sigma_2)$ acting on the logical states can perform any unitary evolution in SU(N). In this representation, we see that the distinguishable states of n anyons (a basis for the Hilbert space) are labeled by binary strings of length n-3. However, it is not possible to have two zeros in a row in this binary representation and this results in the dimension of the Hilbert space for n anyons with a trivial total charge *0* is a Fibonacci number D=2,3,5,8,...

Asymptotically, the number of qubits encoded by each anyon is

$$\log_2\phi = \log_2[(1+\sqrt{5})/2] = .694$$

where ϕ is the "quantum dimension. This counting illustrates that the qubits are a nonlocal property of the anyons and that the topological Hilber space has no particularly natural decomposition as a tensor produce of small subsystems.

In conclusion, the Yang-Lee model demonstrates how a model of non-Abelian anyons can achieve universal quantum computation with arbitrary accuracy.

7 Topological Phases of Matter

While the theory of a topological quantum computer is a beautiful one, the ability to build one is not obviously possible. The theory is fundamentally built on anyonic states which are two dimensional, while we live in a very three-dimensional world. Luckily, quantum mechanics provides the key to solving this.

It is possible to construct quantum states that split into a two-dimensional state and a decoupled one-dimensional state.

$$\Psi(\mathbf{r}) = \Psi_{xy}(x,y)\Psi_z(z)$$

Confining the particle via a potential in the 1-D leads to discretized states. It is also necessary to have an energy gap that separates the 2-Ds ground state from its excited states in order to protect the anyonic state from external perturbations. Strong perturbations will destroy the state so it is essential to build a system with a low sensitivity to its environment.

Particles created in this way, by realizing an emergent two dimensional state in a three dimensional material, are termed quasiparticles. Quasiparticles emerge in systems whose particles interact to create a highly correlated wave function. Topological systems exhibit this behavior and the stability of topological order from external perturbations provides the stability that other candidate quantum computational systems lack.

However, topological systems are not immune to errors in the form temperature. Topological order is stable at zero temperature when perturbed weakly but is not so at finite temperatures. Thus, topological systems do not provide the perfect immunity that they seem to. Several theoretical approaches to solve this have been made by employing schemes based on long-range interactions so it may be possible to realize temperature independent topological order.

There are several candidate topological systems being explored for the purposes of quantum computation. Fractional quantum Hall states are perhaps the best candidate.

Conclusion

Topological systems provide a unique and interesting way to accomplish quantum computing. By encoding information into a topological state, it is naturally protected from the environmental perturbations that cause physical system states to decohere quickly. To perform topological quantum compution, anyon are intwined and information is encoded in the possible outcomes that occur when they are brought together. This information is protected since it is inaccessible when the anyons are kept apart. To perform statistical logical gate operations, anyons are exchanged. Exchange affects the outcome possibilities and thus provides the lever of control to construct universal quantum gates. Fault tolerance results from an ability to keep the anyons intact. Thus, provided it is possible to develop a topological system that can realize these anyon states and a solution to the temperature issue is found, a clean solution to the decoherence problem has been accomplished and a powerful quantum computer may be possible.

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