Quantum Chaos

A Review

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Abstract

The field of quantum chaos explores how chaotic dynamics can manifest in or affect the quantum scale regime. As Schrodinger's equation is linear, a full form of quantum chaos with exponential sensitivity to initial conditions is not believed to exist but it is useful to study the semiclassical regime to better understand the connection between chaotic classical dynamics and the quantum description of a system. This review focuses on this regime and discusses the connection between energy level spectra of quantum systems corresponding to classically chaotic ones. Lastly, the concept of wave chaos is introduced.

1 Introduction

> "The true reason for the prevalence of chaos is that large quantum systems are hard to isolate from from their surroundings. Even the pattern of photons from the Sun destroys the delicate interference underlying the quantum regularity. This effect, of large quantum systems being dramatically sensitive to uncontrolled external influences, is called decoherence. In the classical limit, the quantum suppression of chaos is itself suppressed by decoherence, allowing chaos to re-emerge as a familiar feature of the large- scale world." Michael Berry

Both the fields of quantum mechanics and chaotic dynamics had their birth in the early 20th century and had been largely developed by the end of it. The question of how chaotic dynamics can apply to quantum mechanics is a natural one and underpins the study of quantum chaos. However, chaotic behavior has not been observed to manifest in the quantum regime and most likely will not due to the linearity of Schrodinger's equation. This raises fundamental physical questions about the correspondence principle. Most notably, how is it that chaotic behavior can exist at all in a classical world given that quantum mechanics is the fundamental theory of physics and does not exhibit chaotic behavior? How can the correspondence principle be reconciled with singularities encountered in the semiclassical limit? Exploring questions such as these makes quantum chaos a source for deepening our understanding of fundamental physics even if true quantum chaos does not exist.

Attempting to extend classical chaos into the quantum regime begins in the semiclassical regime where quantum effects begin to become significant to the dynamical behavior of the system. This occurs according to the correspondence principle by taking $\hbar$ to 0. This interface region is interesting as all atomic systems will exhibit chaos when treated classically while the associated wavefunctions and energy spectrums are not rigorously chaotic.

One perspective for understanding quantum chaos arises from considering the physical problems of quantum dynamics as divided between a) proper quantum dynamics described by a specific dynamical variable; the wavefunction Ψ and b) the quantum measurement with the collapse of Ψ. The measurement collapse has a nature of inherent randomness and does create a type of dynamical chaos. The wavefunction obeys the Schrodinger equation and does not exhbit chaotic behavior. True chaos requires that there be exponential sensitivity on initial conditions. However, the evolution of a quantum mechanical state is

$$\Psi(\mathrm{t}) = \mathrm{e}^{-iHt/\hbar}\Psi(0)$$

which has an exponential operator that is unitary. This restricts the divergence of two states that have similar initial conditions.

Another approach to quantum chaos is to focus on Hamiltonian systems described by Hamilton's equations. By quantizing the Hamiltonian, replacing classical position and momentum with their quantum counterparts, one obtains an eigenvalue equation which gives the energy spectrum of the system. The statistical distribution of this spectrum correlates to the dynamics of the classically chaotic system. Thus, a connection between chaos and quantum mechanics arises when chaos manifests itself in the nature of the systems energy level spacing distribution. We make this more formal in a later section.

2 Singularity limits

In determining how classical behavior and chaos can emerge from quantum mechanics in the semiclassical limit, there is a nontrivial issue of a singularity when taking $\hbar$ to 0. To see how this issue arises and how a resolution occurs we can consider a simple example of two coherent beams of light traveling in opposite directions.

$$\Psi_{\pm}(\mathrm{x,t}) = \exp[2\pi \mathrm{i}(\pm \mathrm{x}/\lambda\text{-}\mathrm{vt})]$$

The the total quantum state is a superposition and the resultant wave is the sum of the two. The intensity of this state is

$$\mathrm{I(x)}=|\Psi_{+} + \Psi_{-}|^2 = 4\cos^2(2\pi \mathrm{x}/\lambda)$$

The semiclassical limit of this wave, obtained by taking $\hbar$ to 0, results in taking λ to 0 since $\lambda = \hbar/\mathrm{p}$. Classically, we know the intensity of the resulting wave is I=2 but in the semi-classical limit this is not what is obtained. Rather, the intensity oscillates at faster and faster rates in the limit and ultimately results in a singularity at $\hbar = 0$. This singularity is unavoidable and raises an issue in the connection between the classical and quantum worlds, even in a simplest of scenarios.

However, a reconciliation can be obtained by averaging. This could physically be related to decoherence effects caused by external influences. The average of $\cos^2$ is 1/2 so

$$\langle \mathrm{I(x)} \rangle = 4 \langle \cos^2(2\pi \mathrm{x}/\lambda)\rangle = 4 \text{ x } 1/2 = 2$$

Thus, the correspondence holds but not in the strict case of taking $\hbar = 0$. An averaging process must occur that can be attributed to interactions with the environment. The bridge between quantum and classical phenomenon is not a smooth one and encounters difficulties, such as singularities, which require a remedy to make results agree and for the correspondence principle to hold.

3 Decoherence and Emergence

To understand how chaos can emerge in classical systems, it is useful to use an example of a system that displays chaos but that is relatively isolated from external perturbations. Such an example is found in one of Saturn's moons, Hyperion. Under the gravitational influence of the sun, Saturn and Titan, Hyperion undergoes an erratic rotation about Saturn with an orbit period of $\simeq 5$ days.

Treating Hyperion as a quantum object, it has an angular momentum of $2\pi J/\hbar \simeq 2\mathrm{x}10^{58}$ [2] and it's phase space motion is on a sphere with a radial vector of this length. Since the motion of Hyperion is not known exactly, either classically or quantumly, the state of Hyperion on the sphere is represented by a patch with a certain initial boundary and surface area. As Hyperion evolves in time, each point within this patch will separate exponentially from the others due to the chaotic motion and thus exponential sensitivity to initial conditions. This is well a connected set and by Liouville's theorem the total area enclosed within the boundary is constant in time. Although, the perimeter length will grow exponentially.

The evolution of this system in phase space can be visualized as an initial circular blob that breaks into ever more intricate tendricals with time. Classically this is expected and leads to no difficulties. Quantum mechanically, there is an issue when the tendricals become so fine that they surpass the phase-space fine structure that is limited by $\hbar$. The uncertainty principle puts a lower bound on how well known the state of the system can be so there will always be a "blurriness" radius about any phase space point of order $\hbar$. Therefore, the tendricals cannot have a width less than $\hbar$ and, thus, further branching would lead to an increase in the total area enclosed within the boundary. This would be in contradiction of Louisville's theorem which must hold for the system. Thus, if treated in this quantum mechanical way, the fine-structure will suppress the chaotic divergence of the possible trajectories. The time for this to occur is calculated to be

$$\mathrm{t} \simeq \mathrm{T}_c \log(\mathrm{J/h}) \simeq 37 \text{ years}$$ [2]

where T_c is the chaos time and is $\approx$ 100 days.

This example again highlights the difficulty encountered in the semiclassical limit. Treating the system classically, letting $\hbar$ go to zero, allows for indefinite chaotic motion. Finite $\hbar$ puts a restriction on the divergence of phase-space trajectories and eventually suppresses any further exponential separation. The orbit of Hyperion does not lose it's chaotic nature so why is it not suppressed by quantum mechanics?

Although Hyperion is a relatively isolated system and the main contributors to it's motion are accounted for, it still has environmental interactions which will affect it's ability to be treated as an isolated quantum mechanical system. For example, it interacts with photons from the sun, reflecting them and allowing us to see it from Earth. The total energy of these photons ($4\mathrm{x}10^{-19}$ J) is negligible relative to the rotational energy ($2\mathrm{x}10^{19}$ J) and would thus seem to play no part on the phase space trajectories. However, the photon energy is large relative to the energy spacing of the Hyperion's quantum rotational levels (10^{-39} J) [2]. These photon interactions are enough to cause decoherence of the individual rotational states. Thus, decoherence from external sources suppresses the quantum suppression of chaos.

A single photon impacting Hyperion will cause a change in angular momentum by $\delta J \simeq (h/\lambda)R$ and thus changes the total phase of Hyperion's wavefunction. This phase shift, from a single photon, is sufficient to cause decoherence. This illustrates that a minuscule external disturbance can drastically and quickly affect the kinematics of the quantum state. Through this decoherence, chaos is able to emerge.

"Chaos magnifies *any* uncertainty, but in the quantum case h has a smoothing effect, which would suppress chaos if this suppression were not itself suppressed by externally-induced decoherence, that restores classicality (including chaos if the classical orbits are unstable)." (Berry [2])

4 Emergent Semiclassical Phenomena

In the previous two sections we have seen how the semiclassical limit leads to seemingly disparate results between the quantum regime, with finite $\hbar$ and the classical regime, where $\hbar$ is zero. In order to reconcile the theoretical results in this semiclassical regime, decoherence was needed to average or blur the quantum effects in the transition to a classical picture. Here we consider a third situation in which the transition from classical dynamics to quantum mechanics is deeply dependent upon the regularity of the classical motion, energy level spectra. Out of it emerges one of the most interesting aspects of quantum chaos, the universality classes for classically chaotic quantum spectra.

4.1 Regular and Irregular Motion

To more explicitly see how chaotic motion can appear in the quantum regime, we classify classical motion as either being regular or irregular. Regular motion corresponds to motion that is predictable, such as the one-dimensional oscillator or elliptical orbits or the planets. Trajectories for objects in regular motion separate linearly. Irregular motion corresponds to unpredictable motion such as that of many colliding molecules. It is characterized by sensitivity to initial conditions in which trajectories separate linearly.

The difference between regular and irregular motion can be explicitly seen by considering the geometry of typical trajectories in the systems phase space. These trajectories can be categorized as: integrable, quasi-integrable, ergodic, mixing, K-system, and B-system. For integrable motion, the phase space trajectories corresponding to N constants of motion C(q,p) are restricted to a surface $\sum$ of dimensionality N. If C are smooth enough then $\sum$ is a N-

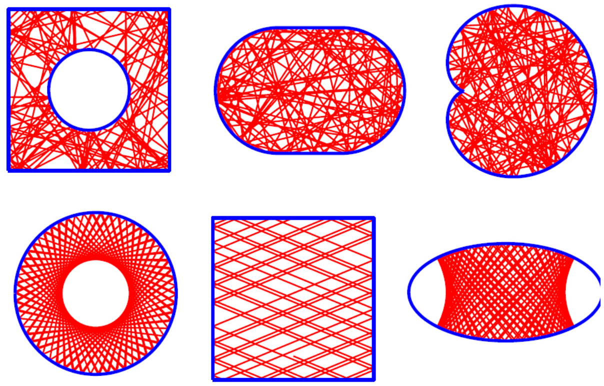

Figure 1: Ergodic motion is displayed in the top three image and non-ergodic in the bottom three (Images from Mueller)

dimensional torus and the trajectories can be determined by elimination and integration. A simple example is harmonic oscillators with N=1. All separable systems are integrable.

Billiards provide a useful visual tool to understand regular and irregular motion. Depending on the wall geometry of a billaird table, motion of a typical billiard either behaves in such a way to create repeating patterns or it behaves in an erratic way. If the billiard behaves erratically, it will eventually pass through every point in the space and is said to be ergodic. Regular motion corresponds to trajectories that create patterns that are not space filling. This is illustrated in Figure 1. Chaotic ergodic trajectories explore all of the points in a $2N$-1 dimensional energy surface defined by *H(p,q)*=constant. To study this chaotic motion it is useful to divide(coarse-graine) phase space into cells (Markov partitions) and study the sequence(Bernoulli shift).

A quasi-integrable system is one which is neither integrable nor ergodic. They have Hamiltonians $H = H_0 + \epsilon H_1$ where H_0 is integrable. If ϵ is not 0 (KAM theory), under perturbation most tori survive but they are distorted. So motion is not ergodic but since some tori are destroyed, it is not integrable. Destroyed tori form a set of finite measure growing with ϵ. Only a finite even number of closed orbits survives the perturbation:, half of which are stable and half of which are unstable. Motion near the unstable orbits is chaotic and fills regions with dimension $2N$-1. The solar system is an example of a quasi-integrable system in which the planets move in Kepler ellipses and asteroids behave chaotically in nearby, unstable orbits.

It is necessary to be very careful in specifying what class of deformation of the boundary B is being considered. A pseudo-integrable system has a boundary B which is a polygon with angles that are rational multiples of π. Here, $\sum$ is not a torus but a multiply handled sphere.

4.2 Wave functions and Wigner functions

We now study how the different types of underlying classical motion manifest themselves in quantum mechanics. Consider the morphologies of eigenfunctions of time-independent Hamiltonian operators whose analogous classical Hamiltonians generate regular or irregular bounded motion. There is strong evidence that the morphologies corresponding to the two sorts of motion are very different, as was first suggested by Einstein. Maslov proposed an association between wave functions and *N*-dimensional surfaces $\sum$ in the $2N$-dimensional phase space (q,p). The wave function ψ need not be an eigenstate of H and $\sum$ need not be one of the tori. It is worth noting that this association is purely geometric. The surface $\sum$ is written as a function $p(q)$ and corresponds to an *N*-parameter ensemble of states(points) in classical phase space.

Define a density on E in which states are uniformly distributed in some coordinate $Q = (Q_1 ... Q_N)$. We wish to associate E with a wave $\psi(q) = a(q)e^{ib(q)}$. ψ is unrestricted by any wave equation and this allows a freedom in choosing a and b. However, there is a natural choice for a and b. For a, we require the wave intensity ω^2 is proportional to density of points in the classical coordinate space. For the phase b, we use de Broglie's rule relating momentum $p(q)$ to the wave vector $k(q)$ of a locally plane wave. This plane wave is given [3]

$$\psi(q) = K \left| det \frac{\partial^2 S(q;P)}{\partial q_i \partial q_j} \right|^{\frac{1}{2}} \exp\left(\frac{i}{\hbar} S(q;P)\right)$$

Regarding ψ as an initial quantum state and letting it evolve according to the Schrodinger equation under dynamics governed by H, after a time time t, ψ will have evolved into $\psi\prime$ and $\sum$ into $\sum\prime$. It can be shown that $\psi\prime$ can be constructed from $\sum\prime$. This correlation persists in time (at least in semiclassical approximation) and so represents a natural association between evolving quantal waves and N-parameter families of classical orbits.

Globalization is a method for constructing the semiclassical energy eigenfunctions and can succeed only if tori exist, i.e. the motion is regular as in integrable systems or throughout most of the phase space in quasi-integrable systems. For irregular and in particular ergodic motion, no tori exist and Maslovs method fails completely. At present, no asymptotic theory for the eigenfunctions corresponding to irregular motion

Wigner introduced the phase-space distribution function $W(q,p)$ corresponding to quantum state $\psi(q)$ that is the Fourier transform of the production of ψ and ψ^* at positions separated by a distance X.

$$W(q,p) = \frac{1}{(2\pi\hbar)^N} \int ... \int d^N X exp(-ipX/\hbar)\psi^*(q - X/2)\psi(q + X/2)$$

W is a quantal generalization of the classical density of points in phase space and has complete formal symmetry in q and p. Most importantly, it contains all of the information about the quantum state. We can describe statistical mixtures of pure states with weights and each semiclassical eigenstate has a Wigner function concentrated on the region explored by a typical orbit over infinite times. Applied to an ergodic system whose orbits fill whole energy surfaces in phase space, each quantum state corresponds to one energy surface, selected by a quantum condition. However, the eigenfunctions are unknown becuase it is unknown how to associate a wave with an energy space in such a way that quantization follows from single-valuedness.

The regular Wigner function for integrable or quasi-integrable systems is [3]

$$W_m(q,p) \approx \frac{1}{(2\pi)^N}\delta[I(q,p) - I_m]$$

where $I(q,p)$ is the action of the torus passing through q,p and m is the set of quantum numbers.

For an ergodic system, the irregular Wigner function is [3]

$$W(q,p) \approx \frac{\delta[E-H(q,P)]}{\int \ldots \int d^N q d^N p \delta[E-H(q,P)]}$$

for an eigenstate with energy E.

Using the regular and irregular Wigner functions, morphological information about $\psi(q)$ is obtained regarding its local average strength and its set of local oscillations. The probability density is obtained by projecting W down p and it is observed that there are caustics at the singularities of the projection of the torus onto q, i.e. on local boundaries of the region explored by the orbit in q-space.

The simple idea underlying the semiclassical eigenfunction hypothesis has led to dramatic predictions about the morphology of wave functions. As $\hbar$ goes to 0, wave functions separate into two universality classes, associated with regular and irregular classical motion. In the regular case, ψ is associated with tori and has anisotropic interference oscillations rising to high intensities ψ^2 on caustics. In the irregular case, psi is associated with chaotic regions in phase space and has a random pattern of oscillation. These universality classes are emergent properties as $\hbar$ goes to zero. Away from the semiclassical limit, it may often be impossible to unambiguously categorize a state as being regular or irregular.

5 Energy Level Spectra

Before Bohr's early work, it was assumed that electrons' atomic orbits would be free to revolve at any distance from the nucleus that they pleased. And thus it would be in a classical world. Luckily we live in a quantum mechanical world, with well defined orbital energies, in which electrons are not free to spiral into the nucleus. In the semiclassical limit, how can a classical energy continuum be reconciled with a quantum discontinuum? Pragmatically, the conflict is inconsequential since the semiclassical regime occurs at high orbital energies where the finite resolution of any spectroscope makes the discrete energy levels indistinguishable. At a more fundamental, philosophical level, this is equivalent to a time-independent version of Zeno's paradox in a sense.

In this semiclassical regime, it has been found that the spectrum for a classically integrable system is qualitatively different from that of a chaotic one. Surprisingly, spectra related to classically chaotic systems display universality in the statistics of the level spacing. Consider the eigenenergies E_n of a perturbed Hermitian Hamiltonian

$$H(\epsilon) = H_0 + \epsilon V$$

The equations of motion are found

$$H(\epsilon)\ |n(\epsilon)\rangle = E_n(\epsilon)|n(\epsilon)\rangle$$

and leads to evolution equations of the level dynamics

$$\dot{V}_{nm} = V_{nm}(V_{nn}\text{-}V_{mm})/(E_m\text{-}E_n) + \textstyle\sum_{l\neq n,m} V_{nl}v_{lm}(1/(E_n\text{-}E_l) \text{ - } 1/(E_m\text{-}E_l))$$

This result is independent of the specific form of H and shows that the dynamics of the energy levels for all Hamiltonian spectra is split into two parts. For sufficiently large ϵ, this can be generalized to "resonance dynamics" that preserve the Liouville volume, the energy and the total coupling strength. Further, these dynamics are integrable and to understand a system with infinitely many degrees of freedom and infinitely many constants of motion we make use of statistical mechanics. The distribution is

$$P = \tfrac{1}{Z}\ \mathrm{e}^{-\beta E - \gamma Q}$$

where Q is coupling strength and Z is the partition function. Integrating out V_{nm} gives the distribution of eigenvalues E_n that is a GOE probability distribution.

Semi-classical approximations create a link between a quantum system and it's classical analog and is the realm in which the qualitative nature, whether chaotic or not, appears in the nature of the quantum mechanical properties.

5.1 Energy Level Spacing

The energy level spacing between adjacent energy levels is $S_i = E_{i+1} - E_i$ and fluctuates about a central energy E. We wish to calculate a distribution function for these spacings. It is useful to first normalized the density of states by making the substitution of $\{E_i\}$ with $\{e_i\}$ where $e_i = \bar{N}(E_i)$ and $s_i = e_{i+1} - e_i$.

The distribution, $P(s)$, in the semiclassical quantum regime for a Hamiltonian for a classical integrable system is universally a Poisson one.

$$P(s) = \mathrm{e}^{-s}$$

For completely chaotic systems, the time-reversible case and non-time-reversible case lead to differing distributions. If the system is complicated then the eigenproblem becomes intractable and the distribution is not analytically obtainable. One way to circumspect this issue is to make use of random matrices.

Random matrix theory was originally used in relation to energy level spectra in 1951 by Wigner in an attempt to understand many-body nuclear systems that were intractable by more direct methods. By treating them statistically, he conjectured that they would have spectra similar to those of the spectra of ensembles of random matrices. To make the random matrices represent physical systems they must meet the criterions of invariance and of independence. These imply Gaussian distributions for individual elements of the random matrix. The ensembles for the time-reversible and non-time-reversible yield energy level spacing distributions

$$P(s) \simeq (\pi/2)se^{-\pi s^2/4} \tag{1}$$

$$P(s) \simeq (32/\pi)s^2 e^{-(4/\pi)s^2} \tag{2}$$

Thus, we get different spectral statistics depending on whether or not the corresponding classical system is chaotic.

6 Wave Chaos

Another area in which the question of whether or not chaos arises is wave dynamics. Waves are spread out and do not have a well defined position and momentum so it is not clear how chaos would manifest in wave phenomena. However, in the high-frequency or semiclassical regime the wavelength of waves is small compared to typical dimensions and this particle-like nature allows a form of chaos to arise. Ray chaos is common in elasticity especially when considering scattering of elastic waves at interface boundaries.

The study of wave propagation in complicated structures can be achieved in the high frequency (or small wavelength)limit by considering the dynamics of rays. The complexity of wave media can be either due to the presence of inhomogeneities (scattering centers) of the wave velocity, or to the geometry of boundaries enclosing a homogeneous medium. It is the latter case that was originally addressed by the field of Quantum Chaos to describe solutions of the Schrodinger equation when the classical limit displays chaos. The Helmholtz equation is the strict formal analog of the Schrodinger equation for electromagnetic or acoustic waves, the geometrical limit of rays being equivalent to the classical limit of particle motion. To qualify this context, the new expression Wave Chaos has naturally emerged. Accordingly, billiards have become geometrical paradigms of wave cavities.

In the first sections of this review, the possibility of quantum chaos was viewed from a perspective of the linearity of Schrodinger's equation. Since Schrodinger's equation is a scalar, linear wave equation, it is in a respect not very different from the wave equations of optics or linear elasticity. Therefore, there is a correlation between quantum chaos and acoustics that allows methods and concepts used and developed in quantum chaos to be applied in the optics and engineering community.

The wave equation to be considered, that is relevant in a variety of physical contexts, is the Helmholtz equation.

$$c^2 \Delta\Phi(\mathbf{r}) + w^2\Phi(\mathbf{r}) = 0; \tag{3}$$

where Φ is a scalar function and c^2 is the wave velocity. To get to the ray picture we let

$$\Phi(\mathbf{r}) = A(\mathbf{r})e^{iS(\mathbf{r})} \tag{4}$$

With this definition of Φ the ray trajectories are obtained

$$\frac{d}{dt}\mathbf{r} = 2c^2\mathbf{k}; \qquad \frac{d}{dt}\mathbf{k} = 2c\mathbf{k}^2\nabla c \tag{5}$$

where $\mathbf{k}$ is the wave number.

There is a formal analogy between the geometrical limit of rays and classical mechanics, and this now allows an application of classical chaos to be applied to ray forms of waves. The simplest form of this chaos is billiard dynamics and depending on shape of the billiard, chaos may or may not arise. In chaotic billiards there is extreme sensitivity to initial conditions. This correlation means that in electromagnetic phenomena where there are high frequency electromagnetic waves inside of cavity structures, a chaotic billiard effect may be observed. The fact that waves can display chaotic behavior has ramifications to a variety of phenomena including seismology, underwater acoustics, shell theory and time-reversal imaging.

6.1 The Trace Formula

To obtain approximate solutions for the response of a stationary wave in the high-frequency regime we consider the Green's function

$$G(\mathbf{r}, \mathbf{r}_0, w) = \sum_n \frac{u_n(\mathbf{r})u_n(\mathbf{r}_0)}{w^2 - w_n^2} \tag{6}$$

Approximating the Green's function as a sum over all geometrical ray contribution we have

$$G(\mathbf{r}, \mathbf{r}_0, w) = \frac{\pi}{w} \frac{1}{(2\pi i)^{(d+1)/2}} \sum_{\mathbf{r} \to \mathbf{r_0}} \sqrt{|D|} exp[iS(\mathbf{r}, \mathbf{r_0}; w) - i\mu \frac{\pi}{2}] \tag{7}$$

where S is the action and μ counts the number of singular points along the classical path on the energy manifold. The trace of the Green function contains the information about the eigenfrequency spectrum $\{w_n\}$

$$g(w) = Tr\, G(w) \sim \sum_n \frac{1}{w^2 - w_n^2} \tag{8}$$

Once integrated over the domain, or trace, this yields the density of eigenvalues

$$d(w) = \sum_n \delta(w - w_n) = -\frac{2w}{\pi} \lim_{\epsilon \to 0+} Img(w + i\epsilon) \tag{9}$$

7 Conclusion

In this review we have seen how the semiclassical limit brings up important fundamental issues regarding the correspondence principle and we have seen how decoherence plays a vital role in this regime. Further, we have seen that the quantum spectrum corresponding to classical systems is directly dependent upon whether or not the behavior of the corresponding classical system is chaotic and leads to differing universal spectral statistics. Lastly, high energy waves behave as rays and chaotic behavior may be observed.

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