1.

Leaf is a vertex in a tree that is incident to only one edge

Just based on the condition $|V| \geq 2$ alone, we can determine that the tree always has **at least** two leaves. The leaves are the vertices in the tree, and in the case of $|V| = 2$ there are exactly two leaves/vertices, which are connected by a single edge. If there were to be less than 2 leaves, the "tree" would just be a single vertice, not a tree. The amount of edges is $|V| - 1$.

2.

Base case: A tree with a single vertex is technically bipartite. Let $V_1 = \{v\} \text{ and } V_2 = \emptyset$. The conditions a and b are thus satisfied

Inductive step: Assuming that every tree with k vertices is bipartite, where $k \geq 1$. Now, lets consider a tree with $k + 1$ vertices. Pink any vertex v in the tree. Remove v and its connecting edge, leaving a tree T' with k vertices. By the assumption, T' is bipartite. Add v back to the tree. Since v is connected to a vertex in T', it must go in the opposite set from that vertex. This way, we can create two sets of vertices V_1 and V_2 for the entire tree. V_1 contains v and the set $V_1{}'$ from T' and V_2 contains $V_2{}'$ from T'. Now lets check conditions for bipartiteness:

a) All vertices in V are in either V_1 or V_2, and no vertex is in both

b) Every edge connects vertices from different sets

By ensuring that v is placed in the opposite set from its connection in T' these conditions are satisfied.

Therefore, the original tree with $k + 1$ vertices is also bipartite.

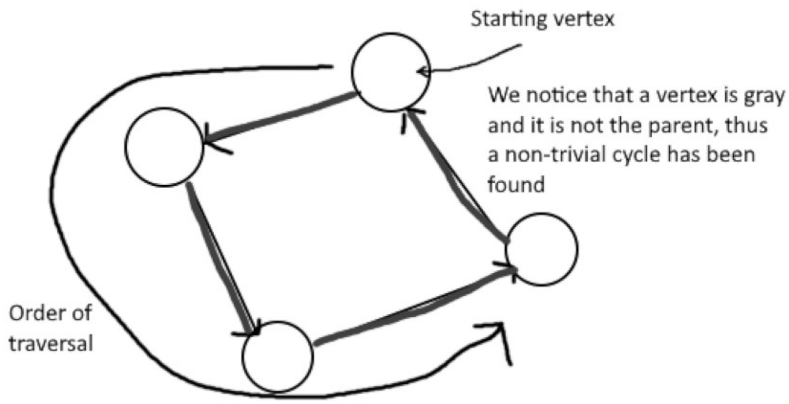
Conclusion: By induction, we've shown that every tree is bipartite.

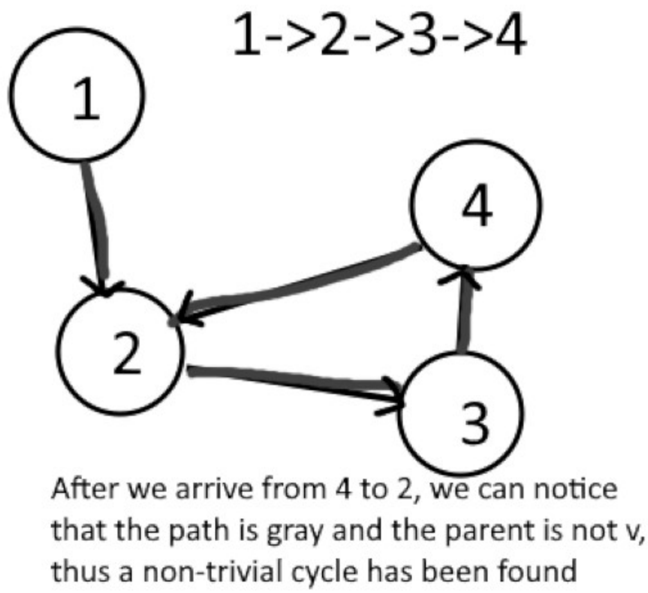
3.

If we introduce an additional parameter to track the parent of each vertex during the traversal, we can recognize whether a directed graph has a non-trivial cycle. Utilizing this information, we can identify back edges, which indicate the presence of a cycle.

By adding an $\text{else if col}[v] = \text{gray and } v \neq \text{parent}$ check alongisde the $\text{if col}[v] = \text{white}$ check, we can determine if a back edge is to be found, indicating that the graph has a non-trivial cycle.

Test cases





4.

To demonstrate that $d(x_0, x_i) \leq \min\{i, k-1\}$, where $d(x_0, x_i)$ represents the shortest distance between vertices x_0 and x_i in the given cycle, we should consider two test cases

Case 1: $i \leq \frac{k}{2}$

In this case, the distance $d(x_0, x_i)$ corresponds to the length of the path along the cycle from x_0 to x_i. Since the cycle has length k, the maximum distance along the cycle is $\frac{2}{k}$ Therefore, $d(x_0, x_i) \leq \frac{k}{2} \leq \min\{i, k-1\}$ for $i \leq \frac{k}{2}$

Case 2: $i > \frac{k}{2}$

In this case, the distance $d(x_0, x_i)$ corresponds to the length of the path along the cycle from x_i back to x_0. Since $i > \frac{k}{2}$, the distance i is greater than $\frac{k}{2}$. Therefore, $d(x_0, x_i) \leq \frac{k}{2} \leq \min\{i, k-1\}$ for $i > \frac{k}{2}$

5.

Lines 6-7

Vertex u is currently being explored (turned to gray), but its neighbor, vertex v, has not yet been visited (white)

Lines 7-8

The exploration can be stopped. Vertex u has been fully explored and all of its neighbors have been visited (black)

6.

In order to prove the claim " A graph is a tree if and only if two different vertices have exactly one simple undirected path between them", we need to satisfy/prove both directions.

1) If the graph is a tree, then there is exactly one simple undirected path bween any two vertices

2) If there is exactly one simple undirected path between any two vertices, the graph is a tree

Proof 1

Assuming T is a tree, which contains vertices u and v. Since T is a tree, there is an unique simple path/edge between u and v. We know that the path is unique and simple, since any other path would create a cycle, violating the definition of a tree.

Proof 2

If a graph satisfies the two conditions (graph is connected and acyclic) we can say that the graph is a tree.

a) connectedness: Since there exists a simple path between any two vertices, the graph is connected by its default nature.

b) acyclic: Suppose the tree contains two vertices that have two simple paths between them. This would contradict our assumption that there is exactly one simple undirected path between any two vertices.

Since both conditions are satisfied, the graph is a tree.