Linear algebra.

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Invertible Matrix Theorem

For n by n matrix A to have an inverse, any (and hence all) must hold

1. A is row equivalent to I_n
2. A has a n pivot positions
3. $Ax = 0$ only has the trivial solution
4. Columns of A form a linearly independent set
5. Linear transformation $x \rightarrowtail Ax$ is one-to-one
6. $Ax = b$ has a unique solution for each column vector $b \in \mathbb{R}^n$
7. Columns of A span $\mathbb{R}^n$
8. Linear transformation $x \rightarrowtail Ax$ is a surjection
9. There exists n by n matrix C such that $CA = I_n$
10. There exists n by n matrix D such that $AD = I_n$
11. Transpose matrix A^T is invertible
12. Columns of A form a basis for $\mathbb{R}^n$
13. Column space of A is equal to $\mathbb{R}^n$
14. Dimension of the column space of A is n
15. Rank of A is n
16. Null space of A is $\{0\}$
17. Dimension of null space is 0
18. 0 fails to be an eigenvalue of A
19. Determinant of A is not zero
20. Orthogonal complement of column space of A is $\{0\}$
21. Orthogonal complement of null space of A is $\mathbb{R}^n$
22. Row space of A is $\mathbb{R}^n$
23. Matrix A has n non-zero singular values

Echelon form

1. All non-zero rows are above any row of all zeros
2. Each leading point (pivot) is in a column to the right of the leading entry in the previous row
3. All entries below a leading entry are zero

Reduced echelon form

1. The matrix is in echelon form
2. The pivot of each nonzero row is 1
3. Each leading 1 is the only nonzero entry in its column

Basic, free variables

If the augmented matrix is in echelon form:

1. Basic variables of a linear system are those corresponding to pivot columns
2. Free variables of a linear system are those that have no pivot in the corresponding column
3. A consistent system is solved by expressing all basic variables in terms of the free variables

If there are no free variables, then the system has a unique solution

Linear combinations

Given vectors $v_1, v_2, ..., v_p$ in $\mathbb{R}^n$ and scalars $c_1, c_2, ..., c_p$,

$$y = c_1 v_1 + ... + c_p v_p$$

is called a linear combination of $v_1, v_2, ..., v_p$ with weights $c_1, c_2, ...c_p$

Spans

$$Span\{v_1, v_2, ..., v_p\}$$

is the set of all linear combinations of $v_1, v_2, ...v_p$ and consists of all vectors that can be written as

$$x_1 v_1 + x_2 v_2 + ... + x_p v_p$$

Matrix-vector product

$$Ax = \begin{bmatrix} a_1 & a_2 & ... & a_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ ... \\ x_n \end{bmatrix} = x_1 a_2 + x_2 a_2 + ... + x_n a_n$$

Solutions of matrix equation

For $m \times n$ A, the following statements are equivalent:

1. For each b in $\mathbb{R}^m$, $Ax = b$ has a solution
2. Each b in $\mathbb{R}^m$ is a linear combination of the columns of A
3. The columns of A span $\mathbb{R}^m$
4. Each row in A has a pivot position

Linear independence

$$x_1 v_1 + x_2 v_2 + ... + x_p v_p = 0$$

only has the trivial solution
If the set $\{v_1, v_2, ..., v_p\}$ in $\mathbb{R}^n$ contains the zero vector, then the set is linearly dependent

Linear transformation

$T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation if

1. $T(u + v) = T(u) + T(v)$
2. $T(cu) = cT(u)$

If T is a linear transformation, then

1. $T(0) = 0$
2. $T(c_1 v_1 + ... + c_p v_p) = c_1 T(v_1) + ... + c_p T(v_p)$

Linear transformation composition

$$(S \circ T)(x) = S(T(x))$$

Transpose matrix properties

1. $(A^T)^T = A$
2. $(A + B)^T = A^T + B^T$
3. $(rA)^T = rA^T$ for any scalar r
4. $(AB)^T = B^T A^T$

Inverse matrices

A square $n \times n$ matrix A is invertible if there is an $n \times n$ matrix C such that

$$CA = I_n \text{ and } AC = I_n$$

System with inverse matrix

$$Ax = b$$

has the unique solution

$$x = A^{-1} b$$

Inverse matrix properties

1. $(A^{-1})^{-1} = A$
2. $(AB)^{-1} = B^{-1} A^{-1}$
3. $(A^T)^{-1} = (A^{-1})^T$

Subspace

A subspace of $\mathbb{R}^n$ is a set W such that:

1. 0 is in W
2. If v and u in W, so is $v + u$
3. If v in W and c in $\mathbb{R}$, then cv is in W

The span of a set of vectors $\{v_1, v_2, ...v_n\}$ in $\mathbb{R}^m$ is a subspace of $\mathbb{R}^m$

Null, column space

$NulA$ of A is the set of all solutions of the homogeneous equation $Ax = 0$. Subspace of $\mathbb{R}^n$
$ColA$ of A is the span of the columns of A. Subspace of $\mathbb{R}^m$

Basis

A basis for subspace W is a set of vectors in W which

1. is linearly independent
2. spans W

The pivot columns of A form a basis for the column space of A

Coordinate vector

Let $\mathcal{B} = \{b_1, ..., b_p\}$ be a basis for a subspace W of $\mathbb{R}^n$ and $x \in W$. Then there are unique weights $c_1, ...c_p$ such that

$$x = c_1 b_1 + c_2 b_2 + ... + c_p b_p$$

Weights $c_1, ...c_p$ form the $\mathcal{B}$-coordinate vector of x:

$$[x]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ ... \\ c_p \end{bmatrix}$$

Rank theorem

Rank of A is dimension of column space of A

$$rank(A) + dim(Nul(A)) = n$$

Cofactor expansion

Across row i:

$$\det A = a_{i1}C_{i1} + ... + a_{in}C_{in}$$

Across column j:

$$\det A = a_{1j}C_{1j} + ... + a_{nj}C_{nj}$$

Row operations, determinants

1. Interchange two rows:
   $\det B = -\det A$
2. Multiply a row by constant k:
   $\det B = k \det A$
3. Adding multiples of rows:
   $\det B = \det A$

Eigenvalues, eigenvectors

Scalar λ is an eigenvalue of A is there exists a nonzero eigenvector $x \in \mathbb{R}^n$ for which

$$Ax = \lambda x$$

Eigenspace E_λ

$E_\lambda = \text{Nul}(A - \lambda I)$ is the set of all solutions of $Ax = \lambda x$, forms a subspace of $\mathbb{R}^n$, and consists of

1. all the eigenvectors with eigenvalue λ
2. the zero vector

Eigenvalue multiplicity

Algebraic multiplicity of eigenvalue λ_0 of A is the number of factors $(\lambda - \lambda_0)$ in the characteristic polynomial.
Geometric multiplicity of eigenvalue λ_0 is the dimension of the corresponding eigenspace E_{λ_0}.

Similarity

A and B similar if there exists P such that $A = PBP^{-1}$.
Similar matrices A and B have the same characteristic polynomial and hence the same eigenvalues with the same multiplicities.
If v is an eigenvector of B with eigenvalue λ, then Pv is an eigenvector of A with the same eigenvalue λ.

Diagonalizable matrix

A diagonalizable if it is similar to diagonal matrix D
D: eigenvalues of A
P: corresponding eigenvectors of A
Diagonalizable if $\text{a.m.}(\lambda_i) = \text{g.m.}(\lambda_i)$ for all λ_i

Real matrices, complex eigenvalues

Suppose $a, b \in \mathbb{R}^n, b \neq 0$ and

$$A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}.$$

Let $z = a + bi = re^{i\theta}$. Then

$$A = r\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix},$$

and the eigenvalues of A are

$$a \pm bi = re^{\pm i\theta}$$

Real 2x2 matrix with non-real eigenvalues

Let A be a real 2×2 matrix with eigenvalues $a \pm bi$, where $b \neq 0$. Then there exists an invertible matrix P such that

$$A = P\begin{bmatrix} a & -b \\ b & a \end{bmatrix}P^{-1}.$$

The matrix P can be constructed as

$$P = \begin{bmatrix} \text{Re}v & \text{Im}v \end{bmatrix},$$

where v is an eigenvector associated to eigenvalue $a - bi$

Differential equations, general

Suppose A is an $n \times n$ matrix with n linearly independent eigenvectors v_j and corresponding eigenvalues λ_j. The general solution to $x' = Ax$ is given by

$$x(t) = c_1 v_1 e^{\lambda_1 t} + ... + c_n v_n e^{\lambda_n t}$$

Complex eigenvalues

Suppose 2×2 real matrix A has complex eigenvalues $\lambda_{1,2} = a \pm bi$ with corresponding eigenvectors v_1 and v_2. Two independent, real solutions of the system

$$x' = Ax$$

are given by

$$x_1(t) = \text{Re}(v_1 e^{\lambda_1 t}) \text{ and } x_2(t) = \text{Im}(v_1 e^{\lambda_1 t})$$

Different orbit types

Let A be a 2×2 matrix. Consider the system $x' = Ax$, where A has real eigenvalues λ_1 and λ_2. The origin is:

1. an attractor/sink if $\lambda_1, \lambda_2 < 0$
2. a repeller/source if $\lambda_1, \lambda_2 > 0$
3. a saddle point if $\lambda_1 < 0 < \lambda_2$

Let A be a 2×2 matrix. Consider the system $x' = Ax$, where A has the eigenvalues $\lambda_{1,2} = a \pm bi, b \neq 0$:

1. if $a > 0$, the trajectories spiral away from the origin
2. if $a < 0$, the trajectories spiral inward towards the origin
3. if $a = 0$, the trajectories are closed curves around the origin

Orthogonal complement

Given a subspace W of $\mathbb{R}^n$, the orthogonal complement of W is the set of all vectors in $\mathbb{R}^n$ that are orthogonal to all vectors in W: W^T

1. The orthogonal complement of W is a subspace of $\mathbb{R}^n$
2. Let $W \subset \mathbb{R}^n$ be a subset, then $(W^T)^T = W$

Fundamental matrix spaces

$(\text{Col}A)^\perp = \text{Nul}A^T$ and $(\text{Col}A^T)^\perp = \text{Nul}A$
The subspace $\text{Col}A^T$ is the Row Space of A

Orthogonal, orthonormal sets

A set of vectors $\{u_1, ..., u_p\}$ in $\mathbb{R}^n$ is called an

1. orthogonal set if $u_i \cdot u_j = 0$ for each pair $i \neq j$
2. orthonormal set if $u_i \cdot u_j = \begin{cases} 0, & \text{if } i \neq j \\ 1, & \text{if } i = j \end{cases}$

Any orthogonal set which doesn't contain the zero vector is linearly independent

Coordinates with respect to orthogonal basis

Let $\mathcal{B} = \{v_1, ..., v_p\}$ be an orthogonal basis for a subspace W in $\mathbb{R}^n$. Then any vector y in W has the following decomposition w.r.t. $\mathcal{B}$:

$$y = \frac{y \cdot v_1}{v_1 \cdot v_1} v_1 + ... + \frac{y \cdot v_p}{v_p \cdot v_p} v_p$$

Denominators disappear for an orthonormal basis

Orthonormal columns

An $m \times n$ matrix U has orthonormal columns iff

$$U^T U = I$$

Let U be an $m \times n$ matrix with orthonormal columns and x, y vectors in $\mathbb{R}^n$:

1. $(Ux) \cdot (Uy) = x \cdot y$
2. $||Ux|| = ||x||$
3. $(Ux) \perp (Uy)$ iff $x \perp y$

$U^{-1} = U^T$

Projection

$$\hat{y} = \text{proj}_L(y) = \frac{y \cdot u}{u \cdot u} u$$

Orthogonal decomposition theorem

$y = \hat{y} + z$ where $\hat{y} = \text{proj}_W(y)$ and $z \in W^\perp$

Subspaces with an orthonormal basis

Let $\{u_1, ..., u_p\}$ be an orthonormal basis of $W \subset \mathbb{R}^n$:

1. $\text{proj}_W(y) = UU^T y$ with $U = [u_1 u_2 ... u_p]$
2. if $P = UU^T$ is the standard matrix of proj_W, then $P^2 = P = P^T$

Best approximation

$$||y - \hat{y}|| \leq ||y - v|| \text{ for all } v \in W$$

Least-squares solution

$$||b - A\hat{x}|| \leq ||b - Ax|| \text{ for all } x \in R^n$$

Distance $||b - A\hat{x}||$ is the least-squares error

Spectral theorem

An $n \times n$ symmetric matrix A has the following properties

1. A has n real eigenvalues, counting multiplicities
2. For each eigenvalue, the geometric multiplicity is equal to the algebraic multiplicity
3. The eigenspaces are mutually orthogonal, in the sense that eigenvectors corresponding to different eigenvalues are orthogonal
4. A is orthogonally diagonalizable