Electricity & Magnetism.

Teemu Weckroth, September 7, 2023

Constants.

Elementary charge $e = 1.6 \cdot 10^{-19}$ C
Coulomb constant $k = 9.0 \times 10^9$ N m^2 C^{-2}
Vacuum permittivity $\epsilon_0 = 8.85 \times 10^{-12}$ C^2 N^{-1} m^{-2}
Vacuum permeability $\mu_0 = 4\pi \times 10^{-7}$ N A^{-2}
Speed of light $c = 3.0 \times 10^8$ m s^{-1}
Electronvolt 1 eV $= 1.6 \times 10^{-19}$ J

$$\epsilon_0 = \frac{1}{4\pi k} = \frac{1}{\mu_0 c^2}$$

Units.

$[Q] = \mathrm{C} = \mathrm{A\,s}$
$[\vec{F}] = \mathrm{N} = \mathrm{kg\,m\,s^{-2}}$
$[\vec{E}] = \mathrm{N\,C^{-1}} = \mathrm{V\,m^{-1}} = \mathrm{kg\,m\,s^{-3}\,A^{-1}}$
$[p] = \mathrm{C\,m} = \mathrm{m\,A\,s}$
$[\vec{\tau}] = \mathrm{N\,m} = \mathrm{kg\,m^2\,s^{-2}}$
$[\Phi_E] = \mathrm{V\,m} = \mathrm{N\,m^2\,C^{-1}} = \mathrm{kg\,m^3\,s^{-3}\,A^{-1}}$
$[V] = \mathrm{V} = \mathrm{J\,C^{-1}} = \mathrm{A\,\Omega} = \mathrm{kg\,m^2\,s^{-3}\,A^{-1}}$
$[I] = \mathrm{A} = \mathrm{C\,s^{-1}} = \mathrm{V\,\Omega^{-1}} = \mathrm{W\,V^{-1}} = \mathrm{A}$
$[R] = \Omega = \mathrm{V\,A^{-1}} = \mathrm{W\,A^{-2}} = \mathrm{V^2\,W^{-1}} = \mathrm{kg\,m^2\,s^{-3}\,A^{-2}}$
$[P] = \mathrm{W} = \mathrm{V\,A} = \mathrm{V^2\,\Omega^{-1}} = \mathrm{A^2\,\Omega} = \mathrm{J\,s^{-1}} = \mathrm{H\,A^2\,s^{-1}} = \mathrm{kg\,m^2\,s^{-3}}$
$[C] = \mathrm{F} = \mathrm{C\,V^{-1}} = \mathrm{A^2\,s^4\,kg^{-1}\,m^{-2}}$
$[U_E] = \mathrm{J} = \mathrm{F\,V^2} = \mathrm{kg\,m^2\,s^{-2}}$
$[\vec{J}] = \mathrm{A\,m^{-2}}$
$[\rho] = \Omega\,\mathrm{m} = \mathrm{kg\,m^3\,s^{-3}\,A^{-2}}$
$[\vec{B}] = \mathrm{T} = \mathrm{Wb\,m^{-2}} = \mathrm{kg\,s^{-2}\,A^{-1}}$
$[\Phi_B] = \mathrm{Wb} = \mathrm{V\,s} = \mathrm{kg\,m^2\,s^{-2}\,A^{-1}}$
$[\mathcal{E}] = \mathrm{V} = \mathrm{J\,C^{-1}} = \mathrm{A\,\Omega} = \mathrm{kg\,m^2\,s^{-3}\,A^{-1}}$
$[L] = \mathrm{H} = \mathrm{T\,m^2\,A^{-1}} = \mathrm{kg\,m^2\,s^{-2}\,A^{-2}}$
$[S] = \mathrm{W\,m^{-2}} = \mathrm{kg\,s^{-3}}$
$[P_{\mathrm{rad}}] = \mathrm{Pa} = \mathrm{kg\,m^{-1}\,s^{-2}}$

Coulomb's law.

The electric force is described by

$$\vec{F}_{12} = k\frac{q_1 q_2}{r^2}\hat{r},$$

where $\vec{F}_{12}$ is the force q_1 exerts on q_2 and $\hat{r}$ is a unit vector pointing from q_1 towards q_2.

Electric field.

The electric field at a point in space is the force per unit charge that a charge q placed at that point would experience:

$$\vec{E} = \frac{\vec{F}}{q}$$

The force on a charge q in an electric field is

$$\vec{F} = q\vec{E}$$

Field of a point charge.

The field of a point charge is radial:

$$\vec{E}_{\mathrm{pointcharge}} = \frac{\vec{F}}{Q} = \frac{kqQ}{Qr^2}\hat{r} = \frac{kq}{r^2}\hat{r}$$

Field of a charge distribution.

From the superposition principle:

$$\vec{E} = \sum \vec{E}_i = \sum \frac{kq_i}{r_i^2}\hat{r}_i$$

Dipole moment.

Product of charge and separation:

$$p = qd$$

Dipoles in electric fields.

For q in $\vec{E}$,

$$\vec{a} = \frac{q\vec{E}}{m}$$

A dipole in an electric field experiences a torque that tends to along the dipole moment with the field:

$$\vec{\tau} = \vec{r} \times \vec{F} = \vec{p} \times \vec{E}$$

Electric flux.

$$\Phi_E = EA\cos\theta = \vec{E} \cdot \vec{A},$$

where $\vec{A}$ is a vector whose magnitude is the surface area A and whose orientation is normal to the surface.

$$d\Phi_E = \vec{E} \cdot d\vec{A}$$

$$\Phi_E = \int \vec{E} \cdot d\vec{A}$$

Gauss's law.

$$\oint \vec{E} \cdot d\vec{A} = \frac{q_{\mathrm{enclosed}}}{\epsilon_0}$$

Electric potential difference.

Describes the energy per unit charge involved in moving charge between two points:

$$\Delta V_{AB} = \frac{\Delta U_{AB}}{q} = -\int_A^B \vec{E} \cdot d\vec{r}$$

In a uniform field:

$$\Delta V_{AB} = -\vec{E} \cdot \Delta\vec{r}$$

Electric potential of a point charge.

$$\Delta V_{AB} = kq\left(\frac{1}{r_A} - \frac{1}{r_B}\right)$$

Taking the zero of potential at infinity gives

$$V_{\mathrm{inf}\,r} = V(r) = \frac{kq}{r}$$

Potential difference of a charge distribution.

For discrete point charges:

$$V(P) = \sum_i \frac{kq_i}{r_i}$$

For a continuous charge distribution:

$$V(P) = \int \frac{kdq}{r}$$

Potential difference and the electric field.

The potential difference involves an integral over the electric field, so the field involves derivatives of the potential:

$$\vec{E} = -\left(\frac{\partial V}{\partial x}\hat{\imath} + \frac{\partial V}{\partial y}\hat{\jmath} + \frac{\partial V}{\partial z}\hat{k}\right)$$

Electrostatic energy.

Each charge pair q_i, q_j contributes energy

$$U_{ij} = k\frac{q_i q_j}{r_{ij}},$$

where r_{ij} is the distance between the charges in the final contribution.

Capacitance.

Charge stored per unit potential difference:

$$C = \frac{Q}{V}$$

The capacitance of a parallel-plate capacitor is

$$C = \epsilon_0\frac{A}{d}$$

Energy stored in a capacitor.

Work involved in moving charge is

$$dW = VdQ = CVdV$$

Work involved in building up a potential difference V is

$$W = \int dW = \int_0^V CVdV = \frac{1}{2}CV^2 = U$$

Dielectric constant κ.

A property of the dielectric material. For a parallel-plate capacitor:

$$C = \kappa\frac{\epsilon_0 A}{d} = \kappa C_0, \quad C_0 = \frac{\epsilon_0 A}{d}$$

Energy in the electric field.

The electric energy density is

$$u_E = \frac{1}{2}\epsilon_0 E^2$$

Electric current.

$$I = \frac{dQ}{dt}$$

Drift velocity.

The current through a cross-sectional area A is

$$I = nqAv_d$$

The drift velocity is

$$v_d = \frac{I}{nqA} = \frac{JA}{nqA} = \frac{J}{nq}$$

Current density.

Current density is the current per unit area

$$\vec{J} = nq\vec{v}_d = \sigma\vec{E} = \frac{\vec{E}}{\rho}$$

Current through an area is the flux of $\vec{J}$ over that area:

$$I = \int_{\text{area}} \vec{J} \cdot d\vec{A}$$

Ohm's law.

Microscopic version:

$$\vec{J} = \frac{\vec{E}}{\rho}$$

Macroscopic version:

$$I = \frac{V}{R}$$

Electric power.

$$P = \frac{dW}{dt} = \frac{dQV}{dt} = \frac{VdQ}{dt} = VI$$

For materials that obey Ohm's law:

$$P = I^2R = \frac{V^2}{R}$$

Magnetic force.

The magnetic field $\vec{B}$ exerts a force on moving electric charges:

$$\vec{F} = q\vec{v} \times \vec{B}$$
$$|F| = qvB\sin\theta$$

Charged particles in magnetic fields.

For a particle moving perpendicular to $\vec{B}$:

$$F = qvB = \frac{mv^2}{r} \qquad r = \frac{mv}{qB}$$

Cyclotron frequency is

$$f = \frac{1}{T} = \frac{v}{2\pi r} = \frac{v}{2\pi(mv/qB)} = \frac{qB}{2\pi m}$$

Magnetic force on a current.

A current-carrying conductor experiences a magnetic force.

$$\vec{F} = q_{\text{tot}}\vec{v} \times \vec{B}$$
$$q_{\text{tot}} = nqAl$$
$$\vec{F} = nqAl\vec{v} \times \vec{B} = I\vec{l} \times \vec{B}$$

Origin of the magnetic field.

The Biot-Savart law gives the magnetic field arising from an infinitesimal current element:

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{Id\vec{L} \times \hat{r}}{r^2}$$
$$\vec{B} = \int d\vec{B} = \int \frac{\mu_0}{4\pi}\frac{Id\vec{L} \times \hat{r}}{r^2}$$

Magnetic dipoles.

A current loop constitutes a magnetic dipole. Its dipole moment is $\mu = IA$. For a N-turn loop, $\mu = NIA$.

$$B = \frac{\mu_0}{2\pi}\frac{\mu}{x^3}$$

Gauss's law for magnetism.

$$\oint \vec{B} \cdot d\vec{A} = 0$$

Torque on a current loop.

$$F_{\text{side}} = IaB$$
$$\tau_{\text{side}} = \frac{1}{2}bF_{\text{side}}\sin\theta = \frac{1}{2}IabB\sin\theta = \frac{1}{2}IAB\sin\theta$$
$$\tau = 2\tau_{\text{side}} = IAB\sin\theta = \mu B\sin\theta$$
$$\vec{\tau} = \vec{\mu} \times \vec{B}$$

Ampere's law.

For steady currents,

$$\oint \vec{B} \cdot d\vec{r} = \mu_0 I_{\text{encircled}}$$

Faraday's law.

Describes induction by relating the emf induced in a circuit to the rate of change of magnetic flux through the circuit:

$$\mathcal{E} = \oint \vec{E} \cdot d\vec{r} = -\frac{d\Phi_B}{dt},$$

where the magnetic flux is given by

$$\Phi_B = \int \vec{B} \cdot d\vec{A}$$

Self-inductance.

Ratio of the magnetic flux through the circuit to the current in the circuit:

$$L = \frac{\Phi_B}{I}$$
$$\frac{d\Phi_B}{dt} = L\frac{dI}{dt}$$

The emf across an inductor is

$$\mathcal{E} = -L\frac{dI}{dt}$$

Self-inductance of a solenoid.

$$B = \mu_0 nI$$
$$\Phi_B = nlBA = nl(\mu nI)A = \mu_0 n^2 IAl$$
$$L = \frac{\Phi_B}{I} = \mu_0 n^2 Al$$

Inductive time constant.

The inductor current starts at zero and builds up with time constant L/R.

$$\mathcal{E}_0 - IR + \mathcal{E}_L = 0$$
$$I = \frac{\mathcal{E}_0}{R}(1 - e^{-Rt/L})$$

Magnetic energy.

The inductor absorbs energy from the circuit, which is stored in the inductor's magnetic field.

$$P = LI\frac{dI}{dt}$$
$$U_B = \int Pdt = \int_0^I LIdI = \frac{1}{2}LI^2$$
$$u_B = \frac{B^2}{2\mu_0}$$

Maxwell's equations.

Law	Mathematical statement
Gauss for $\vec{E}$	$\oint \vec{E} \cdot d\vec{A} = \frac{q}{\epsilon_0}$
Gauss for $\vec{B}$	$\oint \vec{B} \cdot d\vec{A} = 0$
Faraday	$\oint \vec{E} \cdot d\vec{r} = -\frac{d\Phi_B}{dt}$
Ampère	$\oint \vec{B} \cdot d\vec{r} = \mu_0 I + \mu_0\epsilon_0\frac{d\Phi_E}{dt}$

Maxwell's equations in vacuum.

Law	Mathematical statement
Gauss for $\vec{E}$	$\oint \vec{E} \cdot d\vec{A} = 0$
Gauss for $\vec{B}$	$\oint \vec{B} \cdot d\vec{A} = 0$
Faraday	$\oint \vec{E} \cdot d\vec{r} = -\frac{d\Phi_B}{dt}$
Ampère	$\oint \vec{B} \cdot d\vec{r} = \mu_0\epsilon_0\frac{d\Phi_E}{dt}$

Plane electromagnetic waves.

The direction of the electric field defines the direction of the wave's polarisation.

$$\vec{E}(x,t) = E_p\sin(kx - \omega t)\hat{j}$$
$$\vec{B}(x,t) = B_p\sin(kx - \omega t)\hat{k}$$
$$E = cB \qquad B = \frac{E}{c}$$

Law of malus.

A wave of intensity S_0 emerges from a polariser with intensity given by:

$$S = S_0\cos^2\theta$$

Energy in EM waves.

$$dU = (u_E + u_B)Adx = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{B^2}{\mu_0}\right)Adx$$
$$S = \frac{1}{A}\frac{dU}{dt} = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{B^2}{\epsilon_0}\right)c$$
$$S = \frac{1}{2}\left(\epsilon_0 cEB + \frac{EB}{c\epsilon_0}\right) = \frac{EB}{2\mu_0}\left(\epsilon_0\mu_0 c^2 + 1\right) = \frac{EB}{\mu_0}$$

Poynting vector describes rate of energy flow per unit area:

$$\vec{S} = \frac{\vec{E} \times \vec{B}}{\mu_0}$$
$$\overline{S} = \frac{E_p B_p}{2\mu_0} = \frac{c{B_p}^2}{2\mu_0} = \frac{{E_p}^2}{2c\mu_0}$$

Momentum in EM waves.

$$W = dU = Fdr = \frac{dp}{dt}dr = cdp \qquad p = \frac{U}{c}$$
$$P_{\text{rad}} = \frac{F}{A} = \frac{1}{A}\frac{dp}{dt} = \frac{1}{cA}\frac{dU}{dt} = \frac{S}{c}$$