Signals and Systems with Python.

Teemu Weckroth, October 31, 2022

Complex numbers.

$$j := \sqrt{-1}$$
$$e^{ja} = \cos a + j \sin a$$
$$z = a + jb = r \cdot e^{j\theta}$$
$$z^* = a - jb = r \cdot e^{-j\theta}$$
$$r = \sqrt{a^2 + b^2}$$
$$\theta = \arctan\left(\frac{b}{a}\right)$$
$$a = r \cdot \cos\theta$$
$$b = r \cdot \sin\theta$$

For special case $r = 1$:

$$z = e^{j\theta} = \cos\theta + j\sin\theta$$
$$z^* = e^{-j\theta} = \cos\theta - j\sin\theta$$
$$\cos\theta = \frac{1}{2}\left(e^{j\theta} + e^{-j\theta}\right)$$
$$\sin\theta = \frac{1}{2j}\left(e^{j\theta} - e^{-j\theta}\right)$$

Operations with complex numbers.

Take $z_1 = a_1 + jb_1 = r_1 e^{j\theta_1}$ and $z_2 = a_2 + jb_2 = r_2 e^{j\theta_2}$:

$$z_1 + z_2 = (a_1 + b_2) + j(b_1 + b_2)$$
$$z_1 \cdot z_2 = (a_1 a_2 - b_1 b_2) + j(a_1 b_2 + a_1 b_2) = r_1 r_2 e^{j(\theta_1 + \theta_2)}$$
$$\frac{1}{z_1} = \frac{1}{a_1 + jb_1} = \frac{1}{r_1} e^{-j\theta_1}$$
$$\frac{z_2}{z_1} = \frac{r_2}{r_1} e^{j(\theta_2 - \theta_1)}$$
$$|z_1|^2 = z_1 \cdot z_1^*$$

Harmonic functions and phasors.

We often deal with signals of the form

$$x_a(t) = A\cos(\omega t + \phi)$$

or

$$x_b(t) = A_1 \cos(\omega t + \phi_1) + A_2 \cos(\omega t + \phi_2)$$

We can write $x_a(t)$ as

$$x_a(t) = \Re(Ae^{j(\omega t + \phi)}) = \Re(e^{j\omega t} \cdot Ae^{j\phi})$$

and $x_b(t))$ as

$$x_b(t) = \Re(e^{j\omega t} \cdot (A_1 e^{j\phi_1} + A_2 e^{j\phi_2}))$$

Signal with delay.

Signal $y(t)$ delayed by τ can be written as

$$y(t) = k \cdot x(t - \tau)$$

or as a convolution

$$y(t) = x(t) * h(t)$$

with

$$h(t) = k \cdot \delta(t - \tau)$$

Moving average.

The signal can be written as

$$y(t) = k \cdot \frac{1}{T_0} \int_{t-T_0}^{t} x(\alpha) \cdot d\alpha$$

or as a convolution

$$y(t) = x(t) * h(t)$$

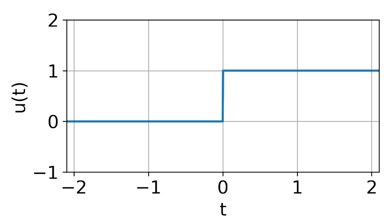
with

$$h(t) = \frac{k}{T_0} \Pi\left(\frac{t - T_0/2}{T_0}\right)$$

Unit step function $u(t)$.

Describes something starting at $t = 0$:

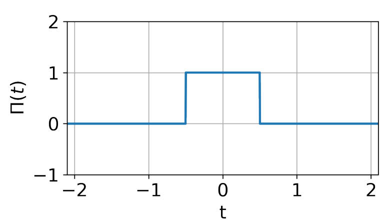
$$u(t) = \begin{cases} 0 & \text{for } t < 0 \\ 1 & \text{for } t \geq 0 \end{cases}$$



Unit pulse function $\Pi(t)$.

Describes something lasting for one second and centred at $t = 0$:

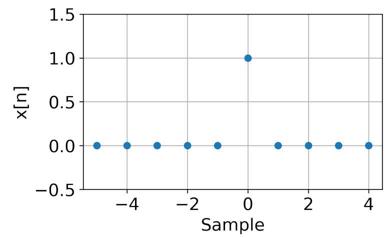
$$\Pi(t) = \begin{cases} 1 & \text{for } |t| \leq \frac{1}{2} \\ 0 & \text{elsewhere} \end{cases}$$



Discrete-time unit impulse $\delta[n]$.

Describes something lasting for one sample and centred at $n = 0$:

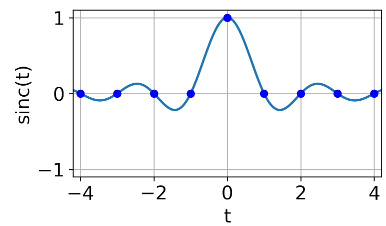
$$\delta[n] = \begin{cases} 1 & \text{for } n = 0 \\ 0 & \text{elsewhere} \end{cases}$$



Sinc function $\mathrm{sinc}(t)$.

$$\mathrm{sinc}(t) = \frac{\sin(\pi \cdot t)}{\pi \cdot t}$$

Sampling the sinc function with sampling intervals $T_s = 1$ gives us the discrete unit pulse $\delta[n] = \mathrm{sinc}(1 \cdot n)$.



Periodic signals.

Periodic signals repeat after period $T_0 = \frac{1}{f_0} = \frac{2\pi}{\omega_0}$:

$$x(t) = x(t + T_0) = x(t + n \cdot T_0)$$

Phasors.

Take

$$x(t) = 2 \cdot \cos\left(2\pi \cdot 10 \cdot t + \frac{\pi}{4}\right)$$

Using Euler's theorem:

$$x(t) = \Re\left(2 \cdot e^{j(2\pi \cdot 10 \cdot t + \frac{\pi}{4})}\right) = \frac{1}{2}\left(2 \cdot e^{j(2\pi \cdot 10 \cdot t + \frac{\pi}{4})} + 2 \cdot e^{-j(2\pi \cdot 10 \cdot t + \frac{\pi}{4})}\right)$$

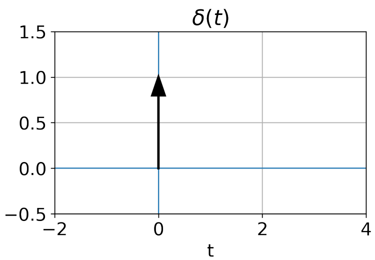
Dirac delta function $\delta(t)$.

Properties that define it:

$$\delta(t) = 0 \text{ for } t \neq 0$$
$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1$$

Consequences:

$$u(t) = \int_{-\infty}^{t} \delta(\tau)\, d\tau$$
$$\int_{-\infty}^{\infty} x(\tau) \cdot \delta(t - \tau)\, d\tau = \int_{-\infty}^{\infty} x(t - \tau) \cdot \delta(\tau)\, d\tau = x(t)$$



Energy and power of a signal.

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt$$

Formally it is

$$E = \lim_{T\to\infty} \int_{-T/2}^{T/2} |x(t)|^2 \, dt$$

For signals with infinite energy:

$$P = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 \, dt$$

For periodic signals:

$$P = \frac{1}{T_0} \int_{<T_0>} |x(t)|^2 \, dt$$

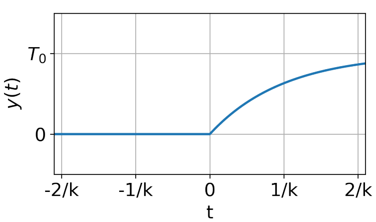
Systems defined by simple ODEs.

For a system described by

$$\frac{dy(t)}{dt} + k \cdot y(t) = k \cdot x(t)$$

where $x(t) = T_0\, u(t)$, the solution $y(t)$ is given by

$$y(t) = T_0 \left(1 - e^{-kt}\right) \cdot u(t)$$



The output of a system where the input is $\delta(n)$ is called the impulse response $h(t)$. In this case:

$$y(t) = h(t) = k \cdot e^{-kt} \cdot u(t)$$

Variant vs invariant systems.

For a time-invariant system with input-output pair x(t) and y(t) and any time-shift τ:

$$x(t-\tau) \to \boxed{\text{Time-invariant system}} \to y(t-\tau)$$

Example.

$$y(t) = \frac{1}{T_0} \cdot \int_{t-T_0}^{t} x(\alpha)\, d\alpha$$

Give the input as signal $x_1(t) = x(t-\tau)$:

$$y_1(t) = \frac{1}{T_0} \cdot \int_{t-T_0}^{t} x_1(\alpha)\, d\alpha$$

Variable change: $\beta = \alpha - \tau$, so that $\alpha = \beta + \tau$ and $d\alpha = d\beta$.
Substitute and change the integration limits:

$$y_1(t) = \frac{1}{T_0} \cdot \int_{(t-\tau)-T_0}^{(t-\tau)} x(\beta)\, d\beta = y(t-\tau)$$

Linear systems.

A system is linear if, for any constants a and b and inputs $x_1(t)$ and $x_2(t)$:

$$\mathcal{H}[a \cdot x_1(t) + b \cdot x_2(t)] = a \cdot \mathcal{H}[x_1(t)] + b \cdot \mathcal{H}[x_2(t)]$$

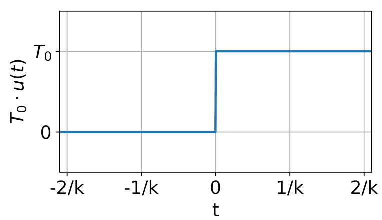
Causal systems.

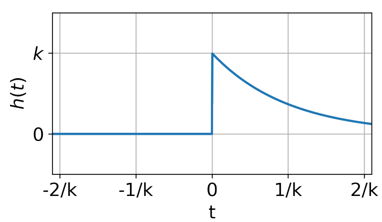
A system is causal if the output signal does not depend on future inputs.
Example: $y(t) = \frac{1}{T_0} \cdot \int_{t-T_0}^{t} x(\alpha)\, d\alpha$.
Counter-example: $\mathcal{H}[x(t)] = x(t+1)$, $\mathcal{H}[x(t)] = \frac{1}{T_0} \cdot \int_{t}^{t+T_0} x(\alpha)\, d\alpha$.

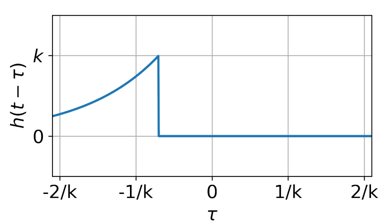
LTI systems, impulse response, and the convolution.

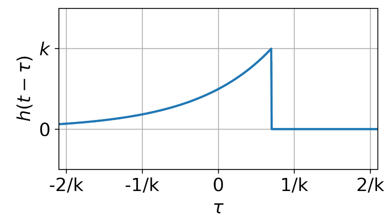
$$\delta(t) \to \boxed{\text{System}} \to h(t)$$

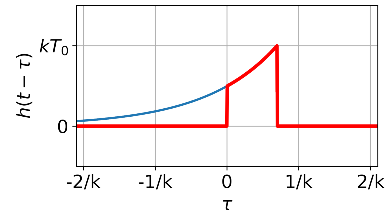
$$x(t) \to \boxed{\text{System}} \to \int_{-\infty}^{\infty} x(\tau) \cdot h(t-\tau)\, d\tau = x(t) * h(t)$$

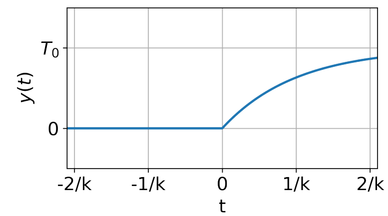












Temporal length of the output = length of input + length of $h(t)$.

Properties of the convolution.

Commutative: $x(t) * y(t) = y(t) * x(t)$
Associative: $x(t) * h_1(t) * h_2(t) = x(t) * (h_1(t) * h_2(t))$
Distributive: $x(t) * h_1(t) + x(t) * h_2(t) = x(t) * (h_1(t) + h_2(t))$

Frequency response.

$$x(t) = e^{j2\pi f_0 t} \to \boxed{\text{H(f)}} \to e^{j2\pi f_0 t} \cdot H(f_0)$$

$$x(t) * h(t) = \int_{-\infty}^{\infty} h(\tau) \cdot e^{j2\pi f_0 (t-\tau)}\, d\tau$$
$$= e^{j2\pi f_0 t} \int_{-\infty}^{\infty} h(\tau) \cdot e^{-j2\pi f_0 \tau}\, d\tau$$

$H(f)$: frequency response. The frequency response is the Fourier transform of the impulse response.

$$H(f) = \int_{-\infty}^{\infty} h(\tau) \cdot e^{-j2\pi f \tau}\, d\tau = \int_{-\infty}^{\infty} h(t) \cdot e^{-j2\pi f t}\, dt$$

Constructing periodic signals.

$$x_0(t) = a_0 + \sum_{k=1}^{n} a_k \cos(k\omega_0 t) + \sum_{k=1}^{n} b_k \sin(k\omega_0 t)$$

$$x_0(t) = \sum_{n=-N}^{N} X_n e^{jn\omega_0 t}$$

Synthesis and analysis equations.

A non-pathological periodic function with period T_0 can be expressed as

$$x(t) = \sum_{n=0}^{\infty} a_n \cos(n\omega_0 t) + \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t),$$

with

$$a_0 = \frac{1}{T_0}\int_{<T_0>} x(t)\,dt = \frac{1}{T_0}\int_0^{T_0} x(t)\,dt = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)\,dt,$$
$$a_n = \frac{2}{T_0}\int_{<T_0>} x(t)\cdot\cos(n\omega_0 t)\,dt,$$
$$b_n = \frac{2}{T_0}\int_{<T_0>} x(t)\cdot\sin(n\omega_0 t)\,dt.$$

a_0 is the mean value. $\cos(n\omega_0 t)$ are even functions of $t (x(t) = x(-t))$. $\sin(n\omega_0 t)$ are odd functions of $t (x(t) = -x(-t))$.

Complex Fourier series.

Since

$$\cos(n\omega_0 t) = \frac{1}{2}e^{-jn\omega_0 t} + \frac{1}{2}e^{jn\omega_0 t}$$

and

$$\sin(n\omega_0 t) = \frac{j}{2}e^{-jn\omega_0 t} - \frac{j}{2}e^{jn\omega_0 t},$$

we can rewrite the Fourier series as

$$x(t) = \sum_{n=0}^{\infty} a_n \cos(n\omega_0 t) + \sum_{n=1}^{\infty} b_n sin(n\omega_0 t) = \sum_{n=-\infty}^{\infty} X_n \cdot e^{jn\omega_0 t}$$

with

$$X_n = \frac{1}{T_0}\int_{<T_0>} x(t)\cdot e^{-jn\omega_0 t}\,dt$$

Symmetry property.

For real-valued signals:

$$X_n = X_{-n}^*$$

Real part is even function of n.
Imaginary part is odd function of n.
Amplitude $|X_n|$ is even function.
Phase is odd.

Differentiation.

$$x(t) \rightarrow \boxed{\frac{d}{dt}} \rightarrow \mathcal{H}[x(t)] = \frac{dx(t)}{dt} = \dot{x}(t)$$

Per definition:

$$h(t) = \dot{\delta}(t)$$

Thus:

$$\mathcal{H}[x(t)] = x(t) * \dot{\delta}(t) = \int_{-\infty}^{\infty} x(\tau)\cdot\dot{\delta}(t-\tau)\,dt = \dot{x}(t)$$

$$e^{jk\omega_0 t} \rightarrow \boxed{\frac{d}{dt}} \rightarrow jk\omega_0 \cdot e^{jk\omega_0 t}$$

$$X_l e^{jl\omega_0 t} + X_k e^{jk\omega_0 t} \rightarrow \boxed{\frac{d}{dt}} \rightarrow jl\omega_0 X_l e^{jl\omega_0 t} + jk\omega_0 X_k e^{jk\omega_0 t}$$

$$\sum_{k=-\infty}^{\infty} X_k e^{jk\omega_0 t} \rightarrow \boxed{\frac{d}{dt}} \rightarrow \sum_{k=-\infty}^{\infty} jk\omega_0 X_k e^{jk\omega_0 t}$$

Thus the derivative of a complex Fourier series is another one with new coefficients

$$X_k' = jk\omega_0 X_k$$

Response of LTI system to periodic signal.

$$\sum_{k=-\infty}^{\infty} X_k e^{jk\omega_0 t} \rightarrow \boxed{\begin{matrix} h(t) \\ H(\omega) \end{matrix}} \rightarrow \sum_{k=-\infty}^{\infty} H(k\omega_0)\cdot X_k e^{jk\omega_0 t}$$

$$\tilde{X}_k = H(k\omega_0)\cdot X_k$$

Parseval's theorem.

$$P = \frac{1}{T_0}\int_0^{T_0} |x(t)|^2\,dt = \frac{1}{T_0}\int_0^{T_0} x(t)\cdot x(t)^*\,dt$$
$$= \sum_{n=-\infty}^{\infty} X_n^* \cdot X_n = \sum_{n=-\infty}^{\infty} |X_n|^2$$

For trigonometric series:

$$P = \frac{1}{T_0}\int_0^{T_0} |x(t)|^2\,dt = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty} a_n^2 + \frac{1}{2}\sum_{n=1}^{\infty} b_n^2$$