Signals and Systems.

Teemu Weckroth, September 7, 2023

Complex numbers.

$$
\begin{aligned}
j &:= \sqrt{-1} \\
e^{ja} &= \cos a + j \sin a \\
z &= a + jb = r \cdot e^{j\theta} \\
z^* &= a - jb = r \cdot e^{-j\theta} \\
r &= \sqrt{a^2 + b^2} \\
\theta &= \arctan\left(\frac{b}{a}\right) \\
a &= r \cdot \cos\theta \\
b &= r \cdot \sin\theta
\end{aligned}
$$

For special case $r = 1$:

$$
\begin{aligned}
z &= e^{j\theta} = \cos\theta + j\sin\theta \\
z^* &= e^{-j\theta} = \cos\theta - j\sin\theta \\
\cos\theta &= \frac{1}{2}\left(e^{j\theta} + e^{-j\theta}\right) \\
\sin\theta &= \frac{1}{2j}\left(e^{j\theta} - e^{-j\theta}\right)
\end{aligned}
$$

Operations with complex numbers.

Take $z_1 = a_1 + jb_1 = r_1 e^{j\theta_1}$ and $z_2 = a_2 + jb_2 = r_2 e^{j\theta_2}$:

$$
\begin{aligned}
z_1 + z_2 &= (a_1 + b_2) + j(b_1 + b_2) \\
z_1 \cdot z_2 &= (a_1 a_2 - b_1 b_2) + j(a_1 b_2 + a_1 b_2) \\
&= r_1 r_2 e^{j(\theta_1 + \theta_2)} \\
\frac{1}{z_1} &= \frac{1}{a_1 + jb_1} = \frac{1}{r_1} e^{-j\theta_1} \\
\frac{z_2}{z_1} &= \frac{r_2}{r_1} e^{j(\theta_2 - \theta_1)} \\
|z_1|^2 &= z_1 \cdot z_1^*
\end{aligned}
$$

Harmonic functions and phasors.

We often deal with signals of the form

$$x_a(t) = A\cos(\omega t + \phi)$$

or

$$x_b(t) = A_1\cos(\omega t + \phi_1) + A_2\cos(\omega t + \phi_2)$$

We can write $x_a(t)$ as

$$x_a(t) = \mathrm{Re}(Ae^{j(\omega t + \phi)}) = \mathrm{Re}(e^{j\omega t} \cdot Ae^{j\phi})$$

and $x_b(t)$) as

$$x_b(t) = \mathrm{Re}(e^{j\omega t} \cdot (A_1 e^{j\phi_1} + A_2 e^{j\phi_2}))$$

Signal with delay.

Signal $y(t)$ delayed by τ can be written as

$$y(t) = k \cdot x(t - \tau)$$

or as a convolution

$$y(t) = x(t) * h(t)$$

with

$$h(t) = k \cdot \delta(t - \tau)$$

Moving average.

The signal can be written as

$$y(t) = k \cdot \frac{1}{T_0} \int_{t-T_0}^{t} x(\alpha) \cdot d\alpha$$

or as a convolution

$$y(t) = x(t) * h(t)$$

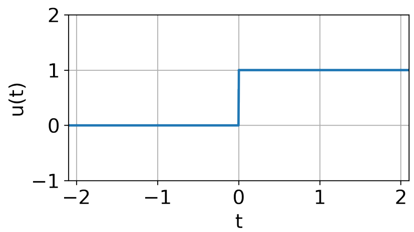
with

$$h(t) = \frac{k}{T_0} \Pi\left(\frac{t - T_0/2}{T_0}\right)$$

Unit step function $u(t)$.

Describes something starting at $t = 0$:

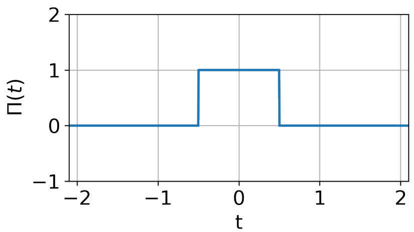
$$u(t) = \begin{cases} 0 & \text{for } t < 0 \\ 1 & \text{for } t \geq 0 \end{cases}$$



Unit pulse function $\Pi(t)$.

Describes something lasting for one second and centred at $t = 0$:

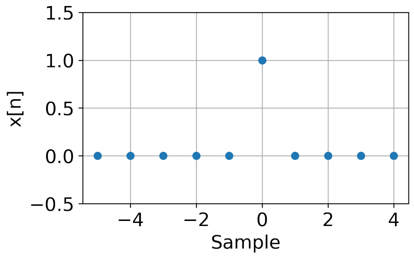
$$\Pi(t) = \begin{cases} 1 & \text{for } |t| \leq \frac{1}{2} \\ 0 & \text{elsewhere} \end{cases}$$



Discrete-time unit impulse $\delta[n]$.

Describes something lasting for one sample and centred at $n = 0$:

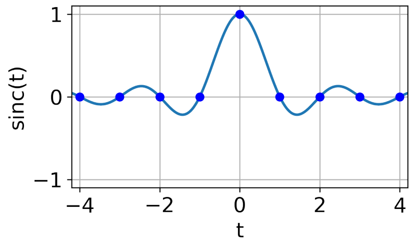
$$\delta[n] = \begin{cases} 1 & \text{for } n = 0 \\ 0 & \text{elsewhere} \end{cases}$$



Sinc function $\mathrm{sinc}(t)$.

$$\mathrm{sinc}(t) = \frac{\sin(\pi \cdot t)}{\pi \cdot t}$$

Sampling the sinc function with sampling intervals $T_s = 1$ gives us the discrete unit pulse $\delta[n] = \mathrm{sinc}(1 \cdot n)$.

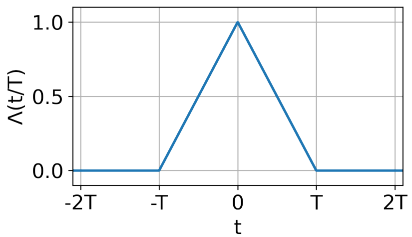


Triangle function $\Lambda\left(\frac{t}{T}\right)$.

$$\Lambda\left(\frac{t}{T}\right) = \begin{cases} 1 - \left|\frac{t}{T}\right| & \text{for } t \in [-T, T] \\ 0 & \text{elsewhere} \end{cases}$$

We encounter this while studying the convolution

$$\frac{1}{T} \cdot \Pi\left(\frac{t}{T}\right) * \Pi\left(\frac{t}{T}\right) = \Lambda\left(\frac{t}{T}\right)$$



Periodic signals.

Periodic signals repeat after period $T_0 = \frac{1}{f_0} = \frac{2\pi}{\omega_0}$:

$$x(t) = x(t + T_0) = x(t + n \cdot T_0)$$

Phasors.

Take

$$x(t) = 2 \cdot \cos\left(2\pi \cdot 10 \cdot t + \frac{\pi}{4}\right)$$

Using Euler's theorem:

$$\begin{aligned} x(t) &= \mathrm{Re}\left\{\left(2 \cdot e^{j(2\pi\cdot 10\cdot t+\frac{\pi}{4})}\right)\right\} \\ &= \frac{1}{2}\left(2 \cdot e^{j(2\pi\cdot 10\cdot t+\frac{\pi}{4})} + 2 \cdot e^{-j(2\pi\cdot 10\cdot t+\frac{\pi}{4})}\right) \end{aligned}$$

Dirac delta function $\delta(t)$.

Properties that define it:

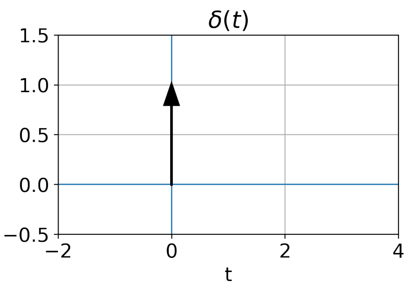
$$\delta(t) = 0 \text{ for } t \neq 0$$

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1$$

Consequences:

$$u(t) = \int_{-\infty}^{t} \delta(\tau)\, d\tau$$

$$\begin{aligned} \int_{-\infty}^{\infty} x(\tau) \cdot \delta(t-\tau)\, d\tau &= \int_{-\infty}^{\infty} x(t-\tau) \cdot \delta(\tau)\, d\tau \\ &= x(t) \end{aligned}$$



Energy and power of a signal.

$$E = \int_{-\infty}^{\infty} |x(t)|^2\, dt$$

Formally it is

$$E = \lim_{T\to\infty} \int_{-T/2}^{T/2} |x(t)|^2\, dt$$

For signals with infinite energy:

$$P = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2\, dt$$

For periodic signals:

$$P = \frac{1}{T_0} \int_{<T_0>} |x(t)|^2\, dt$$

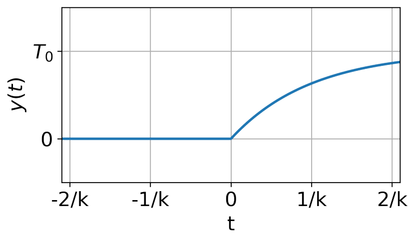
Systems defined by simple ODEs.

For a system described by

$$\frac{dy(t)}{dt} + k \cdot y(t) = k \cdot x(t)$$

where $x(t) = T_0\, u(t)$, the solution $y(t)$ is given by

$$y(t) = T_0\left(1 - e^{-kt}\right) \cdot u(t)$$



The output of a system where the input is $\delta(n)$ is called the impulse response $h(t)$. In this case:

$$y(t) = h(t) = k \cdot e^{-kt} \cdot u(t)$$

Variant vs invariant systems.

For a time-invariant system with input-output pair x(t) and y(t) and any time-shift τ:

$$x(t-\tau) \to \boxed{\text{Time-invariant system}} \to y(t-\tau)$$

Example.

$$y(t) = \frac{1}{T_0} \cdot \int_{t-T_0}^{t} x(\alpha)\, d\alpha$$

Give the input as signal $x_1(t) = x(t-\tau)$:

$$y_1(t) = \frac{1}{T_0} \cdot \int_{t-T_0}^{t} x_1(\alpha)\, d\alpha$$

Variable change: $\beta = \alpha - \tau$, so that $\alpha = \beta + \tau$ and $d\alpha = d\beta$.
Substitute and change the integration limits:

$$y_1(t) = \frac{1}{T_0} \cdot \int_{(t-\tau)-T_0}^{(t-\tau)} x(\beta)\, d\beta = y(t-\tau)$$

Linear systems.

A system is linear if, for any constants a and b and inputs $x_1(t)$ and $x_2(t)$:

$$\mathcal{H}[a \cdot x_1(t) + b \cdot x_2(t)] = a \cdot \mathcal{H}[x_1(t)] + b \cdot \mathcal{H}[x_2(t)]$$

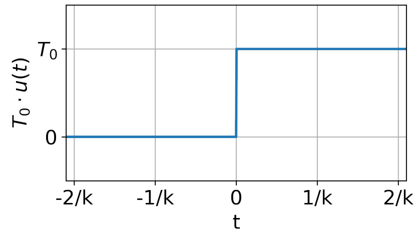
Causal systems.

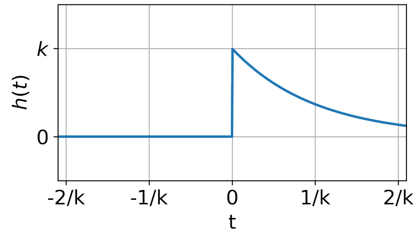
A system is causal if the output signal does not depend on future inputs.
Example: $y(t) = \frac{1}{T_0} \cdot \int_{t-T_0}^{t} x(\alpha)\, d\alpha$.
Counter-example: $\mathcal{H}[x(t)] = x(t+1)$, $\mathcal{H}[x(t)] = \frac{1}{T_0} \cdot \int_{t}^{t+T_0} x(\alpha)\, d\alpha$.

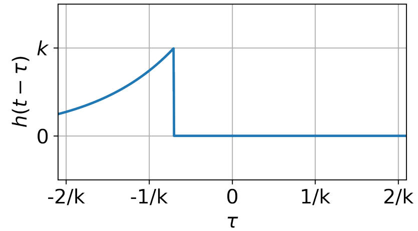
LTI systems, impulse response, and the convolution.

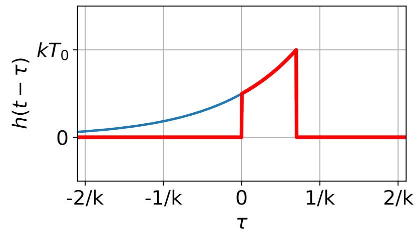
$$\delta(t) \to \boxed{\text{System}} \to h(t)$$

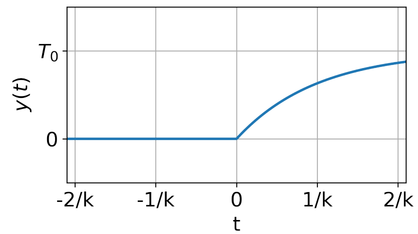
$$x(t) \to \boxed{\text{System}} \to \int_{-\infty}^{\infty} x(\tau) \cdot h(t-\tau)\, d\tau = x(t) * h(t)$$











Temporal length of the output = length of input + length of $h(t)$.

Computing convolutions by hand.

- Sketch functions.
- Identify cases/intervals
- Solve integral as a function of t.

Properties of the convolution.

Commutative: $x(t) * y(t) = y(t) * x(t)$
Associative: $x(t) * h_1(t) * h_2(t) = x(t) * (h_1(t) * h_2(t))$
Distributive: $x(t) * h_1(t) + x(t) * h_2(t) = x(t) * (h_1(t) + h_2(t))$

Frequency response.

$$x(t) = e^{j2\pi f_0 t} \rightarrow \boxed{\text{H(f)}} \rightarrow e^{j2\pi f_0 t} \cdot H(f_0)$$

$$x(t) * h(t) = \int_{-\infty}^{\infty} h(\tau) \cdot e^{j2\pi f_0 (t-\tau)}\, d\tau$$
$$= e^{j2\pi f_0 t} \int_{-\infty}^{\infty} h(\tau) \cdot e^{-j2\pi f_0 \tau}\, d\tau$$

$H(f)$: frequency response. The frequency response is the Fourier transform of the impulse response.

$$H(f) = \int_{-\infty}^{\infty} h(\tau) \cdot e^{-j2\pi f \tau}\, d\tau = \int_{-\infty}^{\infty} h(t) \cdot e^{-j2\pi f t}\, dt$$

Constructing periodic signals.

$$x_0(t) = a_0 + \sum_{k=1}^{n} a_k \cos(k\omega_0 t) + \sum_{k=1}^{n} b_k \sin(k\omega_0 t)$$
$$= \sum_{n=-N}^{N} X_n e^{jn\omega_0 t}$$

Synthesis and analysis equations.

A non-pathological periodic function with period T_0 can be expressed as

$$x(t) = \sum_{n=0}^{\infty} a_n \cos(n\omega_0 t) + \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t)$$

with

$$a_0 = \frac{1}{T_0} \int_{<T_0>} x(t)\, dt = \frac{1}{T_0} \int_0^{T_0} x(t)\, dt = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t)\, dt$$
$$a_n = \frac{2}{T_0} \int_{<T_0>} x(t) \cdot \cos(n\omega_0 t)\, dt$$
$$b_n = \frac{2}{T_0} \int_{<T_0>} x(t) \cdot \sin(n\omega_0 t)\, dt$$

a_0 is the mean value. $\cos(n\omega_0 t)$ are even functions of $t (x(t) = x(-t))$.
$\sin(n\omega_0 t)$ are odd functions of $t (x(t) = -x(-t))$.
A signal $x(t)$ can always be decomposed in an even and an odd component:

$$x(t) = x_{\text{even}}(t) + x_{\text{odd}}(t)$$

Complex Fourier series.

Since

$$\cos(n\omega_0 t) = \frac{1}{2} e^{-jn\omega_0 t} + \frac{1}{2} e^{jn\omega_0 t}$$

and

$$\sin(n\omega_0 t) = \frac{j}{2} e^{-jn\omega_0 t} - \frac{j}{2} e^{jn\omega_0 t},$$

we can rewrite the Fourier series as

$$x(t) = \sum_{n=0}^{\infty} a_n \cos(n\omega_0 t) + \sum_{n=1}^{\infty} b_n sin(n\omega_0 t) = \sum_{n=-\infty}^{\infty} X_n \cdot e^{jn\omega_0 t}$$

with

$$X_n = \frac{1}{T_0} \int_{<T_0>} x(t) \cdot e^{-jn\omega_0 t}\, dt$$

Symmetry property.

For real-valued signals:

$$X_n = X_{-n}^*$$

Real part is even function of n.
Imaginary part is odd function of n.
Amplitude $|X_n|$ is even function.
Phase is odd.
Altogether this is called Hermitian symmetry.

Differentiation.

$$x(t) \rightarrow \boxed{\frac{d}{dt}} \rightarrow \mathcal{H}[x(t)] = \frac{dx(t)}{dt} = \dot{x}(t)$$

Per definition:

$$h(t) = \dot{\delta}(t)$$

Thus:

$$\mathcal{H}[x(t)] = x(t) * \dot{\delta}(t) = \int_{-\infty}^{\infty} x(\tau) \cdot \dot{\delta}(t-\tau)\, dt = \dot{x}(t)$$

$$e^{jk\omega_0 t} \rightarrow \boxed{\frac{d}{dt}} \rightarrow jk\omega_0 \cdot e^{jk\omega_0 t}$$
$$X_l e^{jl\omega_0 t} + X_k e^{jk\omega_0 t} \rightarrow \boxed{\frac{d}{dt}} \rightarrow jl\omega_0 X_l e^{jl\omega_0 t} + jk\omega_0 X_k e^{jk\omega_0 t}$$
$$\sum_{k=-\infty}^{\infty} X_k e^{jk\omega_0 t} \rightarrow \boxed{\frac{d}{dt}} \rightarrow \sum_{k=-\infty}^{\infty} jk\omega_0 X_k e^{jk\omega_0 t}$$

Thus the derivative of a complex Fourier series is another one with new coefficients

$$X_k' = jk\omega_0 X_k$$

Response of LTI system to periodic signal.

$$\sum_{k=-\infty}^{\infty} X_k e^{jk\omega_0 t} \rightarrow \boxed{\begin{matrix} h(t) \\ H(\omega) \end{matrix}} \rightarrow \sum_{k=-\infty}^{\infty} H(k\omega_0) \cdot X_k e^{jk\omega_0 t}$$

Thus the response of a LTI system to a periodic signal is another periodic signal with the same fundamental period and coefficients

$$\tilde{X}_k = H(k\omega_0) \cdot X_k$$

E.g. differentiation: $H(\omega) = j\omega$.

Parseval's theorem.

$$P = \frac{1}{T_0} \int_0^{T_0} |x(t)|^2\, dt = \frac{1}{T_0} \int_0^{T_0} x(t) \cdot x(t)^*\, dt$$
$$= \sum_{n=-\infty}^{\infty} X_n^* \cdot X_n = \sum_{n=-\infty}^{\infty} |X_n|^2$$

For trigonometric series:

$$P = \frac{1}{T_0} \int_0^{T_0} |x(t)|^2\, dt = \frac{1}{T_0} \int_0^{T_0} x(t) \cdot x(t)^*\, dt$$
$$= a_0^2 + \frac{1}{2} \sum_{n=1}^{\infty} a_n^2 + \frac{1}{2} \sum_{n=1}^{\infty} b_n^2$$

Time and frequency.

As the signal becomes shorter in the time domain, we need higher frequencies to synthesize it.

Short/concentrated in time $\leftrightarrow$ long/extended in frequency
Long/extended in time $\leftrightarrow$ short/concentrated in frequency

Fourier transform.

$$\text{Fourier Transform: } X(f) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j2\pi f t}\, dt$$
$$\text{Inverse Fourier Transform: } x(t) = \int_{-\infty}^{\infty} X(f) \cdot e^{j2\pi f t}\, df$$

Duality.

Given the Fourier transform pair

$$\mathcal{F}\{x(t)\} = X(f)$$

we also have the following one:

$$\mathcal{F}\{X(t)\} = x(-f)$$

Convolution and product theorems.

Time domain	Frequency domain
$x(t) * h(t)$	$X(f) \cdot H(f)$
$x(t) \cdot y(t)$	$X(f) * Y(f)$

$Y(f) = H(f) \cdot X(f)$
Fourier transforms are slight extensions of Fourier series:

$$x(t) = \int_{-\infty}^{\infty} X(f) \cdot e^{j2\pi f t}\, df \rightarrow \boxed{\begin{matrix} h(t) \\ H(f) \end{matrix}}$$
$$\rightarrow y(t) = \int_{-\infty}^{\infty} H(f) \cdot X(f) \cdot e^{j2\pi f t}\, df$$

Energy of signal after going through system.

Parseval's theorem:

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} |X(f)|^2 \, df$$

$$x(t) \text{ with } E_x \rightarrow \boxed{\begin{matrix} h(t) \\ H(f) \end{matrix}} \rightarrow y(t) \text{ with } E_y$$

$$E_y = \int_{-\infty}^{\infty} |y(t)|^2 \, dt = \int_{-\infty}^{\infty} |Y(f)|^2 \, df$$

First route: calculate $y(t) = x(t) * h(t)$ and continue from there.
Second route: $Y(f) = H(f) \cdot X(f)$, thus

$$E_y = \int_{-\infty}^{\infty} |H(f)|^2 \, |X(f)|^2 \, df$$

Filters.

Low-pass Butterworth filter example:

$$H_{\text{Butter}}(f) = \frac{1}{B_n\left(j \cdot \frac{f}{f_c}\right)}$$

where $B_n(s)$ are the Butterworth polynomials of order n.
E.g. $B_2(s) = s^2 + \sqrt{2}s + 1$.
In general:

$$|H_{\text{Butter}}(f)|^2 = \frac{1}{1 + \left(\frac{f}{f_c}\right)^{2n}}$$

At f_c (or f_1, f_2) we have $|H(f_c)|^2 = \frac{1}{2}$.

Higher order filter $\rightarrow$ sharper in frequency domain
$\rightarrow$ longer impulse response
$\rightarrow$ more complexity

Fourier transform vs DFT.

Fourier Transform.
Physical world: continuous time signals $x(t)$ for $t \in [-\infty, \infty]$.

$$X(f) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j2\pi f t} \, dt$$

$$x(t) = \int_{-\infty}^{\infty} X(f) \cdot e^{j2\pi f t} \, df$$

$X(f)$ is continuous, $f \in [-\infty, \infty]$.
Discrete Fourier Transform.
Typical digital world: discrete time signals $x[n]$ for $n = 0, 1, ..., N-1$.

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi \frac{k \cdot n}{N}}$$

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j2\pi \frac{k \cdot n}{N}}$$

$X[k]$ is discrete. Discrete angular frequencies $0, \frac{2\pi}{N}, \frac{2\pi \cdot 2}{N}, ..., \frac{2\pi \cdot (N-1)}{N}$.

Discrete Fourier Transform.

Very often $x[n]$ are samples of a continuous time signal:

$$x[n] = x(n \cdot T_s)$$

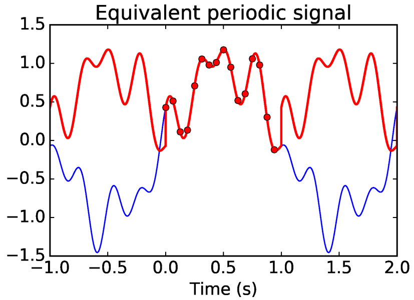
In those cases, the discrete frequencies correspond to continuous time frequencies

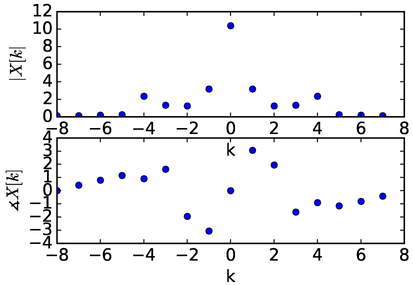
$$f_k = \frac{k}{N} \cdot F_s = \frac{k}{N \cdot T_s} = \frac{k}{T_{\text{meas}}}$$

Frequency resolution:

$$\Delta f = \frac{F_s}{N} = \frac{1}{N \cdot T_s} = \frac{1}{T_{\text{meas}}}$$

DFT treats signals as periodic: we take N samples of a continuous time signal. Since we do not know what happens before or after, we can assume that the signal is periodic.
First $N/2$ values of DFT: positive frequencies.
Second $N/2$ values of DFT: negative frequencies.



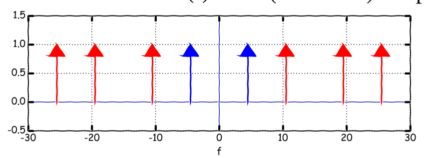


DFT vs FFT.

The Fast Fourier Transform is a family of algorithms to efficiently compute the DFT: number of operations for DFT according to formula is $\mathcal{O}(N^2)$; number of operations for FFT is $\mathcal{O}(N \cdot \log N)$.

Aliasing.

Causes different signals to become indistinguishable (or aliases to one another) when sampled.
Possible aliases for $x(t) = \cos(2\pi \cdot 4.5 \cdot t)$ sampled at $F_s = 15$ Hz:



Nyquist Theorem.

To perfectly reconstruct a signal, the sampling frequency F_s needs to be at least twice the highest frequency W of a given signal:

$$F_s > 2 \cdot W$$

Simple sampling scheme: Sample and Hold.

Sample and Hold: retain values of signal at instants

$$t_n = n \cdot T_s$$

During sampling, this operation gives the Digital to Analog Converter (DAC) time to do the quantization of the signal.

Ideal sampling.

We can model sampling as multiplying a continuous time signal $x(t)$ with a train of Dirac-deltas:

$$x_s(t) = x(t) \cdot \sum_{n=-\infty}^{\infty} T_s \cdot \delta(t - n \cdot T_s)$$

$$= T_s \sum_{n=-\infty}^{\infty} x(n \cdot T_s) \cdot \delta(t - n \cdot T_s)$$

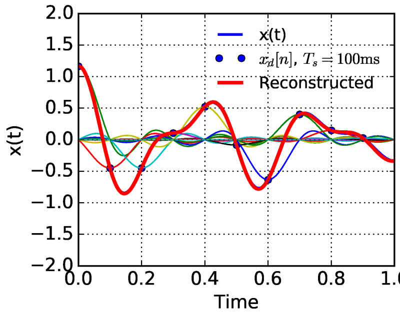
$$\boxed{x_s(t) = x(t) \cdot T_s \sum_{n=-\infty}^{\infty} \delta(t - n \cdot T_s)} \overset{\mathcal{F}}{\leftrightarrow} \boxed{X_s(f) = X(f) * \sum_{k=-\infty}^{\infty} \delta(f - k \cdot F_s)}$$

$$\boxed{x_s(t) = T_s \sum_{n=-\infty}^{\infty} x(n \cdot T_s) \cdot \delta(t - n \cdot T_s)} \overset{\mathcal{F}}{\leftrightarrow} \boxed{X_s(f) = \sum_{k=-\infty}^{\infty} X(f - k \cdot F_s)}$$

Ideal reconstruction.

Given a properly sampled signal ($F_s > 2 \cdot W$) we can reconstruct the original perfectly using a Sinc interpolation (essentially the same as using a low pass filter):

$$x(t) = \sum_{n=-\infty}^{\infty} x[n] \cdot \operatorname{sinc}\left(\frac{t - n \cdot T_s}{T_s}\right)$$



$$x[n] \rightarrow \boxed{\text{DAC}} \xrightarrow[X(f) * \sum_{k=-\infty}^{\infty} \delta(f - k \cdot F_s)]{T_s \sum_{n=-\infty}^{\infty} x[n] \delta(t - nT_s)} \boxed{\text{Ideal LPF}} \xrightarrow{\sum_{n=-\infty}^{\infty} x[n] \operatorname{sinc}(t - nT_s) = x(t)}$$

Various Fourier transforms.

From Fourier series to Fourier transform.

$$x(t) = \sum_{n=-\infty}^{\infty} X_n \cdot e^{jn\omega_0 t}$$
$$X_n = \frac{1}{T_0}\int_{<T_0>} x(t)\cdot e^{-jn\omega_0 t}\,dt$$

Move $\frac{1}{T_0}$ to $x(t)$, use $\omega_0 = 2\pi f_0$, and rewrite.

$$x(t) = \sum_{n=-\infty}^{\infty} X[n]\cdot e^{j2\pi n f_0 t}\frac{1}{T_0}$$
$$X[n] = \int_{<T_0>} x(t)\cdot e^{-j2\pi n f_0 t}\,dt$$

Write $X[n]$ as a continuous function of f and use $f_0 = \frac{1}{T_0}$.

$$x(t) = \sum_{n=-\infty}^{\infty} X(nf_0)\cdot e^{j2\pi n f_0 t} f_0$$
$$X(nf_0) = \int_{<T_0>} x(t)\cdot e^{-j2\pi n f_0 t}\,dt$$

Now: we let $T_0 \to \infty$, or $f_0 \to 0$; we call this df; use $f = n\cdot f_0$; and the summation becomes an integral.

$$\text{Inverse Fourier Transform: } x(t) = \int_{-\infty}^{\infty} X(f)\cdot e^{j2\pi f t}\,df$$
$$\text{Fourier Transform: } X(f) = \int_{-\infty}^{\infty} x(t)\cdot e^{-j2\pi f t}\,dt$$

Fourier transform of Dirac delta.

$$\mathcal{F}\{\delta(t)\} = \int_{-\infty}^{\infty} \delta(t)\cdot e^{-j2\pi f t}\,dt$$

Apply shifting property.

$$\mathcal{F}\{\delta(t)\} = \int_{-\infty}^{\infty} \delta(t)\cdot e^{-j2\pi f t}\,dt = 1$$

Did we know this?

$$x(t) \to \boxed{\delta(t)} \to x(t)$$

We knew the expression of the frequency response:

$$\begin{aligned} H(f) &= \int_{-\infty}^{\infty} h(t)\cdot e^{-j2\pi f t}\,dt \\ &= \int_{-\infty}^{\infty} \delta(t)\cdot e^{-j2\pi f t}\,dt = 1 \end{aligned}$$

Fourier transform of a delay.

$$\begin{aligned} \mathcal{F}\{\delta(t-\tau)\} &= \int_{-\infty}^{\infty} \delta(t-\tau)\cdot e^{-j2\pi f t}\,dt \\ &= \int_{-\infty}^{\infty} \delta(t-\tau) e^{-j2\pi f t}\,dt \\ &= e^{-j2\pi f \tau} \end{aligned}$$

Duality example.

$$X(f) = \delta(F)$$
$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi f t}\,df = \int_{-\infty}^{\infty} \delta(f)\cdot e^{j2\pi f t}\,df = 1$$
$$1 \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\Longleftrightarrow}} \delta(f)$$

What is the Fourier transform of

$$x_1(t) = \frac{1}{10 + j2\pi t}$$

From the Fourier transform pair table:

$$\mathcal{F}\left\{e^{-\alpha t}\cdot u(t)\right\} = \frac{1}{\alpha + j2\pi f} = X(f)$$

Take $\alpha = 10$, then we have $x_1(t) = X(f)$, therefore

$$\mathcal{F}\{x_1(t)\} = \mathcal{F}\{X(f)\} = e^{-\alpha(-f)}\cdot u(-f)$$
$$\mathcal{F}\left\{\frac{1}{10 + j2\pi t}\right\} = e^{10 f}\cdot u(-f)$$

Fourier Transform theorems.

	Time domain	Frequency domain
Superposition	$a_1 x_1(t) + a_2 x_2(t)$	$a_1 X_1(f) + a_2 X_2(f)$
Time delay	$x(t - t_0)$	$e^{-j2\pi f t_0} X(f)$
Scaling	$x(a\cdot t)$	$\frac{1}{\lvert a\rvert} X\left(\frac{f}{a}\right)$
Time reversal	$x(-t)$	$X(-f)$
Frequency translation	$x(t)\cdot e^{j2\pi f_0 t}$	$X(f - f_0)$
Modulation	$\cos 2\pi f_0 t \cdot x(t)$	$\frac{1}{2}X(f-f_0) + \frac{1}{2}X(f+f_0)$
Differentiation	$\frac{d}{dt}x(t)$	$j2\pi f X(f)$
Integration	$\int_{-\infty}^{t} x(\alpha)d\alpha$	$\frac{1}{j2\pi f}X(f) + \frac{1}{2}X(0)\delta(f)$

Fourier transform of unit pulse.

$$\mathcal{F}\left\{\Pi\left(\frac{t}{A}\right)\right\} = A\,\mathrm{sinc}(f\cdot A)$$

Fourier transform of sinc.

$$x(t) = \mathrm{sinc}(2Wt)$$

Use duality to write

$$\begin{aligned} \mathcal{F}\{A\cdot \mathrm{sinc}(t\cdot A)\} &= \Pi\left(-\frac{f}{A}\right) = \Pi\left(\frac{f}{A}\right) \\ \frac{1}{2W}\mathcal{F}\{2W\cdot \mathrm{sinc}(2Wt)\} &= \frac{1}{2W}\Pi\left(\frac{f}{2W}\right) \\ \mathcal{F}\{\mathrm{sinc}(2Wt)\} &= \frac{1}{2W}\Pi\left(\frac{f}{2W}\right) \end{aligned}$$

Fourier transform of train of deltas.

To find the Fourier transform of

$$T_s \sum_{n=-\infty}^{\infty} \delta(t - n\cdot T_s)$$

we are going to use Fourier series:

$$\begin{aligned} T_s \sum_{n=-\infty}^{\infty} \delta(t - n\cdot T_s) &= \sum_{k=-\infty}^{\infty} X_k \cdot e^{j2\pi k F_s t} \\ &= \sum_{k=-\infty}^{\infty} e^{j2\pi k F_s t} \end{aligned}$$

Thus

$$\begin{aligned} \mathcal{F}\left\{T_s \sum_{n=-\infty}^{\infty} \delta(t - n\cdot T_s)\right\} &= \mathcal{F}\left\{\sum_{k=-\infty}^{\infty} e^{j2\pi k F_s t}\right\} \\ &= \sum_{k=-\infty}^{\infty} \delta(f - kF_s) \end{aligned}$$