Mathematics 4: Differential Equations.

Teemu Weckroth, September 7, 2023

Linearity.

An n^{th}-order ordinary differential equation for $y(t)$ is called linear if it is of the form

$$a_n(t)\frac{d^n y}{dt^n} + \ldots + a_2(t)\frac{d^2 y}{dt^2} + a_1(t)\frac{dy}{dt} + a_0(t)y(t) = b(t).$$

And ODE with a term such as y^2 or $y\frac{dy}{dt}$ is called non-linear.

Homogeneity.

The ODE is called homogeneous if $b(t) = 0$ and inhomogeneous if $b(t) \neq 0$.

Autonomy.

A differential equation is called autonomous when it does not depend explicitly on the independent variable (t).
An ODE with a term such as $\sin(t)$ or $t\frac{dy}{dt}$ is called not autonomous.

Direction field.

A direction field for a single differential equation is a graph with:

- the independent variable on the horizontal axis;
- the dependent variable on the vertical axis;
- and arrows which indicate the rate of change of the dependent variable.

Equilibrium point stability.

An equilibrium point is stable if any initial value close to the equilibrium point gives solutions that always remain close to the equilibrium point. Any equilibrium point which is not stable is called unstable.

The harmonic addition theorem.

$$c_1 \cos \omega t + c_2 \sin \omega t = R \cos(\omega t - \delta)$$

$$R = \sqrt{c_1^2 + c_2^2}$$

$$\delta = \begin{cases} \arctan(c_2/c_1) & \text{if } c_1 > 0 \\ \arctan(c_2/c_1) + \pi & \text{if } c_1 < 0 \end{cases}$$

The Laplace Transform.

For a function $f : [0, \infty) \to \mathbb{R}$:

$$\mathscr{L}\{f\}(s) = F(s) = \int_0^\infty e^{-st} f(t)\, dt,$$

provided the integral exists.

Linearity.

For constants $a, b \in \mathbb{R}$:

$$\mathscr{L}\{af(t) + bg(t)\} = a\mathscr{L}\{f(t)\} + b\mathscr{L}\{g(t)\}.$$

Shift property.

For constant $a \in \mathbb{R}$:

$$\mathscr{L}\{f(t)e^{at}\} = \mathscr{L}\{f(t)\}(s-a) = F(s-a)$$

Differentiation rule.

If $\mathscr{L}\{f(t)\} = F(s)$, then $\mathscr{L}\{f'(t)\} = sF(s) - f(0)$.
Consequence:

$$\begin{aligned} \mathscr{L}\{f''(t)\} &= s(sF(s) - f(0)) - f'(0) \\ &= s^2F(s) - sf(0) - f'(0) \end{aligned}$$

The Inverse Laplace Transform.

If $f(t)$ is piecewise continuous and $\mathscr{L}\{f\}(s) = F(s)$ exists, then $f(t)$ is uniquely determined by $F(s)$ by means of the Inverse Laplace Transform

$$f(t) = \mathscr{L}^{-1}\{F(s)\}.$$

Solving linear ODE using Laplace Transforms.

Three steps:

1. Compute Laplace transform of the initial value problem.
2. Compute solutions $Y(s)$ in the s-domain.
3. Compute inverse Laplace transform to find solutions $y(t)$ in the t-domain.

Finding a partial fraction decomposition.

Five steps:

1. Factorize the denominator into linear and quadratic factors.
2. For each factor $(s-a)^k$ introduce k terms
$$\frac{A_1}{s-a}, \frac{A_2}{(s-a)^2}, \ldots, \frac{A_k}{(s-a)^k},$$
and for a factor $((s-a)^2 + b^2)^\ell$ add terms
$$\frac{B_1(s-a) + C_1}{(s-a)^2 + b^2}, \frac{B_2(s-a) + C_2}{((s-a)^2 + b^2)^2}, \ldots, \frac{B_\ell(s-a) + C_\ell}{((s-a)^2 + b^2)^\ell}.$$
3. Put all the terms together into one fraction and compare with the given rational function.
4. Comparing the numerators gives a system of n linear equations for the n unknown parameters.
5. Solve the system and write down the partial fraction decomposition.

Step function.

The function

$$u_a(t) = \begin{cases} 0, & \text{if } 0 \le t < a \\ 1, & \text{if } t \ge a \end{cases}$$

is called the Heaviside function or the unit step function.
E.g.

$$F(t) = \begin{cases} 0, & \text{if } t < 2 \text{ or } t \ge 4 \\ t-2, & \text{if } 2 \le t < 4 \end{cases}$$

$F(t)$ in one line: $F(t) = (t-2) \cdot (u_2(t) - u_4(t))$.

The impulse function.

$$\text{Limit for } \tau \to 0^+ : d_\tau(t) = \begin{cases} \frac{1}{2\tau}, & -\tau < t < \tau \\ 0, & \text{other } t \end{cases}$$

Dirac delta "function" $\delta(t-a)$:

$$\delta(t-a) = \begin{cases} 0, & \text{when } t \neq a \\ \infty, & \text{when } t = a \end{cases}$$

Properties:

$$\int_{-\infty}^{\infty} \delta(t-a)\, dt = 1$$

$$\int_{-\infty}^{\infty} f(t)\delta(t-a)\, dt = f(a)$$

Solving a system of ODE.

If $\mathbf{u}_1(t)$ and $\mathbf{u}_2(t)$ are two linearly independent solutions, then every solution can be written as $\mathbf{u}(t) = c_1\mathbf{u}_1(t) + c_2\mathbf{u}_2(t)$.
The function $\mathbf{u}(t) = \mathbf{v}e^{rt}$ is a solution of $\frac{d\mathbf{u}}{dt} = A\mathbf{u}$ precisely when $\mathbf{v}$ is an eigenvector and r is an eigenvalue of A.

$$\begin{aligned} &\mathbf{u}_1(t) = \mathbf{v}_1 e^{r_1 t},\ \mathbf{u}_2(t) = \mathbf{v}_2 e^{r_2 t} \\ &\to \mathbf{u}(t) = c_1\mathbf{v}_1 e^{r_1 t} + c_2\mathbf{v}_2 e^{r_2 t} \end{aligned}$$

For inhomogeneous system, $\mathbf{z}(t) = \mathbf{z}_c + \mathbf{z}_p$.
Assume $\mathbf{z}_p$ is of the form $\mathbf{z}_p(t) = \mathbf{c}$ (constant).
Solve $\mathbf{c}$ from $\frac{d\mathbf{c}}{dt} = A\mathbf{c} + \mathbf{b}$.

Rewrite to real form.

If $\mathbf{u}_0$ is a complex (non-trivial) solution of $\mathbf{u}' = A\mathbf{u}$, then $\mathrm{Re}(\mathbf{u}_0)$ and $\mathrm{Im}(\mathbf{u}_0)$ are two linearly independent solutions.
If $\mathbf{u}_0(t)$ is a complex solution, then $\mathbf{u}(t) = c_1 \mathrm{Re}(\mathbf{u}_0(t)) + c_2 \mathrm{Im}(\mathbf{u}_0(t))$ is a real solution.

Classification of equilibrium points.

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix}$$

A is a constant 2×2-matrix with eigenvalues r_1 and r_2 and corresponding eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$.
If r_1 and r_2 are real-valued, the solution is given by

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1\mathbf{v}_1 e^{r_1 t} + c_2\mathbf{v}_2 e^{r_2 t}.$$

$r_1 \le r_2 < 0$	stable node
$0 < r_1 \le r_2$	unstable node
$r_1 < 0 < r_2$	saddle point, unstable

If $r_1, r_2 = a \pm bi, (b \neq 0)$, the solution is given by

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \mathbf{c}_1 e^{at} \cos bt + \mathbf{c}_2 e^{at} \sin bt.$$

$a < 0$	stable spiral point
$a > 0$	unstable spiral point
$a = 0$	stable centre

Phase portrait procedure

Four steps:

1. Calculate equilibrium points.
2. Linearise ODE around equilibrium points.
3. Characterise solutions near equilibrium points.
4. Sketch a phase portrait.

Linearisation around equilibrium.

Deviation from equilibrium point (x_0, y_0): $x = x_0 + u, y = y_0 + v$.

$$\begin{cases} x' = F(x,y) \\ y' = G(x,y) \end{cases} \quad \to \quad \begin{cases} u' = F(x_0+u, y_0+v) \\ v' = G(x_0+u, y_0+v) \end{cases}$$

Linearisation:

$$\begin{cases} u' \approx \frac{\partial F}{\partial x}(x_0,y_0)\,u + \frac{\partial F}{\partial y}(x_0,y_0)\,v \\ v' \approx \frac{\partial G}{\partial x}(x_0,y_0)\,u + \frac{\partial G}{\partial y}(x_0,y_0)\,v \end{cases}$$

Linearisation.

Consider $\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} F(x,y) \\ G(x,y) \end{pmatrix}$ and equilibrium point (x_0, y_0).
The linearisation of an autonomous system at (x_0, y_0) is given by

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \frac{\partial F}{\partial x}(x_0,y_0) & \frac{\partial F}{\partial y}(x_0,y_0) \\ \frac{\partial G}{\partial x}(x_0,y_0) & \frac{\partial G}{\partial y}(x_0,y_0) \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$$
$$= J(x_0,y_0)\begin{pmatrix} u \\ v \end{pmatrix},$$

where $\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} x(t) - x_0 \\ y(t) - y_0 \end{pmatrix}$.

Type change after linearisation.

Linearised	Non-linear
Unstable spiral point	Unstable spiral point
Stable spiral point	Stable spiral point
Stable node ($r_1 < r_2 < 0$)	Stable node
($r_1 = r_2 < 0$)	Stable spiral point or stable node
Unstable node ($0 < r_1 < r_2$)	Unstable node
($0 < r_1 = r_2$)	Unstable spiral point or unstable node
Saddle point (unstable)	Saddle point (unstable)
Center point (stable)	(Un)stable spiral or center point (stable)

Van der Pol equation.

$$u'' - \mu(1-u^2)u' + u = 0$$

Lorenz equations.

$$\frac{dX}{dt} = \sigma(-X+Y)$$
$$\frac{dY}{dt} = rX - Y - XZ$$
$$\frac{dZ}{dt} = -bZ + XY$$

Separation of variables.

$$\text{PDE: } \frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

Four steps:

1. Split $u(x,t)$: $u(x,t) = X(x)\cdot T(t)$.
2. Substitute $X(x)T(t)$ in PDE: $\frac{\partial X(x)T(t)}{\partial t} = k\frac{\partial^2 X(x)T(t)}{\partial x^2}$
$\to X(x)\frac{dT(t)}{dt} = k\frac{d^2X(x)}{dx^2}T(t) \to XT' = kX''T$.
3. Divide both side by kXT: $\frac{T'}{kT} = \frac{X''}{X} = \lambda$
4. Rewrite as a set of ODE: $\begin{cases} \frac{X''}{X} = \lambda \\ \frac{T'}{kT} = \lambda \end{cases} \to \begin{cases} X'' - \lambda X = 0 \\ T' - k\lambda T = 0 \end{cases}$

Hyperbolic cosine/sine.

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \sinh x = \frac{e^x - e^{-x}}{2}$$

Non-trivial solutions of a boundary value problem.

$$y'' + \lambda y = 0,\ y(0) = 0,\ y(\pi) = 0,\ \text{with } \lambda \text{ a real number.}$$

Find all numbers λ for which non-trivial solutions $y(x)$ exist.
Consider three cases: $\lambda < 0$, $\lambda = 0$, $\lambda > 0$.

$$\begin{aligned} \lambda < 0 &: \text{suppose } \lambda = -\mu^2 (\mu > 0),\ \text{so } y'' - \mu^2 y = 0. \\ &\quad \text{Only trivial solution.} \\ \lambda = 0 &: y'' = 0. \\ &\quad \text{Only trivial solution.} \\ \lambda > 0 &: \text{suppose } \lambda = \mu^2 (\mu > 0),\ \text{so } y'' + \mu^2 y = 0. \\ &\quad y(x) = c_2 \sin(kx),\ \text{with } \lambda = \mu^2 = k^2 = 1, 4, 9, 16, \ldots. \end{aligned}$$

A periodic function.

$f(x)$ is periodic if there exists a T such that $f(x+T) = f(x)$ for all x.
T is called a period of the function $f(x)$.
Convention: it is sufficient to consider the function on the interval $[-L, L]$, where $T = 2L$.

Orthogonal functions.

For integers m and n:

$$\int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)\,dx = \begin{cases} 0, & m \neq n \\ L, & m = n \end{cases}$$

$$\int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)\,dx = 0,\ \text{for each } m, n$$

$$\int_{-L}^{L} \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)\,dx = \begin{cases} 0, & m \neq n \\ L, & m = n \end{cases}$$

Fourier series.

A periodic function $f(x)$ with period $T = 2L$ can be expressed as a Fourier series:

$$f(x) = c_0 \sum_{m=1}^{\infty}\left(a_m \cos\left(\frac{m\pi x}{L}\right) + b_m \sin\left(\frac{m\pi x}{L}\right)\right).$$

Calculating coefficients $c_0, a_m,$ and b_m.

$$\text{Find } c_0\text{: } \int_{-L}^{L} f(x)\,dx = \int_{-L}^{L} c_0\,dx = 2c_0 L$$

$$\text{Find } a_m\text{: } \int_{-L}^{L} f(x)\cos\left(\frac{m\pi x}{L}\right)\,dx = a_m \int_{-L}^{L} \cos^2\left(\frac{m\pi x}{L}\right)\,dx = a_m L$$

$$\text{Find } b_m\text{: } \int_{-L}^{L} f(x)\sin\left(\frac{m\pi x}{L}\right)\,dx = b_m \int_{-L}^{L} \sin^2\left(\frac{m\pi x}{L}\right)\,dx = b_m L$$

Even and odd function identities.

Even function: $f(x) = f(-x)$ Odd function: $f(x) = -f(-x)$

odd + odd = odd; even + even = even; even + odd = neither
odd · odd = even; even · even = even; even · odd = odd

$$\int_{-L}^{L} \text{odd}\,dx = 0 \qquad \int_{-L}^{L} \text{even}\,dx = 2\int_{0}^{L} \text{even}\,dx$$

Fourier sine and cosine series.

Let $f(x)$ be a function with $T = 2L$.

$$\text{Cosine series: } f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right),\ \text{where}$$
$$a_n = \frac{2}{L}\int_0^L f_{\text{even}}(x)\cos\left(\frac{n\pi x}{L}\right)\,dx.$$
$$\text{Sine series: } f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right),\ \text{where}$$
$$b_n = \frac{2}{L}\int_0^L f_{\text{odd}}(x)\sin\left(\frac{n\pi x}{L}\right)\,dx.$$

$$f_{\text{even}}(x) = \frac{1}{2}(f(x) + f(-x)) \qquad f_{\text{odd}}(x) = \frac{1}{2}(f(x) - f(-x))$$

Solving a partial differential equation.

Five steps:

1. Separation of variables.
2. Solve ODE with two homogeneous conditions.
3. Solve other ODE.
4. Formulate general solution.
5. Apply last condition.

Various Differential Equations.

Heat diffusion equation.

$$\alpha^2 \frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial u(x,t)}{\partial t}$$

Wave equation.

$$a^2 \frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial^2 u(x,t)}{\partial t^2}$$

Laplace equation.

$$\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = 0$$

Filling a cistern.

When the floater moves up, the tap closes.
Tap is completely closed when $h = H$.

t	Time	[s]
$V(t)$	Water volume in cistern	[dm^3]
A	Area of the cistern	[dm^2]

Consider time interval $[t, t+\Delta t]$.
Change in V:

$$\Delta V = V(t+\Delta t) - V(t).$$

For ΔV the following equation holds:

$$\Delta V = c(H-h)\Delta t.$$

We know that

$$\Delta V = A\Delta h,$$

where $\Delta h = h(t+\Delta t) - h(t)$.
Balance equation:

$$A\Delta h = c(H-h)\Delta t.$$

From the equation $A\Delta h = c(H-h)\Delta t$, we can derive the ODE:

$$\begin{aligned} \frac{\Delta h}{\Delta t} &= \frac{c}{A}(H-h) \\ \lim_{\Delta t\to 0}\frac{\Delta h}{\Delta t} &= \lim_{\Delta t\to 0}\frac{c}{A}(H-h) \\ \frac{dh}{dt} &= \frac{c}{A}(H-h) \\ h(0) &= 0 \end{aligned}$$

Periodically forced oscillator.

$$\begin{aligned} m\frac{d^2u}{dt^2} &= F_{\text{ext}} + F_{\text{friction}} + F_{\text{spring}} \\ &= F_0\cos\omega t - \gamma\frac{du}{dt} - ku \end{aligned}$$

u	Displacement of the oscillator
t	Time
m	Mass of the oscillator
γ	friction coefficient (> 0)
k	Spring coefficient (> 0)
F_0	Amplitude driving force
ω	Angular frequency of driving force

Solution without friction.

$$mu'' + ku = F_0\cos\omega t$$

$$\begin{aligned} u(t) &= u_c(t) + u_p(t) \\ &= c_1\cos\omega_0 t + c_2\sin\omega_0 t + \left(\frac{\frac{F_0}{k}}{1-\frac{\omega^2}{\omega_0^2}}\right)\cos\omega t \end{aligned}$$

Natural frequency $\omega_0 = \sqrt{\frac{k}{m}}$.
Solutions with friction.

$$mu'' + \gamma u' + ku = F_0\cos\omega t$$

$$\begin{aligned} u(t) &= u_c(t) + u_p(t) \\ &= u_c(t) + R\cos(\omega t - \delta) \end{aligned}$$

Amount of friction $\Gamma = \frac{\gamma^2}{mk}$.
Natural frequency $\Omega_0 = \sqrt{\frac{k}{m} - \left(\frac{\gamma}{2m}\right)^2} \approx \sqrt{\frac{k}{m}} = \omega_0$, where $0 < \gamma < 2m\Omega_0$.

System of ODE: mass-spring system.

$$my'' + ky' + cy = 0$$

Take $v = y'$.

$$mv' + kv + cy = 0$$

Define the derivatives.

$$\begin{cases} y' = v \\ v' = y'' = -\frac{c}{m}y - \frac{k}{m}v \end{cases}$$

$$\frac{d}{dt}\begin{pmatrix} y \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{c}{m} & -\frac{k}{m} \end{pmatrix}\begin{pmatrix} y \\ v \end{pmatrix}$$