Mathematics and Computers in Simulation 53 (2000) 95–103

MATHEMATICS AND COMPUTERS IN SIMULATION

www.elsevier.nl/locate/matcom

Analytical approximations for real values of the Lambert *W*-function

D.A. Barry[a,*], J.-Y. Parlange[b], L. Li[a], H. Prommer[a], C.J. Cunningham[a], F. Stagnitti[c]

[a] *Contaminated Land Assessment and Remediation Research Centre, The University of Edinburgh, Edinburgh EH9 3JN, UK*

[b] *Department of Agricultural and Biological Engineering, Cornell University, Ithaca, New York 14853-5701, USA*

[c] *School of Ecology and Environment, Deakin University, P.O. Box 423, Warrnambool, Vic. 3280, Australia*

Received 1 April 2000; accepted 5 May 2000

Abstract

The Lambert W is a transcendental function defined by solutions of the equation $W\exp(W) = x$. For real values of the argument, x, the W-function has two branches, W_0 (the principal branch) and W_{-1} (the negative branch). A survey of the literature reveals that, in the case of the principal branch (W_0), the vast majority of W-function applications use, at any given time, only a portion of the branch viz. the parts defined by the ranges $-1 \le W_0 \le 0$ and $0 \le W_0$. Approximations are presented for each portion of W_0, and for W_{-1}. It is shown that the present approximations are very accurate with relative errors down to around 0.02% or smaller. The approximations can be used directly, or as starting values for iterative improvement schemes. © 2000 IMACS. Published by Elsevier Science B.V. All rights reserved.

Keywords: Analytical approximations; Algorithms; Iteration scheme

1. Introduction

The history of the Lambert W-function goes back to the late 18th century with publications by Lambert and Euler [13]. The history of the function and its mathematical characteristics have been studied extensively [13]. The W-function arises in many mathematical and physical applications. Recently, global approximations to the principal real-valued branch of the W-function were constructed [8]. Two relatively simple approximations were presented, one of which had a maximum relative error of about 15% and one with a maximum relative error of about 5%.

Several numerical approximations to the W-function were provided previously [3,4,16,33]. Furthermore, numerical approximations are contained in the computer algebra packages, *Maple* [10] and

* Corresponding author. Tel.: +44-131-650-7204; fax: +44-131-650-7276.
E-mail address: a.barry@ed.ac.uk (D.A. Barry).

PII: S0378-4754(00)00172-5

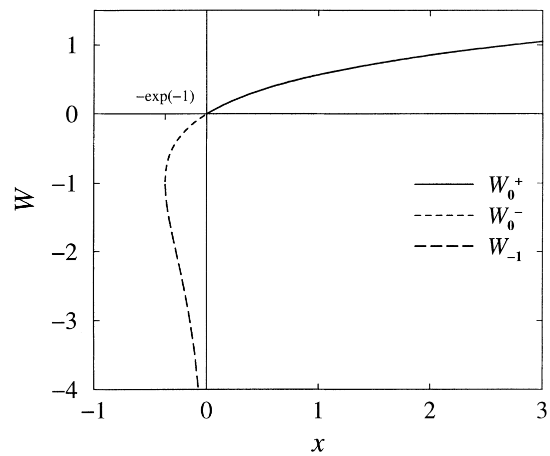

Fig. 1. Branches of the W-function, showing the division into W_{-1}, W_0^- and W_0^+.

Mathematica [41]. These various approximations incorporate iterative schemes to compute W with predefined precision, and use piece-wise approximations to generate initial guesses [8]. There are, however, situations where analytical approximations are useful, for example in the analysis of physical systems where the W-function arises in solutions. Regardless, simple analytical approximations are useful for quickly generating results with known accuracy. Also, they can assist in understanding the physical behaviour or be used to analyse experimental data.

The Lambert W-function is defined by

$$W \exp(W) = x. \tag{1}$$

For real values of W, the range of x is limited to $x \geq -\exp(-1)$. The behaviour of the W-function is depicted in Fig. 1, where its branches are identified. Note that the W_0 and W_{-1} names follow established usage [13]. The range of the lower branch is $-1 \geq W_{-1}$, while the upper branch W_0 is divided into $-1 \leq W_0^- \leq 0$ and $0 \leq W_0^+$.

Over the past several years, the W-function has been used with increasing frequency. In Table 1, a summary of recent applications is presented, along with the branches used in those applications. In the vast majority of cases where the principal branch is used, only sections of the branch are needed in any given instance. Furthermore, W_0 separates naturally into these sections. The widespread use of W suggests that simple yet accurate approximations for these sections would be of practical use. Furthermore, because end points are often important in applications, we pay special attention to constructing approximations for which the relative error vanishes at these locations.

In this paper, we provide easily computable approximations for both real-valued branches of the Lambert W-function, or portions of W-function in the case of W_0. These approximations have maximum relative errors down to around 0.02% or better, depending on the branch considered. They can be used directly, or as starting values to initialise iteration schemes, such as Newton's method (except around $x = -\exp(-1)$, where the gradient is very large), in order to obtain improved estimates of W, should that be necessary.

Table 1
Applications of the real-valued W-function including the branch used

Problem description	Branch of the W-function used	Reference
Water movement in soil	W_{-1} or W_0^- or W_0^+	[5,6]
Enzyme–substrate reactions	W_0^+ or W_0^-	[22,36]
Time of a parachute jump	W_0^+	[29]
Iterated exponentiation	$W_0(x)$, $-\exp(-1) \le x \le \exp(1)$	[13,23]
Jet fuel consumption	W_0^- or W_{-1}	[1,13]
Combustion	W_0^+	[13,30]
Forces in hydrogen ions	W_0^+ or W_0^-	[34,35]
Population growth	W_{-1} and W_0^-	[13]
Roots of trinomials	W_0^+	[21]
Disease spreading	W_0^-	[13]
Recurrences in algorithm analysis	W_0^-	[13,25]
Binary search tree height	W_0^-	[13,15,32]
Hashing with uniform probing	W_0^+	[20]
Hashing methods	W_{-1}	[27]
Optimal wire shapes	W_0^-	[17]
SU(N) gauge theory	W_0^+	[2]
QCD renormalisation	W_0^+ or W_0^- or W_{-1}	[18,19,37]
Star collapse	W_0^+	[14]
Two-body motion	W_0^- and W_{-1}	[28]
Structure learning	W_0^+	[7]
Reaction–diffusion modelling	W_{-1}	[9]
Sample partitioning	W_0^+	[12]
Entropy-constrained scalar quantization	W_0^-	[39]
Redox barrier design	W_0^-	[11]
Photochemical bleaching	W_0^+	[40]
Thin film life time	W_{-1}	[38]
Testing Legendre transform algorithm	W_0^+	[26]
Exponential function approximation	W_{-1} and W_0^-	[33]
Herbivore–plant coexistence	W_0^+	[24]
Photorefractive two-wave mixing	W_0^+	[31]

2. Existing approximations

As already noted above, our interest is in simple analytical approximations. In that case, the piece-wise approximations [4,10,16,41] referred to in Section 1 are, not surprisingly, unsuitable, since they were developed for use by iterative improvement schemes. On the other hand, Boyd [8] has developed several analytical approximations to $W_0(x)$ for all $x \ge -\exp(-1)$. Probably, the most useful of his approximations is

$$W_0^{\mathrm{Boyd}} = W_0^{\mathrm{B}} \left\{ 1 + \frac{[\ln(y) - (7/5)] \exp[-(3/40)(\ln(y) - (7/5))^2]}{10} \right\}, \tag{2}$$

where

$$y = 1 + \exp(1)x \tag{3}$$

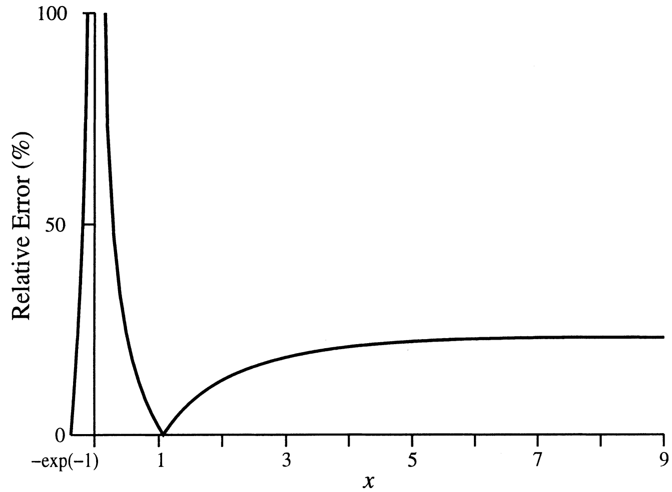

Fig. 2. Relative error of Boyd's approximation, W_0^{Boyd}.

and

$$W_0^{\mathrm{B}} = \tanh\left\{\frac{\sqrt{2y}}{\ln(10) - \ln[\ln(10)]}\right\}\{\ln(y+10) - \ln[\ln(y+10)]\}. \quad (4)$$

In Fig. 2, we plot the relative error, i.e., $|100(1-(W_0^{\mathrm{Boyd}}/W_0))|$%, of this approximation as it varies with $x \geq -\exp(-1)$. Clearly, the relative error of (2) is divergent at $x = 0$. Note that [8] presents results as the 'shifted' relative error, i.e., the plots $(1+W_0^{\mathrm{Boyd}})/(1+W_0)$ versus y. Because of this method of plotting the relative error, the divergence at the origin displayed in Fig. 2 is not apparent in [8]. Indeed, this method of plotting the relative error is not suitable for showing the relative error when the magnitude of W is less than 1. Furthermore, since all Boyd's approximations contain W_0^{B}, they all diverge, in terms of the relative error at least, at $x = 0$. Below, we present approximations for which the relative error vanishes at the origin.

3. Analytic approximations

In this section, we present approximations for W_{-1} and W_0. The latter branch of W can be broken into the two portions W_0^- and W_0^+ since most applications are confined to either of these sections (Table 1). However, in order to account for possible uses of the entire range of W_0, and because end points are often especially important in applications, the approximations below are constrained to give the correct values for W and $\mathrm{d}W/\mathrm{d}x$ at the end points. In particular, the relative error vanishes at $x = 0$.

3.1. Approximation for W_{-1}

A simple approximation for W_{-1} with a maximum relative error of only 0.025% is already available [6]:

$$W_{-1} = -1 - \sigma - \frac{2}{M_1}\left[1 - \frac{1}{1 + ((M_1\sqrt{\sigma/2})/(1 + M_2\sigma\exp(M_3\sqrt{\sigma})))}\right], \tag{5}$$

where

$$\sigma = -1 - \ln(-x), \tag{6}$$

with $M_1 = 0.3361$, $M_2 = -0.0042$ and $M_3 = -0.0201$.

We looked at several modifications to this approximation but obtained only minor improvements at the cost of considerable additional complexity. If further accuracy is required, then existing piece-wise approximations [4] are likely to be more useful.

3.2. Approximation for W_0^-

The relative error of the approximation of [8] diverges, as shown in Fig. 2. However, this portion of W_0 is well approximated by manipulating its series expansion about $x = -\exp(-1)$. The series is in terms of $\eta = 2 + 2\exp(1)x$. This approach was exploited previously [6], and is modified slightly here. The previous approximation has a maximum relative error of less than 0.1%. However, the relative error at $x = 0$ is non-zero, although the correct value of $W_0(0) = 0$ is obtained there. Here, we remove the relative error at this point in an improved approximation.

The first few terms in the expansion about $-\exp(-1)$ are expressed in continued fraction form as

$$W_0^- = -1 + \frac{\sqrt{\eta}}{1 + ((N_1\sqrt{\eta})/(N_2 + \sqrt{\eta}))}, \tag{7}$$

where

$$N_1 = \left(1 - \frac{1}{\sqrt{2}}\right)(N_2 + \sqrt{2}). \tag{8}$$

This definition of N_1 was taken so that the approximation is exact at $x = -\exp(-1)$ and 0. Setting

$$N_2 = 3\sqrt{2} + 6 - \frac{[(2237 + 1457\sqrt{2})\exp(1) - 4108\sqrt{2} - 5764]\eta}{(215 + 199\sqrt{2})\exp(1) - 430\sqrt{2} - 796} \tag{9}$$

ensures that the relative error of (7) vanishes at these limits. The maximum relative error of this approximation is 0.013%. More complicated expressions for N_2 can be used to reduce further the maximum relative error. These expressions can be developed by continuing the expansion in η, and then manipulating the continued fraction version of the expansion.

3.3. Approximation for W_0^+

At present, with the exception of Boyd's approximation, (2), there is no single approximation to W_0^+ that can reasonably be used for all $x \geq 0$. The relative error of (2), as we have seen in Fig. 2, diverges at the origin and, for $x > 1$, has a maximum relative error of around 23%. Below, we present a sequence of approximations for which the relative error disappears at the origin, and improve on Boyd's approximations elsewhere.

Table 2
Ratios of coefficients for approximations based on (10)

Ratio	Value
$\frac{a^0}{a^1}$	1
$\frac{a^1}{a^2}$	2
$\frac{a^2}{a^3}$	$\frac{6}{5}$
$\frac{a^3}{a^4}$	$\frac{50}{47}$
$\frac{a^4}{a^5}$	$\frac{13,254}{12,917}$
$\frac{a^5}{a^6}$	$\frac{333,697,778}{329,458,703}$

As noted by [4], a formal solution to (1) is $W = \ln(x/\ln(x/\cdots))$. We modify this form to force it to be exact at the origin. That is, we consider a sequence of approximations of the form

$$W_0^{+^{n+1}} = \ln\left(\frac{a^{n+1}x}{W_0^{+^n}}\right), \qquad n = 0, 1, \ldots, \tag{10}$$

where n is a counter and not a power. The sequence is initialised with

$$W_0^{+^0} = \ln(1 + a^0 x). \tag{11}$$

The values of the coefficients in (10) are chosen such that the relative error of the approximation disappears at the origin. In particular, for any given $n > 0$, the set $A^n = \{a^i, i = 0, 1, \ldots, n\}$ is unique, and is determined such that (10) produces the first term in the Taylor series of W_0 at $x = 0$. Remarkably, for any given n, the elements of the appropriate set A^n are such that the ratios of successive pairs of a's is fixed. Put another way, although each set A^n consists of a unique series of coefficients a^i, $i = 0, 1, \ldots, n$, the ratios of successive pairs of coefficients are identical for all sets (up to the appropriate n of course). The first few of these ratios is shown in Table 2. The table can easily be extended if necessary.

Given the ratios in Table 2, the set A^n is determined once one of the values in the set is known. Another attractive feature of (10) is that the last value, a^n, of the set A^n is obtained from Table 2 with the coefficient a^{n+1} set equal to 1. This behaviour is related to the preservation of ratios of coefficients in any set A^n. As an example, consider the calculation of $W_0^{+^2}$. From Table 2, $a^2 = 6/5$ (since we take $a^3 = 1$ to calculate $W_0^{+^2}$). Then, again from Table 2, $a^1 = 2a^2 = 12/5$, and so $a^0 = a^1 = 12/5$. Consequently, the expression for $W_0^{+^2}$ is

$$W_0^{+^2} = \ln\left\{\frac{6}{5}\frac{x}{\ln[(12/5)(x/\ln(1 + (12x/5)))]}\right\}. \tag{12}$$

The maximum relative error of the first few approximations calculated from (10) is given in Table 3. The results are encouraging for such a simple approximation, although not as good as we would like. As mentioned above, it is straightforward to continue the iteration in (10); however, the ratios in Table 2

Table 3
Maximum relative errors computed for (10)

$W_0^{+^n}$	Relative error (%)
$W_0^{+^0}$	38.1
$W_0^{+^1}$	7.85
$W_0^{+^2}$	2.39
$W_0^{+^3}$	0.914
$W_0^{+^4}$	0.426
$W_0^{+^5}$	0.230

Table 4
Maximum relative errors computed for (13)

$W_0^{+*^n}$	Relative error (%)
$W_0^{+*^1}$	1.40
$W_0^{+*^2}$	0.196
$W_0^{+*^3}$	0.0844
$W_0^{+*^4}$	0.0371
$W_0^{+*^5}$	0.0157

quickly become unwieldy. We therefore looked at various simple interpolation schemes between the $W_0^{+^n}$'s with a view to producing an improved approximation, W_0^{+*}. Linear interpolation

$$(1+\epsilon)W_0^{+^{n+1}} - \epsilon W_0^{+^n} = W_0^{+*^{n+1}}, \tag{13}$$

was found to produce a significant improvement. The interpolation parameter, ϵ, is determined by forcing (13) to be exact at $x = \exp(1)$ (i.e., where $W = 1$), since this is roughly the location of the maximum relative error for (10). As an example, we consider the case of $n = 1$. For this case, ϵ is given by

$$\epsilon = \frac{\ln\{(6\exp(1))/(5\ln[(12/5)\exp(1)/\ln(1+(12/5)\exp(1))])\}-1}{\ln\{(2\exp(1))/\ln[1+2\exp(1)]\}-\ln\{(6\exp(1))/(5\ln[(12/5)\exp(1)/\ln(1+(12/5)\exp(1))])\}} \cong 0.4586887, \tag{14}$$

while (13) becomes

$$W_0^{+*^2} = (1+\epsilon)\ln\left\{\frac{6x}{5\ln[(12/5)(x/\ln(1+(12/5)x))]}\right\} - \epsilon\ln\left[\frac{2x}{\ln(1+2x)}\right]. \tag{15}$$

For (15), the maximum relative error is 0.196% as shown in Table 4. The results in this table confirm that the combination of (10) and (13) constitutes a simple systematic scheme to estimate W_0^{+*}.

References

[1] J.D. Anderson, Introduction to Flight, 3rd Edition, McGraw-Hill, New York, 1989.
[2] T. Appelquist, A. Ratnaweera, J. Terning, L.C.R. Wijewardhana, Phase structure of an SU(N) gauge theory with N_f flavors, Phys. Rev. D 58 (10) (1998) 105017(1)–105017(11).

[3] D.A. Barry, S.J. Barry, P.J. Culligan-Hensley, Algorithm 743: WAPR: a FORTRAN routine for calculating real values of the W-function, Assoc. Comput. Machinery Trans. Math. Software 21 (2) (1995) 172–181.
[4] D.A. Barry, P.J. Culligan-Hensley, S.J. Barry, Real values of the W-function, Assoc. Comput. Machinery Trans. Math. Software 21 (2) (1995) 161–171.
[5] D.A. Barry, J.-Y. Parlange, Infiltration subject to time-dependent surface ponding: exact results and correspondence to solute transport with non-linear reaction, in: V.P. Singh, B. Kumar (Eds.), Subsurface-Water Hydrology, Vol. 2, Vol. 16 of Water Science and Technology Library, Proceedings of the International Conference on Hydrology and Water Resources, New Delhi, India, December 1993, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1996, pp. 33–48.
[6] D.A. Barry, J.-Y. Parlange, G.C. Sander, M. Sivaplan, A class of exact solutions for Richards' equation, J. Hydrol. 142 (1–4) (1993) 29–46.
[7] M. Beccaria, G. Soliani, Mathematical properties of models of the reaction–diffusion type, Physica A 260 (3/4) (1998) 301–337.
[8] J.P. Body, Global approximations to the principal real-valued branch of the Lambert W-function, Appl. Math. Lett. 11 (6) (1998) 27–31.
[9] M. Brand, Structure learning in conditional probability models via an entropic prior and parameter estimation, Neural Comput. 11 (5) (1999) 1155–1182.
[10] B.W. Char, K.O. Geddes, G.H. Gonnet, B.L. Leong, M.B. Monagan, S.M. Watt, Maple V Library Reference Manual, Springer, New York, 1991.
[11] A. Chilakapati, Optimal design of a subsurface redox barrier, AIChE J. 45 (6) (1999) 1342–1350.
[12] A. Cohen, M. Kam, R. Conn, Partitioning a sample using binary-type questions with ternary feedback, IEEE Trans. Systems, Man, and Cybernetics 25 (10) (1995) 1405–1408.
[13] R.M. Corless, G.H. Gonnet, D.E.G. Hare, D.J. Jeffrey, D.E. Knuth, On the Lambert W-function, Adv. Comput. Math. 5 (4) (1996) 329–359.
[14] A. Das, N. Tariq, D. Aruliah, Spherically symmetric collapse of an anisotropic fluid body into an exotic black hole, J. Math. Phys. 38 (8) (1997) 4202–4227.
[15] L. Devroye, A note on the height of binary search trees, J. Assoc. Comput. Machinery 33 (3) (1986) 489–498.
[16] F.N. Fritsch, R.E. Shafer, W.P. Crowley, Solution of the transcendental equation $we^w = x$, Commun. Assoc. Comput. Machinery 16 (2) (1973) 123–124.
[17] Y. Gao, D.F. Wong, Optimal shape function for a bi-directional wire under Elmore delay model, IEEE Trans. Computer-Aided Design of Integrated Circuits and Systems 18 (7) (1999) 994–999.
[18] E. Gardi, M. Karliner, Relations between observables and the infrared fixed point in QCD, Nucl. Phys. B 529 (1/2) (1998) 383–423.
[19] E. Gardi, M. Karliner, G. Grunberg, Can the QCD running coupling have a causal analyticity structure? J. High Energy Phys. 7 (1998) 136–159.
[20] G.H. Gonnet, Expected length of the longest probe sequence in hash code searching, J. Assoc. Comput. Machinery 28 (2) (1981) 298–304.
[21] G.H. Gonnet, Handbook of Algorithms and Data Structures, International Computer Science Series, Addison-Wesley, London, 1984.
[22] C.T. Goudar, J.R. Sonnod, R.G. Duggleby, Parameter estimation using a direct solution of the integrated Michaelis–Menten equation, Biochem. Biophys. Acta 1429 (2) (1999) 377–383.
[23] D. Gruntz, Automatic differentiation and bisection, MapleTech 4 (1) (1997) 14–19.
[24] P.A. Hambäck, Seasonality, optimal foraging, and prey coexistence, Am. Naturalist 152 (6) (1998) 881–895.
[25] D.E. Knuth, B. Pittel, A recurrence related to trees, Proc. Am. Math. Soc. 105 (2) (1989) 335–349.
[26] Y. Lucet, Faster than the fast Legendre transform, the linear-time Legendre transform, Numer. Algorithms 16 (2) (1997) 171–185.
[27] B.S. Majewski, N.C. Wormald, G. Havas, Z.J. Czech, A family of perfect hashing methods, Comput. J. 39 (6) (1996) 547–554.
[28] R.B. Mann, T. Ohta, Exact solution for the metric and the motion of two bodies in $(1+1)$-dimensional gravity, Phys. Rev. D 55 (8) (1997) 4723–4747.
[29] D.B. Meade, Maple and the parachute problem: modelling with an impact, MapleTech 4 (1) (1997) 67–86.
[30] R.E. O'Malley, Singular Perturbation Methods for Ordinary Differential Equations, Vol. 89 of Applied Mathematical Sciences, Springer, New York, 1991.

[31] H. Rehn, Photorefractive two-wave mixing and the Lambert *W* function, J. Modern Optics 45 (10) (1998) 2197–2199.
[32] J.M. Robson, The height of binary search trees, Aust. Comput. J. 11 (4) (1979) 151–153.
[33] N.N. Schraudolph, A fast, compact approximation of the exponential function, Neural Comput. 11 (4) (1999) 853–862.
[34] T.C. Scott, J.F. Babb, A. Dalgarno, J.D. Morgan, The calculation of exchange forces: general results and specific models, J. Chem. Phys. 99 (4) (1993) 2841–2854.
[35] T.C. Scott, J.F. Babb, A. Dalgarno, J.D. Morgan, Resolution of a paradox in the calculation of exchange forces for ${H_2}^+$, Chem. Phys. Lett. 203 (2/3) (1993) 175–183.
[36] L.A. Segel, M. Slemrod, The quasi-steady-state assumption: a case study in perturbation, SIAM Rev. 31 (3) (1989) 446–477.
[37] D.V. Shirkov, Renormalization group, causality, and nonpower perturbation expansion in QFT, Theoret. Math. Phys. 119 (1) (1999) 438–447.
[38] G.A. Shugai, P.A. Yakubenko, Spatio-temporal instability in free ultra-thin films, Eur. J. Mechanics — B/Fluids 17 (3) (1998) 371–384.
[39] G.J. Sullivan, Efficient scalar quantization of exponential and Laplacian random variables, IEEE Trans. Inform. Theory 42 (5) (1996) 1365–1374.
[40] M.C.J.M. Vissenberg, M.J.M de Jong, Photobleaching and spectral diffusion in disordered media, Phys. Rev. B 56 (11) (1997) 6681–6688.
[41] S. Wolfram, The Mathematica Book, 4th Edition, Cambridge University Press, Cambridge, 1999.