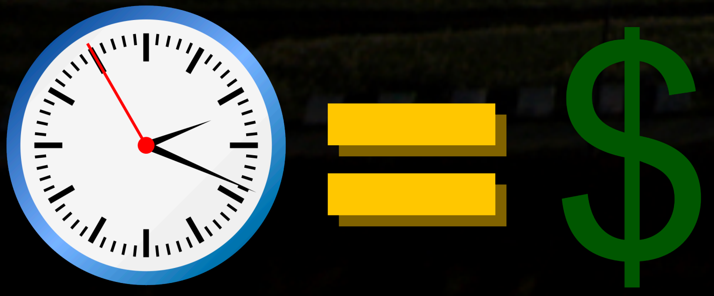
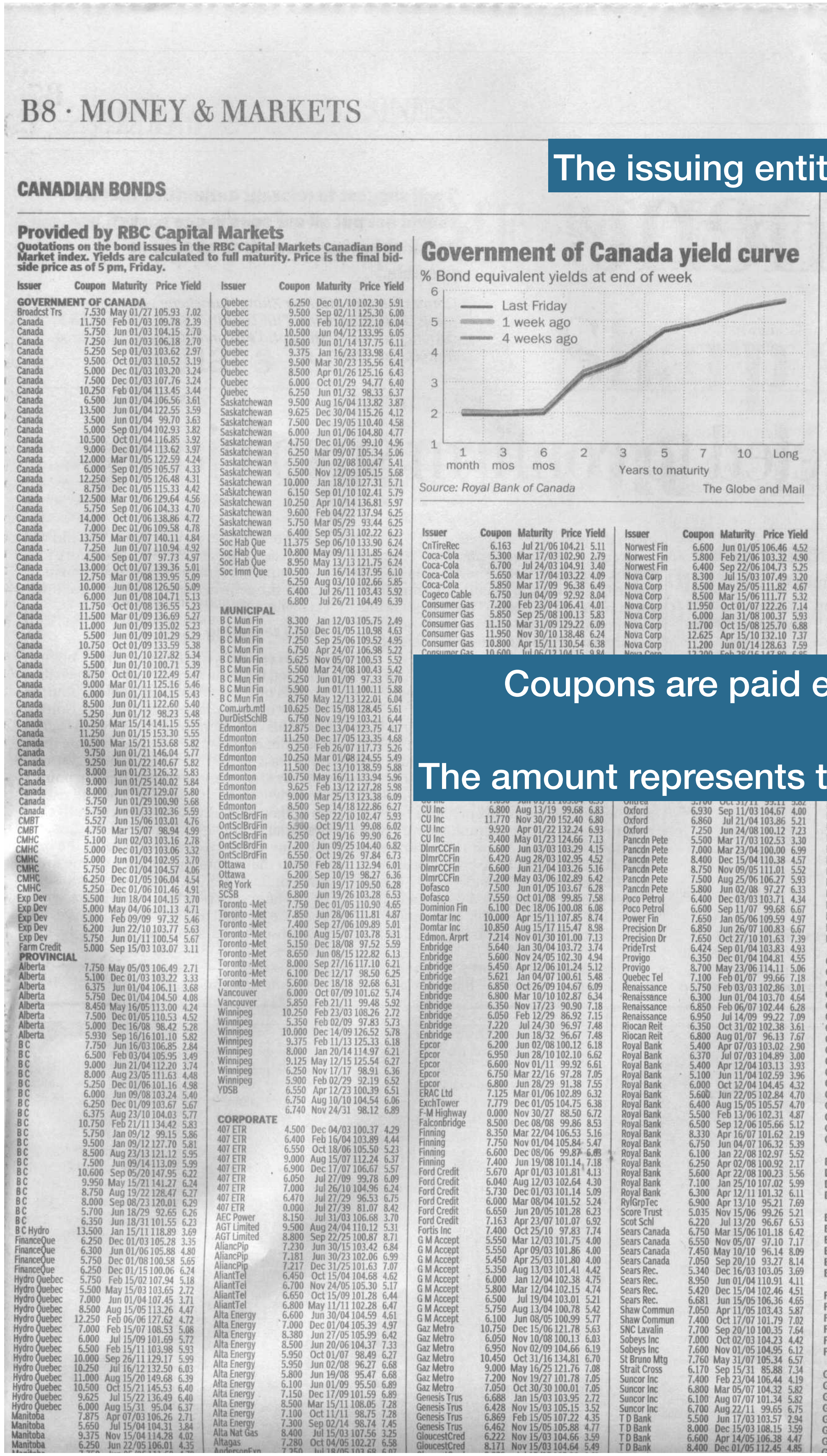
Lecture 3: Time is money



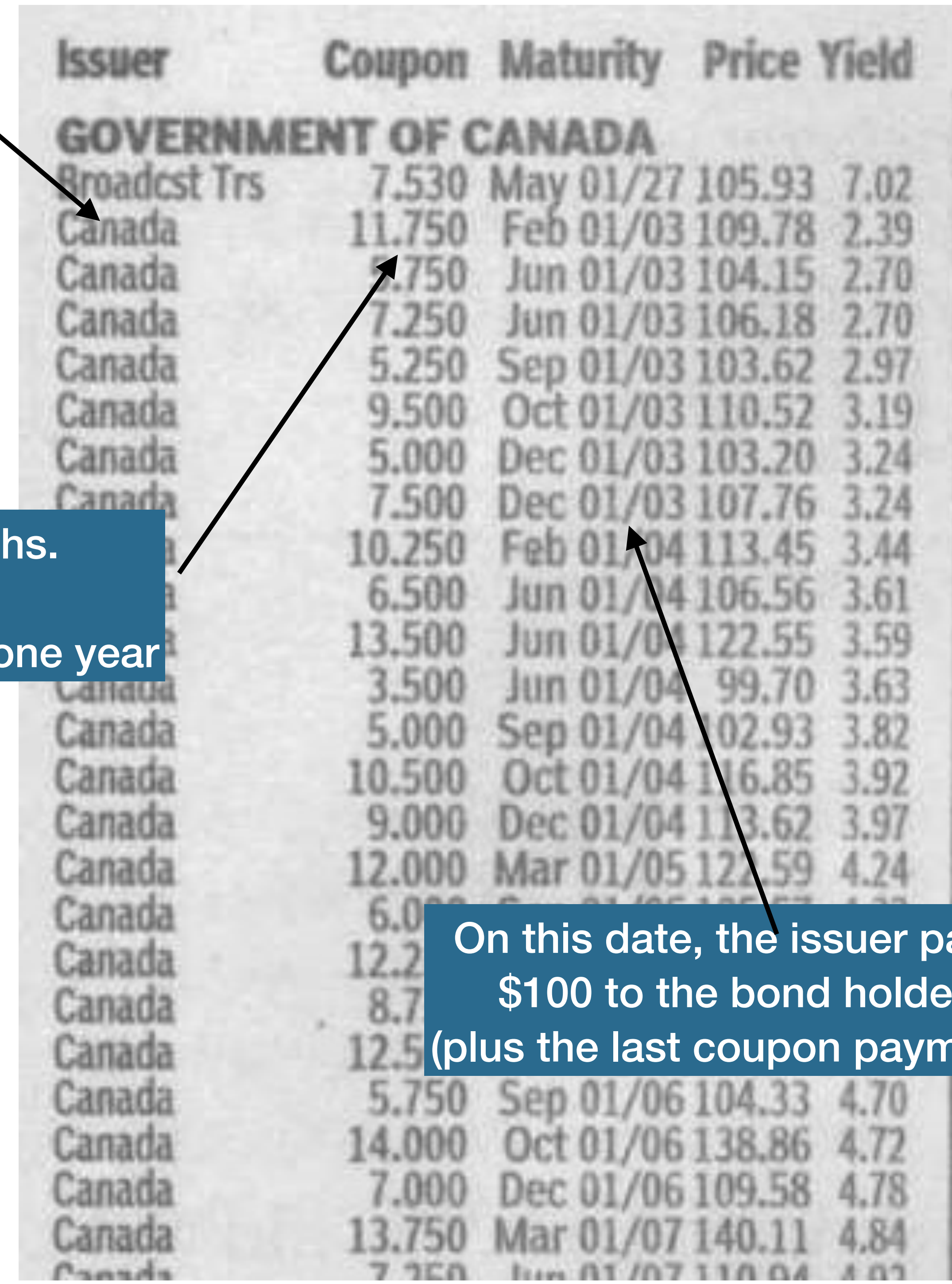
Financial Products

Everybody needs money

Product	Good	Bad
Stock	Lots of upside	Lots of downside
Bond	Safety	No upside
Convertible Bond	Lots of upside	Limited downside



The issuing entity



Issuer	Coupon	Maturity	Price	Yield
GOVERNMENT OF CANADA				
Broadcst Trs	7.530	May 01/27	105.93	7.02
Canada	11.750	Feb 01/03	109.78	2.39
Canada	5.750	Jun 01/03	104.15	2.70
Canada	7.250	Jun 01/03	106.18	2.70
Canada	5.250	Sep 01/03	103.62	2.97
Canada	9.500	Oct 01/03	110.52	3.19
Canada	5.000	Dec 01/03	103.20	3.24
Canada	7.500	Dec 01/03	107.76	3.24
	10.250	Feb 01/04	113.45	3.44
	6.500	Jun 01/04	106.56	3.61
	13.500	Jun 01/04	122.55	3.59
	3.500	Jun 01/04	99.70	3.63
Canada	5.000	Sep 01/04	102.93	3.82
Canada	10.500	Oct 01/04	116.85	3.92
Canada	9.000	Dec 01/04	113.62	3.97
Canada	12.000	Mar 01/05	122.59	4.24
Canada	6.0			
Canada	12.2			
Canada	8.7			
Canada	12.5			
Canada	5.750	Sep 01/06	104.33	4.70
Canada	14.000	Oct 01/06	138.86	4.72
Canada	7.000	Dec 01/06	109.58	4.78
Canada	13.750	Mar 01/07	140.11	4.84

Coupons are paid every 6 months.

The amount represents the total for one year

On this date, the issuer pays $100 to the bond holder (plus the last coupon payment)

Bond prices

Provided by RBC Capital Markets

Quotations on the bond issues in the RBC Capital Markets Canadian Bond Market index. Yields are calculated to full maturity. Price is the final bid-side price as of 5 pm, Friday.

Issuer	Coupon	Maturity	Price	Yield
GOVERNMENT OF CANADA				
Broadcst Trs	7.530	May 01/27	105.93	7.02
Canada	11.750	Feb 01/03	109.78	2.39
Canada	5.750	Jun 01/03	104.15	2.70
Canada	7.250	Jun 01/03	106.18	2.70

Issuer	Coupon	Maturity	Price	Yield
Quebec	6.250	Dec 01/10	102.30	5.91
Quebec	9.500	Sep 02/11	125.30	6.00
Quebec	9.000	Feb 10/12	122.10	6.04
Quebec	10.500	Jun 04/12	133.95	6.05
Quebec	10.500	Jun 01/14	127.75	6.11

Issuer	Coupon	Maturity	Price	Yield
GOVERNMENT OF CANADA				
Broadcst Trs	7.530	May 01/27	105.93	7.02
Canada	11.750	Feb 01/03	109.78	2.39
Canada	5.750	Jun 01/03	104.15	2.70
Canada	7.250	Jun 01/03	106.18	2.70
		p 01/03	103.62	2.97
		t 01/03	110.52	3.19
		c 01/03	103.20	3.24
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		b 01/04	113.45	3.44
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Canada	9.000	Dec 01/04	113.62	3.97
Canada	12.000	Mar 01/05	122.59	4.24
Canada	6.000	Sep 01/05	105.57	4.33
Canada	12.250	Sep 01/05	126.48	4.31
Canada	8.750	Dec 01/05	115.33	4.42
Canada	12.500	Mar 01/06	129.64	4.56
Canada	5.750	Sep 01/06	104.33	4.70
Canada	14.000	Oct 01/06	138.86	4.72
Canada	7.000	Dec 01/06	109.58	4.78
Canada	13.750	Mar 01/07	140.11	4.84

Unlike stocks
bond prices do not trade in an exchange
prices are determine by independent brokers

Clean vs. dirty prices

Issuer	Coupon	Maturity	Price	Yield
GOVERNMENT OF CANADA				
Broadcst Trs	7.530	May 01/27	105.93	7.02
Canada	11.750	Feb 01/03	109.78	2.39
Canada	5.750	Jun 01/03	104.15	2.70
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With annual compounding

$$\text{Dirty price} = \sum_i p_i \,(1+r)^{-t_i}$$
$$= \text{Accrued interest} + \text{Clean price}$$

Number of days since the last coupon payment

$$\text{Accrued interest} = \frac{n}{365} \times \text{Annual Coupon Rate}$$

Compounding

- The definition of interest rates is dependent on the compounding convention.
- For annual compounding, a series of cashflows is discounted

$$P_1 = \sum_i p_i \, (1 + r)^{-t_i}.$$

- If the compounding is n times a year,

$$P_n = \sum_i p_i \, (1 + \tfrac{r}{n})^{-t_i n}.$$

- In the limit, instantaneous compounding, yields

$$P_\infty = \sum_i p_i \, e^{-r \, t_i}.$$

- Instantaneous compounding has many analytical advantages, and from now on, we will use it most of the time.

Zero coupon bonds

Imagine a market where all bonds pay no coupons

- Bonds will contain a single cash flow: a single payment of at the time of maturity
- A bond will be characterized by three variables:
  - The notional, i.e. the payment to occur at maturity. FV
  - The price of the bond
  - The time to maturity
- With these concepts in mind, we define the yield

Bond price PV

Notional FV

$$\underbrace{r(T)}_{\text{yield}} = -\frac{\log(P/N)}{T}$$

Time to maturity

Cashflow valuation formula

The yield curve can then be used to calculate the price of any series of future cashflows:

$$P = \sum_i p_i \, e^{-r(t_i)\, t_i} \, .$$

Bootstrapping

If we return to a world where bonds have coupon payments, we can still recover the yield curve from those, avoiding the zero coupon bonds, as follows:

- For maturities less than 6 months, all coupons are zero-coupon bonds, therefore

$$r(T) = -\frac{\log(P/N)}{T}$$

Valid for 0<T<1

- For maturities between six months and one year, bonds have a coupon payment within six months, and another payment between six months and a year.

Known number, using the previous step

Unknown, can be solved as a one-variable equation

$$P = p_1\, e^{-r(t_1)\cdot t_1} + p_2\, e^{-r(t_2)\cdot t_2}, \qquad 0 < t_1 < \tfrac{1}{2} < t_2 < 1.$$

From the market (Dirty price)

- The process can be extended to infinity, therefore allowing us to calculate the yield curve for all maturities, assuming coupon bearing bonds for all maturities.

Worked out example (FIT)

- The time is January 1, 2000.
- We have the following bonds traded in the market:

Maturity date	Coupon payment	Price
March 31, 2000	4%	100.51
June 30, 2000	6%	101.72
October 1, 2000	6%	102.40
Jan 1, 2001	8%	104.87
June 30, 2001	4%	101.06
Jan 1, 2002	8%	108.61

Dirty prices

- We calculate the dirty prices adding the accrued interest on the coupon payments:

Number of days since the last coupon payment

$$\text{Accrued interest} = \frac{n}{365} \times \text{Annual Coupon Rate}$$

Maturity date	Coupon payment	Price
March 31, 2000	4%	100.51
June 30, 2000	6%	101.72
October 1, 2000	6%	102.40
Jan 1, 2001	8%	104.87
June 30, 2001	4%	101.06
Jan 1, 2002	8%	108.61

Maturity date	Coupon payment	Dirty Price
March 31, 2000	4%	101.50
June 30, 2000	6%	101.72
October 1, 2000	6%	103.88
Jan 1, 2001	8%	104.88
June 30, 2001	4%	101.06
Jan 1, 2002	8%	108.61

Bootstrapping: step 1

- Start with the bond with shortest maturity:

$$101.50 = 102 \cdot e^{-r(0.25)/4} \implies r(0.25) = 2\,\%$$

- Continue with the bond with next shortest maturity:

$$101.72 = 103 \cdot e^{-r(0.5)/2} \implies r(0.5) = 2.5\,\%$$

- $\cdots$ and so on and so forth until we run out of zero coupon bonds.

Step 2:

- We move to the first bond with two payments:

$$103.88 = 3 \cdot e^{-r(0.25)/4} + 103 \cdot e^{-r(0.75)*3/4} \implies r(0.75) = 2.75\,\%$$

- In deriving this value for $r(0.75)$ note that we are using the fact that we already calculated $r(0.25)$ in step 1.

- This point is important, and highlights the importance that the yield curve be the same for all bonds at the same time.

- We continue in the same fashion for the next bond:

$$104.88 = 4 \cdot e^{-r(0.5)/2} + 104 \cdot e^{-r(1)} \implies r(1) = 3\,\%$$

Continued···

- We continue to the next bond, which now has three payments:

$$101.06 = 2 \cdot e^{-r(0.5)/2} + 2 \cdot e^{-r(1)} + 102 \cdot e^{-r(1.5)\cdot 1.5} \implies r(1.5) = 3.25\,\%\,,$$

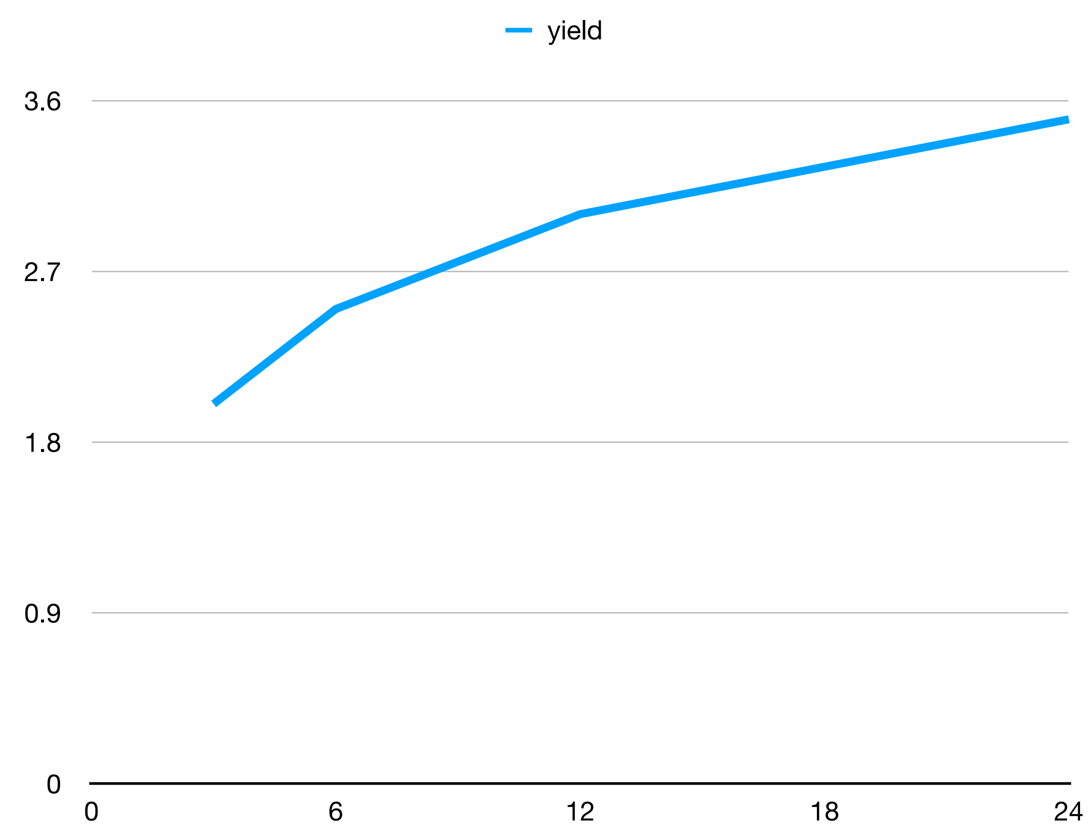
- ··· and so on and so forth···

$$108.61 = 4 \cdot e^{-r(0.5)/2} + 4 \cdot e^{-r(1)} + 4 \cdot e^{-r(1.5)\cdot 1.5} + 104 \cdot e^{-r(2)\cdot 2} \implies r(2) = 3.5\,\%\,,$$

The Yield Curve

- The result is the yield curve, determined by the values we calculated:

Term	rate
3 month	2%
6 month	2.5%
9 month	2.75%
1 year	3%
18 months	3.25%
2 year	3.5 %



Practical issues

- What if we had a bond maturing on December 1, 2000, with coupon payment 4% and dirty price of 100?

$$100 = 2 \cdot e^{-r(4/12)\cdot 4/12} + 102 \cdot e^{-r(10/12)\cdot 10/12}$$

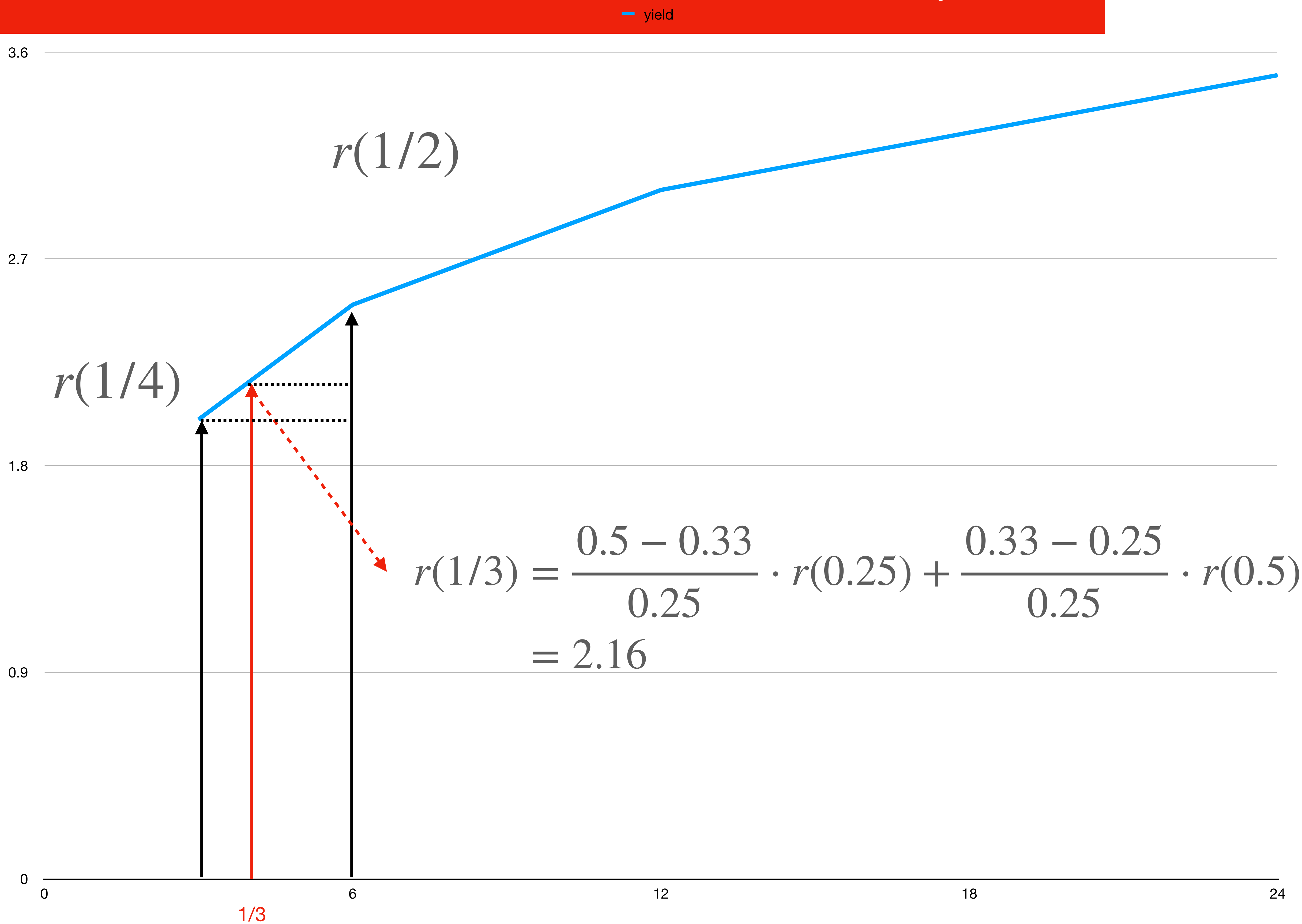
- In this case, we see that suddenly we have two new unknowns, $r(1/3)$ and $r(5/6)$.

- The practical thing to do is to calculate

$$r(1/3) = \frac{0.5 - 0.33}{0.25} \cdot r(0.25) + \frac{0.33 - 0.25}{0.25} r(0.5) = 2.16\,\%\,,$$

- the interpolated value from the known values $r(0.25)$ and $r(0.5)$, and then solve for $r(5/6)$ as usual.

Interpolation



$$r(1/3) = \frac{0.5 - 0.33}{0.25} \cdot r(0.25) + \frac{0.33 - 0.25}{0.25} \cdot r(0.5)$$
$$= 2.16$$

Additional practical issues

Other practical issues that become important in applications are:

- Include for precise day-counts between dates
- Taking into account holidays, which may be different for different jurisdictions
- Use the precise accounting rules for each bond (360/30, 365/30, actual/actual, etc.)⋯
- ⋯ and the correct coupon accrual conventions.

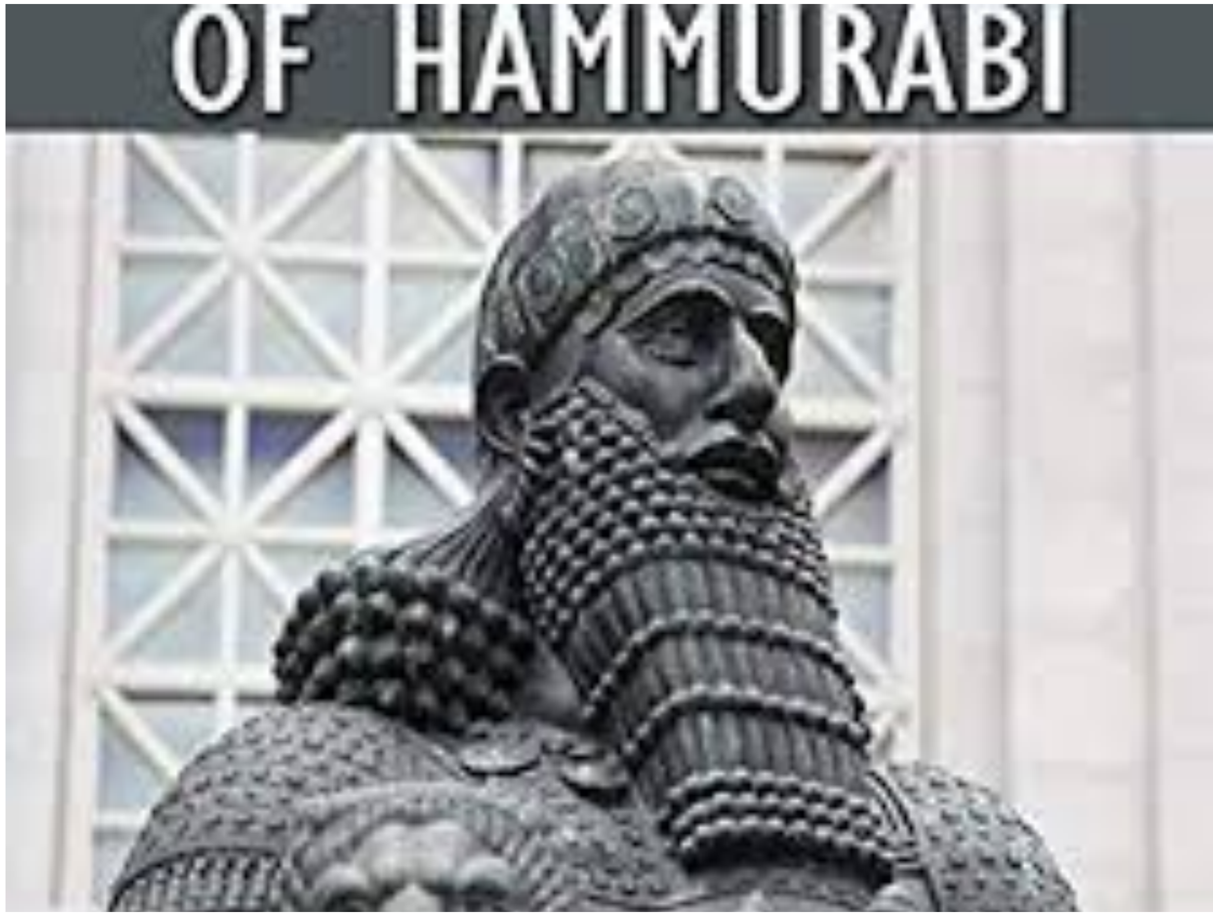
Financial markets

	Income	Risks	Headline Cases
Banking	Fees/interest	Default	Lehman, Bankers Trust
Insurance	Premia	Events	AIG
Asset Management	Management fees	Legal	Madoff
Payment systems	% of transaction	Fraud	FTX
Services	Commissions	-	Andersen

China and Babylonia

- Code of Hammurabi, c. 1750 BC

- If a merchant received a loan to fund his shipment, he would pay the lender an additional sum in exchange for the lender's guarantee to cancel the loan should the shipment be stolen or lost at sea.



First Option Trade

FIT

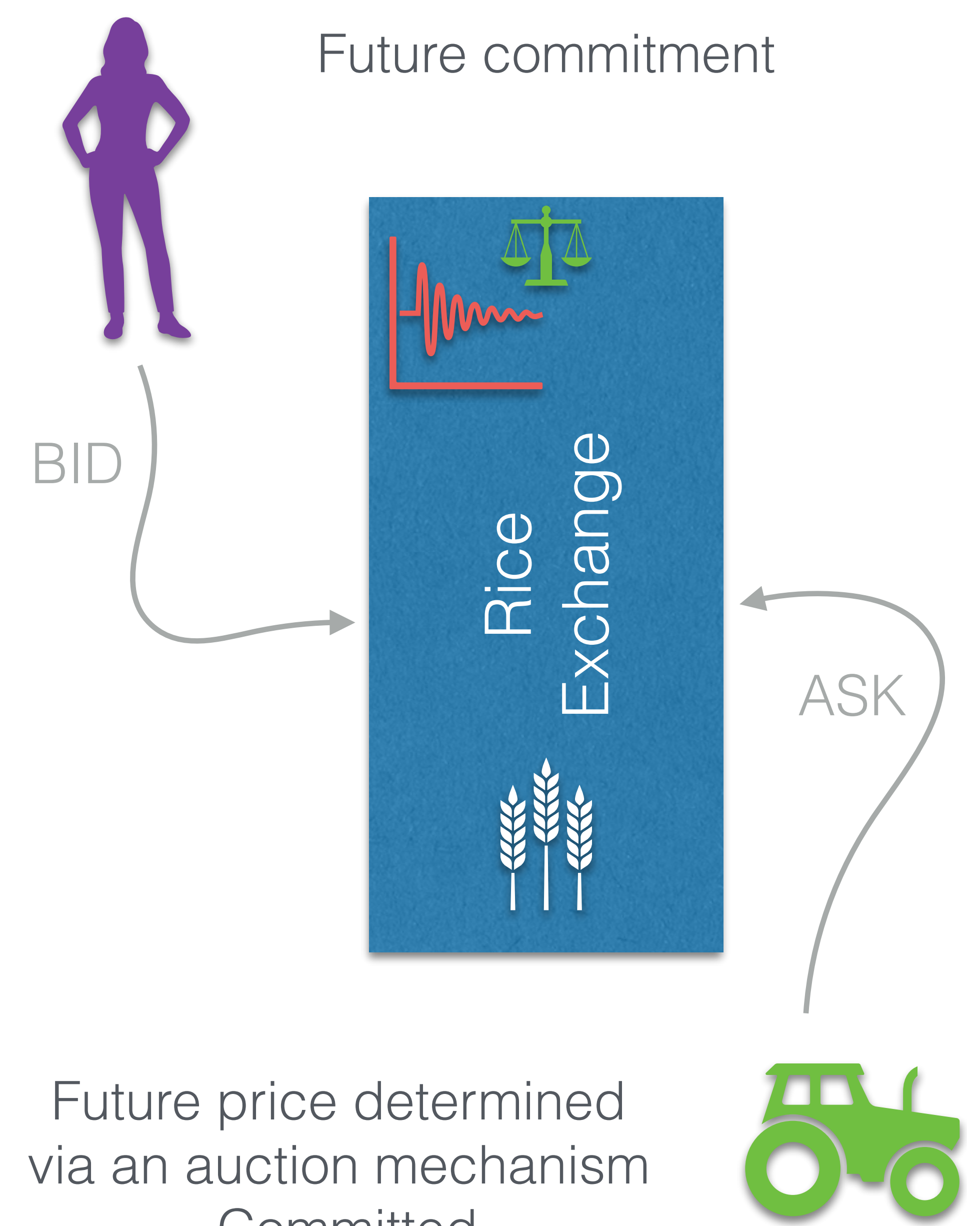
- Aristotle (348 BC - 322 BC), in his book Politics, describes the story of Thales of Miletus (624 BC - 546 BC), who was a great mathematician but poor,

Thales, so the story goes, because of his poverty was taunted with the uselessness of philosophy; but from his knowledge of astronomy he had observed while it was still winter that there was going to be a large crop of olives, so he raised a small sum of money and paid round deposits for the whole of the olive-presses in Miletus and Chios, which he hired at a low rent as nobody was running him up; and when the season arrived, there was a sudden demand for a number of presses at the same time, and by letting them out on what terms he liked he realized a large sum of money, so proving that it is easy for philosophers to be rich if they choose, but this is not what they care about.

- Thales effectively bought a call option - the right, but not the obligation, to use the olive presses.

- Luckily for Thales, the crop was strong, and his options finished in the money.

- The Dōjima Rice Exchange (堂島米市場), Osaka
- Established in **1697**
- Consumers and producers negotiated the future price of rice: **a futures exchange**



Future price determined via an auction mechanism
-Committed
-No default risk

Chicago

- In the 19th century, agricultural products where traded all over the North American midwest
- One of those centres, Chicago, developed the concept of the clearing house.
- The concept was so powerful, that it became the dominant centre, sending all others into oblivion

100 years of quantitative methods

1930 1950 2000 2020

Harry Markowitz realized
no value was given to
RISK

"Portfolio Selection," *Journal of Finance*. 1952
languished on dusty library shelves
only four of the 14 pages contained text

John Burr Williams,
The Theory of Investment Value
value of a stock given by
current value of future dividends.

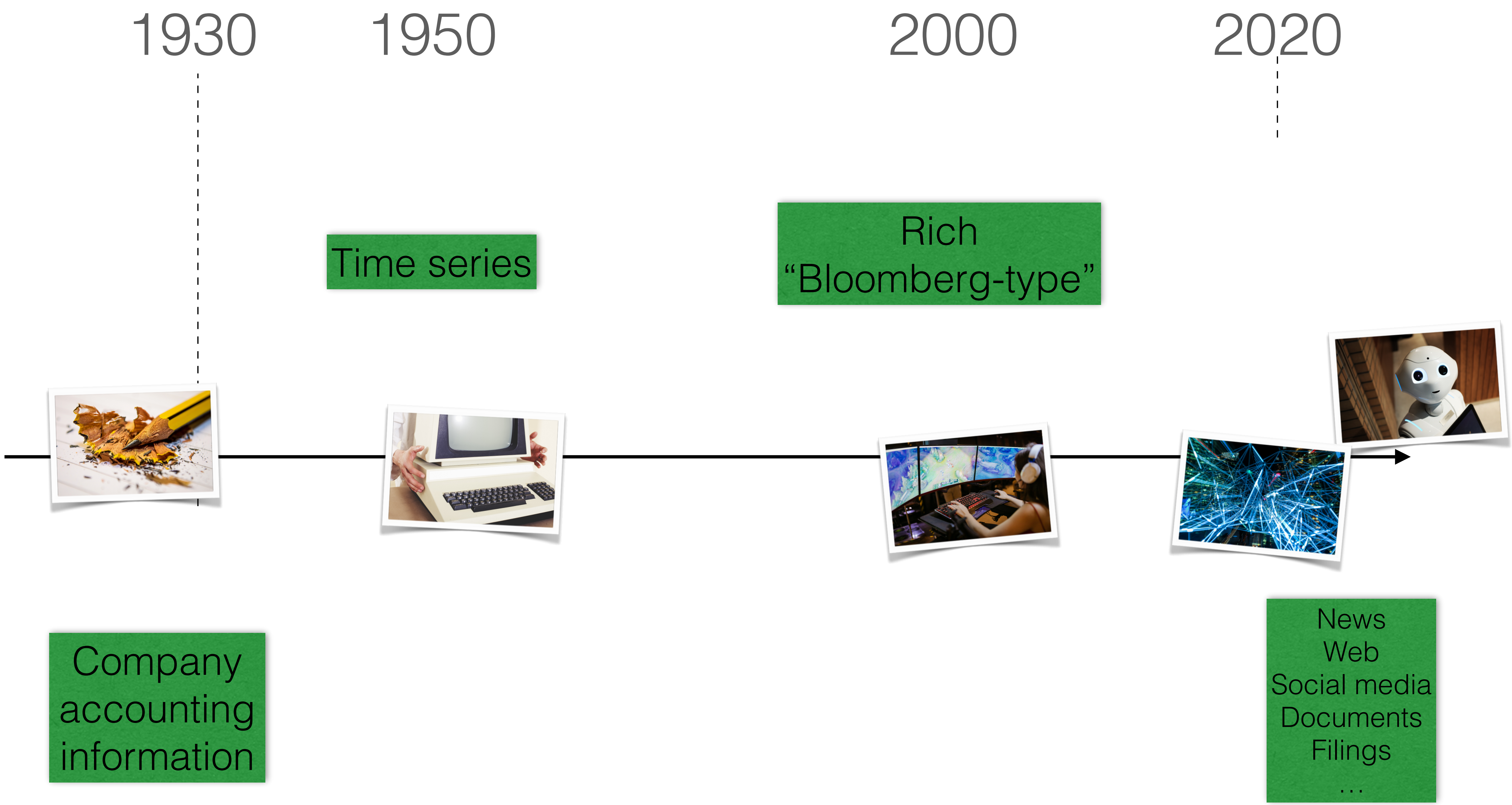
Find a good stock
buy it at the best price.

A plethora of quant methods
started with the ubiquity of the PC

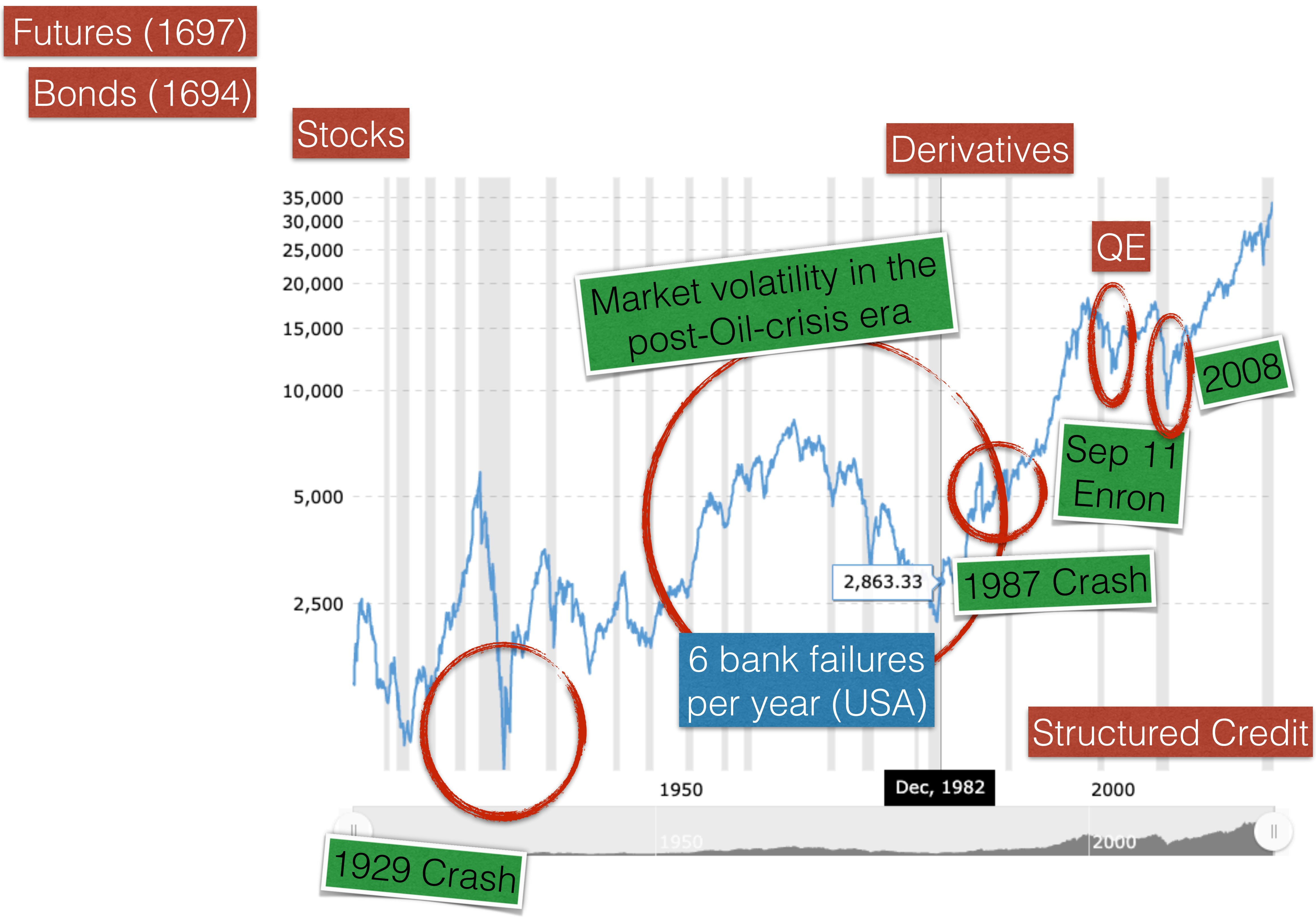
GME

ML
AI
ESG
NLP
Social
Behavioral

100 years of data inputs



100 years of failures



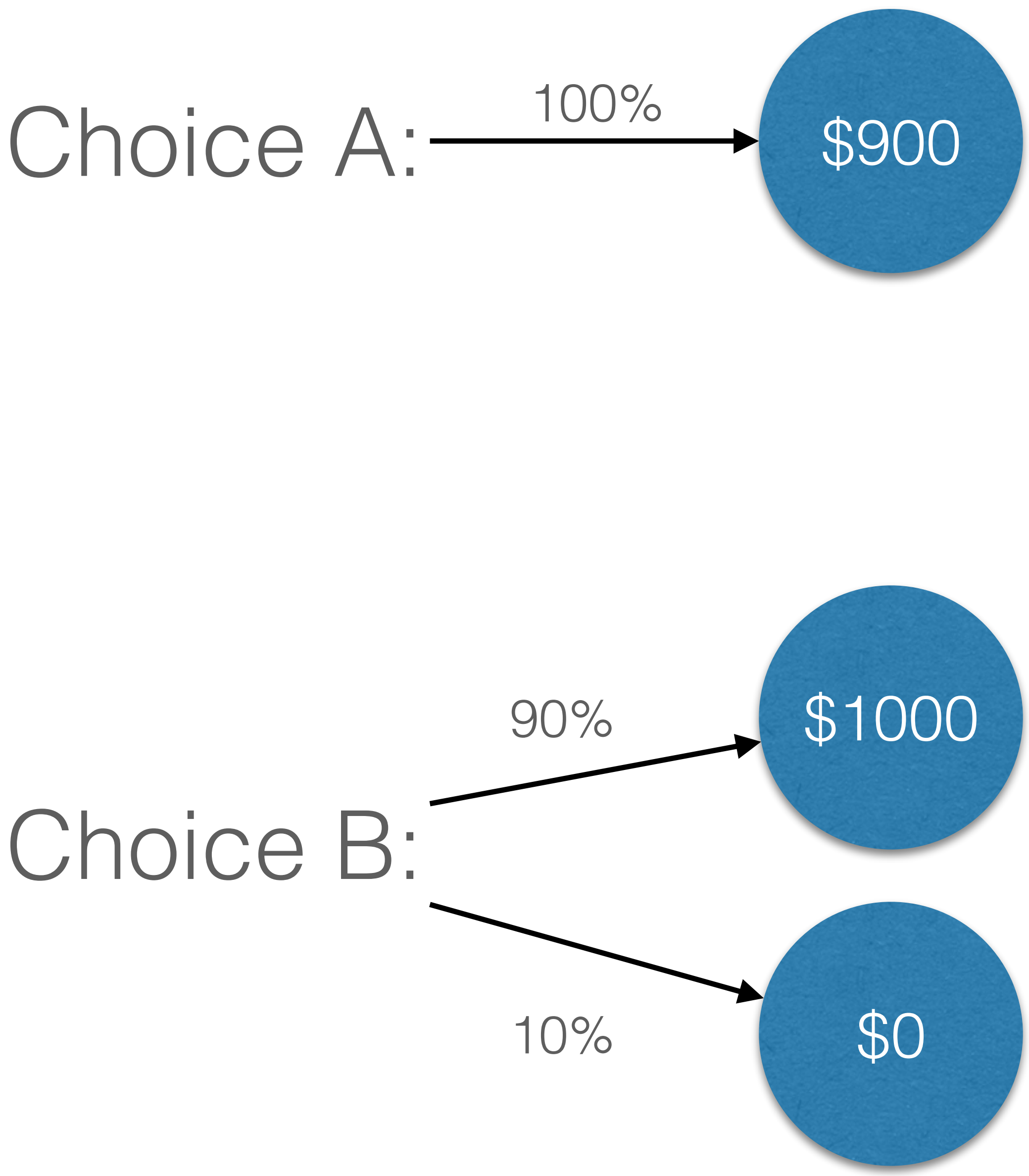
Events & Regulation

Event	Regulation
1929 Crash	Securities ACT (1933)
Enron	SOX (2002)
2008	Dodd-Frank (2010)
?	Data Regulation?
COVID-19	?
Climate Risk	Solvency II

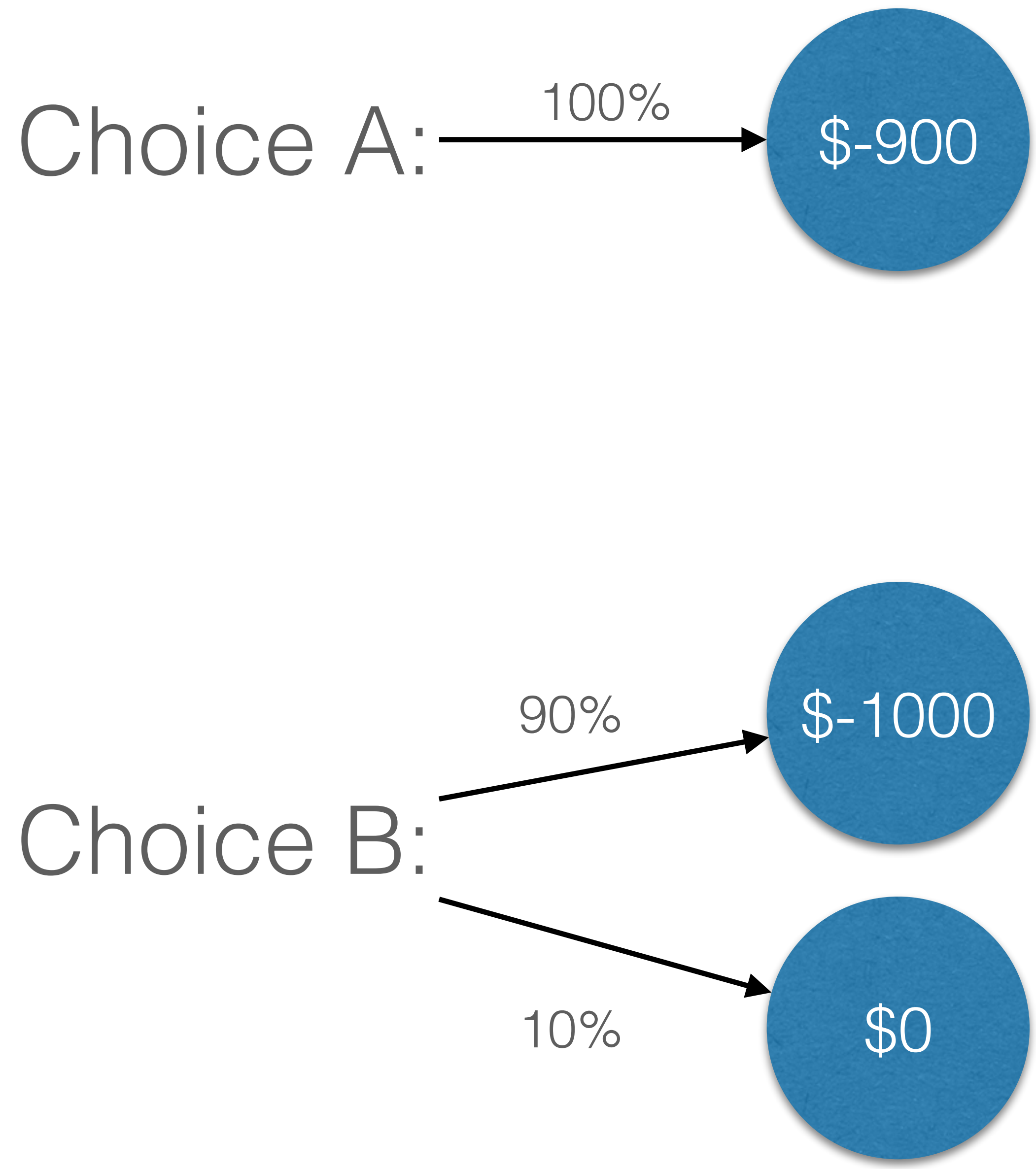
How will data be regulated?

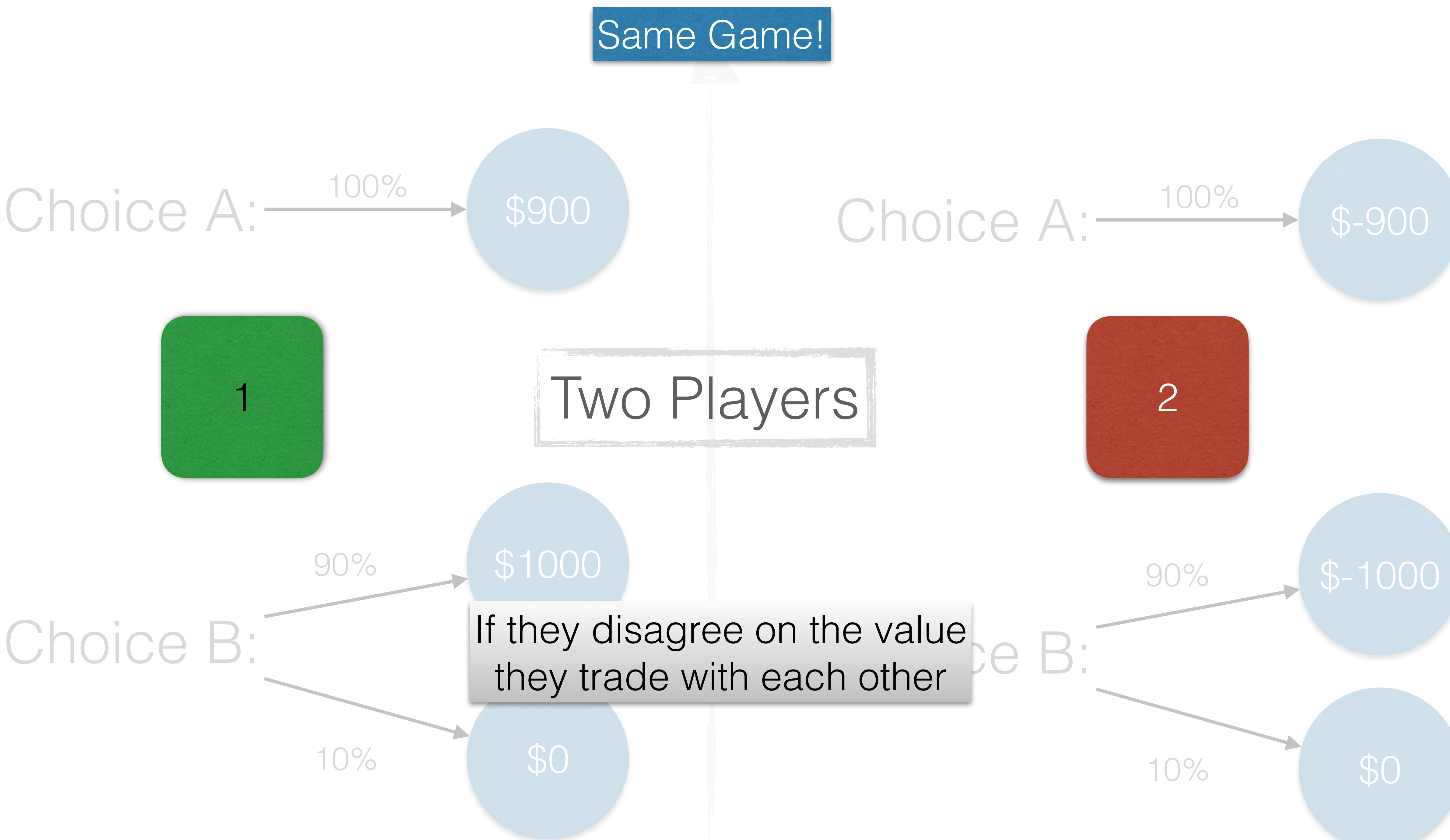
Decision making

Game 1

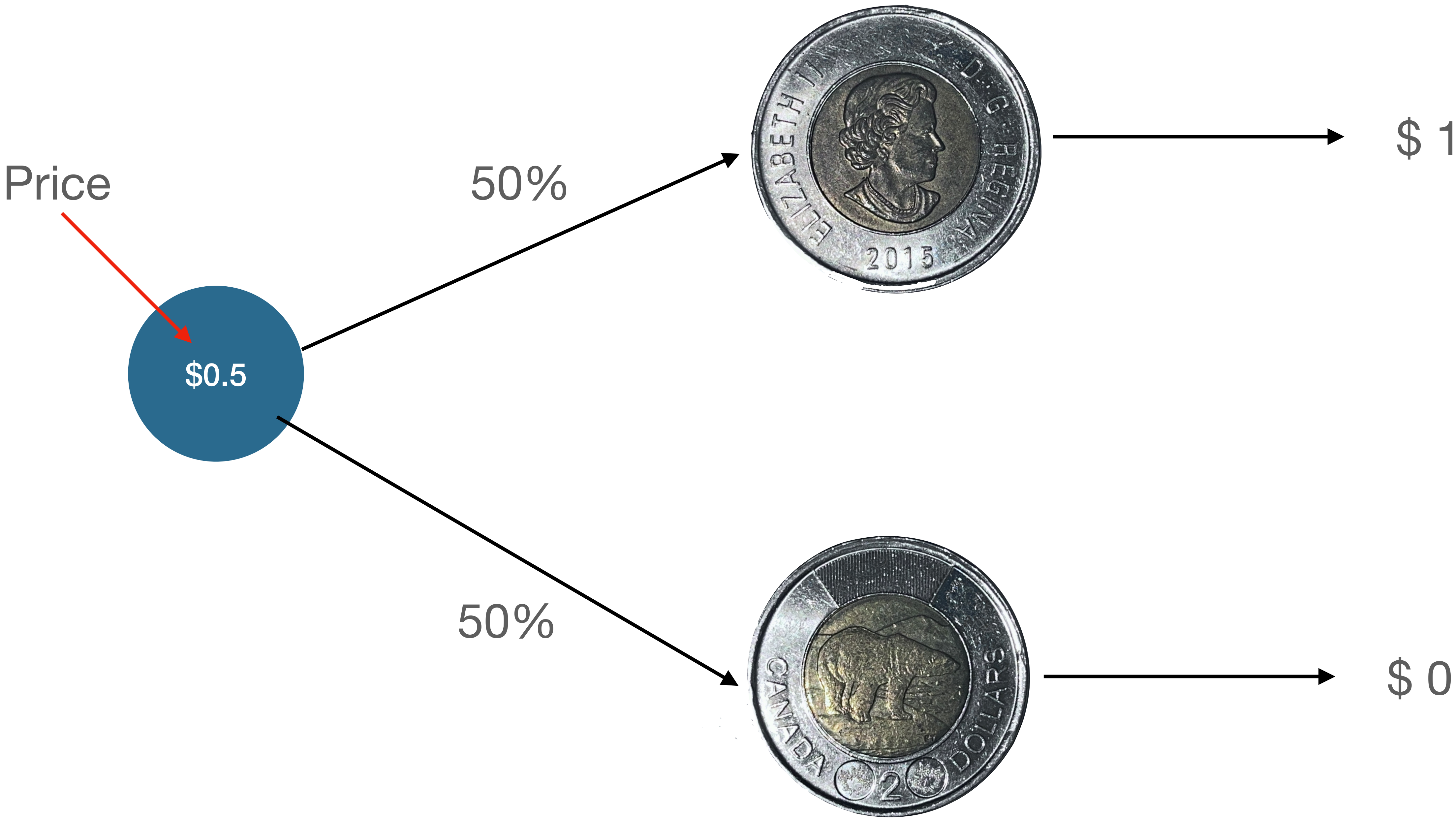


Game 2




Same Game!
Choice A:
100%
$900
1
90%
$1000
Choice B:
10%
$0
Two Players
Choice A:
100%
$-900
2
90%
$-1000
Choice B:
10%
$0
If they disagree on the value
they trade with each other

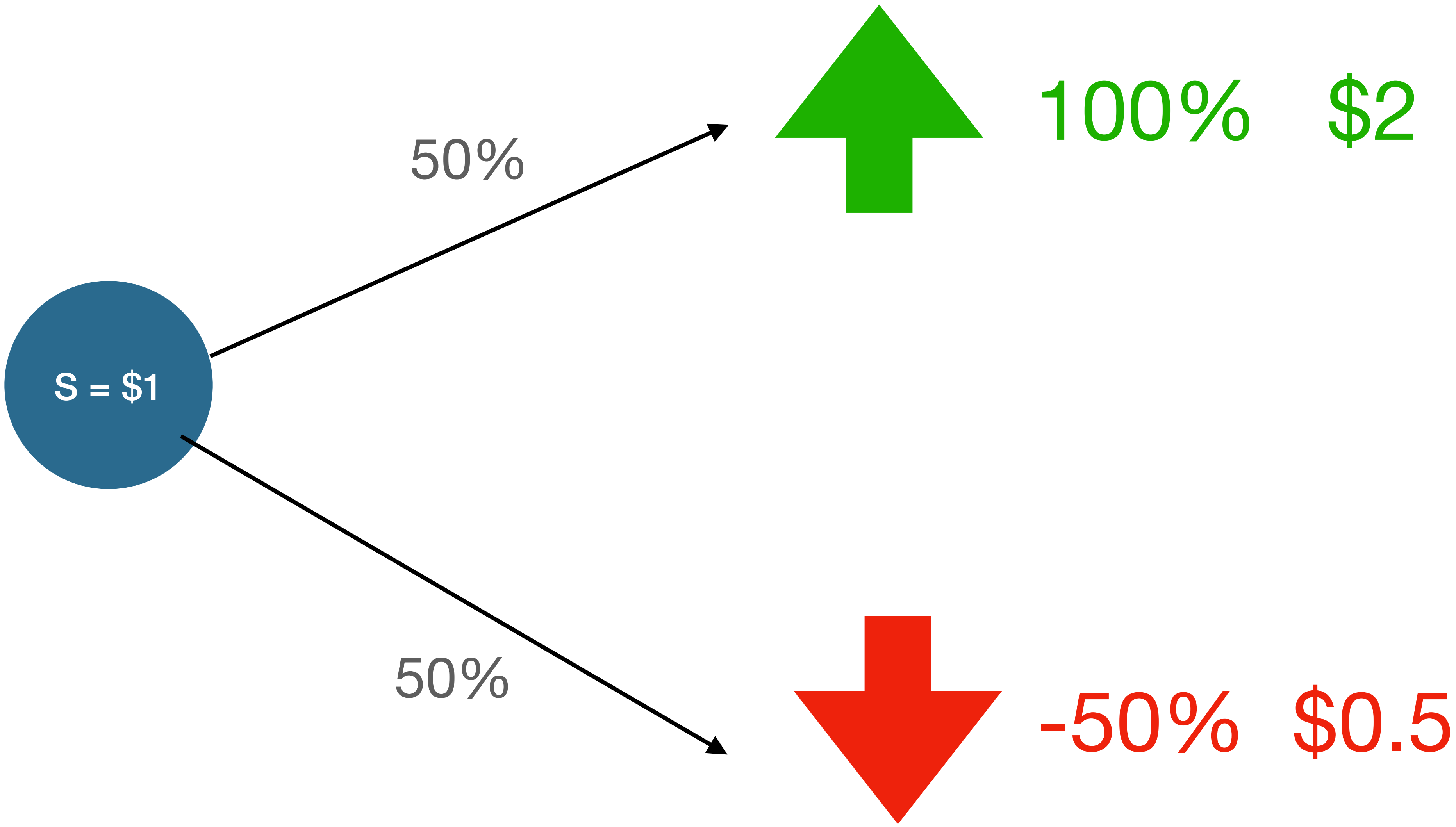
A coin flip



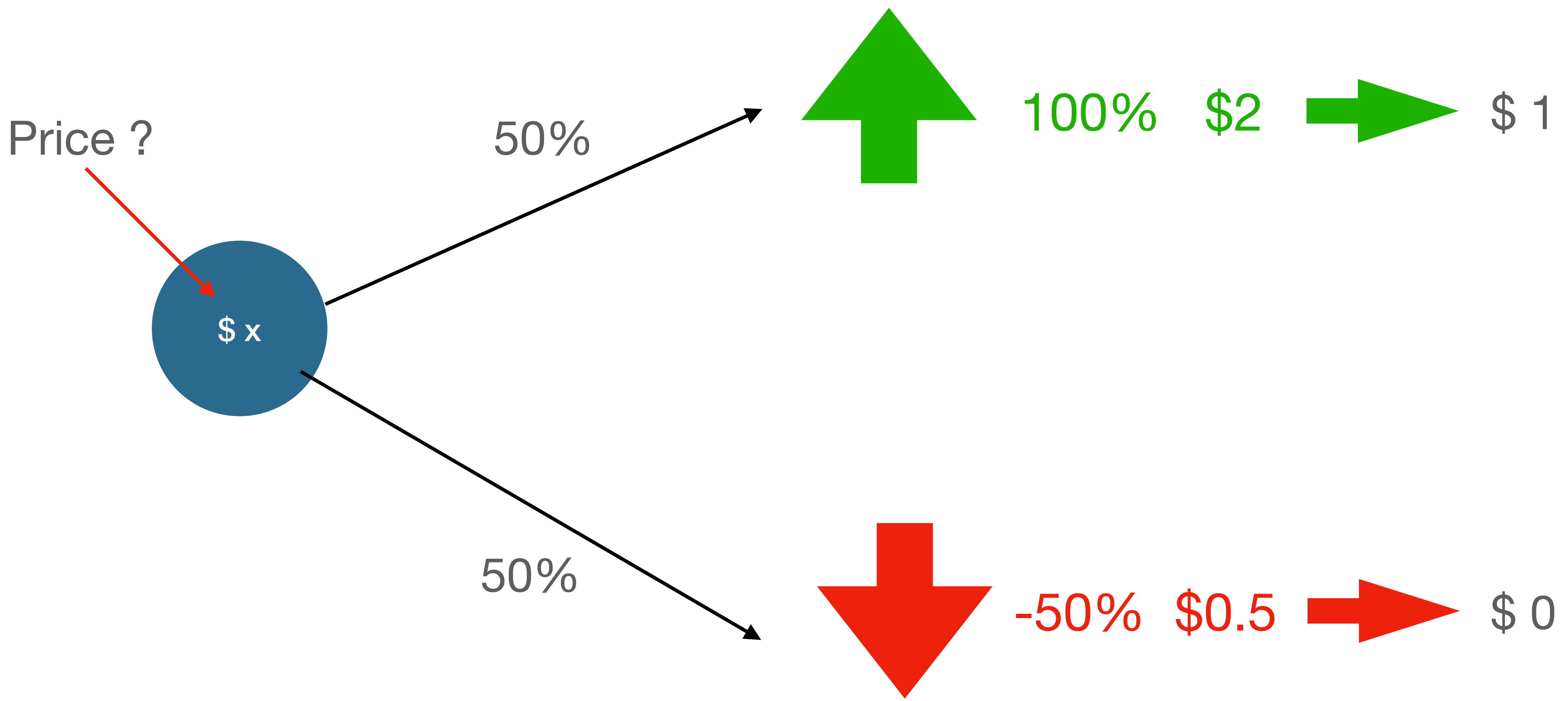
Price is $0.50

- By participating in the game, participants can gain or lose.
- The more they play, the more "fair" the game becomes: losses and gains will converge to 0.
- This is a probabilistic model

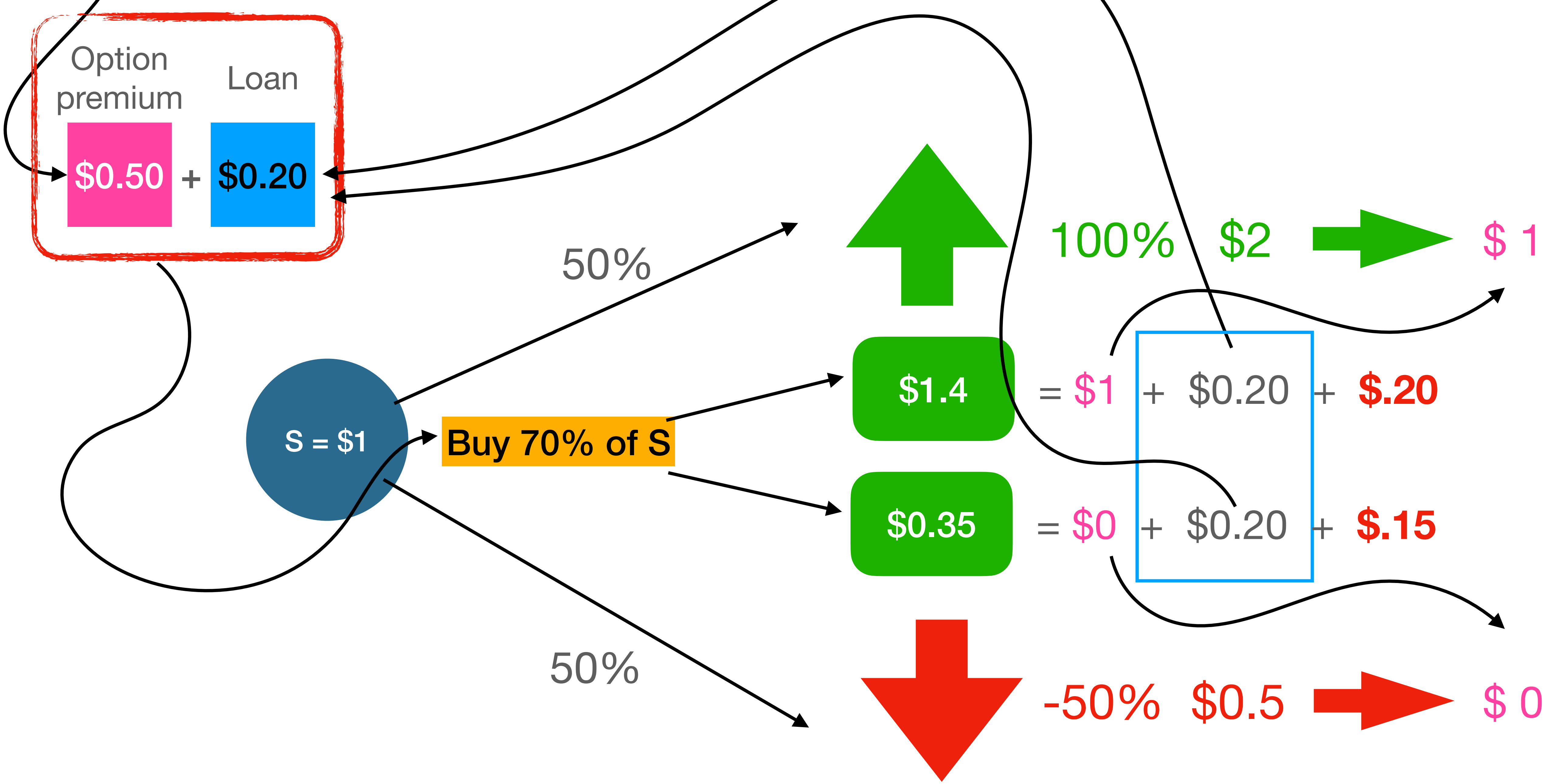
A stock movement



A stock option



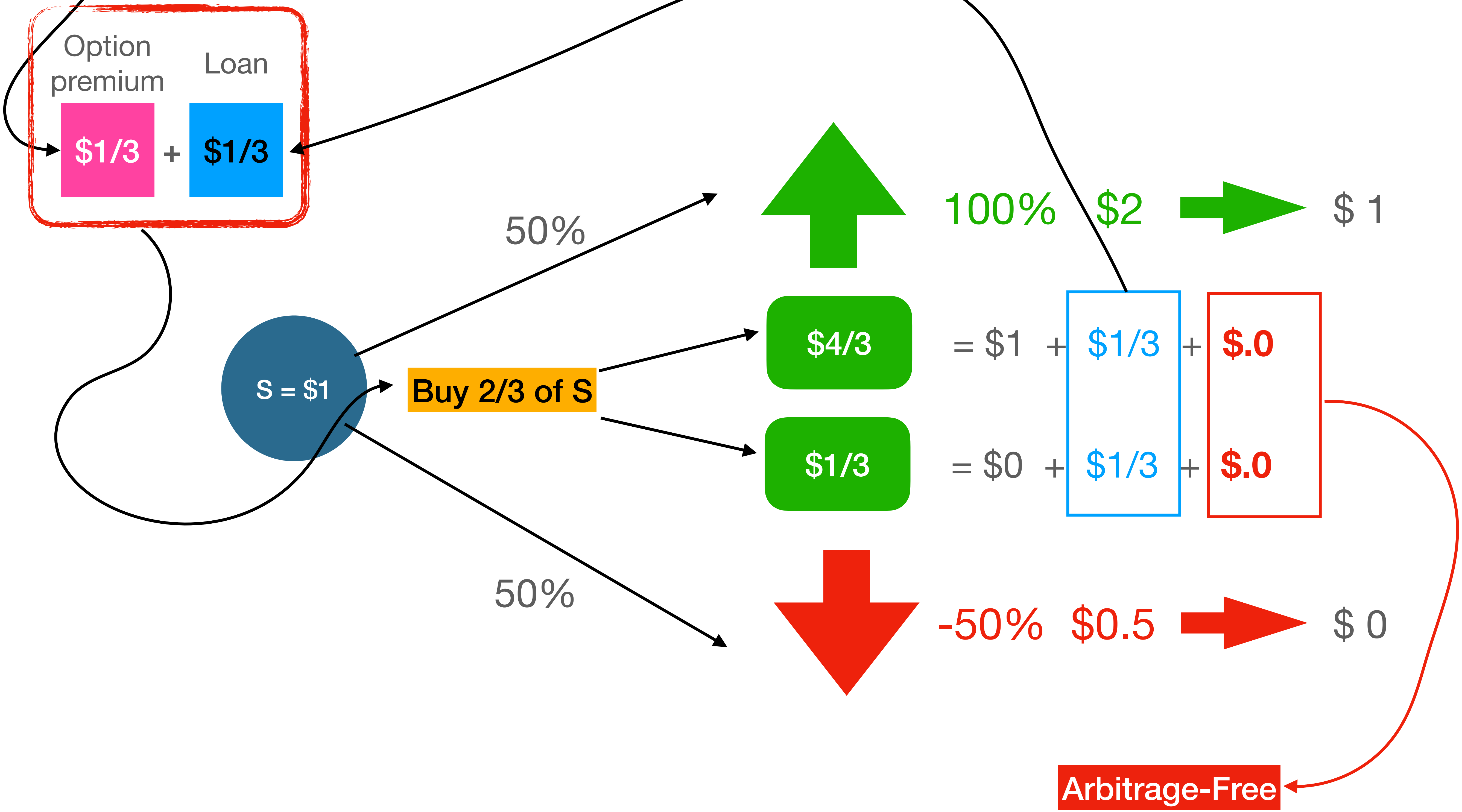
$0.5 is the wrong price



Price is NOT $0.50

- One side always wins: guaranteed
- The more they plan, the more money the issuer will make
- A price of $0.50 leads to arbitrage: **making money for free**
- This is not a probability game, but a **financial** one.

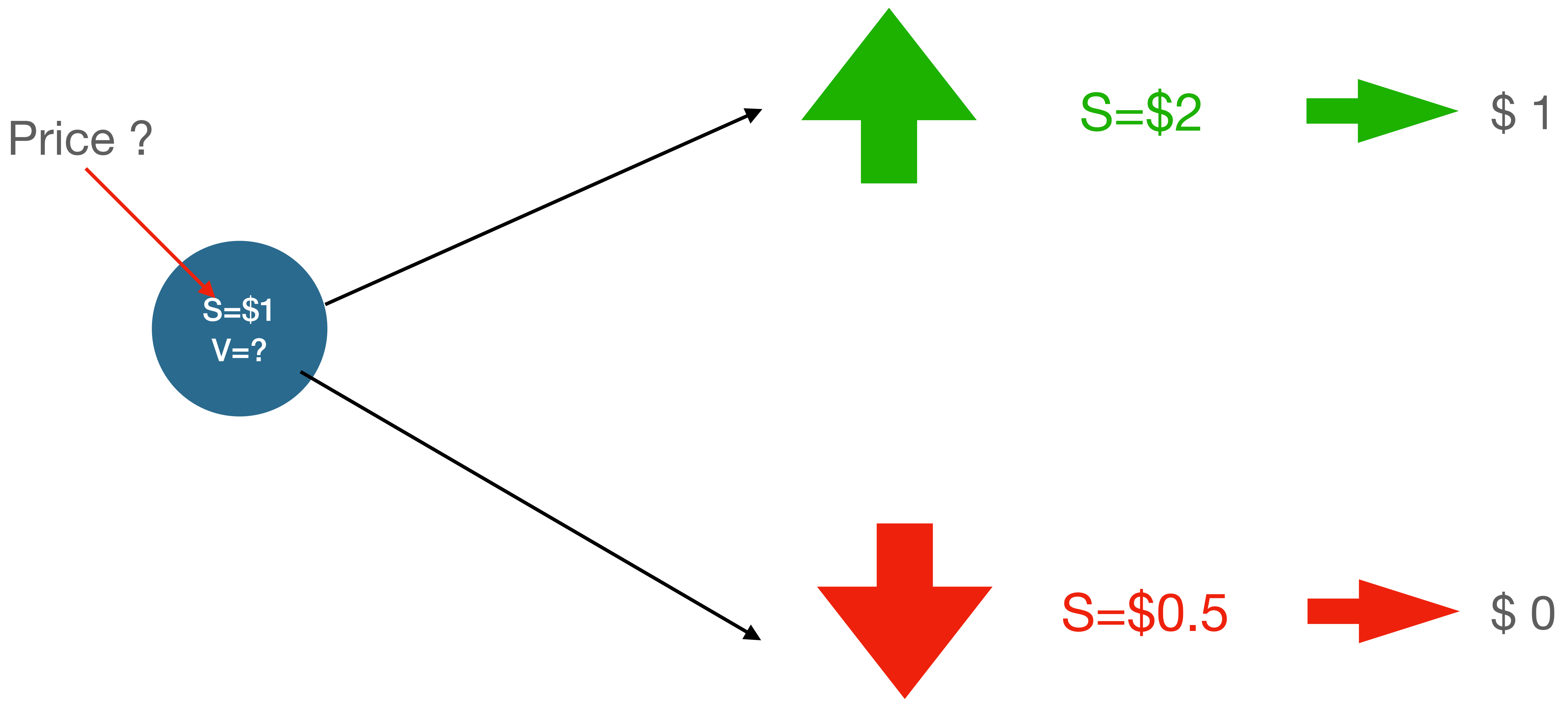
$1/3 is the right price



Fundamental Facts

- The price of the option is independent of the probability of stock movements…
  - … as long as they are not zero
- The replicating portfolio yields exactly the same pay off as the option
- The replicating portfolio is built solving a linear system of equations

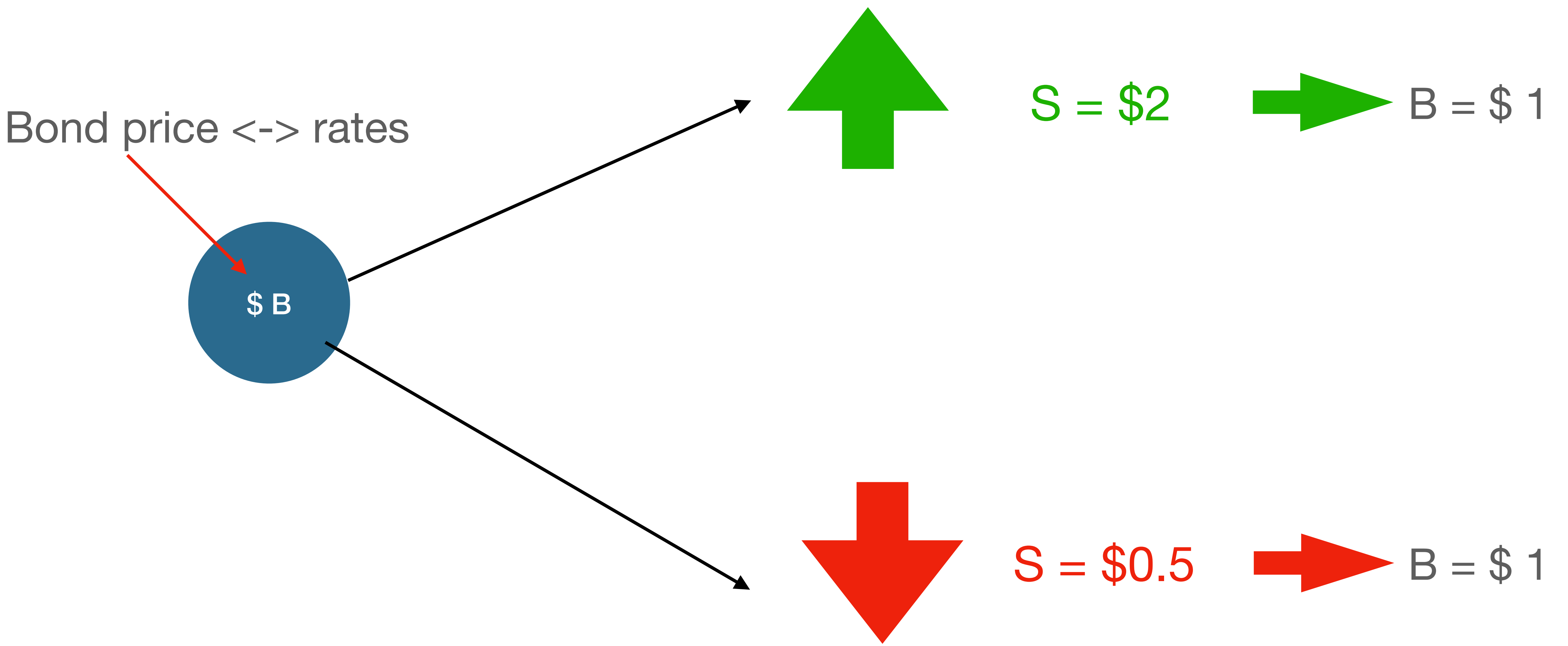
A stock option



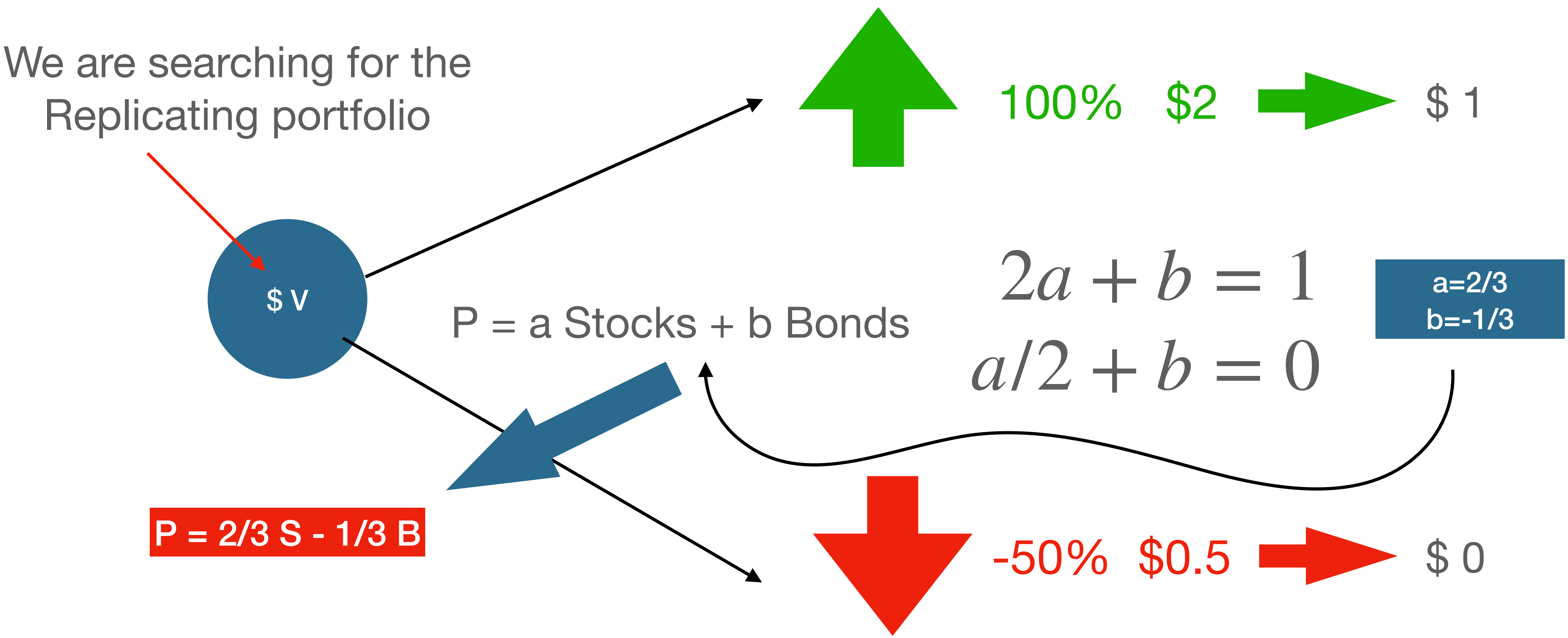
Non-zero interest rates

- If we assume interest rates non-zero, ⋯
- ⋯ or equivalently we assume a cost of borrowing⋯
- ⋯ we can adjust the replicating portfolio, easily.
- Next, we will set up a linear program that gives us the solution.

We track the price of a bond



Replicating portfolio of the Option



Pricing with interest rates

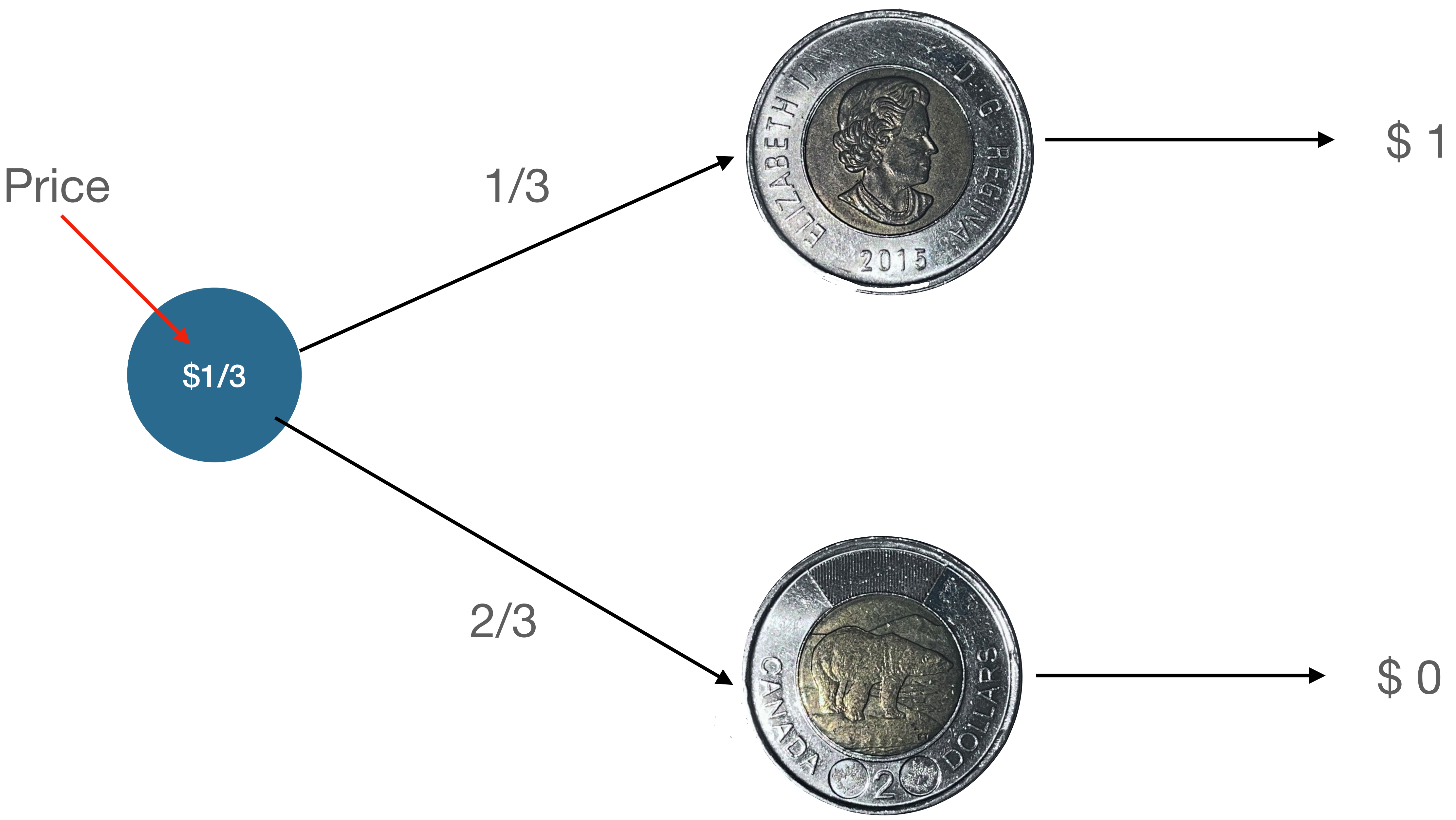
- The replicating portfolio $P = aS + bB$, works for
  - all bond and stock prices today
    - (note we have only used their value at maturity)
  - $\cdots$ therefore it works for all interest rates
- If, as before, S=\$1 and interest rates are 0 (i.e. B=\$1), then

$$V = \$1/3$$

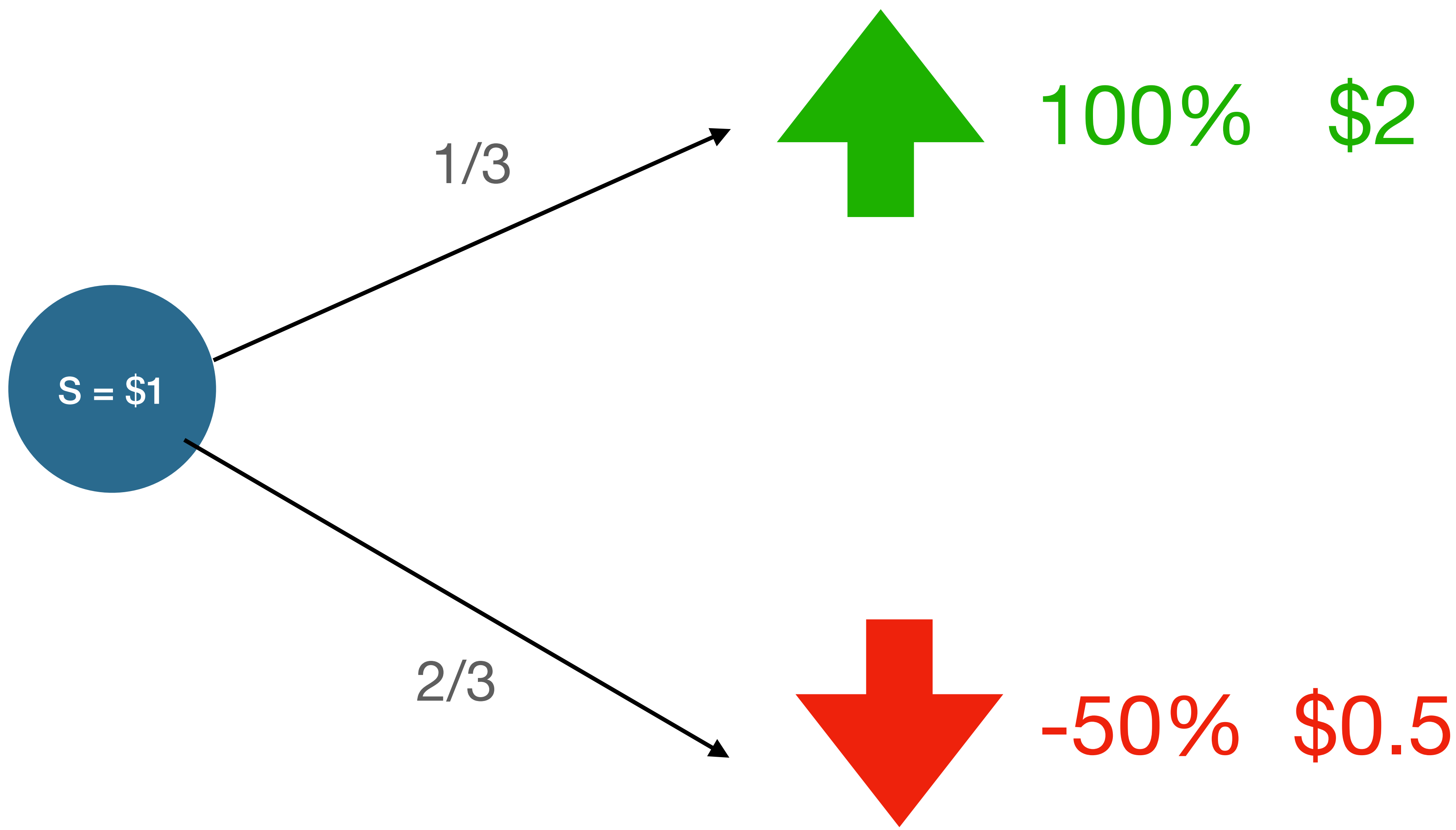
- If, however, rates are higher than 0 (say B=\$0.9 today), then

$$\begin{aligned} V &= \$2/3 - \$1/3B \\ &= \$0.3666 \end{aligned}$$

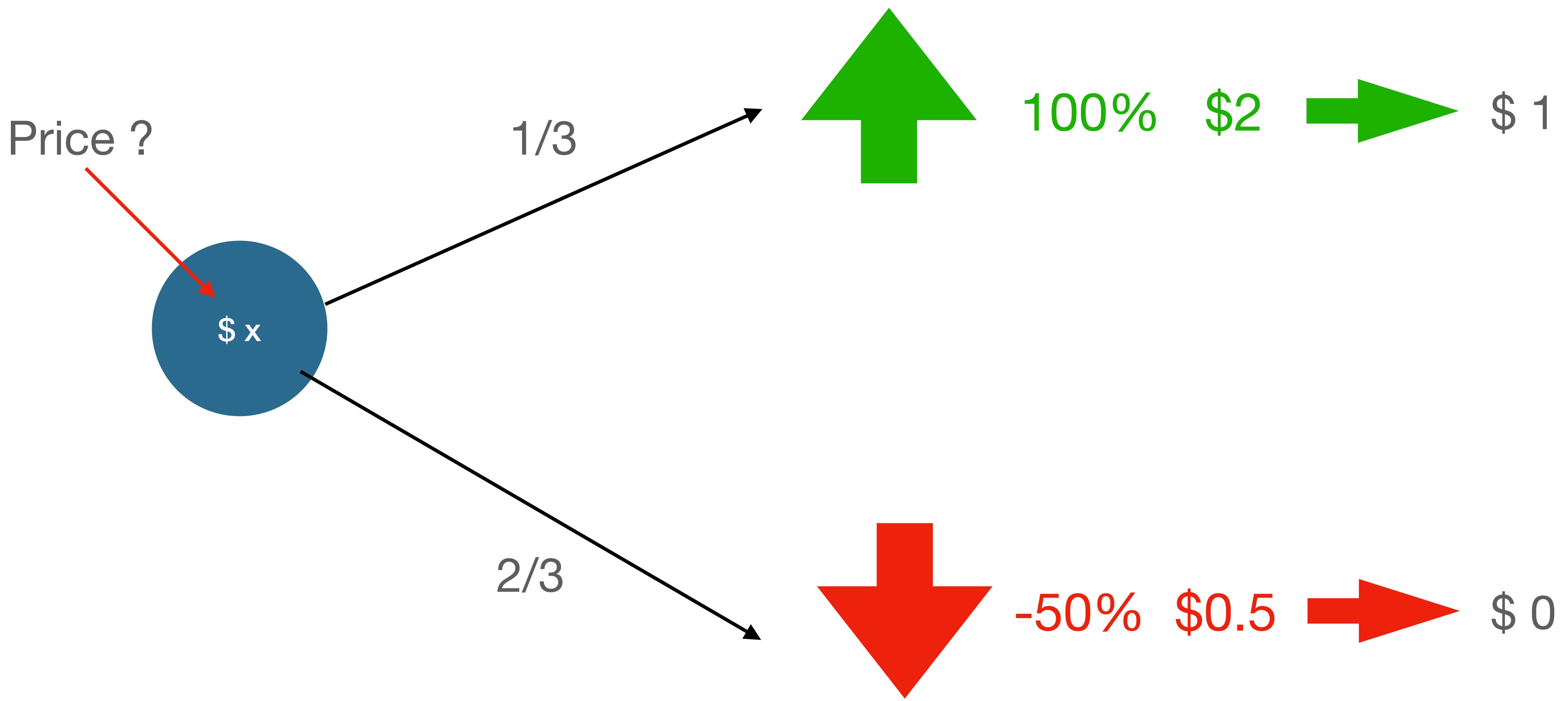
A coin flip



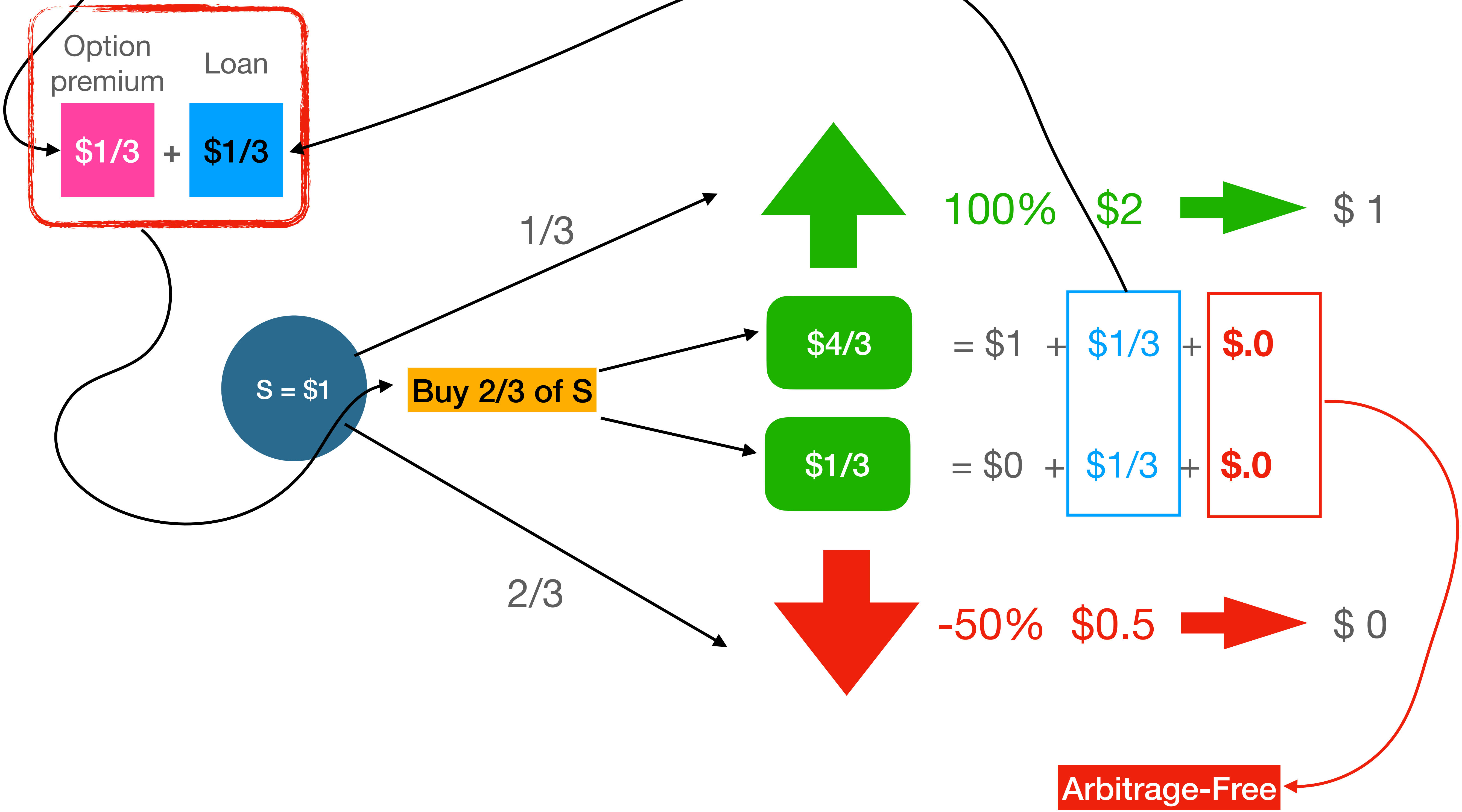
A stock movement



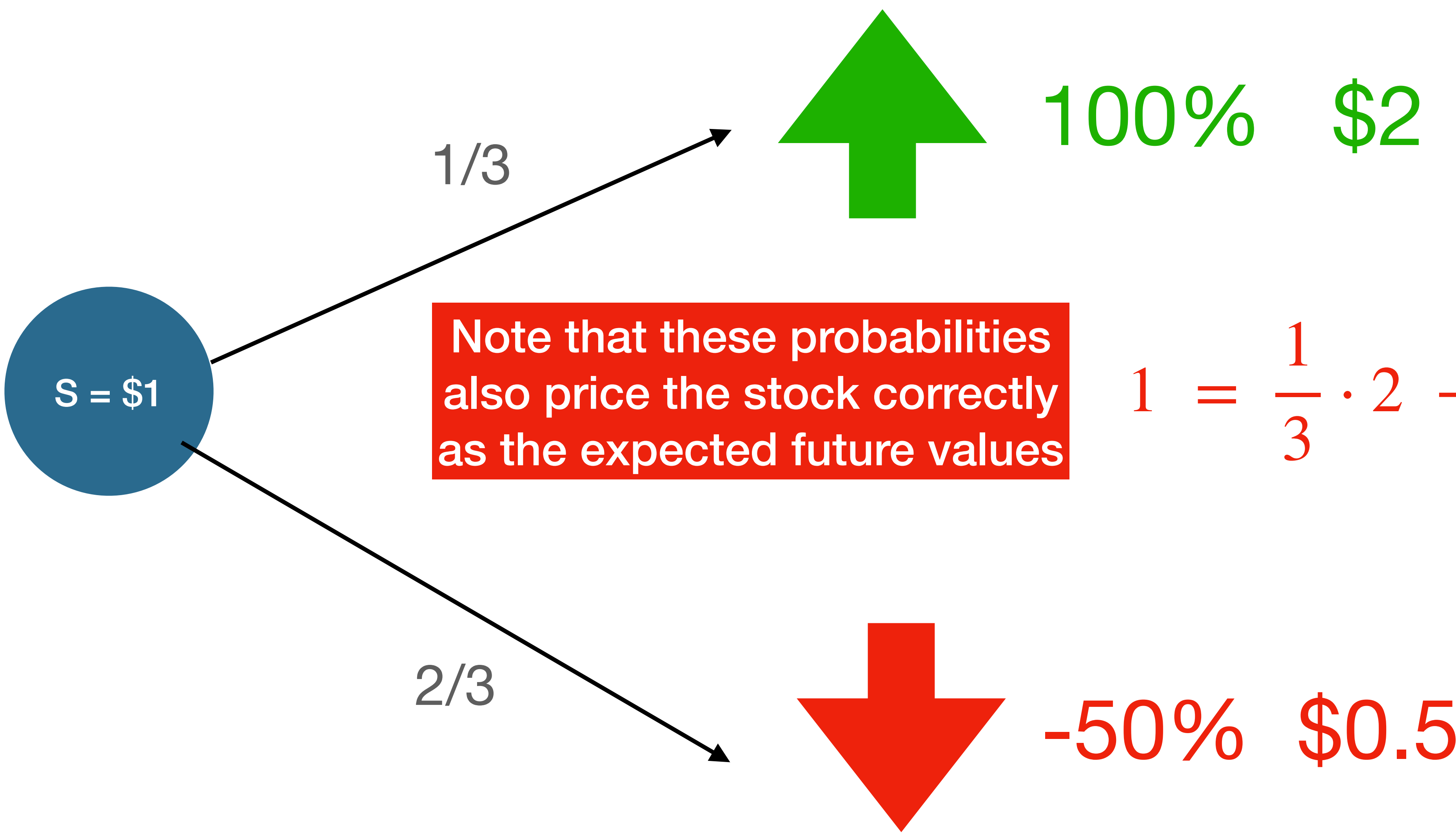
A stock option



$1/3 is the right financial and probabilistic price



A “martingale”

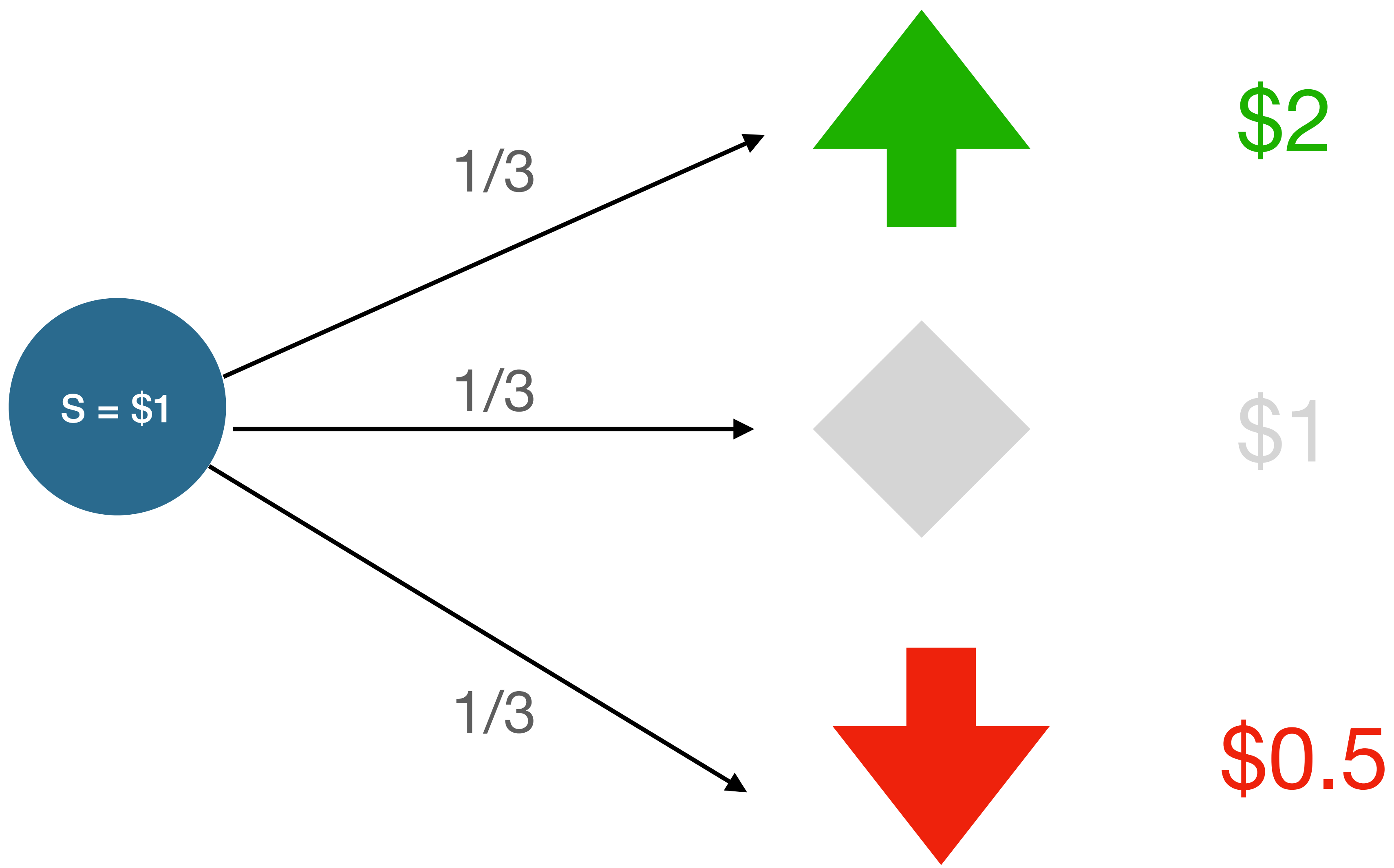


$$1 = \frac{1}{3} \cdot 2 + \frac{2}{3} \cdot 0.5$$

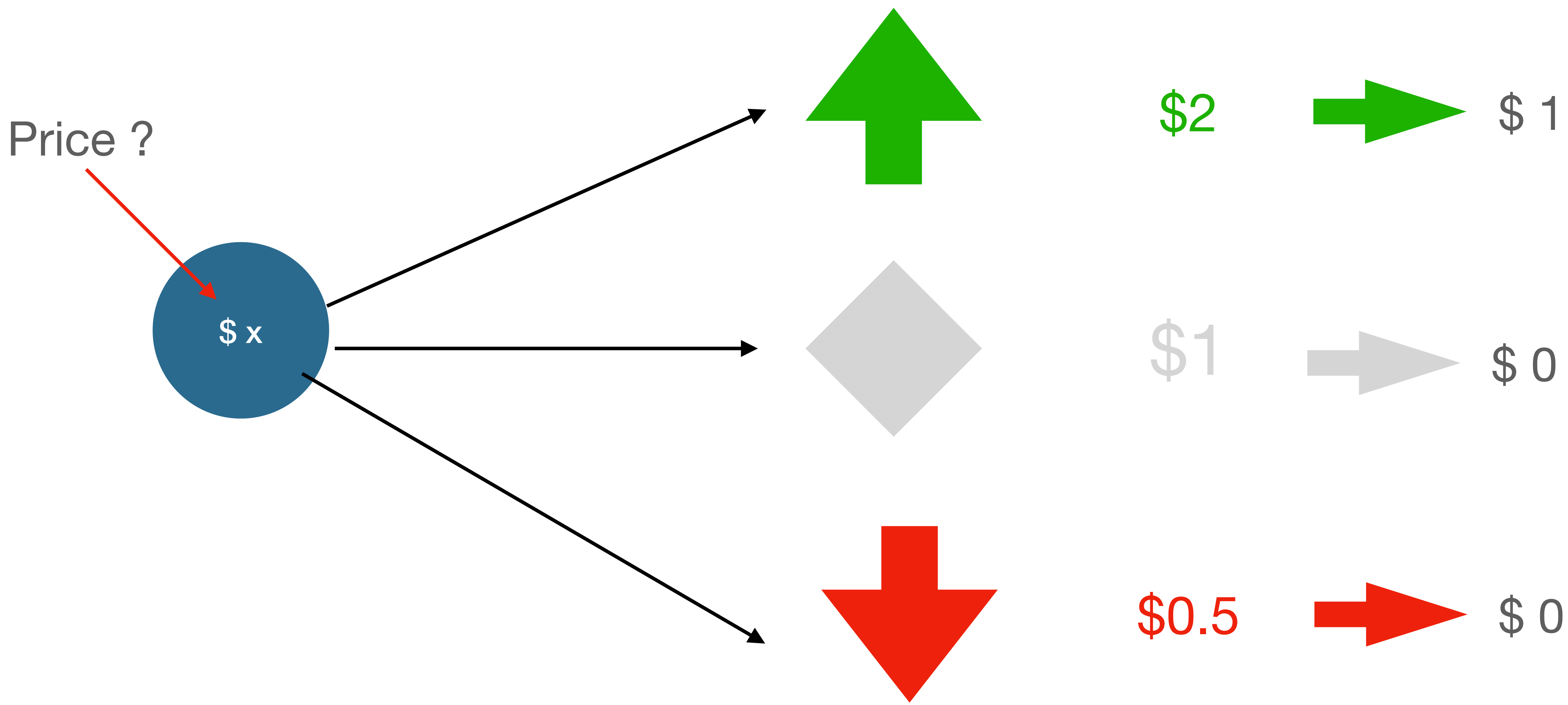
The Risk Neutral Measure

- It refers to the probability of future asset prices which prices all securities observed in the market today as the expected discounted value
- The discounting comes from bond prices:
  - We know their future value for sure
  - Their value today is different, reflecting interest rates
  - The discount factor given by bonds, when applied to other securities, prices markets today correctly knowing future values
- This is risk-neutral pricing, and is equivalent to the price given by portfolio replication.

A stock movement



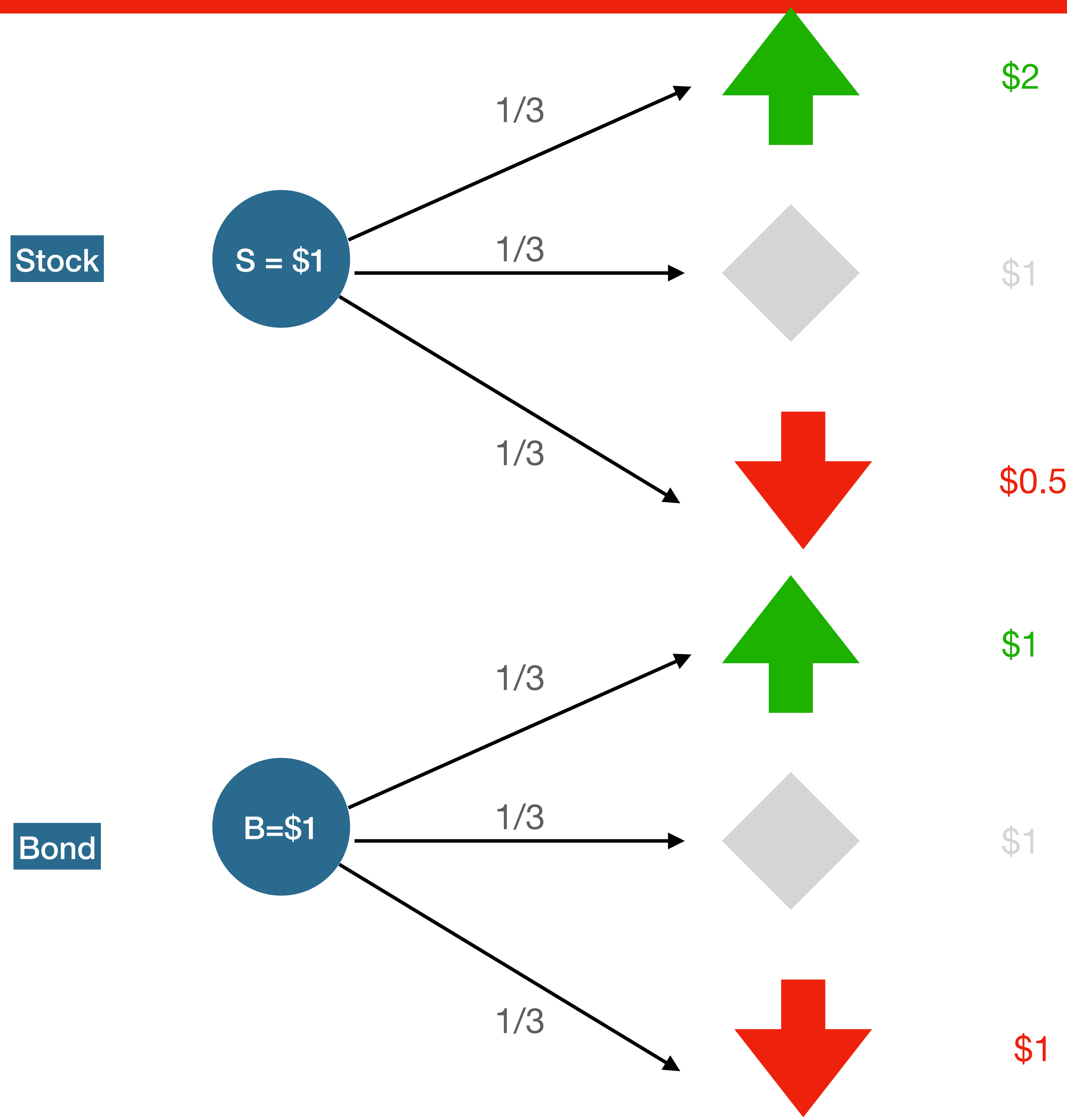
A stock option



Incomplete markets

- The term refers to a situation where we have more possible future scenarios and securities, …
- … and we can therefore **not always** find a replicating portfolio for all possible pay-offs
- The situation is analogous to a linear system with less equations than variables
- In this situation we will not always have a unique price.
- In the next episode, we will present a situation where market can be made complete introducing a single new security price
  - In practice, this is what **liquidity providers** do

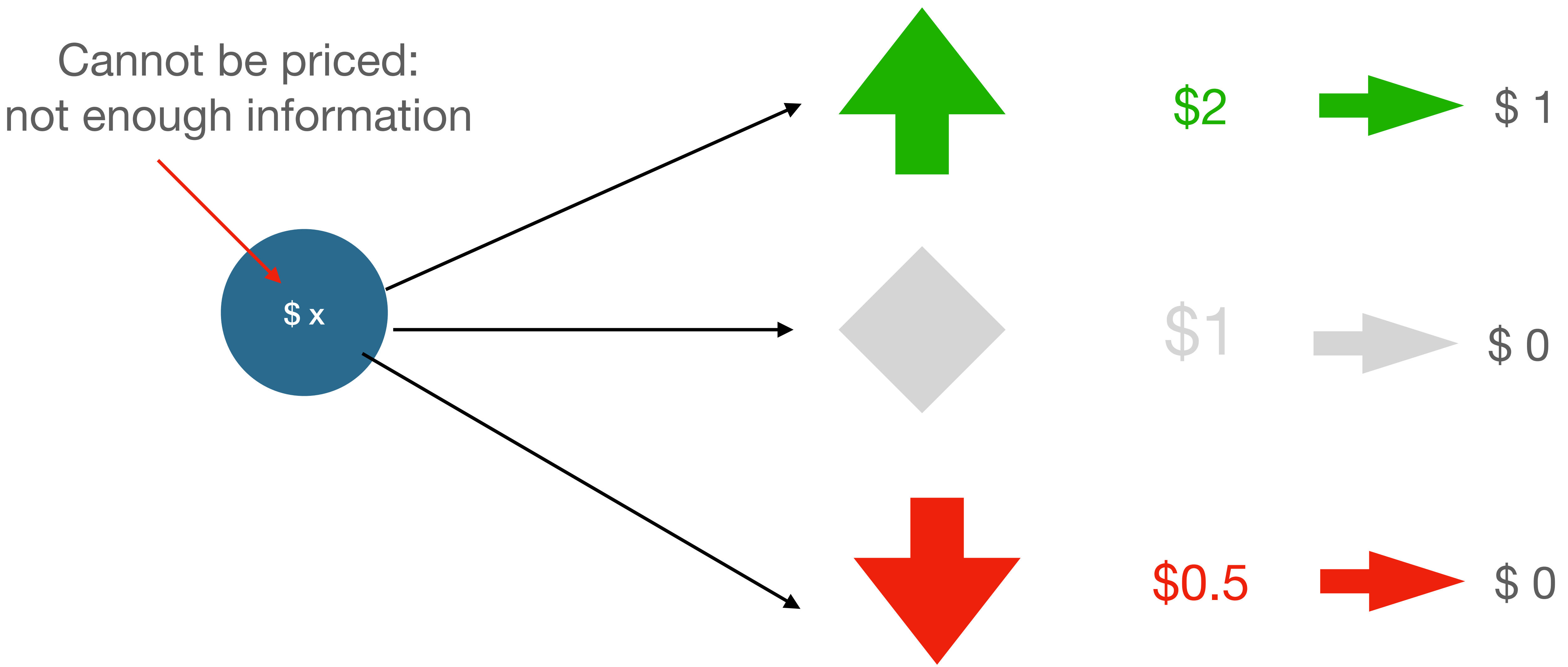
Two instruments:



Here, scenarios for bond prices correspond to the stock movements, the only source of randomness

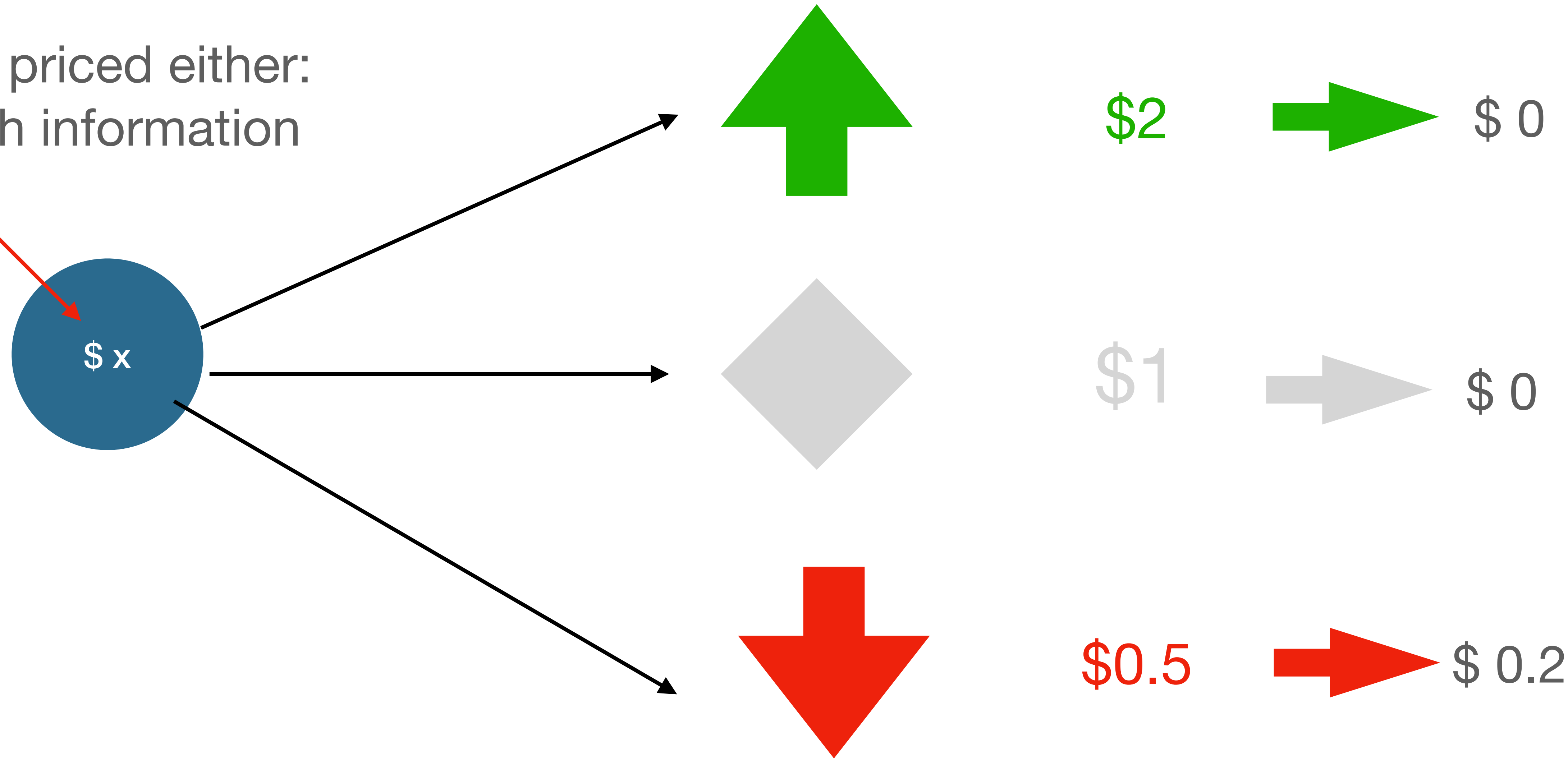
We assume price $1 (0% interest rates) but it can be an y other number

A stock option: O_1



Another stock option: O_2

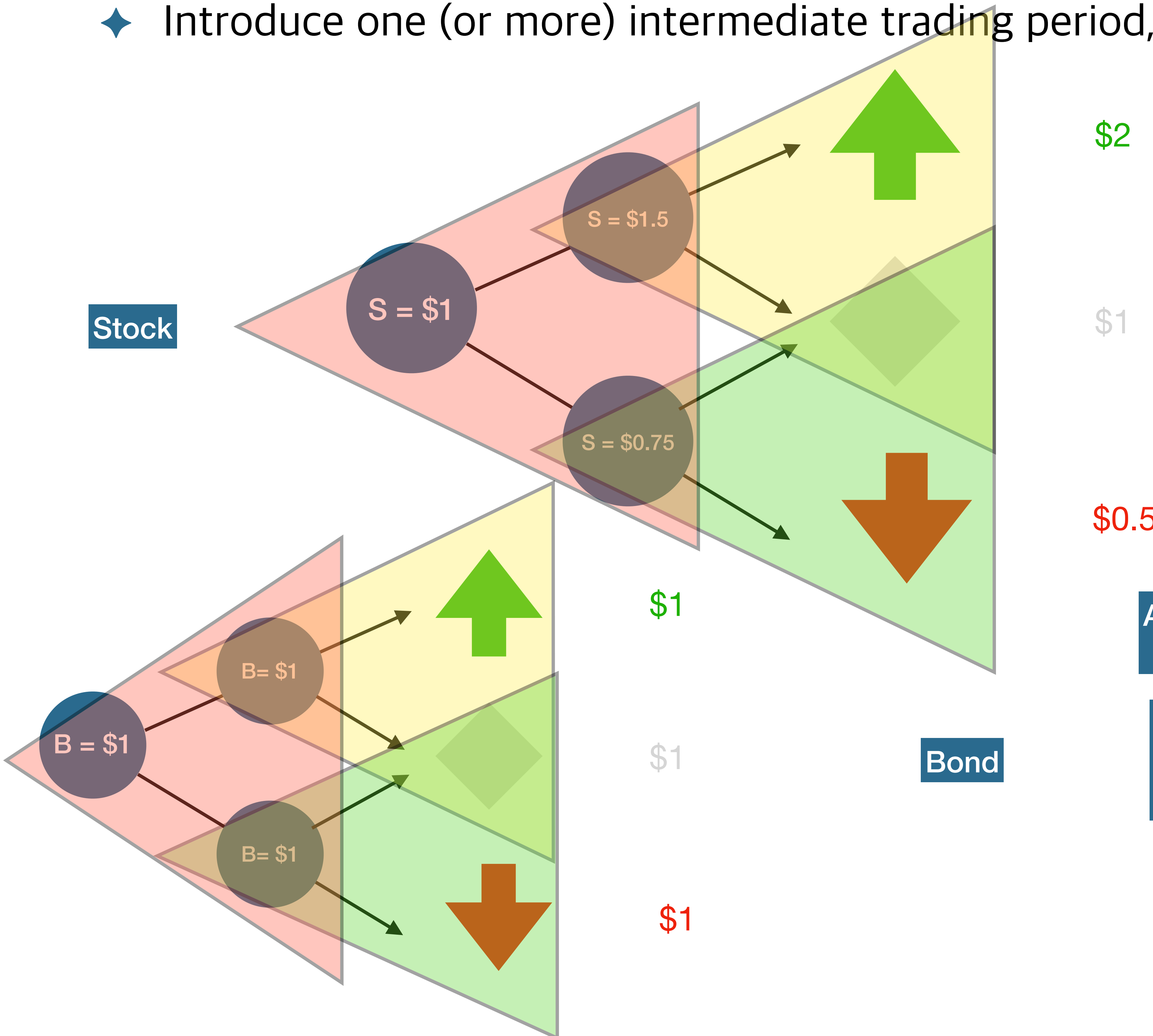
Cannot be priced either:
not enough information



We can produce many more options
without prices from replicating portfolios

First way to resolve the incompleteness

- Introduce one (or more) intermediate trading period, for example



As more trading days are allowed, we can model the market better

We could add “infinite numbers” of trading periods, resulting in “continuous time finance”

Second way to resolve the incompleteness

- If we obtain a single new value from the market···
- ··· for example, we find that option O_1 trades for \$1/3,
- then we can find the price of any other pay-off (option, or any other instrument).

Pricing theory

- Expanding the price trees that securities follow, leads to a complete market:
  - With unique prices
  - With replicating portfolio which need to be rebalanced at every step of the way
  - The result is the **dynamic asset pricing theory**
  - It can be extended to continuous time pricing
  - The Black-Scholes-Merton theory is a special case.
- Expanding the universe of security prices leads to
  - **Capital asset pricing model (CAPM)**
  - Prices are all reflections of each others'
- We will be studying each of these separately

The event space

- Consider a single period market, with possible events represented by a set

$$\Omega = \{\omega_1, \omega_2, \ldots, \omega_n\}$$

- Example 1: in our example from Episode 1, $\omega_1 =$ stock goes up, $\omega_2 =$ stock goes down.

- Example 2: in our example of incomplete markets, $\omega_1 =$ stock goes up, $\omega_2 =$ stock stays at \$1, $\omega_3 =$ stock goes down.

- If we had two stocks, each of which can go up or down, we could have $\omega_1 =$ both stocks up, $\omega_2 =$ both stocks go down, $\omega_3 =$ one goes up, the other goes down, $\omega_4 =$ one goes down, the other up.

- $\cdots$ and so on and so forth.

- Remember that, for a probability to be well constructed, we need the event space Ω to include all possible intersections and unions of its events (what is often called a $\sigma-$algebra), but we are not going to emphasize that.

The pay-off space

- If Ω represents the source of randomness, we can now consider the possible future prices as a random variable.
- In other words, if we are given securities $S_0, S_1, \ldots, S_m$, each will have a future value, or pay-off under each of our events ω_i, creating the pay-off matrix of all such values, organized into rows and columns:

$$p_{i,j}, \quad i = 1, \ldots, m, \quad j = 0, \ldots, n$$

- Here, S_0 is often considered to be the bond, i.e., the security that has the same values under all scenarios ω_i.
- The matrix $P = \{p_{i,j}\}$ is the pay-off matrix.
- We also consider the cost vector of all such securities

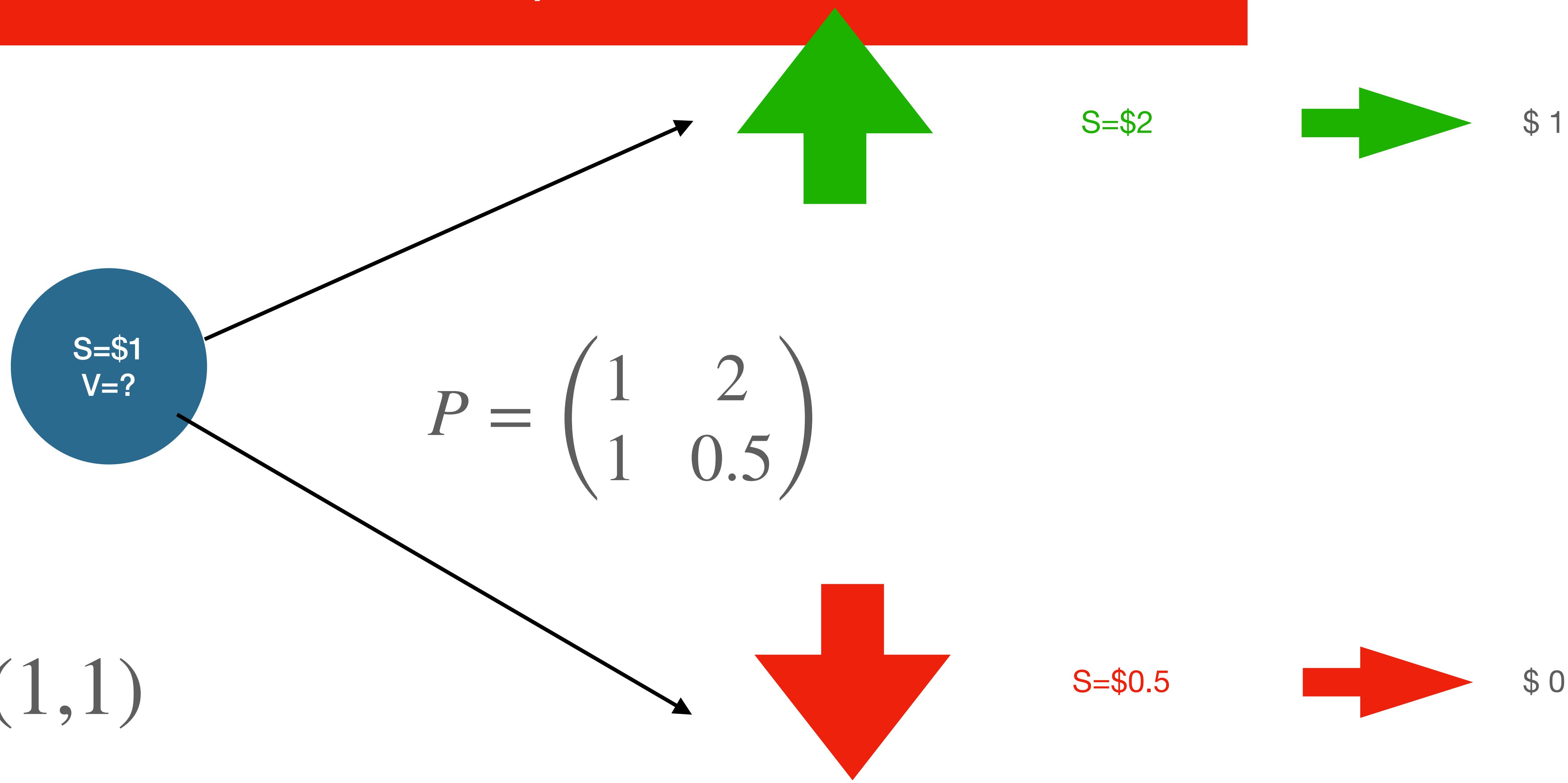
$$\vec{q} = (q_0, q_1, \ldots, q_m)$$

- representing the value today of all such securities.

Our set up assumes then:

- we have a set of financial instruments whose price is observed today and whose future price is known as a random value.
- We are assuming the number of events and the number of securities is finite, but this is not important: with enough math, it can be generalized to infinite probability spaces and infinite sets of securities, but we are not going to present that here.
- The pay-off matrix is P, and the cost vector is $\vec{q}$.
- Note that the financial instruments we can price span a vector space of pay-offs: all those pay-offs are linear combination of such instruments, therefore they can all be priced today.
- If there is a pay-off which is not in that linear span, it cannot be priced, and we have an incomplete market.

Example 1, with 0 interest rates

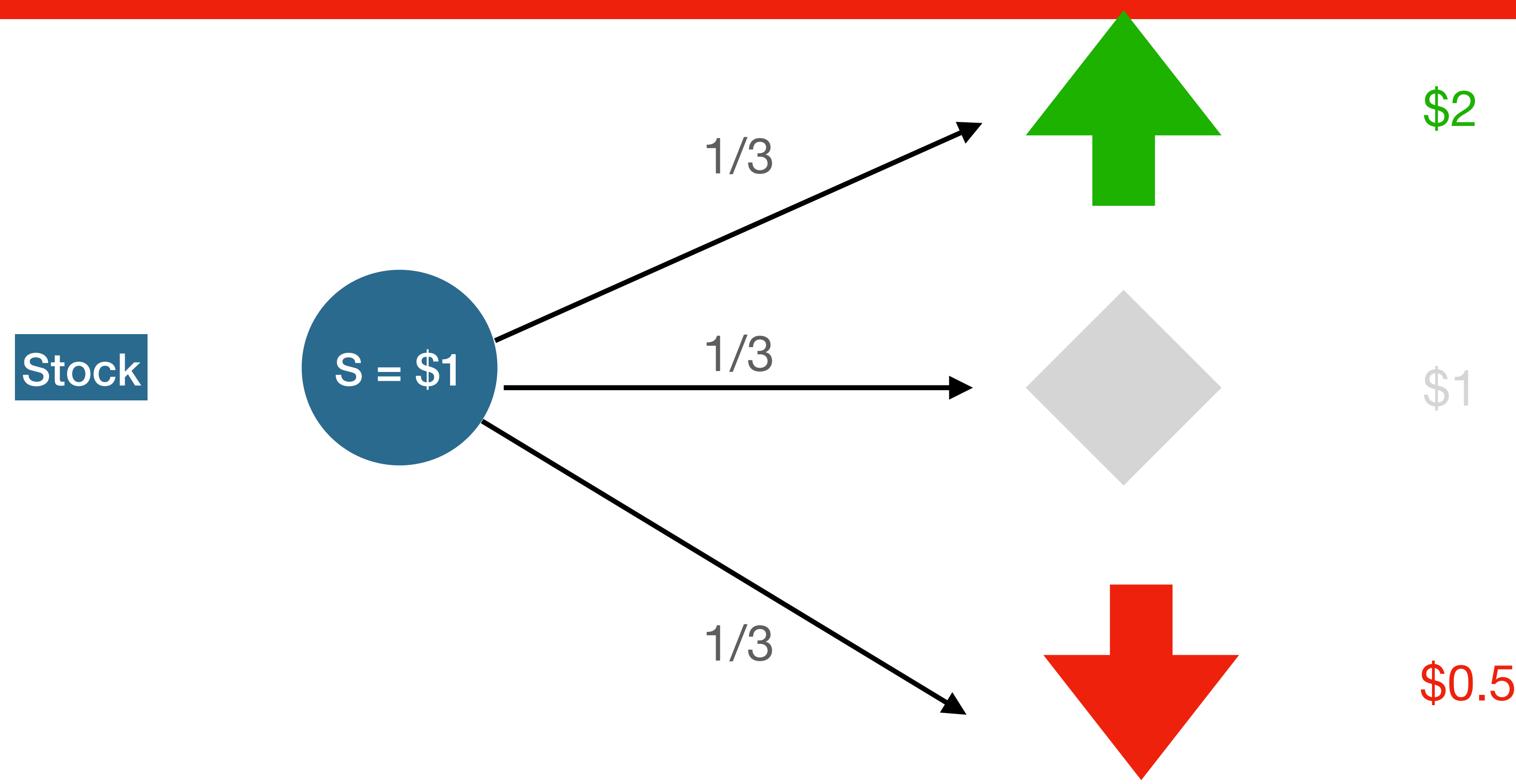


$$P = \begin{pmatrix} 1 & 2 \\ 1 & 0.5 \end{pmatrix}$$

$$\vec{q} = (1,1)$$

span $\begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 2 \\ 0.5 \end{pmatrix} = \mathbb{R}^2 \rightarrow$ complete market + replicating portfolio

Example 2.



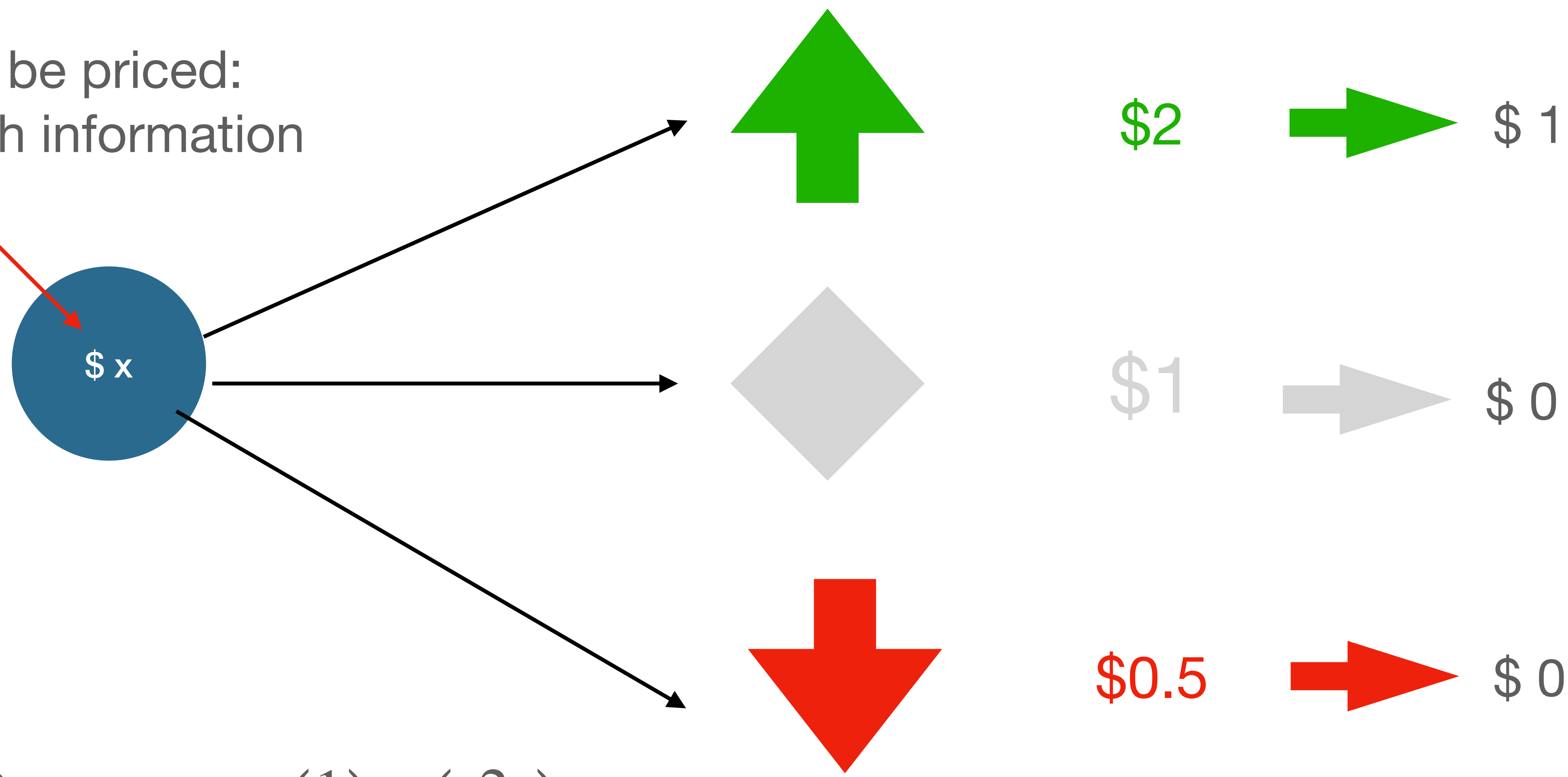
$$P = \begin{pmatrix} 1 & 2 \\ 1 & 1 \\ 1 & 0.5 \end{pmatrix}$$

$$\vec{q} = (B, 1)$$

B is given by interest rates
If rates are 0%, B=1

Example 2.

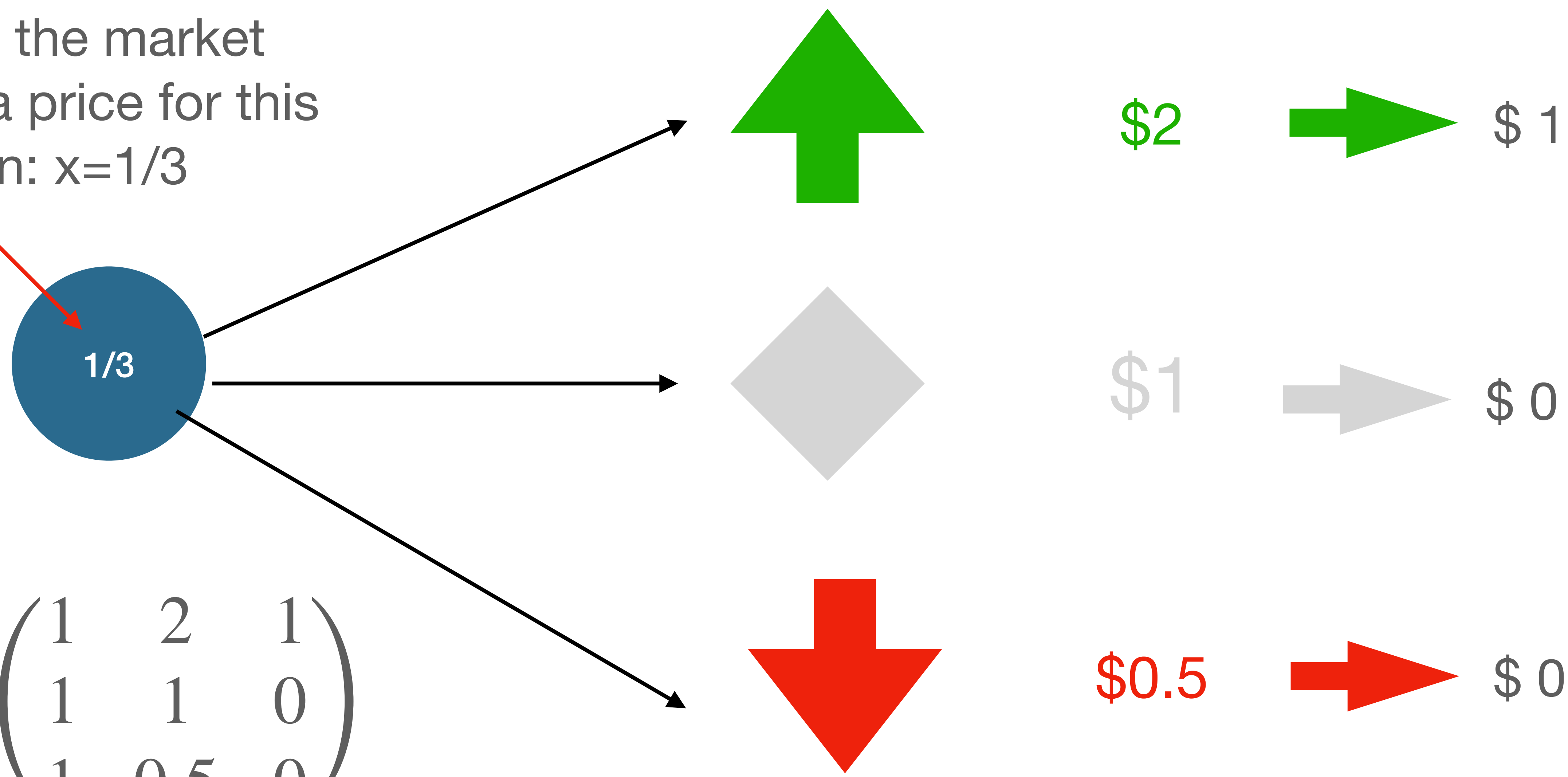
Cannot be priced:
not enough information



$$\begin{pmatrix}1\\0\\0\end{pmatrix} \notin \text{span}\begin{pmatrix}1\\1\\1\end{pmatrix}\begin{pmatrix}2\\1\\0.5\end{pmatrix}$$

Example 3 with 0 rates.

Assume the market publishes a price for this option: x=1/3

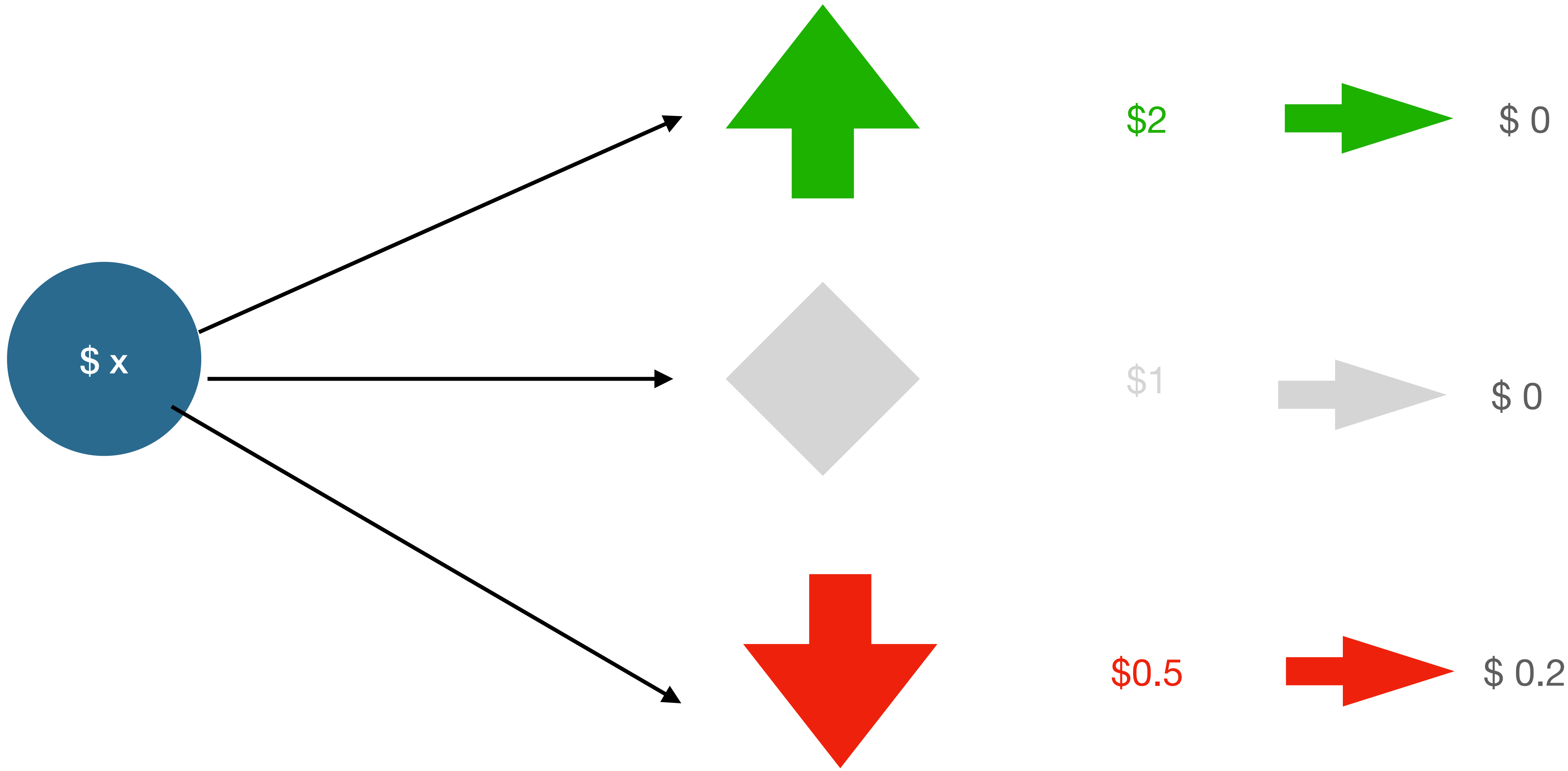


$$P = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 0 \\ 1 & 0.5 & 0 \end{pmatrix}$$

$$\vec{q} = (1,1,1/3)$$

market is augmented with the option value and becomes complete

Pricing another stock option: O_2



$$\text{span}\begin{pmatrix}1\\1\\1\end{pmatrix}\begin{pmatrix}2\\1\\0.5\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix} = \mathbb{R}^3$$

All pay-offs can be priced, and replicating portfolios exist

General method for pricing and replicating

In a market with priced instruments S_i, with pay-off matrix P and cost vector $\vec{q}$,

- a security with pay-off vector v will have a replicating portfolio

$$V = (v_1, \ldots, v_m)$$

when

$$V \cdot P = v$$

- Therefore, the replicating portfolio is

$$V = v \cdot P^{-1}$$

- and the price of the derivative with such pay-off is

$$\{\text{Price of } v\} = v \cdot P^{-1} \cdot q^T$$

Expected discounted pay-off

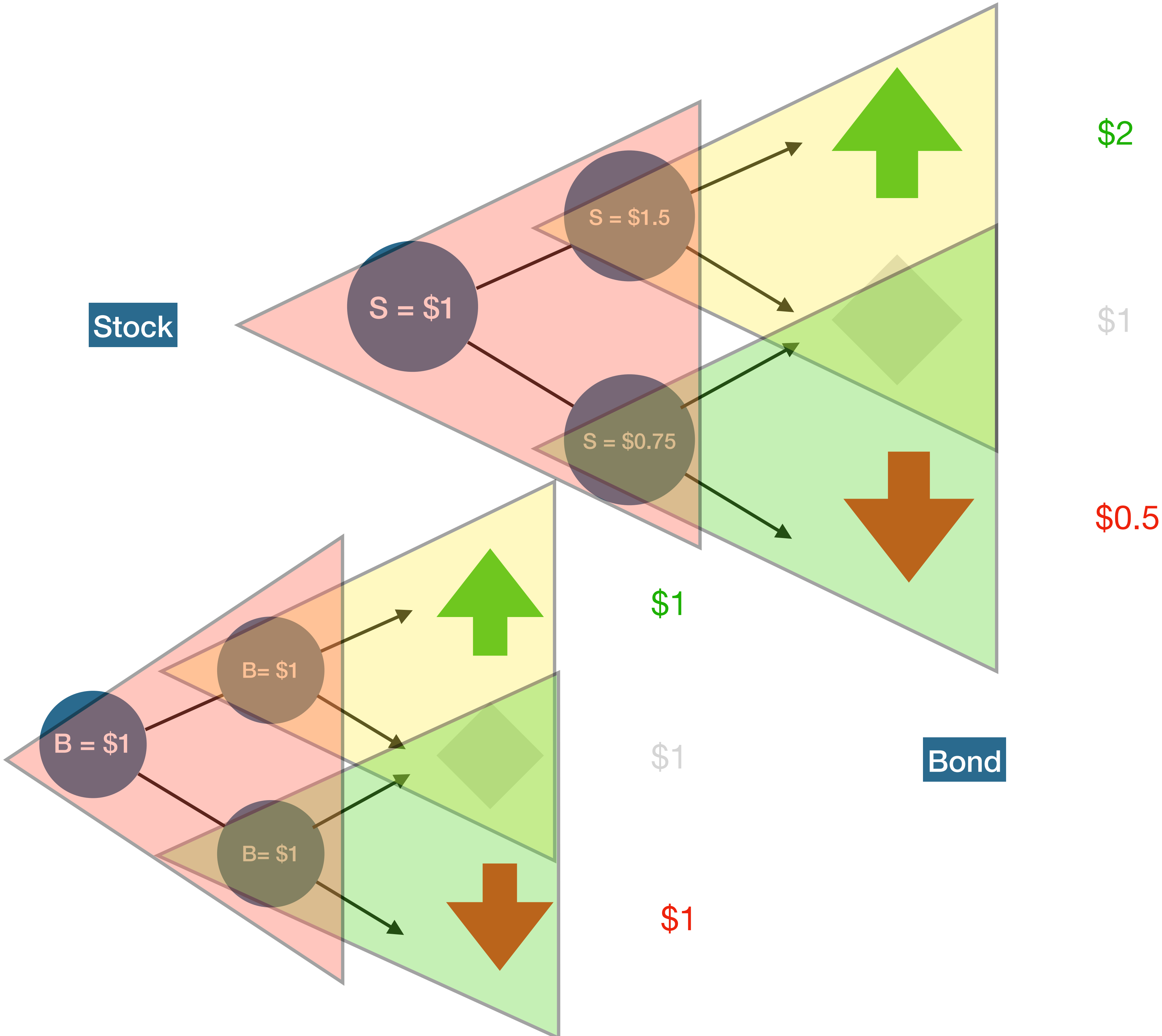
Price of v: $v \cdot \boxed{P^{-1} \cdot q^T}$

For some column vector $\begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix} = P^{-1} \cdot v^T$

- If $p_1 + \cdots + p_n = 1$ one could think they represent probabilities⋯
- ⋯then we could say the price of a derivative is the **expected pay-off**, under this measure.

- But typically they don't:
- if you consider the price of the bond ($v_i = 1$), ⋯
- ⋯we can see that $p_1 + \cdots + p_n = B$,
- For that reason, we call this expression **expected discounted pay-off.**

Market information

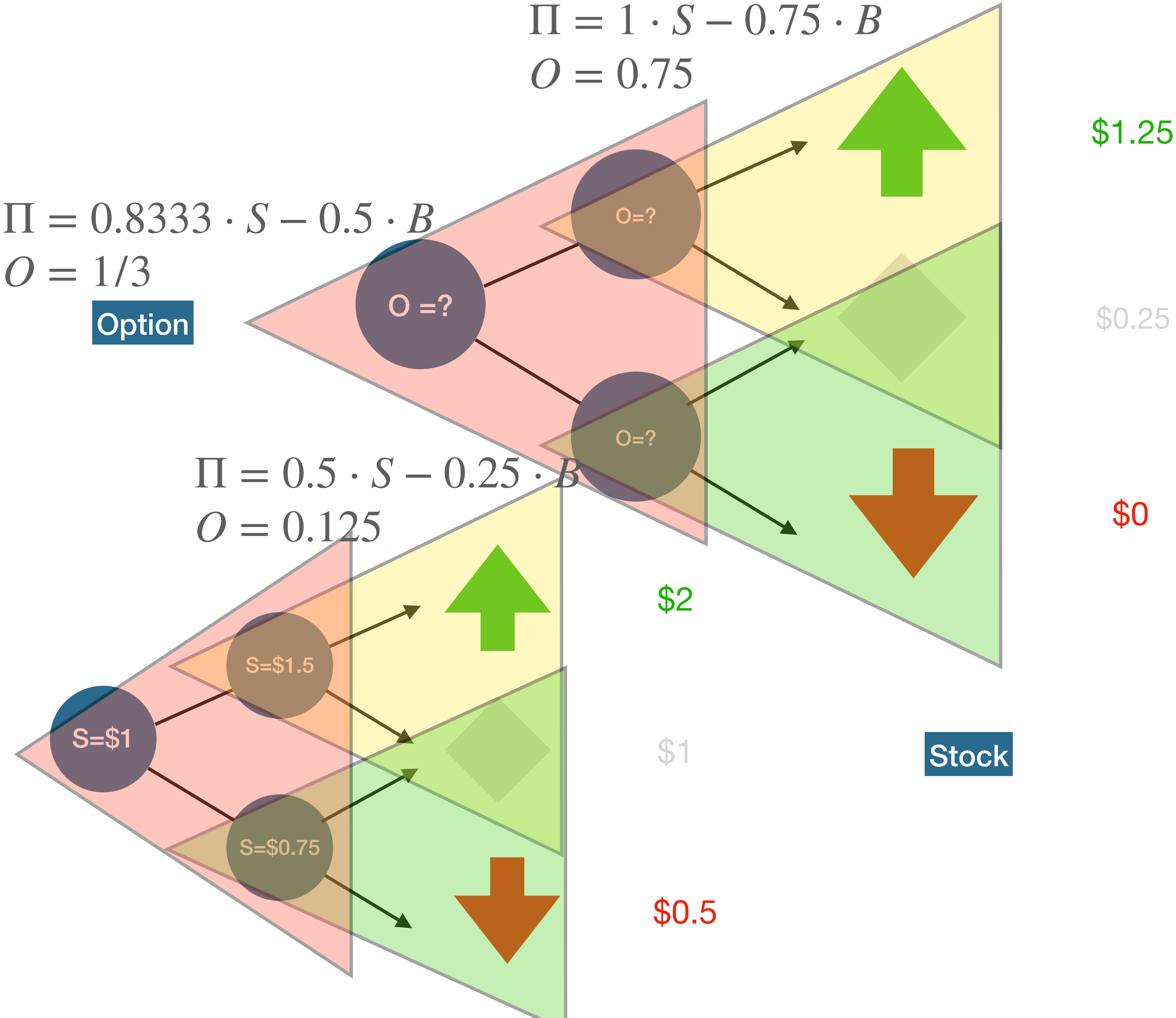


Option pricing

$$\Pi = 1 \cdot S - 0.75 \cdot B$$
$$O = 0.75$$

$$\Pi = 0.8333 \cdot S - 0.5 \cdot B$$
$$O = 1/3$$

$$\Pi = 0.5 \cdot S - 0.25 \cdot B$$
$$O = 0.125$$



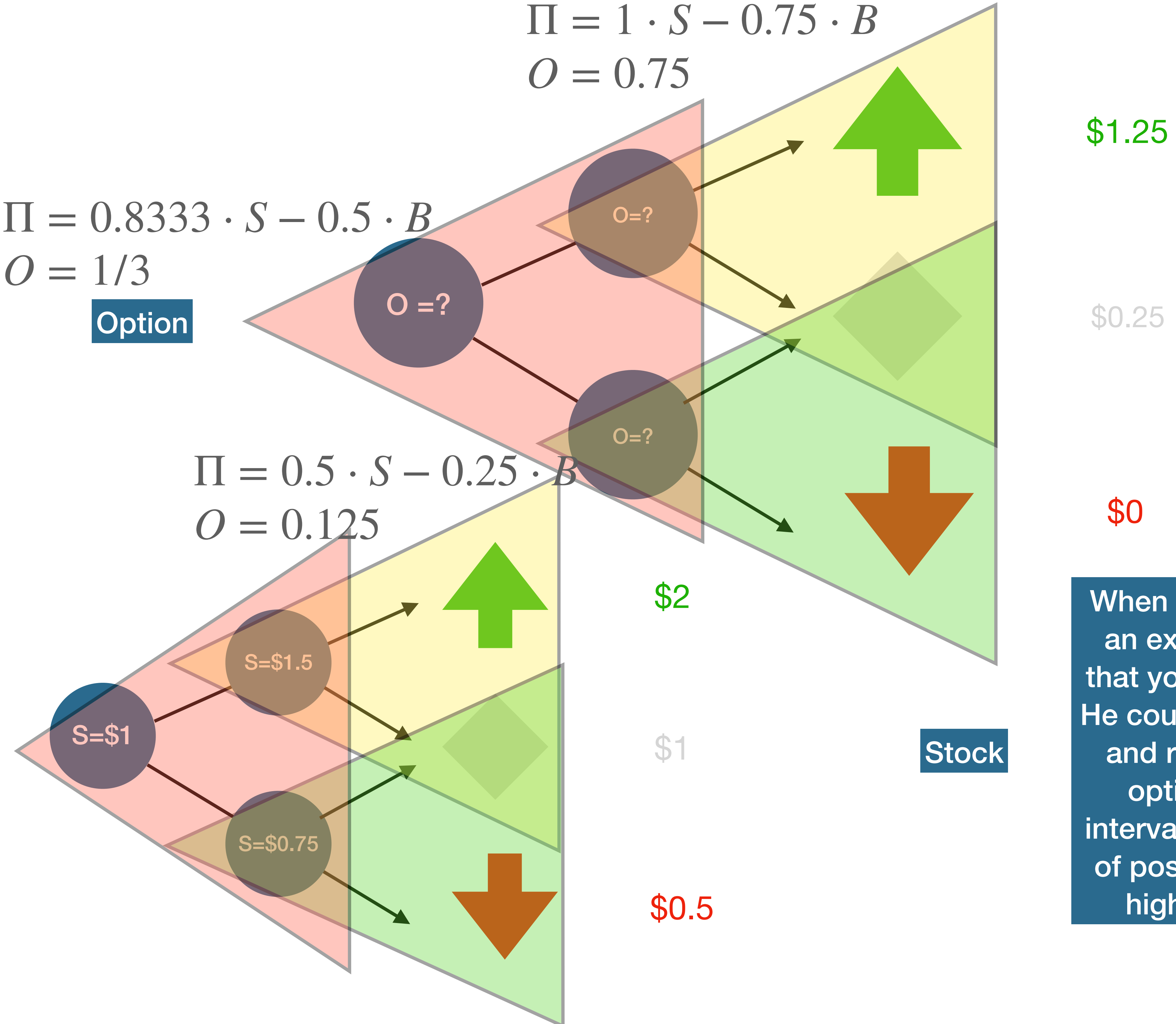
$$O = (S - 0.75)_+$$

The good student: a true story

$$\Pi = 1 \cdot S - 0.75 \cdot B$$
$$O = 0.75$$

$$\Pi = 0.8333 \cdot S - 0.5 \cdot B$$
$$O = 1/3$$

$$\Pi = 0.5 \cdot S - 0.25 \cdot B$$
$$O = 0.125$$



$$O = (S - 0.75)_+$$

When one of my students, who was an excellent programmer, learned that you can do multi period pricing. He could not wait for the next lecture and rushed to write code to price options at smaller and smaller intervals, with a larger and larger set of possible future prices, trading at higher and higher frequencies.

This is what he did:

- Codified each stock movement from S_t to the next period S_{t+1} considering percentage increments equal to $(1 \pm u)$,

$$S_{t+1} = S_t \cdot (1 \pm u)$$

- He knew that another possibility could have been to model the price tree differently

$$S_{t+1} \in \{(1+u) \cdot S_t \quad , \quad (1+u)^{-1} \cdot S_t\}$$

- but, being a good student, also knew that, as $u \to 0$, both are approximately the same; he was right.

- He did not care, in hist first continuous time self-driven exercise, to make u time dependent, or dependent on other things, such as stock price. He was also right in his choice.

- He also picked a one-year time horizon, to simplify his initial code. Another good choice.

- He decided to code his tree with a user defined number N, representing the number of trading periods in that one year. Good idea.

- As for u, he suspected that value would depend on the stock volatility, which he created as another user-defined parameter σ, so he coded $u = \sigma / N$.

Coding the option price

- For each branch, he has the risk neutral probability of up and down moves equal to 1/2, because

$$S_t = 0.5 \cdot S_t \cdot (1 + u) + 0.5 \cdot S_t \cdot (1 - u)$$

- Therefore, at each node, he can code the backward propagation as

$$O_t = 0.5 \cdot O_{t+1}^{\uparrow} + 0.5 \cdot O_{t+1}^{\downarrow}$$

- This recursive algorithm eventually gives the option price today O_0.

- The replicating portfolio can also be obtained easily.

A problem

- He write this program quickly, it is easy, and was excited to run it and see it converge as $N \to \infty$.
- Surprise: it either did not converge, or covered always to 0.
- Why?
- The central limit theorem holds the key to this behaviour.

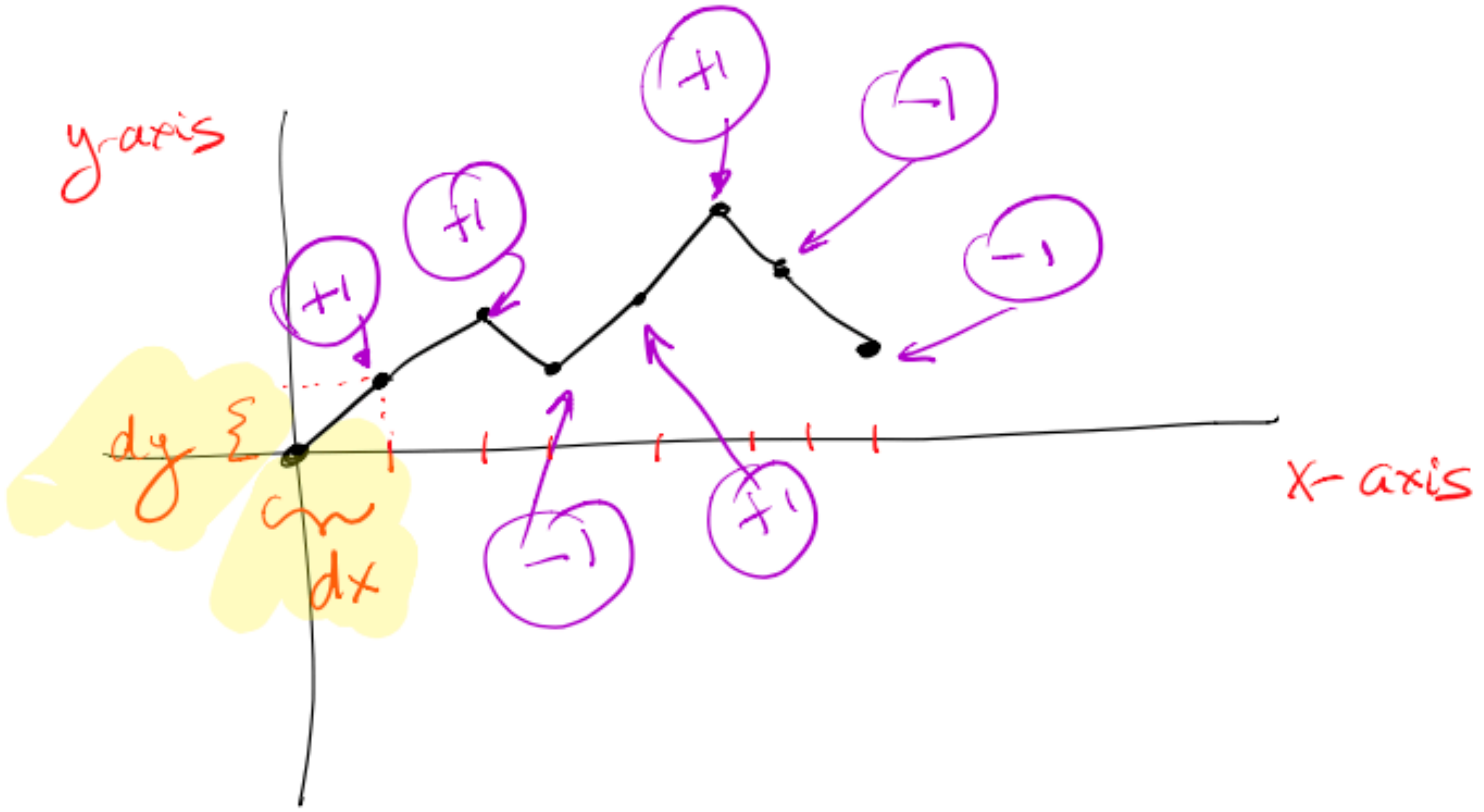
Random walks

- The passage to continuum of a discrete tree has an important dimensional constraint.
- Let's work with $x_t = \log S_t$, it will be easier to explain..
- Note that $x_{t+1} = x_t + X$, with X being a coin flip with outcome $\pm u$.
- Therefore, we have

$$x_t = x_0 + \sum_{i=1}^{t} X_i$$

- where the X_i is an i.i.d sequence of coin flips with mean 0 and std u.
- Therefore, by the Central Limit Theorem, x_t converges to a Gaussian distribution with mean x_0 and standard deviation $u \cdot \sqrt{N}$. **This is the problem**
- Because he had scaled $u \sim 1/N$, x_t does not converge, and his computer program gave unreal answers.
- The fix was simple, but deep: $u = \sigma/\sqrt{N}$.

Coin flips



Central limit theorem

If the random outcomes are linked to independent identically distributed random variables X_i, then the path location after n steps is equal to

$$s_n = \sum_{k=1}^{n} X_k$$

Our interest here is how S_n behaves as n grows, to determine the location of the path after many random steps are taken.

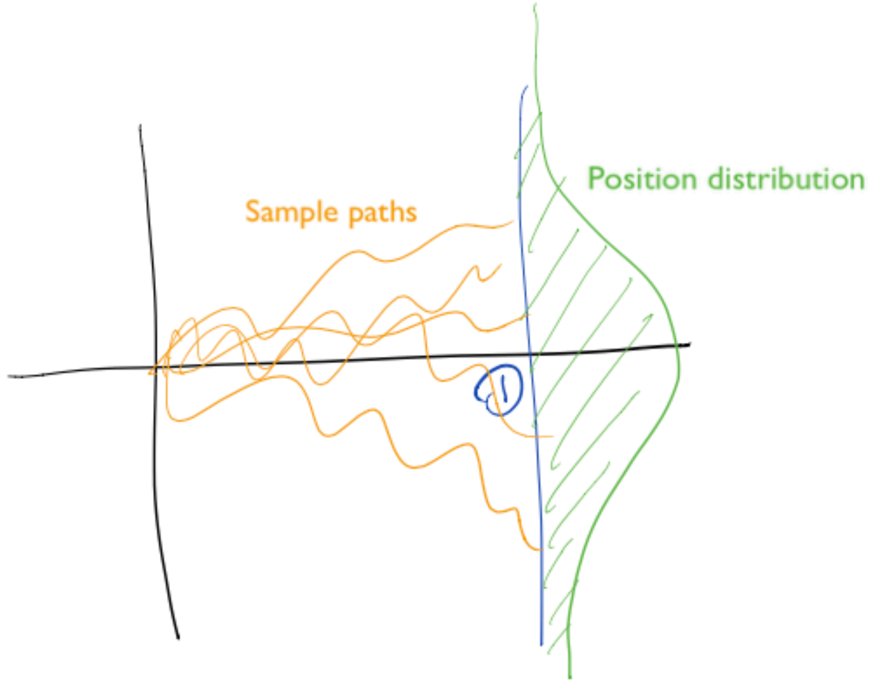
Assuming step increments in the x-axis equal to dx, and step increments in the y-axis equal to dy, the final location of the path is

$$(n \cdot dx \, , \, s_n \cdot dy)$$

Central Limit Theorem

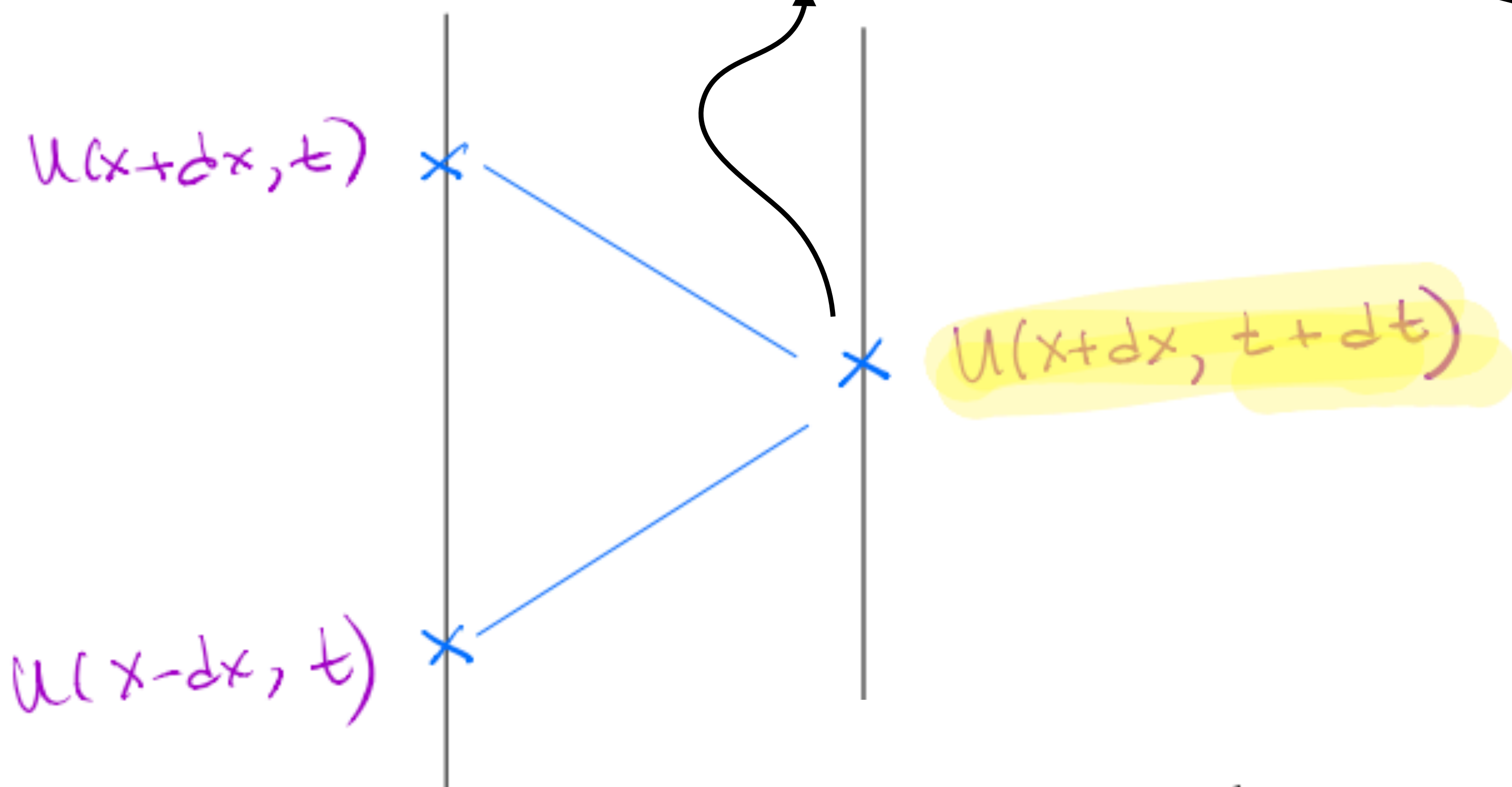
$$\lim_{n\to\infty} \mathrm{Prob}\{n^{-1/2}S_n \le \lambda\} \to \int_{-\infty}^{\lambda} e^{-x^2/2}\frac{dx}{\sqrt{2\pi}}$$

This is equivalent to the fact that, if **dx=1/n**, and **dy=1/√n**, then S_n converges to a normal (0,1) distribution on the vertical axis x=1.



Einstein’s theory

$$u(x\,,t+dt)=\frac{1}{2}[u(x+dx,t)\;+\;u(x-dx,t)]$$



$$u(x\,,t+dt)-u(x,t)=\frac{1}{2}[u(x+dx,t)\;+\;u(x-dx,t)-2u(x,t)]$$

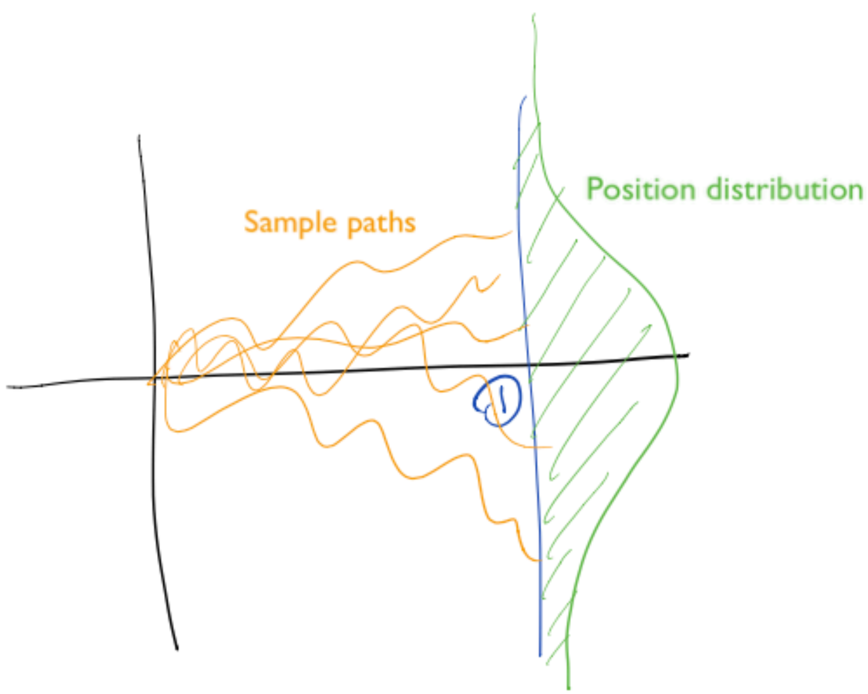
The Heat Equation

$$\frac{\partial u}{\partial t}\cdot dt=\frac{1}{2}\frac{\partial^2 u}{\partial x^2}\cdot dx^2$$

Brownian motion

Brownian motion moves up or down with probability 0.5, by an amount of $\sqrt{dt}$:

$$dW_t = \pm\sqrt{dt}, \qquad \mathbb{E}(dW_t) = 0.$$



It is distributed at time t according to

$$P(x,t) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{x^2}{2t}\right).$$

Ito’s processes

A process of the type

$$X_t = X_0 + \int_0^t \sigma_s \, dW_s + \int_0^t \mu_s \, ds$$

will be written down as

$$dX_t = \sigma_t \, dW_t \; + \; \mu_t \, dt$$

SDE appear when the terms σ and μ above are made X dependent, as

$$dX_t = \sigma(X_t, t) \, dW_t + \mu(X_t, t) \, dt.$$

They are also called Ito processes.

Ito's lemma

Ito's Lemma is the Chain Rule for stochastic processes

$$dX_t = \sigma_t \, dW_t \; + \; \mu_t \, dW_t$$

$$df(X_t, t) = \frac{\partial_x f}{\partial x} \underbrace{\cdot \, dX_t \; + \; \frac{\partial_t f}{\partial t} \cdot dt}_{\text{classic}} \; + \; \underbrace{\frac{1}{2}\sigma_t^2 \, dt}_{\text{stochastic}} \cdot \partial_x^2 f$$

$$\frac{dS_t}{S_t} = \mu \cdot dt + \sigma \cdot dW_t$$

stochastic model for stocks

Random walks

- The passage to continuum of a discrete tree has an important dimensional constraint.
- If we define a **random walk** as $x_{t+1} = x_t + X$, with X being a coin flip with outcome $\pm u$, we have

$$x_t = x_0 + \sum_{i=1}^{t} X_i$$

- where the X_i is an i.i.d sequence of coin flips with mean 0 and std u.
- Therefore, by the Central Limit Theorem, x_t converges to a Gaussian distribution with mean x_0 and standard deviation $u \cdot \sqrt{N}$.
- If we keep our time window fixed, we need: $u = \sigma/\sqrt{N}$ **and** $t = 1/N$.
- Therefore, we if we have the fundamental relationship

$$dx = \sigma\sqrt{dt}$$

Ito processes

- An Ito process is a general random walk of the form

$$dx_t = \mu\, dt + \sigma\, dW_t$$

- where the brownian process dW_t is an infinitesimal coin flip as we saw earlier, giving values $\pm\sqrt{dt}$ each with probability 1/2,

- and the drift term μdt adds an infinitesimal deterministic direction to the walk at each step.

- Ito processes can be very complex, with both μ and σ taking functional forms, but we are ignoring this here at this time.

- Ito processes describe price evolution of securities, but also many other random processes in nature.

The stock price process

- For stocks and many other securities, with prices given by S_t, we establish the model

$$dx_t = \mu \, dt + \sigma \, dW_t \, . \qquad x_t = \log S_t$$

- This gives us a price process for S_t which is realistic in the following ways:
    - Is always positive: $S_t \geq 0$
    - Makes the changes in price of S_t proportional to the value of S_t itself

	Fund balance	Simple Return	Logarithmic return
January	$1		0.00%
February	$2	100%	69.31%
March	$1	-50%	-69.31%
April	$2	100%	69.31%
May	$1	-50%	-69.31%
June	$2	100%	69.31%
July	$1	-50%	-69.31%
August	$2	100%	69.31%
September	$1	-50%	-69.31%
October	$2	100%	69.31%
November	$1	-50%	-69.31%
December	$2	100%	69.31%
Average return		25%	0
Standard Deviation		75%	70%

Why that big a discrepancy?

The stock price process

- For stocks and many other securities, with prices given by S_t, we establish the model

$$dx_t = \mu\, dt + \sigma\, dW_t\,. \qquad x_t = \log S_t$$

- Question: is S_t an Ito process itself?
  - If so, which one is it?
- More general question:
  - Is $F(x_t, t)$ an Ito process for reasonable functions F?
  - and if so, which one is it?
- Note that the general question answers our specific one for stock prices setting $F(x_t, t) = e^{x_t}$.

The stochastic chain rule

- With our basic Ito process given by

$$dx_t = \mu\, dt + \sigma\, dW_t$$

- an application of the chain rule yields:

$$\begin{aligned} dF(x_t, t) &= \frac{\partial F}{\partial x} \cdot dx_t + \frac{\partial F}{\partial t} \cdot dt + \frac{1}{2} \frac{\partial^2 F}{\partial x^2} \cdot |\, dx_t|^2 \\ &= \left(\frac{\partial F}{\partial x} \cdot \mu + \frac{\partial F}{\partial t} + \frac{\sigma^2}{2} \cdot \frac{\partial^2 F}{\partial x^2} \right) dt + \sigma \cdot \frac{\partial F}{\partial x} \cdot dW_t \end{aligned}$$

- The term in red are stochastic in nature, and arise from the fact that

$$|\, dx_t|^2 \sim dt$$

The log price process

- With our stock price process

$$dx_t = \mu\, dt + \sigma\, dW_t. \qquad x_t = \log S_t$$

- We apply Ito's lemma with $S_t = F(x_t, t) = e^{x_t}$ to get

$$dS_t = \left(e^{x_t} \cdot \mu + \frac{\sigma^2}{2} \cdot e^{x_t} \right) + e^{x_t} \cdot \sigma \cdot dW_t$$

$$= S_t \left(\mu + \frac{\sigma^2}{2} \right) + \sigma \cdot S_t \cdot dW_t$$

- This gives us the **stock price process**

$$\frac{dS_t}{S_t} = \left(\mu + \frac{\sigma^2}{2} \right) dt + \sigma \cdot dW_t$$

Log-returns

- Recall our example

- it shows a remarkable difference between
  - the returns of a fund and
  - the log-return of the same fund

	Fund balance	Simple Return	Logarithmic return
January	\$1		0.00%
February	\$2	100%	69.31%
March	\$1	-50%	-69.31%
April	\$2	100%	69.31%
May	\$1	-50%	-69.31%
June	\$2	100%	69.31%
July	\$1	-50%	-69.31%
August	\$2	100%	69.31%
September	\$1	-50%	-69.31%
October	\$2	100%	69.31%
November	\$1	-50%	-69.31%
December	\$2	100%	69.31%
Average return		25%	0
Standard Deviation		75%	70%

- This is due to the fact that

$$d(\log S_t) = \frac{dS_t}{S_t} - \frac{\sigma^2}{2}$$

- In the example, $\sigma = 0.75$, so $\frac{\sigma^2}{2} = 28.1$

- Since the returns are not gaussian, the prediction of Ito's lemma is impressive.

Financial "false friends"

- The world of quantitative finance is full of false friends
  - .. things that look correct but are in fact misleading.
- The power of mathematics is..
  - .. not so much with the ability to discover new things..
  - .. as it is with avoiding false leads.

FIT

Köszönöm תודה Спасибі

Thanks ありがとう

εὐχαριστώ 감사해요 Gracias cảm ơn

Kösz Dankon

Teşekkürler Danke متشكرم

Merci Grazie

tack

rahmat Takk

谢谢

ਧੰਨਵਾਦ

მადლობა

Gratias dzięki

спасибо धन्यवादा ขอบใจ

شكرًا asante dankie շնորհակալություն