The Black Scholes theory

Recap of Random walks

- The passage to continuum of a discrete tree has an important dimensional constraint.
- If we define a **random walk** as $x_{t+1} = x_t + X$, with X being a coin flip with outcome $\pm u$, we have

$$x_t = x_0 + \sum_{i=1}^{t} X_i$$

- where the X_i is an i.i.d sequence of coin flips with mean 0 and std u.
- Therefore, by the Central Limit Theorem, x_t converges to a Gaussian distribution with mean x_0 and standard deviation $u \cdot \sqrt{N}$.
- If we keep our time window fixed, we need: $u = \sigma/\sqrt{N}$ **and** $t = 1/N$.
- Therefore, we if we have the fundamental relationship

$$dx = \sigma\sqrt{dt}$$

Ito processes

- An Ito process is a general random walk of the form

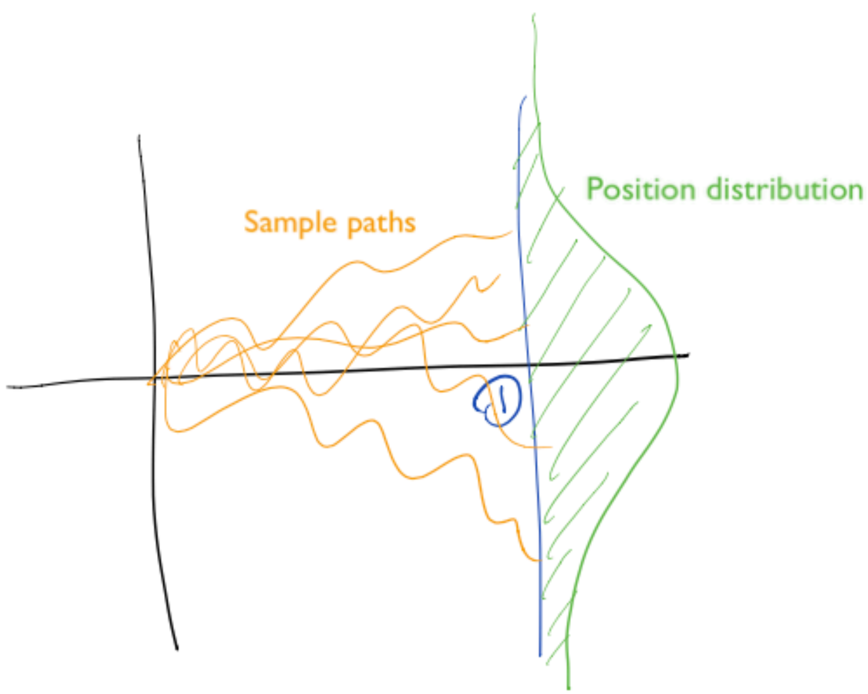
$$dx_t = \mu\, dt + \sigma\, dW_t$$

- where the brownian process dW_t is an infinitesimal coin flip as we saw earlier, giving values $\pm\sqrt{dt}$ each with probability 1/2,
- and the drift term μdt adds an infinitesimal deterministic direction to the walk at each step.
- Ito processes can be very complex, with both μ and σ taking functional forms, but we are ignoring this here at this time.
- Ito processes describe price evolution of securities, but also many other random processes in nature.

Stochastic calculus

Brownian motion moves up or down with probability 0.5, by an amount of $\sqrt{dt}$:

$$dW_t = \pm\sqrt{dt}, \qquad \mathbb{E}(dW_t) = 0.$$



It is distributed at time t according to

$$P(x,t) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{x^2}{2t}\right).$$

The stock price process

- For stocks and many other securities, with prices given by S_t, we establish the model

$$dx_t = \mu \, dt + \sigma \, dW_t \, . \qquad x_t = \log S_t$$

- This gives us a price process for S_t which is realistic in the following ways:
  - Is always positive: $S_t \geq 0$
  - Makes the changes in price of S_t proportional to the value of S_t itself

The stock price process

- For stocks and many other securities, with prices given by S_t, we establish the model

$$dx_t = \mu \, dt + \sigma \, dW_t \, . \qquad x_t = \log S_t$$

- Question: is S_t an Ito process itself?
  - If so, which one is it?
- More general question:
  - Is $F(x_t, t)$ an Ito process for reasonable functions F?
  - and if so, which one is it?
- Note that the general question answers our specific one for stock prices setting $F(x_t, t) = e^{x_t}$.

The stochastic chain rule: Ito's Lemma

- With our basic Ito process given by

$$dx_t = \mu \, dt + \sigma \, dW_t$$

- an application of the chain rule yields:

$$\begin{aligned} dF(x_t, t) &= \frac{\partial F}{\partial x} \cdot dx_t + \frac{\partial F}{\partial t} \cdot dt + \frac{1}{2} \frac{\partial^2 F}{\partial x^2} \cdot |dx_t|^2 \\ &= \left(\frac{\partial F}{\partial x} \cdot \mu + \frac{\partial F}{\partial t} + \frac{\sigma^2}{2} \cdot \frac{\partial^2 F}{\partial x^2} \right) dt + \sigma \cdot \frac{\partial F}{\partial x} \cdot dW_t \end{aligned}$$

- The term in red are stochastic in nature, and arise from the fact that

$$|dx_t|^2 \sim dt$$

The log price process

- With our stock price process

$$dx_t = \mu\, dt + \sigma\, dW_t . \qquad x_t = \log S_t$$

  - We apply Ito's lemma with $S_t = F(x_t, t) = e^{x_t}$ to get

$$dS_t = \left(e^{x_t} \cdot \mu + \frac{\sigma^2}{2} \cdot e^{x_t} \right) dt + e^{x_t} \cdot \sigma \cdot dW_t$$

$$= S_t \left(\mu + \frac{\sigma^2}{2} \right) dt + \sigma \cdot S_t \cdot dW_t$$

- This gives us the **stock price process**

$$\frac{dS_t}{S_t} = \left(\mu + \frac{\sigma^2}{2} \right) dt + \sigma \cdot dW_t$$

Log-returns

- Recall our example
  - it shows a remarkable difference between
    - the returns of a fund and
    - the log-return of the same fund
- This is due to the fact that

$$d(\log S_t) = \frac{dS_t}{S_t} - \frac{\sigma^2}{2}$$

- In the example, $\sigma = 0.70$, so $\frac{\sigma^2}{2} = 24.5\,\%$.
- Since the returns are not gaussian, the prediction of Ito's lemma is impressive.

	Fund balance	Simple Return	Logarithmic return
January	$1		0.00%
February	$2	100%	69.31%
March	$1	-50%	-69.31%
April	$2	100%	69.31%
May	$1	-50%	-69.31%
June	$2	100%	69.31%
July	$1	-50%	-69.31%
August	$2	100%	69.31%
September	$1	-50%	-69.31%
October	$2	100%	69.31%
November	$1	-50%	-69.31%
December	$2	100%	69.31%
Average return		25%	0
Standard Deviation		75%	70%

Financial “false friends”

- The world of quantitative finance is full of false friends
  - .. things that look correct but are in fact misleading.
- The power of mathematics is..
  - .. not so much with the ability to discover new things..
  - .. as it is with avoiding false leads.

The Black Scholes theory

The Black-Scholes Theory

Assume a derivative (i.e. European option) on stock with Ito process

$$\frac{dS_t}{S_t} = \mu \cdot dt + \sigma \cdot dW_t$$

and payoff $f_T(S)$, at maturity time T.

We will find a price function $f(S, t)$ and a trading strategy holding

- $a(S, t)$ stocks at time t,
- $b(S, t)$ bonds at time t

such that

- $f(S, t) = a(S, t) \cdot S_t + b(S, t) \cdot B_t$
- $f(S, T) = f_T(S)$

The financial argument

Assume the price function exists. Consider the choice of

$$a(S,t) = \partial_S f(S,t)$$

Then, the portfolio $\Pi = f - a \cdot S \; - \; b \cdot B$ evolves as

=0; self-financing condition

$$d_t\Pi = d_t f - d_t S \cdot a - b \cdot dB_t - (d_t a \cdot S + d_t b \cdot B)$$

By Ito's Lemma

$$= \left(\partial_t f + \frac{1}{2}\sigma^2 S^2 \partial_S^2 f\right) dt - b \cdot r \cdot B\, dt + \underbrace{\partial_S f \cdot d_t S \; - \; a \cdot d_t S}_{=\,0 \text{ by construction of } a}$$

$$= \left(\partial_t f + \frac{1}{2}\sigma^2 S^2 \partial_S^2 f - brB\right) dt$$

This shows that the portfolio Π is a bond, and therefore

$$d_t\Pi = r \cdot \Pi\, dt$$

The mathematical argument

Replacing the stock part of the replicating portfolio

$$a(S,t) = -\partial_S f(S,t)$$

into the portfolio $\Pi = f - a \cdot S - b \cdot B$, the Black-Scholes equation becomes

$$r \cdot f - r \cdot \partial_S f \cdot S = \partial_t f + \frac{1}{2}\sigma^2 S^2 \partial_S^2 f$$

We rewrite it to resemble a diffusion equation as

$$\begin{cases} \partial_t f & = r \cdot f - r \cdot \partial_S f \cdot S - \frac{1}{2}\sigma^2 S^2 \partial_S^2 f \\ f(S,T) & = f_T(S) \end{cases}$$

Backward diffusion

Terminal condition

The Black-Scholes price

- The solution to the Black-Scholes equation can be seen to be given by

Discounted... ...pay-off

$$f(S,t) \;=\; \underbrace{e^{-r(T-t)}}_{\text{Discounted...}} \int_{-\infty}^{\infty} f_0\left(\underbrace{S_0\, e^{(r-\frac{\sigma^2}{2})(T-t)\,+\,x}}_{\text{Log-normal price}}\right) P_\sigma(x, T-t)\, dx$$

Log-normal price

...expected...

- with the Gaussian probability density

$$P_\sigma(x,t) \;=\; \frac{1}{\sqrt{2\pi t\sigma^2}}\,\exp\left(-\frac{x^2}{2t\sigma^2}\right).$$

The Black-Scholes formula

The price of a call option with strike K, current price S_0 and volatility σ, at time t is given by

$$V(t, K, \sigma, r) = S_0 \cdot N(d_1) - K \cdot e^{-r(T-t)} N(d_2)$$

with the cumulative distribution of the gaussian

$$N(d) = \int_{-\infty}^{d} e^{-x^2/2} \frac{dx}{\sqrt{2\pi}},$$

and

$$d_1 = \frac{\ln(S_0/K) + (r + \frac{1}{2}\sigma^2)\,(T-t)}{\sigma\sqrt{T-t}},$$

$$d_2 = \frac{\ln(S_0/K) + (r - \frac{1}{2}\sigma^2)\,(T-t)}{\sigma\sqrt{T-t}}.$$

European options

European Options expire at a preset future time.

Their pay-off $P(S, T)$ depends on the price of the underlying S_T at expiration and a strike price K

$$P(S_T) = (S_T - K)_+.$$

Put options have pay-off given by

$$P(S_T) = (K - S_T)_+ .$$

American options

American Options can be exercised at any time prior to expiration T.

Their pay-off $P(S,t)$ depends on the price of the underlying S_T at expiration and a strike price K, but can happen at any time $t \leq T$.

$$P(S_t, t) = (S_t - K)_+.$$

Put options have pay-off given by

$$P(S_t, t) = (K - S_t)_+ .$$

SPY Options

February 2024

CALLS						Expires Feb 16, 2024	PUTS					
LAST	CHG	BID	ASK	VOL	OPEN INT.	STRIKE	LAST	CHG	BID	ASK	VOL	OPEN INT.
290.53	7.26	291.08	291.43	1	374	210.00	0.01	0.00	0.00	0.01	1	10,039
258.22	0.00	286.13	286.42	0	1	215.00	0.01	0.00	0.00	0.01	0	4,715
257.36	0.00	281.10	281.43	0	10	220.00	0.01	0.00	0.00	0.01	0	6,923
0.00	0.00	276.11	276.44	0		225.00	0.01	0.00	0.00	0.01	0	773
221.54	0.00	271.11	271.45	0	1	230.00	0.01	0.00	0.00	0.01	0	5,692
238.22	0.00	266.16	266.45	0	1	235.00	0.01	0.00	0.00	0.01	0	2,230
0.00	0.00	261.17	261.46	0		240.00	0.01	0.00	0.00	0.01	0	6,211
241.24	0.00	256.14	256.46	0	1	245.00	0.01	0.00	0.00	0.01	0	11,208
236.26	0.00	251.15	251.48	0	7	250.00	0.01	0.00	0.00	0.01	10	14,128
205.07	0.00	246.15	246.49	0		255.00	0.01	0.00	0.00	0.01	0	2,082
226.33	0.00	241.20	241.48	0	11	260.00	0.01	0.00	0.00	0.01	0	9,402
0.00	0.00	236.17	236.50	0		265.00	0.01	0.00	0.00	0.01	184	13,417
203.60	0.00	231.18	231.51	0	16	270.00	0.01	0.00	0.00	0.01	0	8,858
197.79	0.00	226.18	226.51	0	25	275.00	0.01	0.00	0.00	0.01	0	1,876
198.38	0.00	221.19	221.52	0	5	280.00	0.01	0.00	0.00	0.01	0	12,617
204.64	0.00	216.20	216.53	0	1	285.00	0.01	0.00	0.00	0.01	0	14,119
183.32	0.00	211.21	211.53	0	4	290.00	0.01	0.00	0.00	0.01	0	11,235
193.94	0.00	206.22	206.54	0	6	295.00	0.02	0.00	0.00	0.01	404	5,979

Köszönöm תודה Спасибі

Thanks ありがとう

Gracias

ευχαριστώ 감사해요 cảm ơn

Kösz

Dankon

Teşekkürler

Danke متشكرم

Merci

Grazie

tack

rahmat

Takk

谢谢

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Gratias

dzięki

спасибо

धन्यवादा

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شكرًا

dankie

asante

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