Q-1 (a) $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

$$= \frac{1}{\sqrt{2}}\left(\begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix}\right) = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$$

$|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$

$$= \frac{1}{\sqrt{2}}\left(\begin{pmatrix}1\\0\end{pmatrix} - \begin{pmatrix}0\\1\end{pmatrix}\right) = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$$

(b) $|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$ —①

$|-\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$ —②

Adding ① & ②

$|0\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$

$|1\rangle = \frac{1}{\sqrt{2}}\{|+\rangle - |-\rangle\}$ Subtracting ② from ①

(c) $|\psi\rangle = \begin{pmatrix}1/\sqrt{3}\\ -i\sqrt{\frac{2}{3}}\end{pmatrix}$

$$= \begin{pmatrix}1/\sqrt{3}\\0\end{pmatrix} + \begin{pmatrix}0\\ -i\sqrt{\frac{2}{3}}\end{pmatrix} = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\0\end{pmatrix} + \left(-i\sqrt{\frac{2}{3}}\right)\begin{pmatrix}0\\1\end{pmatrix}$$

$$= \frac{1}{\sqrt{3}}|0\rangle - i\sqrt{\frac{2}{3}}|1\rangle$$

(d) From (b)

$|\psi\rangle = \frac{1}{\sqrt{3}}\left(\frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)\right) - i\sqrt{\frac{2}{3}}\left(\frac{1}{\sqrt{2}}\{|+\rangle - |-\rangle\}\right)$

$= \frac{1}{\sqrt{3}}\cdot\frac{1}{\sqrt{2}}(|+\rangle + |-\rangle) - i\sqrt{\frac{2}{3}}\cdot\frac{1}{\sqrt{2}}|+\rangle - \frac{i}{\sqrt{3}}|-\rangle$

$= \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}+i\right)|+\rangle + \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}-i\right)|-\rangle$

Q-2(a)

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

$$|+\rangle \otimes |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

$$= \frac{1}{2}|0\rangle \otimes |0\rangle - \frac{1}{2}|0\rangle \otimes |1\rangle + \frac{1}{2}|1\rangle \otimes |0\rangle - \frac{1}{2}|1\rangle \otimes |1\rangle$$

$$= \frac{1}{2}|00\rangle - \frac{1}{2}|01\rangle + \frac{1}{2}|10\rangle - \frac{1}{2}|11\rangle$$

(b) Since,

$$|00\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad |01\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad |10\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad |11\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Substituting these vectors in (a)

$$|+\rangle \otimes |-\rangle = \frac{1}{2}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$$

(c) $|\psi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ i/2\sqrt{2} \\ 1/2\sqrt{2} \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \frac{i}{2\sqrt{2}}\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \frac{1}{2\sqrt{2}}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

$$= \frac{1}{\sqrt{2}}|00\rangle + \frac{i}{2\sqrt{2}}|10\rangle + \frac{1}{2\sqrt{2}}|11\rangle$$

Q-3 (a)

$$|\Psi\rangle = (X \otimes H)(|0\rangle \otimes |+\rangle)$$

$$= X|0\rangle \otimes H|+\rangle$$

$$X|0\rangle = |1\rangle$$

Now,

$$X|0\rangle = |1\rangle$$

$$H|+\rangle = H \cdot H|0\rangle \qquad (\text{since } |+\rangle = H \cdot |0\rangle)$$

$$= H^2|0\rangle = |0\rangle \qquad (\text{since } H \text{ is unitary})$$

$$\therefore \; |\Psi\rangle = |1\rangle \otimes |0\rangle = |10\rangle$$

(b) $X \otimes H$

$$X \otimes H \; |00\rangle = |1+\rangle$$
$$X \otimes H \; |01\rangle = |1-\rangle$$
$$X \otimes H \; |10\rangle = |0+\rangle$$
$$X \otimes H \; |11\rangle = |0-\rangle$$

Using these & eq's, the matrix $X \otimes H$ will be

$$\begin{bmatrix} 0 & 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 & 0 \end{bmatrix}$$

since the matrix is a linear map of basis vectors.

(c)

$$|0\rangle \otimes |+\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \\ 0 \end{bmatrix}$$

(d)

$$|1\rangle \otimes |0\rangle = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \\ 0 \end{bmatrix}$$

Q-4 (a) Since A is a linear map.

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

(b) For A to be unitary, A^T should be A^{-1}

$$A^T = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A \cdot A^T = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Hence, A is unitary.

(c) Since A is unitary A^{-1} is A^T

$$A^T = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad |111\rangle$$

$A^T |00\rangle = |11\rangle$

$A^T |01\rangle = |10\rangle$

$A^T |10\rangle = |00\rangle$

$A^T |11\rangle = |01\rangle$

Q-5(a) $|\Psi_0\rangle = H|0\rangle|00\rangle = |000\rangle$

$|\Psi_1\rangle = H|0\rangle|00\rangle$

$\quad = \left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right)|00\rangle = \frac{1}{\sqrt{2}}|000\rangle + \frac{1}{\sqrt{2}}|100\rangle$

$|\Psi_2\rangle = \frac{1}{\sqrt{2}}|000\rangle + \frac{1}{\sqrt{2}}|101\rangle$

$|\Psi_3\rangle = \frac{1}{\sqrt{2}}|000\rangle + \frac{1}{\sqrt{2}}|111\rangle$

(b) $|\Psi_3\rangle = \frac{1}{\sqrt{2}}|000\rangle + \frac{1}{\sqrt{2}}|111\rangle$

Now, $|0\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle) \qquad |1\rangle = \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle)$

Substituting $\frac{1}{\sqrt{2}}|+\rangle$ by a and $\frac{1}{\sqrt{2}}|-\rangle$ by b

$|\Psi_3\rangle = \frac{1}{\sqrt{2}}\left[(a+b)\cdot(a+b)\cdot(a+b) + (a-b)\cdot(a-b)\cdot(a-b)\right]$

$= \frac{1}{\sqrt{2}}\left[(a\cdot a + a\cdot b + b\cdot a + b\cdot b)(a+b) + (a\cdot a - a\cdot b - b\cdot a + b\cdot b)(a-b)\right]$

$= \frac{1}{\sqrt{2}}\left[\begin{array}{l} a\cdot a\cdot a + a\cdot b\cdot a + a\cdot b\cdot b\cdot a + b\cdot a\cdot b + b\cdot b\cdot a + b\cdot b\cdot b \\ a\cdot a\cdot a - a\cdot b\cdot a + a\cdot b\cdot b - b\cdot a\cdot a + b\cdot a\cdot b + b\cdot b\cdot a - b\cdot b\cdot b \end{array}\right]$

$= \frac{1}{\sqrt{2}}\left[2\, a\cdot a\cdot a + 2\, a\cdot b\cdot b + 2\, b\cdot a\cdot b + 2\, b\cdot b\cdot a\right]$

$= \frac{1}{\sqrt{2}}\left[\begin{array}{l} 2\cdot\frac{1}{2\sqrt{2}}|+++\rangle + 2\cdot\frac{1}{2\sqrt{2}}|+\rangle|-\rangle|-\rangle + 2\cdot\frac{1}{2\sqrt{2}}|-\rangle|+\rangle|-\rangle + 2\cdot\frac{1}{2\sqrt{2}}|-\rangle|-\rangle|+\rangle \end{array}\right]$

$= \frac{1}{2}|+++\rangle + \frac{1}{2}|+\rangle|-\rangle|-\rangle + \frac{1}{2}|-\rangle|+\rangle|-\rangle + \frac{1}{2}|-\rangle|-\rangle|+\rangle$

(c) $\frac{1}{\sqrt{2}}|\Psi_3\rangle = \frac{1}{\sqrt{2}}|000\rangle + \frac{1}{\sqrt{2}}|111\rangle$

$P(|000\rangle) = \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}$

$P(|111\rangle) = \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}$