1. (a) $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

(b) $|0\rangle = \frac{1}{\sqrt{2}} (|+\rangle + |-\rangle)$

$|1\rangle = \frac{1}{\sqrt{2}} (|+\rangle - |-\rangle)$

(c) $|\psi\rangle = \frac{1}{\sqrt{3}} |0\rangle - i\sqrt{\frac{2}{3}} |1\rangle$

(d) Using values of $|0\rangle$ & $|1\rangle$ from (b)

$|\psi\rangle = \frac{1}{\sqrt{3}} \left(\frac{1}{\sqrt{2}}(|+\rangle + |-\rangle) \right) - i\sqrt{\frac{2}{3}} \left(\frac{1}{\sqrt{2}} (|+\rangle - |-\rangle) \right)$

$= \frac{1}{\sqrt{6}} |+\rangle + \frac{1}{\sqrt{6}} |-\rangle - \frac{i}{\sqrt{3}} |+\rangle + \frac{i}{\sqrt{3}} |-\rangle$

$= \left(\frac{1}{\sqrt{6}} - \frac{i}{\sqrt{3}} \right) |+\rangle + \left(\frac{1}{\sqrt{6}} + \frac{i}{\sqrt{3}} \right) |-\rangle$

2. (a) Substituting $|+\rangle = \frac{1}{\sqrt{2}} |0\rangle + \frac{1}{\sqrt{2}} |1\rangle$

& $|-\rangle = \frac{1}{\sqrt{2}} |0\rangle - \frac{1}{\sqrt{2}} |1\rangle$

$|+\rangle \otimes |-\rangle = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle)$

$= \frac{1}{2} (|00\rangle - |01\rangle + |10\rangle - |11\rangle)$

(b) $|+\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad |-\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$|+\rangle \otimes |-\rangle = \frac{1}{2} \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$

(c) $|\psi\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{i}{2\sqrt{2}} \\ \frac{1}{2\sqrt{2}} \end{pmatrix}$

Mapping each element to $|00\rangle, |01\rangle, |10\rangle$ & $|11\rangle$

$$|\psi\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{i}{2\sqrt{2}}|10\rangle + \frac{1}{2\sqrt{2}}|11\rangle$$

3. (a) $(X \otimes H)(|0\rangle \otimes |+\rangle)$

X acts on $|0\rangle$ & H acts on $|+\rangle$. ~~Hence~~ Hence, superposition is: $(X|0\rangle) \otimes (H \otimes |+\rangle)$

X flips $|0\rangle$ & H ~~ac~~ reverses $|+\rangle$ to $|0\rangle$

$\therefore (X \otimes H)(|0\rangle \otimes |+\rangle) = |1\rangle \otimes |0\rangle = |10\rangle$

(b) $X \otimes H = \begin{pmatrix} 0 \cdot H & 1 \cdot H \\ 1 \cdot H & 0 \cdot H \end{pmatrix} = \begin{pmatrix} 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \end{pmatrix}$

(c) $|0\rangle \otimes |+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

(d) $|-\rangle \otimes |0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}$

4. (a) Since, the outputs on the operation of matrix A are standard basis vectors, the matrix's ~~is a linear m~~ columns linearly map the standard basis.

Column 1 $= (A|00\rangle)^T = (|10\rangle)^T$

Column 2 $= (A|01\rangle)^T = (|11\rangle)^T$

Column 3 $= (A|10\rangle)^T = (|01\rangle)^T$

Column 4 $= (A|11\rangle)^T = (|00\rangle)^T$

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

(b) For unitary, $A^* = A^{-1}$ or $A \cdot A^* = I$

$A^* = A^T$ since A is real.

$$A^T = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$A \cdot A^T = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Hence, A is unitary.

(c) Since A is unitary, A^* is A^{-1}

The maps for A^{-1} are:

$A^{-1}|00\rangle = |11\rangle$

$A^{-1}|01\rangle = |10\rangle$

$A^{-1}|10\rangle = |00\rangle$

$A^{-1}|11\rangle = |01\rangle$

5. (a) $|\psi_0| = |0{:}00\rangle$

$|\psi_1| = (H|0\rangle) \otimes |00\rangle$

$= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |00\rangle$

$= \frac{1}{\sqrt{2}}(|000\rangle + |100\rangle)$

$|\psi_2| = \frac{1}{\sqrt{2}}(|00\rangle \otimes CNOT|00\rangle + |10\rangle \otimes CNOT|10\rangle)$

$= \frac{1}{\sqrt{2}}(|000\rangle + |101\rangle)$

$|\psi_3| = \frac{1}{\sqrt{2}}(|0\rangle \otimes CNOT|0{:}0| \otimes |0\rangle + |1\rangle \otimes CNOT(|0\rangle \otimes |1\rangle)$

$= \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$

(b) In standard bases: $\frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$

In Hadamard bases

$|0\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$

$|1\rangle = \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle)$

Writing $|+\rangle$ as a

$|-\rangle$ as b

$$\frac{1}{\sqrt{2}} \left(|000\rangle + |111\rangle \right)$$

$$= \frac{1}{2\sqrt{2}} \left(\begin{array}{l} (a+b) \otimes (a+b) \otimes (a+b) + \\ (a-b) \otimes (a-b) \otimes (a-b) \end{array} \right)$$

Expanding $(a+b) \otimes (a+b) \otimes (a+b)$

$$= (|aa\rangle + |ab\rangle + |ba\rangle + |bb\rangle) \otimes (a+b)$$

$$= |aaa\rangle + |aab\rangle + |aba\rangle + |abb\rangle + |baa\rangle + |bab\rangle + |bba\rangle + |bbb\rangle \quad — ①$$

Similarly $(a-b) \otimes (a-b) \otimes (a-b)$

$$= (|aa\rangle - |ab\rangle - |ba\rangle + |bb\rangle) \otimes (a-b)$$

$$= |aaa\rangle - |aab\rangle - |aba\rangle + |abb\rangle - |baa\rangle + |bab\rangle + |bba\rangle - |bbb\rangle \quad — ②$$

Adding ① & ②

$$\frac{1}{2\cdot\sqrt{2}} (|000\rangle + |111\rangle) = \frac{1}{\sqrt{2}} \left(|aaa\rangle + |abb\rangle + |bab\rangle + |bba\rangle \right)$$

$$= \frac{1}{\sqrt{2}} \left(|{+}{+}{+}\rangle + |{+}{-}{-}\rangle + |{-}{+}{-}\rangle + |{-}{-}{+}\rangle \right)$$

(c) Measuring any qubit will be 0 or 1 with equal probability and measuring any 1 qubit collapses other qubits in the same state.

6. (a) Since $|e^{i\alpha}| = 1$

Since measurement probability are magnitude of each state and

$$|ab| = |a|.|b|$$

for $a, b \in \mathbb{C}$

$\therefore$ measurement probabilities are unchanged.

(b) (i) $|\psi_\varphi| = \frac{1}{\sqrt{2}}(|0\rangle + e^{i\varphi}|1\rangle)$

$$P(0) = \left(\frac{1}{\sqrt{2}}\right)^2$$

$$P(|1\rangle) = \left(\frac{1}{\sqrt{2}}\right)^2 \cdot |e^{i\varphi}|^2 = \frac{1}{2}$$

(ii) $H\,|\psi_\varphi| = \frac{1}{\sqrt{2}}(H|0\rangle + e^{i\varphi} H|1\rangle)$

$$= \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle + \frac{e^{i\varphi}}{\sqrt{2}}(|0\rangle - |1\rangle)\right)$$

$$= \frac{1}{2}\left((1+e^{i\varphi})|0\rangle + (1-e^{i\varphi})|1\rangle\right)$$

Now, when $e^{i\varphi} = 1$, we get $|0\rangle$ deterministically — ①

when $e^{i\varphi} = -1$, we get $|1\rangle$ deterministically — ②

① when $\varphi \in 2k\pi$ where $k \in$ Whole numbers.

② when $\varphi \in (2k+1)\pi$ where $k \in$ whole numbers.