ORTA DOĞU TEKNİK ÜNİVERSİTESİ

Lecture Notes 1
Electrical and Electronics Engineering - 5670402
Discrete Time Systems
Fall, 2018-2019

1. Continuous & Discrete Time Signals

A continuous time bilateral signal is a mapping defined by $f : \mathbb{R} \mapsto \mathbb{R}$ (or for unilateral case $f : \mathbb{R}^+ \mapsto \mathbb{R}$). Examples

$$f(t) = \sin(t)\ ,\ f(t) = e^t\ ,\ f(t) = u(t)\ ,\ f(t) = \delta(t)\ ,\ \text{where } t \in \mathbb{R}$$

A discrete time bilateral signal is a mapping defined by $g : \mathbb{Z} \mapsto \mathbb{R}$ (or for unilateral case $g : \mathbb{Z}^+ \mapsto \mathbb{R}$). Examples

$$f[n] = \sin[n]\ ,\ f[n] = 5^n\ ,\ f[n] = u[n]\ ,\ f[n] = \delta[n]\ ,\ \text{where } n \in \mathbb{Z}$$

Graphical Examples

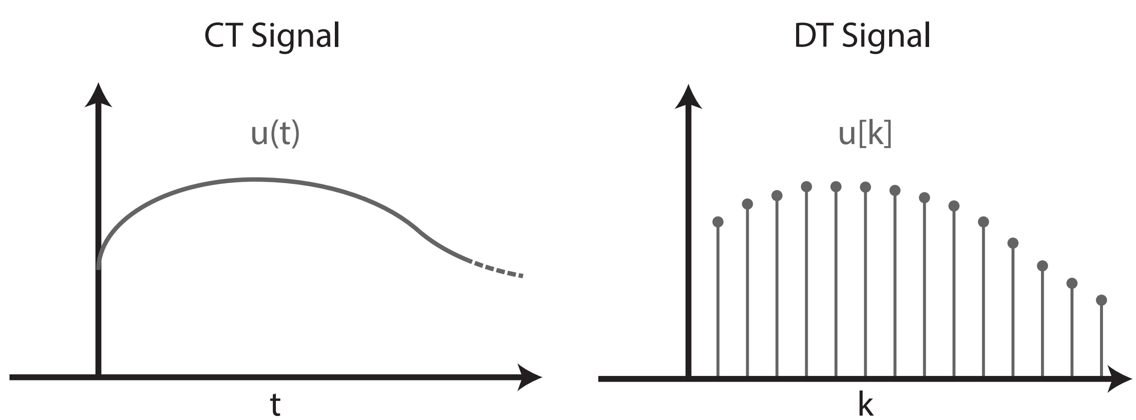


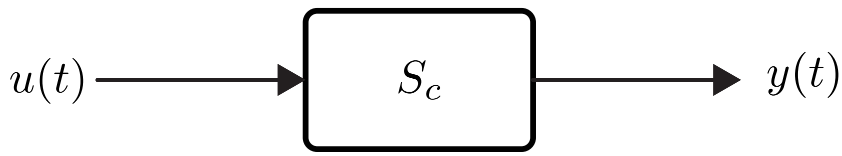
Figure 1. CT vs DT Signal

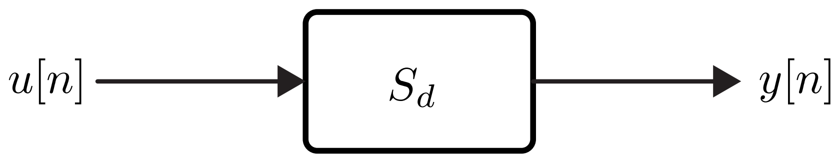
2. Continuous & Discrete Time Dynamical Systems

The system is modeled as a *mapping* from a set of input signals, $u(t)$ or $u[n]$, to a set of output signals, $y(t)$ or $y[n]$. We may represent continuous time and discrete time maps as

$$\begin{aligned}\text{Continuous}: &\quad y(t) = (S_c\, u)(t)\\ \text{Discrete}: &\quad y[n] = (S_d\, u)[n]\end{aligned}$$

Operation is performed on the entire input signal, $u(\cdot)$ or $u[\cdot]$ where the mappings S_c and S_d yield the signals $y(\cdot)$ and $y[\cdot]$.





2.1. **Properties of Input–Output Systems.**

- **Linearity**

  A continuous time system is **linear** if and only if

$$\begin{aligned}(S_c\,(\alpha u_1 + \beta u_2))(t) = \alpha(S_c\, u_1)(t) + \beta(S_c\, u_2)(t)\\ \forall \alpha,\ \beta,\ u_1(\cdot),\ \&\ u_2(\cdot)\end{aligned}$$

  A discrete time system is **linear** if and only if

$$\begin{aligned}(S_d\,(\alpha u_1 + \beta u_2))[n] = \alpha(S_d\, u_1)[n] + \beta(S_d\, u_2)[n]\\ \forall \alpha,\ \beta,\ u_1[\cdot],\ \&\ u_2[\cdot]\end{aligned}$$

- **Time Invariance**

  Let σ_T be the time-shift operator as

$$\sigma_T\, u(t) = u(t - T)$$

  Then a continuous time system is time-invariant if and only if

$$(S_c\, \sigma_T\, u))(t) = y(t-T)\ \forall T \in \mathbb{R}^+,\ \text{where}\ (S_c\, u))(t) = y(t)$$

  Similarly for discrete time systems

$$(S_d\,\sigma_k\,u))[n] = y[n-k]\;\forall k \in \mathbb{Z}^+,\text{ where }(S_d\,u))[n] = y[n]$$

- **Memoryless Systems:**

  A continuous time system is memoryless if and only if $y(t_0)$ only depends on $u(t_0)$ for all $t \in \mathbb{R}$.

  A discrete time system is memoryless if and only if $y[n_0]$ only depends on $u[n_0]$ for all $t \in \mathbb{Z}$.

- **Causality:**

  We say the system is causal if the output does not depend on future values of the input. Mathematically we can show causality using the *truncation* operator, P_T. For continuous systems *truncation* is defined as

  $$(P_T\,u)(t) = \begin{cases} u(t) & \text{for } t \leq T \leq 0 \\ 0 & \text{otherwise} \end{cases}$$

  for discrete systems

  $$(P_k\,u)[n] = \begin{cases} u[n] & \text{for } n \leq k \leq 0 \\ 0 & \text{otherwise} \end{cases}$$

  then the system, S (continuous or discrete), is said to be causal if $P_T S = P_T S P_T$

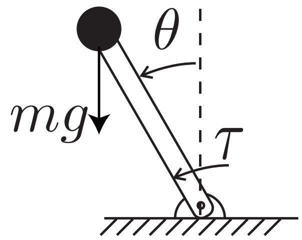
- **Finite & Infinite Dimensional Systems:**

  A continuous dynamical system, S_c, is finite dimensional if there exist an ODE in u, y that models S_c.

  A discrete time dynamical system, S_d, is finite dimensional if there exist an Ordinary Difference Equation in u, y that models S_d.

2.2. **Examples.**

(1) $u(t) = \tau(t),\ y(t) = \theta(t)$



Non-linear, Time-invariant, causal, system has memory, and finite dimensional

(2) Now let's assume that $g = 0$, what happens?

Linear, Time-invariant, causal, system has memory, and finite dimensional

(3) $y(t) = \int_0^t (t-s)^3 u(s) ds$

Linear, causal, system has memory, finite dimensional,

Since the system is causal it may be OK to assume that the set of inputs are limited to causal signals and the convolution is unilateral. In this case the system is *time-invariant*.

However if non-causal signals are allowed then the system becomes *time-varying*.

3. Representations of Dynamical Systems

3.1. Differential & Difference Equations.

- **Continuous Time Systems - ODEs**

Linear Time Invariant System (LTI)

$$a_n y^{(n)} + ... + a_1 y' + a_0 y = b_n u^{(n)} + ... + b_1 u' + b_0 u$$

Linear Time Varying System (LTV)

$$a_n(t) y^{(n)} + ... + a_1(t) y' + a_0(t) y = b_n(t) u^{(n)} + ... + b_1(t) u' + b_0(t) u$$

Non-linear Time Invariant System

$$y^{(n)} = f(y^{(n-1)}, ..., y', y, u^{(n)}, ..., u', u)$$

Non-linear Time Varying System

$$y^{(n)} = f(y^{(n-1)}, ..., y', y, u^{(n)}, ..., u', u, t)$$

- **Discrete Time Systems - Difference Equations**

Discrete-Time Linear Time Invariant System (LTI)

$$a_n y[k] + a_{n-1} y[k-1] + ... + a_0 y[k-n] = b_n u[k] + ... + b_0 u[k-n]$$

Discrete-Time Linear Time Varying System (LTV)

$$a_n[k] y[k] + a_{n-1}[k] y[k-1] + ... + a_0[k] y[k-n] = b_n[k] u[k] + ... + b_0[k] u[k-n]$$

Non-linear Time Invariant System

$$y[k] = f(y[k-1], ..., y[k-n], u[k], ..., u[k-n])$$

Non-linear Time Varying System

$$y[k] = f(y[k-1], ..., y[k-n], u[k], ..., u[k-n], k)$$

Discussion

- When an ODE representation becomes *memoryless*?
- When a difference equation representation becomes *memoryless*?
- What about infinite dimensional systems?
- What about *causality*?

3.2. State-Space Representation of Dynamical Systems.

- **Continuous-Time Dynamical Systems**

  Linear Time Invariant Systems

$$\begin{aligned}
&\text{Let } x(t) \in \mathbb{R}^n \ , \ y(t) \in \mathbb{R} \ , \ u(t) \in \mathbb{R}, \\
&\dot{x}(t) = Ax(t) + Bu(t), \\
&y(t) = Cx(t) + Du(t), \\
&\text{where } A \in \mathbb{R}^{n \times n} \ , \ B \in \mathbb{R}^{n \times 1} \ , \ C \in \mathbb{R}^{1 \times n} \ , \ D \in \mathbb{R}
\end{aligned}$$

  Linear Time Varying Systems

$$\begin{aligned}
&\text{Let } x(t) \in \mathbb{R}^n \ , \ y(t) \in \mathbb{R} \ , \ u(t) \in \mathbb{R}, \\
&\dot{x}(t) = A(t)x(t) + B(t)u(t), \\
&y(t) = C(t)x(t) + D(t)u(t), \\
&\text{where } A(t) \in \mathbb{R}^{n \times n} \ , \ B(t) \in \mathbb{R}^{n \times 1} \ , \ C(t) \in \mathbb{R}^{1 \times n} \ , \ D(t) \in \mathbb{R}
\end{aligned}$$

  Non-Linear Time Invariant Systems

$$\begin{aligned}
&\text{Let } x(t) \in \mathbb{R}^n \ , \ y(t) \in \mathbb{R} \ , \ u(t) \in \mathbb{R}, \\
&\dot{x}(t) = F(x(t), u(t)), \\
&y(t) = H(x(t), u(t)),
\end{aligned}$$

  Non-Linear Time Varying Systems

$$\begin{aligned}
&\text{Let } x(t) \in \mathbb{R}^n \ , \ y(t) \in \mathbb{R} \ , \ u(t) \in \mathbb{R}, \\
&\dot{x}(t) = F(x(t), u(t), t), \\
&y(t) = H(x(t), u(t), t),
\end{aligned}$$

- **Discrete-Time Dynamical Systems**

  Linear Time Invariant Systems

$$\begin{aligned}
&\text{Let } x[n] \in \mathbb{R}^n \ , \ y[n] \in \mathbb{R} \ , \ u[n] \in \mathbb{R}, \\
&x[n+1] = Ax[n] + Bu[n], \\
&y[n] = Cx[n] + Du[n], \\
&\text{where } A \in \mathbb{R}^{n \times n} \ , \ B \in \mathbb{R}^{n \times 1} \ , \ C \in \mathbb{R}^{1 \times n} \ , \ D \in \mathbb{R}
\end{aligned}$$

Linear Time Varying Systems

$$\begin{aligned} \text{Let } x[n] &\in \mathbb{R}^n\ ,\ y[n] \in \mathbb{R}\ ,\ u[n] \in \mathbb{R}, \\ x[n+1] &= A[n]x[n] + B[n]u[n], \\ y[n] &= C[n]x[n] + D[n]u[n], \\ \text{where } A[n] &\in \mathbb{R}^{n\times n}\ ,\ B[n] \in \mathbb{R}^{n\times 1}\ ,\ C[n] \in \mathbb{R}^{1\times n}\ ,\ D[n] \in \mathbb{R} \end{aligned}$$

Non-Linear Time Invariant Systems

$$\begin{aligned} \text{Let } x[n] &\in \mathbb{R}^n\ ,\ y[n] \in \mathbb{R}\ ,\ u[n] in \mathbb{R}, \\ x[n+1] &= F(x[n], u[n]), \\ y[n] &= H(x[n], u[n]), \end{aligned}$$

Non-Linear Time Varying Systems

$$\begin{aligned} \text{Let } x[n] &\in \mathbb{R}^n\ ,\ y[n] \in \mathbb{R}\ ,\ u[n] \in \mathbb{R}, \\ x[n+1] &= F(x[n], u[n], n), \\ y[n] &= H(x[n], u[n], n), \end{aligned}$$

Discussion

- When a state-space representation becomes *memoryless*?
- What about infinite dimensional systems?
- What about *causality*?

3.3. **Impulse-Response Representation of Dynamical Systems.**

- **Continuous-Time Dynamical Systems**

  Linear Time Invariant Systems

$$y(t) = h(t) * u(t) = \int_{-\infty}^{\infty} h(t-\tau)u(\tau)d\tau$$

  Linear Time Varying Systems

$$y(t) = \int_{-\infty}^{\infty} h(t,\tau)u(\tau)d\tau$$

  where $h(t,\tau)$ is called time-varying impulse response function.

- **Discrete-Time Dynamical Systems**

  Linear Time Invariant Systems

$$y[n] = \sum_{k=-\infty}^{\infty} h[n-k]u[k]$$

Linear Time Varying Systems

$$y[n] = \sum_{k=-\infty}^{\infty} h[n,k]u[k]$$

Discussion

- Under what condition(s) an impulse response representation becomes *memoryless*?
- Under what condition(s) an impulse response representation becomes *causal*?
- What about finite and infinite dimensional systems?
- What are the differences between continuous time and discrete time impulse response?

3.4. **Transfer Functions Representation of Dynamical Systems.**

- **Continuous-Time Dynamical Systems**
  Linear Time Invariant Systems

$$Y(s) = G(s)U(s), \text{ where,}$$
$$Y(s) = \mathcal{L}\{y(t)\}\ , \&\ U(s) = \mathcal{L}\{u(t)\}$$

- **Discrete-Time Dynamical Systems**
  Linear Time Invariant Systems

$$Y(z) = G(z)U(z), \text{ where,}$$
$$Y(z) = \mathcal{Z}\{y[n]\}\ , \&\ U(z) = \mathcal{Z}\{u[n]\}$$

  Discussion

  – Can we model/represent non-linear systems using transfer functions?
  – Can we model/represent linear time-varying systems using transfer functions?
  – What about finite and infinite dimensional systems?
  – What about *causality*?

Lecture 2

Lecturer: Asst. Prof. M. Mert Ankarali

*Big Picture of EE402

In this course, the main focus will be on continuous-time systems that are controlled (sampled and actuated) by a digital computer interface. Such a discrete-time control system consists of four major parts as illustrated in Fig. 2.2,

1. *The plant* is a continuous-time dynamical system
2. The Analog-to-Digital Converter (ADC)
3. The Controller (μP), a microprocessor/microcontroller with a "real-time" OS
4. The Digital-to-Analog Converter (DAC)

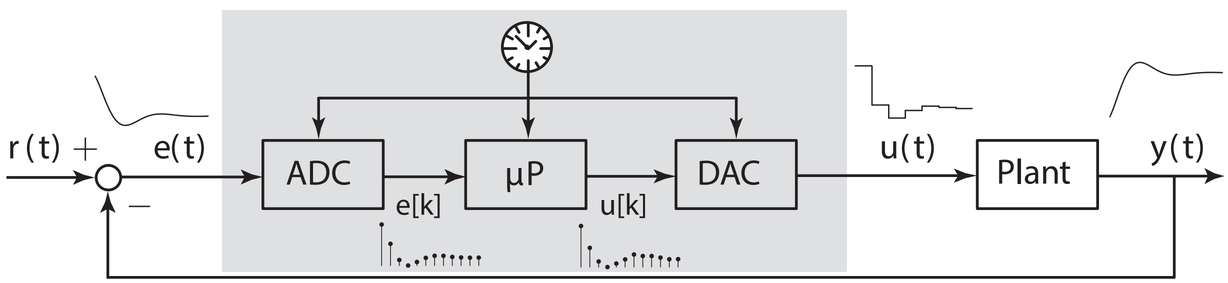


Figure 2.1: Block diagram of a digital control system

Most of the time the plant is modeled as a "smooth" continuous dynamical system. In this course our focus is LTI systems thus, we will assume that (unless otherwise is given) that the plant is a continuous LTI plant model with a transfer function of $G_c(s)$ for which both the input and output are continuous time signals.

The "digital" blocks of the closed-loop block diagram structure are ADC, The Controller, and DAC. It is generally assumed (design preference) that all blocks shares a common "hard real time" clock.

A general ADC is a device that converts an analog signal to a digital signal. In this course we will model the ADC block as an *ideal sampler* for which the input is a continious-time signal, $e(t)$ and the output is a discrete-time signal, $e[k]$, where the relating between the continuous- and discrete-time signals are given as

$$e(kT) = e[k], \ k \in \mathbb{Z}^+,$$

where constant T is the *sampling time*.

The microcontoller/microprocessor processes some set of digital input signals to produce some set of digital output signals. The outputs are defined at only certain instances defined by the real-time clock. In this course, we will model the μP block as an ideal discrete-time LTI system for which the both the input and output are discrete-time signals, with a transfer function of $G_c(z)$.

The DAC is a device that converts a digital signal to an analog signal. In this course we assume that it is an ideal *Hold* element for which the input signal is a discrete-time signal, where as output is a continuous-time signal. The most commonly used *Hold* system is ZOH (Zero-Order-Hold) which is a mapping defined by the following relation

$$u(t) = u[k],\ \text{for} t \in [kT, (k+1)T)$$

Higher-order holds available but seldom used.

The idealized and simplified block-diagram structure is given in Fig.

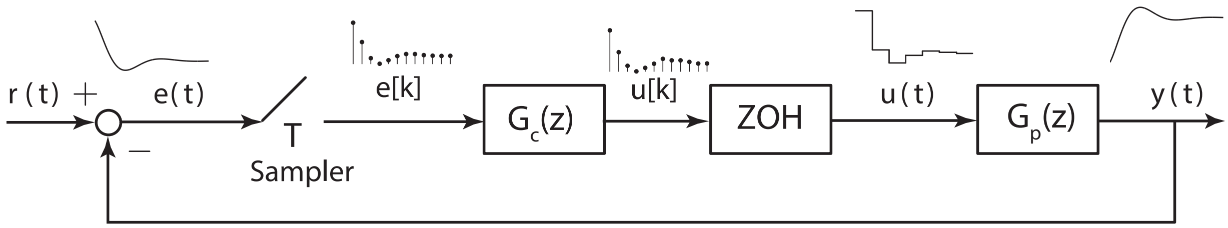


Figure 2.2: Block diagram of an LTI discrete-time control system

Major challenge: Loop contains both continuous-time and discrete-time parts.

Sampling

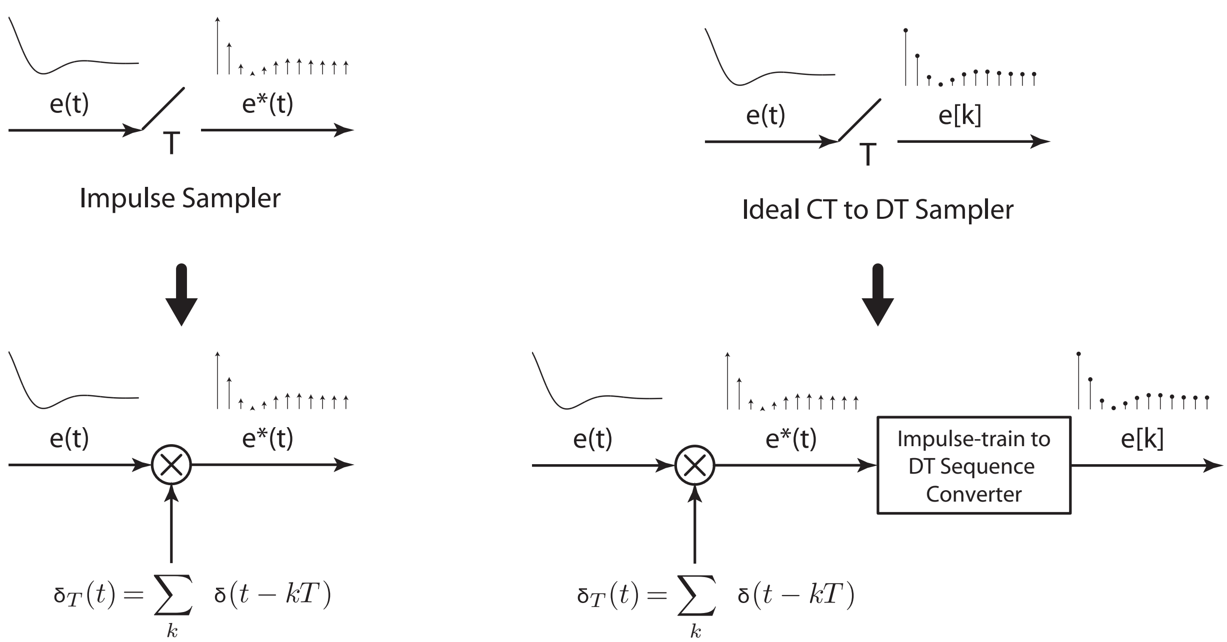


Figure 2.3: Two different ideal samplers

Fig. 2.3 illustrates two different ideal sampler (both of them will be covered in this course. Firs column is an *impulse sampler* for which the output is a continuous signal but it is composed of trains of impulses (impulse train). Second one is an ideal complete CT-to-DT sampler which convertes the impulse train into DT sequence.

The output of the impulse sampler, $x * (t)$, can be represented with the following infinte summations

$$
\begin{aligned}
x^*(t) &= \sum_{k=0}^{\infty} x(kT)\delta(t-kT) = \sum_{k=0}^{\infty} x[k]\delta(t-kT) \\
&\text{or} \\
x^*(t) &= x(0)\delta(t) + x(T)\delta(t-T) + \cdots + x(kT)\delta(t-kT) + \cdots \\
&= x[0]\delta(t) + x[1]\delta(t-T) + \cdots + x[k]\delta(t-kT) + \cdots
\end{aligned}
$$

Now let's consider the Laplace transform of $x^*(t)$

$$
\begin{aligned}
X^*(s) = \mathcal{L}\{x^*(t)\} &= \mathcal{L}\left\{\sum_{k=0}^{\infty} x(kT)\delta(t-kT)\right\} = \sum_{k=0}^{\infty} x(kT)\mathcal{L}\left\{\delta(t-kT)\right\} \\
&= \sum_{k=0}^{\infty} x(kT) \int_{t=0}^{\infty} \delta(t-kT)e^{-st}dt = \sum_{k=0}^{\infty} x(kT)e^{-skT} = \sum_{k=0}^{\infty} x[k]e^{-skT}
\end{aligned}
$$

Now let's define a map in complex domain such that

$$
z = e^{Ts} \text{ or } s = \frac{1}{T}\ln z
$$

Then we have

$$
X^*(s)|_{s=(1/T)lnz} = \sum_{k=0}^{\infty} x[k]z^{-k}
$$

where

$$
X(z) = \mathcal{Z}\{x[k]\} = \sum_{k=0}^{\infty} x[k]z^{-k}
$$

Z-transform

Z-transform of a (causal) discrete time signal $x[k]$ is given by

$$
X(z) = \mathcal{Z}\{x[k]\} = \sum_{k=0}^{\infty} x[k]z^{-k}
$$

If $x[k]$ is a sampled signal from a continuous time signal $x(t)$ with a sampling time of T, we (abuse of notation) also use the following notation

$$
X(z) = \mathcal{Z}\{x(kT)\} = \mathcal{Z}\{x^*(t)\}
$$

Z-transforms of elementary functions

We assume that all signals are causal thus $t \in \mathbb{R}^+$ and $k \in \mathbb{Z}^+$

Unit-step function $x(t) = 1$ and thus $x(kT) = x[k] = 1$, the Z-transform is given by

$$X(z) = \sum_{k=0}^{\infty} z^{-k} = \frac{1}{1 - z^{-1}} = \frac{z}{z-1}$$

Unit-ramp function $x(t) = t$ and thus $x(kT) = x[k] = kT$, the Z-transform is given by

$$\begin{aligned} X(z) &= \sum_{k=0}^{\infty} kTz^{-k} = T(z^{-1} + 2z^{-2} + 3z^{-3} + \cdots) = Tz(z^{-2} + 2z^{-3} + 3z^{-4} + \cdots) \\ &= Tz\frac{d}{dz}\left(\int (z^{-2} + 2z^{-3} + 3z^{-4} + \cdots)dz\right) = Tz\frac{d}{dz}\left(-(z^{-1} + z^{-2} + z^{-3} + \cdots)\right) \\ &= Tz\frac{d}{dz}\left(\frac{-1}{z-1}\right) = \frac{Tz}{(z-1)^2} = \frac{Tz^{-1}}{(1-z^{-1})^2} \end{aligned}$$

Exponential sequence $x[k] = a^k$

$$X(z) = \sum_{k=0}^{\infty} a^k z^{-k} = \sum_{k=0}^{\infty} \left(\frac{z}{a}\right)^{-k} = \frac{1}{1 - az^{-1}} = \frac{z}{z-a}$$

Exponential function $x(t) = e^{bt}$ and thus $x(kT) = x(k) = e^{bTk}$

$$X(z) = \sum_{k=0}^{\infty} e^{bTk} z^{-k} = \sum_{k=0}^{\infty} (e^{bT})^k z^{-k} = \frac{1}{1 - e^{bT}z^{-1}} = \frac{z}{z - e^{bT}}$$

Cosine function $x(t) = \cos(\omega t)$, and thus $x(kT) = x(k) = \cos(\omega Tk)$

$$\begin{aligned} \cos(\omega Tk) = \frac{1}{2}\left(e^{j\omega Tk} + e^{-j\omega Tk}\right) X(z) &= \frac{1}{2}\left(\frac{z}{z - e^{j\omega T}} + \frac{z}{z - e^{-j\omega T}}\right) = \frac{1}{2}\frac{z(z - e^{-j\omega T}) + z(z - e^{j\omega T})}{(z - e^{-j\omega T})(z - e^{j\omega T})} \\ &= \frac{1}{2}\frac{2z^2 - z(e^{-j\omega T} + e^{j\omega T})}{z^2 - z(e^{-j\omega T} + e^{j\omega T}) + 1} = \frac{z^2 - z\cos(\omega T)}{z^2 - z2\cos(\omega T) + 1} \\ &= \frac{1 - z^{-1}\cos(\omega T)}{1 - z^{-1}2\cos(\omega T) + z^{-2}} \end{aligned}$$

Properties and Theorems of the Z-transform

Linearity

$$x(k) = \alpha f(k) + \beta g(k) \ \rightarrow X(z) = \alpha F(z) + \beta G(z) \ , \forall \ \alpha, \ \beta, \ f(k), \ \& \ g(k)$$

Multiplication by a^k

$$\begin{aligned} \mathcal{Z}\{a^k x[k]\} &= \sum_{k=0}^{\infty} a^k x[k] z^{-k} = \sum_{k=0}^{\infty} x[k](z/a)^{-k} \\ \mathcal{Z}\{a^k x[k]\} &= X(z/a) \end{aligned}$$

Complex translation theorem Let $y(t) = e^{-at}x(t)$ and $X(z) = \ddagger\{x(kT)\}$, then

$$\mathcal{Z}\{y(kT)\} = \mathcal{Z}\{e^{-aTk}x(kT)\} = X(e^{aT}z)$$

Shifting theorem Let $x(t)$ be a causal CT signal, thus we have $x(t) = 0$ for $t < 0$. Similarly sampled DT signal has the property $x[nk] = 0$ for $k < 0$. For the sake of simplicity lets work on the sampled (i.e. DT) signal. Let

$$\mathcal{Z}\{x^*(t)\} = \mathcal{Z}\{x[k]\} = X(z)$$

Shifting right by N (Causal shifting): Let $y[k] = x[k-N]$, then

$$\mathcal{Z}\{y[k]\} = \sum_{k=0}^{\infty} y[k]z^{-k} = \sum_{k=0}^{\infty} x[k-N]z^{-k} = \sum_{k=N}^{\infty} x[k-N]z^{-k}$$

Let $k = m + N$ then

$$\begin{aligned}\mathcal{Z}\{y[k]\} &= \sum_{m=0}^{\infty} x[m]z^{-(m+N)} = z^{-N}\sum_{m=0}^{\infty} x[m]z^{-m}\\ \mathcal{Z}\{x[k-N]\} &= z^{-N}X(z)\end{aligned}$$

Shifting left by N (Non-causal shifting) & Bilateral Z transform: Let $y[k] = x[k+N]$,

$$\begin{aligned}\mathcal{Z}\{x[k+N]\} &= \sum_{k=-\infty}^{\infty} x[k+N]z^{-k} = \sum_{m=-\infty}^{\infty} x[m]z^{-(m-N)} = z^{N}\sum_{m=-\infty}^{\infty} x[m]z^{-m}\\ \mathcal{Z}\{x[k+N]\} &= z^{N}X(z)\end{aligned}$$

Shifting left by N (Non-causal shifting) & Unilateral Z transform: Let $y[k] = x[k+N]$,

$$\mathcal{Z}\{x[k+N]\} = \sum_{k=0}^{\infty} x[k+N]z^{-k}$$

Let $k = m - N$ then

$$\begin{aligned}\mathcal{Z}\{x[k+N]\} &= \sum_{m=N}^{\infty} x[m]z^{-(m-N)} = z^{N}\sum_{m=N}^{\infty} x[m]z^{-m} = z^{N}\left(\sum_{k=0}^{\infty} x[k]z^{-k} - \sum_{k=0}^{N-1} x[k]z^{-k}\right)\\ \mathcal{Z}\{x[k+N]\} &= z^{N}\left(X(z) - \sum_{k=0}^{N-1} x[k]z^{-k}\right)\end{aligned}$$

From this equation we can obtain

$$\begin{aligned}\mathcal{Z}\{x[k+1]\} &= zX(z) - zx[0]\\ \mathcal{Z}\{x[k+2]\} &= z^2X(z) - z^2x[0] - zx[1]\\ &\vdots\end{aligned}$$

Example 1. Let $u[k]$ be the unit-step function. Compute $\mathcal{Z}\{u[k-1]\}$ both directly and using the shifting property.

$$\mathcal{Z}\{u[k-1]\} = \frac{z^{-1}}{1 - z^{-1}}$$

Example 2. Let $y[k] = \sum_{n=0}^{k} x[k]$ where $k \in \mathbb{Z}^+$. Compute $Y(z)$ in terms of $X(z)$ using the shifting theorem.

$$Y(z) = \frac{1}{1 - z^{-1}} X(z)$$

Initial Value Theorem Let $X(z) = \mathcal{Z}\{x[n]\}$ and if the following limit exists, then the initial value of $x[0]$ or $x(0)$ is given by

$$x[0] = \lim_{z \to \infty} X(z)$$

Indeed the proof is very easy

$$\lim_{z \to \infty} X(z) = \lim_{z \to \infty} \left[\sum_{k=0}^{\infty} x(k) z^{-k} \right] = \lim_{z \to \infty} \left[x(0) + x(1) z^{-1} + x(2) z^{-2} + \cdots \right] = x(0)$$

Final Value Theorem

Let's assume that $x(kT)$ or $x[k]$ is a convergent sequence (DT signal). Then the final value theorem states that

$$\lim_{k \to \infty} x[k] = \lim_{z \to 1} (1 - z^{-1}) X(z)$$

Complex Differentiation Theorem

Consider

$$\begin{aligned}
\frac{d}{dz} X(z) &= \frac{d}{dz} \left[\sum_{k=0}^{\infty} x[k] z^{-k} \right] = \sum_{k=0}^{\infty} x[k] \frac{d}{dz} z^{-k} = \sum_{k=0}^{\infty} (-k) x[k] z^{-k-1} \\
-z \frac{d}{dz} X(z) &= \sum_{k=0}^{\infty} k x[k] z^{-k} \\
-z \frac{d}{dz} X(z) &= \mathcal{Z}\{k x[k]\}
\end{aligned}$$

In general

$$(-z)^m \frac{d}{dz^m} X(z) = \mathcal{Z}\{k^m x[k]\}$$

Example 3. Find the Z-transform of the unit ramp function, $r[k] = k \; k \in \mathbb{Z}^+$ by applying the Complex Differentiation Theorem to the Z-transform of the unit step function.

Real Convolution Theorem Let $f[k]$ and $g[k]$ are causal signals and associated Z transforms are $F(z)$ and $G(z)$ respectively. The DT convolution operator is defined as

$$f[n] * g[n] = \sum_{k=0}^{n} f[n-k] g[k]$$

Real Convolution Theorem states that

$$\mathcal{Z}\{f[n] * g[n]\} = F(z) G(z)$$

Proof

$$\mathcal{Z}\{f[n]*g[n]\} = \sum_{n=0}^{\infty}\left[\sum_{k=0}^{n} f[n-k]g[k]\right]z^{-n}$$

Since we know that $f[m]=0$ for $m<0$, we can stretch the upper limit of the sum as

$$\mathcal{Z}\{f[n]*g[n]\} = \sum_{n=0}^{\infty}\left[\sum_{k=0}^{\infty} f[n-k]g[k]\right]z^{-n} = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} f[n-k]g[k]z^{-n}$$

Let $n=m+k$ then

$$\begin{aligned}\mathcal{Z}\{f[n]*g[n]\} &= \sum_{k=0}^{\infty}\sum_{m=-k}^{\infty} f[m]g[k]z^{-m}z^{-k} = \sum_{k=0}^{\infty} g[k]z^{-k}\sum_{m=0}^{\infty} f[m]z^{-m} \\ \mathcal{Z}\{f[n]*g[n]\} &= F(z)G(z)\end{aligned}$$

The Inverse Z-transform

1. Direct division method
2. Z-tranform tables & partial-fraction expansion
3. “Simulation” method
4. Inversion integral method

Direct division

Direct division (or long division) method uses the fact that $X(z)$ can be expressed as

$$X(z) = \sum_{k=0}^{\infty} x[k]z^{-k} = x(0) + x(1)z^{-1} + x(2)z^{-2} + x(3)z^{-3} + \cdots$$

The goal is finding the power series expansion of $X(z)$ using the long division approach. Here we assume that $X(z)$ cane be represnted as a ratio of two polynomials in z (or z^{-1})

$$X(z) = \frac{b_0z^m + b_1z^{m-1} + \cdots + b_mz^{-n}}{1 + a_1z^{n-1} + \cdots + a_n} = \frac{b_0z^{-n+m} + b_1z^{-n+m-1} + \cdots + b_m}{z^n + a_1z^{-1} + \cdots + a_nz^{-n}}$$

For the direct division method it is easer to work when the polynomials are written interms of powers of z^{-1}.

Example 4. Find the inverse Z-transform of $X(z) = \frac{z^{-1}}{1-2z^{-1}+z^{-2}}$.

$$
\begin{array}{r|l}
z^{-1} & 1 - 2z^{-1} + z^{-2} \\ \cline{2-2}
z^{-1} - 2z^{-2} + z^{-3} & z^{-1} + 2z^{-2} + 3z^{-3} + 4z^{-4} + \ldots \\ \hline
2z^{-2} - z^{-3} & \\
2z^{-2} - 4z^{-3} + 2z^{-4} & \\ \hline
3z^{-3} - 2z^{-4} & \\
3z^{-3} - 6z^{-4} + 3z^{-5} & \\ \hline
4z^{-4} - 3z^{-5} & \\
\vdots &
\end{array}
$$

Thus,

$$
\begin{aligned}
&X(z) = 0 + 1z^{-1} + 2z^{-2} + 3z^{-3} + 4z^{-4} + \ldots \\
&\quad\downarrow \\
&x[k] = 0\delta[k] + 1\delta[k-1] + 2\delta[k-2] + 3\delta[k-3] + 4\delta[k-4] + \ldots = k
\end{aligned}
$$

Partial Fraction Expansion

In most applications $X(z)$ can be re-written in terms of poles and zeros as

$$X(z) = b_0 \frac{(z - z_1)\cdots(z - z_m)}{(z - p_1)\cdots(z - p_n)} \quad (m \leq n)$$

Specific (but extremely common) case

$$\frac{X(z)}{z} = \sum_{i=1}^{n} \frac{a_i}{(z - p_i)}$$

where all poles are distinct and simple order. We can compute each a_i using

$$a_i = \lim_{z \to p_i} \left[(z - p_i) \frac{X(z)}{z} \right]$$

Example 5. Find the inverse Z-transform $X(z) = \frac{(1-b)z}{(z-1)(z-b)}$. Solution:

$$
\begin{aligned}
\frac{X(z)}{z} &= \frac{(1-b)}{(z-1)(z-b)} = \frac{a_1}{z-1} + \frac{a_2}{z-b} \\
a_1 &= \lim_{z \to 1} \left[(z-1) \frac{X(z)}{z} \right] = 1 \\
a_2 &= \lim_{z \to b} \left[(z-b) \frac{X(z)}{z} \right] = -1 \\
X(z) &= \frac{z}{z-1} - \frac{z}{z-b} \\
x[k] &= 1 - b^k
\end{aligned}
$$

Now let's assume that $\frac{X(z)}{z}$ has double pole at p_1 and all other poles are distinct

$$\frac{X(z)}{z} = \frac{c_1}{z-p_1} + \frac{c_2}{(z-p_1)^2} + \cdots$$

It is easy to show that

$$c_2 = \lim_{z\to p_1}\left[(z-p_1)^2\frac{X(z)}{z}\right]$$

It is also possible to show that

$$c_1 = \lim_{z\to p_1}\left\{\frac{d}{dz}\left[(z-p_1)^2\frac{X(z)}{z}\right]\right\}$$

Example 6. Find the inverse Z-transform $X(z) = \frac{2z^2-3z}{(z-1)^2}$. Solution:

$$\begin{aligned}\frac{X(z)}{z} &= \frac{c_1}{z-1} + \frac{c_2}{(z-1)^2} \\ c_1 &= \lim_{z\to 1}\frac{d}{z}\left[(z-1)^2\frac{X(z)}{z}\right] = 2 \\ c_2 &= \lim_{z\to 1}\left[(z-1)^2\frac{X(z)}{z}\right] = -1 \\ x[k] &= 2-k\end{aligned}$$

Example 7. Find the inverse Z-transform $X(z) = \frac{(1-b)}{(z-1)(z-b)}$. Solution:

$$\begin{aligned}X(z) &= \frac{(1-b)}{(z-1)(z-b)} = \frac{a_1}{z-1} + \frac{a_2}{z-b} \\ a_1 &= \lim_{z\to 1}[(z-1)X(z)] = 1 \\ a_2 &= \lim_{z\to b}[(z-b)X(z)] = -1 \\ X(z) &= z^{-1}\left(\frac{z}{z-1} - \frac{z}{z-b}\right) \\ x[k] &= [1-b^{k-1}]u[k-1]\end{aligned}$$

Example 8. Find the inverse Z-transform $X(z) = \frac{z^2-2}{(z-1)(z-2)}$. Solution:

$$\begin{aligned}X(z) &= \frac{z^2-2}{z^2-3z+2} = 1 + \frac{3z-4}{z^2-3z+2} \\ X(z) &= 1 + \frac{a_1}{z-1} + \frac{a_2}{z-2} \\ a_1 &= \lim_{z\to 1}[(z-1)X(z)] = 1 \\ a_2 &= \lim_{z\to 2}[(z-2)X(z)] = 2 \\ X(z) &= 1 + \frac{1}{z-1} + \frac{2}{z-2} = \frac{1}{z-1} + \frac{z}{z-2} \\ x[k] &= 1 + 2^k - \delta[k]\end{aligned}$$

EE402 - Discrete Time Systems **Spring 2018**

Lecture 3

Lecturer: Asst. Prof. M. Mert Ankarali

Difference Equations

We have covered that in discrete-time domain we have difference equations that replaces differential equations. Since we are mainly interested in LTI systems, that are represented by linear constant coefficient difference equations. Let $x[k]$ and $y[b]$ be the input and output respectively, an LTI difference equation can be expressed as

$$a_0 y[k] + a_1 y[k-1] + ... + a_N y[k-N] = b_0 x[k] + ... + b_M x[k-M]$$
$$\sum_{n=1}^{N} a_n y[k-n] = \sum_{n=1}^{M} b_n x[k-n]$$

Unlike ODEs difference equations are very easy to solve computationally or simulate in Computer environment. Let's consider the following first-order difference equation

$$y[k] = \frac{1}{2} y[k-1] + x[k] \quad , x[k] = 0 \; \& \; y[k] = 0, \; \text{for} k < 0$$

Let's "simulate" the difference equation for $x[k] = \delta[k]$.

$$\begin{aligned} y[0] &= \frac{1}{2} y[-1] + x[0] = 0 + 1 = 1 \\ y[1] &= \frac{1}{2} y[0] + x[1] = \frac{1}{2} + 0 = \frac{1}{2} \\ y[2] &= \frac{1}{2}\frac{1}{2} = \frac{1}{4} \\ y[3] &= \frac{1}{2}\frac{1}{4} = \frac{1}{8} \\ &\vdots \\ y[k] &= \left(\frac{1}{2}\right)^k \end{aligned}$$

Now let's simulate for $x[k] = u[k]$

$$
\begin{aligned}
y[0] &= 0 + 1 = 1 \\
y[1] &= \frac{1}{2} + 1 \\
y[2] &= \frac{1}{4} + \frac{1}{2} + 1 \\
y[3] &= \frac{1}{8} + \frac{1}{4} + \frac{1}{2} + 1 \\
&\vdots \\
y[k] &= \frac{1}{2^k} + \cdots + \frac{1}{2} + 1 = 2 - \left(\frac{1}{2}\right)^k
\end{aligned}
$$

This is a great method for "simulating" using a computational approach, but in general it may be very hard to get a closed from expression. The most basic solution method is solving the difference equation directly in time domain by trying to find a "basis" for the solution space similar to the operation in ODEs. We try sequences/signals of the form $\lambda^k\ ,\ k > 0$ to find a solution form for the homogeneous equation. Let's apply this method for the first-order difference equation above

$$
\begin{aligned}
y[k] = \lambda^k \ \rightarrow y[k] - \frac{1}{2}y[k-1] &= 0 \\
\lambda^k - \frac{\lambda^{k-1}}{2} &= 0 \\
\lambda^{k-1}\left(\lambda - \frac{1}{2}\right) &= 0 \\
\lambda - \frac{1}{2} &= 0
\end{aligned}
$$

Where the last equation is the characteristic equation of the difference equation. Since the characteristic equation has one root only, we obtain a solution of the form

$$
y[k] = y_h[k] + y_p[k] = C\left(\frac{1}{2}\right)^k + y_p[k]
$$

Let's assume that for $x[k] = u[k]$ particular solution has the form $y_p[k] = A\ fork > 0$ then

$$
A = \frac{1}{2}A + 1 \ \rightarrow A = 2
$$

Now let's find C using the fact that $y[k] = 0$ for $k < 0$

$$
\begin{gathered}
y[0] = \frac{1}{2}y[-1] + x[0] \ \rightarrow \ y[0] = 1 \\
1 = C\left(\frac{1}{2}\right)^0 + 2 \rightarrow C = -1
\end{gathered}
$$

Then the solution can be written as

$$
y[k] = -\left(\frac{1}{2}\right)^k + 2
$$

Example 1.1 Find the general form of the homogeneous solution for the following difference equation

$$
y[k] - 3y[k-1] + 2y[k-2] = x[k]
$$

Solution:

$$
\begin{aligned}
&\lambda^2 - 3\lambda + 2 = 0 \\
&\lambda_1 = 1 \text{ \& } \lambda_2 = 2 \\
&y[k] = C_1 + C_2 2^k \ , k > 0
\end{aligned}
$$

Example 1.2 Now let's assume that $y[k] = 0$ for $k < 0$ and $x[k] = 3^k$, then find $y[k]$ for $k \geq 0$.

Solution: First let's find a particular solution. Let's assume that $y_p[k] = A3^k$, then

$$
\begin{aligned}
A3^k - 3A3^{k-1} + 2A3^{k-2} = 3^k \ \rightarrow \ A = 9/2 \\
y_p[k] = 4.5 \ 3^k
\end{aligned}
$$

Now let's try to find C_1 and C_2

$$
\begin{aligned}
&y[k] - 3y[k-1] + 2y[k-2] = x[k] \\
&y[0] = x[0] \ \rightarrow \ C_1 + C_2 = -3.5 \\
&y[1] - 3y[0] = x[1] \ \rightarrow \ C_1 + C_2 2 = -7.5 \\
&C_1 = 0.5 \text{ \& } C_2 = -4 \\
&y[k] = 0.5 - 4\,2^k + 4.5\,3^k \ , k > 0
\end{aligned}
$$

What about repeated roots? Possible mini project question

Example 2 Find the general form of the homogeneous solution for the following difference equation

$$
y[k] + 4y[k-2] = x[k]
$$

Solution:

$$
\begin{aligned}
&\lambda^2 + 4 = 0 \ \rightarrow \ \lambda_{1,2} = \pm 2j \\
&y[k] = C_1(2j)^k + C_2(-2j)^k = C_1 2^k e^{j\pi k} + C_2 2^k e^{-j\pi k} \\
&y[k] = \bar{C}_1 2^k \frac{e^{j\pi k} + e^{j\pi k}}{2} + \bar{C}_2 2^k \frac{e^{j\pi k} - e^{j\pi k}}{2j} \\
&y[k] = \bar{C}_1 2^k \cos(\pi k) + \bar{C}_2 2^k \sin(\pi k)
\end{aligned}
$$

How we can generalize this to arbitrary complex conjugate roots? Possible mini project question

What is the home message? Similar to ODEs time domain solution of difference equations is generally "messy".

Z-transform & Difference Equations

Difference Equations to Z-transform

Let's consider the following difference equation with $y[n]$ and $x[n]$ be the strictly causal input–output pair.

$$
a_0 y[k] + a_1 y[k-1] + ... + a_N y[k-N] = b_0 x[k] + ... + b_M x[k-M]
$$

Now let's assume that $\mathcal{Z}\{x[k]\} = X(z)$ and $\mathcal{Z}\{y[k]\} = Y(z)$. If we take the Z-transform fo the both sides of the equation by applying the shifting theorem we obtain

$$\begin{aligned} a_0 Y(z) + a_1 z^{-1} Y(z) + ... + a_N z^{-N} Y(z) &= b_0 X(z) + ... + b_M z^{-M} X(z) \\ (a_0 + a_1 z^{-1} + ... + a_N z^{-N}) Y(z) &= (b_0 + b_1 z^{-1} + ... + b_M z^{-M}) X(z) \\ \frac{Y(z)}{X(z)} = G(z) &= \frac{b_0 + b_1 z^{-1} + ... + b_M z^{-M}}{a_0 + a_1 z^{-1} + ... + a_N z^{-N}} \\ &= z^{N-M} \frac{b_0 z^M + b_1 z^{M-1} + ... + b_M}{a_0 z^N + a_1 z^{N-1} + ... + a_N} \end{aligned}$$

Under "zero initial conditions" if we can find $X(z)$ then simply $Y(z) = G(z)X(z)$. After that we can take the inverse z-transform and compute $y[k]$.

Example 3.1 Compute $y[k]$ using the Z-transform method

$$\begin{aligned} y[k] &= \frac{1}{2} y[k-1] + x[k] \\ y[k] &= 0, \text{ for } k < 0 \;\&\; x[k] = \delta[k] \end{aligned}$$

Solution:

$$\begin{aligned} Y(z) &= \frac{1}{2} Y(z) z^{-1} + X(z) \;\rightarrow\; \frac{Y(z)}{X(z)} = G(z) = \frac{z}{z - 1/2} \\ Y(z) &= \frac{z}{z-1/2} \;\rightarrow\; y[k] = (\frac{1}{2})^k \end{aligned}$$

Example 3.2 Now let's compute $y[k]$ for $x[k] = u[k]$

$$\begin{aligned} Y(z) &= G(z)X(z) \;\rightarrow\; Y(z) = \frac{z^2}{(z-1/2)(z-1)} \\ Y(z) &= -\frac{z}{z-1/2} + 2\frac{z}{z-1} \\ y[k] &= 2 - \left(\frac{1}{2}\right)^k \end{aligned}$$

Example 4 For the following difference equation, compute $y[k]$ for $x[k] = x[k] = 3^k u[k]$

$$y[k] - 3y[k-1] + 2y[k-2] = x[k]$$

Solution:

$$\begin{aligned} Y(z)(1 - 3z^{-1} + 2z^{-2}) = X(z) \;&\rightarrow\; G(z) = \frac{z^2}{z^2 - 3z + 2} = \frac{z^2}{(z-1)(z-2)} \\ Y(z) &= \frac{z^3}{(z-1)(z-2)(z-3)} = 0.5\frac{z}{z-1} - 4\frac{z}{z-2} + 4.5\frac{z}{z-3} \\ y[k] &= \left(0.5 - 4\; 2^k + 4.5\; 3^k\right) u[k] \end{aligned}$$

Z-transform to Difference Equations

Sometimes the Z-domain transfer function of a system is given, and we may be supposed to find the difference equation representation. Let's assume that we have a general transfer function that can be represented in terms of ratio of two polynomials in z or z^{-1} as given below

$$\frac{Y(z)}{X(z)} = G(z) = \frac{b_0 + b_1 z^{-1} + ... + b_M z^{-M}}{a_0 + a_1 z^{-1} + ... + a_N z^{-N}} = z^{N-M} \frac{b_0 z^M + b_1 z^{M-1} + ... + b_M}{a_0 z^N + a_1 z^{N-1} + ... + a_N}$$

In his case, I prefer to work with the polynomials that are written in terms of z^{-1}. Let's manipulate the Z-domain equation to obtain

$$Y(z)(a_0 + a_1 z^{-1} + ... + a_N z^{-N}) = X(z)(b_0 + b_1 z^{-1} + ... + b_M z^{-M})$$
$$a_0 Y(z) + a_1 z^{-1} Y(z) + ... + a_N z^{-N} Y(z) = b_0 X(z) + b_1 z^{-1} X(z) + ... + b_M z^{-M} X(z)$$

Let's assume that $\mathcal{Z}^{-1}\{Y(z)\} = y[k]$ and $\mathcal{Z}^{-1}\{X(z)\} = x[k]$. If we take the inverse Z-transform of both sides by applying the shifting theorem we obtain

$$a_0 y[k] + a_1 y[k-1] + ... + a_N z^{-N} y[k-N] = b_0 x[k] + b_1 x[k-1] + ... + b_M x[k-M]$$

We can use this conversion to "simulate" a given discrete time transfer function or realizing the given system (it may be a filter or controller) to implement on an embedded platform.

It can also be used for computationally finding the inverse Z-transform of a given z-domain rational function. The next example will illustrate this feature.

Example 5 Find a computational solution for the inverse Z-transform of $H(z) = \frac{z^{-1}}{1-2z^{-1}+z^{-2}}$ by using the conversion from Z-domain transfer function to difference equation concept.

Solution: Let's assume that $H(z)$ is a "transfer function" not an arbitrary z-domain function. Then $\mathcal{Z}^{-1}\{H(z)\} = h(t)$ becomes the impulse response of the "system". Thus we can assume some imaginary input-output pair $y[n]$ and $x[n]$ where

$$\frac{Y(z)}{X(z)} = H(z)$$

If we can find a difference equation realization for $H(z)$ then we can simulate the difference equation by assuming $x[k] = \delta[k]$ (i.e. unit impulse input). So let's find a realization for the given $H(z)$ as

$$\frac{Y(z)}{X(z)} = \frac{z^{-1}}{1 - 2z^{-1} + z^{-2}}$$
$$Y(z) - 2z^{-1} Y(z) + z^{-2} Y(z) = z^{-1} X(z)$$
$$y[k] - 2y[k-1] + y[k-2] = x[k-1]$$

Now let's simulate the above equation for $x[k] = \delta[k]$

$$\begin{aligned}
y[k] &= 2y[k-1] - y[k-2] + x[k-1] \\
y[0] &= 2y[-1] - y[-2] + x[-1] = 0 \\
y[1] &= 2y[0] - y[-1] + x[0] = 1 \\
y[2] &= 2y[1] - y[0] + x[1] = 2 \\
y[3] &= 2y[2] - y[1] + x[2] = 3 \\
y[4] &= 4 \\
&\cdots \\
y[k] &= k
\end{aligned}$$

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Lecture 3

Lecturer: Asst. Prof. M. Mert Ankarali

Realization of Discrete Time Systems / Filters / Controllers

In this lecture we will cover some basic block-diagram realization techniques for discrete time systems/filters/controllers which are represented by Z-Domain transfer functions or difference equations.

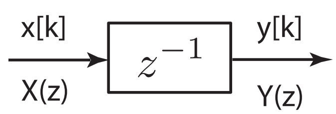
Block-diagram realizations are extremely useful practically for the purpose of implementing the system/filter/controller on a real embedded system of a hardware platform. The goal can be programming the discrete filter/controller on a microcontroller, or embedding the structure on an fpga module a block diagram representation is always helpful.

In the future weeks of the course, we will also actively use block-diagram representation to obtain a state-space realization of the given discrete-time system.

The most fundamental block for a discrete-time system is the unit delay operator

$$y[k] = x[k-1]$$
$$Y(z) = z^{-1}X(z)$$

which is represented by the following block-diagram



In this lecture our goal is to realize different kinds of discrete time transfer functions using this this fundamental block as the main brick.

Realization of FIR Systems

A third order (order is fixed for the sake of clarity) FIR (Finite impulse response) discrete time system has the following difference equation and transfer function

$$y[k] = b_0x[k] + b_1x[k-1] + b_2x[k-2] + b_3x[k-3]$$
$$Y(z) = \left(b_0 + b_1z^{-1} + b_2z^{-2} + b_3z^{-3}\right) X(z)$$

From inspection it is easy to see that we need at least three unit delay blocks (and memory elements) to construct a full realization. Below the block-diagram realization of a third order FIR is given in Fig. 3.1

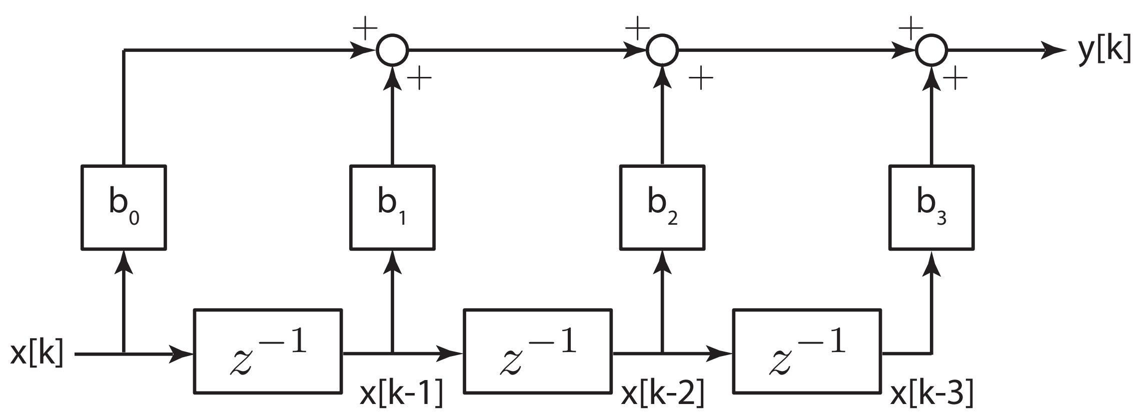


Figure 3.1: Block diagram realization of a third order FIR system

Realization of IIR Systems with Static Numerator Dynamics

In this part we will analyze a special case if IIR (Infinite impulse response) systems where the input dynamics is static, i.e. there is no direct delayed term of the input in the difference equation. Again for the sake of clarity let's assume that the discrete system is third order. For such a system, the difference equation and the transfer function can be written as

$$y[k] = -a_1x[k-1] - a_2y[k-2] - a_3y[k-3] + b_0x[k]$$
$$Y(z) = \frac{b_0}{1 + a_1z^{-1} + a_2z^{-2} + a_3z^{-3}}X(z)$$

Similar to the FIR case we also need minimum three delay blocks to realize this system, However, now delay blocks are in the feedback-loop. The block digram representation of the given IIR system is given in Fig. 3.2.

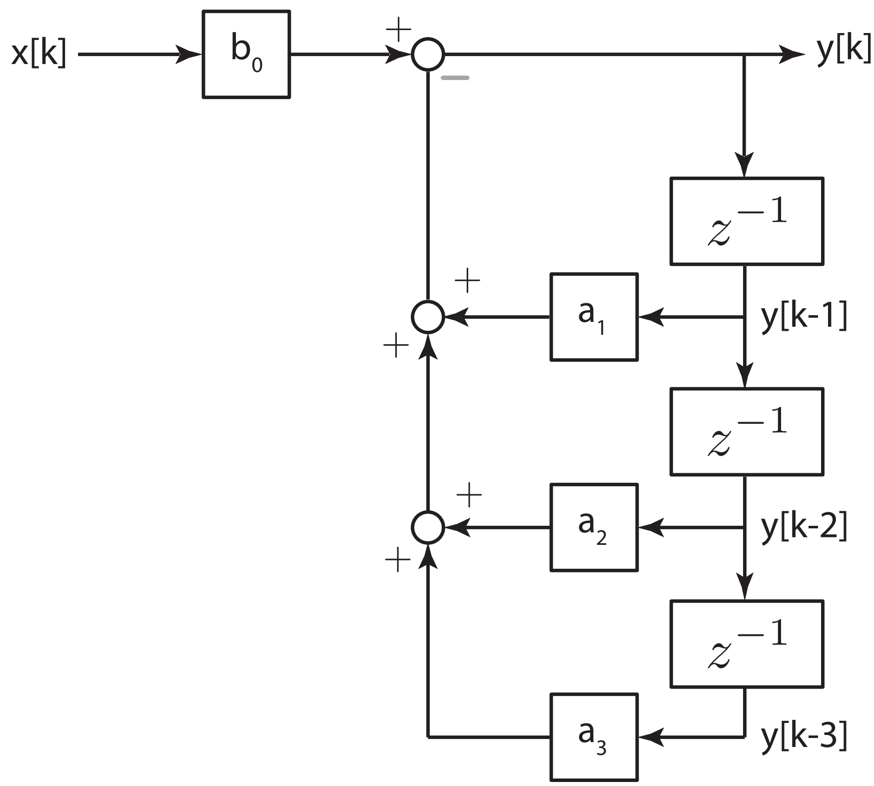


Figure 3.2: Block diagram realization of a third order IIR system with static numerator/input dynamics

Realization of General Discrete-Time Systems

Now we will attempt to obtain a realization of more general discrete-time systems/filters/controllers. Again for the sake oc clarity let's keep the order (maximum order of z^{-1} in the equation) of the system to 3. A general third order discrete time system can be expressed with following difference equation and transfer function

$$y[k] = -a_1 y[k-1] - a_2 y[k-2] - a_3 y[k-3] + b_0 x[k] + b_1 x[k-1] + b_2 x[k-2] + b_3 x[k-3]$$
$$Y(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3}}{1 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3}} X(z)$$

Non-minimal Realization / Direct Programming

One way of realizing a discrete-time system is simply combining the block diagrams of special cases (i.e. FIR and IIR with static input dynamics) given in Figures 3.1 & 3.2. The block diagram realization obtained with this method can be observed in Fig. 3.3

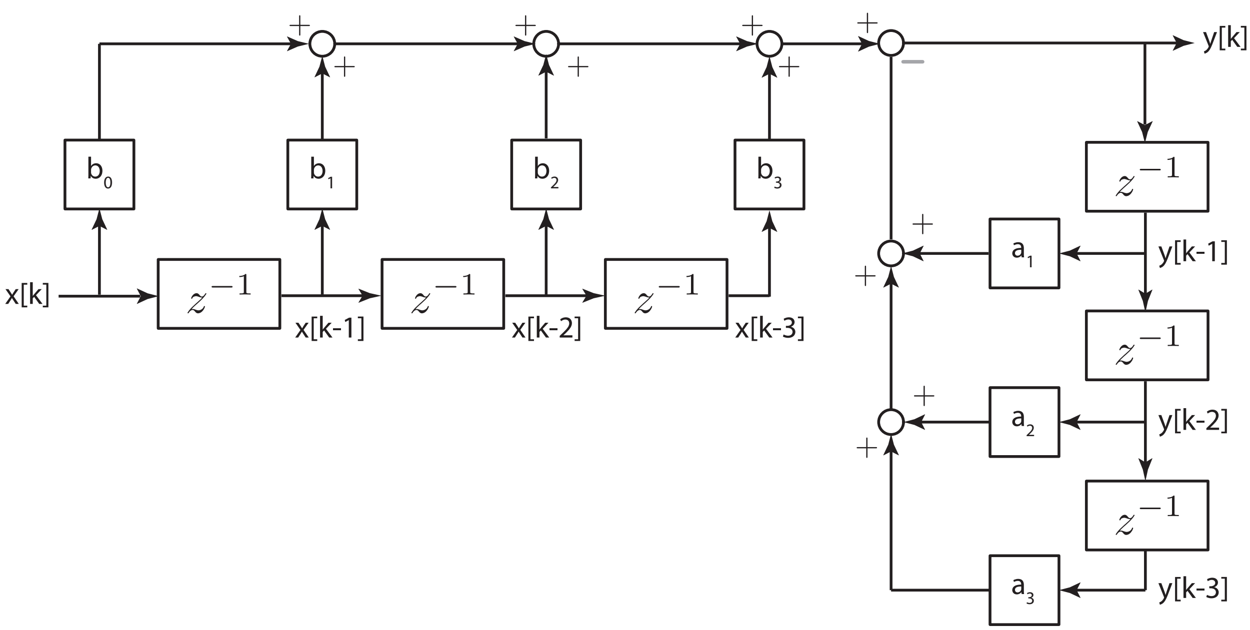


Figure 3.3: Direct/non-minimal realization of a third order discrete time dynamical system

The obvious problem in this realization is that even though the transfer function represents a third order discrete-dynamical system, the "order" of the block diagram is indeed 6 not 3. Because there exist 6 different memory blocks, i.e. delay elements. If we forexample obtain a state-space model from this block diagram we would obtain a system with 6 states.

Canonical Realization I / Standard Programming

In this method of realization, we will use the fact the system is LTI. Let's consider the transfer function of the system and let's perform some LTI operations.

$$
\begin{aligned}
Y(z) &= \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3}}{1 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3}} X(z) \\
&= \left(b_0 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3}\right) \frac{1}{1 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3}} X(z) \\
&= G_2(z) G_1(z) X(z) \text{ where} \\
G_1(z) &= \frac{H(z)}{X(z)} = \frac{1}{1 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3}} \\
G_2(z) &= \frac{Y(z)}{H(z)} = b_0 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3}
\end{aligned}
$$

As you can see we introduced an intermediate variable $h[k]$ with a Z-transform of $H(z)$, First transfer function, which is an IIR system with static input dynamics operates on $x[n]$ and produces an output. Second transfer function operates on $h[n]$ and produces output $x[n]$. If we write the difference equations of both systems we obtain

$$
\begin{aligned}
h[k] &= -a_1 h[y-1] - a_2 h[k-2] - a_3 h[k-3] + x[k] \\
y[k] &= b_0 x[k] + b_1 h[k-1] + b_2 h[k-2] + h_3 x[k-3]
\end{aligned}
$$

As it can be seen that the delay/shifting operations are only performed on the signal $h[k]$ and maximum delay operation is by 3 samples. Basically if we utilize this structure we can draw a minimal block diagram representation as given in Fig. 3.4. If we obtain a state-space model from this block diagram, the form will be in *controllable canonical form*. We will cover this later in the semester. Thus we can call this representation also as *controllable canonical realization*.

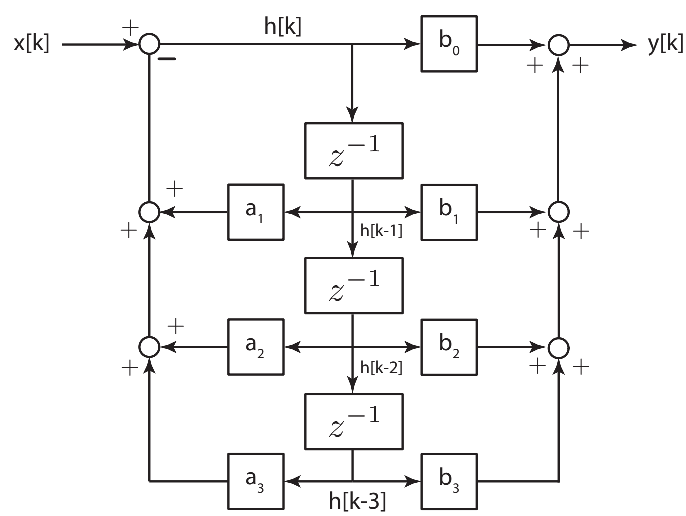


Figure 3.4: A minimal block diagram realization of a discrete time system obtained with standard programming (Canonical representation I)

Canonical Realization II

In this method will obtain a different minimal realization. The process will be different and this the block diagram will have different topology. Let's start with the transfer function and perform some grouping based on the delay elements.

$$Y(z)(1 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3}) = (b_0 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3})X(z)$$
$$Y(z) = b_0 X(z) + z^{-1}\left(b_1 X(z) - a_1 Y(z)\right) + z^{-2}\left(b_2 X(z) - a_2 Y(z)\right) + z^{-3}\left(b_3 X(z) - a_3 Y(z)\right)$$
$$Y(z) = b_0 X(z) + z^{-1}\left\{\left(b_1 X(z) - a_1 Y(z)\right) + z^{-1}\left[\left(b_2 X(z) - a_2 Y(z)\right) + z^{-1}\left(b_3 X(z) - a_3 Y(z)\right)\right]\right\}$$

As you can see we have only z^{-1} terms in the representation there is a special toplogy embedded inside the expression. If we convert it to the block diagram form we obtain the structure given in Fig. 3.5. If we obtain a state-space model from this block diagram, the form will be in *observable canonical form.* Thus we can call this representation also as *observable canonical realization.* This form and representation is the dual of the previous representation.

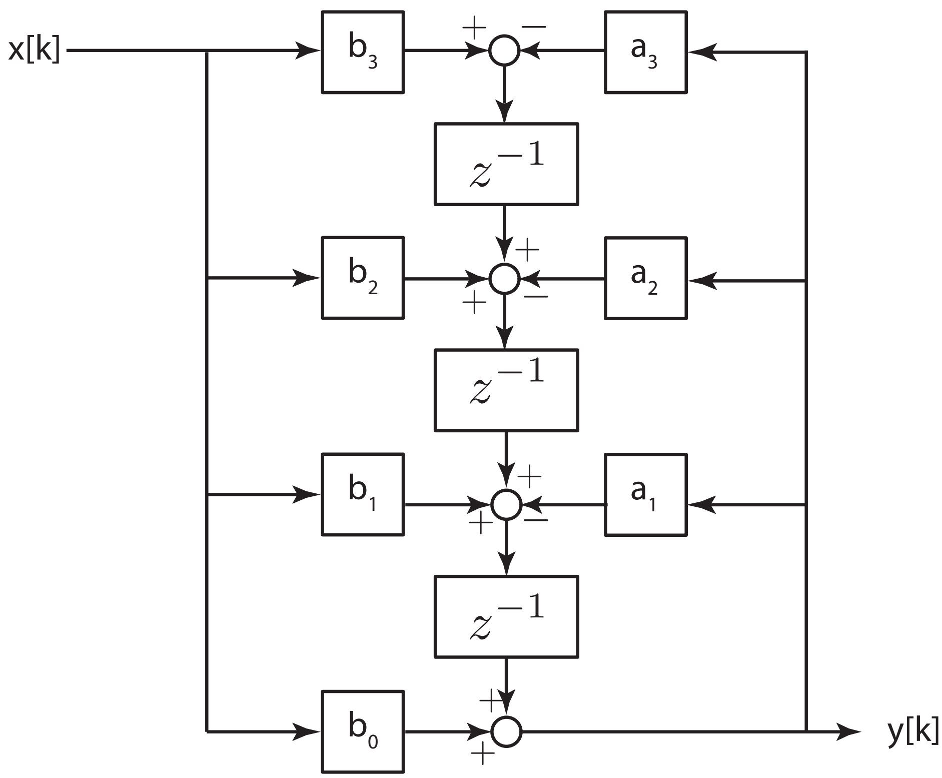


Figure 3.5: A minimal block diagram realization of the 3^{rd} order discrete time system obtained with Canonical representation II

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Lecture 5

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Let's remember the idealized and simplified block-diagram structure a discrete-time control system (See Fig. 5.1)

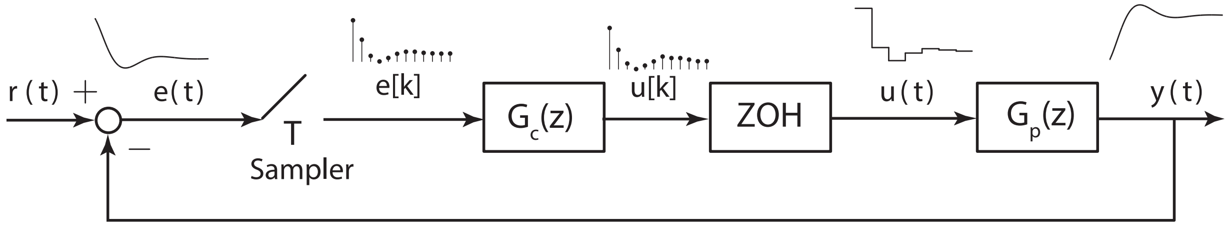


Figure 5.1: Block diagram of an LTI discrete-time control system

Loop contains both continuous-time and discrete-time signals and blocks.

- We can treat the system as a completely discrete-time system. We technically restrict ourselves into sampled time instants (which may be just fine)
- Alternatively, we can use continuous time signals (as much as possible) and deal with starred versions of signals and starred Laplace transform.

Sampling - Review

Fig. 5.2 illustrates an ideal *impulse sampler* and an ideal complete CT-to-DT sampler.

The output of the impulse sampler, $x^*(t)$, can be represented with the following infinte summations

$$x^*(t) = \sum_{k=0}^{\infty} x(kT)\delta(t-kT) = \sum_{k=0}^{\infty} x[k]\delta(t-kT)$$

Now let's consider the Laplace transform of $x^*(t)$

$$X^*(s) = \mathcal{L}\{x^*(t)\} = \sum_{k=0}^{\infty} x(kT) \int_{t=0}^{\infty} \delta(t-kT)e^{-st}dt = \sum_{k=0}^{\infty} x(kT)e^{-skT} = \sum_{k=0}^{\infty} x[k]e^{-skT}$$

Now let's define a map in complex domain such that $z = e^{Ts}$ or $s = \frac{1}{T}\ln z$. Then we have

$$X^*(s)|_{s=(1/T)lnz} = \sum_{k=0}^{\infty} x[k]z^{-k}$$

where

$$X(z) = \mathcal{Z}\{x[k]\} = \sum_{k=0}^{\infty} x[k]z^{-k}$$

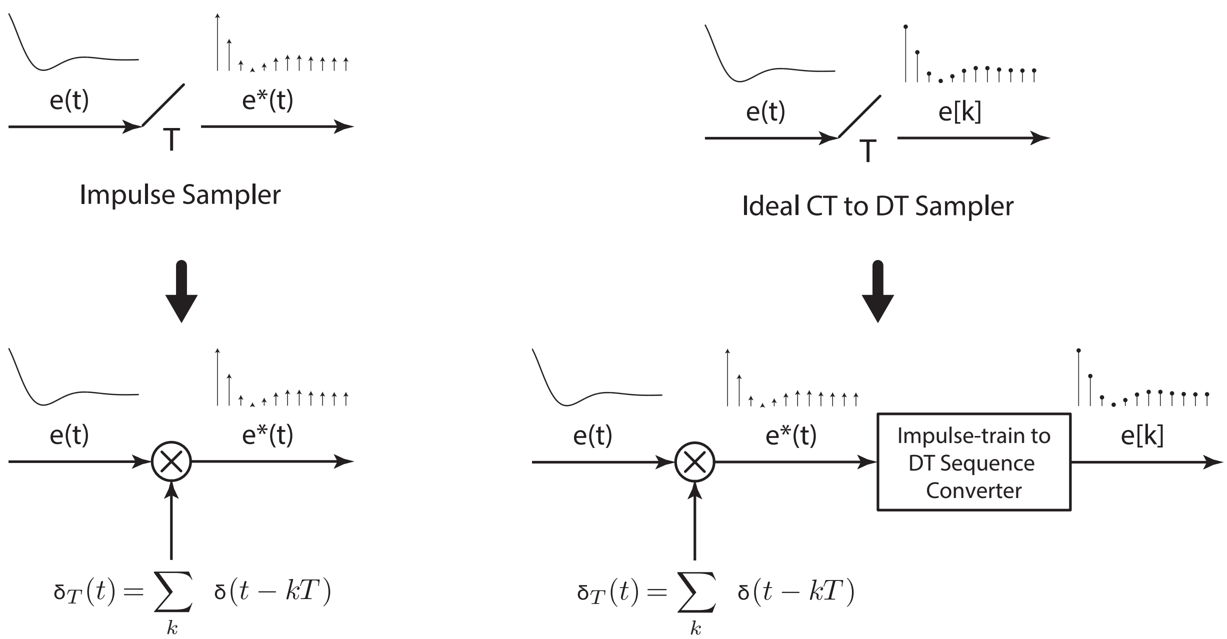
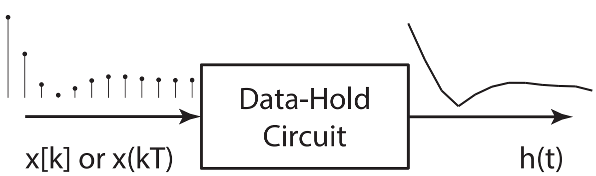


Figure 5.2: Two different ideal samplers

Data Hold Operation

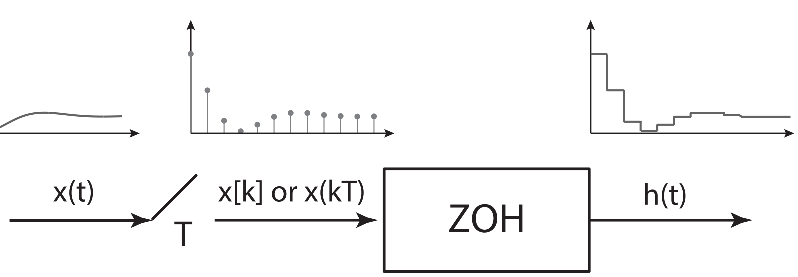
Data-Hold operation is an idealized model of a DAC device which converts a digital signal to an analog signal. In terms of the terminology used in this class, Data-Hold operation is the process of obtaining a CT signal $h(t)$ from a DT sequence. A general data-hold operation block circuit is shown below



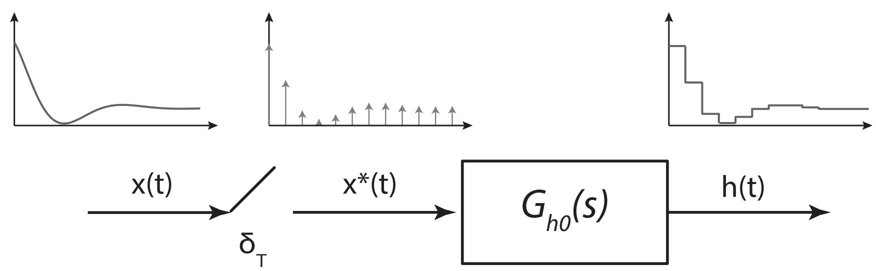
Simplest and most dominantly used (I have never seen a practical usage of other hold operations) hold circuit/operation is the zero-order-hold (ZOH). Basically, at each time instant kT ZOH "samples" the input $x[k]$ or $x(kT)$ and "holds" this value at the output until the next sampling event. Mathematically,

$$h(kT + t) = x(kT) = x[k], \text{ for } 0 \geq t < T$$

The figure below illustrates a serios connection of an ideal CT-DT sampler and an ideal ZOH block.



If we model the sampler using an ideal impulse sampler (not CT-DT converter) then it becomes more convenient to model the ZOH with a CT transfer function as shown with the block diagram below



Let's assume that $x(t)$ is a strictly causal signal, then from the definition of ZOH we can express $h(t)$ in terms of $x(t)$ (or $x^*(t)$, $x[k]$, $x(kT)$) as

$$h(t) = x(0)\,[u(t) - u(t-T)] + x(T)\,[u(t-T) - u(t-2T)] + x(2T)\,[u(t-2T) - u(t-3T)] + \cdots$$

$$h(t) = \sum_{k=0}^{\infty} x(kT)\,[u(t-kT) - u(t-(k+1)T)]$$

If we take the Laplace transform of $h(t)$, we obtain

$$\begin{aligned}
\mathcal{L}\{h(t)\} &= \sum_{k=0}^{\infty} x(kT)\mathcal{L}\left\{[u(t-kT) - u(t-(k+1)T)]\right\} \\
&= \sum_{k=0}^{\infty} x(kT)\left[\frac{e^{-kTs}}{s} - \frac{e^{-(k+1)Ts}}{s}\right] \\
&= \frac{1-e^{-Ts}}{s}\sum_{k=0}^{\infty} x(kT)e^{-kTs} = \frac{1-e^{-Ts}}{s}\sum_{k=0}^{\infty} x[k]e^{-kTs} \\
H(s) &= \frac{1-e^{-Ts}}{s}X^*(s) = G_{h0}(s)X^*(s) \\
G_{h0}(s) &= \frac{1-e^{-Ts}}{s}
\end{aligned}$$

Z-transform & ZOH

When analyzing the discrete time control systems, we will (frequently) need to compute the Z-transform of sampled signals, for which the Laplace transform involves the term $\frac{1-e^{-Ts}}{s}$.

Let $\mathcal{L}\{x(t)\} = X(s) = \frac{1-e^{-Ts}}{s}G(s)$. Now let's analyze the z-transform of the sampled version of the signal, i.e. $X(z) = \mathcal{Z}\{x^*(t)\}$. First let's find $x(t)$ from $X(z)$

$$\begin{aligned}
x(t) = \mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\{X(s)\} &= \mathcal{L}^{-1}\left\{\frac{1-e^{-Ts}}{s}G(s)\right\} = \mathcal{L}^{-1}\left\{(1-e^{-Ts})\frac{G(s)}{s}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{G(s)}{s}\right\} - \mathcal{L}^{-1}\left\{e^{-Ts}\frac{G(s)}{s}\right\}
\end{aligned}$$

Let $\hat{g}(t) = \mathcal{L}^{-1}\left\{\frac{G(s)}{s}\right\}$ then

$$x(t) = \hat{g}(t) - \hat{g}(t-T)$$

$x(kT)$ and $x[k]$ takes the form

$$\begin{aligned} x(kT) &= \hat{g}(kT) - \hat{g}(kT - T) \\ x[k] &= \hat{g}[k] - \hat{g}[k-1] \end{aligned}$$

Then $X(z)$ takes the form

$$X(z) = \left(1 - z^{-1}\right) \hat{G}(z)$$

where $\hat{G}(z) = \mathcal{Z}\left\{\left[\mathcal{L}^{-1}\left\{\frac{G(s)}{s}\right\}\right]^*\right\}$. In the textbook this notation is shortened to have $\hat{G}(z) = \mathcal{Z}\left\{\frac{G(s)}{s}\right\}$. After that we have

$$X(z) = \left(1 - z^{-1}\right) \mathcal{Z}\left\{\frac{G(s)}{s}\right\}$$

Example 1. Obtain the z transform of $x(kT)$ where $T = 1$ $X(s)$ is given as

$$X(s) = \frac{1 - e^{-s}}{s} \frac{1}{s+1}$$

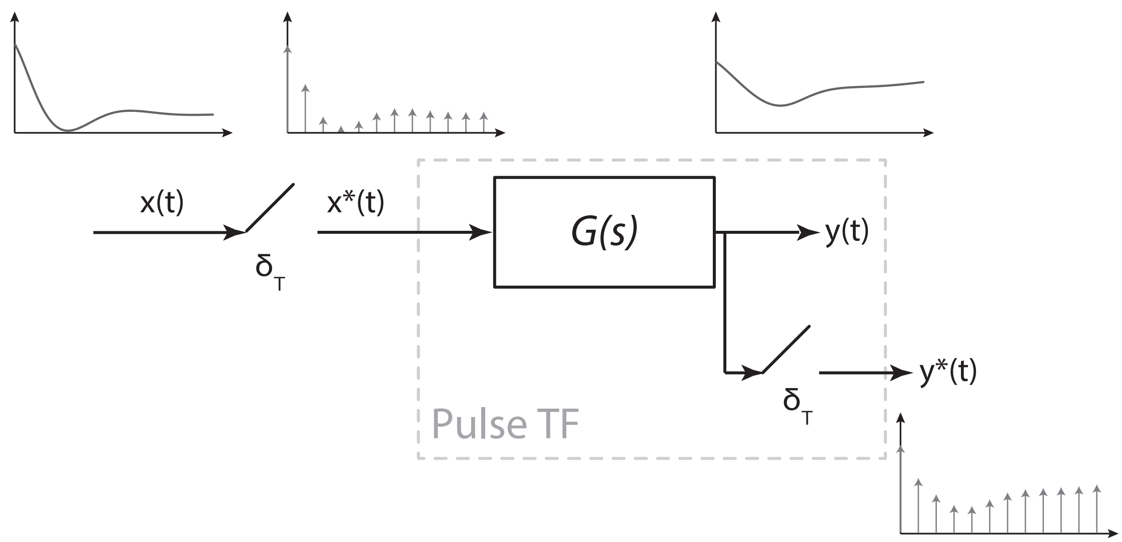
Solution:

$$\begin{aligned} X(z) &= \left(1 - z^{-1}\right) \mathcal{Z}\left\{\frac{1}{s(s+1)}\right\} = \left(1 - z^{-1}\right) \mathcal{Z}\left\{\frac{1}{s} - \frac{1}{s+1}\right\} \\ &= \frac{z-1}{z}\left(\frac{z}{z-1} - \frac{z}{z - e^{-1}}\right) = 1 - \frac{z-1}{z - e^{-1}} \\ X(z) &= \frac{1 - e^{-1}}{z - e^{-1}} \end{aligned}$$

Pulse Transfer Function

The CT transfer function relates the Laplace transform of the continious-time output, $y(t)$ and $Y(s)$, to that of the continious-time input $x(t)$ and $X(s)$.

The Pulse Transfer Function will relate the Z transform of the output, $y*(t)$ (or $y(kT)$) and $Y(z)$, to that of the sampled input, $x^*(t)$ (or $x(kT)$) and $X(z)$. Note that without a feedback-loop the sampling at the output remains purely synthetic. The figure below illustrates the signals and associated transfer function blocks:



Let $g(t)$ be the impulse response of the transfer function $G(s)$, then we know that

$$y(t) = \int_0^t g(t-\tau)x^*(\tau)d\tau$$

$$y(t) = \int_0^t g(t-\tau)\sum_{k=0}^{\infty} x(kT)\delta(\tau - kT)d\tau$$

Let $t = nT + \hat{t}$ where $\hat{t} \in [0, T)$ then

$$y(nT+\hat{t}) = \int_0^{nT+\hat{t}} g(nT+\hat{t}-\tau)\sum_{k=0}^{\infty} x(kT)\delta(\tau - kT)d\tau$$

$$y(nT+\hat{t}) = \sum_{k=0}^{n} x(kT)\int_0^{nT+\hat{t}} g(nT+\hat{t}-\tau)\delta(\tau - kT)d\tau$$

$$y(nT+\hat{t}) = \sum_{k=0}^{n} x(kT)g((n-k)T+\hat{t})$$

Let $\epsilon = 0$, then

$$y(nT) = \sum_{k=0}^{n} x(kT)g((n-k)T)$$

$$y[n] = \sum_{k=0}^{n} x[k]g[(n-k)]$$

In other words

$$y(nT) = x(nT) * g(nT) = g(nT) * x(nT)$$
$$y[n] = x[n] * g[n] = g[n] * x[n]$$

The result is pretty interesting: the impulse response of the "discretized" system is equal to the signal which is obtained by sampling the impulse response function of original the continuous time system.

If we take the Z transform of the equation given by the convolution (remember the properties of Z-transform) we obtain

$$Y(z) = G(z)X(z) \ \rightarrow \ G(z) = \frac{Y(z)}{X(z)}$$

where $G(z)$ is called the **Pulse Transfer Function of the DT System**. Note that

$$G(z) = \sum_{k=0}^{\infty} g(kT) z^{-k}$$

If we know $g(t)$ and $G(s)$, given the sampling time T, we can compute $g[k]$ and $G(z)$.

Can we say something extra regarding the starred Laplace transforms and Z transforms of y, x, and g?

$$\begin{aligned} Y(s) &= G(s)X^*(s) \ \rightarrow Y^*(s) = [G(s)X^*(s)]^* \\ Y(z) &= G(z)X(z) \end{aligned}$$

Given that starred Laplace transform is the z-transform where z is evaluated e^{TS} we can conclude that

$$Y^*(s) = [G(s)X^*(s)]^* = G^*(s)X^*(s)$$

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Lecture 6

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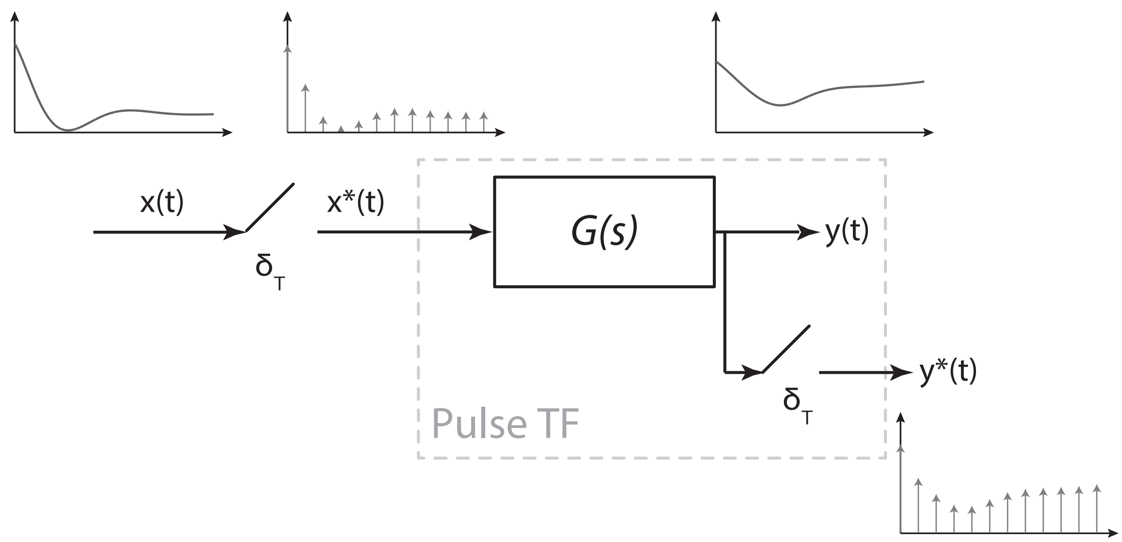
Pulse Transfer Functions

For the following block diagram representation we have the following time domain input–output representations

$$y(t) = g(t) * x^*(t) \quad \text{CT} - \text{convolution}$$
$$y[n] = g[n] * x[n] \quad \text{DT} - \text{convolution}$$



We then showed that Input–output relation between $x[n]$ and $y[n]$ in Z-domain can be represented with the following Pulse Transfer Function equation

$$Y(z) = G(z)X(z) \;\rightarrow\; G(z) = \frac{Y(z)}{X(z)}$$

If we take the Laplace transform of the CT-convolution equation

$$Y(s) = G(s)X^*(s) \;\rightarrow Y^*(s) = [G(s)X^*(s)]^*$$

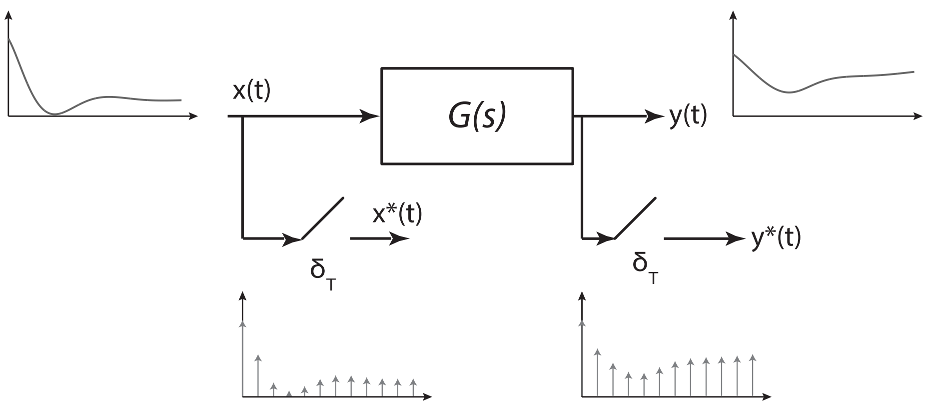
Now let's analyze $Y^*(s)$

$$Y(s) = G(s)X^*(s) \;\rightarrow Y^*(s) = [G(s)X^*(s)]^*$$
$$Y(z) = G(z)X(z)$$

Given that starred Laplace transform is the z-transform where z is evaluated e^{Ts} we can conclude that

$$Y(z) = G(z)X(z) \;\rightarrow\; Y^*(s)|_{z=e^{Ts}} = G^*(s)|_{z=e^{Ts}} X^*(s)|_{z=e^{Ts}}$$
$$Y^*(s) = [G(s)X^*(s)]^* = G^*(s)X^*(s)$$

Now let's consider the following system for which there is no sampler of the continuous time block $G(s)$ at the input.

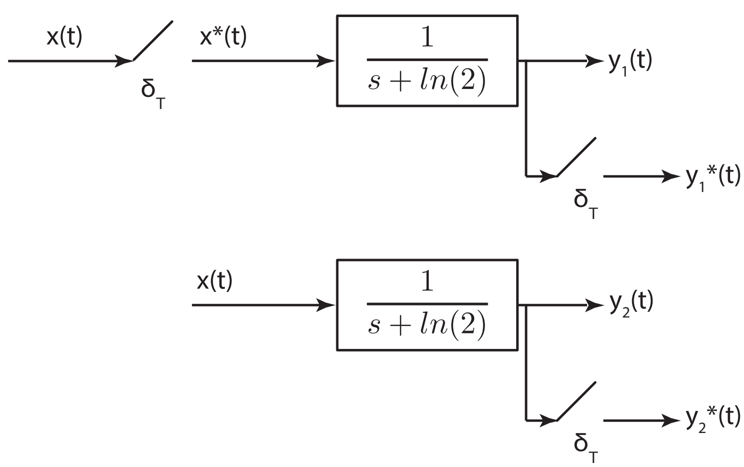


$$Y(s) = G(s)X(s) \;\rightarrow\; Y^*(s) = [G(s)X(s)]^*$$
$$Y^*(s) = [G(s)X(s)]^* = [GX(s)]^* \neq G^*(s)X^*(s)$$

where $GX(s) := G(s)X(s)$. In z-domain we have

$$Y(z) = \mathcal{Z}\{GX(s)\} = GX(z) \neq G(z)X(z)$$

Example: Given that $x(t) = u(t)$ and $T = 1$, compute $Y_1(z)$ and $Y_2(z)$ for the following example in the figure



Solution: Let's start with $Y_1(z)$.

$$\begin{aligned} Y_1(z) &= G(z)X(z) \\ &= \frac{z^2}{(z-1)(z-0.5)} \\ &= \frac{2z}{z-1} - \frac{z}{z-0.5} \end{aligned}$$

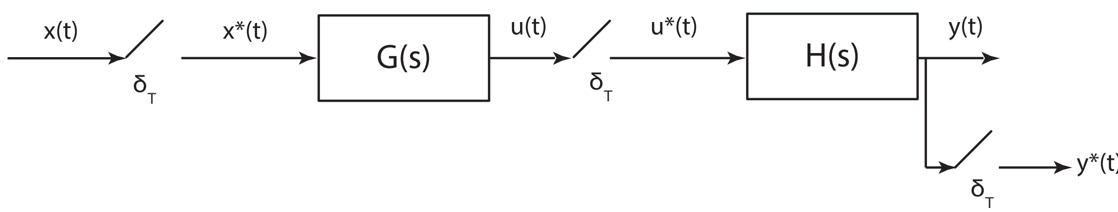
Now let's compute $Y_2(z)$

$$\begin{aligned} Y_2(z) &= GX(z) \\ &= \mathcal{Z}\left\{\frac{1}{s(s+ln(2))}\right\} \\ &= \mathcal{Z}\left\{\frac{1/ln(2)}{s} - \frac{1/ln(2)}{s+ln(2)}\right\} \\ &= \frac{\ln(2)\,z}{z-1} - \frac{\ln(2)\,z}{z-0.5} \end{aligned}$$

We can conclude that $Y_1(z) \neq Y_2(z)$, and thus $y_1(t) \neq y_2(t)$.

Pulse Transfer Functions of Cascaded Elements

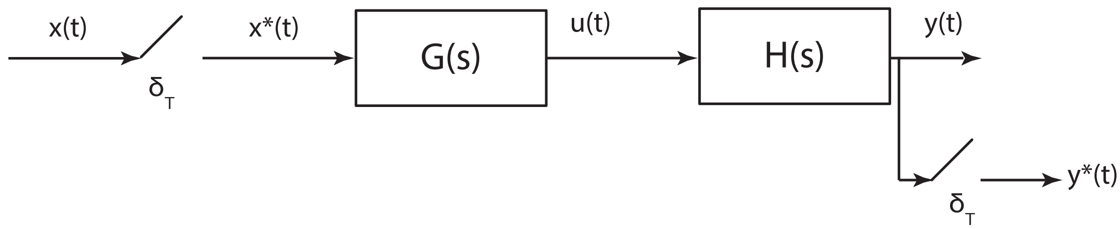
Consider the cascaded system shown in Figure below



Pulse transfer function of this system can be computed

$$\begin{aligned} U^*(s) &= G^*(s)X^*(s) \quad Y^*(s) = H^*(s)U^*(s) \\ Y^*(s) &= H^*(s)G^*(s)X^*(s) \\ Y(z) &= H(z)G(z)X(z) \\ \frac{Y(z)}{X(z)} &= H(z)G(z) \end{aligned}$$

Now, let's consider the following cascaded system

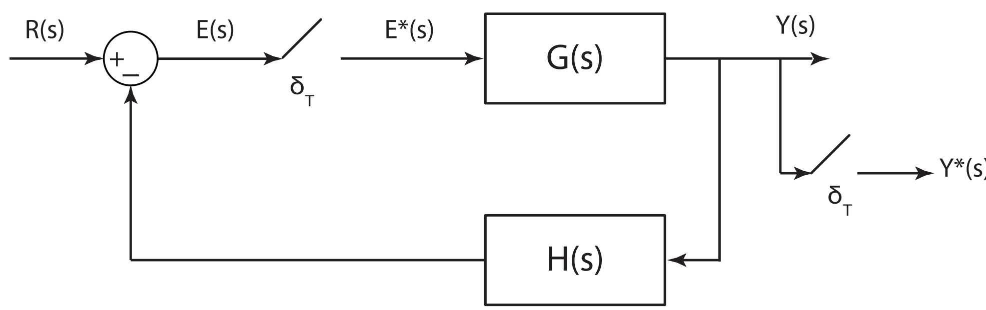


Pulse transfer function of this new system can be computed

$$\begin{aligned}
Y(s) &= H(s)G(s)X^*(s) \\
Y(s) &= HG(s)X^*(s) \quad \text{where } HG(s) = H(s)G(s) \\
Y^*(s) &= [HG(s)X^*(s)]^* = HG^*(s)X^*(s) \\
Y(z) &= HG(z)X(z) \quad \text{where } HG(z) = \mathcal{Z}\{HG(s)\} \\
\frac{Y(z)}{X(z)} &= HG(z) \neq H(z)G(z)
\end{aligned}$$

Pulse Transfer Functions of Closed Loop Systems

SYS1: Consider the following closeed-loop system



$$\begin{aligned}
E(s) &= R(s) - H(s)Y(s) \quad , \quad Y(s) = G(s)E^*(s) \\
E(s) &= R(s) - H(s)G(s)E^*(s) \\
&= R(s) - HG(s)E^*(s)
\end{aligned}$$

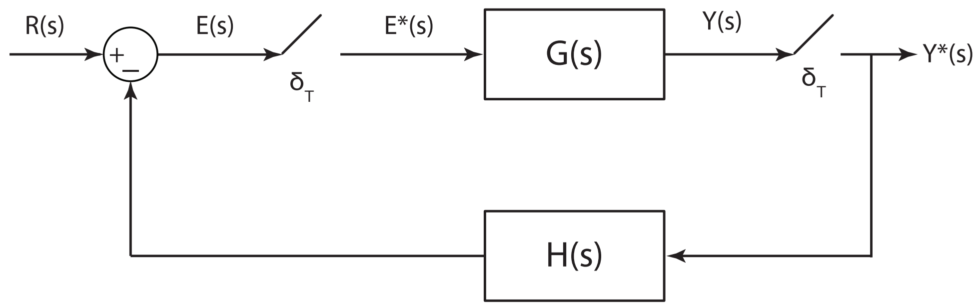
Let's take the starred Laplace transform of both sides

$$\begin{aligned}
E^*(s) &= R^*(s) - HG^*(s)E^*(s) \\
E^*(s) &= \frac{R^*(s)}{1 + HG^*(s)}
\end{aligned}$$

Since $Y^*(s) = G^*(s)E^*(s)$ we can conclude

$$\begin{aligned}
Y^*(s) &= \frac{G^*(s)}{1 + HG^*(s)} R^*(s) \\
Y(z) &= \frac{G(z)}{1 + HG(z)} R(z) \\
\frac{Y(z)}{X(z)} &= \frac{G(z)}{1 + HG(z)}
\end{aligned}$$

SYS2: Now let's consider a slightly different topology



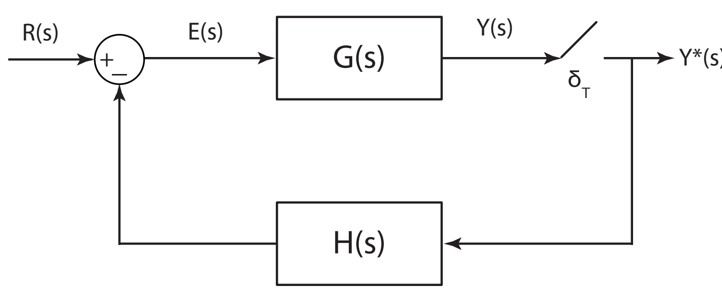
$$
\begin{aligned}
E(s) &= R(s) - H(s)Y^*(s) \\
Y(s) &= G(s)E^*(s) \quad , \quad Y^*(s) = G^*(s)E^*(s) \\
E(s) &= R(s) - H(s)G^*(s)E^*(s) \\
E^*(s) &= R^*(s) - H^*(s)G^*(s)E^*(s) \\
E^*(s) &= \frac{R^*(s)}{1 + H^*(s)G^*(s)}
\end{aligned}
$$

Then we can have

$$
\begin{aligned}
Y^*(s) &= \frac{G^*(s)}{1 + H^*(s)G^*(s)}R^*(s) \\
Y(z) &= \frac{G(z)}{1 + H(z)G(z)}R(z) \\
\frac{Y(z)}{R(z)} &= \frac{G(z)}{1 + H(z)G(z)}
\end{aligned}
$$

It can be seen that the pulse transfer function of the closed-loop systems are different.

SYS3: Now consider another closed-loop system.

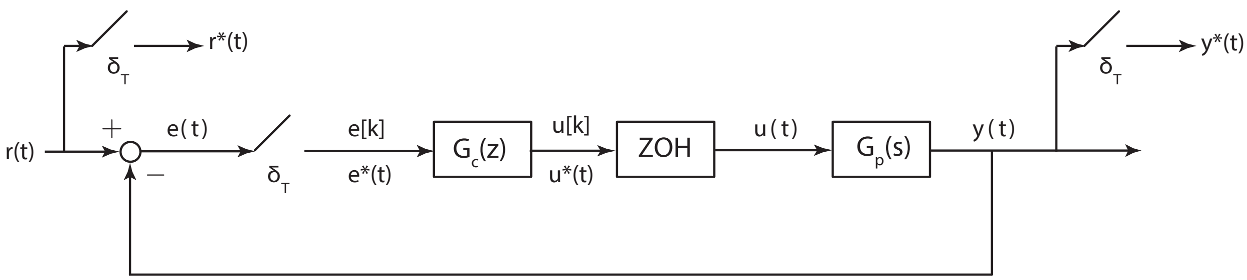


$$
\begin{aligned}
E(s) &= R(s) - H(s)Y^*(s) \qquad Y(s) = G(s)E(s) \\
Y(s) &= G(s)R(s) - G(s)H(s)Y^*(s) \\
Y*(s) &= GR^*(s) - GH^*(s)Y^*(s) \\
Y*(s) &= \frac{GR^*(s)}{1 + GH^*(s)} \\
Y(z) &= \frac{GR(z)}{1 + GH(z)}
\end{aligned}
$$

We can't even find a direct transfer function from $R^*(s)$ to $Y^*(s)$, or equivalently from $R(z)$ to $Y(z)$.

Pulse Transfer Function of A Closed Loop DT Control System

Consider the fundamental DT-control system below



Previously we derived the transfer function of ZOH, thus we can combine ZOH and plant TF into a single TF block $G(s)$ as

$$\frac{Y(s)}{E^*(s)} = G(s) = \frac{1-e^{-sT}}{s} G_p(s)$$

The controller is actually implemented in an hardware/software platfrom which indeed works in discrete-time. However, we can find a starred version of $G_c(z)$ by

$$G_c^*(s) = G_c(z)|_{z=e^{Ts}}$$

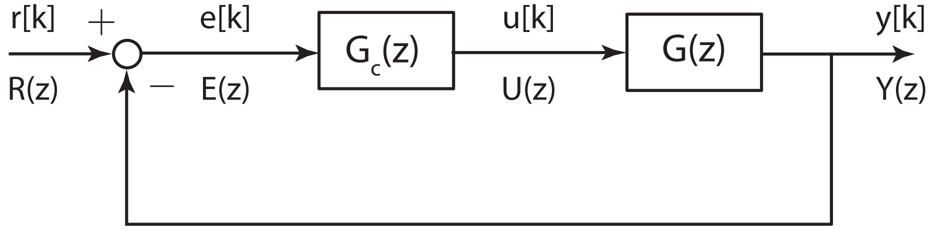
Now let's derive the PTF

$$\begin{aligned}
E(s) &= Y(s) - R(s) \quad , \quad Y(s) = G(s)G_c^*(s)E^*(s) \\
E^*(s) &= Y^*(s) - R^*(s) \quad , \quad Y(s) = G^*(s)G_c^*(s)E^*(s) \\
E^*(s) &= G^*(s)G_c^*(s)E^*(s) - R^*(s) \\
E^*(s) &= \frac{R^*(s)}{1+G^*(s)G_c^*(s)}
\end{aligned}$$

Then the PTF in starred Laplace domain and z domain can be find as

$$\begin{aligned}
\frac{Y^*(s)}{R^*(s)} &= \frac{G^*(s)G_c^*(s)}{1+G^*(s)G_c^*(s)} \\
\frac{Y(z)}{R(z)} &= \frac{G(z)G_c(z)}{1+G(z)G_c(z)}
\end{aligned}$$

Note if we only care the signal flow in the sampled instants we can re-draw the block diagram such that all time domain signals are in DT and all systems are represented in Z-domain. The fundamental block diagram can be re-drawn as



It is obvious that this discretized block diagram is much simpler and neater. Let's show that the transfer function of this DT system is equal to the PTF of the hybrid system above (CT & DT combined)

$$E(z) = R(z) - Y(z) \quad , \quad Y(z) = G(z)G_c(z)E(z)$$
$$E(z) = \frac{1}{1 + G(z)G_c(z)} R(z)$$
$$\frac{Y(z)}{R(z)} = \frac{G(z)G_c(z)}{1 + G(z)G_c(z)}$$

Example: Let $G_p(s) = \frac{1}{s+1}$, $T = 1$, $G_c(z) = K$ (Discrete P Controller) . First find PTF (in z-domain).
Solution: First let's find $G(z)$

$$G(z) = (1 - z^{-1})\mathcal{Z}\left\{\frac{G_p(s)}{s}\right\} = \frac{1 - e^{-1}}{z - e^{-1}}$$

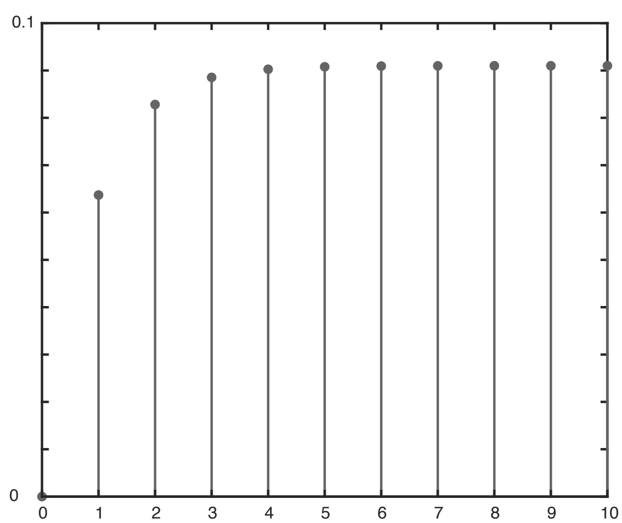
which we already knew from the Lecture Notes 4. Now let's compute the closed-loop PTF, $T(z)$.

$$T(z) = \frac{G_c(z)G(z)}{1 + G_c(z)G(z)} = \frac{K(1 - e^{-1})}{z + K - (K + 1)e^{-1}}$$

Let $\mathbf{K = 0.1}$, then compute the step-response of the closed-loop PTF

$$Y(z) = R(z)T(z) = \frac{0.063\, z}{(z - 1)(z - 0.3)}$$
$$y[k] = \mathcal{Z}^{-1}[Y(z)] = \left[0.09 - 0.09(0.3)^k\right] u[k]$$

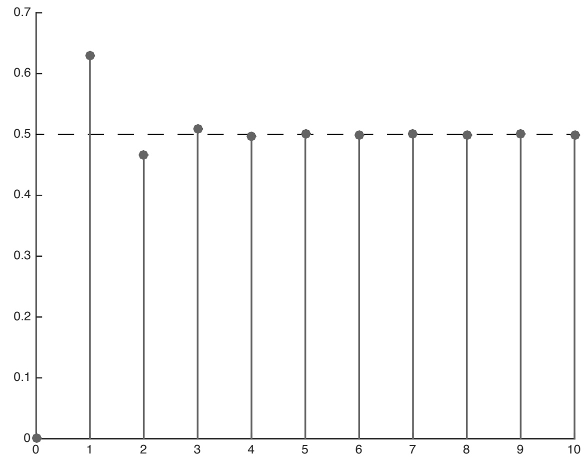
If we plot the step response we obtain the following plot



Now, Let $\mathbf{K = 1}$, then compute the step-response of the closed-loop PTF

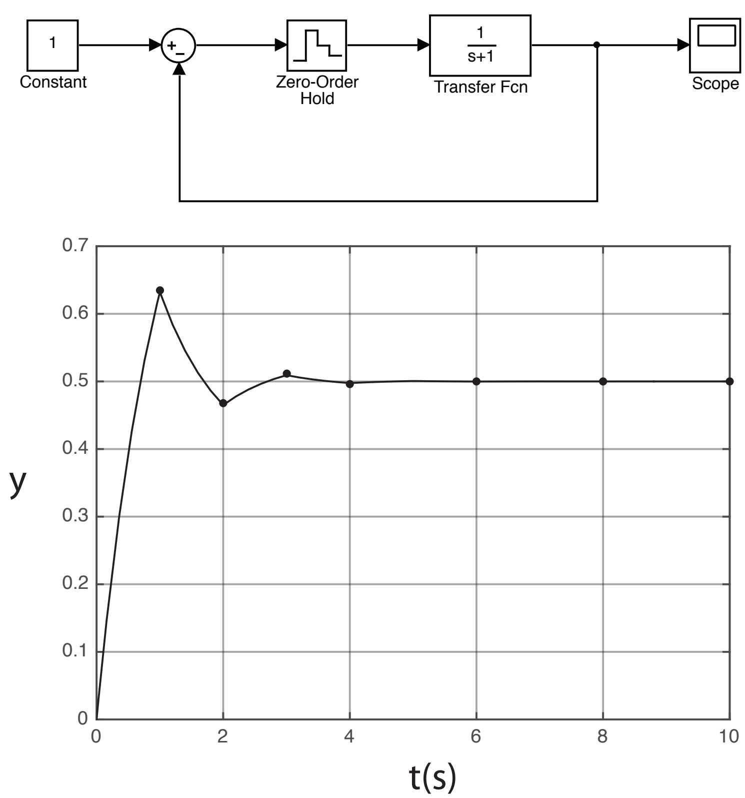
$$Y(z) = R(z)T(z) = \frac{z}{z - 1}\frac{0.63}{z + 0.26} = \frac{0.63\, z}{(z - 1)(z + 0.26)}$$
$$y[k] = \mathcal{Z}^{-1}[Y(z)] = \left[0.5 - 0.5(-0.26)^k\right] u[k]$$

If we plot the step response we obtain the following plot



What about inter-sample behavior?

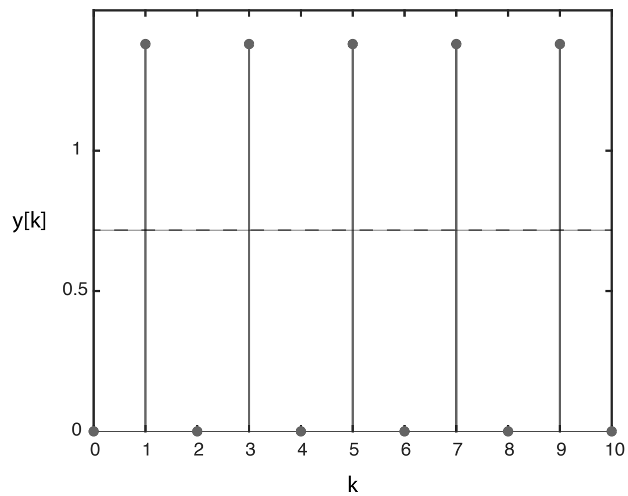
We can simulate the system and analyze the behavior. The figure below shows the Simulink model as well as the output.



Now let $K = \frac{1+e^{-1}}{1-e^{-1}}$, then $Y(z)$ and $y[k]$ takes the form

$$Y(z) = \frac{1.37z}{(z-1)(z+1)}$$
$$y[k] = 0.69 - 0.69(-1)^n$$

The graph of $y[k]$ is illustrated below. It can be seen that the output shows an oscillatory behavior.



Now let $K = \frac{e^{-1}}{1-e^{-1}}$, then $Y(z)$ and $y[k]$ takes the form

$$Y(z) = \frac{0.37}{z-1}$$
$$y[k] = 0.37u[k-1]$$

Output converges to its steady state value in "finite time" (dead-beat behavior/controller).

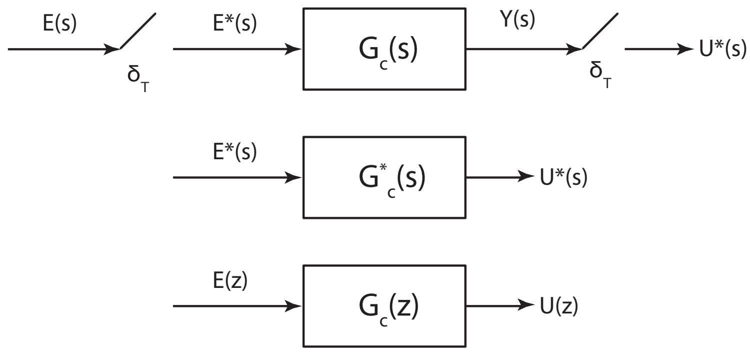
Take home message: It can be seen that even if the plant is a simple first order transfer function, depending on the value of K, we can observe very interesting behavior in the closed-loop DT system.

Pulse Transfer Function of a Digital PID Controller

In this section, we will try to obtain a from for the digital PID controller. The continuous transfer function of a PID is given as

$$G_{PID}(s) = K_P + K_D s + fracK_I s$$

One idea is to start from continuous PID form and then "discretize" it. One way of computing a discrete controller, $G_c(z)$, is using the star operation for discretization. This operation is illustrated in the Figure below.



Based in this approach if possible $G_c(z)$ simply commuted as

$$G_c(z) = \mathcal{Z}\{G_c(s)\}$$

Let's start with PI controller.

Digitization of PI Controller: It is a well known fact that the PI Controller is in the form

$$G_{PI}(s) = K_P + \frac{K_I}{s}$$

The discretization simply gives

$$\begin{aligned} G_{PI}(z) &= \mathcal{Z}\{G_{PI}(s)\} \\ &= K_P + K_I \frac{1}{1 - z^{-1}} \\ &= \frac{(K_I + K_P) + K_P z^{-1}}{1 - z^{-1}} \\ &= \frac{b_0 + b_1 z^{-1}}{1 - z^{-1}} \end{aligned}$$

Now let's discretize PID controller which has the following CT transfer function

$$G_{PI}(s) = K_P + K_D s + \frac{K_I}{s}$$

The problem is $K_D s$ term is non-causal. Let us approximate the effect of $K_D s$ in time domain and then perform a discretization. A causal approximate derivative can be find by computing the backward difference.

$$\frac{dx(t)}{dt} \approx \frac{x(t) - x(t - \Delta t)}{\Delta t}$$

Now let's compute the approximate derivative term at the sampling instants and let $\Delta t = T$ we have

$$\frac{dx(t)}{dt}|_{t=kT} \approx \frac{x(kT) - x((k-1)T)}{T}$$

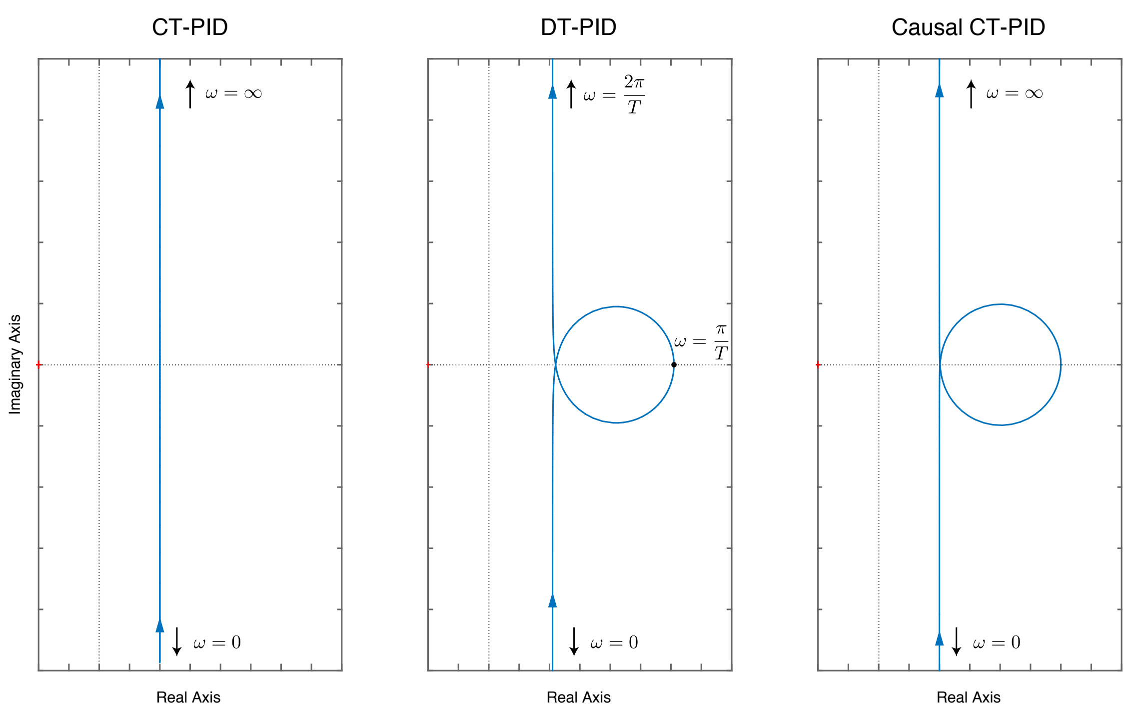
If we take the z-transform we can simply obtain a transfer function for this FIR filter as

$$G_D(z) = \frac{K_D}{T}(1 - z^{-1})$$

Note that instead of K_D/T, we can just use K_D for the gain. If we combine PI and D terms we obtain the following pulse transfer function fir the digital PID controller.

$$\begin{aligned} G_{PID}(z) &= K_P + K_I \frac{1}{1-z^{-1}} + K_D\left(1-z^{-1}\right) \\ &= \frac{K_P + K_D + K_I - (K_P + 2K_D)z^{-1} + K_D z^{-2}}{1-z^{-1}} \\ &= \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1-z^{-1}} \end{aligned}$$

The Figure below illustrates the frequency response characteristics of an ideal CT-PID, a DT-PID, as well as an approximate causal CT-PID controllers. Qualitatively, at low frequencies all controllers behaves similar however for "high" frequencies there are significant differences between the CT-PID and DT-PID (as well as approximate causal CT-PID). Remarkably, for these DT-PID and approximate causal CT-PID controllers the frequency response polar polts are qualitatively similar. This shows that if we choose the right parameters digitization of derivative term has a similar effect as implementing an approximate analog derivative circuit.



Note that frequency response function for CT and DT systems are found by the Fourier (CT or DT) transforms of the impulse response functions, or simply they can be computed from the s-domain or z-domain transfer functions

$$\begin{aligned} \text{CT}: &\quad G_c(s)|_{s=j\omega} = G_c(j\omega) \\ \text{DT}: &\quad G_d(z)|_{z=e^{j\omega}} = G_d(e^{j\omega}) \end{aligned}$$

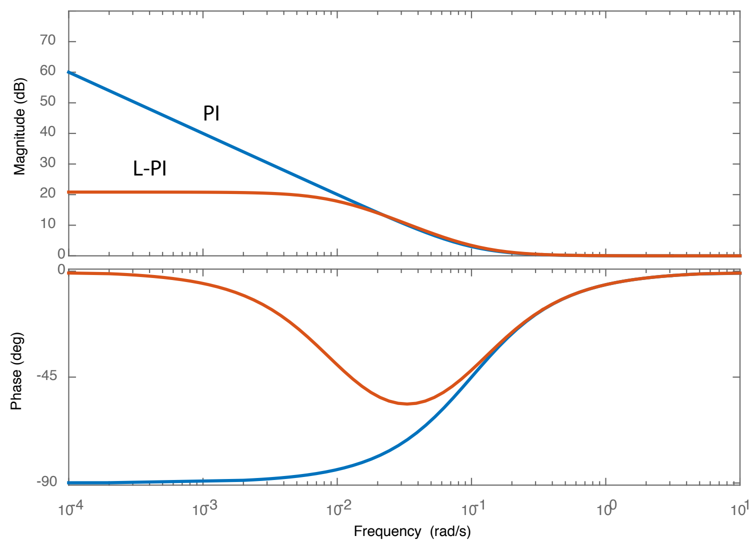
Note that in DT case ω stands for DT frequency. Sometimes $G_d(j\omega)$ or $G_d(\omega)$ used instead of $G_d(e^{j\omega})$. We will cover the Frequency respons later in the class.

PI & PID Controllers with Leaky Integrator

Some times for some practical and other considerations instead of a true integrator (or accumulator) a leaky version is used. A leaky PI controller is in the form

$$\begin{aligned} G_{L-PI}(s) &= K_P + K_I \frac{1}{s+\alpha} \\ &= K_P \frac{s + (\alpha + K_I/K_P)}{s+\alpha} \end{aligned}$$

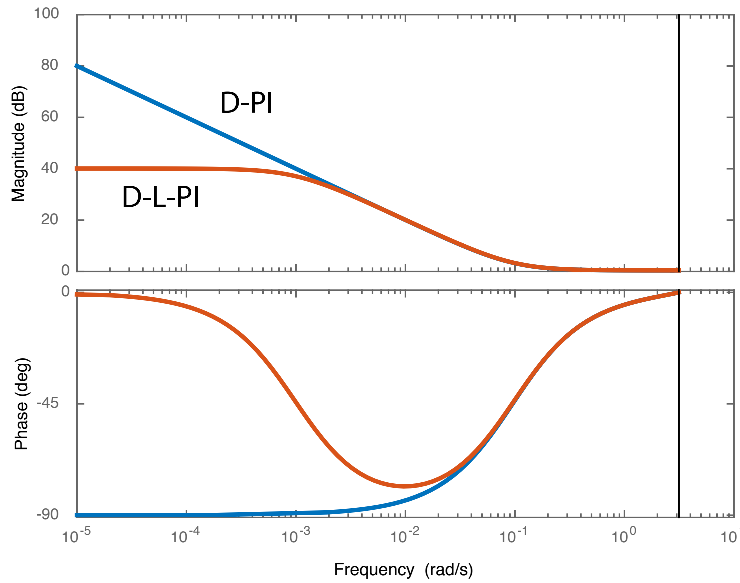
where $\alpha > 0$ and $\alpha \approx 0$ (considering the bandwith of the closed loop system). It can be seen that a leaky-PI controller has the same from with the compensator controller that we covered in 302. If we obscure the frequency response characteristics of both classical and leaky PI controllers, we observe that the behavior is quite different at low frequencies but similar at high frequencies.



If we discretize this CT controller using the star operation approach, we obtain

$$\begin{aligned} G_{L-PI}(z) = \mathcal{Z}\{G_{L-PI}(s)\} &= K_P + \frac{K_I}{1 + e^{-\alpha T} z^{-1}} \\ &= K_P + \frac{K_I}{1 + \beta z^{-1}} \\ &= \frac{(K_P + K_I) + K_P \beta z^{-1}}{1 + \beta z^{-1}} \\ &= \frac{b_0 + b_1 z^{-1}}{1 + a_1 z^{-1}} \end{aligned}$$

where similar to the CT case, $\beta < 1$ and $\beta \approx 1$. Similar to the CT case, this DT transfer function has one zero and one pole and it has the equivalent from with a DT-compensator controller. The bode plots of DT-PI and DT-Leaky-PI controllers are illustrated in the Figure below. It can be seen that at low frequencies the differences are significant, but at high frequencies the responses between classical and leaky PI controllers are very similar.



One interesting result is that both for classical and leaky cases, CT and DT frequency responses are qualitatively very similar.

EE402 - Discrete Time Systems	**Spring 2018**

Lecture 7

Lecturer: Asst. Prof. M. Mert Ankarali

Mapping Between s & z Planes

When the (uniform) impulse sampling is involved in the process, then we know that s and z variables are related with

$$z = e^{Ts}$$

which is a mapping from complex plane to complex plane, i.e. $M : \mathbb{C} \mapsto \mathbb{C}$, where $M(s) = e^{Ts}$. We will analyze different cases of this mapping and their relevance and importance.

Morever, if s_p is a pole of $G_p(s)$, then it is straightforward to show that $z_p = e^{Ts_p}$ is a pole of $G(z) = \mathcal{Z}\{G_p(s)\}$ (as well as $G(z) = (1 - z^{-1})\mathcal{Z}\{G(s)/s\}$).

Since poles of an LTI system (CT or DT) are the major features that defines the stability and some other performance metrics, analyzing this mapping is very critical for analyzing discrete time control systems.

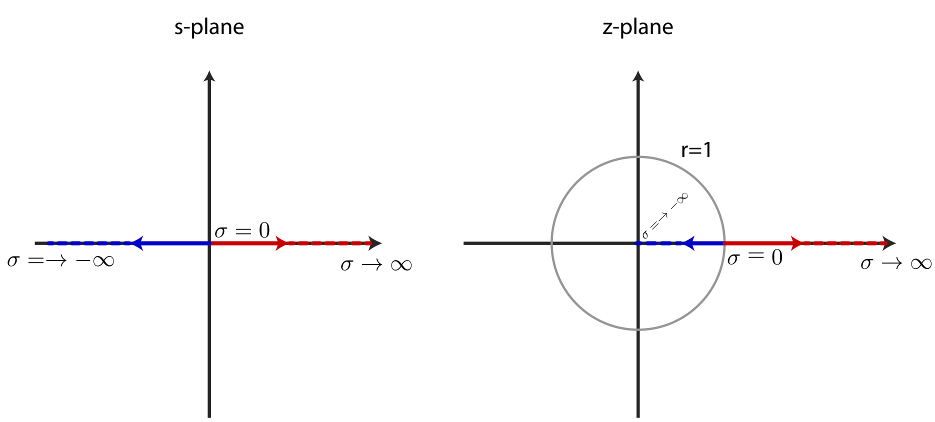
Mapping of real line (left half and right half): When s is purely real (i.e. when the roots of the CT plant are real) we have

$$\begin{aligned} s &= \sigma \quad , \ \sigma \in \mathbb{R} \\ z &= e^{Ts} = e^{T\sigma} \quad , \ z \in \mathbb{R}^+ \end{aligned}$$

It is also easy to see the difference between left half and right half of the real line

$$\begin{aligned} &\text{If } \sigma \leq= 0 \quad \rightarrow \quad z = e^{\sigma T} \in [0, 1] \\ &\text{If } \sigma \geq= 0 \quad \rightarrow \quad z = e^{\sigma T} \in [1, \infty) \end{aligned}$$

Mapping of both left and right real lines to z-plane is illustrated in the Figure below. It can be seen that when $\sigma > 0$, $z > 1$, and similarly when $\sigma < 0$, $z < 1$. Technically on both planes red curves belong to "unstable" behaviors, where as blue curves belong to "stable" behaviors.



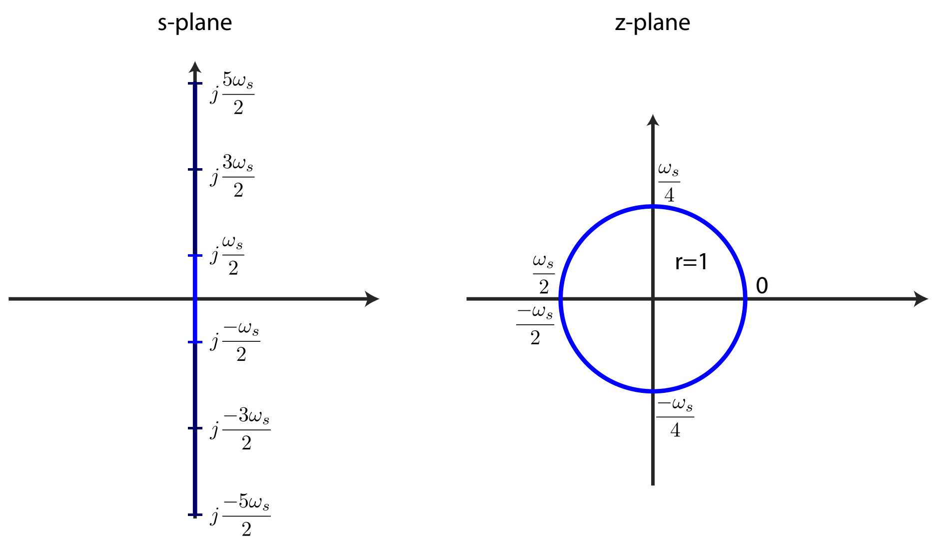
Mapping of imaginary axis: When s is purely imaginary (i.e. when the roots of the CT plant are critically stable) we have

$$
\begin{aligned}
s &= j\omega \quad , \ \omega \in \mathbb{R} \\
z &= e^{Ts} = e^{T\omega j} \\
|z| &= 1 \\
\angle z &= T\omega = T\omega + 2\pi k \quad , \ k \in \mathbb{Z}
\end{aligned}
$$

This means that mapping of the imaginary axis is not 1-1, since multiple points on s plane can correspond to a single point on the z plane

$$e^{T\omega j} = e^{T\omega j + 2\pi k} \quad \rightarrow \quad M(\omega j) = M((\omega + 2\pi/T)j) = M((\omega + \omega_s)j)$$

where $\omega_s = 2\pi T$ is the sampling frequency. It can be seen that imaginary axis on the s plane is mapped to the unit circle on the z plane. However as $\omega \to \infty$ or $\omega \to -\infty$, the mapping circles the unit circle multiple (indeed infinite) times. This mapping is illustrated in the Figure below. The light blue section on the s plane (which covers the points in the imaginary axis between $[-\omega_s/2, \omega_s/2]$) is called the primary section/strip and fully mapped to the unit cicle. Dark blue sections are called complementary sections/strips and they are also individually fully mapped onto the unit circle.



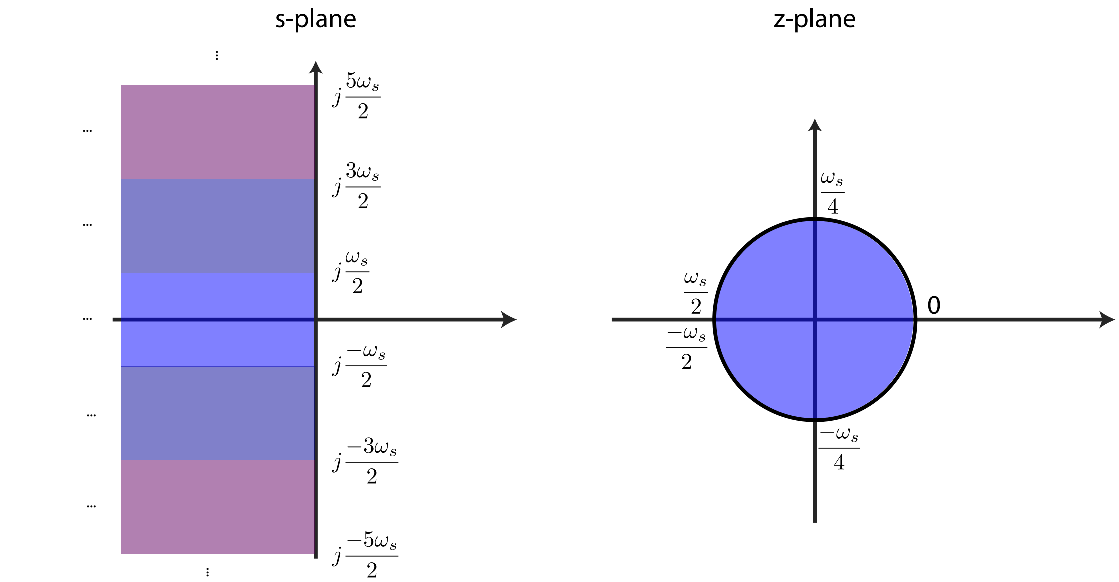
Mapping of open left-half plane: Now let's generalize a little, and consider the mapping of the whole open left-half plane.

$$
\begin{aligned}
s &= \sigma + j\omega \quad , \ \sigma < 0 \\
z &= e^{Ts} = e^{T\sigma} e^{T\omega j} = e^{T\sigma} e^{(T\omega + 2\pi k)j} \\
|z| &= e^{T\sigma} < 1 \\
\angle z &= T\omega + 2\pi k \quad , \ k \in \mathbb{Z}
\end{aligned}
$$

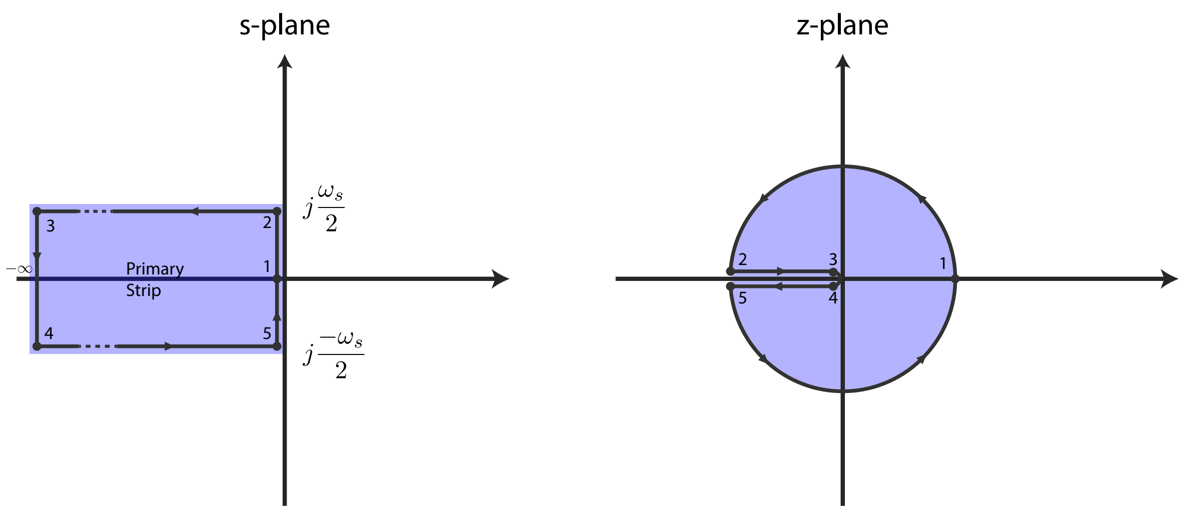
Obviously, this mapping is also not 1-1, and "periodic" in ω, i.e.

$$M(\sigma, \omega) = M(\sigma, (\omega + 2\pi T)j) = M(\sigma, (\omega + \omega_s)j)$$

Mapping of OLH on s-plane to z-plane is illustrated in Figure below.



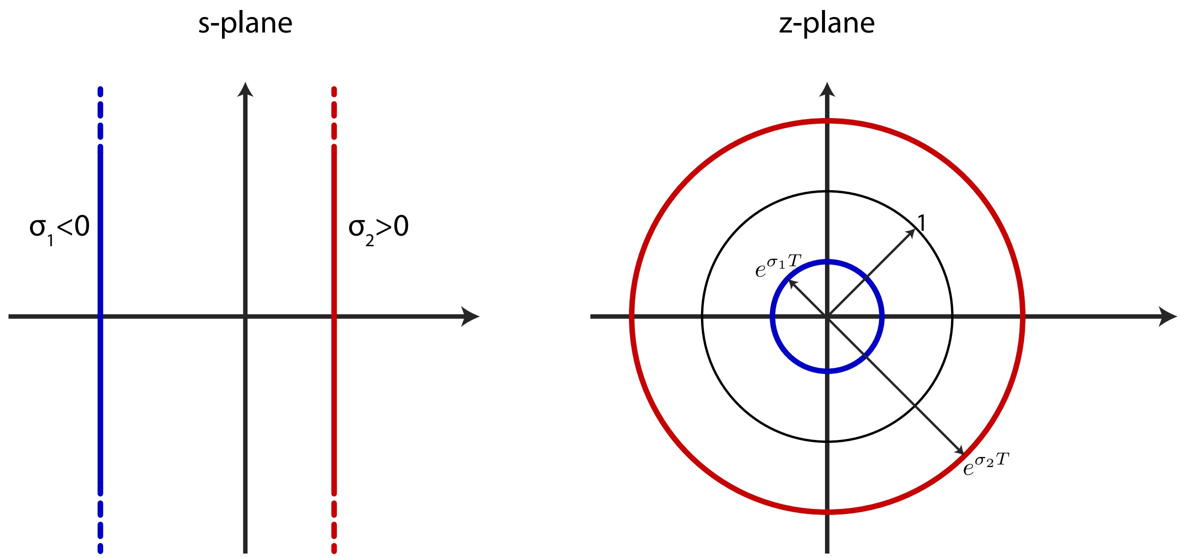
In the primary strip, if we trace the path that is defined by the sequence of points 1-2-3-4-5-1 in the s plane as shown in the Figure below, than this path is mapped in to the z-plane as shown in the Figure. The mapping forms a different path again associated with mapped point sequence 1-2-3-4-5-1.



Mapping of constant attenuation line: In the s-plane, it corresponds to the line for which σ is constant. Constant σ in s-plane corresponds to a constant radius in the z-plane. Thus line is mapped to a circle with a radius of $e^{\sigma T}$.

$$z = e^{\sigma T} e^{\omega T j} = e^{\sigma T} \angle \omega T$$
$$R = e^{\sigma T} = \text{Constant}$$

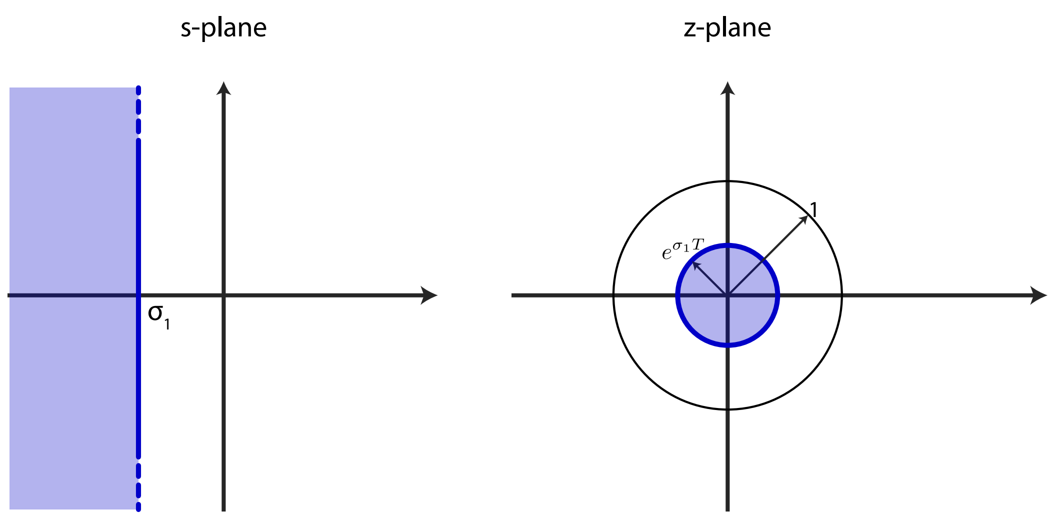
Figure below illustrates mapping of one constant attenuation line in open left half plane and one open right half plane.



Max convergence/settling time region: For a stable CT LTI system convergence/settling time is defined by the real part of the pole. Thus on the s-plane we have the following condition

$$\mathrm{Re}(s) < \sigma_1 < 0$$

In the z-plane it is mapped to region enclosed by the circle with radius $r = e^{\sigma_1 T}$. This mapping is illustrated below.

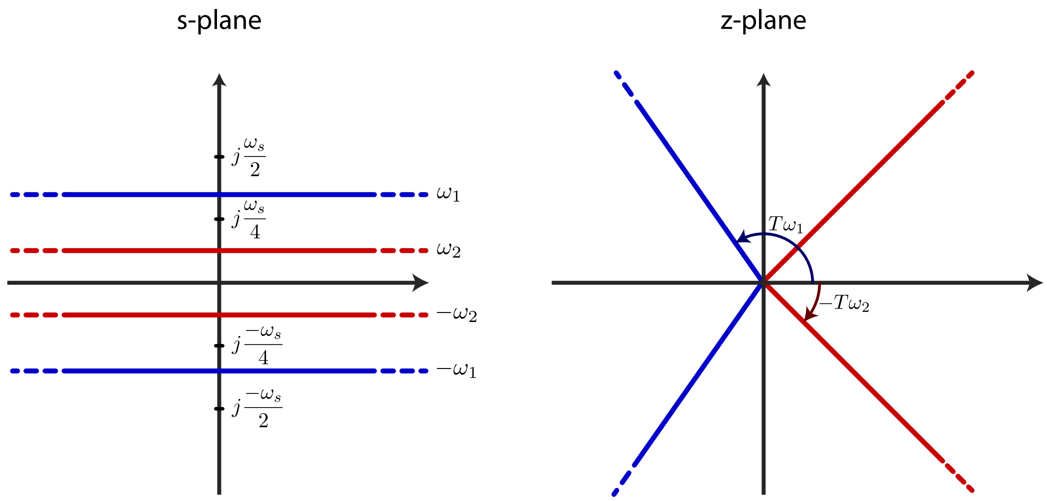


Constant frequency loci: A constant frequency locus $\omega = \omega_1$ in s-plane is mapped to a constant angle line in z-plane, for which the angle is equal to $\omega_1 T$.

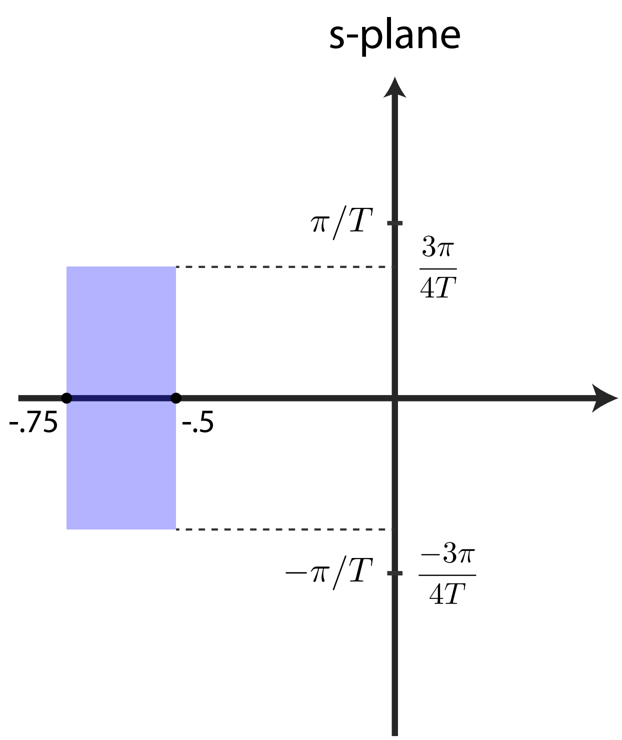
$$z = e^{\sigma T} e^{\omega T j} = e^{\sigma T} \angle \omega T$$

$$\angle \omega T = \text{Constant}$$

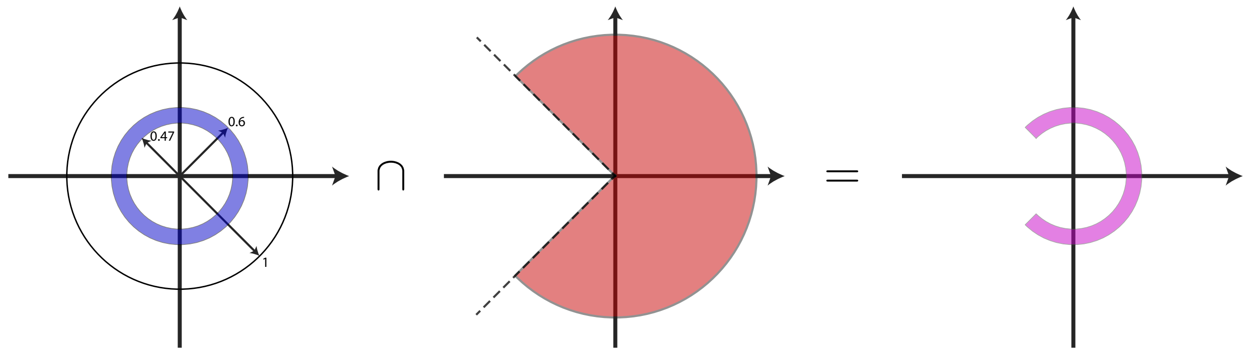
Different constant frequency lines (all inside the primary strip) and their mappings are illustrated in Figure below.



Example: Find the mapping of the area defined inside s-plane illustrated in Figure below with $T = 1$.



Solution: The solution is illustrated in the Figure below



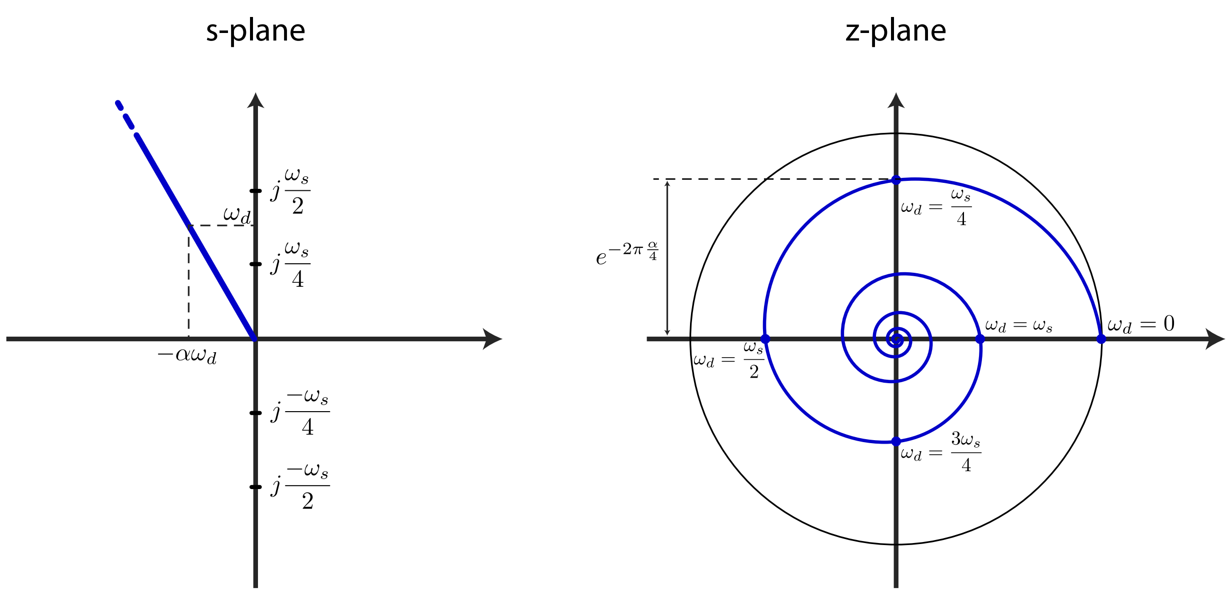
Constant damping loci: A constant damping loci is a line in s-plane passing through the origin and the angle between the line and the Real (or Imaginary) axis defines the damping ratio. In s-plane we have the following relations

$$
\begin{aligned}
s &= \sigma + \omega j = -\zeta\omega_n + \omega_n\sqrt{1-\zeta^2}j \\
&= -\alpha\omega_d + j\omega_d \\
\omega_d &= \omega_n\sqrt{1-\zeta^2} \\
\alpha &= \frac{\zeta}{\sqrt{1-\zeta^2}} = \text{Constant}
\end{aligned}
$$

The mapping of this line to the z-plane yields

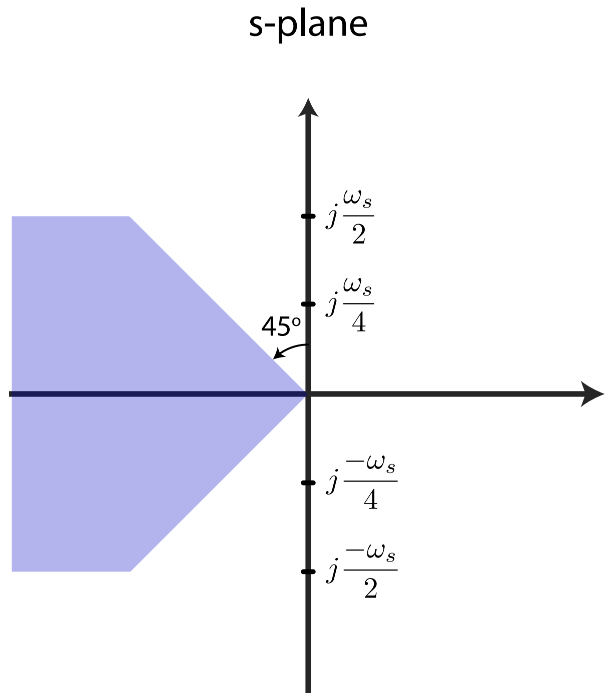
$$
\begin{aligned}
z &= e^{-\alpha\omega_d T + jT\omega_d} = e^{-\alpha\omega_d T} + e^{j\omega_d T} \\
z &= e^{-2\pi\alpha\frac{\omega_d}{\omega_s}} + e^{j2\pi\frac{\omega_d}{\omega_s}} \\
|z| &= e^{-2\pi\alpha\frac{\omega_d}{\omega_s}} \\
\angle z &= 2\pi\frac{\omega_d}{\omega_s}
\end{aligned}
$$

The curve in z-plane corresponds to a spiral shape as seen in Figure below.

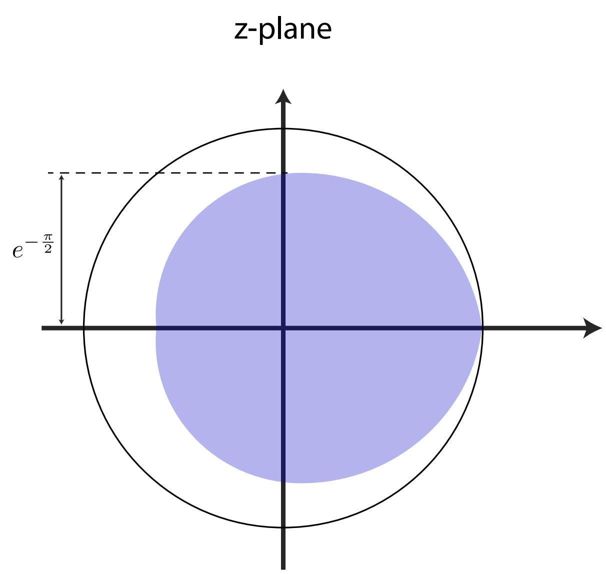


Note that for real systems we need to have also complex conjugates of both the line in s-plane and spiral in z-plane.

Example: Find the mapping of the area defined inside s-plane illustrated in Figure below.



Solution: The solution is illustrated in the Figure below



Bilinear Transformation

A very common transformation used for design and analysis of discrete time control systems, digitized filters etc. is the bilinear transformation. It is a 1-1 mapping between complex z-plane and another complex plane $\bar{s}$ which is an "imaginary" s-domain plane.

The bilinear transformation is defined by

$$z = \frac{1 + \frac{T}{2}\bar{s}}{1 - \frac{T}{2}\bar{s}}$$
$$\bar{s} = \frac{2}{T}\frac{z-1}{z+1}$$

Let's analyze $|z| < 1$

$$|z| = \left|\frac{1 + \frac{T}{2}\bar{s}}{1 - \frac{T}{2}\bar{s}}\right| = \frac{|1 + \frac{T}{2}\bar{s}|}{|1 - \frac{T}{2}\bar{s}|}$$
$$|z| < 1 \implies \left|1 + \frac{T}{2}\bar{s}\right| < \left|1 - \frac{T}{2}\bar{s}\right|$$

Let $\bar{s} = \bar{\sigma} + j\bar{\omega}$

$$\left|1 + \frac{T}{2}(\bar{\sigma} + j\bar{\omega})\right| < \left|1 - \frac{T}{2}(\bar{\sigma} + j\bar{\omega})\right|$$
$$\left(\frac{T}{2}\bar{\sigma} + 1\right)^2 + \left(\frac{T}{2}\bar{\omega}\right)^2 < \left(\frac{T}{2}\bar{\sigma} - 1\right)^2 + \left(\frac{T}{2}\bar{\omega}\right)^2$$
$$\left(\frac{T}{2}\bar{\sigma} + 1\right)^2 < \left(\frac{T}{2}\bar{\sigma} - 1\right)^2 \implies \bar{\sigma} < 0$$

In other words we have the following relation

$$|z| < 1 \iff \mathrm{Re}\{\bar{s}\} < 0$$

which imples that tha area inside unit circle of z-plane (stable region) is mapped to whole open left half plane of $\bar{s}$-plane.

Now let's consider the mapping of unit-cirlce on z-plane onto the $\bar{s}$-plane.

$$|z| = 1 \implies \left(\frac{T}{2}\bar{\sigma} + 1\right)^2 = \left(\frac{T}{2}\bar{\sigma} - 1\right)^2 \implies \bar{\sigma} = 0$$

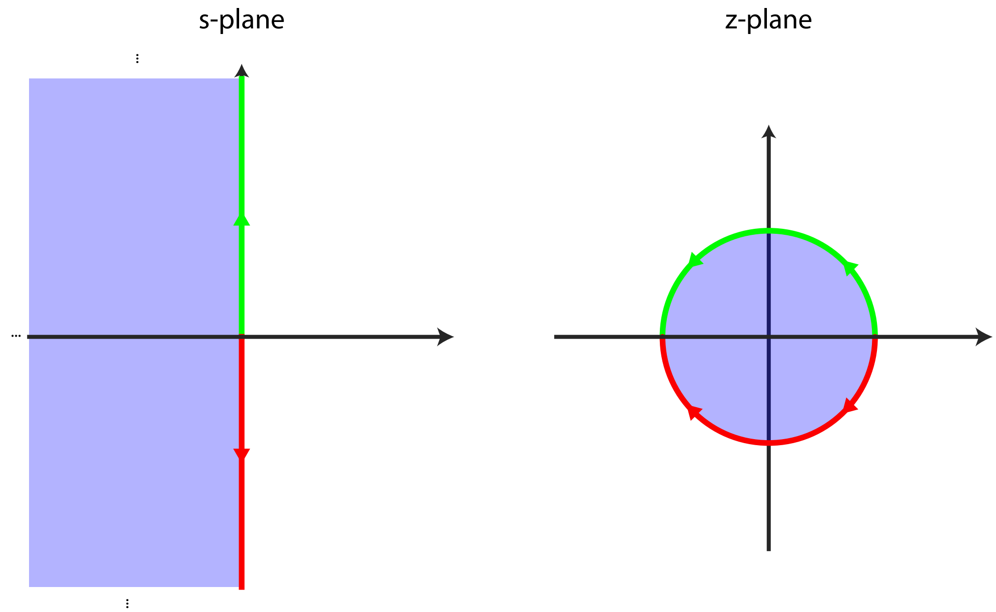
which imples that points on the unit circle are mapped to the imaginary axis on the $\bar{s}$-plane. Now let's analyze this mapping further:

$$|z| = 1 \implies z = e^{j\omega_d} \ , \omega_d \in [-pi, pi]$$
$$\bar{s} = \frac{2}{T}\frac{e^{j\omega_d} - 1}{e^{j\omega_d} + 1} = \frac{2}{T}\frac{(e^{j\omega_d} - 1)(e^{-j\omega_d} + 1)}{(e^{j\omega_d} + 1)(e^{-j\omega_d} + 1)} = j\frac{2}{T}\frac{\sin\omega_d}{1 + \cos\omega_d} = j\frac{2}{T}\frac{\sin(\omega_d/2)}{\cos(\omega_d/2)}$$
$$\bar{\omega} = \frac{2}{T}\tan(\omega_d/2)$$

where $\bar{\omega}$ is the artificial frequency of the artificial CT system. If we analyze the frequency mapping we can easly see that

$$\omega_d : \ 0 \rightarrow \pi \quad \implies \quad \bar{\omega} : \ 0 \rightarrow \infty$$
$$\omega_d : \ 0 \rightarrow -\pi \quad \implies \quad \bar{\omega} : \ 0 \rightarrow -\infty$$

Bilinear transformation and its basic mapping properties are illustrated in the Figure below.



Blue transparent region corresponds to open left half plane in $\bar{s}$-plane and area inside the unit circle in z-plane. Red and green lines/curves illustrate the frequency paths in the respected planes.

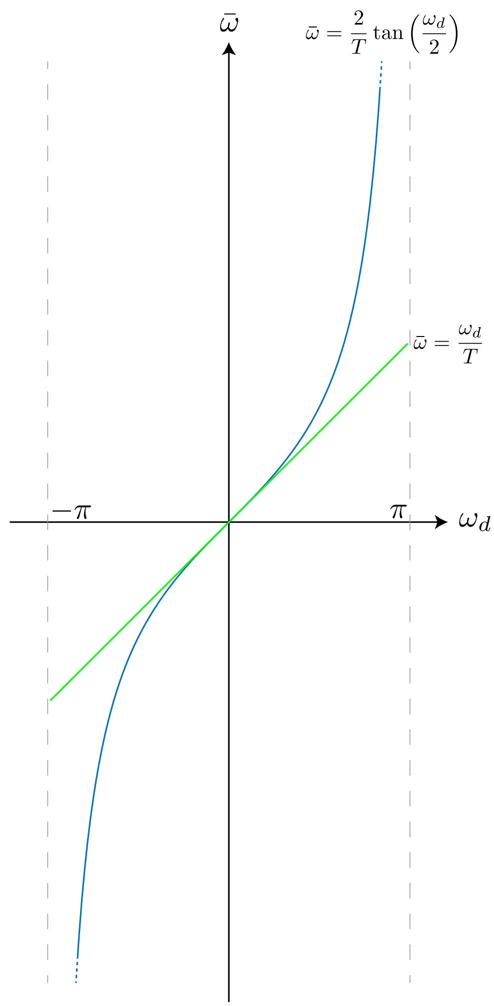
Let's analyze the behavior of the frequency mapping around $\omega_d = 0$

$$\bar{\omega} \approx \left[\frac{d\bar{\omega}}{\omega_d}\right]_{\omega_d} \omega_d = \omega_d / T$$
$$\bar{\omega} \approx \omega_c$$

where $\bar{\omega}$, ω_d, and ω_c are the artificial CT-frequency, frequency in DT domain, and actual frequency in CT domain respectively. The most important relation is that at low frequencies $\bar{\omega} \approx \omega_c$.

In this linear region, the bilinear transformation behaves nicely and we can roughly say that $G(s) \approx G(\bar{s})$. This means that bilinear transformation can be effectively used for design of control systems, or measuring relative performance metrics for the system of interest. However, as ω_d increases there is a highly non-linear frequency wrapping (compression) effect which can make both the design and analysis challenging and sometimes even meaningless.

The Figure below illustrates the relation between ω_d and ω as well as the linear approximation.


$\bar{\omega}$
$\bar{\omega} = \frac{2}{T}\tan\left(\frac{\omega_d}{2}\right)$
$\bar{\omega} = \frac{\omega_d}{T}$
$-\pi$
π
ω_d

Stability of Discrete Time Control Systems

For an LTI discrete time dynamical system which can be represented with a rational transfer function, closed loop poles determine the stability characteristics of the system.

- If all poles of the system are located strictly inside the unit-circle then the system is **(asymptotically) stable**. Asymptotically stability systems are also **BIBO stable**.
- If there exist some *simple* (non-repeated) poles on the unit circle and all remaining poles are located inside the unit circle, then the system is **critically/marginally stable**. Note that critically/marginally stable systems are **BIBO unstable**.
- If there exist at least one repeated pole on the the unit circle, then the system is **unstable**, of course also **BIBO unstable**.
- If there exist at least one pole outside of the unit circle, then the system is **unstable**, of course also **BIBO unstable**.

Jury Stability Test

Jury stability test similar to the Routh-Hurwitz in CT systems, can define the stability of a DT system given the characteristic equation which is in the form

$$D(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n$$

without loss of generality we will assume that $a_0 > 0$.

First Order: When $n = 1$, $D(z)$ takes the form

$$D(z) = a_0 z + a_1$$

DT System is stable if

$$|a_1| < a_0$$

Second Order: When $n = 2$, $D(z)$ takes the form

$$D(z) = a_0 z^2 + a_1 z + a_2$$

DT System is stable if

$$|a_2| < a_0$$
$$D(1) > 0$$
$$D(-1) > 0$$

Third Order: When $n = 3$, $D(z)$ takes the form

$$D(z) = a_0 z^3 + a_1 z^2 + a_2 z + a_3$$

We need to construct the Jury table

Row	z^0	z^1	z^2	z^3
1	a_3	a_2	a_1	a_0
2	a_0	a_1	a_2	a_3
3	b_2	b_1	b_0	

where

$$b_0 = \begin{vmatrix} a_3 & a_2 \\ a_0 & a_1 \end{vmatrix} \quad , \quad b_1 = \begin{vmatrix} a_3 & a_1 \\ a_0 & a_2 \end{vmatrix} \quad , \quad b_2 = \begin{vmatrix} a_3 & a_0 \\ a_0 & a_3 \end{vmatrix}$$

Then DT system is stable if

$$\begin{aligned} |\mathbf{a_3}| &< a_0 \\ D(1) &> 0 \\ -D(-1) &> 0 \\ |b_2| &> |b_0| \end{aligned}$$

General Case: The jury table for systems with order n has $2n-3$ rows and it has the from below

Row	z^0	z^1	z^2	$\cdots$	z^{n-2}	z^{n-1}	z^n
1	a_n	a_{n-1}	a_{n-2}	$\cdots$	a_2	a_1	a_0
2	a_0	a_1	a_2	$\cdots$	a_{n-2}	a_{n-1}	a_n
3	b_{n-1}	b_{n-2}	b_{n-3}	$\cdots$	b_1	b_0	
4	b_0	b_1	b_2	$\cdots$	b_{n-2}	b_{n-1}	
5	c_{n-2}	c_{n-3}	c_{n-3}	$\cdots$	c_0		
6	c_0	c_1	c_2	$\cdots$	c_{n-2}		
$\vdots$	$\vdots$						
2n - 3	q_2	q_1	q_0				

where

$$b_k = \begin{vmatrix} a_n & a_{n-1-k} \\ a_0 & a_{k+1} \end{vmatrix} \quad , \quad k \in \{0, 1, \cdots, n-1\}$$

$$c_k = \begin{vmatrix} b_n & b_{n-2-k} \\ b_0 & b_{k+1} \end{vmatrix} \quad , \quad k \in \{0, 1, \cdots, n-2\}$$

$$q_k = \begin{vmatrix} p_3 & p_{2-k} \\ p_0 & p_{k+1} \end{vmatrix} \quad , \quad k \in \{0, 1, 3\}$$

Then DT system is stable if

$$
\begin{aligned}
|a_n| &< a_0 \\
D(1) &> 0 \\
(-1)^n D(-1) &> 0 \\
|b_{n-1}| &> |b_0| \\
|c_{n-2}| &> |c_0| \\
&\cdots \\
|q_2| &> |q_0|
\end{aligned}
$$

Example: Using Jury test, find if the following characteristic equation is stable or not

$$G(z) = \frac{0.02z^{-1} + 0.03z^{-2} + 0.02z^{-3}}{1 - 3z^{-1} + 4z^{-2} - 2z^{-3} + 0.5z^{-4}}$$

Solution: This is a 4^{th} order system for which the characteristic equation is

$$
\begin{aligned}
D(z) &= a_0 z^4 + a_1 z^3 + a_2 z^2 + a_3 z + a_4 \\
&= 1z^4 + -3z^3 + 4z^2 + -2z + 0.5
\end{aligned}
$$

Jury table for a $n = 4$ system has the form

Row	z^0	z^1	z^2	z^3	z^4
1	a_4	a_3	a_2	a_1	a_0
2	a_0	a_1	a_2	a_3	a_4
3	b_3	b_2	b_1	b_0	
4	b_0	b_1	b_2	b_3	
5	c_2	c_1	c_0		

Before computing the whole Jury table let's check conditions one-by-one

- Check if $|a_4| < a_0$

$$0.5 < 1 \quad \text{OK}$$

- Check if $D(1) > 0$

$$D(1) = 1 - 3 + 4 - 2 + 0.5 = 0.5 > 0 \quad \text{OK}$$

- Check if $(-1)^4 D(-1) > 0$

$$D(-1) = 1 + 3 + 4 + 2 + 0.5 = 10.5 > 0 \quad \text{OK}$$

- Let's compute b_0 and b_3 and check if $|b_3| > |b_0|$

$$
\begin{aligned}
b_0 &= \begin{vmatrix} a_4 & a_3 \\ a_0 & a_1 \end{vmatrix} = \begin{vmatrix} 0.5 & -2 \\ 1 & -3 \end{vmatrix} = 0.5 \\
b_3 &= \begin{vmatrix} a_4 & a_0 \\ a_0 & a_4 \end{vmatrix} = \begin{vmatrix} 0.5 & 1 \\ 1 & 0.5 \end{vmatrix} = -0.75 \\
|b_3| &= 0.75 > 0.5 = |b_0| \quad \text{OK}
\end{aligned}
$$

- Let's compute b_1 and b_2

$$b_1 = \begin{vmatrix} a_4 & a_2 \\ a_0 & a_2 \end{vmatrix} = \begin{vmatrix} 0.5 & 4 \\ 1 & 4 \end{vmatrix} = -2$$

$$b_2 = \begin{vmatrix} a_4 & a_1 \\ a_0 & a_3 \end{vmatrix} = \begin{vmatrix} 0.5 & -3 \\ 1 & -2 \end{vmatrix} = 2$$

- Let's compute c_0 and c_2 and check if $|c_2| > |c_0|$

$$c_0 = \begin{vmatrix} b_3 & b_2 \\ b_0 & b_1 \end{vmatrix} = \begin{vmatrix} -0.75 & 2 \\ 0.5 & -2 \end{vmatrix} = 0.5$$

$$c_2 = \begin{vmatrix} b_3 & b_0 \\ b_0 & b_3 \end{vmatrix} = \begin{vmatrix} -0.75 & 0.5 \\ 0.5 & -0.75 \end{vmatrix} = 0.3125$$

$$|c_2| = 0.3125 \not> 0.5 = |c_0| \quad \text{NOTOK}$$

Bilinear Transformation & Routh-Hurwitz Test

In Lecture 7, we showed that bilinear transformation has a 1-1 mapping between stable regions in z-plane and s-plane, as well as unstable regions in z-plane and s-plane. As a way of testing stability, we can transform the the characterstic polynomial using bilinear transformation, then we can apply Routh-Hurwitz test.

Routh-Hurwitz is simpler and easier then the Jury test, however amount of computation needed for transformation generally shadows the relative computational advantage of Routh-Hurwitz.

We know that the bilinear transformation has the form

$$z = \frac{1 + \frac{T}{2}\bar{s}}{1 - \frac{T}{2}\bar{s}}$$

Since we only consider the test of stability, for the sake of simplicity it is reasonable to assume that $T = 2$. Then, the transformation of a general $D(z)$ looks like

$$D(\bar{s}) = D(z)\big|_{z=\frac{1+\bar{s}}{1-\bar{s}}} = a_0 \left(\frac{1+\bar{s}}{1-\bar{s}}\right)^n + a_1 \left(\frac{1+\bar{s}}{1-\bar{s}}\right)^{n-1} + \cdots + a_{n-1}\left(\frac{1+\bar{s}}{1-\bar{s}}\right) + a_n$$

Then clearing the fractions by multiplying both sides by $(1-\bar{s})^n$, we obtain

$$Q(\bar{s}) = b_0\bar{s}^n + b_1\bar{s}^{n-1} + \cdots + b_{n-1}\bar{s} + b_n$$

Testing the stability on $Q(s)$ using Routh-Hurwitz will yield the stability condition of the original DT system.

Example: Consider the following characterstic equation of a DT system

$$D(z) = (z-1)*(z-2) = z^2 - 3z + 2 \tag{8.1}$$

Test the stability (already known) using Bilinear Transformation and Routh-Hurwitz.

Solution:

$$\begin{aligned} D(\bar{s}) = D(z)\big|_{z=\frac{1+\bar{s}}{1-\bar{s}}} &= \left(\frac{1+\bar{s}}{1-\bar{s}}\right)^2 - 3\left(\frac{1+\bar{s}}{1-\bar{s}}\right) + 2 \\ Q(\bar{s}) &= (1+\bar{s})^2 - 3(1+\bar{s})(1-\bar{s}) + 2\,(1-\bar{s})^2 \\ &= (1 + 2\bar{s} + \bar{s}^2) - 3(1-\bar{s}^2) + 2(1 - 2\bar{s} + \bar{s}^2) \\ &= 6\bar{s}^2 - 2\bar{s} \end{aligned}$$

This artificial CT system is unstable since one coefficient is negative and one coefficient is equal to zero. It is clear from this example that just for testing stability Bilinear transformation is not very useful.

EE402 - Discrete Time Systems	Spring 2018

Lecture 9

Lecturer: Asst. Prof. M. Mert Ankarali

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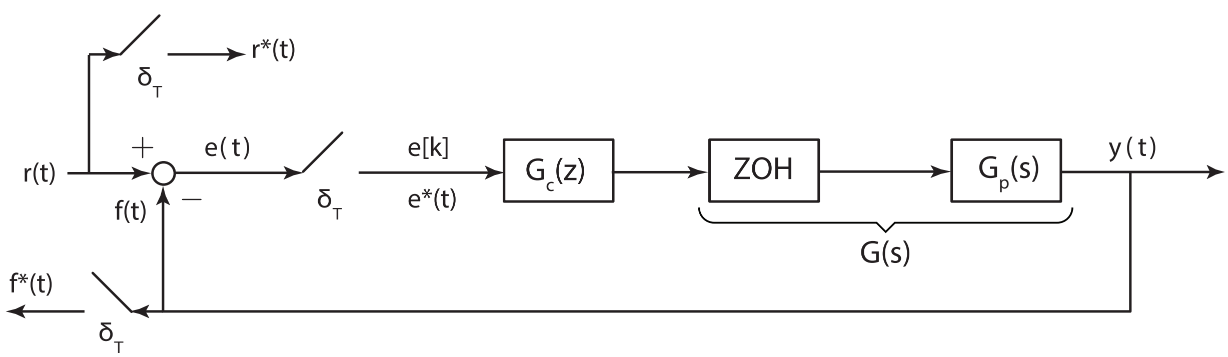
Steady-Sate (DC) Response Analysis

Let's remember the final value theorem. Given a discrete time signal $x[k]$ and its z-transform $X(z)$, if $x[k]$ is convergent sequence final value theorem states that

$$\lim_{k\to\infty} x[k] = \lim_{z\to 1}\left[\left(1 - z^{-1}\right) X(z)\right]$$
$$x_{ss} = \lim_{z\to 1}\left[\frac{z-1}{z} X(z)\right]$$



Now let's find the pulse transfer function from the reference signal $r[k]$ to the error signal $e[k]$, to further analyze the steady-state error response.

$$E(z) = R(z) - E(z)\left(G_c(z)G(z)\right), \quad \text{where } G(z) = \mathcal{Z}\{G(s)\}$$
$$\frac{E(z)}{R(z)} = \frac{1}{1 + G_c(z)G(z)}$$

Note that $G_c(z)G(z)$ is the pulse transfer function from the error signal $E(z)$ to the signal which is fed to the negative terminal of the main difference operator, i.e. $F(z)$. This transfer function is called feed-forward or open-loop pulse transfer function of the closed-loop digital control system. For this system,

$$\frac{F(z)}{E(z)} = G_{OL} = G_c(z)G(z)$$

Then $E(z)$ can be written as

$$E(z) = R(z)\frac{1}{1 + G_{OL}(z)}$$

It is obvious that first requirement on m steady-state error performance is that closed-loop system have to be stable. Now let's analyze specific but fundamental input scenarios.

Unit-Step Input

We know that $r[k] = u[k]$ and $R(z) = \frac{1}{1-z^{-1}}$ then we have

$$\begin{aligned} e_{ss} &= \lim_{z\to 1}\left[\left(1-z^{-1}\right)R(z)\frac{1}{1+G_{OL}(z)}\right] \\ &= \lim_{z\to 1}\left[\left(1-z^{-1}\right)\frac{1}{1-z^{-1}}\frac{1}{1+G_{OL}(z)}\right] \\ e_{ss} &= \frac{1}{1+\lim_{z\to 1} G_{OL}(z)} \end{aligned}$$

If the DC gain of the system (also called static error constant) is constant, i.e. $G_{OL}(1) = K_{DC}$ then the steady state error can be computed as

$$e_{ss} = \frac{1}{1+K_{DC}}$$

It is obvious that

$$\begin{aligned} e_{ss} &\neq 0 \quad \text{if} \quad |K_{DC}| < \infty \\ e_{ss} &\to 0 \quad \text{if} \quad K_{DC} \to \infty \end{aligned}$$

Based on these results, we can have the following conclusions

- If $G_{OL}(1) = 0$, then $e_{ss} = 1$. These are **type *negative*** systems, and we the steady-state error of step response type signals are always 100%.
- If $G_{OL}(1) = K_{DC}$, $0 < |K_{DC}| < \infty$, then $e_{ss} = 1/(1+K_{DC})$. These are **type 0** systems. We observe a bounded steady-state error and it is possible to reduce the by increasing the static gain constant K_P.
- If $G_{OL}(1) = \infty$, then $e_{ss} = 0$. These are **type *positive*** systems. The steady-state error is perfectly zero for such systems.

Now let's generalize the *type* of systems. An *N type* closed loop system has the following form of open-loop pulse transfer function

$$\begin{aligned} G_{OL}(z) &= \frac{1}{(z-1)^N} G_{DC}(z) \\ |G_{DC}(1)| &= K_{DC} \quad \text{where } 0 < |K_{DC}| < \infty \end{aligned}$$

It is easy to see that for unit-step response

- Type $N < 0$: $e_{ss} = 1$ (or $e_{ss} = 100\%$)
- Type $N = 0$: $e_{ss} = 1/(1+K_{DC})$
- Type $N > 0$: $e_{ss} = 0$

Unit-Ramp Input

We know that $r[k] = ku[k]$ and $R(z) = \frac{z^{-1}}{(1-z^{-1})^2}$ then we have

$$
\begin{aligned}
e_{ss} &= \lim_{z\to 1}\left[\left(1-z^{-1}\right)R(z)\frac{1}{1+G_{OL}(z)}\right]\\
&= \lim_{z\to 1}\left[\left(1-z^{-1}\right)\frac{z^{-1}}{(1-z^{-1})^2}\frac{1}{1+\frac{1}{(z-1)^N}G_{DC}(z)}\right]\\
&= \lim_{z\to 1}\left[\frac{1}{z-1}\frac{1}{1+\frac{1}{(z-1)^N}G_{DC}(z)}\right]\\
&= \lim_{z\to 1}\left[\frac{1}{(z-1)+\frac{1}{(z-1)^{N-1}}G_{DC}(z)}\right]\\
e_{ss} &= \frac{1}{\lim_{z\to 1}\left[\frac{1}{(z-1)^{N-1}}G_{DC}(z)\right]}
\end{aligned}
$$

Based on this result we can have the following steady-state error conditions for the unit-ramp input based on the type condition of the system

- Type $N < 1$: $e_{ss} \to \infty$
- Type $N = 1$: $e_{ss} = \frac{1}{K_{DC}}$
- Type $N > 1$: $e_{ss} = 0$

Unit-Quadratic (Acceleration) Input

We know that $r[k] = \frac{1}{2}k^2u[k]$ and $R(z) = \frac{z^{-1}(1+z^{-1})}{2(1-z^{-1})^3}$ then we have

$$
\begin{aligned}
e_{ss} &= \lim_{z\to 1}\left[\left(1-z^{-1}\right)R(z)\frac{1}{1+G_{OL}(z)}\right]\\
&= \lim_{z\to 1}\left[\left(1-z^{-1}\right)\frac{z^{-1}(1+z^{-1})}{2(1-z^{-1})^3}\frac{1}{1+\frac{1}{(z-1)^N}G_{DC}(z)}\right]\\
&= \lim_{z\to 1}\left[\frac{(z+1)}{2(z-1)^2}\frac{1}{1+\frac{1}{(z-1)^N}G_{DC}(z)}\right]\\
&= \lim_{z\to 1}\left[\frac{(z+1)/2}{(z-1)^2+\frac{1}{(z-1)^{N-2}}G_{DC}(z)}\right]\\
e_{ss} &= \frac{1}{\lim_{z\to 1}\left[\frac{1}{(z-1)^{N-2}}G_{DC}(z)\right]}
\end{aligned}
$$

- Type $N < 2$: $e_{ss} \to \infty$
- Type $N = 2$: $e_{ss} = \frac{1}{K_{DC}}$
- Type $N > 2$: $e_{ss} = 0$

Example 1: $G(z) = \frac{z-1}{z-0.5}$ and $G_C(z) = K$. Compute the steady-state erro to unit-step, unit-ramp, a and unit-quadratic inputs.

Solution:

$$G_{OL}(z) = K\frac{z-1}{z-0.5} = \frac{1}{(z-1)^{-1}}\frac{K}{z-0.5}$$
$$G_{DC}(1) = 2K \quad ,\text{Type } -1$$

Then the steady-state errors are computed as

- Unit-step: $e_{ss} = 1$
- Unit-ramp: $e_{ss} = \infty$
- Unit-acceleration: $e_{ss} = \infty$

Example 2: $G(z) = \frac{z-1}{z-0.5}$ and $G_C(z) = K\frac{z}{z-1}$. Compute the steady-state erro to unit-step, unit-ramp, a and unit-quadratic inputs.

Solution:

$$G_{OL}(z) = \frac{Kz}{z-0.5}$$
$$G_{DC}(1) = 2K \quad ,\text{Type } 0$$

Then the steady-state errors are computed as

- Unit-step: $e_{ss} = \frac{1}{1+2K}$
- Unit-ramp: $e_{ss} = \infty$
- Unit-acceleration: $e_{ss} = \infty$

Example 3: $G(z) = \frac{z-1}{z-0.5}$ and $G_C(z) = K\frac{z^2}{(z-1)^2}$. Compute the steady-state erro to unit-step, unit-ramp, a and unit-quadratic inputs.

Solution:

$$G_{OL}(z) = \frac{Kz^2}{(z-1)(z-0.5)} = \frac{1}{z-1}\frac{Kz^2}{z-0.5}$$
$$G_{DC}(1) = 2K \quad ,\text{Type } 1$$

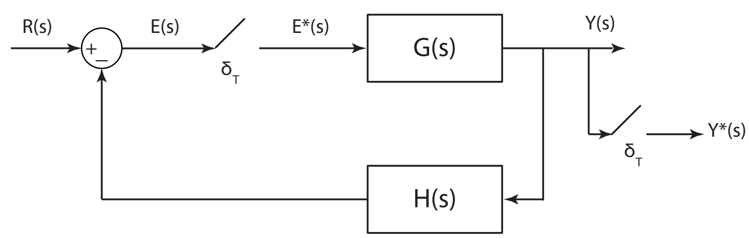
Then the steady-state errors are computed as

- Unit-step: $e_{ss} = 0$
- Unit-ramp: $e_{ss} = \frac{1}{2K}$
- Unit-acceleration: $e_{ss} = \infty$

Open-Loop Transfer Function for Different Topologies

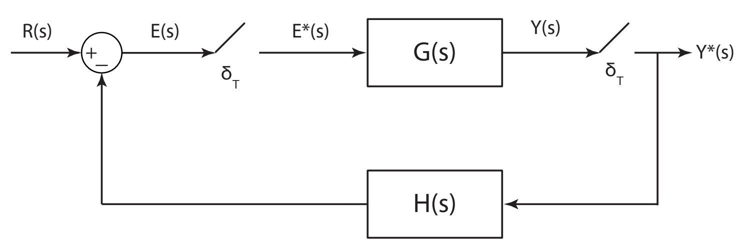
When computing the steady-state error it is important to carefully analyze the topology of the control system.

Compute the $G_{OL}(z)$ for the following system



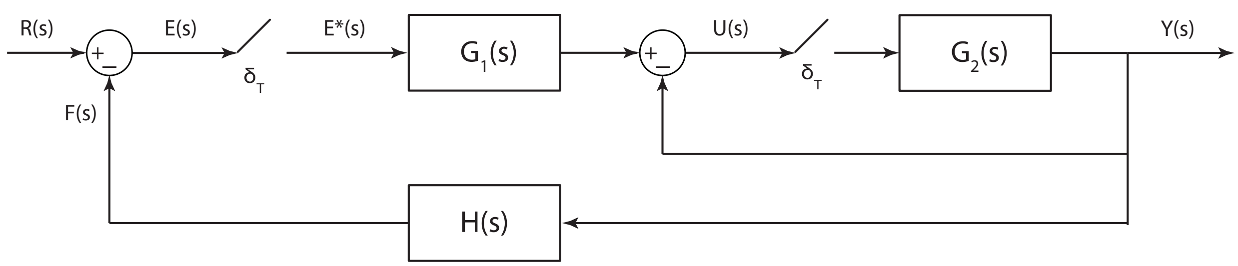
$$F(s) = E^*(s)G(s)H(s)$$
$$F^*(s) = E^*(s)[G(s)H(s)]^* = E^*(s)GH^*(s)$$
$$G_{OL}(z) = GH(z)$$

Now let's compte the $G_{OL}(z)$ for the following system



$$F(s) = [E^*(s)G(s)] * H(s) = E^*(s)G^*(s)H(s)$$
$$F^*(s) = E^*(s)G^*(s)H^*(s)$$
$$G_{OL}(z) = G(z)H(z)$$

Now let's compte the $G_{OL}(z)$ for the following system



From last week we know that

$$U^*(s) = \frac{G_1^*(s)}{1 + G_2^*(s) + G_1^*(s)GH^*(s)} R^*(s)$$

Then we can

$$U(s) = E^*(s)G_1(s) - U^*(s)G_2(s) \quad \rightarrow \quad U^*(s) = E^*(s)G_1^*(s) - U^*(s)G_2^*(s)$$

$$U^*(s) = \frac{G_1^*(s)}{1 + G_2^*(s)} E^*(s)$$

$$E^*(s) = \frac{1 + G_2^*(s)}{1 + G_2^*(s) + G_1^*(s)GH^*(s)} R^*(s)$$

$$\frac{E(z)}{R(z)} = \frac{1 + G_2(z)}{1 + G_2(z) + G_1(z)GH(z)}$$

This transfer function form does not (directly) fit to the form we analyzed, i.e. $\frac{E(z)}{R(z)} = \frac{1}{1+G_{OL}(z)}$, so we can not directly used the conditions and formulaes for this form. One way of comuting the steady-state errors is directly applying the final-value theorem.

The other way is we can simply convert the computed pulse transer function $E(z)/R(z)$ such that it fits the form $\frac{E(z)}{R(z)} = \frac{1}{1+G_{OL}(z)}$. If we carefully analyze the transer function we can obtain

$$\frac{E(z)}{R(z)} = \frac{1}{1 + \frac{G_1(z)G_2H(z)}{1+G_2(z)}}$$

$$G_{OL}(z) = \frac{G_1(z)G_2H(z)}{1 + G_2(z)}$$

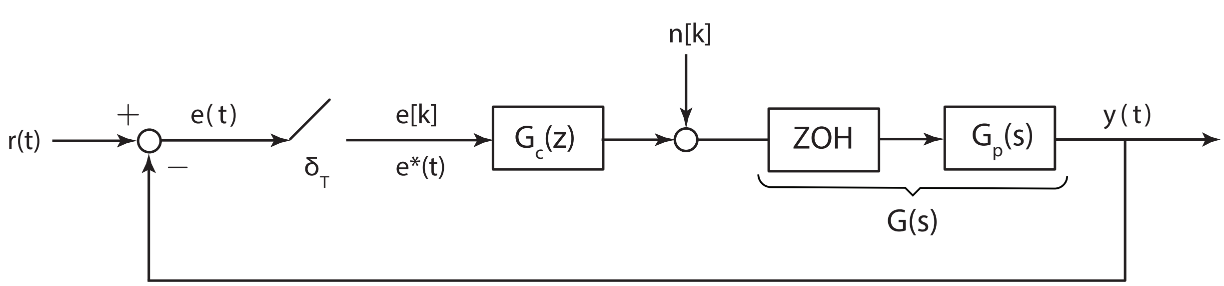
It is also possible to derive $G_{OL}(z)$ via direct computation of $F(z)/E(z)$.

Response to Disturbances

When analyzing response of a system in addition to the desired response to the reference input, it is also important to analyze the response (both steady-state, transient, and frequency) to unwanted disturbances and noises.

Process Disturbance/Uncertainty/Noise

Let's analyze a type of important disturbance on the fundamental discrete-time block diagram topology.



In order to analyze the response to the disturbance $n[k]$, we assume $r[k] = 0$ (which is just fine due to the linearity). Let's first find the pulse transfer function from $N(z)$ to $Y(z)$.

$$\begin{aligned}
Y(z) &= (-Y(z)G_c(z) + N(z))G(z) \\
\frac{Y(z)}{N(z)} &= \frac{G(z)}{1 + G_c(z)G(z)}
\end{aligned}$$

Technically, we want $\frac{Y(z)}{N(z)} = 0$, while also tracking the reference signal. Since it is not perfectly possible to achieve $\frac{Y(z)}{N(z)} = 0$ while satisfying other constraints, we want $\frac{Y(z)}{N(z)}$ to be "small". If $|G_C(z)G(z)| \gg 1$ then we have

$$\frac{Y(z)}{N(z)} \approx \frac{1}{G_c(z)}$$

Now let's consider a specific type of disturbance. An important class of process disturbance/uncertainty is in the form of DC bias, i.e. $n(t) = Nu(t)$ and $N(z) = \frac{N}{1-z^{-1}}$. Let's analyze DC steady state response using final value theorem.

$$\begin{aligned}
y_{ss} &= \lim_{z\to 1} \left[\left(1 - z^{-1}\right) Y(z)\right] = \lim_{z\to 1} \left[\left(1 - z^{-1}\right) N(z) \frac{G(z)}{1 + G_c(z)G(z)}\right] \\
&= \lim_{z\to 1} \left[\left(1 - z^{-1}\right) \frac{N}{1 - z^{-1}} \frac{G(z)}{1 + G_c(z)G(z)}\right] = \lim_{z\to 1} \left[N \frac{G(z)}{1 + G_c(z)G(z)}\right] \\
&= N \frac{\lim_{z\to 1} G(z)}{1 + \lim_{z\to 1} G_C(z)G(z)}
\end{aligned}$$

Let's analyze the steady-state disturbance response

- If plant is a type < 0 system (high pass filter plant) then $G(1) = 0$ and $y_{ss} = 0$.
- If plant is a type 0 system, then

  $$y_{ss} = \frac{NG(1)}{1 + G(1) \lim_{z\to 1} G_C(z)}$$

  Now let's analyze the response based on the type of $G_c(z)$

  - Type < 0, then

    $$y_{ss} = NG(1)$$

    In this case, controller has no control on the steady-state disturbance rejection performance.
  - Type 0, then

    $$y_{ss} = \frac{NG(1)}{1 + G(1)G_C(1)}$$

Obviously in order to "filter" the disturbance we should select a $G_C(z)$ such that $|G_C(1)G(1)| \gg 1$ then

$$y_{ss} = \frac{N}{G_C(1)}$$

Large gain $G_C(z)$ can effectively filter the disturbance (but not completely).

  – Type > 0, then

$$\begin{aligned} y_{ss} &= \frac{NG(1)}{1 + G(1)\lim_{z\to 1} G_C(z)} \\ &= 0 \end{aligned}$$

Integral action on $G_C(z)$ can perfectly reject the DC disturbance on steady state.

- If plant is a type $m > 0$ then

$$\begin{aligned} y_{ss} &= \frac{N \lim_{z\to 1} G(z)}{1 + \lim_{z\to 1} G(z)G_C(z)} \\ \lim_{z\to 1} G(z) &= \infty \end{aligned}$$

  Depending on the type of $G_C(z)$, we can conclude that

  – Type < 0, $y_{ss} = \infty$
  – Type 0, $y_{ss} = C$, where $0 < C < \infty$
  – Type > 0, $y_{ss} = 0$

Example 4: $G(z) = \frac{z-1}{z-0.5}$ and $G_C(z) = K$. Compute the steady-state response to a unit step process disturbance/noise.

Solution:

$$y_{ss} = \frac{\lim_{z\to 1} \frac{z-1}{z-0.5}}{1 + \lim_{z\to 1} K\frac{z-1}{z-0.5}} = 0$$

Plant perfectly rejects disturbance.

Example 5: $G(z) = \frac{z}{z-0.5}$ and $G_C(z) = K$. Compute the steady-state response to a unit step process disturbance/noise.

Solution:

$$y_{ss} = \frac{\lim_{z\to 1} \frac{z}{z-0.5}}{1 + \lim_{z\to 1} K\frac{z}{z-0.5}} = \frac{2}{1+2K}$$

Large gain K can be effective solution to reject disturbance

Example 6: $G(z) = \frac{z}{z-0.5}$ and $G_C(z) = K_P + K_I\frac{z}{z-1}$. Compute the steady-state response to a unit step process disturbance/noise.

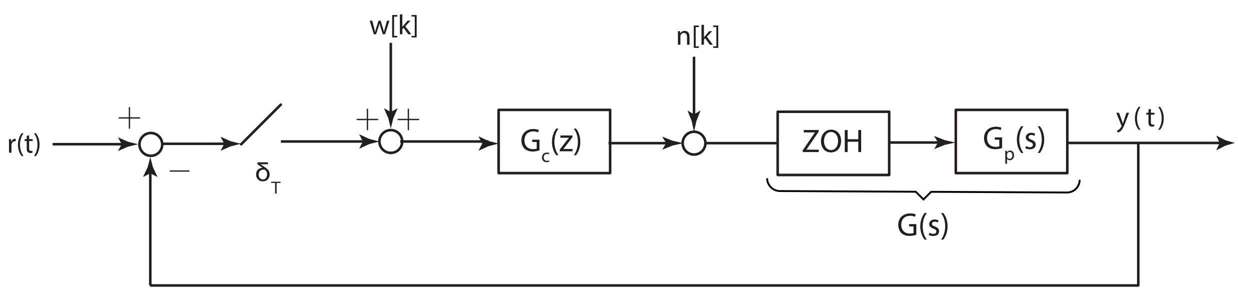
Solution:

$$y_{ss} = \frac{2}{1 + \lim_{z\to 1} 2\left(K_P + K_I \frac{z}{z-1}\right)} = 0$$

A PI controller can perfectly reject the DC provess disturbance.

Measurement Disturbance/Uncertainty/Noise

Let's analyze a different type of important disturbance on the fundamental discrete-time block diagram topology.



In order to analyze the response to the disturbance $w[k]$, we assume $r[k] = 0$ and $n[k] = 0$

$$\begin{aligned} Y(z) &= (W(z) - Y(z))G_c(z)G(z) \\ \frac{Y(z)}{W(z)} &= \frac{G(z)G_C(z)}{1 + G_c(z)G(z)} \\ &= \frac{G_{OL}(z)}{1 + G_{OL}(z)} \end{aligned}$$

Technically, we want $\frac{Y(z)}{N(z)} = 0$, while also tracking the reference signal. Thus, practically we should design $G_C(z)$ such that $|G_C(z)G(z)| \ll 1$, to eliminate measurement noises/disturbances. (??????)

$$\frac{Y(z)}{N(z)} \approx G_{OL}(z)$$

This "requirement" obviously contradicts with requirements on stead-state tracking error performance and process noise/disturbance rejection performance. Most well known limitation of feedback control systems.

Now let's consider a specific type of measurement noise, i.e. DC measurement bias. $w(t) = Wu(t)$ and $W(z) = \frac{W}{1-z^{-1}}$. Let's analyze DC steady state response using final value theorem.

$$\begin{aligned} y_{ss} &= \lim_{z\to 1}\left[(1 - z^{-1})R(z)\frac{G_{OL}(z)}{1 + G_{OL}(z)}\right] = \lim_{z\to 1}\left[(1 - z^{-1})\frac{W}{1 - z^{-1}}\frac{G_{OL}(z)}{1 + G_{OL}(z)}\right] \\ &= \lim_{z\to 1}\left[\frac{WG_{OL}(z)}{1 + G_{OL}(z)}\right] \end{aligned}$$

Using the form $G_{OL}(z) = \frac{1}{(z-1)^N} G_{DC}(z)$, y_{ss} takes the form

$$y_{ss} = \lim_{z \to 1} \left[\frac{W G_{DC}(1) \frac{1}{(z-1)^N}}{1 + G_{DC}(1) \frac{1}{(z-1)^N}} \right]$$

Based on the type of the open-loop transfer function, $G_{OL}(z)$, we can conclude

- Type $N < 0$: $y_{ss} = 0$. Perfect rejection of measurement bias, but we know that this is unacceptable from reference tracking point of view.
- Type $N = 0$

$$y_{ss} = \frac{W G_{DC}(1)}{1 + G_{DC}(1)}$$

  It seems that in order to "filter" the measurement bias $G_{DC}(1)$ should be selected very small.
- Type $N > 0$

$$y_{ss} = W$$

  The disturbance is directly transfered to the output.

Example 7: $G(z) = \frac{z-1}{z-0.5}$ and $G_C(z) = K$. Compute the steady-state response to a unit step measurement disturbance/noise.

Solution:

$$G_{OL}(z) = K \frac{z-1}{z-0.5} \quad \text{Type } -1$$
$$y_{ss} = 0$$

Plant perfectly rejects measurement bias.

Example 8: $G(z) = \frac{z}{z-0.5}$ and $G_C(z) = K$. Compute the steady-state response to a unit step process disturbance/noise.

Solution:

$$G_{OL}(z) = K \frac{z}{z-0.5} \quad \text{Type } 0$$
$$y_{ss} = \frac{2K}{1+2K}$$

Small gain K can be effective solution to reject measurement bias.

EE402 - Discrete Time Systems **Spring 2018**

Lecture 10

Lecturer: Asst. Prof. M. Mert Ankarali

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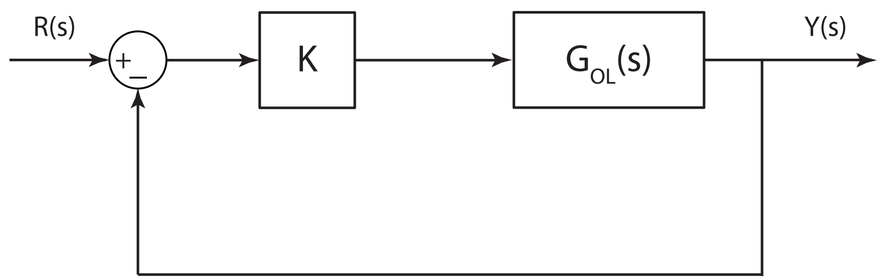
Root Locus

For continuous time systems the root locus diagram illustrates the location of roots/poles of a closed loop LTI systems, with respect to gain parameter K (can be considered as a P controller). The basic closed-loop topology is used for deriving the root-locus rules, however we know that many different topologies can be reduced to this from.



The closed loop transfer function of this basic control system is

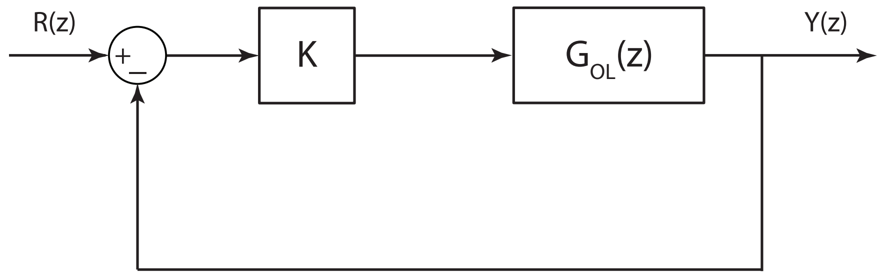
$$\frac{Y(s)}{R(s)} = \frac{KG_{OL}(s)}{1 + KG_{OL}(s)}$$

where the poles of the closed loop system are the roots of the characteristic equation

$$\begin{aligned} 1 + KG_{OL}(s) &= 0 \\ 1 + K\frac{n(s)}{d(s)} &= 0 \end{aligned}$$

In 302 we learned the rules such that we can derive the qualitative and quantitive structure of root locus paths for **positive** gain K that solves the equation above.

In discrete time systems, similar to the CT case we use the root locus diagram to illustrates the location of roots/poles of a closed loop DT-LTI systems, with respect to a gain parameter K (can be considered as a P controller). The basic discrete-time closed-loop topology is used for deriving the root-locus rules, however we know that many different DT topologies can be reduced to this from.



The closed loop transfer function of this basic control system is

$$\frac{Y(z)}{R(z)} = \frac{KG_{OL}(z)}{1 + KG_{OL}(z)}$$

where the poles of the closed loop system are the roots of the characteristic equation

$$1 + KG_{OL}(z) = 0$$
$$1 + K\frac{n(z)}{d(z)} = 0$$

I think, it is obvious that fundamental equation that relates the gain K and roots/poles is exactly same for both CT and DT systems. This means that same rules are directly applied for CT systems.

However, even if we have same diagram for CT and DT systems the meaning and interpretation of the diagram is fundamentally different. Because, the effects of pole locations are different in CT and DT systems.

Angle and Magnitude Conditions

Let's analyze the characteristic equation

$$KG_{OL}(z) = -1 \quad \text{,or} \quad K\frac{n(z)}{d(z)} = -1$$
$$|KG_{OL}(z)| = 1 \quad \text{,or} \quad \left|K\frac{n(z)}{d(z)}\right| = 1$$
$$\angle[KG_{OL}(z)] = \pi(2k+1),\ k \in \mathbb{Z} \quad \text{,or} \quad \angle\left[K\frac{n(z)}{d(z)}\right] = \pi(2k+1),\ k \in \mathbb{Z}$$

For a given K, z values that satisfy both magnitude and angle conditions are located on the root loci.

Rules and procedure for constructing root loci

1. Characteristic equation, zeros and poles of the Open-Loop pulse transfer function.

$$1 + KG_{OL}(z) = 0$$
$$1 + K\frac{n(z)}{d(z)} = 0$$
$$1 + K\frac{(z - z_1)\cdots(z - z_M)}{(z - p_1)\cdots(z - p_N)} = 0$$

2. Root loci has N separate branches.

3. Root loci starts from poles of $G_{OL}(z)$ and

   (a) M branches terminates at the zeros of $G_{OL}(z)$

   (b) N branches terminates at ∞ (implicit zeros of $G_{OL}(z)$)

   It is relatively easy to undestand this

$$\begin{aligned} d(z) + Kn(z) &= 0 \\ K \to 0 \ \to \ d(z) &= 0 \\ K \to \infty \ \to \ n(z) &= 0 \end{aligned}$$

4. Root loci on the real axis determined by open-loop zeros and poles. $z = \sigma \in \mathbb{R}$ then,

$$\begin{aligned} |KG_{OL}(\sigma)| &= 1 \\ \text{Sign}[G_{OL}(\sigma)] &= -1 \end{aligned}$$

   We can always find a K that satisfy the magnitude condition,so angle condition will determine which parts of real axis belong to the root locus.

   We can first see that complex conjugate zero/pole pairs has not effect, then for the remaining ones we can derive the following condition

$$\text{Sign}[G_{OL}(\sigma)] = \prod_{i=1}^{M} \text{Sign}[\sigma - z_i] \prod_{j=1}^{N} \text{Sign}[\sigma - p_j] = -1$$

   which means that for ODD number of poles + zeros Sign$[\sigma - p_i]$ and Sign$[\sigma - z_i]$ must be negative for satisfying this condition for that particular σ to be on the root-locus. We can summarize the rule as

   If the test point σ on real axis has ODD numbers of poles and zeros in its right, then this point is located on the root-locus.

5. Asymptotes

   - $N - M$ branches goes to infinity. Thus, there exist $N - M$ many asymptotes
   - For large z we can have the following approximation

$$\begin{aligned} & K\frac{(z - z_1)\cdots(z - z_M)}{(z - p_1)\cdots(z - p_N)} \approx \frac{K}{z^3} \\ & \angle\left[\frac{K}{z^{N-M}}\right] = -(N - M)\angle[z] = \pi(2k + 1), \ k \in \mathbb{Z} \\ & \phi_a = \frac{\pm\pi(2k + 1)}{N - M}, \ k \in \{1, \cdots, N - M\} \end{aligned}$$

   - Real axis intercept σ_a can be computed as

$$\sigma_a = \frac{\sum p_i - \sum z_i}{N - M}$$

     This can be derived via a different approximation (see textbook)

6. Breakaway and break-in points on real axis. When z is real $z = \sigma,\ \sigma \in \mathbb{R}$, we can have

$$1 + KG_{OL}(\sigma) = 0 \quad \rightarrow \quad K(\sigma) = \frac{-1}{G_{OL}(\sigma)}$$

Note that break-in and breakaway points corresponds to double roots. Thus, σ_b is a break-away or break-in point if

$$\left[\frac{dK(\sigma)}{\sigma}\right]_{\sigma=\sigma_b} = 0$$
$$K(\sigma) > 0$$

7. Angle of departure (or arrival) from open-loop complex conjugate poles (or to open-loop complex conjugate zeros)

Let's assume that p^* is a complex conjugate pole of $G_{OL}(z)$, then let's define a $P(z)$ such that

$$P^*(z) = (z - p^*)G_{OL}(z)$$

We know that for $K = 0$, the root locus is located at p^*. If we add a very small $K = \delta K$, then pole/root locus moves to $p^* + \delta z$. If we evaluate the phase condition of the root locus on this new "unknown" point

$$\begin{aligned}
\angle[KG_{OL}(z)]_{z=p*+\delta z} &= \pm\pi \\
\angle\left[\frac{P^*(z)}{z - p^*}\right]_{z=p^*+\delta z} &= \pm\pi \\
\angle\left[P^*(p^* + \delta z)\right] - \angle\left[\delta z\right] &= \pm\pi \\
\theta_d = \angle\left[\delta z\right] &= \pm\pi + \angle\left[P^*(p^*)\right]
\end{aligned}$$

Geometrically speaking, $\angle\left[P^*(p^*)\right]$ stands for **# of angles from the zeros to this specific pole** – **# of angles from all other remaining poles to this specific pole**.

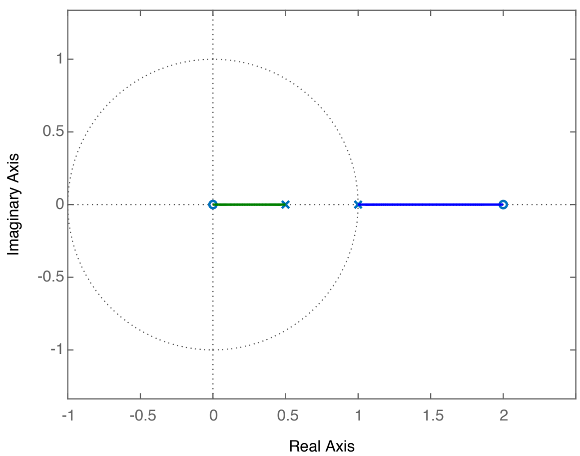
A similar condition can be derived for angle of arrival to complex conjugate zeros.

$$\theta_a = \pm\pi - \angle\left[P^*(z^*)\right]$$
$$P^*(z) = (z - z^*)G_{OL}(z)$$

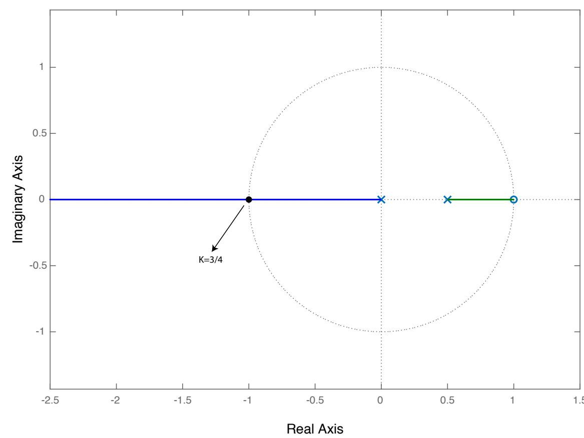
where z^* is a complex conjugate zero of $G_{OL}(z)$.

Examples

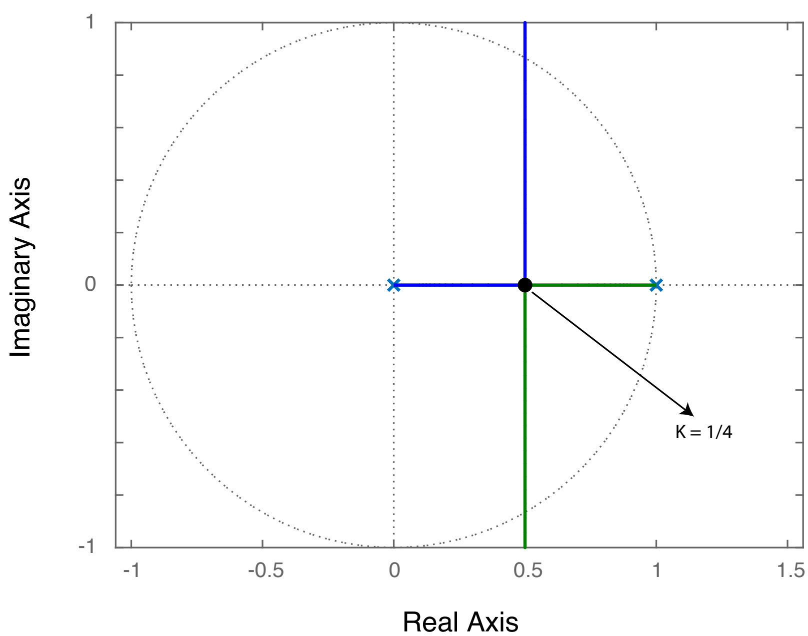
$$G_{OL}(z) = \frac{z(z-2)}{(z-1)(z-0.5)}$$



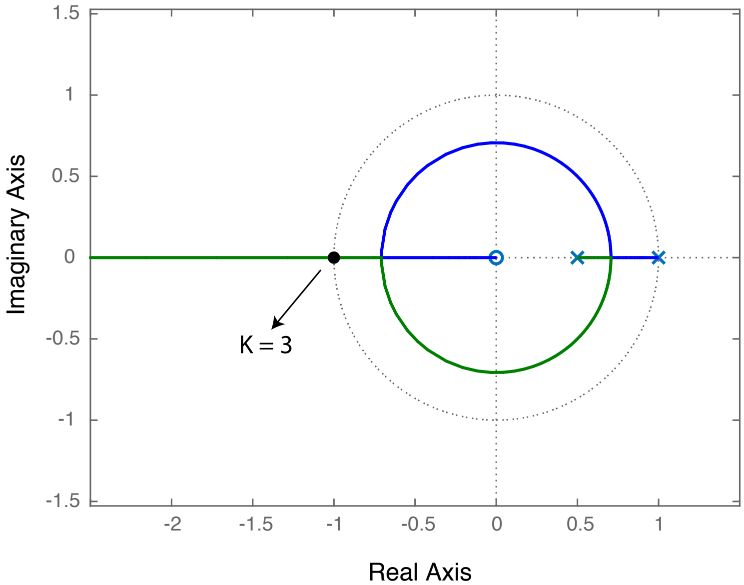
$$G_{OL}(z) = \frac{z-1}{z(z-0.5)}$$



$$G_{OL}(z) = \frac{1}{z(z-1)}$$



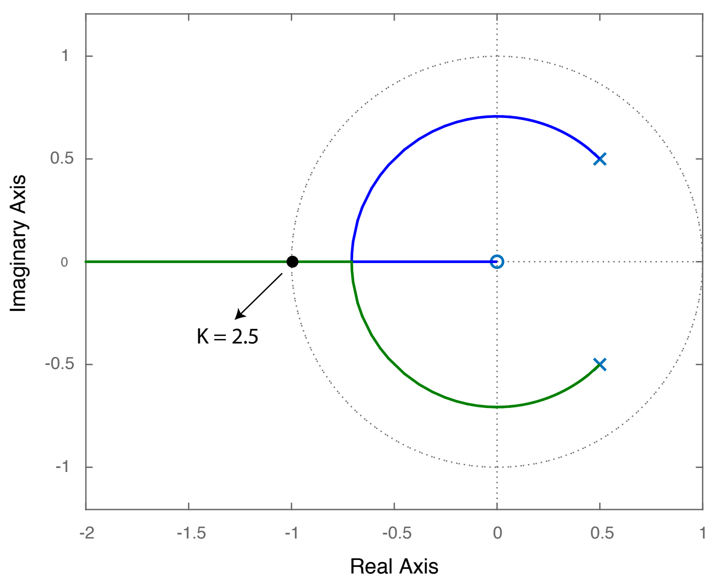
$$G_{OL}(z) = \frac{z}{(z-0.5)(z-1)}$$



$$\sigma_{ba} = \frac{\sqrt{2}}{2} \quad , \quad \sigma_{bi} = -\frac{\sqrt{2}}{2}$$

K = 0.086 K = 2.9

$$G_{OL}(z) = \frac{z}{z^2 - z + 1/2}$$



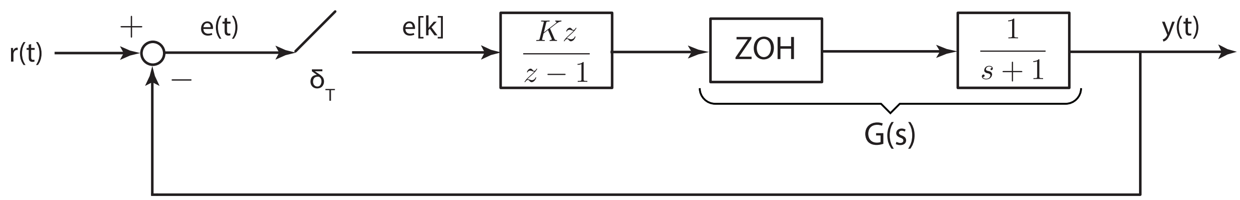
$$\theta_d = \pm\frac{3\pi}{4}$$

$$\sigma_{bi} = -\frac{\sqrt{2}}{2}$$

K = 2.4

Root-Locus of Digital Control Systems

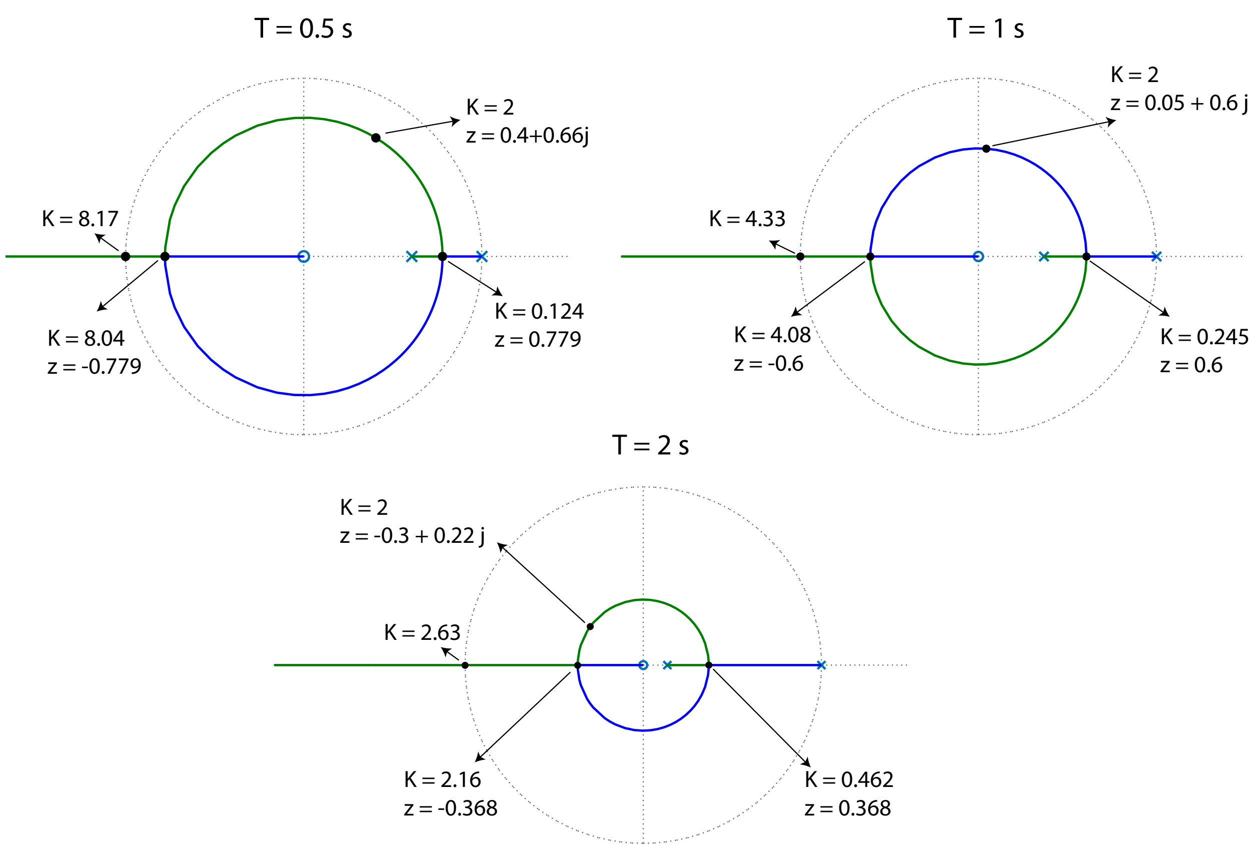
Let us draw the root-locus diagrams for the discrete time control system below for $T = 0.5s$, $T = 1s$, and $T = 2s$.



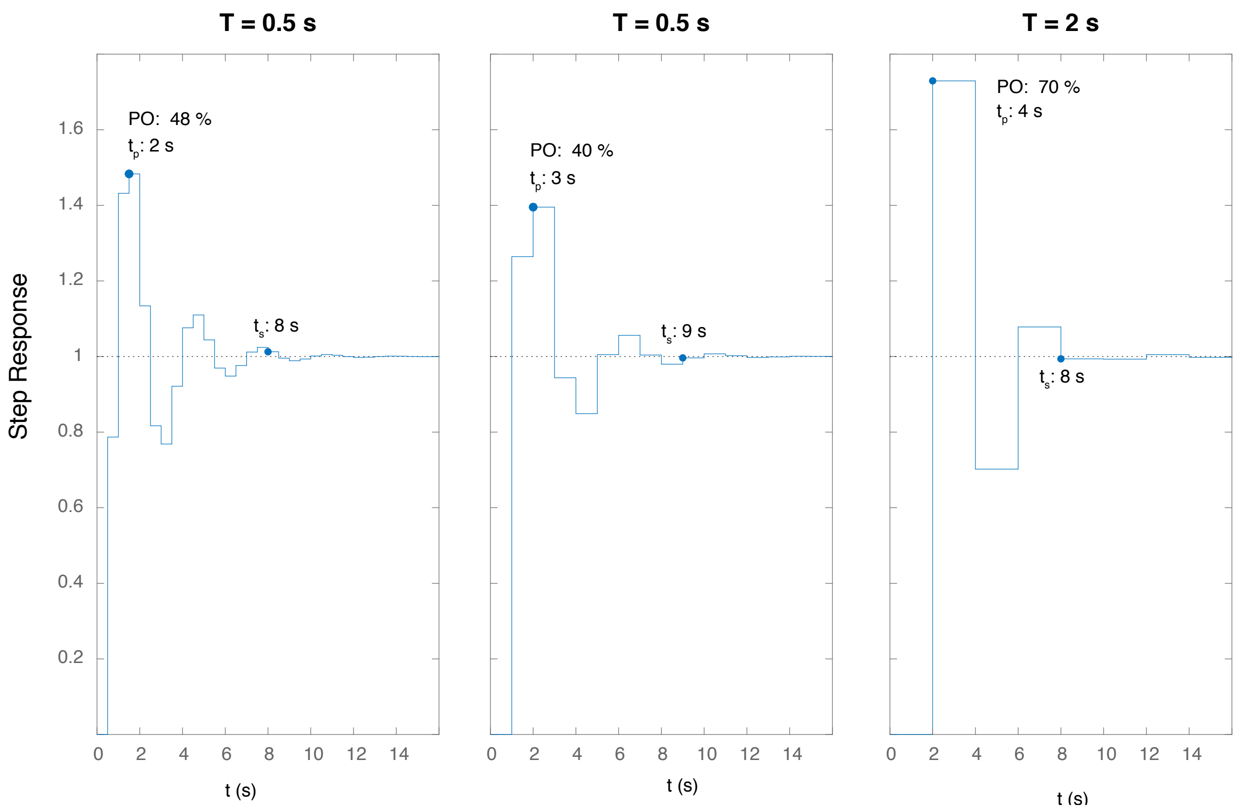
Open-loop pulse transfer functions cane be obtained as

$$G_{0.5} = K\ 0.3935\frac{z}{(z-1)(z-0.6065)}$$
$$G_1 = K\ 0.6321\frac{z}{(z-1)(z-0.3679)}$$
$$G_2 = K\ 0.8647\frac{z}{(z-1)(z-0.1353)}$$

Root-locus plots for all there cases are illustrated in the Figure below.



This Figure below compares DT step responses of all three sampling time cases, where x-axis is the actual time.



If we compare three different responses, we can clearly see that in terms of over-shoo, $T = 2s$ has the worst performance, where as $T = 1s$ seems to be a little better than $T = 0.5\ s$. However, if one "draws" the CT-response by simulating the whole hybrid system, he/she can see that the over-shoot for $T = 1\ s$ indeed larger than $T = 0.5s$. Due to the "sampling rate" we can not capture the over-shoot difference clearly between $T = 0.5\ s$ and $T = 1\ s$. We will talk about it in the next section.

Another similarity between these responses is that settling times of DT-system responses seem to be very close. We know that settling time for a CT-system mainly depends on the real part of the dominant pole, i.e. $\sigma = \text{Re}\{s\}$. Let's compute $\sigma_{0.5}$, σ_1,and σ_2 by taking into account the $e^{Ts} = z$ mapping.

$$
\begin{aligned}
\sigma_{0.5} &= \frac{\ln(|z|)}{0.5} \approx -0.5 \\
\sigma_1 &= \frac{\ln(|z|)}{1} \approx -0.5 \\
\sigma_2 &= \frac{\ln(|z|)}{2} \approx -0.5
\end{aligned}
$$

It can be seen that indeed the real part of the mapped CT pole locations are approximately same for same gain $K = 2$.

Now let's evaluate the steady-state error performances for both sampling times. All three open-loop transfer functions are Type 1 systems and thus for unit step response $e_{ss} = 0$, which is also observable from the step response plots. Let's compute e_{ss} for unit ramp response.

$$e_{0.5} = \frac{1}{2}$$
$$e_1 = \frac{1}{2}$$
$$e_2 = \frac{1}{2}$$

This can give an illusion that they have the same performance. However we evaluated response to the discrete time unit step input which is $r[k] = k[uk]$. However, in order to compare fairly, we need to compute the steady state error to sampled continuous time unit ramp function. $r(t) = t$ & $r(kT) = kT$, thus we have

$$r_{0.5}[k] = 0.5k$$
$$r_1[k] = 1k$$
$$r_2[k] = 2k$$

If we re-evaluate the stead-state errors, we obtain

$$e_{0.5} = \frac{1}{4}$$
$$e_1 = \frac{1}{2}$$
$$e_2 = 1$$

Now it is clear that as T increases the steady-state error also increases.

Now let's compare different cases, where each response show a critically damped behavior. Critically dampled locations are obtained for the following K values.

$$K_{0.5} = 0.124$$
$$K_1 = 0.245$$
$$K_2 = 0.462$$

Now let's compare responses using different specifications

- Obviously no oscillations and 0% overs-shoot for all three cases
- The CT-time mapped pole locations can be computed as

  $$\sigma_{0.5} \approx \sigma_1 \approx \sigma_2 \approx -0.5$$

  which implies that settling time performance is similar.
- Since all systems are Type 1, the stead-state errors to unit step is zero for all cases.
- If we compute the steady-state errors to sampled CT unit ramp input for all three cases we obtain

  $$e_{0.5} \approx e_1 \approx e_2 \approx 4$$

  Quite interestingly even the steady-state errors are approximately similar.

This shows that for the specific location where all systems show a critically damped behavior the response performance (stability, overshoot, stead-state error) is quite similar and almost independent on T.

However if one needs to improve the steady-state error performance (to ramp like inputs) obviously lower T (or higher sampling rate) is better both from the perspective of stability and steady-state error performance.

Specs for discrete time pole locations

1. Absolutely we want the poles to be located inside the unit-circle for stability.

2. From the perspective of DT-control systems (CT-systems that are controlled by DT-controllers), we want the frequency of oscillations to be sufficiently smaller than the Nyquist frequency $\omega_s/2$. As a rule of thumb 8-10 samples per cycle are required. Let's assume that require 8 samples per cycle then we have the following condition on z

$$\angle z \in [-\pi/4, \pi/4]$$

3. For a CT system if the system has a dominant second order (or first order) characteristic, then we know that settling time is approximately given by, $T_s = \frac{4}{\sigma}$, where $-\sigma$ is the real part of the CT-pole. Let's assume that the requirement is the settling time of the step response of the system will be less than $\bar{T}$. Then considering the $z = e^{Ts}$ mapping we have the following condition.

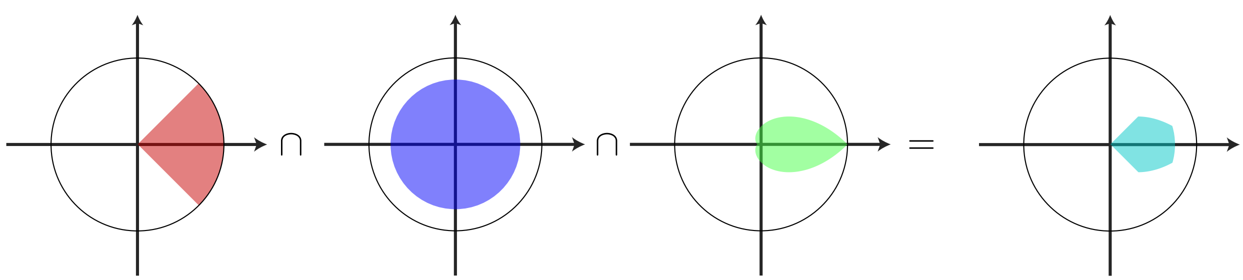
$$T_s \leq \bar{T} \;\rightarrow\; \sigma \geq \frac{4}{\bar{T}}$$
$$|z| \leq e^{-4\frac{T}{\bar{T}}}$$

4. Another important requirement for CT systems is the overshoot and damping coefficient. We want damping to be high enough such that overshoot is reasonable. A rule of thumb for damping coefficient is that $\zeta \geq 1/\sqrt{2}$ (However different requirements can also be specified). Based on this we have the following condition on s

$$s = -\alpha\omega_d + \omega_d j$$
$$\alpha = \frac{\zeta}{\sqrt{1-\zeta^2}}$$
$$\alpha \geq 1$$

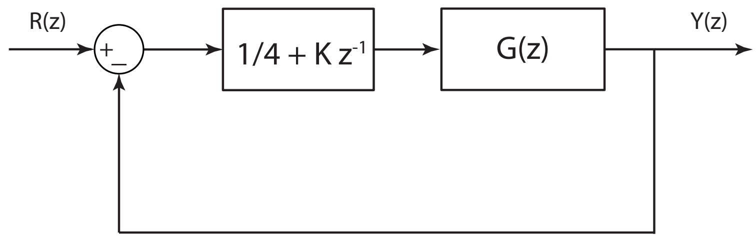
   In Lecture 7, we already illustrated the region where $\alpha \geq 1$.

These specs and their combined desired pole location region is illustrated in the Figure below.



Root-locus with respect to different parameters

Let's consider the following purely DT system where plant has a transfer function of $G(z)$ and controller has a first order FIR filter (low-pass) $G_c(z) = 1 + Az^{-1}$ form. We wonder the location of closed-loop poles with respect to the parameter A which does not directly fit to the classical form.



Let's first compute the closed-loop PTF and analyze the characteristic equation.

$$\frac{Y(z)}{R(z)} = \frac{(0.25 + Az^{-}1)\, G(z)}{1 + (0.25 + Az^{-}1)\, G(z)}$$
$$1 + \left(0.25 + Az^{-1}\right) G(z) = 0$$
$$1 + 0.25G(z) + Az^{-1}G(z) = 0$$

If we divide the characteristic equation by $1 + 0.25G(z)$ we obtain

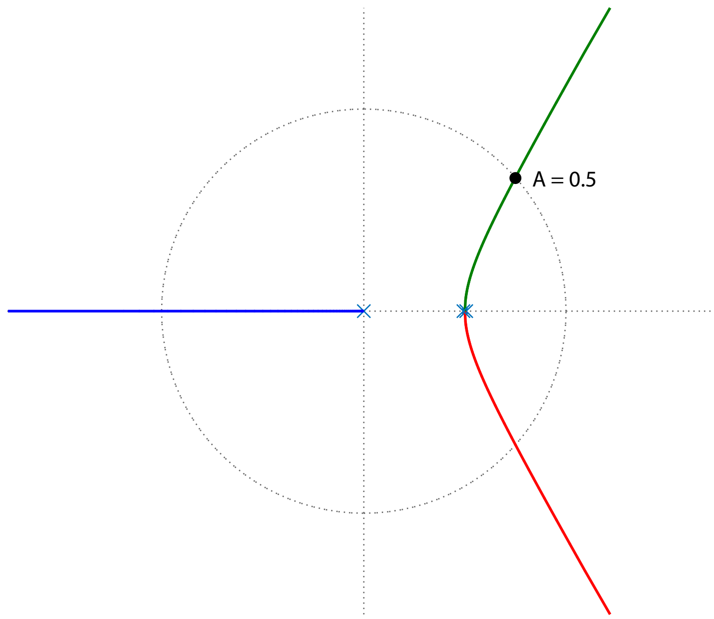
$$1 + A\frac{z^{-1}G(z)}{1 + 0.25G(z)} = 0$$
$$1 + A\bar{G}_{OL}(z) = 0$$

Now if we consider as $\bar{G}_{OL}(z)$ as the open-loop transfer function and draw the root-locus then we would derive the dependence of the roots to the parameter A.

Let's assume that $G(z) = \frac{1}{z(z-1)}$. Then for this system, we can compute

$$\bar{G}_{OL}(z) = \frac{z^{-1}G(z)}{1 + 0.25G(z)} = \frac{\frac{1}{z^2(z-1)}}{1 + \frac{0.25}{z(z-1)}} = \frac{\frac{1}{z^2(z-1)}}{\frac{z^2-z+0.25}{z(z-1)}}$$
$$= \frac{1}{z(z^2 - z + 0.25)}$$

Root-locus of the system w.r.t parameter A is given below. It can be seen that as A increases, dominant system poles deviates from the origin and eventually becomes unstable at $A = 0.5$. Technically this is a simple low-pass filter which may be inevitable in many closed-loop control systems. However, as we decrease the cut-off frequency (by increasing A) we push the poles towards the unit circle thus making the system less stable.


A = 0.5

EE402 - Discrete Time Systems	**Spring 2018**

Lecture 11

Lecturer: Asst. Prof. M. Mert Ankarali

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Frequency Response in Discrete Time Control Systems

Let's assume $u[k]$, $y[k]$, and $G(z)$ represents the input, output, and transfer function representation of an input-output discrete time system.

In order to characterize frequency response of a discret system, the test signal is

$$u[k] = e^{j\omega k}$$

which is an artificial complex periodic signal with a DT domain frequency of ω. The z-transform of $u[k]$ takes the form

$$U(z) = \mathcal{Z}\{e^{j\omega k}\} = \frac{z}{z - e^{j\omega}}$$

Response of the system in z-domain is given by

$$Y(z) = G(z)U(z) = G(z)\frac{z}{z - e^{j\omega}}$$

Assuming that $G(z)$ is a rational transfer function we can perform a partial fraction expansion

$$\begin{aligned} Y(z) &= \frac{az}{z - e^{j\omega}} + [\text{terms due to the poles of } G(z)] \\ a &= \lim_{z \to e^{j\omega}} \left[(z - e^{j\omega})\frac{Y(z)}{z}\right] = G(e^{j\omega}) \\ Y(z) &= \frac{G(e^{j\omega})z}{z - e^{j\omega}} + [\text{terms due to the poles of } G(z)] \end{aligned}$$

Taking the inverse z-transform yields

$$y(t) = G(e^{j\omega})e^{j\omega k} + \mathcal{Z}^{-1}\left[\text{terms due to the poles of } G(z)\right]$$

If we assume that the system is "stable" or system is a part of closed loop system and closed loop behavior is stable then at steady state we have

$$\begin{aligned} y_{ss}[k] &= G(e^{j\omega})e^{j\omega k} \\ &= |G(e^{j\omega})|e^{i\omega k + \angle G(e^{j\omega})} \\ &= Me^{i\omega k + \theta} \end{aligned}$$

In other words complex periodic signal is scaled and phase shifted based on the following operators

$$\begin{aligned} M &= |G(e^{j\omega})| \\ \theta &= \angle G(e^{j\omega} \end{aligned}$$

It is very easy to show that for a general real time domain signal $u[k] = \sin(\omega k + \phi)$, the output $y[k]$ at steady state is computed via

$$y_{ss}[k] = M \sin(\omega k + \phi + \theta)$$

If there is sampling involved in the system the following relation between DT frequency and CT frequency exists $\omega_d = \omega_c T$

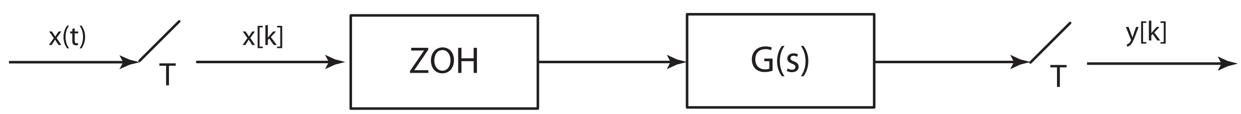
Similar to CT systems we utilize bode polts (or FRF function polts) to analyze DT systems and design filter/controllers. Main difference between CT and DT bode polt is that while the frequency goes to infinity for CT bode polts, for DT systems the frequency goes up-to π rad or $\omega_s/2$ (i.e. Nyquist frequency). Given the bode polt one can extract the magnitude scale and phase difference with respect to any input frequency.

Example: Let's assume that we have the following CT plant transfer function

$$G(s) = \frac{1}{s+1}$$

The pulse transfer function of the following discretized system can be computed as

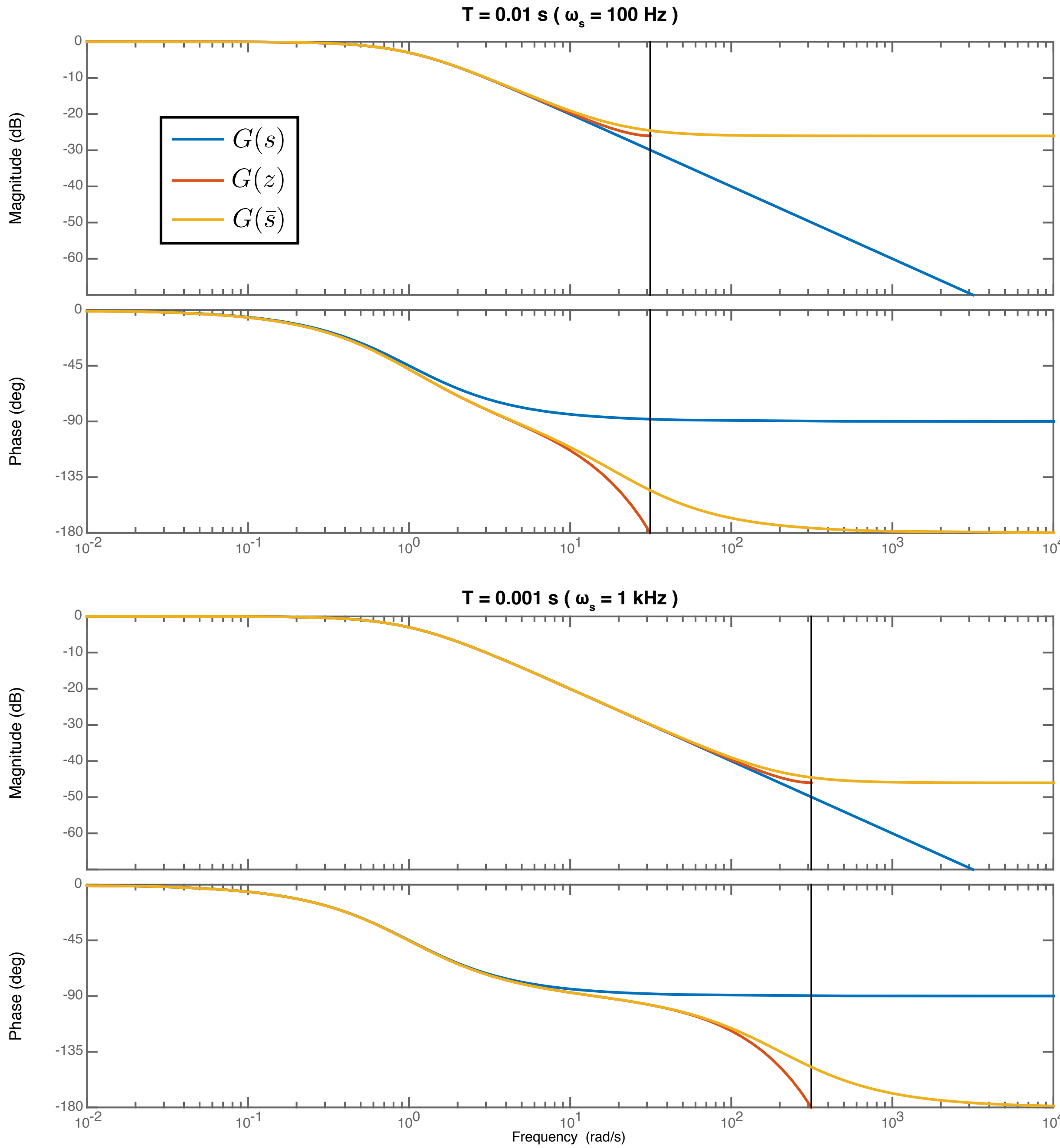
$$G(z) = \mathcal{Z}\left[\frac{1-e^{-Ts}}{s}\frac{1}{s+1}\right]$$



We can also transform this DT transfer function to an artificial CT form using Bilinear-Tustin transformation.

$$G(\bar{s}) = G(z)\big|_{z=\frac{1+(T/2)\bar{s}}{1-(T/2)\bar{s}}}$$

Now let's draw the bode plots of $G(s)$, $G(z)$, and $G(\bar{s})$ for both $T = 0.01s$ and $T = 0.001s$.


T = 0.01 s (ω_s = 100 Hz)
Magnitude (dB)
Phase (deg)
G(s)
G(z)
G(s̄)
0
-10
-20
-30
-40
-50
-60
-45
-90
-135
-180
10^-2
10^-1
10^0
10^1
10^2
10^3
10^4
T = 0.001 s (ω_s = 1 kHz)
Frequency (rad/s)

Phase and Gain Margins

We already know that a binary stability metric is not enough the characterize the system performance and that we need metrics to evaluate how stable the system is and its robustness to perturbations. Using root-locus techniques we talked about some "good" pole regions which provides some specifications about stability and closed-loop performance.

Another common and powerful method is the use gain and phase margins based on the Frequency Response Functions of a closed-loop topology. Phase and gain margins are derived from the Nyquists stability criterion and it is relatively easy compute them from the Bode diagrams.

Gain Margin

The *gain margin*, g_m, of a system is defined as the smallest amount that the open loop gain can be increased before the closed loop system goes unstable. For a system, whose "open-loop" phase response starts from an angle $> -180^o$ at $\omega = 0$, the gain margin can be computed based on the smallest frequency where the phase of the loop transfer function $G_{OL}(s)$ is 180^o. Let ω_{pc} represent this frequency, called the phase crossover frequency. Then the gain margin for the system is given by

$$g_m = \frac{1}{|G_{OL}(j\omega_{wc})} \quad \text{or} \quad G_m = -20\log_{10}|G_{OL}(j\omega_{wc})|$$

where G_m is the gain margin in dB scale.

If the phase response never crosses the -180^o line gain margin is simply ∞.

Phase Margin

The *phase margin* is the amount of "phase lag" required to reach the (Nyquist) stability limit. Let ω_{gc} be the gain crossover frequency, the smallest frequency where the loop transfer function $G_{OL}(s)$ has unit magnitude. Then for a system for which the gain response at $\omega = 0$ is larger than 1 and gain decreases and eventually crosses the unity gain line, the phase margin is given by

$$\phi_m = \pi + \angle G_{OL}(j\omega_{gc})$$

When the gain and phase plots shows monotonic like behaviors gain and phase margins becomes more meaningful in terms of closed-loop performance.

So far we have only talked about stability margins for a CT control system. Indeed, the Nyquist stability criterion and associated phase and gain margin definitions are almost exactly same, if we consider FRF functions. Let $G_{OL}(z)$ be the open-loop pulse transfer function of a discrete time control system, then the gain and phase margins are computed as

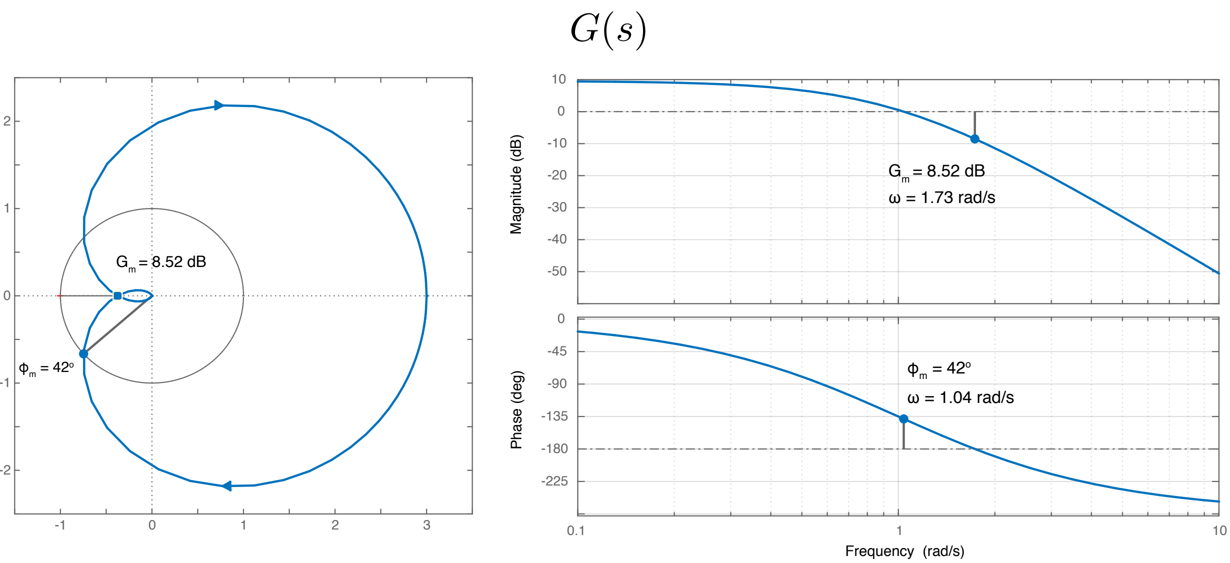
$$g_m = \frac{1}{|G_{OL}(e^{j\omega_{wc}})} \quad \text{or} \quad G_m = -20\log_{10}|G_{OL}(e^{j\omega_{wc}})|$$
$$\phi_m = \pi + \angle G_{OL}(e^{j\omega_{gc}})$$

where as definitions of gain and phase crossover frequencies are exactly same.

Example: Let's consider a CT plant transfer function

$$G(s) = \frac{3}{(s+1)^3}$$

Nyquist plot, bode diagrams are illustrated in the given Figure below



The phase crosover frequency and gain margin for the CT open-loop transfer function is given below

$$\begin{aligned} \omega_{pc} &= 1.73 \text{ rad/s} \\ g_m &= \frac{1}{|G(j\omega_{pc}|} = 2.67 \\ G_m &= 8.5dB \end{aligned}$$

On the other hand gain crosover frequency and phase margin for $G(s)$ is computed as

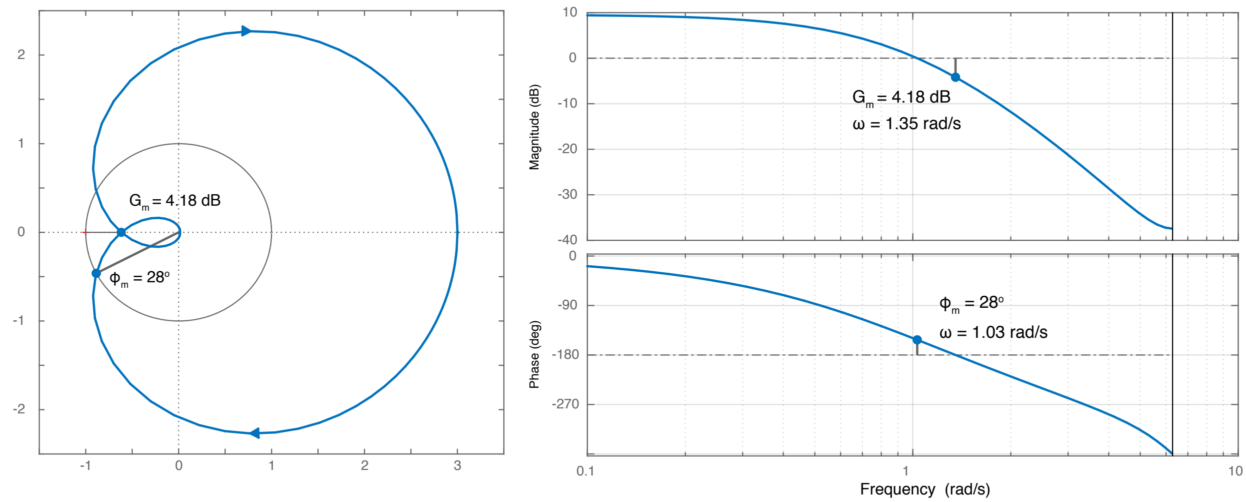
$$\begin{aligned} \omega_{gc} &= 1.04 \text{ rad/s} \\ \phi_m &= \pi + \angle G(j\omega_{pc}| = 42^0 \end{aligned}$$

Now let's assume that this plant transfer function is controlled via a unity gain digital feedback controller and a ZOH operator, where sampling time is $T = 0.5\ s$. Open-loop pulse transfer function can be found as

$$G(z) = \left[\frac{1-e^{-Ts}}{s} G(s)\right]$$

The Nyquist and bode polts for this DT open-loop puls transfer function is illustrated below

$G(z)$



Note that instead of DT frequency $\omega_d \in [0, pi)$, the x-axis illustrates the actual frequency $\omega = \omega_d/T$. Is very easy to associate DT and CT frequencies, and main advantage of actual frequency is that CT and DT versions of bode plots becomes directly comparable.

The phase crosover frequency and gain margin for the DT open-loop transfer function is given below

$$\begin{aligned}
\omega_{pc} &= 1.35 \text{ rad/s} \\
g_m &= \frac{1}{|G(e^{j\omega_{pc}T}|} = 1.62 \\
G_m &= 4.18dB
\end{aligned}$$

On the other hand gain crosover frequency and phase margin for $G(s)$ is computed as

$$\begin{aligned}
\omega_{gc} &= 1.03 \text{ rad/s} \\
\phi_m &= \pi + \angle G(j\omega_{pc}| = 28^0
\end{aligned}$$

If we compare the CT and DT versions of the same system, we can see that both gain margin and phase margin of the original CT system is better, and we can conclude that discretization reduces the "stability". Another interesting result is that while there is a significant change in phase-crossover frequency, the change in gain crosover frequency is minimal.

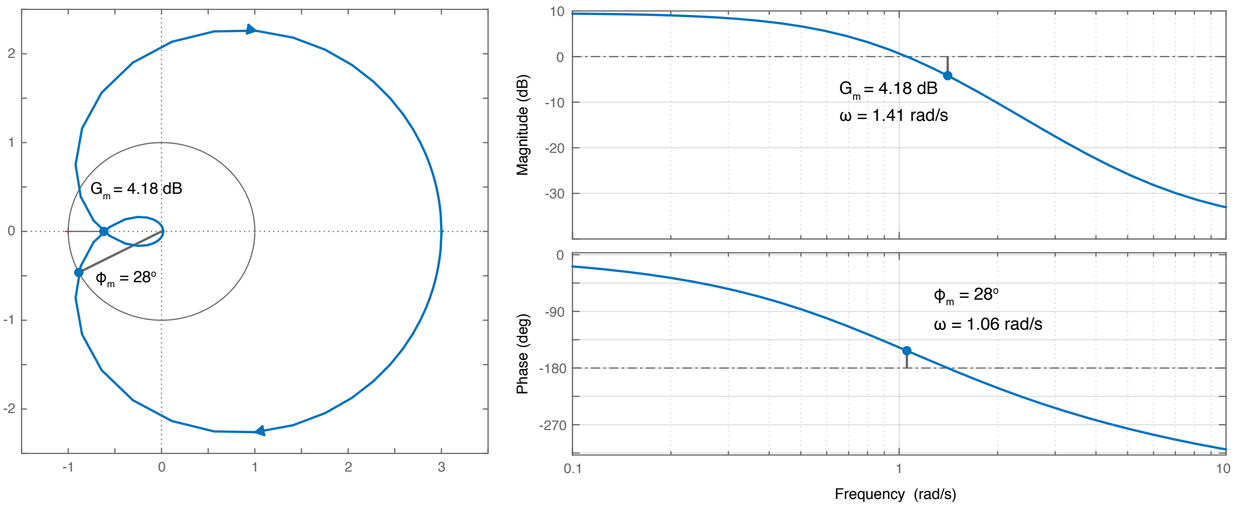
Now let's transfrom the $G(z)$ to a artificial CT system using Bilinear-Tustin transformation:

$$G(\bar{s}) = G(z)|_{z=\frac{1+(T/2)\bar{s}}{1-(T/2)\bar{s}}}$$

We know that the relation between the frequency of this artificial system, $\bar{\omega}$, and frequencies of the actual system and discretized system are given by

$$\bar{\omega} = \frac{2}{T}\tan(\frac{\omega_d}{2}) = \frac{2}{T}\tan(\frac{\omega T}{2})$$

$G(\bar{s})$



The phase crosover frequency and gain margin for this artificial CT open-loop transfer function is given below

$$
\begin{aligned}
\omega_{pc} &= 1.41 \text{rad/s} \\
g_m &= \frac{1}{|G(e^{j\omega_{pc}T}|} = 1.62 \\
G_m &= 4.18dB
\end{aligned}
$$

On the other hand gain crosover frequency and phase margin for $G(s)$ is computed as

$$
\begin{aligned}
\omega_{gc} &= 1.06 \text{ rad/s} \\
\phi_m &= \pi + \angle G(j\omega_{pc}| = 28^0
\end{aligned}
$$

If we compare the phase and gain margin and associated crosover frequencies we can see that stability margins of DT and Tustin-Transformed are almost same. It is an expected result, because the core purpose of Tustin transformation is transforming a DT system to CT from by preserving the stability and other importan characteristics.

Tustin transformation has two basic advantages

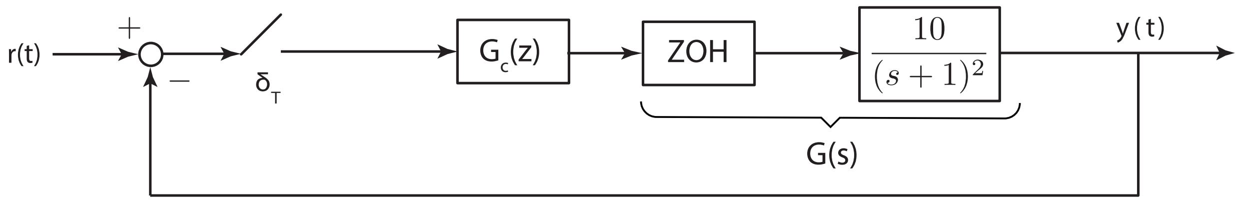
- It "preserves" the stability and robustness characteristics of the digital control system.
- Even though bode polts of DT systems are perfectly valid (and useful), due to periodicity in ω_d the some of the simplicities and advantages of CT bode plots are lost. Since Tustin transformed system is a CT representation, we still can use these simplicities and other advantages.

There two basic disadvantages of Tustin transformation

- It has significant computational costs. However in a computer environment these costs can be negligible.
- If one designs a controller in Tustin from, and then back-transformes the controller in DT form. For some class of controller the quantization effects can become important and deadly.

Lead-Compensator Design for DT Control Systems

Let's consider the DT control system below. Let's assume that $T = 0.1s$



The goal is designing a DT phase-lead compensator such that the Phase-Margin is of the controlled system is in the desired range of $\phi_m \in [50^o \, 60^o]$.

Due to nice proporties if CT bode plots, lead-compensator design procedure is handled in the Bilinear-Tustin transformed domain. Below we will summarize the process for the example plant.

1. Compute discretized plant transfer function $G(z)$

$$\begin{aligned} G(z) &= \mathcal{Z}\left[\frac{1-e^{-Ts}}{s} G_P(s)\right] \\ &= \mathcal{Z}\left[\frac{1-e^{-Ts}}{s} \frac{10}{(s+1)^2}\right] \\ &= \frac{0.047z + 0.044}{z^2 - 1.8z + 0.82} \end{aligned}$$

2. Compute the Bilinear-Tustin transformation of $G(z)$

$$\begin{aligned} G(\bar{s}) &= G(z)\big|_{z=\frac{1+(T/2)\bar{s}}{1-(T/2)\bar{s}}} \\ &= \frac{-0.0008s^2 - 0.48s + 9.98}{s^2 + 2s + 0.998} \end{aligned}$$

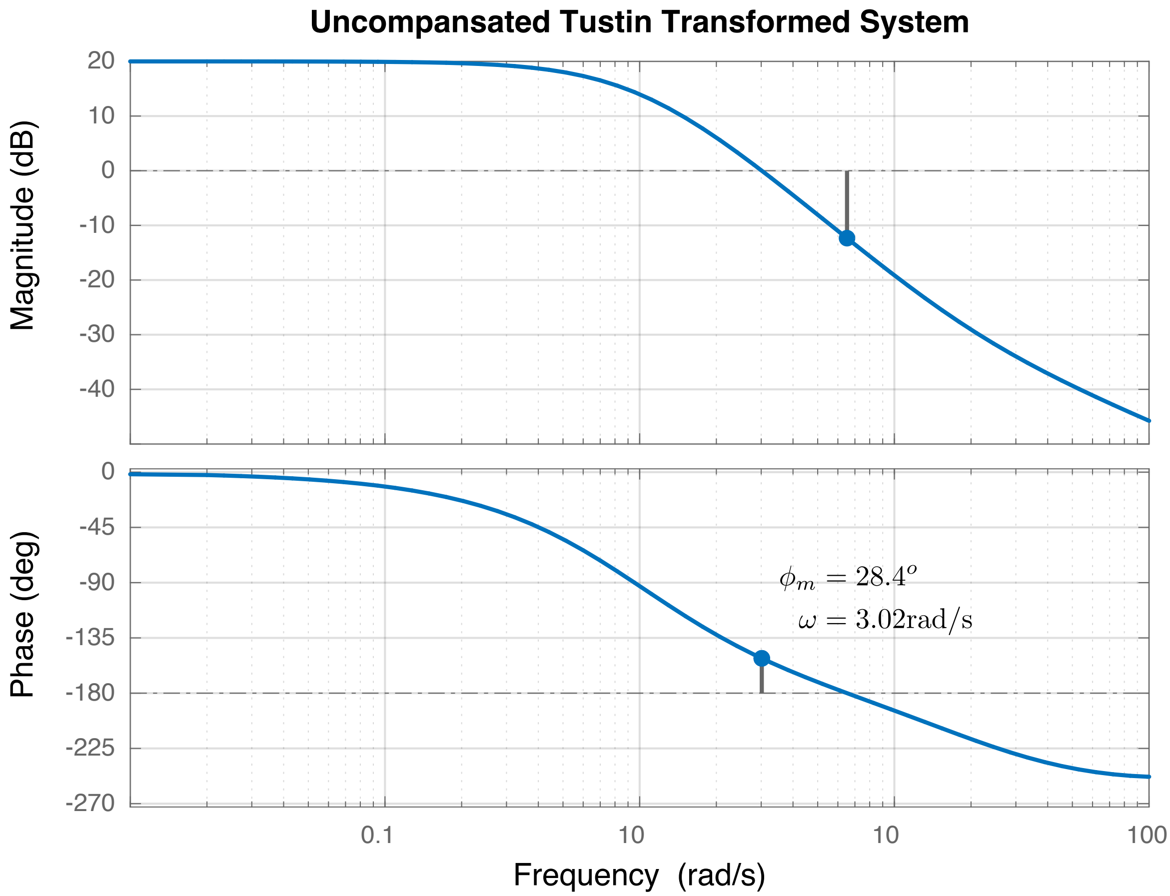
3. Now the goal is designing a lead compensator for $G(\bar{s})$. Lead-compensator has the form.

$$G_l(\bar{s}) = K_{\text{lead}} \frac{T_l \ a \ s + 1}{T/a \ s + 1}$$

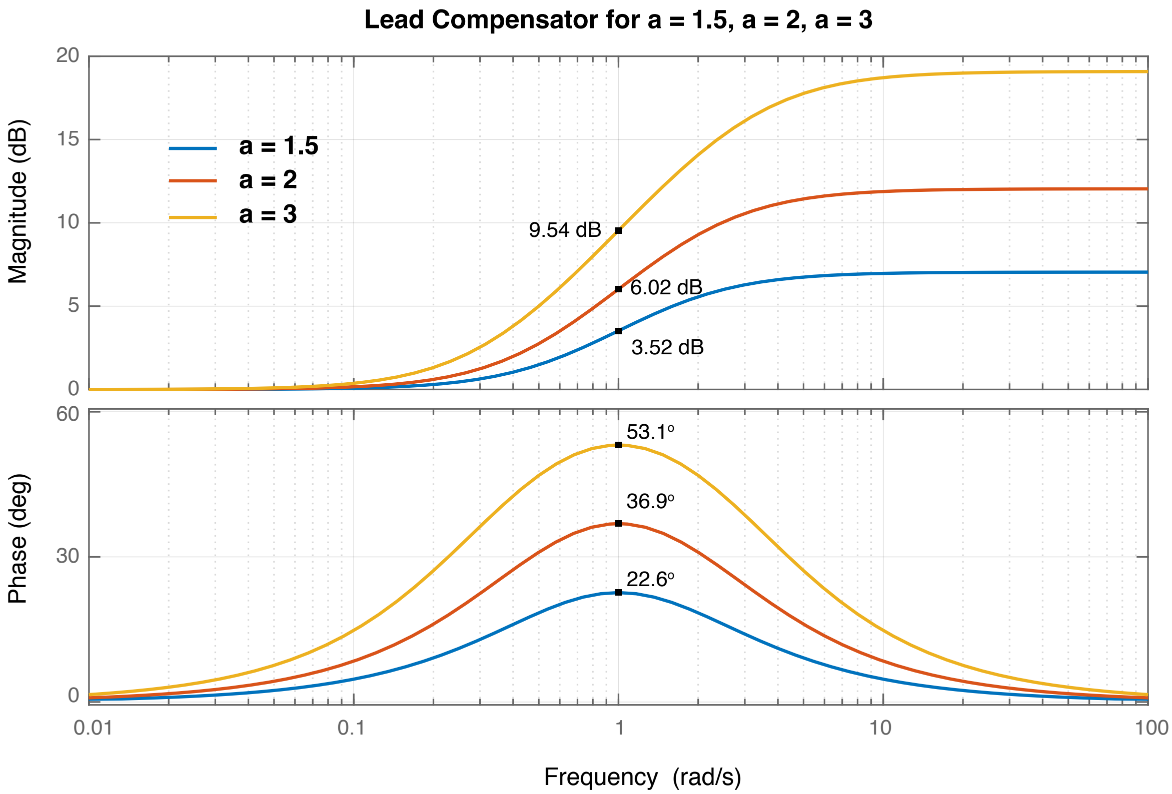
   where gain K_l (generally) computed based on steady state requirments. This can be computed either in z-domain or s-bar domain. Let's assume that we are OK with steady-state performance and $K_l = 1$.

4. Compute the ϕ_m of the un-compensated system and find the required $\Delta\phi_m$ such that the compensated system meets the specifications.

$$\begin{aligned} \phi_m &\approx 28^o \\ \Delta\phi_m &= [22^o \ , \ 32^o] \end{aligned}$$



5. For $a = 1.5$, $a = 2$, and $a = 3$, bode plots of a phase-lead compensator is illustrated with $T_l = 1$ in the figure below.



It can be seen that ϕ_{max} is larger for large a. ϕ_{max} (or a) can also be computed using the following relation.

$$\sin \phi_{max} = \frac{a^2 - 1}{a^2 + 1}$$

As your remember from EE302 Lead compensator should have a lead angle that is above $\approx 5 - 15^o$ the reqired $\Delta\phi_m$. Based on this we compute/find a. Based on the bode polts, it seems that $a = 2$ may supply the required additonal phase-margin. One should see that ϕ_{max} is not affected from the choice of T_l.

6. Now our goal is to copmpute T_l, where $1/T_l$ corresponds to the center freqeuncy of the copmensator. One possible choice is choosing T such that $1/T_l = \bar{\omega}_{gc}$, i.e. gain croesover frequency. However, at center frequency the lead copmensator shifts the bode magnitude by

$$G_{\text{center}} = 20\log_{10}(1/a)$$

which causes a shift in gain-croesover frequency. For example, for $a = 2$, $G_{\text{center}} \approx 6\ dB$. For this reason, a "better" choice is choosing T such that center freqeuncy of the lead-compensator coincides with the frequency where $|G(j\bar{\omega})|$ crosses $-20\ \log_{10}(1/a)$, i.e. we compute T_l such that

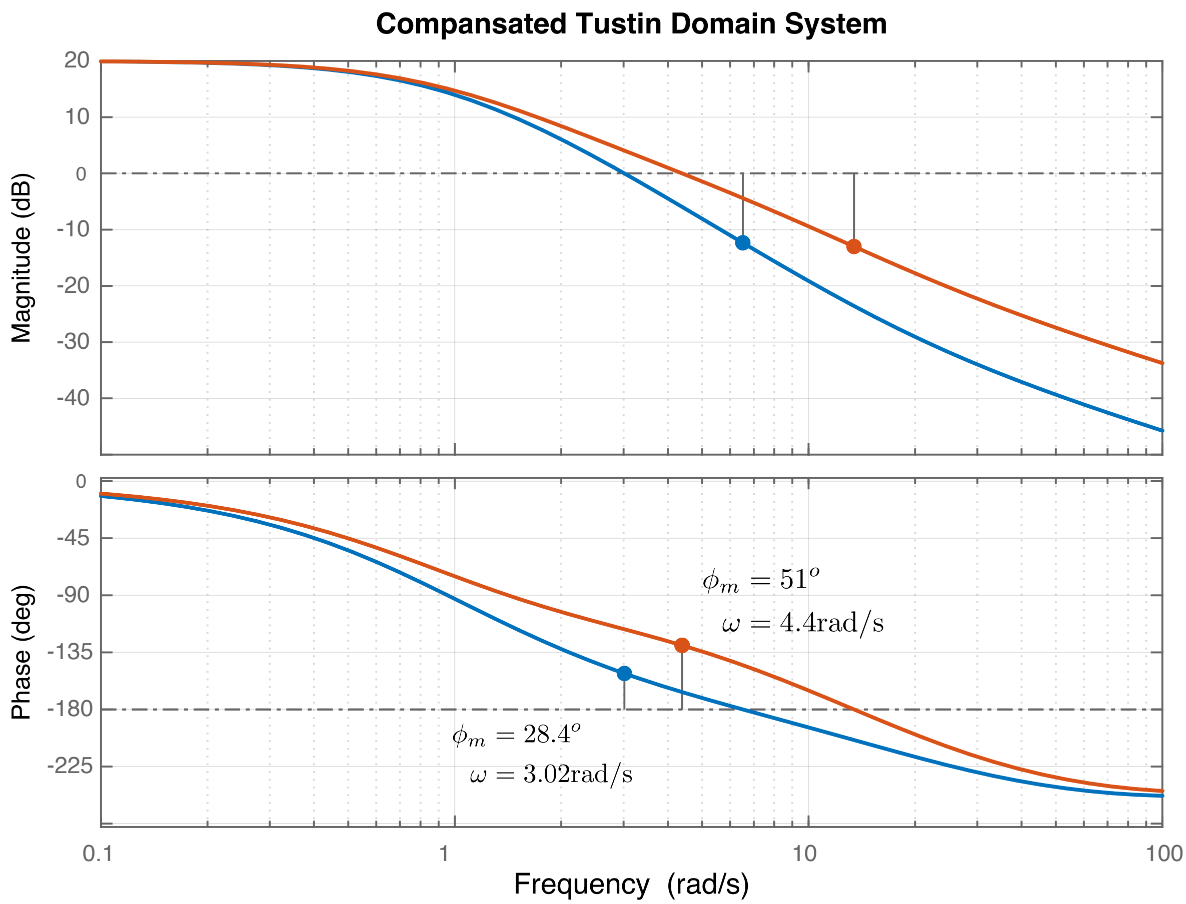
$$|G(j1/T_l)| = -20\ \log_{10}(1/a)$$

From the bode polts for $a = 2$, the freqeuncy for which $G(\bar{s})$ croesses the $-6dB$ line is approximately 4.45 rad/s. Thus we choose $T = 0.225s$. Check if the lead-compensator meets the phase-margin reqeuirment. Otherwise, repeat the process with a higher $\delta\phi$ angle.

In our example, the resultant tustin domain lead compensator has the form.

$$G_l(\bar{s}) = \frac{0.45s + 1}{0.1125s + 1}$$

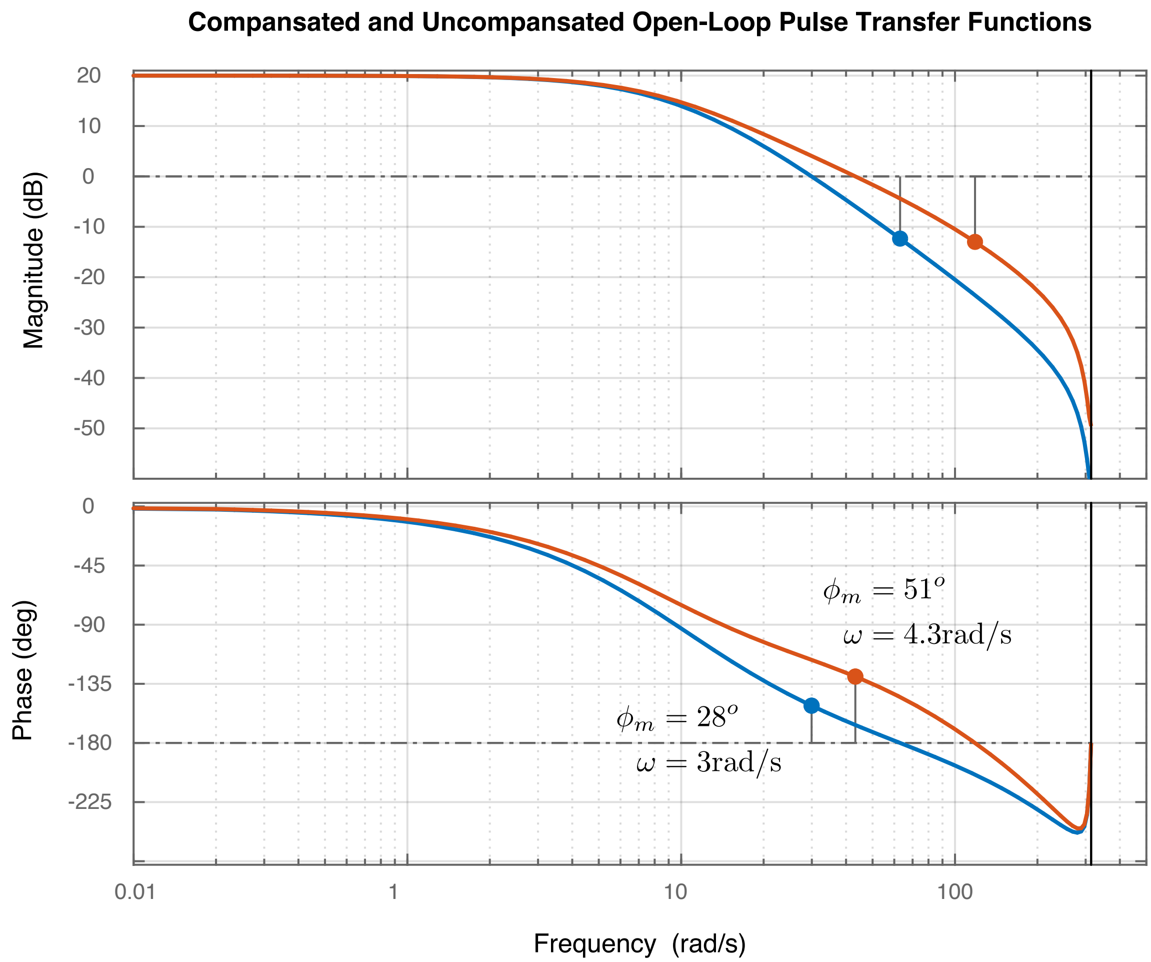
The Figure below illiustrates the bode plots of both (tustin domain) compensated and uncompansated systems. Compensated systems has a phase margin of $\phi_m = 51^o$ which meets the requirements.



7. Transform the $\bar{s}$-domain lead-compensator to z-domain.

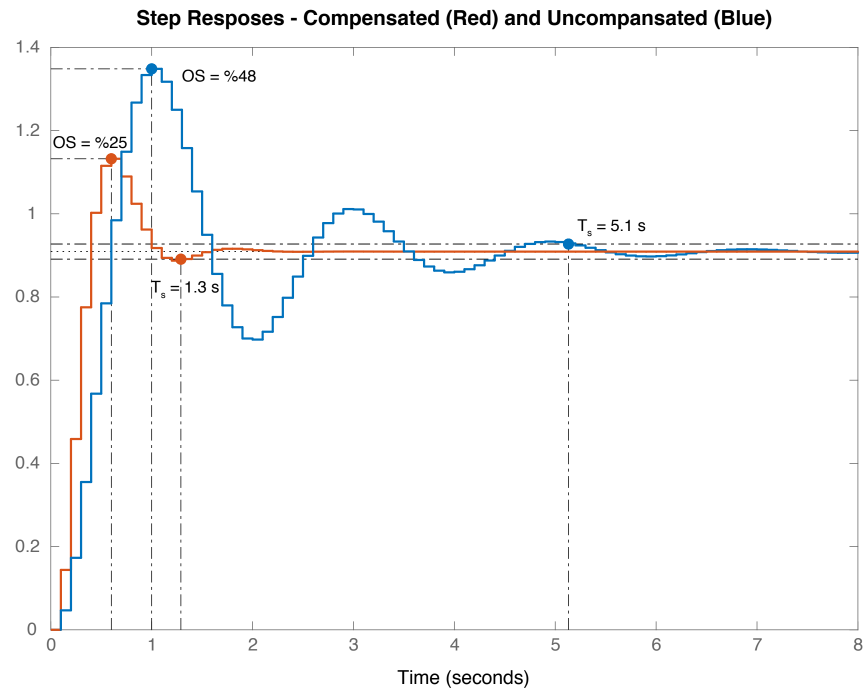
$$G_l(z) = G_l(\bar{s})\big|_{\bar{s}=\frac{2}{T}\frac{z-1}{z+1}}$$
$$G_l(z) = \frac{3.08z - 2.46}{z - 0.385}$$

8. Check if the discrete-time compensator meets the the phse-margin requirments.



It can be seen that the designed compensator in z-domain also meets the phase-margin specifications.

In Figure below, we compare the closed-loop step responses of both uncompensated and compensated pulse transfer functions.


Step Resposes - Compensated (Red) and Uncompansated (Blue)
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
1
2
3
4
5
6
7
8
Time (seconds)
OS = %48
OS = %25
T_s = 5.1 s
T_s = 1.3 s

EE402 - Discrete Time Systems **Spring 2018**

Lecture 12

Lecturer: Asst. Prof. M. Mert Ankarali

State-Space Representation of DT Systems

State-space representation of a (causal & finite dimensional) LTI CT system is given by

$$
\begin{aligned}
\text{Let } & x(t) \in \mathbb{R}^n \ , \ y(t) \in \mathbb{R}^m \ , \ u(t) \in \mathbb{R}^r, \\
\dot{x}(t) &= Ax(t) + Bu(t), \\
y(t) &= Cx(t) + Du(t), \\
\text{where } & A \in \mathbb{R}^{n \times n} \ , \ B \in \mathbb{R}^{n \times r} \ , \ C \in \mathbb{R}^{m \times n} \ , \ D \in \mathbb{R}^{m \times r}
\end{aligned}
$$

State-space representation of a (causal & finite dimensional) LTI DT system is given by

$$
\begin{aligned}
\text{Let } & x[k] \in \mathbb{R}^n \ , \ y[k] \in \mathbb{R}^m \ , \ u[k] \in \mathbb{R}^r, \\
x[k+1] &= Gx[k] + Hu[k], \\
y[k] &= Cx[k] + Du[k], \\
\text{where } & G \in \mathbb{R}^{n \times n} \ , \ H \in \mathbb{R}^{n \times r} \ , \ C \in \mathbb{R}^{m \times n} \ , \ D \in \mathbb{R}^{m \times r}
\end{aligned}
$$

Depending on the values of m and r we have

- $m = r = 1$, the system represents a SISO system
- $m > 1 \ , \ r < 1$, the system represents a MIMO system
- $m = 1 \ , \ r > 1$, the system represents a MISO system
- $m > 1 \ , \ r = 1$, the system represents a SIMO system

for both CT and DT cases.

State property of CT state-space models: Given the initial time, t_0 and state $x(t_0)$ and input $u(t)$ for $t_0 \geq t < t_f$ (with t_0 & t_f arbitrary), we can compute the output $y(t)$ for $t_0 \geq t < t_f$ and the state $x(t)$ for $t_0 < t \geq t_f$.

State property of DT state-space models: Given the state vector $x[k]$ and input $u[k]$ at an arbitrary time k, we can compute the the present output, $y[k]$, and next state $x[k+1]$.

Note that both definitions are not limited to LTI state-space models. Nonlinear and time-varying state-space models also are based on this definition.

When a state-space representation includes minimum number of state variables, the representation is called minimal.

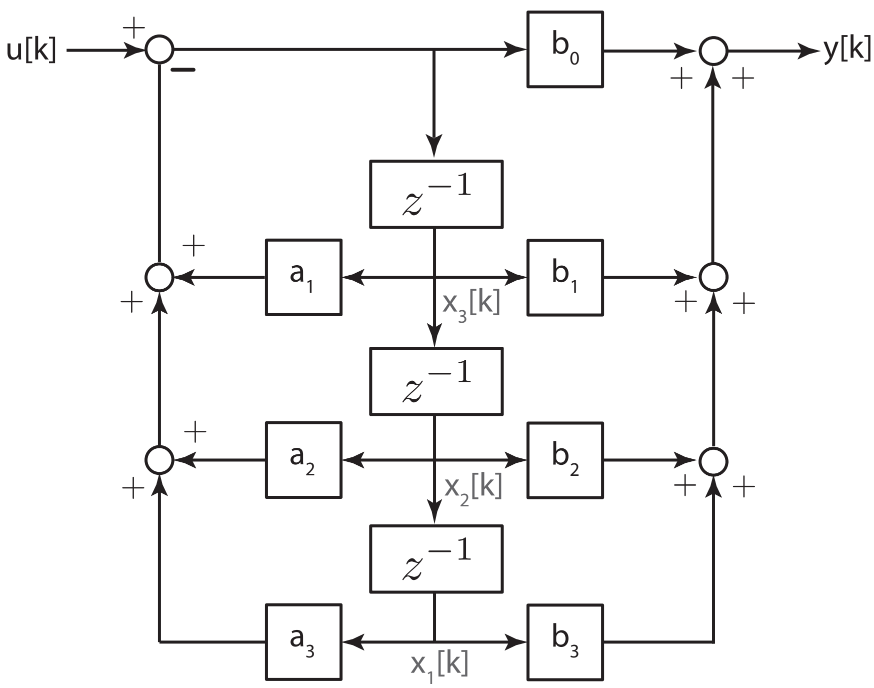
Canonical State-Space Realizations of (SISO) DT Systems

Reachable/Controllable Canonical Form

For the sake of clarity let's assume that the system that we would like to represent is a third order DT system with the following difference equation and transfer function

$$y[k] = -a_1 y[k-1] - a_2 y[k-2] - a_3 y[k-3] + b_0 x[k] + b_1 x[k-1] + b_2 x[k-2] + b_3 x[k-3]$$
$$Y(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3}}{1 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3}} X(z)$$

We know that following block diagram realizes this system structure with minimum number of delay elements and it is a canonical realization. Delay operation is directly related with state and state evolution concept.



If we label the signals as given in the Figure, state evolution equations can be derived as

$$\begin{aligned} X_1(z) &= X_2(z)z^{-1} &&\rightarrow \quad x_1[k+1] = x_2[k] \\ X_2(z) &= X_3(z)z^{-1} &&\rightarrow \quad x_2[k+1] = x_3[k] \\ X_3(z) &= \left(U(z) - (X_1(z)a_3 + X_2(z)a_2 + X_3(z)a_1)\right) z^{-1} &&\rightarrow \quad x_3[k+1] = u[k] + x_1[k](-a_3) + x_2[k](-a_2) + x_3[k](-a_1) \end{aligned}$$

where as output equation can be derived as

$$\begin{aligned} y[k] &= b_1 x_3[k] + b_2 x_2[k] + b_3 x_1[k] b_0 u[k] - b_0 \left(a_1 x_3[k] + a_2 x_2[k] + a_3 x_1[k]\right) \\ &= b_0 u[k] + (b_3 - b_0 a_3) x_1[k] + +(b_2 - b_0 a_2) x_2[k] + +(b_1 - b_0 a_1) x_3[k] \end{aligned}$$

If we gather these equations, we can obtain the state space form

$$\mathbf{x}[k+1] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix} \mathbf{x}[k] + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u[k]$$
$$y[k] = \begin{bmatrix} (b_3 - b_0 a_3) & (b_2 - b_0 a_2) & (b_1 - b_0 a_1) \end{bmatrix} + b_0 u[k]$$

where

$$\mathbf{x} = \begin{bmatrix} x_1[k] \\ x_2[k] \\ x_1[k] \end{bmatrix} \quad , \quad A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix} \quad , \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$
$$C = \begin{bmatrix} (b_3 - b_0 a_3) & (b_2 - b_0 a_2) & (b_1 - b_0 a_1) \end{bmatrix} \quad , \quad D = b_0$$

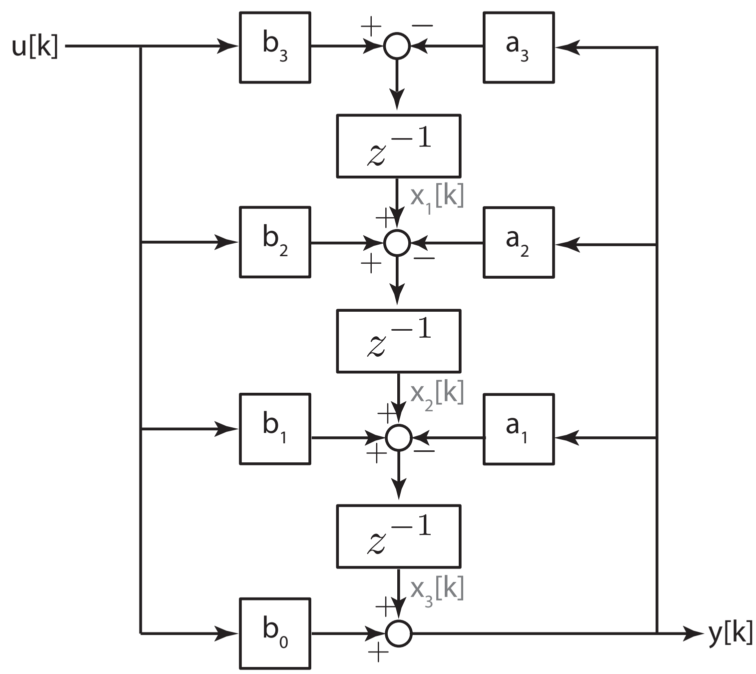
The form obtained with this approach is called reachable/controllable canonical form.

For a general n^{th} order system reachable/controllable canonical form has the following A ,B ,C ,& D matrices

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \quad , \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$
$$C = \begin{bmatrix} (b_n - b_0 a_n) & (b_{n-1} - b_0 a_{n-1}) & \cdots & (b_2 - b_0 a_2) & (b_1 - b_0 a_1) \end{bmatrix} \quad , \quad D = b_0$$

Observable Canonical Form

We also learnt a different type of canonical minimal realization which is illustrated in the Figure below



If we label the signals as given in the Figure, state evolution equations can be derived as

$$\begin{aligned} X_1(z) &= (b_3 U(z) - a_3 (U(z) b_0 + X_3(z))) z^{-1} &\rightarrow\quad x_1[k+1] &= b_3 u[k] - a_3 (u[k] b_0 + x_3[k]) \\ X_2(z) &= (b_2 U(z) + X_1(z) - a_2 (U(z) b_0 + X_3(z))) z^{-1} &\rightarrow\quad x_2[k+1] &= b_2 u[k] + x_1[k] - a_2 (u[k] b_0 + x_3[k]) \\ X_3(z) &= (b_1 U(z) + X_2(z) - a_1 (U(z) b_0 + X_3(z))) z^{-1} &\rightarrow\quad x_3[k+1] &= b_1 u[k] + x_2[k] - a_1 (u[k] b_0 + x_3[k]) \end{aligned}$$

where as output equation can be derived as

$$y[k] = b_0 u[k] + x_3[k]$$

If we combine the state and output equations, we can obtain the state space form as

$$\mathbf{x}[k+1] = \begin{bmatrix} 0 & 0 & -a_3 \\ 1 & 0 & -a_2 \\ 0 & 1 & -a_1 \end{bmatrix} \mathbf{x}[k] + \begin{bmatrix} b_3 - b_0 a_3 \\ b_2 - b_0 a_2 \\ b_1 - b_0 a_1 \end{bmatrix} u[k]$$
$$y[k] = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} + b_0 u[k]$$

where

$$\mathbf{x} = \begin{bmatrix} x_1[k] \\ x_2[k] \\ x_1[k] \end{bmatrix} \quad , \quad A = \begin{bmatrix} 0 & 0 & -a_3 \\ 1 & 0 & -a_2 \\ 0 & 1 & -a_1 \end{bmatrix} \quad , \quad B = \begin{bmatrix} b_3 - b_0 a_3 \\ b_2 - b_0 a_2 \\ b_1 - b_0 a_1 \end{bmatrix} \quad , \quad C = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \quad , \quad D = b_0$$

The form obtained with this approach is called observable canonical form.

For a general n^{th} order system observable canonical form has the following A , B , C , & D matrices

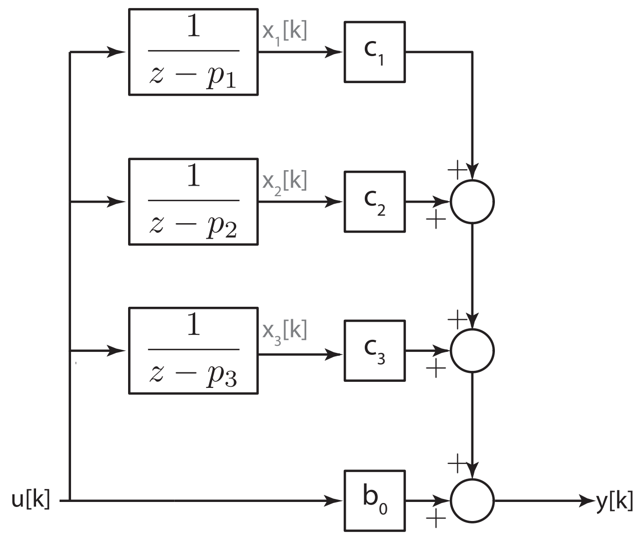
$$A = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_n \\ 1 & 0 & \cdots & 0 & -a_{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -a_2 \\ 0 & 0 & \cdots & 1 & -a_1 \end{bmatrix} \quad , \quad B = \begin{bmatrix} (b_n - b_0 a_n) \\ (b_{n-1} - b_0 a_{n-1}) \\ \vdots \\ (b_2 - b_0 a_2) \\ (b_1 - b_0 a_1) \end{bmatrix}$$
$$C = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \quad , \quad D = b_0$$

Diagonal Canonical Form

If the pulse transfer function of the system has distinct poles, we can expand it using partial fraction expansion

$$Y(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3}}{1 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3}} X(z)$$
$$= b_0 + \frac{c_1}{z - p_1} + \frac{c_2}{z - p_2} \frac{c_3}{z - p_3}$$

It is possible to realize this expanded form in block diagram form as given in the Figure below.



Now let's concentrate on the candidate "state variables" and try to write state evaluation equations

$$
\begin{aligned}
X_1(z) &= \frac{1}{z-p_1}U(z) \quad \rightarrow \quad x_1[k+1] = p_1x_1[k] + u[k] \\
X_2(z) &= \frac{1}{z-p_2}U(z) \quad \rightarrow \quad x_2[k+1] = p_2x_2[k] + u[k] \\
X_3(z) &= \frac{1}{z-p_3}U(z) \quad \rightarrow \quad x_3[k+1] = p_3x_3[k] + u[k]
\end{aligned}
$$

where as output equation can be derived as

$$y[k] = b_0u[k] + c_1x_1[k] + c_2x_2[k] + c_3x_3[k]$$

If we combine the state and output equations, we can obtain the state space form as

$$
\begin{aligned}
\mathbf{x}[k+1] &= \begin{bmatrix} p_1 & 0 & 0 \\ 0 & p_2 & 0 \\ 0 & 0 & p_3 \end{bmatrix}\mathbf{x}[k] + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}u[k] \\
y[k] &= \begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix} + b_0u[k]
\end{aligned}
$$

where

$$\mathbf{x} = \begin{bmatrix} x_1[k] \\ x_2[k] \\ x_1[k] \end{bmatrix} \quad , \quad A = \begin{bmatrix} p_1 & 0 & 0 \\ 0 & p_2 & 0 \\ 0 & 0 & p_3 \end{bmatrix} \quad , \quad B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad , \quad C = \begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix} \quad , \quad D = b_0$$

The form obtained with this approach is called diagonal canonical form. Obviously, this form is not aplicable for systems that has repeated roots.

For a general n^{th} order system with distinct roots diagonal canonical form has the following A , B , C , & D

matrices

$$A = \begin{bmatrix} p_1 & 0 & \cdots & 0 & 0 \\ 0 & p_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & p_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & p_n \end{bmatrix} \quad , \quad B = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{bmatrix}$$
$$C = \begin{bmatrix} c_1 & c_2 & \cdots & c_{n-1} & c_n \end{bmatrix} \quad , \quad D = b_0$$

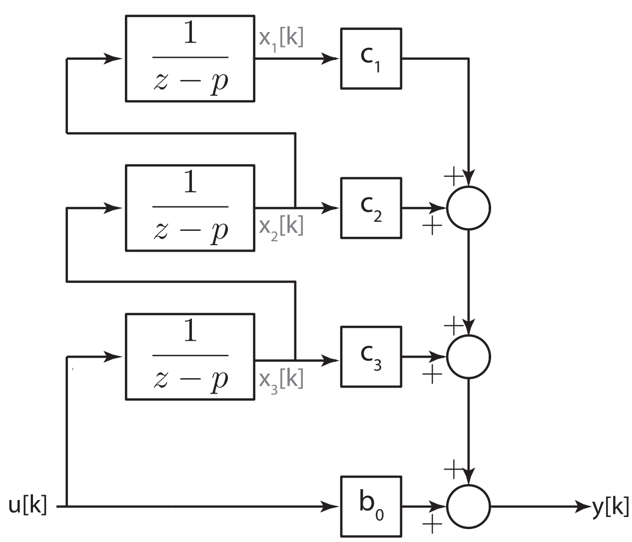
Jordan Canonical Form

Generalization of diagonal canonical farm is called Jordan canonical form which handles repeated roots.

In Jordan form the distinct roots has the same structure with Diagonal canonical form. Let's assume that the 3^{rd} order pulse transfer function has three repeated roots. In this case, we can expand it using partial fraction expansion

$$\begin{aligned} Y(z) &= \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3}}{1 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3}} X(z) \\ &= b_0 + \frac{c_1}{(z-p)^3} + \frac{c_2}{(z-p)^2} \frac{c_3}{z-p} \end{aligned}$$

It is possible to realize this expanded form in block diagram form as given in the Figure below.



Now let's concentrate on the candidate "state variables" and try to write state evaluation equations

$$
\begin{aligned}
X_1(z) &= \frac{1}{z-p}X_2(z) \quad \rightarrow \quad x_1[k+1] = p\ x_1[k] + x_2[k] \\
X_2(z) &= \frac{1}{z-p}X_3(z) \quad \rightarrow \quad x_2[k+1] = p\ x_2[k]x_3[k] \\
X_3(z) &= \frac{1}{z-p}U(z) \quad \rightarrow \quad x_3[k+1] = p\ x_3[k] + u[k]
\end{aligned}
$$

where as output equation can be derived as

$$y[k] = b_0u[k] + c_1x_1[k] + c_2x_2[k] + c_3x_3[k]$$

If we combine the state and output equations, we can obtain the state space form as

$$
\begin{aligned}
\mathbf{x}[k+1] &= \begin{bmatrix} p & 1 & 0 \\ 0 & p & 1 \\ 0 & 0 & p \end{bmatrix}\mathbf{x}[k] + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}u[k] \\
y[k] &= \begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix} + b_0u[k]
\end{aligned}
$$

where

$$\mathbf{x} = \begin{bmatrix} x_1[k] \\ x_2[k] \\ x_1[k] \end{bmatrix} \quad , \quad A = \begin{bmatrix} p & 1 & 0 \\ 0 & p & 1 \\ 0 & 0 & p \end{bmatrix} \quad , \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad , \quad C = \begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix} \quad , \quad D = b_0$$

$A,\ B,\ \&\ C$ forms a Jordan block.

For a general n^{th} order system a Jordan block with m repeated roots inside a stat-space representation in Jordan canonical form looks like

$$
A = \left[\begin{array}{c|ccccc|c}
\ddots & & & & & & \\
\hline
 & \bar{p} & 1 & \cdots & 0 & 0 & \\
 & 0 & \bar{p} & \cdots & 0 & 0 & \\
 & & & \ddots & & & \\
 & 0 & 0 & \cdots & \bar{p} & 1 & \\
 & 0 & 0 & \cdots & 0 & \bar{p} & \\
\hline
 & & & & & & \ddots
\end{array}\right] \quad , \quad B = \left[\begin{array}{c} \vdots \\ \hline 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ \hline \vdots \end{array}\right]
$$

$$C = \left[\begin{array}{c|ccccc|c} \cdots & c_1 & c_2 & \cdots & c_{n-1} & c_n & \cdots \end{array}\right]$$

Similarity Transformations

Consider the state-space representation of a given DT system

$$
\begin{aligned}
x[k+1] &= Gx[k] + Hu[k] \\
y[k] &= Cx[k] + Du[k]
\end{aligned}
$$

Let's define a new "state-vector" $\hat{x}$ such that

$$
\begin{aligned}
x[k] &= P\hat{x}[k] \quad \text{where} \\
P \in \mathbb{R}^{n\times n} & \quad \det(P) \neq 0
\end{aligned}
$$

Then we can transform the state-space equations using P as

$$P^{-1}\hat{x}[k+1] = GP^{-1}\hat{x}[k] + Hu[k] \quad , \quad y[k] = CP^{-1}x[k] + Du[k]$$
$$\hat{x}[k+1] = PGP^{-1}\hat{x}[k] + PHu[k] \quad , \quad y[k] = CP^{-1}x[k] + Du[k]$$

The “new” state-space representation is obtained as

$$\begin{aligned}\hat{x}[k+1] &= \hat{G}\hat{x}[k] + \hat{H}u[k] \\ y[k] &= \hat{C}x[k] + \hat{D}u[k] \\ \hat{G} &= PGP^{-1} \ , \ \hat{H} = PH \ , \ \hat{C} = CP^{-1} \ , \ \hat{D} = D\end{aligned}$$

Since there exist infinitely many non-singular $n \times n$ matrices, for a given LTI DT systems, there exist infinitely many different but equivalent state-space representations.

Example: Show that $A \in \mathbb{R}^{n\times n}$ and $P^{-1}AP$, where $P\mathbb{R}^{n\times n}$ and $\det(P) \neq= 0$, have the same characteristic equation

Solution:

$$\begin{aligned}\det\left(\lambda I - P^{-1}AP\right) &= \det\left(\lambda P^{-1}IP - P^{-1}AP\right) \\ &= \det\left(P^{-1}\left(\lambda I - A\right)P\right) \\ &= \det\left(P^{-1}\right)\det\left(\lambda I - A\right)\det\left(P\right) \\ &= \det\left(P^{-1}\right)\det\left(P\right)\det\left(\lambda I - A\right) \\ \det\left(\lambda I - P^{-1}AP\right) &= \det\left(\lambda I - A\right)\end{aligned}$$

Obtaining Transfer Functions from a State-Space Representation

Let’s consider the following general state-space form

$$\begin{aligned}x[k+1] &= Gx[k] + Hu[k] \\ y[k] &= Cx[k] + Du[k]\end{aligned}$$

In order to obtain transfer function form, we assume that initial conditions are zero. Under this assumption, let’s take z-transform of both equations

$$\begin{aligned}zX(z) &= GX(z) + HU(z) \quad , \quad Y(z) = CX(z) + DU(z) \\ \left(zI - G\right)X(z) &= HU(z) \\ X(z) &= \left(zI - G\right)^{-1}HU(z) \\ Y(z) &= C\left(zI - G\right)^{-1}HU(z) + DU(z) \\ Y(z) &= \left[C\left(zI - G\right)^{-1}H + D\right]U(z) \\ T(z) &= \left[C\left(zI - G\right)^{-1}H + D\right]\end{aligned}$$

If the system is a SISO system, then $T(z)$ is a transfer function, where as for MIMO case $T(z)$ becomes a *transfer function matrix*. Note that $\left(zI - G\right)^{-1}$ is invertible for all $z \in \mathbb{C}$ except the eigenvalues of G.

Example: Let p be a pole of $T(z)$, show that p is also an eigenvalue of G.

Solution: Let

$$T(z) = \frac{n(z)}{d(z)}$$

If p is a pole of $T(z)$, then $d(z)|_p = 0$. Now let's analyze the dependence of $T(z)$ to the state-space form.

$$\begin{aligned}
T(z) &= \left[C\left(zI - G\right)^{-1} H + D\right] \\
\left(zI - G\right)^{-1} &= \frac{\mathrm{Adj}\left(zI - G\right)}{\det\left(zI - G\right)} \\
T(z) &= \frac{C\mathrm{Adj}\left(zI - G\right) H + D\det\left(zI - G\right)}{\det\left(zI - G\right)}
\end{aligned}$$

If p is a pole of $T(z)$, then

$$\det\left(zI - G\right)|_{z=p} = 0$$

Obviously p is an eigenvalue of G.

Invariance of Transfer Functions Under Similarity Transformation

Consider the two different state-space representations

$$\begin{aligned}
x[k+1] &= Gx[k] + Hu[k] \qquad & \hat{x}[k+1] &= \hat{G}\hat{x}[k] + \hat{H}u[k] \\
y[k] &= Cx[k] + Du[k] \qquad & y[k] &= \hat{C}x[k] + \hat{D}u[k]
\end{aligned}$$

where they are related with the following similarity transformation

$$x[k] = P\hat{x}[k]\ ,\ \hat{G} = PGP^{-1}\ ,\ \hat{H} = PH\ ,\ \hat{C} = CP^{-1}\ ,\ \hat{D} = D$$

Let's compute the transfer function for the second representation

$$\begin{aligned}
\hat{T}(z) &= \left[\hat{C}\left(zI - \hat{G}\right)^{-1} \hat{H} + \hat{D}\right] \\
&= \left[CP^{-1}\left(zI - PGP^{-1}\right)^{-1} PH + D\right] \\
&= \left[CP^{-1}\left(P\left(zI - G\right)P^{-1}\right)^{-1} PH + D\right] \\
&= \left[CP^{-1}P\left(zI - G\right)^{-1} P^{-1}PH + D\right] \\
&= \left[C\left(zI - G\right)^{-1} H + D\right] \\
\hat{T}(z) &= T(z)
\end{aligned}$$

Solution of Discrete-Time State-Space Equations

Let's first assume that $u[k] = 0$, and find un-driven (homogeneous) response.

$$\begin{aligned} x[k+1] &= Gx[k] \\ y[k] &= Cx[k] \end{aligned}$$

Unlike CT systems we can compute the response iteratively

$$\begin{aligned} x[1] &= Gx[0] \\ x[2] &= Gx[1] = G^2 x[0] \\ x[3] &= Gx[2] = G^3 x[0] \\ &\vdots \\ x[k] &= Gx[k-1] = G^k x[0] \quad , \quad y[k] = CG^k x[0] \end{aligned}$$

It is easy to see that for $k, p \in \mathbb{Z}$ where $k > p$

$$x[k] = G^{k-p} x[p]$$

Let $\Psi(k) = G^k$, then this matrix of functions solves the homogeneous difference equation

$$x[k] = Gx[k]$$

$$\begin{aligned} x[k] &= \Psi[k[x[0] \\ x[k] &= \Psi[k-p]x[p] \\ x[k+m] &= \Psi[k+m-m]x[m] = \Psi[k] \end{aligned}$$

Let $\Psi[k]$ is called the state-transition matrix. Now let's consider input-only state response (i.e. $x[0] = 0$).

$$x[k+1] = Gx[k] + Hu[k]$$

$$\begin{aligned}
x[1] &= Hu[0] \\
x[2] &= Gx[1] + Hu[1] = GHu[0] + Hu[1] \\
x[3] &= Gx[2] + Hu[2] = G^2Hu[0] + GHu[1] + Hu[2] \\
x[4] &= Gx[3] + Hu[3] = G^3Hu[0] + G^2Hu[1] + GHu[2] + Hu[3] \\
&\vdots \\
x[k] &= Gx[k-1] + Hu[k-1] \\
&= G^{k-1}Hu[0] + G^{k-2}Hu[1] + \cdots + GHu[k-2] + Hu[k-1] \\
&= \left[\begin{array}{c|c|c|c|c} G^{k-1}H & G^{k-2}H & \cdots & GH & H \end{array}\right] \begin{bmatrix} u[0] \\ u[1] \\ \vdots \\ u[k-2] \\ u[k-1] \end{bmatrix} \\
&= \sum_{j=0}^{k-1} G^{k-j-1}Hu[j] \\
&= \sum_{j=0}^{k-1} G^{j}Hu[k-j-1]
\end{aligned}$$

Given that $\Psi[k] = G^k$

$$\begin{aligned}
x[k] &= \sum_{j=0}^{k-1} \Psi[k-j-1]Hu[j] \\
&= \sum_{j=0}^{k-1} \Psi[j]Hu[k-j-1]
\end{aligned}$$

If we combine homegeneous and driven responses we can simply obtain

$$\begin{aligned}
x[k] &= \Psi[k]x[0] + \sum_{j=0}^{k-1} \Psi[k-j-1]Hu[j] \\
&= \Psi[k]x[0] + \sum_{j=0}^{k-1} \Psi[j]Hu[k-j-1]
\end{aligned}$$

whereas output at time k has the form

$$\begin{aligned}
y[k] &= C\Psi[k]x[0] + C\left(\sum_{j=0}^{k-1} \Psi[k-j-1]Hu[j]\right) Du[k] \\
&= C\Psi[k]x[0] + C\left(\sum_{j=0}^{k-1} \Psi[j]Hu[k-j-1]\right) Du[k]
\end{aligned}$$

Z-domain Solution of State-Space Equations

We already computed the transfer function under zero initial conditions.

$$Y(z) = \left[C\,(zI-G)^{-1}\,H + D\right]U(z)$$

Now let's compute the response to initial condition in Z-domain.

$$\begin{aligned}\mathcal{Z}\left[x[k+1]\right] &= \mathcal{Z}\left[Gx[k]\right]\\ zX(z) - zx[0] &= GX(z)\\ (zI-G)\,X(z) &= zX(z)\\ X(z) &= z\,(zI-G)^{-1}\,x[0]\end{aligned}$$

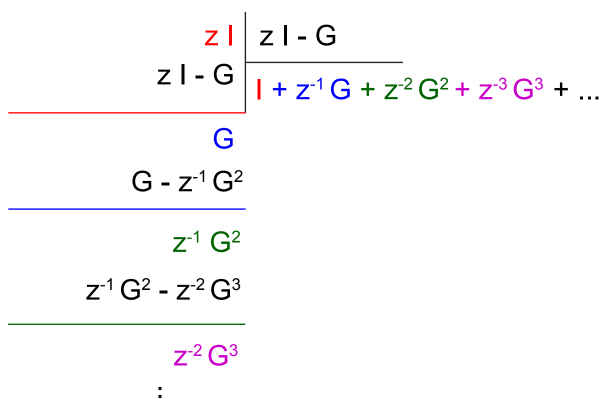
Similarly $Y(z)$ takes the form

$$Y(z) = zC\,(zI-G)^{-1}\,x[0]$$

We can also observe that

$$\begin{aligned}\mathcal{Z}\left[\Psi[k]\right] = \mathcal{Z}[G^k] &= z\,(zI-G)^{-1}\\ \mathcal{Z}^{-1}[z\,(zI-G)^{-1}] = \Psi[k] &= G^k\end{aligned}$$

If we expand $z\,(zI-G)^{-1}$ by long "division" we can also observe the relation in z-domain and time doman expressions froma different perspective



$$\begin{aligned}z\,(zI-G)^{-1} &= I + z^{-1}G + z^{-2}G^2 + z^{-3}G^3 + \cdots\\ \mathcal{Z}\left[z\,(zI-G)^{-1}\right] &= \{I\ ,\ G\ ,\ G^2\ ,\ G^3\ ,\ \cdots\}\end{aligned}$$

Example: Consider the following state-space representation

$$x[k+1] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & -1 \end{bmatrix} x[k] + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u[k]$$
$$y[k] = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} x[k]$$

- Compute the closed form expression $\Psi[k]$ using the time expression

  Solution: The state-space representation is in Diagonal canonical form

$$\Psi[k] = G^k = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & -1 \end{bmatrix}^k = \begin{bmatrix} 1^k & 0 & 0 \\ 0 & (1/2)^k & 0 \\ 0 & 0 & (-1)^k \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & (1/2)^k & 0 \\ 0 & 0 & (-1)^k \end{bmatrix}$$

- Compute the closed form expression $\Psi[k]$ using the z-domain solution method

  Solution:

$$\begin{aligned}
\Psi[k] &= \mathcal{Z}^{-1}\left[z\,(zI-G)^{-1}\right] \\
&= \mathcal{Z}^{-1}\left[z\left(\begin{bmatrix} z-1 & 0 & 0 \\ 0 & (z-1/2) & 0 \\ 0 & 0 & z+1 \end{bmatrix}\right)^{-1}\right] \\
&= \mathcal{Z}^{-1}\left[\begin{bmatrix} \frac{z}{z-1} & 0 & 0 \\ 0 & \frac{z}{z-1/2} & 0 \\ 0 & 0 & \frac{z}{z+1} \end{bmatrix}\right] \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & (1/2)^k & 0 \\ 0 & 0 & (-1)^k \end{bmatrix} \quad \text{for} k \geq 0
\end{aligned}$$

- Compute the impulse response of the system from the time domain solution

  Solution:

$$\begin{aligned}
x[k] &= G^{k-1}Hu[0] \quad \text{for } k > 0 \\
y[k] &= CG^{k-1}H \quad \text{for } k > 0 \\
&= \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & (1/2)^{k-1} & 0 \\ 0 & 0 & (-1)^{k-1} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ (1/2)^{k-1} \\ (-1)^{k-1} \end{bmatrix} \\
y[k] &= 1 + 2(1/2)^{k-1} + 3(-1)^{k-1} \quad \text{for } k > 0
\end{aligned}$$

- Compute the transfer function $\frac{Y(z)}{U(z)}$

Solution:

$$
\begin{aligned}
T(z) &= C\,(zI - G)^{-1}\,H \\
&= \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} z-1 & 0 & 0 \\ 0 & z-1/2 & 0 \\ 0 & 0 & z+1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{z-1} & 0 & 0 \\ 0 & \frac{1}{z-1/2} & 0 \\ 0 & 0 & \frac{1}{z+1} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{z-1} \\ \frac{1}{z-1/2} \\ \frac{1}{z+1} \end{bmatrix} \\
T(z) &= \frac{1}{z-1} + \frac{2}{z-1/2} + \frac{3}{z+1}
\end{aligned}
$$

- Compute the inverse Z-transform of the transfer function

  Solution:

$$
\begin{aligned}
t[k] &= \mathcal{Z}^{-1}\left[\frac{1}{z-1} + \frac{2}{z-1/2} + \frac{3}{z+1}\right] \\
&= \left(1 + 2(1/2)^{k-1} + 3(-1)^{k-1}\right) h[k-1]
\end{aligned}
$$

where $h[k]$ is the unit step function

EE402 - Discrete Time Systems **Spring 2018**

Lecture 13

Lecturer: Asst. Prof. M. Mert Ankarali

Matrix Exponential, e^{At}

Let's first review the matrix exponential, e^{At}. Let $t \in \mathbb{R}$ and $A \in \mathbb{R}^{nxn}$, then e^{At} defined as

$$e^{At} := I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots$$
$$e^{At} := \sum_{k=0}^{\infty} \frac{t^k}{k!}A^k$$

which converges for all $t \in \mathbb{R}$ and $A \in \mathbb{R}^{nxn}$.

Now let's review some properties

- **Claim:**

$$\frac{d}{dt}e^{At} = Ae^{At} = e^{At}A$$

  Proof:

$$\begin{aligned}
\frac{d}{dt}e^{At} &= \frac{d}{dt}\left(\sum_{k=0}^{\infty} \frac{t^k}{k!}A^k\right) = \sum_{k=0}^{\infty} k\frac{t^{k-1}}{k!}A^k = \sum_{k=1}^{\infty} \frac{t^{k-1}}{(k-1)!}A^k \\
&= \sum_{n=0}^{\infty} \frac{t^n}{n!}A^{n+1} = A\sum_{n=0}^{\infty} \frac{t^n}{n!}A^n = \left(\sum_{n=0}^{\infty} \frac{t^n}{n!}A^n\right)A \\
\frac{d}{dt}e^{At} &= Ae^{At} = e^{At}A
\end{aligned}$$

- **Claim:** Let $t_1, t_2 \in$ R then

$$e^{At_1}e^{At_2} = e^{At_2}e^{At_1} = e^{A(t_1+t_2)}$$

  Proof:

$$e^{At_1}e^{At_2} = \left(\sum_{k=0}^{\infty} \frac{t_1^k}{k!}A^k\right)\left(\sum_{j=0}^{\infty} \frac{t_2^j}{j!}A^j\right) = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} \frac{t_1^k}{k!}\frac{t_2^j}{j!}A^{k+j}$$

  Let $n = k + j$ and $j = n - k$, then

$$e^{At_1}e^{At_2} = \sum_{k=0}^{\infty}\sum_{n=k}^{\infty} \frac{t_1^k\ t_2^{n-k}}{k!(n-k)!}A^n = \sum_{k=0}^{\infty}\sum_{n=k}^{\infty} \frac{t_1^k\ t_2^{n-k}}{n!}\frac{n!}{k!(n-k)!}A^n \quad = \sum_{k=0}^{\infty}\sum_{n=k}^{\infty} \frac{t_1^k\ t_2^{n-k}}{n!}\begin{pmatrix} n \\ k \end{pmatrix}A^n$$

Since for $n < k$, $\begin{pmatrix} n \\ k \end{pmatrix} = 0$,

$$e^{At_1} e^{At_2} = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{t_1^k \; t_2^{n-k}}{n!} \begin{pmatrix} n \\ k \end{pmatrix} A^n = \sum_{n=0}^{\infty} \frac{A^n}{n!} \sum_{k=0}^{\infty} t_1^k \; t_2^{n-k} \begin{pmatrix} n \\ k \end{pmatrix}$$

Using *binomial* theorem we find

$$\begin{aligned} e^{At_1} e^{At_2} &= \sum_{n=0}^{\infty} \frac{A^n}{n!} (t_1 + t_2)^n \\ e^{At_1} e^{At_2} &= e^{A(t_1+t_2)} \end{aligned}$$

Now let $t_1 = t$ and $t_2 = -t$, then we have

$$e^{At} e^{-At} = e^{A(t-t)} = I \quad \rightarrow \quad \left(e^{At}\right)^{-1} = e^{-At}$$

- **Claim:** Let $A, B \in \mathrm{R}^{n\times n}$ and $AB = BA$, then

  $$e^{At} e^{Bt} = e^{Bt} e^{At} \qquad\qquad = e^{(A+B)t)}$$

  Proof: Mini Project 6

  Note that if $AB \neq BA$ then

  $$e^{At} e^{Bt} \neq e^{(A+B)t)}$$

- **Claim:** Let $P \in \mathrm{R}^{n\times n}$ and $\det(P) \neq 0$, then

  $$e^{\left(P^{-1}AP\right)t} = P^{-1} e^{At} P$$

  Proof: Mini Project 6

Solution of CT State-Space Equations

CT state-space representation has the form

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ \dot{y}(t) &= Cx(t) + Du(t) \end{aligned}$$

where Let $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}^p$, $u(t) \in \mathbb{R}^q$

First consider the homogeneous solution, i.e. $u(t) = 0$ and $x(0) = x_0$.

$$\begin{aligned} \dot{x}(t) &= Ax(t) \quad , \quad x(0) = x_0 \\ \dot{y}(t) &= Cx(t) \end{aligned}$$

Let's test if $x(t) = e^{At}x_0$ is a solution of the homogeneous equation

$$\begin{aligned} x(0) &= e^{A0}x_0 = x_0 \\ \dot{x}(t) - Ax(t) &= \left(Ae^{At}\right)x_0 - Ae^{At}x_0 = 0 \end{aligned}$$

Now let's compute the forced response. First let's analyze the following derivative

$$\frac{d}{dt}\left[e^{-At}x(t)\right] = e^{-At}\dot{x}(t) - e^{-At}Ax(t) = e^{-At}\left[\dot{x}(t) - Ax(t)\right]$$

Now using this relation let's solve the state-space equations

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ \dot{x}(t) - Ax(t) &= Bu(t) \\ e^{-At}\left[\dot{x}(t) - Ax(t)\right] &= e^{-At}Bu(t) \\ \frac{d}{dt}\left[e^{-At}x(t)\right] &= e^{-At}Bu(t) \\ e^{-At}x(t) &= x(0) + \int_0^t e^{-A\tau}Bu(\tau)d\tau \\ x(t) &= e^{-At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau \end{aligned}$$

Thus the solution of a system in state-space form can be written as

$$\begin{aligned} x(t) &= e^{-At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau \\ y(t) &= Ce^{-At}x_0 + \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t) \end{aligned}$$

The function $\Psi(t) = e^{At}$ is called the state-transition matrix of the system.

Example: Let's assume that system is a SISO system and $u(t) = \delta(t)$ (unit-impulse function) and $x_0 = 0$, compute the impulse response of the system, i.e. $y(t) = h(t)$,

$$\begin{aligned} h(t) &= \int_0^t Ce^{A(t-\tau)}B\delta(\tau)d\tau + D\delta(t) \\ &= Ce^{At}B + D\delta(t) \end{aligned}$$

S-Domain Solution of CT State-Space Equations

First take the Laplace transform of state evaluation equation

$$\begin{aligned}
\mathcal{L}\left[\dot{x}(t)\right] &= \mathcal{L}\left[Ax(t) + Bu(t)\right] \\
sX(s) - x(0) &= AX(s) + BU(s) \\
[sI - A]X(s) &= x(0) + BU(s) \\
X(s) &= [sI - A]^{-1}x_0 + [sI - A]^{-1}BU(s) \\
Y(s) &= C[sI - A]^{-1}x_0 + \left[C[sI - A]^{-1}B + D\right]U(s)
\end{aligned}$$

If we relate time and s-domain solutions we obtain

$$\begin{aligned}
e^{At} &= \mathcal{L}^{-1}\left[[sI - A]^{-1}\right] \\
h(t) &= \mathcal{L}^{-1}\left[C[sI - A]^{-1}B + D\right]
\end{aligned}$$

Discretization of CT State-Space Equations

Consider the CT system with the given state-space representation

$$\begin{aligned}
\dot{x}(t) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t) + Du(t)
\end{aligned}$$

and suppose that the input is piece-wise constant over intervals of length T. That is

$$u(t) = u[k] \quad , \quad t \in (kT \;\; (k+1)T)$$

i.e. input of the system is the output of a ZOH operator. Let's derive the the DT state-space equations with respect to the sampled-state $x[k] = x(kT)$.

Let's start with the state evolution equation

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

It is obvious that due to time-invariant the initial time of the equation above can be generalized as

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^t e^{A(t-\tau)}Bu(\tau)d\tau \quad , \quad t > t_0$$

Now let $t_0 = kT$ and $t = (k+1)T$,

$$x((k+1)T) = e^{AT}x(kT) + \int_{kT}^{(k+1)T} e^{A((k+1)T-\tau)}Bu(\tau)d\tau$$

Since the $u(t) = u[k]$ in this time interval

$$x[k+1] = e^{AT}x[k] + \int_{kT}^{(k+1)T} e^{A((k+1)T-\tau)}Bd\tau \; u[k]$$

Let $\lambda = (k+1)T - \tau$, then

$$\begin{aligned} x[k+1] &= e^{AT}x[k] - \int\limits_{T}^{0} e^{A\lambda}Bd\lambda\ u[k] \\ &= \left[e^{AT}\right]x[k] + \left[\left(\int\limits_{0}^{T} e^{A\lambda}d\lambda\right)B\right]u[k] \end{aligned}$$

Given that

$$x[k+1] = Gx[k] + Hu[k]$$

G and H matrices can be extracted as

$$\begin{aligned} G &= e^{AT} \\ H &= \left(\int\limits_{0}^{T} e^{A\lambda}d\lambda\right)B \end{aligned}$$

Claim: If A is invertable then we also have

$$H = A^{-1}\left(e^{AT} - I\right)B = \left(e^{AT} - I\right)A^{-1}B$$

Claim: Mini Project 6

Now let's consider the output equation

$$\begin{aligned} y(t) &= Cx(t) + Dx(t) \\ y(kT) &= Cx(kT) + Dx(kT) \\ y[k] &= Cx[k] + Dx[k] \end{aligned}$$

It can be seen that output equation matrices are not affected from the discretization.

Example: Consider the following CT-plant transfer function.

$$\frac{Y(s)}{U(s)} = \frac{1}{s} + \frac{1}{s + \ln(2)}$$

Find a CT state-space representation for this system **Solution:**

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} 0 & 0 \\ 0 & -\ln(2) \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 1 & 1 \end{bmatrix} x(t) \end{aligned}$$

Compute the state-transition matrix

Solution:

$$\begin{aligned} e^{At} &= \begin{bmatrix} e^{0t} & 0 \\ 0 & e^{-\ln(2)t} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 0.5^{t} \end{bmatrix} \end{aligned}$$

Discretize the CT State-Space equation under zero hold operation and ideal sampling of the defined state variables, with $T = 1s$.

Solution:

$$G = e^{AT} = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}$$

$$H = \left(\int_0^T e^{A\lambda} d\lambda \right) B = \left(\int_0^1 \begin{bmatrix} 1 & 0 \\ 0 & 0.5^\lambda \end{bmatrix} d\lambda \right) \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \int_0^1 \begin{bmatrix} 1 \\ 0.5^\lambda \end{bmatrix} d\lambda$$

$$H = \begin{bmatrix} 1 \\ 0.721 \end{bmatrix}$$

Full DT state-space formulation takes the form

$$x[k+1] = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix} x[k] + \begin{bmatrix} 1 \\ 0.721 \end{bmatrix} u[k]$$
$$y[k] = \begin{bmatrix} 1 & 1 \end{bmatrix} x[k]$$

Compute the DT pulse transfer function $Y(z)/U(z)$ **Solution:**

$$\begin{aligned} \frac{Y(z)}{R(z)} &= C\,[zI - G]^{-1}\,H \\ &= \begin{bmatrix} 1 & 1 \end{bmatrix} [zI - G]^{-1} \begin{bmatrix} 1 \\ 0.721 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} z-1 & 0 \\ 0 & z-0.5 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0.721 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{z-1} & 0 \\ 0 & \frac{1}{z-0.5} \end{bmatrix} \begin{bmatrix} 1 \\ 0.721 \end{bmatrix} \\ &= \frac{1}{z-1} + \frac{0.721}{z-0.5} \end{aligned}$$

Now discretize $Y(s)/U(s)$ directly under ZOH operation

Solution:

$$\frac{Y(z)}{U(z)} = \mathcal{Z}\left[\frac{1 - e^{-s}}{s} \frac{Y(s)}{U(s)} \right] \frac{1}{z-1} + \frac{0.721}{z-0.5}$$

EE402 - Discrete Time Systems **Spring 2018**

Lecture 14

Lecturer: Asst. Prof. M. Mert Ankarali

Asymptotic Stability of LTI Systems

Asymptotic Stability of CT-LTI Systems

Let's consider the state-representation of an CT LTI system

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

Given CT-LTI system is called *asymptotically stable* if, with $u(t) = 0]$ and $\forall x(0) \in \mathbb{R}^n$, we have

$$\lim_{t\to\infty} ||x(t)|| = 0$$

Example: Let's assume that system matrix A is diagonalizable and all eigenvalues are real. Find a necessary and sufficient condition such that the state-space representation is asymptotically stable.

Solution: When $u(t) = 0$, we have

$$\dot{x}(t) = Ax(t)$$
$$x(t) = e^{At}x_0$$

Since A is diagonalizable, we know that

$$A = P \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} P^{-1}$$
$$e^{At} = P \begin{bmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & \cdots & 0 \\ \vdots & & \ddots & \\ 0 & & & e^{\lambda_n t} \end{bmatrix} P^{-1}$$

Let $Px_0 = [e_1 \ e_2 \ \cdots \ e_n]^T, e_i \in \mathbb{R}$, then $x(t)$ can be expressed as

$$x(t) = P \begin{bmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & \cdots & 0 \\ \vdots & & \ddots & \\ 0 & & & e^{\lambda_n t} \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix}$$
$$x(t) = P \left(\begin{bmatrix} e_1 e^{\lambda_1 t} \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ e_2 e^{\lambda_2 t} \\ \vdots \\ 0 \end{bmatrix} + \cdots \begin{bmatrix} 0 \\ 0 \\ \vdots \\ e_n e^{\lambda_n t} \end{bmatrix} \right)$$

Then it is easy to see that

$$\forall x_0 \in \mathbb{R}^n, \lim_{t\to\infty} ||x(t)|| = 0 \iff \forall i \in \{1, \ \cdots, \ n\}, \lambda_i < 0$$

Asymptotic Stability of DT-LTI Systems

Now, let's consider the state-representation of an DT LTI system

$$\begin{aligned} x[k+1] &= Gx[k] + Hu[k] \\ y[k] &= Cx[k] + Du[k] \end{aligned}$$

Given DT-LTI system is called *asymptotically stable* if, with $u[k] = 0]$ and $\forall x(0) \in \mathbb{R}^n$, we have

$$\lim_{k\to\infty} ||x[k]|| = 0$$

Example: Let's assume that system matrix G is diagonalizable and all eigenvalues are real. Find a necessary and sufficient condition such that the state-space representation is asymptotically stable.

Solution: When $u[k] = 0$, we have

$$\begin{aligned} x[k+1] &= Gx[k] \\ x(t) &= G^k x_0 \end{aligned}$$

Since G is diagonalizable, we know that

$$G = P \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} P^{-1}$$
$$G^k = P \begin{bmatrix} \lambda_1^k & 0 & \cdots & 0 \\ 0 & \lambda_2^l & \cdots & 0 \\ \vdots & & \ddots & \\ 0 & & & \lambda_n^k \end{bmatrix} P^{-1}$$

Let $Px_0 = [e_1 \ e_2 \ \cdots \ e_n]^T, e_i \in \mathbb{R}$, then $x[k]$ can be expressed as

$$x[k] = P\left(\begin{bmatrix} e_1\lambda_1^k \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ e_2\lambda_2^k \\ \vdots \\ 0 \end{bmatrix} + \cdots \begin{bmatrix} 0 \\ 0 \\ \vdots \\ e_n\lambda_n^k \end{bmatrix}\right)$$

Then it is easy to see that

$$\forall x_0 \in \mathbb{R}^n, \lim_{k\to\infty} ||x[k]|| = 0 \iff \forall i \in \{1,\ \cdots,\ n\}, |\lambda_i| < 1$$

BIBO Stability of LTI Systems

A CT-system written in state-space form is stable if and only if the poles of $H(s) = C\,[sI - A]^{-1}\,B + D$ are located strictly in the open left half plane.

A DT-system written in state-space form is stable if and only if the poles of $H(z) = C\,[zI - G]^{-1}\,H + D$ are located strictly inside the unit circle.

We did not mention weather the system is SISO or MIMO.

- Do you think that BIBO notion is valid for MIMO systems?
- Do you think that checking "poles" of $H(s)$ or $H(z)$ is a valid method for checking stability?
- What are the poles of $H(s)$ and $H(z)$?

Example: Consider the following state-space form of a CT system

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$
$$y(t) = \begin{bmatrix} 1 & -1 \end{bmatrix} x(t)$$

Is this system asymptotically stable?

Solution: Let's compute the eigenvalues of A

$$\det\left(\begin{bmatrix} \lambda & -1 \\ -1 & \lambda \end{bmatrix}\right) = \lambda^2 - 1$$
$$\lambda_{1,2} = \pm 1$$

Thus the system is NOT Asymptotically Stable.

Is this system BIBO stable?

Solution: Let's compute the $H(s)$

$$
\begin{aligned}
H(s) &= \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} s & -1 \\ -1 & s \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} s & 1 \\ 1 & s \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{1}{s^2-1} \\
&= \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} s & 1 \\ 1 & s \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{1}{s^2-1} \\
&= \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ s \end{bmatrix} \frac{1}{s^2-1} \\
&= \frac{-(s-1)}{s^2-1} \\
&= \frac{-1}{s+1}
\end{aligned}
$$

Indeed, the system is BIBO Stable.

In conclusion

- Asymptotic Stability $\rightarrow$ BIBO Stability
- BIBO Stability $\not\rightarrow$ Asymptotic Stability

Lecture 15

Lecturer: Asst. Prof. M. Mert Ankarali

Reachability & Controllability

Reachability & Controllability of CT Systems

For LTI a continuous time state-space representation

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

- A state x_d is said to be **reachable** if there exist a finite time interval $t \in [0, t_f]$ and an input signal defined on this interval, $u(t)$, that transfers the state vector $x(t)$ from the origin (i.e. $x(0) = 0$) to the state x_d within this time interval, i.e. $x(t_f) = x_d$.
- A state x_d is said ti be **controllable** if there exist a finite time interval $t \in [0, t_f]$ and an input signal defined on this interval, $u(t)$, that transfers the state vector $x(t)$ from the initial state x_d (i.e. $x(0) = x_d$) to the origin this time interval, i.e. $x(t_f) = 0$.

- The set $\mathcal{R}$ of all reachable states is a linear (sub)space: $\mathcal{R} \subset \mathbb{R}^n$
- The set $\mathcal{C}$ of all controllable states is a linear (sub)space: $\mathcal{C} \subset \mathbb{R}^n$

For CT systems $x_d \in \mathcal{R}$ if and only if $x_d \in \mathcal{C}$, the Reachability and Controllability conditions are equivalent.

- If the reachable (or controllable) set is the entire state space, i.e., if $\mathcal{R} = \mathbb{R}^n$, then the system is called reachable (or controllable).

One way of testing reachability/controllability is checking the rank (or the range space) the of reachability/controllability matrix

$$\mathbf{M} = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$$

A CT system is reachable/controllable if and only of

$$\text{rank}(\mathbf{M}) = n$$

or equivalently

$$\text{Ra}(\mathbf{M}) = \mathbb{R}^n$$

Reachability & Controllability of DT Systems

For LTI a discrete time state-space representation

$$\begin{aligned} x[k+1] &= Ax[k] + Bu[k] \\ y[k] &= Cx[k] + Du[k] \end{aligned}$$

- A state x_d is said to be **reachable**, if there exist an input sequence, $u[k]$, that transfers the state vector $x[k]$ from the origin (i.e. $x[0] = 0$) to the state x_d in finite number of steps, i.e. $x[k] = x_d$ for some $k \in \mathbb{Z}^+$.
- A state x_d is said ti be **controllable**, if there exist an input sequence, $u[k]$, that transfers the state vector $x[k]$ from the initial state x_d (i.e. $x[0] = x_d$) to the origin in finite number of steps, i.e. $x[k] = 0$ for some $k \in \mathbb{Z}^+$

- The set $\mathcal{R}$ of all reachable states is a linear (sub)space: $\mathcal{R} \subset \mathbb{R}^n$
- The set $\mathcal{C}$ of all controllable states is a linear (sub)space: $\mathcal{C} \subset \mathbb{R}^n$

Unlike from CT systems the Reachability and Controllability conditions are not equivalent.

- $x_d \in \mathcal{R} \Rightarrow x_d \in \mathcal{C}$
- $x_d \in \mathcal{C} \not\Rightarrow x_d \in \mathcal{R}$
- $\mathcal{R} \subset \mathcal{C}$

Thus Reachability implies Controllability but Controllability does not necessarily implies Reachability. For this reason, the term of Reachability is generally preferred for DT systems.

- If the reachable set is the entire state space, i.e., if $\mathcal{R} = \mathbb{R}^n$, then the system is called Reachable (and automatically Controllable).
- If the controllable set is the entire state space, i.e., if $\mathcal{C} = \mathbb{R}^n$, then the system is called Controllable. But there is no guarantee of Reachability.

Example: Consider the following autonomous system

$$x[k+1] = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x[k]$$

What can we infer about the Reachability and Controllability of this system.

Solution: Since this is an autonomous system, obviously

$$B = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Thus input has no affect on the states. If $x[0] = 0$, then $x[k] = 0,\ \forall k > 0$. Thus the system is obviously NOT Reachable.

Now let's compute $x[2]$ for a general $x[0] = x_0$,

$$x[2] = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}^2 x_0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} x_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Obviously $\forall x_0 \in \mathbb{R}^n$ $x[2] = 0$, thus all state-space is Controllable.

Test of Reachability on DT Systems

When $x[0] = 0$, the solution of $x[k]$ is given by

$$
\begin{aligned}
x[k] &= \sum_{j=0}^{k-1} G^{k-j-1} H u[j] \\
&= \left[\begin{array}{c|c|c|c|c} G^{k-1}H & G^{k-2}H & \cdots & GH & H \end{array}\right] \begin{bmatrix} u[0] \\ u[1] \\ \vdots \\ u[k-2] \\ u[k-1] \end{bmatrix}
\end{aligned}
$$

Let

$$
\begin{aligned}
\mathbf{M}_k &= \left[\begin{array}{c|c|c|c|c} G^{k-1}H & G^{k-2}H & \cdots & GH & H \end{array}\right] \\
\mathbf{U}_k &= \begin{bmatrix} u[0] \\ u[1] \\ \vdots \\ u[k-2] \\ u[k-1] \end{bmatrix}
\end{aligned}
$$

then if a state x_d is reachable at k steps, it should satisfy the following equation for some $\mathcal{U}_k$.

$$\mathbf{M}_k \mathbf{U}_k = x_d$$

In order this matrix equation to have a solution x_d should be in the range space of $\mathbf{M}_k$.

$$x_d \in \mathrm{Ra}(\mathbf{M}_k)$$

It is fairly easy to see that

$$\mathrm{Ra}(\mathbf{M}_k) \subset \mathrm{Ra}(\mathbf{M}_{k+1})$$

Thus increasing k increases the chance of x_d being in the reachable subset.

Theorem: For $k < n < l$

$$\mathrm{Ra}(\mathbf{M}_k) \subset \mathrm{Ra}(\mathbf{M}_n) = \mathrm{Ra}(\mathbf{M}_l)$$

Proof: In order to prove this Theorem, we need to use a different well-known theorem.

Cayley-Hamilton Theorem states that every square matrix satisfies its own characteristic equation. In other words, Let $A \in \mathbb{R}^{n\times n}$, and let $p(\lambda)$ be the characteristic equation defined as

$$
\begin{aligned}
p(\lambda) &= \det(\lambda I - A) \\
&= \lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n\lambda \\
&= 0
\end{aligned}
$$

Then by Cayley-Hamilton theorem we conclude that

$$p(G) = G^n + a_1 G^{n-1} + \cdots + a_{n-1}G + a_n I = 0$$

Using this we can see easily that

$$G^n B = -a_1 G^{n-1} B - \cdots - a_{n-1} G B - a_n I$$

Now lets observe M_{n+1}

$$\mathbf{M}_{n+1} = \left[\begin{array}{c|c|c|c|c} G^n H & G^{n-1} H & \cdots & GH & H \end{array} \right]$$

If we follow the Cayley-Hamilton theorem and associated derivations, we can see that the first column $G^n H$ is a linear combination of other columns this it can not increase the rank of the matrix.

This the reachability matrix is defined as

$$\mathbf{M} = \left[\begin{array}{c|c|c|c|c} G^{n-1} H & G^{n-2} H & \cdots & GH & H \end{array} \right]$$

where n is the dimension of the state-space.

The DT system is called reachable if

$$\text{rank}(\mathbf{M}) = n$$

or equivalently

$$\text{Ra}(\mathbf{M}) = \mathbb{R}^n$$

Observability

It turns out it is more natural to think in terms of "un-observability" as reflected in the following definitions.

- For CT systems, a state x_o of a finite dimensional linear dynamical system is said to be unobservable, if with $x(0) = x_o$ and for every $u(t)$ we get the same $y(t)$ as we would with $x(0) = 0$.
- For DT systems, a state x_o of a finite dimensional linear dynamical system is said to be unobservable, if with $x[0] = x_o$ and for every $u[k]$ we get the same $y[k]$ as we would with $x[0] = 0$.

In other words, for both CT and DT systems an unobservable initial condition cannot be distinguished from the zero initial condition.

The set $\bar{\mathcal{O}}$ of all unobservable states is a linear (sub)space: $\bar{\mathcal{O}} \subset \mathbb{R}^n$

- If the unobservable set only contains the origin, i.e., if $\bar{\mathcal{O}} = \{0\}$,
- If the dimension of unobservable subspace is equal to 0, $\dim = \left(\bar{\mathcal{O}}\right) = 0$,
- If any initial condition, $x(0)$ or $x[0]$, can be uniquely determined from input-output measurement,

then the system is called Observable.

Test of Observability on CT Systems

One way of testing Observability of CT systems is checking the rank (or the range space, or null space) the of the Observability matrix

$$\mathbf{O} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

A CT system is Observable if and only of

$$\text{rank}(\mathbf{O}) = n$$

or equivalently

$$\text{Ra}(\mathbf{O}) = \mathbb{R}^n$$

or equivalently

$$\dim\left(\mathcal{N}(\mathbf{O})\right) = 0$$

Test of Observability on DT Systems

Without loss of generality, let's assume that $u[k] = 0$. Under this assumption, we know that

$$y[k] = CG^k x_0$$

Based on this solution we can write

$$\begin{aligned} y[0] &= Cx_0 \\ y[1] &= CGx_0 \\ y[2] &= CG^2 x_0 \\ &\vdots \\ y[k] &= CA^k x_0 \end{aligned}$$

If we combine these equations matrix form we obtain

$$\begin{bmatrix} y[0] \\ y[1] \\ y[2] \\ \vdots \\ y[k] \end{bmatrix} = \begin{bmatrix} C \\ CG \\ CG^2 \\ \vdots \\ CG^k \end{bmatrix} x_0$$

Let

$$\mathbf{Y}_k = \begin{bmatrix} y[0] \\ y[1] \\ y[2] \\ \vdots \\ y[k] \end{bmatrix} \quad , \quad \mathbf{O}_k = \begin{bmatrix} C \\ CG \\ CG^2 \\ \vdots \\ CG^k \end{bmatrix}$$

Then the equation takes the simple form $\mathbf{Y}_k = \mathbf{O}_k x_0$. If x_0 is an unobservable state, then for-all k we should have $\mathbf{O}_k x_0 = 0$, or equivalently $x_0 \in \mathcal{N}(\mathbf{O}_k)$ (Null-space).

From this point, we can conclude that, the DT system is observable if and only if,

$$\forall k \in \mathbb{Z}, \dim\left(\mathcal{N}\left(\mathbf{O}_k\right)\right) = 0$$

However we don't need to test all $k \in \mathbb{Z}$. First of all it should be obvious that we should take k as large as possible to guarantee weather x_0 is unobservable ot not. Formally speaking,

$$\mathcal{N}\left(\mathbf{O}_{k+1}\right) \subset \mathcal{N}\left(\mathbf{O}_k\right)$$

However from Cayley-Hamilton theorem, we know that CA^n can be written as a linear combination of $\{CA^{n-1},\ CA^{n-2},\ \cdots, CA,\ C\}$, thus we have

$$\mathcal{N}\left(\mathbf{O}_{n+1}\right) = \mathcal{N}\left(\mathbf{O}_n\right)$$

For this reason it is necessary and sufficient to test $\mathbf{O}_n$ for observability. In conclusion, observability matrix is defined as

$$\mathbf{O} = \begin{bmatrix} C \\ CG \\ CG^2 \\ \vdots \\ CG^{n-1} \end{bmatrix}$$

The DT system is called Observable if

$$\operatorname{rank}\left(\mathbf{O}\right) = n$$

or equivalently

$$\operatorname{Ra}\left(\mathbf{O}\right) = \mathbb{R}^n$$

$$\dim\left(\mathcal{N}(\mathbf{O})\right) = 0$$

Lecture 16

Lecturer: Asst. Prof. M. Mert Ankarali

State-Feedback & Pole Placement

Given a discrete-time state-evolution equation

$$x[k+1] = Gx[k] + Hu[k]$$

If direct measurements of all of the states of the system (e.g. $y[k] = x[k]$) are available, the most popular control method is the linear state feedback control,

$$u[k] = -Kx[k]$$

which can be thought as a generalization of P controller to the vector form. Under this control law, without any reference signal, the system becomes an autonomous system

$$\begin{aligned} x[k+1] &= Gx[k] + H(-Kx[k]) \\ x[k+1] &= (G - HK)\, x[k] \end{aligned}$$

The system matrix of this new autonomous system is $\hat{G} = G - HK$. Important questions is how to choose K. Note that

$$\begin{aligned} K &\in \mathbb{R}^n \quad \text{Single} - \text{Input} \\ K &\in \mathbb{R}^{n \times p} \quad \text{Multi} - \text{Input} \end{aligned}$$

As in all of the control desgn techniques, the most critical criterion is stability, thus we want all of the eigenvalues to be strictly inside the unit-circle. However, we know that there could be different requirements on the poles/eigenvalues of the system.

The fundamental principle of "pole-placement" design is that we first define a desired closed-loop eigenvalue set $\mathcal{E}^* = \{\lambda_1^*, \ \cdots \ , \ \lambda_1^*\}$, and then if possible we choose K^* such that the closed-loop eigenvalues match the desired ones.

The necessary and sufficient condition on arbitrary pole-placement is that the system should be Reachable.

In Pole-Placement, first step is computing the desired characteristic polynomial.

$$\begin{aligned} \mathcal{E}^* &= \{\lambda_1^*, \ \cdots \ , \ \lambda_n^*\} \\ p^*(z) &= (z - \lambda_1^*) \cdots (z - \lambda_n^*) \\ &= z^n + a_1^* z^{n-1} + \cdots + a_{n-1}^* z + a_n^* \end{aligned}$$

Then we tune K such that

$$\det\left(zI - (G - HK)\right) = p^*(z)$$

Direct Design of State-Feedback Gain

If n is small, the most efficient method could be the direct design.

Example: Consider the following DT system

$$x[k+1] = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} x[k] + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u[k]$$

Design a state-feedback rule such that poles are located at $\lambda_{1,2} = 0$ (Dead-beat gain)

Solution: Desired characteristic equation can be computed as

$$p^*(z) = z^2$$

Let $K = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$, then the characteristic equation of $\hat{G}$ can be computed as

$$\begin{aligned} \det\left(zI - (G - HK)\right) &= \det\left(\begin{bmatrix} z - 1 + k_1 & k_1 \\ k_2 & z - 2 + k_2 \end{bmatrix}\right) \\ &= z^2 + z(k_1 + k_2 - 3) + (2 - 2k_1 - k_2) \end{aligned}$$

If we match the equations

$$\begin{aligned} z^2 + z(k_1 + k_2 - 3) + (2 - 2k_1 - k_2) &= z^2 \\ k_1 + k_2 &= 3 \\ 2k_1 + k_2 &= 2 \\ k_1 &= -1 \\ k_2 &= 4 \end{aligned}$$

This $K = \begin{bmatrix} -1 & 4 \end{bmatrix}$. Now let's compute $\hat{G}^k$

$$\begin{aligned} \hat{G} &= \begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix} \\ \hat{G}^2 &= \begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ &\vdots \end{aligned}$$

It can be seen that closed-loop system rejects all initial condition perturbations in 2 steps.

Design of State-Feedback Gain Using Reachable Canonical Form

Let's assume that the state-space representation is in controllable canonical form and we have access to the all states of this form

$$x[k+1] = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} x[k] + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u[k]$$

Let $K = \begin{bmatrix} k_n & \cdots & k_1 \end{bmatrix}$, then autonomous system takes the form

$$x[k+1] = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} x[k] - \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} k_n & \cdots & k_1 \end{bmatrix} x[k]$$

$$x[k+1] = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -(a_n + k_n) & -(a_{n-1} + k_{n-1}) & -(a_{n-2} + k_{n-2}) & \cdots & -(a_1 + k_1) \end{bmatrix} x[k]$$

Let $p^*(z) = z^2 + a_1^* z + \cdots + a_{n-1}^* z + a_n^*$, then K can be computed as

$$K = \begin{bmatrix} (a_n^* - a_n) & \cdots & (a_1^* - a_1) \end{bmatrix}$$

However, what if the system is not in Reachable canonical form. We can find a transformation which finds the Reachable canonical representation.

The reachability matrix of a state-space representation is given as

$$M = \begin{bmatrix} H \mid GH \mid \cdots \mid G^{n-1}H \end{bmatrix}$$

Let's define a transformation matrix T as follows:

$$T = MW \quad , \quad x[k] = T\hat{x}[k]$$
$$\hat{x}[k+1] = \left[T^{-1}GT\right] \hat{x}[k] + T^{-1}Hu[k]$$

where

$$W = \begin{bmatrix} a_{n-1} & a_{n-2} & \cdots & a_1 & 1 \\ a_{n-2} & a_{n-3} & \cdots & 1 & 0 \\ \vdots & \vdots & \iddots & & \vdots \\ a_1 & 1 & & & \\ 1 & 0 & \cdots & & 0 \end{bmatrix}$$

Then it is given that

$$T^{-1}GT = \hat{G} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix}$$

$$T^{-1}H = \hat{H} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Let's compute $T\hat{H}$

$$
\begin{aligned}
T\hat{H} = MW\hat{H} &= M \begin{bmatrix} a_{n-1} & a_{n-2} & \cdots & a_1 & 1 \\ a_{n-2} & a_{n-3} & \cdots & 1 & 0 \\ \vdots & \vdots & ⋰ & & \vdots \\ a_1 & 1 & & & \\ 1 & 0 & \cdots & & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \\
&= \left[\begin{array}{c|c|c|c} H & GH & \cdots & G^{n-1}H \end{array}\right] \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \\
&= H
\end{aligned}
$$

A similar approach (but longer) can be used to show that $T^{-1}GT = \hat{G}$. We know how to design a state-feedback gain $\hat{K}$ for the Reachable canonical form. Given $\hat{K}$ $u[k]$ is given as

$$
\begin{aligned}
u[k] &= -\hat{K}\hat{x}[k] \\
&= -\hat{K}T^{-1}\hat{x}[k] \\
K &= \hat{K}T^{-1}
\end{aligned}
$$

Example 2: Consider the following DT system

$$
x[k+1] = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} x[k] + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u[k]
$$

Design a state-feedback rule using the Reachable canonical form approach, such that poles are located at $\lambda_{1,2} = 0$ (Dead-beat gain)

Solution: Characteristic equation of G can be derived as

$$
\det\left(\begin{bmatrix} z-1 & 0 \\ 0 & z-2 \end{bmatrix}\right) = z^2 - 3z + 2
$$

The Reachability matrix can be computed as

$$
M = \left[\begin{array}{c|c} H & GH \end{array}\right] = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}
$$

The matrix W can be computed as

$$
W = \begin{bmatrix} -3 & 1 \\ 1 & 0 \end{bmatrix}
$$

Transformation matrix, T and its inverse T^{-1} can be computed as

$$
\begin{aligned}
T = MW &= \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -3 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -1 & 1 \end{bmatrix} \\
T^{-1} &= \begin{bmatrix} -1 & 1 \\ -1 & 2 \end{bmatrix}
\end{aligned}
$$

Given that desired characteristic polynomial is $p^*(z) = z^2$, $\hat{K}$ of reachable canonical from can be computed as

$$\begin{aligned} \hat{K} &= \begin{bmatrix} -a_2 & -a_1 \end{bmatrix} \\ &= \begin{bmatrix} -2 & 3 \end{bmatrix} \end{aligned}$$

Finally K can be computed as

$$\begin{aligned} K = \hat{K}T^{-1} &= \begin{bmatrix} -2 & 3 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ -1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 4 \end{bmatrix} \end{aligned}$$

As expected this is the same result with the one found in Example 1 (Direct-Method).

EE402 - Discrete Time Systems	**Spring 2018**

Lecture 17

Lecturer: Asst. Prof. M. Mert Ankarali

Discrete-time Luenberger Observer

In general the state, $x[k]$ of a system is not accessible and *observers, estimators, filters*) have to be used to extract this information. The output, $y[k]$, represents the measurements which is a function of $x[k]$ and $u[k]$.

$$\begin{aligned} x[k+1] &= Gx[k] + Hu[k] \\ y[k] &= Cx[k] + Du[k] \end{aligned}$$

A Luenberger observers is built using a "simulated" model of the system and the errors caused by the mismatched initial conditions $x_0 \neq \hat{x}_0$ (or other types of perturbations) are reduced by introducing output error feedback.

Let's assume that the states of the simulated system is $\hat{x}[k]$, then the state space equation of this synthetic system takes the form

$$\begin{aligned} \hat{x}[k+1] &= \hat{G}x[k] + \hat{H}u[k] \\ \hat{y}[k] &= \hat{C}x[k] + \hat{D}u[k] \end{aligned}$$

Note that since $u[k]$ is the input that is controlled it is assumed to be known. If $x[0] = \hat{x}[0]$ and when there is no model mismatch or uncertainty in the system then we expect that $x[k] = \hat{x}[k]$ and $y[k] = \hat{y}[k]$ for all k. When $x[0] = \hat{x}[0]$, then we observe a difference between the measured and predicted output $y[k] \neq \hat{y}[k]$. The core idea in Luenberger observers is feeding the error in the output prediction $y[k] - \hat{y}[k]$ to the system via a linear feedback gain.

$$\begin{aligned} \hat{x}[k+1] &= G\hat{x}[k] + Hu[k] + L\left(y[k] - \hat{y}[k]\right) \\ \hat{y}[k] &= Cx[k] + Du[k] \end{aligned}$$

In order to understand how a Luenberger observer works and ot choose a proper observer gain L, we define an error signal $e[k] = x[k] - \hat{x}[k]$. The dynamics w.r.t $e[k]$ can be derived as

$$\begin{aligned} e[k+1] &= x[k+1] - \hat{x}[k+1] \\ &= \left(Gx[k] + Hu[k]\right) - \left(G\hat{x}[k] + Hu[k] + L\left(y[k] - \hat{y}[k]\right)\right) \\ e[k+1] &= \left(G - LC\right) e[k] \end{aligned}$$

where $e[0] = x[0] - \hat{x}[0]$ denotes the error in the initial condition.

If the matrix $(G - LC)$ is stable then the errors initial condition will diminish eventually. Moreover, in order to have a good observer/estimator performance the observer convergence should be sufficiently fast.

Observer Gain & Pole Placement

Similar to the state-feedback gain design, the fundamental principle of "pole-placement" Observer design is that we first define a desired closed-loop eigenvalue set and compute associated desired characteristic polynomial.

$$\begin{aligned}
\mathcal{E}^* &= \{\lambda_1^*,\ \cdots,\ \lambda_n^*\} \\
p^*(z) &= (z-\lambda_1^*)\cdots(z-\lambda_n^*) \\
&= z^n + a_1^* z^{n-1} + \cdots + a_{n-1}^* z + a_n^*
\end{aligned}$$

The necessary and sufficient condition on arbitrary observer pole-placement is that the system should be fully Observable. Then we tune L such that

$$\det\left(zI - (G - LC)\right) = p^*(z)$$

Direct Design of Observer Gain

If n is small, the most efficient method could be the direct design.

Example: Consider the following DT system

$$\begin{aligned}
x[k+1] &= \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} x[k] + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u[k] \\
y[k] &= \begin{bmatrix} 1 & -1 \end{bmatrix} u[k]
\end{aligned}$$

Design an observer such that estimater poles are located at $\lambda_{1,2} = 0$ (Dead-beat Observer)

Solution: Desired characteristic equation can be computed as

$$p^*(z) = z^2$$

Let $L = \begin{bmatrix} l_2 \\ l_1 \end{bmatrix}$, then the characteristic equation of $(G - LC)$ can be computed as

$$\begin{aligned}
\det\left(zI - (G - LC)\right) &= \det\left(\begin{bmatrix} z - 1 + l_2 & -l_2 \\ l_1 & z - 2 - l_1 \end{bmatrix}\right) \\
&= z^2 + z(l_2 - l_1 - 3) + (l_1 - 2l_2 + 2)
\end{aligned}$$

If we match the equations

$$\begin{aligned}
l_2 - l_1 &= 3 \\
-l_1 + 2l_2 &= 2 \\
l_2 &= -1 \\
l_1 &= -4
\end{aligned}$$

Thus $L = \begin{bmatrix} -1 \\ -4 \end{bmatrix}$

Design of Observer Gain Using Reachable Canonical Form

Let's assume that the state-space representation is in observable canonical form

$$G = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_n \\ 1 & 0 & \cdots & 0 & -a_{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -a_2 \\ 0 & 0 & \cdots & 1 & -a_1 \end{bmatrix} \quad , \quad H = \begin{bmatrix} (b_n - b_0 a_n) \\ (b_{n-1} - b_0 a_{n-1}) \\ \vdots \\ (b_2 - b_0 a_2) \\ (b_1 - b_0 a_1) \end{bmatrix}$$
$$C = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \quad , \quad D = b_0$$

Let $L = \begin{bmatrix} l_n \\ \vdots \\ l_1 \end{bmatrix}$, then $(G - LC)$ takes the form

$$(G - LC) = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_n \\ 1 & 0 & \cdots & 0 & -a_{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -a_2 \\ 0 & 0 & \cdots & 1 & -a_1 \end{bmatrix} - \begin{bmatrix} l_n \\ \vdots \\ l_1 \end{bmatrix} \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 & \cdots & 0 & -(a_n + l_n) \\ 1 & 0 & \cdots & 0 & -(a_{n-1} + l_{n-1}) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -(a_2 + l_n) \\ 0 & 0 & \cdots & 1 & -(a_1 + l_n) \end{bmatrix}$$

The characterstic equation of $G - LC$ is simply given as

$$p(z) = z^n + (a_1 - l_1)z^{n-1} + \cdots + (a_{n-1} - l_{n-1})z + (a_n - l_n)$$

Let's assume that desired $p^*(z)$ is equal to

$$p(z) = z^n + a_1^* z^{n-1} + \cdots + a_{n-1}^* z + a_n^*$$

Then, the observer gain L is computed as

$$L^* = \begin{bmatrix} a_n^* - a_n \\ \vdots \\ a_1^* - a_1 \end{bmatrix}$$

If the system is not in Observable canonical form, we can find a transformation that outputs the Observable canonical form representation The Observability matrix of a state-space representation is given as

$$O = \begin{bmatrix} C \\ CG \\ \vdots \\ CG^{-1} \end{bmatrix}$$

Let's define a transformation matrix Q as follows:

$$\begin{aligned} Q &= (WO)^{-1} \quad , \quad x[k] = Q\bar{x}[k] \\ \bar{x}[k+1] &= \left[Q^{-1}GQ\right]\bar{x}[k] + Q^{-1}Hu[k] \\ y[k] &= CQ\bar{x}[k] + Du[k] \end{aligned}$$

where

$$W = \begin{bmatrix} a_{n-1} & a_{n-2} & \cdots & a_1 & 1 \\ a_{n-2} & a_{n-3} & \cdots & 1 & 0 \\ \vdots & \vdots & \iddots & & \vdots \\ a_1 & 1 & & & \\ 1 & 0 & \cdots & & 0 \end{bmatrix}$$

Then it is given that

$$\bar{G} = Q^{-1}GQ = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_n \\ 1 & 0 & \cdots & 0 & -a_{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -a_2 \\ 0 & 0 & \cdots & 1 & -a_1 \end{bmatrix}$$

$$\bar{C} = CQ = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

Let's compute $\bar{C}Q^{-1}$

$$\begin{aligned} \bar{C}Q^{-1} &= \bar{C}WO \\ &= \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{n-1} & a_{n-2} & \cdots & a_1 & 1 \\ a_{n-2} & a_{n-3} & \cdots & 1 & 0 \\ \vdots & \vdots & \iddots & & \vdots \\ a_1 & 1 & & & \\ 1 & 0 & \cdots & & 0 \end{bmatrix} O^T \\ &= \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} C & CG & \cdots & CG^{-1} \end{bmatrix} \\ &= C \end{aligned}$$

A similar approach (but longer) can be used to show that $Q^{-1}GQ = \bar{G}$.

We know how to design a observer gain $\bar{L}$ for the Observable canonical form. Given $\bar{L}$, Observer gain w.r.t. original state-space representation is computed as

$$L = Q\bar{L}$$

Example 2: Consider the following DT system

$$\begin{aligned} x[k+1] &= \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} x[k] + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u[k] \\ y[k] &= \begin{bmatrix} 1 & -1 \end{bmatrix} u[k] \end{aligned}$$

Design an observer using the Observable canonical form such that estimater poles are located at $\lambda_{1,2} = 0$ (Dead-beat Observer)

Solution: Characteristic equation of G can be derived as

$$\det\left(\begin{bmatrix} z-1 & 0 \\ 0 & z-2 \end{bmatrix}\right) = z^2 - 3z + 2$$

Observability matrix can be computed as

$$O = \begin{bmatrix} 1 & -1 \\ 1 & -2 \end{bmatrix}$$

The matrix W can be computed as

$$W = \begin{bmatrix} -3 & 1 \\ 1 & 0 \end{bmatrix}$$

Transformation matrix Q can be computed as

$$\begin{aligned} Q &= \left(WO^T\right)^{-1} \\ &= \left(\begin{bmatrix} -3 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & -2 \end{bmatrix}\right)^{-1} = \left(\begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix}\right)^{-1} = \begin{bmatrix} -1 & -1 \\ -1 & -2 \end{bmatrix} \end{aligned}$$

Given that desired characteristic polynomial is $p^*(z) = z^2$, $\bar{L}$ of observable canonical from can be computed as

$$\begin{aligned} \bar{L} &= \begin{bmatrix} -a_2 \\ -a_1 \end{bmatrix} \\ &= \begin{bmatrix} -2 \\ 3 \end{bmatrix} \end{aligned}$$

Finally Observer Gain L can be computed as

$$L = Q\bar{L} = \begin{bmatrix} -1 \\ -4 \end{bmatrix}$$

Not surprisingly the result is same with the one found in first example.

Closed-Loop Observer & State-Feedback

In the state-feedback control policy the input is ideally defined by the following law

$$u[k] = -Kx[k]$$

However, as mentioned in Observer lecture, in general we don't have direct access to the all states of the system. In this case, we learnt how to design an Observer/Estimator of the states. In this respect, it is natural to assume that in a closed-loop system, the control policy that define the input should depend on the estimated states

$$u[k] = -K\hat{x}[k]$$

However the important question how this coupling affect the closed-loop behavior, and even deeper question can be even use such a policy. The advantage of LTI systems is that state-feedback gain, and observer gain can be seperatelly designed and we guarntee a stable closed-loop performance. In this section, we will analyze the coupled system Equations of motion for the closed-loop observer & state-feedback based control system is given below

$$
\begin{aligned}
x[k+1] &= Gx[k] + Hu[k] \\
y[k] &= Cx[k] + Du[k] \\
\hat{x}[k+1] &= G\hat{x}[k] + Hu[k] + L\left(y[k] - \hat{y}[k]\right) \\
\hat{y}[k] &= C\hat{x}[k] + Du[k] \\
u[k] &= -K\hat{x}[k]
\end{aligned}
$$

If we eliminate $u[k]$ and $\hat{y}[k]$ we obtain following dynamical representation

$$
\begin{aligned}
x[k+1] &= Gx[k] - HK\hat{x}[k] \\
\hat{x}[k+1] &= G\hat{x}[k] - HK\hat{x}[k] + LC\left(x[k] - \hat{x}[k]\right) \\
y[k] &= Cx[k] - DK\hat{x}[k]
\end{aligned}
$$

Now let's replace $\hat{x}[k]$ with $e[k] = x[k] - \hat{x}[k]$

$$
\begin{aligned}
x[k+1] &= (G - HK)x[k] + HKe[k] \\
e[k+1] &= (G - LC)e[k] \\
y[k] &= (C - DK)x[k] + DKe[k]
\end{aligned}
$$

Now let's defina a state for the whole system, $z[k] = \begin{bmatrix} x[k] \\ e[k] \end{bmatrix}$ then the state-space representation is given by

$$
\begin{aligned}
z[k+1] &= \begin{bmatrix} (G-HK) & HK \\ 0_{n\times n} & (G-LC) \end{bmatrix} z[k] \\
y[k] &= \begin{bmatrix} (C-DK) & DK \end{bmatrix} z[k]
\end{aligned}
$$

The system matrix is in block diagonal form and the eigenvalues of this new system matrix is find by taking the union of eigenvalues of $(G - HK)$ and eigenvalues of $(G - LC)$. Thus a seperate pole-placement can be performed for the state-feedback controller and the observer.

Table of Laplace and Z-transforms

	$X(s)$	$x(t)$	$x(kT)$ or $x(k)$	$X(z)$
1.	–	–	Kronecker delta $\delta_0(k)$ 1 $k = 0$ 0 $k \neq 0$	1
2.	–	–	$\delta_0(n-k)$ 1 $n = k$ 0 $n \neq k$	z^{-k}
3.	$\frac{1}{s}$	$1(t)$	$1(k)$	$\frac{1}{1-z^{-1}}$
4.	$\frac{1}{s+a}$	e^{-at}	e^{-akT}	$\frac{1}{1-e^{-aT}z^{-1}}$
5.	$\frac{1}{s^2}$	t	kT	$\frac{Tz^{-1}}{(1-z^{-1})^2}$
6.	$\frac{2}{s^3}$	t^2	$(kT)^2$	$\frac{T^2z^{-1}(1+z^{-1})}{(1-z^{-1})^3}$
7.	$\frac{6}{s^4}$	t^3	$(kT)^3$	$\frac{T^3z^{-1}(1+4z^{-1}+z^{-2})}{(1-z^{-1})^4}$
8.	$\frac{a}{s(s+a)}$	$1 - e^{-at}$	$1 - e^{-akT}$	$\frac{(1-e^{-aT})z^{-1}}{(1-z^{-1})(1-e^{-aT}z^{-1})}$
9.	$\frac{b-a}{(s+a)(s+b)}$	$e^{-at} - e^{-bt}$	$e^{-akT} - e^{-bkT}$	$\frac{(e^{-aT}-e^{-bT})z^{-1}}{(1-e^{-aT}z^{-1})(1-e^{-bT}z^{-1})}$
10.	$\frac{1}{(s+a)^2}$	te^{-at}	kTe^{-akT}	$\frac{Te^{-aT}z^{-1}}{(1-e^{-aT}z^{-1})^2}$
11.	$\frac{s}{(s+a)^2}$	$(1-at)e^{-at}$	$(1-akT)e^{-akT}$	$\frac{1-(1+aT)e^{-aT}z^{-1}}{(1-e^{-aT}z^{-1})^2}$
12.	$\frac{2}{(s+a)^3}$	t^2e^{-at}	$(kT)^2e^{-akT}$	$\frac{T^2e^{-aT}(1+e^{-aT}z^{-1})z^{-1}}{(1-e^{-aT}z^{-1})^3}$
13.	$\frac{a^2}{s^2(s+a)}$	$at - 1 + e^{-at}$	$akT - 1 + e^{-akT}$	$\frac{[(aT-1+e^{-aT})+(1-e^{-aT}-aTe^{-aT})z^{-1}]z^{-1}}{(1-z^{-1})^2(1-e^{-aT}z^{-1})}$
14.	$\frac{\omega}{s^2+\omega^2}$	$\sin \omega t$	$\sin \omega kT$	$\frac{z^{-1}\sin\omega T}{1-2z^{-1}\cos\omega T+z^{-2}}$
15.	$\frac{s}{s^2+\omega^2}$	$\cos \omega t$	$\cos \omega kT$	$\frac{1-z^{-1}\cos\omega T}{1-2z^{-1}\cos\omega T+z^{-2}}$
16.	$\frac{\omega}{(s+a)^2+\omega^2}$	$e^{-at}\sin \omega t$	$e^{-akT}\sin \omega kT$	$\frac{e^{-aT}z^{-1}\sin\omega T}{1-2e^{-aT}z^{-1}\cos\omega T+e^{-2aT}z^{-2}}$
17.	$\frac{s+a}{(s+a)^2+\omega^2}$	$e^{-at}\cos \omega t$	$e^{-akT}\cos \omega kT$	$\frac{1-e^{-aT}z^{-1}\cos\omega T}{1-2e^{-aT}z^{-1}\cos\omega T+e^{-2aT}z^{-2}}$
18.	–	–	a^k	$\frac{1}{1-az^{-1}}$
19.	–	–	a^{k-1} $k = 1, 2, 3, \ldots$	$\frac{z^{-1}}{1-az^{-1}}$
20.	–	–	ka^{k-1}	$\frac{z^{-1}}{(1-az^{-1})^2}$
21.	–	–	k^2a^{k-1}	$\frac{z^{-1}(1+az^{-1})}{(1-az^{-1})^3}$
22.	–	–	k^3a^{k-1}	$\frac{z^{-1}(1+4az^{-1}+a^2z^{-2})}{(1-az^{-1})^4}$
23.	–	–	k^4a^{k-1}	$\frac{z^{-1}(1+11az^{-1}+11a^2z^{-2}+a^3z^{-3})}{(1-az^{-1})^5}$
24.	–	–	$a^k \cos k\pi$	$\frac{1}{1+az^{-1}}$

$x(t) = 0$ for $t < 0$
$x(kT) = x(k) = 0$ for $k < 0$
Unless otherwise noted, $k = 0, 1, 2, 3, \ldots$

Definition of the Z-transform

$$\mathscr{Z}\{x(k)\} = X(z) = \sum_{k=0}^{\infty} x(k) z^{-k}$$

Important properties and theorems of the Z-transform

	$x(t)$ or $x(k)$	**$\mathcal{Z}\{x(t)\}$ or $\mathcal{Z}\{x(k)\}$**
1.	$ax(t)$	$aX(z)$
2.	$ax_1(t)+bx_2(t)$	$aX_1(z)+bX_2(z)$
3.	$x(t+T)$ or $x(k+1)$	$zX(z)-zx(0)$
4.	$x(t+2T)$	$z^2X(z)-z^2x(0)-zx(T)$
5.	$x(k+2)$	$z^2X(z)-z^2x(0)-zx(1)$
6.	$x(t+kT)$	$z^kX(z)-z^kx(0)-z^{k-1}x(T)-\ldots-zx(kT-T)$
7.	$x(t-kT)$	$z^{-k}X(z)$
8.	$x(n+k)$	$z^kX(z)-z^kx(0)-z^{k-1}x(1)-\ldots-zx(k1-1)$
9.	$x(n-k)$	$z^{-k}X(z)$
10.	$tx(t)$	$-Tz\dfrac{d}{dz}X(z)$
11.	$kx(k)$	$-z\dfrac{d}{dz}X(z)$
12.	$e^{-at}x(t)$	$X(ze^{aT})$
13.	$e^{-ak}x(k)$	$X(ze^{a})$
14.	$a^kx(k)$	$X\left(\dfrac{z}{a}\right)$
15.	$ka^kx(k)$	$-z\dfrac{d}{dz}X\left(\dfrac{z}{a}\right)$
16.	$x(0)$	$\lim_{z\to\infty} X(z)$ if the limit exists
17.	$x(\infty)$	$\lim_{z\to 1}\left[\left(1-z^{-1}\right)X(z)\right]$ if $\left(1-z^{-1}\right)X(z)$ is analytic on and outside the unit circle
18.	$\nabla x(k)=x(k)-x(k-1)$	$\left(1-z^{-1}\right)X(z)$
19.	$\Delta x(k)=x(k+1)-x(k)$	$(z-1)X(z)-zx(0)$
20.	$\sum_{k=0}^{n} x(k)$	$\dfrac{1}{1-z^{-1}}X(z)$
21.	$\dfrac{\partial}{\partial a}x(t,a)$	$\dfrac{\partial}{\partial a}X(z,a)$
22.	$k^mx(k)$	$\left(-z\dfrac{d}{dz}\right)^m X(z)$
23.	$\sum_{k=0}^{n} x(kT)y(nT-kT)$	$X(z)Y(z)$
24.	$\sum_{k=0}^{\infty} x(k)$	$X(1)$