EE402 - Discrete Time Systems
Mini Project 6
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Tuesday

Q1: State Space Realization

a) $G_1(z) = \dfrac{z^2 + 0{,}5z}{z^3 - 2{,}2z^2 + 1{,}52z - 0{,}32} = \dfrac{z^3}{z^3} \cdot \dfrac{z^{-1} + 0{,}5z^{-2}}{1 - 2{,}2z^{-1} + 1{,}52z^{-2} - 0{,}32z^{-3}}$

$$G_1(z) = \frac{z^{-1} + 0{,}5z^{-2}}{1 - 2{,}2z^{-1} + 1{,}52z^{-2} - 0{,}32z^{-3}}$$

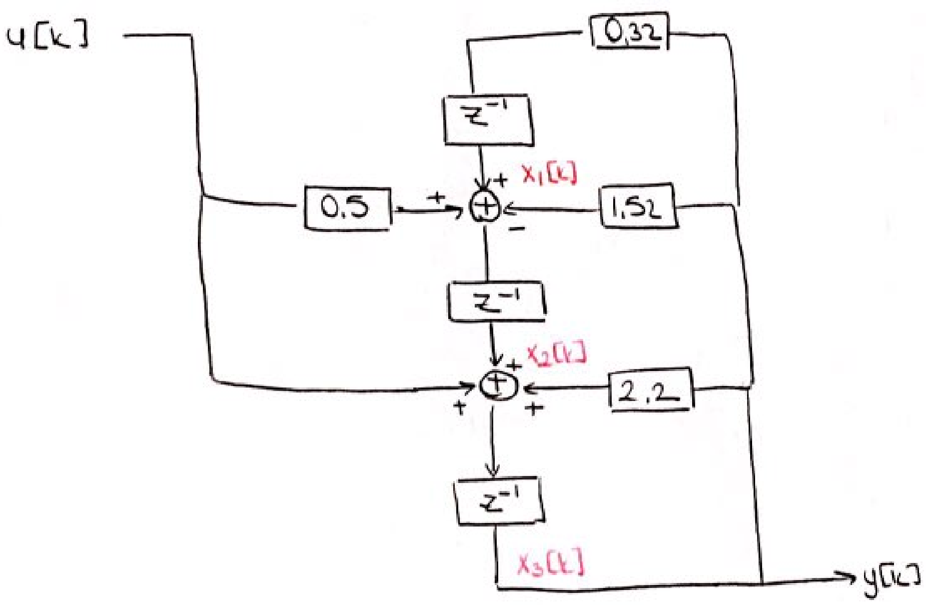
I'd like to first find out its block diagram representation. (Observable Canonical Realization)

$$G_1(z) = \frac{Y(z)}{U(z)} = \frac{z^{-1} + 0{,}5z^{-2}}{1 - 2{,}2z^{-1} + 1{,}52z^{-2} - 0{,}32z^{-3}}$$

$$Y(z)\left[1 - 2.2z^{-1} + 1{,}52z^{-2} - 0{,}32z^{-3}\right] = U(z)\left[z^{-1} + 0{,}5z^{-2}\right]$$

$$Y(z) = 2.2z^{-1}Y(z) + U(z)z^{-1} - 1{,}52z^{-2}Y(z) + 0{,}5z^{-2}U(z) + 0{,}32z^{-3}Y(z)$$

$$Y(z) = \left\{\left[0{,}32Y(z)z^{-1} - 1{,}52Y(z) + 0{,}5U(z)\right]z^{-1} + 2.2Y(z) + U(z)\right\}z^{-1}$$



Define the states ⇒ $x_1[k]$ $x_2[k]$ $x_3[k]$

$X_1(z) = \left[0{,}32\,Y(z)\right]z^{-1} = 0{,}32\,X_3(z)z^{-1} \longrightarrow x_1[k+1] = 0{,}32x_3[k]$

$X_2(z) = \left[X_1(z) - 1{,}52X_3(z) + 0{,}5U(z)\right]z^{-1} \longrightarrow x_2[k+1] = x_1[k] - 1{,}52x_3[k] + 0{,}5u[k]$

$X_3(z) = \left[2{,}2X_3(z) + U(z) + X_2(z)\right]z^{-1} \longrightarrow x_3[k+1] = 2{,}2x_3[k] + u[k] + x_2[k]$

$Y(z) = X_3(z)$

$$\underbrace{\begin{bmatrix} x_1[k+1] \\ x_2[k+1] \\ x_3[k+1] \end{bmatrix}}_{x[k+1]} = \begin{bmatrix} 0 & 0 & 0{,}32 \\ 1 & 0 & -1{,}52 \\ 0 & 1 & -2{,}2 \end{bmatrix} \underbrace{\begin{bmatrix} x_1[k] \\ x_2[k] \\ x_3[k] \end{bmatrix}}_{x[k]} + \begin{bmatrix} 0 \\ 0{,}5 \\ 1 \end{bmatrix} u[k]$$

$$y[k] = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1[k] \\ x_2[k] \\ x_3[k] \end{bmatrix}$$

The results are consistent with observable canonical form.

b) $G_2(z) = \dfrac{z^2 + 0{,}4z - 0{,}12}{z^2 + 0{,}6z - 0{,}4} = \dfrac{z^2}{z^2} \cdot \dfrac{1 + 0{,}4z^{-1} - 0{,}12z^{-2}}{1 + 0{,}6z^{-1} - 0{,}4z^{-2}}$

$$G_2(z) = \frac{1 + 0{,}4z^{-1} - 0{,}12z^{-2}}{1 + 0{,}6z^{-1} - 0{,}4z^{-2}}$$

$$\frac{Y(z)}{U(z)} = G_2(z) = \frac{1 + 0{,}4z^{-1} - 0{,}12z^{-2}}{1 + 0{,}6z^{-1} - 0{,}4z^{-2}}$$

Now, I'd like to use controllable canonical form

$$Y(z) = \underbrace{U(z) \frac{1}{1 + 0{,}6z^{-1} - 0{,}4z^{-2}}}_{W(z)} (1 + 0{,}4z^{-1} - 0{,}12z^{-2})$$

$$W(z) = U(z) \frac{1}{1 + 0{,}6z^{-1} - 0{,}4z^{-2}} \qquad Y(z) = W(z)\left(1 + 0{,}4z^{-1} - 0{,}12z^{-2}\right)$$

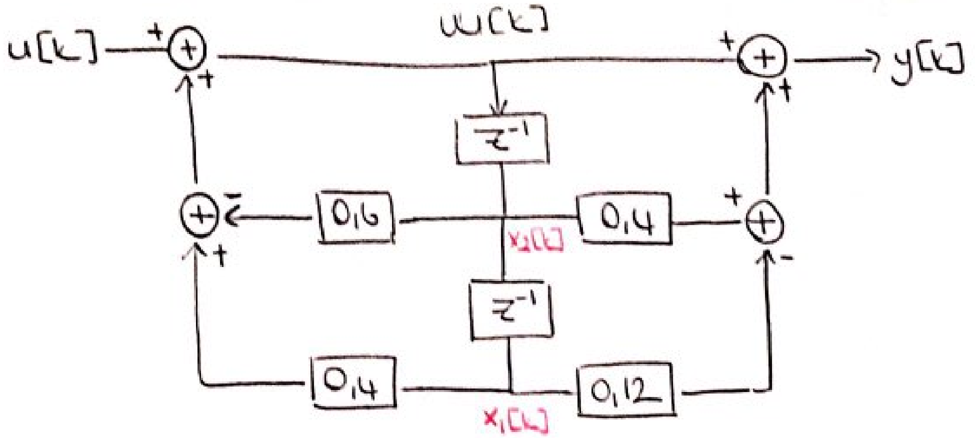
$$W(z) = U(z) - 0{,}6z^{-1}W(z) + 0{,}4z^{-2}W(z)$$

$$Y(z) = W(z) + 0{,}4z^{-1}W(z) - 0{,}12z^{-2}W(z)$$

then we have

$$w[k] = u[k] - 0{,}6w[k-1] + 0{,}4w[k-2]$$

$$y[k] = w[k] + 0{,}4w[k-1] - 0{,}12h[k-2]$$



$X_1(z) = z^{-1} X_2(z) \longrightarrow x_1[k+1] = x_2[k]$

$X_2(z) = [U(z) - 0{,}6 X_2(z) + 0{,}4 X_1(z)]\, z^{-1} \longrightarrow x_2[k+1] = u[k] - 0{,}6 x_2[k] + 0{,}4 x_1[k]$

$Y(z) = 0{,}4 X_2(z) - 0{,}12 X_1(z) + U(z) - 0{,}6 X_2(z) + 0{,}4 X_1(z)$

$Y(z) = -0{,}2 X_2(z) - 0{,}28 X_1(z) + U(z) \longrightarrow y[k] = u[k] - 0{,}28 x_1[k] - 0{,}2 x_2[k]$

Then

$$\begin{bmatrix} x_1[k+1] \\ x_2[k+1] \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0{,}4 & -0{,}6 \end{bmatrix} \begin{bmatrix} x_1[k] \\ x_2[k] \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u[k]$$

$$y[k] = [-0{,}28 \quad -0{,}2] \begin{bmatrix} x_1[k] \\ x_2[k] \end{bmatrix} + u[k]$$

The results are consistent with controllable canonical form.

c) Let's first find the relationship between $Y(z)$ & $U(z)$

$Y(z) = [U(z) - G_2(z) Y(z)]\, G_1(z)$

$$\frac{Y(z)}{U(z)} = \frac{G_1(z)}{1 + G_1(z) G_2(z)}$$

$\Rightarrow$ I don't really think that is the most practical way to solve this problem

Instead, let's use another approach. From the previous calculations, I know the relations between the input, the output and the states of each system.

$$G_1(z): \quad \underline{x}_1[k+1] = A_1 x_1[k] + B_1 u_1[k]$$
$$y_1[k] = C_1 x_1[k]$$
$$G_2(z): \quad x_2[k+1] = A_2 x_2[k] + B_2 u_2[k]$$
$$y_2[k] = C_2 x_2[k] + \underbrace{D_2}_{1} u_2[k]$$

we know every one of the matrices

Now for $G_1(z)$: input $= u[k] - y_2[k]$ output $= y_1[k]$

for $G_2(z)$: input $= y_1[k]$ output $= y_2[k]$

Define $x[k] = \begin{bmatrix} x_1[k] \\ x_2[k] \end{bmatrix} = \begin{bmatrix} x_{11}[k] \\ x_{12}[k] \\ x_{13}[k] \\ x_{21}[k] \\ x_{22}[k] \end{bmatrix}$

For our feedback system:

$$x_1[k+1] = A_1 x_1[k] + B_1 (u[k] - y_2[k])$$
$$= A_1 x_1[k] + B_1 u[k] - B_1 C_2 x_2[k] - B_1 \underbrace{u_2[k]}_{C_1 x_1[k]}$$
$$y_1[k] = C_1 x_1[k]$$
$$x_2[k+1] = A_2 x_2[k] + B_2 y_1[k] = A_2 x_2[k] + B_2 C_1 x_1[k]$$
$$y_2[k] = C_2 x_2[k] + u_2[k]$$

Then $x[k+1] = \begin{bmatrix} A_1 - B_1 C_1 & -B_1 C_2 \\ B_2 C_1 & A_2 \end{bmatrix} x[k] + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u[k]$

$$y[k] = y_1[k] = \begin{bmatrix} C_1 & 0 \end{bmatrix} x[k]$$

$$A_1 - B_1 C_1 = \begin{bmatrix} 0 & 0 & 0{,}32 \\ 1 & 0 & -1{,}52 \\ 0 & 1 & -2{,}2 \end{bmatrix} - \begin{bmatrix} 0 \\ 0{,}5 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0{,}32 \\ 1 & 0 & -1{,}52 \\ 0 & 1 & -2{,}2 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0{,}5 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0{,}32 \\ 1 & 0 & -2{,}02 \\ 0 & 1 & -3{,}2 \end{bmatrix}$$

$$B_1 C_2 = \begin{bmatrix} 0 \\ 0{,}5 \\ 1 \end{bmatrix} \begin{bmatrix} -0{,}8 & -0{,}2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -0{,}4 & -0{,}1 \\ -0{,}8 & -0{,}2 \end{bmatrix}$$

$$B_2 C_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$x[k+1] = \begin{bmatrix} 0 & 0 & 0{,}32 & 0 & 0 \\ 1 & 0 & -2{,}02 & 0{,}4 & 0{,}1 \\ 0 & 1 & -3{,}2 & 0{,}8 & 0{,}2 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0{,}4 & -0{,}6 \end{bmatrix} x[k] + \begin{bmatrix} 0 \\ 0{,}5 \\ 1 \\ 0 \\ 0 \end{bmatrix} u[k]$$

$$y[k] = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \end{bmatrix} x[k]$$

Q2 : Investigation of the matrix exponential

a) $\det(P) \neq 0 \Rightarrow P$ is invertible, P^{-1} exists

$$e^{(P^{-1}AP)t} = P^{-1} e^{At} P$$

we know that $e^{At} = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k = I + At + \frac{1}{2!} A^2 t^2 + \dots$

then $e^{(P^{-1}AP)t} = I + (P^{-1}AP)t + \frac{1}{2!} (P^{-1}AP)^2 t^2 + \dots$

$$e^{(P^{-1}AP)t} = \sum_{k=0}^{\infty} \frac{t^k}{k!} (P^{-1}AP)^k$$

Let's investigate $(P^{-1}AP)^k$ for $k \geq 1$.

$$(P^{-1}AP)^k = (P^{-1}A\underbrace{P)(P^{-1}}AP) \dots (P^{-1}A\underbrace{P)(P^{-1}}AP)$$

$$= P^{-1} A^k P$$

then $e^{(P^{-1}AP)t} = \sum_{k=0}^{\infty} \frac{t^k}{k!} (P^{-1}AP)^k = I + (P^{-1}AP)t + \frac{1}{2!} (P^{-1}A^2P) t^2 + \dots$

$$e^{(P^{-1}AP)t} = \sum_{k=0}^{\infty} \frac{t^k}{k!} P^{-1} A^k P$$

$$= P^{-1} \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k \, P = P^{-1} e^{At} P \,//$$

b) $\det(A) \neq 0$, A is invertible, A^{-1} exists

We'd like to show that

$$\left(\int_0^T e^{A\lambda} d\lambda\right) = A^{-1}(e^{AT} - I) = (e^{AT} - I)A^{-1}$$

$$e^{A\lambda} = I + A\lambda + \frac{1}{2!}A^2\lambda^2 + \dots$$

$$\int_0^T e^{A\lambda} d\lambda \Rightarrow \int_0^T I\, d\lambda + \int_0^T A\lambda\, d\lambda + \dots$$

$$= \left[I\lambda + A\frac{\lambda^2}{2} + \frac{1}{2!}A^2\frac{\lambda^3}{3} + \dots\right]_0^T = \left[\lambda + A\frac{\lambda^2}{2} + \frac{1}{3!}A^2\lambda^3 + \dots\right]_0^T$$

$$= T + A\frac{T^2}{2} + \frac{1}{3!}A^2T^3 + \dots$$

$$e^{AT} = I + AT + \frac{1}{2!}A^2T^2 + \dots \qquad e^{AT} - I = AT + \frac{1}{2!}A^2T^2 + \frac{1}{3!}A^3T^3 + \dots$$

then $\int_0^T e^{A\lambda} d\lambda = A^{-1}\left[AT + \frac{1}{2!}A^2T^2 + \frac{1}{3!}A^3T^3 + \dots\right]$

$$\Rightarrow \int_0^T e^{A\lambda} d\lambda = A^{-1}[e^{AT} - I] = [e^{AT} - I]A^{-1} //$$

$$[e^{AT} - I]A^{-1} = \left[AT + \frac{1}{2!}A^2T^2 + \frac{1}{3!}A^3T^3 + \dots\right]A^{-1} = T + \frac{1}{2!}AT^2 + \frac{1}{3!}A^2T^3 + \dots$$

$$\Rightarrow (e^{AT} - I)A^{-1} = A^{-1}(e^{AT} - I)$$

c) For a matrix A

eigenvalues $\Rightarrow$ $\det(\lambda I - A) = 0$ $\quad \boxed{\lambda}$: eigenvalues

eigenvectors $\Rightarrow$ $(A - \lambda I)\vartheta = 0$ $\quad \boxed{\vartheta}$: eigenvectors

we know that $A\vartheta = \lambda\vartheta$

$$e^{At} = I + At + \frac{1}{2!}A^2t^2 + \dots = \sum_{k=0}^{\infty}\frac{t^k}{k!}A^k$$

$$A\vartheta = \lambda\vartheta$$

$$A^2\vartheta = A(A\vartheta) = A\lambda\vartheta = \lambda A\vartheta = \lambda\lambda\vartheta = \lambda^2\vartheta$$

Similarly $A^3\vartheta = \lambda^3\vartheta$ $\quad A^4\vartheta = \lambda^4\vartheta \dots$

then define the eigenvalues $\tilde{\lambda}$ & the eigenvectors $\tilde{\vartheta}$ for e^{At}.

$$e^{At}\tilde{\vartheta} = \tilde{\lambda}\tilde{\vartheta} \Rightarrow \sum_{k=0}^{\infty}\frac{t^k}{k!}A^k\tilde{\vartheta}$$

Observe $\Rightarrow$ if $\tilde{\vartheta} = \vartheta$ then $A^k \tilde{\vartheta} = A^k \vartheta = \lambda^k \vartheta$

$$\sum_{k=0}^{\infty} \frac{t^k}{k!} A^k \tilde{\vartheta} = \sum_{k=0}^{\infty} \frac{t^k}{k!} \lambda^k \vartheta = \tilde{\lambda} \tilde{\vartheta}$$

$\tilde{\lambda} = \sum_{k=0}^{\infty} \frac{t^k}{k!} \lambda^k$ $\quad \tilde{\vartheta} = \vartheta$ are good candidates for eigenvalues & eigenfunctions of e^{At}

$\tilde{\lambda} = e^{\lambda t}$ & $\tilde{\vartheta} = \vartheta$ //

d) $e^{At} = \mathcal{L}^{-1}\{[sI - A]^{-1}\}$

$$A = \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix} \qquad sI - A = \begin{bmatrix} s-\sigma & -\omega \\ \omega & s-\sigma \end{bmatrix}$$

$$[sI - A]^{-1} = \frac{1}{(s-\sigma)^2 + \omega^2} \begin{bmatrix} s-\sigma & \omega \\ -\omega & s-\sigma \end{bmatrix} = \begin{bmatrix} \frac{s-\sigma}{(s-\sigma)^2 + \omega^2} & \frac{\omega}{(s-\sigma)^2 + \omega^2} \\ \frac{-\omega}{(s-\sigma)^2 + \omega^2} & \frac{s-\sigma}{(s-\sigma)^2 + \omega^2} \end{bmatrix}$$

$$e^{At} = \mathcal{L}^{-1}\{[sI - A]^{-1}\} = \begin{bmatrix} e^{\sigma t}\cos\omega t & e^{\sigma t}\sin\omega t \\ -e^{\sigma t}\sin\omega t & e^{\sigma t}\cos\omega t \end{bmatrix}$$

e) $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\quad$ eigenvalues $\Rightarrow$ $\det|\lambda I - A| = 0$

$$\begin{vmatrix} \lambda & -1 \\ -1 & \lambda \end{vmatrix} = 0 \qquad \lambda^2 - 1 = 0$$

$$\left.\begin{aligned} \lambda_1 &= 1 \\ \lambda_2 &= -1 \end{aligned}\right\} \text{two different distinct eigenvalues}$$

A is diagonalizable.

eigenvectors: $\lambda_1 = 1$ $\quad (A - \lambda_1 I)\vartheta_1 = 0$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \vartheta_{11} \\ \vartheta_{12} \end{bmatrix} = 0 \qquad -\vartheta_{11} + \vartheta_{12} = 0 \qquad \bar{\vartheta}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\lambda_2 = -1$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \vartheta_{21} \\ \vartheta_{22} \end{bmatrix} = 0 \qquad \vartheta_{21} + \vartheta_{22} = 0 \qquad \bar{\vartheta}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

define $\quad P = [\bar{\vartheta}_1 \quad \bar{\vartheta}_2] = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ $\qquad P^{-1} = \frac{-1}{2} \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$

then $e^{At} = P \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} P^{-1}$ due to $P^{-1}AP = D$

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{-1}{2} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = D$$

$$D = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{-1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \rightarrow \text{diagonal}$$

$$e^{At} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} e^{t} & 0 \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{-1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} e^{t} & e^{-t} \\ e^{t} & -e^{-t} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{-1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}e^{t} + \frac{1}{2}e^{-t} & \frac{1}{2}e^{t} - \frac{1}{2}e^{-t} \\ \frac{1}{2}e^{t} - \frac{1}{2}e^{-t} & \frac{1}{2}e^{t} + \frac{1}{2}e^{-t} \end{bmatrix} //$$

Q3: Discretization of CT State-Space Equations

$$\dot{x}(t) = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}}_{A} x(t) + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u(t)$$

a) From the discretization procedure

$x[k+1] = Gx[k] + Hu[k]$ where $u(t) = u[k]$, $t \in (kT, (k+1)T)$

$G = e^{AT}$ $\quad H = \left(\int_0^T e^{A\lambda} d\lambda\right) B$

$$e^{At} = \mathcal{L}^{-1}\left[[sI-A]^{-1}\right]$$

$$= \mathcal{L}^{-1}\left\{\begin{bmatrix} s & -1 \\ 0 & s+1 \end{bmatrix}^{-1}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s(s+1)}\begin{bmatrix} s+1 & 1 \\ 0 & s \end{bmatrix}\right\}$$

$$= \mathcal{L}^{-1}\left\{\begin{bmatrix} \frac{1}{s} & \frac{1}{s(s+1)} \\ 0 & \frac{1}{s+1} \end{bmatrix}\right\} = \begin{bmatrix} 1 & 1-e^{-t} \\ 0 & e^{-t} \end{bmatrix}$$

then $e^{AT} = \begin{bmatrix} 1 & 1-e^{-T} \\ 0 & e^{-T} \end{bmatrix} = G$

$$H = \left(\int_0^T e^{A\lambda} d\lambda\right) B = \left(\int_0^T \begin{bmatrix} 1 & 1-e^{-\lambda} \\ 0 & e^{-\lambda} \end{bmatrix} d\lambda\right) \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \int_0^T \begin{bmatrix} 1-e^{-\lambda} \\ e^{-\lambda} \end{bmatrix} d\lambda$$

$$H = \begin{bmatrix} \lambda + e^{-\lambda} \\ -e^{-\lambda} \end{bmatrix}_0^T = \begin{bmatrix} T + e^{-T} - 1 \\ -e^{-T} + 1 \end{bmatrix}$$

then $x[k+1] = \begin{bmatrix} 1 & 1-e^{-T} \\ 0 & e^{-T} \end{bmatrix} x[k] + \begin{bmatrix} T+e^{-T}-1 \\ -e^{-T}+1 \end{bmatrix} u[k]$

b) Now we have the approximation

$$e^{AT} = I + AT$$

Then $e^{AT} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & T \\ 0 & -T \end{bmatrix} = \begin{bmatrix} 1 & T \\ 0 & 1-T \end{bmatrix}$

$$\tilde{G} = e^{AT}$$

$$\tilde{H} = \left(\int_0^T e^{A\lambda} d\lambda\right) B = \left(\int_0^T \begin{bmatrix} 1 & \lambda \\ 0 & 1-\lambda \end{bmatrix} d\lambda\right) \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \int_0^T \begin{bmatrix} \lambda \\ 1-\lambda \end{bmatrix} d\lambda = \begin{bmatrix} \frac{\lambda^2}{2} \\ \lambda - \frac{\lambda^2}{2} \end{bmatrix}_0^T = \begin{bmatrix} \frac{T^2}{2} \\ T - \frac{T^2}{2} \end{bmatrix}$$

$$x[k+1] \approx \begin{bmatrix} 1 & T \\ 0 & 1-T \end{bmatrix} x[k] + \begin{bmatrix} T^2/2 \\ T - T^2/2 \end{bmatrix} u[k]$$

c) Comment:

$T = 0,1$ $\quad x[k+1] = \begin{bmatrix} 1 & 1-e^{0,1} \\ 0 & e^{-0,1} \end{bmatrix} x[k] + \begin{bmatrix} 0,1 + e^{-0,1} - 1 \\ -e^{-0,1} + 1 \end{bmatrix} u[k]$

$$= \begin{bmatrix} 1 & 0,095 \\ 0 & 0,905 \end{bmatrix} x[k] + \begin{bmatrix} 0,004 \\ 0,095 \end{bmatrix} u[k]$$

$$x[k+1] \approx \begin{bmatrix} 1 & 0,1 \\ 0 & 0,9 \end{bmatrix} x[k] + \begin{bmatrix} 0,01 \\ 0,09 \end{bmatrix} u[k]$$

for $T = 0,1$ the approximation gives a result very close the real value. At this value of T, approximation can be used.

$T=1$ $\quad x[k+1] = \begin{bmatrix} 1 & 1-e^{-1} \\ 0 & e^{-1} \end{bmatrix} x[k] + \begin{bmatrix} 1+e^{-1}-1 \\ -e^{-1}+1 \end{bmatrix} u[k]$

$$x[k+1] = \begin{bmatrix} 1 & 0{,}632 \\ 0 & 0{,}368 \end{bmatrix} x[k] + \begin{bmatrix} 0{,}368 \\ 0{,}632 \end{bmatrix} u[k] = G\,x[k] + H\,u[k]$$

$$x[k+1] \approx \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} x[k] + \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} u[k] = \tilde{G}\,x[k] + \tilde{H}\,u[k]$$

for $T=1$ the approximation is not close to the real matrices of the system. Therefore we can say that the $e^{AT} = I + AT$ approximation does not give an approximation to our system as T gets larger.

For large T, $e^{AT} = I + AT$ approximation cannot be used for our system.

$$e^{AT} = I + AT + \underbrace{\frac{1}{2!}A^2T^2 + \frac{1}{3!}A^3T^3 + \dots}$$

as T gets larger the effects of the remaining terms increase. Therefore they cannot be omitted.

& for $T<1 \Rightarrow T > T^2 > T^3$

However for $T>1 \Rightarrow T^3 > T^2 > T \Rightarrow$ their effect became more important.

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Q4 : Stability

a) DT system

$$x[k+1] = \begin{bmatrix} 0 & 1 \\ \alpha & 2\alpha - \frac{1}{2} \end{bmatrix} x[k] + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u[k]$$

i. Asymptotic Stability

when $u[k]=0$ & $\forall x(0) \in \mathbb{R}^n$ if we have $\lim_{k\to\infty} \|x[k]\| = 0$

then our system is asymptotic stable

$u[k]=0$ $\quad x[k+1] = \underbrace{\begin{bmatrix} 0 & 1 \\ \alpha & 2\alpha - \frac{1}{2} \end{bmatrix}}_{G} x[k]$

I'd like to first find out whether G matrix is diagonalizable or not.

$$\det|\lambda I - G| = \begin{vmatrix} \lambda & -1 \\ -\alpha & \lambda + \frac{1}{2} - 2\alpha \end{vmatrix} = \lambda^2 + \frac{\lambda}{2} - 2\alpha\lambda - \alpha$$

$= \lambda^2 + \lambda\left(\frac{1}{2} - 2\alpha\right) - \alpha = 0$ $\Rightarrow$ $\lambda_1 = \frac{-1}{2}$ $\quad \lambda_2 = 2\alpha$ $\quad$ G is diagonalizable

$\lambda \quad -2\alpha$

$\lambda \quad \frac{1}{2}$

when $2\alpha \neq -\frac{1}{2}$ $\quad \alpha \neq -\frac{1}{4}$

We need to check $\alpha = -\frac{1}{4}$ for whether it is diagonalizable there or not

$$G = P \begin{bmatrix} -1/2 & 0 \\ 0 & 2\alpha \end{bmatrix} P^{-1}$$

let's find the eigenvalues

$(G - \lambda I)\vartheta = 0$

$(G + \frac{1}{2} I)\bar{\vartheta}_1 = 0$ $\quad \begin{bmatrix} \frac{1}{2} & 1 \\ \alpha & 2\alpha \end{bmatrix} \begin{bmatrix} \vartheta_{11} \\ \vartheta_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad \frac{\vartheta_{11}}{2} + \vartheta_{12} = 0$, $\alpha\vartheta_{11} + 2\alpha\vartheta_{12} = 0$ $\quad \bar{\vartheta}_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$(G - 2\alpha I)\bar{\vartheta}_2 = 0$ $\quad \begin{bmatrix} -2\alpha & 1 \\ \alpha & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} \vartheta_{21} \\ \vartheta_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad -2\alpha\vartheta_{21} + \vartheta_{22} = 0$, $\alpha\vartheta_{21} - \frac{1}{2}\vartheta_{22} = 0$

$$\bar{\vartheta}_2 = \begin{bmatrix} 1 \\ 2\alpha \end{bmatrix}$$

then $P = \begin{bmatrix} \bar{\vartheta}_1 & \bar{\vartheta}_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & 2\alpha \end{bmatrix}$

$$P^{-1} = \frac{1}{4\alpha+1} \begin{bmatrix} 2\alpha & -1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} \frac{2\alpha}{4\alpha+1} & \frac{-1}{4\alpha+1} \\ \frac{1}{4\alpha+1} & \frac{2}{4\alpha+1} \end{bmatrix}$$

$$G = P \begin{bmatrix} -1/2 & 0 \\ 0 & 2\alpha \end{bmatrix} P^{-1}$$

$$= \begin{bmatrix} -1 & 2\alpha \\ \frac{1}{2} & 4\alpha^2 \end{bmatrix} \begin{bmatrix} \frac{2\alpha}{4\alpha+1} & \frac{-1}{4\alpha+1} \\ \frac{1}{4\alpha+1} & \frac{2}{4\alpha+1} \end{bmatrix} = \begin{bmatrix} 0 & \frac{1+4\alpha}{4\alpha+1} \\ \frac{\alpha+4\alpha^2}{4\alpha+1} & \frac{-1/2+8\alpha^2}{4\alpha+1} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ \alpha & 2\alpha-\frac{1}{2} \end{bmatrix}$$

$\Rightarrow$ Our asymptotic condition is $|\lambda_i|<1$ for $\forall i$

$\lambda_1 = \frac{-1}{2}$ $\quad |\lambda_1| < 1$ ✓

$\lambda_2 = 2\alpha$ $\quad |2\alpha| < 1$ $\Rightarrow$ $\boxed{-\frac{1}{2} < \alpha < \frac{1}{2}}$

However around $\alpha = -\frac{1}{4}$ there could be a problem.

when $\alpha = -\frac{1}{4}$ $\quad P = \begin{bmatrix} 2 & 1 \\ -1 & -\frac{1}{2} \end{bmatrix}$ $\quad \det(P) = -1+1 = 0$

$P \rightarrow$ not invertible

$G \rightarrow$ not diagonalizable when $\alpha = -\frac{1}{4}$

then our condition becomes $\Rightarrow$ $\boxed{-\frac{1}{2} < \alpha < \frac{1}{2},\ \alpha \neq -\frac{1}{4}}$

ii. BIBO Stability

If the poles of $H(z) = C[zI-G]^{-1}H + D$ are strictly inside the unit circle $\Rightarrow$ BIBO stable

$x[k+1] = Gx[k] + Hu[k]$

$y[z] = Cx[k]$

$$H(z) = C(zI - G)^{-1}H + D$$

$$zI - G = \begin{bmatrix} z & -1 \\ -\alpha & z - 2\alpha + \frac{1}{2} \end{bmatrix}$$

$$(zI - G)^{-1} = \frac{1}{z^2 - 2\alpha z + \frac{z}{2} - \alpha} \begin{bmatrix} z - 2\alpha + \frac{1}{2} & 1 \\ \alpha & z \end{bmatrix}$$

$$C(zI - G)^{-1} = \begin{bmatrix} -2 & 1 \end{bmatrix} \frac{1}{z^2 - 2\alpha z + \frac{z}{2} - \alpha} \begin{bmatrix} z - 2\alpha + \frac{1}{2} & 1 \\ \alpha & z \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-2z + 4\alpha - 1 + \alpha}{z^2 - 2\alpha z + \frac{z}{2} - \alpha} & \frac{-2 + z}{z^2 - 2\alpha z + \frac{z}{2} - \alpha} \end{bmatrix}$$

$$H(z) = \begin{bmatrix} \frac{-2z + 5\alpha - 1}{z^2 - 2\alpha z + \frac{z}{2} - \alpha} & \frac{z - 2}{z^2 - 2\alpha z + \frac{z}{2} - \alpha} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{z - 2}{z^2 - 2\alpha z + \frac{z}{2} - \alpha} \end{bmatrix}$$

poles $\Rightarrow \left(z + \frac{1}{2}\right)(z - 2\alpha) \Rightarrow z_1 = -\frac{1}{2} \quad z_2 = 2\alpha$

Stability (BIBO) $\Rightarrow |z_1| < 1 \quad |z_2| < 1 \Rightarrow |2\alpha| < 1$

$\left|-\frac{1}{2}\right| < 1$ ✓ $\qquad \boxed{-\frac{1}{2} < \alpha < +\frac{1}{2}}$

& Comment: We already knew it was BIBO Stable since asymptotic stability implies BIBO stability.

Observe when $\alpha = 1 \Rightarrow$ pole-zero cancellation happens

Then $\boxed{-\frac{1}{2} < \alpha < \frac{1}{2}, \quad \alpha = 1}$

b) $\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ \alpha & 2\alpha - \frac{1}{2} \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$

$y(t) = \begin{bmatrix} -2 & 1 \end{bmatrix} x(t)$

i) Asymptotic Stability

$u(t) = 0 \Rightarrow \dot{x}(t) = A\,x(t)$

$\det(\lambda I - A) = \begin{vmatrix} \lambda & -1 \\ -\alpha & \lambda - 2\alpha + \frac{1}{2} \end{vmatrix} = 0$ $\quad \lambda_1 = -\frac{1}{2} \quad \lambda_2 = 2\alpha$

$\alpha = -\frac{1}{4} \Rightarrow$ A may be undiagonalizable

then $A = P \begin{bmatrix} -1/2 & 0 \\ 0 & 2\alpha \end{bmatrix} P$

$P = \begin{bmatrix} 2 & 1 \\ -1 & 2\alpha \end{bmatrix} \qquad P^{-1} = \begin{bmatrix} \frac{2\alpha}{4\alpha+1} & \frac{1}{4\alpha+1} \\ \frac{1}{4\alpha+1} & \frac{2}{4\alpha+1} \end{bmatrix}$ $\quad$ since $A = G$

Now our asymptotic stable condition is $\lambda_i < 0$

$\lambda_1 = -\frac{1}{2} < 0 \qquad \lambda_2 = 2\alpha < 0 \qquad \alpha < 0$

when $\alpha = -\frac{1}{4}$ A is no longer diagonalizable

$\boxed{\alpha < 0, \ \alpha \neq -\frac{1}{4}}$

ii) BIBO Stability $\Rightarrow$ poles located at left half plane

$H(s) = C(sI - A)^{-1}B + D$ $\quad$ observe every matrices are the same with part a (change $H(z)$ to $H(s)$)

$H(s) = \dfrac{s-2}{s^2 - 2\alpha s + \frac{s}{2} - \alpha}$

poles $\Rightarrow$ $s_1 = -\frac{1}{2} \quad s_2 = 2\alpha \quad s_1 < 0$ ✓ $\quad s_2 < 0$ --- $2\alpha < 0 \quad \alpha < 0$

when $\alpha = 1 \Rightarrow$ pole-zero cancellation happens

$\boxed{\alpha < 0 \ , \ \alpha = 1}$ //