EE402 Discrete Time Systems
MP-2

1. (a) $x[k] = 9k2^k - 4^k + 3$

It is known that one-sided Z-Transform of $x[k] = 1$ is $X(z) = \mathcal{Z}\{1\} = \dfrac{1}{1-z^{-1}}$ using the properties and linearity of z-transform

- $\mathcal{Z}\{a^k x[k]\} = X(z/a)$
- $\mathcal{Z}\{kx[k]\} = -z\frac{d}{dz}X(z)$

The z-transform of given function can be found as follows;

$$\mathcal{Z}\{a^k\} = \frac{1}{1-az^{-1}}$$

$$X(z) = -9z\frac{d}{dz}(\frac{1}{1-2z^{-1}}) - \frac{1}{1-4z^{-1}} + 3\frac{1}{1-z^{-1}}$$

$$\boxed{X(z) = \frac{18z^{-1}}{(1-2z^{-1})^{-2}} - \frac{1}{1-4z^{-1}} + 3\frac{1}{1-z^{-1}}}$$

(b) $x[k] = \sum_{h=0}^{k} a^h$ where a is a constant

In this part, we can use, causal shifting property of Z-transform, i.e.,

- $\mathcal{Z}\{x[k-N]\} = z^N X(z)$

Thus, the z-transform of given function can be found as follows;

$$x[k-1] = \sum_{h=0}^{k-1} a^{h-1}$$

$$x[k] - x[k-1] = a^k$$

$$X(z) - z^{-1}X(z) = \mathcal{Z}\{a^k\}$$

$$X(z)(1-z^{-1}) = \mathcal{Z}\{a^k\}$$

$$X(z) = \frac{\mathcal{Z}\{a^k\}}{(1-z^{-1})}$$

$$\mathcal{Z}\{a^k\} = \frac{1}{1-az^{-1}}$$

$$\boxed{X(z) = \frac{1}{(1-z^{-1})(1-az^{-1})}}$$

(c) $x[k] = k(k-1)...(k-h+1)a^{k-h}$

- for $h = 0$, $x[k] = a^k$, $\mathcal{Z}\{x[k]\} = X_0(z) = \frac{1}{1-az^{-1}}$
- for $h = 1$, $x[k] = ka^{k-1}$, $\mathcal{Z}\{x[k]\} = X_1(z) = a^{-1}(-z\frac{d}{dz})X_0(z) = \frac{z^{-1}}{(1-az^{-1})^2}$
- for $h = 2$, $x[k] = k(k-1)a^{k-2}$, $\mathcal{Z}\{x[k]\} = X_2(z) = a^{-1}(-z\frac{d}{dz}-1)X_1(z) = \frac{z^{-1}(1+az^{-1})}{(1+az^{-1})^3}$
- ...

iteratively, it can be observed that

$$\boxed{X(z) = \frac{z^{-h}}{(1-az^{-1})^{h+1}}h!}$$

(d) It can be seen from the given graph at the mini-project that, the function $x[n]$ can be written in terms of ramp functions $r[k] = k$;

$$\boxed{x[k] = r[k-2] - r[k-5]}$$

using the linearity and casual time shift properties of z-transform

- $\mathcal{Z}\{x[k-N]\} = z^N X(z)$

and complex differentiation theorem , i.e.,

- $\mathcal{Z}\{kx[k]\} = -z\frac{d}{dz}X(z)$

The z-transform of given function can be found as follows;

$$X(z) = z^{-2}\frac{z^{-1}}{(1-z^{-1})^2} - z^{-5}\frac{z^{-1}}{(1-z^{-1})^2}$$

$$\boxed{X(z) = \frac{z^{-3}(1-z^{-3})}{(1-z^{-1})^2}}$$

(e)

$$X(s) = \frac{1-e^{Ts}}{s}\frac{1}{(s+a)^2} = \frac{1-e^{Ts}}{s}G(s)$$

$$x(t) = \mathcal{L}^{-1}\{(\frac{1-e^{Ts}}{s})G(s)\} = \mathcal{L}^{-1}\{(\frac{1}{s})G(s)\} - \mathcal{L}^{-1}\{(\frac{e^{Ts}}{s})G(s)\}$$

Lets as assume $\hat{G}(s) = \frac{G(s)}{s}$,

$$x(t) = \hat{g}(t) - \hat{g}(t-T)$$

$$\mathcal{Z}\{x(kT)\} = \mathcal{Z}\{\hat{g}(kT) - \hat{g}(kT-T) = \mathcal{Z}\{\hat{g}[k] - \hat{g}[k-1]\} = (1-z^{-1})\hat{G}(z)$$

$$\hat{G}(z) = \mathcal{Z}\{\mathcal{L}^{-1}\{\frac{G(s)}{s}\}\} = \mathcal{Z}\{\mathcal{L}^{-1}\{\frac{1}{s(s+a)^2}\}\}$$

To find inverse Laplace transform, we can use partial fraction expansion for the expression,

$$\frac{1}{s(s+a)^2} = \frac{a_1}{s} + \frac{a_2}{s+a} + \frac{a_3}{(s+a)^2}$$

With some calculation, coefficients are found to be,

$$\boxed{a_1 = \frac{1}{a^2}}, \boxed{a_2 = -\frac{1}{a^2}}, \boxed{a_3 = -\frac{1}{a}}$$

Thus,

$$\hat{g}(t) = a_1 + a_2 e^{-at} + a_3 t e^{-t}$$

$$\hat{g}(t) = \frac{1}{a^2} - \frac{1}{a^2} e^{-at} - \frac{1}{a} t e^{-t}$$

$$\hat{G}(z) = \mathcal{Z}\{\frac{1}{a^2} - \frac{1}{a^2} e^{-akT} - \frac{1}{a} kT e^{-kT}\}$$

$$\hat{G}(z) = \frac{1}{a^2}\frac{1}{1-z^{-1}} - \frac{1}{a^2}\frac{1}{1-e^{-aT}z^{-1}} - \frac{1}{a}\frac{Te^{-aT}z^{-1}}{(1-e^{-aT}z^{-1})^2}$$

$$X(z) = (1-z^{-1})(\frac{1}{a^2}\frac{1}{1-z^{-1}} - \frac{1}{a^2}\frac{1}{1-e^{-aT}z^{-1}} - \frac{1}{a}\frac{Te^{-aT}z^{-1}}{(1-e^{-aT}z^{-1})^2})$$

$$\boxed{X(z) = (\frac{1}{a^2} - \frac{1}{a^2}\frac{(1-z^{-1})}{1-e^{-aT}z^{-1}} - \frac{1}{a}\frac{(1-z^{-1})(Te^{-aT}z^{-1})}{(1-e^{-aT}z^{-1})^2})}$$

2.

$$X(z) = \frac{z^{-1}}{(1-z^{-1})(1+1.3z^{-1}+0.4z^{-2})^2}$$

$$X(z) = \frac{z^4}{(z-1)(z^2+1.3+0.4z^2)^2}$$

(a) Using partial fraction expansion,

$$\frac{X(z)}{z} = \frac{a_1}{z-1} + \frac{a_2}{(z+0.5)^2} + \frac{a_3}{z+0.5} + \frac{a_4}{(z+0.8)^2} + \frac{a_5}{z+0.8}$$

$$a_1 = lim_{z\to 1}\frac{X(z)(z-1)}{z} = \frac{100}{729}$$

$$a_2 = lim_{z\to -0.5}\frac{X(z)(z+0.5)^2}{z} = \frac{100}{729}$$

$$a_3 = lim_{z\to -0.5}\frac{X(z)(z+0.5)}{z} = \frac{-100}{9}$$
$$a_4 = lim_{z\to -0.8}\frac{X(z)(z+0.8)^2}{z} = \frac{-320}{81}$$
$$a_5 = lim_{z\to -0.8}\frac{X(z)(z+0.8)}{z} = \frac{8000}{729}$$

Thus, $x[k]$ becomes

$$\boxed{x[k] = \frac{100}{729} - \frac{50}{27}k(-0.5)^k - \frac{100}{9}(-0.5)^k - \frac{320}{81}k(-0.8)^k + \frac{8000}{729}(-0.8)^k}$$

Matlab code simulating the result can be found at Appendix A, at lines 46-54.

(b) Matlab code simulating the result can be found at Appendix A, at lines 1-45.

(c) Matlab code simulating the result can be found at Appendix A, at lines 56-71.

(d) Matlab code simulating the result can be found at Appendix A, at lines 72-80.

(e) Matlab code simulating the result can be found at Appendix A, at lines 81-126. The resulting $x[k]s$ can be seen at *Figure 1*.

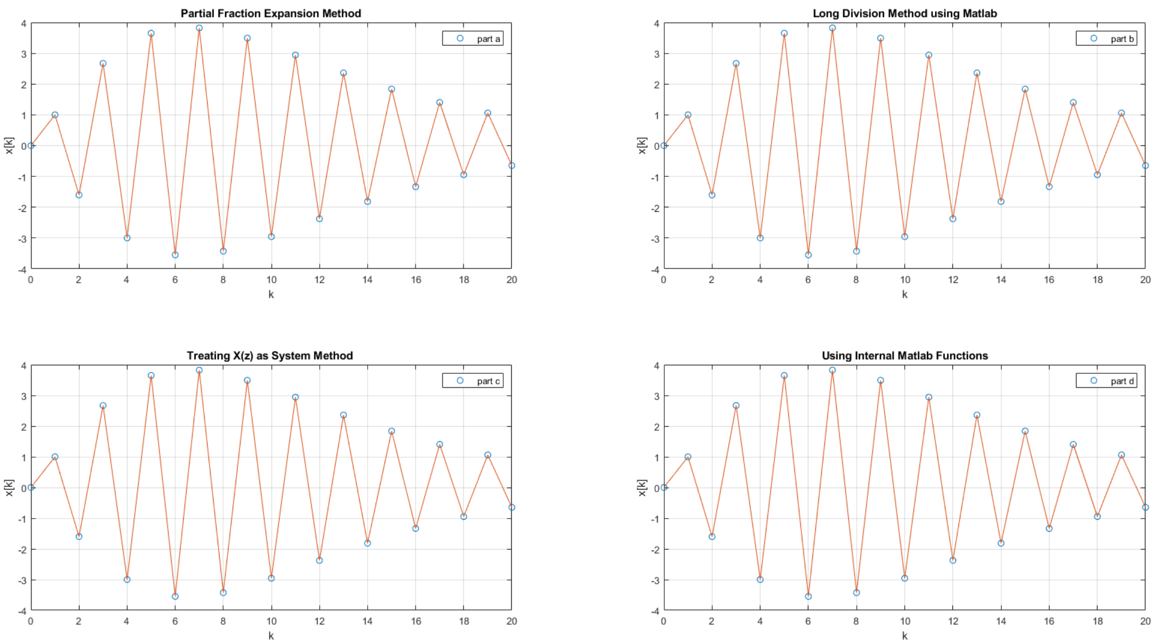

Figure 1: $x[k]$ plots using different methodologies

3. $y[k] - 7y[k-1] + 12y[k-2] = x[k]$, $y[k] = 0$ for $k < 0$ and $x[k] = 52^k + k$, $x[k] = 0$ for $k < 0$

(a) Assume $x[k] = 0$ to find homogeneous solution, it is known that, $y_h[k]$ will be in the form A^k.

$$A^k - 7A^{k-1} + 12A^{k-2} = 0$$
$$A^{k-2}\,(\,(A-4)(A-3)) = 0$$

Thus, it can be observed that, $y_h[k]$ will be in the form $a_1 4^k + a_2 3^k$.
To find the the $y_p[k]$, assume it to be in a form close to $x[n]$,i.e., in a form of $b_1 2^k + b_2 k + b_3$. Coefficients can be found by inserting it into the original equation.

$$b_1 2^k + b_2 k + b_3 - 7[b_1 2^{k-1} + b_2(k-1) + b_3] + 12[b_1 2^{k-2} + b_2(k-2) + b_3] = 52^k + k$$

by some arrangement in the equation, coefficients can be found as

$$\boxed{b_1 = 10}\,,\ \boxed{b_2 = \frac{1}{6}}\,,\ \boxed{b_3 = \frac{17}{36}}$$

Thus, $y[n]$ will be in the form $a_1 4^k + a_2 3^k + 102^k + \frac{1}{6}k + \frac{17}{36}$, to find the unknown coefficients, original equation can be iterated starting from $k = 0$.

$$y[0] = 7y[-1] - 12y[-2] + x[0] = 0 - 0 + 5 = a_1 + a_2 + 10 + \frac{17}{36}$$
$$y[1] = 7y[0] - 12y[-1] + x[1] = 35 - 0 + 11 = 4a_1 + 3a_2 + 20 + \frac{1}{6} + \frac{17}{36}$$

handling the algebraic equations, unknown coefficient can be found to be as

$$\boxed{a_1 = \frac{376}{9}}\,,\ \boxed{a_2 = \frac{-189}{4}}$$

$$\boxed{y[k] = \frac{376}{9}4^k - \frac{189}{4}3^k + 102^k + \frac{1}{6}k + \frac{17}{36}}$$

Matlab code simulating the result can be found at Appendix B, at lines 1-12.

(b) The difference equation can be transferred the z-domain in the following form,

$$Y(z)\left[1 - 7z^{-1} + 12z^{-2}\right] = X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - 7z^{-1} + 12z^{-2}}$$

with

$$X(z) = 5\frac{1}{1 - 2z^{-1}} + \frac{z^{-1}}{(1 - z^{-1})^2}$$

$Y(z)$ can be found as,

$$\begin{aligned} Y(z) &= \frac{z^{-1}}{(1-2z^{-1})^2(1-7z^{-1}+12z^{-2})} + \frac{5}{(1-2z^{-1})(1-7z^{-1}+12z^{-2})} \\ &= \frac{z^3}{(z-3)(z-4)(z-1)^2} + \frac{5z^3}{(z-3)(z-4)(z-2)} \end{aligned} \tag{1}$$

using partial fraction expansion

$$Y(z) = \left[\frac{a_1}{z-3} + \frac{a_2}{z-4} + \frac{a_3}{(z-1)^2} + \frac{a_4}{z-1}\right] + \left[\frac{b_1}{z-3} + \frac{b_2}{z-4} + \frac{b_3}{z-2}\right]$$

with some calculation coefficients can be found to be

$$\boxed{a_1 = -9/4}\ ,\ \boxed{a_2 = 16/9}\ ,\ \boxed{a_3 = 1/6}\ ,\ \boxed{a_4 = 17/36}$$

$$\boxed{b_1 = -45}\ ,\ \boxed{b_2 = 40}\ ,\ \boxed{b_3 = 10}$$

Then, $y[k]$ becomes

$$\boxed{y[k] = \frac{-189}{4}3^k + \frac{376}{9}4^k + 102^k + \frac{1}{6}k + \frac{17}{36}}$$

Matlab code simulating the result can be found at Appendix B, at lines 13-24.

(c) Matlab code simulating the result can be found at Appendix B, at lines 25-34.

(d) Matlab code simulating the result can be found at Appendix B, at lines 36-68.The resulting $x[k]s$ can be seen at *Figure 2*.

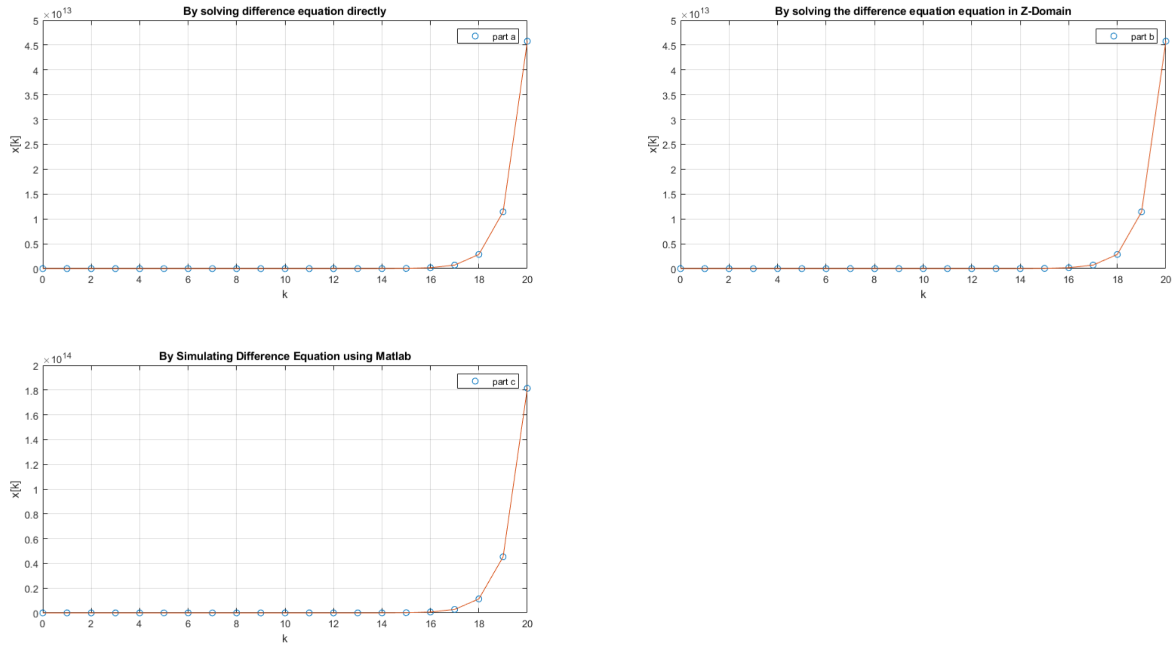

Figure 2: $y[k]$ plots using different methodologies

4.

$$G(z) = \frac{4 - 0.02z^{-1} + 0.3z^{-2} + 0.02z^{-3}}{2 - 6z^{-1} + 2z^{-2} - z^{-3} + z^{-4}} = \frac{Y(z)}{X(z)}$$

Assume a middle variable $H(z)$

$$G_1(z) = \frac{H(z)}{X(z)}$$

$$G_2(z) = \frac{Y(Z)}{H(z)}$$

with

$$G_1(z) = 4 - 0.02z^{-1} + 0.3z^{-2} + 0.02z^{-3}$$
$$G_2(z) = 2 - 6z^{-1} + 2z^{-2} - z^{-3} + z^{-4}$$

from there, it can be observed that if the inverse z-transform is performed on both equations,

$$\boxed{x[k] = 2h[k] - 6h[k-1] + 2h[k-2] - 8h[k-3] + 0.2h[k-4]}$$

$$\boxed{y[k] = 4h[k] - 0.02h[k-1] + 0.3h[k-2] + 0.02h[k-3]}$$

The resulting block diagram constructed at Simulink can be seen at *Figure* ??

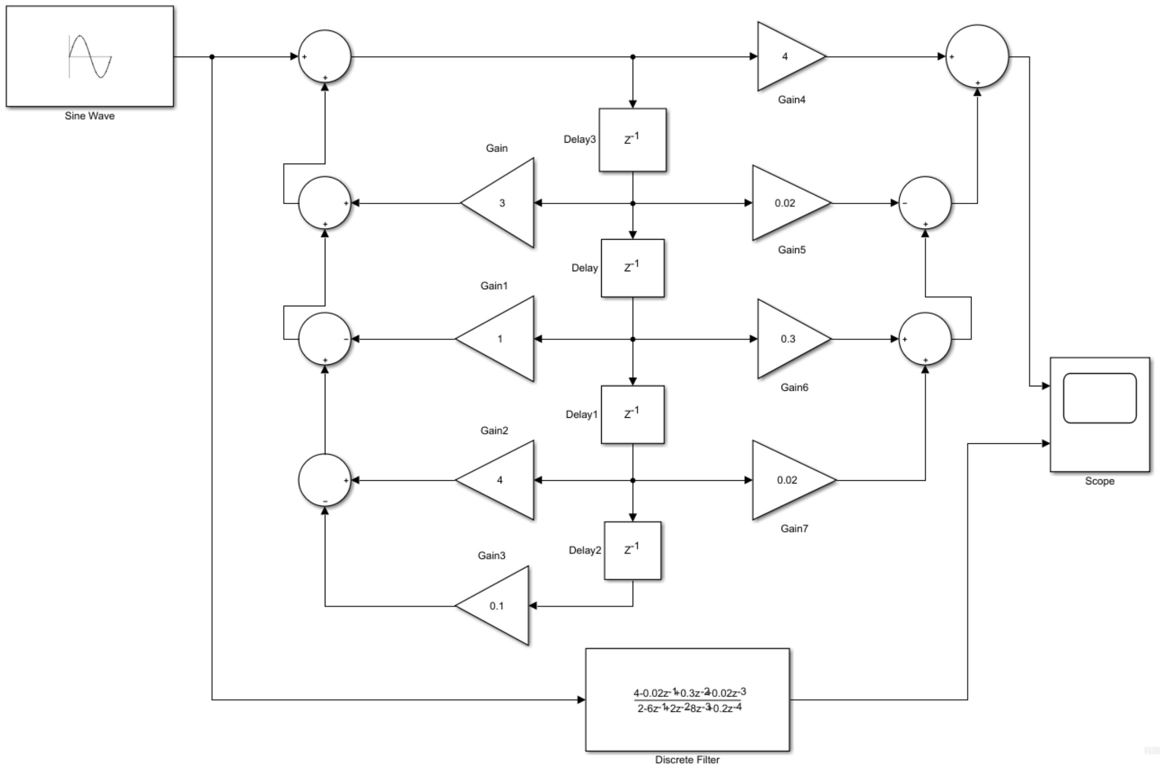

Figure 3: Block Diagram of system given at question 4

Differences in the simulation results can be observed form *Figure 4 to 5.*

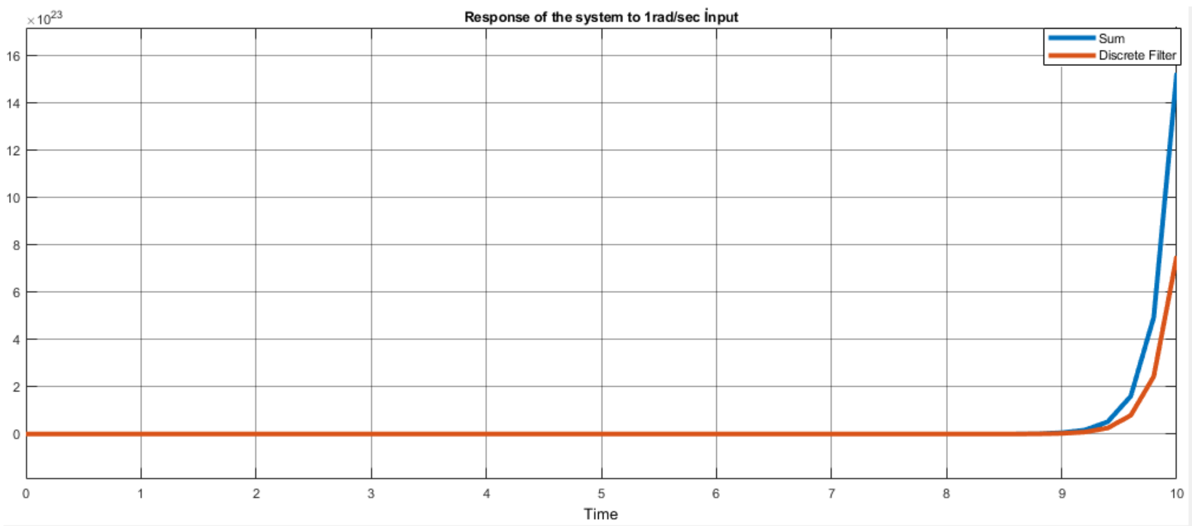

Figure 4: Response of the system to 1rad/sec Input

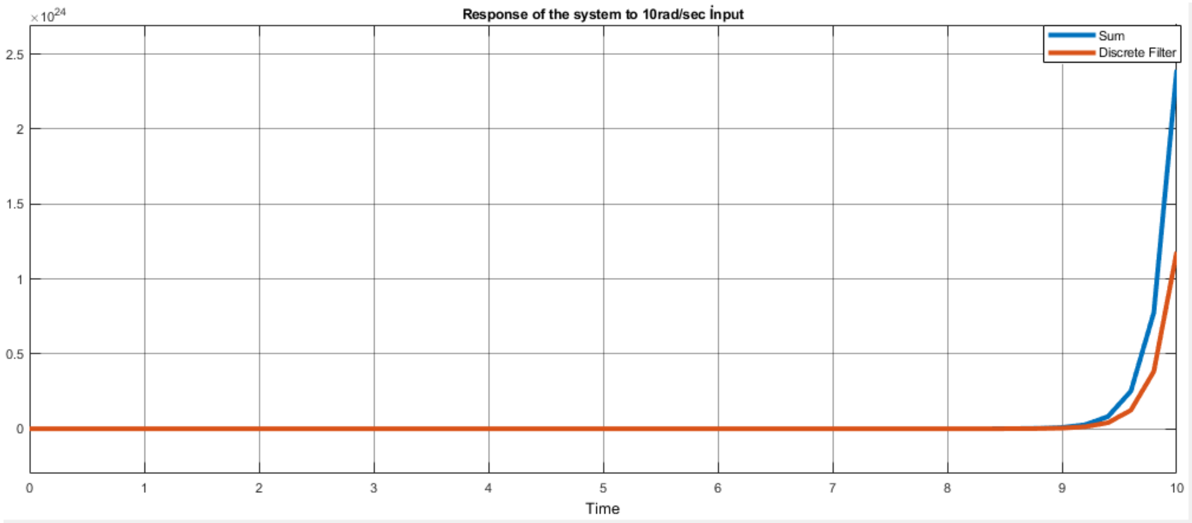

Figure 5: Response of the system to 10rad/sec Input

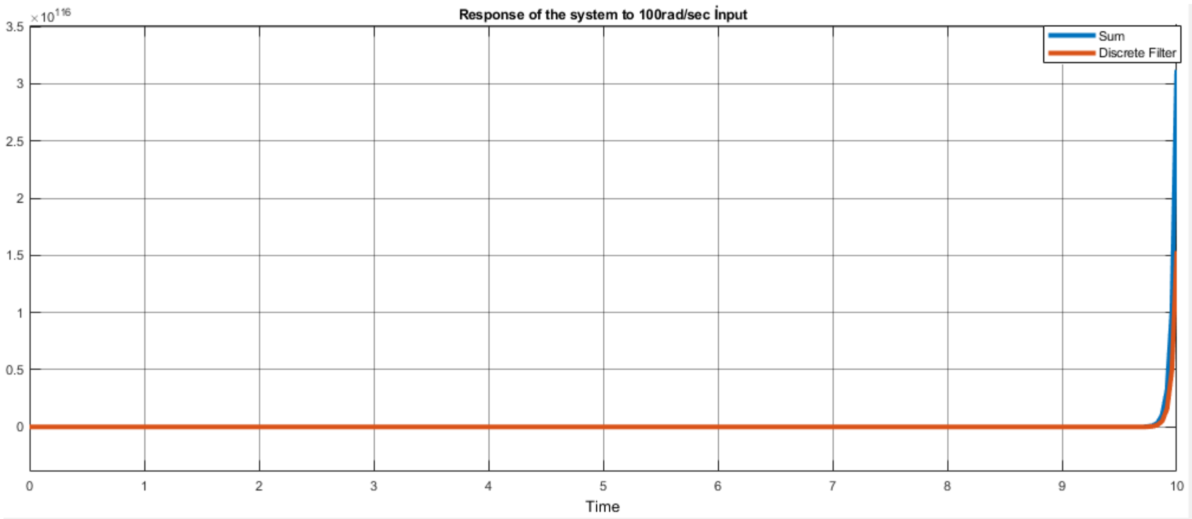

Figure 6: Response of the system to 5jjj0rad/sec Input

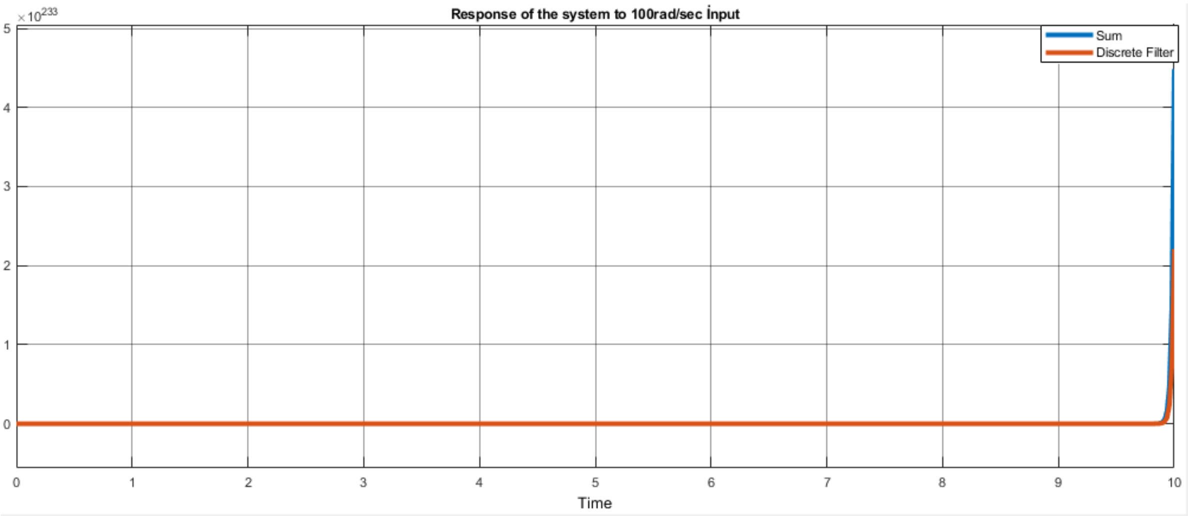

Figure 7: Response of the system to 100rad/sec Input

As the frequency is increased, the results became more similar. The internal block is not functioning well enough. However, our design is functioning better than our expectations.

Appendices

A Source Code For Question 2

```
%% %%%%%%%%%%%%%% part b %%%%%%%%%%%%%%%%%%%%
    clear
    clc
    num = [0 1 zeros(1,21) ]
    num_lenght= length(num)
    den = [1 1.6 -0.11 -1.45 -0.88 -0.16 zeros(1,17)]
    den_lenght= length(den)
    %ss= tf(num,den)

    i=1
    a=0
    while i<num_lenght
        if num(i)== 0
            i=i+1
            a=a+1
        else
            w=i
            while w>1
                den2(w-1)=0
                w=w-1
            end
            q=i
            den_lenght2=length(den2)
            while q<den_lenght+den_lenght2+1
                if q-den_lenght2<1
                    q=q+1
                else
                    den2(q)=(num(i)/(den(1)))*den(q-den_lenght2
                        )
                    q=q+1
                end
            end
            x_b(i)=num(i)
            u=i
            while u<length(num)
                num(u)=num(u)-den2(u)
                u=u+1
```

```
                end
                i=i+1
                clearvars den2
            end

        end
        x_b(i-1)=[]
        %% x[-1] is left untouched since the internal matlab func at part d leaves it, x_b(1)=[] removes it
        %% %%%%%%%%%%%% end of part b %%%%%%%%%%%%%%%%%
        %% %%%%%%%%%%%%%%% part a %%%%%%%%%%%%%%%%%%%%%
        i=1
        while i<21
            x(1)=0
            x_a(i+1)=(100/729)+(-50/27)*i*(-0.5)^(i)-(100/9)*(-0.5)^(i)+(-320/81)*i*(-0.8)^(i)+(8000/729)*(-0.8)^(i)
            i=i+1
        end
        v=[0 20 -1 1]
        axis(v)

        k=0:20
        %% %%%%%%%%%%%% end of part a %%%%%%%%%%%%%%%%%
        %% %%%%%%%%%%%%%%% part c %%%%%%%%%%%%%%%%%%%%%
        i=1
        x_c=zeros(1,5) % Initialize the array with k<0 values
        while i<21
            m=i+5
            x_c(m)=(-1.6)*x_c(m-1)+(0.11)*x_c(m-2)+(1.45)*x_c(m-3)+(0.88)*x_c(m-4)+(0.16)*x_c(m-5)+mydelta(m-1)
            i=i+1
        end
        x_c(1)=[] % Delete x[-5]
        x_c(1)=[] % Delete x[-4]
        x_c(1)=[] % Delete x[-3]
        x_c(1)=[] % Delete x[-2]
        %% x[-1] is left untouched since the internal matlab func at part d leaves it, x_b(1)=[] removes it
        %% %%%%%%%%%%%% end of part c %%%%%%%%%%%%%%%%%
        %% %%%%%%%%%%%%%%% part d %%%%%%%%%%%%%%%%%%%%%
        num = [0 1 0 0 0 0 ]

```

```
    den = [1 1.6 -0.11 -1.45 -0.88 -0.16]

    %% Inverse Z Part
    x=[1 zeros(1,20)]
    x_d=filter(num,den,x) %filter to finde inverse z transfor
    %% %%%%%%%%%%%% end of part d %%%%%%%%%%%%%%%%%%%
    %% %%%%%%%%%%%%%%%% part e %%%%%%%%%%%%%%%%%%%%%%%
    %% plot x_a
    subplot(2,2,1)
    v=[0 20 -1 1]
    axis(v)
    k=0:20
    plot(k,x_a,'o',k,x_a,'-')
    legend('part a')
    grid
    title('Partial Fraction Expansion Method')
    xlabel('k')
    ylabel('x[k]')
    %% plot x_b
    subplot(2,2,2);
    v=[0 20 -1 1]
    axis(v)
    k=0:20
    plot(k,x_b,'o',k,x_b,'-')
    legend('part b')
    grid
    title('Long Division Method using Matlab')
    xlabel('k')
    ylabel('x[k]')
    %% plot x_c
    subplot(2,2,3);
    v=[0 20 -1 1]
    axis(v)
    k=0:20
    plot(k,x_c,'o',k,x_c,'-')
    legend('part c')
    grid
    title('Treating X(z) as System Method ')
    xlabel('k')
    ylabel('x[k]')
    %% plot x_d
    subplot(2,2,4);
```

```
    v=[0 20 -1 1]
    axis(v)
    k=0:20
    plot(k,x_d,'o',k,x_d,'-')
    legend('part d')
    grid
    title('Using Internal Matlab Functions')
    xlabel('k')
    ylabel('x[k]')
    %% %%%%%%%%%%%% end of part e %%%%%%%%%%%%%%%%%%
```

```
function y = mydelta(x)
    if x == 5
        y=1
    else
        y=0
    end
end
```

B Source Code For Question 3

```
%% %%%%%%%%%%%%%%% part a %%%%%%%%%%%%%%%%%%%%%%
    i=1
    while i<21
        x(1)=0
        x_a(i+1)=-(189/4)*(3)^(i)+(376/9)*(4)^(i)+10*(2)^(i)
            +(1/6)*(i)+17/36
        i=i+1
    end
    v=[0 20 -1 1]
    axis(v)

    k=0:20
%% %%%%%%%%%%%%% end of part a %%%%%%%%%%%%%%%%%%
%% %%%%%%%%%%%%%%% part b %%%%%%%%%%%%%%%%%%%%%%
    i=1
    while i<21
        x(1)=0
        x_b(i+1)=-(189/4)*(3)^(i)+(376/9)*(4)^(i)+10*(2)^(i)
            +(1/6)*(i)+17/36
        i=i+1
    end
    v=[0 20 -1 1]
    axis(v)

    k=0:20
%% %%%%%%%%%%%%% end of part b %%%%%%%%%%%%%%%%%%
%% %%%%%%%%%%%%%%% part c %%%%%%%%%%%%%%%%%%%%%%
    i=1
    x_c=zeros(1,3) % Initialize the array with k<0 values
    while i<21
        m=i+3
        x_c(m)=(7)*x_c(m-1)-(12)*x_c(m-2)+5*2^m+m
        i=i+1
    end
    x_c(1)=[] % Delete x[-3]
    x_c(1)=[] % Delete x[-2]
%% %%%%%%%%%%%%% end of part c %%%%%%%%%%%%%%%%%%
%% plot x_a
    subplot(2,2,1)
    v=[0 20 -1 1]
```

```
    axis(v)
    k=0:20
    plot(k,x_a,'o',k,x_a,'-')
    legend('part a')
    grid
    title('By solving difference equation directly')
    xlabel('k')
    ylabel('x[k]')
%% plot x_b
    subplot(2,2,2)
    v=[0 20 -1 1]
    axis(v)
    k=0:20
    plot(k,x_b,'o',k,x_b,'-')
    legend('part b')
    grid
    title('By solving the difference equation equation in Z-
      Domain')
    xlabel('k')
    ylabel('x[k]')
%% plot x_c
    subplot(2,2,3)
    v=[0 20 -1 1]
    axis(v)
    k=0:20
    plot(k,x_c,'o',k,x_c,'-')
    legend('part c')
    grid
    title('By Simulating Difference Equation using Matlab')
    xlabel('k')
    ylabel('x[k]')
```