EE402 - Discrete Time Systems **Spring 2018**

Lecture 16

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State-Feedback & Pole Placement

Given a discrete-time state-evolution equation

$$x[k+1] = Gx[k] + Hu[k]$$

If direct measurements of all of the states of the system (e.g. $y[k] = x[k]$) are available, the most popular control method is the linear state feedback control,

$$u[k] = -Kx[k]$$

which can be thought as a generalization of P controller to the vector form. Under this control law, without any reference signal, the system becomes an autonomous system

$$\begin{aligned} x[k+1] &= Gx[k] + H(-Kx[k]) \\ x[k+1] &= (G - HK)\, x[k] \end{aligned}$$

The system matrix of this new autonomous system is $\hat{G} = G - HK$. Important questions is how to choose K. Note that

$$\begin{aligned} K &\in \mathbb{R}^n \quad \text{Single} - \text{Input} \\ K &\in \mathbb{R}^{n \times p} \quad \text{Multi} - \text{Input} \end{aligned}$$

As in all of the control desgn techniques, the most critical criterion is stability, thus we want all of the eigenvalues to be strictly inside the unit-circle. However, we know that there could be different requirements on the poles/eigenvalues of the system.

The fundamental principle of "pole-placement" design is that we first define a desired closed-loop eigenvalue set $\mathcal{E}^* = \{\lambda_1^*, \ \cdots, \ \lambda_1^*\}$, and then if possible we choose K^* such that the closed-loop eigenvalues match the desired ones.

The necessary and sufficient condition on arbitrary pole-placement is that the system should be Reachable.

In Pole-Placement, first step is computing the desired characteristic polynomial.

$$\begin{aligned} \mathcal{E}^* &= \{\lambda_1^*, \ \cdots, \ \lambda_n^*\} \\ p^*(z) &= (z - \lambda_1^*) \cdots (z - \lambda_n^*) \\ &= z^n + a_1^* z^{n-1} + \cdots + a_{n-1}^* z + a_n^* \end{aligned}$$

Then we tune K such that

$$\det\left(zI - (G - HK)\right) = p^*(z)$$

Direct Design of State-Feedback Gain

If n is small, the most efficient method could be the direct design.

Example: Consider the following DT system

$$x[k+1] = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} x[k] + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u[k]$$

Design a state-feedback rule such that poles are located at $\lambda_{1,2} = 0$ (Dead-beat gain)

Solution: Desired characteristic equation can be computed as

$$p^*(z) = z^2$$

Let $K = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$, then the characteristic equation of $\hat{G}$ can be computed as

$$\begin{aligned} \det\left(zI - (G - HK)\right) &= \det\left(\begin{bmatrix} z-1+k_1 & k_1 \\ k_2 & z-2+k_2 \end{bmatrix}\right) \\ &= z^2 + z(k_1 + k_2 - 3) + (2 - 2k_1 - k_2) \end{aligned}$$

If we match the equations

$$\begin{aligned} z^2 + z(k_1 + k_2 - 3) + (2 - 2k_1 - k_2) &= z^2 \\ k_1 + k_2 &= 3 \\ 2k_1 + k_2 &= 2 \\ k_1 &= -1 \\ k_2 &= 4 \end{aligned}$$

This $K = \begin{bmatrix} -1 & 4 \end{bmatrix}$. Now let's compute $\hat{G}^k$

$$\begin{aligned} \hat{G} &= \begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix} \\ \hat{G}^2 &= \begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ &\vdots \end{aligned}$$

It can be seen that closed-loop system rejects all initial condition perturbations in 2 steps.

Design of State-Feedback Gain Using Reachable Canonical Form

Let's assume that the state-space representation is in controllable canonical form and we have access to the all states of this form

$$x[k+1] = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} x[k] + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u[k]$$

Let $K = \begin{bmatrix} k_n & \cdots & k_1 \end{bmatrix}$, then autonomous system takes the form

$$x[k+1] = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} x[k] - \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} k_n & \cdots & k_1 \end{bmatrix} x[k]$$

$$x[k+1] = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -(a_n + k_n) & -(a_{n-1} + k_{n-1}) & -(a_{n-2} + k_{n-2}) & \cdots & -(a_1 + k_1) \end{bmatrix} x[k]$$

Let $p^*(z) = z^2 + a_1^* z + \cdots + a_{n-1}^* z + a_n^*$, then K can be computed as

$$K = \begin{bmatrix} (a_n^* - a_n) & \cdots & (a_1^* - a_1) \end{bmatrix}$$

However, what if the system is not in Reachable canonical form. We can find a transformation which finds the Reachable canonical representation.

The reachability matrix of a state-space representation is given as

$$M = \begin{bmatrix} H \mid GH \mid \cdots \mid G^{n-1}H \end{bmatrix}$$

Let's define a transformation matrix T as follows:

$$T = MW \quad , \quad x[k] = T\hat{x}[k]$$
$$\hat{x}[k+1] = \left[T^{-1}GT\right]\hat{x}[k] + T^{-1}Hu[k]$$

where

$$W = \begin{bmatrix} a_{n-1} & a_{n-2} & \cdots & a_1 & 1 \\ a_{n-2} & a_{n-3} & \cdots & 1 & 0 \\ \vdots & \vdots & \iddots & & \vdots \\ a_1 & 1 & & & \\ 1 & 0 & \cdots & & 0 \end{bmatrix}$$

Then it is given that

$$T^{-1}GT = \hat{G} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix}$$

$$T^{-1}H = \hat{H} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Let's compute $T\hat{H}$

$$
\begin{aligned}
T\hat{H} = MW\hat{H} &= M \begin{bmatrix} a_{n-1} & a_{n-2} & \cdots & a_1 & 1 \\ a_{n-2} & a_{n-3} & \cdots & 1 & 0 \\ \vdots & \vdots & \iddots & & \vdots \\ a_1 & 1 & & & \\ 1 & 0 & \cdots & & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \\
&= \left[\begin{array}{c|c|c|c} H & GH & \cdots & G^{n-1}H \end{array}\right] \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \\
&= H
\end{aligned}
$$

A similar approach (but longer) can be used to show that $T^{-1}GT = \hat{G}$. We know how to design a state-feedback gain $\hat{K}$ for the Reachable canonical form. Given $\hat{K}$ $u[k]$ is given as

$$
\begin{aligned}
u[k] &= -\hat{K}\hat{x}[k] \\
&= -\hat{K}T^{-1}\hat{x}[k] \\
K &= \hat{K}T^{-1}
\end{aligned}
$$

Example 2: Consider the following DT system

$$
x[k+1] = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} x[k] + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u[k]
$$

Design a state-feedback rule using the Reachable canonical form approach, such that poles are located at $\lambda_{1,2} = 0$ (Dead-beat gain)

Solution: Characteristic equation of G can be derived as

$$
\det\left(\begin{bmatrix} z-1 & 0 \\ 0 & z-2 \end{bmatrix}\right) = z^2 - 3z + 2
$$

The Reachability matrix can be computed as

$$
M = \left[\begin{array}{c|c} H & GH \end{array}\right] = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}
$$

The matrix W can be computed as

$$
W = \begin{bmatrix} -3 & 1 \\ 1 & 0 \end{bmatrix}
$$

Transformation matrix, T and its inverse T^{-1} can be computed as

$$
\begin{aligned}
T = MW &= \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -3 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -1 & 1 \end{bmatrix} \\
T^{-1} &= \begin{bmatrix} -1 & 1 \\ -1 & 2 \end{bmatrix}
\end{aligned}
$$

Given that desired characteristic polynomial is $p^*(z) = z^2$, $\hat{K}$ of reachable canonical from can be computed as

$$\begin{aligned}\hat{K} &= \begin{bmatrix} -a_2 & -a_1 \end{bmatrix} \\ &= \begin{bmatrix} -2 & 3 \end{bmatrix}\end{aligned}$$

Finally K can be computed as

$$\begin{aligned}K = \hat{K}T^{-1} &= \begin{bmatrix} -2 & 3 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ -1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 4 \end{bmatrix}\end{aligned}$$

As expected this is the same result with the one found in Example 1 (Direct-Method).