EE402 - Discrete Time Systems **Spring 2018**

Lecture 10

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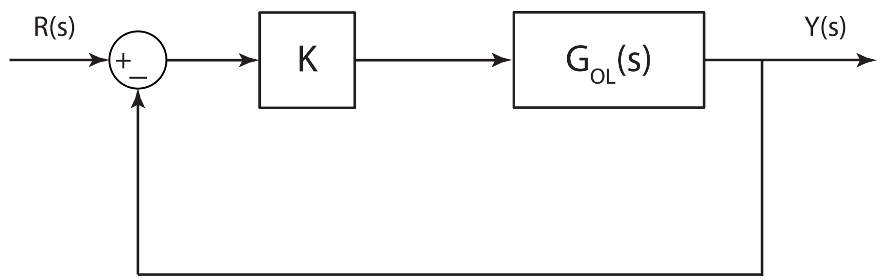
Root Locus

For continuous time systems the root locus diagram illustrates the location of roots/poles of a closed loop LTI systems, with respect to gain parameter K (can be considered as a P controller). The basic closed-loop topology is used for deriving the root-locus rules, however we know that many different topologies can be reduced to this from.



The closed loop transfer function of this basic control system is

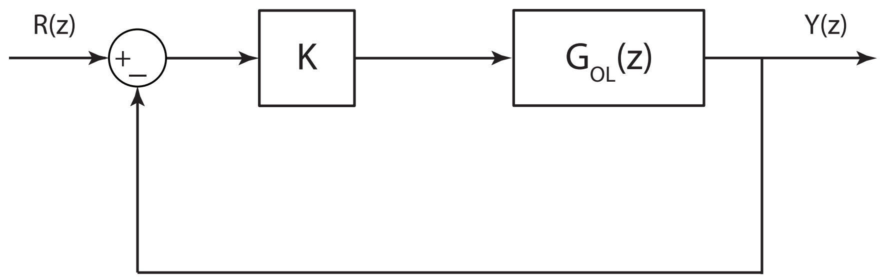
$$\frac{Y(s)}{R(s)} = \frac{KG_{OL}(s)}{1 + KG_{OL}(s)}$$

where the poles of the closed loop system are the roots of the characteristic equation

$$1 + KG_{OL}(s) = 0$$
$$1 + K\frac{n(s)}{d(s)} = 0$$

In 302 we learned the rules such that we can derive the qualitative and quantitive structure of root locus paths for **positive** gain K that solves the equation above.

In discrete time systems, similar to the CT case we use the root locus diagram to illustrates the location of roots/poles of a closed loop DT-LTI systems, with respect to a gain parameter K (can be considered as a P controller). The basic discrete-time closed-loop topology is used for deriving the root-locus rules, however we know that many different DT topologies can be reduced to this from.



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I think, it is obvious that fundamental equation that relates the gain K and roots/poles is exactly same for both CT and DT systems. This means that same rules are directly applied for CT systems.

However, even if we have same diagram for CT and DT systems the meaning and interpretation of the diagram is fundamentally different. Because, the effects of pole locations are different in CT and DT systems.

Angle and Magnitude Conditions

Let's analyze the characteristic equation

$$KG_{OL}(z) = -1 \quad ,\text{or} \quad K\frac{n(z)}{d(z)} = -1$$
$$|KG_{OL}(z)| = 1 \quad ,\text{or} \quad \left|K\frac{n(z)}{d(z)}\right| = 1$$
$$\angle[KG_{OL}(z)] = \pi(2k+1),\ k \in \mathbb{Z} \quad ,\text{or} \quad \angle\left[K\frac{n(z)}{d(z)}\right] = \pi(2k+1),\ k \in \mathbb{Z}$$

For a given K, z values that satisfy both magnitude and angle conditions are located on the root loci.

Rules and procedure for constructing root loci

1. Characteristic equation, zeros and poles of the Open-Loop pulse transfer function.

$$1 + KG_{OL}(z) = 0$$
$$1 + K\frac{n(z)}{d(z)} = 0$$
$$1 + K\frac{(z - z_1)\cdots(z - z_M)}{(z - p_1)\cdots(z - p_N)} = 0$$

2. Root loci has N separate branches.

3. Root loci starts from poles of $G_{OL}(z)$ and

   (a) M branches terminates at the zeros of $G_{OL}(z)$

   (b) N branches terminates at ∞ (implicit zeros of $G_{OL}(z)$)

   It is relatively easy to undestand this

$$\begin{aligned} d(z) + Kn(z) &= 0 \\ K \to 0 \ \to \ d(z) &= 0 \\ K \to \infty \ \to \ n(z) &= 0 \end{aligned}$$

4. Root loci on the real axis determined by open-loop zeros and poles. $z = \sigma \in \mathbb{R}$ then,

$$\begin{aligned} |KG_{OL}(\sigma)| &= 1 \\ \text{Sign}[G_{OL}(\sigma)] &= -1 \end{aligned}$$

   We can always find a K that satisfy the magnitude condition,so angle condition will determine which parts of real axis belong to the root locus.

   We can first see that complex conjugate zero/pole pairs has not effect, then for the remaining ones we can derive the following condition

$$\text{Sign}[G_{OL}(\sigma)] = \prod_{i=1}^{M} \text{Sign}[\sigma - z_i] \prod_{j=1}^{N} \text{Sign}[\sigma - p_j] = -1$$

   which means that for ODD number of poles + zeros $\text{Sign}[\sigma - p_i]$ and $\text{Sign}[\sigma - z_i]$ must be negative for satisfying this condition for that particular σ to be on the root-locus. We can summarize the rule as

   If the test point σ on real axis has ODD numbers of poles and zeros in its right, then this point is located on the root-locus.

5. Asymptotes

   - $N - M$ branches goes to infinity. Thus, there exist $N - M$ many asymptotes
   - For large z we can have the following approximation

$$\begin{aligned} &K \frac{(z - z_1) \cdots (z - z_M)}{(z - p_1) \cdots (z - p_N)} \approx \frac{K}{z^3} \\ &\angle \left[\frac{K}{z^{N-M}} \right] = -(N - M)\angle[z] = \pi(2k + 1), \ k \in \mathbb{Z} \\ &\phi_a = \frac{\pm\pi(2k + 1)}{N - M}, \ k \in \{1, \cdots, N - M\} \end{aligned}$$

   - Real axis intercept σ_a can be computed as

$$\sigma_a = \frac{\sum p_i - \sum z_i}{N - M}$$

     This can be derived via a different approximation (see textbook)

6. Breakaway and break-in points on real axis. When z is real $z = \sigma,\ \sigma \in \mathbb{R}$, we can have

$$1 + KG_{OL}(\sigma) = 0 \quad \rightarrow \quad K(\sigma) = \frac{-1}{G_{OL}(\sigma)}$$

Note that break-in and breakaway points corresponds to double roots. Thus, σ_b is a break-away or break-in point if

$$\left[\frac{dK(\sigma)}{\sigma}\right]_{\sigma=\sigma_b} = 0$$
$$K(\sigma) > 0$$

7. Angle of departure (or arrival) from open-loop complex conjugate poles (or to open-loop complex conjugate zeros)

Let's assume that p^* is a complex conjugate pole of $G_{OL}(z)$, then let's define a $P(z)$ such that

$$P^*(z) = (z - p^*)G_{OL}(z)$$

We know that for $K = 0$, the root locus is located at p^*. If we add a very small $K = \delta K$, then pole/root locus moves to $p^* + \delta z$. If we evaluate the phase condition of the root locus on this new "unknown" point

$$\begin{aligned}
\angle[KG_{OL}(z)]_{z=p*+\delta z} &= \pm\pi \\
\angle\left[\frac{P^*(z)}{z - p^*}\right]_{z=p^*+\delta z} &= \pm\pi \\
\angle\left[P^*(p^* + \delta z)\right] - \angle\left[\delta z\right] &= \pm\pi \\
\theta_d = \angle\left[\delta z\right] &= \pm\pi + \angle\left[P^*(p^*)\right]
\end{aligned}$$

Geometrically speaking, $\angle\left[P^*(p^*)\right]$ stands for **# of angles from the zeros to this specific pole** – **# of angles from all other remaining poles to this specific pole**.

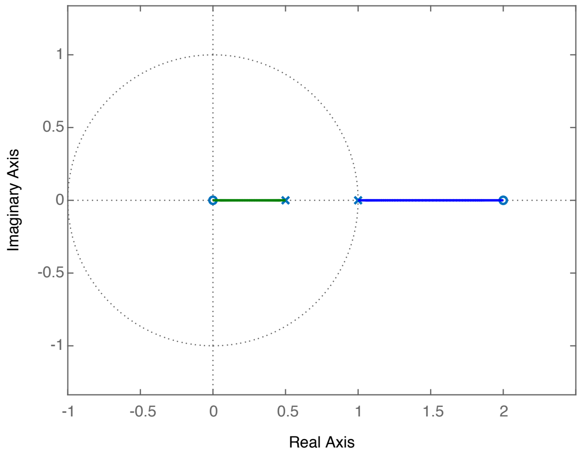
A similar condition can be derived for angle of arrival to complex conjugate zeros.

$$\begin{aligned}
\theta_a &= \pm\pi - \angle\left[P^*(z^*)\right] \\
P^*(z) &= (z - z^*)G_{OL}(z)
\end{aligned}$$

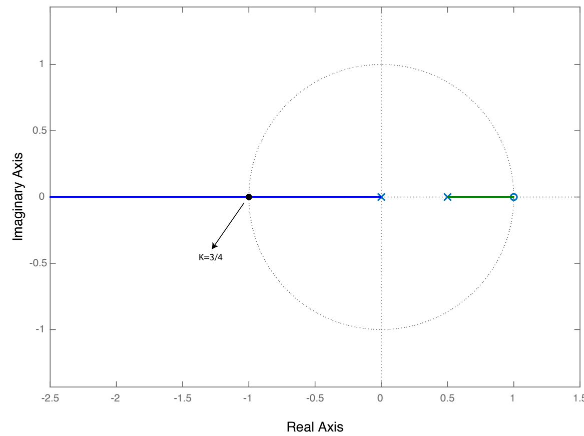
where z^* is a complex conjugate zero of $G_{OL}(z)$.

Examples

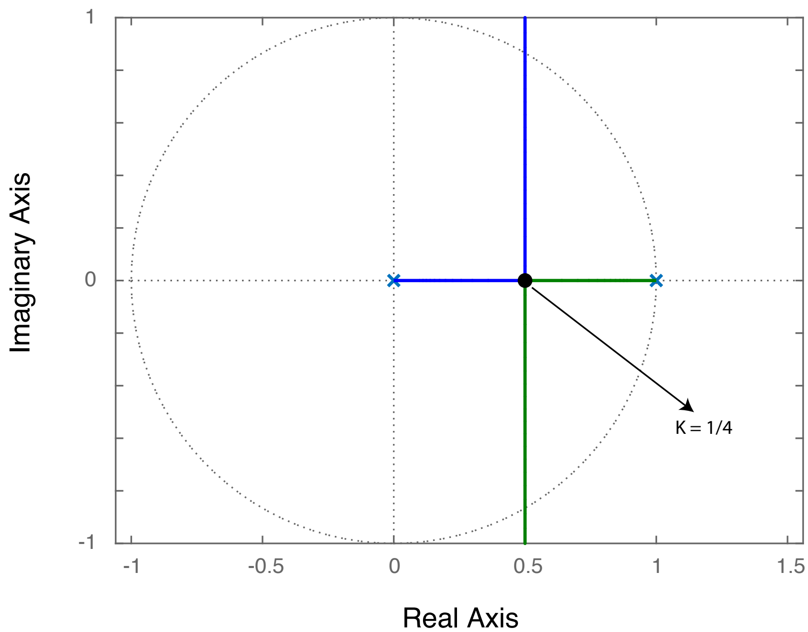
$$G_{OL}(z) = \frac{z(z-2)}{(z-1)(z-0.5)}$$



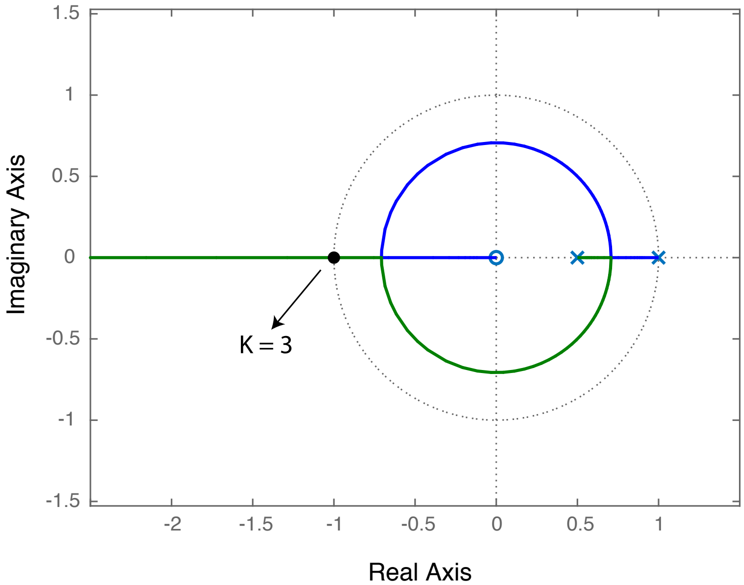
$$G_{OL}(z) = \frac{z-1}{z(z-0.5)}$$



$$G_{OL}(z) = \frac{1}{z(z-1)}$$



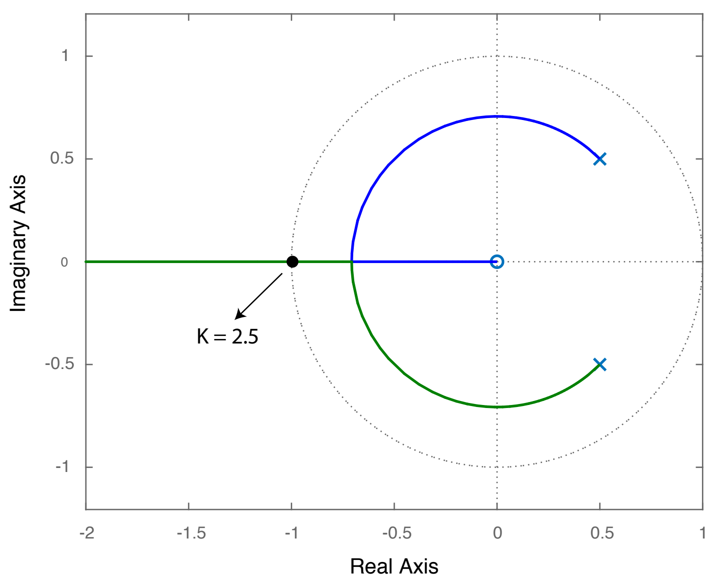
$$G_{OL}(z) = \frac{z}{(z-0.5)(z-1)}$$



$$\sigma_{ba} = \frac{\sqrt{2}}{2} \quad , \quad \sigma_{bi} = -\frac{\sqrt{2}}{2}$$

K = 0.086 K = 2.9

$$G_{OL}(z) = \frac{z}{z^2 - z + 1/2}$$



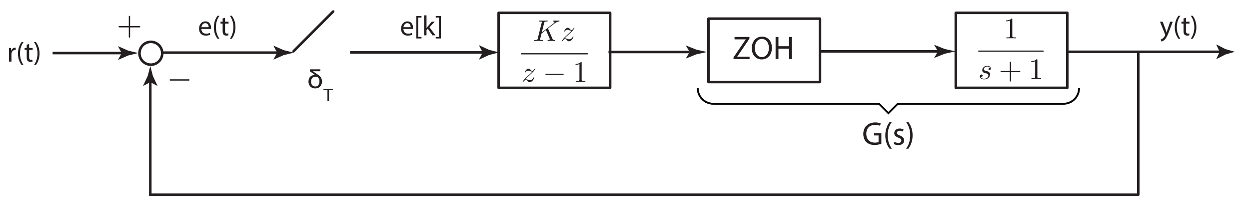
$$\theta_d = \pm\frac{3\pi}{4}$$

$$\sigma_{bi} = -\frac{\sqrt{2}}{2}$$

K = 2.4

Root-Locus of Digital Control Systems

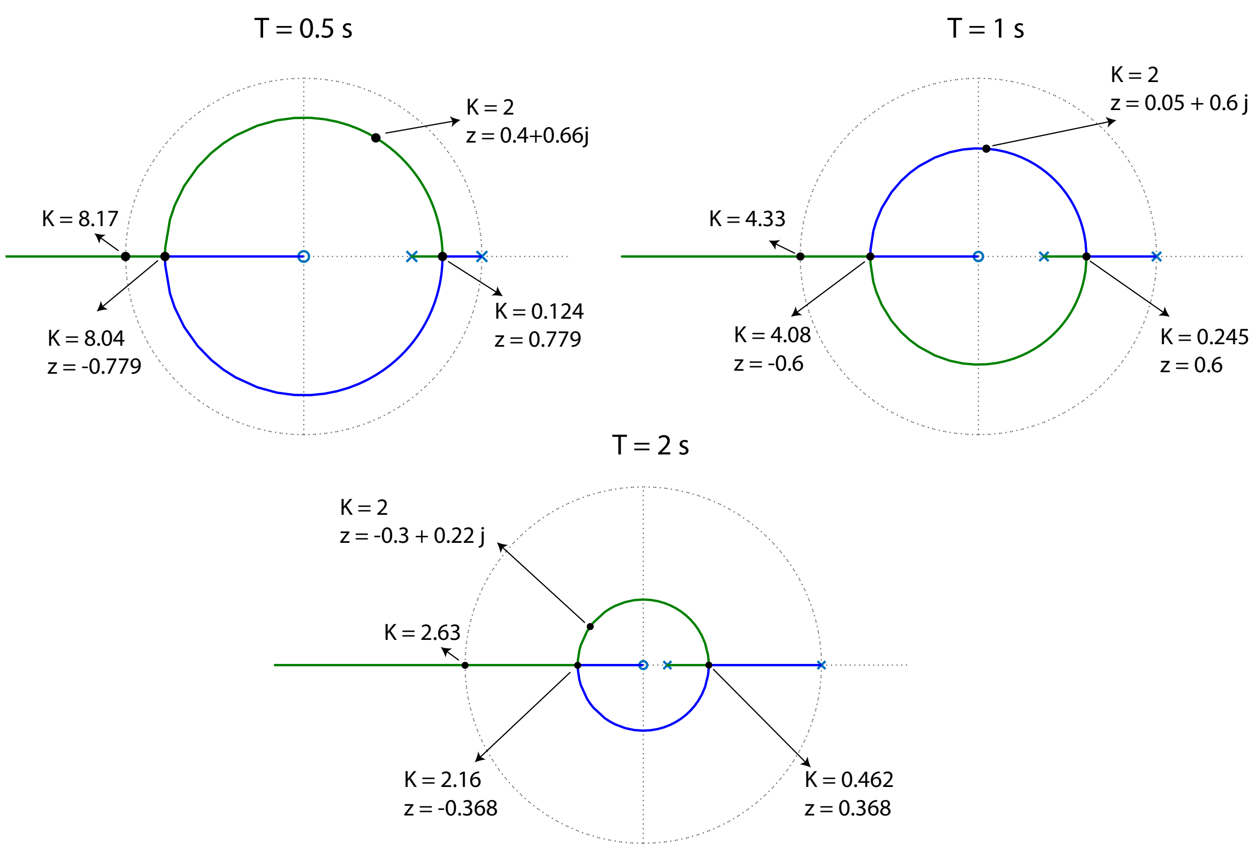
Let us draw the root-locus diagrams for the discrete time control system below for $T = 0.5s$, $T = 1s$, and $T = 2s$.



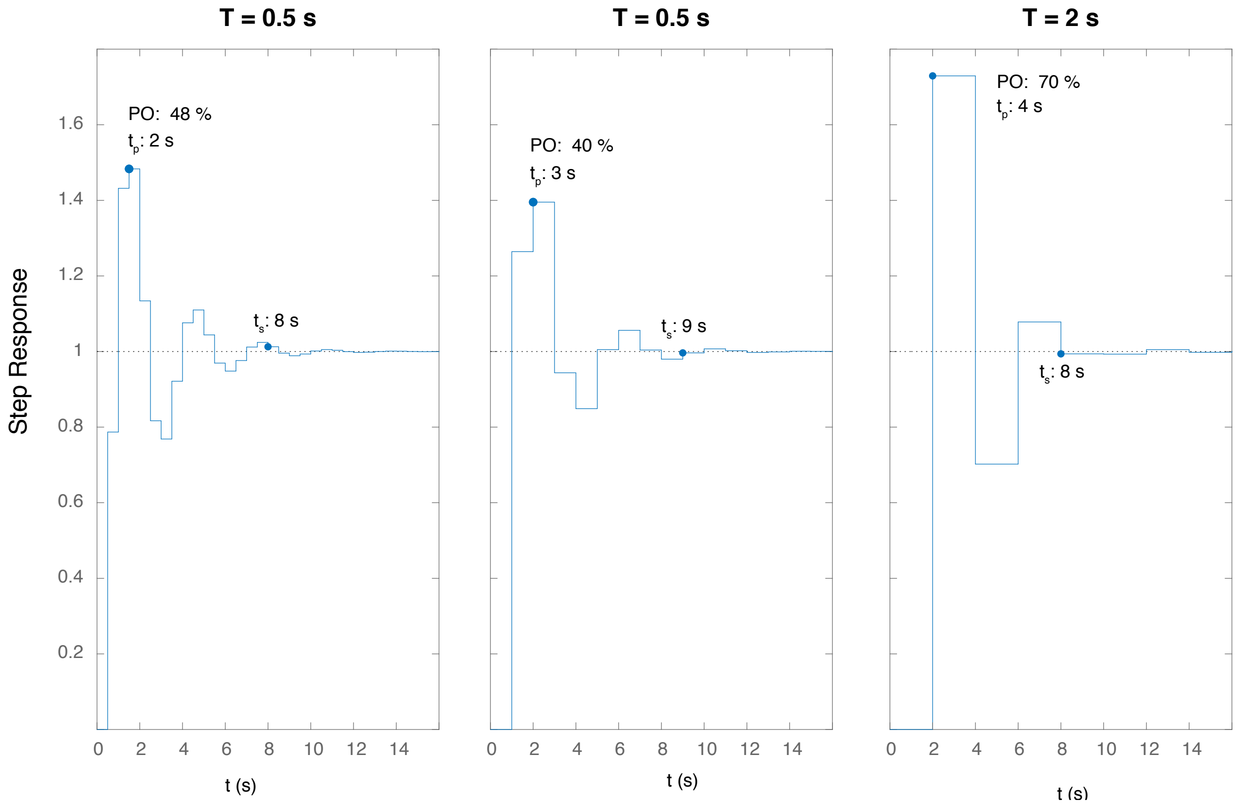
Open-loop pulse transfer functions cane be obtained as

$$G_{0.5} = K\ 0.3935 \frac{z}{(z-1)(z-0.6065)}$$
$$G_1 = K\ 0.6321 \frac{z}{(z-1)(z-0.3679)}$$
$$G_2 = K\ 0.8647 \frac{z}{(z-1)(z-0.1353)}$$

Root-locus plots for all there cases are illustrated in the Figure below.



This Figure below compares DT step responses of all three sampling time cases, where x-axis is the actual time.



If we compare three different responses, we can clearly see that in terms of over-shoo, $T = 2s$ has the worst performance, where as $T = 1s$ seems to be a little better than $T = 0.5\ s$. However, if one "draws" the CT-response by simulating the whole hybrid system, he/she can see that the over-shoot for $T = 1\ s$ indeed larger than $T = 0.5s$. Due to the "sampling rate" we can not capture the over-shoot difference clearly between $T = 0.5\ s$ and $T = 1\ s$. We will talk about it in the next section.

Another similarity between these responses is that settling times of DT-system responses seem to be very close. We know that settling time for a CT-system mainly depends on the real part of the dominant pole, i.e. $\sigma = \text{Re}\{s\}$. Let's compute $\sigma_{0.5}$, σ_1,and σ_2 by taking into account the $e^{Ts} = z$ mapping.

$$\sigma_{0.5} = \frac{\ln(|z|)}{0.5} \approx -0.5$$
$$\sigma_1 = \frac{\ln(|z|)}{1} \approx -0.5$$
$$\sigma_2 = \frac{\ln(|z|)}{2} \approx -0.5$$

It can be seen that indeed the real part of the mapped CT pole locations are approximately same for same gain $K = 2$.

Now let's evaluate the steady-state error performances for both sampling times. All three open-loop transfer functions are Type 1 systems and thus for unit step response $e_{ss} = 0$, which is also observable from the step response plots. Let's compute e_{ss} for unit ramp response.

$$e_{0.5} = \frac{1}{2}$$
$$e_1 = \frac{1}{2}$$
$$e_2 = \frac{1}{2}$$

This can give an illusion that they have the same performance. However we evaluated response to the discrete time unit step input which is $r[k] = k[uk]$. However, in order to compare fairly, we need to compute the steady state error to sampled continuous time unit ramp function. $r(t) = t$ & $r(kT) = kT$, thus we have

$$r_{0.5}[k] = 0.5k$$
$$r_1[k] = 1k$$
$$r_2[k] = 2k$$

If we re-evaluate the stead-state errors, we obtain

$$e_{0.5} = \frac{1}{4}$$
$$e_1 = \frac{1}{2}$$
$$e_2 = 1$$

Now it is clear that as T increases the steady-state error also increases.

Now let's compare different cases, where each response show a critically damped behavior. Critically dampled locations are obtained for the following K values.

$$K_{0.5} = 0.124$$
$$K_1 = 0.245$$
$$K_2 = 0.462$$

Now let's compare responses using different specifications

- Obviously no oscillations and 0% overs-shoot for all three cases
- The CT-time mapped pole locations can be computed as

  $$\sigma_{0.5} \approx \sigma_1 \approx \sigma_2 \approx -0.5$$

  which implies that settling time performance is similar.
- Since all systems are Type 1, the stead-state errors to unit step is zero for all cases.
- If we compute the steady-state errors to sampled CT unit ramp input for all three cases we obtain

  $$e_{0.5} \approx e_1 \approx e_2 \approx 4$$

  Quite interestingly even the steady-state errors are approximately similar.

This shows that for the specific location where all systems show a critically damped behavior the response performance (stability, overshoot, stead-state error) is quite similar and almost independent on T.

However if one needs to improve the steady-state error performance (to ramp like inputs) obviously lower T (or higher sampling rate) is better both from the perspective of stability and steady-state error performance.

Specs for discrete time pole locations

1. Absolutely we want the poles to be located inside the unit-circle for stability.
2. From the perspective of DT-control systems (CT-systems that are controlled by DT-controllers), we want the frequency of oscillations to be sufficiently smaller than the Nyquist frequency $\omega_s/2$. As a rule of thumb 8-10 samples per cycle are required. Let's assume that require 8 samples per cycle then we have the following condition on z

$$\angle z \in [-\pi/4, \pi/4]$$

3. For a CT system if the system has a dominant second order (or first order) characteristic, then we know that settling time is approximately given by, $T_s = \frac{4}{\sigma}$, where $-\sigma$ is the real part of the CT-pole. Let's assume that the requirement is the settling time of the step response of the system will be less than $\bar{T}$. Then considering the $z = e^{Ts}$ mapping we have the following condition.

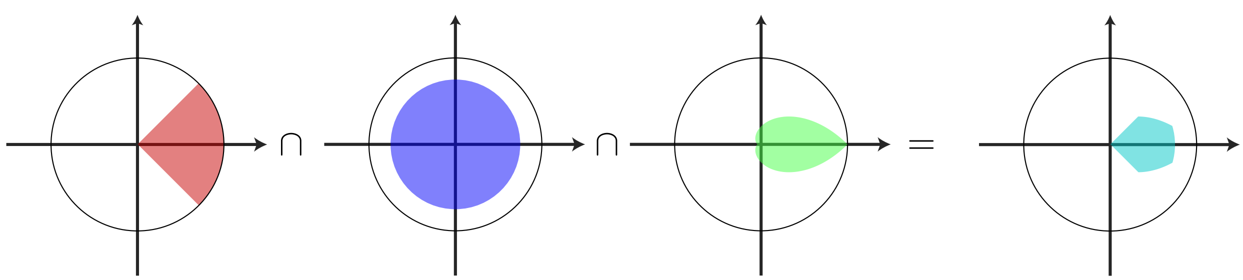
$$T_s \leq \bar{T} \;\rightarrow\; \sigma \geq \frac{4}{\bar{T}}$$
$$|z| \leq e^{-4\frac{T}{\bar{T}}}$$

4. Another important requirement for CT systems is the overshoot and damping coefficient. We want damping to be high enough such that overshoot is reasonable. A rule of thumb for damping coefficient is that $\zeta \geq 1/\sqrt{2}$ (However different requirements can also be specified). Based on this we have the following condition on s

$$s = -\alpha\omega_d + \omega_d j$$
$$\alpha = \frac{\zeta}{\sqrt{1-\zeta^2}}$$
$$\alpha \geq 1$$

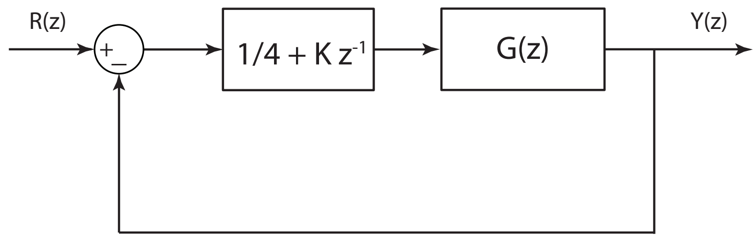
   In Lecture 7, we already illustrated the region where $\alpha \geq 1$.

These specs and their combined desired pole location region is illustrated in the Figure below.



Root-locus with respect to different parameters

Let's consider the following purely DT system where plant has a transfer function of $G(z)$ and controller has a first order FIR filter (low-pass) $G_c(z) = 1 + Az^{-1}$ form. We wonder the location of closed-loop poles with respect to the parameter A which does not directly fit to the classical form.



Let's first compute the closed-loop PTF and analyze the characteristic equation.

$$\frac{Y(z)}{R(z)} = \frac{(0.25 + Az^{-}1)\,G(z)}{1 + (0.25 + Az^{-}1)\,G(z)}$$

$$1 + \left(0.25 + Az^{-1}\right) G(z) = 0$$

$$1 + 0.25G(z) + Az^{-1}G(z) = 0$$

If we divide the characteristic equation by $1 + 0.25G(z)$ we obtain

$$1 + A\frac{z^{-1}G(z)}{1 + 0.25G(z)} = 0$$

$$1 + A\bar{G}_{OL}(z) = 0$$

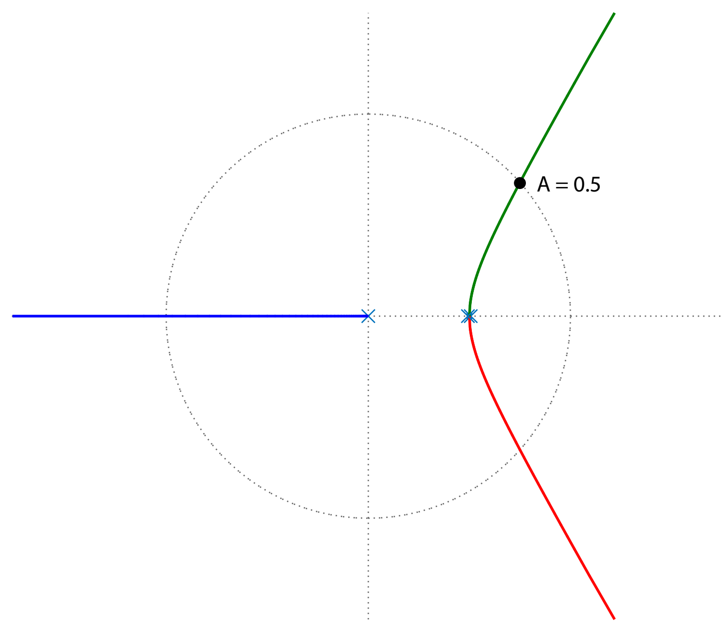
Now if we consider as $\bar{G}_{OL}(z)$ as the open-loop transfer function and draw the root-locus then we would derive the dependence of the roots to the parameter A.

Let's assume that $G(z) = \frac{1}{z(z-1)}$. Then for this system, we can compute

$$\bar{G}_{OL}(z) = \frac{z^{-1}G(z)}{1 + 0.25G(z)} = \frac{\frac{1}{z^2(z-1)}}{1 + \frac{0.25}{z(z-1)}} = \frac{\frac{1}{z^2(z-1)}}{\frac{z^2-z+0.25}{z(z-1)}}$$

$$= \frac{1}{z(z^2 - z + 0.25)}$$

Root-locus of the system w.r.t parameter A is given below. It can be seen that as A increases, dominant system poles deviates from the origin and eventually becomes unstable at $A = 0.5$. Technically this is a simple low-pass filter which may be inevitable in many closed-loop control systems. However, as we decrease the cut-off frequency (by increasing A) we push the poles towards the unit circle thus making the system less stable.


A = 0.5