ORTA DOĞU TEKNİK ÜNİVERSİTESİ

Lecture Notes 1
Electrical and Electronics Engineering - 5670402
Discrete Time Systems
Fall, 2018-2019

1. Continuous & Discrete Time Signals

A continuous time bilateral signal is a mapping defined by $f : \mathbb{R} \mapsto \mathbb{R}$ (or for unilateral case $f : \mathbb{R}^+ \mapsto \mathbb{R}$). Examples

$$f(t) = \sin(t)\ ,\ f(t) = e^t\ ,\ f(t) = u(t)\ ,\ f(t) = \delta(t)\ ,\ \text{where } t \in \mathbb{R}$$

A discrete time bilateral signal is a mapping defined by $g : \mathbb{Z} \mapsto \mathbb{R}$ (or for unilateral case $g : \mathbb{Z}^+ \mapsto \mathbb{R}$). Examples

$$f[n] = \sin[n]\ ,\ f[n] = 5^n\ ,\ f[n] = u[n]\ ,\ f[n] = \delta[n]\ ,\ \text{where } n \in \mathbb{Z}$$

Graphical Examples

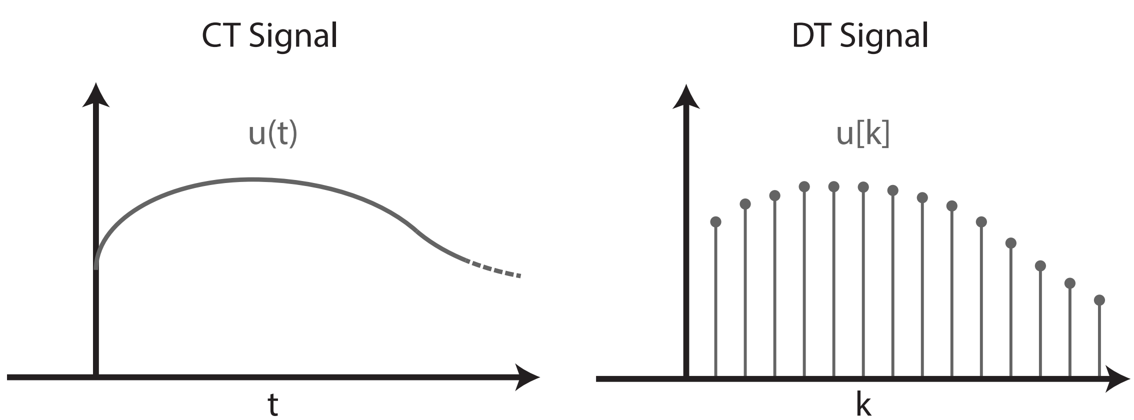


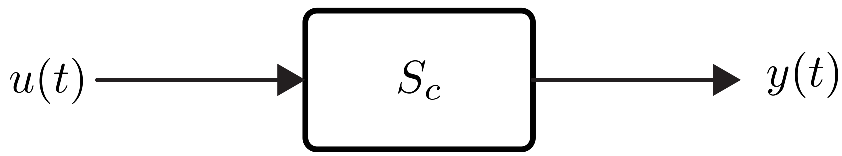
FIGURE 1. CT vs DT Signal

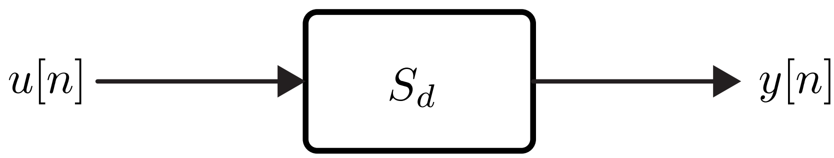
2. Continuous & Discrete Time Dynamical Systems

The system is modeled as a *mapping* from a set of input signals, $u(t)$ or $u[n]$, to a set of output signals, $y(t)$ or $y[n]$. We may represent continuous time and discrete time maps as

$$\begin{aligned} &\text{Continuous}: \quad y(t) = (S_c\, u)(t) \\ &\text{Discrete}: \quad y[n] = (S_d\, u)[n] \end{aligned}$$

Operation is performed on the entire input signal, $u(\cdot)$ or $u[\cdot]$ where the mappings S_c and S_d yield the signals $y(\cdot)$ and $y[\cdot]$.





2.1. **Properties of Input–Output Systems.**

- **Linearity**

  A continuous time system is **linear** if and only if

$$\begin{aligned} (S_c\,(\alpha u_1 + \beta u_2))(t) &= \alpha(S_c\, u_1)(t) + \beta(S_c\, u_2)(t) \\ &\forall \alpha,\ \beta,\ u_1(\cdot),\ \&\ u_2(\cdot) \end{aligned}$$

  A discrete time system is **linear** if and only if

$$\begin{aligned} (S_d\,(\alpha u_1 + \beta u_2))[n] &= \alpha(S_d\, u_1)[n] + \beta(S_d\, u_2)[n] \\ &\forall \alpha,\ \beta,\ u_1[\cdot],\ \&\ u_2[\cdot] \end{aligned}$$

- **Time Invariance**

  Let σ_T be the time-shift operator as

$$\sigma_T\, u(t) = u(t-T)$$

  Then a continuous time system is time-invariant if and only if

$$(S_c\, \sigma_T\, u))(t) = y(t-T)\ \forall T \in \mathbb{R}^+,\ \text{where } (S_c\, u))(t) = y(t)$$

  Similarly for discrete time systems

$$(S_d\, \sigma_k\, u))[n] = y[n-k]\; \forall k \in \mathbb{Z}^+,\ \text{where } (S_d\, u))[n] = y[n]$$

- **Memoryless Systems:**

  A continuous time system is memoryless if and only if $y(t_0)$ only depends on $u(t_0)$ for all $t \in \mathbb{R}$.

  A discrete time system is memoryless if and only if $y[n_0]$ only depends on $u[n_0]$ for all $t \in \mathbb{Z}$.

- **Causality:**

  We say the system is causal if the output does not depend on future values of the input. Mathematically we can show causality using the *truncation* operator, P_T. For continuous systems *truncation* is defined as

$$(P_T\, u)(t) = \begin{cases} u(t) & \text{for } t \leq T \leq 0 \\ 0 & \text{otherwise} \end{cases}$$

  for discrete systems

$$(P_k\, u)[n] = \begin{cases} u[n] & \text{for } n \leq k \leq 0 \\ 0 & \text{otherwise} \end{cases}$$

  then the system, S (continuous or discrete), is said to be causal if $P_T S = P_T S P_T$

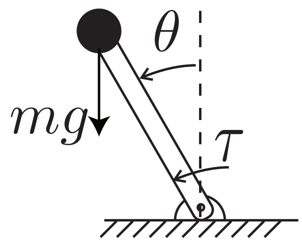
- **Finite & Infinite Dimensional Systems:**

  A continuous dynamical system, S_c, is finite dimensional if there exist an ODE in u, y that models S_c.

  A discrete time dynamical system, S_d, is finite dimensional if there exist an Ordinary Difference Equation in u, y that models S_d.

2.2. **Examples.**

(1) $u(t) = \tau(t),\ y(t) = \theta(t)$



Non-linear, Time-invariant, causal, system has memory, and finite dimensional

(2) Now let's assume that $g = 0$, what happens?

Linear, Time-invariant, causal, system has memory, and finite dimensional

(3) $y(t) = \int_0^t (t-s)^3 u(s) ds$

Linear, causal, system has memory, finite dimensional,

Since the system is causal it may be OK to assume that the set of inputs are limited to causal signals and the convolution is unilateral. In this case the system is *time-invariant*.

However if non-causal signals are allowed then the system becomes *time-varying*.

3. Representations of Dynamical Systems

3.1. Differential & Difference Equations.

- **Continuous Time Systems - ODEs**

Linear Time Invariant System (LTI)

$$a_n y^{(n)} + ... + a_1 y' + a_0 y = b_n u^{(n)} + ... + b_1 u' + b_0 u$$

Linear Time Varying System (LTV)

$$a_n(t) y^{(n)} + ... + a_1(t) y' + a_0(t) y = b_n(t) u^{(n)} + ... + b_1(t) u' + b_0(t) u$$

Non-linear Time Invariant System

$$y^{(n)} = f(y^{(n-1)}, ..., y', y, u^{(n)}, ..., u', u)$$

Non-linear Time Varying System

$$y^{(n)} = f(y^{(n-1)}, ..., y', y, u^{(n)}, ..., u', u, t)$$

- **Discrete Time Systems - Difference Equations**

Discrete-Time Linear Time Invariant System (LTI)

$$a_n y[k] + a_{n-1} y[k-1] + ... + a_0 y[k-n] = b_n u[k] + ... + b_0 u[k-n]$$

Discrete-Time Linear Time Varying System (LTV)

$$a_n[k] y[k] + a_{n-1}[k] y[k-1] + ... + a_0[k] y[k-n] = b_n[k] u[k] + ... + b_0[k] u[k-n]$$

Non-linear Time Invariant System

$$y[k] = f(y[k-1], ..., y[k-n], u[k], ..., u[k-n])$$

Non-linear Time Varying System

$$y[k] = f(y[k-1], ..., y[k-n], u[k], ..., u[k-n], k)$$

Discussion

- When an ODE representation becomes *memoryless*?
- When a difference equation representation becomes *memoryless*?
- What about infinite dimensional systems?
- What about *causality*?

3.2. State-Space Representation of Dynamical Systems.

- **Continuous-Time Dynamical Systems**

  Linear Time Invariant Systems

$$\begin{aligned} \text{Let } x(t) &\in \mathbb{R}^n\ ,\ y(t) \in \mathbb{R}\ ,\ u(t) \in \mathbb{R}, \\ \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t), \\ \text{where } A &\in \mathbb{R}^{n\times n}\ ,\ B \in \mathbb{R}^{n\times 1}\ ,\ C \in \mathbb{R}^{1\times n}\ ,\ D \in \mathbb{R} \end{aligned}$$

  Linear Time Varying Systems

$$\begin{aligned} \text{Let } x(t) &\in \mathbb{R}^n\ ,\ y(t) \in \mathbb{R}\ ,\ u(t) \in \mathbb{R}, \\ \dot{x}(t) &= A(t)x(t) + B(t)u(t), \\ y(t) &= C(t)x(t) + D(t)u(t), \\ \text{where } A(t) &\in \mathbb{R}^{n\times n}\ ,\ B(t) \in \mathbb{R}^{n\times 1}\ ,\ C(t) \in \mathbb{R}^{1\times n}\ ,\ D(t) \in \mathbb{R} \end{aligned}$$

  Non-Linear Time Invariant Systems

$$\begin{aligned} \text{Let } x(t) &\in \mathbb{R}^n\ ,\ y(t) \in \mathbb{R}\ ,\ u(t) \in \mathbb{R}, \\ \dot{x}(t) &= F(x(t), u(t)), \\ y(t) &= H(x(t), u(t)), \end{aligned}$$

  Non-Linear Time Varying Systems

$$\begin{aligned} \text{Let } x(t) &\in \mathbb{R}^n\ ,\ y(t) \in \mathbb{R}\ ,\ u(t) \in \mathbb{R}, \\ \dot{x}(t) &= F(x(t), u(t), t), \\ y(t) &= H(x(t), u(t), t), \end{aligned}$$

- **Discrete-Time Dynamical Systems**

  Linear Time Invariant Systems

$$\begin{aligned} \text{Let } x[n] &\in \mathbb{R}^n\ ,\ y[n] \in \mathbb{R}\ ,\ u[n] \in \mathbb{R}, \\ x[n+1] &= Ax[n] + Bu[n], \\ y[n] &= Cx[n] + Du[n], \\ \text{where } A &\in \mathbb{R}^{n\times n}\ ,\ B \in \mathbb{R}^{n\times 1}\ ,\ C \in \mathbb{R}^{1\times n}\ ,\ D \in \mathbb{R} \end{aligned}$$

Linear Time Varying Systems

$$
\begin{aligned}
\text{Let } x[n] &\in \mathbb{R}^n \ , \ y[n] \in \mathbb{R} \ , \ u[n] \in \mathbb{R}, \\
x[n+1] &= A[n]x[n] + B[n]u[n], \\
y[n] &= C[n]x[n] + D[n]u[n], \\
\text{where } A[n] &\in \mathbb{R}^{n\times n} \ , \ B[n] \in \mathbb{R}^{n\times 1} \ , \ C[n] \in \mathbb{R}^{1\times n} \ , \ D[n] \in \mathbb{R}
\end{aligned}
$$

Non-Linear Time Invariant Systems

$$
\begin{aligned}
\text{Let } x[n] &\in \mathbb{R}^n \ , \ y[n] \in \mathbb{R} \ , \ u[n] in \mathbb{R}, \\
x[n+1] &= F(x[n], u[n]), \\
y[n] &= H(x[n], u[n]),
\end{aligned}
$$

Non-Linear Time Varying Systems

$$
\begin{aligned}
\text{Let } x[n] &\in \mathbb{R}^n \ , \ y[n] \in \mathbb{R} \ , \ u[n] \in \mathbb{R}, \\
x[n+1] &= F(x[n], u[n], n), \\
y[n] &= H(x[n], u[n], n),
\end{aligned}
$$

Discussion

- When a state-space representation becomes *memoryless*?
- What about infinite dimensional systems?
- What about *causality*?

3.3. Impulse-Response Representation of Dynamical Systems.

- **Continuous-Time Dynamical Systems**

  Linear Time Invariant Systems

$$
y(t) = h(t) * u(t) = \int_{-\infty}^{\infty} h(t-\tau)u(\tau)d\tau
$$

  Linear Time Varying Systems

$$
y(t) = \int_{-\infty}^{\infty} h(t,\tau)u(\tau)d\tau
$$

  where $h(t,\tau)$ is called time-varying impulse response function.

- **Discrete-Time Dynamical Systems**

  Linear Time Invariant Systems

$$y[n] = \sum_{k=-\infty}^{\infty} h[n-k]u[k]$$

Linear Time Varying Systems

$$y[n] = \sum_{k=-\infty}^{\infty} h[n,k]u[k]$$

Discussion

- Under what condition(s) an impulse response representation becomes *memoryless*?
- Under what condition(s) an impulse response representation becomes *causal*?
- What about finite and infinite dimensional systems?
- What are the differences between continuous time and discrete time impulse response?

3.4. **Transfer Functions Representation of Dynamical Systems.**

- **Continuous-Time Dynamical Systems**

  Linear Time Invariant Systems

$$Y(s) = G(s)U(s), \text{ where,}$$
$$Y(s) = \mathcal{L}\{y(t)\}\ , \&\ U(s) = \mathcal{L}\{u(t)\}$$

- **Discrete-Time Dynamical Systems**

  Linear Time Invariant Systems

$$Y(z) = G(z)U(z), \text{ where,}$$
$$Y(z) = \mathcal{Z}\{y[n]\}\ , \&\ U(z) = \mathcal{Z}\{u[n]\}$$

  Discussion

  - Can we model/represent non-linear systems using transfer functions?
  - Can we model/represent linear time-varying systems using transfer functions?
  - What about finite and infinite dimensional systems?
  - What about *causality*?

EE402 - Discrete Time Systems	**Spring 2018**

Lecture 2

Lecturer: Asst. Prof. M. Mert Ankarali

*Big Picture of EE402

In this course, the main focus will be on continuous-time systems that are controlled (sampled and actuated) by a digital computer interface. Such a discrete-time control system consists of four major parts as illustrated in Fig. 2.2,

1. *The plant* is a continuous-time dynamical system
2. The Analog-to-Digital Converter (ADC)
3. The Controller (μP), a microprocessor/microcontroller with a "real-time" OS
4. The Digital-to-Analog Converter (DAC)

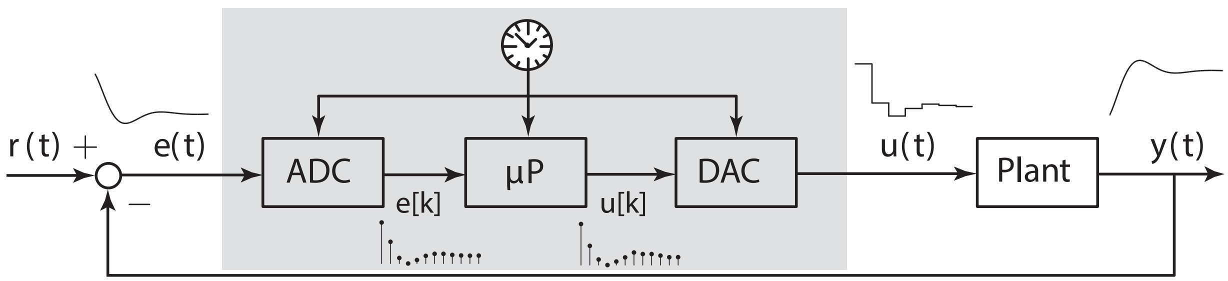


Figure 2.1: Block diagram of a digital control system

Most of the time the plant is modeled as a "smooth" continuous dynamical system. In this course our focus is LTI systems thus, we will assume that (unless otherwise is given) that the plant is a continuous LTI plant model with a transfer function of $G_c(s)$ for which both the input and output are continuous time signals.

The "digital" blocks of the closed-loop block diagram structure are ADC, The Controller, and DAC. It is generally assumed (design preference) that all blocks shares a common "hard real time" clock.

A general ADC is a device that converts an analog signal to a digital signal. In this course we will model the ADC block as an *ideal sampler* for which the input is a continious-time signal, $e(t)$ and the output is a discrete-time signal, $e[k]$, where the relating between the continuous- and discrete-time signals are given as

$$e(kT) = e[k], \; k \in \mathbb{Z}^+,$$

where constant T is the *sampling time*.

The microcontoller/microprocessor processes some set of digital input signals to produce some set of digital output signals. The outputs are defined at only certain instances defined by the real-time clock. In this course, we will model the μP block as an ideal discrete-time LTI system for which the both the input and output are discrete-time signals, with a transfer function of $G_c(z)$.

The DAC is a device that converts a digital signal to an analog signal. In this course we assume that it is an ideal *Hold* element for which the input signal is a discrete-time signal, where as output is a continuous-time signal. The most commonly used *Hold* system is ZOH (Zero-Order-Hold) which is a mapping defined by the following relation

$$u(t) = u[k], \text{ for} t \in [kT, (k+1)T)$$

Higher-order holds available but seldom used.

The idealized and simplified block-diagram structure is given in Fig.

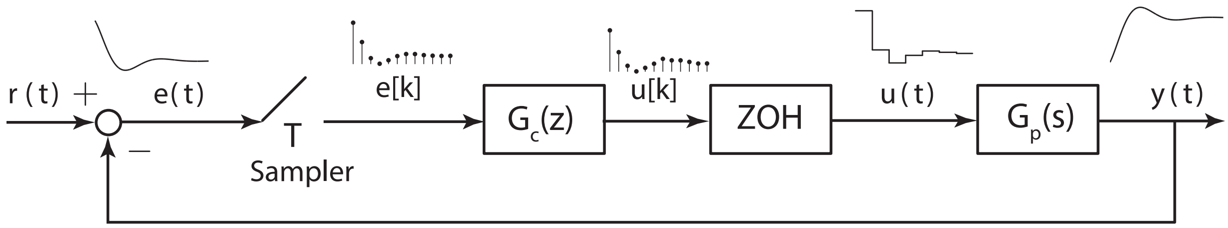


Figure 2.2: Block diagram of an LTI discrete-time control system

Major challenge: Loop contains both continuous-time and discrete-time parts.

Sampling

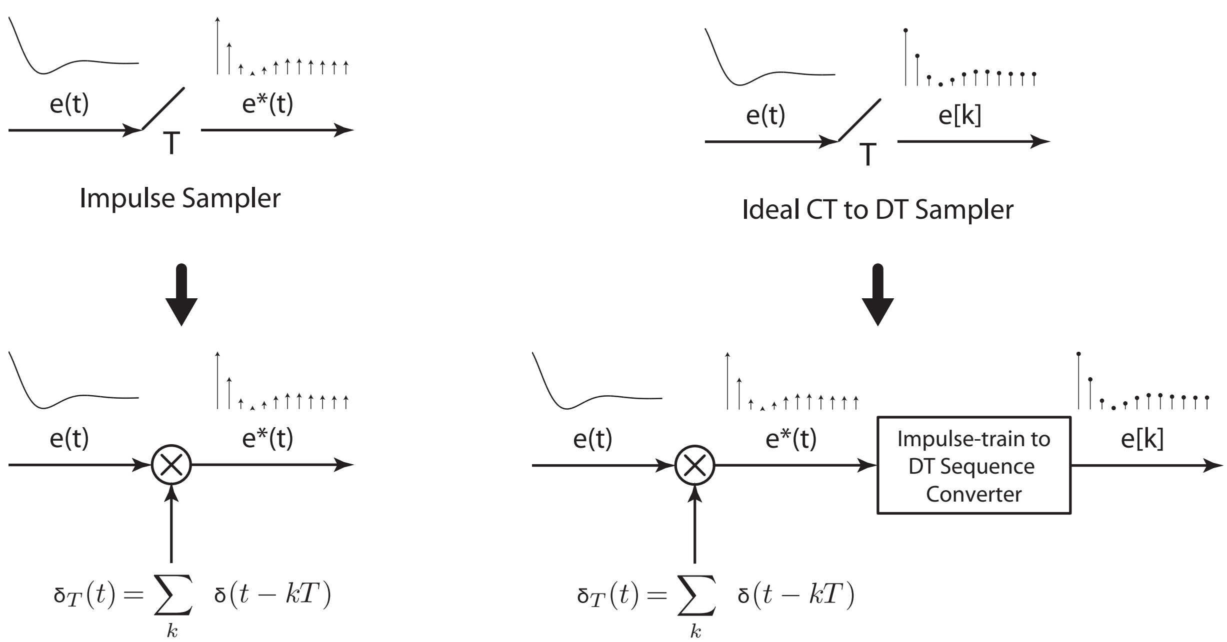


Figure 2.3: Two different ideal samplers

Fig. 2.3 illustrates two different ideal sampler (both of them will be covered in this course. Firs column is an *impulse sampler* for which the output is a continuous signal but it is composed of trains of impulses (impulse train). Second one is an ideal complete CT-to-DT sampler which convertes the impulse train into DT sequence.

The output of the impulse sampler, $x*(t)$, can be represented with the following infinte summations

$$
\begin{aligned}
x^*(t) &= \sum_{k=0}^{\infty} x(kT)\delta(t-kT) = \sum_{k=0}^{\infty} x[k]\delta(t-kT) \\
&\text{or} \\
x^*(t) &= x(0)\delta(t) + x(T)\delta(t-T) + \cdots + x(kT)\delta(t-kT) + \cdots \\
&= x[0]\delta(t) + x[1]\delta(t-T) + \cdots + x[k]\delta(t-kT) + \cdots
\end{aligned}
$$

Now let's consider the Laplace transform of $x^*(t)$

$$
\begin{aligned}
X^*(s) = \mathcal{L}\{x^*(t)\} &= \mathcal{L}\left\{\sum_{k=0}^{\infty} x(kT)\delta(t-kT)\right\} = \sum_{k=0}^{\infty} x(kT)\mathcal{L}\left\{\delta(t-kT)\right\} \\
&= \sum_{k=0}^{\infty} x(kT)\int_{t=0}^{\infty} \delta(t-kT)e^{-st}dt = \sum_{k=0}^{\infty} x(kT)e^{-skT} = \sum_{k=0}^{\infty} x[k]e^{-skT}
\end{aligned}
$$

Now let's define a map in complex domain such that

$$z = e^{Ts} \text{ or } s = \frac{1}{T}\ln z$$

Then we have

$$X^*(s)|_{s=(1/T)lnz} = \sum_{k=0}^{\infty} x[k]z^{-k}$$

where

$$X(z) = \mathcal{Z}\{x[k]\} = \sum_{k=0}^{\infty} x[k]z^{-k}$$

Z-transform

Z-transform of a (causal) discrete time signal $x[k]$ is given by

$$X(z) = \mathcal{Z}\{x[k]\} = \sum_{k=0}^{\infty} x[k]z^{-k}$$

If $x[k]$ is a sampled signal from a continuous time signal $x(t)$ with a sampling time of T, we (abuse of notation) also use the following notation

$$X(z) = \mathcal{Z}\{x(kT)\} = \mathcal{Z}\{x^*(t)\}$$

Z-transforms of elementary functions

We assume that all signals are causal thus $t \in \mathbb{R}^+$ and $k \in \mathbb{Z}^+$

Unit-step function $x(t) = 1$ and thus $x(kT) = x[k] = 1$, the Z-transform is given by

$$X(z) = \sum_{k=0}^{\infty} z^{-k} = \frac{1}{1 - z^{-1}} = \frac{z}{z-1}$$

Unit-ramp function $x(t) = t$ and thus $x(kT) = x[k] = kT$, the Z-transform is given by

$$\begin{aligned}
X(z) &= \sum_{k=0}^{\infty} kTz^{-k} = T(z^{-1} + 2z^{-2} + 3z^{-3} + \cdots) = Tz(z^{-2} + 2z^{-3} + 3z^{-4} + \cdots) \\
&= Tz\frac{d}{dz}\left(\int (z^{-2} + 2z^{-3} + 3z^{-4} + \cdots)dz\right) = Tz\frac{d}{dz}\left(-(z^{-1} + z^{-2} + z^{-3} + \cdots)\right) \\
&= Tz\frac{d}{dz}\left(\frac{-1}{z-1}\right) = \frac{Tz}{(z-1)^2} = \frac{Tz^{-1}}{(1-z^{-1})^2}
\end{aligned}$$

Exponential sequence $x[k] = a^k$

$$X(z) = \sum_{k=0}^{\infty} a^k z^{-k} = \sum_{k=0}^{\infty} \left(\frac{z}{a}\right)^{-k} = \frac{1}{1 - az^{-1}} = \frac{z}{z-a}$$

Exponential function $x(t) = e^{bt}$ and thus $x(kT) = x(k) = e^{bTk}$

$$X(z) = \sum_{k=0}^{\infty} e^{bTk} z^{-k} = \sum_{k=0}^{\infty} (e^{bT})^k z^{-k} = \frac{1}{1 - e^{bT}z^{-1}} = \frac{z}{z - e^{bT}}$$

Cosine function $x(t) = \cos(\omega t)$, and thus $x(kT) = x(k) = \cos(\omega Tk)$

$$\begin{aligned}
\cos(\omega Tk) = \frac{1}{2}\left(e^{j\omega Tk} + e^{-j\omega Tk}\right) X(z) &= \frac{1}{2}\left(\frac{z}{z - e^{j\omega T}} + \frac{z}{z - e^{-j\omega T}}\right) = \frac{1}{2}\frac{z(z - e^{-j\omega T}) + z(z - e^{j\omega T})}{(z - e^{-j\omega T})(z - e^{j\omega T})} \\
&= \frac{1}{2}\frac{2z^2 - z(e^{-j\omega T} + e^{j\omega T})}{z^2 - z(e^{-j\omega T} + e^{j\omega T}) + 1} = \frac{z^2 - z\cos(\omega T)}{z^2 - z2\cos(\omega T) + 1} \\
&= \frac{1 - z^{-1}\cos(\omega T)}{1 - z^{-1}2\cos(\omega T) + z^{-2}}
\end{aligned}$$

Properties and Theorems of the Z-transform

Linearity

$$x(k) = \alpha f(k) + \beta g(k) \;\rightarrow X(z) = \alpha F(z) + \beta G(z)\;, \forall\; \alpha,\; \beta,\; f(k),\; \&\; g(k)$$

Multiplication by a^k

$$\begin{aligned}
\mathcal{Z}\{a^k x[k]\} &= \sum_{k=0}^{\infty} a^k x[k] z^{-k} = \sum_{k=0}^{\infty} x[k](z/a)^{-k} \\
\mathcal{Z}\{a^k x[k]\} &= X(z/a)
\end{aligned}$$

Complex translation theorem Let $y(t) = e^{-at}x(t)$ and $X(z) = \ddagger\{x(kT)\}$, then

$$\mathcal{Z}\{y(kT)\} = \mathcal{Z}\{e^{-aTk}x(kT)\} = X(e^{aT}z)$$

Shifting theorem Let $x(t)$ be a causal CT signal, thus we have $x(t) = 0$ for $t < 0$. Similarly sampled DT signal has the property $x[nk] = 0$ for $k < 0$. For the sake of simplicity lets work on the sampled (i.e. DT) signal. Let

$$\mathcal{Z}\{x^*(t)\} = \mathcal{Z}\{x[k]\} = X(z)$$

Shifting right by N (Causal shifting): Let $y[k] = x[k - N]$, then

$$\mathcal{Z}\{y[k]\} = \sum_{k=0}^{\infty} y[k]z^{-k} = \sum_{k=0}^{\infty} x[k-N]z^{-k} = \sum_{k=N}^{\infty} x[k-N]z^{-k}$$

Let $k = m + N$ then

$$\begin{aligned}
\mathcal{Z}\{y[k]\} &= \sum_{m=0}^{\infty} x[m]z^{-(m+N)} = z^{-N}\sum_{m=0}^{\infty} x[m]z^{-m} \\
\mathcal{Z}\{x[k-N]\} &= z^{-N}X(z)
\end{aligned}$$

Shifting left by N (Non-causal shifting) & Bilateral Z transform: Let $y[k] = x[k + N]$,

$$\begin{aligned}
\mathcal{Z}\{x[k+N]\} &= \sum_{k=-\infty}^{\infty} x[k+N]z^{-k} = \sum_{m=-\infty}^{\infty} x[m]z^{-(m-N)} = z^{N}\sum_{m=-\infty}^{\infty} x[m]z^{-m} \\
\mathcal{Z}\{x[k+N]\} &= z^{N}X(z)
\end{aligned}$$

Shifting left by N (Non-causal shifting) & Unilateral Z transform: Let $y[k] = x[k + N]$,

$$\mathcal{Z}\{x[k+N]\} = \sum_{k=0}^{\infty} x[k+N]z^{-k}$$

Let $k = m - N$ then

$$\begin{aligned}
\mathcal{Z}\{x[k+N]\} &= \sum_{m=N}^{\infty} x[m]z^{-(m-N)} = z^{N}\sum_{m=N}^{\infty} x[m]z^{-m} = z^{N}\left(\sum_{k=0}^{\infty} x[k]z^{-k} - \sum_{k=0}^{N-1} x[k]z^{-k}\right) \\
\mathcal{Z}\{x[k+N]\} &= z^{N}\left(X(z) - \sum_{k=0}^{N-1} x[k]z^{-k}\right)
\end{aligned}$$

From this equation we can obtain

$$\begin{aligned}
\mathcal{Z}\{x[k+1]\} &= zX(z) - zx[0] \\
\mathcal{Z}\{x[k+2]\} &= z^2X(z) - z^2x[0] - zx[1] \\
&\vdots
\end{aligned}$$

Example 1. Let $u[k]$ be the unit-step function. Compute $\mathcal{Z}\{u[k - 1]\}$ both directly and using the shifting property.

$$\mathcal{Z}\{u[k-1]\} = \frac{z^{-1}}{1 - z^{-1}}$$

Example 2. Let $y[k] = \sum_{n=0}^{k} x[k]$ where $k \in \mathbb{Z}^+$. Compute $Y(z)$ in terms of $X(z)$ using the shifting theorem.

$$Y(z) = \frac{1}{1 - z^{-1}} X(z)$$

Initial Value Theorem Let $X(z) = \mathcal{Z}\{x[n]\}$ and if the following limit exists, then the initial value of $x[0]$ or $x(0)$ is given by

$$x[0] = \lim_{z \to \infty} X(z)$$

Indeed the proof is very easy

$$\lim_{z \to \infty} X(z) = \lim_{z \to \infty} \left[\sum_{k=0}^{\infty} x(k) z^{-k} \right] = \lim_{z \to \infty} \left[x(0) + x(1) z^{-1} + x(2) z^{-2} + \cdots \right] = x(0)$$

Final Value Theorem

Let's assume that $x(kT)$ or $x[k]$ is a convergent sequence (DT signal). Then the final value theorem states that

$$\lim_{k \to \infty} x[k] = \lim_{z \to 1} (1 - z^{-1}) X(z)$$

Proof: Let's take the Z transform of $x[k] - x[k-1]$

$$\begin{aligned}
\mathcal{Z}\{x[k] - x[k-1]\} &= \sum_{k=0}^{\infty} \left(x[k] - x[k-1] \right) z^{-k} \\
X(z) - X(z) z^{-1} &= \left(x[0] \left(1 - z^{-1}\right) + x[1] \left(z^{-1} - z^{-2}\right) + x[2] \left(z^{-2} - z^{-3}\right) + x[3] \left(z^{-3} - z^{-4}\right) + \cdots \right) + \lim_{k \to \infty} x[k] z^{-k} \\
\lim_{z \to 1} X(z) \left(1 - z^{-1}\right) &= (0 + 0 + \cdots) + \lim_{z \to 1} \lim_{k \to \infty} x[k] z^{-k} \\
\lim_{z \to 1} X(z) \left(1 - z^{-1}\right) &= \lim_{k \to \infty} x[k]
\end{aligned}$$

Complex Differentiation Theorem

Consider

$$\begin{aligned}
\frac{d}{dz} X(z) &= \frac{d}{dz} \left[\sum_{k=0}^{\infty} x[k] z^{-k} \right] = \sum_{k=0}^{\infty} x[k] \frac{d}{dz} z^{-k} = \sum_{k=0}^{\infty} (-k) x[k] z^{-k-1} \\
-z \frac{d}{dz} X(z) &= \sum_{k=0}^{\infty} k x[k] z^{-k} \\
-z \frac{d}{dz} X(z) &= \mathcal{Z}\{k x[k]\}
\end{aligned}$$

In general

$$(-z)^m \frac{d}{dz^m} X(z) = \mathcal{Z}\{k^m x[k]\}$$

Example 3. Find the Z-transform of the unit ramp function, $r[k] = k$, $k \in \mathbb{Z}^+$ by applying the Complex Differentiation Theorem to the Z-transform of the unit step function.

Real Convolution Theorem Let $f[k]$ and $g[k]$ are causal signals and associated Z transforms are $F(z)$ and $G(z)$ respectively. The DT convolution operator is defined as

$$f[n] * g[n] = \sum_{k=0}^{n} f[n-k]g[k]$$

Real Convolution Theorem states that

$$\mathcal{Z}\{f[n] * g[n]\} = F(z)G(z)$$

Proof

$$\mathcal{Z}\{f[n] * g[n]\} = \sum_{n=0}^{\infty} \left[\sum_{k=0}^{n} f[n-k]g[k] \right] z^{-n}$$

Since we know that $f[m] = 0$ for $m < 0$, we can stretch the upper limit of the sum as

$$\mathcal{Z}\{f[n] * g[n]\} = \sum_{n=0}^{\infty} \left[\sum_{k=0}^{\infty} f[n-k]g[k] \right] z^{-n} = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} f[n-k]g[k]z^{-n}$$

Let $n = m + k$ then

$$\begin{aligned} \mathcal{Z}\{f[n] * g[n]\} &= \sum_{k=0}^{\infty} \sum_{m=-k}^{\infty} f[m]g[k]z^{-m}z^{-k} = \sum_{k=0}^{\infty} g[k]z^{-k} \sum_{m=0}^{\infty} f[m]z^{-m} \\ \mathcal{Z}\{f[n] * g[n]\} &= F(z)G(z) \end{aligned}$$

The Inverse Z-transform

1. Direct division method
2. Z-tranform tables & partial-fraction expansion
3. "Simulation" method
4. Inversion integral method

Direct division

Direct division (or long division) method uses the fact that $X(z)$ can be expressed as

$$X(z) = \sum_{k=0}^{\infty} x[k]z^{-k} = x(0) + x(1)z^{-1} + x(2)z^{-2} + x(3)z^{-3} + \cdots$$

The goal is finding the power series expansion of $X(z)$ using the long division approach. Here we assume that $X(z)$ cane be represnted as a ratio of two polynomials in z (or z^{-1})

$$X(z) = \frac{b_0 z^m + b_1 z^{m-1} + \cdots + b_m}{z^n + a_1 z^{n-1} + \cdots + a_n} = \frac{b_0 z^{-n+m} + b_1 z^{-n+m-1} + \cdots + b_m z^{-n}}{1 + a_1 z^{-1} + \cdots + a_n z^{-n}}$$

For the direct division method it is easer to work when the polynomials are written interms of powers of z^{-1}.

Example 4. Find the inverse Z-transform of $X(z) = \frac{z^{-1}}{1-2z^{-1}+z^{-2}}$.

$$
\begin{array}{r|l}
z^{-1} & 1 - 2z^{-1} + z^{-2} \\
\cline{2-2}
z^{-1} - 2z^{-2} + z^{-3} & z^{-1} + 2z^{-2} + 3z^{-3} + 4z^{-4} + \ldots \\
\cline{1-1}
2z^{-2} - z^{-3} & \\
2z^{-2} - 4z^{-3} + 2z^{-4} & \\
\cline{1-1}
3z^{-3} - 2z^{-4} & \\
3z^{-3} - 6z^{-4} + 3z^{-5} & \\
\cline{1-1}
4z^{-4} - 3z^{-5} & \\
\vdots &
\end{array}
$$

Thus,

$$
\begin{array}{l}
X(z) = 0 + 1z^{-1} + 2z^{-2} + 3z^{-3} + 4z^{-4} + \ldots \\
\quad \downarrow \\
x[k] = 0\delta[k] + 1\delta[k-1] + 2\delta[k-2] + 3\delta[k-3] + 4\delta[k-4] + \ldots = k
\end{array}
$$

Partial Fraction Expansion

In most applications $X(z)$ can be re-written in terms of poles and zeros as

$$X(z) = b_0 \frac{(z-z_1)\cdots(z-z_m)}{(z-p_1)\cdots(z-p_n)} \quad (m \leq n)$$

Specific (but extremely common) case

$$\frac{X(z)}{z} = \sum_{i=1}^{n} \frac{a_i}{(z-p_i)}$$

where all poles are distinct and simple order. We can compute each a_i using

$$a_i = \lim_{z \to p_i} \left[(z-p_i) \frac{X(z)}{z} \right]$$

Example 5. Find the inverse Z-transform $X(z) = \frac{(1-b)z}{(z-1)(z-b)}$. Solution:

$$\begin{aligned}
\frac{X(z)}{z} &= \frac{(1-b)}{(z-1)(z-b)} = \frac{a_1}{z-1} + \frac{a_2}{z-b} \\
a_1 &= \lim_{z\to 1}\left[(z-1)\frac{X(z)}{z}\right] = 1 \\
a_2 &= \lim_{z\to b}\left[(z-b)\frac{X(z)}{z}\right] = -1 \\
X(z) &= \frac{z}{z-1} - \frac{z}{z-b} \\
x[k] &= 1 - b^k
\end{aligned}$$

Now let's assume that $\frac{X(z)}{z}$ has double pole at p_1 and all other poles are distinct

$$\frac{X(z)}{z} = \frac{c_1}{z-p_1} + \frac{c_2}{(z-p_1)^2} + \cdots$$

It is easy to show that

$$c_2 = \lim_{z\to p_1}\left[(z-p_1)^2\frac{X(z)}{z}\right]$$

It is also possible to show that

$$c_1 = \lim_{z\to p_1}\left\{\frac{d}{dz}\left[(z-p_1)^2\frac{X(z)}{z}\right]\right\}$$

Example 6. Find the inverse Z-transform $X(z) = \frac{2z^2-3z}{(z-1)^2}$. Solution:

$$\begin{aligned}
\frac{X(z)}{z} &= \frac{c_1}{z-1} + \frac{c_2}{(z-1)^2} \\
c_1 &= \lim_{z\to 1}\frac{d}{z}\left[(z-1)^2\frac{X(z)}{z}\right] = 2 \\
c_2 &= \lim_{z\to 1}\left[(z-1)^2\frac{X(z)}{z}\right] = -1 \\
x[k] &= 2 - k
\end{aligned}$$

Example 7. Find the inverse Z-transform $X(z) = \frac{(1-b)}{(z-1)(z-b)}$. Solution:

$$\begin{aligned}
X(z) &= \frac{(1-b)}{(z-1)(z-b)} = \frac{a_1}{z-1} + \frac{a_2}{z-b} \\
a_1 &= \lim_{z\to 1}[(z-1)X(z)] = 1 \\
a_2 &= \lim_{z\to b}[(z-b)X(z)] = -1 \\
X(z) &= z^{-1}\left(\frac{z}{z-1} - \frac{z}{z-b}\right) \\
x[k] &= [1-b^{k-1}]u[k-1]
\end{aligned}$$

Example 8. Find the inverse Z-transform $X(z) = \frac{z^2-2}{(z-1)(z-2)}$. Solution:

$$\begin{aligned}
X(z) &= \frac{z^2-2}{z^2-3z+2} = 1 + \frac{3z-4}{z^2-3z+2} \\
X(z) &= 1 + \frac{a_1}{z-1} + \frac{a_2}{z-2} \\
a_1 &= \lim_{z\to 1}\left[(z-1)X(z)\right] = 1 \\
a_2 &= \lim_{z\to 2}\left[(z-2)X(z)\right] = 2 \\
X(z) &= 1 + \frac{1}{z-1} + \frac{2}{z-2} = \frac{1}{z-1} + \frac{z}{z-2} \\
x[k] &= 1 + 2^k - \delta[k]
\end{aligned}$$

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Lecture 3

Lecturer: Asst. Prof. M. Mert Ankarali

Difference Equations

In discrete-time domain, we have difference equations that replaces differential equations. We are mainly interested in LTI systems, that are represented by linear constant coefficient difference equations. Let $x[k]$ and $y[k]$ be the input and output respectively, then an LTI difference equation can be expressed as

$$a_0 y[k] + a_1 y[k-1] + ... + a_N y[k-N] = b_0 x[k] + ... + b_M x[k-M]$$
$$\sum_{n=1}^{N} a_n y[k-n] = \sum_{n=1}^{M} b_n x[k-n]$$

Unlike ODEs difference equations are very easy to solve computationally or simulate in computer environment. Let's consider the following first-order difference equation

$$y[k] = \frac{1}{2} y[k-1] + x[k] \quad , x[k] = 0 \ \& \ y[k] = 0, \ \text{for} k < 0$$

Let's "simulate" the difference equation for $x[k] = \delta[k]$.

$$\begin{aligned}
y[0] &= \frac{1}{2} y[-1] + x[0] = 0 + 1 = 1 \\
y[1] &= \frac{1}{2} y[0] + x[1] = \frac{1}{2} + 0 = \frac{1}{2} \\
y[2] &= \frac{1}{2}\frac{1}{2} = \frac{1}{4} \\
y[3] &= \frac{1}{2}\frac{1}{4} = \frac{1}{8} \\
&\vdots \\
y[k] &= \left(\frac{1}{2}\right)^k
\end{aligned}$$

Now let's simulate for $x[k] = u[k]$

$$\begin{aligned}
y[0] &= 0 + 1 = 1 \\
y[1] &= \frac{1}{2} + 1 \\
y[2] &= \frac{1}{4} + \frac{1}{2} + 1 \\
y[3] &= \frac{1}{8} + \frac{1}{4} + \frac{1}{2} + 1 \\
&\vdots \\
y[k] &= \frac{1}{2^k} + \cdots + \frac{1}{2} + 1 = 2 - \left(\frac{1}{2}\right)^k
\end{aligned}$$

This is a great method for "simulating" using a computational approach, but in general it may be very hard to get a closed from expression. The most basic solution method is solving the difference equation directly in time domain by trying to find a "basis" for the solution space similar to the operation in ODEs. We try sequences/signals of the form λ^k , $k > 0$ to find a solution form for the homogeneous equation. Let's apply this method for the first-order difference equation above

$$\begin{aligned} y[k] = \lambda^k \ \rightarrow y[k] - \frac{1}{2}y[k-1] &= 0 \\ \lambda^k - \frac{\lambda^{k-1}}{2} &= 0 \\ \lambda^{k-1}\left(\lambda - \frac{1}{2}\right) &= 0 \\ \lambda - \frac{1}{2} &= 0 \end{aligned}$$

Where the last equation is the characteristic equation of the difference equation. Since the characteristic equation has one root only, we obtain a solution of the form

$$y[k] = y_h[k] + y_p[k] = C\left(\frac{1}{2}\right)^k + y_p[k]$$

Let's assume that for $x[k] = u[k]$ particular solution has the form $y_p[k] = A$ for $k > 0$ then

$$A = \frac{1}{2}A + 1 \ \rightarrow A = 2$$

Now let's find C using the fact that $y[k] = 0$ for $k < 0$

$$\begin{aligned} y[0] = \frac{1}{2}y[-1] + x[0] \ &\rightarrow \ y[0] = 1 \\ 1 = C\left(\frac{1}{2}\right)^0 + 2 &\rightarrow C = -1 \end{aligned}$$

Then the solution can be written as

$$y[k] = -\left(\frac{1}{2}\right)^k + 2$$

Example 1.1 Find the general form of the homogeneous solution for the following difference equation

$$y[k] - 3y[k-1] + 2y[k-2] = x[k]$$

Solution:

$$\begin{aligned} &\lambda^2 - 3\lambda + 2 = 0 \\ &\lambda_1 = 1 \ \& \ \lambda_2 = 2 \\ &y[k] = C_1 + C_2 2^k \ , k > 0 \end{aligned}$$

Example 1.2 Now let's assume that $y[k] = 0$ for $k < 0$ and $x[k] = 3^k$, then find $y[k]$ for $k \geq 0$.

Solution: First let's find a particular solution. Let's assume that $y_p[k] = A3^k$, then

$$\begin{aligned} A3^k - 3A3^{k-1} + 2A3^{k-2} = 3^k \ &\rightarrow \ A = 9/2 \\ y_p[k] = 4.5\ 3^k& \end{aligned}$$

Now let's try to find C_1 and C_2

$$
\begin{aligned}
& y[k] - 3y[k-1] + 2y[k-2] = x[k] \\
& y[0] = x[0] \;\rightarrow\; C_1 + C_2 = -3.5 \\
& y[1] - 3y[0] = x[1] \;\rightarrow\; C_1 + C_2 2 = -7.5 \\
& C_1 = 0.5 \;\&\; C_2 = -4 \\
& y[k] = 0.5 - 4\,2^k + 4.5\,3^k \;, k > 0
\end{aligned}
$$

What about repeated roots? Possible mini project question

Example 2 Find the general form of the homogeneous solution for the following difference equation

$$y[k] + 4y[k-2] = x[k]$$

Solution:

$$
\begin{aligned}
& \lambda^2 + 4 = 0 \;\rightarrow\; \lambda_{1,2} = \pm 2j \\
& y[k] = C_1(2j)^k + C_2(-2j)^k = C_1 2^k e^{j\pi k} + C_2 2^k e^{-j\pi k} \\
& y[k] = \bar{C}_1 2^k \frac{e^{j\pi k} + e^{j\pi k}}{2} + \bar{C}_2 2^k \frac{e^{j\pi k} - e^{j\pi k}}{2j} \\
& y[k] = \bar{C}_1 2^k \cos(\pi k) + \bar{C}_2 2^k \sin(\pi k)
\end{aligned}
$$

How we can generalize this to arbitrary complex conjugate roots? Possible mini project question

What is the home message? Similar to ODEs time domain solution of difference equations is generally "messy".

Z-transform & Difference Equations

Difference Equations to Z-transform

Let's consider the following difference equation with $y[n]$ and $x[n]$ be the strictly causal input–output pair.

$$a_0 y[k] + a_1 y[k-1] + ... + a_N y[k-N] = b_0 x[k] + ... + b_M x[k-M]$$

Now let's assume that $\mathcal{Z}\{x[k]\} = X(z)$ and $\mathcal{Z}\{y[k]\} = Y(z)$. If we take the Z-transform fo the both sides of the equation by applying the shifting theorem we obtain

$$
\begin{aligned}
a_0 Y(z) + a_1 z^{-1} Y(z) + ... + a_N z^{-N} Y(z) &= b_0 X(z) + ... + b_M z^{-M} X(z) \\
(a_0 + a_1 z^{-1} + ... + a_N z^{-N}) Y(z) &= (b_0 + b_1 z^{-1} + ... + b_M z^{-M}) X(z) \\
\frac{Y(z)}{X(z)} = G(z) &= \frac{b_0 + b_1 z^{-1} + ... + b_M z^{-M}}{a_0 + a_1 z^{-1} + ... + a_N z^{-N}} \\
&= z^{N-M} \frac{b_0 z^M + b_1 z^{M-1} + ... + b_M}{a_0 z^N + a_1 z^{N-1} + ... + a_N}
\end{aligned}
$$

Under "zero initial conditions" if we can find $X(z)$ then simply $Y(z) = G(z)X(z)$. After that we can take the inverse z-transform and compute $y[k]$.

Example 3.1 Compute $y[k]$ using the Z-transform method

$$y[k] = \frac{1}{2}y[k-1] + x[k]$$
$$y[k] = 0, \text{ for } k < 0 \;\&\; x[k] = \delta[k]$$

Solution:

$$Y(z) = \frac{1}{2}Y(z)z^{-1} + X(z) \;\rightarrow\; \frac{Y(z)}{X(z)} = G(z) = \frac{z}{z-1/2}$$
$$Y(z) = \frac{z}{z-1/2} \;\rightarrow\; y[k] = \left(\frac{1}{2}\right)^k$$

Example 3.2 Now let's compute $y[k]$ for $x[k] = u[k]$

$$Y(z) = G(z)X(z) \;\rightarrow\; Y(z) = \frac{z^2}{(z-1/2)(z-1)}$$
$$Y(z) = -\frac{z}{z-1/2} + 2\frac{z}{z-1}$$
$$y[k] = 2 - \left(\frac{1}{2}\right)^k$$

Example 4 For the following difference equation, compute $y[k]$ for $x[k] = x[k] = 3^k u[k]$

$$y[k] - 3y[k-1] + 2y[k-2] = x[k]$$

Solution:

$$Y(z)(1 - 3z^{-1} + 2z^{-2}) = X(z) \;\rightarrow\; G(z) = \frac{z^2}{z^2 - 3z + 2} = \frac{z^2}{(z-1)(z-2)}$$
$$Y(z) = \frac{z^3}{(z-1)(z-2)(z-3)} = 0.5\frac{z}{z-1} - 4\frac{z}{z-2} + 4.5\frac{z}{z-3}$$
$$y[k] = \left(0.5 - 4\;2^k + 4.5\;3^k\right)u[k]$$

Z-transform to Difference Equations

Sometimes the Z-domain transfer function of a system is given, and we may be supposed to find the difference equation representation. Let's assume that we have a general transfer function that can be represented in terms of ratio of two polynomials in z or z^{-1} as given below

$$\frac{Y(z)}{X(z)} = G(z) = \frac{b_0 + b_1 z^{-1} + ... + b_M z^{-M}}{a_0 + a_1 z^{-1} + ... + a_N z^{-N}} = z^{N-M}\frac{b_0 z^M + b_1 z^{M-1} + ... + b_M}{a_0 z^N + a_1 z^{N-1} + ... + a_N}$$

In his case, I prefer to work with the polynomials that are written in terms of z^{-1}. Let's manipulate the Z-domain equation to obtain

$$Y(z)(a_0 + a_1 z^{-1} + ... + a_N z^{-N}) = X(z)(b_0 + b_1 z^{-1} + ... + b_M z^{-M})$$
$$a_0 Y(z) + a_1 z^{-1} Y(z) + ... + a_N z^{-N} Y(z) = b_0 X(z) + b_1 z^{-1} X(z) + ... + b_M z^{-M} X(z)$$

Let's assume that $\mathcal{Z}^{-1}\{Y(z)\} = y[k]$ and $\mathcal{Z}^{-1}\{X(z)\} = x[k]$. If we take the inverse Z-transform of both sides by applying the shifting theorem we obtain

$$a_0 y[k] + a_1 y[k-1] + ... + a_N z^{-N} y[k-N] = b_0 x[k] + b_1 x[k-1] + ... + b_M x[k-M]$$

We can use this conversion to "simulate" a given discrete time transfer function or realizing the given system (it may be a filter or controller) to implement on an embedded platform.

It can also be used for computationally finding the inverse Z-transform of a given z-domain rational function. The next example will illustrate this feature.

Example 5 Find a computational solution for the inverse Z-transform of $H(z) = \frac{z^{-1}}{1-2z^{-1}+z^{-2}}$ by using the conversion from Z-domain transfer function to difference equation concept.

Solution: Let's assume that $H(z)$ is a "transfer function" not an arbitrary z-domain function. Then $\mathcal{Z}^{-1}\{H(z)\} = h(t)$ becomes the impulse response of the "system". Thus we can assume some imaginary input-output pair $y[n]$ and $x[n]$ where

$$\frac{Y(z)}{X(z)} = H(z)$$

If we can find a difference equation realization for $H(z)$ then we can simulate the difference equation by assuming $x[k] = \delta[k]$ (i.e. unit impulse input). So let's find a realization for the given $H(z)$ as

$$\begin{aligned} \frac{Y(z)}{X(z)} &= \frac{z^{-1}}{1-2z^{-1}+z^{-2}} \\ Y(z) - 2z^{-1}Y(z) + z^{-2}Y(z) &= z^{-1}X(z) \\ y[k] - 2y[k-1] + y[k-2] &= x[k-1] \end{aligned}$$

Now let's simulate the above equation for $x[k] = \delta[k]$

$$\begin{aligned} y[k] &= 2y[k-1] - y[k-2] + x[k-1] \\ y[0] &= 2y[-1] - y[-2] + x[-1] = 0 \\ y[1] &= 2y[0] - y[-1] + x[0] = 1 \\ y[2] &= 2y[1] - y[0] + x[1] = 2 \\ y[3] &= 2y[2] - y[1] + x[2] = 3 \\ y[4] &= 4 \\ &\cdots \\ y[k] &= k \end{aligned}$$

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Lecture 4

Lecturer: Asst. Prof. M. Mert Ankarali

Realization of Discrete Time Systems / Filters / Controllers

In this lecture we will cover some basic block-diagram realization techniques for discrete time systems/filters/controllers which are represented by Z-Domain transfer functions or difference equations.

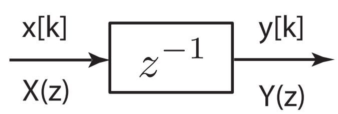
Block-diagram realizations are extremely useful practically for the purpose of implementing the system/filter/controller on a real embedded system of a hardware platform. The goal can be programming the discrete filter/controller on a microcontroller, or embedding the structure on an fpga module a block diagram representation is always helpful.

In the future weeks of the course, we will also actively use block-diagram representation to obtain a state-space realization of the given discrete-time system.

The most fundamental block for a discrete-time system is the unit delay operator

$$y[k] = x[k-1]$$
$$Y(z) = z^{-1} X(z)$$

which is represented by the following block-diagram



In this lecture our goal is to realize different kinds of discrete time transfer functions using this this fundamental block as the main brick.

Realization of FIR Systems

A third order (order is fixed for the sake of clarity) FIR (Finite impulse response) discrete time system has the following difference equation and transfer function

$$
\begin{aligned}
y[k] &= b_0x[k] + b_1x[k-1] + b_2x[k-2] + b_3x[k-3] \\
Y(z) &= \left(b_0 + b_1z^{-1} + b_2z^{-2} + b_3z^{-3}\right) X(z)
\end{aligned}
$$

From inspection it is easy to see that we need at least three unit delay blocks (and memory elements) to construct a full realization. Below the block-diagram realization of a third order FIR is given in Fig. **??**

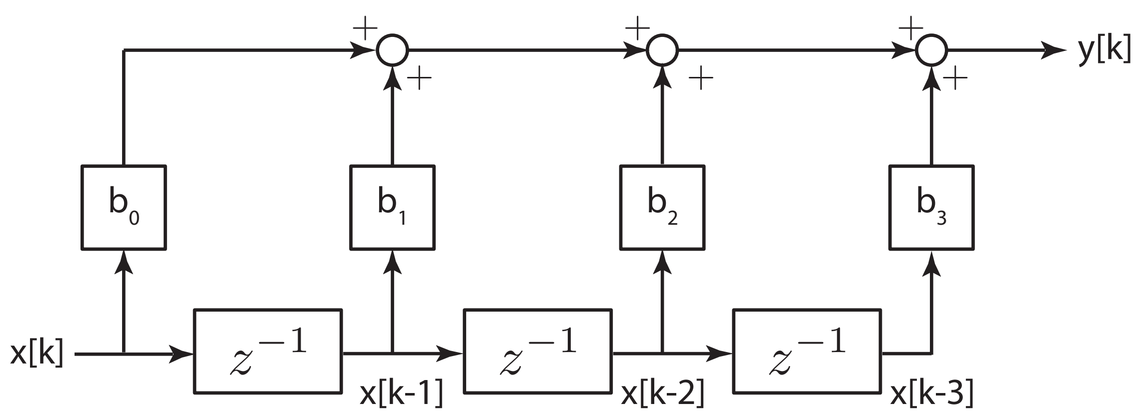


Figure 4.1: Block diagram realization of a third order FIR system

Realization of IIR Systems with Static Numerator Dynamics

In this part we will analyze a special case if IIR (Infinite impulse response) systems where the input dynamics is static, i.e. there is no direct delayed term of the input in the difference equation. Again for the sake of clarity let's assume that the discrete system is third order. For such a system, the difference equation and the transfer function can be written as

$$y[k] = -a_1x[k-1] - a_2y[k-2] - a_3y[k-3] + b_0x[k]$$
$$Y(z) = \frac{b_0}{1 + a_1z^{-1} + a_2z^{-2} + a_3z^{-3}}X(z)$$

Similar to the FIR case we also need minimum three delay blocks to realize this system, However, now delay blocks are in the feedback-loop. The block digram representation of the given IIR system is given in Fig. **??**.

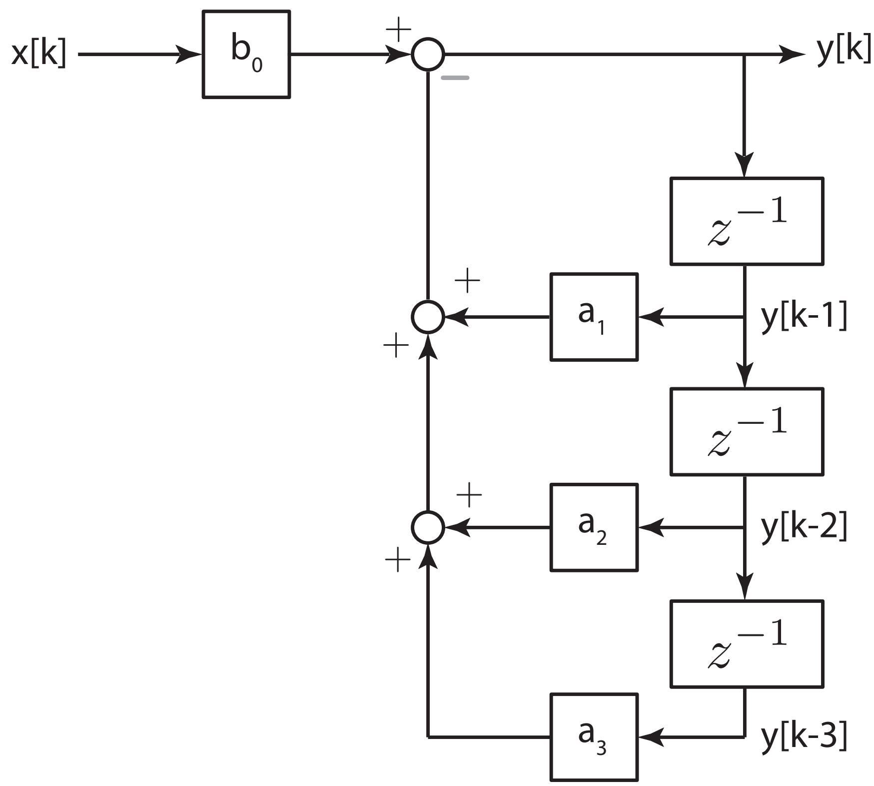


Figure 4.2: Block diagram realization of a third order IIR system with static numerator/input dynamics

Realization of General Discrete-Time Systems

Now we will attempt to obtain a realization of more general discrete-time systems/filters/controllers. Again for the sake oc clarity let's keep the order (maximum order of z^{-1} in the equation) of the system to 3. A general third order discrete time system can be expressed with following difference equation and transfer function

$$y[k] = -a_1 y[k-1] - a_2 y[k-2] - a_3 y[k-3] + b_0 x[k] + b_1 x[k-1] + b_2 x[k-2] + b_3 x[k-3]$$
$$Y(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3}}{1 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3}} X(z)$$

Non-minimal Realization / Direct Programming

One way of realizing a discrete-time system is simply combining the block diagrams of special cases (i.e. FIR and IIR with static input dynamics) given in Figures **??** & **??**. The block diagram realization obtained with this method can be observed in Fig. **??**

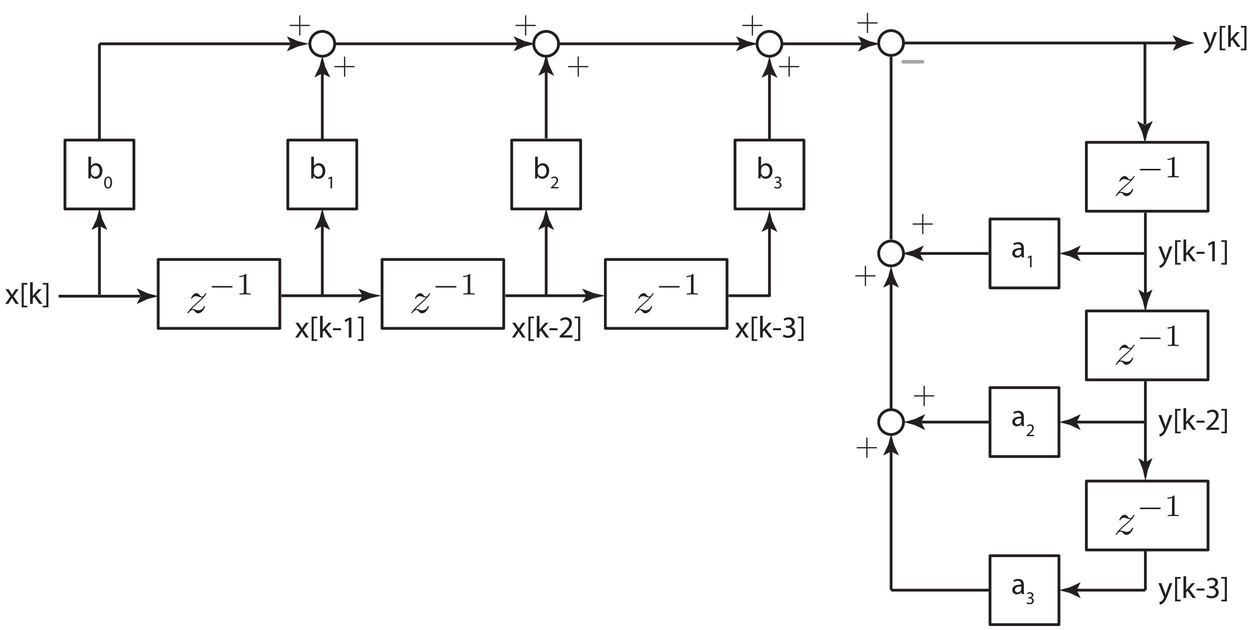


Figure 4.3: Direct/non-minimal realization of a third order discrete time dynamical system

The obvious problem in this realization is that even though the transfer function represents a third order discrete-dynamical system, the "order" of the block diagram is indeed 6 not 3. Because there exist 6 different memory blocks, i.e. delay elements. If we forexample obtain a state-space model from this block diagram we would obtain a system with 6 states.

Canonical Realization I / Standard Programming

In this method of realization, we will use the fact the system is LTI. Let's consider the transfer function of the system and let's perform some LTI operations.

$$
\begin{aligned}
Y(z) &= \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3}}{1 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3}} X(z) \\
&= \left(b_0 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3}\right) \frac{1}{1 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3}} X(z) \\
&= G_2(z) G_1(z) X(z) \text{ where} \\
G_1(z) &= \frac{H(z)}{X(z)} = \frac{1}{1 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3}} \\
G_2(z) &= \frac{Y(z)}{H(z)} = b_0 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3}
\end{aligned}
$$

As you can see we introduced an intermediate variable $h[k]$ with a Z-transform of $H(z)$, First transfer function, which is an IIR system with static input dynamics operates on $x[n]$ and produces an output. Second transfer function operates on $h[n]$ and produces output $x[n]$. If we write the difference equations of both systems we obtain

$$
\begin{aligned}
h[k] &= -a_1 h[y-1] - a_2 h[k-2] - a_3 h[k-3] + x[k] \\
y[k] &= b_0 x[k] + b_1 h[k-1] + b_2 h[k-2] + h_3 x[k-3]
\end{aligned}
$$

As it can be seen that the delay/shifting operations are only performed on the signal $h[k]$ and maximum delay operation is by 3 samples. Basically if we utilize this structure we can draw a minimal block diagram representation as given in Fig. **??**. If we obtain a state-space model from this block diagram, the form will be in *controllable canonical form*. We will cover this later in the semester. Thus we can call this representation also as *controllable canonical realization*.

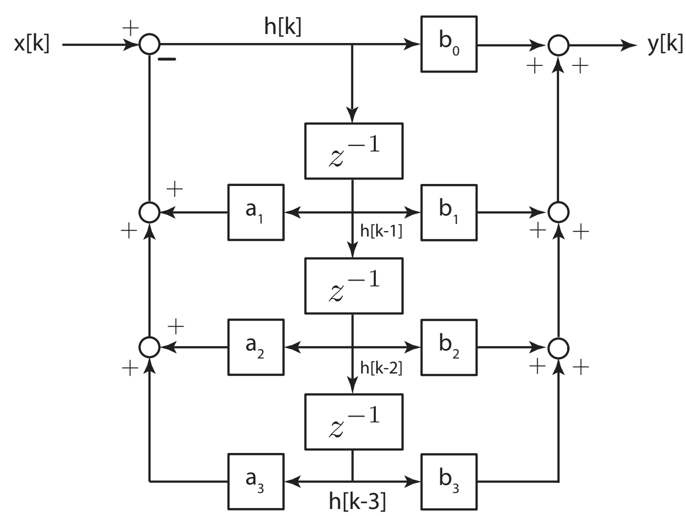


Figure 4.4: A minimal block diagram realization of a discrete time system obtained with standard programming (Canonical representation I)

Canonical Realization II

In this method will obtain a different minimal realization. The process will be different and this the block diagram will have different topology. Let's start with the transfer function and perform some grouping based on the delay elements.

$$Y(z)(1 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3}) = (b_0 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3})X(z)$$
$$Y(z) = b_0 X(z) + z^{-1}\left(b_1 X(z) - a_1 Y(z)\right) + z^{-2}\left(b_2 X(z) - a_2 Y(z)\right) + z^{-3}\left(b_3 X(z) - a_3 Y(z)\right)$$
$$Y(z) = b_0 X(z) + z^{-1}\left\{(b_1 X(z) - a_1 Y(z)) + z^{-1}\left[(b_2 X(z) - a_2 Y(z)) + z^{-1}\left(b_3 X(z) - a_3 Y(z)\right)\right]\right\}$$

As you can see we have only z^{-1} terms in the representation there is a special toplogy embedded inside the expression. If we convert it to the block diagram form we obtain the structure given in Fig. **??**. If we obtain a state-space model from this block diagram, the form will be in *observable canonical form.* Thus we can call this representation also as *observable canonical realization.* This form and representation is the dual of the previous representation.

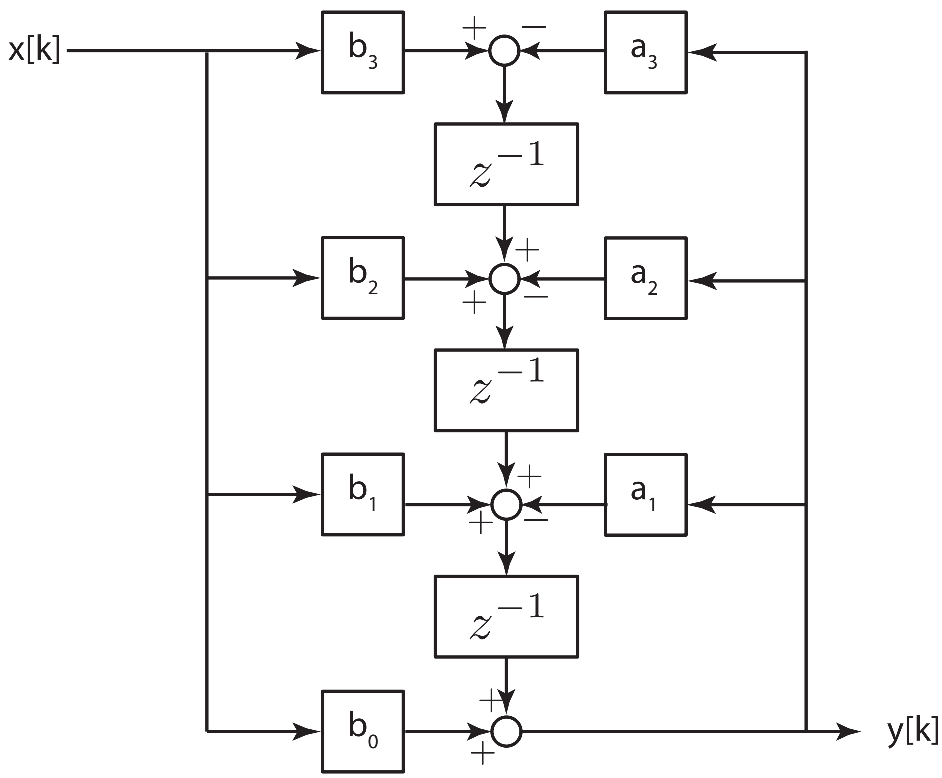


Figure 4.5: A minimal block diagram realization of the 3^{rd} order discrete time system obtained with Canonical representation II

EE402 - Discrete Time Systems	Spring 2018

Lecture 5

Lecturer: Asst. Prof. M. Mert Ankarali

Let's remember the idealized and simplified block-diagram structure a discrete-time control system (See Fig. 5.1)

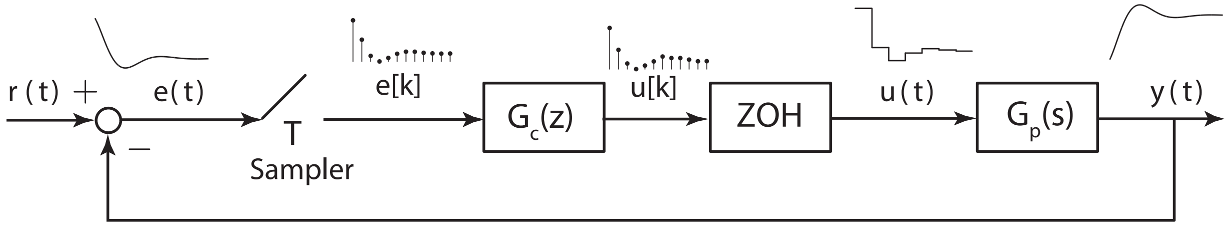


Figure 5.1: Block diagram of an LTI discrete-time control system

Loop contains both continuous-time and discrete-time signals and blocks.

- We can treat the system as a completely discrete-time system. We technically restrict ourselves into sampled time instants (which may be just fine)
- Alternatively, we can use continuous time signals (as much as possible) and deal with starred versions of signals and starred Laplace transform.

Sampling - Review

Fig. 5.2 illustrates an ideal *impulse sampler* and an ideal complete CT-to-DT sampler.

The output of the impulse sampler, $x^*(t)$, can be represented with the following infinte summations

$$x^*(t) = \sum_{k=0}^{\infty} x(kT)\delta(t-kT) = \sum_{k=0}^{\infty} x[k]\delta(t-kT)$$

Now let's consider the Laplace transform of $x^*(t)$

$$X^*(s) = \mathcal{L}\{x^*(t)\} = \sum_{k=0}^{\infty} x(kT) \int_{t=0}^{\infty} \delta(t-kT)e^{-st}dt = \sum_{k=0}^{\infty} x(kT)e^{-skT} = \sum_{k=0}^{\infty} x[k]e^{-skT}$$

Now let's define a map in complex domain such that $z = e^{Ts}$ or $s = \frac{1}{T}\ln z$. Then we have

$$X^*(s)|_{s=(1/T)lnz} = \sum_{k=0}^{\infty} x[k]z^{-k}$$

where

$$X(z) = \mathcal{Z}\{x[k]\} = \sum_{k=0}^{\infty} x[k]z^{-k}$$

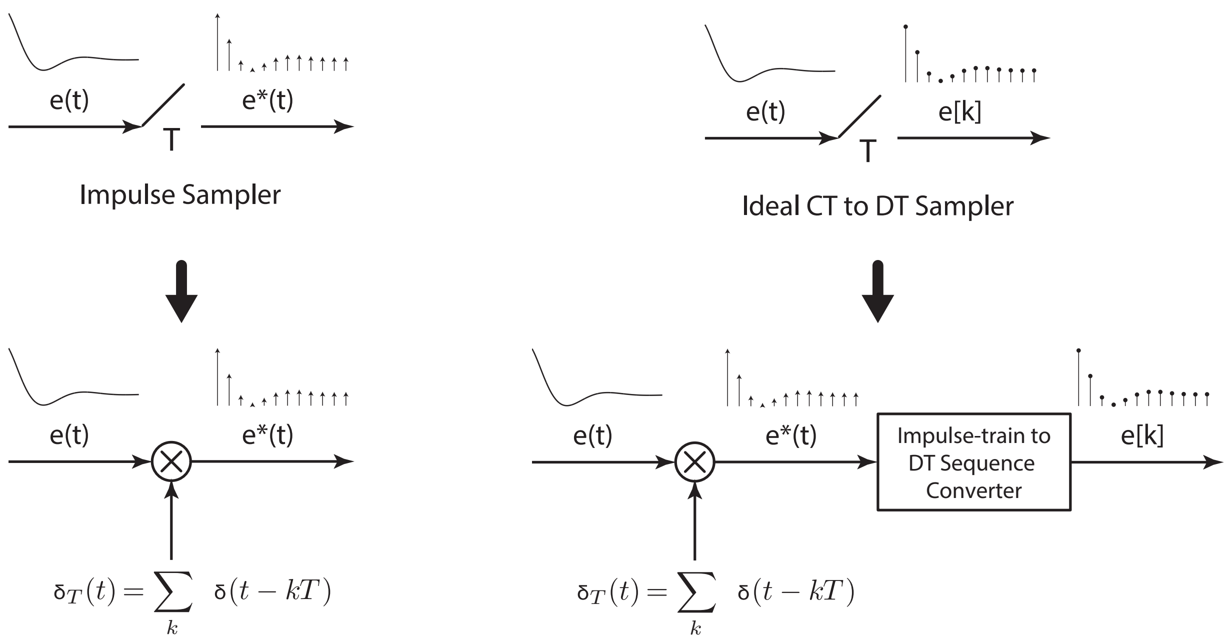
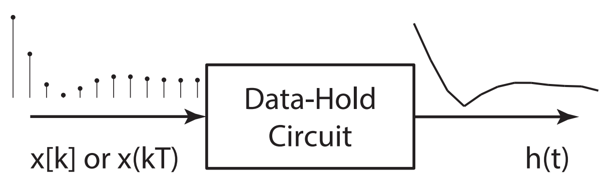


Figure 5.2: Two different ideal samplers

Data Hold Operation

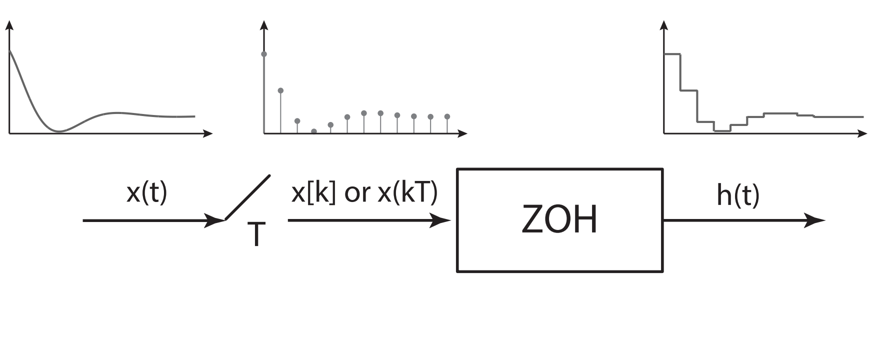
Data-Hold operation is an idealized model of a DAC device which converts a digital signal to an analog signal. In terms of the terminology used in this class, Data-Hold operation is the process of obtaining a CT signal $h(t)$ from a DT sequence. A general data-hold operation block circuit is shown below



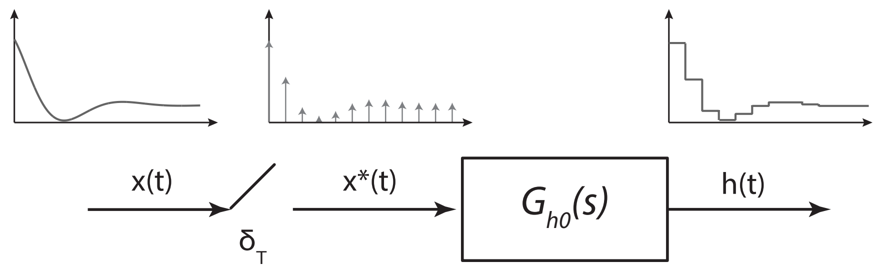
Simplest and most dominantly used (I have never seen a practical usage of other hold operations) hold circuit/operation is the zero-order-hold (ZOH). Basically, at each time instant kT ZOH "samples" the input $x[k]$ or $x(kT)$ and "holds" this value at the output until the next sampling event. Mathematically,

$$h(kT + t) = x(kT) = x[k], \text{ for } 0 \geq t < T$$

The figure below illustrates a serios connection of an ideal CT-DT sampler and an ideal ZOH block.



If we model the sampler using an ideal impulse sampler (not CT-DT converter) then it becomes more convenient to model the ZOH with a CT transfer function as shown with the block diagram below



Let's assume that $x(t)$ is a strictly causal signal, then from the definition of ZOH we can express $h(t)$ in terms of $x(t)$ (or $x^*(t)$, $x[k]$, $x(kT)$) as

$$h(t) = x(0)\left[u(t) - u(t-T)\right] + x(T)\left[u(t-T) - u(t-2T)\right] + x(2T)\left[u(t-2T) - u(t-3T)\right] + \cdots$$
$$h(t) = \sum_{k=0}^{\infty} x(kT)\left[u(t-kT) - u(t-(k+1)T)\right]$$

If we take the Laplace transform of $h(t)$, we obtain

$$\begin{aligned}
\mathcal{L}\{h(t)\} &= \sum_{k=0}^{\infty} x(kT)\mathcal{L}\left\{\left[u(t-kT) - u(t-(k+1)T)\right]\right\} \\
&= \sum_{k=0}^{\infty} x(kT)\left[\frac{e^{-kTs}}{s} - \frac{e^{-(k+1)Ts}}{s}\right] \\
&= \frac{1-e^{-Ts}}{s}\sum_{k=0}^{\infty} x(kT)e^{-kTs} = \frac{1-e^{-Ts}}{s}\sum_{k=0}^{\infty} x[k]e^{-kTs} \\
H(s) &= \frac{1-e^{-Ts}}{s}X^*(s) = G_{h0}(s)X^*(s) \\
G_{h0}(s) &= \frac{1-e^{-Ts}}{s}
\end{aligned}$$

Z-transform & ZOH

When analyzing the discrete time control systems, we will (frequently) need to compute the Z-transform of sampled signals, for which the Laplace transform involves the term $\frac{1-e^{-Ts}}{s}$.

Let $\mathcal{L}\{x(t)\} = X(s) = \frac{1-e^{-Ts}}{s}G(s)$. Now let's analyze the z-transform of the sampled version of the signal, i.e. $X(z) = \mathcal{Z}\{x^*(t)\}$. First let's find $x(t)$ from $X(z)$

$$\begin{aligned}
x(t) = \mathcal{L}^{-1}\{X(s)\} &= \mathcal{L}^{-1}\left\{\frac{1-e^{-Ts}}{s}G(s)\right\} = \mathcal{L}^{-1}\left\{(1-e^{-Ts})\frac{G(s)}{s}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{G(s)}{s}\right\} - \mathcal{L}^{-1}\left\{e^{-Ts}\frac{G(s)}{s}\right\}
\end{aligned}$$

Let $\hat{g}(t) = \mathcal{L}^{-1}\left\{\frac{G(s)}{s}\right\}$ then

$$x(t) = \hat{g}(t) - \hat{g}(t-T)$$

$x(kT)$ and $x[k]$ takes the form

$$x(kT) = \hat{g}(kT) - \hat{g}(kT - T)$$
$$x[k] = \hat{g}[k] - \hat{g}[k-1]$$

Then $X(z)$ takes the form

$$X(z) = \left(1 - z^{-1}\right)\hat{G}(z)$$

where $\hat{G}(z) = \mathcal{Z}\left\{\left[\mathcal{L}^{-1}\left\{\frac{G(s)}{s}\right\}\right]^*\right\}$. In the textbook this notation is shortened to have $\hat{G}(z) = \mathcal{Z}\left\{\frac{G(s)}{s}\right\}$. After that we have

$$X(z) = \left(1 - z^{-1}\right)\mathcal{Z}\left\{\frac{G(s)}{s}\right\}$$

Example 1. Obtain the z transform of $x(kT)$ where $T = 1$ and $X(s)$ is given as

$$X(s) = \frac{1 - e^{-s}}{s}\frac{1}{s+1}$$

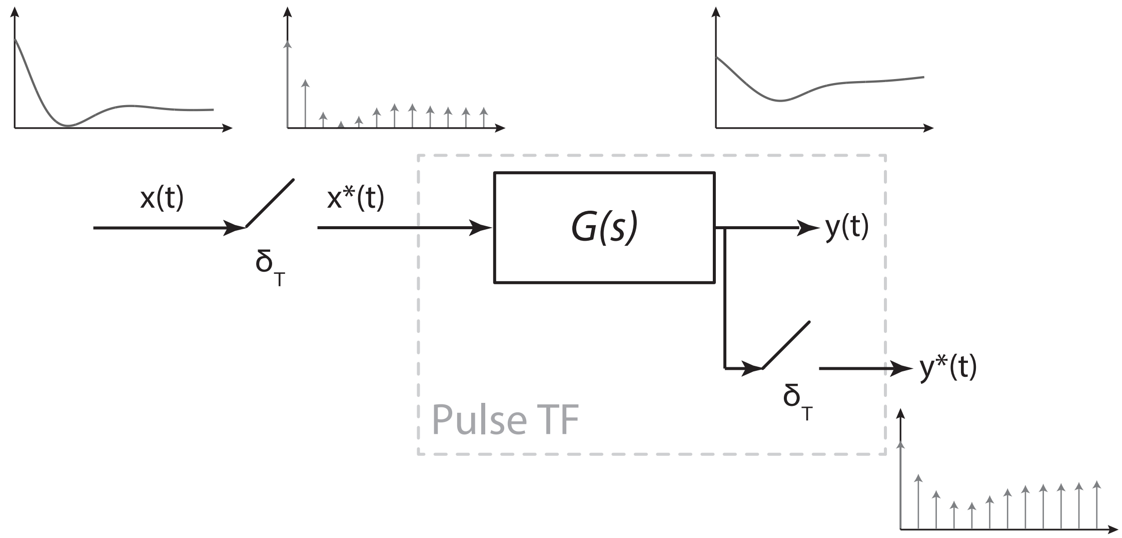
Solution:

$$\begin{aligned}
X(z) &= \left(1 - z^{-1}\right)\mathcal{Z}\left\{\frac{1}{s(s+1)}\right\} = \left(1 - z^{-1}\right)\mathcal{Z}\left\{\frac{1}{s} - \frac{1}{s+1}\right\} \\
&= \frac{z-1}{z}\left(\frac{z}{z-1} - \frac{z}{z-e^{-1}}\right) = 1 - \frac{z-1}{z-e^{-1}} \\
X(z) &= \frac{1-e^{-1}}{z-e^{-1}}
\end{aligned}$$

Pulse Transfer Function

The CT transfer function relates the Laplace transform of the continious-time output, $y(t)$ and $Y(s)$, to that of the continious-time input $x(t)$ and $X(s)$.

The Pulse Transfer Function will relate the Z transform of the output, $y^*(t)$ (or $y(kT)$) and $Y(z)$, to that of the sampled input, $x^*(t)$ (or $x(kT)$) and $X(z)$. Note that without a feedback-loop the sampling at the output remains purely synthetic. The figure below illustrates the signals and associated transfer function blocks:



Let $g(t)$ be the impulse response of the transfer function $G(s)$, then we know that

$$y(t) = \int_0^t g(t-\tau)x^*(\tau)d\tau$$

$$y(t) = \int_0^t g(t-\tau)\sum_{k=0}^{\infty} x(kT)\delta(\tau - kT)d\tau$$

Let $t = nT + \hat{t}$ where $\hat{t} \in [0, T)$ then

$$y(nT+\hat{t}) = \int_0^{nT+\hat{t}} g(nT+\hat{t}-\tau)\sum_{k=0}^{\infty} x(kT)\delta(\tau - kT)d\tau$$

$$y(nT+\hat{t}) = \sum_{k=0}^{n} x(kT)\int_0^{nT+\hat{t}} g(nT+\hat{t}-\tau)\delta(\tau - kT)d\tau$$

$$y(nT+\hat{t}) = \sum_{k=0}^{n} x(kT)g((n-k)T+\hat{t})$$

Let $\hat{t} = 0$, then

$$y(nT) = \sum_{k=0}^{n} x(kT)g((n-k)T)$$

$$y[n] = \sum_{k=0}^{n} x[k]g[(n-k)]$$

In other words

$$y(nT) = x(nT) * g(nT) = g(nT) * x(nT)$$
$$y[n] = x[n] * g[n] = g[n] * x[n]$$

The result is pretty interesting: the impulse response of the "discretized" system is equal to the signal which is obtained by sampling the impulse response function of original the continuous time system.

If we take the Z transform of the equation given by the convolution (remember the properties of Z-transform) we obtain

$$Y(z) = G(z)X(z) \;\rightarrow\; G(z) = \frac{Y(z)}{X(z)}$$

where $G(z)$ is called the **Pulse Transfer Function of the DT System**. Note that

$$G(z) = \sum_{k=0}^{\infty} g(kT)z^{-k}$$

If we know $g(t)$ and $G(s)$, given the sampling time T, we can compute $g[k]$ and $G(z)$.

Can we say something extra regarding the starred Laplace transforms and Z transforms of y, x, and g?

$$\begin{aligned} &Y(s) = G(s)X^*(s) \;\rightarrow Y^*(s) = [G(s)X^*(s)]^* \\ &Y(z) = G(z)X(z) \end{aligned}$$

Given that starred Laplace transform is the z-transform where z is evaluated e^{TS} we can conclude that

$$Y^*(s) = [G(s)X^*(s)]^* = G^*(s)X^*(s)$$

EE402 - Discrete Time Systems **Spring 2018**

Lecture 6

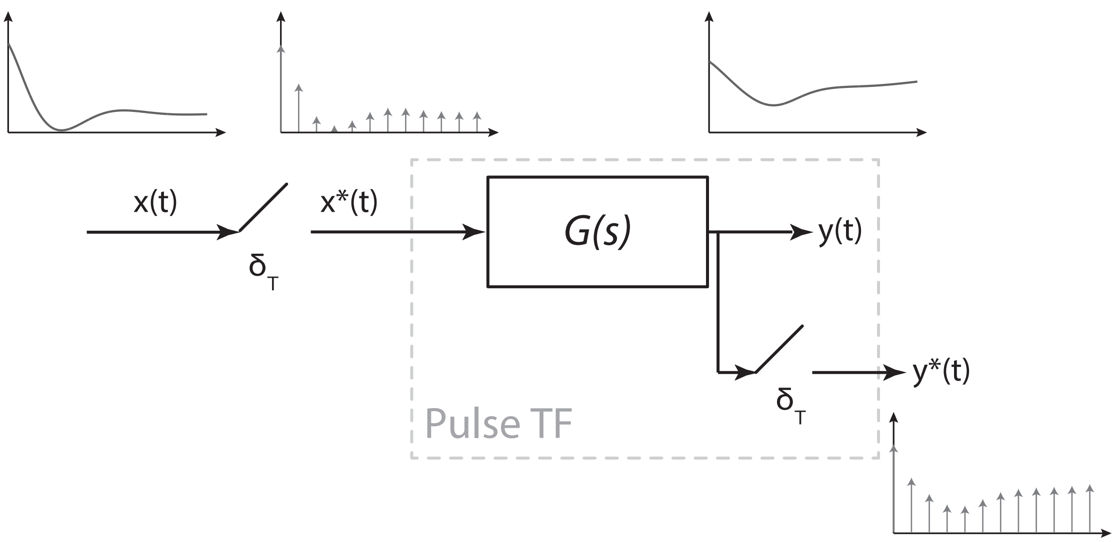
Lecturer: Asst. Prof. M. Mert Ankarali

...

Pulse Transfer Functions

For the following block diagram representation we have the following time domain input–output representations

$$\begin{aligned} y(t) &= g(t) * x^*(t) \quad \text{CT} - \text{convolution} \\ y[n] &= g[n] * x[n] \quad \text{DT} - \text{convolution} \end{aligned}$$



We then showed that Input–output relation between $x[n]$ and $y[n]$ in Z-domain can be represented with the following Pulse Transfer Function equation

$$Y(z) = G(z)X(z) \ \rightarrow \ G(z) = \frac{Y(z)}{X(z)}$$

If we take the Laplace transform of the CT-convolution equation

$$Y(s) = G(s)X^*(s) \ \rightarrow Y^*(s) = [G(s)X^*(s)]^*$$

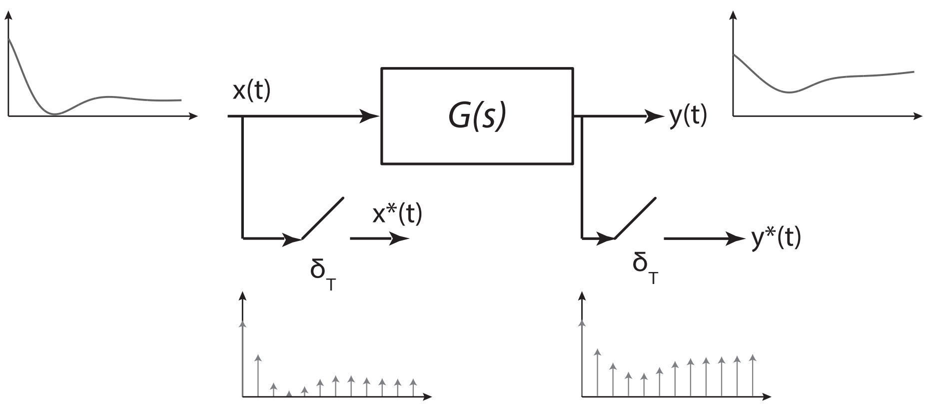
Now let's analyze $Y^*(s)$

$$\begin{aligned} &Y(s) = G(s)X^*(s) \ \rightarrow Y^*(s) = [G(s)X^*(s)]^* \\ &Y(z) = G(z)X(z) \end{aligned}$$

Given that starred Laplace transform is the z-transform where z is evaluated e^{Ts} we can conclude that

$$Y(z) = G(z)X(z) \;\rightarrow\; Y^*(s)|_{z=e^{Ts}} = G^*(s)|_{z=e^{Ts}} X^*(s)|_{z=e^{Ts}}$$
$$Y^*(s) = [G(s)X^*(s)]^* = G^*(s)X^*(s)$$

Now let's consider the following system for which there is no sampler of the continuous time block $G(s)$ at the input.

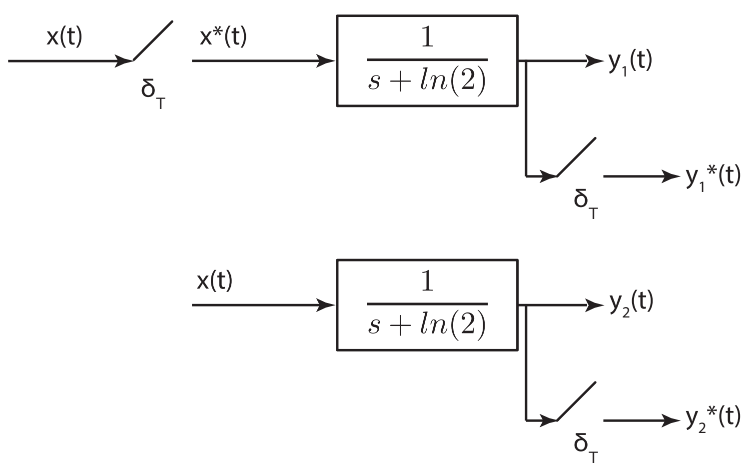


$$Y(s) = G(s)X(s) \;\rightarrow\; Y^*(s) = [G(s)X(s)]^*$$
$$Y^*(s) = [G(s)X(s)]^* = [GX(s)]^* \neq G^*(s)X^*(s)$$

where $GX(s) := G(s)X(s)$. In z-domain we have

$$Y(z) = \mathcal{Z}\{GX(s)\} = GX(z) \neq G(z)X(z)$$

Example: Given that $x(t) = u(t)$ and $T = 1$, compute $Y_1(z)$ and $Y_2(z)$ for the following example in the figure



Solution: Let's start with $Y_1(z)$.

$$\begin{aligned} Y_1(z) &= G(z)X(z) \\ &= \frac{z^2}{(z-1)(z-0.5)} \\ &= \frac{2z}{z-1} - \frac{z}{z-0.5} \end{aligned}$$

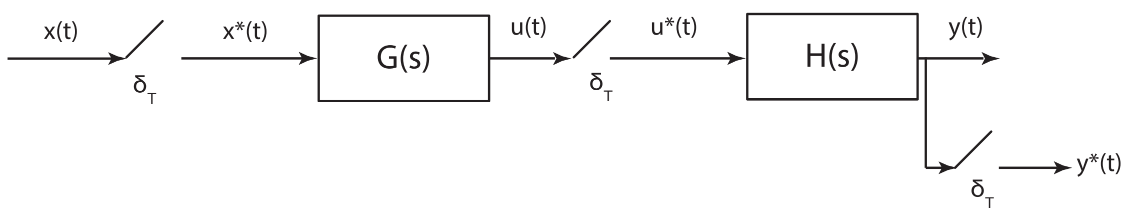
Now let's compute $Y_2(z)$

$$\begin{aligned} Y_2(z) &= GX(z) \\ &= \mathcal{Z}\left\{\frac{1}{s(s+\ln(2))}\right\} \\ &= \mathcal{Z}\left\{\frac{1/\ln(2)}{s} - \frac{1/\ln(2)}{s+\ln(2)}\right\} \\ &= \frac{1}{\ln(2)}\left(\frac{z}{z-1} - \frac{z}{z-0.5}\right) \end{aligned}$$

We can conclude that $Y_1(z) \neq Y_2(z)$, and thus $y_1(t) \neq y_2(t)$.

Pulse Transfer Functions of Cascaded Elements

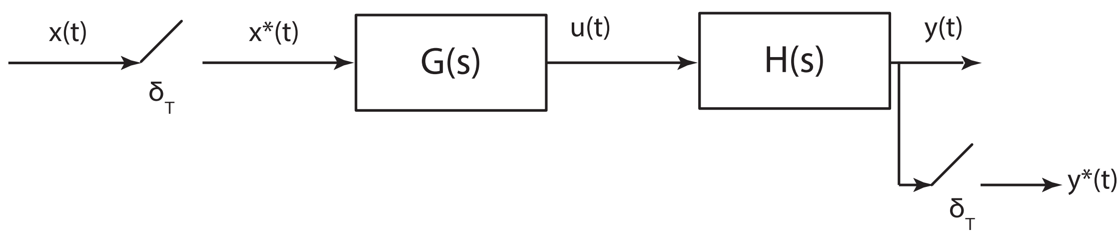
Consider the cascaded system shown in Figure below



Pulse transfer function of this system can be computed

$$\begin{aligned} U^*(s) &= G^*(s)X^*(s) \quad , \; Y^*(s) = H^*(s)U^*(s) \\ Y^*(s) &= H^*(s)G^*(s)X^*(s) \\ Y(z) &= H(z)G(z)X(z) \\ \frac{Y(z)}{X(z)} &= H(z)G(z) \end{aligned}$$

Now, let's consider the following cascaded system

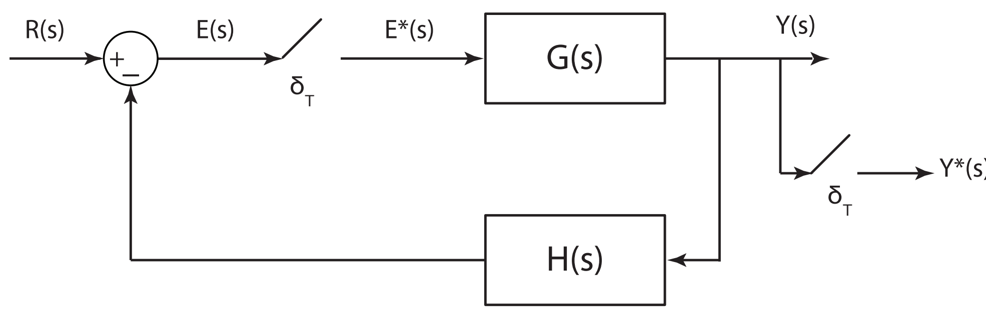


Pulse transfer function of this new system can be computed

$$
\begin{aligned}
Y(s) &= H(s)G(s)X^*(s) \\
Y(s) &= HG(s)X^*(s) \quad \text{where } HG(s) = H(s)G(s) \\
Y^*(s) &= [HG(s)X^*(s)]^* = HG^*(s)X^*(s) \\
Y(z) &= HG(z)X(z) \quad \text{where } HG(z) = \mathcal{Z}\{HG(s)\} \\
\frac{Y(z)}{X(z)} &= HG(z) \neq H(z)G(z)
\end{aligned}
$$

Pulse Transfer Functions of Closed Loop Systems

SYS1: Consider the following closeed-loop system



$$
\begin{aligned}
E(s) &= R(s) - H(s)Y(s) \quad , \quad Y(s) = G(s)E^*(s) \\
E(s) &= R(s) - H(s)G(s)E^*(s) \\
&= R(s) - HG(s)E^*(s)
\end{aligned}
$$

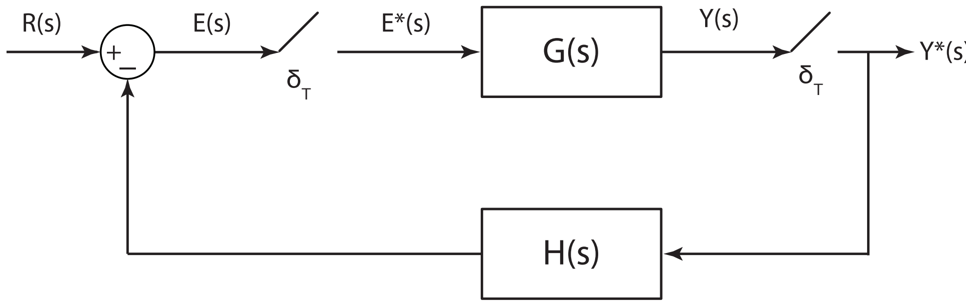
Let's take the starred Laplace transform of both sides

$$
\begin{aligned}
E^*(s) &= R^*(s) - HG^*(s)E^*(s) \\
E^*(s) &= \frac{R^*(s)}{1 + HG^*(s)}
\end{aligned}
$$

Since $Y^*(s) = G^*(s)E^*(s)$ we can conclude

$$
\begin{aligned}
Y^*(s) &= \frac{G^*(s)}{1 + HG^*(s)} R^*(s) \\
Y(z) &= \frac{G(z)}{1 + HG(z)} R(z) \\
\frac{Y(z)}{X(z)} &= \frac{G(z)}{1 + HG(z)}
\end{aligned}
$$

SYS2: Now let's consider a slightly different topology



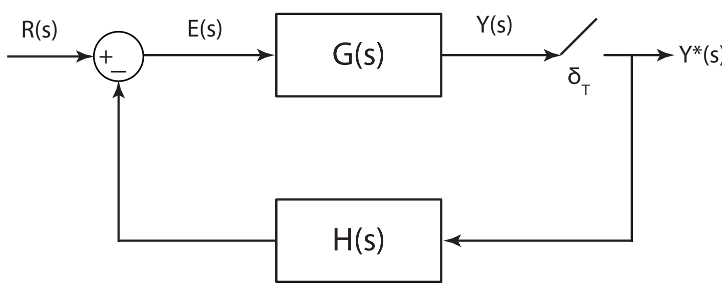
$$
\begin{aligned}
E(s) &= R(s) - H(s)Y^*(s) \\
Y(s) &= G(s)E^*(s) \quad , \quad Y^*(s) = G^*(s)E^*(s) \\
E(s) &= R(s) - H(s)G^*(s)E^*(s) \\
E^*(s) &= R^*(s) - H^*(s)G^*(s)E^*(s) \\
E^*(s) &= \frac{R^*(s)}{1 + H^*(s)G^*(s)}
\end{aligned}
$$

Then we can have

$$
\begin{aligned}
Y^*(s) &= \frac{G^*(s)}{1 + H^*(s)G^*(s)} R^*(s) \\
Y(z) &= \frac{G(z)}{1 + H(z)G(z)} R(z) \\
\frac{Y(z)}{R(z)} &= \frac{G(z)}{1 + H(z)G(z)}
\end{aligned}
$$

It can be seen that the pulse transfer function of the closed-loop systems are different.

SYS3: Now consider another closed-loop system.

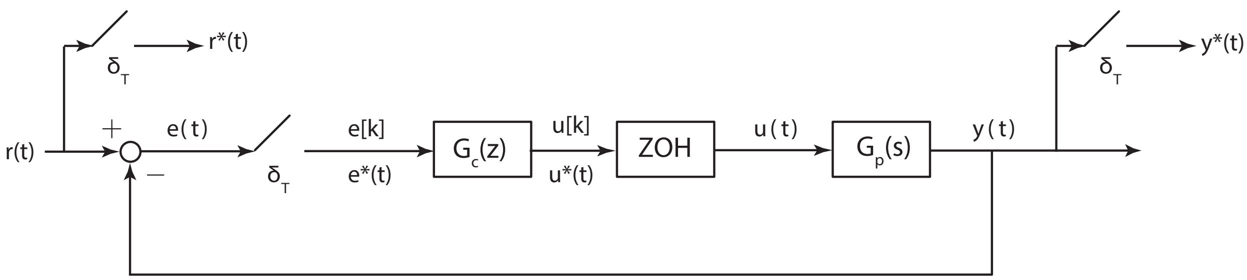


$$
\begin{aligned}
E(s) &= R(s) - H(s)Y^*(s) \qquad Y(s) = G(s)E(s) \\
Y(s) &= G(s)R(s) - G(s)H(s)Y^*(s) \\
Y^*(s) &= GR^*(s) - GH^*(s)Y^*(s) \\
Y^*(s) &= \frac{GR^*(s)}{1 + GH^*(s)} \\
Y(z) &= \frac{GR(z)}{1 + GH(z)}
\end{aligned}
$$

We can't even find a direct transfer function from $R^*(s)$ to $Y^*(s)$, or equivalently from $R(z)$ to $Y(z)$.

Pulse Transfer Function of A Closed Loop DT Control System

Consider the fundamental DT-control system below



Previously we derived the transfer function of ZOH, thus we can combine ZOH and plant TF into a single TF block $G(s)$ as

$$\frac{Y(s)}{U^*(s)} = G(s) = \frac{1-e^{-sT}}{s} G_p(s)$$

The controller is actually implemented in an hardware/software platfrom which indeed works in discrete-time. However, we can find a starred version of $G_c(z)$ by

$$G_c^*(s) = G_c(z)|_{z=e^{Ts}}$$

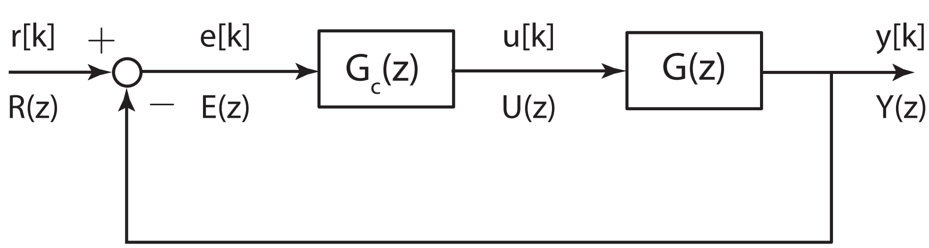
Now let's derive the PTF

$$\begin{aligned}
E(s) &= Y(s) - R(s) \quad , \quad Y(s) = G(s)G_c^*(s)E^*(s) \\
E^*(s) &= Y^*(s) - R^*(s) \quad , \quad Y(s) = G^*(s)G_c^*(s)E^*(s) \\
E^*(s) &= G^*(s)G_c^*(s)E^*(s) - R^*(s) \\
E^*(s) &= \frac{R^*(s)}{1+G^*(s)G_c^*(s)}
\end{aligned}$$

Then the PTF in starred Laplace domain and z domain can be find as

$$\begin{aligned}
\frac{Y^*(s)}{R^*(s)} &= \frac{G^*(s)G_c^*(s)}{1+G^*(s)G_c^*(s)} \\
\frac{Y(z)}{R(z)} &= \frac{G(z)G_c(z)}{1+G(z)G_c(z)}
\end{aligned}$$

Note if we only care the signal flow in the sampled instants we can re-draw the block diagram such that all time domain signals are in DT and all systems are represented in Z-domain. The fundamental block diagram can be re-drawn as



It is obvious that this discretized block diagram is much simpler and neater. Let's show that the transfer function of this DT system is equal to the PTF of the hybrid system above (CT & DT combined)

$$E(z) = R(z) - Y(z) \quad , \quad Y(z) = G(z)G_c(z)E(z)$$
$$E(z) = \frac{1}{1 + G(z)G_c(z)} R(z)$$
$$\frac{Y(z)}{R(z)} = \frac{G(z)G_c(z)}{1 + G(z)G_c(z)}$$

Example: Let $G_p(s) = \frac{1}{s+1}$, $T = 1$, $G_c(z) = K$ (Discrete P Controller) . First find PTF (in z-domain).
Solution: First let's find $G(z)$

$$G(z) = (1 - z^{-1})\mathcal{Z}\left\{\frac{G_p(s)}{s}\right\} = \frac{1 - e^{-1}}{z - e^{-1}}$$

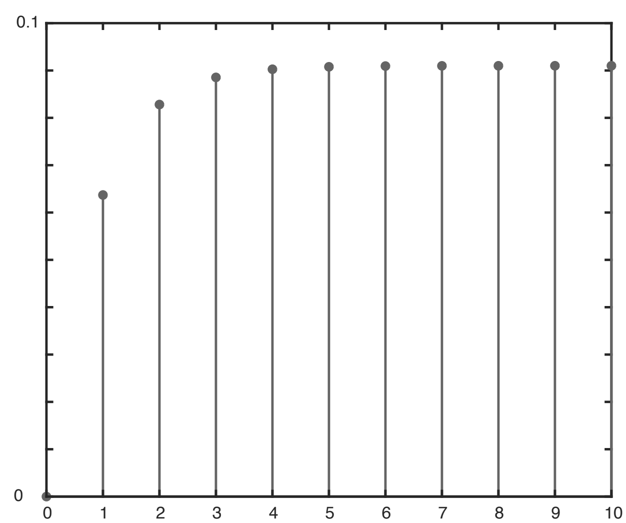
which we already knew from the Lecture Notes 4. Now let's compute the closed-loop PTF, $T(z)$.

$$T(z) = \frac{G_c(z)G(z)}{1 + G_c(z)G(z)} = \frac{K(1 - e^{-1})}{z + K - (K+1)e^{-1}}$$

Let $\mathbf{K = 0.1}$, then compute the step-response of the closed-loop PTF

$$Y(z) = R(z)T(z) = \frac{0.063\, z}{(z-1)(z-0.3)}$$
$$y[k] = \mathcal{Z}^{-1}[Y(z)] = \left[0.09 - 0.09(0.3)^k\right] u[k]$$

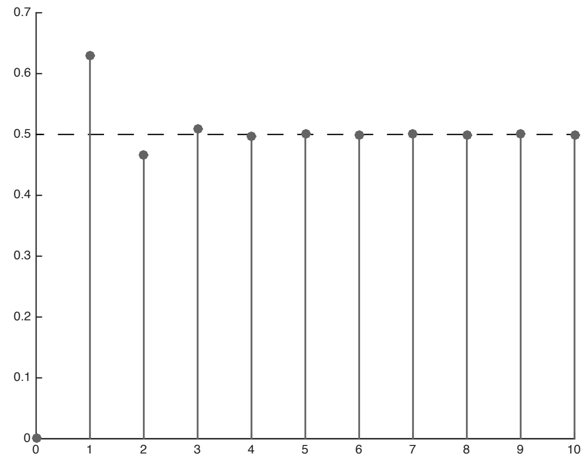
If we plot the step response we obtain the following plot



Now, Let $\mathbf{K = 1}$, then compute the step-response of the closed-loop PTF

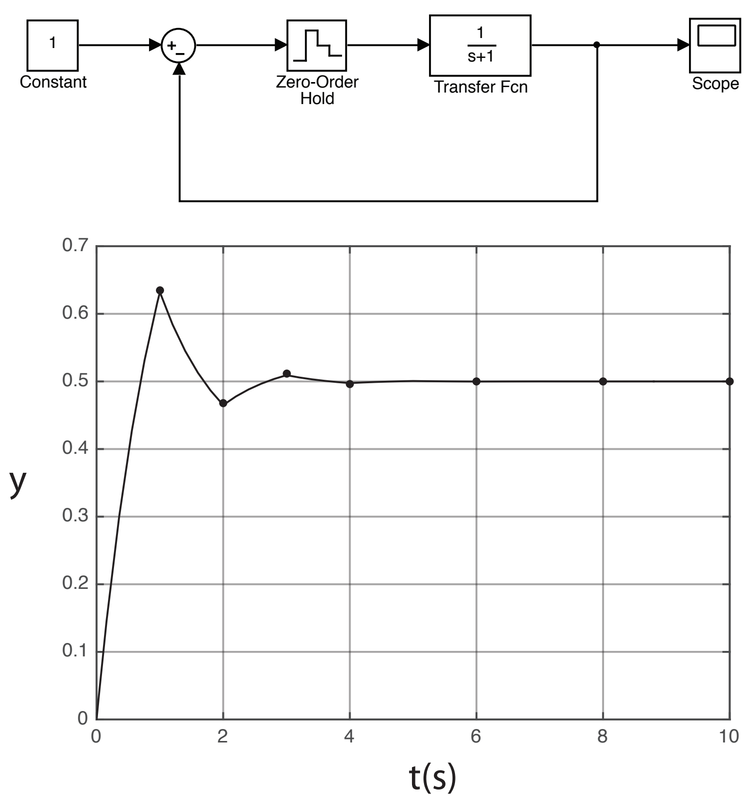
$$Y(z) = R(z)T(z) = \frac{z}{z-1}\frac{0.63}{z+0.26} = \frac{0.63\, z}{(z-1)(z+0.26)}$$
$$y[k] = \mathcal{Z}^{-1}[Y(z)] = \left[0.5 - 0.5(-0.26)^k\right] u[k]$$

If we plot the step response we obtain the following plot



What about inter-sample behavior?

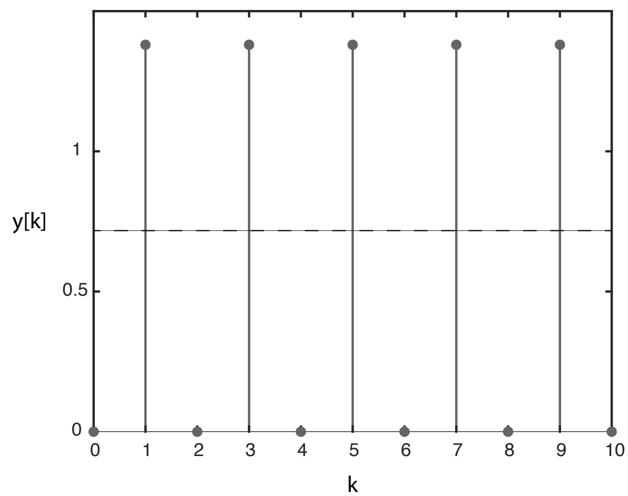
We can simulate the system and analyze the behavior. The figure below shows the Simulink model as well as the output.



Now let $K = \frac{1+e^{-1}}{1-e^{-1}}$, then $Y(z)$ and $y[k]$ takes the form

$$Y(z) = \frac{1.37z}{(z-1)(z+1)}$$
$$y[k] = 0.69 - 0.69(-1)^n$$

The graph of $y[k]$ is illustrated below. It can be seen that the output shows an oscillatory behavior.



Now let $K = \frac{e^{-1}}{1-e^{-1}}$, then $Y(z)$ and $y[k]$ takes the form

$$Y(z) = \frac{0.37}{z-1}$$
$$y[k] = 0.37u[k-1]$$

Output converges to its steady state value in "finite time" (dead-beat behavior/controller).

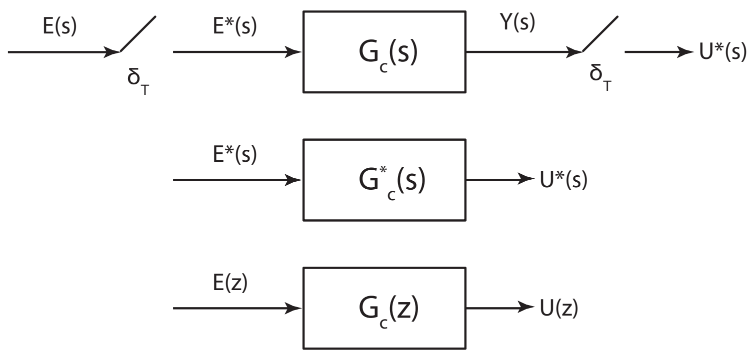
Take home message: It can be seen that even if the plant is a simple first order transfer function, depending on the value of K, we can observe very interesting behavior in the closed-loop DT system.

Pulse Transfer Function of a Digital PID Controller

In this section, we will try to obtain a from for the digital PID controller. The continuous transfer function of a PID is given as

$$G_{PID}(s) = K_P + K_D s + \frac{K_I}{s}$$

One idea is to start from continuous PID form and then "discretize" it. One way of deriving a discrete controller, $G_c(z)$, is using the star operation for discretization. This operation is illustrated in the Figure below.



Based in this approach if possible $G_c(z)$ simply commuted as

$$G_c(z) = \mathcal{Z}\{G_c(s)\}$$

Let's start with PI controller.

Digitization of PI Controller: It is a well known fact that the PI Controller is in the form

$$G_{PI}(s) = K_P + \frac{K_I}{s}$$

The discretization simply gives

$$\begin{aligned}
G_{PI}(z) &= \mathcal{Z}\{G_{PI}(s)\} \\
&= K_P + K_I \frac{1}{1 - z^{-1}} \\
&= \frac{(K_I + K_P) - K_P z^{-1}}{1 - z^{-1}} \\
&= \frac{b_0 + b_1 z^{-1}}{1 - z^{-1}}
\end{aligned}$$

Now let's discretize PID controller which has the following CT transfer function

$$G_{PID}(s) = K_P + K_D s + \frac{K_I}{s}$$

The problem is $K_D s$ term is non-causal. Let us approximate the effect of $K_D s$ in time domain and then perform a discretization. A causal approximate derivative can be find by computing the backward difference.

$$\frac{dx(t)}{dt} \approx \frac{x(t) - x(t - \Delta t)}{\Delta t}$$

Now let's compute the approximate derivative term at the sampling instants and let $\Delta t = T$ we have

$$\frac{dx(t)}{dt}\Big|_{t=kT} \approx \frac{x(kT) - x((k-1)T)}{T}$$

If we take the z-transform we can simply obtain a transfer function for this FIR filter as

$$G_D(z) = \frac{K_D}{T}(1 - z^{-1})$$

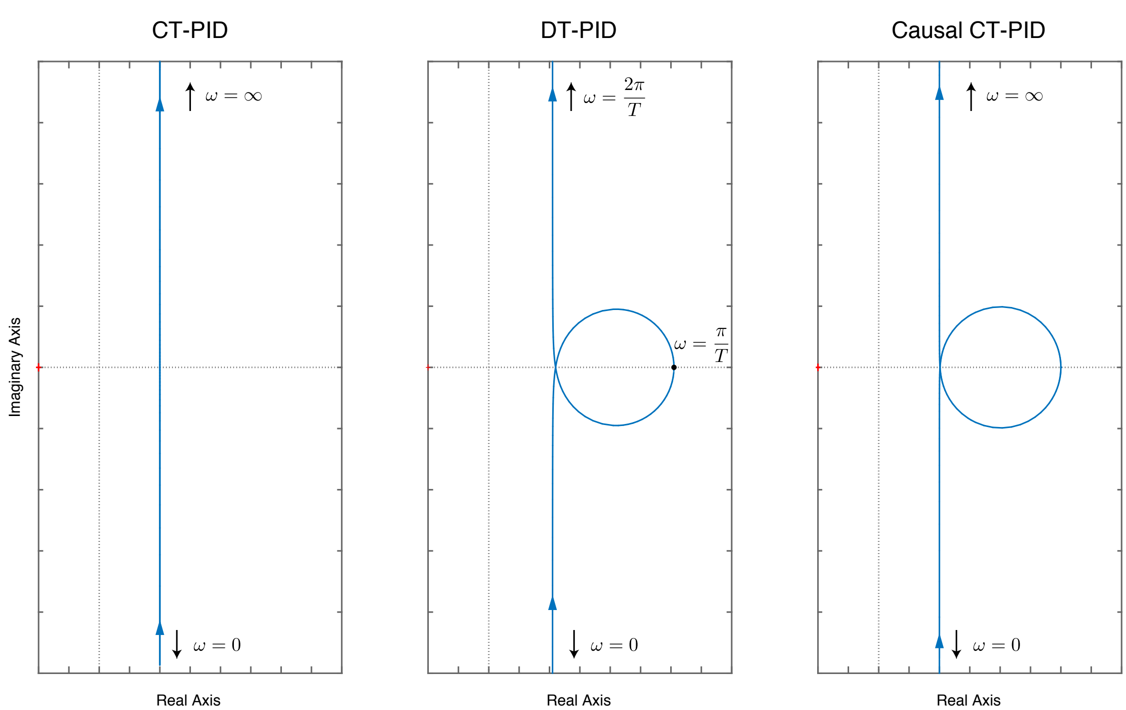
Note that instead of K_D/T, we can just use K_D for the gain. If we combine PI and D terms we obtain the following pulse transfer function for the digital PID controller.

$$\begin{aligned} G_{PID}(z) &= K_P + K_I \frac{1}{1 - z^{-1}} + K_D \left(1 - z^{-1}\right) \\ &= \frac{K_P + K_D + K_I - (K_P + 2K_D)z^{-1} + K_D z^{-2}}{1 - z^{-1}} \\ &= \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 - z^{-1}} \end{aligned}$$

The Figure below illustrates the frequency response characteristics of an ideal CT-PID, a DT-PID, as well as an approximate causal CT-PID controllers. The transfer function of a causal CT-PID has the form below

$$G_{PID}(s) \approx K_P + K_D \frac{s}{\gamma s + 1} + \frac{K_I}{s} \quad \text{where,}$$
$$\gamma > 0 \ \& \ \gamma \approx 0$$

Qualitatively, at low and "high" frequencies all controllers behaves similar. However, for some intermediate range of frequencies there are significant differences between the CT-PID and DT-PID (as well as approximate causal CT-PID). Remarkably, for this frequency range DT-PID and approximate causal CT-PID controllers' the frequency response polar polts are qualitatively similar. This shows that if we choose the right parameters, digitization of derivative term has a similar effect as implementing an approximate analog derivative circuit.



Note that frequency response function for CT and DT systems are found by the Fourier (CT or DT) transforms of the impulse response functions, or simply they can be computed from the s-domain or z-domain transfer functions

$$\begin{aligned} \text{CT}: \quad & G_c(s)|_{s=j\omega} = G_c(j\omega) \\ \text{DT}: \quad & G_d(z)|_{z=e^{j\omega}} = G_d(e^{j\omega}) \end{aligned}$$

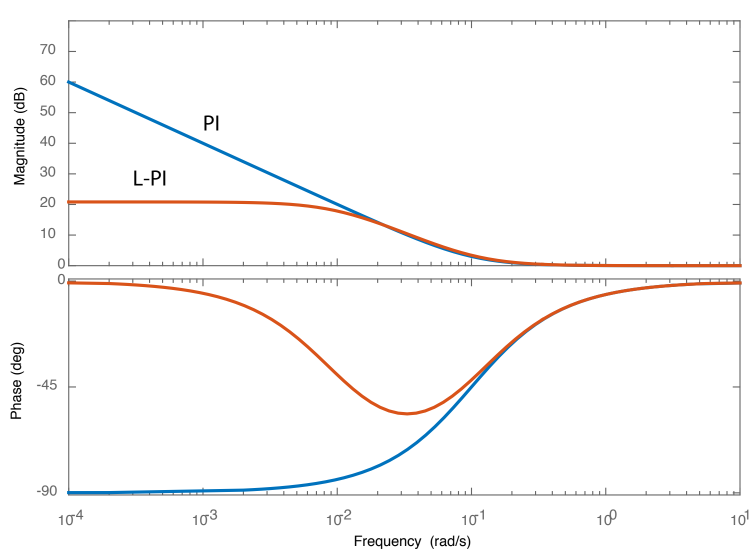
Note that in DT case ω stands for DT frequency. Sometimes $G_d(j\omega)$ or $G_d(\omega)$ used instead of $G_d(e^{j\omega})$. We will cover the Frequency respons later in the class.

PI & PID Controllers with Leaky Integrator

Some times for some practical and other considerations instead of a true integrator (or accumulator) a leaky version is used. A leaky PI controller is in the form

$$\begin{aligned} G_{L-PI}(s) &= K_P + K_I \frac{1}{s+\alpha} \\ &= K_P \frac{s + (\alpha + K_I/K_P)}{s+\alpha} \end{aligned}$$

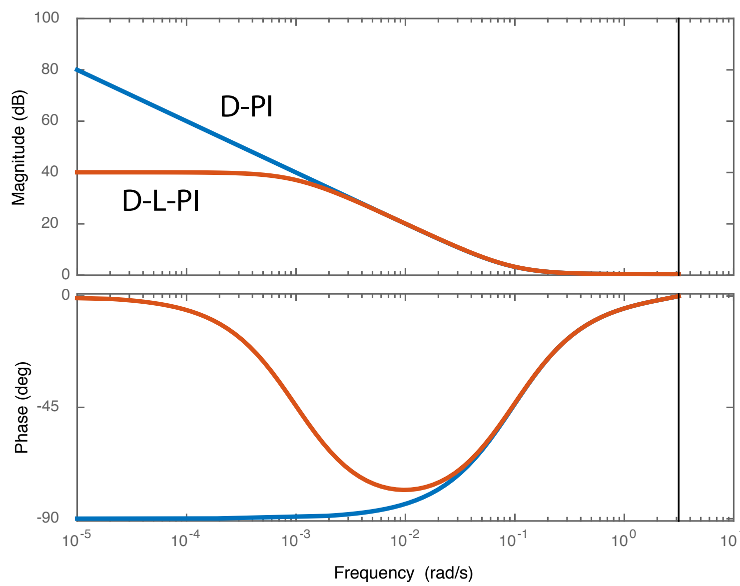
where $\alpha > 0$ and $\alpha \approx 0$ (considering the bandwith of the closed loop system). It can be seen that a leaky-PI controller has the same from with the compensator controller that we covered in 302. If we obscure the frequency response characteristics of both classical and leaky PI controllers, we observe that the behavior is quite different at low frequencies but similar at high frequencies.



If we discretize this CT controller using the star operation approach, we obtain

$$\begin{aligned}
G_{L-PI}(z) = \mathcal{Z}\{G_{L-PI}(s)\} &= K_P + \frac{K_I}{1 - e^{-\alpha T} z^{-1}} \\
&= K_P + \frac{K_I}{1 - \beta z^{-1}} \\
&= \frac{(K_P + K_I) - K_P \beta z^{-1}}{1 - \beta z^{-1}} \\
&= \frac{b_0 + b_1 z^{-1}}{1 + a_1 z^{-1}}
\end{aligned}$$

where similar to the CT case, $\beta < 1$ and $\beta \approx 1$. Similar to the CT case, this DT transfer function has one zero and one pole and it has the equivalent from with a DT-compensator controller. The bode plots of DT-PI and DT-Leaky-PI controllers are illustrated in the Figure below. It can be seen that at low frequencies the differences are significant, but at high frequencies the responses between classical and leaky PI controllers are very similar.



One interesting result is that both for classical and leaky cases, CT and DT frequency responses are qualitatively very similar.

EE402 - Discrete Time Systems **Spring 2018**

Lecture 7

Lecturer: Asst. Prof. M. Mert Ankarali

Mapping Between s & z Planes

When the (uniform) impulse sampling is involved in the process, then we know that s and z variables are related with

$$z = e^{Ts}$$

which is a mapping from complex plane to complex plane, i.e. $M : \mathbb{C} \mapsto \mathbb{C}$, where $M(s) = e^{Ts}$. We will analyze different cases of this mapping and their relevance and importance.

Morever, if s_p is a pole of $G_p(s)$, then it is straightforward to show that $z_p = e^{Ts_p}$ is a pole of $G(z) = \mathcal{Z}\{G_p(s)\}$ (as well as $G(z) = (1 - z^{-1})\mathcal{Z}\{G(s)/s\}$).

Since poles of an LTI system (CT or DT) are the major features that defines the stability and some other performance metrics, analyzing this mapping is very critical for analyzing discrete time control systems.

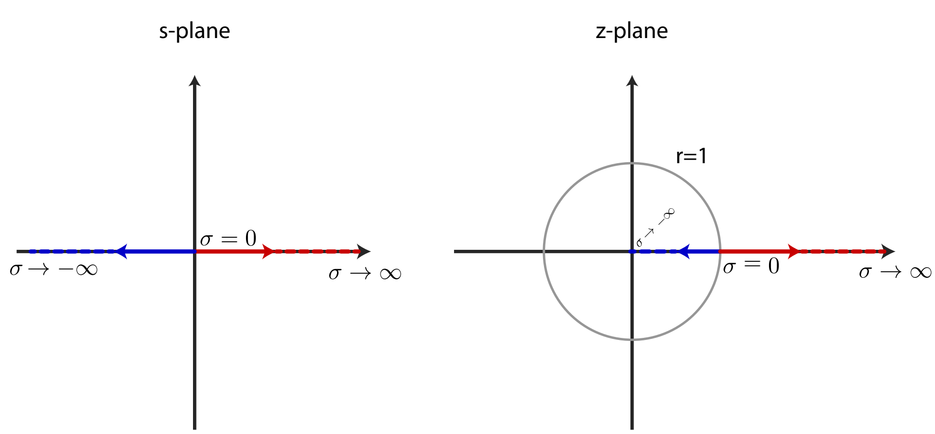
Mapping of real line (left half and right half): When s is purely real (i.e. when the roots of the CT plant are real) we have

$$s = \sigma \quad , \ \sigma \in \mathbb{R}$$
$$z = e^{Ts} = e^{T\sigma} \quad , \ z \in \mathbb{R}^+$$

It is also easy to see the difference between left half and right half of the real line

$$\text{If } \sigma \leq = 0 \quad \rightarrow \quad z = e^{\sigma T} \in [0, 1]$$
$$\text{If } \sigma \geq = 0 \quad \rightarrow \quad z = e^{\sigma T} \in [1, \infty)$$

Mapping of both left and right real lines to z-plane is illustrated in the Figure below. It can be seen that when $\sigma > 0$, $z > 1$, and similarly when $\sigma < 0$, $z < 1$. Technically on both planes red curves belong to "unstable" behaviors, where as blue curves belong to "stable" behaviors.



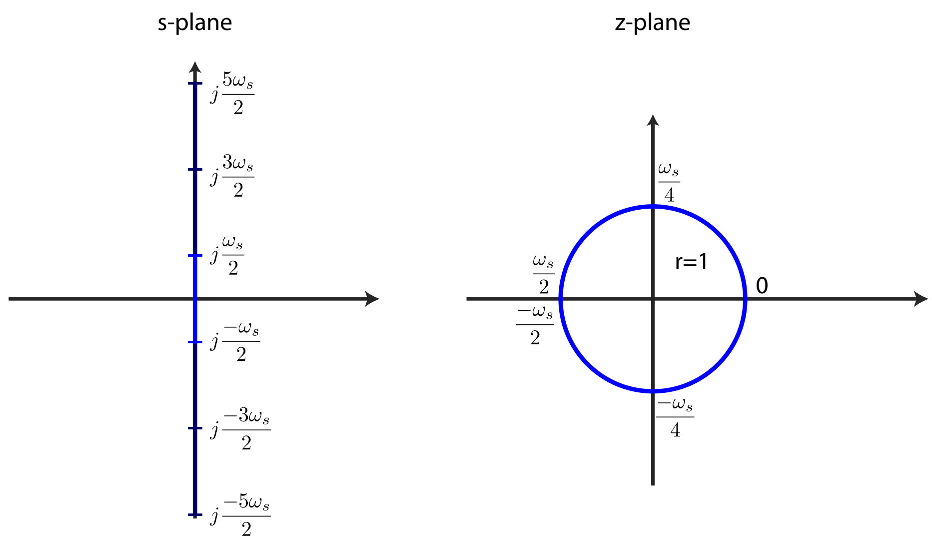
Mapping of imaginary axis: When s is purely imaginary (i.e. when the roots of the CT plant are critically stable) we have

$$\begin{aligned} s &= j\omega \quad , \ \omega \in \mathbb{R} \\ z &= e^{Ts} = e^{T\omega j} \\ |z| &= 1 \\ \angle z &= T\omega = T\omega + 2\pi k \quad , \ k \in \mathbb{Z} \end{aligned}$$

This means that mapping of the imaginary axis is not 1-1, since multiple points on s plane can correspond to a single point on the z plane

$$e^{T\omega j} = e^{(T\omega + 2\pi k)j} \quad \rightarrow \quad M(\omega j) = M((\omega + 2\pi k/T)j) = M((\omega + \omega_s k)j) \ , \ k \in \mathbb{Z}$$

where $\omega_s = 2\pi/T$ is the sampling frequency. It can be seen that imaginary axis on the s plane is mapped to the unit circle on the z plane. However as $\omega \to \infty$ or $\omega \to -\infty$, the mapping circles the unit circle multiple (indeed infinite) times. This mapping is illustrated in the Figure below. The light blue section on the s plane (which covers the points in the imaginary axis between $[-\omega_s/2, \omega_s/2])$ is called the primary section/strip and fully mapped to the unit cicle. Dark blue sections are called complementary sections/strips and they are also individually fully mapped onto the unit circle.



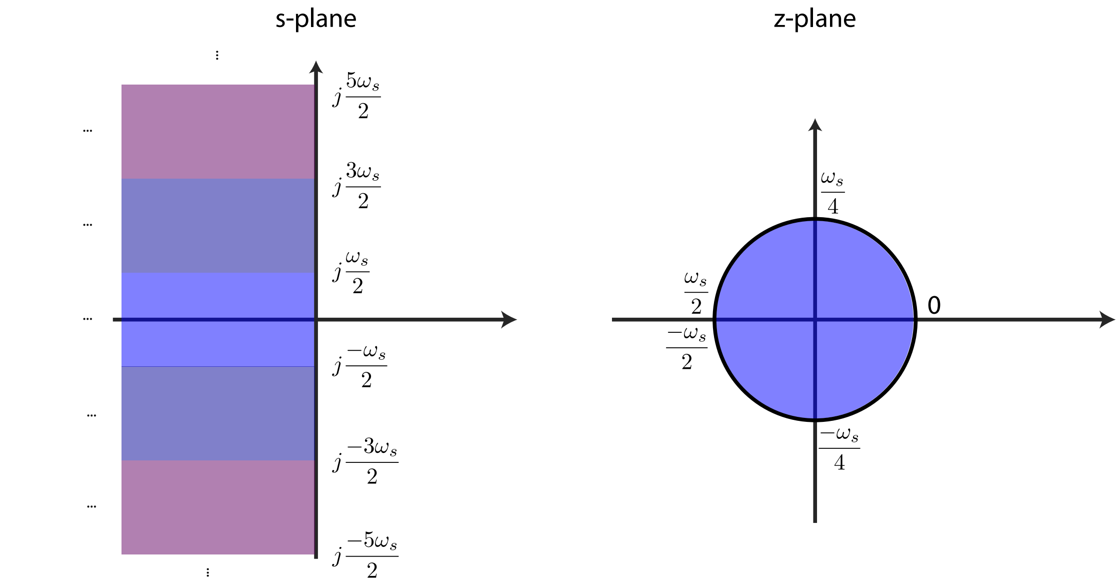
Mapping of open left-half plane: Now let's generalize a little, and consider the mapping of the whole open left-half plane.

$$\begin{aligned} s &= \sigma + j\omega \quad , \ \sigma < 0 \\ z &= e^{Ts} = e^{T\sigma} e^{T\omega j} = e^{T\sigma} e^{(T\omega + 2\pi k)j} \\ |z| &= e^{T\sigma} < 1 \\ \angle z &= T\omega + 2\pi k \quad , \ k \in \mathbb{Z} \end{aligned}$$

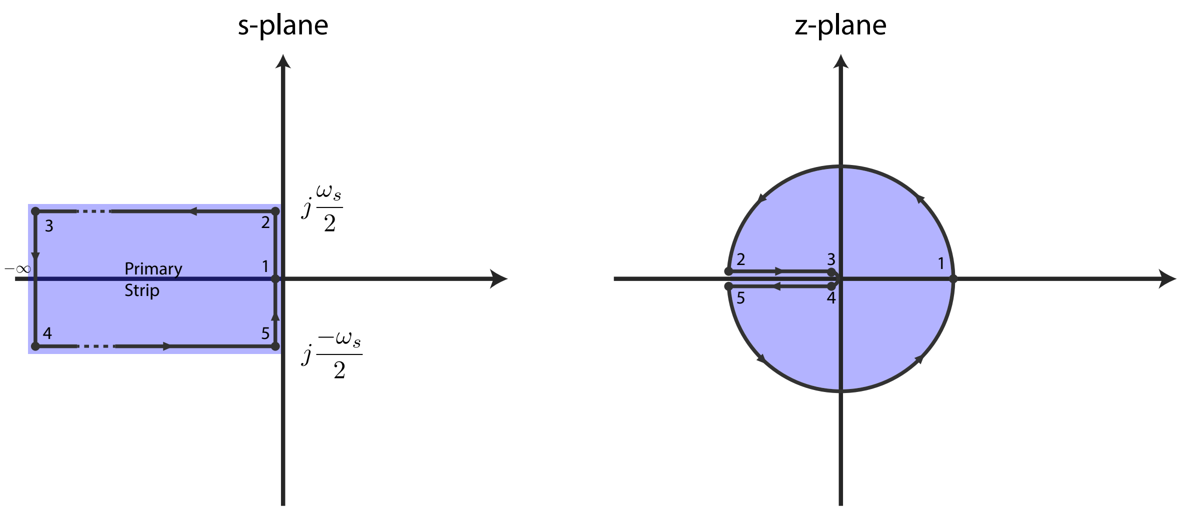
Obviously, this mapping is also not 1-1, and "periodic" in ω, i.e.

$$M(\sigma + \omega j) = M(\sigma + (\omega + 2\pi T)j) = M(\sigma + (\omega + \omega_s)j)$$

Mapping of OLH on s-plane to z-plane is illustrated in Figure below.



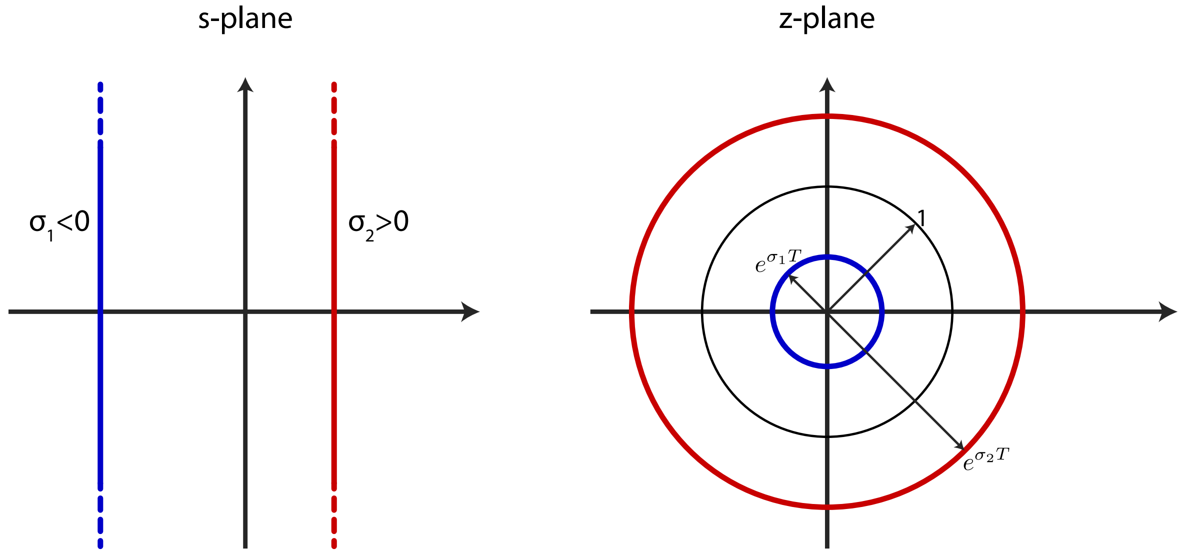
In the primary strip, if we trace the path that is defined by the sequence of points 1-2-3-4-5-1 in the s plane as shown in the Figure below, than this path is mapped in to the z-plane as shown in the Figure. The mapping forms a different path again associated with mapped point sequence 1-2-3-4-5-1.



Mapping of constant attenuation line: In the s-plane, it corresponds to the line for which σ is constant. Constant σ in s-plane corresponds to a constant radius in the z-plane. Thus line is mapped to a circle with a radius of $e^{\sigma T}$.

$$z = e^{\sigma T} e^{\omega T j} = e^{\sigma T} \angle \omega T$$
$$R = e^{\sigma T} = \text{Constant}$$

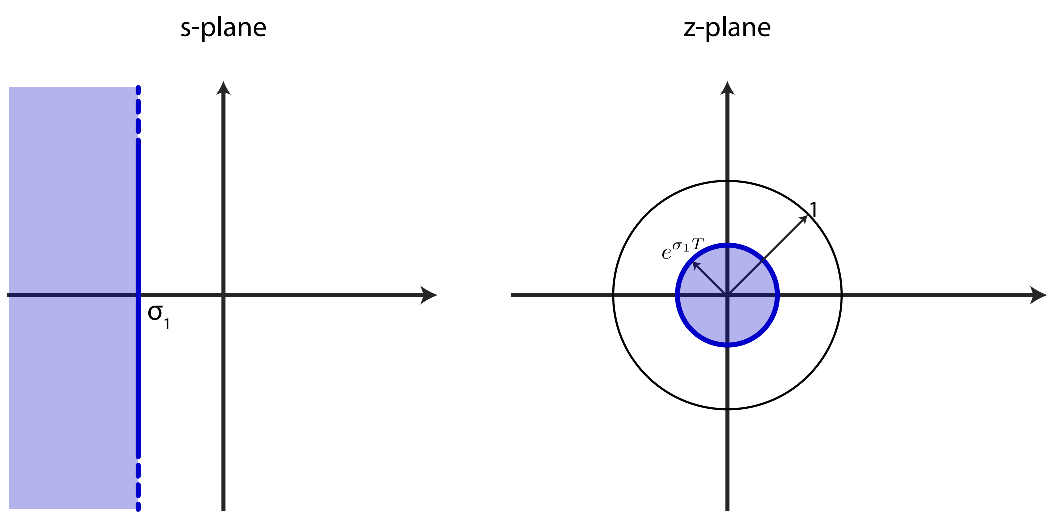
Figure below illustrates mapping of one constant attenuation line in open left half plane and one open right half plane.



Max convergence/settling time region: For a stable CT LTI system convergence/settling time is defined by the real part of the pole. Thus on the s-plane we have the following condition

$$\mathrm{Re}(s) < \sigma_1 < 0$$

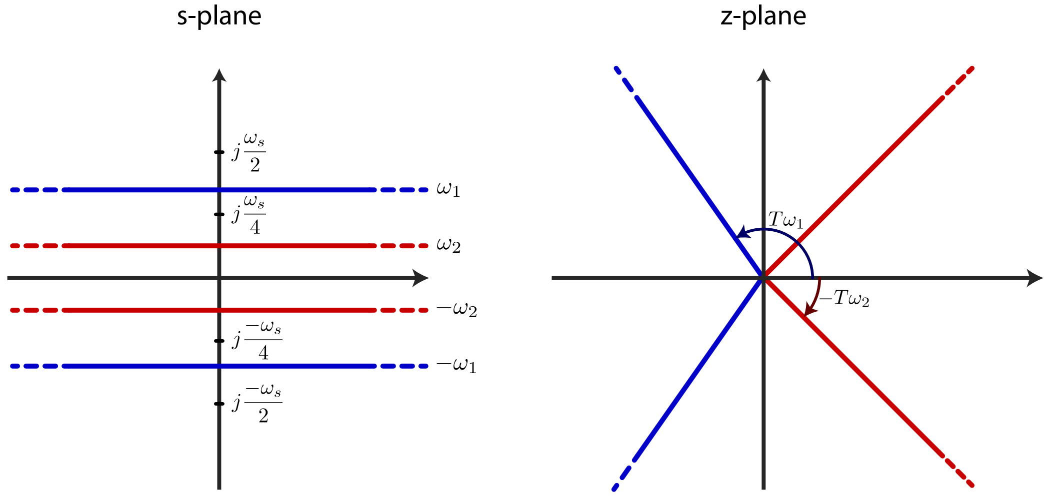
In the z-plane it is mapped to region enclosed by the circle with radius $r = e^{\sigma_1 T}$. This mapping is illustrated below.



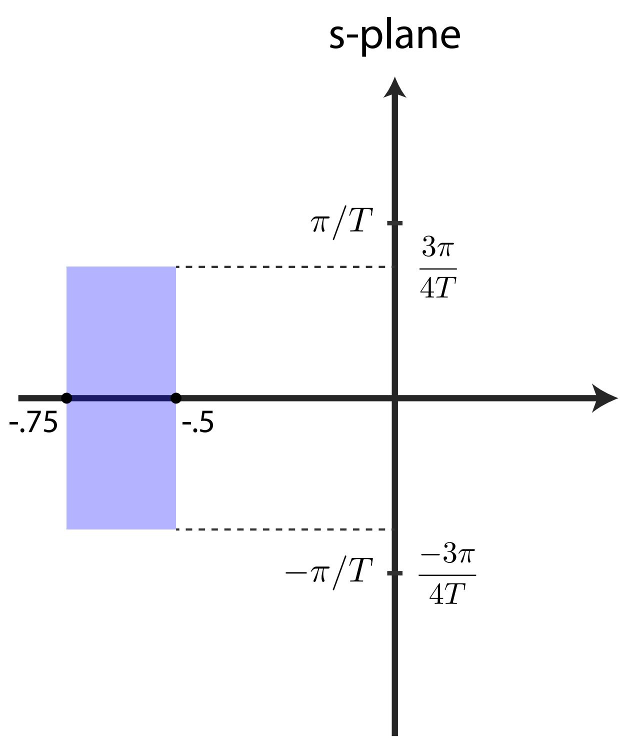
Constant frequency loci: A constant frequency locus $\omega = \omega_1$ in s-plane is mapped to a constant angle line in z-plane, for which the angle is equal to $\omega_1 T$.

$$z = e^{\sigma T} e^{\omega T j} = e^{\sigma T} \angle \omega T$$
$$\angle \omega T = \text{Constant}$$

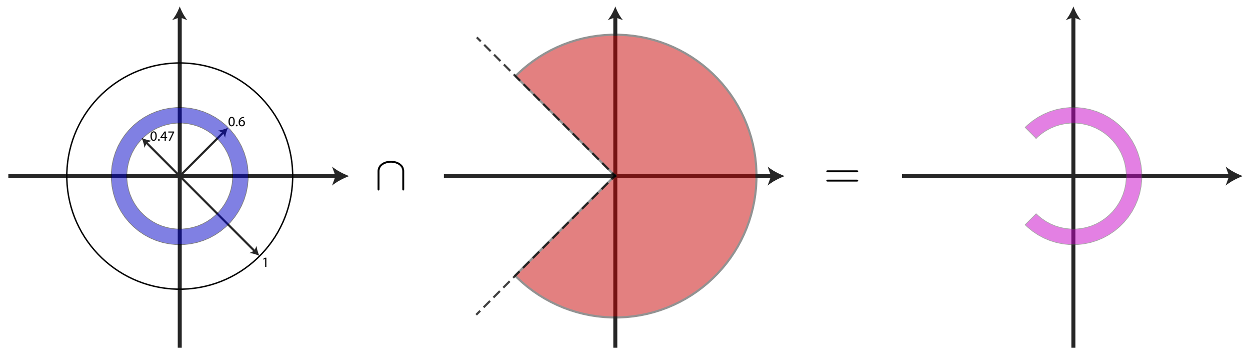
Different constant frequency lines (all inside the primary strip) and their mappings are illustrated in Figure below.



Example: Find the mapping of the area defined inside s-plane illustrated in Figure below with $T = 1$.



Solution: The solution is illustrated in the Figure below



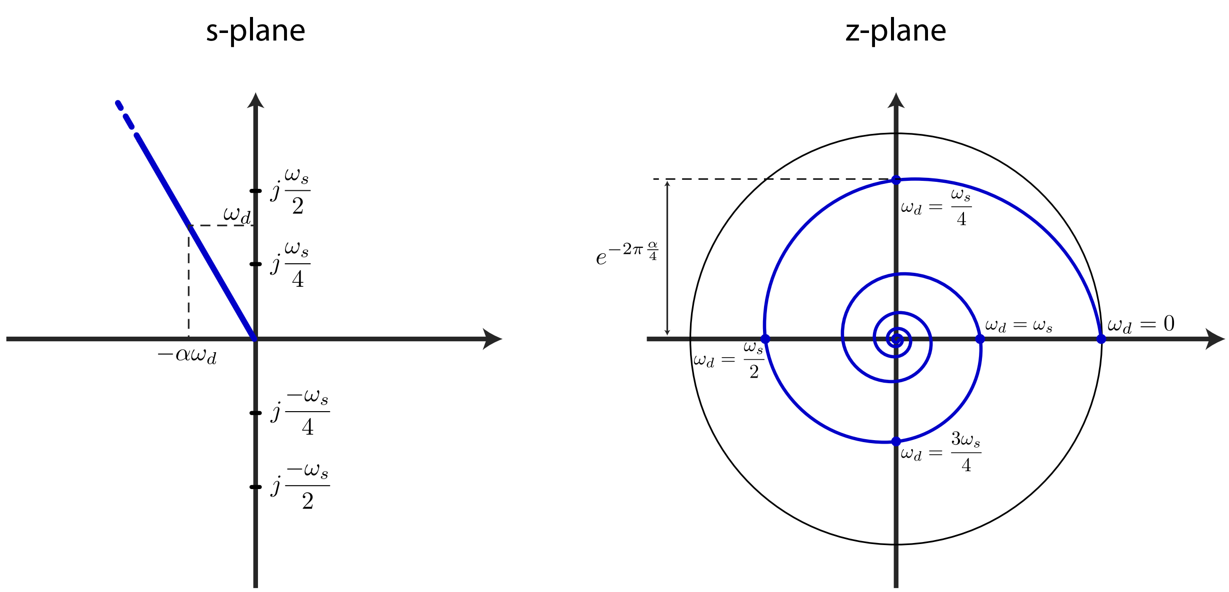
Constant damping loci: A constant damping loci is a line in s-plane passing through the origin and the angle between the line and the Real (or Imaginary) axis defines the damping ratio. In s-plane we have the following relations

$$\begin{aligned} s &= \sigma + \omega j = -\zeta\omega_n + \omega_n\sqrt{1-\zeta^2}j \\ &= -\alpha\omega_d + j\omega_d \\ \omega_d &= \omega_n\sqrt{1-\zeta^2} \\ \alpha &= \frac{\zeta}{\sqrt{1-\zeta^2}} = \text{Constant} \end{aligned}$$

The mapping of this line to the z-plane yields

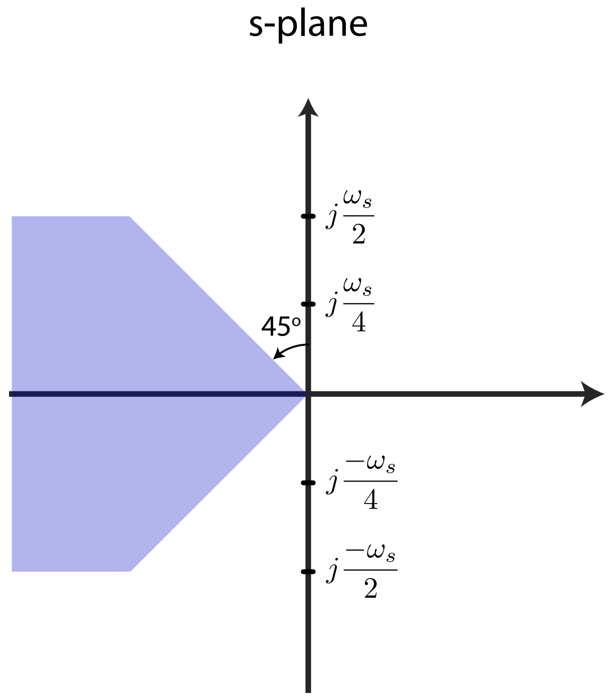
$$\begin{aligned} z &= e^{-\alpha\omega_d T + jT\omega_d} = e^{-\alpha\omega_d T} e^{j\omega_d T} \\ z &= e^{-2\pi\alpha\frac{\omega_d}{\omega_s}} e^{j2\pi\frac{\omega_d}{\omega_s}} \\ |z| &= e^{-2\pi\alpha\frac{\omega_d}{\omega_s}} \\ \angle z &= 2\pi\frac{\omega_d}{\omega_s} \end{aligned}$$

The curve in z-plane corresponds to a spiral shape as seen in Figure below.

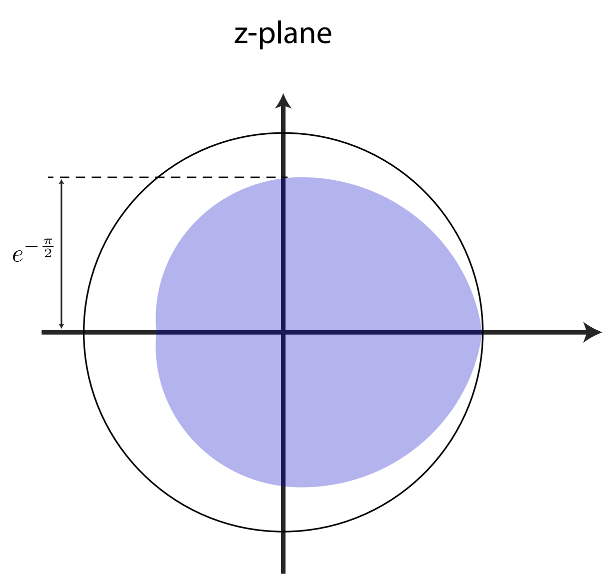


Note that for real systems we need to have also complex conjugates of both the line in s-plane and spiral in z-plane.

Example: Find the mapping of the area defined inside s-plane illustrated in Figure below.



Solution: The solution is illustrated in the Figure below



Bilinear Transformation

A very common transformation used for design and analysis of discrete time control systems, digitized filters etc. is the bilinear transformation. It is a 1-1 mapping between complex z-plane and another complex plane $\bar{s}$ which is an "imaginary" s-domain plane.

The bilinear transformation is defined by

$$z = \frac{1+\frac{T}{2}\bar{s}}{1-\frac{T}{2}\bar{s}}$$
$$\bar{s} = \frac{2}{T}\frac{z-1}{z+1}$$

Let's analyze $|z| < 1$

$$|z| = \left|\frac{1+\frac{T}{2}\bar{s}}{1-\frac{T}{2}\bar{s}}\right| = \frac{|1+\frac{T}{2}\bar{s}|}{|1-\frac{T}{2}\bar{s}|}$$
$$|z| < 1 \implies \left|1+\frac{T}{2}\bar{s}\right| < \left|1-\frac{T}{2}\bar{s}\right|$$

Let $\bar{s} = \bar{\sigma} + j\bar{\omega}$

$$\left|1+\frac{T}{2}(\bar{\sigma}+j\bar{\omega})\right| < \left|1-\frac{T}{2}(\bar{\sigma}+j\bar{\omega})\right|$$
$$\left(\frac{T}{2}\bar{\sigma}+1\right)^2 + \left(\frac{T}{2}\bar{\omega}\right)^2 < \left(\frac{T}{2}\bar{\sigma}-1\right)^2 + \left(\frac{T}{2}\bar{\omega}\right)^2$$
$$\left(\frac{T}{2}\bar{\sigma}+1\right)^2 < \left(\frac{T}{2}\bar{\sigma}-1\right)^2 \implies \bar{\sigma} < 0$$

In other words we have the following relation

$$|z| < 1 \iff \mathrm{Re}\{\bar{s}\} < 0$$

which imples that tha area inside unit circle of z-plane (stable region) is mapped to whole open left half plane of $\bar{s}$-plane.

Now let's consider the mapping of unit-cirlce on z-plane onto the $\bar{s}$-plane.

$$|z| = 1 \implies \left(\frac{T}{2}\bar{\sigma}+1\right)^2 = \left(\frac{T}{2}\bar{\sigma}-1\right)^2 \implies \bar{\sigma} = 0$$

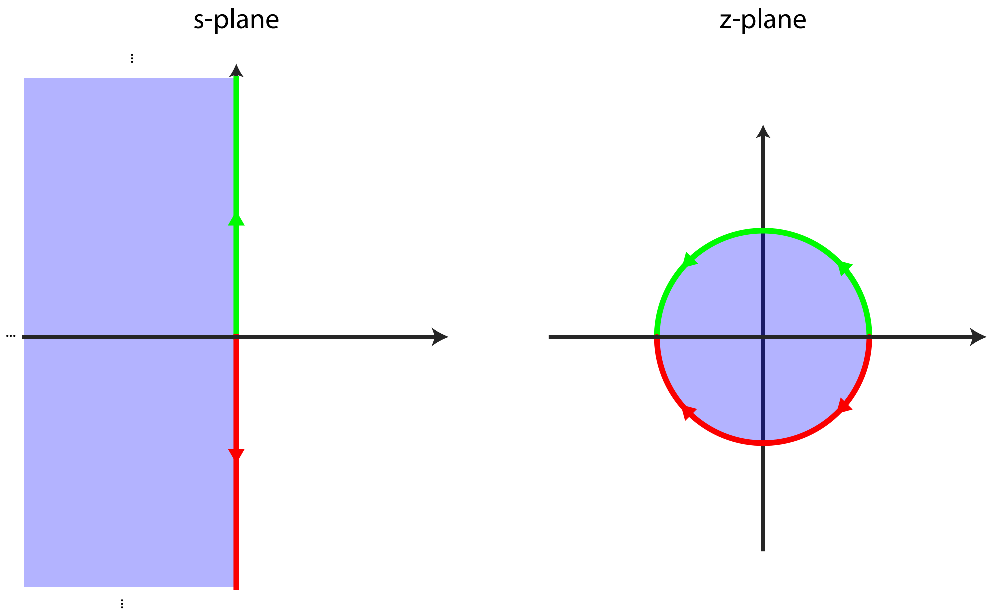
which imples that points on the unit circle are mapped to the imaginary axis on the $\bar{s}$-plane. Now let's analyze this mapping further:

$$|z| = 1 \implies z = e^{j\omega_d} \;, \omega_d \in [-\pi, \pi]$$
$$\bar{s} = \frac{2}{T}\frac{e^{j\omega_d}-1}{e^{j\omega_d}+1} = \frac{2}{T}\frac{(e^{j\omega_d}-1)(e^{-j\omega_d}+1)}{(e^{j\omega_d}+1)(e^{-j\omega_d}+1)} = j\frac{2}{T}\frac{\sin\omega_d}{1+\cos\omega_d} = j\frac{2}{T}\frac{\sin(\omega_d/2)}{\cos(\omega_d/2)}$$
$$\bar{\omega} = \frac{2}{T}\tan(\omega_d/2)$$

where $\bar{\omega}$ is the artificial frequency of the artificial CT system. If we analyze the frequency mapping we can easly see that

$$\omega_d : \; 0 \to \pi \quad \implies \quad \bar{\omega} : \; 0 \to \infty$$
$$\omega_d : \; 0 \to -\pi \quad \implies \quad \bar{\omega} : \; 0 \to -\infty$$

Bilinear transformation and its basic mapping properties are illustrated in the Figure below.



Blue transparent region corresponds to open left half plane in $\bar{s}$-plane and area inside the unit circle in z-plane. Red and green lines/curves illustrate the frequency paths in the respected planes.

Let's analyze the behavior of the frequency mapping around $\omega_d = 0$

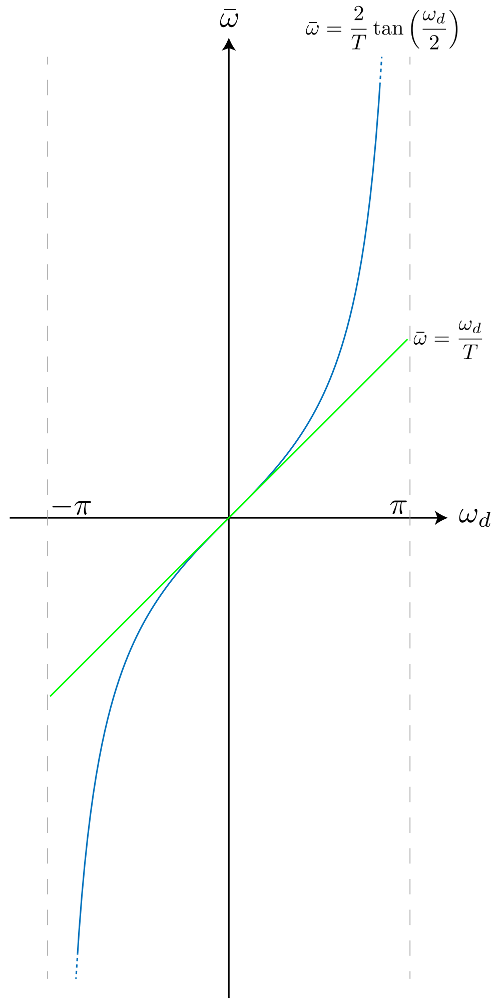
$$\bar{\omega} \approx \left[\frac{d\bar{\omega}}{d\omega_d}\right]_{\omega_d} \omega_d = \omega_d / T$$

$$\bar{\omega} \approx \omega_c$$

where $\bar{\omega}$, ω_d, and ω_c are the artificial CT-frequency, frequency in DT domain, and actual frequency in CT domain respectively. The most important relation is that at low frequencies $\bar{\omega} \approx \omega_c$.

In this linear region, the bilinear transformation behaves nicely and we can roughly say that $G(s) \approx G(\bar{s})$. This means that bilinear transformation can be effectively used for design of control systems, or measuring relative performance metrics for the system of interest. However, as ω_d increases there is a highly non-linear frequency wrapping (compression) effect which can make both the design and analysis challenging and sometimes even meaningless.

The Figure below illustrates the relation between ω_d and ω as well as the linear approximation.



$\bar{\omega}$
$\bar{\omega} = \frac{2}{T}\tan\left(\frac{\omega_d}{2}\right)$
$\bar{\omega} = \frac{\omega_d}{T}$
$-\pi$
π
ω_d

Lecture 8

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Stability of Discrete Time Control Systems

For an LTI discrete time dynamical system which can be represented with a rational transfer function, closed loop poles determine the stability characteristics of the system.

- If all poles of the system are located strictly inside the unit-circle then the system is **(asymptotically) stable**. Asymptotically stability systems are also **BIBO stable**.
- If there exist some *simple* (non-repeated) poles on the unit circle and all remaining poles are located inside the unit circle, then the system is **critically/marginally stable**. Note that critically/marginally stable systems are **BIBO unstable**.
- If there exist at least one repeated pole on the the unit circle, then the system is **unstable**, of course also **BIBO unstable**.
- If there exist at least one pole outside of the unit circle, then the system is **unstable**, of course also **BIBO unstable**.

Jury Stability Test

Jury stability test similar to the Routh-Hurwitz in CT systems, can define the stability of a DT system given the characteristic equation which is in the form

$$D(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n$$

without loss of generality we will assume that $a_0 > 0$.

First Order: When $n = 1$, $D(z)$ takes the form

$$D(z) = a_0 z + a_1$$

DT System is stable if

$$|a_1| < a_0$$

Second Order: When $n = 2$, $D(z)$ takes the form

$$D(z) = a_0 z^2 + a_1 z + a_2$$

DT System is stable if

$$|a_2| < a_0$$
$$D(1) > 0$$
$$D(-1) > 0$$

Third Order: When $n = 3$, $D(z)$ takes the form

$$D(z) = a_0 z^3 + a_1 z^2 + a_2 z + a_3$$

We need to construct the Jury table

Row	z^0	z^1	z^2	z^3
1	a_3	a_2	a_1	a_0
2	a_0	a_1	a_2	a_3
3	b_2	b_1	b_0	

where

$$b_0 = \begin{vmatrix} a_3 & a_2 \\ a_0 & a_1 \end{vmatrix} \quad , \quad b_1 = \begin{vmatrix} a_3 & a_1 \\ a_0 & a_2 \end{vmatrix} \quad , \quad b_2 = \begin{vmatrix} a_3 & a_0 \\ a_0 & a_3 \end{vmatrix}$$

Then DT system is stable if

$$\begin{aligned} |\mathbf{a_3}| &< a_0 \\ D(1) &> 0 \\ -D(-1) &> 0 \\ |b_2| &> |b_0| \end{aligned}$$

General Case: The jury table for systems with order n has $2n - 3$ rows and it has the from below

Row	z^0	z^1	z^2	$\cdots$	z^{n-2}	z^{n-1}	z^n
1	a_n	a_{n-1}	a_{n-2}	$\cdots$	a_2	a_1	a_0
2	a_0	a_1	a_2	$\cdots$	a_{n-2}	a_{n-1}	a_n
3	b_{n-1}	b_{n-2}	b_{n-3}	$\cdots$	b_1	b_0	
4	b_0	b_1	b_2	$\cdots$	b_{n-2}	b_{n-1}	
5	c_{n-2}	c_{n-3}	c_{n-3}	$\cdots$	c_0		
6	c_0	c_1	c_2	$\cdots$	c_{n-2}		
$\vdots$	$\vdots$						
2n - 3	q_2	q_1	q_0				

where

$$b_k = \begin{vmatrix} a_n & a_{n-1-k} \\ a_0 & a_{k+1} \end{vmatrix} \quad , \quad k \in \{0, 1, \cdots, n-1\}$$

$$c_k = \begin{vmatrix} b_n & b_{n-2-k} \\ b_0 & b_{k+1} \end{vmatrix} \quad , \quad k \in \{0, 1, \cdots, n-2\}$$

$$q_k = \begin{vmatrix} p_3 & p_{2-k} \\ p_0 & p_{k+1} \end{vmatrix} \quad , \quad k \in \{0, 1, 3\}$$

Then DT system is stable if

$$
\begin{aligned}
|a_n| &< a_0 \\
D(1) &> 0 \\
(-1)^n D(-1) &> 0 \\
|b_{n-1}| &> |b_0| \\
|c_{n-2}| &> |c_0| \\
&\dots \\
|q_2| &> |q_0|
\end{aligned}
$$

Example: Using Jury test, find if the following characteristic equation is stable or not

$$G(z) = \frac{0.02z^{-1} + 0.03z^{-2} + 0.02z^{-3}}{1 - 3z^{-1} + 4z^{-2} - 2z^{-3} + 0.5z^{-4}}$$

Solution: This is a 4^{th} order system for which the characteristic equation is

$$
\begin{aligned}
D(z) &= a_0 z^4 + a_1 z^3 + a_2 z^2 + a_3 z + a_4 \\
&= 1z^4 + -3z^3 + 4z^2 + -2z + 0.5
\end{aligned}
$$

Jury table for a $n = 4$ system has the form

Row	z^0	z^1	z^2	z^3	z^4
1	a_4	a_3	a_2	a_1	a_0
2	a_0	a_1	a_2	a_3	a_4
3	b_3	b_2	b_1	b_0	
4	b_0	b_1	b_2	b_3	
5	c_2	c_1	c_0		

Before computing the whole Jury table let's check conditions one-by-one

- Check if $|a_4| < a_0$

$$0.5 < 1 \quad \text{OK}$$

- Check if $D(1) > 0$

$$D(1) = 1 - 3 + 4 - 2 + 0.5 = 0.5 > 0 \quad \text{OK}$$

- Check if $(-1)^4 D(-1) > 0$

$$D(-1) = 1 + 3 + 4 + 2 + 0.5 = 10.5 > 0 \quad \text{OK}$$

- Let's compute b_0 and b_3 and check if $|b_3| > |b_0|$

$$
\begin{aligned}
b_0 &= \begin{vmatrix} a_4 & a_3 \\ a_0 & a_1 \end{vmatrix} = \begin{vmatrix} 0.5 & -2 \\ 1 & -3 \end{vmatrix} = 0.5 \\
b_3 &= \begin{vmatrix} a_4 & a_0 \\ a_0 & a_4 \end{vmatrix} = \begin{vmatrix} 0.5 & 1 \\ 1 & 0.5 \end{vmatrix} = -0.75 \\
|b_3| &= 0.75 > 0.5 = |b_0| \quad \text{OK}
\end{aligned}
$$

- Let's compute b_1 and b_2

$$b_1 = \begin{vmatrix} a_4 & a_2 \\ a_0 & a_2 \end{vmatrix} = \begin{vmatrix} 0.5 & 4 \\ 1 & 4 \end{vmatrix} = -2$$
$$b_2 = \begin{vmatrix} a_4 & a_1 \\ a_0 & a_3 \end{vmatrix} = \begin{vmatrix} 0.5 & -3 \\ 1 & -2 \end{vmatrix} = 2$$

- Let's compute c_0 and c_2 and check if $|c_2| > |c_0|$

$$c_0 = \begin{vmatrix} b_3 & b_2 \\ b_0 & b_1 \end{vmatrix} = \begin{vmatrix} -0.75 & 2 \\ 0.5 & -2 \end{vmatrix} = 0.5$$
$$c_2 = \begin{vmatrix} b_3 & b_0 \\ b_0 & b_3 \end{vmatrix} = \begin{vmatrix} -0.75 & 0.5 \\ 0.5 & -0.75 \end{vmatrix} = 0.3125$$
$$|c_2| = 0.3125 \not> 0.5 = |c_0| \quad \text{NOT OK}$$

Final Jury Table is also given below

Row	z^0	z^1	z^2	z^3	z^4
1	$a_4 = 0.5$	$a_3 = -2$	$a_2 = 4$	$a_1 = -3$	$a_0 = 1$
2	$a_0 = 1$	$a_1 = -3$	$a_2 = 4$	$a_3 = -2$	$a_4 = 0.5$
3	$b_3 = -0.75$	$b_2 = 2$	$b_1 = -2$	$b_0 = 0.5$	
4	$b_0 = 0.5$	$b_1 = -2$	$b_2 = 2$	$b_3 = -0.75$	
5	$c_2 = 0.3125$	c_1	$c_0 = 0.5$		

Bilinear Transformation & Routh-Hurwitz Test

In Lecture 7, we showed that bilinear transformation has a 1-1 mapping between stable regions in z-plane and s-plane, as well as unstable regions in z-plane and s-plane. As a way of testing stability, we can transform the the characterstic polynomial using bilinear transformation, then we can apply Routh-Hurwitz test.

Routh-Hurwitz is simpler and easier then the Jury test, however amount of computation needed for transformation generally shadows the relative computational advantage of Routh-Hurwitz.

We know that the bilinear transformation has the form

$$z = \frac{1 + \frac{T}{2}\bar{s}}{1 - \frac{T}{2}\bar{s}}$$

Since we only consider the test of stability, for the sake of simplicity it is reasonable to assume that $T = 2$. Then, the transformation of a general $D(z)$ looks like

$$D(\bar{s}) = D(z)\big|_{z=\frac{1+\bar{s}}{1-\bar{s}}} = a_0 \left(\frac{1+\bar{s}}{1-\bar{s}}\right)^n + a_1 \left(\frac{1+\bar{s}}{1-\bar{s}}\right)^{n-1} + \cdots + a_{n-1}\left(\frac{1+\bar{s}}{1-\bar{s}}\right) + a_n$$

Then clearing the fractions by multiplying both sides by $(1-\bar{s})^n$, we obtain

$$Q(\bar{s}) = b_0\bar{s}^n + b_1\bar{s}^{n-1} + \cdots + b_{n-1}\bar{s} + b_n$$

Testing the stability on $Q(s)$ using Routh-Hurwitz will yield the stability condition of the original DT system.

Example: Consider the following characterstic equation of a DT system

$$D(z) = (z-1) * (z-2) = z^2 - 3z + 2 \tag{8.1}$$

Test the stability (already known) using Bilinear Transformation and Routh-Hurwitz.

Solution:

$$\begin{aligned}
D(\bar{s}) &= D(z)|_{z=\frac{1+\bar{s}}{1-\bar{s}}} = \left(\frac{1+\bar{s}}{1-\bar{s}}\right)^2 - 3\left(\frac{1+\bar{s}}{1-\bar{s}}\right) + 2 \\
Q(\bar{s}) &= (1+\bar{s})^2 - 3(1+\bar{s})(1-\bar{s}) + 2\,(1-\bar{s})^2 \\
&= (1+2\bar{s}+\bar{s}^2) - 3(1-\bar{s}^2) + 2(1-2\bar{s}+\bar{s}^2) \\
&= 6\bar{s}^2 - 2\bar{s}
\end{aligned}$$

This artificial CT system is unstable since one coefficient is negative and one coefficient is equal to zero. It is clear from this example that just for testing stability Bilinear transformation is not very useful.