EE 402 - HW 6

Berivan Işık
2093938

1) a) $G_1(z) = \dfrac{z^2+0.5z}{z^3-2.2z^2+1.52z-0.32}$

$$= \frac{z^{-1}+0.5z^{-2}}{1-2.2z^{-1}+1.52z^{-2}-0.32z^{-3}}$$

$$x[k+1] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -0.32 & -1.52 & 2.2 \end{bmatrix} x[k] + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u[k]$$

$$y[k] = \begin{bmatrix} 0 & 0.5 & 1 \end{bmatrix} x[k]$$

Controllable canonical form

b) $G_2(z) = \dfrac{z^2+0.4z-0.12}{z^2+0.6z-0.4}$

$$G_2(z) = \frac{1+0.4z^{-1}-0.12z^{-2}}{1+0.6z^{-1}-0.4z^{-2}}$$

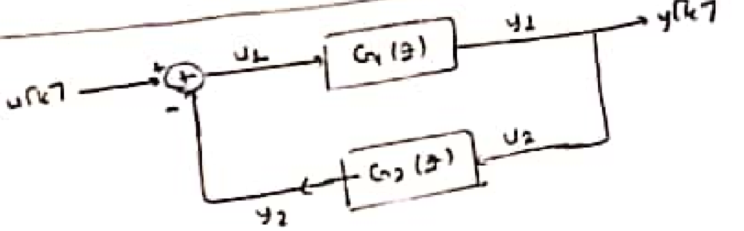
$$x[k+1] = \begin{bmatrix} 0 & 1 \\ 0.4 & -0.6 \end{bmatrix} x[k] + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u[k]$$

$$y[k] = \begin{bmatrix} (-0.12+0.4) & (0.4-0.6) \end{bmatrix} x[k]$$

$$y[k] = \begin{bmatrix} 0.28 & -0.2 \end{bmatrix} x[k]$$

Controllable canonical form

c)



$x_1[k+1] = G_1 x_1[k] + H_1 u_1[k]$
$y_1[k] = C_1 x_1[k] + D_1 u_1[k]$
$x_2[k+1] = G_2 x_2[k] + H_2 u_2[k]$
$y_2[k] = C_2 x_2[k] + D_2 u_2[k]$

$u_1 = u - y_2$

$y_1 = u_2$

$$x[k+1] = \begin{bmatrix} x_1[k+1] \\ x_2[k+1] \end{bmatrix} = \begin{pmatrix} G_1 x_1[k] + H_1(u[k]-y_2[k]) \\ G_2 x_2[k] + H_2\, y[k] \end{pmatrix} = \begin{pmatrix} G_1 x_1[k] + H_1 u[k] - H_1(C_2 x_2[k] + D_2 u_2[k]) \\ G_2 x_2[k] + H_2(C_1 x_1[k] + D_1 u_1[k]) \end{pmatrix}$$

$$x[k+1] = \begin{pmatrix} G_1 & -H_1C_2 \\ H_2C_1 & G_2 \end{pmatrix} \begin{pmatrix} x_1[k] \\ x_2[k] \end{pmatrix} + \begin{pmatrix} H_1 - H_1D_2 \\ H_2D_1 \end{pmatrix} u[k] \quad \ldots (1)$$

$$y[k] = y_1[k] = C_1 x_1[k] + D_1 u_1[k]$$

$$= C_1 x_1[k] + D_1 u[k] - D_1 y_2[k]$$

$$y_1[k] = C_1 x_1[k] + D_1 u[k] - D_1 C_2 x_2[k] - D_1 D_2 u_2[k]$$

$$y_1[k] = C_1 x_1[k] - D_1 C_2 x_2[k] + D_1 u[k] - D_1 D_2 y_1[k]$$

$$y_1[k] = C_1 x_1[k] - D_1 C_2 x_2[k] + D_1 u[k]$$

since $D_1 D_2$ is scalar

$$(1 + D_1 D_2) y_1[k] = C_1 x_1[k] - D_1 C_2 x_2[k] + D_1 u[k]$$

$$y[k] = (1 + D_1 D_2)^{-1} C_1 x_1[k] - (1 + D_1 D_2)^{-1} D_1 C_2 x_2[k] + (1 + D_1 D_2)^{-1} D_1 u[k]$$

$$y[k] = \frac{1}{1+D_1 D_2} \begin{bmatrix} C_1 & -D_1 C_2 \end{bmatrix} \begin{bmatrix} x_1[k] \\ x_2[k] \end{bmatrix} + \frac{1}{1+D_1 D_2} D_1 u[k] \quad \ldots (2)$$

combining (1) and (2) with state space representations in part (a) and (b)

$$x[k+1] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -0.32 & -1.52 & 2.2 \\ & & \end{bmatrix}$$

Scanned by CamScanner

2) a) $e^{(P^{-1}AP)t} \overset{?}{=}$

$$e^{P^{-1}APt} = I + (P^{-1}AP)t + \frac{1}{2!}(P^{-1}A\underbrace{PP^{-1}}_{I}AP)t^2 + \frac{1}{3!}(P^{-1}A\underbrace{PP^{-1}}_{I}A\underbrace{PP^{-1}}_{I}AP)t^3 + \dots$$

$$= P^{-1}P + P^{-1}APt + \frac{1}{2!}P^{-1}A^2Pt^2 + \frac{1}{3!}P^{-1}A^3Pt^3 + \dots$$

$$= P^{-1}(I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \dots)P$$

$$\boxed{e^{(P^{-1}AP)t} = P^{-1}e^{At}P}$$

b) $$e^{At} = I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \dots$$

$$\int_0^T e^{A\lambda}d\lambda = \left(\lambda I + A\frac{\lambda^2}{2} + \frac{1}{3!}A^2\lambda^3 + \frac{1}{4!}A^3\lambda^4 + \dots\right)\Big|_0^T$$

$$= \left(T.I + A\frac{T^2}{2!} + A^2\frac{T^3}{3!} + A^3\frac{T^4}{4!} + \dots\right)$$

If $\det(A) \neq 0$ A^{-1} exists, then,

$$= A^{-1}.A\left(T.I + A\frac{T^2}{2!} + A^2\frac{T^3}{3!} + A^3\frac{T^4}{4!} + \dots\right)$$

$$= A^{-1}\left(AT + A^2\frac{T^2}{2!} + A^3\frac{T^3}{3!} + A^4\frac{T^4}{4!} + \dots\right)$$

$$\boxed{\int_0^T e^{A\lambda}d\lambda = A^{-1}(e^{AT} - I) = (e^{AT} - I)A^{-1}}$$

↳ polynomials of a matrix commute

c) $$e^{At} = I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \frac{1}{4!}A^4t^4 + \dots$$

If $Av = \lambda v \Rightarrow$ $$e^{At}v = v + Avt + \frac{1}{2!}A^2vt + \frac{1}{3!}A^3vt^3 + \frac{1}{4!}A^4vt^4 + \dots$$

$$= v + \lambda vt + \frac{1}{2!}A\lambda vt + \frac{1}{3!}A^2\lambda vt^3 + \frac{1}{4!}A^3\lambda vt^4 + \dots$$

$$= v + \lambda vt + \frac{1}{2!}\lambda^2 vt + \frac{1}{3!}\lambda^3 vt^3 + \frac{1}{4!}\lambda^4 vt^4 + \dots$$

$$= v + \lambda vt + \frac{1}{2!}\lambda^2 vt^2 \frac{1}{3!}\lambda^3 vt^3 + \dots)v$$

$$= (1 + \lambda t + \frac{1}{2!}\lambda^2t^2 + \frac{1}{3!}\lambda^3t^3 + \dots)v$$

$$= e^{\lambda t}v$$

If (v, λ) is an eigenvector - eigenvalue pair of A.. then $(v, e^{\lambda t})$ is an eigenvector - eigenvalue pair of e^{At}.

d)

$$sI - A = \begin{bmatrix} s-\sigma & -\omega \\ +\omega & s-\sigma \end{bmatrix} \qquad d(s) = (s-\sigma)^2 + \omega^2$$

$$\lambda_1 = \sigma + j\omega$$

$$\lambda_2 = \sigma - j\omega$$

$$r(s) = c_0 + c_1 s \qquad r(A) = e^{At}$$

$$r(\lambda_1) = c_0 + c_1(\sigma + j\omega) = e^{\sigma t} e^{j\omega t} \qquad 2j\omega c_1 = e^{\sigma t}\left(e^{j\omega t} - e^{-j\omega t}\right)$$

$$r(\lambda_2) = c_0 + c_1(\sigma - j\omega) = e^{\sigma t} e^{-j\omega t} \qquad c_1 = \frac{1}{\omega} e^{\sigma t} \sin\omega t$$

$$c_0(\sigma - j\omega) + c_1(\sigma^2 + \omega^2) = e^{\sigma t} e^{j\omega t}(\sigma - j\omega)$$

$$c_0(\sigma + j\omega) + c_1(\sigma^2 + \omega^2) = e^{\sigma t} e^{-j\omega}(\sigma + j\omega)$$

$$2j\omega c_0 = e^{\sigma t}\left((\sigma - j\omega)e^{j\omega t} - (\sigma + j\omega)e^{-j\omega t}\right)$$

$$2j\omega c_0 = e^{\sigma t}\left(\sigma\left(e^{j\omega t} - e^{-j\omega t}\right) - j\omega\left(e^{j\omega t} + e^{-j\omega t}\right)\right)$$

$$c_0 = \frac{e^{\sigma t}}{\omega}\left(\sigma \sin\omega t - \omega \cos\omega t\right)$$

$$c_0 = e^{\sigma t}\left(\frac{\sigma}{\omega}\sin\omega t - \cos\omega t\right)$$

$$r(s) = e^{\sigma t}\left(\frac{\sigma}{\omega}\sin\omega t - \cos\omega t\right) + \frac{1}{\omega} e^{\sigma t} \sin\omega t \; s$$

$$e^{At} = \begin{bmatrix} e^{\sigma t}\left(\frac{\sigma}{\omega}\sin\omega t - \cos\omega t\right) + \frac{\sigma}{\omega} e^{\sigma t}\sin\omega t & e^{\sigma t}\sin\omega t \\ -e^{\sigma t}\sin\omega t & e^{\sigma t}\left(\frac{2\sigma}{\omega}\sin\omega t - \cos\omega t\right) \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} e^{\sigma t}\left(\frac{2\sigma}{\omega}\sin\omega t - \cos\omega t\right) & e^{\sigma t}\sin\omega t \\ -e^{\sigma t}\sin\omega t & e^{\sigma t}\left(\frac{2\sigma}{\omega}\sin\omega t - \cos\omega t\right) \end{bmatrix}$$

c) $sI - A = \begin{bmatrix} s & -1 \\ -1 & s \end{bmatrix}$

$d(s) = s^2 - 1 = (s-1)(s+1)$

$\lambda_1 = -1 \quad \lambda_2 = 1$

$r(s) = c_0 + c_1 s$

$r(A) = e^{At}$

$$\left.\begin{aligned} r(\lambda_1) &= c_0 - c_1 = e^{-t} \\ r(\lambda_2) &= c_0 + c_1 = e^{t} \end{aligned}\right\}$$

$c_0 = \frac{e^{-t} + e^{t}}{2}$

$c_1 = e^{t} - \frac{e^{-t} + e^{t}}{2}$

$c_1 = \frac{e^{t} - e^{-t}}{2}$

$$r(s) = \frac{e^{-t} + e^{t}}{2} + \frac{e^{t} - e^{-t}}{2} s$$

$$e^{At} = \begin{bmatrix} \frac{e^{-t} + e^{t}}{2} & \frac{e^{t} - e^{-t}}{2} \\ \frac{e^{t} - e^{-t}}{2} & \frac{e^{-t} + e^{t}}{2} \end{bmatrix}$$

b) i.) Find eigenvalues.

From part (a) $d(s) = (s+\frac{1}{2})(s-2\alpha)$

For stability in CT; $Re\{\lambda\} < 0$

$\lambda_1 = -\frac{1}{2} < 0$ ✓

$\lambda_2 = 2\alpha \Rightarrow Re\{2\alpha\} < 0$

$\boxed{Re\{\alpha\} < 0}$

for asymptotical stability.

(ii) $T(s) = C(sI-A)^{-1}B$

$= \begin{bmatrix} -2 & 1 \end{bmatrix} \begin{bmatrix} s & -1 \\ -\alpha & s-2\alpha+\frac{1}{2} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$= \frac{1}{(s+\frac{1}{2})(s-2\alpha)} \begin{bmatrix} -2 & 1 \end{bmatrix} \begin{bmatrix} s-2\alpha+\frac{1}{2} & 1 \\ \alpha & s \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$= \frac{1}{(s+\frac{1}{2})(s-2\alpha)} \begin{bmatrix} -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ s \end{bmatrix}$

$T(s) = \frac{s-2}{(s+\frac{1}{2})(s-2\alpha)}$

• Case 1: $\alpha \neq 1$: $A = -\frac{1}{2} < 0$ ✓

$p_2 = 2\alpha$ $Re\{2\alpha\} < 0$ $\boxed{Re\{\alpha\} < 0.}$

• Case 2: $\alpha = 1$: $T(s) = \frac{1}{s+\frac{1}{2}}$ $p_1 = -\frac{1}{2} < 0$ ✓

For $\boxed{\{Re\{\alpha\} \in (-\infty, 0)\} \cup \{\alpha = 1\}}$, system is BIBO stable.

4) a)

i) Find eigen values

$$sI-G=\begin{bmatrix} s & -1 \\ -\alpha & s-2\alpha+\frac{1}{2} \end{bmatrix}$$

$$d(s)=\left(s^2-2\alpha s+\frac{1}{2}s-\alpha\right)$$
$$=\left(s^2+\left(\frac{1}{2}-2\alpha\right)s-\alpha\right)$$
$$=\left(s+\frac{1}{2}\right)(s-2\alpha)$$

$\lambda_1=-\frac{1}{2}$ $\quad |\lambda_1|<1$ ✓

$\lambda_2=2\alpha$ $\quad |2\alpha|<1$ $\quad |\alpha|<\frac{1}{2}$ $\Rightarrow$ $\boxed{-\frac{1}{2}<\alpha<\frac{1}{2}}$

for asymptotical stability

ii) Find poles;

$$T(s)=C(zI-G)^{-1}H$$

$$=\begin{bmatrix}-1 & 1\end{bmatrix}\begin{bmatrix} z & -1 \\ -\alpha & z-2\alpha+\frac{1}{2} \end{bmatrix}^{-1}\begin{bmatrix}0\\1\end{bmatrix}$$

$$=\begin{bmatrix}-1 & 1\end{bmatrix}\frac{1}{z^2+\left(\frac{1}{2}-2\alpha\right)z-\alpha}\begin{bmatrix} z-2\alpha+\frac{1}{2} & 1 \\ \alpha & z \end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}$$

$$=\frac{1}{\left(z+\frac{1}{2}\right)(z-2\alpha)}\begin{bmatrix}-1 & 1\end{bmatrix}\begin{bmatrix}1\\z\end{bmatrix}$$

$$T(s)=\frac{-1+z}{\left(z+\frac{1}{2}\right)(z-2\alpha)}$$

* There is no pole zero cancellation.

• Case 1: $\alpha\neq 1$

$\Rightarrow$ same range

$p_1=-\frac{1}{2}$ $\quad \left|-\frac{1}{2}\right|<1$ ✓

$p_2=2\alpha$ $\quad |2\alpha|<1$ $\quad |\alpha|<\frac{1}{2}$

$\boxed{-\frac{1}{2}<\alpha<\frac{1}{2}}$

for BIBO stability.

• Case 2: $\alpha=1\Rightarrow$ $T(s)=\frac{z-2}{\left(z+\frac{1}{2}\right)(z-2)}=\frac{1}{z+\frac{1}{2}}$ $\quad |p_1|=\left|-\frac{1}{2}\right|<1$ ✓

$\Rightarrow$ system is BIBO stable for $\boxed{\alpha=1}$

$$\Rightarrow\ \boxed{\alpha\in\left(-\frac{1}{2},\frac{1}{2}\right)\cup\{1\}}$$