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Frequency Response in Discrete Time Control Systems

Let's assume $u[k]$, $y[k]$, and $G(z)$ represents the input, output, and transfer function representation of an input-output discrete time system.

In order to characterize frequency response of a discret system, the test signal is

$$u[k] = e^{j\omega k}$$

which is an artificial complex periodic signal with a DT domain frequency of ω. The z-transform of $u[k]$ takes the form

$$U(z) = \mathcal{Z}\{e^{j\omega k}\} = \frac{z}{z - e^{j\omega}}$$

Response of the system in z-domain is given by

$$Y(z) = G(z)U(z) = G(z)\frac{z}{z - e^{j\omega}}$$

Assuming that $G(z)$ is a rational transfer function we can perform a partial fraction expansion

$$\begin{aligned}
Y(z) &= \frac{az}{z - e^{j\omega}} + [\text{terms due to the poles of } G(z)] \\
a &= \lim_{z \to e^{j\omega}} \left[(z - e^{j\omega}) \frac{Y(z)}{z} \right] = G(e^{j\omega}) \\
Y(z) &= \frac{G(e^{j\omega})z}{z - e^{j\omega}} + [\text{terms due to the poles of } G(z)]
\end{aligned}$$

Taking the inverse z-transform yields

$$y(t) = G(e^{j\omega})e^{j\omega k} + \mathcal{Z}^{-1} [\text{terms due to the poles of } G(z)]$$

If we assume that the system is "stable" or system is a part of closed loop system and closed loop behavior is stable then at steady state we have

$$\begin{aligned}
y_{ss}[k] &= G(e^{j\omega})e^{j\omega k} \\
&= |G(e^{j\omega})|e^{i\omega k + \angle G(e^{j\omega})} \\
&= Me^{i\omega k + \theta}
\end{aligned}$$

In other words complex periodic signal is scaled and phase shifted based on the following operators

$$\begin{aligned}
M &= |G(e^{j\omega})| \\
\theta &= \angle G(e^{j\omega}
\end{aligned}$$

It is very easy to show that for a general real time domain signal $u[k] = \sin(\omega k + \phi)$, the output $y[k]$ at steady state is computed via

$$y_{ss}[k] = M \sin(\omega k + \phi + \theta)$$

If there is sampling involved in the system the following relation between DT frequency and CT frequency exists $\omega_d = \omega_c T$

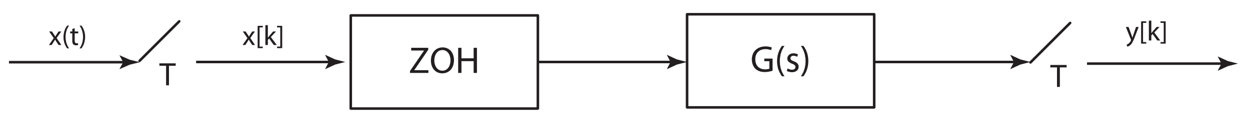
Similar to CT systems we utilize bode polts (or FRF function polts) to analyze DT systems and design filter/controllers. Main difference between CT and DT bode polt is that while the frequency goes to infinity for CT bode polts, for DT systems the frequency goes up-to π rad or $\omega_s/2$ (i.e. Nyquist frequency). Given the bode polt one can extract the magnitude scale and phase difference with respect to any input frequency.

Example: Let's assume that we have the following CT plant transfer function

$$G(s) = \frac{1}{s+1}$$

The pulse transfer function of the following discretized system can be computed as

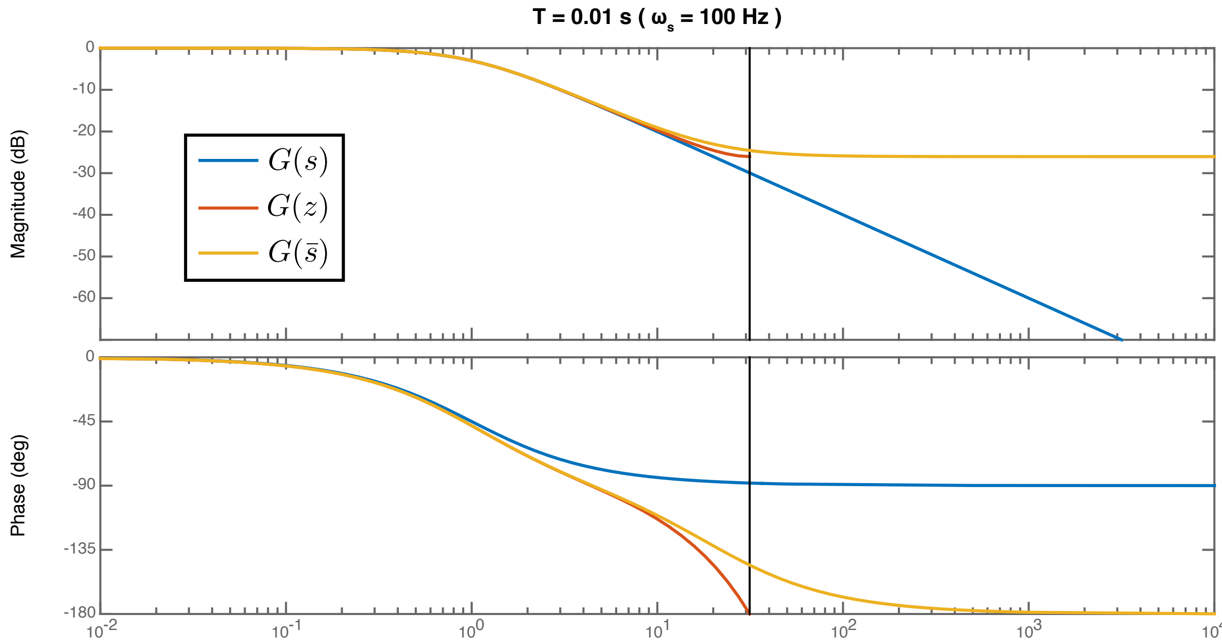
$$G(z) = \mathcal{Z}\left[\frac{1-e^{-Ts}}{s}\frac{1}{s+1}\right]$$

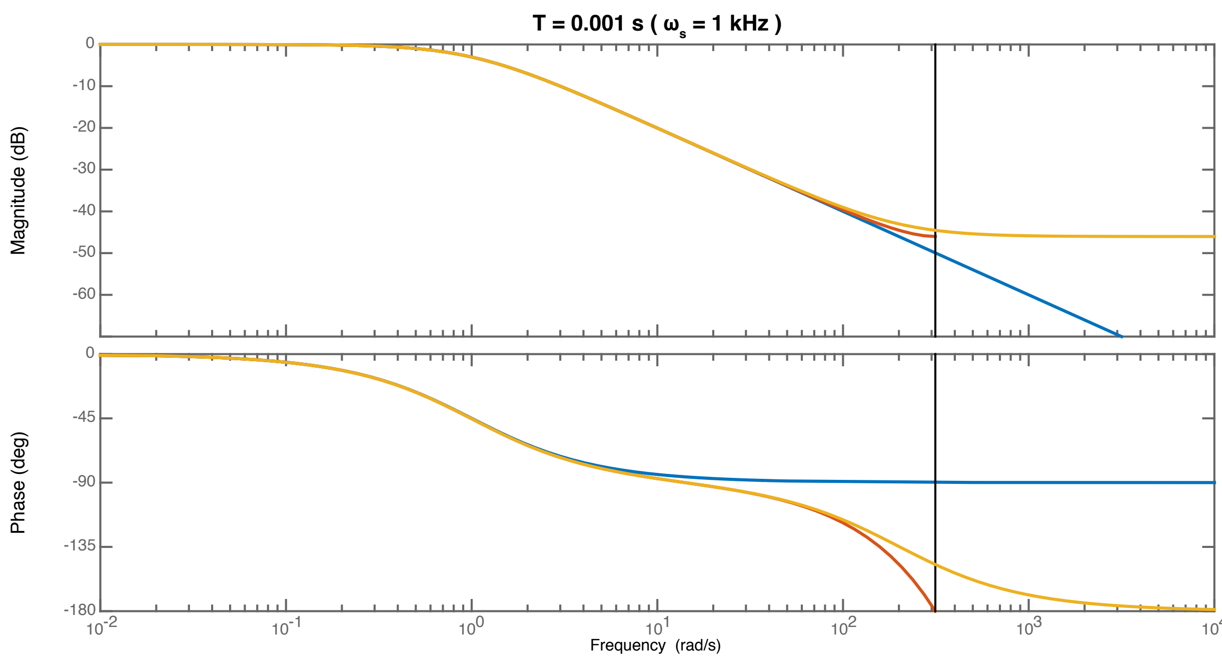


We can also transform this DT transfer function to an artificial CT form using Bilinear-Tustin transformation.

$$G(\bar{s}) = G(z)\big|_{z=\frac{1+(T/2)\bar{s}}{1-(T/2)\bar{s}}}$$

Now let's draw the bode plots of $G(s)$, $G(z)$, and $G(\bar{s})$ for both $T = 0.01s$ and $T = 0.001s$.


T = 0.01 s (ωs = 100 Hz)
Magnitude (dB)
Phase (deg)
$G(s)$
$G(z)$
$G(\bar{s})$
0
-10
-20
-30
-40
-50
-60
-45
-90
-135
-180
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}
10^{4}


T = 0.001 s (ωs = 1 kHz)
Magnitude (dB)
Phase (deg)
0
-10
-20
-30
-40
-50
-60
-45
-90
-135
-180
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}
10^{4}
Frequency (rad/s)

Phase and Gain Margins

We already know that a binary stability metric is not enough the characterize the system performance and that we need metrics to evaluate how stable the system is and its robustness to perturbations. Using root-locus techniques we talked about some “good” pole regions which provides some specifications about stability and closed-loop performance.

Another common and powerful method is the use gain and phase margins based on the Frequency Response Functions of a closed-loop topology. Phase and gain margins are derived from the Nyquists stability criterion and it is relatively easy compute them from the Bode diagrams.

Gain Margin

The *gain margin*, g_m, of a system is defined as the smallest amount that the open loop gain can be increased before the closed loop system goes unstable. For a system, whose “open-loop” phase response starts from an angle $> -180^o$ at $\omega = 0$, the gain margin can be computed based on the smallest frequency where the phase of the loop transfer function $G_{OL}(s)$ is 180^o. Let ω_{pc} represent this frequency, called the phase crossover frequency. Then the gain margin for the system is given by

$$g_m = \frac{1}{|G_{OL}(j\omega_{wc})} \quad \text{or} \quad G_m = -20\log_{10}|G_{OL}(j\omega_{wc})|$$

where G_m is the gain margin in dB scale.

If the phase response never crosses the -180^o line gain margin is simply ∞.

Phase Margin

The *phase margin* is the amount of “phase lag” required to reach the (Nyquist) stability limit. Let ω_{gc} be the gain crossover frequency, the smallest frequency where the loop transfer function $G_{OL}(s)$ has unit magnitude. Then for a system for which the gain response at $\omega = 0$ is larger than 1 and gain decreases and eventually crosses the unity gain line, the phase margin is given by

$$\phi_m = \pi + \angle G_{OL}(j\omega_{gc})$$

When the gain and phase plots shows monotonic like behaviors gain and phase margins becomes more meaningful in terms of closed-loop performance.

So far we have only talked about stability margins for a CT control system. Indeed, the Nyquist stability criterion and associated phase and gain margin definitions are almost exactly same, if we consider FRF functions. Let $G_{OL}(z)$ be the open-loop pulse transfer function of a discrete time control system, then the gain and phase margins are computed as

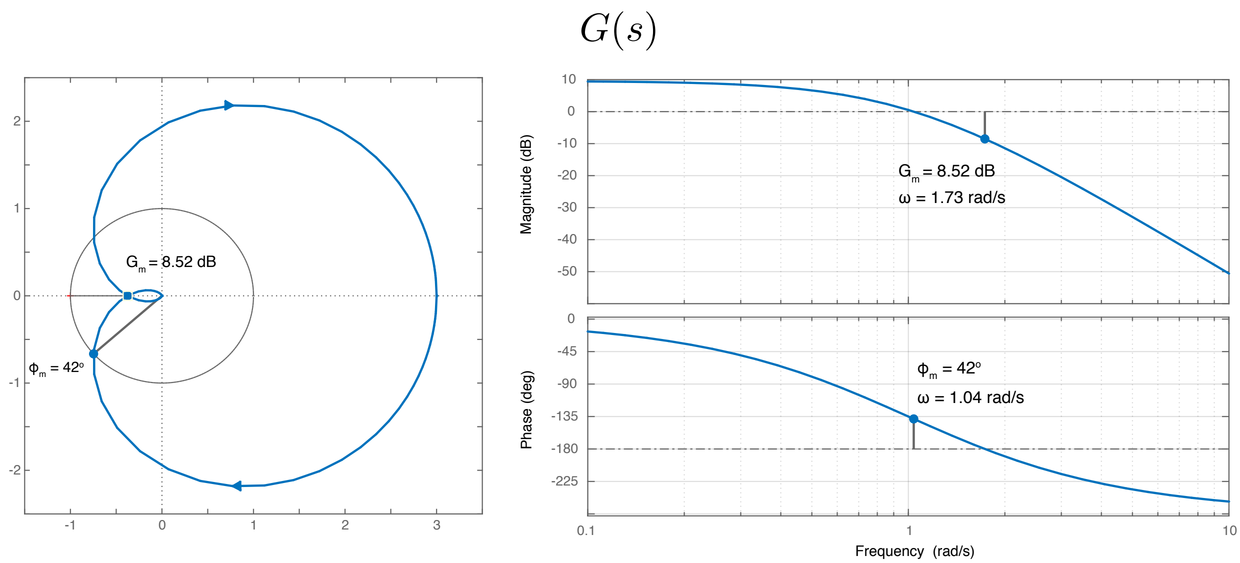
$$g_m = \frac{1}{|G_{OL}(e^{j\omega_{wc}})} \quad \text{or} \quad G_m = -20\log_{10}|G_{OL}(e^{j\omega_{wc}})|$$
$$\phi_m = \pi + \angle G_{OL}(e^{j\omega_{gc}})$$

where as definitions of gain and phase crossover frequencies are exactly same.

Example: Let's consider a CT plant transfer function

$$G(s) = \frac{3}{(s+1)^3}$$

Nyquist plot, bode diagrams are illustrated in the given Figure below



The phase crosover frequency and gain margin for the CT open-loop transfer function is given below

$$\begin{aligned}\omega_{pc} &= 1.73 \text{ rad/s}\\ g_m &= \frac{1}{|G(j\omega_{pc}|} = 2.67\\ G_m &= 8.5dB\end{aligned}$$

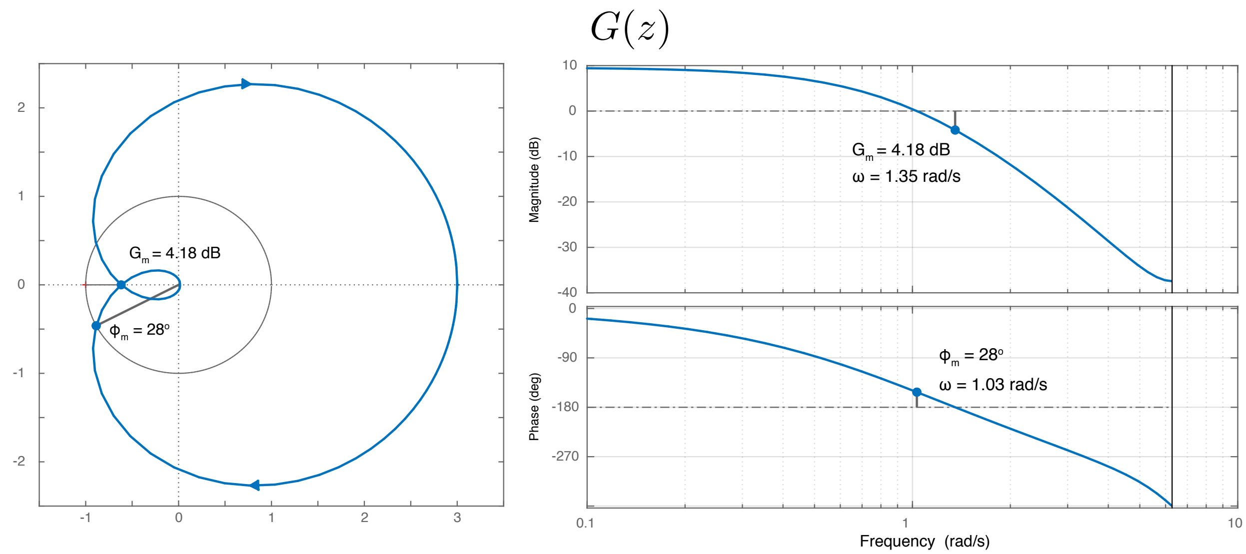
On the other hand gain crosover frequency and phase margin for $G(s)$ is computed as

$$\begin{aligned}\omega_{gc} &= 1.04 \text{ rad/s}\\ \phi_m &= \pi + \angle G(j\omega_{pc}| = 42^0\end{aligned}$$

Now let's assume that this plant transfer function is controlled via a unity gain digital feedback controller and a ZOH operator, where sampling time is $T = 0.5\ s$. Open-loop pulse transfer function can be found as

$$G(z) = \left[\frac{1-e^{-Ts}}{s}G(s)\right]$$

The Nyquist and bode polts for this DT open-loop puls transfer function is illustrated below



Note that instead of DT frequency $\omega_d \in [0, pi)$, the x-axis illustrates the actual frequency $\omega = \omega_d/T$. Is very easy to associate DT and CT frequencies, and main advantage of actual frequency is that CT and DT versions of bode plots becomes directly comparable.

The phase crosover frequency and gain margin for the DT open-loop transfer function is given below

$$\begin{aligned}
\omega_{pc} &= 1.35 \text{ rad/s} \\
g_m &= \frac{1}{|G(e^{j\omega_{pc}T}|} = 1.62 \\
G_m &= 4.18dB
\end{aligned}$$

On the other hand gain crosover frequency and phase margin for $G(s)$ is computed as

$$\begin{aligned}
\omega_{gc} &= 1.03 \text{ rad/s} \\
\phi_m &= \pi + \angle G(j\omega_{pc}| = 28^0
\end{aligned}$$

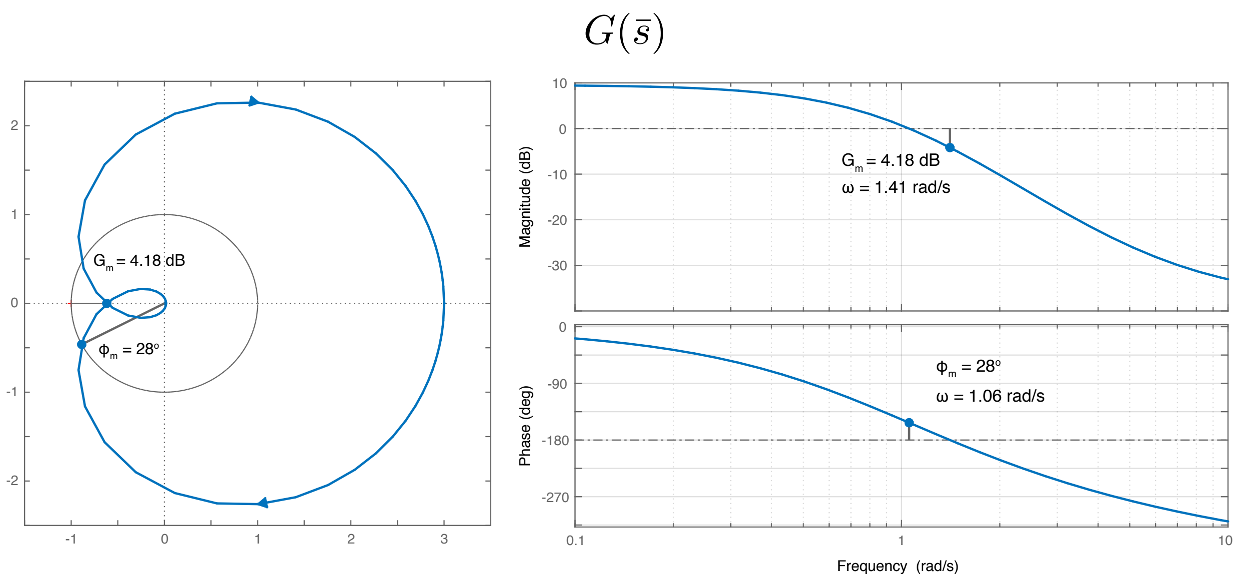
If we compare the CT and DT versions of the same system, we can see that both gain margin and phase margin of the original CT system is better, and we can conclude that discretization reduces the "stability". Another interesting result is that while there is a significant change in phase-crossover frequency, the change in gain crosover frequency is minimal.

Now let's transfrom the $G(z)$ to a artificial CT system using Bilinear-Tustin transformation:

$$G(\bar{s}) = G(z)|_{z=\frac{1+(T/2)\bar{s}}{1-(T/2)\bar{s}}}$$

We know that the relation between the frequency of this artificial system, $\bar{\omega}$, and frequencies of the actual system and discretized system are given by

$$\bar{\omega} = \frac{2}{T}\tan(\frac{\omega_d}{2}) = \frac{2}{T}\tan(\frac{\omega T}{2})$$



The phase crosover frequency and gain margin for this artificial CT open-loop transfer function is given below

$$\begin{aligned}\omega_{pc} &= 1.41\text{rad/s}\\ g_m &= \frac{1}{|G(e^{j\omega_{pc}T}|} = 1.62\\ G_m &= 4.18dB\end{aligned}$$

On the other hand gain crosover frequency and phase margin for $G(s)$ is computed as

$$\begin{aligned}\omega_{gc} &= 1.06 \text{ rad/s}\\ \phi_m &= \pi + \angle G(j\omega_{pc}| = 28^0\end{aligned}$$

If we compare the phase and gain margin and associated crosover frequencies we can see that stability margins of DT and Tustin-Transformed are almost same. It is an expected result, because the core purpose of Tustin transformation is transforming a DT system to CT from by preserving the stability and other importan characteristics.

Tustin transformation has two basic advantages

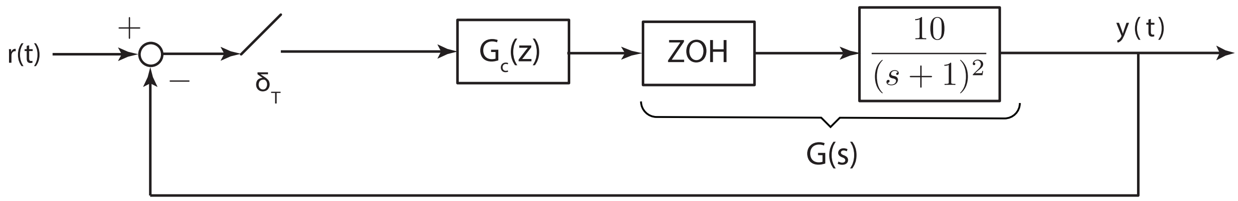
- It "preserves" the stability and robustness characteristics of the digital control system.
- Even though bode polts of DT systems are perfectly valid (and useful), due to periodicity in ω_d the some of the simplicities and advantages of CT bode plots are lost. Since Tustin transformed system is a CT representation, we still can use these simplicities and other advantages.

There two basic disadvantages of Tustin transformation

- It has significant computational costs. However in a computer environment these costs can be negligible.
- If one designs a controller in Tustin from, and then back-transformes the controller in DT form. For some class of controller the quantization effects can become important and deadly.

Lead-Compensator Design for DT Control Systems

Let's consider the DT control system below. Let's assume that $T = 0.1s$



The goal is designing a DT phase-lead compensator such that the Phase-Margin is of the controlled system is in the desired range of $\phi_m \in [50^o\ 60^o]$.

Due to nice proporties if CT bode plots, lead-compensator design procedure is handled in the Bilinear-Tustin transformed domain. Below we will summarize the process for the example plant.

1. Compute discretized plant transfer function $G(z)$

$$\begin{aligned} G(z) &= \mathcal{Z}\left[\frac{1-e^{-Ts}}{s} G_P(s)\right] \\ &= \mathcal{Z}\left[\frac{1-e^{-Ts}}{s} \frac{10}{(s+1)^2}\right] \\ &= \frac{0.047z + 0.044}{z^2 - 1.8z + 0.82} \end{aligned}$$

2. Compute the Bilinear-Tustin transformation of $G(z)$

$$\begin{aligned} G(\bar{s}) &= G(z)\big|_{z=\frac{1+(T/2)\bar{s}}{1-(T/2)\bar{s}}} \\ &= \frac{-0.0008s^2 - 0.48s + 9.98}{s^2 + 2s + 0.998} \end{aligned}$$

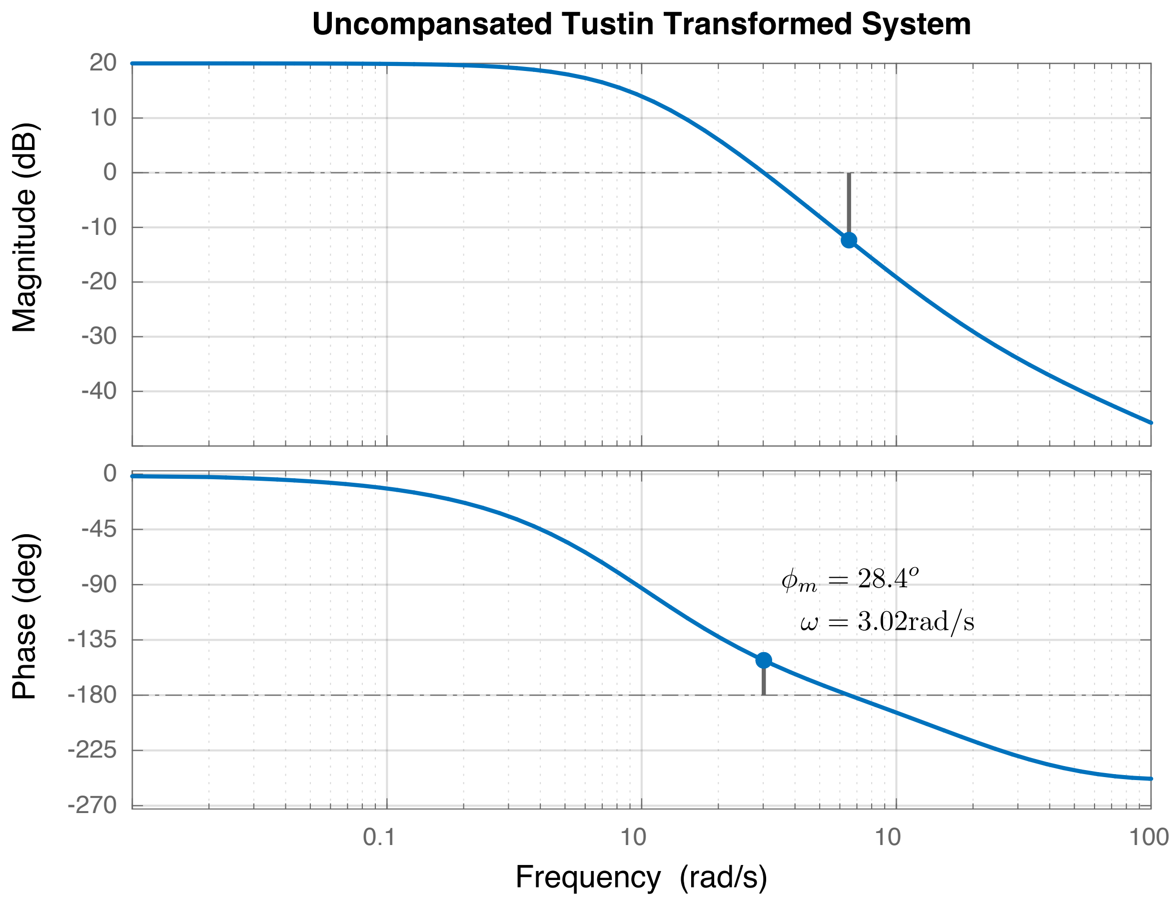
3. Now the goal is designing a lead compensator for $G(\bar{s})$. Lead-compensator has the form.

$$G_l(\bar{s}) = K_{\text{lead}} \frac{T_l\ a\ s + 1}{T/a\ s + 1}$$

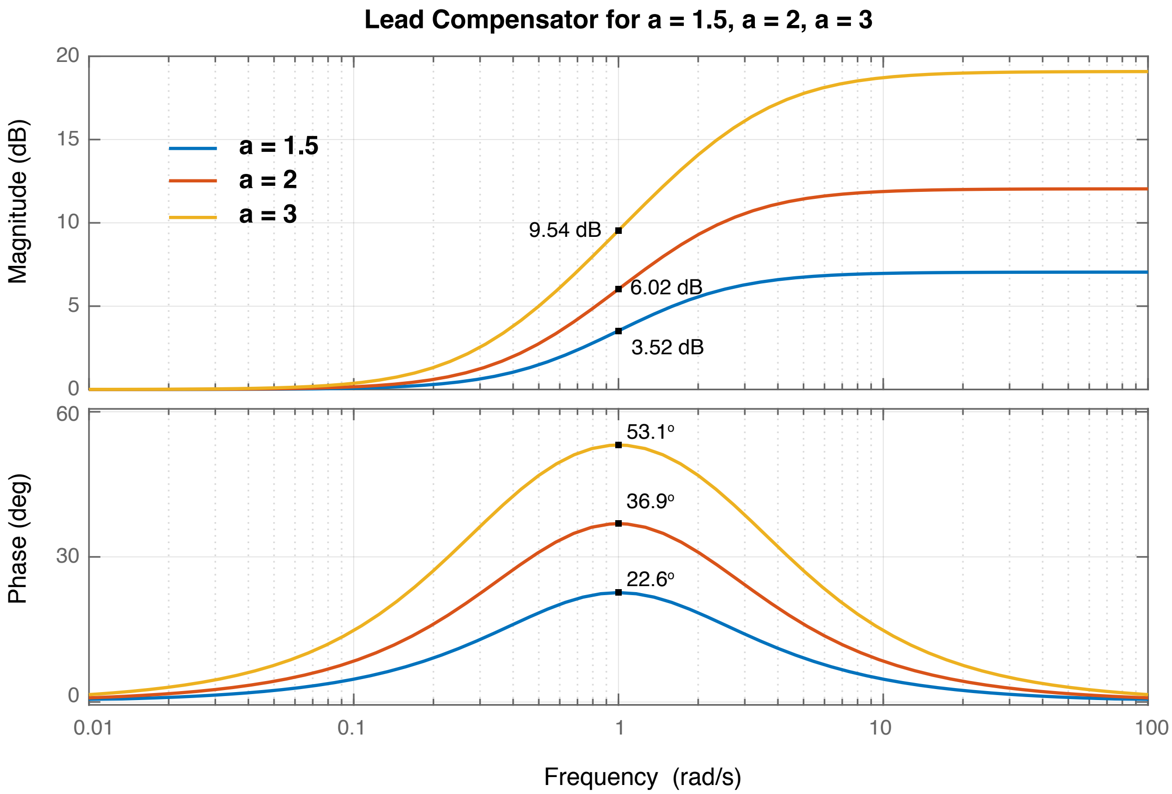
   where gain K_l (generally) computed based on steady state requirments. This can be computed either in z-domain or s-bar domain. Let's assume that we are OK with steady-state performance and $K_l = 1$.

4. Compute the ϕ_m of the un-compensated system and find the required $\Delta\phi_m$ such that the compensated system meets the specifications.

$$\begin{aligned} \phi_m &\approx 28^o \\ \Delta\phi_m &= [22^o\ ,\ 32^o] \end{aligned}$$



5. For $a = 1.5$, $a = 2$, and $a = 3$, bode plots of a phase-lead compensator is illustrated with $T_l = 1$ in the figure below.



It can be seen that ϕ_{max} is larger for large a. ϕ_{max} (or a) can also be computed using the following relation.

$$\sin \phi_{max} = \frac{a^2 - 1}{a^2 + 1}$$

As your remember from EE302 Lead compensator should have a lead angle that is above $\approx 5 - 15^o$ the reqired $\Delta\phi_m$. Based on this we compute/find a. Based on the bode polts, it seems that $a = 2$ may supply the required additonal phase-margin. One should see that ϕ_{max} is not affected from the choice of T_l.

6. Now our goal is to copmpute T_l, where $1/T_l$ corresponds to the center freqeuncy of the copmensator. One possible choice is choosing T such that $1/T_l = \bar{\omega}_{gc}$, i.e. gain croesover frequency. However, at center frequency the lead copmensator shifts the bode magnitude by

$$G_{\text{center}} = 20\log_{10}(1/a)$$

which causes a shift in gain-croesover frequency. For example, for $a = 2$, $G_{\text{center}} \approx 6\ dB$. For this reason, a "better" choice is choosing T such that center freqeuncy of the lead-compensator coincides with the frequency where $|G(j\bar{\omega})|$ crosses $-20\ \log_{10}(1/a)$, i.e. we compute T_l such that

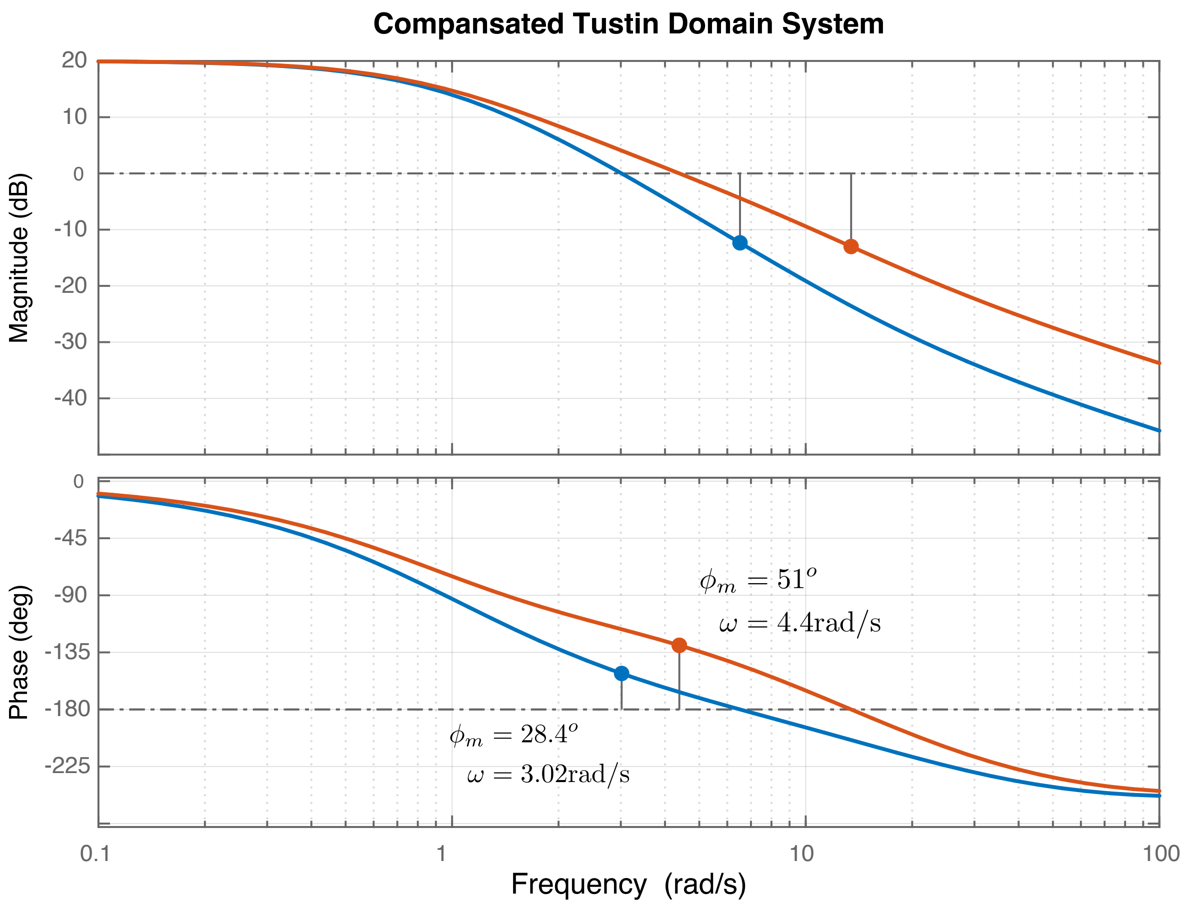
$$|G(j1/T_l)| = -20\ \log_{10}(1/a)$$

From the bode polts for $a = 2$, the freqeuncy for which $G(\bar{s})$ croesses the $-6dB$ line is approximately 4.45 rad/s. Thus we choose $T = 0.225s$. Check if the lead-compensator meets the phase-margin reqeuirment. Otherwise, repeat the process with a higher $\delta\phi$ angle.

In our example, the resultant tustin domain lead compensator has the form.

$$G_l(\bar{s}) = \frac{0.45s + 1}{0.1125s + 1}$$

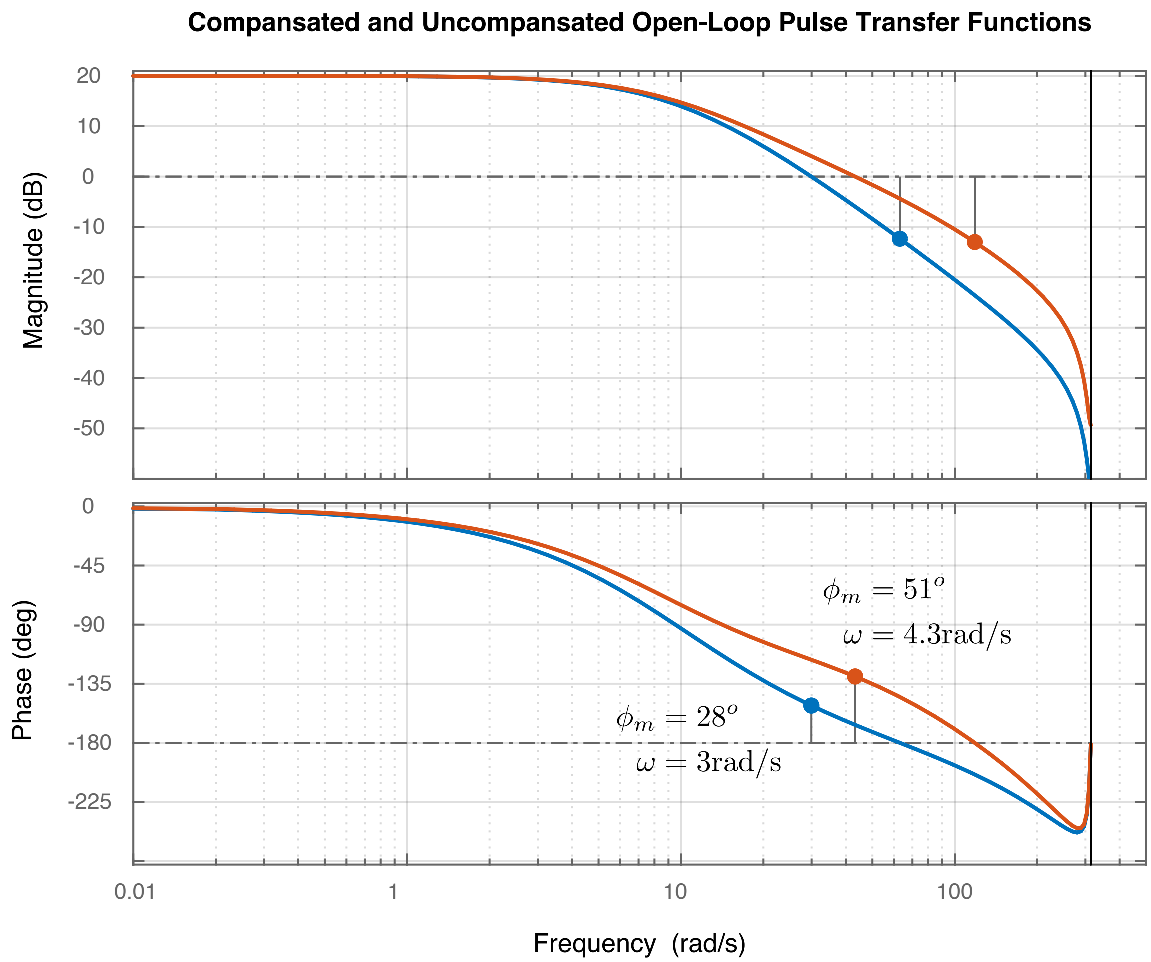
The Figure below illiustrates the bode plots of both (tustin domain) compensated and uncompansated systems. Compensated systems has a phase margin of $\phi_m = 51^o$ which meets the requirements.



7. Transform the $\bar{s}$-domain lead-compensator to z-domain.

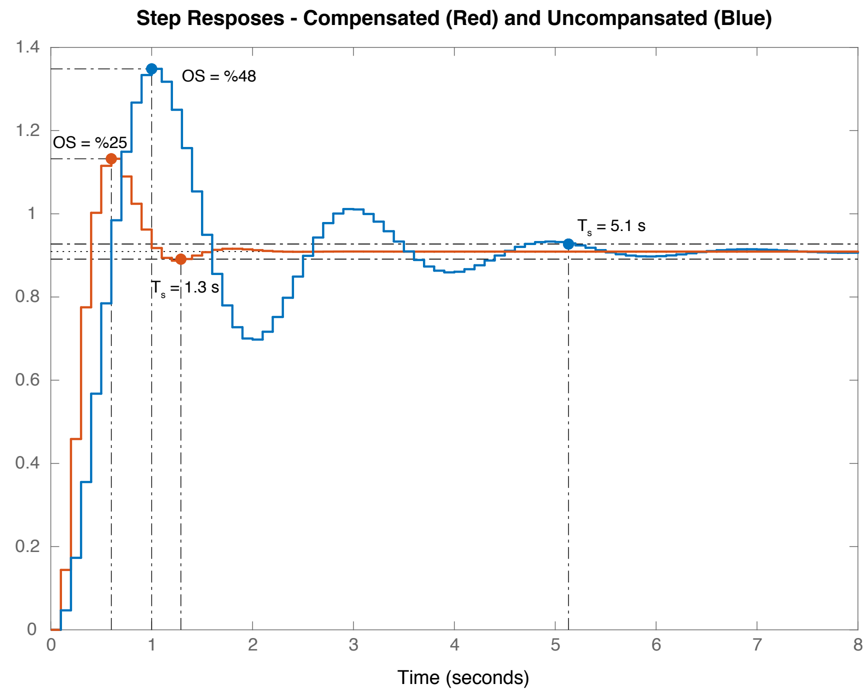
$$
\begin{aligned}
G_l(z) =& G_l(\bar{s})|_{\bar{s}=\frac{2}{T}\frac{z-1}{z+1}} \\
G_l(z) =& \frac{3.08z - 2.46}{z - 0.385}
\end{aligned}
$$

8. Check if the discrete-time compensator meets the the phse-margin requirments.



It can be seen that the designed compensator in z-domain also meets the phase-margin specifications.

In Figure below, we compare the closed-loop step responses of both uncompensated and compensated pulse transfer functions.


Step Resposes - Compensated (Red) and Uncompansated (Blue)
OS = %48
OS = %25
T_s = 5.1 s
T_s = 1.3 s
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0 1 2 3 4 5 6 7 8
Time (seconds)