Lecture 17

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Discrete-time Luenberger Observer

In general the state, $x[k]$ of a system is not accessible and *observers, estimators, filters*) have to be used to extract this information. The output, $y[k]$, represents the measurements which is a function of $x[k]$ and $u[k]$.

$$\begin{aligned} x[k+1] &= Gx[k] + Hu[k] \\ y[k] &= Cx[k] + Du[k] \end{aligned}$$

A Luenberger observers is built using a "simulated" model of the system and the errors caused by the mismatched initial conditions $x_0 \neq \hat{x}_0$ (or other types of perturbations) are reduced by introducing output error feedback.

Let's assume that the states of the simulated system is $\hat{x}[k]$, then the state space equation of this synthetic system takes the form

$$\begin{aligned} \hat{x}[k+1] &= \hat{G}x[k] + \hat{H}u[k] \\ \hat{y}[k] &= \hat{C}x[k] + \hat{D}u[k] \end{aligned}$$

Note that since $u[k]$ is the input that is controlled it is assumed to be known. If $x[0] = \hat{x}[0]$ and when there is no model mismatch or uncertainty in the system then we expect that $x[k] = \hat{x}[k]$ and $y[k] = \hat{y}[k]$ for all k. When $x[0] = \hat{x}[0]$, then we observe a difference between the measured and predicted output $y[k] \neq \hat{y}[k]$. The core idea in Luenberger observers is feeding the error in the output prediction $y[k] - \hat{y}[k]$ to the system via a linear feedback gain.

$$\begin{aligned} \hat{x}[k+1] &= G\hat{x}[k] + Hu[k] + L\left(y[k] - \hat{y}[k]\right) \\ \hat{y}[k] &= Cx[k] + Du[k] \end{aligned}$$

In order to understand how a Luenberger observer works and ot choose a proper observer gain L, we define an error signal $e[k] = x[k] - \hat{x}[k]$. The dynamics w.r.t $e[k]$ can be derived as

$$\begin{aligned} e[k+1] &= x[k+1] - \hat{x}[k+1] \\ &= (Gx[k] + Hu[k]) - (G\hat{x}[k] + Hu[k] + L\left(y[k] - \hat{y}[k]\right)) \\ e[k+1] &= (G - LC)\, e[k] \end{aligned}$$

where $e[0] = x[0] - \hat{x}[0]$ denotes the error in the initial condition.

If the matrix $(G - LC)$ is stable then the errors initial condition will diminish eventually. Moreover, in order to have a good observer/estimator performance the observer convergence should be sufficiently fast.

Observer Gain & Pole Placement

Similar to the state-feedback gain design, the fundamental principle of "pole-placement" Observer design is that we first define a desired closed-loop eigenvalue set and compute associated desired characteristic polynomial.

$$\begin{aligned}\mathcal{E}^* &= \{\lambda_1^*, \ \cdots, \ \lambda_n^*\} \\ p^*(z) &= (z-\lambda_1^*)\cdots(z-\lambda_n^*) \\ &= z^n + a_1^* z^{n-1} + \cdots + a_{n-1}^* z + a_n^*\end{aligned}$$

The necessary and sufficient condition on arbitrary observer pole-placement is that the system should be fully Observable. Then we tune L such that

$$\det(zI - (G - LC)) = p^*(z)$$

Direct Design of Observer Gain

If n is small, the most efficient method could be the direct design.

Example: Consider the following DT system

$$\begin{aligned}x[k+1] &= \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} x[k] + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u[k] \\ y[k] &= \begin{bmatrix} 1 & -1 \end{bmatrix} u[k]\end{aligned}$$

Design an observer such that estimater poles are located at $\lambda_{1,2} = 0$ (Dead-beat Observer)

Solution: Desired characteristic equation can be computed as

$$p^*(z) = z^2$$

Let $L = \begin{bmatrix} l_2 \\ l_1 \end{bmatrix}$, then the characteristic equation of $(G - LC)$ can be computed as

$$\begin{aligned}\det(zI - (G - LC)) &= \det\left(\begin{bmatrix} z-1+l_2 & -l_2 \\ l_1 & z-2-l_1 \end{bmatrix}\right) \\ &= z^2 + z(l_2 - l_1 - 3) + (l_1 - 2l_2 + 2)\end{aligned}$$

If we match the equations

$$\begin{aligned}l_2 - l_1 &= 3 \\ -l_1 + 2l_2 &= 2 \\ l_2 &= -1 \\ l_1 &= -4\end{aligned}$$

Thus $L = \begin{bmatrix} -1 \\ -4 \end{bmatrix}$

Design of Observer Gain Using Reachable Canonical Form

Let's assume that the state-space representation is in observable canonical form

$$G = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_n \\ 1 & 0 & \cdots & 0 & -a_{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -a_2 \\ 0 & 0 & \cdots & 1 & -a_1 \end{bmatrix} \quad , \quad H = \begin{bmatrix} (b_n - b_0 a_n) \\ (b_{n-1} - b_0 a_{n-1}) \\ \vdots \\ (b_2 - b_0 a_2) \\ (b_1 - b_0 a_1) \end{bmatrix}$$
$$C = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \quad , \quad D = b_0$$

Let $L = \begin{bmatrix} l_n \\ \vdots \\ l_1 \end{bmatrix}$, then $(G - LC)$ takes the form

$$(G - LC) = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_n \\ 1 & 0 & \cdots & 0 & -a_{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -a_2 \\ 0 & 0 & \cdots & 1 & -a_1 \end{bmatrix} - \begin{bmatrix} l_n \\ \vdots \\ l_1 \end{bmatrix} \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 & \cdots & 0 & -(a_n + l_n) \\ 1 & 0 & \cdots & 0 & -(a_{n-1} + l_{n-1}) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -(a_2 + l_n) \\ 0 & 0 & \cdots & 1 & -(a_1 + l_n) \end{bmatrix}$$

The characterstic equation of $G - LC$ is simply given as

$$p(z) = z^n + (a_1 - l_1)z^{n-1} + \cdots + (a_{n-1} - l_{n-1})z + (a_n - l_n)$$

Let's assume that desired $p^*(z)$ is equal to

$$p(z) = z^n + a_1^* z^{n-1} + \cdots + a_{n-1}^* z + a_n^*$$

Then, the observer gain L is computed as

$$L^* = \begin{bmatrix} a_n^* - a_n \\ \vdots \\ a_1^* - a_1 \end{bmatrix}$$

If the system is not in Observable canonical form, we can find a transformation that outputs the Observable canonical form representation The Observability matrix of a state-space representation is given as

$$O = \begin{bmatrix} C \\ CG \\ \vdots \\ CG^{-1} \end{bmatrix}$$

Let's define a transformation matrix Q as follows:

$$\begin{aligned} Q &= (WO)^{-1} \quad , \quad x[k] = Q\bar{x}[k] \\ \bar{x}[k+1] &= \left[Q^{-1}GQ\right]\bar{x}[k] + Q^{-1}Hu[k] \\ y[k] &= CQ\bar{x}[k] + Du[k] \end{aligned}$$

where

$$W = \begin{bmatrix} a_{n-1} & a_{n-2} & \cdots & a_1 & 1 \\ a_{n-2} & a_{n-3} & \cdots & 1 & 0 \\ \vdots & \vdots & \iddots & & \vdots \\ a_1 & 1 & & & \\ 1 & 0 & \cdots & & 0 \end{bmatrix}$$

Then it is given that

$$\begin{aligned} \bar{G} = Q^{-1}GQ &= \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_n \\ 1 & 0 & \cdots & 0 & -a_{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -a_2 \\ 0 & 0 & \cdots & 1 & -a_1 \end{bmatrix} \\ \bar{C} = CQ &= \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \end{aligned}$$

Let's compute $\bar{C}Q^{-1}$

$$\begin{aligned} \bar{C}Q^{-1} &= \bar{C}WO \\ &= \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{n-1} & a_{n-2} & \cdots & a_1 & 1 \\ a_{n-2} & a_{n-3} & \cdots & 1 & 0 \\ \vdots & \vdots & \iddots & & \vdots \\ a_1 & 1 & & & \\ 1 & 0 & \cdots & & 0 \end{bmatrix} O^T \\ &= \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} C & CG & \cdots & CG^{-1} \end{bmatrix} \\ &= C \end{aligned}$$

A similar approach (but longer) can be used to show that $Q^{-1}GQ = \bar{G}$.

We know how to design a observer gain $\bar{L}$ for the Observable canonical form. Given $\bar{L}$, Observer gain w.r.t. original state-space representation is computed as

$$L = Q\bar{L}$$

Example 2: Consider the following DT system

$$\begin{aligned} x[k+1] &= \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} x[k] + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u[k] \\ y[k] &= \begin{bmatrix} 1 & -1 \end{bmatrix} u[k] \end{aligned}$$

Design an observer using the Observable canonical form such that estimater poles are located at $\lambda_{1,2} = 0$ (Dead-beat Observer)

Solution: Characteristic equation of G can be derived as

$$\det\left(\begin{bmatrix} z-1 & 0 \\ 0 & z-2 \end{bmatrix}\right) = z^2 - 3z + 2$$

Observability matrix can be computed as

$$O = \begin{bmatrix} 1 & -1 \\ 1 & -2 \end{bmatrix}$$

The matrix W can be computed as

$$W = \begin{bmatrix} -3 & 1 \\ 1 & 0 \end{bmatrix}$$

Transformation matrix Q can be computed as

$$\begin{aligned} Q &= \left(WO^T\right)^{-1} \\ &= \left(\begin{bmatrix} -3 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & -2 \end{bmatrix}\right)^{-1} = \left(\begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix}\right)^{-1} = \begin{bmatrix} -1 & -1 \\ -1 & -2 \end{bmatrix} \end{aligned}$$

Given that desired characteristic polynomial is $p^*(z) = z^2$, $\bar{L}$ of observable canonical from can be computed as

$$\begin{aligned} \bar{L} &= \begin{bmatrix} -a_2 \\ -a_1 \end{bmatrix} \\ &= \begin{bmatrix} -2 \\ 3 \end{bmatrix} \end{aligned}$$

Finally Observer Gain L can be computed as

$$L = Q\bar{L} = \begin{bmatrix} -1 \\ -4 \end{bmatrix}$$

Not surprisingly the result is same with the one found in first example.

Closed-Loop Observer & State-Feedback

In the state-feedback control policy the input is ideally defined by the following law

$$u[k] = -Kx[k]$$

However, as mentioned in Observer lecture, in general we don't have direct access to the all states of the system. In this case, we learnt how to design an Observer/Estimator of the states. In this respect, it is natural to assume that in a closed-loop system, the control policy that define the input should depend on the estimated states

$$u[k] = -K\hat{x}[k]$$

However the important question how this coupling affect the closed-loop behavior, and even deeper question can be even use such a policy. The advantage of LTI systems is that state-feedback gain, and observer gain can be seperatelly designed and we guarntee a stable closed-loop performance. In this section, we will analyze the coupled system Equations of motion for the closed-loop observer & state-feedback based control system is given below

$$\begin{aligned}
x[k+1] &= Gx[k] + Hu[k] \\
y[k] &= Cx[k] + Du[k] \\
\hat{x}[k+1] &= G\hat{x}[k] + Hu[k] + L\left(y[k] - \hat{y}[k]\right) \\
\hat{y}[k] &= C\hat{x}[k] + Du[k] \\
u[k] &= -K\hat{x}[k]
\end{aligned}$$

If we eliminate $u[k]$ and $\hat{y}[k]$ we obtain following dynamical representation

$$\begin{aligned}
x[k+1] &= Gx[k] - HK\hat{x}[k] \\
\hat{x}[k+1] &= G\hat{x}[k] - HK\hat{x}[k] + LC\left(x[k] - \hat{x}[k]\right) \\
y[k] &= Cx[k] - DK\hat{x}[k]
\end{aligned}$$

Now let's replace $\hat{x}[k]$ with $e[k] = x[k] - \hat{x}[k]$

$$\begin{aligned}
x[k+1] &= (G - HK)x[k] + HKe[k] \\
e[k+1] &= (G - LC)e[k] \\
y[k] &= (C - DK)x[k] + DKe[k]
\end{aligned}$$

Now let's defina a state for the whole system, $z[k] = \begin{bmatrix} x[k] \\ e[k] \end{bmatrix}$ then the state-space representation is given by

$$\begin{aligned}
z[k+1] &= \begin{bmatrix} (G - HK) & HK \\ 0_{n\times n} & (G - LC) \end{bmatrix} z[k] \\
y[k] &= \begin{bmatrix} (C - DK) & DK \end{bmatrix} z[k]
\end{aligned}$$

The system matrix is in block diagonal form and the eigenvalues of this new system matrix is find by taking the union of eigenvalues of $(G - HK)$ and eigenvalues of $(G - LC)$. Thus a seperate pole-placement can be performed for the state-feedback controller and the observer.