

## Lecture 9

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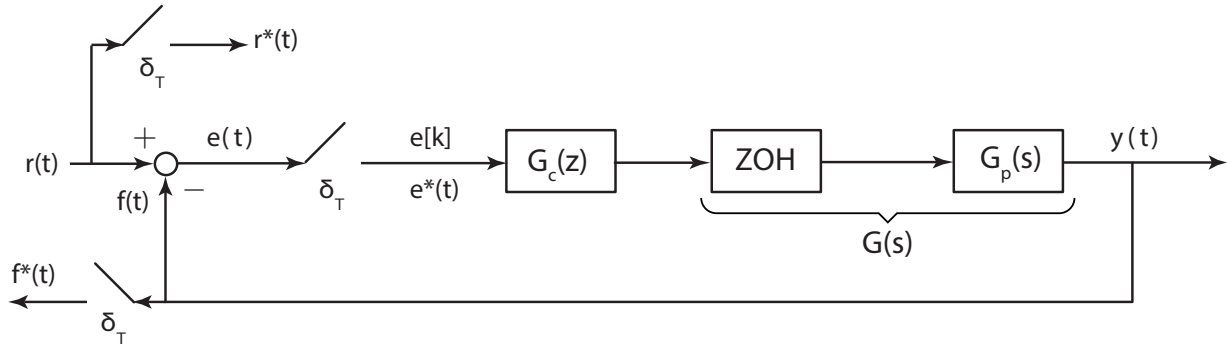
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## Steady-State (DC) Response Analysis

Let's remember the final value theorem. Given a discrete time signal  $x[k]$  and its z-transform  $X(z)$ , if  $x[k]$  is convergent sequence final value theorem states that

$$\lim_{k \rightarrow \infty} x[k] = \lim_{z \rightarrow 1} [(1 - z^{-1}) X(z)]$$

$$x_{ss} = \lim_{z \rightarrow 1} \left[ \frac{z-1}{z} X(z) \right]$$



Now let's find the pulse transfer function from the reference signal  $r[k]$  to the error signal  $e[k]$ , to further analyze the steady-state error response.

$$E(z) = R(z) - E(z) (G_c(z)G(z)), \quad \text{where } G(z) = \mathcal{Z}\{G(s)\}$$

$$\frac{E(z)}{R(z)} = \frac{1}{1 + G_c(z)G(z)}$$

Note that  $G_c(z)G(z)$  is the pulse transfer function from the error signal  $E(z)$  to the signal which is fed to the negative terminal of the main difference operator, i.e.  $F(z)$ . This transfer function is called feed-forward or open-loop pulse transfer function of the closed-loop digital control system. For this system,

$$\frac{F(z)}{E(z)} = G_{OL} = G_c(z)G(z)$$

Then  $E(z)$  can be written as

$$E(z) = R(z) \frac{1}{1 + G_{OL}(z)}$$

It is obvious that first requirement on steady-state error performance is that closed-loop system have to be stable. Now let's analyze specific but fundamental input scenarios.

## Unit-Step Input

We know that  $r[k] = u[k]$  and  $R(z) = \frac{1}{1-z^{-1}}$  then we have

$$\begin{aligned} e_{ss} &= \lim_{z \rightarrow 1} \left[ (1 - z^{-1}) R(z) \frac{1}{1 + G_{OL}(z)} \right] \\ &= \lim_{z \rightarrow 1} \left[ (1 - z^{-1}) \frac{1}{1 - z^{-1}} \frac{1}{1 + G_{OL}(z)} \right] \\ e_{ss} &= \frac{1}{1 + \lim_{z \rightarrow 1} G_{OL}(z)} \end{aligned}$$

If the DC gain of the system (also called static error constant) is constant, i.e.  $G_{OL}(1) = K_{DC}$  then the steady state error can be computed as

$$e_{ss} = \frac{1}{1 + K_{DC}}$$

It is obvious that

$$\begin{aligned} e_{ss} &\neq 0 \quad \text{if} \quad |K_{DC}| < \infty \\ e_{ss} &\rightarrow 0 \quad \text{if} \quad K_{DC} \rightarrow \infty \end{aligned}$$

Based on these results, we can have the following conclusions

- If  $G_{OL}(1) = 0$ , then  $e_{ss} = 1$ . These are **type negative** systems, and the steady-state error of step response type signals are always 100%.
- If  $G_{OL}(1) = K_{DC}$ ,  $0 < |K_{DC}| < \infty$ , then  $e_{ss} = 1/(1 + K_{DC})$ . These are **type 0** systems. We observe a bounded steady-state error and it is possible to reduce it by increasing the static gain constant  $K_P$ .
- If  $G_{OL}(1) = \infty$ , then  $e_{ss} = 0$ . These are **type positive** systems. The steady-state error is perfectly zero for such systems.

Now let's generalize the *type* of systems. An  $N$  *type* closed loop system has the following form of open-loop pulse transfer function

$$\begin{aligned} G_{OL}(z) &= \frac{1}{(z - 1)^N} G_{DC}(z) \\ |G_{DC}(1)| &= K_{DC} \quad \text{where } 0 < |K_{DC}| < \infty \end{aligned}$$

It is easy to see that for unit-step response

- Type  $N < 0$ :  $e_{ss} = 1$  (or  $e_{ss} = 100\%$ )
- Type  $N = 0$ :  $e_{ss} = 1/(1 + K_{DC})$
- Type  $N > 0$ :  $e_{ss} = 0$

## Unit-Ramp Input

We know that  $r[k] = ku[k]$  and  $R(z) = \frac{z^{-1}}{(1-z^{-1})^2}$  then we have

$$\begin{aligned}
e_{ss} &= \lim_{z \rightarrow 1} \left[ (1 - z^{-1}) R(z) \frac{1}{1 + G_{OL}(z)} \right] \\
&= \lim_{z \rightarrow 1} \left[ (1 - z^{-1}) \frac{z^{-1}}{(1 - z^{-1})^2} \frac{1}{1 + \frac{1}{(z-1)^N} G_{DC}(z)} \right] \\
&= \lim_{z \rightarrow 1} \left[ \frac{1}{z-1} \frac{1}{1 + \frac{1}{(z-1)^N} G_{DC}(z)} \right] \\
&= \lim_{z \rightarrow 1} \left[ \frac{1}{(z-1) + \frac{1}{(z-1)^{N-1}} G_{DC}(z)} \right] \\
e_{ss} &= \frac{1}{\lim_{z \rightarrow 1} \left[ \frac{1}{(z-1)^{N-1}} G_{DC}(z) \right]}
\end{aligned}$$

Based on this result we can have the following steady-state error conditions for the unit-ramp input based on the type condition of the system

- Type  $N < 1$ :  $e_{ss} \rightarrow \infty$
- Type  $N = 1$ :  $e_{ss} = \frac{1}{K_{DC}}$
- Type  $N > 1$ :  $e_{ss} = 0$

### Unit-Quadratic (Acceleration) Input

We know that  $r[k] = \frac{1}{2}k^2u[k]$  and  $R(z) = \frac{z^{-1}(1+z^{-1})}{2(1-z^{-1})^3}$  then we have

$$\begin{aligned}
e_{ss} &= \lim_{z \rightarrow 1} \left[ (1 - z^{-1}) R(z) \frac{1}{1 + G_{OL}(z)} \right] \\
&= \lim_{z \rightarrow 1} \left[ (1 - z^{-1}) \frac{z^{-1}(1 + z^{-1})}{2(1 - z^{-1})^3} \frac{1}{1 + \frac{1}{(z-1)^N} G_{DC}(z)} \right] \\
&= \lim_{z \rightarrow 1} \left[ \frac{(z+1)}{2(z-1)^2} \frac{1}{1 + \frac{1}{(z-1)^N} G_{DC}(z)} \right] \\
&= \lim_{z \rightarrow 1} \left[ \frac{(z+1)/2}{(z-1)^2 + \frac{1}{(z-1)^{N-2}} G_{DC}(z)} \right] \\
e_{ss} &= \frac{1}{\lim_{z \rightarrow 1} \left[ \frac{1}{(z-1)^{N-2}} G_{DC}(z) \right]}
\end{aligned}$$

- Type  $N < 2$ :  $e_{ss} \rightarrow \infty$
- Type  $N = 2$ :  $e_{ss} = \frac{1}{K_{DC}}$
- Type  $N > 2$ :  $e_{ss} = 0$

**Example 1:**  $G(z) = \frac{z-1}{z-0.5}$  and  $G_C(z) = K$ . Compute the steady-state error to unit-step, unit-ramp, a and unit-quadratic inputs.

**Solution:**

$$G_{OL}(z) = K \frac{z-1}{z-0.5} = \frac{1}{(z-1)^{-1}} \frac{K}{z-0.5}$$

$$G_{DC}(1) = 2K, \text{ Type } -1$$

Then the steady-state errors are computed as

- Unit-step:  $e_{ss} = 1$
- Unit-ramp:  $e_{ss} = \infty$
- Unit-acceleration:  $e_{ss} = \infty$

**Example 2:**  $G(z) = \frac{z-1}{z-0.5}$  and  $G_C(z) = K \frac{z}{z-1}$ . Compute the steady-state error to unit-step, unit-ramp, a and unit-quadratic inputs.

**Solution:**

$$G_{OL}(z) = \frac{Kz}{z-0.5}$$

$$G_{DC}(1) = 2K, \text{ Type } 0$$

Then the steady-state errors are computed as

- Unit-step:  $e_{ss} = \frac{1}{1+2K}$
- Unit-ramp:  $e_{ss} = \infty$
- Unit-acceleration:  $e_{ss} = \infty$

**Example 3:**  $G(z) = \frac{z-1}{z-0.5}$  and  $G_C(z) = K \frac{z^2}{(z-1)^2}$ . Compute the steady-state error to unit-step, unit-ramp, a and unit-quadratic inputs.

**Solution:**

$$G_{OL}(z) = \frac{Kz^2}{(z-1)(z-0.5)} = \frac{1}{z-1} \frac{Kz^2}{z-0.5}$$

$$G_{DC}(1) = 2K, \text{ Type } 1$$

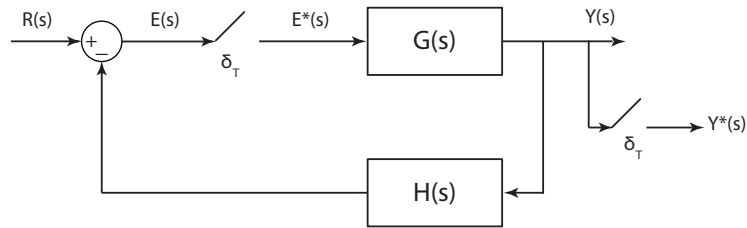
Then the steady-state errors are computed as

- Unit-step:  $e_{ss} = 0$
- Unit-ramp:  $e_{ss} = \frac{1}{2K}$
- Unit-acceleration:  $e_{ss} = \infty$

## Open-Loop Transfer Function for Different Topologies

When computing the steady-state error it is important to carefully analyze the topology of the control system.

Compute the  $G_{OL}(z)$  for the following system

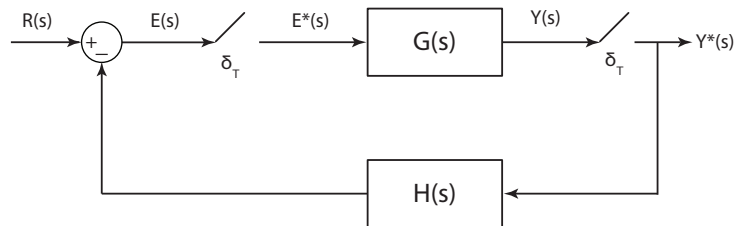


$$F(s) = E^*(s)G(s)H(s)$$

$$F^*(s) = E^*(s)[G(s)H(s)]^* = E^*(s)GH^*(s)$$

$$G_{OL}(z) = GH(z)$$

Now let's compute the  $G_{OL}(z)$  for the following system

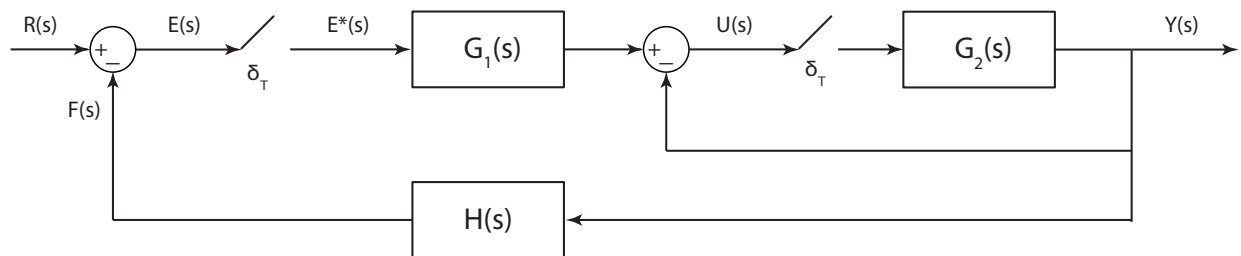


$$F(s) = [E^*(s)G(s)] * H(s) = E^*(s)G^*(s)H(s)$$

$$F^*(s) = E^*(s)G^*(s)H^*(s)$$

$$G_{OL}(z) = G(z)H(z)$$

Now let's compute the  $G_{OL}(z)$  for the following system



From last week we know that

$$U^*(s) = \frac{G_1^*(s)}{1 + G_2^*(s) + G_1^*(s)GH^*(s)} R^*(s)$$

Then we can

$$U(s) = E^*(s)G_1(s) - U^*(s)G_2(s) \rightarrow U^*(s) = E^*(s)G_1^*(s) - U^*(s)G_2^*(s)$$

$$U^*(s) = \frac{G_1^*(s)}{1 + G_2^*(s)} E^*(s)$$

$$E^*(s) = \frac{1 + G_2^*(s)}{1 + G_2^*(s) + G_1^*(s)GH^*(s)} R^*(s)$$

$$\frac{E(z)}{R(z)} = \frac{1 + G_2(z)}{1 + G_2(z) + G_1(z)GH(z)}$$

This transfer function form does not (directly) fit to the form we analyzed, i.e.  $\frac{E(z)}{R(z)} = \frac{1}{1 + G_{OL}(z)}$ , so we can not directly use the conditions and formulae for this form. One way of computing the steady-state errors is directly applying the final-value theorem.

The other way is we can simply convert the computed pulse transfer function  $E(z)/R(z)$  such that it fits the form  $\frac{E(z)}{R(z)} = \frac{1}{1 + G_{OL}(z)}$ . If we carefully analyze the transfer function we can obtain

$$\begin{aligned} \frac{E(z)}{R(z)} &= \frac{1}{1 + \frac{G_1(z)G_2H(z)}{1 + G_2(z)}} \\ G_{OL}(z) &= \frac{G_1(z)G_2H(z)}{1 + G_2(z)} \end{aligned}$$

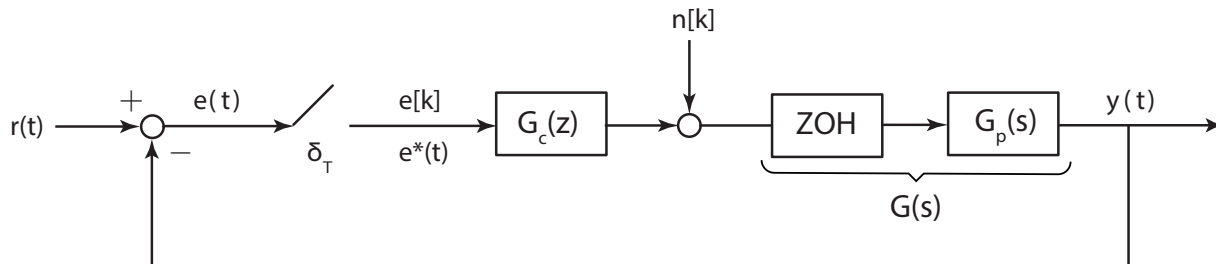
It is also possible to derive  $G_{OL}(z)$  via direct computation of  $F(z)/E(z)$ .

## Response to Disturbances

When analyzing response of a system in addition to the desired response to the reference input, it is also important to analyze the response (both steady-state, transient, and frequency) to unwanted disturbances and noises.

### Process Disturbance/Uncertainty/Noise

Let's analyze a type of important disturbance on the fundamental discrete-time block diagram topology.



In order to analyze the response to the disturbance  $n[k]$ , we assume  $r[k] = 0$  (which is just fine due to the linearity). Let's first find the pulse transfer function from  $N(z)$  to  $Y(z)$ .

$$Y(z) = (-Y(z)G_c(z) + N(z))G(z)$$

$$\frac{Y(z)}{N(z)} = \frac{G(z)}{1 + G_c(z)G(z)}$$

Technically, we want  $\frac{Y(z)}{N(z)} = 0$ , while also tracking the reference signal. Since it is not perfectly possible to achieve  $\frac{Y(z)}{N(z)} = 0$  while satisfying other constraints, we want  $\frac{Y(z)}{N(z)}$  to be "small". If  $|G_C(z)G(z)| \gg 1$  then we have

$$\frac{Y(z)}{N(z)} \approx \frac{1}{G_c(z)}$$

Now let's consider a specific type of disturbance. An important class of process disturbance/uncertainty is in the form of DC bias, i.e.  $n(t) = Nu(t)$  and  $N(z) = \frac{N}{1-z^{-1}}$ . Let's analyze DC steady state response using final value theorem.

$$\begin{aligned} y_{ss} &= \lim_{z \rightarrow 1} [(1 - z^{-1}) Y(z)] = \lim_{z \rightarrow 1} \left[ (1 - z^{-1}) N(z) \frac{G(z)}{1 + G_c(z)G(z)} \right] \\ &= \lim_{z \rightarrow 1} \left[ (1 - z^{-1}) \frac{N}{1 - z^{-1}} \frac{G(z)}{1 + G_c(z)G(z)} \right] = \lim_{z \rightarrow 1} \left[ N \frac{G(z)}{1 + G_c(z)G(z)} \right] \\ &= N \frac{\lim_{z \rightarrow 1} G(z)}{1 + \lim_{z \rightarrow 1} G_c(z)G(z)} \end{aligned}$$

Let's analyze the steady-state disturbance response

- If plant is a type  $< 0$  system (high pass filter plant) then  $G(1) = 0$  and  $y_{ss} = 0$ .
- If plant is a type 0 system, then

$$y_{ss} = \frac{NG(1)}{1 + G(1)\lim_{z \rightarrow 1} G_c(z)}$$

Now let's analyze the response based on the type of  $G_c(z)$

- Type  $< 0$ , then

$$y_{ss} = NG(1)$$

In this case, controller has no control on the steady-state disturbance rejection performance.

- Type 0, then

$$y_{ss} = \frac{NG(1)}{1 + G(1)G_C(1)}$$

Obviously in order to “filter” the disturbance we should select a  $G_C(z)$  such that  $|G_C(1)G(1)| \gg 1$  then

$$y_{ss} = \frac{N}{G_C(1)}$$

Large gain  $G_C(z)$  can effectively filter the disturbance (but not completely).

- Type  $> 0$ , then

$$\begin{aligned} y_{ss} &= \frac{NG(1)}{1 + G(1) \lim_{z \rightarrow 1} G_C(z)} \\ &= 0 \end{aligned}$$

Integral action on  $G_C(z)$  can perfectly reject the DC disturbance on steady state.

- If plant is a type  $m > 0$  then

$$\begin{aligned} y_{ss} &= \frac{N \lim_{z \rightarrow 1} G(z)}{1 + \lim_{z \rightarrow 1} G(z)G_C(z)} \\ \lim_{z \rightarrow 1} G(z) &= \infty \end{aligned}$$

Depending on the type of  $G_C(z)$ , we can conclude that

- Type  $< 0$ ,  $y_{ss} = \infty$
- Type 0,  $y_{ss} = C$ , where  $0 < C < \infty$
- Type  $> 0$ ,  $y_{ss} = 0$

**Example 4:**  $G(z) = \frac{z-1}{z-0.5}$  and  $G_C(z) = K$ . Compute the steady-state response to a unit step process disturbance/noise.

**Solution:**

$$y_{ss} = \frac{\lim_{z \rightarrow 1} \frac{z-1}{z-0.5}}{1 + \lim_{z \rightarrow 1} K \frac{z-1}{z-0.5}} = 0$$

Plant perfectly rejects disturbance.

**Example 5:**  $G(z) = \frac{z}{z-0.5}$  and  $G_C(z) = K$ . Compute the steady-state response to a unit step process disturbance/noise.

**Solution:**

$$y_{ss} = \frac{\lim_{z \rightarrow 1} \frac{z}{z-0.5}}{1 + \lim_{z \rightarrow 1} K \frac{z}{z-0.5}} = \frac{2}{1 + 2K}$$

Large gain  $K$  can be effective solution to reject disturbance

**Example 6:**  $G(z) = \frac{z}{z-0.5}$  and  $G_C(z) = K_P + K_I \frac{z}{z-1}$ . Compute the steady-state response to a unit step process disturbance/noise.



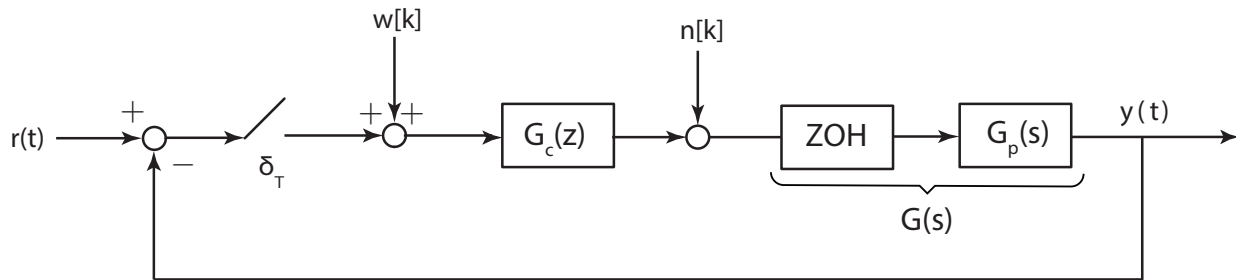
**Solution:**

$$y_{ss} = \frac{2}{1 + \lim_{z \rightarrow 1} 2 \left( K_P + K_I \frac{z}{z-1} \right)} = 0$$

A PI controller can perfectly reject the DC process disturbance.

## Measurement Disturbance/Uncertainty/Noise

Let's analyze a different type of important disturbance on the fundamental discrete-time block diagram topology.



In order to analyze the response to the disturbance  $w[k]$ , we assume  $r[k] = 0$  and  $n[k] = 0$

$$\begin{aligned} Y(z) &= (W(z) - Y(z))G_c(z)G(z) \\ \frac{Y(z)}{W(z)} &= \frac{G(z)G_c(z)}{1 + G_c(z)G(z)} \\ &= \frac{G_{OL}(z)}{1 + G_{OL}(z)} \end{aligned}$$

Technically, we want  $\frac{Y(z)}{N(z)} = 0$ , while also tracking the reference signal. Thus, practically we should design  $G_C(z)$  such that  $|G_C(z)G(z)| \ll 1$ , to eliminate measurement noises/disturbances. (?????)

$$\frac{Y(z)}{N(z)} \approx G_{OL}(z)$$

This “requirement” obviously contradicts with requirements on steady-state tracking error performance and process noise/disturbance rejection performance. Most well known limitation of feedback control systems.

Now let's consider a specific type of measurement noise, i.e. DC measurement bias.  $w(t) = Wu(t)$  and  $W(z) = \frac{W}{1-z^{-1}}$ . Let's analyze DC steady state response using final value theorem.

$$\begin{aligned} y_{ss} &= \lim_{z \rightarrow 1} \left[ (1 - z^{-1})R(z) \frac{G_{OL}(z)}{1 + G_{OL}(z)} \right] = \lim_{z \rightarrow 1} \left[ (1 - z^{-1}) \frac{W}{1 - z^{-1}} \frac{G_{OL}(z)}{1 + G_{OL}(z)} \right] \\ &= \lim_{z \rightarrow 1} \left[ \frac{WG_{OL}(z)}{1 + G_{OL}(z)} \right] \end{aligned}$$

Using the form  $G_{OL}(z) = \frac{1}{(z-1)^N} G_{DC}(z)$ ,  $y_{ss}$  takes the form

$$y_{ss} = \lim_{z \rightarrow 1} \left[ \frac{W G_{DC}(1) \frac{1}{(z-1)^N}}{1 + G_{DC}(1) \frac{1}{(z-1)^N}} \right]$$

Based on the type of the open-loop transfer function,  $G_{OL}(z)$ , we can conclude

- Type  $N < 0$ :  $y_{ss} = 0$ . Perfect rejection of measurement bias, but we know that this is unacceptable from reference tracking point of view.
- Type  $N = 0$

$$y_{ss} = \frac{W G_{DC}(1)}{1 + G_{DC}(1)}$$

It seems that in order to “filter” the measurement bias  $G_{DC}(1)$  should be selected very small.

- Type  $N > 0$

$$y_{ss} = W$$

The disturbance is directly transferred to the output.

**Example 7:**  $G(z) = \frac{z-1}{z-0.5}$  and  $G_C(z) = K$ . Compute the steady-state response to a unit step measurement disturbance/noise.

**Solution:**

$$G_{OL}(z) = K \frac{z-1}{z-0.5} \quad \text{Type } -1$$

$$y_{ss} = 0$$

Plant perfectly rejects measurement bias.

**Example 8:**  $G(z) = \frac{z}{z-0.5}$  and  $G_C(z) = K$ . Compute the steady-state response to a unit step process disturbance/noise.

**Solution:**

$$G_{OL}(z) = K \frac{z}{z-0.5} \quad \text{Type } 0$$

$$y_{ss} = \frac{2K}{1+2K}$$

Small gain  $K$  can be effective solution to reject measurement bias.