EE402 - Discrete Time Systems **Spring 2018**

Lecture 7

Lecturer: Asst. Prof. M. Mert Ankarali

Mapping Between s & z Planes

When the (uniform) impulse sampling is involved in the process, then we know that s and z variables are related with

$$z = e^{Ts}$$

which is a mapping from complex plane to complex plane, i.e. $M : \mathbb{C} \mapsto \mathbb{C}$, where $M(s) = e^{Ts}$. We will analyze different cases of this mapping and their relevance and importance.

Morever, if s_p is a pole of $G_p(s)$, then it is straightforward to show that $z_p = e^{Ts_p}$ is a pole of $G(z) = \mathcal{Z}\{G_p(s)\}$ (as well as $G(z) = (1 - z^{-1})\mathcal{Z}\{G(s)/s\}$).

Since poles of an LTI system (CT or DT) are the major features that defines the stability and some other performance metrics, analyzing this mapping is very critical for analyzing discrete time control systems.

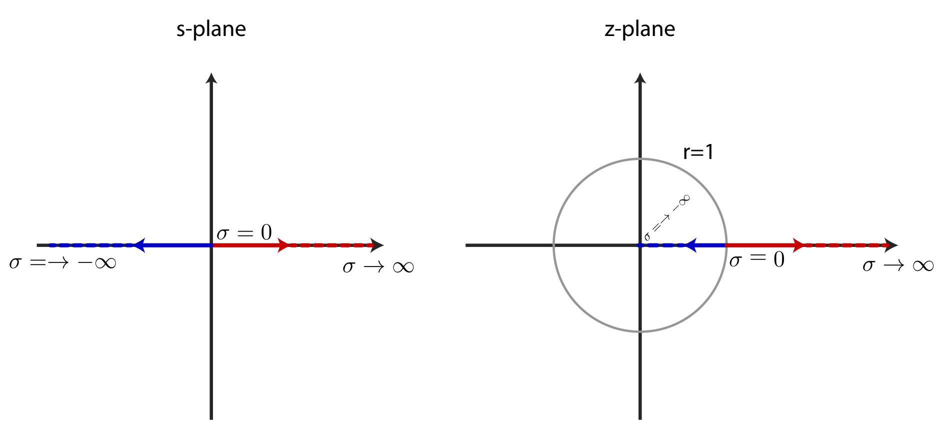
Mapping of real line (left half and right half): When s is purely real (i.e. when the roots of the CT plant are real) we have

$$s = \sigma \quad , \ \sigma \in \mathbb{R}$$
$$z = e^{Ts} = e^{T\sigma} \quad , \ z \in \mathbb{R}^+$$

It is also easy to see the difference between left half and right half of the real line

$$\text{If } \sigma \leq = 0 \quad \rightarrow \quad z = e^{\sigma T} \in [0, 1]$$
$$\text{If } \sigma \geq = 0 \quad \rightarrow \quad z = e^{\sigma T} \in [1, \infty)$$

Mapping of both left and right real lines to z-plane is illustrated in the Figure below. It can be seen that when $\sigma > 0$, $z > 1$, and similarly when $\sigma < 0$, $z < 1$. Technically on both planes red curves belong to "unstable" behaviors, where as blue curves belong to "stable" behaviors.



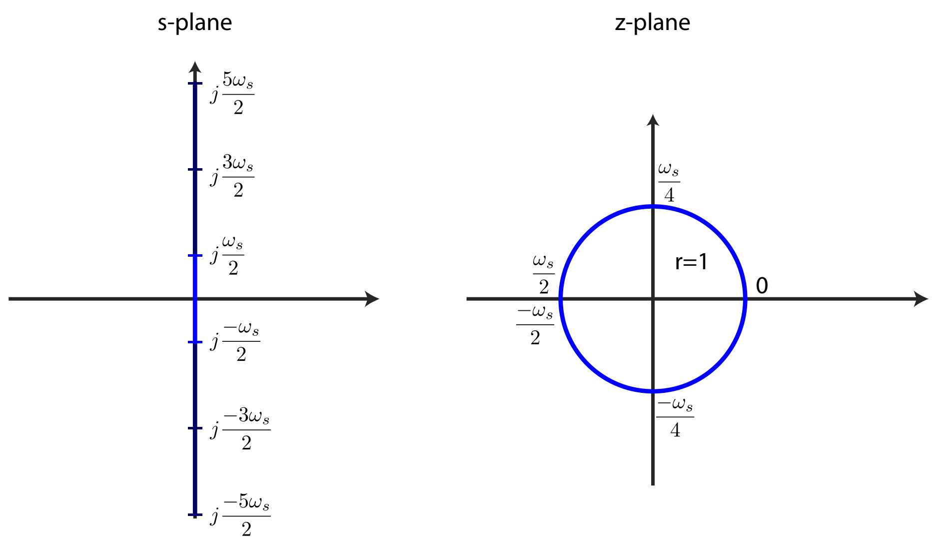
Mapping of imaginary axis: When s is purely imaginary (i.e. when the roots of the CT plant are critically stable) we have

$$\begin{aligned} s &= j\omega \quad , \ \omega \in \mathbb{R} \\ z &= e^{Ts} = e^{T\omega j} \\ |z| &= 1 \\ \angle z &= T\omega = T\omega + 2\pi k \quad , \ k \in \mathbb{Z} \end{aligned}$$

This means that mapping of the imaginary axis is not 1-1, since multiple points on s plane can correspond to a single point on the z plane

$$e^{T\omega j} = e^{T\omega j + 2\pi k} \quad \rightarrow \quad M(\omega j) = M((\omega + 2\pi/T)j) = M((\omega + \omega_s)j)$$

where $\omega_s = 2\pi T$ is the sampling frequency. It can be seen that imaginary axis on the s plane is mapped to the unit circle on the z plane. However as $\omega \to \infty$ or $\omega \to -\infty$, the mapping circles the unit circle multiple (indeed infinite) times. This mapping is illustrated in the Figure below. The light blue section on the s plane (which covers the points in the imaginary axis between $[-\omega_s/2, \omega_s/2]$) is called the primary section/strip and fully mapped to the unit cicle. Dark blue sections are called complementary sections/strips and they are also individually fully mapped onto the unit circle.



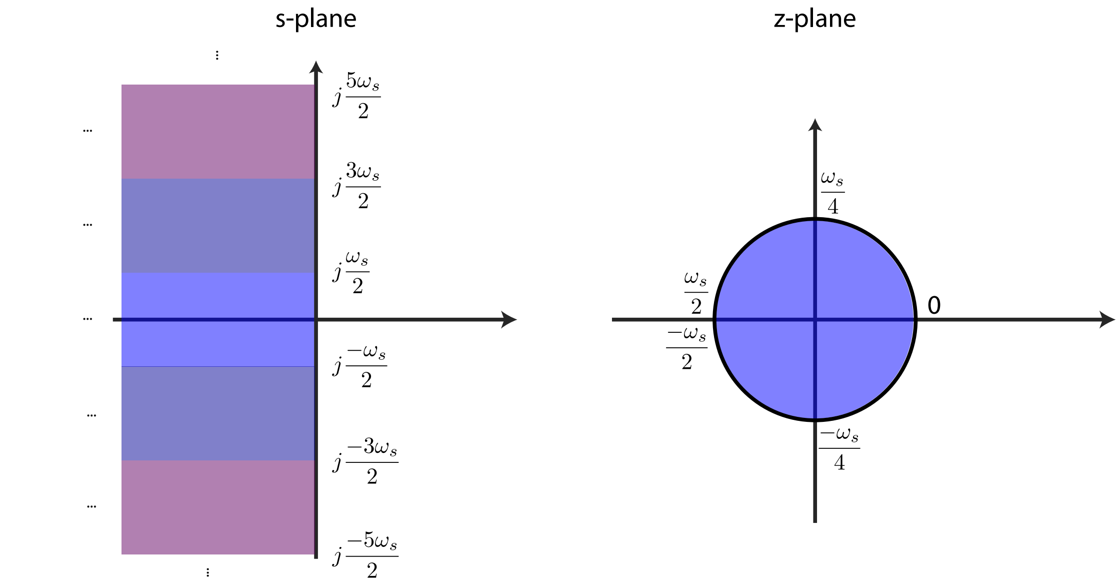
Mapping of open left-half plane: Now let's generalize a little, and consider the mapping of the whole open left-half plane.

$$\begin{aligned} s &= \sigma + j\omega \quad , \ \sigma < 0 \\ z &= e^{Ts} = e^{T\sigma} e^{T\omega j} = e^{T\sigma} e^{(T\omega + 2\pi k)j} \\ |z| &= e^{T\sigma} < 1 \\ \angle z &= T\omega + 2\pi k \quad , \ k \in \mathbb{Z} \end{aligned}$$

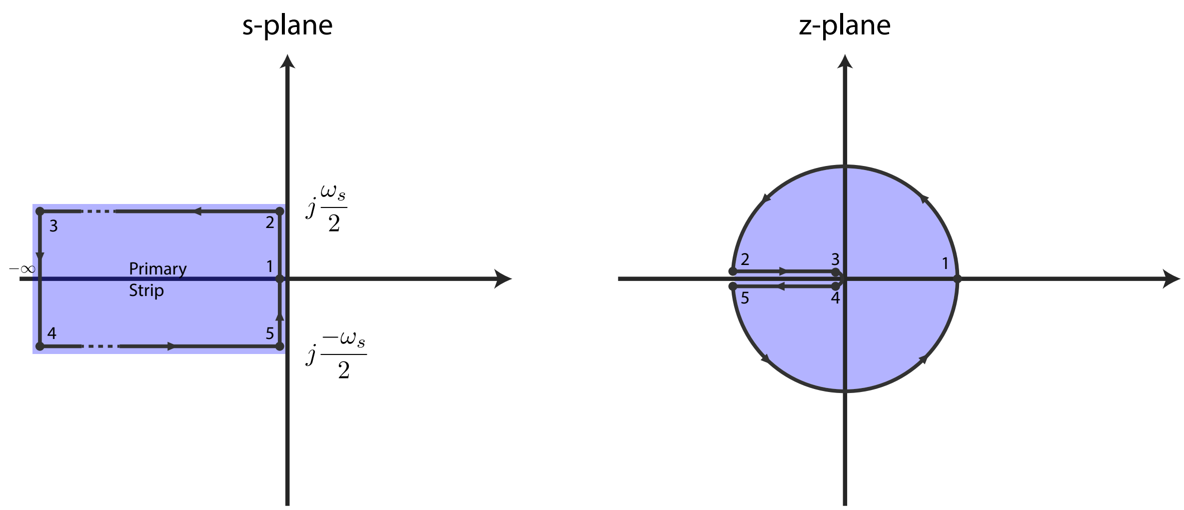
Obviously, this mapping is also not 1-1, and "periodic" in ω, i.e.

$$M(\sigma, \omega) = M(\sigma, (\omega + 2\pi T)j) = M(\sigma, (\omega + \omega_s)j)$$

Mapping of OLH on s-plane to z-plane is illustrated in Figure below.



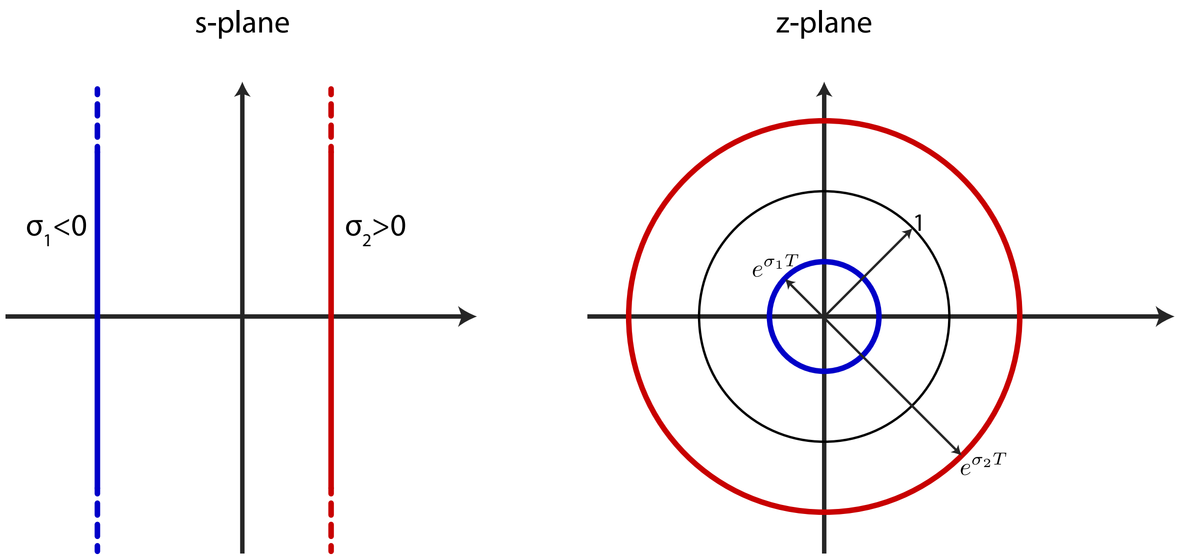
In the primary strip, if we trace the path that is defined by the sequence of points 1-2-3-4-5-1 in the s plane as shown in the Figure below, than this path is mapped in to the z-plane as shown in the Figure. The mapping forms a different path again associated with mapped point sequence 1-2-3-4-5-1.



Mapping of constant attenuation line: In the s-plane, it corresponds to the line for which σ is constant. Constant σ in s-plane corresponds to a constant radius in the z-plane. Thus line is mapped to a circle with a radius of $e^{\sigma T}$.

$$z = e^{\sigma T} e^{\omega T j} = e^{\sigma T} \angle \omega T$$
$$R = e^{\sigma T} = \text{Constant}$$

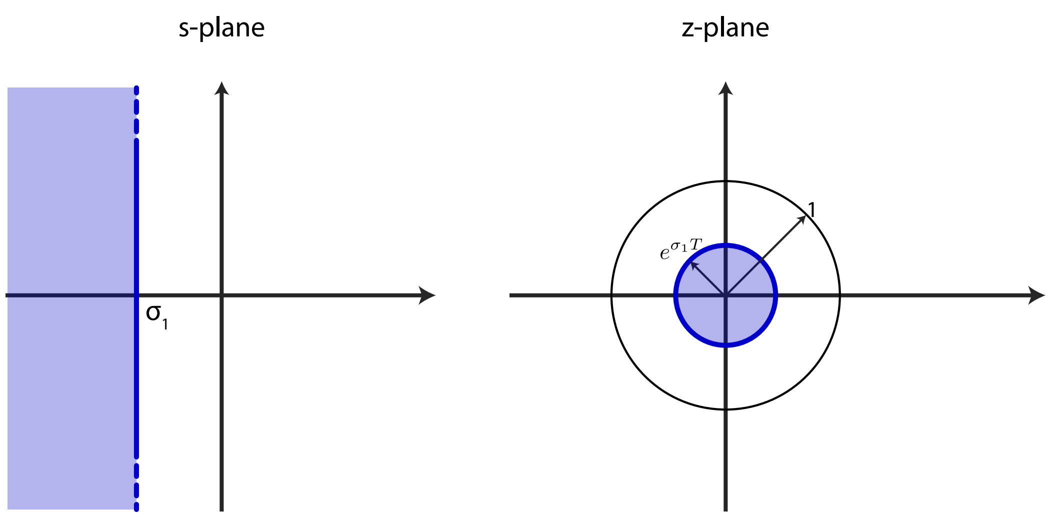
Figure below illustrates mapping of one constant attenuation line in open left half plane and one open right half plane.



Max convergence/settling time region: For a stable CT LTI system convergence/settling time is defined by the real part of the pole. Thus on the s-plane we have the following condition

$$\mathrm{Re}(s) < \sigma_1 < 0$$

In the z-plane it is mapped to region enclosed by the circle with radius $r = e^{\sigma_1 T}$. This mapping is illustrated below.

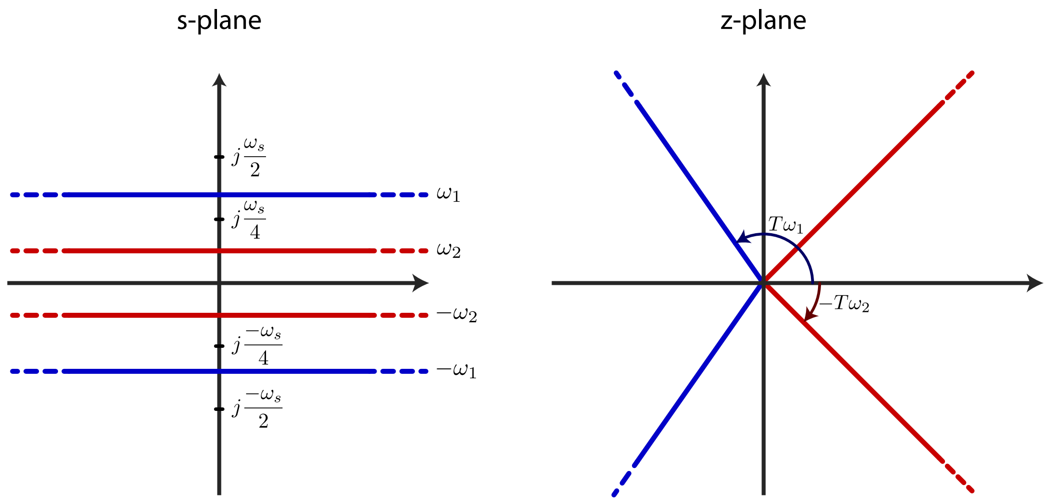


Constant frequency loci: A constant frequency locus $\omega = \omega_1$ in s-plane is mapped to a constant angle line in z-plane, for which the angle is equal to $\omega_1 T$.

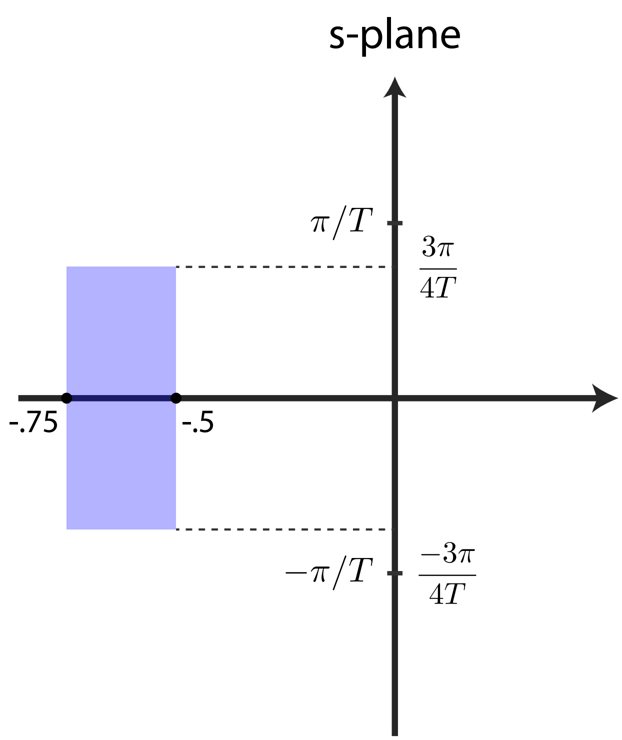
$$z = e^{\sigma T} e^{\omega T j} = e^{\sigma T} \angle \omega T$$

$$\angle \omega T = \text{Constant}$$

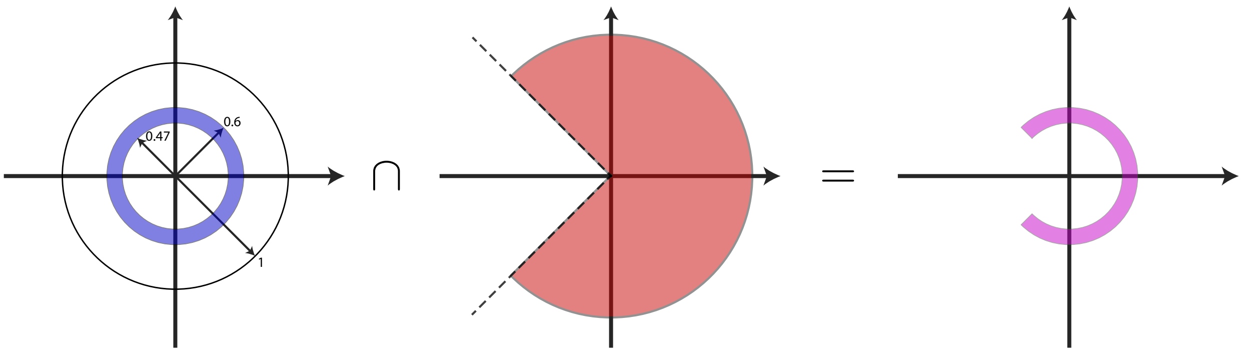
Different constant frequency lines (all inside the primary strip) and their mappings are illustrated in Figure below.



Example: Find the mapping of the area defined inside s-plane illustrated in Figure below with $T = 1$.



Solution: The solution is illustrated in the Figure below



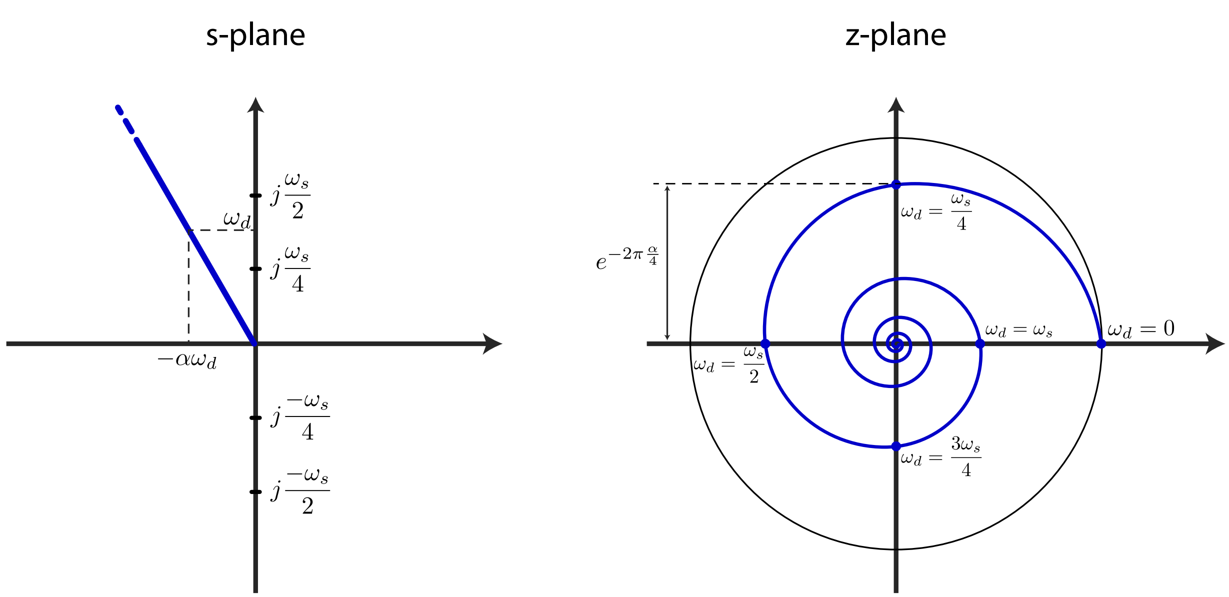
Constant damping loci: A constant damping loci is a line in s-plane passing through the origin and the angle between the line and the Real (or Imaginary) axis defines the damping ratio. In s-plane we have the following relations

$$\begin{aligned} s &= \sigma + \omega j = -\zeta\omega_n + \omega_n\sqrt{1-\zeta^2}j \\ &= -\alpha\omega_d + j\omega_d \\ \omega_d &= \omega_n\sqrt{1-\zeta^2} \\ \alpha &= \frac{\zeta}{\sqrt{1-\zeta^2}} = \text{Constant} \end{aligned}$$

The mapping of this line to the z-plane yields

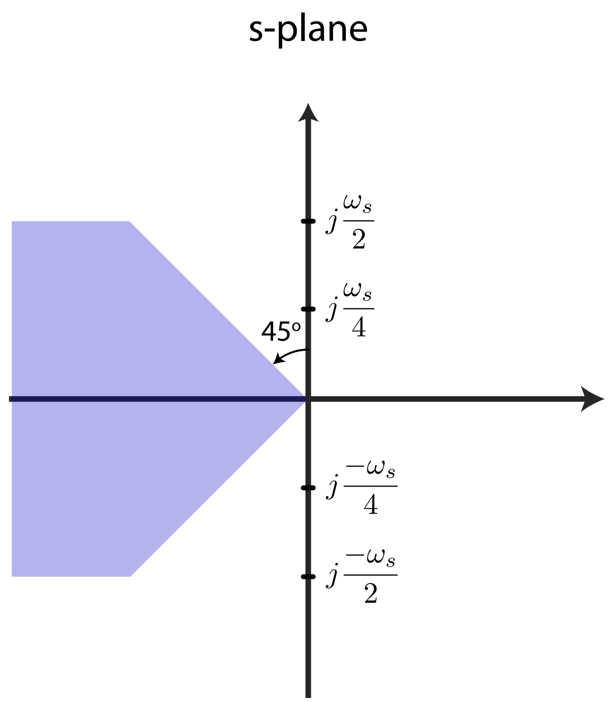
$$\begin{aligned} z &= e^{-\alpha\omega_d T + jT\omega_d} = e^{-\alpha\omega_d T} + e^{j\omega_d T} \\ z &= e^{-2\pi\alpha\frac{\omega_d}{\omega_s}} + e^{j2\pi\frac{\omega_d}{\omega_s}} \\ |z| &= e^{-2\pi\alpha\frac{\omega_d}{\omega_s}} \\ \angle z &= 2\pi\frac{\omega_d}{\omega_s} \end{aligned}$$

The curve in z-plane corresponds to a spiral shape as seen in Figure below.

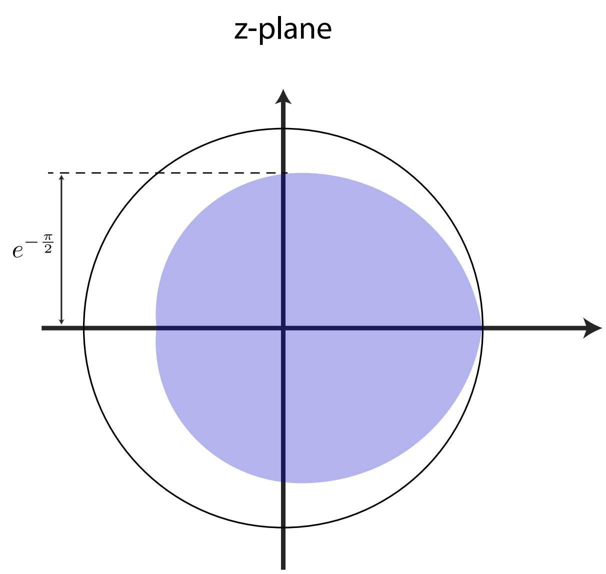


Note that for real systems we need to have also complex conjugates of both the line in s-plane and spiral in z-plane.

Example: Find the mapping of the area defined inside s-plane illustrated in Figure below.



Solution: The solution is illustrated in the Figure below



Bilinear Transformation

A very common transformation used for design and analysis of discrete time control systems, digitized filters etc. is the bilinear transformation. It is a 1-1 mapping between complex z-plane and another complex plane $\bar{s}$ which is an "imaginary" s-domain plane.

The bilinear transformation is defined by

$$z = \frac{1+\frac{T}{2}\bar{s}}{1-\frac{T}{2}\bar{s}}$$
$$\bar{s} = \frac{2}{T}\frac{z-1}{z+1}$$

Let's analyze $|z| < 1$

$$|z| = \left|\frac{1+\frac{T}{2}\bar{s}}{1-\frac{T}{2}\bar{s}}\right| = \frac{|1+\frac{T}{2}\bar{s}|}{|1-\frac{T}{2}\bar{s}|}$$
$$|z| < 1 \implies \left|1+\frac{T}{2}\bar{s}\right| < \left|1-\frac{T}{2}\bar{s}\right|$$

Let $\bar{s} = \bar{\sigma} + j\bar{\omega}$

$$\left|1+\frac{T}{2}(\bar{\sigma}+j\bar{\omega})\right| < \left|1-\frac{T}{2}(\bar{\sigma}+j\bar{\omega})\right|$$
$$\left(\frac{T}{2}\bar{\sigma}+1\right)^2 + \left(\frac{T}{2}\bar{\omega}\right)^2 < \left(\frac{T}{2}\bar{\sigma}-1\right)^2 + \left(\frac{T}{2}\bar{\omega}\right)^2$$
$$\left(\frac{T}{2}\bar{\sigma}+1\right)^2 < \left(\frac{T}{2}\bar{\sigma}-1\right)^2 \implies \bar{\sigma} < 0$$

In other words we have the following relation

$$|z| < 1 \iff \mathrm{Re}\{\bar{s}\} < 0$$

which imples that tha area inside unit circle of z-plane (stable region) is mapped to whole open left half plane of $\bar{s}$-plane.

Now let's consider the mapping of unit-cirlce on z-plane onto the $\bar{s}$-plane.

$$|z| = 1 \implies \left(\frac{T}{2}\bar{\sigma}+1\right)^2 = \left(\frac{T}{2}\bar{\sigma}-1\right)^2 \implies \bar{\sigma} = 0$$

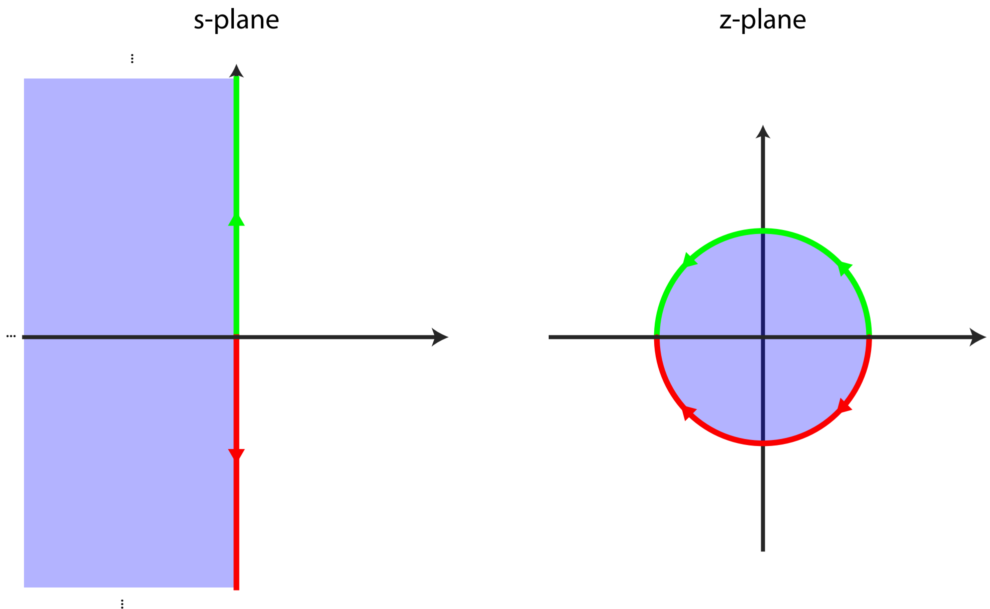
which imples that points on the unit circle are mapped to the imaginary axis on the $\bar{s}$-plane. Now let's analyze this mapping further:

$$|z| = 1 \implies z = e^{j\omega_d} \ , \omega_d \in [-pi, pi]$$
$$\bar{s} = \frac{2}{T}\frac{e^{j\omega_d}-1}{e^{j\omega_d}+1} = \frac{2}{T}\frac{(e^{j\omega_d}-1)(e^{-j\omega_d}+1)}{(e^{j\omega_d}+1)(e^{-j\omega_d}+1)} = j\frac{2}{T}\frac{\sin\omega_d}{1+\cos\omega_d} = j\frac{2}{T}\frac{\sin(\omega_d/2)}{\cos(\omega_d/2)}$$
$$\bar{\omega} = \frac{2}{T}\tan(\omega_d/2)$$

where $\bar{\omega}$ is the artificial frequency of the artificial CT system. If we analyze the frequency mapping we can easly see that

$$\omega_d : \ 0 \to \pi \quad \implies \quad \bar{\omega} : \ 0 \to \infty$$
$$\omega_d : \ 0 \to -\pi \quad \implies \quad \bar{\omega} : \ 0 \to -\infty$$

Bilinear transformation and its basic mapping properties are illustrated in the Figure below.



Blue transparent region corresponds to open left half plane in $\bar{s}$-plane and area inside the unit circle in z-plane. Red and green lines/curves illustrate the frequency paths in the respected planes.

Let's analyze the behavior of the frequency mapping around $\omega_d = 0$

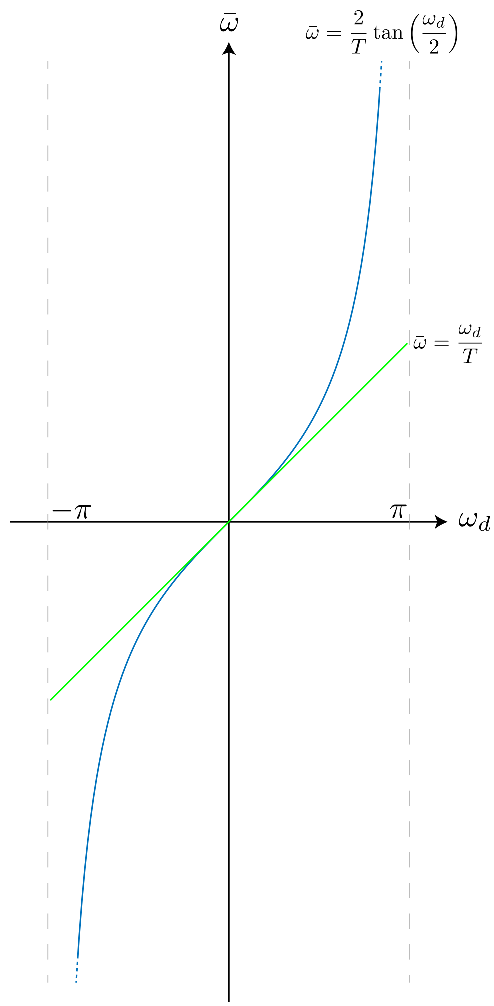
$$\bar{\omega} \approx \left[\frac{d\bar{\omega}}{\omega_d}\right]_{\omega_d} \omega_d = \omega_d/T$$

$$\bar{\omega} \approx \omega_c$$

where $\bar{\omega}$, ω_d, and ω_c are the artificial CT-frequency, frequency in DT domain, and actual frequency in CT domain respectively. The most important relation is that at low frequencies $\bar{\omega} \approx \omega_c$.

In this linear region, the bilinear transformation behaves nicely and we can roughly say that $G(s) \approx G(\bar{s})$. This means that bilinear transformation can be effectively used for design of control systems, or measuring relative performance metrics for the system of interest. However, as ω_d increases there is a highly non-linear frequency wrapping (compression) effect which can make both the design and analysis challenging and sometimes even meaningless.

The Figure below illustrates the relation between ω_d and ω as well as the linear approximation.



$\bar{\omega}$
$\bar{\omega} = \frac{2}{T}\tan\left(\frac{\omega_d}{2}\right)$
$\bar{\omega} = \frac{\omega_d}{T}$
$-\pi$
π
ω_d