EE402 Discrete Time Systems
MP-2

1. (a) $x[k] = 9k2^k - 4^k + 3$

   It is known that one-sided Z-Transform of $x[k] = 1$ is $X(z) = \mathcal{Z}\{1\} = \dfrac{1}{1 - z^{-1}}$

   using the properties and linearity of z-transform

   - $\mathcal{Z}\{a^k x[k]\} = X(z/a)$
   - $\mathcal{Z}\{kx[k]\} = -z\frac{d}{dz}X(z)$

   The z-transform of given function can be found as follows;

   $$\mathcal{Z}\{a^k\} = \frac{1}{1 - az^{-1}}$$

   $$X(z) = -9z\frac{d}{dz}\left(\frac{1}{1 - 2z^{-1}}\right) - \frac{1}{1 - 4z^{-1}} + 3\frac{1}{1 - z^{-1}}$$

   $$\boxed{X(z) = \frac{18z^{-1}}{(1 - 2z^{-1})^{-2}} - \frac{1}{1 - 4z^{-1}} + 3\frac{1}{1 - z^{-1}}}$$

   (b) $x[k] = \sum_{h=0}^{k} a^h$ where a is a constant

   In this part, we can use, causal shifting property of Z-transform, i.e.,

   - $\mathcal{Z}\{x[k - N]\} = z^N X(z)$

   Thus, the z-transform of given function can be found as follows;

   $$x[k - 1] = \sum_{h=0}^{k-1} a^{h-1}$$

   $$x[k] - x[k - 1] = a^k$$

   $$X(z) - z^{-1}X(z) = \mathcal{Z}\{a^k\}$$

   $$X(z)(1 - z^{-1}) = \mathcal{Z}\{a^k\}$$

   $$X(z) = \frac{\mathcal{Z}\{a^k\}}{(1 - z^{-1})}$$

   $$\mathcal{Z}\{a^k\} = \frac{1}{1 - az^{-1}}$$

   $$\boxed{X(z) = \frac{1}{(1 - z^{-1})(1 - az^{-1})}}$$

(c) $x[k] = k(k-1)...(k-h+1)a^{k-h}$

- for $h = 0$, $x[k] = a^k$, $\mathcal{Z}\{x[k]\} = X_0(z) = \frac{1}{1-az^{-1}}$
- for $h = 1$, $x[k] = ka^{k-1}$, $\mathcal{Z}\{x[k]\} = X_1(z) = a^{-1}(-z\frac{d}{dz})X_0(z) = \frac{z^{-1}}{(1-az^{-1})^2}$
- for $h = 2$, $x[k] = k(k-1)a^{k-2}$, $\mathcal{Z}\{x[k]\} = X_2(z) = a^{-1}(-z\frac{d}{dz}-1)X_1(z) = \frac{z^{-1}(1+az^{-1})}{(1+az^{-1})^3}$
- ...

iteratively, it can be observed that

$$X(z) = \frac{z^{-h}}{(1-z^{-1})^{h+1}}h!$$

(d) It can be seen from the given graph at the mini-project that, the function $x[n]$ can be written in terms of ramp functions $r[k] = k$;

$$\boxed{x[k] = r[k-2] - r[k-5]}$$

using the linearity and casual time shift properties of z-transform

- $\mathcal{Z}\{x[k-N]\} = z^N X(z)$

and complex differentiation theorem , i.e.,

- $\mathcal{Z}\{kx[k]\} = -z\frac{d}{dz}X(z)$

The z-transform of given function can be found as follows;

$$X(z) = z^{-2}\frac{z^{-1}}{1-z^{-1^2}} - z^{-5}\frac{z^{-1}}{1-z^{-1^2}}$$

$$\boxed{X(z) = \frac{z^{-3}(1-z^{-3})}{1-z^{-1^2}}}$$

(e)

$$X(s) = \frac{1-e^{Ts}}{s}\frac{1}{(s+a)^2} = \frac{1-e^{Ts}}{s}G(s)$$

$$x(t) = \mathcal{L}^{-1}\{(\frac{1-e^{Ts}}{s})G(s)\} = \mathcal{L}^{-1}\{(\frac{1}{s})G(s)\} - \mathcal{L}^{-1}\{(\frac{e^{Ts}}{s})G(s)\}$$

Lets as assume $\hat{G}(s) = \frac{G(s)}{s}$,

$$x(t) = \hat{g}(t) - \hat{g}(t-T)$$

$$\mathcal{Z}\{x(kT)\} = \mathcal{Z}\{\hat{g}(kT) - \hat{g}(kT-T)\} = \mathcal{Z}\{\hat{g}[k] - \hat{g}[k-1]\} = (1-z^{-1})\hat{G}(z)$$

$$\hat{G}(z) = \mathcal{Z}\{\mathcal{L}^{-1}\{\frac{G(s)}{s}\}\} = \mathcal{Z}\{\mathcal{L}^{-1}\{\frac{1}{s(s+a)^2}\}\}$$

To find inverse Laplace transform, we can use partial fraction expansion for the expression,

$$\frac{1}{s(s+a)^2} = \frac{a_1}{s} + \frac{a_2}{s+a} + \frac{a_3}{(s+a)^2}$$

With some calculation, coefficients are found to be,

$$\boxed{a_1 = \frac{1}{a^2}} \, , \, \boxed{a_2 = -\frac{1}{a^2}} \, , \, \boxed{a_3 = -\frac{1}{a}}$$

Thus,

$$\hat{g}(t) = a_1 + a_2 e^{-at} + a_3 t e^{-t}$$

$$\hat{g}(t) = \frac{1}{a^2} - \frac{1}{a^2} e^{-at} - \frac{1}{a} t e^{-t}$$

$$\hat{G}(z) = \mathcal{Z}\{\frac{1}{a^2} - \frac{1}{a^2} e^{-akT} - \frac{1}{a} kT e^{-kT}\}$$

$$\hat{G}(z) = \frac{1}{a^2}\frac{1}{1-z^{-1}} - \frac{1}{a^2}\frac{1}{1-e^{-aT}z^{-1}} - \frac{1}{a}\frac{Te^{-aT}z^{-1}}{(1-e^{-aT}z^{-1})^2}$$

$$X(z) = (1-z^{-1})(\frac{1}{a^2}\frac{1}{1-z^{-1}} - \frac{1}{a^2}\frac{1}{1-e^{-aT}z^{-1}} - \frac{1}{a}\frac{Te^{-aT}z^{-1}}{(1-e^{-aT}z^{-1})^2})$$

$$\boxed{X(z) = (\frac{1}{a^2} - \frac{1}{a^2}\frac{(1-z^{-1})}{1-e^{-aT}z^{-1}} - \frac{1}{a}\frac{(1-z^{-1})(Te^{-aT}z^{-1})}{(1-e^{-aT}z^{-1})^2})}$$

2.

$$X(z) = \frac{z^{-1}}{(1-z^{-1})(1+1.3z^{-1}+0.4z^{-2})^2}$$

$$X(z) = \frac{z^4}{(z-1)(z^2+1.3+0.4z^2)^2}$$

(a) Using partial fraction expansion,

$$\frac{X(z)}{z} = \frac{a_1}{z-1} + \frac{a_2}{(z+0.5)^2} + \frac{a_3}{z+0.5} + \frac{a_4}{(z+0.8)^2} + \frac{a_5}{z+0.8}$$

$$a_1 = lim_{z\to 1}\frac{X(z)(z-1)}{z} = \frac{100}{729}$$

$$a_2 = lim_{z\to -0.5}\frac{X(z)(z+0.5)^2}{z} = \frac{100}{729}$$

$$a_3 = lim_{z \to -0.5} \frac{X(z)(z+0.5)}{z} = \frac{-100}{9}$$

$$a_4 = lim_{z \to -0.8} \frac{X(z)(z+0.8)^2}{z} = \frac{-320}{81}$$

$$a_5 = lim_{z \to -0.8} \frac{X(z)(z+0.8)}{z} = \frac{8000}{729}$$

Thus, $x[k]$ becomes

$$x[k] = \frac{100}{729} - \frac{50}{27}k(-0.5)^k - \frac{100}{9}(-0.5)^k - \frac{320}{81}k(-0.8)^k + \frac{8000}{729}(-0.8)^k$$

Matlab code simulating the result can be found at Appendix A, at lines 46-54.

(b) Matlab code simulating the result can be found at Appendix A, at lines 1-45.

(c) Matlab code simulating the result can be found at Appendix A, at lines 56-71.

(d) Matlab code simulating the result can be found at Appendix A, at lines 72-80.

(e) Matlab code simulating the result can be found at Appendix A, at lines 81-126. The resulting $x[k]s$ can be seen at *Figure 1*.
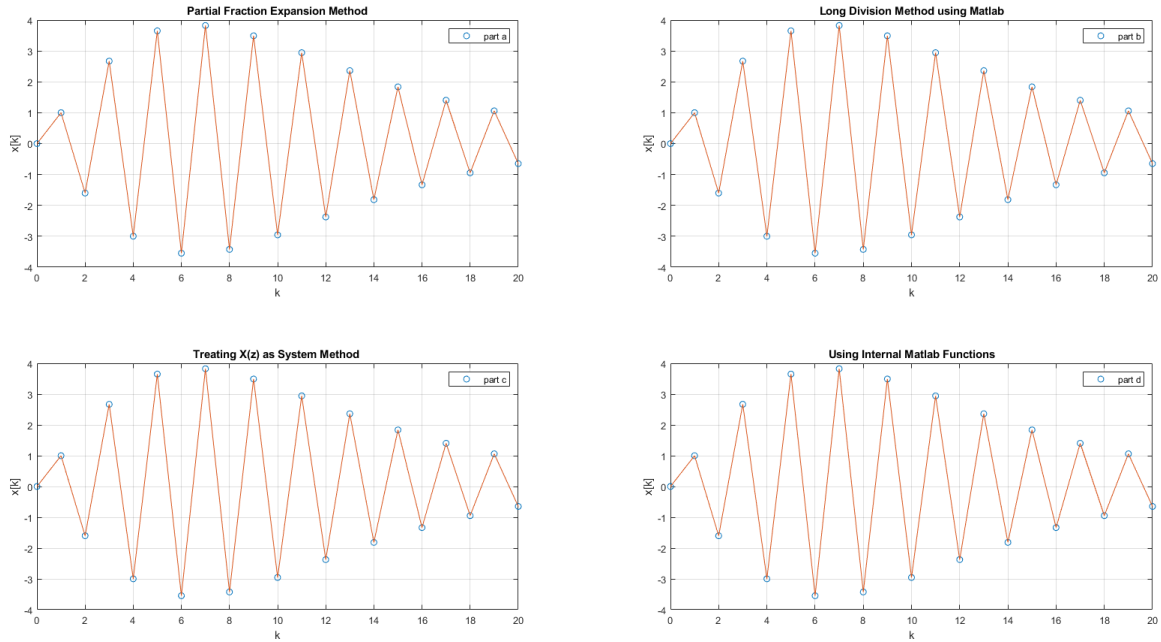

Figure 1: $x[k]$ plots using different methodologies

3. $y[k] - 7y[k-1] + 12y[k-2] = x[k]$, $y[k] = 0$ for $k < 0$ and $x[k] = 52^k + k$, $x[k] = 0$ for $k < 0$

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(a) Assume $x[k] = 0$ to find homogeneous solution, it is known that, $y_h[k]$ will be in the form A^k.

$$A^k - 7A^{k-1} + 12A^{k-2} = 0$$

$$A^{k-2} \left((A-4)(A-3) \right) = 0$$

Thus, it can be observed that, $y_h[k]$ will be in the form $a_1 4^k + a_2 3^k$.

To find the the $y_p[k]$, assume it to be in a form close to $x[n]$,i.e., in a form of $b_1 2^k + b_2 k + b_3$. Coefficients can be found by inserting it into the original equation.

$$b_1 2^k + b_2 k + b_3 - 7[b_1 2^{k-1} + b_2(k-1) + b_3] + 12[b_1 2^{k-2} + b_2(k-2) + b_3] = 52^k + k$$

by some arrangement in the equation, coefficients can be found as

$$\boxed{b_1 = 10} \ , \ \boxed{b_2 = \frac{1}{6}} \ , \ \boxed{b_3 = \frac{17}{36}}$$

Thus, $y[n]$ will be in the form $a_1 4^k + a_2 3^k + 102^k + \frac{1}{6}k + \frac{17}{36}$, to find the unknown coefficients, original equation can be iterated starting from $k = 0$.

$$y[0] = 7y[-1] - 12y[-2] + x[0] = 0 - 0 + 5 = a_1 + a_2 + 10 + \frac{17}{36}$$

$$y[1] = 7y[0] - 12y[-1] + x[1] = 35 - 0 + 11 = 4a_1 + 3a_2 + 20 + \frac{1}{6} + \frac{17}{36}$$

handling the algebraic equations, unknown coefficient can be found to be as

$$\boxed{a_1 = \frac{376}{9}} \ , \ \boxed{a_2 = \frac{-189}{4}}$$

$$\boxed{y[k] = \frac{376}{9} 4^k - \frac{189}{4} 3^k + 102^k + \frac{1}{6}k + \frac{17}{36}}$$

Matlab code simulating the result can be found at Appendix B, at lines 1-12.

(b) The difference equation can be transferred the z-domain in the following form,

$$Y(z) \left[1 - 7z^{-1} + 12z^{-2} \right] = X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - 7z^{-1} + 12z^{-2}}$$

with

$$X(z) = 5\frac{1}{1 - 2z^{-1}} + \frac{z^{-1}}{(1 - z^{-1})^2}$$

$Y(z)$ can be found as,

$$Y(z) = \frac{z^{-1}}{(1-2z^{-1})^2(1-7z^{-1}+12z^{-2})} + \frac{5}{(1-2z^{-1})(1-7z^{-1}+12z^{-2})}$$
$$= \frac{z^3}{(z-3)(z-4)(z-1)^2} + \frac{5z^3}{(z-3)(z-4)(z-2)} \tag{1}$$

using partial fraction expansion

$$Y(z) = \left[\frac{a_1}{z-3} + \frac{a_2}{z-4} + \frac{a_3}{(z-1)^2} + \frac{a_4}{z-1}\right] + \left[\frac{b_1}{z-3} + \frac{b_2}{z-4} + \frac{b_3}{z-2}\right]$$

with some calculation coefficients can be found to be

$$\boxed{a_1 = -9/4} \;,\; \boxed{a_2 = 16/9} \;,\; \boxed{a_3 = 1/6} \;,\; \boxed{a_4 = 17/36}$$

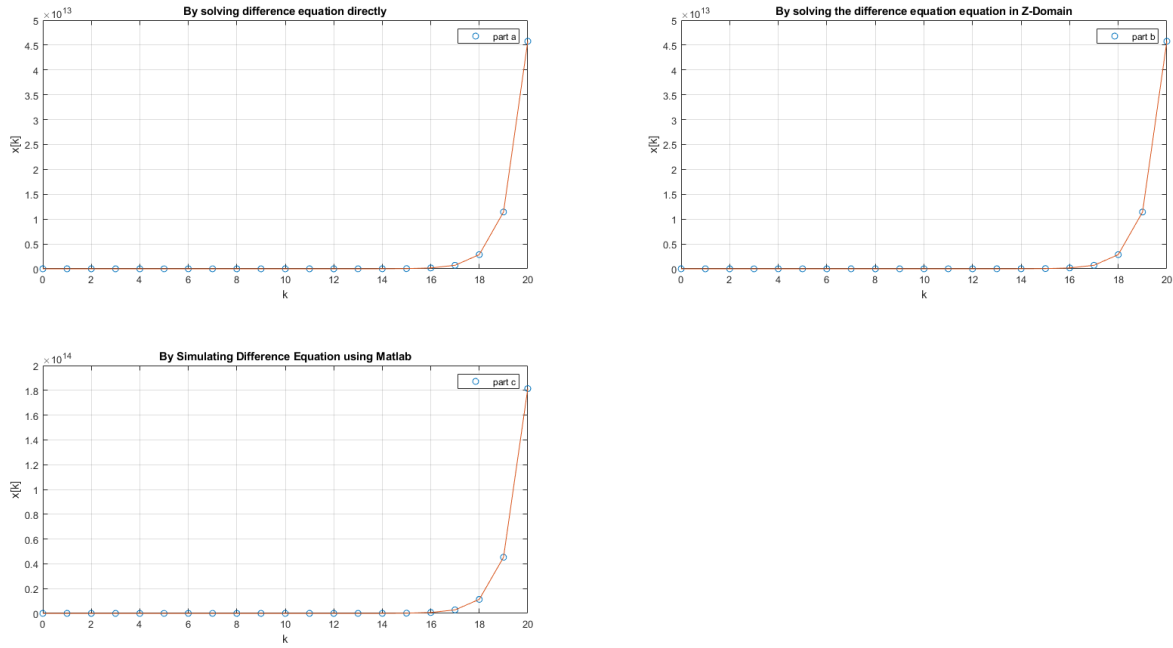$$\boxed{b_1 = -45} \;,\; \boxed{b_2 = 40} \;,\; \boxed{b_3 = 10}$$

Then, $y[k]$ becomes

$$\boxed{y[k] = \frac{-189}{4}3^k + \frac{376}{9}4^k + 102^k + \frac{1}{6}k + \frac{17}{36}}$$

Matlab code simulating the result can be found at Appendix B, at lines 13-24.

(c) Matlab code simulating the result can be found at Appendix B, at lines 25-34.

(d) Matlab code simulating the result can be found at Appendix B, at lines 36-68.The resulting $x[k]s$ can be seen at *Figure 2*.

Figure 2: $y[k]$ plots using different methodologies

4.

$$G(z) = \frac{4 - 0.02z^{-1} + 0.3z^{-2} + 0.02z^{-3}}{2 - 6z^{-1} + 2z^{-2} - z^{-3} + z^{-4}} = \frac{Y(z)}{X(z)}$$

Assume a middle variable $H(z)$

$$G_1(z) = \frac{H(z)}{X(z)}$$

$$G_2(z) = \frac{Y(Z)}{H(z)}$$

with

$$G_1(z) = 4 - 0.02z^{-1} + 0.3z^{-2} + 0.02z^{-3}$$
$$G_2(z) = 2 - 6z^{-1} + 2z^{-2} - z^{-3} + z^{-4}$$

from there, it can be observed that if the inverse z-transform is performed on both equations,

$$\boxed{x[k] = 2h[k] - 6h[k-1] + 2h[k-2] - 8h[k-3] + 0.2h[k-4]}$$

$$\boxed{y[k] = 4h[k] - 0.02h[k-1] + 0.3h[k-2] + 0.02h[k-3]}$$

The resulting block diagram constructed at Simulink can be seen at *Figure* **??**
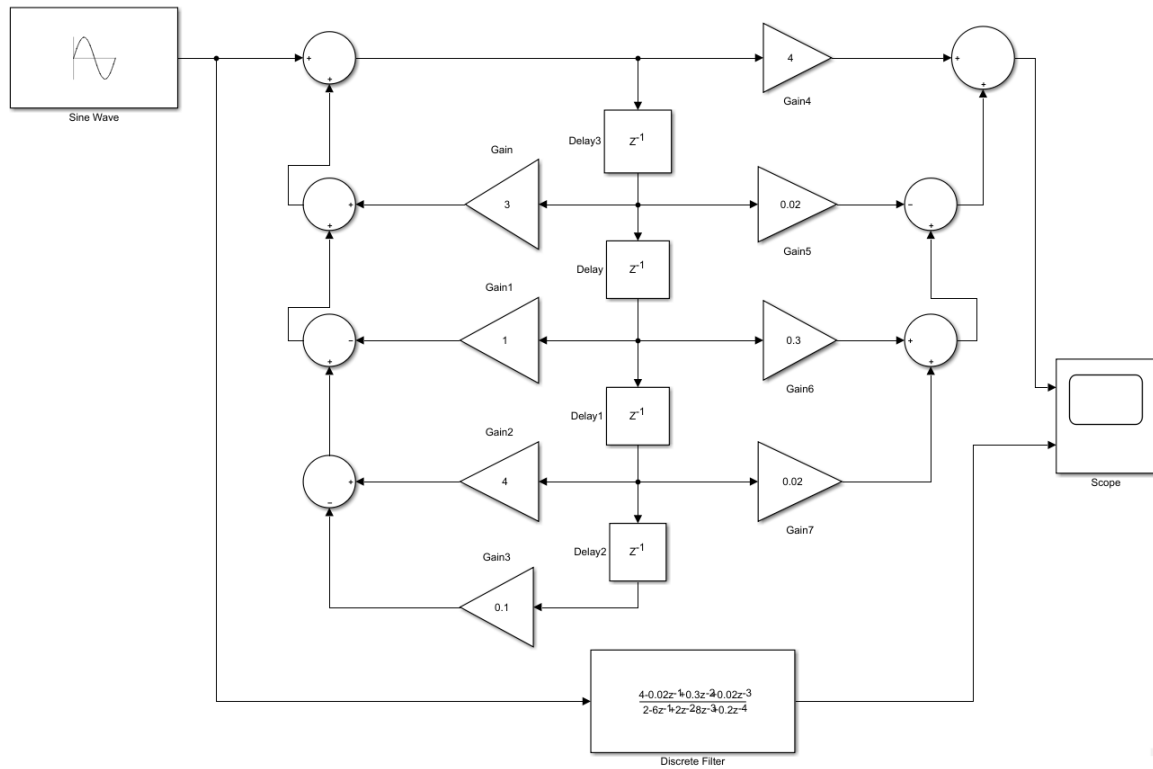
7

Figure 3: Block Diagram of system given at question 4

Differences in the simulation results can be observed form *Figure 4 to 5.*
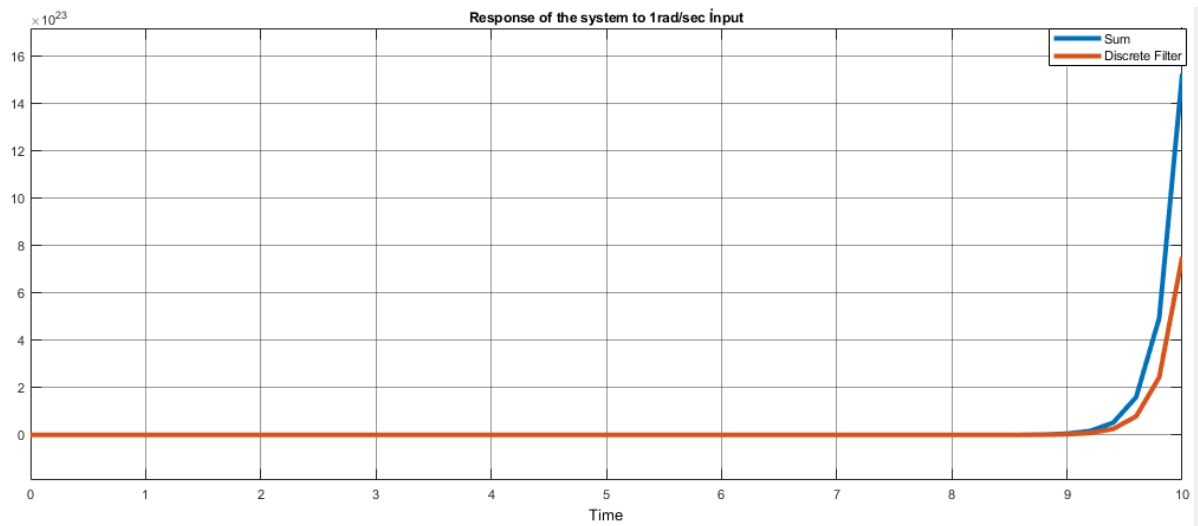

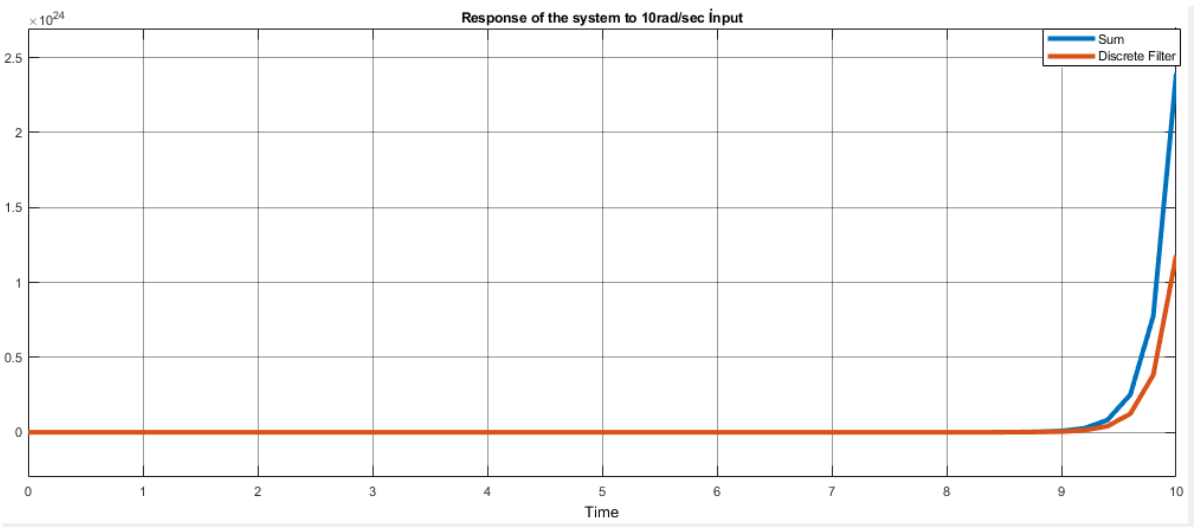Figure 4: Response of the system to 1rad/sec Input

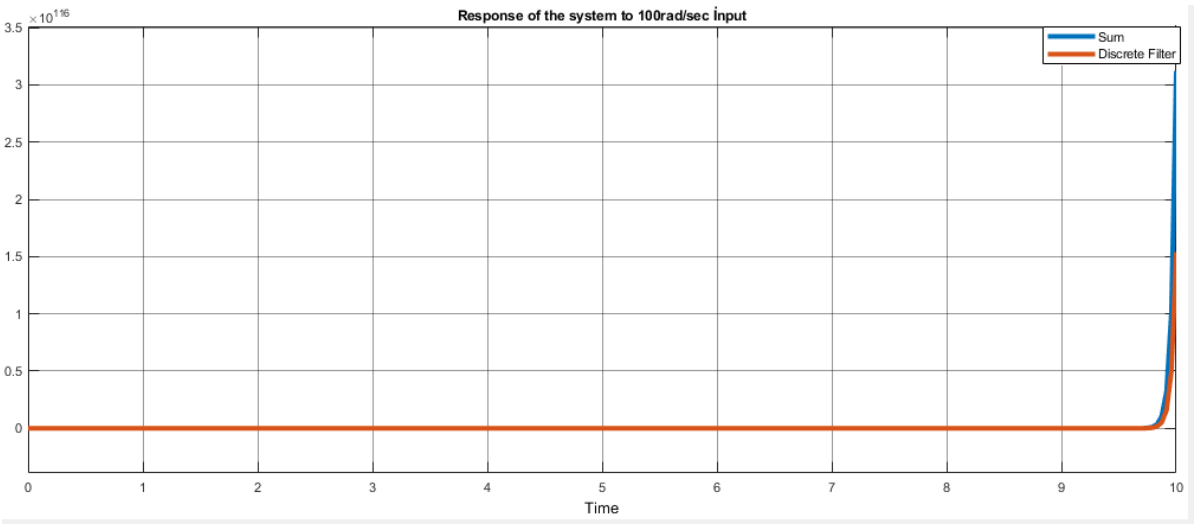Figure 5: Response of the system to 10rad/sec Input

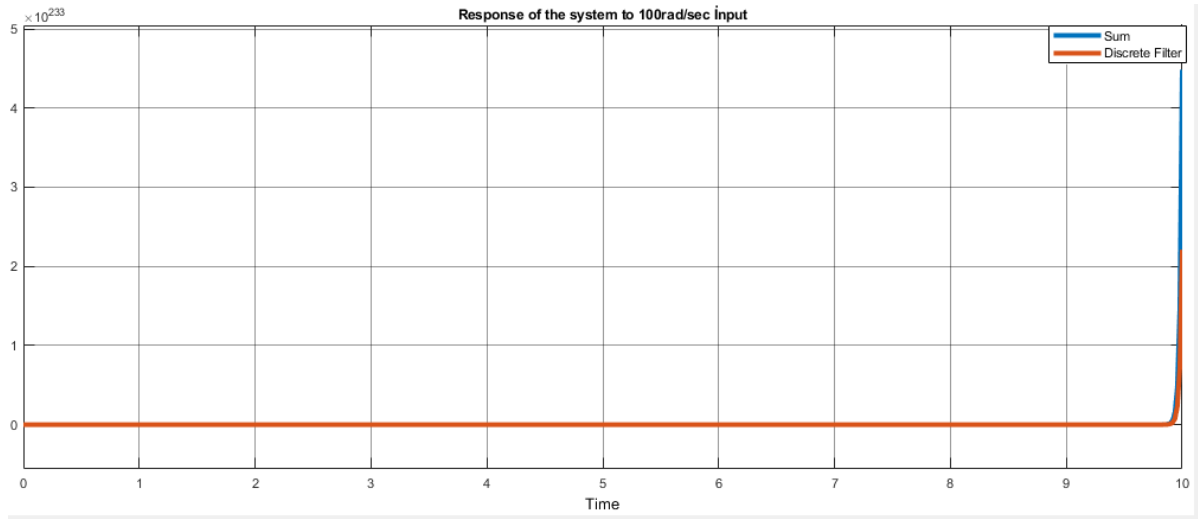Figure 6: Response of the system to 5jjj0rad/sec Input

Figure 7: Response of the system to 100rad/sec Input

As the frequency is increased, the results became more similar. The internal block is not functioning well enough. However, our design is functioning better than our expectations.

Appendices

A   Source Code For Question 2

```matlab
%% %%%%%%%%%%%%%%%% part b %%%%%%%%%%%%%%%%%%%%%%%%%
    clear
    clc
    num = [0 1 zeros(1,21) ]
    num_lenght= length(num)
    den = [1 1.6 -0.11 -1.45 -0.88 -0.16 zeros(1,17)]
    den_lenght= length(den)
    %ss= tf(num,den)


    i=1
    a=0
    while i<num_lenght
        if num(i)== 0
            i=i+1
            a=a+1
        else
            w=i
            while w>1
                den2(w-1)=0
                w=w-1
            end
            q=i
            den_lenght2=length(den2)
            while q<den_lenght+den_lenght2+1
                if q-den_lenght2<1
                    q=q+1
                else
                    den2(q)=(num(i)/(den(1)))*den(q-den_lenght2
                        )
                    q=q+1
                end
            end
            x_b(i)=num(i)
            u=i
            while u<length(num)
                num(u)=num(u)-den2(u)
                u=u+1
```

```matlab
37                    end
38                    i=i+1
39                    clearvars den2
40                end

42        end
43        x_b(i-1)=[]
44        %% x[-1] is left untouched since the internal matlab func
              at part d leaves it, x_b(1)=[] removes it
45        %% %%%%%%%%%%%% end of part b %%%%%%%%%%%%%%%%%%%
46        %% %%%%%%%%%%%%%% part a %%%%%%%%%%%%%%%%%%%%%
47        i=1
48        while i<21
49            x(1)=0
50            x_a(i+1)=(100/729)+(-50/27)*i*(-0.5)^(i)-(100/9)*(-0.5)
                  ^(i)+(-320/81)*i*(-0.8)^(i)+(8000/729)*(-0.8)^(i)
51            i=i+1
52        end
53        v=[0 20 -1 1]
54        axis(v)

56        k=0:20
57        % %%%%%%%%%%%% end of part a %%%%%%%%%%%%%%%%%%%%
58        %% %%%%%%%%%%%%%% part c %%%%%%%%%%%%%%%%%%%%%
59        i=1
60        x_c=zeros(1,5) % Initialize the array with k<0 values
61        while i<21
62            m=i+5
63            x_c(m)=(-1.6)*x_c(m-1)+(0.11)*x_c(m-2)+(1.45)*x_c(m-3)
                  +(0.88)*x_c(m-4)+(0.16)*x_c(m-5)+mydelta(m-1)
64            i=i+1
65        end
66        x_c(1)=[] % Delete x[-5]
67        x_c(1)=[] % Delete x[-4]
68        x_c(1)=[] % Delete x[-3]
69        x_c(1)=[] % Delete x[-2]
70        %% x[-1] is left untouched since the internal matlab func
              at part d leaves it, x_b(1)=[] removes it
71        %% %%%%%%%%%%%% end of part c %%%%%%%%%%%%%%%%%%%
72        %% %%%%%%%%%%%%%% part d %%%%%%%%%%%%%%%%%%%%%
73        num = [0 1 0 0 0 0 ]

```

12

```
75    den = [1  1.6  -0.11  -1.45  -0.88  -0.16]
76
77    %% Inverse Z Part
78    x=[1 zeros(1,20)]
79    x_d=filter(num,den,x) %filter to finde inverse z transfor
80    %% %%%%%%%%%%%%% end of part d %%%%%%%%%%%%%%%%%%%
81    %% %%%%%%%%%%%%%%% part e %%%%%%%%%%%%%%%%%%%%
82    %% plot x_a
83    subplot(2,2,1)
84    v=[0 20 -1 1]
85    axis(v)
86    k=0:20
87    plot(k,x_a,'o',k,x_a,'-')
88    legend('part a')
89    grid
90    title('Partial Fraction Expansion Method')
91    xlabel('k')
92    ylabel('x[k]')
93    %% plot x_b
94    subplot(2,2,2);
95    v=[0 20 -1 1]
96    axis(v)
97    k=0:20
98    plot(k,x_b,'o',k,x_b,'-')
99    legend('part b')
100   grid
101   title('Long Division Method using Matlab')
102   xlabel('k')
103   ylabel('x[k]')
104   %% plot x_c
105   subplot(2,2,3);
106   v=[0  20 -1 1]
107   axis(v)
108   k=0:20
109   plot(k,x_c,'o',k,x_c,'-')
110   legend('part c')
111   grid
112   title('Treating X(z) as System Method ')
113   xlabel('k')
114   ylabel('x[k]')
115   %% plot x_d
116   subplot(2,2,4);
```

```matlab
117        v=[0  20  −1  1]
118        axis(v)
119        k=0:20
120        plot(k,x_d,'o',k,x_d,'−')
121        legend('part d')
122        grid
123        title('Using Internal Matlab Functions')
124        xlabel('k')
125        ylabel('x[k]')
126        %% %%%%%%%%%%%%%%% end of part e %%%%%%%%%%%%%%%%%%%%%
```

```matlab
1  function y = mydelta(x)
2        if x == 5
3              y=1
4        else
5              y=0
6        end
7  end
```

B   Source Code For Question 3

```matlab
%% %%%%%%%%%%%%%%%% part a %%%%%%%%%%%%%%%%%%%%%%%%
    i=1
    while i<21
        x(1)=0
        x_a(i+1)=-(189/4)*(3)^(i)+(376/9)*(4)^(i)+10*(2)^(i)
            +(1/6)*(i)+17/36
        i=i+1
    end
    v=[0 20 -1 1]
    axis(v)

    k=0:20
%% %%%%%%%%%%%%%% end of part a %%%%%%%%%%%%%%%%%%%%%
%% %%%%%%%%%%%%%%%% part b %%%%%%%%%%%%%%%%%%%%%%%%%
    i=1
    while i<21
        x(1)=0
        x_b(i+1)=-(189/4)*(3)^(i)+(376/9)*(4)^(i)+10*(2)^(i)
            +(1/6)*(i)+17/36
        i=i+1
    end
    v=[0 20 -1 1]
    axis(v)

    k=0:20
%% %%%%%%%%%%%% end of part b %%%%%%%%%%%%%%%%%%%%%%%
%% %%%%%%%%%%%%%%%% part c %%%%%%%%%%%%%%%%%%%%%%%%%
    i=1
    x_c=zeros(1,3) % Initialize the array with k<0 values
    while i<21
        m=i+3
        x_c(m)=(7)*x_c(m-1)-(12)*x_c(m-2)+5*2^m+m
        i=i+1
    end
    x_c(1)=[] % Delete x[-3]
    x_c(1)=[] % Delete x[-2]
%% %%%%%%%%%%%% end of part c %%%%%%%%%%%%%%%%%%%%%%%
%% plot x_a
    subplot(2,2,1)
    v=[0 20 -1 1]
```

```matlab
39        axis(v)
40        k=0:20
41        plot(k,x_a,'o',k,x_a,'-')
42        legend('part a')
43        grid
44        title('By solving difference equation directly')
45        xlabel('k')
46        ylabel('x[k]')
47 %% plot x_b
48        subplot(2,2,2)
49        v=[0 20 -1 1]
50        axis(v)
51        k=0:20
52        plot(k,x_b,'o',k,x_b,'-')
53        legend('part b')
54        grid
55        title('By solving the difference equation equation in Z-
              Domain')
56        xlabel('k')
57        ylabel('x[k]')
58 %% plot x_c
59        subplot(2,2,3)
60        v=[0 20 -1 1]
61        axis(v)
62        k=0:20
63        plot(k,x_c,'o',k,x_c,'-')
64        legend('part c')
65        grid
66        title('By Simulating Difference Equation using Matlab')
67        xlabel('k')
68        ylabel('x[k]')
```