

Lecture 14

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Asymptotic Stability of LTI Systems

Asymptotic Stability of CT-LTI Systems

Let's consider the state-representation of an CT LTI system

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t)\end{aligned}$$

Given CT-LTI system is called *asymptotically stable* if, with $u(t) = 0$ and $\forall x(0) \in \mathbb{R}^n$, we have

$$\lim_{t \rightarrow \infty} \|x(t)\| = 0$$

Example: Let's assume that system matrix A is diagonalizable and all eigenvalues are real. Find a necessary and sufficient condition such that the state-space representation is asymptotically stable.

Solution: When $u(t) = 0$, we have

$$\begin{aligned}\dot{x}(t) &= Ax(t) \\ x(t) &= e^{At}x_0\end{aligned}$$

Since A is diagonalizable, we know that

$$\begin{aligned}A &= P \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} P^{-1} \\ e^{At} &= P \begin{bmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & \cdots & 0 \\ \vdots & & \ddots & \\ 0 & & & e^{\lambda_n t} \end{bmatrix} P^{-1}\end{aligned}$$

Let $Px_0 = [e_1 \ e_2 \ \cdots \ e_n]^T$, $e_i \in \mathbb{R}$, then $x(t)$ can be expressed as

$$x(t) = P \begin{bmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & \cdots & 0 \\ \vdots & & \ddots & \\ 0 & & & e^{\lambda_n t} \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix}$$

$$x(t) = P \left(\begin{bmatrix} e_1 e^{\lambda_1 t} \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ e_2 e^{\lambda_2 t} \\ \vdots \\ 0 \end{bmatrix} + \cdots + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ e_n e^{\lambda_n t} \end{bmatrix} \right)$$

Then it is easy to see that

$$\forall x_0 \in \mathbb{R}^n, \lim_{t \rightarrow \infty} \|x(t)\| = 0 \iff \forall i \in \{1, \dots, n\}, \lambda_i < 0$$

Asymptotic Stability of DT-LTI Systems

Now, let's consider the state-representation of an DT LTI system

$$\begin{aligned} x[k+1] &= Gx[k] + Hu[k] \\ y[k] &= Cx[k] + Du[k] \end{aligned}$$

Given DT-LTI system is called *asymptotically stable* if, with $u[k] = 0$ and $\forall x(0) \in \mathbb{R}^n$, we have

$$\lim_{k \rightarrow \infty} \|x[k]\| = 0$$

Example: Let's assume that system matrix G is diagonalizable and all eigenvalues are real. Find a necessary and sufficient condition such that the state-space representation is asymptotically stable.

Solution: When $u[k] = 0$, we have

$$\begin{aligned} x[k+1] &= Gx[k] \\ x(t) &= G^k x_0 \end{aligned}$$

Since G is diagonalizable, we know that

$$G = P \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} P^{-1}$$

$$G^k = P \begin{bmatrix} \lambda_1^k & 0 & \cdots & 0 \\ 0 & \lambda_2^k & \cdots & 0 \\ \vdots & & \ddots & \\ 0 & & & \lambda_n^k \end{bmatrix} P^{-1}$$

Let $Px_0 = [e_1 \ e_2 \ \cdots \ e_n]^T, e_i \in \mathbb{R}$, then $x[k]$ can be expressed as

$$x[k] = P \left(\begin{bmatrix} e_1 \lambda_1^k \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ e_2 \lambda_2^k \\ \vdots \\ 0 \end{bmatrix} + \cdots + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ e_n \lambda_n^k \end{bmatrix} \right)$$

Then it is easy to see that

$$\forall x_0 \in \mathbb{R}^n, \lim_{k \rightarrow \infty} \|x[k]\| = 0 \iff \forall i \in \{1, \dots, n\}, |\lambda_i| < 1$$

BIBO Stability of LTI Systems

A CT-system written in state-space form is stable if and only if the poles of $H(s) = C[sI - A]^{-1}B + D$ are located strictly in the open left half plane.

A DT-system written in state-space form is stable if and only if the poles of $H(z) = C[zI - G]^{-1}H + D$ are located strictly inside the unit circle.

We did not mention whether the system is SISO or MIMO.

- Do you think that BIBO notion is valid for MIMO systems?
- Do you think that checking “poles” of $H(s)$ or $H(z)$ is a valid method for checking stability?
- What are the poles of $H(s)$ and $H(z)$?

Example: Consider the following state-space form of a CT system

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 1 & -1 \end{bmatrix} x(t) \end{aligned}$$

Is this system asymptotically stable?

Solution: Let's compute the eigenvalues of A

$$\det \left(\begin{bmatrix} \lambda & -1 \\ -1 & \lambda \end{bmatrix} \right) = \lambda^2 - 1$$

$$\lambda_{1,2} = \pm 1$$

Thus the system is NOT Asymptotically Stable.

Is this system BIBO stable?

Solution: Let's compute the $H(s)$

$$\begin{aligned}
H(s) &= \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} s & -1 \\ -1 & s \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} s & 1 \\ 1 & s \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{1}{s^2 - 1} \\
&= \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} s & 1 \\ 1 & s \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{1}{s^2 - 1} \\
&= \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ s \end{bmatrix} \frac{1}{s^2 - 1} \\
&= \frac{-(s-1)}{s^2 - 1} \\
&= \frac{-1}{s+1}
\end{aligned}$$

Indeed, the system is BIBO Stable.

In conclusion

- Asymptotic Stability \rightarrow BIBO Stability
- BIBO Stability \nrightarrow Asymptotic Stability