EE402 - Discrete Time Systems	**Spring 2018**

Lecture 6

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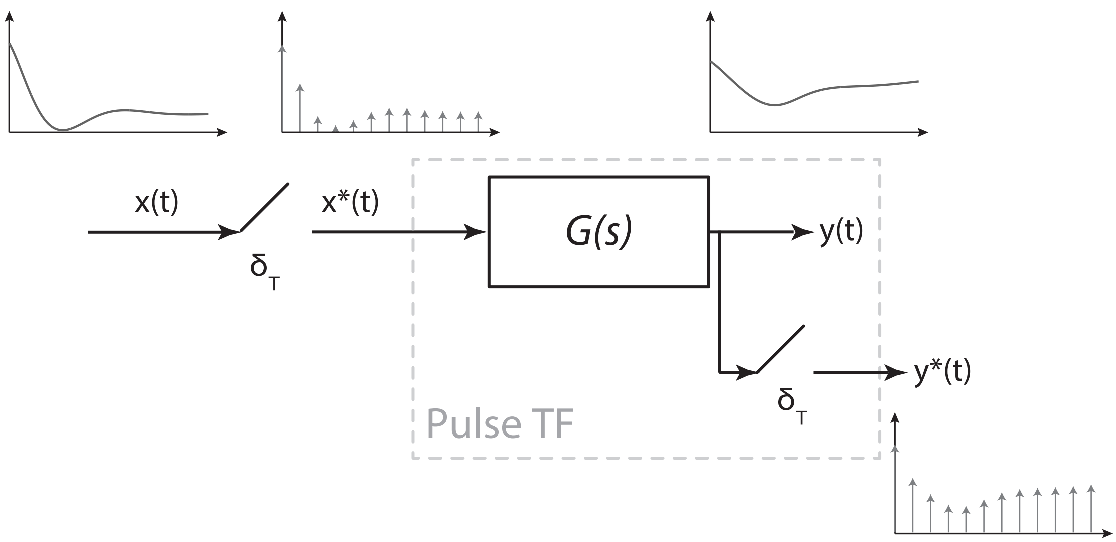
Pulse Transfer Functions

For the following block diagram representation we have the following time domain input–output representations

$$y(t) = g(t) * x^*(t) \quad \text{CT} - \text{convolution}$$
$$y[n] = g[n] * x[n] \quad \text{DT} - \text{convolution}$$



We then showed that Input–output relation between $x[n]$ and $y[n]$ in Z-domain can be represented with the following Pulse Transfer Function equation

$$Y(z) = G(z)X(z) \;\rightarrow\; G(z) = \frac{Y(z)}{X(z)}$$

If we take the Laplace transform of the CT-convolution equation

$$Y(s) = G(s)X^*(s) \;\rightarrow Y^*(s) = [G(s)X^*(s)]^*$$

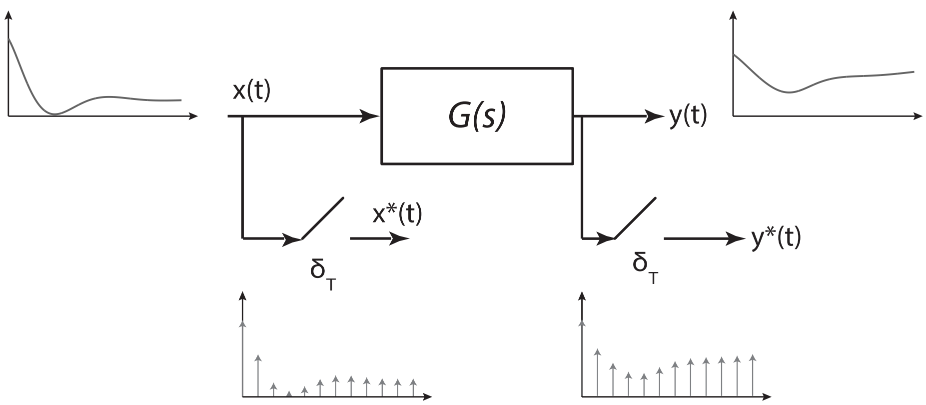
Now let's analyze $Y^*(s)$

$$Y(s) = G(s)X^*(s) \;\rightarrow Y^*(s) = [G(s)X^*(s)]^*$$
$$Y(z) = G(z)X(z)$$

Given that starred Laplace transform is the z-transform where z is evaluated e^{Ts} we can conclude that

$$Y(z) = G(z)X(z) \;\rightarrow\; Y^*(s)|_{z=e^{Ts}} = G^*(s)|_{z=e^{Ts}} X^*(s)|_{z=e^{Ts}}$$
$$Y^*(s) = [G(s)X^*(s)]^* = G^*(s)X^*(s)$$

Now let's consider the following system for which there is no sampler of the continuous time block $G(s)$ at the input.

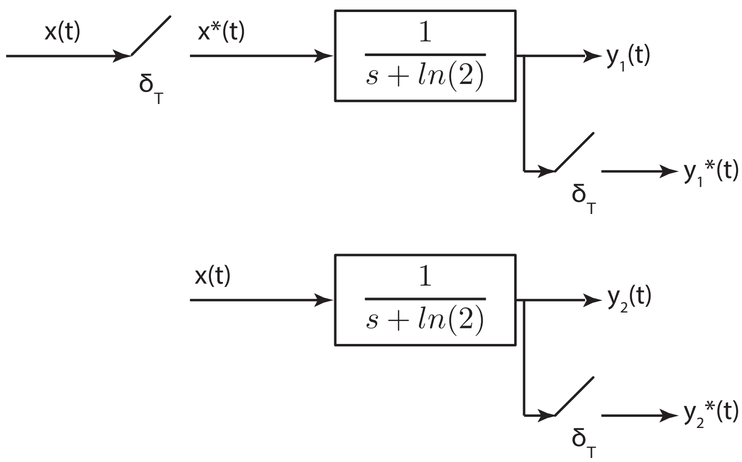


$$Y(s) = G(s)X(s) \;\rightarrow\; Y^*(s) = [G(s)X(s)]^*$$
$$Y^*(s) = [G(s)X(s)]^* = [GX(s)]^* \neq G^*(s)X^*(s)$$

where $GX(s) := G(s)X(s)$. In z-domain we have

$$Y(z) = \mathcal{Z}\{GX(s)\} = GX(z) \neq G(z)X(z)$$

Example: Given that $x(t) = u(t)$ and $T = 1$, compute $Y_1(z)$ and $Y_2(z)$ for the following example in the figure



Solution: Let's start with $Y_1(z)$.

$$\begin{aligned} Y_1(z) &= G(z)X(z) \\ &= \frac{z^2}{(z-1)(z-0.5)} \\ &= \frac{2z}{z-1} - \frac{z}{z-0.5} \end{aligned}$$

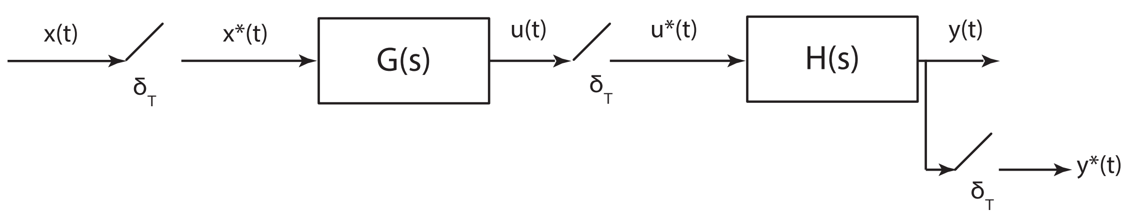
Now let's compute $Y_2(z)$

$$\begin{aligned} Y_2(z) &= GX(z) \\ &= \mathcal{Z}\left\{\frac{1}{s(s+ln(2))}\right\} \\ &= \mathcal{Z}\left\{\frac{1/ln(2)}{s} - \frac{1/ln(2)}{s+ln(2)}\right\} \\ &= \frac{\ln(2)\, z}{z-1} - \frac{\ln(2)\, z}{z-0.5} \end{aligned}$$

We can conclude that $Y_1(z) \neq Y_2(z)$, and thus $y_1(t) \neq y_2(t)$.

Pulse Transfer Functions of Cascaded Elements

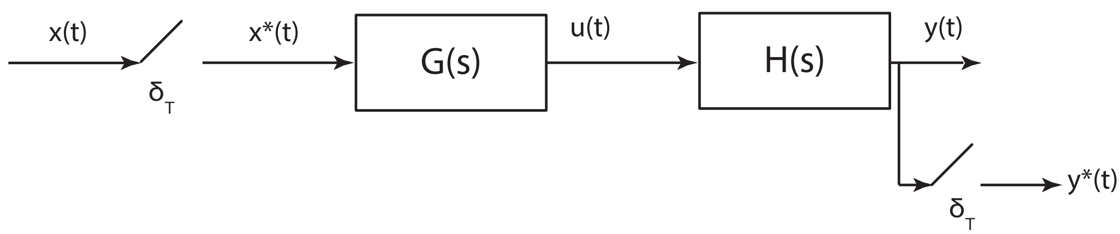
Consider the cascaded system shown in Figure below



Pulse transfer function of this system can be computed

$$\begin{aligned} U^*(s) &= G^*(s)X^*(s) \quad Y^*(s) = H^*(s)U^*(s) \\ Y^*(s) &= H^*(s)G^*(s)X^*(s) \\ Y(z) &= H(z)G(z)X(z) \\ \frac{Y(z)}{X(z)} &= H(z)G(z) \end{aligned}$$

Now, let's consider the following cascaded system

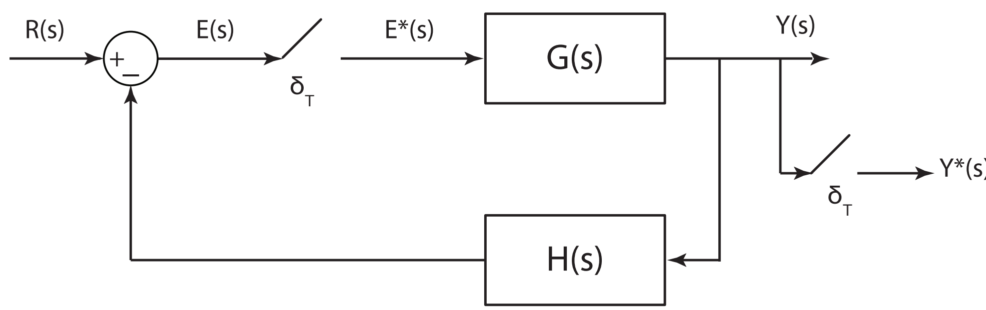


Pulse transfer function of this new system can be computed

$$
\begin{aligned}
Y(s) &= H(s)G(s)X^*(s) \\
Y(s) &= HG(s)X^*(s) \quad \text{where } HG(s) = H(s)G(s) \\
Y^*(s) &= [HG(s)X^*(s)]^* = HG^*(s)X^*(s) \\
Y(z) &= HG(z)X(z) \quad \text{where } HG(z) = \mathcal{Z}\{HG(s)\} \\
\frac{Y(z)}{X(z)} &= HG(z) \neq H(z)G(z)
\end{aligned}
$$

Pulse Transfer Functions of Closed Loop Systems

SYS1: Consider the following closeed-loop system



$$
\begin{aligned}
E(s) &= R(s) - H(s)Y(s) \quad , \quad Y(s) = G(s)E^*(s) \\
E(s) &= R(s) - H(s)G(s)E^*(s) \\
&= R(s) - HG(s)E^*(s)
\end{aligned}
$$

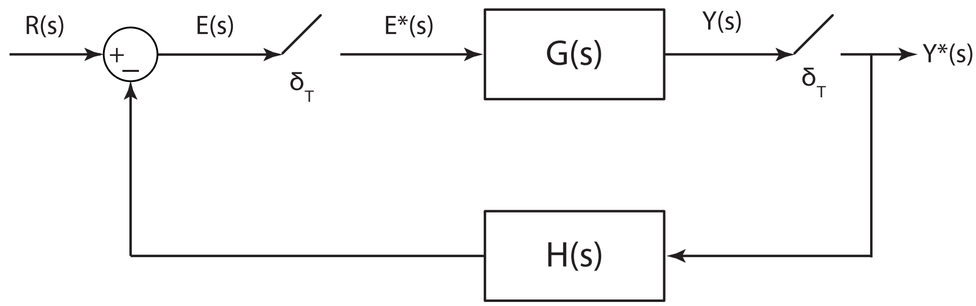
Let's take the starred Laplace transform of both sides

$$
\begin{aligned}
E^*(s) &= R^*(s) - HG^*(s)E^*(s) \\
E^*(s) &= \frac{R^*(s)}{1 + HG^*(s)}
\end{aligned}
$$

Since $Y^*(s) = G^*(s)E^*(s)$ we can conclude

$$
\begin{aligned}
Y^*(s) &= \frac{G^*(s)}{1 + HG^*(s)} R^*(s) \\
Y(z) &= \frac{G(z)}{1 + HG(z)} R(z) \\
\frac{Y(z)}{X(z)} &= \frac{G(z)}{1 + HG(z)}
\end{aligned}
$$

SYS2: Now let's consider a slightly different topology



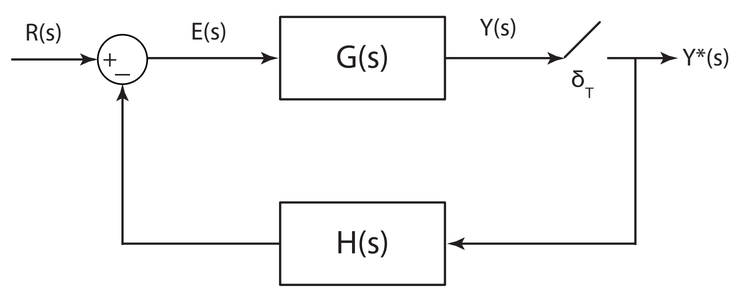
$$
\begin{aligned}
E(s) &= R(s) - H(s)Y^*(s) \\
Y(s) &= G(s)E^*(s) \quad , \quad Y^*(s) = G^*(s)E^*(s) \\
E(s) &= R(s) - H(s)G^*(s)E^*(s) \\
E^*(s) &= R^*(s) - H^*(s)G^*(s)E^*(s) \\
E^*(s) &= \frac{R^*(s)}{1 + H^*(s)G^*(s)}
\end{aligned}
$$

Then we can have

$$
\begin{aligned}
Y^*(s) &= \frac{G^*(s)}{1 + H^*(s)G^*(s)} R^*(s) \\
Y(z) &= \frac{G(z)}{1 + H(z)G(z)} R(z) \\
\frac{Y(z)}{R(z)} &= \frac{G(z)}{1 + H(z)G(z)}
\end{aligned}
$$

It can be seen that the pulse transfer function of the closed-loop systems are different.

SYS3: Now consider another closed-loop system.

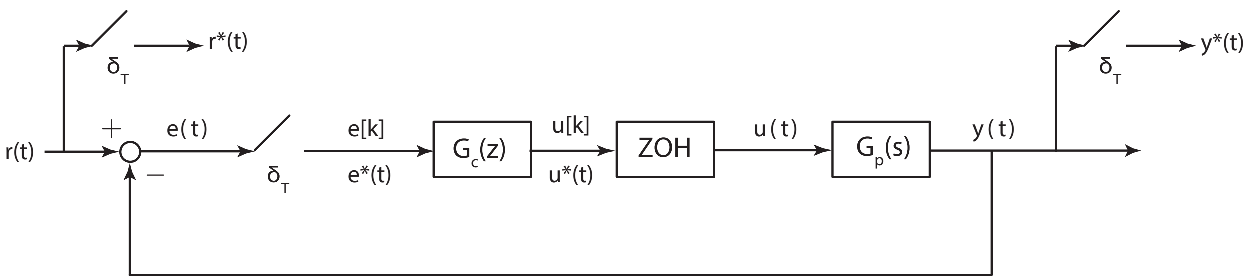


$$
\begin{aligned}
E(s) &= R(s) - H(s)Y^*(s) \qquad Y(s) = G(s)E(s) \\
Y(s) &= G(s)R(s) - G(s)H(s)Y^*(s) \\
Y*(s) &= GR^*(s) - GH^*(s)Y^*(s) \\
Y*(s) &= \frac{GR^*(s)}{1 + GH^*(s)} \\
Y(z) &= \frac{GR(z)}{1 + GH(z)}
\end{aligned}
$$

We can't even find a direct transfer function from $R^*(s)$ to $Y^*(s)$, or equivalently from $R(z)$ to $Y(z)$.

Pulse Transfer Function of A Closed Loop DT Control System

Consider the fundamental DT-control system below



Previously we derived the transfer function of ZOH, thus we can combine ZOH and plant TF into a single TF block $G(s)$ as

$$\frac{Y(s)}{E^*(s)} = G(s) = \frac{1-e^{-sT}}{s}G_p(s)$$

The controller is actually implemented in an hardware/software platfrom which indeed works in discrete-time. However, we can find a starred version of $G_c(z)$ by

$$G_c^*(s) = G_c(z)|_{z=e^{Ts}}$$

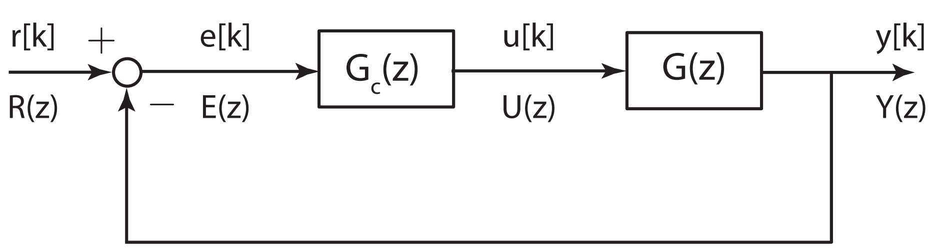
Now let's derive the PTF

$$\begin{aligned} E(s) &= Y(s) - R(s) \quad , \quad Y(s) = G(s)G_c^*(s)E^*(s) \\ E^*(s) &= Y^*(s) - R^*(s) \quad , \quad Y(s) = G^*(s)G_c^*(s)E^*(s) \\ E^*(s) &= G^*(s)G_c^*(s)E^*(s) - R^*(s) \\ E^*(s) &= \frac{R^*(s)}{1+G^*(s)G_c^*(s)} \end{aligned}$$

Then the PTF in starred Laplace domain and z domain can be find as

$$\begin{aligned} \frac{Y^*(s)}{R^*(s)} &= \frac{G^*(s)G_c^*(s)}{1+G^*(s)G_c^*(s)} \\ \frac{Y(z)}{R(z)} &= \frac{G(z)G_c(z)}{1+G(z)G_c(z)} \end{aligned}$$

Note if we only care the signal flow in the sampled instants we can re-draw the block diagram such that all time domain signals are in DT and all systems are represented in Z-domain. The fundamental block diagram can be re-drawn as



It is obvious that this discretized block diagram is much simpler and neater. Let's show that the transfer function of this DT system is equal to the PTF of the hybrid system above (CT & DT combined)

$$E(z) = R(z) - Y(z) \quad , \quad Y(z) = G(z)G_c(z)E(z)$$
$$E(z) = \frac{1}{1+G(z)G_c(z)}R(z)$$
$$\frac{Y(z)}{R(z)} = \frac{G(z)G_c(z)}{1+G(z)G_c(z)}$$

Example: Let $G_p(s) = \frac{1}{s+1}$, $T = 1$, $G_c(z) = K$ (Discrete P Controller) . First find PTF (in z-domain).
Solution: First let's find $G(z)$

$$G(z) = (1-z^{-1})\mathcal{Z}\left\{\frac{G_p(s)}{s}\right\} = \frac{1-e^{-1}}{z-e^{-1}}$$

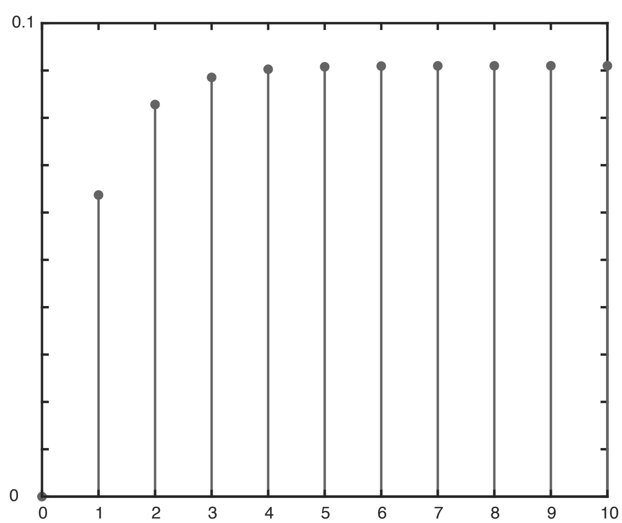
which we already knew from the Lecture Notes 4. Now let's compute the closed-loop PTF, $T(z)$.

$$T(z) = \frac{G_c(z)G(z)}{1+G_c(z)G(z)} = \frac{K(1-e^{-1})}{z+K-(K+1)e^{-1}}$$

Let $\mathbf{K = 0.1}$, then compute the step-response of the closed-loop PTF

$$Y(z) = R(z)T(z) = \frac{0.063\,z}{(z-1)(z-0.3)}$$
$$y[k] = \mathcal{Z}^{-1}[Y(z)] = \left[0.09 - 0.09(0.3)^k\right]u[k]$$

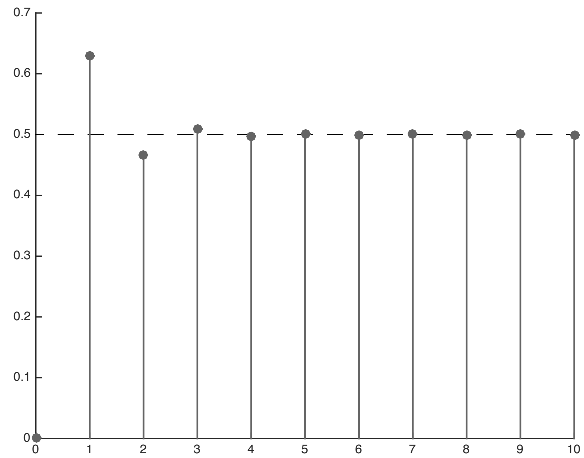
If we plot the step response we obtain the following plot



Now, Let $\mathbf{K = 1}$, then compute the step-response of the closed-loop PTF

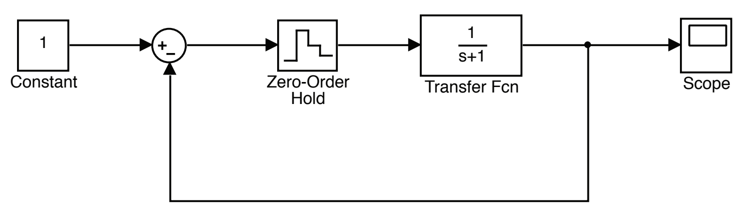
$$Y(z) = R(z)T(z) = \frac{z}{z-1}\frac{0.63}{z+0.26} = \frac{0.63\,z}{(z-1)(z+0.26)}$$
$$y[k] = \mathcal{Z}^{-1}[Y(z)] = \left[0.5 - 0.5(-0.26)^k\right]u[k]$$

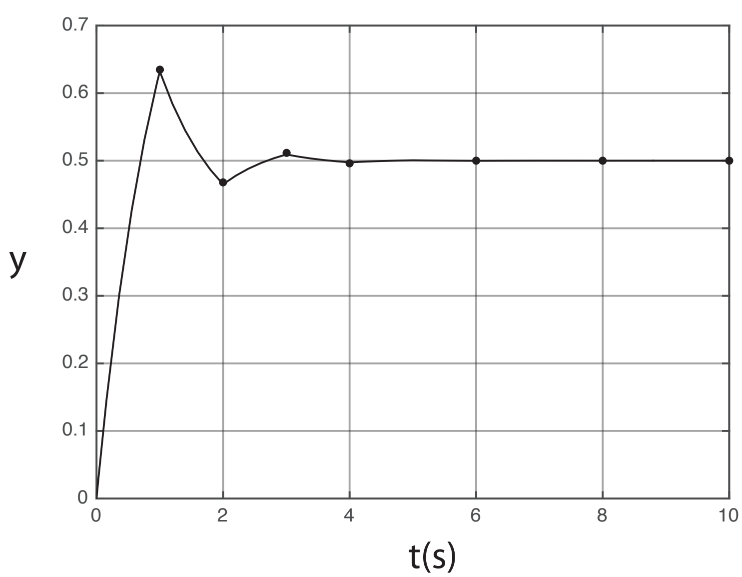
If we plot the step response we obtain the following plot



What about inter-sample behavior?

We can simulate the system and analyze the behavior. The figure below shows the Simulink model as well as the output.

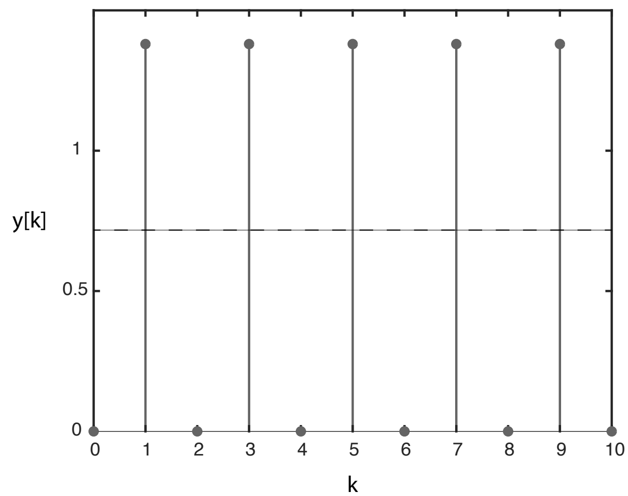




Now let $K = \frac{1+e^{-1}}{1-e^{-1}}$, then $Y(z)$ and $y[k]$ takes the form

$$Y(z) = \frac{1.37z}{(z-1)(z+1)}$$
$$y[k] = 0.69 - 0.69(-1)^n$$

The graph of $y[k]$ is illustrated below. It can be seen that the output shows an oscillatory behavior.



Now let $K = \frac{e^{-1}}{1-e^{-1}}$, then $Y(z)$ and $y[k]$ takes the form

$$Y(z) = \frac{0.37}{z-1}$$
$$y[k] = 0.37u[k-1]$$

Output converges to its steady state value in "finite time" (dead-beat behavior/controller).

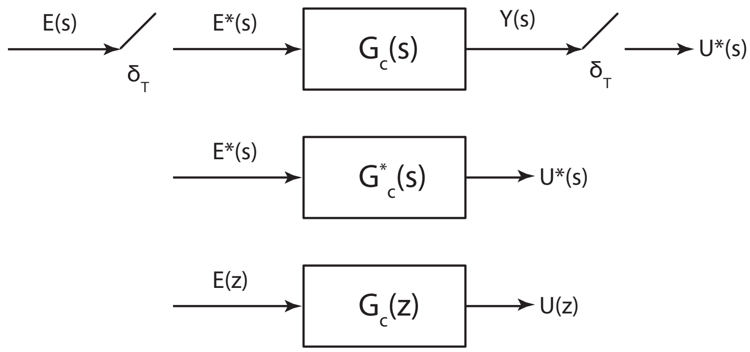
Take home message: It can be seen that even if the plant is a simple first order transfer function, depending on the value of K, we can observe very interesting behavior in the closed-loop DT system.

Pulse Transfer Function of a Digital PID Controller

In this section, we will try to obtain a from for the digital PID controller. The continuous transfer function of a PID is given as

$$G_{PID}(s) = K_P + K_D s + fracK_I s$$

One idea is to start from continuous PID form and then "discretize" it. One way of computing a discrete controller, $G_c(z)$, is using the star operation for discretization. This operation is illustrated in the Figure below.



Based in this approach if possible $G_c(z)$ simply commuted as

$$G_c(z) = \mathcal{Z}\{G_c(s)\}$$

Let's start with PI controller.

Digitization of PI Controller: It is a well known fact that the PI Controller is in the form

$$G_{PI}(s) = K_P + \frac{K_I}{s}$$

The discretization simply gives

$$\begin{aligned} G_{PI}(z) &= \mathcal{Z}\{G_{PI}(s)\} \\ &= K_P + K_I \frac{1}{1 - z^{-1}} \\ &= \frac{(K_I + K_P) + K_P z^{-1}}{1 - z^{-1}} \\ &= \frac{b_0 + b_1 z^{-1}}{1 - z^{-1}} \end{aligned}$$

Now let's discretize PID controller which has the following CT transfer function

$$G_{PI}(s) = K_P + K_D s + \frac{K_I}{s}$$

The problem is $K_D s$ term is non-causal. Let us approximate the effect of $K_D s$ in time domain and then perform a discretization. A causal approximate derivative can be find by computing the backward difference.

$$\frac{dx(t)}{dt} \approx \frac{x(t) - x(t - \Delta t)}{\Delta t}$$

Now let's compute the approximate derivative term at the sampling instants and let $\Delta t = T$ we have

$$\frac{dx(t)}{dt}|_{t=kT} \approx \frac{x(kT) - x((k-1)T)}{T}$$

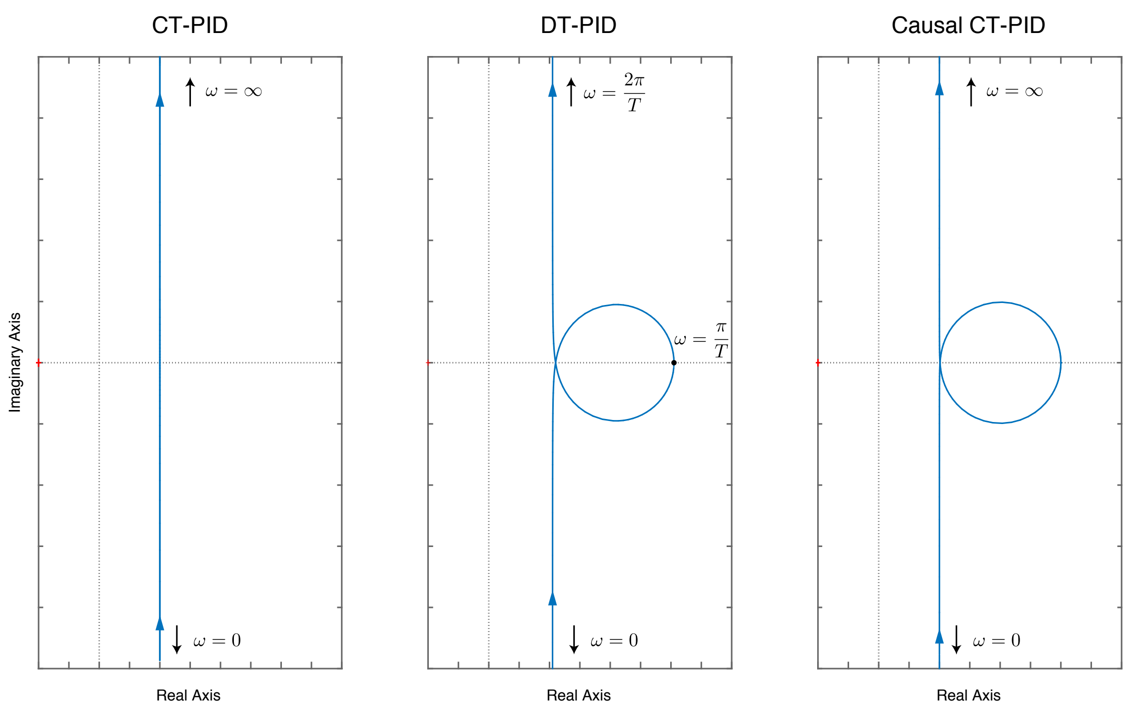
If we take the z-transform we can simply obtain a transfer function for this FIR filter as

$$G_D(z) = \frac{K_D}{T}(1 - z^{-1})$$

Note that instead of K_D/T, we can just use K_D for the gain. If we combine PI and D terms we obtain the following pulse transfer function fir the digital PID controller.

$$\begin{aligned} G_{PID}(z) &= K_P + K_I \frac{1}{1-z^{-1}} + K_D\left(1-z^{-1}\right) \\ &= \frac{K_P + K_D + K_I - (K_P + 2K_D)z^{-1} + K_D z^{-2}}{1-z^{-1}} \\ &= \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1-z^{-1}} \end{aligned}$$

The Figure below illustrates the frequency response characteristics of an ideal CT-PID, a DT-PID, as well as an approximate causal CT-PID controllers. Qualitatively, at low frequencies all controllers behaves similar however for "high" frequencies there are significant differences between the CT-PID and DT-PID (as well as approximate causal CT-PID). Remarkably, for these DT-PID and approximate causal CT-PID controllers the frequency response polar polts are qualitatively similar. This shows that if we choose the right parameters digitization of derivative term has a similar effect as implementing an approximate analog derivative circuit.



Note that frequency response function for CT and DT systems are found by the Fourier (CT or DT) transforms of the impulse response functions, or simply they can be computed from the s-domain or z-domain transfer functions

$$\begin{aligned} \text{CT}: &\quad G_c(s)|_{s=j\omega} = G_c(j\omega) \\ \text{DT}: &\quad G_d(z)|_{z=e^{j\omega}} = G_d(e^{j\omega}) \end{aligned}$$

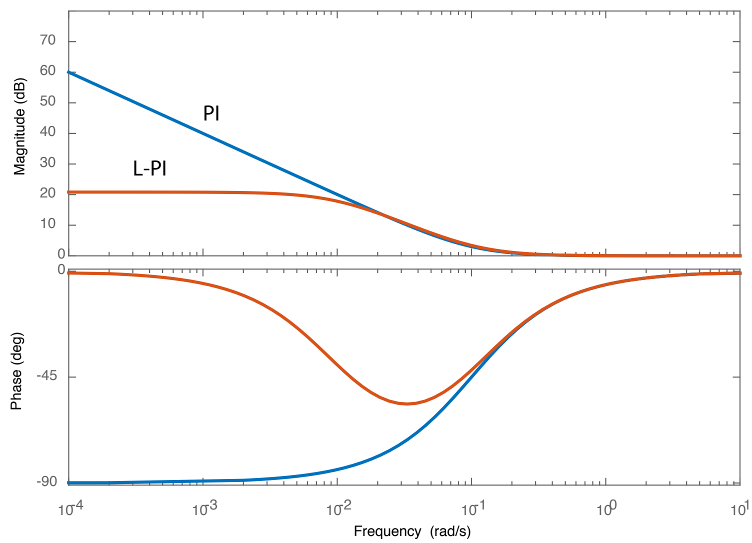
Note that in DT case ω stands for DT frequency. Sometimes $G_d(j\omega)$ or $G_d(\omega)$ used instead of $G_d(e^{j\omega})$. We will cover the Frequency respons later in the class.

PI & PID Controllers with Leaky Integrator

Some times for some practical and other considerations instead of a true integrator (or accumulator) a leaky version is used. A leaky PI controller is in the form

$$\begin{aligned} G_{L-PI}(s) &= K_P + K_I \frac{1}{s+\alpha} \\ &= K_P \frac{s + (\alpha + K_I/K_P)}{s+\alpha} \end{aligned}$$

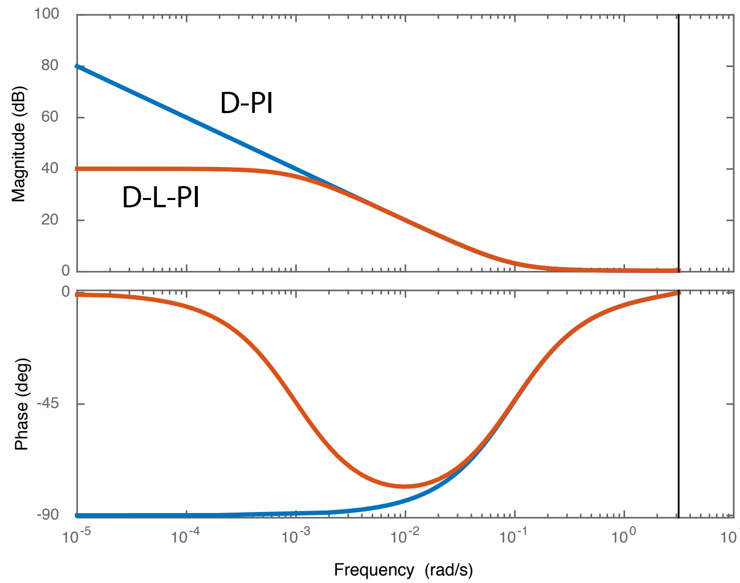
where $\alpha > 0$ and $\alpha \approx 0$ (considering the bandwith of the closed loop system). It can be seen that a leaky-PI controller has the same from with the compensator controller that we covered in 302. If we obscure the frequency response characteristics of both classical and leaky PI controllers, we observe that the behavior is quite different at low frequencies but similar at high frequencies.



If we discretize this CT controller using the star operation approach, we obtain

$$\begin{aligned} G_{L-PI}(z) = \mathcal{Z}\{G_{L-PI}(s)\} &= K_P + \frac{K_I}{1 + e^{-\alpha T} z^{-1}} \\ &= K_P + \frac{K_I}{1+\beta z^{-1}} \\ &= \frac{(K_P + K_I) + K_P \beta z^{-1}}{1+\beta z^{-1}} \\ &= \frac{b_0 + b_1 z^{-1}}{1 + a_1 z^{-1}} \end{aligned}$$

where similar to the CT case, $\beta < 1$ and $\beta \approx 1$. Similar to the CT case, this DT transfer function has one zero and one pole and it has the equivalent from with a DT-compensator controller. The bode plots of DT-PI and DT-Leaky-PI controllers are illustrated in the Figure below. It can be seen that at low frequencies the differences are significant, but at high frequencies the responses between classical and leaky PI controllers are very similar.



One interesting result is that both for classical and leaky cases, CT and DT frequency responses are qualitatively very similar.