MIDDLE EAST TECHNICAL UNIVERSITY

DEPARTMENT OF ELECTRICAL AND ELECTRONICS ENGINEERING

PROCESS CONTROL LABORATORY

EXPERIMENT 4: LUMPED APPROXIMATIONS

I. Objective

In this experiment a thermal diffusion process is investigated and its distributed system model is derived. Temperature variations in a cylindrical solid bar, which is modeled as a distributed system, is observed. Then, an electrical lumped parameter circuit which approximates the dynamics is simulated.

II. Information

<u>Heat Conduction in a Solid:</u>

Consider a slab of solid conducting material of infinite length, as shown in Figure 1.

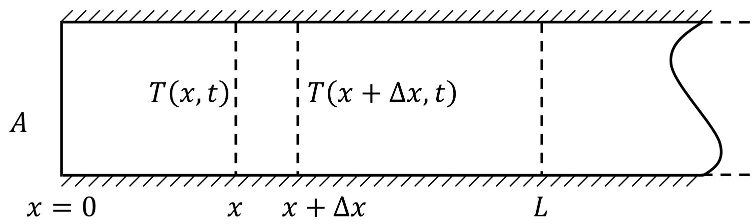

Figure 1. Heat conduction into a solid bar

Let the input to this system be the temperature at the left face, $x = 0$, which is some arbitrary function of time. The output will be the temperature at the position $x = L$ or at an arbitrary position $x < L$. Thermal conductivity, heat capacity, mass density and cross-sectional area of the conduction material are constant and independent of temperature. Assume that the environment temperature is also constant. Initially, the slab is at a uniform steady-state temperature.

We first write an energy balance over a differential length Δx of the slab:

$$\text{Heat Accumulation} = \text{Heat inflow} - \text{Heat outflow}$$

or in mathematical terms

$$\rho A \Delta x c \frac{\partial T(x,t)}{\partial t} = Aq(x,t) - Aq(x+\Delta x,t)$$

where

$T(x,t)$: Temperature in the slab (as a function of position and time) $[^\circ\mathrm{C}]$

$q(x,t)$: Heat flux by conduction (as a function of position and time) $[\mathrm{W/m^2}]$

ρ : Mass density of the conduction material $[\mathrm{kg/m^3}]$

A : Cross-sectional area of the slab $[\mathrm{m^2}]$

c : Heat capacity $[\mathrm{J/kg^\circ C}]$.

The flow of heat by conduction follows Fourier's law:

$$q(x,t) = -\lambda \frac{\partial T(x,t)}{\partial x}$$

where

λ : Thermal conductivity coefficient $[\mathrm{W/m^\circ C}]$.

Combining the energy balance and Fourier's law,

$$\rho A \Delta x c \frac{\partial T(x,t)}{\partial t} = \lambda A \left(-\frac{\partial T(x,t)}{\partial x}\bigg|_{x=x} + \frac{\partial T(x,t)}{\partial x}\bigg|_{x=x+\Delta x} \right).$$

Divide both sides by $\rho A \Delta x c$ to get

$$\frac{\partial T(x,t)}{\partial t} = \frac{\lambda}{\rho c} \frac{\frac{\partial T(x,t)}{\partial x}\Big|_{x=x+\Delta x} - \frac{\partial T(x,t)}{\partial x}\Big|_{x=x}}{\Delta x}.$$

Taking the limit $\Delta x \to 0$ yields the fundamental equation describing conduction in a solid:

$$\frac{\partial T(x,t)}{\partial t} = \alpha \frac{\partial^2 T(x,t)}{\partial x^2}$$

where $\alpha \triangleq \frac{\lambda}{\rho c}$ is defined as the heat conductivity factor.

In order to solve this partial differential equation, we need to specify an initial condition and some boundary conditions. Since, initially, the system is at rest, the initial condition is:

$$T(x,0) = T_{\mathrm{e}} \quad \text{for } x > 0$$

where T_{e} is the temperature of the environment.

We are driving with an input function $V(t)$ on the left. That is

$$T(0,t) = V(t)\,.$$

Also for a finite time, the temperature on the far right will not be effected. That is,

$$\lim_{x\to\infty} T(x,t) = T_{\mathrm{e}}\,.$$

<u>Solution of the Heat Conduction Equation:</u>

A little bit simpler problem is when we define $\eta(x,t) \triangleq T(x,t) - T_{\mathrm{e}}$. In this case, since $\eta(x,0) = 0$, we can talk about the transfer function of the system. The problem involving η becomes

$$\frac{\partial \eta(x,t)}{\partial t} = \alpha \frac{\partial^2 \eta(x,t)}{\partial x^2}$$

with initial and boundary conditions as

$$\eta(x,0) = 0$$

$$\eta(0,t) = \gamma(t)$$

$$\lim_{x\to\infty} \eta(x,t) = 0$$

where $\gamma(t) \triangleq V(t) - T_{\mathrm{e}}$.

Taking the Laplace transform of both sides of the partial differential equation

$$s\tilde{T}(x,s) = \alpha \frac{\partial^2 \tilde{T}(x,s)}{\partial x^2}$$

where

$$\tilde{T}(x,s) \triangleq \int_0^\infty \eta(x,t)e^{-st}\mathrm{d}t$$

is the Laplace transform of $\eta(x,t)$ with respect to its second argument. Now, if you were to solve the above ordinary DE in Laplace domain by letting a candidate solution as e^{xb}, you can find that its homogeneous solution is of the following form.

$$\tilde{T}(x,s) = A_1 e^{-x\sqrt{\frac{s}{\alpha}}} + A_2 e^{x\sqrt{\frac{s}{\alpha}}}$$

where A_1 and A_2 are constants to be determined. Now, the initial infinite slab assumption implies that temperature should not change at sufficiently large values of x (that is, $\lim_{x\to\infty} \eta(x,t) = 0$). This requires its Laplace transform to obey a similar boundary condition: $\lim_{x\to\infty} \tilde{T}(x,s) = 0$, which gives $A_2 = 0$. Moreover taking $x = 0$, we can see that $A_1 = \tilde{T}(0,s)$.

Therefore, the following transfer function is obtained:

$$G(s) = \frac{\tilde{T}(x,s)}{\tilde{T}(0,s)} = e^{-x\sqrt{\frac{s}{\alpha}}}.$$

With an arbitrary input at one end $\eta(0,t) = \gamma(t)$, we have

$$\tilde{T}(x,s) = \tilde{V}(s)e^{-x\sqrt{\frac{s}{\alpha}}}$$

where $\tilde{V}(s)$ is the Laplace transform of $\gamma(t)$.

<u>Step response:</u>

For a given additional temperature M, if $\eta(0,t) = Mu(t)$ where $u(t)$ is the unit step function, we have

$$\tilde{T}(x,s) = \frac{M}{s}e^{-x\sqrt{\frac{s}{\alpha}}}.$$

In time domain (to keep things neat, proof of this is given in the appendix),

$$\eta(x,t) = M\,\mathrm{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right)$$

where erfc is the complementary error function defined as

$$\mathrm{erfc}\,x \triangleq 1 - \frac{2}{\sqrt{\pi}}\int_0^x e^{-u^2}\mathrm{d}u = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-u^2}\mathrm{d}u.$$

For $M = 1$, a plot of $\eta(x,t)$ versus the normalized time variable $\theta \triangleq (\alpha/x^2)t$ is shown in Figure 2 for a fixed value of x.

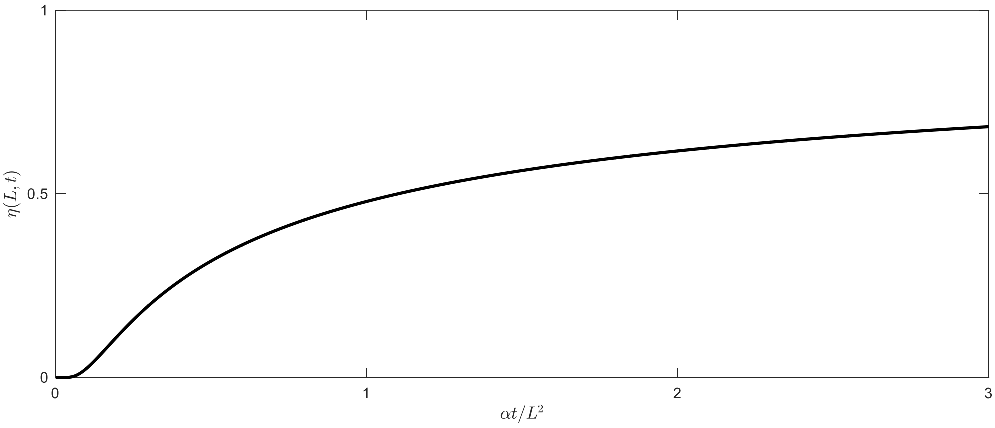

Figure 2. Response of temperature to a unit step change at $x = 0$

Reverting back to actual temperature,

$$T(x,t) = T_\mathrm{e} + (M - T_\mathrm{e})\,\mathrm{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right) \quad \text{for } x > 0 \text{ and } t > 0\,.$$

<u>Lumped approximation</u>:

For this problem we have an analytical solution available for $T(x,t)$. However, for most of the time, for other processes, we cannot obtain an analytical solution. Therefore, we rely on approximate numerical solutions.

Here we approximate the partial differential equation and turn it into a set of ordinary differential equations.

For an infinitesimal Δx, we can write

$$\frac{\partial^2 T(x,t)}{\partial x^2} = \frac{T(x-\Delta x,t) - 2T(x,t) + T(x+\Delta x,t)}{(\Delta x)^2}$$

and therefore

$$\frac{1}{\alpha}\frac{\partial T(x,t)}{\partial t} = \frac{T(x-\Delta x,t) - 2T(x,t) + T(x+\Delta x,t)}{(\Delta x)^2}\,.$$

Let L denote the location x where you will be taking your measurements. Let's divide the length L into n lumped elements. The temperature distribution inside a lumped element is assumed to be uniform. For a given n we set $\Delta x = L/n$ and for $r = 0, 1, \ldots, n$ we define $T_r(t) \triangleq T(r\Delta x, t)$. Dividing the length L into n lumped elements means that we assume, for $r = 1, 2, \ldots, n$,

$$T(x,t) = T_r(t) \quad \text{for} \quad (r-1)\Delta x < x \leq r\Delta x\,.$$

Then we have the following set of ordinary differential equations

$$\frac{1}{\alpha}\frac{\mathrm{d}T_r(t)}{\mathrm{d}t} = \frac{T_{r-1}(t) - 2T_r(t) + T_{r+1}(t)}{(\Delta x)^2} \quad \text{for } r = 1, 2, \ldots, n\,.$$

The above set then have $n + 1$ unknowns in n equations. Here $T_0(t)$ is defined by out input and is not counted as an unknown. If we assume $T_{n+1} = T_n$ then we can solve for the unknowns $T_1(t), T_2(t), \ldots, T_n(t)$.

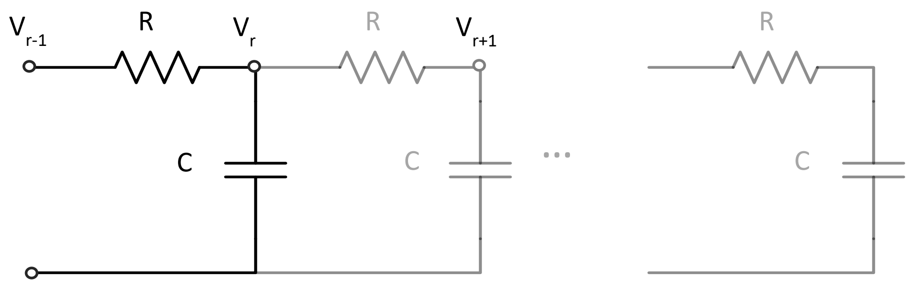

Figure 3. Simulation of the lumped approximate model

One element can be simulated by using an RC circuit as shown in Figure 3. The input-output relation of this circuit is as the following:

$$RC\frac{\mathrm{d}V_r(t)}{\mathrm{d}t} = V_{r-1}(t) - 2V_r(t) + V_{r+1}(t)\,.$$

Thus the solid bar can be modeled by using infinite number of cascaded RC stages with infinitesimal time constants $\tau = (\Delta x)^2/\alpha$.

Since this process is a slow one, a proper time scaling will be useful for the electrical circuit. The time constants of the lumped approximations are chosen to be smaller than the actual time constant.

III. Preliminary Work

1. Find and plot the variation of temperature $T(L,t)$ versus t for a unit step change in temperature at $x = 0$. You may use Table 1 for the complementary error function. You may also use MATLAB in which erfc is implemented as built-in. The necessary data for computations are given below.

Length of the aluminum cylinder	$L = 5\text{cm}$
Cross-sectional area	$A = 4.9\text{cm}^2$
Density of aluminum	$\rho = 2.70\,\text{gr/cm}^3$
Thermal conductivity	$k = 0.497\,\text{cal/s.cm.°C}$
Heat capacity of aluminum	$c = 0.215\,\text{cal/gr.°C}$

Table 1. Complementary error function evaluated at some values

x	$\mathrm{erfc}\left(\frac{1}{2\sqrt{x}}\right)$	x	$\mathrm{erfc}\left(\frac{1}{2\sqrt{x}}\right)$
0	0	1	0.4795
0.1	0.0253	1.5	0.5637
0.2	0.1138	2	0.6171
0.3	0.1967	2.5	0.6547
0.4	0.2636	3	0.6831
0.5	0.3173	3.5	0.7055

2. For $n = 1, 2, 3, 5$ obtain the lumped parameter model and submit the electrical diagram indicating the values of all component values. Let the time constant of the electrical model be 100,000 times smaller than that of the system.

3. Using the lumped parameter model, for each n, plot $T(\Delta x, t)$ vs t for a unit step change at $x = 0$ in a same figure together with the analytical response. Comment on the approximation.

IV. Experimental Procedure

1. To improve your general understanding of background information, find $G(s) = \frac{\tilde{T}(x,s)}{\tilde{T}(0,s)}$, where $\tilde{T}(x,s) = \mathcal{L}\{\eta(x,t)\}$, starting from the equation below. Use your own words for the explanations.

$$\frac{\partial \eta(x,t)}{\partial t} = \alpha \frac{\partial^2 \eta(x,t)}{\partial x^2}$$

2. Consider the experimental setup shown in Figure 4. Turn on the hot plate. Before placing the aluminum cylinder onto the hot plate, wait until the temperature of the hot plate surface reaches a steady state value of 60°C. This steady state value can be adjusted using the knob on the front panel of the heater and it will be used as the unit step temperature input.

The two thermocouples are placed approximately at 5 and 10 centimeters away from the bottom surface. Connect their outputs to the recorder. Rotate the knob of the recorder to make simultaneous temperature measurements from both sensors, as T_1 and T_2. Start the data logging software named Bs82-**52x**____. Either from the menu bar or push buttons below it, click the Link button to connect to the multimeter. This should bring up live measurements to the window Digital Meter. From the menu bar of the graphical recorder window, click on Acquisition→Set interval to set acquisition period as 5 seconds; and click on Acquisition→Start to start recording the measurements. Place the bottom surface of the brass cylinder onto the hot plate immediately and record the temperature variation at the surface until the steady state is reached. Continue with the other steps of the experimental procedure as you collect data. After reaching a steady state value, click on Acquisition→Stop. Again from the menu bar of the graphical recorder window click on File→Export. This should bring up a pop-up window. Click on Browse, navigate to desktop, enter a filename and click Save, bringing you back to the pop up window. Choose "All Pages" setting under page scope and "Text/Data Only" under export. Click on Export button. Take a copy of this file. To visualize one set of data obtained, you may click on View→Fit all data under the menu bar of the graphical recorder.

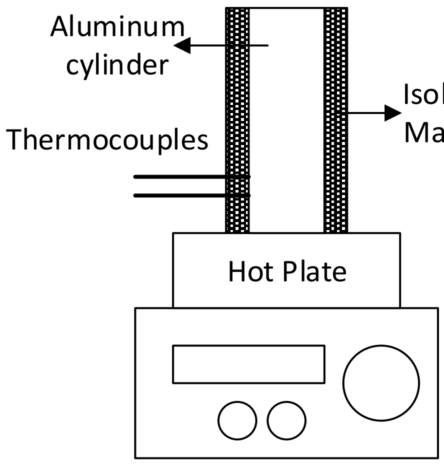

Figure 4. Experimental setup

3. In this part you will simulate the first, second, third and fifth order RC circuits using Simulink. Have a look at the appendix for transfer function representation of interacting RC circuits. Apply a step voltage input at $t = 0$ and measure the voltage V_1. Also add a function block from Simulink library called MATLAB function, which lets you enter custom equations. Fill it as in Figure 7, and finally give current simulation time as its input with the function block called Clock. For a better visualization, connect the outputs of each measurement to a single scope. Also add a legend with proper labeling.

Set the simulation time as 1e-3 and set the maximum solver step size as 1e-6 from Simulation→Configuration Parameters→Solver.

4. Repeat part (2) for V_2 .
5. Repeat part (2) for V_n .

V. Results and Discussion

1. Plot the temperature data you have collected from the setup. If you intend to use MATLAB, have a look at the experimental procedure of Experiment 1 for the steps of importation. Does the actual response curve fit the theoretical one that was found in preliminary work part 1? If not, explain any cause of differences.
2. You have used n number of interacting RC stages to approximate this system. Discuss why non-interacting RC stages would not model this system as well as the interacting one. Think of the analogy between electrical circuits & heat conduction, and use this analogy as a basis for your discussion.
3. What is the effect of the order of the lumped model on the degree of accuracy of approximation?
4. What is the effect of taking measurements near $x = L$? What is the effect of taking measurements far from $x = L$? Discuss the reasons.

VI. Conclusions

Write down your general conclusions and comments on this experiment.

VII. References

Donald R. Coughanowr, Steven E. LeBlanc. Process Systems Analysis and Control, Third Edition. McGraw-Hill 2009, Chapter 20.3 page 458.

VIII. Appendix

a) Relative Temperature $\eta(x,t)$ & Its Laplace Transform

From the previous explanations, we have the following two relationships.

$$\eta(x,t) = M\,\mathrm{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right) = \frac{2}{\sqrt{\pi}}\int_{x/2\sqrt{\alpha t}}^{\infty} e^{-u^2}\mathrm{du}$$

$$\tilde{T}(x,s) = \mathcal{L}\{\eta(x,t)\} = \int_0^{\infty} \eta(x,t)e^{-st}dt$$

Let $l = \frac{x}{2\sqrt{\alpha}}$. Then, we have the following set of equations. While proceeding to the second line, note that integration order has been changed with $u = \frac{l}{\sqrt{t}}$ or equivalently $t = \frac{l^2}{u^2}$. The last integral involves certain manipulations, which will be shown subsequently.

$$\tilde{T}(x,s) = \mathcal{L}\left\{M\,\mathrm{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right)\right\} = \mathcal{L}\left\{M\,\mathrm{erfc}\left(\frac{l}{\sqrt{t}}\right)\right\} = \mathcal{L}\left\{\frac{2M}{\sqrt{\pi}}\int_{\frac{l}{\sqrt{t}}}^{\infty} e^{-u^2}\mathrm{du}\right\} = \frac{2M}{\sqrt{\pi}}\int_0^{\infty} e^{-st}\int_{\frac{l}{\sqrt{t}}}^{\infty} e^{-u^2}\mathrm{du}\,dt$$

$$= \frac{2M}{\sqrt{\pi}}\int_0^{\infty} e^{-u^2}\int_{\frac{l^2}{u^2}}^{\infty} e^{-st}\mathrm{dt}\,du = \frac{2M}{s\sqrt{\pi}}\int_0^{\infty} e^{-u^2-\frac{l^2 s}{u^2}}du = \frac{2M}{s\sqrt{\pi}}\frac{\sqrt{\pi}}{2}e^{-2l\sqrt{s}} = \frac{M}{s}e^{-x\sqrt{\frac{s}{\alpha}}}.$$

What remains now is to show that $\int_0^\infty e^{-u^2-\frac{l^2 s}{u^2}}du=\frac{\sqrt{\pi}}{2}e^{-2l\sqrt{s}}$.

$$\int_0^\infty e^{-u^2-\frac{l^2 s}{u^2}}du = e^{-2l\sqrt{s}}\int_0^\infty e^{-\left(u-\frac{l\sqrt{s}}{u}\right)^2}du = e^{-2l\sqrt{s}}I$$

Letting $z = l\sqrt{s}/u$, $dz = -\frac{l\sqrt{s}}{u^2}du = -\frac{z^2}{l\sqrt{s}}du$, the integral I becomes

$$I = -\int_\infty^0 \frac{l\sqrt{s}}{z^2}e^{-\left(z-\frac{l\sqrt{s}}{z}\right)^2}dz = \int_0^\infty \frac{l\sqrt{s}}{z^2}e^{-(z-\frac{l\sqrt{s}}{z})^2}dz.$$

Summing the two under a common integral variable z yields $2I$ as shown below. Using the final change of variables as $v = z - \frac{l\sqrt{s}}{z}$, $dv = \left(1+\frac{l\sqrt{s}}{z^2}\right)dz$,

$$2I = \int_0^\infty \left(1+\frac{l\sqrt{s}}{z^2}\right)e^{-(z-\frac{l\sqrt{s}}{z})^2}dz = \int_{-\infty}^\infty e^{-v^2}dv = \sqrt{\pi}$$

where we have used the well-known Gaussian integral at the last stage. Then, it is obvious that

$$\int_0^\infty e^{-u^2-\frac{l^2 s}{u^2}}du = Ie^{-2l\sqrt{s}} = \frac{\sqrt{\pi}}{2}e^{-2l\sqrt{s}}$$

which completes the proof.

b) Simulation Model

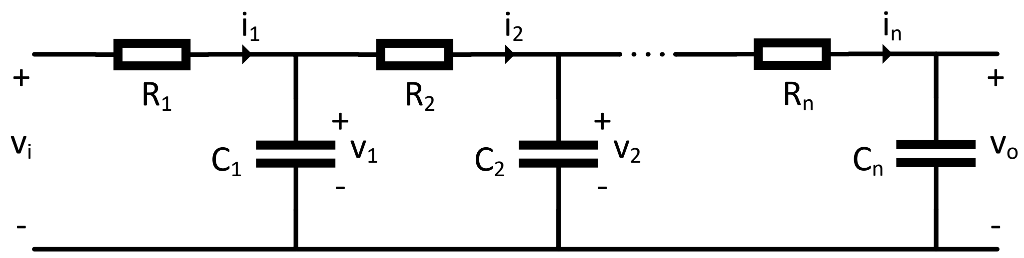

Figure 5. nth order RC circuit

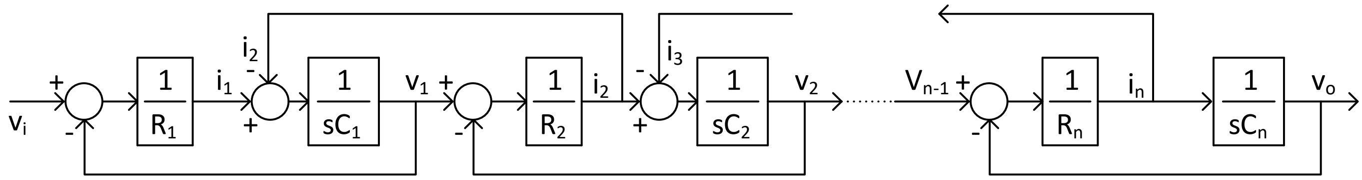

Figure 6. Transfer function representation of nth order RC circuit

Consider the RC circuit shown in Figure 5. It can be simulated with basic transfer functions as in Figure 6. $v_i - v_1$ is the voltage across R_1, dividing it by R_1 gives the current through the resistance. When you apply KCL at node 1, current through C_1 is obtained as $i_1 - i_2$, and when that is integrated and is scaled by $1/C_1$ the voltage across C_1 is obtained as v_1. This is continued recursively until the last step. Note that if you were to remove current feedbacks, the case would be equivalent to placing buffers between stages of RC network, where the transfer function of the overall system would be equal to the multiplication of each individual transfer function.

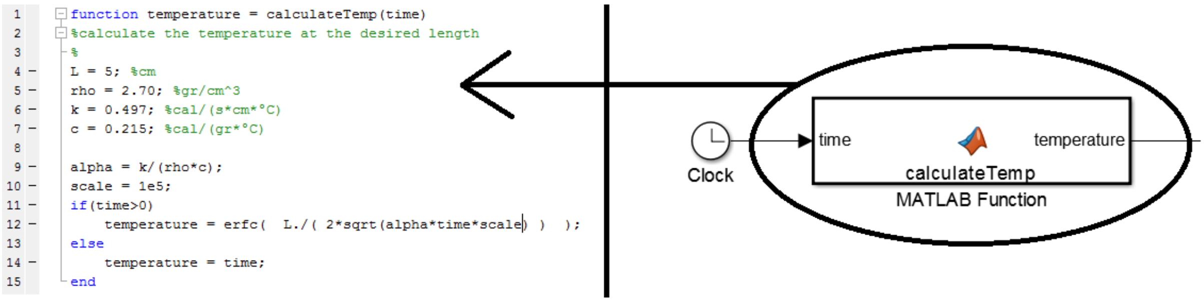

Figure 7. Custom function to calculate temperature