EE402 Discrete Time Systems
MP-3

1. Let us define;

$$G_1(s) = \frac{s}{s^2 - 1}$$
$$G_2(s) = \frac{1}{s}$$
$$G_{ZOH}(s) = \frac{1 - z^{-1}}{s}$$
$$G_X(s) = G_{ZOH}(s)G_1(s)$$
$$G_Y(s) = G_{ZOH}(s)G_1(s)G_2(s)$$

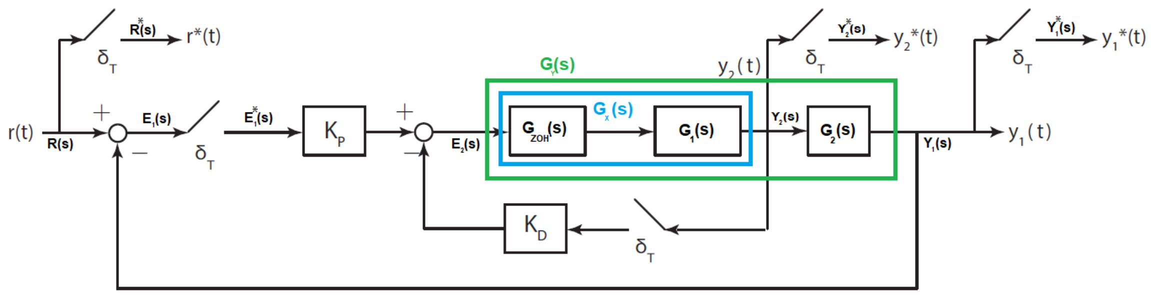


Figure 1: Block Diagram Representation of the System

(a)

$$E(s) = R(s) - Y_1(s)$$
$$E^*(s) = R^*(s) - Y_1^*(s)$$

$$Y_1(s) = E^*(s)K_pG_Y(s)$$
$$Y_1^*(s) = E^*(s)K_pG_Y^*(s)$$

$$\frac{Y_1^*(s)}{R^*(s)} = K_pG_Y^*(s)$$

$$E^*(s) = R^*(s) - E^*(s)K_pG_Y^*(s)$$

$$\frac{E^*(s)}{R^*(s)} = \frac{1}{1 + K_pG_Y^*(s)}$$

$$\frac{Y_1^*(s)}{R^*(s)} = \frac{E^*(s)}{R^*(s)}\frac{Y_1^*(s)}{E^*(s)} = \frac{K_pG_Y^*(s)}{1 + K_pG_Y^*(s)}$$

Thus, with $s = \frac{1}{T}ln(z)$

$$\boxed{\frac{Y_1(z)}{R(z)} = \frac{K_p G_Y(z)}{1 + K_p G_Y(z)}}$$

$$Y_2(s) = E^*(s) K_p G_X(s)$$
$$Y_2^*(s) = E^*(s) K_p G_X^*(s)$$

$$\frac{Y_2^*(s)}{R^*(s)} = \frac{E^*(s)}{R^*(s)} \frac{Y_2^*(s)}{E^*(s)} = \frac{K_p G_X^*(s)}{1 + K_p G_Y^*(s)}$$

Thus, with $s = \frac{1}{T}ln(z)$

$$\boxed{\frac{Y_2(z)}{R(z)} = \frac{K_p G_X(z)}{1 + K_p G_Y(z)}}$$

Let us now find $G_X(z)$ and $G_Y(s)$ to find the pulse transfer functions,

$$G_Y(z) = \mathcal{Z}\{G_Y(s)\} = \mathcal{Z}\{\mathcal{L}^{-1}\{G_{ZOH}(s)G_1(s)G_2(s)\}^*\}$$

with $G_{ZOH}(s) = \dfrac{1-e^{-Ts}}{s}$ and $G_1(s)G_2(s) = \dfrac{1}{s^2-1}$ and $\boxed{T = 0.5}$

$$\boxed{G_Y(z) = \left((1-z^{-1})\mathcal{Z}\{\frac{G_1(s)G_2(s)}{s}\}\right)}$$

$$\begin{aligned}
\mathcal{Z}\left\{\frac{G_1(s)G_2(s)}{s}\right\} &= \mathcal{Z}\left\{\frac{1}{s(s^2-1)}\right\} \\
&= \mathcal{Z}\left\{\frac{-1}{s} + \frac{1}{2(s+1)} + \frac{1}{2(s-1)}\right\} \\
&= \frac{-1}{1-z^{-1}} + \frac{1}{2(1-e^{0.5}z^{-1})} + \frac{1}{2(1-e^{-0.5}z^{-1})}
\end{aligned}$$

$$\begin{aligned}
G_Y(z) &= -1 + \frac{1-z^{-1}}{2(1-e^{0.5}z^{-1})} + \frac{1-z^{-1}}{2(1-e^{-0.5}z^{-1})} \\
&= \frac{-2(1-e^{0.5}z^{-1})(1-e^{-0.5}z^{-1}) + (1-z^{-1})(1-e^{-0.5}z^{-1})}{2(1-e^{0.5}z^{-1})(1-e^{-0.5}z^{-1})} \\
&+ \frac{(1-z^{-1})(1-e^{0.5}z^{-1})}{2(1-e^{0.5}z^{-1})(1-e^{-0.5}z^{-1})} \\
G_Y(z) &= \frac{(e^{0.5}+e^{-0.5})z^{-1} + (e^{0.5}+e^{-0.5})z^{-2}}{2(1-(e^{0.5}+e^{-0.5})z^{-1}+z^{-2})}
\end{aligned}$$

$$\begin{aligned}
G_Y(z) &= \frac{\left[e^{0.5}+e^{-0.5}\right](z^{-1}+z^{-2})}{2(1-(e^{0.5}+e^{-0.5})z^{-1}+z^{-2})} \\
G_Y(z) &\approx \frac{2.255(z^{-1}+z^{-2})}{2(1-2.255z^{-1}+z^{-2})}
\end{aligned}$$

$$\boxed{G_Y(z) \approx \frac{1.1275(z+1)}{z^2-2.255z+1}}$$

$$\boxed{G_X(z) = \left((1-z^{-1})\mathcal{Z}\{\frac{G_1(s)}{s}\}\right)}$$

$$\begin{aligned}
\mathcal{Z}\left\{\frac{G_1(s)}{s}\right\} &= \mathcal{Z}\left\{\frac{1}{s^2-1}\right\} \\
&= \mathcal{Z}\left\{\frac{1}{s-1} - \frac{1}{s+1}\right\} \\
&= \frac{1}{1-e^{0.5}z^{-1}} - \frac{1}{1-e^{-0.5}z^{-1}} = \frac{\left[e^{0.5}-e^{-0.5}\right]z^{-1}}{(1-e^{0.5}z^{-1})(1-e^{-0.5}z^{-1})} \\
&= \frac{\left[e^{0.5}-e^{-0.5}\right]z^{-1}}{1-\left[e^{0.5}+e^{-0.5}\right]z^{-1}+z^{-2}}
\end{aligned}$$

$$\begin{aligned}
G_X(z) &= \frac{\left[e^{0.5}-e^{-0.5}\right]z^{-1}\left(1-z^{-1}\right)}{1-\left[e^{0.5}+e^{-0.5}\right]z^{-1}+z^{-2}} \\
&= \frac{\left[e^{0.5}-e^{-0.5}\right]\left(z^{-1}-z^{-2}\right)}{1-\left[e^{0.5}+e^{-0.5}\right]z^{-1}+z^{-2}} \\
&= \frac{\left[e^{0.5}-e^{-0.5}\right]\left(z-1\right)}{z^2-\left[e^{0.5}+e^{-0.5}\right]z+1}
\end{aligned}$$

$$\boxed{G_X(z) \approx \frac{1.04\,(z-1)}{z^2-2.255z+1}}$$

Let us now put what we have found into pulse transfer function form.

$$\begin{aligned}\frac{Y_1(z)}{R(z)} &= \frac{K_p G_Y(z)}{1+K_p G_Y(z)} \\ &= \frac{K_p\left(\dfrac{1.1275(z+1)}{z^2-2.255z+1}\right)}{1+K_p\left(\dfrac{1.1275(z+1)}{z^2-2.255z+1}\right)} \\ &= \frac{1.1275K_p(z+1)}{z^2-2.255z+1+K_p(1.1275(z+1))}\end{aligned}$$

$$\boxed{\frac{Y_1(z)}{R(z)} = \frac{1.1275K_p(z+1)}{z^2+(1.1275K_p-2.255)z+(1+1.1275K_p)}}$$

$$\begin{aligned}\frac{Y_2(z)}{R(z)} &= \frac{K_p G_X(z)}{1+K_p G_Y(z)} \\ &= \frac{K_p\left(\dfrac{1.04\,(z-1)}{z^2-2.255z+1}\right)}{1+K_p\left(\dfrac{1.1275(z+1)}{z^2-2.255z+1}\right)} \\ &= \frac{1.04K_p(z-1)}{z^2-2.255z+1+1.1275K_p(z+1)}\end{aligned}$$

$$\boxed{\frac{Y_2(z)}{R(z)} = \frac{1.04K_p1(z-1)}{z^2+(1.1275K_p-2.255)z+(1+1.1275K_p)}}$$

(b) To ensure stability conditions for K_p, Let us use Jury Conditions;

$$D(s) = z^2 + (1.1275K_p - 2.255)z + (1 + 1.1275K_p)$$

With coefficients,

$$\boxed{a_0 = 1}\ ,\ \boxed{a_1 = 1.1275K_p - 2.255}\ ,\ \boxed{a_2 = 1 + 1.1275K_p}$$

The characteristic polynomial $D(s)$ should satisfy the following conditions according to Jury stability conditions;

- $a_0 \ > \ |a_2|$

$$1 \ > \ 1 + 1.1275K_p > -1$$
$$\boxed{0 > K_p > -1.7736}$$

- $D(1) > 0$

$$1 + 1.1275K_p - 2.255 + 1 + 1.1275K_p > 0$$
$$2.225K_p - 1.1275 > 0$$
$$\boxed{K_p > 0.4}$$

- $D(-1) > 0$

$$1 - 1.1275K_p + 2.255 + 1 + 1.1275K_p > 0$$
$$\boxed{2.1275 > 0}$$

It can be concluded that there are no K_p value that makes the system stable.

(c) Given that $K_p = 2$, the unit step response can be calculated as follows;

$$R(z) = \frac{1}{1 - z^{-1}} = \frac{z}{z-1}$$

$$\begin{aligned} Y_1(z) &= \frac{1.1275K_p(z+1)}{z^2 - 2.255z + 1 + 1.1275K_p(z+1)} R(z) \\ &= \frac{2.225(z^2+z)}{(z^2+3.225)(z-1)} \end{aligned}$$

$$\boxed{Y_1(z) = \frac{2.225(z^2+z)}{(z^2+3.225)(z-1)}}$$

$$\boxed{y_1(t) = \mathcal{Z}^{-1}\{Y_1(z)\}}$$

$$\begin{aligned} Y_2(z) &= \frac{1.04K_p(z-1)}{z^2 + (1.1275K_p - 2.255)z + (1 + 1.1275K_p)} R(z) \\ &= \frac{2.08z(z-1)}{(z^2+3.225)(z-1)} \\ &= \frac{2.08z}{z^2+3.225} \end{aligned}$$

$$\boxed{Y_2(z) = \frac{2.08z}{z^2+3.225}}$$

$$\boxed{y_2(t) = \mathcal{Z}^{-1}\{Y_2(z)\}}$$

(d) The *Figures 2 , 3* shows the step responses of $y_1(t)$ and $y_2(t)$ respectively. The source code for that operation can be seen at **Appendix A**.

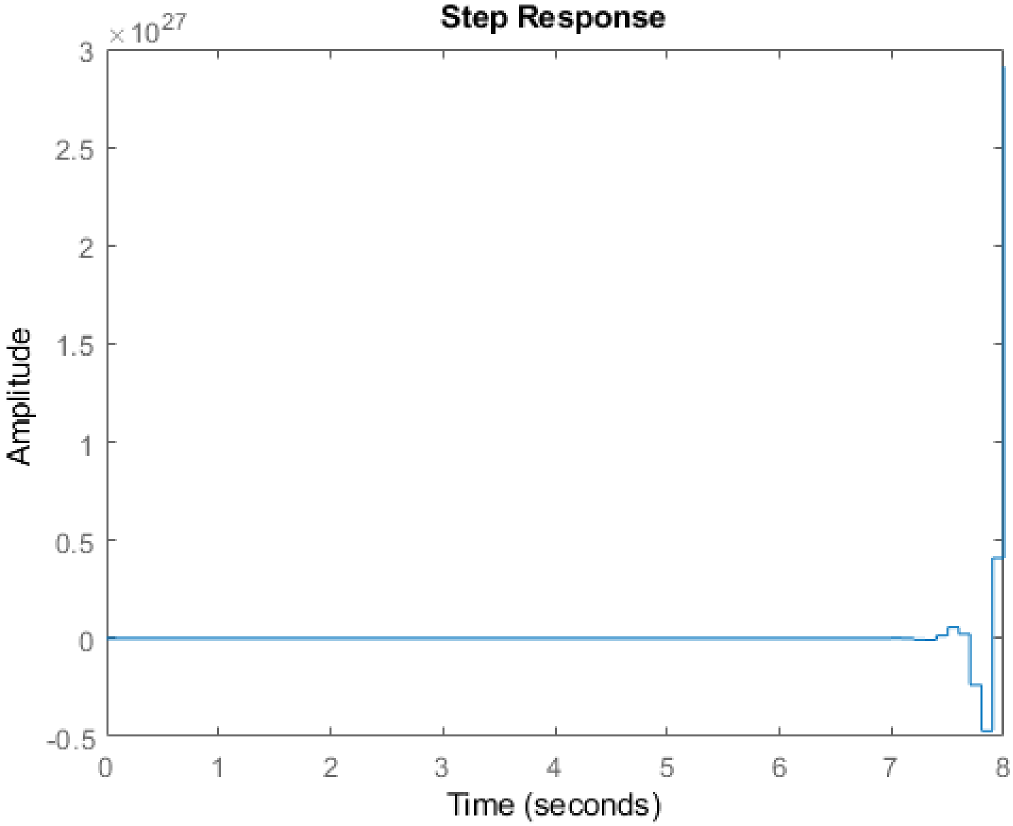


Figure 2: Step Response for the $y_1(t)$ with $K_P = 2$ and $K_D = 0$

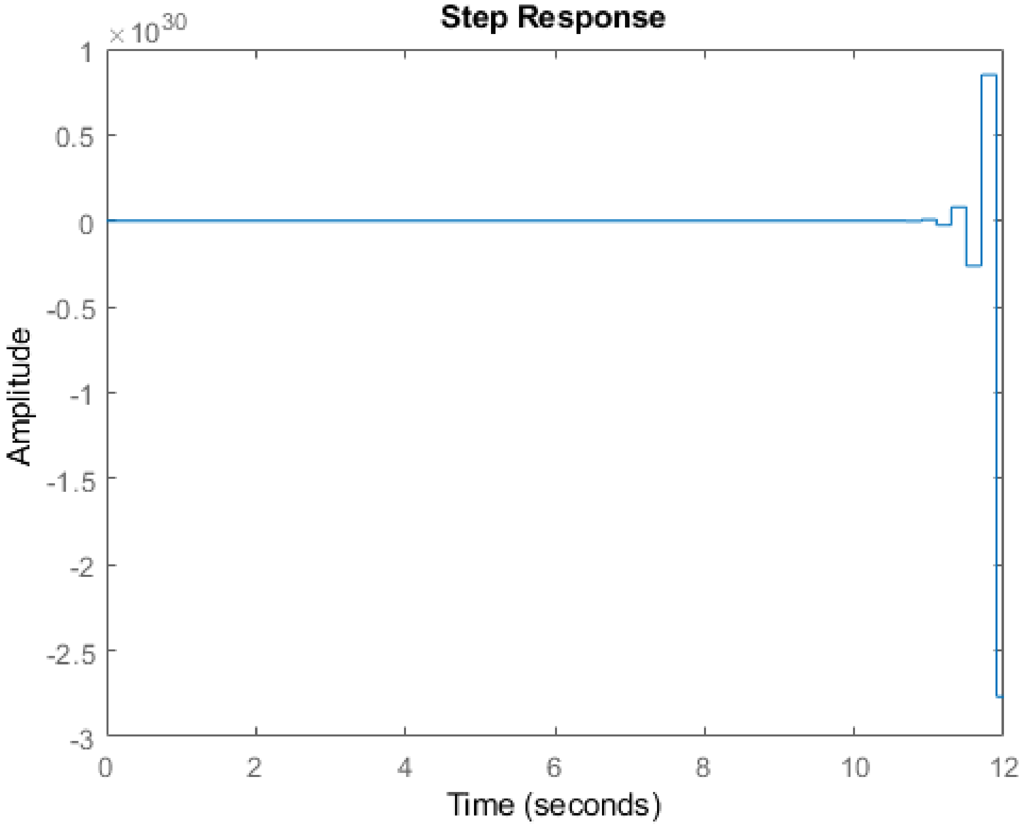


Figure 3: Step Response for the $y_2(t)$ with $K_P = 2$ and $K_D = 0$

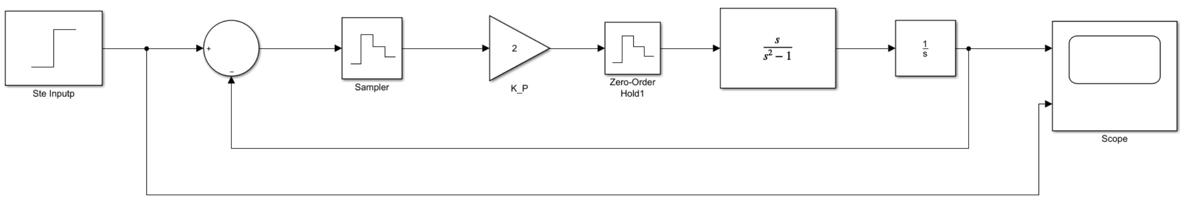
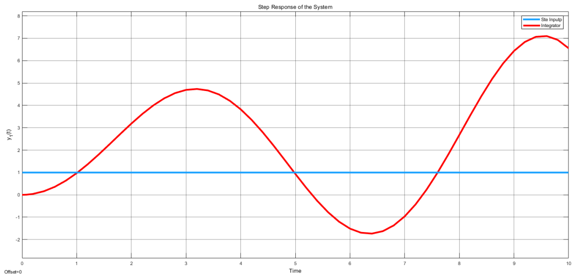
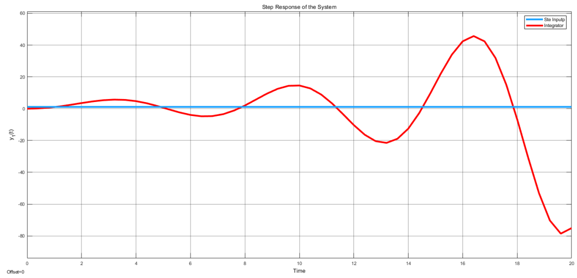


Figure 4: Simulink Model for the given system with $K_P = 2$ and $K_D = 0$

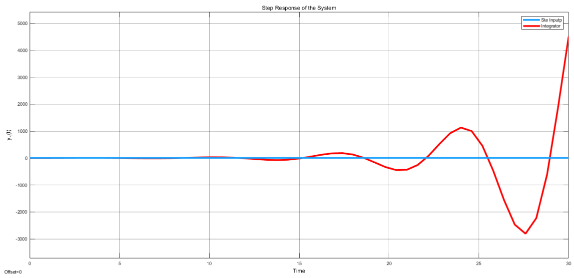
The *Figure 5* shows the step response of the given system with simulink. The Simulink model can be seen at *Figure 4*.



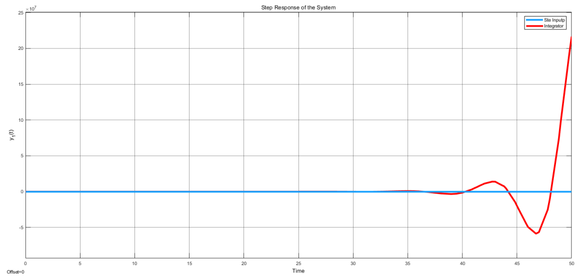
(a) Step Response of the given system till $t = 10\ sec$



(b) Step Response of the given system till $t = 20\ sec$



(c) Step Response of the given system till $t = 30\ sec$



(d) Step Response of the given system till $t = 50\ sec$

Figure 5: Step Responses of the given system with varying end times

It can be observed from the figures also that the system shows very unstable behaviour with given K_p value.

(e) Let us analyse the system now with a $K_d \neq 0$

$$\begin{aligned} E(s) &= R(s) - Y_1(s) \\ E^*(s) &= R^*(s) - Y_1^*(s) \end{aligned}$$

$$\begin{aligned} Y_2(s) &= \left[E^*(s)K_p - K_D Y_2^*(s)\right] G_X(s) \\ Y_2^*(s) &= \left[E^*(s)K_p - K_D Y_2^*(s)\right] G_X^*(s) \end{aligned}$$

$$\frac{Y_2^*(s)}{E^*(s)} = \frac{K_p G_X^*(s)}{1 + K_D G_X^*(s)}$$

$$\begin{aligned} Y_1(s) &= \left[E^*(s)K_p - K_D Y_2^*(s)\right] G_Y(s) \\ Y_1^*(s) &= \left[E^*(s)K_p - K_D Y_2^*(s)\right] G_Y^*(s) \\ &= E^*(s)\left[K_p - K_D \frac{K_p G_X^*(s)}{1 + K_D G_X^*(s)}\right] G_Y^*(s) \\ &= E^*(s)\left[\frac{K_p}{1 + K_D G_X^*(s)}\right] G_Y^*(s) \end{aligned}$$

$$\frac{Y_1^*(s)}{E^*(s)} = \frac{K_P G_Y^*(s)}{1 + K_D G_X^*(s)}$$

$$E^*(s) = R^*(s) - E^*(s)\frac{K_P G_Y^*(s)}{1 + K_D G_X^*(s)}$$

$$R^*(s) = E^*(s)\left(1 + \frac{K_P G_Y^*(s)}{1 + K_D G_X^*(s)}\right)$$

$$\begin{aligned} \frac{E^*(s)}{R^*(s)} &= \frac{1}{1 + \dfrac{K_P G_Y^*(s)}{1 + K_D G_X^*(s)}} \\ &= \frac{1 + K_D G_X^*(s)}{1 + K_D G_X^*(s) + K_P G_Y^*(s)} \end{aligned}$$

$$\frac{Y_1^*(s)}{R^*(s)} = \frac{Y_1^*(s)}{E^*(s)}\frac{E^*(s)}{R^*(s)} = \frac{K_P G_Y^*(s)}{1 + K_D G_X^*(s) + K_P G_Y^*(s)}$$

$$\frac{Y_2^*(s)}{R^*(s)} = \frac{Y_2^*(s)}{E^*(s)}\frac{E^*(s)}{R^*(s)} = \frac{K_p G_X^*(s)}{1 + K_D G_X^*(s) + K_P G_Y^*(s)}$$

Thus, with $s = \frac{1}{T} ln(z)$

$$\boxed{\frac{Y_1(z)}{R(z)} = \frac{K_P G_Y(z)}{1 + K_D G_X(z) + K_P G_Y(z)}}$$

$$\boxed{\frac{Y_2(z)}{R(z)} = \frac{K_p G_X(z)}{1 + K_D G_X(z) + K_P G_Y(z)}}$$

Remember that, the $G_x(z)$ and $G_Y(z)$ were found earlier as

$$\boxed{G_Y(z) \approx \frac{1.1275(z+1)}{z^2 - 2.255z + 1}}$$

$$\boxed{G_X(z) \approx \frac{1.04\,(z-1)}{z^2 - 2.255z + 1}}$$

$$\begin{aligned}
\frac{Y_1(z)}{R(z)} &= \frac{K_P G_Y(z)}{1 + K_D G_X(z) + K_P G_Y(z)} \\
&= \frac{K_P\left(\dfrac{1.1275(z+1)}{z^2 - 2.255z + 1}\right)}{1 + K_D\left(\dfrac{1.04\,(z-1)}{z^2 - 2.255z + 1}\right) + K_P\left(\dfrac{1.1275(z+1)}{z^2 - 2.255z + 1}\right)} \\
&= \frac{1.1275K_p(z+1)}{z^2 - 2.255z + 1 + 1.04K_D(z-1) + 1.1275K_P(z+1)}
\end{aligned}$$

$$\boxed{\frac{Y_1(z)}{R(z)} = \frac{1.1275K_p(z+1)}{z^2 + (1.04K_D + 1.1275K_P - 2.255)z + (1 - 1.04K_D + 1.1275K_P)}}$$

$$\begin{aligned}
\frac{Y_2(z)}{R(z)} &= \frac{K_p G_X(z)}{1 + K_D G_X(z) + K_P G_Y(z)} \\
&= \frac{K_p\left(\dfrac{1.04\,(z-1)}{z^2 - 2.255z + 1}\right)}{1 + K_D\left(\dfrac{1.04\,(z-1)}{z^2 - 2.255z + 1}\right) + K_P\left(\dfrac{1.1275(z+1)}{z^2 - 2.255z + 1}\right)} \\
&= \frac{1.04K_p(z-1)}{z^2 - 2.255z + 1 + 1.04K_D(z-1) + 1.1275K_P(z+1)}
\end{aligned}$$

$$\boxed{\frac{Y_2(z)}{R(z)} = \frac{1.04K_p(z-1)}{z^2 + (1.04K_D + 1.1275K_P - 2.255)z + (1 - 1.04K_D + 1.1275K_P)}}$$

(f) Let us use Jury stability test again

$$\begin{aligned} D(z) &= z^2 + (1.04K_D + 1.1275K_P - 2.255)z + (1 - 1.04K_D + 1.1275K_P) \\ &= z^2 + 1.04K_D z + (3.255 - 1.04K_D) \end{aligned}$$

With coefficients,

$$\boxed{a_0 = 1}\ ,\ \boxed{a_1 = 1.04K_D}\ ,\ \boxed{a_2 = 3.255 - 1.04K_D}$$

The characteristic polynomial $D(z)$ should satisfy the following conditions according to Jury stability conditions;

- $a_0 \ > \ |a_2|$

$$1 \ > \ 3.255 - 1.04K_D > -1$$
$$\boxed{4.09 > K_D > 2.16}$$

- $D(1) > 0$

$$1 + 1.04K_D + 3.255 - 1.04K_D > 0$$
$$\boxed{4.255 > 0}$$

- $D(-1) > 0$

$$1 - 1.04K_D + 3.255 - 1.04K_D > 0$$
$$4.255 - 2.08K_D > 0$$
$$\boxed{2.04 > K_D}$$

It can be concluded that there are no K_D value that makes the system stable.

(g) Let us choose $K_D = 2$ to operate close to stability conditions.

$$\frac{Y_1(z)}{R(z)} = \frac{2.255(z+1)}{z^2 + 2.08z + 1.175}$$

$$\boxed{Y_1(z) = \frac{2.255z(z+1)}{(z^2 + 2.08z + 1.175)(z-1)}}$$

$$\frac{Y_2(z)}{R(z)} = \frac{2.08(z-1)}{z^2+2.08z+1.175}$$

$$\boxed{Y_2(z) = \frac{2.08z}{z^2+2.08z+1.175}}$$

(h) The *Figures 6 , 7* shows the step responses of $y_1(t)$ and $y_2(t)$ respectively. The source code for that operation can be seen at **Appendix A**.

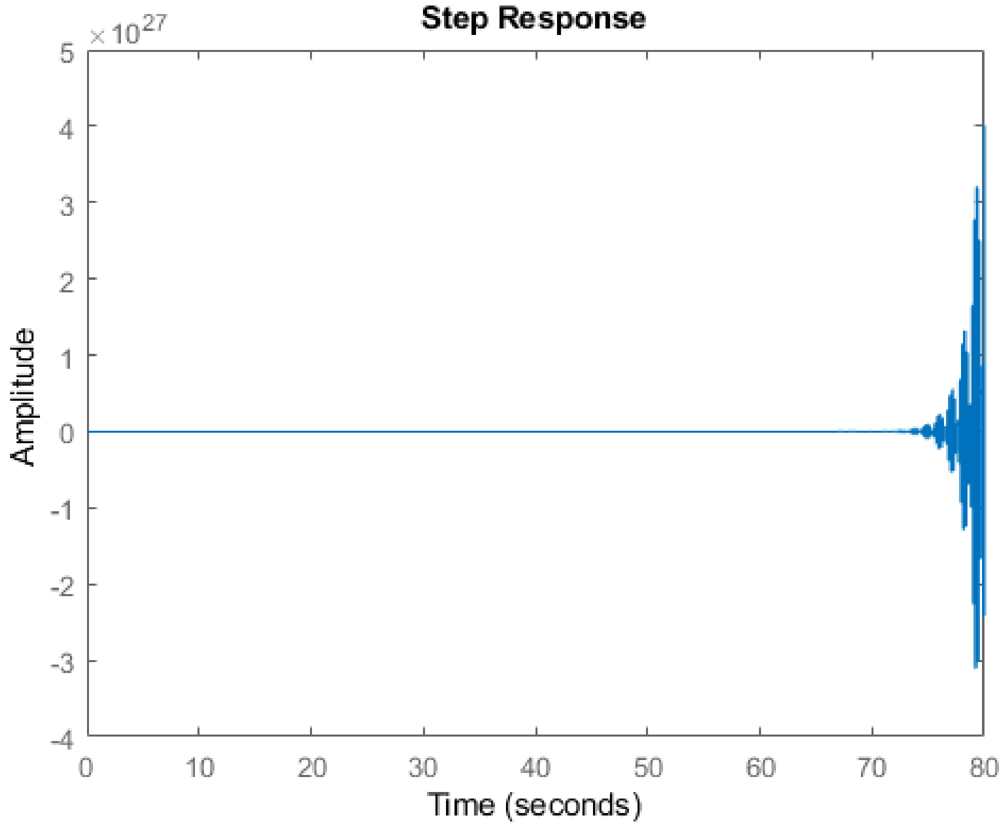


Figure 6: Step Response for the $y_1(t)$ with $K_P = 2$ and $K_D = 2$

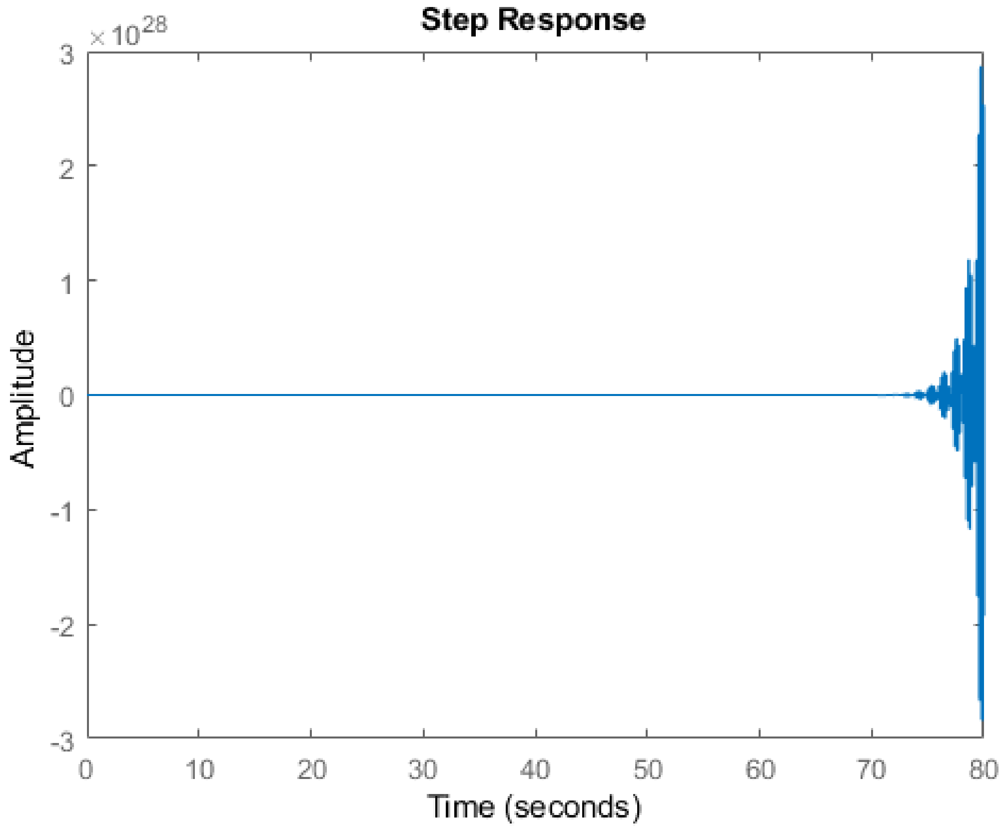


Figure 7: Step Response for the $y_2(t)$ with $K_P = 2$ and $K_D = 2$

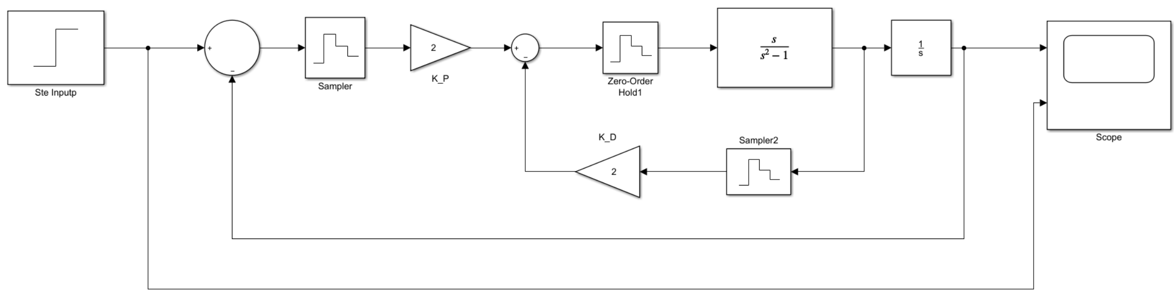
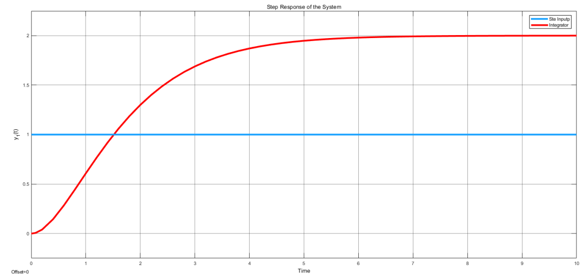
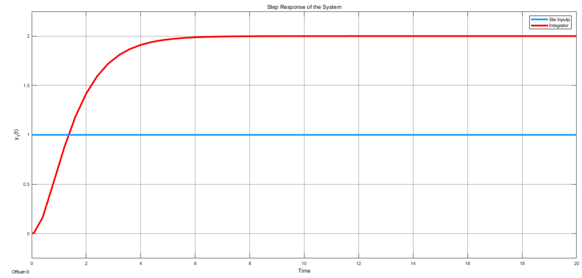


Figure 8: Simulink Model for the given system with $K_P = 2$ and $K_D = 2$

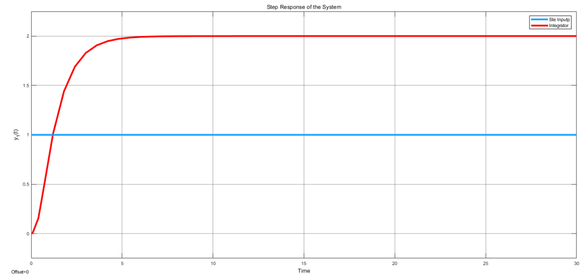
The *Figure 9* shows the step response of the given system with simulink. The Simulink model can be seen at *Figure 8*. Although the System acts very stable at the very beginning of the operation, the step response goes further stability limits as $t \to \infty$ as can be seen from the *Figure 10*.



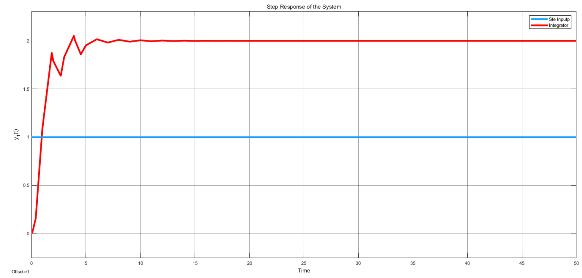
(a) Step Response of the given system till $t = 10\ sec$



(b) Step Response of the given system till $t = 20\ sec$



(c) Step Response of the given system till $t = 30\ sec$



(d) Step Response of the given system till $t = 50\ sec$

Figure 9: Step Responses of the given system with varying end times

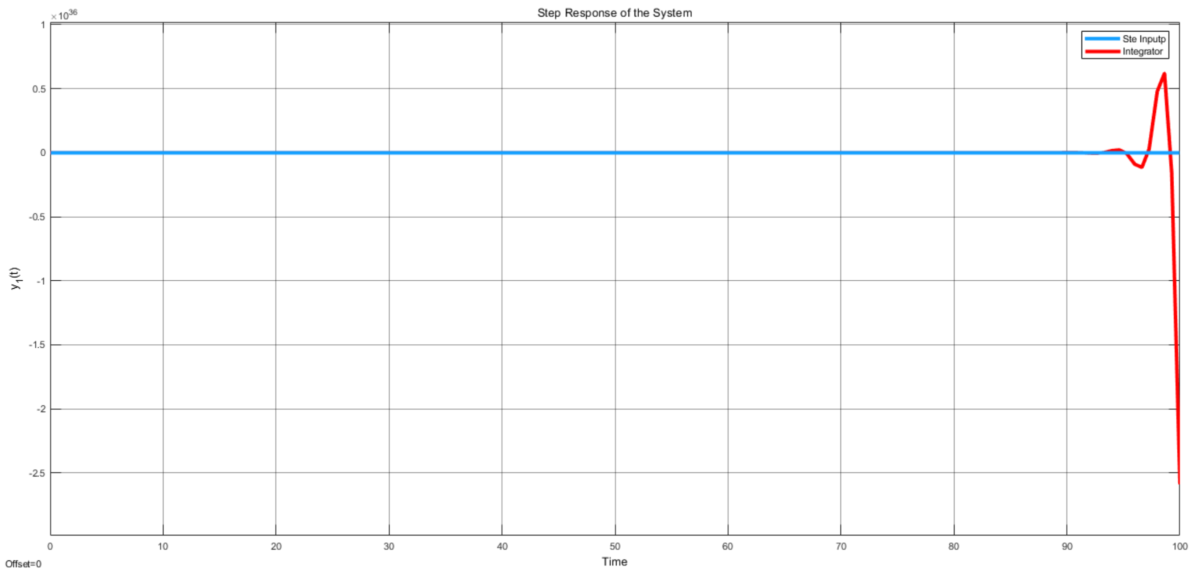


Figure 10: Step Response of the given system till $t = 100\ sec$

Appendices

A Source Code for Matlab Part

```
%% First Part
y_1a = tf ([2.255 2.255],[1 -1.56 4.815],0.1)

step(y_1a)

%% -
y_2a=tf ([2.08 0],[1 0 3.255],0.1)

step(y_2a)

%% Second Part
y_1b = tf ([2.225 2.225 0],[1 1.08 -0.905 -1.175] ,0.1)

step(y_1b)

%% -
y_2b = tf ([2.08 0],[1 2.08 1.175],0.1)

step(y_2b)
```

```
>> y_2a=tf ([2.08 0],[1 0 3.255],0.1)

step(y_2a)

y_2a =

    2.08 z
  -----------
  z^2 + 3.255

Sample time: 0.1 seconds
Discrete-time transfer function.

>> y_1a = tf ([2.255 2.255],[1 -1.56 4.815],0.1)

step(y_1a)

y_1a =

    2.255 z + 2.255
  --------------------
  z^2 - 1.56 z + 4.815

Sample time: 0.1 seconds
Discrete-time transfer function.
```

(a) The output for the source code for first part

```
>> y_2a=tf ([2.08 0],[1 0 3.255],0.1)

step(y_2a)

y_2a =

    2.08 z
  -----------
  z^2 + 3.255

Sample time: 0.1 seconds
Discrete-time transfer function.

>> y_1a = tf ([2.255 2.255],[1 -1.56 4.815],0.1)

step(y_1a)

y_1a =

    2.255 z + 2.255
  --------------------
  z^2 - 1.56 z + 4.815

Sample time: 0.1 seconds
Discrete-time transfer function.
```

(b) The output for the source code for 2nd part

Figure 11: The output for the source code