EE402 Discrete Time Systems
MP-3

1. Let us define;

$$G_1(s) = \frac{s}{s^2-1}$$
$$G_2(s) = \frac{1}{s}$$
$$G_{ZOH}(s) = \frac{1-z^{-1}}{s}$$
$$G_X(s) = G_{ZOH}(s)G_1(s)$$
$$G_Y(s) = G_{ZOH}(s)G_1(s)G_2(s)$$

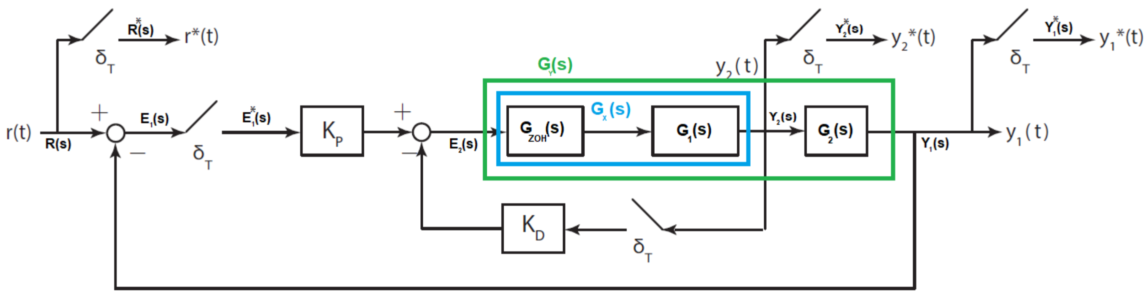


Figure 1: Block Diagram Representation of the System

(a)

$$E(s) = R(s) - Y_1(s)$$
$$E^*(s) = R^*(s) - Y_1^*(s)$$

$$Y_1(s) = E^*(s)K_pG_Y(s)$$
$$Y_1^*(s) = E^*(s)K_pG_Y^*(s)$$

$$\frac{Y_1^*(s)}{R^*(s)} = K_pG_Y^*(s)$$

$$E^*(s) = R^*(s) - E^*(s)K_pG_Y^*(s)$$

$$\frac{E^*(s)}{R^*(s)} = \frac{1}{1+K_pG_Y^*(s)}$$

$$\frac{Y_1^*(s)}{R^*(s)} = \frac{E^*(s)}{R^*(s)}\frac{Y_1^*(s)}{E^*(s)} = \frac{K_pG_Y^*(s)}{1+K_pG_Y^*(s)}$$

Thus, with $s = \frac{1}{T}ln(z)$

$$\boxed{\frac{Y_1(z)}{R(z)} = \frac{K_pG_Y(z)}{1+K_pG_Y(z)}}$$

$$Y_2(s) = E^*(s)K_pG_X(s)$$
$$Y_2^*(s) = E^*(s)K_pG_X^*(s)$$

$$\frac{Y_2^*(s)}{R^*(s)} = \frac{E^*(s)}{R^*(s)}\ \frac{Y_2^*(s)}{E^*(s)} = \frac{K_pG_X^*(s)}{1+K_pG_Y^*(s)}$$

Thus, with $s = \frac{1}{T}ln(z)$

$$\boxed{\frac{Y_2(z)}{R(z)} = \frac{K_pG_X(z)}{1+K_pG_Y(z)}}$$

Let us now find $G_X(z)$ and $G_Y(s)$ to find the pulse transfer functions,

$$G_Y(z) = \mathcal{Z}\{G_Y(s)\} = \mathcal{Z}\{\mathcal{L}^{-1}\{G_{ZOH}(s)G_1(s)G_2(s)\}^*\}$$

with $G_{ZOH}(s) = \dfrac{1-e^{-Ts}}{s}$ and $G_1(s)G_2(s) = \dfrac{1}{s^2-1}$ and $\boxed{T = 0.5}$

$$\boxed{G_Y(z) = \left((1-z^{-1})\mathcal{Z}\{\frac{G_1(s)G_2(s)}{s}\}\right)}$$

$$\begin{aligned}
\mathcal{Z}\left\{\frac{G_1(s)G_2(s)}{s}\right\} &= \mathcal{Z}\left\{\frac{1}{s(s^2-1)}\right\} \\
&= \mathcal{Z}\left\{\frac{-1}{s} + \frac{1}{2(s+1)} + \frac{1}{2(s-1)}\right\} \\
&= \frac{-1}{1-z^{-1}} + \frac{1}{2(1-e^{0.5}z^{-1})} + \frac{1}{2(1-e^{-0.5}z^{-1})}
\end{aligned}$$

$$\begin{aligned}
G_Y(z) &= -1 + \frac{1-z^{-1}}{2(1-e^{0.5}z^{-1})} + \frac{1-z^{-1}}{2(1-e^{-0.5}z^{-1})} \\
&= \frac{-2(1-e^{0.5}z^{-1})(1-e^{-0.5}z^{-1}) + (1-z^{-1})(1-e^{-0.5}z^{-1})}{2(1-e^{0.5}z^{-1})(1-e^{-0.5}z^{-1})} \\
&+ \frac{(1-z^{-1})(1-e^{0.5}z^{-1})}{2(1-e^{0.5}z^{-1})(1-e^{-0.5}z^{-1})} \\
G_Y(z) &= \frac{(e^{0.5}+e^{-0.5}-2)z^{-1} + (e^{0.5}+e^{-0.5}-2)z^{-2}}{2(1-(e^{0.5}+e^{-0.5})z^{-1}+z^{-2})}
\end{aligned}$$

$$\begin{aligned}
G_Y(z) &= \frac{\left[e^{0.5}+e^{-0.5}-2\right](z^{-1}+z^{-2})}{2(1-(e^{0.5}+e^{-0.5})z^{-1}+z^{-2})} \\
G_Y(z) &\approx \frac{0.255(z^{-1}+z^{-2})}{2(1-2.255z^{-1}+z^{-2})}
\end{aligned}$$

$$\boxed{G_Y(z) \approx \frac{0.1275(z+1)}{z^2-2.255z+1}}$$

$$\boxed{G_X(z) = \left((1 - z^{-1})\mathcal{Z}\{\frac{G_1(s)}{s}\}\right)}$$

$$\mathcal{Z}\left\{\frac{G_1(s)}{s}\right\} = \mathcal{Z}\left\{\frac{1}{s^2 - 1}\right\}$$

$$= \mathcal{Z}\left\{\frac{1/2}{s-1} - \frac{1/2}{s+1}\right\}$$

$$= \frac{1/2}{1 - e^{0.5}z^{-1}} - \frac{1/2}{1 - e^{-0.5}z^{-1}} = \frac{\left[e^{0.5} - e^{-0.5}\right] z^{-1}}{2(1 - e^{0.5}z^{-1})(1 - e^{-0.5}z^{-1})}$$

$$= \frac{\left[e^{0.5} - e^{-0.5}\right] z^{-1}}{2(1 - \left[e^{0.5} + e^{-0.5}\right] z^{-1} + z^{-2})}$$

$$G_X(z) = \frac{\left[e^{0.5} - e^{-0.5}\right] z^{-1} \left(1 - z^{-1}\right)}{2(1 - \left[e^{0.5} + e^{-0.5}\right] z^{-1} + z^{-2})}$$

$$= \frac{\left[e^{0.5} - e^{-0.5}\right] \left(z^{-1} - z^{-2}\right)}{2(1 - \left[e^{0.5} + e^{-0.5}\right] z^{-1} + z^{-2})}$$

$$= \frac{\left[e^{0.5} - e^{-0.5}\right] (z - 1)}{2(z^2 - \left[e^{0.5} + e^{-0.5}\right] z + 1)}$$

$$\boxed{G_X(z) \approx \frac{0.52\,(z-1)}{z^2 - 2.255z + 1}}$$

Let us now put what we have found into pulse transfer function form.

$$\begin{aligned}
\frac{Y_1(z)}{R(z)} &= \frac{K_p G_Y(z)}{1+K_p G_Y(z)} \\
&= \frac{K_p\left(\dfrac{0.1275(z+1)}{z^2-2.255z+1}\right)}{1+K_p\left(\dfrac{0.1275(z+1)}{z^2-2.255z+1}\right)} \\
&= \frac{0.1275K_p(z+1)}{z^2-2.255z+1+K_p(0.1275(z+1))}
\end{aligned}$$

$$\boxed{\frac{Y_1(z)}{R(z)} = \frac{0.1275K_p(z+1)}{z^2+(0.1275K_p-2.255)z+(1+1.0275K_p)}}$$

$$\begin{aligned}
\frac{Y_2(z)}{R(z)} &= \frac{K_p G_X(z)}{1+K_p G_Y(z)} \\
&= \frac{K_p\left(\dfrac{0.52\,(z-1)}{z^2-2.255z+1}\right)}{1+K_p\left(\dfrac{0.1275(z+1)}{z^2-2.255z+1}\right)} \\
&= \frac{0.52K_p(z-1)}{z^2-2.255z+1+1.0275K_p(z+1)}
\end{aligned}$$

$$\boxed{\frac{Y_2(z)}{R(z)} = \frac{0.52K_p1(z-1)}{z^2+(0.1275K_p-2.255)z+(1+0.1275K_p)}}$$

(b) To ensure stability conditions for K_p, Let us use Jury Conditions;

$$D(s) = z^2 + (0.1275K_p - 2.255)z + (1 + 0.1275K_p)$$

With coefficients,

$$\boxed{a_0 = 1}\ ,\ \boxed{a_1 = 0.1275K_p - 2.255}\ ,\ \boxed{a_2 = 1 + 0.1275K_p}$$

The characteristic polynomial $D(s)$ should satisfy the following conditions according to Jury stability conditions;

- $a_0 \ > \ |a_2|$

$$1 \ > \ 1 + 0.1275K_p > -1$$
$$\boxed{0 > K_p > -7.843}$$

- $D(1) > 0$

$$1 + 0.1275K_p - 2.255 + 1 + 0.1275K_p > 0$$
$$0.255K_p - 1.1275 > 0$$
$$\boxed{K_p > 5.01}$$

- $D(-1) > 0$

$$1 - 0.1275K_p + 2.255 + 1 + 0.1275K_p > 0$$
$$\boxed{2.1275 > 0}$$

It can be concluded that there are no K_p value that makes the system stable.

(c) Given that $K_p = 2$, the unit step response can be calculated as follows;

$$R(z) = \frac{1}{1 - z^{-1}} = \frac{z}{z - 1}$$

$$\begin{aligned} Y_1(z) &= \frac{0.1275K_p(z+1)}{z^2 - 2.255z + 1 + 0.1275K_p(z+1)}R(z) \\ &= \frac{0.255(z^2 + z)}{(z^2 - 2z + 1.255)(z - 1)} \end{aligned}$$

$$\boxed{Y_1(z) = \frac{2.255(z^2 + z)}{(z^2 + 3.255)(z - 1)}}$$

$$\boxed{y_1[k] = \mathcal{Z}^{-1}\{Y_1(z)\}}$$

$$
\begin{aligned}
Y_2(z) &= \frac{0.52K_p(z-1)}{z^2 + (0.1275K_p - 2.255)z + (1 + 0.1275K_p)} R(z) \\
&= \frac{1.04z(z-1)}{(z^2 - 2z + 1.255)(z-1)} \\
&= \frac{1.04z}{z^2 + -2z + 1.255}
\end{aligned}
$$

$$
\boxed{Y_2(z) = \frac{1.04z}{z^2 - 2z + 1.255}}
$$

$$
\boxed{y_2(t) = \mathcal{Z}^{-1}\{Y_2(z)\}}
$$

(d) The *Figure 2* shows the step responses of $y_1(t)$ and $y_2(t)$ for $k_{final} = 50$ and $k_{final} = 100$. The source code for that operation can be seen at **Appendix A**.

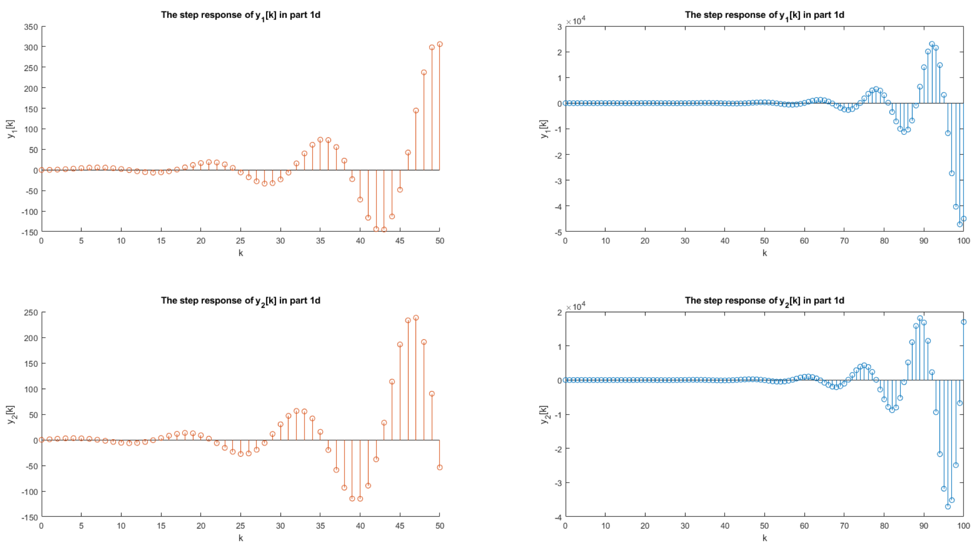


Figure 2: Step Response for the $y_1(t)$ with $K_P = 2$ and $K_D = 0$

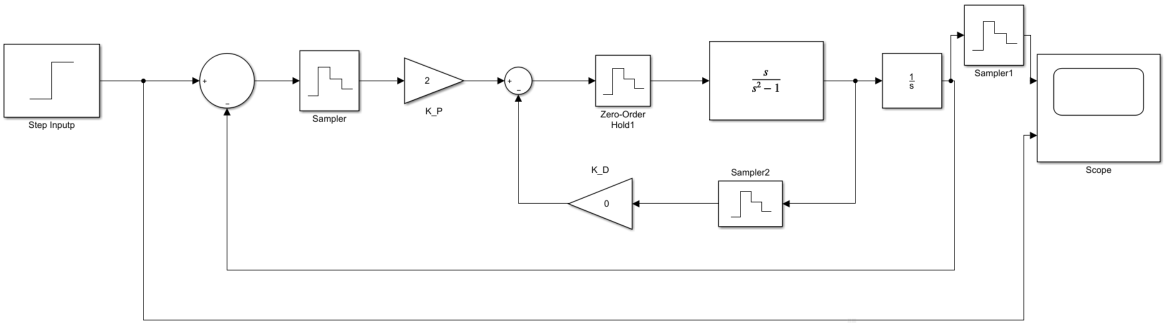
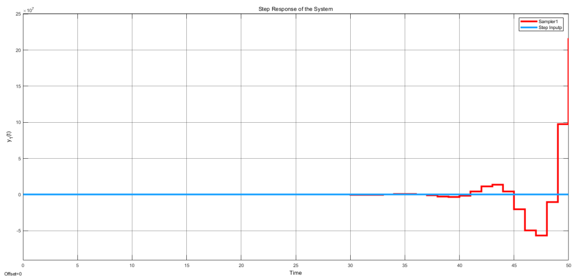
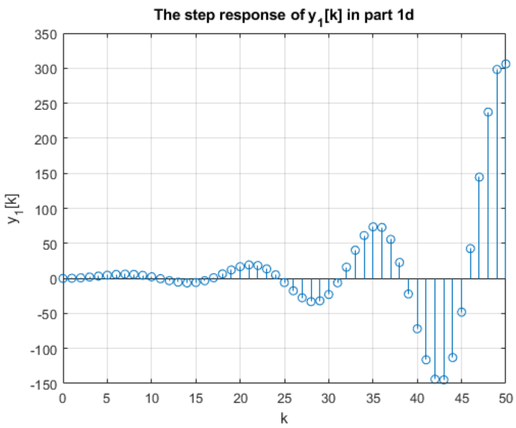


Figure 3: Simulink Model for the given system with $K_P = 2$ and $K_D = 0$

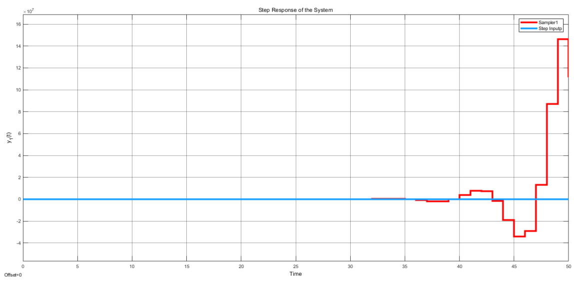
The *Figures 5a , 5b* shows the step response of the given system with simulink. The Simulink model can be seen at *Figure 3*.



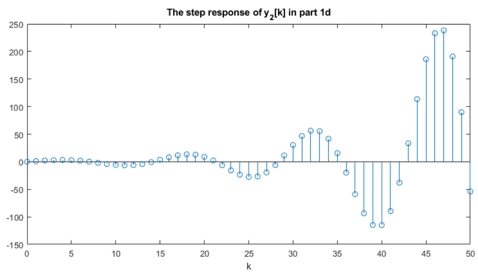
(a) Step Response of the $y_1[k]$ till $k = 50$ with Simulink



(b) Step Response of the $y_1[k]$ till $k = 50$



(a) Step Response of the $y_2[k]$ till $k = 50$ with Simulink



(b) Step Response of the $y_2[k]$ till $k = 50$

It can be observed from the figures also that the system shows very unstable behaviour with given K_p value.

(e) Let us analyse the system now with a $K_d \neq 0$

$$\begin{aligned} E(s) &= R(s) - Y_1(s) \\ E^*(s) &= R^*(s) - Y_1^*(s) \end{aligned}$$

$$\begin{aligned} Y_2(s) &= \left[E^*(s)K_p - K_D Y_2^*(s)\right] G_X(s) \\ Y_2^*(s) &= \left[E^*(s)K_p - K_D Y_2^*(s)\right] G_X^*(s) \\ \frac{Y_2^*(s)}{E^*(s)} &= \frac{K_p G_X^*(s)}{1 + K_D G_X^*(s)} \end{aligned}$$

$$\begin{aligned} Y_1(s) &= \left[E^*(s)K_p - K_D Y_2^*(s)\right] G_Y(s) \\ Y_1^*(s) &= \left[E^*(s)K_p - K_D Y_2^*(s)\right] G_Y^*(s) \\ &= E^*(s)\left[K_p - K_D \frac{K_p G_X^*(s)}{1 + K_D G_X^*(s)}\right] G_Y^*(s) \\ &= E^*(s)\left[\frac{K_p}{1 + K_D G_X^*(s)}\right] G_Y^*(s) \\ \frac{Y_1^*(s)}{E^*(s)} &= \frac{K_P G_Y^*(s)}{1 + K_D G_X^*(s)} \end{aligned}$$

$$\begin{aligned} E^*(s) &= R^*(s) - E^*(s)\frac{K_P G_Y^*(s)}{1 + K_D G_X^*(s)} \\ R^*(s) &= E^*(s)\left(1 + \frac{K_P G_Y^*(s)}{1 + K_D G_X^*(s)}\right) \\ \frac{E^*(s)}{R^*(s)} &= \frac{1}{1 + \dfrac{K_P G_Y^*(s)}{1 + K_D G_X^*(s)}} \\ &= \frac{1 + K_D G_X^*(s)}{1 + K_D G_X^*(s) + K_P G_Y^*(s)} \end{aligned}$$

$$\frac{Y_1^*(s)}{R^*(s)} = \frac{Y_1^*(s)}{E^*(s)}\frac{E^*(s)}{R^*(s)} = \frac{K_P G_Y^*(s)}{1 + K_D G_X^*(s) + K_P G_Y^*(s)}$$

$$\frac{Y_2^*(s)}{R^*(s)} = \frac{Y_2^*(s)}{E^*(s)}\frac{E^*(s)}{R^*(s)} = \frac{K_p G_X^*(s)}{1 + K_D G_X^*(s) + K_P G_Y^*(s)}$$

Thus, with $s = \frac{1}{T} ln(z)$

$$\boxed{\frac{Y_1(z)}{R(z)} = \frac{K_P G_Y(z)}{1 + K_D G_X(z) + K_P G_Y(z)}}$$

$$\boxed{\frac{Y_2(z)}{R(z)} = \frac{K_p G_X(z)}{1 + K_D G_X(z) + K_P G_Y(z)}}$$

Remember that, the $G_x(z)$ and $G_Y(z)$ were found earlier as

$$\boxed{G_Y(z) \approx \frac{0.1275(z+1)}{z^2 - 2.255z + 1}}$$

$$\boxed{G_X(z) \approx \frac{0.52\,(z-1)}{z^2 - 2.255z + 1}}$$

$$\begin{aligned}
\frac{Y_1(z)}{R(z)} &= \frac{K_P G_Y(z)}{1 + K_D G_X(z) + K_P G_Y(z)} \\
&= \frac{K_P \left(\dfrac{0.1275(z+1)}{z^2 - 2.255z + 1}\right)}{1 + K_D \left(\dfrac{0.52\,(z-1)}{z^2 - 2.255z + 1}\right) + K_P \left(\dfrac{0.1275(z+1)}{z^2 - 2.255z + 1}\right)} \\
&= \frac{0.1275 K_p (z+1)}{z^2 - 2.255z + 1 + 0.52 K_D (z-1) + 0.1275 K_P (z+1)}
\end{aligned}$$

$$\boxed{\frac{Y_1(z)}{R(z)} = \frac{0.1275 K_p (z+1)}{z^2 + (0.52K_D + 0.1275K_P - 2.255)z + (1 - 0.52K_D + 0.1275K_P)}}$$

$$\begin{aligned}
\frac{Y_2(z)}{R(z)} &= \frac{K_p G_X(z)}{1 + K_D G_X(z) + K_P G_Y(z)} \\
&= \frac{K_p \left(\dfrac{0.52\,(z-1)}{z^2 - 2.255z + 1}\right)}{1 + K_D \left(\dfrac{0.52\,(z-1)}{z^2 - 2.255z + 1}\right) + K_P \left(\dfrac{0.1275(z+1)}{z^2 - 2.255z + 1}\right)} \\
&= \frac{0.52 K_p (z-1)}{z^2 - 2.255z + 1 + 0.52 K_D (z-1) + 0.1275 K_P (z+1)}
\end{aligned}$$

$$\boxed{\frac{Y_2(z)}{R(z)} = \frac{0.52K_p(z-1)}{z^2 + (0.52K_D + 0.1275K_P - 2.255)z + (1 - 0.52K_D + 0.1275K_P)}}$$

(f) Let us use Jury stability test again

$$\begin{aligned} D(z) &= z^2 + (0.52K_D + 0.1275K_P - 2.255)z + (1 - 0.52K_D + 0.1275K_P) \\ &= z^2 + (-2 + 0.52K_D)z + (1.255 - 0.52K_D) \end{aligned}$$

With coefficients,

$$\boxed{a_0 = 1}\ ,\ \boxed{a_1 = -2 + 0.52K_D}\ ,\ \boxed{a_2 = 1.255 - 0.52K_D}$$

The characteristic polynomial $D(z)$ should satisfy the following conditions according to Jury stability conditions;

- $a_0 \ > \ |a_2|$

$$1 \ > \ 1.255 - 0.52K_D > -1$$
$$\boxed{4.27 > K_D > 0.49}$$

- $D(1) > 0$

$$1 - 2 + 0.52K_D + 1.255 - 0.52K_D > 0$$
$$\boxed{0.255 > 0}$$

- $D(-1) > 0$

$$1 + 2 - 0.52K_D + 1.255 - 0.52K_D > 0$$
$$4.255 - 1.04K_D > 0$$
$$\boxed{4.06 > K_D}$$

It can be concluded that K_D should satisfy the following

$$\boxed{4.06 > K_D > 0.49}$$

(g) Let us choose $K_D = 2$

$$\frac{Y_1(z)}{R(z)} = \frac{0.255(z+1)}{z^2 - 0.93 + 0.255}$$

$$\boxed{Y_1(z) = \frac{0.255z(z+1)}{(z^2 - 0.93 + 0.255)(z-1)}}$$

$$\frac{Y_2(z)}{R(z)} = \frac{1.04(z-1)}{z^2 - 0.93 + 0.255}$$

$$\boxed{Y_2(z) = \frac{1.04z}{z^2 - 0.93 + 0.255}}$$

(h) The *Figure 6* shows the step responses of $y_1(t)$ and $y_2(t)$ respectively. The source code for that operation can be seen at **Appendix A**.

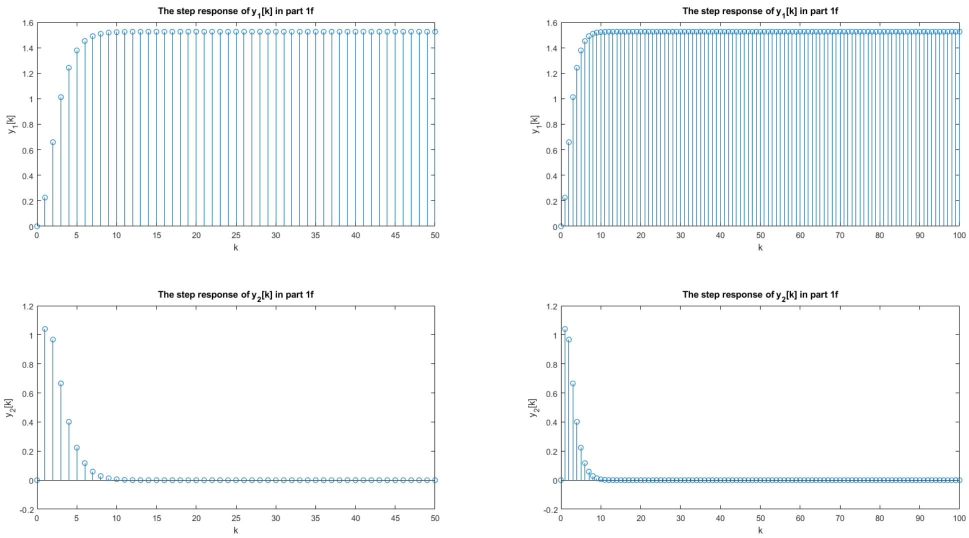


Figure 6: Step Response for the $y_1(t)$ with $K_P = 2$ and $K_D = 2$

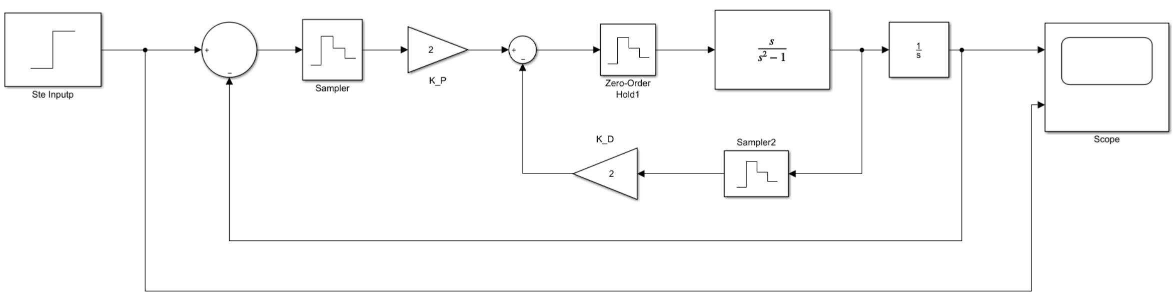
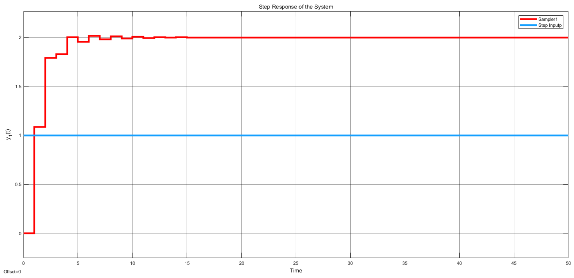
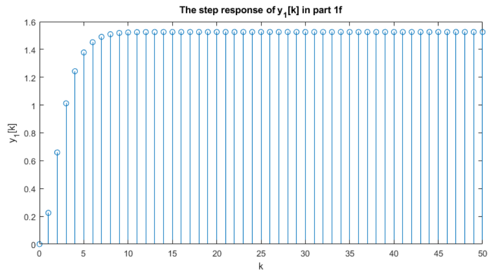


Figure 7: Simulink Model for the given system with $K_P = 2$ and $K_D = 2$

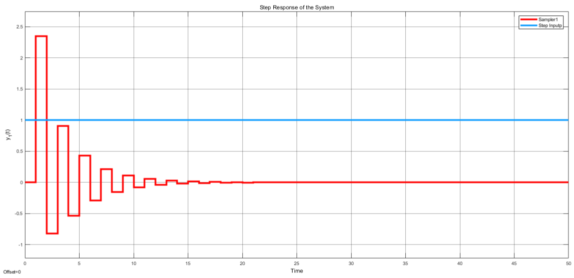
The *Figure* **??** shows the step response of the given system with simulink. The Simulink model can be seen at *Figure 7*. Although the System acts very stable at the very beginning of the operation, the step response goes further stability limits as $t \to \infty$ as can be seen from the *Figure* **??**.



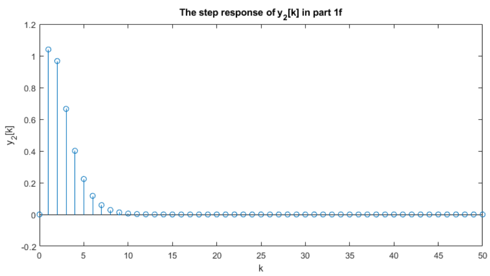
(a) Step Response of the $y_1[k]$ till $k = 50$ with Simulink



(b) Step Response of the $y_1[k]$ till $k = 50$



(a) Step Response of the $y_2[k]$ till $k = 50$ with Simulink



(b) Step Response of the $y_2[k]$ till $k = 50$

Appendices

A Source Code for Matlab Part

```
%% First Part Kp=2 / Kd=0
syms z
Y_1a = 0.225*(z^2+z)/((z^2-2*z+1.225)*(z-1))
y_1a = iztrans(Y_1a)
subs(y_1a,0:50)
subplot(2,2,1)
stem(0:50,ans)
title('The step response of y_1[k] in part 1d')
xlabel('k')
ylabel('y_1[k]')
subs(y_1a,0:100)
subplot(2,2,2)
stem(0:100,ans)
title('The step response of y_1[k] in part 1d')
xlabel('k')
ylabel('y_1[k]')

%%

syms z
Y_2a = 1.04*z/(z^2-2*z+1.225)
y_2a = iztrans(Y_2a)
subs(y_2a,0:50)
subplot(2,2,3)
stem(0:50,ans)
title('The step response of y_2[k] in part 1d')
xlabel('k')
ylabel('y_2[k]')
subs(y_2a,0:100)
subplot(2,2,4)
stem(0:100,ans)
title('The step response of y_2[k] in part 1d')
xlabel('k')
ylabel('y_2[k]')


%% Second Part Kp=2 / Kd=2
```

```

syms z
Y_2a = 0.225*(z^2+z)/((z^2-0.93*z+0.225)*(z-1))
y_2a = iztrans(Y_2a)
subs(y_2a,0:50)
subplot(2,2,1)
stem(0:50,ans)
title('The step response of y_1[k] in part 1f')
xlabel('k')
ylabel('y_1[k]')
subs(y_2a,0:100)
subplot(2,2,2)
stem(0:100,ans)
title('The step response of y_1[k] in part 1f')
xlabel('k')
ylabel('y_1[k]')

%%

syms z
Y_2b = 1.04*z/(z^2-0.93*z+0.225)
y_2b = iztrans(Y_2b)
subs(y_2b,0:50)
subplot(2,2,3)
stem(0:50,ans)
title('The step response of y_2[k] in part 1f')
xlabel('k')
ylabel('y_2[k]')
subs(y_2b,0:100)
subplot(2,2,4)
stem(0:100,ans)
title('The step response of y_2[k] in part 1f')
xlabel('k')
ylabel('y_2[k]')
```