Measurement & Control Instrumentations

M.Sc. Production Engineering Course
Department of Production Engineering & Metallurgy
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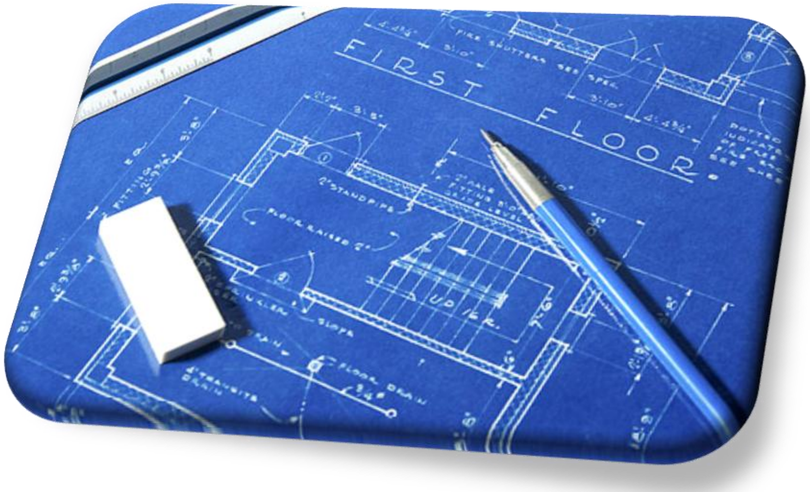
Lecture 2

Dynamic Characteristics of Measurement Systems

If the input signal *I to an element is changed suddenly, from one value to another,*
then the output signal *O will not instantaneously change to its new value.*

<u>*For* example</u>, if the temperature input to a thermocouple is suddenly changed from 25 °C to 100 °C, some time will elapse before the e.m.f. output completes the change from 1 mV to 4 mV.

The ways in which an element responds to sudden input changes are termed its **dynamic characteristics**, and these are most conveniently summarized using a **transfer function *G(s)*.**



First-order elements

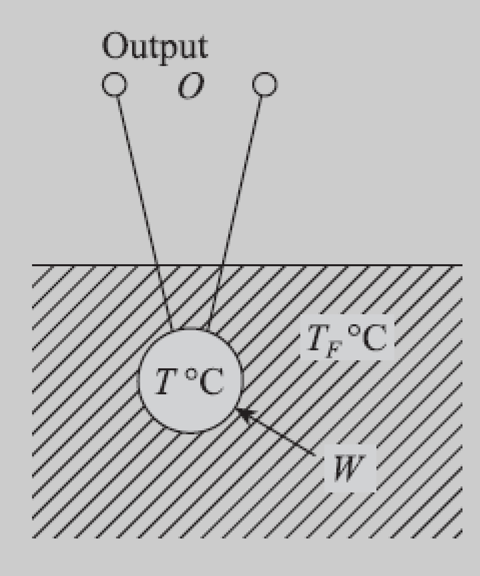
A good example of a first-order element is provided by a **temperature sensor** with an electrical output signal, e.g. a thermocouple or thermistor.

The bare element is placed inside a fluid . Initially at time *t = 0, the sensor temperature is equal to the fluid temperature, i.e.* $T(0) = T_F(0)$. *If the fluid temperature is suddenly raised at t = 0, the sensor is no longer in* a steady state, and its dynamic behavior is described by the **heat balance equation:**

rate of heat inflow − rate of heat outflow = rate of change of sensor heat content

Assuming that $T_F > T$, *then the rate of heat outflow will be zero, and the rate of heat* inflow *W will be proportional to the temperature difference* $(T_F - T)$.

$$W = UA(T_F - T) \qquad W \text{ in watts}$$



Where: ***U*** *is the overall heat transfer coefficient between fluid and sensor* ($W\ m^{-2}\ °C^{-1}$)

A *is the effective heat transfer area* (m^2).

The increase of heat content of the sensor is

$$MC[T - T(0)] \qquad \text{in joules,}$$

Where: *M is the sensor mass (kg).*

C is the specific heat of the sensor material ($J\ kg^{-1}\ °C^{-1}$).

Thus, assuming *M and C are constants:*

$$\text{rate of increase of sensor heat content} = MC \frac{\mathrm{d}}{\mathrm{d}t}[T - T(0)]$$

Defining $\Delta T = T - T(0)$
$\Delta T_F = T_F - T_F(0)$

to be the deviations in temperatures from initial steady-state conditions, the differential equation describing the sensor temperature changes is:

$$UA(\Delta T_F - \Delta T) = MC\frac{\mathrm{d}\Delta T}{\mathrm{d}t} \quad \text{i.e.} \quad \frac{MC}{UA}\frac{\mathrm{d}\Delta T}{\mathrm{d}t} + \Delta T = \Delta T_F$$

This is a **linear differential equation** in which dΔ*T/dt and* Δ*T are multiplied by* constant coefficients; the equation is **first order** because **dΔ*T/dt is the highest*** derivative present.
The quantity *MC/UA has the dimensions of time:*

$$\frac{\mathrm{kg} \times \mathrm{J} \times \mathrm{kg}^{-1} \times {}^\circ\mathrm{C}^{-1}}{\mathrm{W} \times \mathrm{m}^{-2} \times {}^\circ\mathrm{C}^{-1} \times \mathrm{m}^2} = \frac{\mathrm{J}}{\mathrm{W}} = \text{seconds}$$

and is referred to as the **time constant τ for the system.** The differential equation is now:

$$\boxed{\tau\frac{\mathrm{d}\Delta T}{\mathrm{d}t} + \Delta T = \Delta T_F}$$

Linear first-order differential equation

The transfer function based on the *Laplace transform* of the differential equation provides a convenient framework for studying the *dynamics of multi-element systems*.

Definition of Laplace transform

$$\bar{f}(s) = \int_0^\infty e^{-st} f(t)\, dt$$

where *s is a complex variable of the form σ + jω where j =* $\sqrt{-1}$.

Table below gives Laplace transforms for some common standard functions *f(t).*

Laplace transform for

$$\tau \frac{d\Delta T}{dt} + \Delta T = \Delta T_F$$

is

$$\tau[s\Delta\bar{T}(s) - \Delta T(0-)] + \Delta\bar{T}(s) = \Delta\bar{T}_F(s)$$

where ΔT(0−) is the temperature deviation at initial conditions prior to t = 0.
By definition, *ΔT(0−) = 0, giving:*

$$\tau s\Delta\bar{T}(s) + \Delta\bar{T}(s) = \Delta\bar{T}_F(s)$$

i.e.

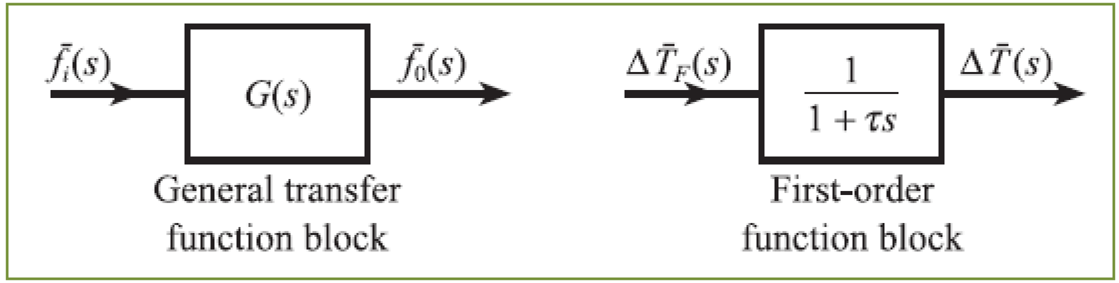
$$(\tau s + 1)\Delta\bar{T}(s) = \Delta\bar{T}_F(s)$$

The transfer function *G(s)* *of an element is defined as the* ***ratio of the Laplace transform*** **of the output to the Laplace transform of the input, provided the initial conditions are zero**. Thus:

$$G(s) = \frac{\bar{f}_0(s)}{\bar{f}_i(s)}$$

Definition of element transfer function

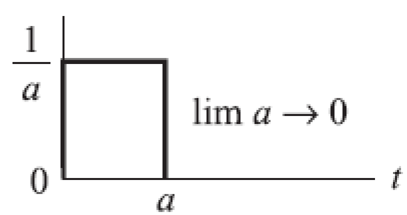
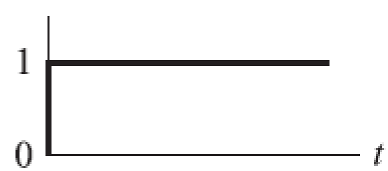
Transfer function representation



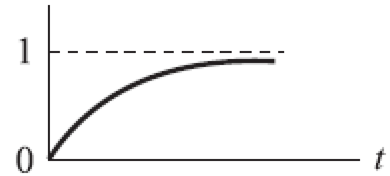
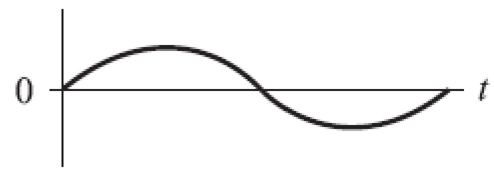
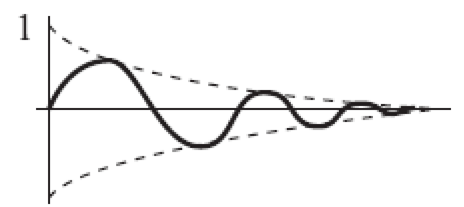
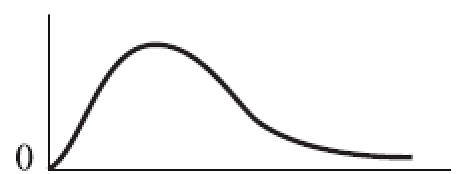
Transfer function for a first-order element

$$G(s) = \frac{\Delta \bar{T}(s)}{\Delta \bar{T}_F(s)} = \frac{1}{1+\tau s}$$

Laplace transforms of common time functions *f(t)*.

Function	Symbol	Graph	Transform
1st derivative	$\frac{\mathrm{d}}{\mathrm{d}t} f(t)$		$s\bar{f}(s) - f(0-)$
2nd derivative	$\frac{\mathrm{d}^2}{\mathrm{d}t^2} f(t)$		$s^2\bar{f}(s) - sf(0-) - \dot{f}(0-)$
Unit impulse	$\delta(t)$		1
Unit step	$\mu(t)$		$\frac{1}{s}$

Dr. Laith Abdullah Mohammed

Exponential decay	$\exp(-\alpha t)$		$\dfrac{1}{s+\alpha}$
Exponential growth	$1-\exp(-\alpha t)$		$\dfrac{\alpha}{s(s+\alpha)}$
Sine wave	$\sin \omega t$		$\dfrac{\omega}{s^2+\omega^2}$
Phase-shifted sine wave	$\sin(\omega t+\phi)$		$\dfrac{\omega \cos \phi + s \sin \phi}{s^2+\omega^2}$
Exponentially damped sine wave	$\exp(-\alpha t)\sin \omega t$		$\dfrac{\omega}{(s+\alpha)^2+\omega^2}$
Ramp with exponential decay	$t\exp(-\alpha t)$		$\dfrac{1}{(s+\alpha)^2}$

[a] Initial conditions are at $t = 0-$, just prior to $t = 0$.

$$\boxed{G(s) = \frac{\Delta\bar{T}(s)}{\Delta\bar{T}_F(s)} = \frac{1}{1+\tau s}}$$

The above transfer function only relates changes in sensor temperature to changes in fluid temperature. The **overall relationship between changes in sensor output signal *O* and fluid *temperature*** *is:*

$$\boxed{\frac{\Delta\bar{O}(s)}{\Delta\bar{T}_F(s)} = \frac{\Delta O}{\Delta T}\frac{\Delta\bar{T}(s)}{\Delta\bar{T}_F(s)}}$$

where $\Delta O/\Delta T$ *is the* ***steady-state sensitivity of the temperature sensor.***

For an ideal element *$\Delta O/\Delta T$ will be equal to the slope K of the* ***ideal straight line.***

For non-linear elements, subject to small temperature fluctuations, we can take $\Delta O/\Delta T = dO/dT$, *The* derivative being evaluated at the steady-state temperature *$T(0-)$ around which the* fluctuations are taking place. Thus for a copper–constantan thermocouple measuring small fluctuations in temperature around 100 °C,
$\Delta E/\Delta T$ is found by evaluating dE/dT *at 100 °C to give* $\Delta E/\Delta T = 35\ \mu V\ °C^{-1}$.

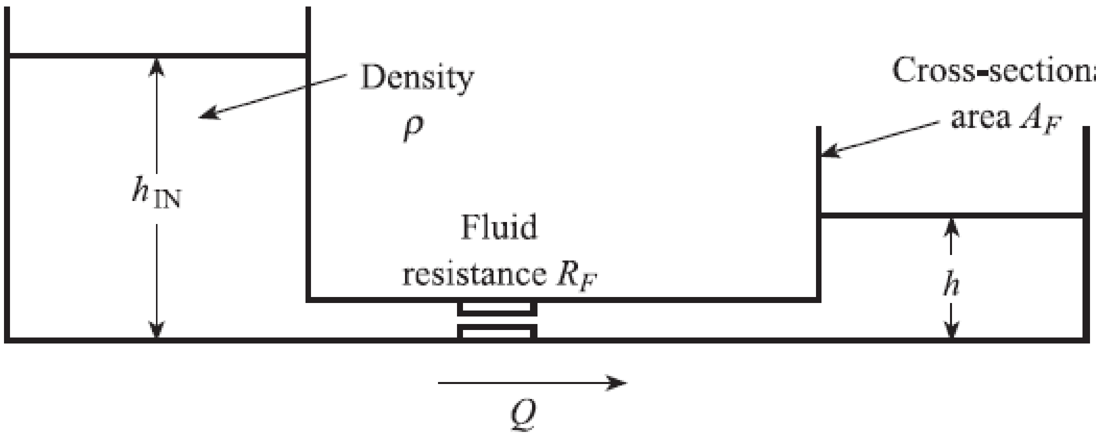
Thus if the time constant of the thermocouple is 10s the overall dynamic relationship between changes in e.m.f. and fluid temperature is:

$$\frac{\Delta\bar{E}(s)}{\Delta\bar{T}_F(s)} = 35 \times \frac{1}{1+10s}$$

Analogous fluidic, electrical and mechanical elements, which are also described by a first-order transfer function *G(s) = 1/(1 + τs)*.

Analogous first-order elements

Fluidic



$$\text{Volume flow rate } Q = \frac{1}{R_F}(P_{\text{IN}} - P)$$

$$\text{Pressures } P_{\text{IN}} = h_{\text{IN}}\rho g,\ P = h\rho g$$

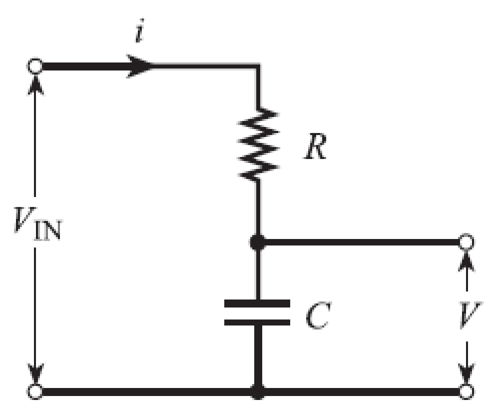
$$A_F \frac{\mathrm{d}h}{\mathrm{d}t} = Q = \frac{\rho g}{R_F}(h_{\text{IN}} - h)$$

$$\frac{A_F R_F}{\rho g}\frac{\mathrm{d}h}{\mathrm{d}t} + h = h_{\text{IN}}$$

i.e.

$$\tau_F \frac{\mathrm{d}h}{\mathrm{d}t} + h = h_{\text{IN}},\ \tau_F = \frac{A_F R_F}{\rho g}$$

Electrical



$$V_{\text{IN}} - V = iR$$

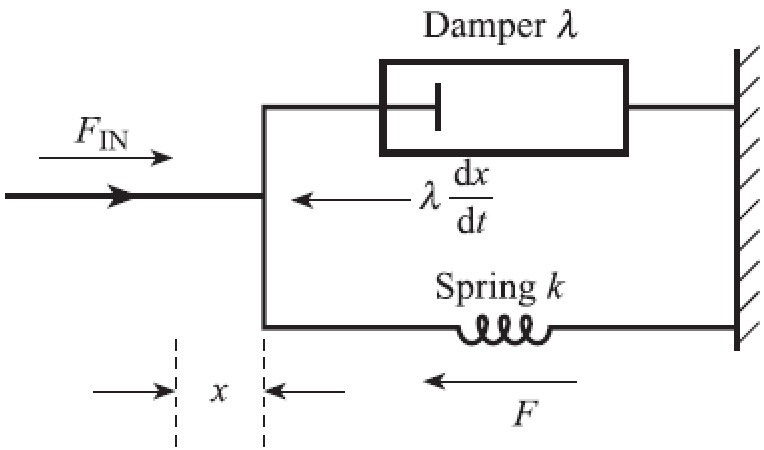
$$\text{Charge } q = CV,\ \text{current } i = \frac{\mathrm{d}q}{\mathrm{d}t} = \frac{C\mathrm{d}V}{\mathrm{d}t}$$

$$RC\frac{\mathrm{d}V}{\mathrm{d}t} + V = V_{\text{IN}}$$

i.e.

$$\tau_E \frac{\mathrm{d}V}{\mathrm{d}t} + V = V_{\text{IN}},\ \tau_E = RC$$

Mechanical



$$F_{IN} - F = \lambda \frac{dx}{dt}, \quad \lambda \text{ N s m}^{-1} = \text{damping constant}$$

$$\text{Displacement } x = \frac{F}{k}, \quad k \text{ N m}^{-1} = \text{spring stiffness}$$

$$\frac{\lambda}{k}\frac{dF}{dt} + F = F_{IN}$$

$$\tau_M \frac{dF}{dt} + F = F_{IN}, \ \tau_M = \frac{\lambda}{k}$$

Thermal	$\tau_{Th} = \dfrac{MC}{UA} = R_{Th}C_{Th};$	$R_{Th} = \dfrac{1}{UA},$	$C_{Th} = MC$
Fluidic	$\tau_F = \dfrac{A_F R_F}{\rho g} = R_F C_F;$	$R_F = R_F,$	$C_F = \dfrac{A_F}{\rho g}$
Electrical	$\tau_E = RC = R_E C_E;$	$R_E = R,$	$C_E = C$
Mechanical	$\tau_M = \dfrac{\lambda}{k} = R_M C_M;$	$R_M = \lambda,$	$C_M = \dfrac{1}{k}$

All four elements are characterized by ‘resistance’ and ‘capacitance’ as illustrated in the table.

- Temperature, pressure, voltage and force are analogous ‘**driving**’ or effort variables;
- heat flow rate, volume flow rate, current and velocity are analogous ‘**driven**’ or flow variables.

Identification of the dynamics of an element

In order to identify the transfer function *G(s) of an element, standard input signals* should be used. The two most commonly used standard signals are step and sine wave.

Step response:

The Laplace transform of a step of unit height *u(t) is* ***g (s) = 1/s****. Thus if a first-order element with G(s) = 1/(1 + τ s) is subject to a unit step* input signal, the Laplace transform of the element output signal is:

$$\boxed{\bar{f}_o(s) = G(s)\bar{f}_i(s) = \frac{1}{(1+\tau s)s}}$$

Expressing in partial fractions

$$\boxed{\bar{f}_o(s) = \frac{1}{(1+\tau s)s} = \frac{A}{(1+\tau s)} + \frac{B}{s}}$$

Equating coefficients of constants gives $B = 1$, and equating coefficients of s gives $0 = A + B\tau$, i.e. $A = -\tau$. Thus:

$$\bar{f}_o(s) = \frac{1}{s} - \frac{\tau}{(1+\tau s)} = \frac{1}{s} - \frac{1}{(s+1/\tau)}$$

Using Table in reverse, i.e. finding a time signal $f(t)$ corresponding to a transform $\bar{f}(s)$, we have:

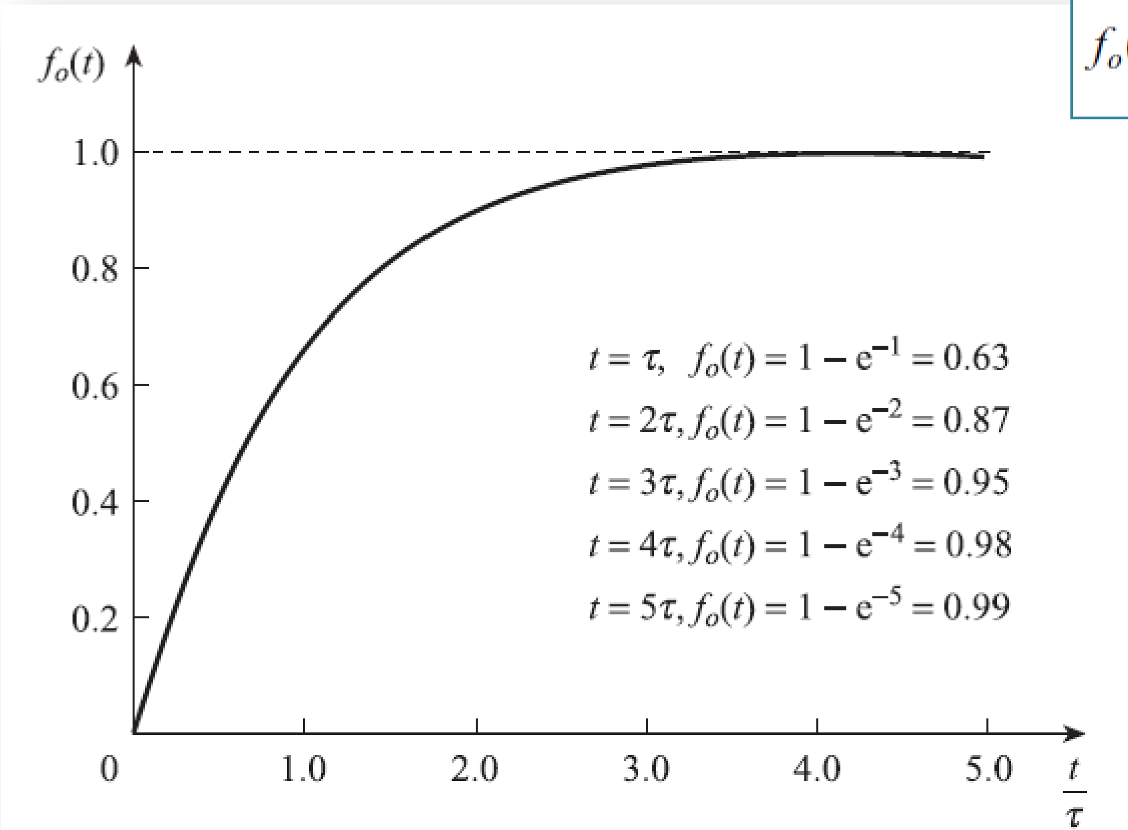
$$f_o(t) = u(t) - \exp\left(\frac{-t}{\tau}\right)$$

and since $u(t) = 1$ for $t > 0$:

$$\boxed{f_o(t) = 1 - \exp\left(\frac{-t}{\tau}\right)}$$

Response of first-order element to unit step

$$f_o(t) = 1 - \exp\left(\frac{-t}{\tau}\right)$$



Response of a first-order element to a unit step.

Example of using *Response of first-order element to unit step equation*

consider the temperature sensor of Initially the temperature of the sensor is equal to that of the fluid, i.e. $T(0-) = T_F(0-) = 25\ °C$, say. If T_F is suddenly raised to $100\ °C$, then this represents a step change ΔT_F of height $75\ °C$. The corresponding *change* in sensor temperature is given by $\Delta T = 75(1 - e^{-t/\tau})$ and the actual temperature T of the sensor at time t is given by:

$\Delta T = T - T(0)$

$$T(t) = 25 + 75(1 - e^{-t/\tau})$$

Thus at time $t = \tau$, $T = 25 + (75 \times 0.63) = 72.3\ °C$. By measuring the time taken for T to rise to $72.3\ °C$ we can find the time constant τ of the element.