EE 407
Homework 4

Due 23:55, 25.12.2017

1 Introduction

In this homework, you will be evaluating several Padé approximations, followed by model-based design methods. Distributed parameter systems as well as feedforward/cascade controllers will also be examined.

2 Questions

1. **Padé Approximation**
In mathematics, a Padé approximant is the "best" approximation of a function by a rational function of given order. This definition is obtained from Wikipedia article titled Padé approximant. Finding the approximation requires you to first find the Taylor series expansion of any given function around a point x_0 and then equate the coefficients of Padé polynomial to that of the Taylor series. Now, as an example, let's prove the equation given in lectures. Let $m > 0$ denote the order of the numerator and $n \geq 1$ denote the order of the denominator of the approximant $R_{[m/n]}$ (that is, $R_{[m/n]} = \frac{\sum_{j=0}^{m} a_j s^j}{1+\sum_{j=1}^{n} b_j s^j}$). For a 1 by 1 approximant $R_{[1/1]}$, using the first $m+n+1 = 3$ coefficients from Taylor series expansion around $s = 0$,

$$\begin{aligned}
&e^{-\theta s} = 1 - \theta s + \frac{\theta^2 s^2}{2!} - \ldots \cong \frac{a_0 + a_1 s}{1 + b_1 s} \\
&a_0 + a_1 s = (1 + b_1 s)(1 - \theta s + \frac{\theta^2}{2} s^2) + \text{ higher order terms of order greater than } s^2 \\
&a_0 = 1, a_1 = b_1 - \theta, 0 = \frac{\theta^2}{2} - b_1 \theta \\
&\implies b_1 = \theta/2 \text{ and } a_1 = -\theta/2
\end{aligned} \tag{1}$$

   (a) Calculate the $R_{[2/2]}$ Padé approximation of $e^{-\theta s}$ around $s = 0$ using $m+n+1$ coefficients of the Taylor series expansion around $s = 0$.

This time we need $n+m+1=5$ coefficients from Taylor series,

$$e^{-\theta s} = 1 - \theta s + \frac{\theta^2 s^2}{2} - \frac{\theta^3 s^3}{6} + \frac{\theta^4 s^4}{24} \ldots \cong \frac{a_0 + a_1 s + a_2 s}{1 + b_1 s + b_2 s}$$

$$a_0 + a_1 s + a_2 s^2 = (1 + b_1 s + b_2 s^2)(1 - \theta s + \frac{\theta^2 s^2}{2} - \frac{\theta^3 s^3}{6} + \frac{\theta^4 s^4}{24}) + \text{ h.o.t of order greater than } s^4$$

$$a_0 = 1$$

$$a_1 = b_1 - \theta$$

$$a_2 = b_2 - b_1\theta + \theta^2/2$$

Then, for the denumerator coefficients,

$$\begin{aligned} 0 &= -b_2\theta + b_1\frac{\theta^2}{2} - \frac{\theta^3}{6} \\ 0 &= b_2\frac{\theta^2}{2} - b_1\frac{\theta^3}{6} + \frac{\theta^4}{24} \\ &\implies b_1 = \theta/2 \text{ and } b_2 = \theta^2/12 \\ &\implies a_1 = -\theta/2 \text{ and } a_2 = \theta^2/12 \end{aligned}$$

(b) Calculate the $R_{[0/1]}$ and $R_{[0/2]}$ Padé approximations of $e^{-\theta s}$ as well.

These can be easily obtained as follows.

$$\begin{aligned} e^{-\theta s} &= \frac{1}{e^{\theta s}} = \frac{1}{1 + \theta s + \frac{\theta^2 s^2}{2} + \ldots} \\ R_{[0/1]} &= \frac{1}{1 + \theta s} \\ R_{[0/2]} &= \frac{1}{1 + \theta s + \frac{\theta^2 s^2}{2}} \end{aligned}$$

(c) Plot the magnitude and phase responses of $R_{[0/1]}$, $R_{[0/2]}$, $R_{[1/1]}$ and $R_{[2/2]}$ for a fixed $\theta = 1$(sec). Compare them to those of the e^{-s}. Justify which one is expected to represent the original system better?

Magnitude and phase responses are given in Fig. 1. All pass systems ($R_{[1/1]}$ and $R_{[2/2]}$) exactly match the magnitude response at all frequencies. $R_{[1/1]}$ is a better match than $R_{[0/1]}$, and so is $R_{[2/2]}$ than $R_{[0/2]}$ in terms of phase responses. Therefore, $R_{[1/1]}$ is expected to outperform $R_{[0/1]}$, and $R_{[2/2]}$ is expected to outperform $R_{[0/2]}$ while representing the original system. No immediate result can be drawn while comparing $R_{[1/1]}$ and $R_{[0/2]}$.

(d) Plot the step responses of those five systems. Does your justification hold from the previous step? How do the systems with non-minimum phase zeros behave arount $t = 0$?

Step responses are given in Fig. 2. As expected, $R_{[1/1]}$ outperforms $R_{[0/1]}$, and $R_{[2/2]}$ outperforms $R_{[0/2]}$ while representing the original system. $R_{[0/1]}$ and $R_{[0/2]}$ try to

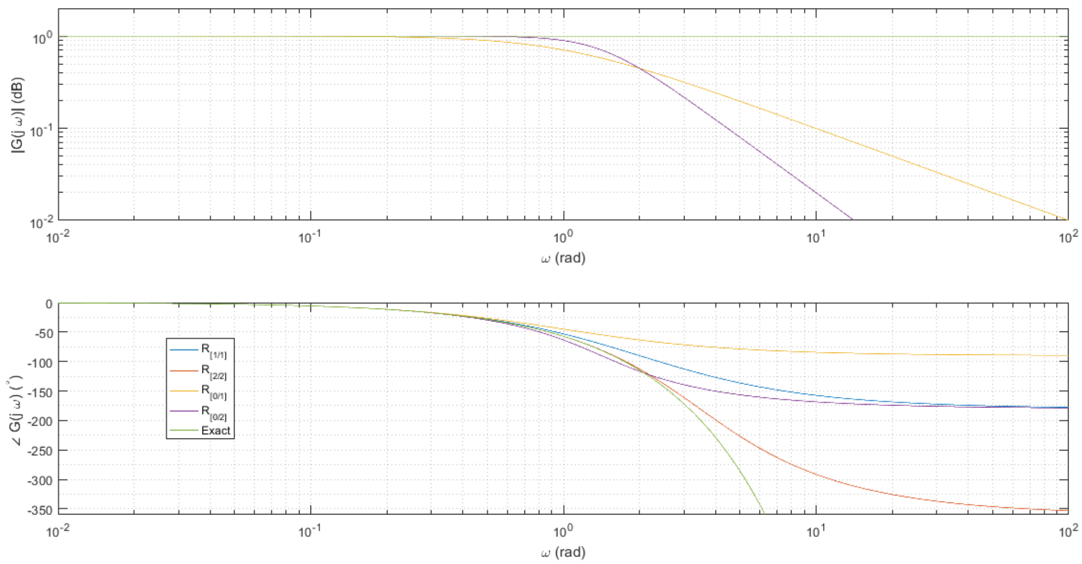

Figure 1

fit all the data equivalently. All pass systems, on the other hand, initially wiggle around 0 thanks to their non-minimum phase zeros and then try to fit the data at steady state with their poles. The # of zero crossings performed by these n-by-n Padé approximations is equal to the # of non-minimum phase zeros they possess. One additional note is that it is not necessary that all systems with non-minimum phase zeros is to perform an initial undershoot; but, it is guaranteed to do so provided that the # of RHP zeros is odd. This last note is not a part of the required answer but is provided for you not to jump to crude conclusions.

2. **Model-Based Design Methods**

   You have learnt two model based controller design methods in your classes. Both methods are closely related and may lead to the same controller parameters if design parameters are specified consistently. In the first part of this problem, you are required to find the IMC controller parameters for two systems as a verification of DS PID parameters. In the second part, you are required to design a PID controller using the DS method and observe the behavior of the plant under uncertainties.

   (a) Derive the PI controller parameters using Internal Model Control design method for the following plant. Assume that $r = 1$.

$$\tilde{G}_p(s) = K_p \frac{1}{\tau_p s + 1} \tag{2}$$

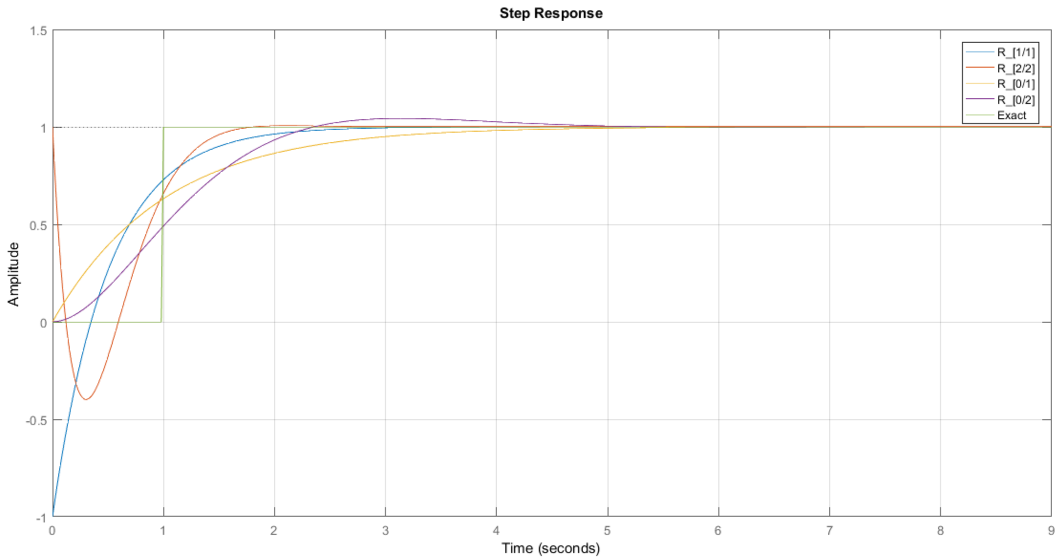

Figure 2

$$
\begin{aligned}
\tilde{G}_p(s) &= K_p \frac{1}{\tau_p s + 1} \implies \tilde{G}_{p-}(s) = K_p \frac{1}{\tau_p s + 1} \;\&\; \tilde{G}_{p+}(s) = 1 \\
G_c^*(s) &= \frac{1}{\tilde{G}_{p-}(s)} \frac{1}{\tau_c s + 1} = \frac{\tau_p s + 1}{K_p(\tau_c s + 1)} \\
G_c(s) &= \frac{G_c^*(s)}{1 - G_c^*(s)\tilde{G}_p(s)} = \frac{\frac{\tau_p s+1}{K_p(\tau_c s+1)}}{1 - \frac{1}{(\tau_c s+1)}} \\
&= \frac{\tau_p s + 1}{K_p \tau_c s} = \frac{\tau_p}{\tau_c K_p}\left(1 + \frac{1}{\tau_p s}\right) \implies K_p = \frac{\tau_p}{\tau_c K_p} \;\&\; T_i = \tau_p
\end{aligned}
$$

(b) Derive the PID controller parameters using Internal Model Control design method for the following plant. Assume that $r = 1$.

$$
\tilde{G}_p(s) = K_p \frac{1}{(\tau_1 s + 1)(\tau_2 s + 1)} \tag{3}
$$

$$
\begin{aligned}
\tilde{G}_p(s) &= K_p \frac{1}{(\tau_1 s+1)(\tau_2 s+1)} \implies \tilde{G}_{p-}(s) = K_p \frac{1}{(\tau_1 s+1)(\tau_2 s+1)} \;\&\; \tilde{G}_{p+}(s) = 1 \\
G_c^*(s) &= \frac{1}{\tilde{G}_{p-}(s)} \frac{1}{\tau_c s+1} = \frac{(\tau_1 s+1)(\tau_2 s+1)}{K_p(\tau_c s+1)} \\
G_c(s) &= \frac{G_c^*(s)}{1-G_c^*(s)\tilde{G}_p(s)} = \frac{\frac{(\tau_1 s+1)(\tau_2 s+1)}{K_p(\tau_c s+1)}}{1-\frac{1}{(\tau_c s+1)}} \\
&= \frac{(\tau_1 s+1)(\tau_2 s+1)}{K_p \tau_c s} = \frac{\tau_1+\tau_2}{\tau_c K_p}\left(1+\frac{1}{(\tau_1+\tau_2)s}+\frac{\tau_1\tau_2}{\tau_1+\tau_2}s\right) \\
&\implies K_c = \frac{\tau_1+\tau_2}{K_p\tau_c}, T_i = \tau_1+\tau_2 \;\&\; T_d = \frac{\tau_1\tau_2}{\tau_1+\tau_2}
\end{aligned}
$$

(c) Derive the PID controller parameters using Internal Model Control design method for the following plant. Assume that $r = 1$.

$$
\tilde{G}_p(s) = K_p \frac{(-\beta s+1)}{(\tau_1 s+1)(\tau_2 s+1)}, \beta > 0
$$

$$
\begin{aligned}
\tilde{G}_p(s) &= K_p \frac{(-\beta s+1)}{(\tau_1 s+1)(\tau_2 s+1)} \implies \tilde{G}_{p-}(s) = K_p \frac{1}{(\tau_1 s+1)(\tau_2 s+1)} \;\&\; \tilde{G}_{p+}(s) = (-\beta s+1) \\
G_c^*(s) &= \frac{1}{\tilde{G}_{p-}(s)} \frac{1}{\tau_c s+1} = \frac{(\tau_1 s+1)(\tau_2 s+1)}{K_p(\tau_c s+1)} \\
G_c(s) &= \frac{G_c^*(s)}{1-G_c^*(s)\tilde{G}_p(s)} = \frac{\frac{(\tau_1 s+1)(\tau_2 s+1)}{K_p(\tau_c s+1)}}{1-\frac{(-\beta s+1)}{(\tau_c s+1)}} \\
&= \frac{(\tau_1 s+1)(\tau_2 s+1)}{K_p (\tau_c+\beta) s} = \frac{\tau_1+\tau_2}{(\tau_c+\beta) K_p}\left(1+\frac{1}{(\tau_1+\tau_2)s}+\frac{\tau_1\tau_2}{\tau_1+\tau_2}s\right) \\
&\implies K_c = \frac{\tau_1+\tau_2}{K_p(\tau_c+\beta)}, T_i = \tau_1+\tau_2 \;\&\; T_d = \frac{\tau_1\tau_2}{\tau_1+\tau_2}
\end{aligned}
$$

Now, consider the following process.

$$
\tilde{G}_p(s) = \frac{K}{(10s+1)(5s+1)} \tag{4}
$$

where $K = 1$. Find the PID controller parameters using Direct Synthesis mehod with $\tau_c = 5$ (min).

(d) Simulate the system for the perfect model.

Using the result of part b, $K_c = \frac{\tau_1+\tau_2}{K\tau_c} = \frac{15}{1\cdot 5} = 3$, $T_i = \tau_1 + \tau_2 = 15$ min & $T_d = \frac{\tau_1\tau_2}{\tau_1+\tau_2} = 3.33$ (min). Step response of the controlled system can be found in 3, which is identical to the step response of a first order plant with unity gain & $\tau_c = 5$ min.

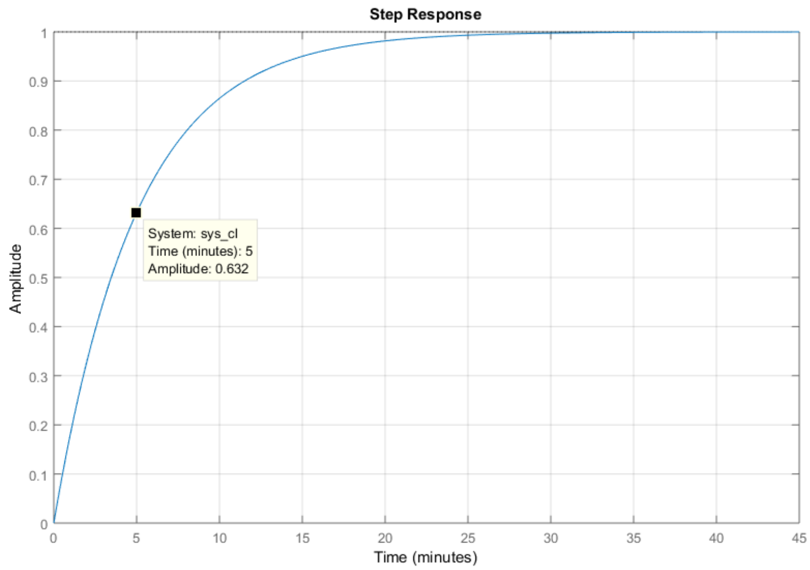

Figure 3

(e) Suppose that K changes unexpectedly from 1 to $1+\alpha$. Find the range of α for which the closed loop system is stable.

Let's find the denominator of the closed loop system. As you can see below the two roots are always with negative real parts, the only root that depends on α is given by $s_3 = -0.2(1+\alpha)$, which is negative provided that $\alpha > -1$ (when $\alpha < -1$ you would be controlling a reverse acting system with a direct acting controller, and it would naturally be unstable). Different than the derivation below, you could perform a pole-zero cancellation at the first stage and find exactly the same solution. What you're doing with IMC/DS desing methods is canceling the original poles & zeros of the system and introducing new roots at desired locations by manipulating PID parameters.

$$d(s) = 1 + \frac{KK_c\left(1 + \frac{1}{T_i s} + T_d s\right)}{(10s+1)(5s+1)} = 1 + K\frac{3\frac{50s^2+15s+1}{15s}}{(10s+1)(5s+1)} = 0$$

$$d_1(s) = (50s^2+15s+1)15s + (1+\alpha)3(50s^2+15s+1) = (50s^2+15s+1)(15s+3+3\alpha) = 0$$

(f) What is the limiting value of τ_c if you are given that $|\alpha| < 0.2$?

Using a similar approach to the previous step, $\tau_c = 0$ can be found as the limiting value for stability. This is a natural result of having the closed loop step response as $(1-e^{-t(1+\alpha)/\tau_c})u(t)$ (except for $\tau_c = 0$). When $\tau_c < 0$, step response grows exponentially without a bound.

$$d(s) = 1 + \frac{\cancel{15}}{1\cdot\tau_c}\frac{\cancel{50s^2+15s+1}}{\cancel{15}s}\frac{1}{\cancel{(10s+1)(5s+1)}}(1+\alpha) = 0$$

$$d_1(s) = \tau_c s + (1+\alpha) \implies s_3 = -(1+\alpha)/\tau_c$$

3. **Distributed Parameter Systems**

Consider the following normalized partial differential equation

$$\frac{\partial T}{\partial x} + \frac{\partial T}{\partial t} = 0 \tag{5}$$

subject to the following conditions.

$$T(x,0) = T_e \text{ for } 0 < x \leq 1 \text{ (initial condition)}$$
$$T(0,t) = V(t) + T_e \text{ for } t > 0 \text{ (boundary condition)}$$

Let $\eta(x,t) = T(x,t) - T_e$. Rewrite the differential equation & boundary conditions. Then use them to find T(x,t) using Laplace transform. Note that as the DE is normalized algebraic equations between x and t are allowed (e.g. x/t).

The differential equation and the conditions become

$$\frac{\partial \eta}{\partial x} + \frac{\partial \eta}{\partial t} = 0$$
$$\eta(x,0) = 0 \text{ for } 0 < x \leq 1 \text{ (initial condition)}$$
$$\eta(0,t) = V \text{ for } t > 0 \text{ (boundary condition)}$$

Taking the Laplace transform with $\mathcal{L}\{\eta(x,t)\} = T^*(x,s)$ and $\mathcal{L}\{\frac{\partial \eta(x,t)}{\partial t}\} = sT^*(x,s) - \cancelto{0}{\eta(x,0)} = sT^*(x,s)$, and trying the candidate solution $T^*(x,s) = ae^{rx}$,

$xae^{rx} + se^{rx} = 0 \implies r = -s.$

Finally, applying the boundary condition to find a and going back to time domain,

$T^*(x,s) = ae^{-sx}, T^*(0,s) = \mathcal{L}\{V\} = V^* = a, \eta(x,t)\mathcal{L}^{-1}\{V^*e^{-sx}\} = V(t) * \delta(t-x) = V(t-x)$

$\implies T(x,t) = V(t-x) + T_e$

4. **Feedforward Control**

For the feedforward control structure given in Fig. 4, let $G_p = \frac{1}{(2s+1)(3s+1)}$, $G_d = \frac{1}{s+1}$.

(a) Find the ideal G_{ff} in order to have the transfer function $D(s)/Y(s) = 0$.

The closed loop transfer function & its time constant can be found as shown below. The inner loop is made five times as fast with a cascade controller. **Note:** In an initial version of this homework the inner loop measuring device was wrongly annotated as G_{m1}. As this correction was done on the last day (which you might not have noticed) both results will be accepted as correct.

$Y = DG_d + G_p(DG_{ff} + K(R - Y)) \implies Y(1 + KG_p) = D(G_d + G_pG_{ff}) + RKG_p$

In order to have $Y/D = 0$ we need $G_{ff} = -G_d/G_p = -\dfrac{(3s+1)(2s+1)}{s+1}$

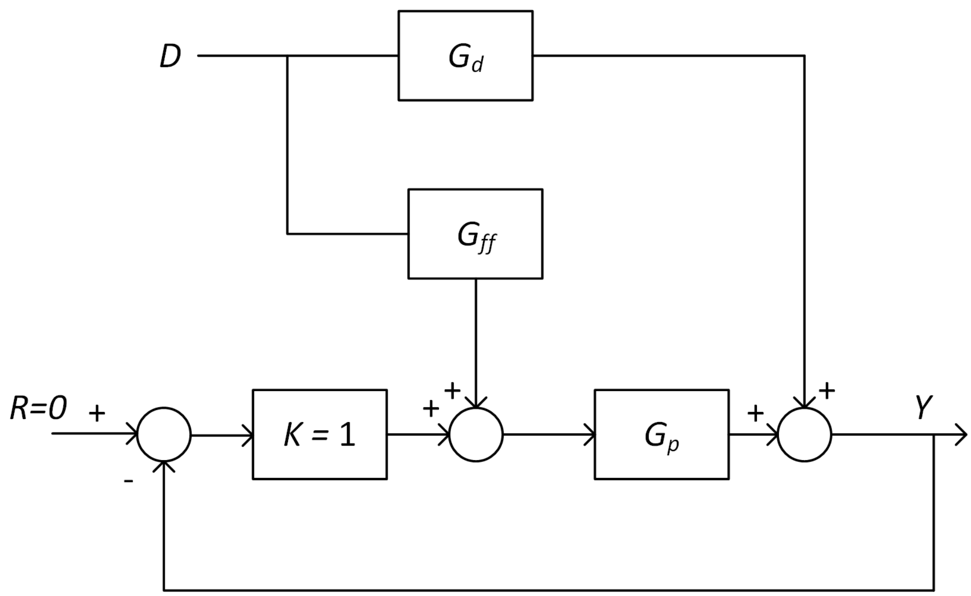

Figure 4

(b) Discuss why the transfer function you found above is not physically realizable (an improper transfer function is unrealizable, but why?).

Several arguments can be made here. One of them would be that the differentiation operator is not causal (in order to calculate the exact slope at time instant t_0 we would need the values of a function at $t_0 - \epsilon$ & $t_0 + \epsilon$) and every physically realizable system has to be causal. When the transfer function is not proper, you would obtain first and perhaps higher order differentiation operations when you perform polynomial division.

(c) Approximate the numerator by a first order transfer function & find G_{ff}. Use the same approach as what you do when you approximate pure time delay as $e^{-\theta_p s} = 1 - \theta_p s + \frac{(\theta_p s)^2}{2!} - \ldots \cong 1 - \theta_p s$.

Using the first two terms of the Taylor series approximation of the numerator $(3s + 1)(2s + 1)$ (T.S. approximation of a polynomial is nothing but itself), feedforward controller can simply be obtained as $G_{ff} = -\frac{5s+1}{s+1}$.

(d) Simulate the system for a unit step disturbance with and without the feedforward controller. Comment on the effect of the feedforward controller.

Simulation result can be seen in Fig. 5. Without the feedforward controller, a P-only controller would not be able to eliminate the steady state error because of the disturbance.

5. **Cascade Control**

Consider the cascade control structure given in Fig. 6. Let $G_v = \frac{5}{s+1}$, $G_p = \frac{4}{(4s+1)(2s+1)}$, $G_{c2} = K_{c2} = 4$, $G_{m1} = 0.05$ and $G_{m2} = 0.2$, where all time constants have the units of minutes.

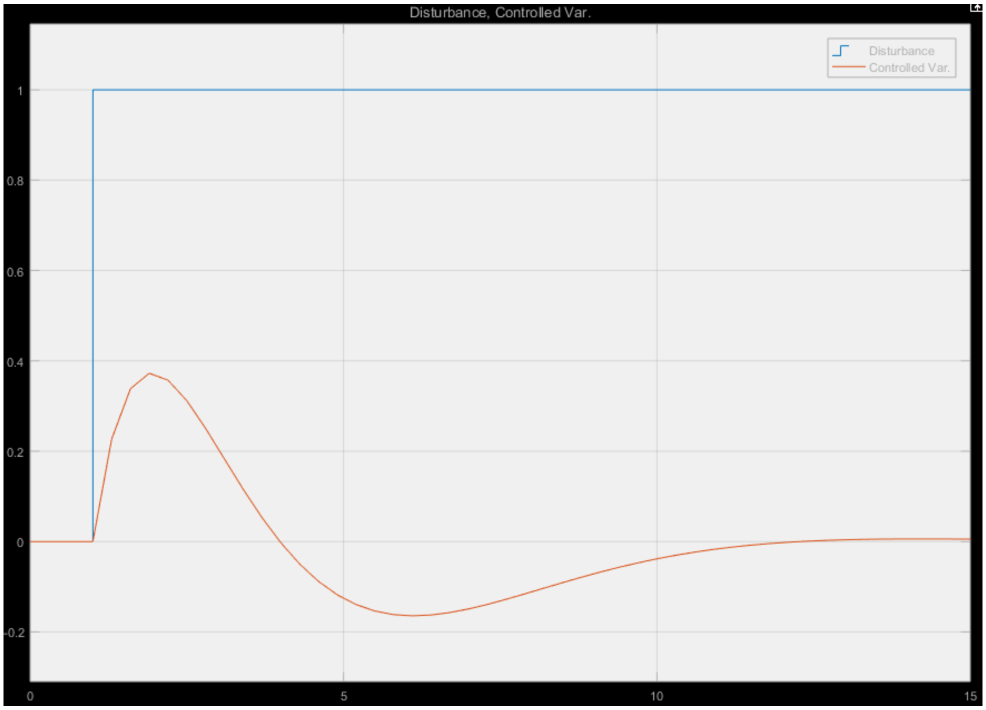

Figure 5

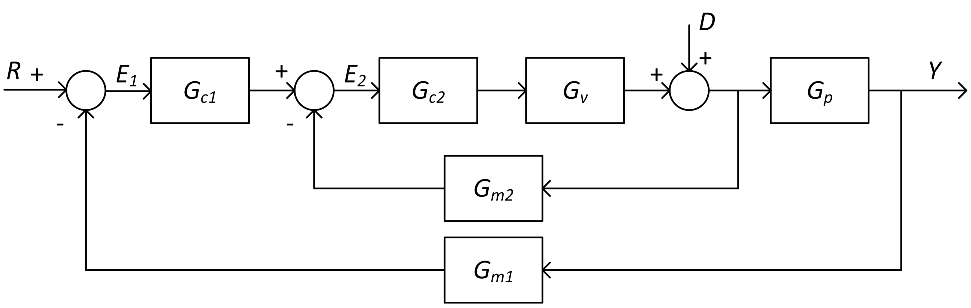

Figure 6

(a) Compare the open and closed loop time constants of the inner loop.

The closed loop transfer function & its time constant can be found as shown below. The inner loop is made five times as fast with a cascade controller. **Note:** In an initial version of this homework the inner loop measuring device was wrongly annotated as G_{m1}. As this correction was done on the last day (which you might not have noticed) both results will be accepted as correct.

$$G_{cl}(s) = \frac{G_{c2}G_v}{1 + G_{c2}G_vG_{m2}} = \frac{4\frac{5}{s+1}}{1 + 4 \cdot 0.2\frac{5}{s+1}} = \frac{20}{s+5} = \frac{4}{0.2s+1}$$

$$\implies \tau_{cl} = 0.2 \text{ min}$$

(b) Find the proportional only gain using Ziegler-Nichols continuous cycling method.

$$\mathrm{den}(s) = 1 + K_{c1}\frac{4}{0.2s+1}\frac{4}{(4s+1)(2s+1)}0.05 = 1 + \frac{4K_{c1}}{(s+5)(4s+1)(2s+1)}$$

$$\mathrm{den}_1(s) = 8s^3 + 46s^2 + 31s + 5 + 4K_{c1,c} = 0$$

Applying Routh-Hurwitz stability criterion, critical gain can be found as

$$K_{c1,c} = \frac{46 \cdot 31 - 40}{32} \cong 43.31 \implies K_{c1} \cong 21.66$$

(c) Find the proportional only gain with the same method, but this time without the inner controller (that is, set $G_{m2} = 0$ and $K_{c2} = 1$).

$$\mathrm{den}(s) = 1 + K_{c1}\frac{5}{s+1}\frac{4}{8s^2+6s+1}0.05$$

$$\mathrm{den}_1(s) = 8s^3 + 14s^2 + 7s + 1 + K_{c1,c} = 0$$

Applying Routh-Hurwitz stability criterion, critical gain can be found as

$$K_{c1,c} = \frac{14 \cdot 7 - 8}{8} = 11.25 \implies K_{c1} \cong 5.63$$

(d) Find the steady state error values (E_1) for a unit step change in the disturbance D for both systems (with and without the inner controller).

Let's start by finding the transfer function $G(s) = \frac{E_1(s)}{D(s)}$. Let R_2 denote the set point of the inner loop and Y_2 its controlled variable. Then,

$$\begin{aligned}
&Y_2 = G_{c2}G_vE_2 + D = G_{c2}G_v(R_2 - G_{m2}Y_2) + D \\
\implies &Y_2 = \frac{G_{c2}G_v}{G_{c2}G_vG_{m2}+1}R_2 + \frac{1}{G_{c2}G_vG_{m2}+1}D \triangleq G_1R_2 + G_2D \\
&E_1 = R - G_{m1}Y = R - G_{m1}G_pY_2 = R - G_{m1}G_p\Big[G_1R_2 + G_2D\Big] \\
&\quad = R - G_{m1}G_p\Big[G_1G_{c1}E_1 + G_2D\Big] \\
&E_1 = \frac{1}{1+G_1G_{c1}G_{m1}G_p}R - \frac{G_{m1}G_pG_2}{1+G_1G_{c1}G_{m1}G_p}D \\
&\quad = G'(s)R - \frac{G_{m1}G_p}{1+G_{c2}G_vG_{m2}+G_{c1}G_{c2}G_vG_{m1}G_p}D
\end{aligned}$$

We know that the system is stable; hence, $E_{1,ss}$ exists and we can use the final value theorem to find $E_{1,ss}$.

$$\begin{aligned}
E_{1,ss} &= \lim_{t\to\infty} E_1(t) = \lim_{s\to 0} s\frac{1}{s}\left(-\frac{G_{m1}G_p}{1+G_{c2}G_vG_{m2}+G_{c1}G_{c2}G_vG_{m1}G_p}\right) \\
&= -\lim_{s\to 0}(\frac{G_{m1}G_p}{1+G_{c2}G_vG_{m2}+G_{c1}G_{c2}G_vG_{m1}G_p} \\
&\cong -2.2 \cdot 10^{-3} \text{ with the inner loop controller} \\
&\cong -30.2 \cdot 10^{-3} \text{ without the inner loop controller}
\end{aligned}$$

(e) Simulate the systems to verify your results. You may assume that $R = 0$.

Simulation results that verify previous results are given in Fig. 7.

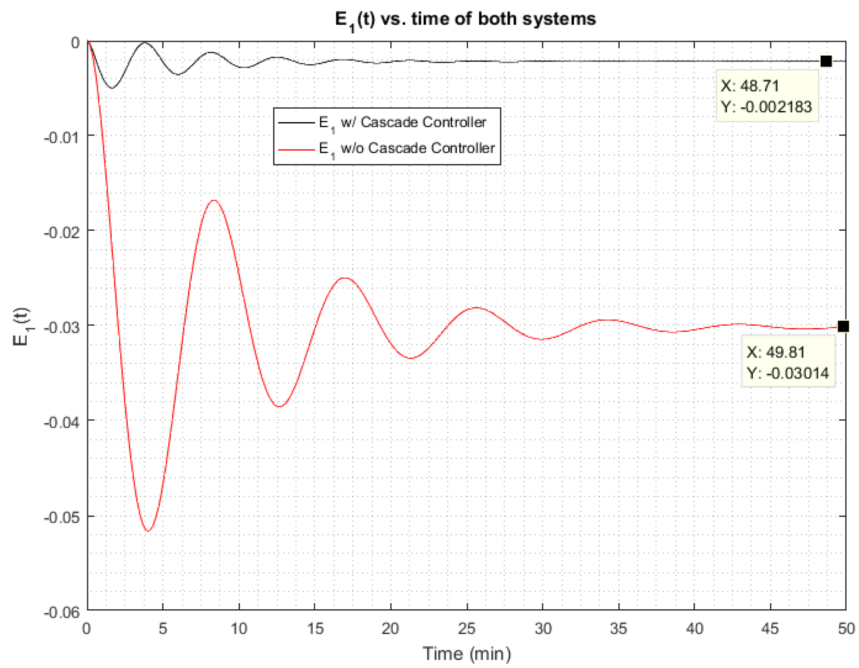

Figure 7

(f) Compare the two systems in terms of stability, disturbance rejection performance in the inner loop and speed.

System with a cascade controller is more stable as it shows less oscillatory behaviour for a unit step disturbance, and it does so even when it has much greater outer loop gain. Effect of the disturbance is mostly eliminated in the inner loop. Thanks to this, and thanks to having a larger gain in the outer loop, $E_{1,ss}$ is much less for the cascaded structure. For this same reason, it is much faster in eliminating the effect of inner loop disturbances.