PROCESS CONTROL

A First Course with MATLAB®

PAO C. CHAU

Process Control

A First Course with MATLAB

Pao C. Chau
University of California, San Diego

CAMBRIDGE
UNIVERSITY PRESS

PUBLISHED BY THE PRESS SYNDICATE OF THE UNIVERSITY OF CAMBRIDGE
The Pitt Building, Trumpington Street, Cambridge, United Kingdom

CAMBRIDGE UNIVERSITY PRESS
The Edinburgh Building, Cambridge CB2 2RU, UK
40 West 20th Street, New York, NY 10011-4211, USA
477 Williamstown Road, Port Melbourne, VIC 3207, Australia
Ruiz de Alarcón 13, 28014 Madrid, Spain
Dock House, The Waterfront, Cape Town 8001, South Africa

http://www.cambridge.org

First published 2002

Printed in the United States of America

Typeface Times Roman 10.25/12.5 pt.   *System* LaTeX 2_ε [TB]

A catalog record for this book is available from the British Library.

Library of Congress Cataloging in Publication Data
Chau, Pao C.
Process control : a first course with MATLAB / Pao C. Chau.
p. cm. – (Cambridge series in chemical engineering)
Includes bibliographical references and index.
ISBN 0-521-80760-3 – ISBN 0-521-00255-9 (pb.)
1. Chemical process control – Data processing. 2. MATLAB. I Title. II. Series.
TP155.75 .C42 2002
660′.2815 – dc21   2001052567

ISBN 0 521 80760 3 hardback
ISBN 0 521 00255 9 paperback

Table of Contents

The best approach to control is to think of it as applied mathematics. Virtually everything we do in this introductory course is related to the properties of first and second order differential equations, and with different techniques in visualizing the solutions.

Chemical Process Control: A First Course with MATLAB

Pao C. Chau

University of California, San Diego

Preface

This is an introductory text written from the perspective of a student. The major concern is not how much material we cover, but rather, how to present the most important and basic concepts that one should grasp in a first course. If your instructor is using some other text that you are struggling to understand, we hope we can help you too. The material here is the result of a process of elimination. The writing and examples are succinct and self-explanatory, and the style is purposely unorthodox and conversational. To a great extent, the style, content, and the extensive use of footnotes are molded heavily by questions raised in class. I left out very few derivation steps. If they were, the missing steps are provided as hints in the Review Problems at the back of each chapter. I strive to eliminate those "easily obtained" results that baffle many of us. Most students should be able to read the material on their own. You just need basic knowledge in differential equations, and it helps if you have taken a course on writing material balances. With the exception of chapters 4, 9, and 10, which should be skipped in a quarter-long course, it also helps if you proceed chapter by chapter. The presentation of material is not intended for someone to just jump right in the middle of the text. We place a very strong emphasis on developing analytical skills. To keep pace with the modern computer era, we also take a coherent and integrated approach to using a computational tool. We believe in active learning. When you read the chapters, it is very important that you have MATLAB with its Control Toolbox to experiment and test the examples firsthand.

Notes to Instructors

There are probably more introductory texts in control than other engineering disciplines. It is arguable whether we need another control text. As we move into the era of hundred dollar textbooks, I believe we can lighten the economic burden, and with the Internet, assemble a new generation of modularized texts that soften the printing burden by off loading selected material to the Web. Still a key resolve is to scale back on the scope of a text to the most crucial basics. How much students can, or be enticed to, learn is inversely proportional to the number of pages that they have to read—akin to diminished magnitude and increased lag in frequency response. So as textbooks become thicker over the years in attempts to reach out to students and are excellent resources from the perspective of instructors, these texts are by no means more effective pedagogical tools. This project was started as a set of review notes when I found students having trouble identifying the key concepts in these expansive texts. I also found these texts in many circumstances deter students from active learning and experimenting on their own.

At this point, the contents are scaled down to fit a one-semester course. On a quarter system, Chapters 4, 9, and 10 can be omitted. With the exception of two chapters (4 and 9) on state space models, the organization has "evolved" to become very classical. The syllabus is chosen such that students can get to tuning PID controllers before they lose interest. Furthermore, discrete-time analysis has been discarded. If there is to be one introductory course in the undergraduate curriculum, it is very important to provide an exposure to state space models as a bridge to a graduate level course. The last chapter on mutliloop systems is a collection of topics that are usually handled by several chapters in a formal text. This chapter is written such that only the most crucial concepts are illustrated and that it could be incorporated comfortably in a one-semester curriculum. For schools with the luxury of two control courses in the curriculum, this last chapter should provide a nice introductory transition. Because the material is so restricted, we emphasize that this is a "first course" textbook, lest a student might mistakenly ignore the immense expanse of the control field. We also have omitted appendices and extensive references. As a modularized tool, we use our Web Support to provide references, support material, and detailed MATLAB plots and results.

Homework problems are also handled differently. At the end of each chapter are short, mostly derivation type, problems which we call Review Problems. Hints or solutions are provided for these exercises. To enhance the skill of problem solving, we take the extreme approach, more so than

Stephanopoulos (1984), of collecting major homework problems at the back and not at the end of each chapter. Our aim is to emphasize the need to understand and integrate knowledge, a virtue that is endearing to ABET, the engineering accreditation body in the United States. These problems do not even specify the associated chapter as many of them involve different techniques. A student has to determine the appropriate route of attack. An instructor may find it aggravating to assign individual parts of a problem, but when all the parts are solved, we hope the exercise would provide a better perspective to how different ideas are integrated.

To be an effective teaching tool, this text is intended for experienced instructors who may have a wealth of their own examples and material, but writing an introductory text is of no interest to them. The concise coverage conveniently provides a vehicle with which they can take a basic, minimalist set of chapters and add supplementary material that they deem appropriate. Even without supplementary material, however, this text contains the most crucial material and there should not be a need for an additional expensive, formal text.

While the intended teaching style relies heavily on the use of MATLAB, the presentation is very different from texts which prepare elaborate M-files and even menu-driven interfaces. One of the reasons why MATLAB is such a great tool is that it does not have a steep learning curve. Students can quickly experiment on their own. Spoon-feeding with our misguided intention would only destroy the incentive to explore and learn on one's own. To counter this pitfall, strong emphasis is placed on what one can accomplish easily with only a few MATLAB statements. MATLAB is introduced as walk-through tutorials that encourage students to enter commands on their own. As strong advocates of active learning, we do not duplicate MATLAB results. Students, again, are encouraged to execute the commands themselves. In case help is needed, our Web Support, however, has the complete set of MATLAB results and plots. This organization provides a more coherent discourse on how one can make use of different features of MATLAB, not to mention saving significant printing costs. Finally, we can revise the tutorials easily to keep up with the continual upgrade of MATLAB. At this writing, the tutorials are based on MATLAB version 5.3, and the object-oriented functions in the Control Toolbox version 4.2. Simulink version 3.0 is also utilized, but its scope is limited to simulating more complex control systems.

As a first course text, the development of models is limited to stirred-tanks, stirred tank heater, and a few other examples that are used extensively and repeatedly throughout the chapters. Our philosophy is one step back in time. The focus is the theory and the building of a foundation that may help to solve other problems. The design is also to be able to launch into the topic of tuning controllers before students may lose interest. The coverage of Laplace transform is not entirely a concession to remedial mathematics. The examples are tuned to illustrate immediately how pole positions may relate to time domain response. Furthermore, students tend to be confused by the many different design methods. As much as I can, especially in the controller design chapters, the same examples are used throughout. The goal is to help a student understand how the same problem can be solved by different techniques.

We have given up the pretense that we can cover controller design and still have time to do all the plots manually. We rely on MATLAB to construct the plots. For example, we take a unique approach to root locus plots. We do not ignore it like some texts do, but we also do not go into the hand sketching details. The same can be said with frequency response analysis. On the whole, we use root locus and Bode plots as computational and pedagogical tools in ways that can help to understand the choice of different controller designs. Exercises that may help such thinking are in the MATLAB tutorials and homework problems.

Finally, I have to thank Costas Pozikidris and Florence Padgett for encouragement and support on this project, Raymond de Callafon for revising the chapters on state space models, and Allan Cruz for proofreading. Last but not least, Henry Lim combed through the manuscript and made numerous insightful comments. His wisdom is sprinkled throughout the text.

Web Support (MATLAB outputs of text examples and MATLAB sessions, references, and supplementary notes) is available at the CENG 120 homepage. Go to `http://courses.ucsd.edu` and find CENG 120.

❖ 1. *Introduction*

Control systems are tightly intertwined in our daily lives, so much that we take them for granted. They may be as low-tech and unglamorous as our flush toilet. Or they may be as high-tech as electronic injection in our cars. In fact, there is more than a handful of computer control systems in a typical car that we now drive. Everything from the engine to transmission, shock absorber, brakes, pollutant emission, temperature and so forth, there is an embedded microprocessor controller keeping an eye out for us. The more gadgetry, the more tiny controllers pulling the trick behind our backs.[1] At the lower end of consumer electronic devices, we can bet on finding at least one embedded microcontroller.

In the processing industry, controllers play a crucial role in keeping our plants running—virtually everything from simply filling up a storage tank to complex separation processes, and to chemical reactors.

As an illustration, let us take a look at a bioreactor (Fig. 1.1). To find out if the bioreactor is operating properly, we monitor variables such as temperature, pH, dissolved oxygen, liquid level, feed flow rate, and the rotation speed of the impeller. In some operations, we may also measure the biomass and the concentration of a specific chemical component in the liquid or the composition of the gas effluent. In addition, we may need to monitor the foam head and make sure it does not become too high. We most likely need to monitor the steam flow and pressure during the sterilization cycles. We should note that the schematic diagram is far from complete. By the time we have added enough details to implement all the controls, we may not recognize the bioreactor. We certainly do not want to scare you with that. On the other hand, this is what makes control such a stimulating and challenging field.

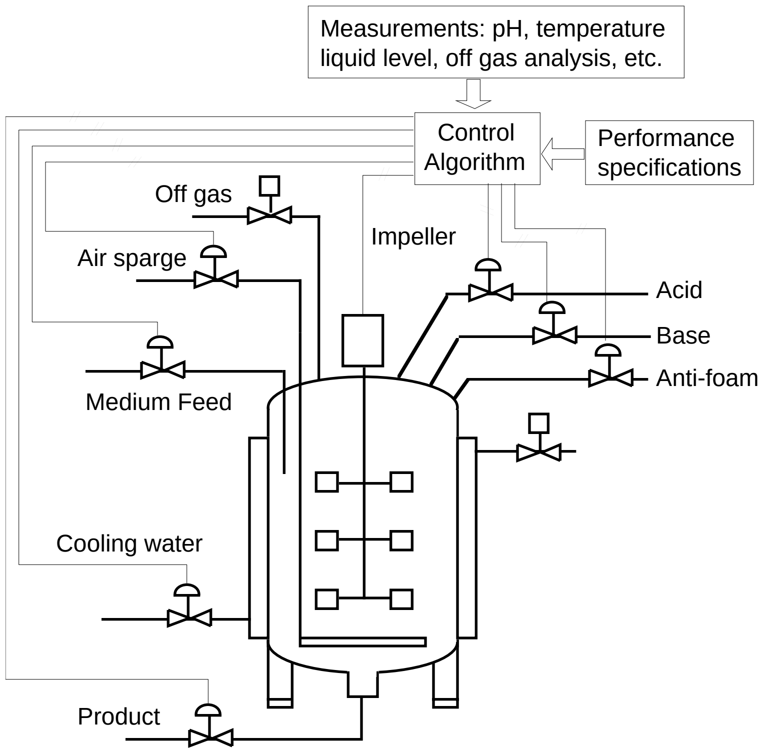

Measurements: pH, temperature liquid level, off gas analysis, etc.
Control Algorithm
Performance specifications
Off gas
Air sparge
Impeller
Acid
Base
Anti-foam
Medium Feed
Cooling water
Product

Figure 1.1. Schematic diagram of instrumentation associated with a fermentor. The steam sterilization system and all sensors and transmitters are omitted for clarity. Solid lines represent process streams. Hairlines represent information flow.

[1] In the 1999 Mercedes-Benz S-Class sedan, there are about 40 "electronic control units" that control up to 170 different variables.

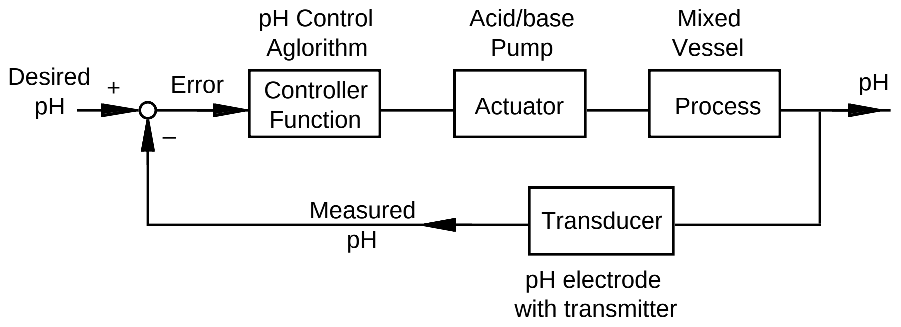


Figure 1.2. A block diagram representation of a single-input single-output negative feedback system. Labels within the boxes are general. Labels outside the boxes apply to the simplified pH control discussion.

For each quantity that we want to maintain at some value, we need to ensure that the bioreactor is operating at the desired conditions. Let's use the pH as an example. In control calculations, we commonly use a **block diagram** to represent the problem (Fig. 1.2). We will learn how to use mathematics to describe each of the blocks. For now, the focus is on some common terminology.

To consider pH as a **controlled variable**, we use a pH electrode to measure its value and, with a transmitter, send the signal to a controller, which can be a little black box or a computer. The controller takes in the pH value and compares it with the desired pH, what we call the **set point** or **reference**. If the values are not the same, there is an **error**, and the controller makes proper adjustments by manipulating the acid or the base pump—the **actuator**.[2] The adjustment is based on calculations using a **control algorithm**, also called the control law. The error is calculated at the summing point where we take the desired pH minus the measured pH. Because of how we calculate the error, this is a **negative feedback** mechanism.

This simple pH control scenario is what we call a single-input single-output (**SISO**) system; the single input is the set point and the output is the pH value.[3] This simple feedback mechanism is also what we called a **closed-loop**. This single loop system ignores the fact that the dynamics of the bioreactor involves complex interactions among different variables. If we want to take a more comprehensive view, we will need to design a multiple-input multiple-output (**MIMO**), or **multivariable**, system. When we invoke the term **system**, we are referring to the **process** [4] (the bioreactor here), the **controller**, and all other instrumentation such as **sensors**, **transmitters**, and **actuators** (like valves and pumps) that enable us to control the pH.

When we change a specific operating condition, meaning the set point, we would like, for example, the pH of the bioreactor to follow our command. This is what we call **servo** control. The pH value of the bioreactor is subjected to external **disturbances** (also called **load** changes), and the task of suppressing or rejecting the effects of disturbances is called **regulatory** control. Implementation of a controller may lead to instability, and the issue of system **stability** is a major concern. The control system also has to be **robust** such that it is not overly sensitive to changes in process parameters.

[2] In real life, bioreactors actually use on-off control for pH.

[3] We'll learn how to identify input and output variables, how to distinguish between manipulated variables, disturbances, measured variables and so forth. Do not worry about remembering all the terms here. We'll introduce them properly later.

[4] In most of the control world, a process is referred to as a **plant**. We stay with "process" because in the process industry, a plant carries the connotation of the entire manufacturing or processing facility.

What are some of the issues when we design a control system? In the first place, we need to identify the role of various variables. We need to determine what we need to control, what we need to manipulate, what are the sources of disturbances, and so forth. We then need to state our design objective and specifications. It may make a difference whether we focus on the servo or the regulator problem, and we certainly want to make clear, quantitatively, the desired response of the system. To achieve these goals, we have to select the proper control strategy and controller. To implement the strategy, we also need to select the proper sensors, transmitters, and actuators. After all is done, we have to know how to tune the controller. Sounds like we are working with a musical instrument, but that's the jargon.

The design procedures depend heavily on the dynamic model of the process to be controlled. In more advanced model-based control systems, the action taken by the controller actually depends on the model. Under circumstances where we do not have a precise model, we perform our analysis with approximate models. This is the basis of a field called "system identification and parameter estimation." Physical insight that we may acquire in the act of model building is invaluable in problem solving.

While we laud the virtue of dynamic modeling, we will not duplicate the introduction of basic conservation equations. It is important to recognize that all of the processes that we want to control, e.g. bioreactor, distillation column, flow rate in a pipe, a drug delivery system, etc., are what we have learned in other engineering classes. The so-called model equations are conservation equations in heat, mass, and momentum. We need force balance in mechanical devices, and in electrical engineering, we consider circuits analysis. The difference between what we now use in control and what we are more accustomed to is that control problems are *transient* in nature. Accordingly, we include the time derivative (also called accumulation) term in our balance (model) equations.

What are some of the mathematical tools that we use? In *classical* control, our analysis is based on linear ordinary differential equations with constant coefficients—what is called **linear time invariant** (**LTI**). Our models are also called **lumped-parameter** models, meaning that variations in space or location are not considered. Time is the only independent variable. Otherwise, we would need partial differential equations in what is called **distributed**-parameter models. To handle our linear differential equations, we rely heavily on **Laplace transform**, and we invariably rearrange the resulting algebraic equation into the so-called **transfer functions**. These algebraic relations are presented graphically as block diagrams (as in Fig. 1.2). However, we rarely go as far as solving for the time-domain solutions. Much of our analysis is based on our understanding of the roots of the characteristic polynomial of the differential equation—what we call the **poles**.

At this point, we should disclose a little secret. Just from the terminology, we may gather that control analysis involves quite a bit of mathematics, especially when we go over stability and frequency response methods. That is one reason why we delay introducing these topics. Nonetheless, we have to accept the prospect of working with mathematics. We would be lying if we say that one can be good in process control without sound mathematical skills.

It may be useful to point out a few topics that go beyond a first course in control. With certain processes, we cannot take data continuously, but rather in certain selected slow intervals (c.f. titration in freshmen chemistry). These are called **sampled-data** systems. With computers, the analysis evolves into a new area of its own—**discrete-time** or **digital** control systems. Here, differential equations and Laplace transform do not work anymore. The mathematical techniques to handle discrete-time systems are difference equations and ***z*-transform**. Furthermore, there are **multivariable** and **state space** control, which we will encounter a brief introduction. Beyond the introductory level are optimal control, nonlinear control, adaptive control, stochastic control, and fuzzy logic control. Do not lose the perspective that control is an immense field. Classical control appears insignificant, but we have to start some where and onward we crawl.

❖ 2. *Mathematical Preliminaries*

Classical process control builds on linear ordinary differential equations and the technique of Laplace transform. This is a topic that we no doubt have come across in an introductory course on differential equations—like two years ago? Yes, we easily have forgotten the details. We will try to refresh the material necessary to solve control problems. Other details and steps will be skipped. We can always refer back to our old textbook if we want to answer long forgotten but not urgent questions.

What are we up to?

- The properties of Laplace transform and the transforms of some common functions. We need them to construct a table for doing **inverse transform**.
- Since we are doing inverse transform using a look-up table, we need to break down any given transfer functions into smaller parts which match what the table has—what is called **partial fractions**. The time-domain function is the sum of the inverse transform of the individual terms, making use of the fact that Laplace transform is a linear operator.
- The time-response characteristics of a model can be inferred from the **poles**, *i.e.*, the roots of the characteristic polynomial. This observation is independent of the input function and singularly the most important point that we must master before moving onto control analysis.
- After Laplace transform, a differential equation of deviation variables can be thought of as an input-output model with **transfer functions**. The causal relationship of changes can be represented by **block diagrams**.
- In addition to transfer functions, we make extensive use of steady state gain and time constants in our analysis.
- Laplace transform is only applicable to *linear* systems. Hence, we have to **linearize** nonlinear equations before we can go on. The procedure of linearization is based on a first order Taylor series expansion.

2.1 A simple differential equation model

We first provide an impetus of solving differential equations in an approach unique to control analysis. The mass balance of a well-mixed tank can be written (see Review Problems) as

$$\tau \frac{dC}{dt} = C_{in} - C \,, \quad \text{with } C(0) = C_o$$

where C is the concentration of a component, C_{in} is the inlet concentration, C_o is the initial concentration, and τ is the space time. In classical control problems, we invariably rearrange the equation as

$$\tau \frac{dC}{dt} + C = C_{in} \tag{2-1}$$

and further redefine variables $C' = C - C_o$ and $C'_{in} = C_{in} - C_o$.[1] We designate C' and C'_{in} as

[1] At steady state, $0 = C^s_{in} - C^s$, and if $C^s_{in} = C_o$, we can also define $C'_{in} = C_{in} - C^s_{in}$. We'll come back to this when we learn to linearize equations. We'll see that we should choose $C_o = C^s$.

deviation variables—they denote how a quantity deviates from the original value at t = 0.[1] Since C_o is a constant, we can rewrite Eq. (2-1) as

$$\tau \frac{dC'}{dt} + C' = C'_{in} , \qquad \text{with } C'(0) = 0 \tag{2-2}$$

Note that the equation now has a zero initial condition. For reference, the solution to Eq. (2-2) is [2]

$$C'(t) = \frac{1}{\tau} \int_0^t C'_{in}(z)\, e^{-(t-z)/\tau}\, dz \tag{2-3}$$

If C'_{in} is zero, we have the trivial solution $C' = 0$. It is obvious from Eq. (2-2) immediately. For a more interesting situation in which C' is nonzero, or for C to deviate from the initial C_o, C'_{in} must be nonzero, or in other words, C_{in} is different from C_o. In the terminology of differential equations, the right hand side C'_{in} is named the *forcing function*. In control, it is called the *input*. Not only C'_{in} is nonzero, it is under most circumstances a function of time as well, $C'_{in} = C'_{in}(t)$.

In addition, the time dependence of the solution, meaning the exponential function, arises from the left hand side of Eq. (2-2), the linear differential operator. In fact, we may recall that the left hand side of (2-2) gives rise to the so-called characteristic equation (or characteristic polynomial).

Do not worry if you have forgotten the significance of the characteristic equation. We will come back to this issue again and again. We are just using this example as a prologue. Typically in a class on differential equations, we learn to transform a *linear* ordinary equation into an *algebraic* equation in the *Laplace-domain*, solve for the transformed dependent variable, and finally get back the *time-domain* solution with an inverse transformation.

In classical control theory, we make extensive use of Laplace transform to analyze the dynamics of a system. The key point (and at this moment the trick) is that we will try to predict the time response *without* doing the inverse transformation. Later, we will see that the answer lies in the roots of the characteristic equation. This is the basis of classical control analyses. Hence, in going through Laplace transform again, it is not so much that we need a remedial course. Your old differential equation textbook would do fine. The key task here is to pitch this mathematical technique in light that may help us to apply it to control problems.

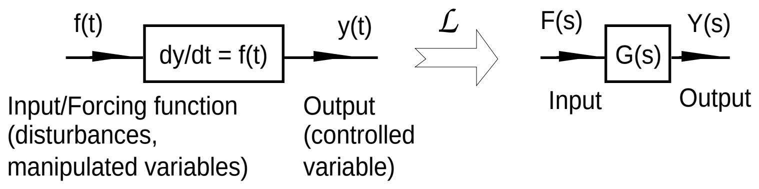

f(t)
dy/dt = f(t)
y(t)
Input/Forcing function (disturbances, manipulated variables)
Output (controlled variable)
ℒ
F(s)
G(s)
Y(s)
Input
Output

Figure 2.1. Relationship between time domain and Laplace domain.

[1] Deviation variables are analogous to **perturbation** variables used in chemical kinetics or in fluid mechanics (linear hydrodynamic stability). We can consider deviation variable as a measure of how far it is from steady state.

[2] When you come across the term convolution integral later in Eq. (4-10) and wonder how it may come about, take a look at the form of Eq. (2-3) again and think about it. If you wonder about where (2-3) comes from, review your old ODE text on integrating factors. We skip this detail since we will not be using the time domain solution in Eq. (2-3).

2.2 Laplace transform

Let us first state a few important points about the application of Laplace transform in solving differential equations (Fig. 2.1). After we have formulated a model in terms of a *linear or linearized* differential equation, dy/dt = f(y), we can solve for y(t). Alternatively, we can transform the equation into an algebraic problem as represented by the function G(s) in the Laplace domain and solve for Y(s). The time domain solution y(t) can be obtained with an inverse transform, but we rarely do so in control analysis.

What we argue (of course it is true) is that the Laplace-domain function Y(s) must contain the same information as y(t). Likewise, the function G(s) contains the same dynamic information as the original differential equation. We will see that the function G(s) can be "clean" looking if the differential equation has zero initial conditions. That is one of the reasons why we always pitch a control problem in terms of deviation variables.[1] We can now introduce the definition.

The Laplace transform of a function $f(t)$ is defined as

$$\mathcal{L}[f(t)] = \int_0^\infty f(t)\, e^{-st}\, dt \tag{2-4}$$

where s is the transform variable.[2] To complete our definition, we have the inverse transform

$$f(t) = \mathcal{L}^{-1}[F(s)] = \frac{1}{2\pi j}\int_{\gamma - j\infty}^{\gamma + j\infty} F(s)\, e^{st}\, ds \tag{2-5}$$

where γ is chosen such that the infinite integral can converge.[3] Do not be intimidated by (2-5). In a control class, we never use the inverse transform definition. Our approach is quite simple. We construct a table of the Laplace transform of some common functions, and we use it to do the inverse transform using a look-up table.

An important property of the Laplace transform is that it is a *linear* operator, and contribution of individual terms can simply be added together (superimposed):

$$\mathcal{L}[a\, f_1(t) + b\, f_2(t)] = a\,\mathcal{L}[f_1(t)] + b\,\mathcal{L}[f_2(t)] = aF_1(s) + bF_2(s) \tag{2-6}$$

Note:
The linear property is one very important reason why we can do partial fractions and inverse transform using a look-up table. This is also how we analyze more complex, but linearized, systems. Even though a text may not state this property explicitly, we rely heavily on it in classical control.

We now review the Laplace transform of some common functions—mainly the ones that we come across frequently in control problems. We do not need to know all possibilities. We can consult a handbook or a mathematics textbook if the need arises. (A summary of the important ones is in Table 2.1.) Generally, it helps a great deal if you can do the following common ones

---

1 But! What we measure in an experiment is the "real" variable. We have to be careful when we solve a problem which provides real data.

2 There are many acceptable notations of Laplace transform. We choose to use a capitalized letter, and where confusion may arise, we further add (s) explicitly to the notation.

3 If you insist on knowing the details, they can be found on our *Web Support*.

without having to look up a table. The same applies to simple algebra such as partial fractions and calculus such as linearizing a function.

1. **A constant**

$$f(t) = a, \qquad F(s) = \frac{a}{s} \tag{2-7}$$

The derivation is:

$$\mathcal{L}[a] = a\int_0^\infty e^{-st}\,dt = -\frac{a}{s}e^{-st}\Big|_0^\infty = a\left[0 + \frac{1}{s}\right] = \frac{a}{s}$$

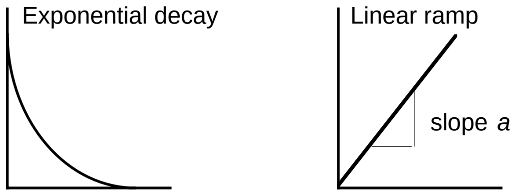


Figure 2.2. Illustration of exponential and ramp functions.

2. **An exponential function** (Fig. 2.2)

$$f(t) = e^{-at} \text{ with } a > 0, \qquad F(s) = \frac{1}{(s + a)} \tag{2-9}$$

$$\mathcal{L}[e^{-at}] = a\int_0^\infty e^{-at}\,e^{-st}\,dt = \frac{-1}{(s + a)}\,e^{-(a+s)t}\Big|_0^\infty = \frac{1}{(s + a)}$$

3. **A ramp function** (Fig. 2.2)

$$f(t) = at \text{ for } t \geq 0 \text{ and } a = \text{constant}, \qquad F(s) = \frac{a}{s^2} \tag{2-8}$$

$$\mathcal{L}[at] = a\int_0^\infty t\,e^{-st}\,dt = a\left[-t\,\frac{1}{s}\,e^{-st}\Big|_0^\infty + \int_0^\infty \frac{1}{s}\,e^{-st}\,dt\right] = \frac{a}{s}\int_0^\infty e^{-st}\,dt = \frac{a}{s^2}$$

4. **Sinusoidal functions**

$$f(t) = \sin\omega t, \qquad F(s) = \frac{\omega}{(s^2 + \omega^2)} \tag{2-10}$$

$$f(t) = \cos\omega t, \qquad F(s) = \frac{s}{(s^2 + \omega^2)} \tag{2-11}$$

We make use of the fact that $\sin \omega t = \frac{1}{2j}(e^{j\omega t} - e^{-j\omega t})$ and the result with an exponential function to derive

$$\mathcal{L}[\sin \omega t] = \frac{1}{2j}\int_0^\infty (e^{j\omega t} - e^{-j\omega t})\,e^{-st}\,dt = \frac{1}{2j}\left[\int_0^\infty e^{-(s-j\omega)t}\,dt - \int_0^\infty e^{-(s+j\omega)t}\,dt\right]$$

$$= \frac{1}{2j}\left[\frac{1}{s - j\omega} - \frac{1}{s + j\omega}\right] = \frac{\omega}{s^2 + \omega^2}$$

The Laplace transform of cosωt is left as an exercise in the Review Problems. If you need a review on complex variables, our *Web Support* has a brief summary.

5. **Sinusoidal function with exponential decay**

$$f(t) = e^{-at}\sin\omega t, \qquad F(s) = \frac{\omega}{(s+a)^2 + \omega^2} \tag{2-12}$$

Making use of previous results with the exponential and sine functions, we can pretty much do this one by inspection. First, we put the two exponential terms together inside the integral:

$$\begin{aligned}\int_0^\infty \sin\omega t\, e^{-(s+a)t}\,dt &= \frac{1}{2j}\left[\int_0^\infty e^{-(s+a-j\omega)t}\,dt - \int_0^\infty e^{-(s+a+j\omega)t}\,dt\right]\\ &= \frac{1}{2j}\left[\frac{1}{(s+a)-j\omega} - \frac{1}{(s+a)+j\omega}\right]\end{aligned}$$

The similarity to the result of sinωt should be apparent now, if it was not the case with the LHS.

6. **First order derivative**, df/dt,
$$\mathcal{L}\left[\frac{df}{dt}\right] = sF(s) - f(0) \tag{2-13}$$

and the **second order derivative**,
$$\mathcal{L}\left[\frac{d^2f}{dt^2}\right] = s^2F(s) - sf(0) - f'(0) \tag{2-14}$$

We have to use integration by parts here,

$$\mathcal{L}\left[\frac{df}{dt}\right] = \int_0^\infty \frac{df}{dt}\,e^{-st}\,dt = f(t)e^{-st}\Big|_0^\infty + s\int_0^\infty f(t)\,e^{-st}\,dt = -f(0) + sF(s)$$

and

$$\mathcal{L}\left[\frac{d^2f}{dt^2}\right] = \int_0^\infty \frac{d}{dt}\left(\frac{df}{dt}\right)e^{-st}\,dt = \frac{df}{dt}e^{-st}\Big|_0^\infty + s\int_0^\infty \frac{df}{dt}\,e^{-st}\,dt = -\frac{df}{dt}\Big|_0 + s\left[sF(s) - f(0)\right]$$

We can extend these results to find the Laplace transform of higher order derivatives. The key is that if we use deviation variables in the problem formulation, all the initial value terms will drop out in Eqs. (2-13) and (2-14). This is how we can get these “clean-looking” transfer functions later.

7. **An integral**,
$$\mathcal{L}\left[\int_0^t f(t)\,dt\right] = \frac{F(s)}{s} \tag{2-15}$$

We also need integration by parts here

$$\int_0^\infty \left[\int_0^t f(t)\,dt\right] e^{-st}\,dt = -\frac{1}{s}e^{-st}\int_0^t f(t)\,dt\,\Big|_0^\infty + \frac{1}{s}\int_0^\infty f(t)\,e^{-st}\,dt = \frac{F(s)}{s}$$

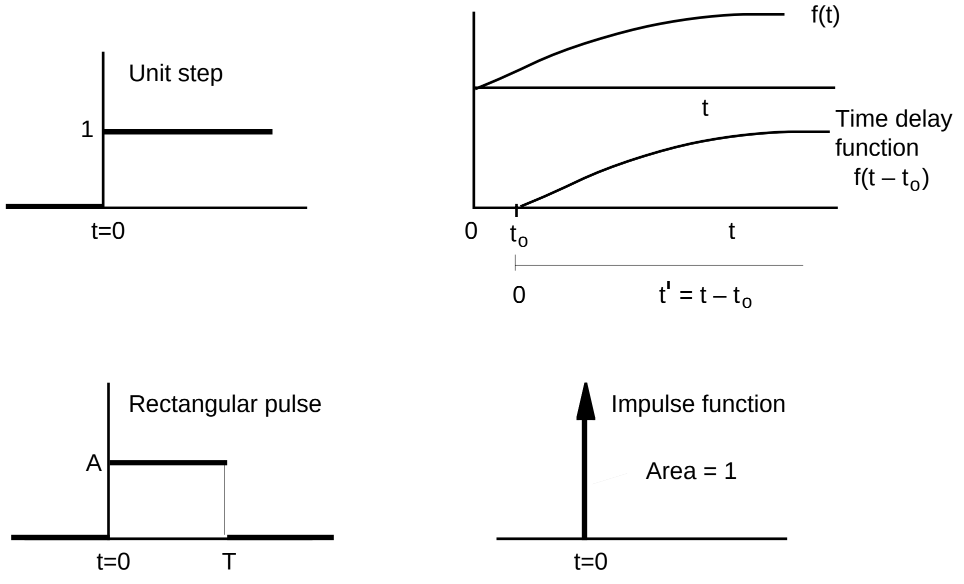


Figure 2.3. Depiction of unit step, time delay, rectangular, and impulse functions.

2.3 Laplace transforms common to control problems

We now derive the Laplace transform of functions common in control analysis.

1. **Step function**

$$f(t) = Au(t), \qquad F(s) = \frac{A}{s} \tag{2-16}$$

We first define the **unit step function** (also called the Heaviside function in mathematics) and its Laplace transform:[1]

$$u(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases} ; \qquad \mathcal{L}[u(t)] = U(s) = \frac{1}{s} \tag{2-17}$$

The Laplace transform of the unit step function (Fig. 2.3) is derived as follows:

$$\mathcal{L}[u(t)] = \lim_{\varepsilon \to 0^+} \int_{\varepsilon}^{\infty} u(t)\, e^{-st}\, dt = \int_{0^+}^{\infty} e^{-st}\, dt = \frac{-1}{s}\, e^{-st}\Big|_0^{\infty} = \frac{1}{s}$$

With the result for the unit step, we can see the results of the Laplace transform of any step function f(t) = Au(t).

$$f(t) = A\,u(t) = \begin{cases} A & t > 0 \\ 0 & t < 0 \end{cases} ; \qquad \mathcal{L}[Au(t)] = \frac{A}{s}$$

The Laplace transform of a step function is essentially the same as that of a constant in (2-7). When you do the inverse transform of A/s, which function you choose depends on the context of the problem. Generally, a constant is appropriate under most circumstances.

[1] Strictly speaking, the step function is discontinuous at t = 0, but many engineering texts ignore it and simply write u(t) = 1 for $t \geq 0$.

2. **Dead time function** (Fig. 2.3)

$$f(t - t_o), \qquad \mathcal{L}[f(t - t_o)] = e^{-st_o} F(s) \qquad (2\text{-}18)$$

The dead time function is also called the **time delay**, **transport lag**, translated, or time shift function (Fig. 2.3). It is defined such that an original function f(t) is "shifted" in time t_o, and no matter what f(t) is, its value is set to zero for $t < t_o$. This time delay function can be written as:

$$f(t - t_o) = \begin{cases} 0 & , t - t_o < 0 \\ f(t - t_o) & , t - t_o > 0 \end{cases} = f(t - t_o)\, u(t - t_o)$$

The second form on the far right is a more concise way to say that the time delay function $f(t - t_o)$ is defined such that it is zero for $t < t_o$. We can now derive the Laplace transform.

$$\mathcal{L}[f(t - t_o)] = \int_0^\infty f(t - t_o)\, u(t - t_o)\, e^{-st}\, dt = \int_{t_o}^\infty f(t - t_o)\, e^{-st}\, dt$$

and finally,

$$\int_{t_o}^\infty f(t - t_o)\, e^{-st}\, dt = e^{-st_o} \int_{t_o}^\infty f(t - t_o)\, e^{-s(t - t_o)}\, d(t - t_o) = e^{-st_o} \int_0^\infty f(t')\, e^{-st'}\, dt' = e^{-st_o} F(s)$$

where the final integration step uses the time shifted axis $t' = t - t_o$.

3. **Rectangular pulse function** (Fig. 2.3)

$$f(t) = \begin{cases} 0 & t < 0 \\ A & 0 < t < T \\ 0 & t > T \end{cases} = A[u(t) - u(t - T)] \,, \qquad \mathcal{L}[f(t)] = \frac{A}{s}[1 - e^{-sT}] \qquad (2\text{-}19)$$

The rectangular pulse can be generated by subtracting a step function with dead time T from a step function. We can derive the Laplace transform using the formal definition

$$\mathcal{L}[f(t] = \int_0^\infty f(t)\, e^{-st}\, dt = A \int_{0^+}^T e^{-st}\, dt = A \frac{-1}{s} e^{-st}\Big|_0^T = \frac{A}{s}[1 - e^{-sT}]$$

or better yet, by making use of the results of a step function and a dead time function

$$\mathcal{L}[f(t] = \mathcal{L}[A\, u(t) - A\, u(t - T)] = \frac{A}{s} - e^{-sT}\frac{A}{s}$$

4. **Unit rectangular pulse function**

$$f(t) = \begin{cases} 0 & t < 0 \\ 1/T & 0 < t < T \\ 0 & t > T \end{cases} = \frac{1}{T}[u(t) - u(t - T)] \,, \qquad \mathcal{L}[f(t)] = \frac{1}{sT}[1 - e^{-sT}] \qquad (2\text{-}20)$$

This is a prelude to the important impulse function. We can define a rectangular pulse such that the area is unity. The Laplace transform follows that of a rectangular pulse function

$$\mathcal{L}[f(t] = \mathcal{L}\left[\frac{1}{T} u(t) - \frac{1}{T} u(t - T)\right] = \frac{1}{T\,s}[1 - e^{-sT}]$$

5. **Impulse function** (Fig. 2.3)

$$\mathcal{L}[\delta(t)] = 1, \text{ and } \mathcal{L}[A\delta(t)] = A \tag{2-21}$$

The (unit) impulse function is called the **Dirac** (or simply delta) function in mathematics.[1] If we suddenly dump a bucket of water into a bigger tank, the impulse function is how we describe the action mathematically. We can consider the impulse function as the unit rectangular function in Eq. (2-20) as T shrinks to zero while the height 1/T goes to infinity:

$$\delta(t) = \lim_{T \to 0} \frac{1}{T}[u(t) - u(t - T)]$$

The area of this "squeezed rectangle" nevertheless remains at unity:

$$\lim_{T \to 0} (T \frac{1}{T}) = 1 \text{, or in other words } \int_{-\infty}^{\infty} \delta(t)\, dt = 1$$

The impulse function is rarely defined in the conventional sense, but rather via its important property in an integral:

$$\int_{-\infty}^{\infty} f(t)\,\delta(t)\, dt = f(0) \text{, and } \int_{-\infty}^{\infty} f(t)\,\delta(t - t_o)\, dt = f(t_o) \tag{2-22}$$

The Laplace transform of the impulse function is obtained easily by taking the limit of the unit rectangular function transform (2-20) with the use of L'Hospital's rule:

$$\mathcal{L}[\delta(t] = \lim_{T \to 0} \frac{1 - e^{-sT}}{T\,s} = \lim_{T \to 0} \frac{s\,e^{-sT}}{s} = 1$$

From this result, it is obvious that $\mathcal{L}[A\delta(t)] = A$.

2.4 Initial and final value theorems

We now present two theorems which can be used to find the values of the time-domain function at two extremes, $t = 0$ and $t = \infty$, without having to do the inverse transform. In control, we use the final value theorem quite often. The initial value theorem is less useful. As we have seen from our very first example in Section 2.1, the problems that we solve are defined to have exclusively zero initial conditions.

Initial Value Theorem: $$\lim_{s \to \infty} [sF(s)] = \lim_{t \to 0} f(t) \tag{2-23}$$

Final Value Theorem: $$\lim_{s \to 0} [sF(s)] = \lim_{t \to \infty} f(t) \tag{2-24}$$

The final value theorem is valid provided that a final value exists. The proofs of these theorems are straightforward. We will do the one for the final value theorem. The proof of the initial value theorem is in the Review Problems.

Consider the definition of the Laplace transform of a derivative. If we take the limit as s approaches zero, we find

[1] In mathematics, the unit rectangular function is defined with a height of 1/2T and a width of 2T from –T to T. We simply begin at t = 0 in control problems. Furthermore, the impulse function is the time derivative of the unit step function.

$$\lim_{s \to 0} \int_0^{\infty} \frac{df(t)}{dt}\, e^{-st}\, dt = \lim_{s \to 0} \left[s\,F(s) - f(0) \right]$$

If the infinite integral exists,[1] we can interchange the limit and the integration on the left to give

$$\int_0^{\infty} \lim_{s \to 0} \frac{df(t)}{dt}\, e^{-st}\, dt = \int_0^{\infty} df(t) = f(\infty) - f(0)$$

Now if we equate the right hand sides of the previous two steps, we have

$$f(\infty) - f(0) = \lim_{s \to 0} \left[s\,F(s) - f(0) \right]$$

We arrive at the final value theorem after we cancel the f(0) terms on both sides.

✎ **Example 2.1**: Consider the Laplace transform $F(s) = \dfrac{6\,(s-2)\,(s+2)}{s\,(s+1)\,(s+3)\,(s+4)}$. What is f(t=∞)?

$$\lim_{s \to 0} s\, \frac{6\,(s-2)\,(s+2)}{s\,(s+1)\,(s+3)\,(s+4)} = \frac{6\,(-2)\,(2)}{(3)\,(4)} = -2$$

✎ **Example 2.2**: Consider the Laplace transform $F(s) = \dfrac{1}{(s-2)}$. What is f(t=∞)?

Here, $f(t) = e^{2t}$. There is no upper bound for this function, which is in violation of the existence of a final value. The final value theorem does not apply. If we insist on applying the theorem, we will get a value of zero, which is meaningless.

✎ **Example 2.3**: Consider the Laplace transform $F(s) = \dfrac{6\,(s^2-4)}{(s^3+s^2-4s-4)}$. What is f(t=∞)?

Yes, another trick question. If we apply the final value theorem without thinking, we would get a value of 0, but this is meaningless. With MATLAB, we can use

```
   roots([1 1 -4 -4])
```

to find that the polynomial in the denominator has roots –1, –2, and +2. This implies that f(t) contains the term e^{2t}, which increases without bound.

As we move on, we will learn to associate the time exponential terms to the roots of the polynomial in the denominator. From these examples, we can gather that to have a meaningful, *i.e.*, finite bounded value, the roots of the polynomial in the denominator must have *negative real parts*. This is the basis of stability, which will formerly be defined in Chapter 7.

[1] This is a key assumption and explains why Examples 2.2 and 2.3 do not work. When a function has no bound—what we call unstable later—the assumption is invalid.

2.5 Partial fraction expansion

Since we rely on a look-up table to do reverse Laplace transform, we need the skill to reduce a complex function down to simpler parts that match our table. In theory, we should be able to "break up" a ratio of two polynomials in *s* into simpler partial fractions. If the polynomial in the denominator, p(s), is of an order higher than the numerator, q(s), we can derive [1]

$$F(s) = \frac{q(s)}{p(s)} = \frac{\alpha_1}{(s + a_1)} + \frac{\alpha_2}{(s + a_2)} + \ldots \frac{\alpha_i}{(s + a_i)} + \ldots \frac{\alpha_n}{(s + a_n)} \tag{2-25}$$

where the order of p(s) is n, and the a_i are the negative values of the roots of the equation p(s) = 0. We then perform the inverse transform term by term:

$$f(t) = \mathcal{L}^{-1}[F(s)] = \mathcal{L}^{-1}\left[\frac{\alpha_1}{(s + a_1)}\right] + \mathcal{L}^{-1}\left[\frac{\alpha_2}{(s + a_2)}\right] + \ldots \mathcal{L}^{-1}\left[\frac{\alpha_i}{(s + a_i)}\right] + \ldots \mathcal{L}^{-1}\left[\frac{\alpha_n}{(s + a_n)}\right] \tag{2-26}$$

This approach works because of the linear property of Laplace transform.

The next question is how to find the partial fractions in Eq. (2-25). One of the techniques is the so-called **Heaviside expansion**, a fairly straightforward algebraic method. We will illustrate three important cases with respect to the roots of the polynomial in the denominator: (1) distinct real roots, (2) complex conjugate roots, and (3) multiple (or repeated) roots. In a given problem, we can have a combination of any of the above. Yes, we need to know how to do them all.

2.5.1 Case 1: p(s) has distinct, real roots

Example 2.4: Find f(t) of the Laplace transform $F(s) = \frac{6s^2 - 12}{(s^3 + s^2 - 4s - 4)}$.

From Example 2.3, the polynomial in the denominator has roots –1, –2, and +2, values that will be referred to as poles later. We should be able to write F(s) as

$$\frac{6s^2 - 12}{(s + 1)(s + 2)(s - 2)} = \frac{\alpha_1}{(s + 1)} + \frac{\alpha_2}{(s + 2)} + \frac{\alpha_3}{(s - 2)}$$

The Heaviside expansion takes the following idea. Say if we multiply both sides by (s + 1), we obtain

$$\frac{6s^2 - 12}{(s + 2)(s - 2)} = \alpha_1 + \frac{\alpha_2}{(s + 2)}(s + 1) + \frac{\alpha_3}{(s - 2)}(s + 1)$$

which should be satisfied by any value of s. Now if we choose s = –1, we should obtain

$$\alpha_1 = \left.\frac{6s^2 - 12}{(s + 2)(s - 2)}\right|_{s = -1} = 2$$

Similarly, we can multiply the original fraction by (s + 2) and (s – 2), respectively, to find

$$\alpha_2 = \left.\frac{6s^2 - 12}{(s + 1)(s - 2)}\right|_{s = -2} = 3$$

and

---

[1] If the order of q(s) is higher, we need first carry out "long division" until we are left with a partial fraction "residue." Thus the coefficients α_i are also called residues. We then expand this partial fraction. We would encounter such a situation only in a mathematical problem. The models of real physical processes lead to problems with a higher order denominator.

$$\alpha_3 = \left.\frac{6s^2 - 12}{(s+1)(s+2)}\right|_{s=2} = 1$$

Hence, $F(s) = \frac{2}{(s+1)} + \frac{3}{(s+2)} + \frac{1}{(s-2)}$, and using a look-up table would give us

$$f(t) = 2e^{-t} + 3e^{-2t} + e^{2t}$$

When you use MATLAB to solve this problem, be careful when you interpret the results. The computer is useless unless we know what we are doing. We provide only the necessary statements.[1] For this example, all we need is:

```
[a,b,k]=residue([6 0 -12],[1 1 -4 -4])
```

✎ **Example 2.5**: Find f(t) of the Laplace transform $F(s) = \frac{6s}{(s^3 + s^2 - 4s - 4)}$.

Again, the expansion should take the form

$$\frac{6s}{(s+1)(s+2)(s-2)} = \frac{\alpha_1}{(s+1)} + \frac{\alpha_2}{(s+2)} + \frac{\alpha_3}{(s-2)}$$

One more time, for each term, we multiply the denominators on the right hand side and set the resulting equation to its root to obtain

$$\alpha_1 = \left.\frac{6s}{(s+2)(s-2)}\right|_{s=-1} = 2,\ \alpha_2 = \left.\frac{6s}{(s+1)(s-2)}\right|_{s=-2} = -3\text{ , and }\alpha_3 = \left.\frac{6s}{(s+1)(s+2)}\right|_{s=2} = 1$$

The time domain function is

$$f(t) = 2e^{-t} - 3e^{-2t} + e^{2t}$$

Note that f(t) has the identical functional dependence in time as in the first example. Only the coefficients (residues) are different.

The MATLAB statement for this example is:

```
[a,b,k]=residue([6 0],[1 1 -4 -4])
```

✎ **Example 2.6**: Find f(t) of the Laplace transform $F(s) = \frac{6}{(s+1)(s+2)(s+3)}$.

This time, we should find

$$\alpha_1 = \left.\frac{6}{(s+2)(s+3)}\right|_{s=-1} = 3\ ,\ \alpha_2 = \left.\frac{6}{(s+1)(s+3)}\right|_{s=-2} = -6\ ,\ \alpha_3 = \left.\frac{6}{(s+1)(s+2)}\right|_{s=-3} = 3$$

The time domain function is

---

1 Starting from here on, it is important that you go over the MATLAB sessions. Explanation of `residue()` is in Session 2. While we do not print the computer results, they can be found on our *Web Support*.

$$f(t) = 3e^{-t} - 6e^{-2t} + 3e^{-3t}$$

The e^{-2t} and e^{-3t} terms will decay faster than the e^{-t} term. We consider the e^{-t} term, or the pole at $s = -1$, as more dominant.

We can confirm the result with the following MATLAB statements:

```
p=poly([-1 -2 -3]);
[a,b,k]=residue(6,p)
```

Note:

(1) The time dependence of the time domain solution is derived entirely from the roots of the polynomial in the denominator (what we will refer to later as the **poles**). The polynomial in the numerator affects only the coefficients α_i. This is one reason why we make qualitative assessment of the dynamic response characteristics entirely based on the poles of the characteristic polynomial.

(2) Poles that are closer to the origin of the complex plane will have corresponding exponential functions that decay more slowly in time. We consider these poles more dominant.

(3) We can generalize the Heaviside expansion into the fancy form for the coefficients

$$\alpha_i = (s + a_i)\left.\frac{q(s)}{p(s)}\right|_{s = -a_i}$$

but we should always remember the simple algebra that we have gone through in the examples above.

2.5.2 Case 2: p(s) has complex roots [1]

Example 2.7: Find f(t) of the Laplace transform $F(s) = \frac{s+5}{s^2 + 4s + 13}$.

We first take the painful route just so we better understand the results from MATLAB. If we have to do the chore by hand, we much prefer the completing the perfect square method in Example 2.8. Even without MATLAB, we can easily find that the roots of the polynomial $s^2 + 4s + 13$ are $-2 \pm 3j$, and F(s) can be written as the sum of

$$\frac{s+5}{s^2+4s+13} = \frac{s+5}{[s-(-2+3j)]\,[s-(-2-3j)]} = \frac{\alpha}{s-(-2+3j)} + \frac{\alpha^*}{s-(-2-3j)}$$

We can apply the same idea formally as before to find

$$\alpha = \left.\frac{s+5}{[s-(-2-3j)]}\right|_{s=-2+3j} = \frac{(-2+3j)+5}{(-2+3j)+2+3j} = \frac{(j+1)}{2j} = \frac{1}{2}(1-j)$$

and its complex conjugate is

$$\alpha^* = \frac{1}{2}(1+j)$$

The inverse transform is hence

[1] If you need a review of complex variable definitions, see our *Web Support*. Many steps in Example 2.7 require these definitions.

$$f(t) = \frac{1}{2}(1-j)\,e^{(-2+3j)t} + \frac{1}{2}(1+j)\,e^{(-2-3j)t}$$
$$= \frac{1}{2}e^{-2t}\left[(1-j)\,e^{j3t} + (1+j)\,e^{-j3t}\right]$$

We can apply Euler's identity to the result:

$$f(t) = \frac{1}{2}e^{-2t}\left[(1-j)(\cos 3t + j\sin 3t) + (1+j)(\cos 3t - j\sin 3t)\right]$$
$$= \frac{1}{2}e^{-2t}\left[2(\cos 3t + \sin 3t)\right]$$

which we further rewrite as

$$f(t) = \sqrt{2}\,e^{-2t}\sin(3t+\phi) \quad \text{where} \quad \phi = \tan^{-1}(1) = \pi/4 \text{ or } 45°$$

The MATLAB statement for this example is simply:

```
[a,b,k]=residue([1 5],[1 4 13])
```

Note:

(1) Again, the time dependence of f(t) is affected only by the roots of p(s). For the general complex conjugate roots $-a \pm bj$, the time domain function involves e^{-at} and (cos bt + sin bt). The polynomial in the numerator affects only the constant coefficients.

(2) We seldom use the form (cos bt + sin bt). Instead, we use the phase lag form as in the final step of Example 2.7.

✎ **Example 2.8**: Repeat Example 2.7 using a look-up table.

In practice, we seldom do the partial fraction expansion of a pair of complex roots. Instead, we rearrange the polynomial p(s) by noting that we can complete the squares:

$$s^2 + 4s + 13 = (s+2)^2 + 9 = (s+2)^2 + 3^2$$

We then write F(s) as

$$F(s) = \frac{s+5}{s^2+4s+13} = \frac{(s+2)}{(s+2)^2+3^2} + \frac{3}{(s+2)^2+3^2}$$

With a Laplace transform table, we find

$$f(t) = e^{-2t}\cos 3t + e^{-2t}\sin 3t$$

which is the answer with very little work. Compared with how messy the partial fraction was in Example 2.7, this example also suggests that we want to leave terms with conjugate complex roots as one second order term.

2.5.3 Case 3: p(s) has repeated roots

✎ **Example 2.9**: Find f(t) of the Laplace transform $F(s) = \dfrac{2}{(s+1)^3(s+2)}$.

The polynomial p(s) has the roots –1 repeated three times, and –2. To keep the numerator of each partial fraction a simple constant, we will have to expand to

$$\frac{2}{(s+1)^3(s+2)} = \frac{\alpha_1}{(s+1)} + \frac{\alpha_2}{(s+1)^2} + \frac{\alpha_3}{(s+1)^3} + \frac{\alpha_4}{(s+2)}$$

To find α_3 and α_4 is routine:

$$\alpha_3 = \left.\frac{2}{(s+2)}\right|_{s=-1} = 2\,, \text{ and } \quad \alpha_4 = \left.\frac{2}{(s+1)^3}\right|_{s=-2} = -2$$

The problem is with finding α_1 and α_2. We see that, say, if we multiply the equation with (s+1) to find α_1, we cannot select s = –1. What we can try is to multiply the expansion with $(s+1)^3$

$$\frac{2}{(s+2)} = \alpha_1(s+1)^2 + \alpha_2(s+1) + \alpha_3 + \frac{\alpha_4(s+1)^3}{(s+2)}$$

and then differentiate this equation with respect to s:

$$\frac{-2}{(s+2)^2} = 2\,\alpha_1(s+1) + \alpha_2 + 0 + [\alpha_4 \text{ terms with } (s+1)]$$

Now we can substitute s = –1 which provides $\alpha_2 = -2$.

We can be lazy with the last α_4 term because we know its derivative will contain (s + 1) terms and they will drop out as soon as we set s = –1. To find α_1, we differentiate the equation one more time to obtain

$$\frac{4}{(s+2)^3} = 2\,\alpha_1 + 0 + 0 + [\alpha_4 \text{ terms with } (s+1)]$$

which of course will yield $\alpha_1 = 2$ if we select s = –1. Hence, we have

$$\frac{2}{(s+1)^3(s+2)} = \frac{2}{(s+1)} + \frac{-2}{(s+1)^2} + \frac{2}{(s+1)^3} + \frac{-2}{(s+2)}$$

and the inverse transform via table-lookup is

$$f(t) = 2\left[\left(1 - t + \frac{t^2}{2}\right)e^{-t} - e^{-2t}\right]$$

We can also arrive at the same result by expanding the entire algebraic expression, but that actually takes more work(!) and we will leave this exercise in the Review Problems.

The MATLAB command for this example is:

```
p=poly([-1 -1 -1 -2]);
[a,b,k]=residue(2,p)
```

Note:
In general, the inverse transform of repeated roots takes the form

$$\mathcal{L}^{-1}\left[\frac{\alpha_1}{(s+a)} + \frac{\alpha_2}{(s+a)^2} + \ldots \quad \frac{\alpha_n}{(s+a)^n}\right] = \left[\alpha_1 + \alpha_2 t + \frac{\alpha_3}{2!}t^2 + \ldots \quad \frac{\alpha_n}{(n-1)!}t^{n-1}\right]e^{-at}$$

The exponential function is still based on the root s = –a, but the actual time dependence will decay slower because of the ($\alpha_2 t$ + …) terms.

2.6 Transfer function, pole, and zero

Now that we can do Laplace transform, let us return to our very first example. The Laplace transform of Eq. (2-2) with its zero initial condition is $(\tau s + 1)C'(s) = C'_{in}(s)$, which we rewrite as

$$\frac{C'(s)}{C'_{in}(s)} = \frac{1}{\tau s + 1} = G(s) \tag{2-27}$$

We define the right hand side as G(s), our ubiquitous **transfer function**. It relates an input to the output of a model. Recall that *we* use *deviation variables*. The input is the *change* in the inlet concentration, $C'_{in}(t)$. The output, or response, is the resulting *change* in the tank concentration, C'(t).

✎ **Example 2.10**: What is the time domain response C'(t) in Eq. (2-27) if the change in inlet concentration is (a) a unit step function, and (b) an impulse function?

(a) With a unit step input, $C'_{in}(t) = u(t)$, and $C'_{in}(s) = 1/s$. Substitution in (2-27) leads to

$$C'(s) = \frac{1}{\tau s + 1}\,\frac{1}{s} = \frac{1}{s} + \frac{-\tau}{\tau s + 1}$$

After inverse transform via table look-up, we have $C'(t) = 1 - e^{-t/\tau}$. The change in tank concentration eventually will be identical to the unit step change in inlet concentration.

(b) With an impulse input, $C'_{in}(s) = 1$, and substitution in (2-27) leads to simply

$$C'(s) = \frac{1}{\tau s + 1}\ ,$$

and the time domain solution is $C'(t) = \frac{1}{\tau}\, e^{-t/\tau}$. The effect of the impulse eventually will decay away.

Finally, you may want to keep in mind that the results of this example can also be obtained via the general time-domain solution in Eq. (2-3).

The key of this example is to note that *irrespective of the input*, the time domain solution contains the time dependent function $e^{-t/\tau}$, which is associated with the root of the polynomial in the denominator of the transfer function.

The inherent dynamic properties of a model are embedded in the characteristic polynomial of the differential equation. More specifically, the dynamics is related to the roots of the characteristic polynomial. In Eq. (2-27), the characteristic equation is $\tau s + 1 = 0$, and its root is $-1/\tau$. In a general sense, that is without specifying what C'_{in} is and without actually solving for C'(t), we can infer that C'(t) must contain a term with $e^{-t/\tau}$. We refer the root $-1/\tau$ as the **pole** of the transfer function G(s).

We can now state the definitions more generally. For an ordinary differential equation [1]

---

[1] Yes, we try to be general and use an *n*-th order equation. If you have trouble with the development in this section, think of a second order equation in all the steps:

$$a_2 y^{(2)} + a_1 y^{(1)} + a_o y = b_1 x^{(1)} + b_o x$$

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y^{(1)} + a_o y = b_m x^{(m)} + b_{m-1} x^{(m-1)} + \dots + b_1 x^{(1)} + b_o x \qquad (2\text{-}28)$$

with n > m and zero initial conditions $y^{(n-1)} = \dots = y = 0$ at t = 0, the corresponding Laplace transform is

$$\frac{Y(s)}{X(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \dots + b_1 s + b_o}{a_n s^n + a_{n-1} s^{n-1} + \dots \qquad + a_1 s + a_o} = G(s) \quad = \frac{Q(s)}{P(s)} \qquad (2\text{-}29)$$

Generally, we can write the transfer function as the ratio of two polynomials in *s*.[1] When we talk about the mathematical properties, the polynomials are denoted as *Q*(*s*) and *P*(*s*), but the *same* polynomials are denoted as *Y*(*s*) and *X*(*s*) when the focus is on control problems or transfer functions. The orders of the polynomials are such that *n* > *m* for physical realistic processes.[2]

We know that G(s) contains information on the dynamic behavior of a model as represented by the differential equation. We also know that the denominator of G(s) is the characteristic polynomial of the differential equation. The roots of the characteristic equation, P(s) = 0: p_1, p_2,... p_n, are the **poles** of G(s). When the poles are real and negative, we also use the time constant notation:

$$p_1 = -\frac{1}{\tau_1}\,,\ p_2 = -\frac{1}{\tau_2}\,, \dots \quad ,\ p_n = -\frac{1}{\tau_n}$$

The poles reveal qualitatively the dynamic behavior of the model differential equation. The "roots of the characteristic equation" is used interchangeably with "poles of the transfer function."

For the general transfer function in (2-29), the roots of the polynomial Q(s), *i.e.*, of Q(s) = 0, are referred to as the **zeros**. They are denoted by z_1, z_2,... z_m, or in time constant notation,

$$z_1 = -\frac{1}{\tau_a}\,,\ z_2 = -\frac{1}{\tau_b}\,, \dots \quad ,\ z_m = -\frac{1}{\tau_m}$$

We can factor Eq. (2-29) into the so-called **pole-zero form**:

$$G(s) = \frac{Q(s)}{P(s)} = \left(\frac{b_m}{a_n}\right) \frac{(s - z_1)(s - z_2) \dots (s - z_m)}{(s - p_1)(s - p_2) \dots \quad (s - p_n)} \qquad (2\text{-}30)$$

If all the roots of the two polynomials are real, we can factor the polynomials such that the transfer function is in the **time constant form**:

$$G(s) = \frac{Q(s)}{P(s)} = \left(\frac{b_o}{a_o}\right) \frac{(\tau_a s + 1)(\tau_b s + 1) \dots (\tau_m s + 1)}{(\tau_1 s + 1)(\tau_2 s + 1) \dots \quad (\tau_n s + 1)} \qquad (2\text{-}31)$$

Eqs. (2-30) and (2-31) will be a lot less intimidating when we come back to using examples in Section 2.8. These forms are the mainstays of classical control analysis.

Another important quantity is the **steady state gain**.[3] With reference to a general differential equation model (2-28) and its Laplace transform in (2-29), the steady state gain is defined as the

---

All the features about poles and zeros can be obtained from this simpler equation.

[1] The exception is when we have dead time. We'll come back to this term in Chapter 3.

[2] For real physical processes, the orders of polynomials are such that n ≥ m. A simple explanation is to look at a so-called lead-lag element when n = m and $y^{(1)} + y = x^{(1)} + x$. The LHS, which is the dynamic model, must have enough complexity to reflect the change of the forcing on the RHS. Thus if the forcing includes a rate of change, the model must have the same capability too.

[3] This quantity is also called the **static gain** or **dc gain** by electrical engineers. When we talk about the model of a process, we also use the term **process gain** quite often, in distinction to a system gain.

final *change* in y(t) relative to a unit *change* in the input x(t). Thus an easy derivation of the steady state gain is to take a unit step input in x(t), or X(s) = 1/s, and find the final value in y(t):

$$y(\infty) = \lim_{s \to 0}[s\, G(s)\, X(s)] = \lim_{s \to 0}[s\, G(s)\, \frac{1}{s}] = \frac{b_o}{a_o} \tag{2-32}$$

The steady state gain is the ratio of the two constant coefficients. Take note that the steady state gain value is based on the transfer function only. From Eqs. (2-31) and (2-32), one easy way to "spot" the steady state gain is to look at a transfer function in the time constant form.

Note:

(1) When we talk about the poles of G(s) in Eq. (2-29), the discussion is *regardless* of the input x(t). Of course, the actual response y(t) also depends on x(t) or X(s).

(2) Recall from the examples of partial fraction expansion that the polynomial Q(s) in the numerator, or the zeros, affects only the coefficients of the solution y(t), but not the time dependent functions. That is why for qualitative discussions, we focus only on the poles.

(3) For the time domain function to be made up only of exponential terms that decay in time, *all* the poles of a transfer function must have negative real parts. (This point is related to the concept of stability, which we will address formally in Chapter 7.)

2.7 Summary of pole characteristics

We now put one and one together. The key is that we can "read" the poles—telling what the *form* of the time-domain function is. We should have a pretty good idea from our exercises in partial fractions. Here, we provide the results one more time in general notation. Suppose we have taken a characteristic polynomial, found its roots and completed the partial fraction expansion, this is what we expect in the time-domain for each of the terms:

A. **Real distinct poles**

Terms of the form $\frac{c_i}{s - p_i}$, where the pole p_i is a real number, have the time-domain function $c_i\, e^{p_i t}$. Most often, we have a negative real pole such that $p_i = -a_i$ and the time-domain function is $c_i\, e^{-a_i t}$.

B. **Real poles, repeated m times**

Terms of the form

$$\left[\frac{c_{i,1}}{(s - p_i)} + \frac{c_{i,2}}{(s - p_i)^2} + \ldots + \frac{c_{i,m}}{(s - p_i)^m}\right]$$

with the root p_i repeated m times have the time-domain function

$$\left[c_{i,1} + c_{i,2}\, t + \frac{c_{i,3}}{2!}\, t^2 + \ldots + \frac{c_{i,m}}{(m-1)!}\, t^{m-1}\right] e^{p_i t}\,.$$

When the pole p_i is negative, the decay in time of the entire response will be slower (with respect to only one single pole) because of the terms involving time in the bracket. This is the reason why we say that the response of models with repeated roots (e.g., tanks-in-series later in Section 3.4) tends to be slower or "sluggish."

C. **Complex conjugate poles**

Terms of the form $\frac{c_i}{s - p_i} + \frac{c^*_i}{s - p^*_i}$, where $p_i = \alpha + j\beta$ and $p^*_i = \alpha - j\beta$ are the complex poles, have time-domain function $c_i\, e^{p_i t} + c^*_i\, e^{p^*_i t}$ of which form we seldom use. Instead, we rearrange them to give the form [some constant] x $e^{\alpha t}\sin(\beta t + \phi)$ where ϕ is the phase lag.

It is cumbersome to write the partial fraction with complex numbers. With complex conjugate poles, we commonly combine the two first order terms into a second order term. With notations that we will introduce formally in Chapter 3, we can write the second order term as

$$\frac{as + b}{\tau^2 s^2 + 2\zeta\tau\, s + 1},$$

where the coefficient ζ is called the damping ratio. To have complex roots in the denominator, we need $0 < \zeta < 1$. The complex poles p_i and p^*_i are now written as

$$p_i,\ p^*_i \ = -\frac{\zeta}{\tau} \pm j\,\frac{\sqrt{1 - \zeta^2}}{\tau} \quad \text{with } 0 < \zeta < 1$$

and the time domain function is usually rearranged to give the form

$$[\text{some constant}] \text{ x } e^{-\zeta t/\tau} \sin\left(\frac{\sqrt{1 - \zeta^2}}{\tau}\, t + \phi\right)$$

where again, ϕ is the phase lag.

D. **Poles on the imaginary axis**

If the real part of a complex pole is zero, then $p = \pm\omega j$. We have a purely sinusoidal behavior with frequency ω. If the pole is zero, it is at the origin and corresponds to the integrator 1/s. In time domain, we'd have a constant, or a step function.

E. If a pole has a **negative real part**, it is in the left-hand plane (LHP). Conversely, if a pole has a positive real part, it is in the right-hand plane (RHP) and the time-domain solution is definitely *unstable*.

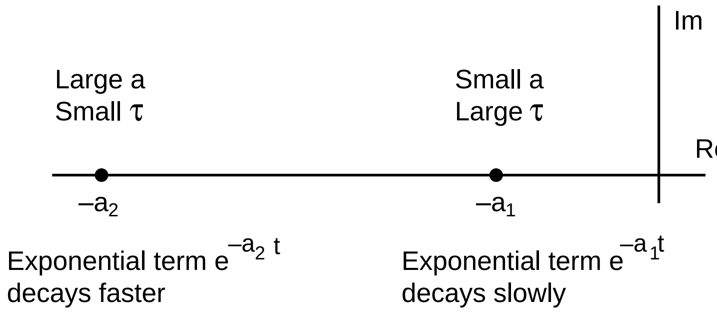

Im
Large a
Small τ
Small a
Large τ
Re
–a2
–a1
Exponential term e^{–a2 t} decays faster
Exponential term e^{–a1 t} decays slowly

Figure 2.4. Depiction of poles with small and large time constants.

Note: Not all poles are born equal!
The poles closer to the origin are dominant.

It is important to understand and be able to identify dominant poles if they exist. This is a skill that is used later in what we call model reduction. This is a point that we first observed in Example 2.6. Consider the two terms such that $0 < a_1 < a_2$ (Fig. 2.4),

$$Y(s) = \frac{c_1}{(s - p_1)} + \frac{c_2}{(s - p_2)} + \ldots = \frac{c_1}{(s + a_1)} + \frac{c_2}{(s + a_2)} + \ldots = \frac{c_1/a_1}{(\tau_1 s + 1)} + \frac{c_2/a_2}{(\tau_2 s + 1)} + \ldots$$

Their corresponding terms in the time domain are

$$y(t) = c_1 e^{-a_1 t} + c_2 e^{-a_2 t} + \ldots = c_1 e^{-t/\tau_1} + c_2 e^{-t/\tau_2} + \ldots$$

As time progresses, the term associated with τ_2 (or a_2) will decay away faster. We consider the term with the larger time constant τ_1 as the dominant pole. [1]

Finally, for a complex pole, we can relate the damping ratio ($\zeta < 1$) with the angle that the pole makes with the real axis (Fig. 2.5). Taking the absolute values of the dimensions of the triangle, we can find

$$\theta = \tan^{-1}\left[\frac{\sqrt{1-\zeta^2}}{\zeta}\right] \qquad (2\text{-}33)$$

and more simply

$$\theta = \cos^{-1}\zeta \qquad (2\text{-}34)$$

Eq. (2-34) is used in the root locus method in Chapter 7 when we design controllers.

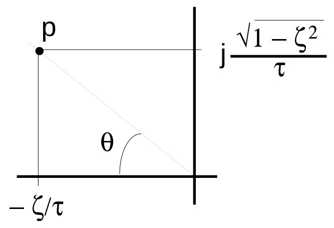


Figure 2.5. Complex pole angular position on the complex plane.

[1] Our discussion is only valid if τ_1 is "sufficiently" larger than τ_2. We could establish a criterion, but at the introductory level, we shall keep this as a qualitative observation.

2.8 Two transient model examples

We now use two examples to review how deviation variables relate to the actual ones, and that we can go all the way to find the solutions.

2.8.1 A Transient Response Example

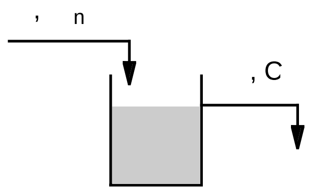


Figure 2.6. A constant volume continuous flow well-mixed vessel.

We routinely test the mixing of continuous flow stirred-tanks (Fig. 2.6) by dumping some kind of inert tracer, say a dye, into the tank and see how they get "mixed up." In more dreamy moments, you can try the same thing with cream in your coffee. However, you are a highly paid engineer, and a more technical approach must be taken. The solution is simple. We can add the tracer in a well-defined "manner," monitor the effluent, and analyze how the concentration changes with time. In chemical reaction engineering, you will see that the whole business is elaborated into the study of residence time distributions.

In this example, we have a stirred-tank with a volume V_1 of 4 m^3 being operated with an inlet flow rate Q of 0.02 m^3/s and which contains an inert species at a concentration C_{in} of 1 gmol/m^3. To test the mixing behavior, we purposely turn the knob which doses in the tracer and jack up its concentration to 6 gmol/m^3 (without increasing the total flow rate) for a duration of 10 s. The effect is a rectangular pulse input (Fig. 2.7).

What is the pulse response in the effluent? If we do not have the patience of 10 s and dump all the extra tracer in at one shot, what is the impulse response?

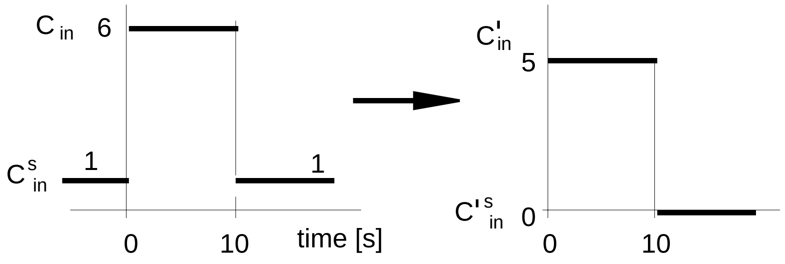


Figure 2.7. A rectangular pulse in real and deviation variables.

The model equation is a continuous flow stirred-tank without any chemical reaction:

$$V_1 \frac{dC_1}{dt} = Q\,(C_{in} - C_1)$$

In terms of space time τ_1, it is written as

$$\tau_1 \frac{dC_1}{dt} = C_{in} - C_1 \quad \text{where} \quad \tau_1 = \frac{V_1}{Q} = \frac{4}{0.02} = 200 \text{ s} \tag{2-35}$$

The initial condition is $C(0) = C_1^s$, where C_1^s is the value of the steady state solution. The inlet concentration is a function of time, $C_{in} = C_{in}(t)$, and will become our input. We present the analytical results here and will do the simulations with MATLAB in the Review Problems.

At steady state, Eq. (2-35) is [1]

$$0 = C_{in}^s - C_1^s \tag{2-36}$$

As suggested in Section 2.1, we define deviation variables

$$C'_1 = C_1 - C_1^s \quad \text{and} \quad C'_{in} = C_{in} - C^s_{in}$$

and combining Eqs. (2-35) and (2-36) would give us

$$\tau_1 \frac{d\,C'_1}{d\,t} = C'_{in} - C'_1$$

with the zero initial condition C'(0) = 0. We further rewrite the equation as:

$$\tau_1 \frac{d\,C'_1}{d\,t} + C'_1 = C_{in} \tag{2-37}$$

to emphasize that C'_{in} is the input (or forcing function). The Laplace transform of (2-37) is

$$\frac{C'_1(s)}{C'_{in}(s)} = \frac{1}{\tau_1 s + 1} \tag{2-38}$$

where the RHS is the transfer function. Here, it relates changes in the inlet concentration to changes in the tank concentration. This is a convenient form with which we can address different input scenarios.

Now we have to fit the pieces together for this problem. Before the experiment, that is, at steady state, we must have

$$C^s_{in} = C_1{}^s = 1 \tag{2-39}$$

Hence the rectangular pulse is really a perturbation in the inlet concentration:

$$C'_{in} = \begin{cases} 0 & t < 0 \\ 5 & 0 < t < 10 \\ 0 & t > 10 \end{cases}$$

This input can be written succinctly as

$$C'_{in} = 5\,[u(t) - u(t-10)]$$

which then can be applied to Eq. (2-37). Alternatively, we apply the Laplace transform of this input

$$C'_{in}(s) = \frac{5}{s}\,[1 - e^{-10\,s}]$$

and substitute it in Eq. (2-38) to arrive at

$$C'_1(s) = \frac{1}{(\tau_1 s + 1)}\,\frac{5}{s}\,[1 - e^{-10\,s}] \tag{2-40}$$

---

[1] At steady state, it is obvious from (2-35) that the solution must be identical to the inlet concentration. You may find it redundant that we add a superscript *s* to C^s_{in}. The action is taken to highlight the particular value of $C_{in}(t)$ that is needed to maintain the steady state and to make the definitions of deviation variables a bit clearer.

Inverse transform of (2-40) gives us the time-domain solution for $C'_1(t)$:

$$C'_1(t) = 5[1 - e^{-t/\tau_1}] - 5[1 - e^{-(t-10)/\tau_1}]\,u(t-10)$$

The most important time dependence of e^{-t/τ_1} arises only from the pole of the transfer function in Eq. (2-38). Again, we can "spell out" the function if we want to:

For $t < 10$ $\quad C'_1(t) = 5[1 - e^{-t/\tau_1}]$

and $t > 10$ $\quad C'_1(t) = 5[1 - e^{-t/\tau_1}] - 5[1 - e^{-(t-10)/\tau_1}] = 5\left[e^{-(t-10)/\tau_1} - e^{-t/\tau_1}\right]$

In terms of the actual variable, we have

for $t < 10$ $\quad C_1(t) = C_1^s + C'_1 = 1 + 5[1 - e^{-t/\tau_1}]$

and $t > 10$ $\quad C_1(t) = 1 + 5\left[e^{-(t-10)/\tau_1} - e^{-t/\tau_1}\right]$

We now want to use an impulse input of equivalent "strength," *i.e.*, same amount of inert tracer added. The amount of additional tracer in the rectangular pulse is

$$5\left[\frac{\text{gmol}}{\text{m}^3}\right] 0.02 \left[\frac{\text{m}^3}{\text{s}}\right] 10\ [\text{s}] = 1 \text{ gmol}$$

which should also be the amount of tracer in the impulse input. Let the impulse input be $C'_{in} = M\delta(t)$. Note that $\delta(t)$ has the unit of time^{-1} and M has a funny and physically meaningless unit, and we calculate the magnitude of the input by matching the quantities

$$1\ [\text{gmol}] = \int_0^\infty 0.02 \left[\frac{\text{m}^3}{\text{s}}\right] M \left[\frac{\text{gmol.s}}{\text{m}^3}\right] \delta(t) \left[\frac{1}{\text{s}}\right] dt\ [\text{s}] = 0.02M \quad \text{or} \quad M = 50 \left[\frac{\text{gmol. s}}{\text{m}^3}\right]$$

Thus

$$C'_{in}(t) = 50\delta(t) \ , \quad C'_{in}(s) = 50$$

and for an impulse input, Eq. (2-38) is simply

$$C'_1(s) = \frac{50}{(\tau_1 s + 1)} \tag{2-41}$$

After inverse transform, the solution is

$$C'_1(t) = \frac{50}{\tau_1} e^{-t/\tau_1}$$

and in the real variable,

$$C_1(t) = 1 + \frac{50}{\tau_1} e^{-t/\tau_1}$$

We can do a mass balance based on the outlet

$$Q \int_0^\infty C'_1(t)\,dt = 0.02\,\frac{50}{\tau_1} \int_0^\infty e^{-t/\tau_1}\,dt = 1\ [\text{gmol}]$$

Hence mass is conserved and the mathematics is correct.

We now raise a second question. If the outlet of the vessel is fed to a second tank with a volume V_2 of 3 m^3 (Fig. 2.8), what is the time response at the exit of the second tank? With the second tank, the mass balance is

$$\tau_2 \frac{dC_2}{dt} = (C_1 - C_2) \quad \text{where} \quad \tau_2 = \frac{V_2}{Q}$$

or

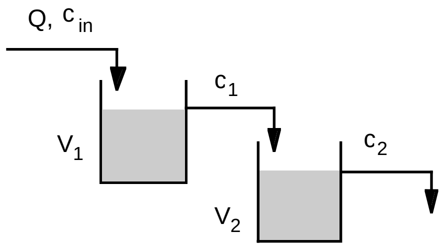


Figure 2.8. Two well-mixed vessels in series.

$$\tau_2 \frac{dC_2}{dt} + C_2 = C_1 \tag{2-42}$$

where C_1 and C_2 are the concentrations in tanks one and two respectively. The equation analogous to Eq. (2-37) is

$$\tau_2 \frac{dC'_2}{dt} + C'_2 = C'_1 \tag{2-43}$$

and its Laplace transform is

$$C'_2(s) = \frac{1}{\tau_2 s + 1} C'_1(s) \tag{2-44}$$

With the rectangular pulse to the first vessel, we use the response in Eq. (2-40) and substitute in (2-44) to give

$$C'_2(s) = \frac{5\,(1 - e^{-10\,s})}{s\,(\tau_1 s + 1)\,(\tau_2 s + 1)}$$

With the impulse input, we use the impulse response in Eq. (2-41) instead, and Eq. (2-44) becomes

$$C'_2(s) = \frac{50}{(\tau_1 s + 1)\,(\tau_2 s + 1)}$$

from which $C'_2(t)$ can be obtained via proper table look-up. The numerical values

$$\tau_1 = \frac{4}{0.02} = 200 \text{ s} \quad \text{and} \quad \tau_2 = \frac{3}{00.2} = 150 \text{ s}$$

can be used. We will skip the inverse transform. It is not always instructive to continue with an algebraic mess. To sketch the time response, we'll do that with MATLAB in the Review Problems.

2.8.2 A stirred tank heater

Temperature control in a stirred-tank heater is a common example (Fig. 2.9). We will come across it many times in later chapters. For now, we present the basic model equation, and use it as a review of transfer functions.

The heat balance, in standard heat transfer notations, is

$$\rho C_p V \frac{dT}{dt} = \rho C_p Q\,(T_i - T) + UA\,(T_H - T) \tag{2-45}$$

where U is the overall heat transfer coefficient, A is the heat transfer area, ρ is fluid density, C_p is the heat capacity, and V is the volume of the vessel. The inlet temperature $T_i = T_i(t)$ and steam coil temperature $T_H = T_H(t)$ are functions of time and are presumably given. The initial condition is $T(0) = T^s$, the steady state temperature.

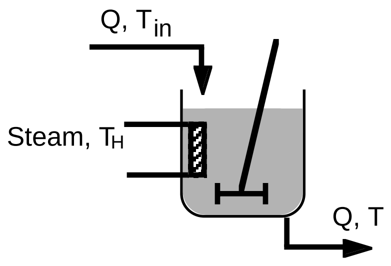


Figure 2.9. A continuous flow stirred-tank heater.

Before we go on, let us emphasize that what we find below are nothing but different algebraic manipulations of the same heat balance. First, we rearrange (2-45) to give

$$\left(\frac{V}{Q}\right)\frac{dT}{dt} = (T_i - T) + \frac{UA}{\rho C_p Q}(T_H - T)$$

The second step is to define

$$\tau = \frac{V}{Q} \quad \text{and} \quad \kappa = \frac{UA}{\rho C_p Q}$$

which leads to

$$\tau\frac{dT}{dt} + (1 + \kappa)T = T_i + \kappa T_H \tag{2-46}$$

At steady state,

$$(1 + \kappa)\,T^s = T_i^s + \kappa\,T_H^s \tag{2-47}$$

We now define deviation variables:

$$T' = T - T^s \;;\quad T'_i = T_i - T_i^s \;;\quad T'_H = T_H - T_H^s$$

and
$$\frac{dT'}{dt} = \frac{d(T - T^s)}{dt} = \frac{dT}{dt}$$

Subtract Eq. (2.47) from the transient equation in Eq. (2-46) would give

$$\tau\frac{dT}{dt} + (1 + \kappa)\,(T - T^s) = (T_i - T_i^s) + \kappa\,(T_H - T_H^s)$$

or in deviation variables,

$$\tau\frac{dT'}{dt} + (1 + \kappa)\,T' = T'_i + \kappa T'_H \tag{2-48}$$

The initial condition is T'(0) = 0. Eq. (2-48) is identical in form to (2-46). This is typical of linear equations. Once you understand the steps, you can jump from (2-46) to (2-48), skipping over the formality.

From here on, we will *omit the apostrophe* (') where it would not cause confusion, as it goes without saying that we work with deviation variables. We now further rewrite the same equation as

$$\frac{dT}{dt} + aT = K_iT_i + K_HT_H \tag{2-48a}$$

where

$$a = \frac{(1+\kappa)}{\tau} \quad ; \quad K_i = \frac{1}{\tau} \quad ; \quad K_H = \frac{\kappa}{\tau}$$

Laplace transform gives us

$$s\,T(s) + a\,T(s) \;=\; K_i\,T_i(s) + K_H\,T_H(s)$$

Hence Eq. (2-48a) becomes

$$T(s) \;=\; \left(\frac{K_i}{s+a}\right) T_i(s) + \left(\frac{K_H}{s+a}\right) T_H(s) \;=\; G_d(s)T_i(s) + G_p(s)T_H(s) \tag{2-49a}$$

where

$$G_d(s) \;=\; \frac{K_i}{s+a} \quad ; \quad G_p(s) \;=\; \frac{K_H}{s+a}$$

Of course, $G_d(s)$ and $G_p(s)$ are the transfer functions, and they are in pole-zero form. Once again(!), we are working with deviation variables. The interpretation is that *changes* in the inlet temperature and the steam temperature lead to *changes* in the tank temperature. The effects of the inputs are additive and mediated by the two transfer functions.

Are there more manipulated forms of the same old heat balance? You bet. In fact, we very often rearrange Eq. (2-48), writing without the apostrophes, as

$$\tau_p \frac{dT}{dt} + T \;=\; K_d\,T_i + K_p\,T_H \tag{2-48b}$$

where [1] $\quad \tau_p = \frac{1}{a} = \frac{\tau}{(1+\kappa)} \quad ; \quad K_d = \frac{K_i}{a} = \frac{1}{(1+\kappa)} \quad ; \quad K_p = \frac{K_H}{a} = \frac{\kappa}{(1+\kappa)}$

After Laplace transform, the model equation is

$$T(s) \;=\; G_d(s)T_i(s) + G_p(s)T_H(s) \tag{2-49b}$$

which is identical to Eq. (2-49a) except that the transfer functions are in the time constant form

$$G_d(s) \;=\; \frac{K_d}{\tau_p s + 1} \qquad \text{and} \qquad G_p(s) \;=\; \frac{K_p}{\tau_p s + 1}$$

In this rearrangement, τ_p is the process time constant, and K_d and K_p are the steady state gains.[2] The denominators of the transfer functions are *identical*; they both are from the LHS of the differential equation—the characteristic polynomial that governs the inherent dynamic characteristic of the process.

---

[1] If the heater is well designed, κ $(=UA/\rho C_pQ)$ should be much larger than 1. The steady state gain K_p approaches unity, meaning changing the steam temperature is an effective means of changing the tank temperature. In contrast, K_d is very small, and the tank temperature is insensitive to changes in the inlet temperature.

At first reading, you'd find the notations confusing—and in some ways we did this on purpose. This is as bad as it gets once you understand the different rearrangements. So go through each step slowly.

[2] K_i and K_H in (2-49a) are referred to as gains, but *not* the steady state gains. The process time constant is also called a *first-order lag* or *linear lag*.

Let us try one simple example. Say if we keep the inlet temperature constant at our desired steady state, the statement in deviation variable (without the apostrophe) is

$$T_i(t) = 0\ , \text{ and } \ T_i(s) = 0$$

Now we want to know what happens if the steam temperature increases by 10 °C. This change in deviation variable is

$$T_H = Mu(t) \ \text{ and } \ T_H(s) = \frac{M}{s}\ , \ \text{ where } \ M = 10\ °C$$

We can write

$$T(s) = \left(\frac{K_p}{\tau_p s + 1}\right)\frac{M}{s} \tag{2-50}$$

After partial fraction expansion,

$$T(s) = MK_p\left[\frac{1}{s} - \frac{\tau_p}{\tau_p s + 1}\right]$$

Inverse transform via table look-up gives our time-domain solution for the deviation in T:[1]

$$T(t) = MK_p\left[1 - e^{-t/\tau_p}\right] \tag{2-51}$$

Keep a mental imprint of the shape of this first order step response as shown in Fig. 2.10. As time progresses, the exponential term decays away, and the temperature approaches the new value MK_p. Also illustrated in the figure is the much used property that at $t = \tau_p$, the normalized response is 63.2%.

After this exercise, let's hope that we have a better appreciation of the different forms of a transfer function. With one, it is easier to identify the pole positions. With the other, it is easier to extract the steady state gain and time constants. It is very important for us to learn how to interpret qualitatively the dynamic response from the pole positions, and to make physical interpretation with the help of quantities like steady state gains, and time constants.

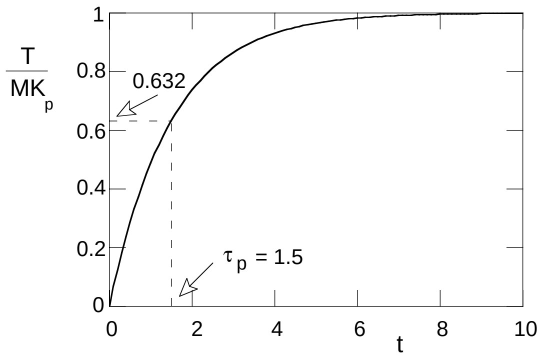


Figure 2.10. Illustration of a first order response (2-51) normalized by MK_p. The curve is plotted with $\tau_p = 1.5$ [arbitrary time unit]. At $t = \tau_p$, the normalized response is 63.2%.

2.9 Linearization of nonlinear equations

Since Laplace transform can only be applied to a linear differential equation, we must "fix" a nonlinear equation. The goal of control is to keep a process running at a specified condition (the steady state). For the most part, if we do a good job, the system should only be slightly perturbed from the steady state such that the dynamics of returning to the steady state is a first order decay, *i.e.*, a linear process. This is the cornerstone of classical control theory.

What we do is a freshmen calculus exercise in first order Taylor series expansion about the

[1] Note that if we had chosen also $T_H = 0$, $T(t) = 0$ for all t, *i.e.*, nothing happens. Recall once again from Section 2.1 that this is a result of how we define a problem using deviation variables.

steady state and reformulating the problem in terms of deviation variables. We will illustrate with one simple example. Consider the differential equation which models the liquid level *h* in a tank with cross-sectional area *A*,

$$A\frac{dh}{dt} = Q_{in}(t) - \beta h^{1/2} \tag{2-52}$$

The initial condition is $h(0) = h_s$, the steady state value. The inlet flow rate Q_{in} is a function of time. The outlet is modeled as a nonlinear function of the liquid level. Both the tank cross-section A, and the coefficient β are constants.

We next expand the nonlinear term about the steady state value h_s (also our initial condition by choice) to provide [1]

$$A\frac{dh}{dt} = Q_{in} - \beta\left[h_s^{1/2} + \frac{1}{2}h_s^{-1/2}(h - h_s)\right] \tag{2-53}$$

At steady state, we can write the differential equation (2-52) as

$$0 = Q_{in}^s - \beta\, h_s^{1/2} \tag{2-54}$$

where h_s is the steady solution, and Q_{in}^s is the particular value of Q_{in} to maintain steady state. If we subtract the steady state equation from the linearized differential equation, we have

$$A\frac{dh}{dt} = \left(Q_{in} - Q_{in}^s\right) - \beta\left[\frac{1}{2}h_s^{-1/2}(h - h_s)\right] \tag{2-55}$$

We now define deviation variables:

$$h' = h - h_s \qquad \text{and} \qquad Q'_{in} = Q_{in} - Q^s_{in}$$

Substitute them into the linearized equation and moving the h' term to the left should give

$$A\frac{dh'}{dt} + \left(\frac{\beta}{2}h_s^{-1/2}\right)h' = Q'_{in}(t) \tag{2-56}$$

with the zero initial condition $h'(0) = 0$.

It is important to note that the initial condition in Eq. (2-52) has to be h_s, the original steady state level. Otherwise, we will not obtain a zero initial condition in (2-56). On the other hand, because of the zero initial condition, the forcing function Q'_{in} must be finite to have a non-trivial solution. Repeating our mantra the umpteenth time, the LHS of (2-56) gives rise to the characteristic polynomial and describes the inherent dynamics. The actual response is subject to the inherent dynamics and the input that we impose on the RHS.

---

[1] We casually ignore the possibility of a more accurate second order expansion. That's because the higher order terms are nonlinear, and we need a linear approximation. Needless to say that with a first order expansion, it is acceptable only if h is sufficiently close to h_s.

In case you forgot, the first order Taylor series expansion can be written as

$$f(x_1,x_2) \approx f(x_{1s},x_{2s}) + \partial f/\partial x_1\big|_{x_{1s},\,x_{2s}}(x_1 - x_{1s}) + \partial f/\partial x_2\big|_{x_{1s},\,x_{2s}}(x_2 - x_{2s})$$

Note:

- Always do the linearization before you introduce the deviation variables.
- As soon as we finish the first-order Taylor series expansion, the equation is linearized. All steps that follow are to clean up the algebra with the understanding that terms of the steady state equation should cancel out, and to change the equation to deviation variables with zero initial condition.

We now provide a more general description. Consider an ordinary differential equation

$$\frac{dy}{dt} = f(y; \mathbf{u}) \quad \text{with} \quad y(0) = y_s \tag{2-57}$$

where $\mathbf{u} = \mathbf{u}(t)$ contains other parameters that may vary with time. If $f(y; \mathbf{u})$ is nonlinear, we approximate with Taylor's expansion:

$$\frac{dy}{dt} \approx f(y_s; \mathbf{u}_s) + \left.\frac{\partial f}{\partial y}\right|_{y_s,\mathbf{u}_s} (y - y_s) + \nabla^T f(y_s; \mathbf{u}_s)\,(\mathbf{u} - \mathbf{u}_s) \tag{2-58}$$

where $\nabla f(y_s; \mathbf{u}_s)$ is a column vector of the partial derivatives of the function with respect to elements in $\mathbf{u}$, $\partial f/\partial u_i$, and evaluated at y_s and $\mathbf{u}_s$. At steady state, (2-57) is

$$0 = f(y_s; \mathbf{u}_s) \tag{2-59}$$

where y_s is the steady state solution, and $\mathbf{u}_s$ the values of parameters needed to maintain the steady state. We again define deviation variables

$$y' = y - y_s \qquad \text{and} \qquad \mathbf{u'} = \mathbf{u} - \mathbf{u}_s$$

and subtract (2-59) from (2-58) to obtain the linearized equation with zero initial condition:

$$\frac{dy'}{dt} + \left[-\left.\frac{\partial f}{\partial y}\right|_{y_s,\mathbf{u}_s} \right] y' = \left[\nabla^T f(y_s; \mathbf{u}_s) \right] \mathbf{u'} \tag{2-60}$$

where we have put quantities being evaluated at steady state conditions in brackets. When we solve a particular problem, they are just constant coefficients after substitution of numerical values.

✎ **Example 2.11**: Linearize the differential equation for the concentration in a mixed vessel: $V\frac{dC}{dt} = Q_{in}(t)C_{in}(t) - Q_{in}(t)C$, where the flow rate and the inlet concentration are functions of time.

A first term Taylor expansion of the RHS leads to the approximation:

$$V\frac{dC}{dt} \approx \left[Q_{in,s}C_{in,s} + C_{in,s}\,(Q_{in} - Q_{in,s}) + Q_{in,s}\,(C_{in} - C_{in,s})\right]$$
$$- \left[Q_{in,s}C_s + C_s\,(Q_{in} - Q_{in,s}) + Q_{in,s}\,(C - C_s)\right]$$

and the steady state equation, without canceling the flow variable, is

$$0 = Q_{in,s}C_{in,s} - Q_{in,s}C_s$$

We subtract the two equations and at the same time introduce deviation variables for the dependent variable C and all the parametric variables to obtain

$$V\frac{dC'}{dt} \approx \left[C_{in,s}\,Q'_{in} + Q_{in,s}\,C'_{in}\right] - \left[C_s\,Q'_{in} + Q_{in,s}\,C'\right]$$

and after moving the C' term to the LHS,

$$V\frac{dC'}{dt} + \left[Q_{in,s}\right] C' = \left[C_{in,s} - C_s\right] Q'_{in} + \left[Q_{in,s}\right] C'_{in}$$

The final result can be interpreted as stating how changes in the flow rate and inlet concentration lead to changes in the tank concentration, as modeled by the dynamics on the LHS. Again, we put the constant coefficients evaluated at steady state conditions in brackets. We can arrive at this result quickly if we understand Eq. (2-60) and apply it carefully.

The final step should also has zero initial condition C'(0) = 0, and we can take the Laplace transform to obtain the transfer functions if they are requested. As a habit, we can define $\tau = V/Q_{in,s}$ and the transfer functions will be in the time constant form.

✎ **Example 2.12**: Linearize the differential equation $\frac{dy}{dt} = -xy - \beta y^2 - \gamma^{y-1}$ where x =x(t).

Each nonlinear term can be approximated as

$$xy \approx x_s y_s + y_s (x - x_s) + x_s (y - y_s) = x_s y_s + y_s x' + x_s y'$$

$$y^2 \approx y_s^2 + 2y_s (y - y_s) = y_s^2 + 2y_s y'$$

$$\gamma^{y-1} \approx \gamma^{y_s - 1} + (\ln \gamma)\, \gamma^{y_s - 1}\, y'$$

With the steady state equation

$$0 = x_s y_s + \beta y_s + \gamma^{y_s - 1} ,$$

and the usual algebraic work, we arrive at

$$\frac{dy'}{dt} + \left[x_s + 2\beta y_s + (\ln \gamma)\, \gamma^{y_s - 1}\right] y' = - y_s x'$$

✎ **Example 2.13**: What is the linearized form of the reaction rate term $r_A = -k(T)\, C_A = -k_o e^{-E/RT} C_A$ where both temperature T and concentration C_A are functions of time?

$$r_A \approx -\left[k_o e^{-E/RT_s} C_{A,s} + k_o e^{-E/RT_s} (C_A - C_{A,s}) + \left(\frac{E}{RT_s^2}\right) k_o e^{-E/RT_s} C_{A,s} (T - T_s)\right]$$

In terms of deviation variables, the linearized approximation is

$$r_A \approx r_{A,s}\left[1 + \frac{1}{C_{A,s}} C'_A + \frac{E}{RT_s^2} T'\right], \quad \text{where} \quad r_{A,s} = -k_o e^{-E/RT_s} C_{A,s}$$

Note:
While our analyses use deviation variables and not the real variables, examples and homework problems can keep bouncing back and forth. The reason is that when we do an experiment, we measure the actual variable, not the deviation variable. You may find this really confusing. All we can do is to be extra careful when we solve a problem.

2.10 Block diagram reduction

The use of block diagrams to illustrate cause and effect relationship is prevalent in control. We use operational blocks to represent transfer functions and lines for unidirectional information transmission. It is a nice way to visualize the interrelationships of various components. Later, they are crucial to help us identify manipulated and controlled variables, and input(s) and output(s) of a system.

Many control systems are complicated looking networks of blocks. The simplest control system looks like Fig. 2.11a. The problem is that many theories in control are based on a simple closed-loop or single-block structure (Fig. 2.11b).

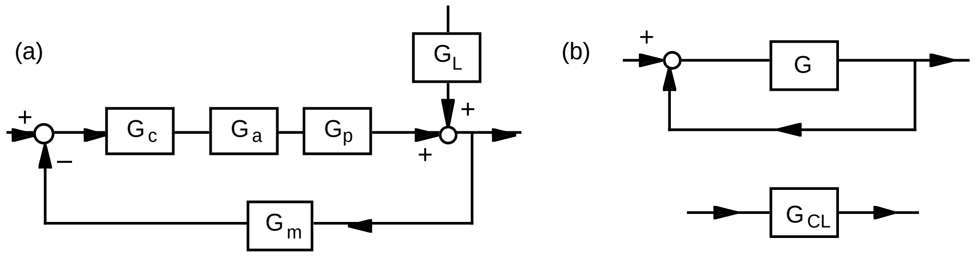


Figure 2.11. (a) Example of a feedback system block diagram; (b) Typical reduced block diagrams.

Hence we must learn how to read a block diagram and *reduce* it to the simplest possible form. We will learn in later chapters how the diagram is related to an actual physical system. First, we will do some simple algebraic manipulation and better yet, do it graphically. It is important to remember that all (graphical) block diagram reduction is a result of formal algebraic manipulation of transfer functions. When all imagination fails, *always refer back to the actual algebraic equations*.[1]

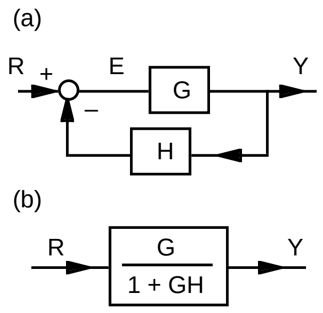


Figure 2.12. (a) Simple negative feedback loop, and (b) its reduced single closed-loop transfer function form.

Of all manipulations, the most important one is the reduction of a feedback loop. Here is the so-called block diagram reduction and corresponding algebra.

For a negative feedback system (Fig. 2.12), we have

$$E = R - HY, \tag{2-61}$$

and

$$Y = GE \tag{2-62}$$

Using (2-61) to substitute for E in (2-62) leads to

$$Y = G[R - HY]$$

which can be rearranged to give, for a **negative feedback** loop,[2]

---

[1] See the *Web Support* for our comment on the Mason's gain formula.

[2] Similarly, we can write for the case of *positive* feedback that E = R + H Y, and Y = G [R + H Y], and we have instead:

$$\frac{Y}{R} = \frac{G}{1 + G\,H} \tag{2-63}$$

The RHS of (2-63) is what we will refer to as the **closed-loop transfer function** in later chapters.

Note:
The important observation is that when we "close" a *negative* feedback loop, the numerator is consisted of the product of all the transfer functions along the forward path. The denominator is 1 *plus* the product of all the transfer functions in the entire feedback loop (*i.e.*, both forward and feedback paths). The denominator is also the characteristic polynomial of the closed-loop system. If we have positive feedback, the sign in the denominator is minus.

Here, we try several examples and take the conservative route of writing out the relevant algebraic relations.[1]

✎ **Example 2.14**. Derive the closed-loop transfer function C/R and C/L for the system as shown in Fig. E2.14.

We identify two locations after the summing points with lower case *e* and *a* to help us.[2] We can write at the summing point below H:

$$a = –C + KR$$

and substitute this relation in the equation for the summing point above H to give

$$e = R + H\,a = R + H\,(KR – C)$$

We substitute the relation for *e* into the equation about G_p to obtain

$$C = L + G_p\,e = L + G_p\,(R + HKR – HC)$$

The final result is a rearrangement to get C out explicitly

$$C = \left[\frac{1}{1 + G_p H}\right] L + \left[\frac{G_p\,(1 + HK)}{1 + G_p H}\right] R$$

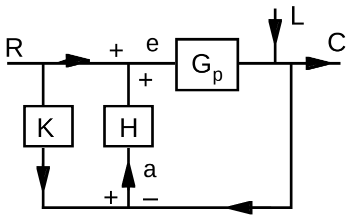


Figure E2.14

$$\frac{Y}{R} = \frac{G(s)}{1 – G(s)H(s)}$$

1 A few more simple examples are in the *Web Support* of this chapter.

2 How do we decide the proper locations? We do not know for sure, but what should help is after a summing point where information has changed. We may also use the location before a branch off point, helping to trace where the information is routed.

✎ **Example 2.15**. Derive the closed-loop transfer function C/R for the system with three overlapping negative feedback loops in Fig. E2.15(a).

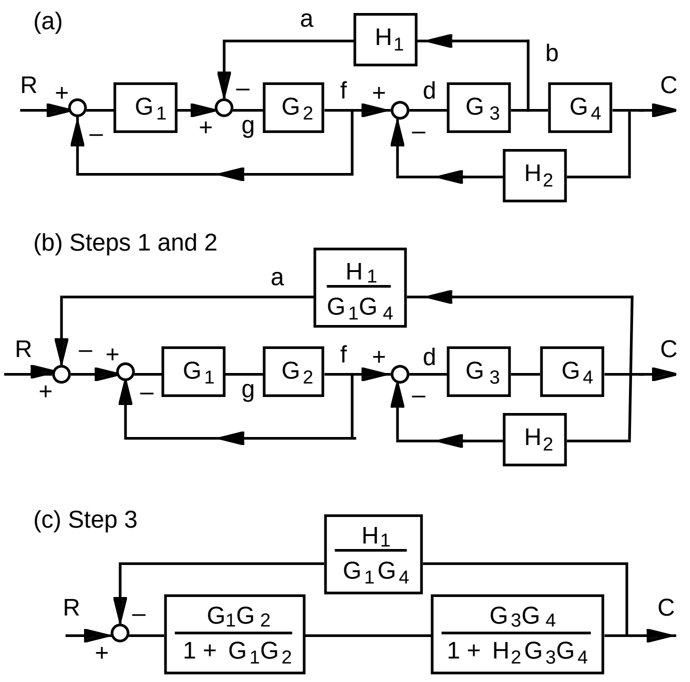


Figure E2.15

The key to this problem is to proceed in steps and "untie" the overlapping loops first. We identify various locations with lower case *a*, *b*, *d*, *f*, and *g* to help us. We first move the branch-off point over to the right side of G_4 (Fig. E2.15b). We may note that we can write

$$a = H_1\,G_3\,d = \frac{H_1}{G_4}\,[G_3G_4]\,d$$

That is, to maintain the same information at the location *a*, we must divide the branch-off information at C by G_4.

Similarly, we note that at the position *g* in Fig. E2.15a,

$$g = G_1\,[R - f] - bH_1 = G_1\,[R - f - \frac{bH_1}{G_1}]$$

That is, if we move the break-in point from *g* out to the left of G_1, we need to divide the information by G_1 prior to breaking in. The block diagram after moving both the branch-off and break-in points are shown as Steps 1 and 2 in Fig. E2.15b. (We could have drawn such that the loops are flush with one another at R.)

Once the loops are no longer overlapping, the block diagram is easy to handle. We first close the two small loops as shown as Step 3 in Fig. E2.15c.

The final result is to close the big loop. The resulting closed-loop transfer function is:

$$\frac{C}{R} = \frac{G_1\,G_2\,G_3\,G_4}{(1 + G_1\,G_2)\,(1 + H_2\,G_3\,G_4) + H_1\,G_2\,G_3}$$

✎ **Example 2.16**. Derive the closed-loop transfer function X_1/U for the block diagram in Fig. E2.16a. We will see this one again in Chapter 4 on state space models. With the integrator $1/s$, X_2 is the Laplace transform of the time derivative of $x_1(t)$, and X_3 is the second order derivative of $x_1(t)$.

As before, we can write down the algebraic equations about the summing point (the comparator) for X_3, and the two equations about the two integrators $1/s$. We should arrive at the result after eliminating X_2 and X_3.

However, we can also obtain the result quickly by recognizing the two feedback loops. We first "close" the inner loop to arrive at Fig. E2.16b. With that, we can see the answer. We "close" this loop to obtain

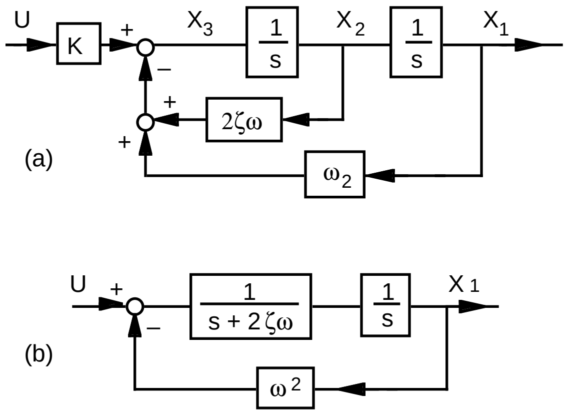


Figure E2.16

$$\frac{X_1}{U} = \frac{1}{s^2 + 2\zeta\omega s + \omega^2}$$

❒ Review Problems

1. Derive Eq. (2.1).
2. (a) Check that when the right hand side is zero, the solution to Eq. (2-2) is zero.
   (b) Derive Eq. (2-3) using the method of integrating factor.
   (c) Derive the solution c(t) in Eq. (2-3) with, respectively, an impulse input, $C'_{in} = \delta(t)$ and a unit step input $C'_{in} = u(t)$. Show that they are identical to when we use the Laplace transform technique as in Example 2.10.
3. Prove the linear property of Laplace transform in (2-6).
4. Derive the Laplace transform of

   (a) $1/(\tau s + 1)$ (b) $\cos \omega t$ (c) $e^{-at} \cos \omega t$
5. Prove the initial value theorem.
6. Show that the inverse transform of $F(s) = \dfrac{6}{(s^3 + s^2 - 4s - 4)}$ is $f(t) = -2e^{-t} + \frac{3}{2}e^{-2t} + \frac{1}{2}e^{2t}$
7. Double check α^* in the complex root Example 2.7 with the Heaviside expansion.
8. Find the inverse Laplace transform of the general expression $Y(s) = \dfrac{c}{s-p} + \dfrac{c^*}{s-p^*}$,

   where $c = a - bj$, and $p = \alpha + \omega j$. Rearrange the result to a sine function with time lag.
9. With respect to the repeated root Example 2.9, show that we could have written

   $2 = \alpha_1 (s + 1)^2(s + 2) + \alpha_2 (s + 1)(s + 2) + \alpha_3 (s + 2) + \alpha_4 (s + 1)^3$

   and after expanding and collecting terms of the same power in *s*, we can form the matrix equation:

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 4 & 1 & 0 & 3 \\ 5 & 3 & 1 & 3 \\ 2 & 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \end{bmatrix}$$

from which we can solve for the coefficients. Yes, this is a chore not worth doing even with MATLAB. The route that we take in the example is far quicker.

10. For a general transfer function G(s) = Y(s)/X(s), how can we find the steady state gain?
11. Do the partial fractions of $\dfrac{s+1}{s^2\,(s^2+4s-5)}$.
12. Find the transfer functions as suggested in Example 2.11.
13. Derive Eqs. (3-33) and (3-34).
14. Do the numerical simulation for Section 2.8.1
15. Regarding Eqs. (2-50) and (2-51) in Section 2.8.2:

    a) What is T(t) as t --> ∞ ? What is the actual temperature that we measure?

    b) What are the effects of K_p on the final tank temperature? What is the significance if K_p approaches unity?

    c) What is the time constant for the process?

Hints:

1. Eq. (2.1) is straight from material balance. With Q denoting the volumetric flow rate and V the volume, the balance equation of some species A as denoted by C with reference to Fig. 2.6 is

   $$V\,\frac{dC}{dt} = QC_{in} - QC$$

   Physically, each term in the equation represents:

   {Accumulation} = {Flow in} – {Flow out}

   Eq. (2.1) is immediately obvious with the definition of space time $\tau = V/Q$.
2. (c) With the impulse input, $C'(t) = \frac{1}{\tau}\,e^{-t/\tau}$, and with the unit step input, $C'(t) = 1 - e^{-t/\tau}$.
3. This is just a simple matter of substituting the definition of Laplace transform.
4. Answers are in the Laplace transform summary table. The key to (a) is to rearrange the function as $\dfrac{1/\tau}{s+1/\tau}$, and the result should then be immediately obvious with Eq. (2-9). The derivation of (b) and (c) is very similar to the case involving $\sin\omega t$.
5. The starting point is again the Laplace transform of a derivative, but this time we take the limit as s —> ∞ and the entire LHS with the integral becomes zero.
6. This follows Examples 2.4 and 2.5.
7. $\alpha^* = \left.\dfrac{s+5}{[s-(-2+3j)]}\right|_{s=-2-3j} = \dfrac{(-2-3j)+5}{(-2-3j)+2-3j} = \dfrac{(1-j)}{-2j} = \dfrac{1}{2}(1+j)$
8. We should have

   $$y(t) = c\,e^{pt} + c^*\,e^{p^*t} = (a-bj)\,e^{(\alpha+\omega j)\,t} + (a+bj)\,e^{(\alpha-\omega j)\,t}\,.$$

We now do the expansion:

$$y(t) = e^{\alpha t}\,[(a - bj)\,(\cos \omega t + j \sin \omega t) + (a + bj)\,(\cos \omega t - j \sin \omega t)]$$

After cancellation:

$$y(t) = 2e^{\alpha t}\,[a \cos \omega t + b \sin \omega t]$$

which of course is

$$y(t) = 2\,e^{\alpha t}\,A \sin (\omega t + \phi)\,,\ \text{ where } A = \sqrt{(a^2 + b^2)},\ \text{and } \phi = \tan^{-1}(a/b)$$

10. Use $X = 1/s$ and the final value theorem. This is what we did in Eq. (2-32). Note that $y(\infty)$ is *not* the same as $g(\infty)$.

11. Use MATLAB to check your algebra.

12. With the time constant defined as $\tau = V/Q_{in,s}$, the steady state gain for the transfer function for the inlet flow rate is $(C_{in,s} - C_s)/Q_{in,s}$, and it is 1 for the inlet concentration transfer function.

13. To find $\tan \theta$ is easy. To find $\cos \theta$, we need to show that the distance from the pole *p* to the origin (the hypotenuse) is $1/\tau$.

14. These are the statements that you can use:

```
tau1=200;
G1=tf(1,[tau1 1]);   % Transfer function of the first vessel

pulselength=10;      % Generate a vector to represent the rectangular pulse
delt=5;              % (We are jumping ahead. These steps are explained
t=0:delt:1000;       %  in MATLAB Session 3.)
u=zeros(size(t));
u(1:pulselength/delt+1)=5;

lsim(G1,u,t);        % The response of the rectangular pulse

hold
y=50*impulse(G1,t);  % Add on the impulse response
plot(t,y)

tau2=150;            % Generate the transfer function for both vessels
G2=tf(1,[tau2 1]);
G=G1*G2;
lsim(G,u,t)          % Add the responses
y=50*impulse(G,t);
plot(t,y)
```

15. (a) As $t \longrightarrow \infty$, the deviation in $T(t)$ should be $10K_p$ °C. We have to know the original steady state temperature in order to calculate the actual temperature.

    (b) Since $K_p < 1$, the final change in temperature will not be 10 °C. But if $K_p = 1$, the final change will be 10 °C. We can make this statement without doing the inverse transform.

    (c) The time constant of the stirred-tank heater is τ_p, not the space time.

Summary of a handful of common Laplace transforms

We may find many more Laplace transforms in handbooks or texts, but we'll stay with the most basic ones. The more complex ones may actually be a distraction to our objective which is to understand pole positions.

The very basic functions:

F(s)	f(t)
$\frac{a}{s}$	a, or au(t)
$\frac{a}{s^2}$	at
$\frac{1}{(s+a)}$	e^{-at}
$\frac{\omega}{(s^2+\omega^2)}$	$\sin \omega t$
$\frac{s}{(s^2+\omega^2)}$	$\cos \omega t$
$\frac{\omega}{(s+a)^2+\omega^2}$	$e^{-at} \sin \omega t$
$\frac{s+a}{(s+a)^2+\omega^2}$	$e^{-at} \cos \omega t$
$s^2F(s) - sf(0) - f'(0)$	$\frac{d^2f}{dt^2}$
$\frac{F(s)}{s}$	$\int_0^t f(t)\,dt$
$e^{-st_o} F(s)$	$f(t - t_o)$
A	$A\delta(t)$

Transfer functions in time constant form:

F(s)	f(t)
$\frac{1}{(\tau s+1)}$	$\frac{1}{\tau} e^{-t/\tau}$
$\frac{1}{(\tau s+1)^n}$	$\frac{1}{\tau^n (n-1)!} t^{n-1} e^{-t/\tau}$
$\frac{1}{s(\tau s+1)}$	$1 - e^{-t/\tau}$
$\frac{1}{(\tau_1 s+1)(\tau_2 s+1)}$	$\frac{e^{-t/\tau_1} - e^{-t/\tau_2}}{\tau_1 - \tau_2}$
$\frac{1}{s(\tau_1 s+1)(\tau_2 s+1)}$	$1 + \frac{\tau_1 e^{-t/\tau_1} - \tau_2 e^{-t/\tau_2}}{\tau_2 - \tau_1}$
$\frac{(\tau_3 s+1)}{(\tau_1 s+1)(\tau_2 s+1)}$	$\frac{1}{\tau_1}\frac{\tau_1-\tau_3}{\tau_1-\tau_2} e^{-t/\tau_1} + \frac{1}{\tau_2}\frac{\tau_2-\tau_3}{\tau_2-\tau_1} e^{-t/\tau_2}$
$\frac{(\tau_3 s+1)}{s(\tau_1 s+1)(\tau_2 s+1)}$	$1 + \frac{\tau_3-\tau_1}{\tau_1-\tau_2} e^{-t/\tau_1} + \frac{\tau_3-\tau_2}{\tau_2-\tau_1} e^{-t/\tau_2}$

Transfer functions in pole-zero form:

F(s)	f(t)
$\frac{1}{(s+a)}$	e^{-at}
$\frac{1}{(s+a)^2}$	$t\,e^{-at}$
$\frac{1}{(s+a)^n}$	$\frac{1}{(n-1)!} t^{n-1} e^{-at}$
$\frac{1}{s(s+a)}$	$\frac{1}{a}(1 - e^{-at})$
$\frac{1}{(s+a)(s+b)}$	$\frac{1}{b-a}(e^{-at} - e^{-bt})$
$\frac{s}{(s+a)^2}$	$(1 - at)\,e^{-at}$
$\frac{s}{(s+a)(s+b)}$	$\frac{1}{b-a}(be^{-bt} - ae^{-at})$
$\frac{1}{s(s+a)(s+b)}$	$\frac{1}{ab}\left[1 + \frac{1}{a-b}(be^{-at} - ae^{-bt})\right]$

❖ 3. *Dynamic Response*

We now derive the time-domain solutions of first and second order differential equations. It is not that we want to do the inverse transform, but comparing the time-domain solution with its Laplace transform helps our learning process. What we hope to establish is a better feel between pole positions and dynamic characteristics. We also want to see how different parameters affect the time-domain solution. The results are useful in control analysis and in measuring model parameters. At the end of the chapter, dead time, reduced order model, and the effect of zeros will be discussed.

What are we up to?

- Even as we speak of time-domain analysis, we invariably still work with Laplace transform. Time-domain and Laplace-domain are inseparable in classical control.
- In establishing the relationship between time-domain and Laplace-domain, we use only first and second order differential equations. That's because we are working strictly with linearized systems. As we have seen in partial fraction expansion, any function can be "broken up" into first order terms. Terms of complex roots can be combined together to form a second order term.
- Repeated roots (of multi-capacity processes) lead to sluggish response. Tanks-in-series is a good example in this respect.
- With higher order models, we can construct approximate reduced-order models based on the identification of dominant poles. This approach is used later in empirical controller tuning relations.
- The dead time transfer function has to be handled differently in classical control, and we'll use the Padé approximation for this purpose.

A brief review is in order: Recall that Laplace transform is a linear operator. The effects of individual inputs can be superimposed to form the output. In other words, an observed output change can be attributed to the individual effects of the inputs. From the stirred-tank heater example in Section 2.8.2 (p. 2-23), we found:

$$T(s) = G_d(s)T_i(s) + G_p(s)T_H(s)$$

We can analyze the *change* in tank temperature as a result of individual *changes* in either inlet or steam temperatures without doing the inverse transform. The compromise is that we do not have the time-domain analytical solution, T(t), and cannot think of time as easily.

We can put the example in more general terms. Let's consider an *n*-th order differential equation and two forcing functions, $x_1(t)$ and $x_2(t)$,

$$a_n\frac{d^ny}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \dots + a_1\frac{dy}{dt} + a_o y = b_1x_1(t) + b_2x_2(t) \tag{3-1}$$

where *y* is the output deviation variable. We also have the zero initial conditions,

$$y(0) = y'(0) = y''(0) = \dots = y^{(n-1)}(0) = 0 \tag{3-2}$$

Laplace transform of the equation leads to

$$Y(s) = G_1(s)X_1(s) + G_2(s)X_2(s) \tag{3-3}$$

where

$$G_1(s) = \frac{b_1}{a_ns^n + a_{n-1}s^{n-1} + \dots + a_1s + a_0}, \text{ and } G_2(s) = \frac{b_2}{a_ns^n + a_{n-1}s^{n-1} + \dots + a_1s + a_0} \tag{3-4}$$

are the two transfer functions for the two inputs $X_1(s)$ and $X_2(s)$.

Take note (again!) that the characteristic polynomials in the denominators of both transfer functions are *identical*. The roots of the characteristic polynomial (the poles) are *independent* of the inputs. It is obvious since they come from the same differential equation (same process or system). The poles tell us what the time-domain solution, y(t), generally would "look" like. A final reminder: no matter how high the order of *n* may be in Eq. (3-4), we can always use partial fractions to break up the transfer functions into first and second order terms.

3.1 First order differential equation models

This section is a review of the properties of a first order differential equation model. Our Chapter 2 examples of mixed vessels, stirred-tank heater, and homework problems of isothermal stirred-tank chemical reactors all fall into this category. Furthermore, the differential equation may represent either a process or a control system. What we cover here applies to any problem or situation as long as it can be described by a linear first order differential equation.

We usually try to identify features which are characteristic of a model. Using the examples in Section 2.8 as a guide, a first order model using deviation variables with one input and with constant coefficients a_1, a_o and b can be written in general notations as [1]

$$a_1 \frac{dy}{dt} + a_o y = bx(t) \quad \text{with } a_1 \neq 0 \text{ and } y(0) = 0 \tag{3-5}$$

The model, as in Eq. (2-2), is rearranged as

$$\tau \frac{dy}{dt} + y = Kx(t) \tag{3-6}$$

where τ is the time constant, and K is the steady state gain.

In the event that we are modeling a process, we would use a subscript *p* ($\tau = \tau_p$, $K = K_p$). Similarly, the parameters would be the system time constant and system steady state gain when we analyze a control system. To avoid confusion, we may use a different subscript for a system.

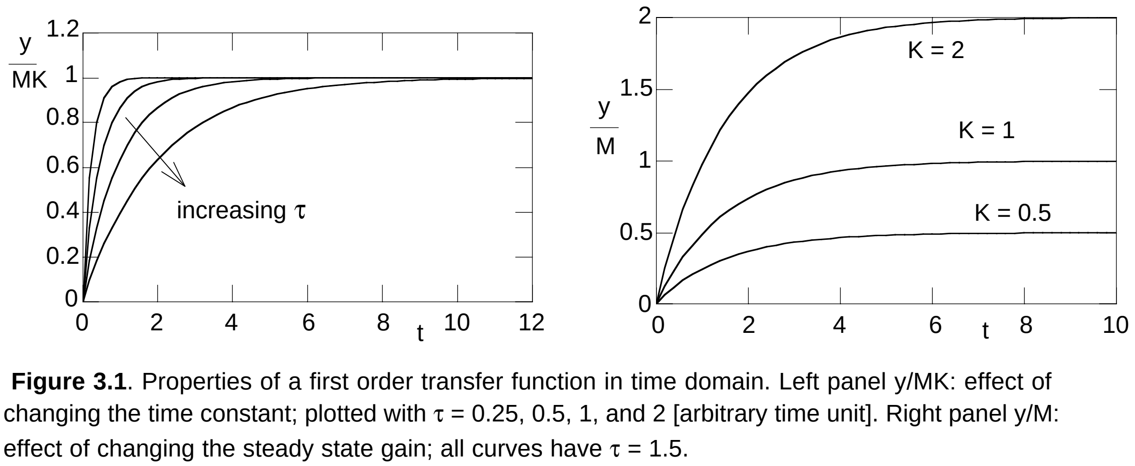


Figure 3.1. Properties of a first order transfer function in time domain. Left panel y/MK: effect of changing the time constant; plotted with τ = 0.25, 0.5, 1, and 2 [arbitrary time unit]. Right panel y/M: effect of changing the steady state gain; all curves have τ = 1.5.

[1] Whether the notation is y or y′ is immaterial. The key is to find the initial condition of the problem statement. If the initial condition is zero, the notation must refer to a deviation variable.

The Laplace transform of Eq. (3-6) is

$$\frac{Y(s)}{X(s)} = G(s) = \frac{K}{\tau s + 1} \tag{3-7}$$

where G(s) denotes the transfer function. There is one real pole at $-1/\tau$. (What does it imply in terms of the time domain function? If you are stuck, refer back to Example 2.10 on page 2-15.)

3.1.1 Step response of a first order model

Consider a step input, $x(t) = Mu(t)$, and $X(s) = M/s$, the output is

$$Y(s) = \frac{K}{(\tau s + 1)} \frac{M}{s} = MK\left[\frac{1}{s} - \frac{\tau}{(\tau s + 1)}\right] \tag{3-8}$$

and inverse transform gives the solution

$$y(t) = MK\,(1 - e^{-t/\tau}) \tag{3-9}$$

We first saw a plot of this function in Fig. 2.10 on page 2-26. The output y(t) starts at zero and increases exponentially to a new steady state MK. A process with this property is called **self-regulating**. The larger the time constant, the slower is the response (Fig. 3.1a).

We can check the result with the final value theorem

$$\lim_{s \to 0}\,[sY(s)] = \lim_{s \to 0}\left[s\,\frac{MK}{s\,(\tau s + 1)}\right] = MK$$

The new steady state is not changed by a magnitude of M, but is scaled by the gain K (Fig. 3.1b). Consequently, we can consider the steady state gain as the ratio of the observed change in output in response to a unit change in an input, y/M. In fact, this is how we measure K. The larger the steady state gain, the more sensitive is the output to changes in the input. As noted in Fig. 2.10, at $t = \tau$, $y(t) = 0.632MK$. This is a result that we often use to estimate the time constant from experimental data.

3.1.2 Impulse response of a first order model

Consider an impulse input, $x(t) = M\delta(t)$, and $X(s) = M$, the output is now

$$Y(s) = \frac{MK}{(\tau s + 1)} \tag{3-10}$$

The time-domain solution, as in Example 2.10, is

$$y(t) = \frac{MK}{\tau}\, e^{-t/\tau} \tag{3-11}$$

which implies that the output rises instantaneously to some value at t = 0 and then decays exponentially to zero.

3.1.3 Integrating process

When the coefficient $a_o = 0$ in the differential equation (3-5), we have

$$\frac{dy}{dt} = \left(\frac{b}{a_1}\right) x(t) \tag{3-12}$$

and

$$\frac{Y(s)}{X(s)} = G(s) = \frac{K_I}{s} \quad \text{where} \quad K_I = \frac{b}{a_1} \tag{3-13}$$

Here, the pole of the transfer function G(s) is at the origin, s = 0. The solution of (3-12), which we could have written down immediately without any transform, is

$$y(t) \; = \; K_I \int_0^t x(t)\, dt \tag{3-14}$$

This is called an **integrating** (also capacitive or non-self-regulating) process. We can associate the name with charging a capacitor or filling up a tank.

We can show that with a step input, the output is a ramp function. When we have an impulse input, the output will not return to the original steady state value, but accumulates whatever we have added. (Both items are exercises in the Review Problems.)

✎ **Example 3.1**: Show that a storage tank with pumps at its inlet and outlet (Fig E3.1) is an integrating process.

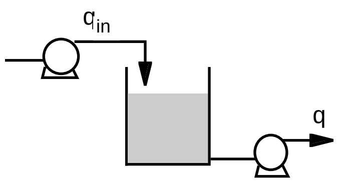


Figure E3.1

At constant density, the mass balance of a continuous flow mixed tank is simply

$$A\frac{dh}{dt} \; = \; q_{in} - q \quad \text{with} \;\; h(0) = h_s$$

where A is the cross-section and h the liquid level. The inlet and outlet flow rates q_{in} and q, as dictated by the pumps, are functions of time but not of the liquid level. At steady state, the flow rates must be the same. Thus we can define deviation variables h' = h – h_s, $q'_{in} = q_{in} - q_s$, and q' = q – q_s, and the mass balance becomes

$$A\frac{dh'}{dt} \; = \; q'_{in} - q' \quad \text{with} \quad h'(0) = 0$$

The general solution is

$$h'(t) \; = \; \frac{1}{A}\int_0^t (q'_{in} - q')\, dt$$

where the change in the liquid level is a simple time integral on the change in flow rates. In terms of Laplace transform, the differential equation leads to

$$H(s) \; = \; \frac{1}{A}\,\frac{Q_{in}(s) - Q(s)}{s}$$

The transfer function has the distinct feature that a pole is at the origin. Since a step input in either q'_{in} or q' would lead to a ramp response in h', there is no steady state gain at all.

To better observe how the tank works like a capacitor, consider the inlet flow rate to be constant and we have a pump only at the outlet. The transfer function is now just

$$H(s) \; = \; \frac{-\,Q(s)}{A\,s}$$

If for some reason the outlet pump slows down, the liquid level in the tank will back up until it overflows. Similarly, if the outlet pump speeds up, the tank will be drained. The tank level will not reach a new steady state with respect to a step change in the input.

3.2 Second order differential equation models

We have not encountered examples with a second order equation, especially one that exhibits oscillatory behavior. One reason is that processing equipment tends to be self-regulating. An oscillatory behavior is most often a result of implementing a controller, and we shall see that in the control chapters. For now, this section provides several important definitions.

A model that leads to a second order differential equation

$$a_2 \frac{d^2y}{dt^2} + a_1 \frac{dy}{dt} + a_o y = b\,x(t), \quad a_2 \neq 0 \text{ and } y(0) = y'(0) = 0 \tag{3-15}$$

is usually rearranged to take the forms

$$\tau^2 \frac{d^2y}{dt^2} + 2\zeta\tau \frac{dy}{dt} + y = K\,x(t)\,, \quad \text{or} \quad \frac{d^2y}{dt^2} + 2\zeta\omega_n \frac{dy}{dt} + \omega_n^2 y = K\omega_n^2 x(t) \tag{3-16}$$

where

$$\tau^2 = \frac{a_2}{a_o} \;;\; 2\zeta\tau = \frac{a_1}{a_o} \;;\; K = \frac{b}{a_o} \;;\; \omega_n = \frac{1}{\tau}$$

The corresponding Laplace transforms are

$$G(s) = \frac{Y(s)}{X(s)} = \frac{K}{\tau^2 s^2 + 2\zeta\tau\, s + 1}, \quad \text{or} \quad \frac{Y(s)}{X(s)} = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{3-17}$$

The notations are τ = **natural period** of oscillation

ω_n = **natural** (undamped) **frequency**

ζ = **damping ratio** (also called damping coefficient or factor)

K = steady state gain

The characteristic polynomial is

$$p(s) = \tau^2 s^2 + 2\zeta\tau\, s + 1 = (s - p_1)\,(s - p_2) \tag{3-18}$$

which provides the poles

$$p_{1,2} = \frac{-2\zeta\tau \pm \sqrt{4\zeta^2\tau^2 - 4\tau^2}}{2\tau^2} = -\frac{\zeta}{\tau} \pm \frac{\sqrt{\zeta^2 - 1}}{\tau} \tag{3-19}$$

A **stable** process (or system) requires $\zeta > 0$ since we need $\tau > 0$ to be physically meaningful. In addition, the two pole positions, and thus time response, take on four possibilities, depending on the value of ζ. The different cases are:

$\zeta > 1$: Two distinct real poles. The case is named **overdamped**. Here, we can factor the polynomial in terms of two time constants τ_1 and τ_2:

$$G(s) = \frac{K}{\tau^2 s^2 + 2\zeta\tau s + 1} = \frac{K}{(\tau_1 s + 1)(\tau_2 s + 1)} \tag{3-20}$$

such that the two real poles are at $-1/\tau_1$ and $-1/\tau_2$. [1]

$\zeta = 1$: Two repeating poles at $-1/\tau$. This case is termed **critically damped**. The natural period τ may be considered the "time constant" in the sense that it is associated with the exponential function. In actual fact, the time response is not strictly exponential, as we saw in Example 2.9 (p. 2-13) and confirmed in the time domain solution in Eq. (3-22).

$0 < \zeta < 1$: Two complex conjugate poles. This situation is considered **underdamped**.

We also write $\sqrt{\zeta^2 - 1} = j\sqrt{1 - \zeta^2}$. It is very important to note that τ is *not* the time constant here. The real part of the pole in Eq. (3-19) is $-\zeta/\tau$ and this is the value that determines the exponential decay, as in Eq. (3- 23). In this sense, the time constant is τ/ζ.

$\zeta = 0$: Two purely imaginary conjugate poles with frequency $\omega_n = 1/\tau$. This is equivalent to an oscillation with no damping and explains why ω_n is referred to as the natural frequency.

3.2.1 Step response time domain solutions

Consider a step input, $x(t) = Mu(t)$ with $X(s) = M/s$, and the different cases with respect to the value of ζ. We can derive the output response $y(t)$ for the different cases. We rarely use these results. They are provided for reference. In the case of the underdamped solution, it is used to derive the characteristic features in the next section.

---

[1] Here, we can derive that

$$\tau^2 = \tau_1\tau_2 \quad \text{and} \quad 2\zeta\tau = (\tau_1 + \tau_2)$$

or

$$\tau = \sqrt{\tau_1\tau_2} \quad \text{and} \quad \zeta = \frac{\tau_1 + \tau_2}{2\sqrt{\tau_1\tau_2}}$$

In this case of having real poles, we can also relate

$$\tau_1 = \frac{\tau}{\zeta - \sqrt{\zeta^2 - 1}} \quad ; \quad \tau_2 = \frac{\tau}{\zeta + \sqrt{\zeta^2 - 1}}$$

(1) $\zeta > 1$, *overdamped*. The response is sluggish compared to critically damped or underdamped.

$$y(t) = MK\left[1 - e^{-\zeta t/\tau}\left(\cosh\frac{\sqrt{\zeta^2-1}}{\tau}\,t + \frac{\zeta}{\sqrt{\zeta^2-1}}\sinh\frac{\sqrt{\zeta^2-1}}{\tau}\,t\right)\right] \tag{3-21}$$

This form is unnecessarily complicated. When we have an overdamped response, we typically use the simple exponential form with the exp($-t/\tau_1$) and exp($-t/\tau_2$) terms. (You'll get to try this in the Review Problems.)

(2) $\zeta = 1$, *critically damped*. The response is the "fastest" without oscillatory behavior.

$$y(t) = MK\left[1 - \left(1 + \frac{t}{\tau}\right)e^{-t/\tau}\right] \tag{3-22}$$

(3) $0 \le \zeta < 1$, *underdamped*. The response is fast initially, then overshoots and decays to steady state with oscillations. The oscillations are more pronounced and persist longer with smaller ζ.

$$y(t) = MK\left\{1 - e^{-\zeta t/\tau}\left[\cos\left(\frac{\sqrt{1-\zeta^2}}{\tau}\,t\right) + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\left(\frac{\sqrt{1-\zeta^2}}{\tau}\,t\right)\right]\right\} \tag{3-23}$$

This equation can be rearranged as

$$y(t) = MK\left[1 - \frac{e^{-\zeta t/\tau}}{\sqrt{1-\zeta^2}}\sin\left(\frac{\sqrt{1-\zeta^2}}{\tau}\,t + \phi\right)\right], \quad \text{where } \phi = \tan^{-1}\left[\frac{\sqrt{1-\zeta^2}}{\zeta}\right] \tag{3-23a}$$

3.2.2 Time-domain features of underdamped step response

From the solution of the *underdamped step response* ($0 < \zeta < 1$), we can derive the following characteristics (Fig. 3.2). They are useful in two respects: (1) fitting experimental data in the measurements of natural period and damping factor, and (2) making control system design specifications with respect to the dynamic response.

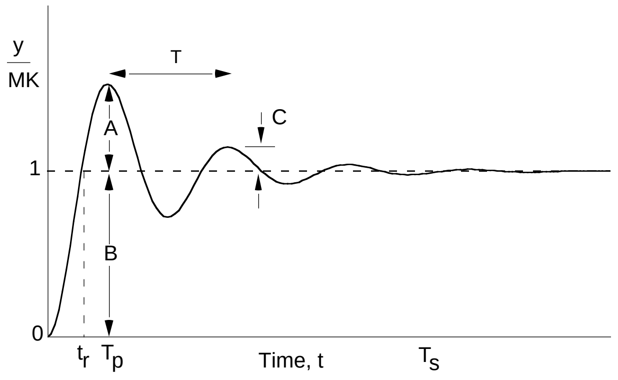


Figure 3.2. Key features in an underdamped response. See text for equations.

1. **Overshoot** (OS)

$$\text{OS} = \left[\frac{A}{B}\right] = \exp\left(\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}\right) \tag{3-24}$$

where *A* and *B* are depicted in Fig. 3.2. We compute only the first or maximum overshoot in the response. The overshoot increases as ζ becomes smaller. The OS becomes zero as ζ approaches 1.

The time to reach the peak value is

$$T_p = \frac{\pi\tau}{\sqrt{1-\zeta^2}} = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} \tag{3-25}$$

This **peak time** is less as ζ becomes smaller and meaningless when $\zeta = 1$. We can also derive the **rise time**—time for $y(t)$ to cross or hit the final value for the first time—as:

$$t_r = \frac{\tau}{\sqrt{1-\zeta^2}}\left(\pi - \cos^{-1}\zeta\right) \tag{3-26}$$

2. Frequency (or **period of oscillation**, T)

$$\omega = \frac{\sqrt{1-\zeta^2}}{\tau} \quad \text{or} \quad T = \frac{2\pi\tau}{\sqrt{1-\zeta^2}} \quad \text{since } \omega = \frac{2\pi}{T} \tag{3-27}$$

Note that $T = 2T_p$ and the unit of the frequency is radian/time.

3. **Settling time**

The real part of a complex pole in (3-19) is $-\zeta/\tau$, meaning that the exponential function forcing the oscillation to decay to zero is $e^{-\zeta t/\tau}$ as in Eq. (3-23). If we draw an analogy to a first order transfer function, the time constant of an underdamped second order function is τ/ζ. Thus to settle within ±5% of the final value, we can choose the settling time as [1]

$$T_s = 3\frac{\tau}{\zeta} = \frac{3}{\zeta\omega_n} \tag{3-28}$$

and if we choose to settle within ±2% of the final value, we can use $T_s = 4\tau/\zeta$.

4. **Decay ratio**

$$\text{DR} = \left[\frac{C}{A}\right] = \exp\left(\frac{-2\pi\zeta}{\sqrt{1-\zeta^2}}\right) = \text{OS}^2 \tag{3-29}$$

The decay ratio is the square of the overshoot and both quantities are functions of ζ only. The definitions of *C* and *A* are shown in Fig. 3.2.

1 Refer to Review Problem 1 to see why we may pick factors of 3 or 4.

3.3 Processes with dead time

Many physio-chemical processes involve a time delay between the input and output. This delay may be due to the time required for a slow chemical sensor to respond, or for a fluid to travel down a pipe. A **time delay** is also called **dead time** or **transport lag**. In controller design, the output will not contain the most current information, and systems with dead time can be difficult to control.

From Eq. (2-18) on page 2-7, the Laplace transform of a time delay is an exponential function. For example, first and second order models with dead time will appear as

$$\frac{Y(s)}{X(s)} = \frac{Ke^{-t_d s}}{\tau s + 1}, \quad \text{and} \quad \frac{Y(s)}{X(s)} = \frac{Ke^{-t_d s}}{\tau^2 s^2 + 2\zeta\tau s + 1}$$

Many classical control techniques are developed to work only with polynomials in *s*, and we need some way to tackle the exponential function.

To handle the time delay, we do not simply expand the exponential function as a Taylor series. We use the so-called **Padé approximation**, which puts the function as a ratio of two polynomials. The simplest is the first order (1/1) Padé approximation:

$$e^{-t_d s} \approx \frac{1 - \frac{t_d}{2} s}{1 + \frac{t_d}{2} s} \qquad \text{(3-30)}$$

This is a form that serves many purposes. The term in the denominator introduces a negative pole in the left-hand plane, and thus probable dynamic effects to the characteristic polynomial of a problem. The numerator introduces a positive zero in the right-hand plane, which is needed to make a problem to become unstable. (This point will become clear when we cover Chapter 7.) Finally, the approximation is more accurate than a first order Taylor series expansion.[1]

There are higher order approximations. For example, the second order (2/2) Padé approximation is

$$e^{-t_d s} \approx \frac{t_d^2 s^2 - 6t_d s + 12}{t_d^2 s^2 + 6t_d s + 12} \qquad \text{(3-31)}$$

Again, this form introduces poles in the left-hand plane and at least one zero is in the right-hand plane. At this point, the important task is to observe the properties of the Padé approximation in numerical simulations.

✎ **Example 3.2**: Using the first order Padé approximation, plot the unit step response of the first order with dead time function:

$$\frac{Y}{X} = \frac{e^{-3s}}{10\,s + 1}$$

Making use of (3-30), we can construct a plot with the approximation:

[1] We will skip the algebraic details. The simple idea is that we can do long division of a function of the form in Eq. (3-30) and match the terms to a Taylor's expansion of the exponential function. If we do, we'll find that the (1/1) Padé approximation is equivalent to a third order Taylor series.

$$\frac{Y}{X} = \frac{(-1.5\ s + 1)}{(10\ s + 1)\ (1.5\ s + 1)}$$

The result is the hairline curve in Fig. E3.2. Note how it dips below zero near t = 0. This behavior has to do with the first order Padé approximation, and we can improve the result with a second order Padé approximation. We will try that in the Review Problems.

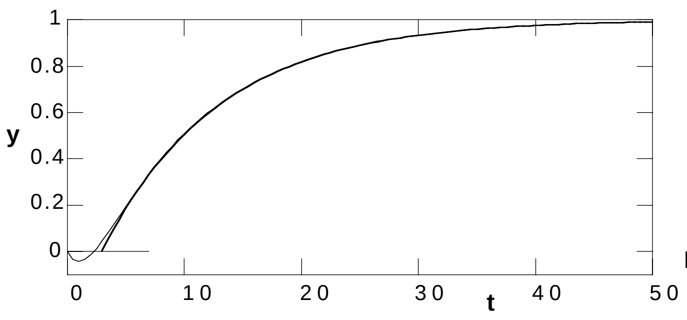


Figure E3.2

Here the observation is that when compared with the original transfer function (the solid curve), the approximation is acceptable at larger times.

How did we generate the solid curve? We computed the result for the first order function and then shifted the curve down three time units ($t_d = 3$). The MATLAB statements are:

```
td=3;
P1=tf([-td/2 1],[td/2 1]);   %First order Padé approximation

t=0:0.5:50;
taup=10;
G1=tf(1,[taup 1]);
y1=step(G1*P1,t); %y1 is first order with Padé approximation of dead time

y2=step(G1,t);
t2=t+td;         %Shift the time axis for the actual time delay function
plot(t,y1,'--', t2,y2);
```

We now move onto a few so-called higher order or complex processes. We should remind ourselves that all linearized higher order systems can be broken down into simple first and second order units. Other so-called "complex" processes like two interacting tanks are just another math problem in coupled differential equations; these problems are still linear. The following sections serve to underscore these points.

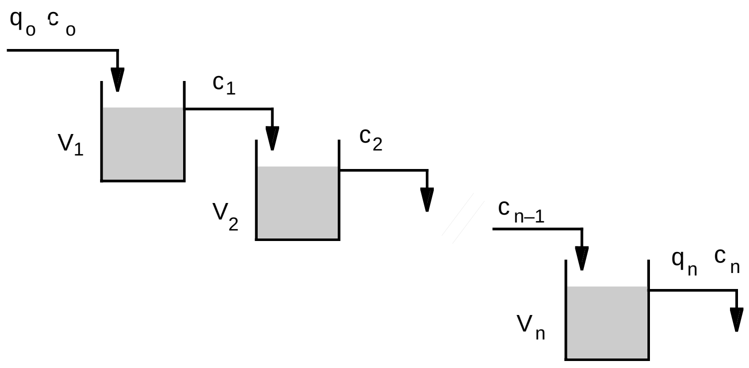


Figure 3.3. Illustration of compartments or tanks-in-series.

3.4 Higher order processes and approximations

Staged processes in chemical engineering or compartmental models in bioengineering give rise to higher order models. The higher order models are due to a cascade of first order elements. Numerical calculation will be used to illustrate that the resulting response becomes more sluggish, thus confirming our analysis in Example 2.9 (p. 2-13). We shall also see how the response may be approximated by a lower order function with dead time. An example of two interacting vessels follows last.

3.4.1 Simple tanks-in-series

Consider a series of well-mixed vessels (or compartments) where the volumetric flow rate and the respective volumes are constant (Fig. 3.3). If we write down the mass balances of the first two vessels as in Section 2.8.1 (p. 2-20), they are: [1]

$$\tau_1 \frac{dc_1}{dt} = c_o - c_1 \tag{3-32}$$

and

$$\tau_2 \frac{dc_2}{dt} = c_1 - c_2 \tag{3-33}$$

where $\tau_1 = V_1/q_o$ and $\tau_2 = V_2/q_o$ are the space times of each vessel. Again following Section 2.8.1, the Laplace transform of the mass balance in deviation variables would be

$$\frac{C_1}{C_o} = \frac{1}{\tau_1 s + 1}, \quad \text{and} \quad \frac{C_2}{C_1} = \frac{1}{\tau_2 s + 1} \tag{3-34}$$

The effect of changes in $c_o(t)$ on the effluent of the second vessel is evaluated as

$$\frac{C_2}{C_o} = \frac{C_2}{C_1}\frac{C_1}{C_o} = \frac{1}{(\tau_2 s + 1)}\,\frac{1}{(\tau_1 s + 1)} \tag{3-35}$$

Obviously, we can generalize to a series of *n* tanks as in

$$\frac{C_n}{C_o} = \frac{1}{(\tau_1 s + 1) \ldots (\tau_{n-1} s + 1)(\tau_n s + 1)} \tag{3-36}$$

1 Many texts illustrate with a model on the change of inlet flow rate. In such a case, we usually need to assume that the outlet flow rate of each vessel is proportional to the liquid level or hydrostatic head. The steady state gains will not be unity.

In this example, the steady state gain is unity, which is intuitively obvious. If we change the color of the inlet with a food dye, all the mixed tanks will have the same color eventually. In addition, the more tanks we have in a series, the longer we have to wait until the *n*-th tank "sees" the changes that we have made in the first one. We say that the more tanks in the series, the more sluggish is the response of the overall process. Processes that are products of first order functions are also called *multicapacity* processes.

Finally, if all the tanks have the same space time, $\tau_1 = \tau_2 = \ldots = \tau$, Eq. (3-36) becomes

$$\frac{C_n}{C_o} = \frac{1}{(\tau s + 1)^n} \tag{3-37}$$

This particular scenario is not common in reality, but is a useful textbook illustration.

✎ **Example 3.3**. Make use of Eq. (3-37), show how the unit step response $C_n(t)$ becomes more sluggish as *n* increases from 1 to 5.

The exercise is almost trivial with MATLAB. To generate Fig. E3.3, the statements are:

```
tau=3;            %Just an arbitrary time constant
G=tf(1,[tau 1]);
step(G);          %First order function unit step response
hold
step(G*G);        %Second order response
step(G*G*G);      %Third order response
step(G*G*G*G);    %Fourth order response
step(G*G*G*G*G); %Fifth order response
```

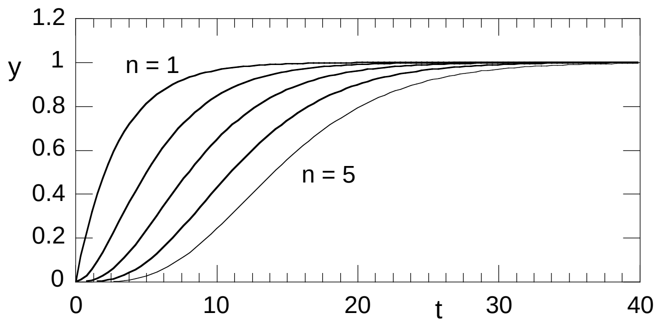


Figure E3.3

It is clear that as *n* increases, the response, as commented in Example 2.9, becomes slower. If we ignore the "data" at small times, it appears that the curves might be approximated with first order with dead time functions. We shall do this exercise in the Review Problems.

3.4.2 Approximation with lower order functions with dead time

Following the lead in Example 3.3, we now make use of the result in Example 2.6 (p. 2-11) and the comments about dominant poles in Section 2.7 (p. 2-17) to see how we may approximate a transfer function.

Let say we have a high order transfer function that has been factored into partial fractions. If there is a large enough difference in the time constants of individual terms, we may try to throw away the small time scale terms and retain the ones with dominant poles (large time constants). This is our **reduced-order model** approximation. From Fig. E3.3, we also need to add a time delay in this approximation. The extreme of this idea is to use *a first order with dead time function*. It obviously cannot do an adequate job in many circumstances. Nevertheless, this simple approximation is all we use when we learn to design controllers with empirical tuning relations.

A second order function with dead time generally provides a better estimate, and this is how we may make a quick approximation. Suppose we have an *n*-th order process which is broken down into *n* first-order processes in series with time constants τ_1, τ_2,... ,τ_n. If we can identify, say, two dominant time constants (or poles) τ_1 and τ_2, we can *approximate* the process as

$$G(s) \approx \frac{Ke^{-t_d s}}{(\tau_1 s + 1)(\tau_2 s + 1)}, \quad \text{where} \quad t_d \approx \sum_{i \neq 1,2}^{n} \tau_i \tag{3-38}$$

The summation to estimate the dead time is over all the other time constants (*i* = 3, 4, etc.). This idea can be extended to the approximation of a first order with dead time function.

✎ **Example 3.4**: Find the simplest lower order approximation of the following transfer function

$$\frac{Y}{X} = \frac{3}{(0.1s + 1)(0.5s + 1)(s + 1)(3s + 1)}$$

In this example, the dominant pole is at –1/3, corresponding to the largest time constant at 3 [time unit]. Accordingly, we may approximate the full order function as

$$\frac{Y}{X} = \frac{3\,e^{-1.6\,s}}{(3s + 1)}$$

where 1.6 is the sum of dead times 0.1, 0.5, and 1. With X representing a unit step input, the response of the full order function (solid curve) and that of the first order with dead time approximation (dotted curve) are shown in Fig. E3.4. The plotting of the dead time function is further approximated by the Padé approximation. Even so, the approximation is reasonable when time is large enough. The pole at –1/3 can indeed be considered as dominant.

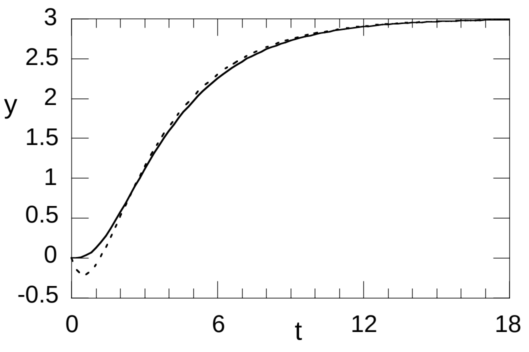


Figure E3.4

The MATLAB statements are:

```
p2=conv([1 1],[3 1]);
p4=conv( conv(p2,[0.5 1]) , [0.1 1] );
G4=tf(3,p4);    %The original full order function
t=0:0.2:18;
y4=step(G4,t);  %Unit step response

td=1+0.1+0.5;   %Approximate dead time
P1=tf([-td/2 1],[td/2 1]);
G1=tf(3,[3 1]);
y1=step(P1*G1,t);
plot(t,y1,t,y4)
```

If we follow Eq. (3-38), we should approximate the original function with a second order function with time constants 1 and 3, and dead time 0.6. We'll find it to be a much better fit and will do this in the Review Problems.

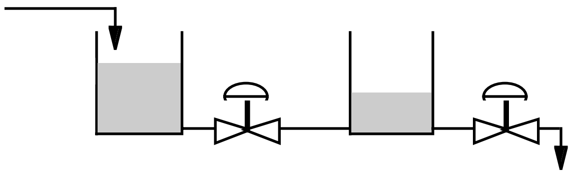


Figure 3.4. Illustration of two tanks interacting in their liquid levels.

3.4.3 Interacting tanks-in-series

To complete our discussion, we include the balance equations for the case when two differential equations may be coupled. The common example is two tanks connected such that there is only a valve between them (Fig. 3.4). Thus the flow between the two tanks depends on the difference in the hydrostatic heads. With constant density, we can write the mass balance of the first tank as

$$A_1 \frac{dh_1}{dt} = q_o - \left(\frac{h_1 - h_2}{R_1} \right) \tag{3-39}$$

Similarly for the second vessel, we have

$$A_2 \frac{dh_2}{dt} = \left(\frac{h_1 - h_2}{R_1} \right) - \frac{h_2}{R_2} \tag{3-40}$$

Here, we model the flow through the valves with resistances R_1 and R_2, both of which are constants. We rearrange the balance equations a bit, and because both equations are linear, we can quickly rewrite them in deviation variables (without the apostrophes):

$$\tau_1 \frac{dh_1}{dt} = -h_1 + h_2 + R_1 q_o \ , \ h_1(0) = 0 \tag{3-41}$$

and

$$\tau_2 \frac{dh_2}{dt} = \frac{R_2}{R_1} h_1 - (1 + \frac{R_2}{R_1})\, h_2 \ , \ h_2(0) = 0 \tag{3-42}$$

where we have defined $\tau_1 = A_1R_1$, and $\tau_2 = A_2R_2$. The Laplace transforms of these equations are

$$(\tau_1 s + 1)\, H_1 - H_2 = R_1 Q_o \tag{3-43}$$

and

$$\frac{-R_2}{R_1} H_1 + (\tau_2 s + 1 + \frac{R_2}{R_1})\, H_2 = 0 \tag{3-44}$$

We have written the equations in a form that lets us apply Cramer's rule. The result is

$$H_1 = \frac{R_1 \tau_2 s + (R_1 + R_2)}{p(s)} Q_o \ , \ \text{and} \ \ H_2 = \frac{R_2}{p(s)} Q_o \tag{3-45}$$

where the characteristic polynomial is

$$p(s) = (\tau_1 s + 1)\,(\tau_2 s + 1 + R_2/R_1) - R_2/R_1 \tag{3-46}$$

We do not need to carry the algebra further. The points that we want to make are clear. First, even the first vessel has a second order transfer function; it arises from the interaction with the second tank. Second, if we expand Eq. (3-46), we should see that the interaction introduces an extra term in the characteristic polynomial, but the poles should remain real and negative.[1] That is, the tank responses remain overdamped. Finally, we may be afraid(!) that the algebra might become hopelessly tangled with more complex models. Indeed, we'd prefer to use state space representation based on Eqs. (3-41) and (3-42). After Chapters 4 and 9, you can try this problem in Homework Problem II.39.

3.5 Effect of zeros in time response

The inherent dynamics is governed by the poles, but the zeros can impart finer "fingerprint" features by modifying the coefficients of each term in the time domain solution. That was the point which we tried to make with the examples in Section 2.5 (p. 2-10). Two common illustrations on the effects of zeros are the lead-lag element and the sum of two functions in parallel.

3.5.1 Lead-lag element

The so-called lead-lag element is a semi-proper function with a first order lead divided by a first order lag:

$$\frac{Y}{X} = \frac{\tau_z s + 1}{\tau s + 1} \qquad (3\text{-}47)$$

We have to wait until the controller chapters to see that this function is the basis of a derivative controller and not till the frequency response chapter to appreciate the terms lead and lag. For now, we take a quick look at its time response.

For the case with a unit step input such that X = 1/s, we have, after partial fraction expansion,

$$Y = \frac{1}{s} + \frac{\tau_z - \tau}{\tau s + 1} \qquad (3\text{-}48)$$

and inverse transform via table look-up yields

$$y(t) = 1 - \left(1 - \frac{\tau_z}{\tau}\right) e^{-t/\tau} \qquad (3\text{-}49)$$

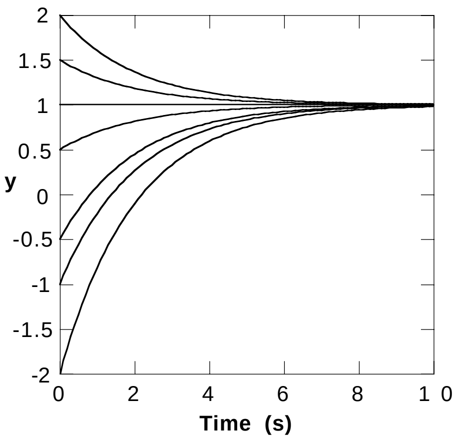


Figure 3.5. Time response of a lead-lag element with τ = 2 s. The curves from top to bottom are plotted with τ_z = 4, 3, 2, 1, –1, –2, and –4 s, respectively.

There are several things that we want to take note of. First, the exponential function is dependent only on τ, or in other words, the pole at $-1/\tau$. Second, with Eq. (3-49), the actual time response depends on whether $\tau < \tau_z$, $\tau > \tau_z$, or $\tau_z < 0$ (Fig. 3.5). Third, when $\tau = \tau_z$, the time response is just a horizontal line at y = 1, corresponding to the input x = u(t). This is also obvious from (3-47) which becomes just Y = X. When a zero equals to a pole, we have what is called a

[1] To see this, you need to go one more step to get

$$p(s) = \tau_1 \tau_2 s^2 + (\tau_1 + \tau_2 + \tau_1 R_2/R_1)\, s + 1$$

and compare the roots of this polynomial with the case with no interaction:

$$p(s) = (\tau_1 + 1)\,(\tau_2 + 1)\,.$$

And note how we have an extra term when the tanks interact.

pole-zero cancellation. Finally, the value of y is nonzero at time zero. We may wonder how that could be the case when we use differential equation models that have zero initial conditions. The answer has to do with the need for the response to match the rate of change term in the input. We'll get a better picture in Chapter 4 when we cover state space models.

3.5.2 Transfer functions in parallel

There are circumstances when a complex process may involve two competing (*i.e.*, opposing) dynamic effects that have different time constants. One example is the increase in inlet temperature to a tubular catalytic reactor with exothermic kinetics. The initial effect is that the exit temperature will momentarily decrease as increased conversion near the entrance region depletes reactants at the distal, exit end. Given time, however, higher reaction rates lead to a higher exit temperature.

To model this highly complex and nonlinear dynamics properly, we need the heat and mass balances. In classical control, however, we would replace them with a linearized model that is the sum of two functions in parallel:

$$\frac{Y}{X} = \frac{K_1}{\tau_1 s + 1} + \frac{K_2}{\tau_2 s + 1} \tag{3-50}$$

We can combine the two terms to give the second order function

$$\frac{Y}{X} = \frac{K\,(\tau_z s + 1)}{(\tau_1 s + 1)\,(\tau_2 s + 1)}\,, \tag{3-51}$$

where

$$K = K_1 + K_2, \text{ and } \quad \tau_z = \frac{K_1 \tau_2 + K_2 \tau_1}{K_1 + K_2}$$

Under circumstances where the two functions represent opposing effects, one of them has a negative steady state gain. In the following illustration, we choose to have $K_2 < 0$.

Based on Eq. (3-51), the time response y(t) should be strictly overdamped. However, this is not necessarily the case if the zero is positive (or $\tau_z < 0$). We can show with algebra how various ranges of K_i and τ_i may lead to different zeros $(-1/\tau_z)$ and time responses. However, we will not do that. (We'll use MATLAB to take a closer look in the Review Problems, though.) The key, once again, is to appreciate the principle of superposition with linear models. Thus we should get a rough idea of the time response simply based on the form in (3-50).

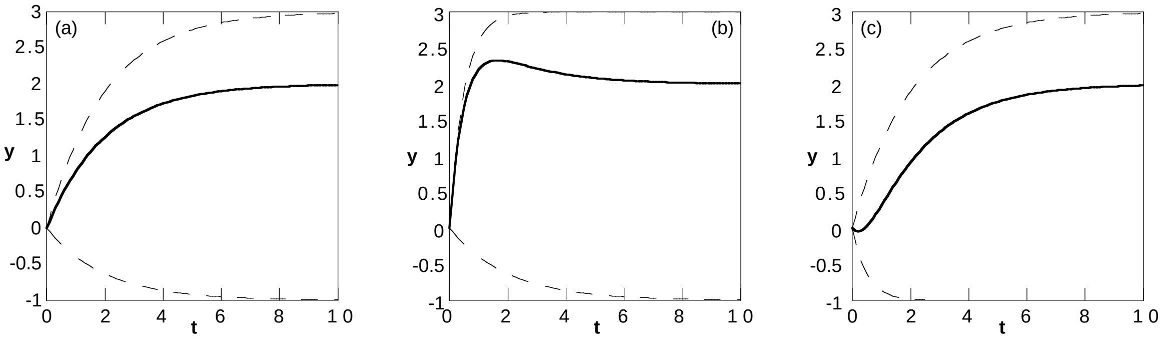


Figure 3.6. Time response calculations with different time constants. In all cases, $K_1 = 3$, $K_2 = -1$, and the individual terms in Eq. (3-50) are in dashed curves. Their superimposed response y is the solid line. (a) $\tau_1 = \tau_2 = 2$; (b) $\tau_1 = 0.5$, $\tau_2 = 2$; (c) $\tau_1 = 2$, $\tau_2 = 0.5$.

The numerical calculation is illustrated in Fig. 3.6. The input is a unit step, X = 1/s, and the two steady state gains are $K_1 = 3$ and $K_2 = -1$ such that $|K_1| > |K_2|$. We consider the three cases where (a) τ_1 is roughly the same as τ_2, (b) $\tau_1 \ll \tau_2$, and (c) $\tau_1 \gg \tau_2$. We should see that the overall response is overdamped in case (a) , but in case (b) we can have an overshoot, and in case (c), an initial inverse response. Note that all three cases have the same overall steady gain of K = 2.[1]

❒ Review Problems

1. With respect to the step response of a first order model in Eq. (3-9), make a table for $y(t)/MK_p$ with t/τ_p = 1, 2, 3, 4 and 5. It is helpful to remember the results for t/τ_p = 1, 3, and 5.
2. It is important to understand that the time constant τ_p of a process, say, a stirred tank is not the same as the space time τ. Review this point with the stirred-tank heater example in Chapter 2. Further, derive the time constant of a continuous flow stirred-tank reactor (CSTR) with a first-order chemical reaction.
3. Write the time response solutions to the integrating process in (3-14) when the input is (a) a unit step and (b) an impulse. How are they different from the solutions to a self-regulating process?
4. Derive the time constant relationships stated in the footnotes of (3-20).
5. With respect to the overdamped solution of a second order equation in (3-21), derive the step response y(t) in terms of the more familiar $\exp(-t/\tau_1)$ and $\exp(-t/\tau_2)$. This is much easier than (3-21) and more useful too!
6. Show that when $\zeta = 0$ (natural period of oscillation, no damping), the process (or system) oscillates with a constant amplitude at the natural frequency ω_n. (The poles are at $\pm\omega_n$) The period is $2\pi\tau$.
7. Use MATLAB to make plots of overshoot and decay ratio as functions of the damping ratio.
8. What is the expected time response when the *real part* of the pole is zero in a second order function? The pole can be just zero or have purely imaginary parts.
9. Plot the unit step response using just the first and second order Padé approximation in Eqs. (3.30) and (3-31). Try also the step response of a first order function with dead time as in Example 3.2. Note that while the approximation to the exponential function itself is not that good, the approximation to the entire transfer function is not as bad, as long as $t_d \ll \tau$. How do you plot the exact solution in MATLAB?
10. Use MATLAB to observe the effect of higher order multicapacity models as in Example 3.3. Try to fit the fifth order case with a first order with dead time function.
11. With respect to Example 3.4, try also a second order with dead time approximation.
12. We do not have a rigorous criterion to determine when a pole is absolutely dominant. Plot the exponential decay with different time constants to get an idea when the terms associated with smaller time constants can be omitted.
13. With MATLAB, try do a unit step response of a lead-lag element in as in Eq. (3-49).
14. Repeat the time response simulation of inverse response in Section 3.5. Calculate the value of zero in each case.

[1] When you repeat this exercise with MATLAB in the Review Problems, check that τ_z is negative in case (c). More commonly, we say that this is the case with a positive zero. After we have learned frequency response, we'll see that this is an example of what we refer to as non-minimum phase.

Hints:

1. $y(t)/MK_p$ at t/τ_p = 1, 2, 3, 4 and 5 are 0.63, 0.86, 0.95, 0.98, and 0.99.

2. The mass balance of a continuous flow stirred-tank reactor (CSTR) with a first-order chemical reaction is very similar to the problem in Section 2.8.1 (p. 2-20). We just need to add the chemical reaction term. The balance written for the reactant A will appear as:

$$V\frac{dC_A}{dt} = q(C_o - C_A) - VkC_A$$

where C_A is the molar concentration of A, V is the reactor volume, q is the volumetric flow rate, C_o is the inlet concentration of A, and k is the first order reaction rate constant. If we define space time $\tau = V/q$, the equation can be rewritten as

$$\tau\frac{dC_A}{dt} + (1 + k\tau)\,C_A = C_o$$

This is a linear equation if k and τ are constants. Now if we follow the logic in Section 2.8.2, we should find that the time constant of a CSTR with a first order reaction is $\tau/(1 + k\tau)$.

3. Part of the answer is already in Example 3.1.

5. This is really an algebraic exercise in partial fractions. The answer hides in Table 2.1.

6. This is obvious from Eq. (3-17) or (3-19).

7. Plot Eq. (3-29) with $0 < \zeta < 1$. You can write a small M-file to do the plotting too.

8. See Question 6.

9. Follow Example 3.2. For the first order approximation, we can try, for example:

```
td=3;             %Use a M-file to re-run with different values
P1=tf([-td/2 1],[td/2 1]);
step(P1);         %Note how the response starts from negative values
t=0:0.1:50;
taup=10;
G1=tf(1,[taup 1]);
y1=step(G1*P1,t);     %y1 is first order with Pade approx of dead time
y2=step(G1,t);        %y2 has no time delay
t2=t+td;
plot(t,y1, t2,y2,'-.');
%Note how the Pade approx has a dip at the beginning
```

10. The MATLAB statements to do the unit step response are already in Example 3.4. You may repeat the computation with a different time constant. The statements to attempt fitting the five tanks-in-series response are:

```
tau=3;
G=tf(1,[tau 1]);
[y5,t]=step(G*G*G*G*G);      %The fifth order calculation
G1=tf(1,[12 1]);
y1=step(G1,t);               %Using a time shift to do the
t1=t+3;                      %first order with dead time plot
plot(t,y5, t1,y1)
```

The choice of the time constant and dead time is meant as an illustration. The fit will not be particularly good in this example because there is no one single dominant pole in the fifth order function with a pole repeated five times. A first order with dead time function will never provide a perfect fit.

11. Both first and second order approximation statements are here.

```
q=3;
```

```
p2=conv([1 1],[3 1]);              % Second order reduced model
p4=conv(conv(p2,[0.5 1]),[0.1 1]);
roots(p4)                          %check
G2=tf(q,p2);
G4=tf(q,p4);
step(G4)
hold
td=0.1+0.5;
P1=tf([-td/2 1],[td/2 1]);
step(P1*G2);                       %Not bad!
td=1+0.1+0.5;
G1=tf(q,[3 1]);                    % First order approximation
step(P1*G1);                       % is not that good in this case
hold off
```

12. Below are the MATLAB statements that we may use for visual comparison of exponential decay with different time constants. In rough engineering calculations, a pole may already exhibit acceptable dominant behavior if other time constants are 1/3 or less.

```
tau=1;
t=0:0.1:5;
f=exp(-t/tau);
plot(t,f)
hold
           %Now add curves with smaller time constants
frac=0.3;
f=exp(-t/(frac*tau));
plot(t,f)
frac=0.2;
f=exp(-t/(frac*tau));
plot(t,f)
frac=0.1;
f=exp(-t/(frac*tau));
plot(t,f)
```

13. Try vary the zero as in

```
tz=3;           %Try vary tz, zero is -1/tz
G=tf([tz 1],[5 1]);
step(G);
```

14. Try vary the values of τ_1 and τ_2. To display the value of zero is trivial.

```
k1=3;   k2=-1;
tau1=2; tau2=0.5;
k=k1+k2;
tz=(k1*tau2+k2*tau1)/k;
G=tf(k*[tz 1], conv([tau1 1],[tau2 1]) );
step(G);
```

❖ 4. *State Space Representation*

The limitation of transfer function representation becomes plain obvious as we tackle more complex problems. For complex systems with multiple inputs and outputs, transfer function matrices can become very clumsy. In the so-called modern control, the method of choice is state space or state variables in time domain—essentially a matrix representation of the model equations. The formulation allows us to make use of theories in linear algebra and differential equations. It is always a mistake to tackle modern control without a firm background in these mathematical topics. For this reason, we will not overreach by doing both the mathematical background and the control together. Without a formal mathematical framework, we will put the explanation in examples as much as possible. The actual state space control has to be delayed until after tackling classical transfer function feedback systems.

What are we up to?

- How to write the state space representation of a model.
- Understand the how a state space representation is related to the transfer function representation.

4.1 State space models

Just as you are feeling comfortable with transfer functions, we now switch gears totally. Nevertheless, we are still working with *linearized* differential equation models in this chapter. Whether we have a high order differential equation or multiple equations, we can always rearrange them into a set of first order differential equations. Bold statements indeed! We will see that when we go over the examples.

With state space models, a set of differential equations is put in the standard matrix form

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \tag{4-1}$$

and

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} \tag{4-2}$$

where **x** is the **state variable** vector, **u** is the input, and **y** is the output. The time derivative is denoted by the overhead dot. In addition, **A** is the process (plant) matrix, **B** is the input matrix, **C** is the output matrix, and **D** is the feed-through matrix. Very few processes (and systems) have an input that has a direct influence on the output. Hence **D** is usually zero.

When we discuss single-input single-output models, u, y, and D are scalar variables. For convenience, we keep the notation for **B** and **C**, but keep in mind that in this case, **B** is a column vector and **C** is a row vector. If **x** is of order n, then **A** is (n x n), **B** is (n x 1), and **C** is (1 x n).[1]

The idea behind the use of Eqs. (4-1) and (4-2) is that we can make use of linear system theories, and complex systems can be analyzed much more effectively. There is no unique way to define the state variables. What we will show is just one of many possibilities.

[1] If you are working with only single-input single-output (SISO) problems, it would be more appropriate to replace the notation **B** by **b** and **C** by $\mathbf{c}^T$, and write d for **D**.

✎ **Example 4.1**: Derive the state space representation of a **second order differential equation** of a form similar to Eq. (3-16) on page 3-5:

$$\frac{d^2y}{dt^2} + 2\zeta\omega_n \frac{dy}{dt} + \omega_n^2 y = Ku(t) \tag{E4-1}$$

We can do blindfolded that the transfer function of this equation, with zero initial conditions, is

$$G_p(s) = \frac{Y(s)}{U(s)} = \frac{K}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{E4-2}$$

Now let's do something different. First, we rewrite the differential equation as

$$\frac{d^2y}{dt^2} = -2\zeta\omega_n \frac{dy}{dt} - \omega_n^2 y + Ku(t)$$

and define state variables [1]

$$x_1 = y \quad \text{and} \quad x_2 = \frac{dx_1}{dt} \tag{E4-3}$$

which allow us to redefine the second order equation as a set of two coupled first order equations. The first equation is the definition of the state variable x_2 in (E4-3); the second equation is based on the differential equation,

$$\frac{dx_2}{dt} = -2\zeta\omega_n x_2 - \omega_n^2 x_1 + Ku(t) \tag{E4-4}$$

We now put the result in a matrix equation:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ K \end{bmatrix} u(t) \tag{E4-5}$$

We further write

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{E4-6}$$

as a statement that x_1 is our output variable. Compare the results with Eqs. (4-1) and (4-2), and we see that in this case,

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{bmatrix} ; \quad \mathbf{B} = \begin{bmatrix} 0 \\ K \end{bmatrix} ; \quad \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix} ; \quad D = 0$$

To find the eigenvalues of **A**, we solve its characteristic equation:

$$|\lambda\mathbf{I} - \mathbf{A}| = \lambda(\lambda + 2\zeta\omega_n) + \omega_n^2 = 0 \tag{E4-7}$$

We can use the MATLAB function `tf2ss()` to convert the transfer function in (E4-2) to state space form:

[1] This exercise is identical to how we handle higher order equations in numerical analysis and would come as no surprise if you have taken a course on numerical methods.

```
z=0.5; wn=1.5;   % Pick two sample numbers for ζ and ωn
p=[1 2*z*wn wn*wn];
[a,b,c,d]=tf2ss(wn*wn,p)
```

However, you will find that the MATLAB result is not identical to (E4-5). It has to do with the fact that there is no unique representation of a state space model. To avoid unnecessary confusion, the differences with MATLAB are explained in MATLAB Session 4.

One important observation that we should make immediately: the characteristic polynomial of the matrix **A** (E4-7) is *identical* to that of the transfer function (E4-2). Needless to say that the eigenvalues of **A** are the poles of the transfer function. It is a reassuring thought that different mathematical techniques provide the same information. It should come as no surprise if we remember our linear algebra.

✎ **Example 4.2**: Draw the block diagram of the state space representation of the **second order differential equation** in the previous example.

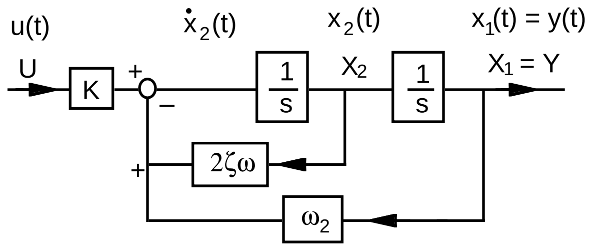


Figure E4.2

The result is in Fig. E4.2. It is quite easy to understand if we take note that the transfer function of an integrator is 1/s. Thus the second order derivative is located prior to the two integrations. The information at the summing point also adds up to the terms of the second order differential equation. The resulting transfer function is identical to (E4-2). The reduction to a closed-loop transfer function was done in Example 2.16 (p. 2-33).

✎ **Example 4.3**: Let's try another model with a slightly more complex input. Derive the state space representation of the differential equation

$$\frac{d^2y}{dt^2} + 0.4\frac{dy}{dt} + y = \frac{du}{dt} + 3u \ , \ \ y(0) = dy/dt(0) = 0, u(0) = 0 \ ,$$

which has the transfer function $\frac{Y}{U} = \frac{s+3}{s^2 + 0.4s + 1}$.

The method that we will follow is more for illustration than for its generality. Let's introduce a variable X_1 between Y and U:

$$\frac{Y}{U} = \frac{X_1}{U}\frac{Y}{X_1} = \left(\frac{1}{s^2 + 0.4s + 1}\right)(s+3)$$

The first part X_1/U is a simple problem itself.

$$\frac{X_1}{U} = \left(\frac{1}{s^2 + 0.4s + 1}\right) \quad \text{is the Laplace transformed of} \quad \frac{d^2x_1}{dt^2} + 0.4\frac{dx_1}{dt} + x_1 = u$$

With Example 4.1 as the hint, we define the state variables $x_1 = x_1$ (*i.e.*, the same), and $x_2 = dx_1/dt$. Using steps similar to Example 4.1, the result as equivalent to Eq. (4-1) is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -0.4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \qquad \text{(E4-8)}$$

As for the second part $Y/X_1 = (s + 3)$, it is the Laplace transformed of $y = \frac{d\,x_1}{d\,t} + 3x_1$. We can use the state variables defined above to rewrite as

$$y = x_2 + 3x_1\,, \quad \text{or in matrix form} \quad y = \begin{bmatrix} 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\,, \qquad \text{(E4-9)}$$

which is the form of Eq. (4-2).

With MATLAB, the statements for this example are:

```
q=[1 3];
p=[1 0.4 1];
roots(p)
[a,b,c,d]=tf2ss(q,p)
eig(a)
```

Comments at the end of Example 4.1 also apply here. The result should be correct, and we should find that both the roots of the characteristic polynomial `p` and the eigenvalues of the matrix `a` are $-0.2 \pm 0.98j$. We can also check by going backward:

```
[q2,p2]=ss2tf(a,b,c,d,1)
```

and the original transfer function is recovered in `q2` and `p2` .

✎ **Example 4.4**: Derive the state space representation of the **lead-lag** transfer function

$$\frac{Y}{U} = \frac{s+2}{s+3}\,.$$

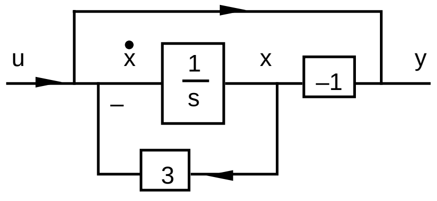


Figure E4.4

We follow the hint in Example 4.3 and write the transfer function as

$$\frac{Y}{U} = \frac{X}{U}\frac{Y}{X} = \frac{1}{s+3}\; s+2$$

From $X/U = 1/(s+3)$, we have $sX = -3X + U$, or in time domain,

$$\frac{dx}{dt} = -\,3x + u \qquad \text{(E4-10)}$$

and from $Y/X = s+2$, we have $Y = sX + 2X$ and substitution for sX leads to

$$Y = (-3X + U) + 2X = -X + U$$

The corresponding time domain equation is

$$y = -x + u \qquad \text{(E4-11)}$$

Thus all the coefficients in Eqs. (4-1) and (4-2) are scalar, with A = –3, B = 1, C = –1, and D = 1. Furthermore, (E4-10) and (E4-11) can be represented by the block diagram in Fig. E4.4.

We may note that the coefficient D is not zero, meaning that with a lead-lag element, an input can have instantaneous effect on the output. Thus while the state variable x has zero initial condition, it is not necessarily so with the output y. This analysis explains the mystery with the inverse transform of this transfer function in Eq. (3-49) on page 3-15.

The MATLAB statement for this example is:

```
[a,b,c,d]=tf2ss([1 2], [1 3])
```

In the next two examples, we illustrate how state space models can handle a multiple-input multiple output (MIMO) problem. We'll show, with a simple example, how to translate information in a block diagram into a state space model. Some texts rely on signal-flow graphs, but we do not need them with simple systems. Moreover, we can handle complex problems easily with MATLAB. Go over MATLAB Session 4 before reading Example 4.7A.

✎ **Example 4.5**: Derive the state space representation of two continuous flow stirred-tank reactors in series (**CSTR-in-series**). Chemical reaction is first order in both reactors. The reactor volumes are fixed, but the volumetric flow rate and inlet concentration are functions of time.

We use this example to illustrate how state space representation can handle complex models. First, we make use of the solution to Review Problem 2 in Chapter 3 (p. 3-18) and write the mass balances of reactant *A* in chemical reactors 1 and 2:

$$V_1 \frac{dc_1}{dt} = q(c_o - c_1) - V_1 k_1 c_1 \tag{E4-12}$$

and

$$V_2 \frac{dc_2}{dt} = q(c_1 - c_2) - V_2 k_2 c_2 \tag{E4-13}$$

Since q and c_o are input functions, the linearized equations in deviation variables and with zero initial conditions are (with all apostrophes omitted in the notations):

$$V_1 \frac{dc_1}{dt} = q_s c_o + (c_{os} - c_{1s})\, q - (q_s + V_1 k_1)\, c_1 \tag{E4-14}$$

and

$$V_2 \frac{dc_2}{dt} = q_s c_1 + (c_{1s} - c_{2s})\, q - (q_s + V_2 k_2)\, c_2 \tag{E4-15}$$

The missing steps are very similar to how we did Example 2.11 (p. 2-28). Divide the equations by the respective reactor volumes and define space times $\tau_1 = V_1/q_s$ and $\tau_2 = V_2/q_s$, we obtain

$$\frac{dc_1}{dt} = \frac{1}{\tau_1} c_o + \left(\frac{c_{os} - c_{1s}}{V_1} \right) q - \left(\frac{1}{\tau_1} + k_1\right) c_1 \tag{E4-16}$$

and

$$\frac{dc_2}{dt} = \frac{1}{\tau_2} c_1 + \left(\frac{c_{1s} - c_{2s}}{V_2} \right) q - \left(\frac{1}{\tau_2} + k_2\right) c_2 \tag{E4-17}$$

Up to this point, the exercise is identical to what we learned in Chapter 2. In fact, we can now take the Laplace transform of these two equations to derive the transfer functions. In state space models, however, we would put the two linearized equations in matrix form. As analogous to Eq. (4-1), we now have

$$\frac{d}{dt}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -(\frac{1}{\tau_1} + k_1) & 0 \\ \frac{1}{\tau_2} & -(\frac{1}{\tau_2} + k_2) \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{\tau_1} & \frac{c_{os} - c_{1s}}{V_1} \\ 0 & \frac{c_{1s} - c_{2s}}{V_2} \end{bmatrix}\begin{bmatrix} c_o \\ q \end{bmatrix}, \qquad \text{(E4-18)}$$

The output **y** in Eq. (4-2) can be defined as

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \qquad \text{(E4-19)}$$

if we are to use two outputs. In SISO problems, we likely would only measure and control c_2, and hence we would define instead

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \qquad \text{(E4-20)}$$

with c_2 as the only output variable.

✎ **Example 4.6**. Derive the transfer function Y/U and the corresponding state space model of the block diagram in Fig. E4.6.

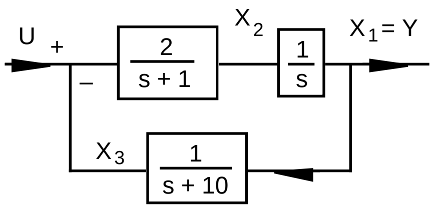


Figure E4.6

From Chapter 2 block diagram reduction, we can easily spot that

$$\frac{Y}{U} = \frac{\dfrac{2}{s\,(s+1)}}{1 + \dfrac{2}{s\,(s+1)\,(s+10)}},$$

which is reduced to

$$\frac{Y}{U} = \frac{2\,(s+10)}{s^3 + 11s^2 + 10s + 2} \qquad \text{(E4-21)}$$

This transfer function has closed-loop poles at –0.29, –0.69, and –10.02. (Of course, we computed them using MATLAB.)

To derive the state space representation, one visual approach is to identify locations in the block diagram where we can assign state variables and write out the individual transfer functions. In this example, we have chosen to use (Fig. E4.6)

$$\frac{X_1}{X_2} = \frac{1}{s}\,; \quad \frac{X_2}{U - X_3} = \frac{2}{s+1}\,; \quad \frac{X_3}{X_1} = \frac{1}{s+10}\,; \text{ and the output equation } Y = X_1$$

We can now rearrange each of the three transfer functions from which we write their time domain equivalent:

$$sX_1 = X_2 \qquad \frac{dx_1}{dt} = x_2 \qquad \text{(E4-22a)}$$

$$sX_2 = -X_2 - 2X_3 + 2U \qquad \frac{dx_2}{dt} = -x_2 - 2x_3 + 2u \qquad \text{(E4-22b)}$$

$$sX_3 = -10X_3 + X_1 \qquad \frac{dx_3}{dt} = x_1 - 10x_3 \qquad \text{(E4-22c)}$$

The rest is trivial. We rewrite the differential equations in matrix form as

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & -2 \\ 1 & 0 & -10 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} u \text{ , and } y = [\,1 \;\; 0 \;\; 0\,] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad \text{(E4-23, 24)}$$

We can check with MATLAB that the model matrix **A** has eigenvalues –0.29, –0.69, and –10.02. They are identical with the closed-loop poles. Given a block diagram, MATLAB can put the state space model together for us easily. To do that, we need to learn some closed-loop MATLAB functions, and we will defer this illustration to MATLAB Session 5.

An important reminder: Eq. (E4-23) has zero initial conditions **x**(0) = 0. This is a direct consequence of deriving the state space representation from transfer functions. Otherwise, Eq. (4-1) is not subjected to this restriction.

4.2 Relation with transfer function models

From the last example, we may see why the primary mathematical tools in modern control are based on linear system theories and time domain analysis. Part of the confusion in learning these more advanced techniques is that the umbilical cord to Laplace transform is not entirely severed, and we need to appreciate the link between the two approaches. On the bright side, if we can convert a state space model to transfer function form, we can still make use of classical control techniques. A couple of examples in Chapter 9 will illustrate how classical and state space techniques can work together.

We can take the Laplace transform of the matrix equation in Eq. (4-1) to give

$$s\mathbf{X}(s) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}U(s) \qquad (4\text{-}3)$$

where the capital **X** does not mean that it is a matrix, but rather it is used in keeping with our notation of Laplace variables. From (4-3), we can extract **X** explicitly as

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{B}U(s) = \Phi(s)\mathbf{B}U(s) \qquad (4\text{-}4)$$

where

$$\Phi(s) = (s\mathbf{I} - \mathbf{A})^{-1} \qquad (4\text{-}5)$$

is the resolvent matrix. More commonly, we refer to the **state transition matrix** (also called the fundamental matrix) which is its inverse transform

$$\Phi(t) = \mathcal{L}^{-1}[(s\mathbf{I} - \mathbf{A})^{-1}] \qquad (4\text{-}6)$$

We will use the same notation Φ for the time function and its Laplace transform, and only add the *t* or *s* dependence when it is not clear in which domain the notation is used.

Setting **D** = 0 and **X**(s) derived in (4-4), the output Y(s) = **CX**(s) becomes

$$Y(s) = \mathbf{C}\Phi(s)\mathbf{B}U(s) \qquad (4\text{-}7)$$

In this case where U and Y are scalar quantities, **CΦB** must also be scalar.[1] In fact, if we make an association between Eq. (4-7) and what we have learned in Chapter 2, **CΦB** is our ubiquitous transfer function. We can rewrite Eq. (4-7) as

$$Y(s) = G_p(s)U(s), \tag{4-7a}$$

where

$$G_p(s) = \mathbf{C}\Phi(s)\mathbf{B} \tag{4-8}$$

Hence, we can view the transfer function as how the Laplace transform of the state transition matrix Φ mediates the input **B** and the output **C** matrices. We may wonder how this output equation is tied to the matrix **A**. With linear algebra, we can rewrite the definition of Φ in Eq. (4-5) as

$$\mathbf{\Phi}(s) = (s\mathbf{I} - \mathbf{A})^{-1} = \frac{\text{adj}\,(s\mathbf{I} - \mathbf{A})}{\det\,(s\mathbf{I} - \mathbf{A})} \tag{4-5a}$$

Substitution of this form in (4-8) provides a more informative view of the transfer function:

$$G_p(s) = \frac{\mathbf{C}\,[\text{adj}\,(s\mathbf{I} - \mathbf{A})]\,\mathbf{B}}{\det\,(s\mathbf{I} - \mathbf{A})} \tag{4-8a}$$

The characteristic polynomial clearly is

$$\det\,(s\mathbf{I} - \mathbf{A}) = 0 \tag{4-9}$$

This is the result that we have arrived at, albeit less formally, in Example 4.1. Again, the poles of G_p are identical to the eigenvalues of the model matrix **A**.

✎ **Example 4.7**: We'll illustrate the results in this section with a numerical version of Example 4.5. Consider again **two CSTR-in-series**, with $V_1 = 1$ m^3, $V_2 = 2$ m^3, $k_1 = 1$ min^{-1}, $k_2 = 2$ min^{-1}, and initially at steady state, $\tau_1 = 0.25$ min, $\tau_2 = 0.5$ min, and inlet concentration $c_{os} = 1$ kmol/m^3. Derive the transfer functions and state transition matrix where both c_o and q are input functions.

With the steady state form of (E4-12) and (E4-13), we can calculate

$$c_{1s} = \frac{c_{os}}{1 + k_1\tau_1} = \frac{1}{1 + 0.25} = 0.8\,, \text{ and } c_{2s} = \frac{c_{1s}}{1 + k_2\tau_2} = \frac{0.8}{1 + 2(0.5)} = 0.4$$

In addition, we find $1/\tau_1 = 4$ min^{-1}, $1/\tau_2 = 2$ min^{-1}, $(1/\tau_1 + k_1) = 5$ min^{-1}, $(1/\tau_2 + k_2) = 4$ min^{-1}, $(c_{os} - c_{1s})/V_1 = 0.2$ kmol/m^6, and $(c_{1s} - c_{2s})/V_2 = 0.2$ kmol/m^6. We substitute these numerical values in (E4-16) and (E4-17), and take the Laplace transform of these equations to obtain (for more general algebraic result, we should take the transform first)

$$C_1(s) = \frac{4}{s + 5}\,C_o(s) + \frac{0.2}{s + 5}\,Q(s) \tag{E4-25}$$

and

$$C_2(s) = \frac{2}{s + 4}\,C_1(s) + \frac{0.2}{s + 4}\,Q(s)$$

[1] From Eq. (4-5), we see that Φ is a (*n* x *n*) matrix. Since **B** is (*n* x 1), and **C** is (1 x *n*), **CΦB** must be (1 x 1).

Further substitution for $C_1(s)$ with (E4-25) in $C_2(s)$ gives

$$C_2(s) = \frac{8}{(s+4)\,(s+5)}\,C_o(s) + \frac{0.2\,(s+7)}{(s+4)\,(s+5)}\,Q(s) \qquad \text{(E4-26)}$$

Equations (E4-25) and (E4-26) provide the transfer functions relating changes in flow rate Q and inlet concentration C_o to changes in the two tank concentrations.

With the state space model, substitution of numerical values in (E4-18) leads to the dynamic equations

$$\frac{d}{dt}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -5 & 0 \\ 2 & -4 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} 4 & 0.2 \\ 0 & 0.2 \end{bmatrix}\begin{bmatrix} c_o \\ q \end{bmatrix} \qquad \text{(E4-27)}$$

With the model matrix **A**, we can derive

$$(s\mathbf{I} - \mathbf{A}) = \begin{bmatrix} s+5 & 0 \\ -2 & s+4 \end{bmatrix},$$

and
$$\mathbf{\Phi}(s) = (s\mathbf{I} - \mathbf{A})^{-1} = \frac{1}{(s+5)(s+4)}\begin{bmatrix} s+4 & 0 \\ 2 & s+5 \end{bmatrix} \qquad \text{(E4-28)}$$

We will consider (E4-19) where both concentrations c_1 and c_2 are outputs of the model. The transfer function in (4-7) is now a matrix

$$\mathbf{G_p}(s) = \mathbf{C}\Phi(s)\mathbf{B} = \frac{1}{(s+5)(s+4)}\begin{bmatrix} s+4 & 0 \\ 2 & s+5 \end{bmatrix}\begin{bmatrix} 4 & 0.2 \\ 0 & 0.2 \end{bmatrix} \qquad \text{(E4-29)}$$

where **C** is omitted as it is just the identity matrix (E4-19).[1] With input $\mathbf{u}(s) = [C_o(s) \quad Q(s)]^T$, we can write the output equation (4-6) as

$$\begin{bmatrix} C_1(s) \\ C_2(s) \end{bmatrix} = \mathbf{C\Phi}(s)\mathbf{Bu}(s) = \frac{1}{(s+5)(s+4)}\begin{bmatrix} 4\,(s+4) & 0.2\,(s+4) \\ 8 & 0.2\,(s+7) \end{bmatrix}\begin{bmatrix} C_o(s) \\ Q(s) \end{bmatrix} \qquad \text{(E4-30)}$$

This is how MATLAB returns the results. We of course can clean up the algebra to give

$$\begin{bmatrix} C_1(s) \\ C_2(s) \end{bmatrix} = \begin{bmatrix} \dfrac{4}{(s+5)} & \dfrac{0.2}{(s+5)} \\ \dfrac{8}{(s+5)(s+4)} & \dfrac{0.2\,(s+7)}{(s+5)(s+4)} \end{bmatrix}\begin{bmatrix} C_o(s) \\ Q(s) \end{bmatrix} \qquad \text{(E4-30a)}$$

which is identical to what we have obtained earlier in (E4-25) and (E4-26). The case of only one output as in (E4-20) is easy and we'll cover that in Example 4.7A.

To wrap things up, we can take the inverse transform of (E4-30a) to get the time domain solutions:

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 4e^{-5t} & 0.2e^{-5t} \\ 8\,(e^{-4t} - e^{-5t}) & 0.2\,(3e^{-4t} - 2e^{-5t}) \end{bmatrix}\begin{bmatrix} c_o \\ q \end{bmatrix} \qquad \text{(E4-31)}$$

---

[1] Be careful with the notation. Upper case C is for concentration in the Laplace domain. The boldface upper case **C** is the output matrix.

✎ **Example 4.7A**: Repeat Example 4.7 using MATLAB.

If you understand the equations in Example 4.7, we are ready to tackle the same problem with MATLAB.

```
t1=0.25; t2=0.5;       % Define the variables
k1=1; k2=2;
V1=1; V2=2;
cos=1;

% Calculate the steady state values. MATLAB should return
c1s=cos/(1+k1*t1);     % 0.8
c2s=c1s/(1+k2*t2);     % 0.4

                       % Coefficients of A and B using (E4-18)
a11=-(1/t1+k1);        % -5
a12=0;
a21=1/t2;
a22=-(1/t2+k2);        % -4
b11=1/t1;
b12=(cos-c1s)/V1;      % 0.2
b21=0;
b22=(c1s-c2s)/V2;      % 0.2

                       % Finally build A and B in (E4-27)
a=[a11 a12; a21 a22];  % [–5 0; 2 4]
b=[b11 b12; b21 b22];  % [4 0.2; 0 0.2]
eig(a)                 % Check that they are -4, -5

c=[1 0; 0 1];          % Define C such that both C1 and C2 are outputs
d=[0 0; 0 0];
```

With all the coefficient matrices defined, we can do the conversion to transfer function. The function `ss2tf()` works with only one designated input variable. Thus, for the *first* input variable C_o, we use

```
                              % MATLAB returns, for input no. 1
[q1,p]=ss2tf(a,b,c,d,1)       % q1=[0 4 16; 0 0 8]
                              % p =[1 9 20] = (s+4)(s+5)
```

The returned vector `p` is obviously the characteristic polynomial. The matrix `q1` is really the first *column* of the transfer function matrix in Eq. (E4-30), denoting the two terms describing the effects of changes in C_o on C_1 and C_2. Similarly, the second column of the transfer function matrix in (E4-30) is associated with changes in the *second* input Q, and can be obtained with:

```
[q2,p]=ss2tf(a,b,c,d,2)       % q2=[0 .2 .8; 0 .2 1.4]
                              % The first term is 0.2(s+4) because
                              % MATLAB retains p=(s+4)(s+5)
```

If C_2 is the only output variable, we define **C** according to the output equation (E4-20). Matrices **A** and **B** remain the same. The respective effects of changes of C_o and Q on C_2 can be obtained with

```
c=[0 1]; d=[0 0];             % C2 is the only output
[q21,p]=ss2tf(a,b,c,d,1)      % Co as input; q21=[0 0 8]
[q22,p]=ss2tf(a,b,c,d,2)      % Q as input; q22=[0 .2 1.4]
```

We should find that the result is the same as the second *row* of (E4-30), denoting the two terms describing the effects of changes in C_o and Q on C_2.

Similarly, if C_1 is the only output variable, we use instead:

```
c=[1 0]; d=[0 0];             % C1 is the only output
[q11,p]=ss2tf(a,b,c,d,1)      % q11=[0 4 16]
[q12,p]=ss2tf(a,b,c,d,2)      % q12=[0 .2 .8]
```

and the result is the first row of (E4-30).

✎ **Example 4.8**: Develop a fermentor model which consists of two mass balances, one for the cell mass (or yeast), C_1, and the other for glucose (or substrate), C_2. We have to forget about the alcohol for now. The cell mass balance (E4-32) has two terms on the right. The first one describes cell growth using the specific growth rate $\mu = \mu(C_2)$. The second term accounts for the loss of cells due to outlet flow Q, which in turn is buried inside the notation D, the dilution rate.

$$\frac{dC_1}{dt} = \mu C_1 - DC_1 \qquad \text{(E4-32)}$$

The specific growth rate and dilution rate are defined as:

$$\mu = \mu(C_2) = \mu_m \frac{C_2}{K_m + C_2}\ , \quad \text{and} \quad D = \frac{Q}{V}$$

The glucose balance has three terms on the right. The first accounts for the consumption by the cells. The last two terms account for the flow of glucose into and out of the fermentor.

$$\frac{dC_2}{dt} = -\frac{\mu C_1}{Y} + D\,(C_{2o} - C_2) \qquad \text{(E4-33)}$$

The maximum specific growth rate μ_m, Monod constant K_m, and cell yield coefficient Y are constants. In (E4-33), C_{2o} is the inlet glucose concentration.

The dilution rate D is dependent on the volumetric flow rate Q and the volume V, and really is the reciprocal of the space time of the fermentor. In this problem, the fermentor volume, V, is fixed, and we vary the flow rate, Q. Hence, it is more logical to use D (and easier to think) as it is proportional to Q.

Our question is to formulate this model under two circumstances: (1) when we only vary the dilution rate, and (2) when we vary both the dilution rate and the amount of glucose input. Derive also the transfer function model in the second case. In both cases, C_1 and C_2 are the two outputs.

To solve this problem, we obviously have to linearize the equations. In vector form, the nonlinear model is

$$\frac{d\mathbf{x}}{dt} = f(\mathbf{x}, D) \qquad \text{(E4-34)}$$

where $\mathbf{x} = [C_1\ C_2]^T$, and

$$f(\mathbf{x}, D) = \begin{bmatrix} f_1(\mathbf{x}, D) \\ f_2(\mathbf{x}, D) \end{bmatrix} = \begin{bmatrix} (\mu(C_2) - D)\,C_1 \\ -\dfrac{\mu(C_2)\,C_1}{Y} + D\,(C_{2o} - C_2) \end{bmatrix} \qquad \text{(E4-35)}$$

We first take the inlet glucose, C_{2o}, to be a constant (*i.e.*, no disturbance) and vary only the dilution rate, D. From the steady state form of (E4-32) and (E4-33), we can derive (without special notations for the steady state values):

$$D\,(C_{2o} - C_2) = \frac{\mu C_1}{Y}, \quad \text{and} \quad C_1 = Y(C_{2o} - C_2) \tag{E4-36}$$

Now we linearize the two equations about the steady state. We expect to arrive at (with the apostrophes dropped from all the deviation variables and partial derivatives evaluated at the steady state):

$$\frac{d}{dt}\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} \dfrac{\partial f_1}{\partial C_1} & \dfrac{\partial f_1}{\partial C_2} \\ \dfrac{\partial f_2}{\partial C_1} & \dfrac{\partial f_2}{\partial C_2} \end{bmatrix}_{s.s} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} + \begin{bmatrix} \dfrac{\partial f_1}{\partial D} \\ \dfrac{\partial f_2}{\partial D} \end{bmatrix}_{s.s} D \tag{E4-37}$$

Using (E4-35) to evaluate the partial derivatives, we should find

$$\frac{d}{dt}\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 0 & C_1\mu' \\ -\dfrac{\mu}{Y} & -\dfrac{C_1}{Y}\mu' - \mu \end{bmatrix}_{s.s.} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} + \begin{bmatrix} -C_1 \\ \dfrac{C_1}{Y} \end{bmatrix}_{s.s.} D \; = \; \mathbf{A}\mathbf{x} + \mathbf{B}D \tag{E4-38}$$

where μ' is the derivative with respect to the substrate C_2:

$$\mu' = \frac{d\mu}{dC_2} = \mu_m \frac{K_m}{(K_m + C_2)^2} \tag{E4-39}$$

All the coefficients in **A** and **B** are evaluated at steady state conditions. From Eq. (E4-32), D = μ at steady state. Hence the coefficient a_{11} in **A** is zero.

To complete the state space model, the output equation is

$$\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} \tag{E4-40}$$

where **C** is the identity matrix.

Now, we'll see what happens with **two inputs**. In practice, we most likely would make a highly concentrated glucose stock and dose it into a main feed stream that contains the other ingredients. What we manipulate is the dosage rate. Consider that the glucose feed stock has a fixed concentration C_{2f} and adjustable feed rate q_f, and the other nutrients are being fed at a rate of q_o. The effective glucose feed concentration is

$$C_{2o} = \frac{q_f\,C_{2f}}{q_f + q_o} = \frac{q_f\,C_{2f}}{Q} \tag{E4-41}$$

where $Q = q_f + q_o$ is the total inlet flow rate, and the dilution rate is

$$D = \frac{Q}{V} = \frac{q_f + q_o}{V} \tag{E4-42}$$

The general fermentor model equation as equivalent to (E4-34) is

$$\frac{d\,\mathbf{x}}{dt} = f(\mathbf{x}, \mathbf{u}) \tag{E4-43}$$

where the state space remains $\mathbf{x} = [C_1\ C_2]^T$, but the input is the vector $\mathbf{u} = [D_o\ D_f]^T$. Here, $D_o = q_o/V$ and $D_f = q_f/V$ are the respective dilution rates associated with the two inlet streams. That is, we vary the main nutrient feed and glucose dosage flow rates to manipulate this system. The function, **f**, is

$$f(\mathbf{x}, \mathbf{u}) = \begin{bmatrix} f_1(\mathbf{x}, \mathbf{u}) \\ f_2(\mathbf{x}, \mathbf{u}) \end{bmatrix} = \begin{bmatrix} \mu(C_2)C_1 - (D_o + D_f)\, C_1 \\ -\dfrac{\mu(C_2)\, C_1}{Y} + D_f C_{2f} - (D_o + D_f)\, C_2 \end{bmatrix} \tag{E4-44}$$

At steady state,

$$\mu = (D_o + D_f) = D \tag{E4-45}$$

and

$$C_1 = Y(C^*_{2o} - C_2), \text{ where } C^*_{2o} = \frac{D_f\, C_{2f}}{D_o + D_f} \tag{E4-46}$$

The equations linearized about the steady state (with the apostrophes dropped from the deviation variables as in E4-38) are

$$\frac{d}{dt}\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 0 & C_1\mu' \\ -\dfrac{\mu}{Y} & -\dfrac{C_1}{Y}\,\mu' - \mu \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} + \begin{bmatrix} -C_1 & -C_1 \\ -C_2 & (C_{2f} - C_2) \end{bmatrix}_{s.s.} \begin{bmatrix} D_o \\ D_f \end{bmatrix} = \mathbf{Ax} + \mathbf{Bu} \tag{E4-47}$$

The output equation remains the same as in (E4-40). Laplace transform of the model equations and rearrangement lead us to

$$\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} D_o \\ D_f \end{bmatrix} \tag{E4-48}$$

where the four open-loop plant transfer functions are:

$$G_{11} = \frac{-\left(\dfrac{C_1}{Y}\right)_{s.s.} s - \left[C_1\left(\dfrac{C_1\,\mu'}{Y} + \mu\right) + C_1\mu' C_2\right]_{s.s.}}{p(s)} \tag{E4-49}$$

$$G_{12} = \frac{-\left(\dfrac{C_1}{Y}\right)_{s.s.} s + \left[\, C_1\mu'\,(C_{2f} - C_2) - C_1\left(\dfrac{C_1\mu'}{Y} + \mu\right)\right]_{s.s.}}{p(s)} \tag{E4-50}$$

$$G_{21} = \frac{-\,(C_2)_{s.s.} s + \left[\dfrac{C_1\mu}{Y}\right]_{s.s.}}{p(s)} \tag{E4-51}$$

$$G_{22} = \frac{(C_{2f} - C_2)_{s.s.} s + \left[\dfrac{C_1\mu}{Y}\right]_{s.s.}}{p(s)} \tag{E4-52}$$

and the characteristic polynomial

$$p(s) = s^2 + \left(\frac{C_1\mu'}{Y} + \mu\right)_{s.s.} s + \left(\frac{C_1\,\mu\,\mu'}{Y}\right)_{s.s.} \tag{E4-53}$$

Until we can substitute numerical values and turn the problem over to a computer, we have to admit that the state space form in (E4-47) is much cleaner to work with.

This completes our "feel good" examples. It may not be too obvious, but the hint is that linear system theory can help us analysis complex problems. We should recognize that state space representation can do everything in classical control and more, and feel at ease with the language of state space representation.

4.3 Properties of state space models

This section contains brief remarks on some transformations and the state transition matrix. We limit the scope to materials that one may draw on introductory linear algebra.

4.3.1 Time-domain solution

We can find the solution to Eq. (4-1), which is simply a set of first order differential equations. As analogous to how Eq. (2-3) on page 2-2 was obtained, we now use the matrix exponential function as the integration factor, and the result is (hints in the Review Problems)

$$\mathbf{x}(t) = e^{\mathbf{A}t}\,\mathbf{x}(0) + \int_0^t e^{-\mathbf{A}(t-\tau)}\,\mathbf{B}u(\tau)\,d\tau \tag{4-10}$$

where the first term on the right evaluates the effect of the initial condition, and the second term is the so-called convolution integral that computes the effect of the input u(t).

The point is that state space representation is general and is not restricted to problems with zero initial conditions. When Eq. (4-1) is homogeneous (*i.e.*, **B**u = 0), the solution is simply

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) \tag{4-11}$$

We can also solve the equation using Laplace transform. Starting again from (4-1), we can find (see Review Problems)

$$\mathbf{x}(t) = \Phi(t)\mathbf{x}(0) + \int_0^t \Phi(t-\tau)\,\mathbf{B}u(\tau)\,d\tau \tag{4-12}$$

where $\Phi(t)$ is the state transition matrix as defined in (4-6). Compare (4-10) with (4-12), and we can see that

$$\Phi(t) = e^{\mathbf{A}t} \tag{4-13}$$

We have shown how the state transition matrix can be derived in a relatively simple problem in Example 4.7. For complex problems, there are numerical techniques that we can use to compute $\Phi(t)$, or even the Laplace transform $\Phi(s)$, but which of course, we shall skip.

One idea (not that we really do that) is to apply the Taylor series expansion on the exponential function of **A**, and evaluate the state transition matrix with

$$\Phi(t) = e^{\mathbf{A}t} = I + \mathbf{A}t + \frac{1}{2!}\mathbf{A}^2t^2 + \frac{1}{3!}\mathbf{A}^3t^3 + \ldots \tag{4-14}$$

Instead of an infinite series, we can derive a closed form expression for the exponential function. For an *n* x *n* matrix **A**, we have

$$e^{\mathbf{A}t} = \alpha_o(t)\mathbf{I} + \alpha_1(t)\mathbf{A} + \alpha_2(t)\mathbf{A}^2 + \ldots + \alpha_{n-1}(t)\mathbf{A}^{n-1} \tag{4-15}$$

The challenge is to find those coefficients $\alpha_i(t)$, which we shall skip.[1]

[1] We only need the general form of (4-15) later in Chapter 9. There are other properties of the state transition matrix that we have skipped, but we have structured our writing such that they are not needed here.

4.3.2 Controllable canonical form

While there is no unique state space representation of a system, there are "standard" ones that control techniques make use of. Given any state equations (and if some conditions are met), it is possible to convert them to these standard forms. We cover in this subsection a couple of important canonical forms.

A tool that we should be familiar with from introductory linear algebra is similarity transform, which allows us to transform a matrix into another one but which retains the same eigenvalues. If a state $\mathbf{x}$ and another $\bar{\mathbf{x}}$ are related via a so-called similarity transformation, the state space representations constructed with $\mathbf{x}$ and $\bar{\mathbf{x}}$ are considered to be equivalent.[1]

For the *n*-th order differential equation: [2]

$$y^{(n)} + a_{n-1}y^{(n-1)} + ... + a_1 y^{(1)} + a_o y = u(t) \tag{4-16}$$

we define

$$x_1 = y,\ \ x_2 = y^{(1)},\ \ x_3 = y^{(2)},\ \ldots,\ \text{and}\ \ x_n = y^{(n-1)} \tag{4-17}$$

The original differential equation can now be reformulated as a set of first order equations:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ &\cdot \\ &\cdot \\ \dot{x}_{n-1} &= x_n \end{aligned} \tag{4-18}$$

and finally

$$\dot{x}_n = -a_o x_1 - a_1 x_2 - \ldots - a_{n-1} x_n + u(t)$$

This set of equation, of course, can be put in matrix form as in Eq. (4-1):

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ -a_o & -a_1 & -a_2 & \ldots & -a_{n-1} \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u = \mathbf{A}\mathbf{x} + \mathbf{B}u \tag{4-19}$$

The output equation equivalent to Eq. (4-2) is

$$y = [\ 1\ 0\ 0\ \ldots\ 0]\ \mathbf{x} = \mathbf{C}\mathbf{x} \tag{4-20}$$

The system of equations in (4-19) and (4-20) is called the **controllable canonical form**

---

[1] That includes transforming a given system to the controllable canonical form. We can say that state space representations are unique up to a similarity transform. As for transfer functions, we can say that they are unique up to scaling in the coefficients in the numerator and denominator. However, the derivation of canonical transforms requires material from Chapter 9 and is not crucial for the discussion here. These details are provided on our *Web Support*.

[2] Be careful when you read the MATLAB manual; it inverts the index of coefficients as in $y^{(n)} + a_1 y^{(n-1)} + ... + a_{n-1} y^{(1)} + a_n y$. Furthermore, we use a simple RHS in the ODE. You'd find more general, and thus messier, derivations in other texts.

(also phase variable canonical form). As the name implies, this form is useful in doing controllability analysis and in doing pole placement system design—topics that we will cover in Chapter 9.

With all the zeros along the leading diagonal, we can find relatively easily that the characteristic equation of **A**, |s**I** – **A**| = 0, is

$$s^n + a_{n-1}s^{n-1} + ... + a_1 s + a_o = 0 \tag{4-21}$$

which is immediately obvious from Eq. (4-16) itself. We may note that the coefficients of the characteristic polynomial are contained in the matrix **A** in (4-19). Matrices with this property are called the *companion* form. When we use MATLAB, its `canon()` function returns a companion matrix which is the transpose of **A** in (4-19); this form is called the *observable canonical form*. We shall see that in MATLAB Session 4.

4.3.3 Diagonal canonical form

Here, we want to transform a system matrix **A** into a diagonal matrix Λ that is made up of the eigenvalues of **A**. In other words, all the differential equations are decoupled after the transformation.

For a given system of equations in (4-1) in which **A** has *distinct eigenvalues*, we should find a transformation with a matrix **P**:

$$\bar{\mathbf{x}} = \mathbf{P}^{-1}\mathbf{x}, \quad \text{or} \quad \mathbf{x} = \mathbf{P}\bar{\mathbf{x}} \tag{4-22}$$

such that

$$\dot{\bar{\mathbf{x}}} = \mathbf{\Lambda}\,\bar{\mathbf{x}} + \bar{\mathbf{B}}\,u \tag{4-23}$$

where now $\bar{\mathbf{B}} = \mathbf{P}^{-1}\mathbf{B}$, and $\Lambda = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ is a diagonal matrix made up of the eigenvalues of **A**. The transformation matrix (also called the modal matrix) **P** is made up of the eigenvectors of **A**. In control, (4-23) is called the *diagonal canonical form*.

If **A** has repeated eigenvalues (multiple roots of the characteristic polynomial), the result, again from introductory linear algebra, is the *Jordan canonical form*. Briefly, the transformation matrix **P** now needs a set of generalized eigenvectors, and the transformed matrix $\mathbf{J} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ is made of Jordan blocks for each of the repeated eigenvalues. For example, if matrix **A** has three repeated eigenvalues λ_1, the transformed matrix should appear as

$$\mathbf{J} = \left[\begin{array}{c:c} \mathbf{J}_{11} & \mathbf{0} \\ \hdashline \mathbf{0} & \mathbf{J}_{22} \end{array}\right] \quad \text{where} \quad \mathbf{J}_{11} = \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{bmatrix} \tag{4-24}$$

and $\mathbf{J}_{22}$ is a diagonal matrix made up of eigenvalues $\lambda_4, ..., \lambda_n$. Since Chapter 9 later will not make use of such cases, we shall leave the details to a second course in modern control.

✎ **Example 4.9**: For a model with the following transfer function

$$\frac{Y}{U} = \frac{1}{(s+3)\,(s+2)\,(s+1)},$$

find the diagonal and observable canonical forms with MATLAB.

The statements to use are:

```
G=zpk([],[-1 -2 -3],1);
S=ss(G);                  % S is the state space system
canon(S)                  % Default is the diagonal form
canon(S,'companion')      % This is the observable companion
```

There is no messy algebra. We can be spoiled! Further details are in MATLAB Session 4.

❐ Review Problems

1. Fill in the gaps in the derivation of (E4-25) and (E4-26) in Example 4.7
2. Write down the dimensions of all the matrixes in (4-6) for the general case of multiple-input and multiple-output models. Take **x** to be (n x 1), **y** (m x 1), and **u** (k x 1). And when y and u are scalar, **CΦB** is a scalar quantity too.
3. Fill in the gaps in the derivation of (4-9) from (4-3a).
4. For the SISO system shown in Fig. R4.4, derive the state space representation. Show that the characteristic equation of the model matrix is identical to the closed-loop characteristic polynomial as derived from the transfer functions.
5. Derive Eq. (4-10).
6. Derive Eq. (4-12).
7. Derive Eq. (4-23).

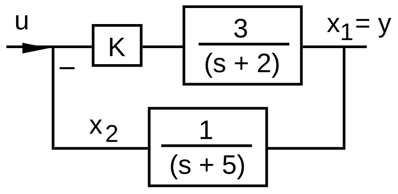


Figure R4.4

Hints:

2. **A** is (n x n), **B** (n x k), **C** (m x n), Φ (n x n), and **CΦB** (m x k).
4. Multiply the K to the transfer function to give a gain of 3K. Then the rest is easier than Example 4.6.
5. We multiply Eq. (4-1) by exp(–**A**t) to give $e^{-\mathbf{A}t}[\dot{\mathbf{x}} - \mathbf{A}\mathbf{x}] = e^{-\mathbf{A}t}\mathbf{B}u$, which is

$$\frac{d}{dt}[e^{-\mathbf{A}t}\mathbf{x}] = e^{-\mathbf{A}t}\mathbf{B}u$$

Integration with the initial condition gives

$$e^{-\mathbf{A}t}\mathbf{x}(t) - \mathbf{x}(0) = \int_0^t e^{-\mathbf{A}\tau}\mathbf{B}u(\tau)\, d\tau$$

which is one step away from Eq. (4-10).

6. The Laplace transform of Eq. (4-1) with nonzero initial conditions is

$$s\mathbf{X} - \mathbf{x}(0) = \mathbf{A}\mathbf{X} + \mathbf{B}U$$

or

$$\mathbf{X} = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{x}(0) + (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}U$$

Substituting in the definition $\Phi(s) = (s\mathbf{I} - \mathbf{A})^{-1}$, we have

$$\mathbf{X} = \Phi(s)\mathbf{x}(0) + \Phi(s)\mathbf{B}U$$

The time domain solution vector is the inverse transform

$$\mathbf{x}(t) = \mathcal{L}^{-1}[\Phi(s)]\mathbf{x}(\mathbf{0}) + \mathcal{L}^{-1}[\Phi(s)\mathbf{B}U]$$

and if we invoke the definition of convolution integral (from calculus), we have Eq. (4-12).

7. We first substitute $\mathbf{x} = \mathbf{P}\bar{\mathbf{x}}$ in Eq. (4-1) to give

$$\mathbf{P}\frac{d}{dt}\bar{\mathbf{x}} = \mathbf{AP}\bar{\mathbf{x}} + \mathbf{B}u$$

Then we multiply the equation by the inverse $\mathbf{P}^{-1}$

$$\frac{d}{dt}\bar{\mathbf{x}} = \mathbf{P}^{-1}\mathbf{AP}\bar{\mathbf{x}} + \mathbf{P}^{-1}\mathbf{B}u$$

which is (4-23).

❖ 5. *Analysis of Single-Loop Control Systems*

We now finally launch into the material on controllers. State space representation is more abstract and it helps to understand controllers in the classical sense first. We will come back to state space controller design later. Our introduction stays with the basics. Our primary focus is to learn how to design and tune a classical PID controller. Before that, we first need to know how to set up a problem and derive the closed-loop characteristic equation.

What are we up to?
- Introduce the basic PID control schemes
- Derive the closed-loop transfer function of a system and understand its properties

5.1 PID controllers

We use a simple liquid level controller to illustrate the concept of a classic feedback control system.[1] In this example (Fig. 5.1), we monitor the liquid level in a vessel and use the information to adjust the opening of an effluent valve to keep the liquid level at some user-specified value (the **set point** or **reference**). In this case, the liquid level is both the **measured variable** and the **controlled variable**—they are the same in a single-input single-output (**SISO**) system. In this respect, the controlled variable is also the output variable of the SISO system. A **system** refers to the **process** which we need to control plus the **controller** and accompanying accessories such as sensors and actuators.[2]

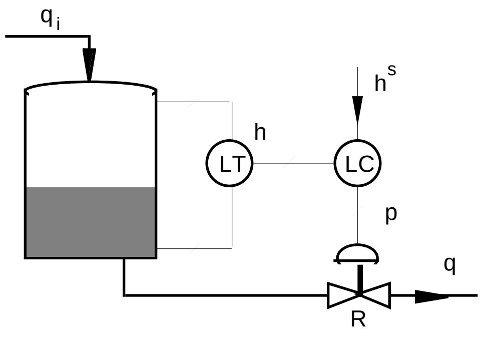


Figure 5.1. Schematic diagram of a liquid level control system.

Let's say we want to keep the liquid level at the set point, h^s, but a sudden surge in the inlet flow rate q_i (the **disturbance** or **load**) increases h such that there is a deviation $h' = h - h^s > 0$. The deviation can be rectified if we open up the valve (or we can think in terms of lowering the flow resistance R). Here, we assume that the level controller will send out an appropriate signal to the valve to accomplish the task. It is logical to think that the signal from the controller, p(t), should be a function of the deviation.

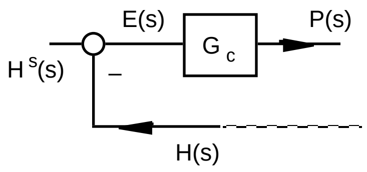


Figure 5.2. Information around the summing point in a negative feedback system.

However, since we are to implement *negative* feedback, we base our decision on the **error** defined as

$$e(t) = h^s(t) - h(t) ,$$

[1] In Fig. 5.1, we use the actual variables because they are what we measure. Regardless of the notations in a schematic diagram, the block diagram in Fig. 5.2 is based on deviation variables and their Laplace transform.

[2] Recall the footnote in Chapter 1: a process is referred to as a **plant** in control engineering.

which is the negative of the deviation (Fig. 5.2). The actual controller output is

$$p(t) = p^s + f[e(t)] = p^s + f[h^s - h(t)] \tag{5-1}$$

where f is some function of e(t), and p^s is the actuating signal at steady state when the deviation is h' = 0; it is commonly referred to as the **controller bias signal**. Our task is to determine plausible choices of the controller function—what is also called control laws. The classical controller functions are explained in the following subsections, but if you need a more physical feel of how a control system is put together now, you may jump ahead and read Section 5.2 first.

5.1.1 Proportional control

The simplest idea is that the compensation signal (actual controller output) is proportional to the error e(t):

$$p(t) = p^s + K_c e(t) = p^s + K_c[h^s - h(t)] \tag{5-2}$$

where K_c is the **proportional gain** of the controller. It is obvious that the value of K_c determines the controller "sensitivity"—how much compensation to enact for a given change in error.

For all commercial devices, the proportional gain is a *positive* quantity. Because we use negative feedback (see Fig. 5.2), the controller output moves in the *reverse* direction of the controlled variable.[1] In the liquid level control example, if the inlet flow is disturbed such that h rises above h^s, then $e < 0$, and that leads to $p < p^s$, *i.e.*, the controller output is decreased. In this case, we of course will have to select or purchase a valve such that a lowered signal means opening the valve (decreasing flow resistance). Mathematically, this valve has a negative steady state gain ($-K_v$).[2]

Now what if our valve has a positive gain? (Increased signal means opening the valve.) In this case, we need a *negative* proportional gain. Commercial devices provide such a "switch" on the controller box to invert the signal. Mathematically, we have changed the sign of the compensation term to: $p = p^s - K_c e$. [3]

By the definition of a control problem, there should be no error at t = 0, *i.e.*, $e^s = 0$, and the deviation variable of the error is simply the error itself:

$$e'(t) = e(t) - e^s = e(t)$$

---

[1] You may come across the terms **reverse and direct acting** and they are extremely confusing. Some authors consider the action between the controller output and the controlled variable, and thus a negative feedback loop with a positive K_c is considered reverse-acting. However, most commercial vendors consider the action between the error (controller input) and the controller output, and now, a controller with a positive K_c is direct-acting, exactly the opposite terminology. We'll avoid using these confusing terms. The important point is to select the proper signs for all the steady state gains, and we'll get back to this issue in Section 5.4.

[2] Take note that from the mass balance of the tank, the process gain associated with the outlet flow rate is also negative. A simple-minded check is that in a negative feedback system, there can only be one *net* negative sign—at the feedback summing point. If one unit in the system has a negative steady state gain, we know something else must have a negative steady state gain too.

[3] We may have introduced more confusion than texts that ignore this tiny detail. To reduce confusion, we will keep K_c a positive number. For problems in which the proportional gain is negative, we use the notation $-K_c$. We can think that the minus sign is likened to having flipped the action switch on the controller.

Hence Eq. (5-2) is a relation between the deviation variables of the error and the controller output:

$$p(t) - p^s = K_c\,[e(t) - e^s]\,, \quad \text{or} \quad p'(t) = K_c e'(t)$$

and the transfer function of a proportional controller is simply

$$G_c(s) = \frac{P(s)}{E(s)} = K_c \tag{5-3}$$

Generally, the proportional gain is dimensionless (*i.e.*, p(t) and e(t) have the same units). Many controller manufacturers use the percent **proportional band**, which is defined as

$$PB = \frac{100}{K_c} \tag{5-4}$$

A high proportional gain is equivalent to a narrow PB, and a low gain is wide PB. We can interpret PB as the range over which the error must change to drive the controller output over its full range.[1]

Before doing any formal analysis, we state a few qualitative features of each type of controller. This is one advantage of classical control. We can make fairly easy physical interpretation of the control algorithm. The analyses that come later will confirm these qualitative observations.

General qualitative features of proportional control

- We expect that a proportional controller will improve or accelerate the response of a process. The larger K_c is, the faster and more sensitive is the change in the compensation with respect to a given error. However, if K_c is too large, we expect the control compensation to overreact, leading to oscillatory response. In the worst case scenario, the system may become unstable.
- There are physical limits to a control mechanism. A controller (like an amplifier) can deliver only so much voltage or current; a valve can deliver only so much fluid when fully opened. At these limits, the control system is **saturated**.[2]
- We expect a system with only a proportional controller to have a **steady state error** (or an **offset**). A formal analysis will be introduced in the next section. This is one simplistic way to see why. Let's say we change the system to a new set point. The proportional controller output, $p = p^s + K_c e$, is required to shift away from the previous bias p^s and move the system to a new steady state. For p to be different from p^s, the error must have a finite non-zero value. [3]
- To tackle a problem, consider a simple proportional controller first. This may be all we need (lucky break!) if the offset is small enough (for us to bear with) and the response is adequately fast. Even if this is not the case, the analysis should help us plan the next step.

[1] In some commercial devices, the proportional gain is defined as the ratio of the percent controller output to the percent controlled variable change [%/%]. In terms of the control system block diagram that we will go through in the next section, we just have to add “gains” to do the unit conversion.

[2] Typical ranges of device outputs are 0–10 V, 0–1 V, 4–20 mA, and 3–15 psi.

[3] The exception is when a process contains integrating action, *i.e.*, 1/s in the transfer functions—a point that we will illustrate later.

5.1.2 Proportional-Integral (PI) control

To eliminate offset, we can introduce integral action in the controller. In other words, we use a compensation that is related to the history of the error:

$$p'(t) = \frac{1}{\tau_I} \int_0^t e'(t)\, dt \quad ; \quad \frac{P(s)}{E(s)} = \frac{1}{s\,\tau_I}$$

where τ_I is the **integral time** constant (**reset time**, or minutes per repeat[1]). Commercial devices may also use $1/\tau_I$ which is called the **reset rate** (repeats per minute).

The integral action is such that we accumulate the error from t = 0 to the present. Thus the integral is not necessarily zero even if the current error is zero. Moreover, the value of the integral will not decrease unless the integrand e'(t) changes its sign. As a result, integral action forces the system to overcompensate and leads to oscillatory behavior, *i.e.*, the closed-loop system will exhibit an underdamped response. If there is too much integral action, the system may become unstable.

In practice, integral action is never used by itself. The norm is a **proportional-integral (PI)** controller. The time-domain equation and the transfer function are:

$$p'(t) = K_c \left[e'(t) + \frac{1}{\tau_I} \int_0^t e'(t)\, dt \right] \quad ; \quad G_c(s) = K_c \left[1 + \frac{1}{\tau_I s} \right] \tag{5-5}$$

If the error cannot be eliminated within a reasonable period, the integral term can become so large that the controller is saturated—a situation referred to as integral or **reset windup**. This may happen during start-up or large set point changes. It may also happen if the proportional gain is too small. Many industrial controllers have "anti-windup" which temporarily halts the integral action whenever the controller output becomes saturated.[2]

On the plus side, the integration of the error allows us to detect and eliminate very small errors. To make a simple explanation of why integral control can eliminate offsets, refer back to our intuitive explanation of offset with only a proportional controller. If we desire $e = 0$ at steady state, and to shift controller output p away from the previous bias p^s, we must have a nonzero term. Here, it is provided by the integral in Eq. (5-5). That is, as time progresses, the integral term takes on a final nonzero value, thus permitting the steady state error to stay at zero.

General qualitative features of PI control

- PI control can eliminate offset. We must use a PI controller in our design if the offset is unacceptably large.

---

[1] Roughly, the reset time is the time that it takes the controller to repeat the proportional action. This is easy to see if we take the error to be a constant in the integral.

[2] Another strategy is to implement the PI algorithm in the so-called **reset feedback** configuration. The basis of internal reset feedback is to rearrange and implement the PI transfer function as

$$1 + \frac{1}{\tau_I s} = \frac{\tau_I s + 1}{\tau_I s} = \frac{1}{\tau_I s / (\tau_I s + 1)} = \frac{1}{1 - 1/(\tau_I s + 1)}$$

Now, the “internal state” of the controller, whether it be electronics or a computer algorithm for integration, will have an upper limit. External reset feedback, on the other hand, makes use of measurements of the manipulated variable. You may find such implementation details in more applied control books.

- The elimination of the offset is usually at the expense of a more underdamped system response. The oscillatory response may have a short rise time, but is penalized by excessive overshoot or exceedingly long settling time. [1]
- Because of the inherent underdamped behavior, we must be careful with the choice of the proportional gain. In fact, we usually lower the proportional gain (or detune the controller) when we add integral control.

5.1.3 Proportional-Derivative (PD) control

We certainly want to respond very differently if the temperature of a chemical reactor is changing at a rate of 100°C/s as opposed to 1°C/s. In a way, we want to "project" the error and make corrections accordingly. In contrast, proportional and integral controls are based on the present and the past. Derivative controller action is based on how fast the error is changing with time (rate action control). We can write

$$p'(t) = \tau_D \frac{de'}{dt} \quad ; \quad \frac{P(s)}{E(s)} = \tau_D s$$

where τ_D is the **derivative time** constant (sometimes just rate time).

Here, the controller output is zero as long as the error stays constant. That is, even if the error is not zero. Because of the proportionality to the rate of change, the controller response is very sensitive to noise. If there is a sudden change in error, especially when we are just changing the set point, the controller response can be unreasonably large—leading to what is called a **derivative kick**.

Derivative action is never used by itself. The simplest implementation is a **proportional-derivative (PD)** controller. The time-domain equation and the transfer function of an "ideal" PD controller are:

$$p'(t) = K_c \left[e'(t) + \tau_D \frac{de'}{dt} \right] \quad ; \quad G_c(s) = K_c \left[1 + \tau_D s \right] \tag{5-6}$$

In practice, we cannot build a pneumatic device or a passive circuit which provides ideal derivative action. Commercial (real!) PD controllers are designed on the basis of a lead-lag element:

$$G_c(s) = K_c \left[\frac{\tau_D s + 1}{\alpha \tau_D s + 1} \right] \tag{5-7}$$

where α is a small number, typically $0.05 \le \alpha \le 0.2$.

In effect, we are adding a very large real pole to the derivative transfer function. Later, after learning root locus and frequency response analysis, we can make more rational explanations, including why the function is called a lead-lag element. We'll see that this is a nice strategy which is preferable to using the ideal PD controller.

To reduce derivative kick (the sudden jolt in response to set point changes), the derivative action can be based on the rate of change of the measured (controlled) variable instead of the rate of change of the error. One possible implementation of this idea is in Fig. 5.3. This way, the derivative control action ignores changes in the reference and just tries to keep the measured variable constant.[2]

---

[1] Some texts use the term "sluggish" here without further qualification. The sluggishness in this case refers to the long settling time, not the initial response.

[2] For review after the chapter on root locus: with the strategy in Fig. 5.3, the closed-loop characteristic polynomial and thus the poles remain the same, but not the zeros. You may also

General qualitative features of derivative control

- PD control is not useful for systems with large dead time or noisy signals.
- The sign of the rate of change in the error could be opposite that of the proportional or integral terms. Thus adding derivative action to PI control may counteract the overcompensation of the integrating action. PD control may improve system response while reducing oscillations and overshoot. (Formal analysis later will show that the problem is more complex than this simple statement.)
- If simple proportional control works fine (in the sense of acceptable offset), we may try PD control. Similarly, we may try PID on top of PI control. The additional stabilizing action allows us to use a larger proportional gain and obtain a faster system response.

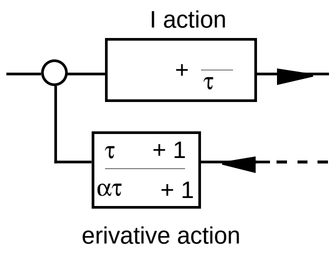


Figure 5.3. Implementation of derivative control on the measured variable.

5.1.4 Proportional-Integral-Derivative (PID) control

Finally, we can put all the components together to make a PID (or 3-mode) controller. The time-domain equation and the transfer function of an "ideal" PID controller are:

$$p'(t) = K_c \left[e'(t) + \frac{1}{\tau_I} \int_0^t e'(t)\,dt + \tau_D \frac{de'}{dt} \right] \tag{5-8a}$$

and

$$G_c(s) = K_c \left[1 + \frac{1}{\tau_I s} + \tau_D s \right] = K_c \frac{\tau_I \tau_D s^2 + \tau_I s + 1}{\tau_I s} \tag{5-8b}$$

We also find rather common that the proportional gain is multiplied into the bracket to give the integral and derivative gains:

$$G_c(s) = K_c + \frac{K_I}{s} + K_D s \tag{5-8c}$$

where $K_I = K_C/\tau_I$, and $K_D = K_C\tau_D$. With a higher order polynomial in the numerator, the ideal PID controller is not considered physically realizable. We nevertheless use this ideal controller in analyses because of the cleaner algebra, and more importantly because we can gain valuable insight with it. You can say the same with the use of the ideal PD controller too.

In real life, different manufacturers implement the "real" PID controller slightly differently.[1] One possibility is to modify the derivative action as

$$G_c(s) = K_c \left[1 + \frac{1}{\tau_I s} + \frac{\tau_D s}{\alpha \tau_D s + 1} \right] = K_c \left[\frac{(\alpha + 1)\tau_D s + 1}{\alpha \tau_D s + 1} + \frac{1}{\tau_I s} \right] \tag{5-9a}$$

wonder how to write the function $G_C(s)$, but it is much easier and more logical just to treat the PI action and derivation action as two function blocks when we analyze a problem.

[1] Not only that, most implementations are based on some form of the digitized control law. An illustration of the positional digital algorithm along with concepts such as bumpless transfer, external rate feedback and bias tracking is in the LabView liquid level simulation module on our *Web Support*.

Another implementation of the actual PID control is to introduce the derivative control in series with PI control:

$$G_c(s) = K_c \left[\frac{\tau_I s + 1}{\tau_I s}\right]\left[\frac{\tau_D s + 1}{\alpha\tau_D s + 1}\right] \qquad (5\text{-}9b)$$

This configuration is also referred to as interacting PID, series PID, or rate-before-reset. To eliminate derivative kick, the derivative lead-lag element is implemented on the measured (controlled) variable in the feedback loop.

5.2 Closed-loop transfer functions

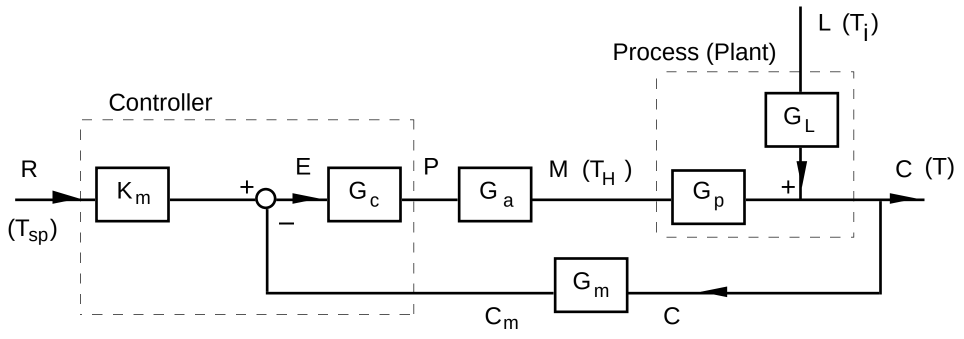


Figure 5.4. Block diagram of a simple SISO closed-loop system.

We first establish the closed-loop transfer functions of a fairly general SISO system. After that, we'll walk through the diagram block by block to gather the thoughts that we must have in synthesizing and designing a control system. An important detail is the units of the physical properties.

5.2.1 Closed-loop transfer functions and characteristic polynomials

Consider the stirred-tank heater again, this time in a closed-loop (Fig. 5.4). The tank temperature can be affected by variables such as the inlet and jacket temperatures and inlet flow rate. Back in Chapter 2, we derived the transfer functions for the inlet and jacket temperatures. In Laplace transform, the change in temperature is given in Eq. (2-49b) on page 2-25 as

$$T(s) = G_L(s)T_i(s) + G_p(s)T_H(s) \qquad (5\text{-}10)$$

This is our process model. Be careful with the context when we use the wording "input." The inlet and jacket temperatures are the inputs to the process, but they are not necessarily the inputs to the system. One of them will become the manipulated variable of the system.

In a SISO system, we manipulate only one variable, so we must make a decision. Since our goal is to control the tank temperature, it would be much more sensible to manipulate the steam temperature T_H instead of the inlet temperature. We can arrive at this decision with physical intuition, or we can base it on the fact that from Chapter 2, the steam temperature has a higher process gain. Hence with respect to the control system, we choose T_H as the **manipulated variable** (M), which is governed by the actuator function G_a and controller signal P. The tank temperature T is the system output (also the controlled variable C). The system input is the set point T_{sp} (or reference R)—our desired steady state tank temperature. [1]

There are other system inputs that can affect our closed-loop response, and we consider them load (or disturbance) variables. In this case, the load variable is the inlet temperature, T_i. Now you may understand why we denote the two transfer functions as G_p and G_L. The important point is that input means different things for the process and the closed-loop system.

For the rest of the control loop, G_c is obviously the controller transfer function. The measuring device (or transducer) function is G_m. While it is not shown in the block diagram, the steady state gain of G_m is K_m. The key is that the summing point can only compare quantities with the same units. Hence we need to introduce K_m on the reference signal, which should have the same units as C. The use of K_m, in a way, performs unit conversion between what we "dial in" and what the controller actually uses in comparative tests. [2]

The next order of business is to derive the closed-loop transfer functions. For better readability, we'll write the Laplace transforms without the *s* dependence explicitly. Around the summing point, we observe that

$$E = K_m R - G_m C$$

Based on the process, we have

$$C = G_p\,(G_a G_c E) + G_L L$$

Substitution for E, the equation becomes

$$C = G_p G_a G_c\,(K_m R - G_m C) + G_L L$$

This step can be rearranged to give

$$C \;=\; \left[\frac{K_m G_c G_a G_p}{1 + G_m G_c G_a G_p}\right] R + \left[\frac{G_L}{1 + G_m G_c G_a G_p}\right] L \;=\; G_{sp} R + G_{load} L \quad , \tag{5-11}$$

which provides us with the closed-loop transfer functions G_{sp} and G_{load}. Based on Eq. (5-11), the inputs to the system are the reference R and the load variable L; the controlled variable is the system output. The first transfer function G_{sp} accounts for the effect of a set point change, and is also called the command tracking function. The second function G_{load} accounts for the effect of changes in disturbance.

The important point is that the dynamics and stability of the **system** are governed by the **closed-loop characteristic polynomial**:

$$1 + G_m G_c G_a G_p \;=\; 0 \tag{5-12}$$

which is the *same* whether we are looking at set point or disturbance changes. As an abbreviation, many books write $G_{OL} = G_m G_c G_a G_p$ and refer to it as the open-loop transfer function as if the

1 There are no standard notations. We could have used Y in place of C for system output. Or replaced G_a by G_v for valve (G_f is also used), G_L and L by G_d and D for disturbance, and G_m by G_T for transducer. We have selected P to denote controller output, more or less for pneumatic.

2 Many texts, especially those in electrical engineering, ignore the need for K_m and the final result is slightly different. They do not have to check the units since all they deal with are electrical signals.

loop were disconnected.[1] We may also refer to $G_cG_aG_p$ as the forward loop transfer function.

Our analyses of SISO systems seldom take into account simultaneous changes in set point and load.[2] We denote the two distinct possibilities as

(1) **Servo** problems: Consider changes in set point with no disturbance (L = 0); $C = G_{sp}R$. Ideally (meaning unlikely to be encountered in reality), we would like to achieve perfect tracking of set point changes: C = R. *Reminder*: we are working with deviation variables.

(2) **Regulator** problems: Consider changes in disturbance with a fixed set point (R = 0); $C = G_{load}L$. The goal is to reject disturbances, *i.e.*, keep the system output at its desired value in spite of load changes. Ideally, we would like to have C = 0, *i.e.*, perfect disturbance rejection.

5.2.2 How do we choose the controlled and manipulated variables?

In homework problems, by and large, the variables are stated. Things will be different when we are on the job. Here are some simple ideas on how we may make the decision:

Choice of controlled variables:

- Those that are dictated by the problem. (For instance, temperature of a refrigerator.)
- Those that are not self-regulating.
- Those that may exceed equipment or process constraints.
- Those that may interact with other variables. (For example, reactor temperature may affect product yield.)

Choice of manipulated variables:

- Those that have a direct effect on the process, especially the output variable.
- Those that have a large steady state gain. (Good sensitivity)
- Those that have no dead time.
- Those that have no interaction with other control loops.

After we have chosen the controlled and manipulated variables, the remaining ones are taken as load variables in a SISO system.

---

[1] Can an open-loop be still a loop? You may wonder what is an open-loop? Often, we loosely refer elements or properties of part of a system as open-loop, as opposed to a complete closed-loop system. You’ll see more of this language in Chapter 7.

[2] In real life, we expect probable simultaneous reference and disturbance inputs. As far as analysis goes, the mathematics is much simpler if we consider one case at a time. In addition, either case *shares the same closed-loop characteristic polynomial*. Hence they should also share the same stability and dynamic response characteristics. Later when we talk about integral error criteria in controller design, there are minor differences, but not sufficient to justify analyzing a problem with simultaneous reference and load inputs.

5.2.3 Synthesis of a single-loop feedback system

We now walk through the stirred-tank heater system once again. This time, we'll take a closer look at the transfer functions and the units (Fig. 5.5).

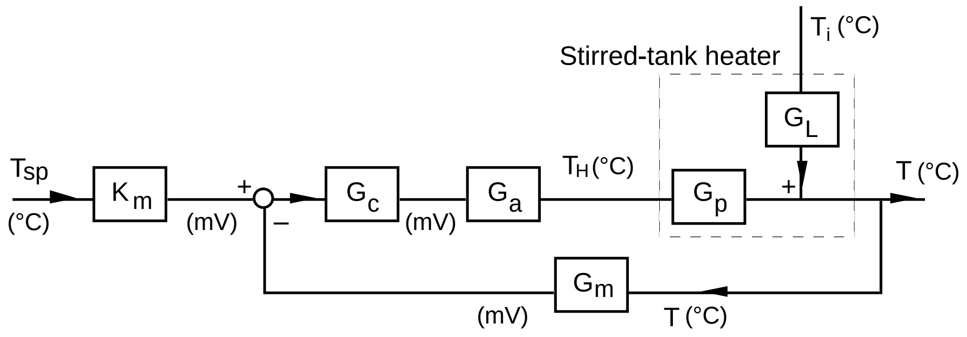


Figure 5.5. Block diagram of a simple SISO closed-loop system with physical units.

◊ Process model

The first item on the agenda is "process identification." We either derive the transfer functions of the process based on scientific or engineering principles, or we simply do a step input experiment and fit the data to a model. Either way, we need to decide what is the controlled variable, which is also the measured variable. We then need to decide which should be the manipulated variable. All remaining variables are delegated to become disturbances.

With the stirred-tank heater, we know quite well by now that we want to manipulate the heating coil temperature to control the tank temperature. The process function G_p is defined based on this decision. In this simple illustration, the inlet temperature is the only disturbance, and the load function is defined accordingly. From Section 2.8.2 and Eq. (2-49b) on page 2-25, we have the first order process model:

$$T = G_L\,T_i + G_p\,T_H = \left(\frac{K_L}{\tau_p s + 1}\right) T_i + \left(\frac{K_p}{\tau_p s + 1}\right) T_H \qquad (5\text{-}13)$$

From that same section, we know that the steady state gain and the time constant are dependent on the values of flow rate, liquid density, heat capacity, heat transfer coefficient, and so on. For the sake of illustration, we are skipping the heat transfer analysis. Let's presume that we have done our homework, substituted in numerical values, and we found $K_p = 0.85$ °C/°C, and $\tau_p = 20$ min.

◊ Signal transmitter

Once we know what to control, we need to find a way to measure the quantity. If the transducer (sensor and transmitter packaged together) is placed far downstream or is too well insulated and the response is slow, the measurement transfer function may appear as

$$\frac{T_m}{T} = G_m = \frac{K_m e^{-t_d s}}{\tau_m s + 1} \qquad (5\text{-}14)$$

where K_m is the measurement gain of the transducer, τ_m is the time constant of the device, and t_d accounts for transport lag. In a worst case scenario, the sensor may be nonlinear, meaning that the measurement gain would change with the range of operating conditions.

With temperature, we can use a thermocouple, which typically has a resolution on the order of 0.05 mV/°C. (We could always use a RTD for better resolution and response time.) That is too small a change in output for most 12-bit analog-digital converters, so we must have an amplifier to boost the signal. This is something we do in a lab, but commercially, we should find off-the-shelf transducers with the sensor and amplifier packaged together. Many of them have a scaled output of, for example, 0-1 V or 4-20 mA.

For the sake of illustration, let’s presume that the temperature transmitter has a built-in amplifier which allows us to have a measurement gain of K_m = 5 mV/°C. Let’s also presume that there is no transport lag, and the thermocouple response is rapid. The measurement transfer function in this case is simply

$$G_m = K_m = 5 \text{ mV/°C}$$

This so-called measurement gain is really the slope of a calibration curve—an idea that we are familiar with. We do a least squares fit if this curve is linear, and find the tangent at the operating point if the curve is nonlinear.

◊ Controller

The amplified signal from the transmitter is sent to the controller, which can be a computer or a little black box. There is not much we can say about the controller function now, except that it is likely a PID controller, or a software application with a similar interface.

A reminder is that a controller has a front panel with physical units such as °C. (Some also have relative scales of 0-100%.) So when we “dial” a change in the set point, the controller needs to convert the change into electrical signals. That’s why K_m is part of the controller in the block diagram (Fig. 5.5).

◊ Actuator/control valve

Last but not least, designing a proper actuator can create the most headaches. We have to find an actuator that can drive the range of the manipulated variable. We also want the device to have a faster response than the process. After that, we have to find a way to interface the controller to the actuator. A lot of work is masked by the innocent-looking notation G_a.

In the stirred-tank heater example, we need to add several comments. We need to consider **safety**. If the system fails, we want to make sure that no more heat is added to the tank. Thus we want a fail-closed valve—meaning that the valve requires energy (or a positive signal change) to open it. In other words, the valve gain is positive. We can check the thinking as follows: If the tank temperature *drops* below the set point, the error increases. With a positive proportional gain, the controller output will *increase*, hence opening up the valve. If the process plant has a power outage, the valve closes and shuts off the steam. But how can the valve shut itself off without power?

This leads to the second comment. One may argue for emergency power or a spring-loaded valve, but to reduce fire hazards, the industrial nominal practice is to use pneumatic (compressed air driven) valves that are regulated by a signal of 3-15 psi. The electrical signal to and from the controller is commonly 4-20 mA. A current signal is less susceptible to noise than voltage signal over longer transmission distances. Hence in a more applied setting, we expect to find a **current-to-pressure transducer** (I/P) situated between the controller output and the valve actuator.

Finally, we have been sloppy in associating the flow rate of steam with the heating coil temperature. The proper analysis that includes a heat balance of the heating medium is in the Review Problems. To side step the actual calculations, we have to make a few more assumptions for the valve gain to illustrate what we need to do in reality:

(1) Assume that we have the proper amplifier or transducer to interface the controller output with the valve, *i.e.*, converting electrical information into flow rate.

(2) We use a valve with linear characteristics such that the flow rate varies linearly with the opening. [1]

(3) The change in steam flow rate can be "translated" to changes in heating coil temperature.

When the steady state gains of all three assumptions are lumped together, we may arrive at a valve gain K_v with the units of °C/mV. For this illustration, let's say the valve gain is 0.6 °C/mV and the time constant is 0.2 min. The actuator controller function would appear as

$$G_v = \frac{K_v}{\tau_v s + 1} = \frac{0.6\ [°C / mV]}{0.2\ s + 1}$$

The closed-loop characteristic equation of the stirred-tank heater system is hence:

$$1 + G_c G_v G_p G_m \; = \; 1 + G_c \frac{(0.6)\,(0.85)\,(5)}{(0.2s + 1)\,(20s + 1)} \; = \; 0$$

We will not write out the entire closed-loop function C/R, or in this case, T/T_{sp}. The main reason is that our design and analysis will be based on only the characteristic equation. The closed-loop function is only handy to do time domain simulation, which can be computed easily using MATLAB. Saying that, we do need to analysis the closed-loop transfer function for several simple cases so we have a better theoretical understanding.

5.3 Closed-loop system response

In this section, we will derive the closed-loop transfer functions for a few simple cases. The scope is limited by how much sense we can make out of the algebra. Nevertheless, the steps that we go through are necessary to learn how to set up problems properly. The analysis also helps us to better understand why a system may have a faster response, why a system may become underdamped, and when there is an offset. When the algebra is clean enough, we can also make observations as to how controller settings may affect the closed-loop system response. The results generally reaffirm the qualitative statements that we've made concerning the characteristics of different controllers.

The actual work is rather cook book-like:

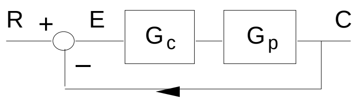


Figure 5.6. Simple unity feedback system.

(1) With a given problem statement, draw the control loop and derive the closed-loop transfer functions.

(2) Pick either the servo or the regulator problem. Reminder: the characteristic polynomial is the same in either case.

(3) With the given choices of G_c (P, PI, PD, or PID), G_p, G_a and G_m, plug their transfer functions into the closed-loop equation. The characteristic polynomial should fall out nicely.

(4) Rearrange the expressions such that we can redefine the parameters as time constants and steady state gains for the closed-loop system.

All analyses follow the same general outline. What we must accept is that there are no handy dandy formulas to plug and chug. We must be competent in deriving the closed-loop transfer function, steady state gain, and other relevant quantities for each specific problem.

[1] In reality, the valve characteristic curve is likely nonlinear and we need to look up the technical specification in the manufacturer's catalog. After that, the valve gain can be calculated from the slope of the characteristic curve at the operating point. See Homework Problem I.33 and the *Web Support*.

In our examples, we will take $G_m = G_a = 1$, and use a servo system with L = 0 to highlight the basic ideas. The algebra tends to be more tractable in this simplified unity feedback system with only G_c and G_p (Fig. 5.6), and the closed-loop transfer function is

$$\frac{C}{R} = \frac{G_c G_p}{1 + G_c G_p} \tag{5-15}$$

which has the closed-loop characteristic equation $1 + G_cG_p = 0$.

✎ **Example 5.1**: Derive the closed-loop transfer function of a system with **proportional control** and a **first order process**. What is the value of the controlled variable at steady state after a unit step change in set point?

In this case, we consider $G_c = K_c$, and $G_p = \dfrac{K_p}{\tau_p s + 1}$, and substitution in Eq. (5-15) leads to [1]

$$\frac{C}{R} = \frac{K_c K_p}{\tau_p s + 1 + K_c K_p}\,. \tag{E5-1}$$

We now divide both the numerator and denominator with $(1 + K_cK_p)$ to obtain

$$\frac{C}{R} = \frac{K_c K_p/(1 + K_c K_p)}{\left[\tau_p/(1 + K_c K_p)\right] s + 1} = \frac{K}{\tau s + 1} \tag{E5-2}$$

where

$$K = \frac{K_c K_p}{1 + K_c K_p} \quad \text{and} \quad \tau = \frac{\tau_p}{1 + K_c K_p}$$

are the closed-loop steady state gain and time constant.

Recall Eq. (5-11), the closed-loop characteristic equation is the denominator of the closed-loop transfer function, and the probable locations of the *closed-loop pole* are given by

$$s = -\,(1 + K_cK_p)/\tau_p.$$

There are two key observations. First, $K < 1$, meaning that the controlled variable will change in magnitude less than a given change in set point, the source of offset. The second is that $\tau < \tau_p$, meaning that the system has a faster response than the open-loop process. The system time constant becomes smaller as we increase the proportional gain. This is consistent with the position of the closed-loop pole, which should "move away" from the origin as K_c increases.

---

[1] You may wonder how transfer functions are related to differential equations. This is a simple illustration. We'll use *y* to denote the controlled variable. The first order process function G_p arises from Eq. (3-6):

$$\tau_p \frac{dy}{dt} + y = K_p x$$

In the unity feedback loop with $G_c = K_c$, we have $x = K_c(r - y)$. Substitution for *x* in the ODE leads to

$$\tau_p \frac{dy}{dt} + y = K_c K_p (r - y)\,, \quad \text{or} \quad \tau_p \frac{dy}{dt} + (1 + K_c K_p)\, y = K_c K_p r$$

It is obvious that (E5-1) is the Laplace transform of this equation. This same idea can be applied to all other systems, but of course, nobody does that. We all work within the Laplace transform domain.

We now take a formal look at the steady state error (offset). Let’s consider a more general step change in set point, R = M/s. The eventual change in the controlled variable, via the final value theorem, is

$$c'(\infty) = \lim_{s \to 0} s \frac{K}{\tau s + 1} \frac{M}{s} = MK$$

The offset is the relative error between the set point and the controlled variable at steady state, *i.e.*, $(r - c_\infty)/r$:

$$e_{ss} = \frac{M - MK}{M} = 1 - K = 1 - \frac{K_c K_p}{1 + K_c K_p} = \frac{1}{1 + K_c K_p} \qquad \text{(E5-3)}$$

We can reduce the offset if we increase the proportional gain.

Let's take another look at the algebra of evaluating the steady state error. The error that we have derived in the example is really the difference between the change in controlled variable and the change in set point in the block diagram (Fig. 5.6). Thus we can write:

$$E = R - C = R\left[1 - \frac{G_c G_p}{1 + G_c G_p}\right] = R\left[\frac{1}{1 + G_c G_p}\right]$$

Now if we have a unit step change R = 1/s, the **steady state error** via the final value theorem is (recall that e = e’)

$$e_{ss} = \lim_{s \to 0} s \frac{1}{1 + G_c G_p} \frac{1}{s} = \frac{1}{1 + \lim\limits_{s \to 0} G_c G_p} = \frac{1}{1 + K_{err}} \ , \quad \text{where } \ K_{err} = \lim_{s \to 0} G_c G_p \qquad \text{(5-16)}$$

We name K_{err} the **position error constant**.[1] For the error to approach zero, K_{err} must approach infinity. In Example 5.1, the error constant and steady state error are

$$K_{err} = \lim_{s \to 0} G_c G_p = \frac{K_c K_p}{\tau_p s + 1} = K_c K_p \ , \ \text{and again } \ e_{ss} = \frac{1}{1 + K_c K_p} \qquad \text{(5-17)}$$

✎ **Example 5.2**: Derive the closed-loop transfer function of a system with **proportional control** and a **second order overdamped process**. If the second order process has time constants 2 and 4 min and process gain 1.0 [units], what proportional gain would provide us with a system with damping ratio of 0.7?

In this case, we consider $G_c = K_c$, and $G_p = \dfrac{K_p}{(\tau_1 s + 1)(\tau_2 s + 1)}$, and substitution in Eq. (5-15) leads to

$$\frac{C}{R} = \frac{K_c K_p}{(\tau_1 s + 1)(\tau_2 s + 1) + K_c K_p} = \frac{K_c K_p/(1 + K_c K_p)}{\left(\dfrac{\tau_1 \tau_2}{1 + K_c K_p}\right) s^2 + \left(\dfrac{\tau_1 + \tau_2}{1 + K_c K_p}\right) s + 1} \qquad \text{(E5-4)}$$

The key is to recognize that the system may exhibit underdamped behavior even though the open-loop process is overdamped. The closed-loop characteristic polynomial can have either real or complex roots, depending on our choice of K_c. (This is much easier to see when we work with

[1] In many control texts, we also find the derivation of the velocity error constant (using $R = s^{-2}$) and acceleration error constant (using $R = s^{-3}$), and a subscript *p* is used on what we call K_{err} here.

root locus later.) For now, we rewrite the closed-loop function as

$$\frac{C}{R} = \frac{K}{\tau^2 s^2 + 2\zeta\tau s + 1} \tag{E5-4a}$$

where the closed-loop steady state gain is $K = \dfrac{K_c K_p}{1 + K_c K_p}$, and the system natural time period and damping ratio are

$$\tau = \sqrt{\frac{\tau_1 \tau_2}{1 + K_c K_p}} \ , \ \text{and} \quad \zeta = \frac{1}{2} \frac{(\tau_1 + \tau_2)}{\sqrt{\tau_1 \tau_2 \, (1 + K_c K_p)}} \tag{E5-5}$$

If we substitute $\zeta = 0.7$, $K_p = 1$, $\tau_1 = 2$ and $\tau_2 = 4$ in the second expression, we should find the proportional gain K_c to be 1.29.

Lastly, we should see immediately that the system steady state gain in this case is the same as that in Example 5.1, meaning that this second order system will have the same steady state error.

In terms of controller design, we can take an entirely analytical approach when the system is simple enough. Of course, such circumstances are not common in real life. Furthermore, we often have to compromise between conflicting criteria. For example, we cannot require a system to have both a very fast rise time and a very short settling time. If we want to provide a smooth response to a set point change without excessive overshoot, we cannot also expect a fast and snappy initial response. As engineers, it is our job to decide.

In terms of design specification, it is not uncommon to use decay ratio as the design criterion. Repeating Eq. (3-29), the decay ratio DR (or the overshoot OS) is a function of the damping ratio:

$$DR = (OS)^2 = \exp\left(\frac{-2\pi\zeta}{\sqrt{1-\zeta^2}}\right) \tag{5-18}$$

We can derive from this equation

$$\zeta^2 = \frac{(\ln DR)^2}{4\pi^2 + (\ln DR)^2} \tag{5-19}$$

If we have a second order system, we can derive an analytical relation for the controller. If we have a proportional controller with a second order process as in Example 5.2, the solution is unique. However, if we have, for example, a PI controller (2 parameters) and a first order process, there are no unique answers since we only have one design equation. We must specify one more design constraint in order to have a well-posed problem.

✎ **Example 5.3**: Derive the closed-loop transfer function of a system with **proportional-integral control** and a **first order process**. What is the offset in this system?

We substitute $G_c = K_c\left(\dfrac{\tau_I s + 1}{\tau_I s}\right)$, and $G_p = \dfrac{K_p}{\tau_p s + 1}$ in Eq. (5-15), and the closed-loop servo transfer function is

$$\frac{C}{R} = \frac{K_c K_p (\tau_I s + 1)}{\tau_I s\,(\tau_p s + 1) + K_c K_p (\tau_I s + 1)} = \frac{(\tau_I s + 1)}{\left(\dfrac{\tau_I \tau_p}{K_c K_p}\right) s^2 + \dfrac{\tau_I (1 + K_c K_p)}{K_c K_p}\, s + 1} \tag{E5-6}$$

There are two noteworthy items. First, the closed-loop system is now second order. The integral action adds another order. Second, the system steady state gain is unity and it will not have an offset. This is a general property of using PI control. (If this is not immediately obvious, try take R = 1/s and apply the final value theorem. We should find the eventual change in the controlled variable to be c'(∞) = 1.)

With the expectation that the second order system may exhibit underdamped behavior, we rewrite the closed-loop function as

$$\frac{C}{R} = \frac{(\tau_I s + 1)}{\tau^2 s^2 + 2\zeta\tau s + 1} \tag{E5-6a}$$

where the system natural time period and damping ratio are

$$\tau = \sqrt{\frac{\tau_I \tau_p}{K_c K_p}}\ ,\ \text{and}\quad \zeta = \frac{1}{2}(1 + K_c K_p)\sqrt{\frac{\tau_I}{K_c K_p \tau_p}} \tag{E5-7}$$

While we have the analytical results, it is not obvious how choices of integral time constant and proportional gain may affect the closed-loop poles or the system damping ratio. (We may get a partial picture if we consider circumstances under which K_cK_p » 1.) Again, we'll defer the analysis to when we cover root locus. We should find that to be a wonderful tool in assessing how controller design may affect system response.

✎ **Example 5.4**: Derive the closed-loop transfer function of a system with **proportional-derivative control** and a **first order process**.

The closed-loop transfer function (5-15) with $G_c = K_c(1 + \tau_D s)$ and $G_p = \dfrac{K_p}{\tau_p s + 1}$ is

$$\frac{C}{R} = \frac{K_c K_p (\tau_D s + 1)}{(\tau_p s + 1) + K_c K_p (\tau_D s + 1)} = \frac{K_c K_p (\tau_D s + 1)}{(\tau_p + K_c K_p \tau_D) s + 1 + K_c K_p} \tag{E5-8}$$

The closed-loop system remains first order and the function is that of a lead-lag element. We can rewrite the closed-loop transfer function as

$$\frac{C}{R} = \frac{K(\tau_D s + 1)}{\tau s + 1} \tag{E5-8a}$$

where the system steady state gain and time constant are

$$K = \frac{K_c K_p}{1 + K_c K_p}\ \text{ and } \tau = \frac{\tau_p + K_c K_p \tau_D}{1 + K_c K_p}\ .$$

The system steady state gain is the same as that with proportional control in Example 5.1. We, of course, expect the same offset with PD control too. The system time constant depends on various parameters. Again, we defer this analysis to when we discuss root locus.

✎ **Example 5.5**: Derive the closed-loop transfer function of a system with **proportional control** and **an integrating process**. What is the offset in this system?

Let's consider $G_c = K_c$, and $G_p = 1/As$, and substitution in Eq. (5-15) leads to

$$\frac{C}{R} = \frac{K_c}{A s + K_c} = \frac{1}{(A/K_c)s + 1} \tag{E5-9}$$

We can see quickly that the system has unity gain and there should be no offset. The point is that integral action can be introduced by the process and we do not need PI control under such circumstances. We come across processes with integral action in the control of rotating bodies and liquid levels in tanks connected to pumps (Example 3.1, p. 3-4).

✎ **Example 5.6**: Provide illustrative closed-loop time response simulations. Most texts have schematic plots to illustrate the general properties of a feedback system. This is something that we can do ourselves using MATLAB. Simulate the observations that we have made in previous examples. Use a unity feedback system.

Consider Example 5.3 again and let's pick τ_p to be 5 min, K_p be 0.8 [unit]. Instead of using the equation that we derived in Example 5.3, we can use the following statements in MATLAB to generate a simulation for the case of a unit step change in the set point. This approach is much faster than using Simulink.

```
kc=1;     %The two tuning parameters to be varied
taui=10;

% The following statements are best saved in an M-file

Gc=tf(kc*[taui 1],[taui 0]);   % The PI controller function
Gp=tf(0.8,[5 1]);              % The process function
Gcl=feedback(Gc*Gp,1)     % Unity closed loop function GcGp/(1 + GcGp)

step(Gcl);                     % Personalize your own plotting
                               % and put a hold for additional curves
```

In these statements, we have used `feedback()` to generate the closed-loop function C/R. The unity feedback loop is indicated by the "1" in the function argument. Try first with $K_c = 1$, and τ_I with values 10, 1, and 0.1. Next, select $\tau_I = 0.1$, and repeat with $K_c = 0.1$, 1, 5, and 10. In both cases, the results should follow the qualitative trends that we anticipate. If we repeat the calculation with a larger integral time $\tau_I = 1$, and use $K_c = 0.1$, 1, 5, 10, and 50, you may find the results to be rather unexpected. However, we do not have enough theory to explain them now. Keep the results in mind and hopefully this is a motivation to explore the later chapters.

We could also modify the M-file by changing the PI controller to a PD or PID controller to observe the effects of changing the derivative time constant. (Help is in MATLAB Session 5.) We'll understand the features of these dynamic simulations better when we cover later chapters. For now, the simulations should give us a qualitative feel on the characteristics of a PID controller and (hopefully) also the feeling that we need a better way to select controller settings.

✎ **Example 5.7**: We have to design a servo-controller for a mixing system. A blue dye for making denim is injected into a stream of water. The injected dye is blended into the pipe flow with the aid of *in situ* static mixers. A photodetector downstream is used to monitor the dye concentration. The analog output of the detector is

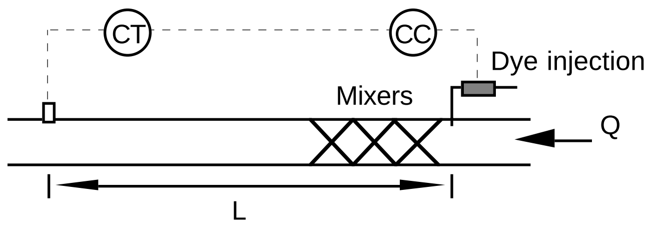


Figure E5.7a

transmitted to a controller, which in turn sends a signal to the dye injection regulating valve. In designing this mixing system, we have to be careful with the location of the photodetector. It has to be downstream enough to ensure a good reading of the mixed stream. However, if the photodetector is too far downstream, the larger transport lag can destabilize the system.

The water flow rate in the pipe is 2 L/s and the pipe has a cross-sectional area of 5 cm^2. The regulating valve is especially designed so that the dye dosage in mL/s varies linearly with the valve position. The regulating valve is thus first order with a time constant of 0.2 s and a steady state gain of 0.6 mL·s^{-1}mV^{-1}. The mixing process itself can also be modeled as first order with a steady state gain of 0.8 ppm·s·mL^{-1}. A previous experiment indicated that a step change in the regulating valve resulted in a response in dye concentration that is 99.3% complete in 20s. The magic photodetector is extremely fast and the response is linear over a large concentration range. The manufacturer provided the calibration as

$$v = 0.3 + 2.6\ [\text{dye}],$$

where the voltage output is in mV and the concentration of the dye is in ppm.

This is a problem that we have to revisit many times in later chapters. For now, draw the block diagram of the dye control system and provide the necessary transfer functions. Identify units in the diagram and any possible disturbances to the system. In addition, we do not know where to put the photodetector at this point. Let's just presume that the photodetector is placed 290 cm downstream.

The block diagram is shown in Fig. E5.7b, where the dye concentration is denoted by C and the set point by R. The flow rate is one probable source of disturbance.

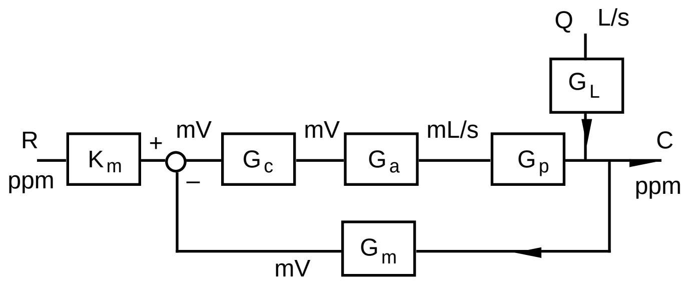


Figure E5.7b

Based on the information given, the transfer functions are

$$G_p = \frac{K_p}{\tau_p s + 1}\,,\ G_v = \frac{K_v}{\tau_v s + 1}\,,\ G_m = K_m e^{-t_d s}\,,$$

and we do not know the steady state gain of the load function G_L. The values of various parameters are K_p = 0.8 ppm·s·mL^{-1}, $\tau_p \approx 20/5 = 4$ s, K_v = 0.6 mL·s^{-1}mV^{-1}, τ_v = 0.2 s, and K_m = 2.6 mV/ppm. The average fluid velocity is 2,000/5 = 400 cm/s. The transport lag is hence t_d = 290/400 = 0.725 s. We presumably will use a PID transfer function for the controller G_C. We'll continue with this problem later.

5.4 Selection and action of controllers

We need to add a few words on the action of controllers. The point to make is that we have to do a bit of physical reasoning when we design a real system. We also have to factor in safety and determine what the controller and actuator may do if there is a system failure—a task that is often omitted in textbook problems.

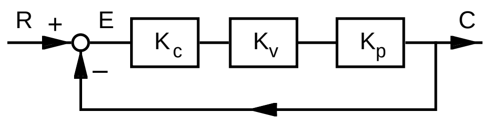


Figure 5.7. Simple system used in the discussion of controller actions. Depending on the situation, K_c, K_v, and K_p can be either positive or negative.

A standard textbook system has a controller with a positive proportional gain. All the other blocks such as the process and actuator have positive steady state gains as well. However, this is not always the case. We will use liquid level control to illustrate the idea. Keep Fig. 5.7 in mind in the discussion below.

Say we want to control the liquid level in a tank by manipulating the *inlet* flow rate (Fig. 5.8). If the liquid level *drops* below the set point, the controller will *increase* its output signal to open up the inlet valve and increase liquid flow. The changes in controlled variable and controller output are in opposite directions. This is a consequence of how the error is defined in a negative feedback system.

In this particular case, we use an **air-to-open** valve, meaning that we need to increase the signal to open up the valve. That is, the valve has a *positive* steady state gain ($+K_v$). A pneumatic air-to-open valve also means that energy is required to keep it open. Under a system failure where power is lost, the valve closes and prevents flooding the tank. We refer to the valve here as a **fail-closed** valve, which is the preferred safety design in Fig. 5.8.

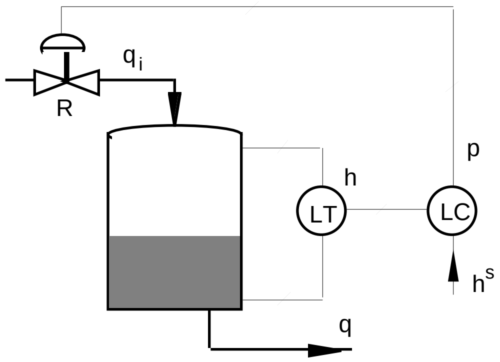


Figure 5.8. Manipulate liquid level with an inlet valve.

The opposite scenario is to use an **air-to-close** valve which has a *negative* steady state gain ($–K_v$); an increase in signal will close the valve.[1] Hence this is a **fail-open** valve, which for safety reasons, is not a wise choice here. Nevertheless, if we insist on installing this air-to-close valve, we will need a controller with a negative gain ($–K_c$). Now if the liquid level *drops*, the controller output signal will also *decrease*, opening up the air-to-close valve.

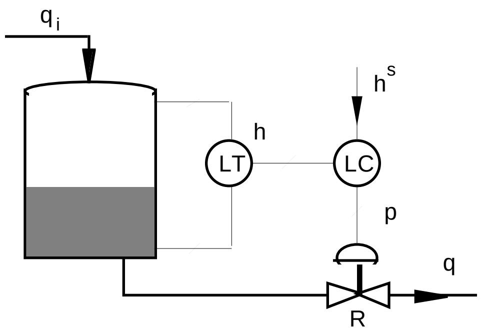


Figure 5.9. Manipulate liquid level with an outlet valve. The process gain is negative in this case.

Let's take a look at the logic when we control the liquid level by manipulating the *outlet* valve (Fig. 5.9). In this case the process gain K_p associated with the outlet flow is negative. If the liquid level drops below the set

[1] This point can easily get lost in the long explanation: An air-to-open valve has a positive gain and is failed-closed. An air-to-close valve has a negative gain ($–K_v$) and is failed-open.

point, we now want to reduce the outlet flow rate by closing up the valve. Again, there are two possible scenarios.

If we use a controller with positive gain ($+K_c$), the controller output increases as the liquid level drops. We can only reduce the flow if we use an air-to-close valve ($-K_v$). In the case of a power outage, the valve will stay open. This fail-open valve can drain the entire tank, an event that we may not want to happen.

On the other hand, we can choose an air-to-open valve ($+K_v$) at the outlet location. Now the only way to reduce the flow rate as the liquid level drops is to "switch" to a controller with a negative gain ($-K_c$). With $-K_c$ as the proportional gain, a drop in liquid level will lead to a decrease in controller output. In this case, we have a fail-closed valve, which is desirable if we do not want perturbations to propagate downstream.

There is no question that the terminology is confusing. Do not let it confuse you. The best strategy is to "walk" through the sign (action) of every single steady state gain of the block diagram, including the process and the sensor, and determine what are the probable and logical signs (actions) of the controller and actuator. As a consistency check, we should note that within the feedback loop, there should only be one net negative sign. There is no getting away from doing some thinking of your own.

5.4.1 A few comments on the choice of controllers

In chemical processes, the most common types of controlled variables are liquid level, flow rate, temperature, pressure, and sometimes concentration. Here are some very general ideas. To fully appreciate these tidbits, we also need to know something about the actual hardware—actuators or control elements—which we find in handbooks or equipment manuals.

Flow control

PI controllers are most common. They eliminate offsets and have acceptable speeds of response in most industrial settings. We usually pick a low to intermediate gain (wide proportional band, $PB \approx 150$) to reduce the effect of noisy signals (from flow turbulence; also why we do not use D control). We also use a low reset time (≈ 0.1 min/repeat; *i.e.* relatively large I action) to get fast set-point tracking.

Level control

We usually only need to keep the liquid level within a certain range around the desired set-point. Speed is not a great concern. Recall that depending on how we implement the outlet pump, we can have a process with integrating action itself. Also, if the outlet flow rate is used as the manipulated variable, the controller setting must be conservative to avoid sudden surges in the exit flow rate. Thus a simple P controller is usually adequate, presuming that we have no complications such as boiling or vaporization. Make sure you check whether the valve that you are using (or buying) is air-to-open or air-to-close.

Pressure control

The control of pressure depends on the situation and cannot be generalized. A couple of examples are:

For the vapor pressure in a flash drum (and thus also vapor flow rate), we need a fast and tight response loop. We need at least a PI controller (c.f. the flow control).

For the top of a distillation column, we usually control the pressure indirectly via the condensation of vapor in the condenser, which in turn is controlled by the amount of cooling water. Heat transfer through a heat exchanger has very slow dynamics. Thus we cannot use PI control. We either use P control, or when response time is important, use PID.

Temperature control

Heat transfer lags can be significant and the nature of the problem can be quite different in various processes. If there is a sensor lag, it is mostly due to heat transfer between the sensor and the fluid medium. (Thermocouples, depending on how we make them, can have very fast response times.) The overall response is sluggish and PI control will make it more so. It is unlikely we can live with any offsets. PID control is the appropriate choice.

Concentration control

The strategy depends on the situation and how we measure the concentration. If we can rely on pH or absorbance (UV, visible, or Infrared spectrometer), the sensor response time can be reasonably fast, and we can make our decision based on the actual process dynamics. Most likely we would be thinking along the lines of PI or PID controllers. If we can only use gas chromatography (GC) or other slow analytical methods to measure concentration, we must consider discrete data sampling control. Indeed, prevalent time delay makes chemical process control unique and, in a sense, more difficult than many mechanical or electrical systems.

In terms of the situation, if we use a PI controller on a slow multi-capacity process, the resulting system response will be even more sluggish. We should use PID control to increase the speed of the closed-loop response (being able to use a higher proportional gain) while maintaining stability and robustness. This comment applies to other cases such as temperature control as well.

❒ Review Problems

1. An alternate and actually very common way of drawing a feedback system block diagram is shown in Fig. R5.1. How is G_L^* related to G_L as in Fig. 5.4?

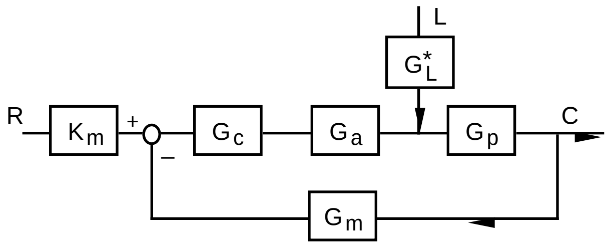


Figure R5.1

2. Try to obtain the closed-loop transfer functions in Eq. (5-11) via observation, *i.e.*, without using the algebraic steps in the text.

3. Consider the liquid flow rate controller in Fig R5.3. We want to keep the flow rate q constant no matter how the upstream pressure fluctuates. Consider if the upstream flow Q drops below the steady state value. How would you choose the regulating valve when you have (a) a positive and (b) a negative proportional gain?

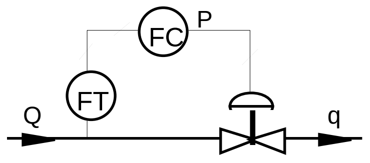


Figure R5.3

4. What is the overshoot and decay ratio if we pick ζ = 0.707? If the decay ratio is 1/4, what is the damping ratio?

5. Refer back to Example 5.1. What is the offset if we consider the regulating problem (R = 0, L = 1/s)?

6. When we developed the model for the stirred tank heater, we ignored the dynamics of the heating coil. Provide a slightly more realistic model which takes into consideration the flow rate of condensing steam.

7. Do the time domain simulations with MATLAB in Example 5.6. Try also with a PD or PID controller.

Hints:

1. $G_L^* = G_L/G_p$.

2. The G_{sp} function is obvious. To obtain the load function G_{load}, set R = 0, and try to visualize the block diagram such that it is unity in the forward path and all the functions are in the feedback loop.

4. OS = 4.32% and DR = 1.87×10^{-3}. When DR = 0.25, $\zeta = 0.215$.

5. Now, $C = \frac{G_L}{1 + G_c G_p} L$, R = 0, and thus $E = R - C = 0 - \frac{G_L}{1 + G_c G_p} L$. With L = 1/s and the final value theorem, $e(\infty) = -\lim_{s \to 0} \frac{G_L}{1 + G_c G_p}$. Substitution of the first order functions and a proportional controller gives $e(\infty) = -\frac{K_L}{1 + K_c K_p}$ which becomes smaller if we increase K_c.

6. How we model the stirred tank heater is subject to the actual situation. At a slightly more realistic level, we may assume that heat is provided by condensing steam and that the coil metal is at the same temperature as the condensing steam. The heat balance and the Laplace transform of the tank remains identical to Chapter 2:

$$\rho C_p V \frac{dT}{dt} = \rho C_p Q\,(T_i - T) + UA\,(T_H - T)\,, \quad \text{and} \quad T = \left[\frac{K_d}{\tau_p s + 1}\right] T_i + \left[\frac{K_p}{\tau_p s + 1}\right] T_H$$

We also need a heat balance for the heating coil, which can be written as

$$M_H C_H \frac{dT_H}{dt} = m_s \lambda - UA\,(T_H - T)$$

where T_H is the temperature, and M_H and C_H are the mass and heat capacity of the heating coil. The steam mass flow rate is m_s, and λ is the heat of condensation. We should obtain the Laplace transform of the form

$$T_H = \left[\frac{1}{\tau_H s + 1}\right] T + \left[\frac{K_S}{\tau_H s + 1}\right] M_S$$

You should be able to fill in the gaps and finish the rest of the work in deriving the transfer functions. In this case, we may want to use the steam mass flow rate as the manipulated variable. The transfer function relating its effect on T will be second order, and the characteristic polynomial does not have the clean form in simpler textbook examples.

7. The basic statements are provided already in the example. For more details, see our *Web Support* for the MATLAB statements and plots.

❖ 6. *Design and Tuning of Single-Loop Control Systems*

We will go through a whole bundle of tuning methods. We only need to "pick" three numbers for a PID controller, but this is one of the most confusing parts of learning control. Different tuning techniques give similar but not identical results. There are no “best” or “absolutely correct” answers. The methods all have pros and cons, and working together, they complement each other. We need to make proper selection and sound judgment—very true to the act (and art) of design.

What are we up to?

- Tune a controller with empirical relations
- Tune a controller with internal model control relations

6.1 Tuning controllers with empirical relations

Let’s presume that we have selected the valves, transducers and even installed a controller. We now need to determine the controller settings—a practice which we call tuning a controller. Trial-and-error tuning can be extremely time consuming (and dumb!), to the extent that it may not be done. A large distillation column can take hours to reach steady state. A chemical reactor may not reach steady state at all if you have a reactor "runaway." Some systems are unstable at high and low feedback gains; they are stable only in some intermediate range. These are reasons why we have to go through all the theories to learn how to design and tune a controller with well educated (or so we hope) guesses.

Empirical tuning roughly involves doing either an open-loop or a closed-loop experiment, and fitting the response to a model. The controller gains are calculated on the basis of this fitted function and some empirical relations. When we use empirical tuning relations, we cannot dictate system dynamic response specifications. The controller settings are seldom optimal and most often require field tuning after installation to meet more precise dynamic response specifications. Empirical tuning may not be appealing from a theoretical viewpoint, but it gives us a quick-and-dirty starting point. Two remarks before we begin.

- Most empirical tuning relations that we use here are based on open-loop data fitted to a **first order with dead time** transfer function. This feature is unique to process engineering where most units are self-regulating. The dead time is either an approximation of multi-stage processes or a result of transport lag in the measurement. With large uncertainties and the need for field tuning, models more elaborate than the first order with dead time function are usually not warranted with empirical tuning.
- Some empirical tuning relations, such as Cohen and Coon, are developed to achieve a one-quarter decay ratio response in handling disturbances. When we apply the settings of these relations to a servo problem, it tends to be very oscillatory and is not what one considers as slightly underdamped.[1] The controller design depends on the specific problem at hand. We certainly need to know how to tune a controller after using empirical tuning relations to select the initial settings.[2]

---

[1] If we assume that an oscillatory system response can be fitted to a second order underdamped function. With Eq. (3-29), we can calculate that with a decay ratio of 0.25, the damping ratio ζ is 0.215, and the maximum percent overshoot is 50%, which is *not* insignificant. (These values came from Revew Problem 4 back in Chapter 5.)

[2] By and large, a quarter decay ratio response is acceptable for disturbances but not desirable for set point changes. Theoretically, we can pick any decay ratio of our liking. Recall Section 2.7 (p. 2-17) that the position of the closed-loop pole lies on a line governed by $\theta = \cos^{-1}\zeta$. In the next chapter, we will locate the pole position on a root locus plot based on a given damping ratio.

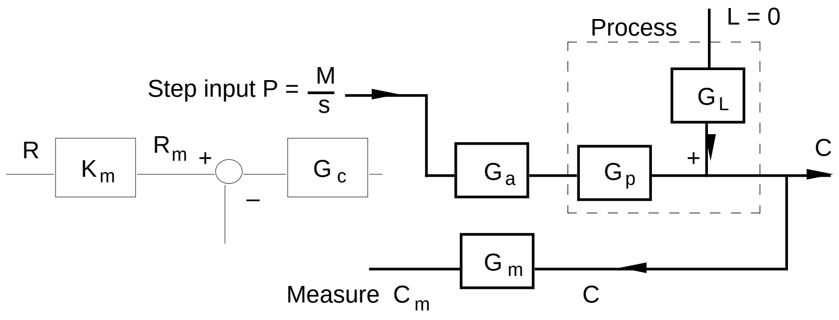


Figure 6.1. Block diagram illustration of an open-loop step test.

6.1.1 Controller settings based on process reaction curve

To make use of empirical tuning relations, one approach is to obtain the so-called *process reaction curve*. We disable the controller and introduce a step change to the actuator. We then measure the **open-loop step response**. This practice can simply be called an open-loop step test. Although we disconnect the controller in the schematic diagram (Fig. 6.1), we usually only need to turn the controller to the “manual” mode in reality. As shown in the block diagram, what we measure is a lumped response, representing the dynamics of the blocks G_a, G_p, and G_m. We denote the lumped function as G_{PRC}, the process reaction curve function:

$$G_{PRC} = \frac{C_m}{P} = G_a\, G_p\, G_m \tag{6-1}$$

From the perspective of doing the experiment, we need the actuator to effect a change in the manipulated variable and the sensor to measure the response.

The measurement of G_{PRC} is how we may design a system if we know little about our process and are incapable of constructing a model (What excuse!). Even if we know what the functions G_a and G_m should be, we do not need them since the controller empirical tuning relations were developed for the lumped function G_{PRC}. On the other hand, if we know precisely what the functions G_a, G_p and G_m are, we may use them to derive G_{PRC} as a reduced-order approximation of the product of $G_aG_pG_m$.

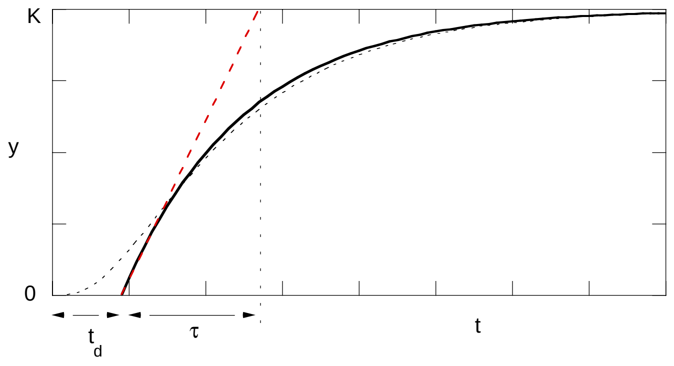


Figure 6.2. Illustration of fitting Eq. (6-2, solid curve) to open-loop step test data representative of self-regulating and multi-capacity processes (dotted curve). The time constant estimation shown here is based on the initial slope and a visual estimation of dead time. The Ziegler-Nichols tuning relation (Table 6.1) also uses the slope through the inflection point of the data (not shown). Alternative estimation methods are provided on our *Web Support*.

The real time data (the process reaction curve) in most processing unit operations take the form of a sigmoidal curve, which is fitted to a first order with dead time function (Fig. 6.2):[1]

$$G_{PRC} = \frac{C_m}{P} \approx \frac{Ke^{-t_d s}}{\tau s + 1} \tag{6-2}$$

One reason why this approximation works is that process unit operations are generally open-loop stable, and many are multi-capacity in nature. Reminder: Underdamped response of the system is due to the controller, which is taken out in the open-loop step test.

Using the first order with dead time function, we can go ahead and determine the controller settings with empirical tuning relations. The most common ones are the **Ziegler-Nichols** relations. In process unit operation applications, we can also use the **Cohen and Coon** or the **Ciancone and Marlin** relations. These relations are listed in the Table of Tuning Relations (Table 6.1).

6.1.2 Minimum error integral criteria

The open-loop test response fitted to a first order with dead time function G_{PRC} can be applied to other tuning relations. One such possibility is a set of relations derived from the minimization of error integrals. Here, we just provide the basic idea behind the use of error integrals.

To derive the tuning equations, we would use the theoretical time-domain closed-loop system response as opposed to a single quantity, such as the decay ratio. The time-domain solution is dependent on the type of controller and the nature of input (set point or disturbance changes) and, in our case, a “process” function that is first order with dead time. We can also calculate the error—the difference between the set point and the controlled variable. We then find controller settings which may minimize the error over time (the error integral), using for instance, Lagrange multipliers as in introductory calculus. Of course, we are not doing the work; the actual analysis is better left for a course in optimal control.

There are different ways to define the error function to be minimized. A few possibilities are as follows:

(1) Integral of the square error (ISE)

$$\text{ISE} = \int_0^\infty \left[e'(t)\right]^2 dt \tag{6-3}$$

The ISE magnifies large errors—squaring a small number (< 1) makes it even smaller. Thus minimization of this integral should help to suppress large, initial errors. The resulting controller setting tends to have a high proportional gain and the system is very underdamped.

(2) Integral of the absolute error (IAE)

$$\text{IAE} = \int_0^\infty \left| e'(t) \right| dt \tag{6-4}$$

The IAE simply integrates the absolute value and puts equal weight to large and small errors.

(3) Integral of time-weighted absolute error (ITAE)

[1] The first order function with dead time is only appropriate for self-regulating and multi-capacity processes. In other controller design methods, we should choose different functions to fit the open-loop test response.

$$\text{ITAE} = \int_0^\infty t\,|e'(t)|\,dt \tag{6-5}$$

The time weighting function puts a heavy penalty on errors that persist for long periods of time. This weighting function also helps to derive controller settings which allow for low settling times.

Before we move on, a few comments and reminders:

- As far as we are concerned, using the error integral criteria is just another empirical method. We are simply using the results of minimization obtained by other people, not to mention that the first order with dead time function is from an open-loop test.
- The controller setting is different depending on which error integral we minimize. Set point and disturbance inputs have different differential equations, and since the optimization calculation depends on the time-domain solution, the result will depend on the type of input. The closed-loop poles are the same, but the zeros, which affect the time-independent coefficients, are not.
- The time integral is from $t = 0$ to $t = \infty$, and we can only minimize it if it is bounded. In other words, we cannot minimize the integral if there is a steady state error. Only PI and PID controllers are applicable to this design method.[1]
- Theoretically, we can minimize the integral using other criteria. On the whole, the controller settings based on minimizing ITAE provide the most conservative controller design, and are highly recommended. This is the only set of tuning relations included in Table 6.1.

6.1.3 Ziegler-Nichols ultimate-cycle method

This empirical method is based on **closed-loop testing** (also called on-line tuning) of processes which are inherently stable, but where the system may become unstable. We use only proportional control in the experiment. If it is not possible to disable the integral and derivative control modes, we set the integral time to its maximum value and the derivative time to its minimum. The proportional gain is slowly increased until the system begins to exhibit sustained oscillations with a given small step set point or load change. The proportional gain and period of oscillation at this point are the **ultimate gain**, K_{cu}, and **ultimate period**, T_u. These two quantities are used in a set of empirical tuning relations developed by Ziegler and Nichols—again listed in Table 6.1.

Two more comments:

- A preview: We can derive the ultimate gain and ultimate period (or frequency) with stability analyses. In Chapter 7, we use the substitution $s = j\omega$ in the closed-loop characteristic equation. In Chapter 8, we make use of what is called the Nyquist stability criterion and Bode plots.
- One may question the meaning of "sustained oscillations." We may gather that the ultimate gain and ultimate period are associated with marginal stability—the instance when the system is just about to become unstable. Of course, we never want to push that far in real life. With large uncertainties involved in empirical tuning and field tuning, it is not necessary to have accurate measurements of K_{cu} and T_u. When we do an experiment, just increase the proportional gain until we achieve a fairly underdamped response.

---

1 If you come across a proportional controller here, it is only possible if the derivation has ignored the steady state error, or shifted the reference such that the so-called offset is zero.

✎ **Example 5.7A**: What would be the PID controller settings for the dye mixing problem in Example 5.7 (p. 5-17)?

Based on what we have obtained in Example 5.7, if we did an open-loop experiment as suggested in Eq. (6-1), our step response would fit well to the function:

$$G_{PRC} = G_a G_p G_m = \frac{(0.8)\,(0.6)\,(2.6)\,e^{-0.725\,s}}{(4s+1)\,(0.2s+1)}$$

However, to use the empirical tuning relations, we need to fit the data to a first order function with dead time. Thus at this stage, we probably would have obtained the approximation:

$$G_{PRC} \approx \frac{1.25\,e^{-0.9\,s}}{4s+1}$$

Here, we assume that the data fitting allows us to recover the time constant of the dominant pole reasonably well, and the dead time is roughly 0.9 s. We are not adding exactly 0.2 to 0.725 as a way to emphasize that in reality, we would be doing data fitting and the result will vary. How good an approximation is depends very much on the relative differences in the time constants. (Try with MATLAB simulation to see how good the approximation is. For the numbers chosen in this example, it is easy to obtain a good approximation.)

Now, with Table 6.1,[1] we can calculate the following PID controller settings:

	K_c	τ_I	τ_D
Cohen-Coon	4.9	2.0	0.31
Ziegler-Nichols	4.3	1.8	0.45
ITAE (Set point)	2.8	3.1	0.31
Ciancone-Marlin (Set point)	1.2	4.4	0.07

All tuning relations provide different results. Generally, the Cohen and Coon relation has the largest proportional gain and the dynamic response tends to be the most underdamped. The Ciancone-Marlin relation provides the most conservative setting, and it uses a very small derivative time constant and a relatively large integral time constant. In a way, their correlation reflects a common industrial preference for PI controllers.

We'll see how they compare in time response simulations when we come back to this problem later in Example 5.7C. A point to be made is that empirical tuning is a very imprecise science. There is no reason to worry about the third significant figure in your tuning parameters. The calculation only serves to provide us with an initial setting with which we begin to do field or computational tuning.

1 Really calculated with our M-file recipe.m, which can be found on our *Web Support*.

While the calculations in the last example may appear as simple plug-and-chug, we should take a closer look at the tuning relations. The Cohen and Coon equations for the proportional gain taken from Table 6.1 are:

P: $$K_cK = \left(\frac{\tau}{t_d} + \frac{1}{3} \right) \tag{6-6}$$

PI: $$K_cK = \left(0.9\,\frac{\tau}{t_d} + \frac{1}{12} \right) \tag{6-7a}$$

PID: $$K_cK = \left(\frac{4}{3}\,\frac{\tau}{t_d} + \frac{1}{4} \right) \tag{6-8a}$$

The choice of the proportional gain is affected by two quantities: the product K_cK, and the ratio of dead time to time constant, t_d/τ. It may not be obvious why the product K_cK is important now, but we shall see how it arises from direct synthesis in the next section and appreciate how it helps determine system stability in Chapter 8.

Under circumstances where the dead time is relatively small, only the first term on the right is important in the three tuning equations. When dead time becomes larger (or τ/t_d smaller), we need to decrease the proportional gain, and this is how the tuning relations are constructed. When we add integral control, we need to decrease K_c. Indeed, in Eq. (6-7a), the τ/t_d term is decreased by 10%, and the constant term is reduced to 1/12. With the implementation of PID control, we can afford to have a larger K_c. This is reflected in (6-8a). We can make similar observations with the Ziegler-Nichols relations in Table 6.1. Furthermore, we may also see in Table 6.1 that if the dead time increases, we should raise the integral time constant.

Table 6.1. Table of tuning relations [1]

A. Tuning relations based on open-loop testing and response fitted to a first order with dead time function

$$G_{PRC} = \frac{Ke^{-t_d s}}{\tau s + 1}$$

Controller	Cohen-Coon	Ziegler-Nichols
P	$K_cK = \left(\frac{\tau}{t_d} + \frac{1}{3}\right)$ (6-6)	$K_cK = \frac{\tau}{t_d}$ (6-9)
PI	$K_cK = \left(0.9\frac{\tau}{t_d} + \frac{1}{12}\right)$ (6-7a)	$K_cK = 0.9\frac{\tau}{t_d}$ (6-10a)
	$\tau_I = t_d \frac{30 + 3(t_d/\tau)}{9 + 20(t_d/\tau)}$ (6-7b)	$\tau_I = 3.3\, t_d$ (6-10b)
PID	$K_cK = \left(\frac{4}{3}\frac{\tau}{t_d} + \frac{1}{4}\right)$ (6-8a)	$K_cK = 1.2\frac{\tau}{t_d}$ (6-11a)
	$\tau_I = t_d \frac{32 + 6(t_d/\tau)}{13 + 8(t_d/\tau)}$ (6-8b)	$\tau_I = 2\, t_d$ (6-11b)
	$\tau_D = t_d \frac{4}{11 + 2(t_d/\tau)}$ (6-8c)	$\tau_D = 0.5\, t_d$ (6-11c)

Minimum ITAE criterion

For **load** change:

$$K_c = \frac{a_1}{K}\left(\frac{\tau}{t_d}\right)^{b_1}, \quad \tau_I = \frac{\tau}{a_2}\left(\frac{t_d}{\tau}\right)^{b_2} \quad \text{and} \quad \tau_D = a_3\tau\left(\frac{t_d}{\tau}\right)^{b_3} \qquad (6\text{-}12)$$

Controller	a_1	b_1	a_2	b_2	a_3	b_3
PI	0.859	0.977	0.674	0.680	–	–
PID	1.357	0.947	0.842	0.738	0.381	0.995

[1] All formulas in Table 6.1, and the PID settings in Table 6.2 later, are implemented in the M-file recipe.m, available from our *Web Support*. The Ciancone and Marlin tuning relations are graphical, and we have omitted them from the tables. The correlation plots, explanations, and the interpolation calculations are provided by our M-file ciancone.m, which is also used by recipe.m.

For **set point** change:

$$K_c = \frac{a_1}{K}\left(\frac{\tau}{t_d}\right)^{b_1} \quad , \quad \tau_I = \frac{\tau}{a_2 - b_2\,(t_d/\tau)} \quad \text{and} \quad \tau_D = a_3\,\tau\left(\frac{t_d}{\tau}\right)^{b_3} \qquad (6\text{-}13)$$

Controller	a_1	b_1	a_2	b_2	a_3	b_3
PI	0.586	0.916	1.03	0.165	–	–
PID	0.965	0.855	0.796	0.147	0.308	0.929

B. Tuning relations based on closed-loop testing and the Ziegler-Nichols ultimate-gain (cycle) method with given ultimate proportional gain K_{cu} and ultimate period T_u.

Ziegler-Nichols ultimate-gain method

Controller						
P	$K_c = 0.5\ K_{cu}$	(6-14)				
PI	$K_c = 0.455\ K_{cu}$	(6-15a)				
	$\tau_I = 0.833\ T_u$	(6-15b)				
PID	Quarter decay		Just a bit of overshoot		No overshoot	
	$K_c = 0.6\ K_{cu}$	(6-16a)	$K_c = 0.33\ K_{cu}$	(6-17a)	$K_c = 0.2\ K_{cu}$	(6-18a)
	$\tau_I = 0.5\ T_u$	(6-16b)	$\tau_I = 0.5\ T_u$	(6-17b)	$\tau_I = 0.5\ T_u$	(6-18b)
	$\tau_D = 0.125\ T_u$	(6-16c)	$\tau_D = 0.333\ T_u$	(6-17c)	$\tau_D = 0.333\ T_u$	(6-18c)

6.2 Direct synthesis and internal model control

We now apply a different philosophy to controller design. Up until now, we have had a preconceived idea of what a controller should be, and we tune it until we have the desired system response. On the other hand, we can be more proactive: we define what our desired closed-loop response should be and design the controller accordingly. The resulting controller is not necessarily a PID controller. This is acceptable with computer based controllers since we are not restricted to off-the-shelf hardware.

In this chapter, however, our objective is more restricted. We will purposely choose simple cases and make simplifying assumptions such that the results are PID controllers. We will see how the method helps us select controller gains based on process parameters (*i.e.*, the process model). The method provides us with a more rational controller design than the empirical tuning relations. Since the result depends on the process model, this method is what we considered a **model-based** design.

6.2.1 Direct synthesis

We consider a servo problem (*i.e.*, L = 0), and set $G_m = G_a = 1$. The closed-loop function is the familiar

$$\frac{C}{R} = \frac{G_c G_p}{1 + G_c G_p} \tag{6-19}$$

which we now rearrange as

$$G_c = \frac{1}{G_p}\left[\frac{C/R}{1 - C/R}\right] \tag{6-20}$$

The implication is that if we define our desired system response C/R, we can derive the appropriate controller function for a specific process function G_p.

A couple of quick observations: First, G_c is the reciprocal of G_p. The poles of G_p are related to the zeros of G_c and vice versa—this is the basis of the so-called *pole-zero cancellation*.[1] Second, the choice of C/R is not entirely arbitrary; it must satisfy the closed-loop characteristic equation:

$$1 + G_c G_p = 1 + \left[\frac{C/R}{1 - C/R}\right] = 0 \tag{6-21}$$

From Eq. (6-20), it is immediately clear that we cannot have an ideal servo response where C/R = 1, which would require an infinite controller gain. Now Eq. (6-21) indicates that C/R cannot be some constant either. To satisfy (6-21), the closed-loop response C/R must be some function of s, meaning that the system cannot respond instantaneously and must have some finite response time.

Let's select a more realistic system response, say, a simple first-order function with unity steady state gain

$$\frac{C}{R} = \frac{1}{\tau_c s + 1} \tag{6-22}$$

---

[1] The controller function will take on a positive pole if the process function has a positive zero. It is not desirable to have an inherently unstable element in our control loop. This is an issue which internal model control will address.

where τ_c is the system time constant, a design parameter that we specify. The unity gain means that we should eliminate offset in the system. Substitution of Eq. (6-22) in (6-20) leads to the controller function:

$$G_c = \frac{1}{G_p}\left[\frac{1}{\tau_c s}\right] \tag{6-23}$$

The closed-loop characteristic equation, corresponding to Eq. (6-21), is

$$1 + \frac{1}{\tau_c s} = 0 \tag{6-24}$$

which really is $1 + \tau_c s = 0$ as dictated by (6-22). The closed-loop pole is at $s = -1/\tau_c$. This result is true no matter what G_p is—as long as we can physically build or program the controller on a computer. Since the system time constant τ_c is our design parameter, it appears that direct synthesis magically allows us to select whatever response time we want. Of course this cannot be the case in reality. There are physical limitations such as saturation.

✎ **Example 6.1**: Derive the controller function for a system with a **first order process** and a system response dictated by Eq. (6-22).

The process transfer function is $G_p = \dfrac{K_p}{\tau_p s + 1}$, and the controller function according to Eq. (6-23) is

$$G_c = \frac{(\tau_p s + 1)}{K_p}\,\frac{1}{\tau_c s} = \frac{\tau_p}{K_p \tau_c}\left(1 + \frac{1}{\tau_p s}\right) \tag{E6-1}$$

which is obviously a PI controller with $K_c = \tau_p/K_p\tau_c$, and $\tau_I = \tau_p$. Note that the proportional gain is inversely proportional to the process gain. Specification of a small system time constant τ_c also leads to a large proportional gain.

A reminder: the controller settings K_c and τ_I are governed by the process parameters and the system response, which we choose. The *one and only tuning parameter* is the system response time constant τ_c.

✎ **Example 6.2**: Derive the controller function for a system with a **second order overdamped process** and system response as dictated by Eq. (6-22).

The process transfer function is $G_p = \dfrac{K_p}{(\tau_1 s + 1)(\tau_2 s + 1)}$, and the controller function according to Eq. (6-23) is

$$G_c = \frac{(\tau_1 s + 1)(\tau_2 s + 1)}{K_p}\,\frac{1}{\tau_c\, s}\,.$$

We may see that this is a PID controller. Nevertheless, there are two ways to manipulate the function. One is to expand the terms in the numerator and factor out $(\tau_1 + \tau_2)$ to obtain

$$G_c = \frac{(\tau_1 + \tau_2)}{K_p \tau_c}\left[1 + \frac{1}{(\tau_1 + \tau_2)}\,\frac{1}{s} + \left(\frac{\tau_1 \tau_2}{\tau_1 + \tau_2}\right) s\right] \tag{E6-2}$$

The proportional gain, integral time and derivative time constants are provided by the respective terms in the transfer function. If you have trouble spotting them, they are summarized in Table 6.2.

The second approach is to consider the controller function as a series-PID such that we write

$$G_c = \frac{\tau_1}{K_p \tau_c}\left(1 + \frac{1}{\tau_1 s}\right)\left(\tau_2 s + 1\right), \quad \text{with } \tau_1 > \tau_2, \tag{E6-3}$$

We can modify the derivative term to be the "real" derivative action as written in Eqs. (5-9a and b) on page 5-7.

Based on experience that the derivative time constant should be smaller than the integral time constant, we should pick the larger time constant as the integral time constant. Thus we select τ_1 to be the integral time constant and τ_2 the derivative time constant. In the limit τ_1 » τ_2, both arrangements (E6-2 and 3) of the controller function are the same.

When dead time is inherent in a process, it is difficult to avoid dead time in the system. Thus we define the system response as

$$\frac{C}{R} = \frac{e^{-\theta s}}{\tau_c s + 1} \tag{6-25}$$

where θ is the dead time in the system. The controller function, via Eq. (6-20), is hence

$$G_c = \frac{1}{G_p}\left[\frac{e^{-\theta s}}{\left(\tau_c s + 1\right) - e^{-\theta s}}\right] \approx \frac{1}{G_p}\left[\frac{e^{-\theta s}}{(\tau_c + \theta)\, s}\right] \tag{6-26}$$

To arrive at the last term, we have used a simple Taylor expansion ($e^{-\theta s} \approx 1 - \theta s$) of the exponential term. This is purposely done to simplify the algebra as shown in the next example. (We could have used the Padé approximation in Eq. (6-26), but the result will not be the simple PI controller.)

✎ **Example 6.3**: Derive the controller function for a system with a **first order process with dead time** and system response as dictated by Eq. (6-25).

The process transfer function is $G_p = \dfrac{K_p e^{-t_d s}}{\tau_p s + 1}$. To apply Eq. (6-26), we make an assumption about the dead time, that $\theta = t_d$. The result is a PI controller:

$$G_c = \frac{\tau_p}{K_p\left(\tau_c + \theta\right)}\left(1 + \frac{1}{\tau_p s}\right) \tag{E6-4}$$

Even though this result is based on what we say is a process function, we could apply (E6-4) as if the derivation is for the first order with dead time function G_{PRC} obtained from an open-loop step test.

This is a question that invariably arises: what is a reasonable choice of the system time constant τ_c? Various sources in the literature have different recommendations. For example, one guideline suggests that we need to ensure $\tau_c > 1.7\theta$ for a PI controller, and $\tau_c > 0.25\theta$ for a PID controller. A reasonably conservative choice has been programmed into the M-file reciepe.m available from our *Web Support*. The important reminder is that we should have a habit of checking the τ_c setting with time response simulation and tuning analysis.

In contrast to Eq. (6-22), we can dictate a second order underdamped system response:

$$\frac{C}{R} = \frac{1}{\tau^2 s^2 + 2\zeta\tau s + 1} \tag{6-27}$$

where τ and ζ are the system natural period and damping ratio yet to be determined. Substitution of (6-27) in Eq. (6-20) leads to

$$G_c = \frac{1}{G_p}\left[\frac{1}{\tau^2 s^2 + 2\zeta\tau s}\right] \tag{6-28}$$

which is a slightly more complicated form than (6-23). Again, with simplified cases, we can arrive at PID type controllers.

✎ **Example 6.4**: Derive the controller function for a system with a **second order overdamped process** but an **underdamped system** response as dictated by Eq. (6-27).

The process transfer function is $G_p = \dfrac{K_p}{(\tau_1 s + 1)(\tau_2 s + 1)}$, and the controller function according to Eq. (6-28) is

$$G_c = \frac{(\tau_1 s + 1)(\tau_2 s + 1)}{K_p \tau s (\tau s + 2\zeta)}.$$

We now define $\tau_f = \tau/2\zeta$, and G_c becomes

$$G_c = \frac{(\tau_1 s + 1)(\tau_2 s + 1)}{2\zeta K_p \tau s (\tau_f s + 1)}$$

Suppose that τ_2 is associated with the slower pole ($\tau_2 > \tau_1$), we now require $\tau_f = \tau_2$ such that the pole and zero cancel each other. The result is a PI controller:

$$G_c = \frac{1}{2\zeta K_p \tau} \frac{(\tau_1 s + 1)}{s}$$

With our definition of τ_f and the requirement $\tau_f = \tau_2$, we can write $\tau = 2\zeta\tau_2$, and the final form of the controller is

$$G_c = \frac{\tau_1}{4K_p \zeta^2 \tau_2}\left(1 + \frac{1}{\tau_1 s}\right) = K_c\left(1 + \frac{1}{\tau_I s}\right) \tag{E6-5}$$

The integral time constant is $\tau_I = \tau_1$, and the term multiplying the terms in the parentheses is the proportional gain K_c. In this problem, the system damping ratio ζ is the only tuning parameter.

6.2.2 Pole-zero cancellation

We used the term "pole-zero cancellation" at the beginning of this section. We should say a few more words to better appreciate the idea behind direct synthesis. Pole-zero cancellation is also referred to as **cancellation compensation** or dominant pole design. Of course, it is unlikely to have perfect pole-zero cancellation in real life, and this discussion is more toward helping our theoretical understanding.

The idea is that we may cancel the (undesirable open-loop) poles of our process and replace them with a desirable closed-loop pole. Recall in Eq. (6-20) that G_C is sort of the reciprocal of G_p. The zeros of G_C are by choice the poles of G_p. The product of G_CG_p cancels everything out—hence the term pole-zero cancellation. To be redundant, we can rewrite the general design equation as

$$G_c G_p = \left[\frac{C/R}{1 - C/R} \right] \tag{6-20a}$$

That is, no matter what G_p is, we define G_C such that their product is dictated entirely by a function (the RHS) in terms of our desired system response (C/R). For the specific closed-loop response as dictated by Eq. (6-22), we can also rewrite Eq. (6-23) as

$$G_c G_p = \left[\frac{1}{\tau_c s} \right] \tag{6-23a}$$

Since the system characteristic equation is $1 + G_cG_p = 0$, our closed-loop poles are only dependent on our design parameter τ_c. A closed-loop system designed on the basis of pole-zero cancellation has drastically different behavior than a system without such cancellation.

Let's try to illustrate using a system with a PI controller and a first order process function, and the simplification that $G_m = G_a = 1$. The closed-loop characteristic equation is

$$1 + G_cG_p = 1 + K_c \left(\frac{\tau_I s + 1}{\tau_I s} \right) \frac{K_p}{\tau_p s + 1} = 0 \tag{6-29}$$

Under normal circumstances, we would pick a τ_I which we deem appropriate. Now if we pick τ_I to be identical to τ_p, the zero of the controller function cancels the pole of the process function. We are left with only one open-loop pole at the origin. Eq. (6-29), when $\tau_I = \tau_p$, is reduced to

$$1 + \frac{K_cK_p}{\tau_p s} = 0, \quad \text{or} \quad s = -\frac{K_c K_p}{\tau_p} .$$

There is now only one real and negative closed-loop pole (presuming $K_c > 0$). This situation is exactly what direct synthesis leads us to.

Recall from Example 6.1 that based on the chosen C/R in Eq. (6-22), the PI controller function is

$$G_c = K_c \left(\frac{\tau_I s + 1}{\tau_I s} \right) = \frac{\tau_p}{K_p \tau_c} \left(\frac{\tau_p s + 1}{\tau_p s} \right)$$

where $\tau_I = \tau_p$ and $K_C = \tau_p/K_p\tau_C$. Substitution of K_C one step back in the characteristic equation will shows that the closed-loop pole is indeed at $s = -1/\tau_C$. The product G_CG_p is also consistent with Eq. (6-23a) and τ_C.

6.2.3 Internal model control (IMC)

A more elegant approach than direct synthesis is internal model control (IMC). The premise of IMC is that in reality, we only have an approximation of the actual process. Even if we have the correct model, we may not have accurate measurements of the process parameters. Thus the imperfect model should be factored as part of the controller design.

In the block diagram implementing IMC (Fig. 6.3a), our conventional controller G_c consists of the (theoretical) model controller G^*_c and the approximate function $\tilde{G}_p$. Again, our objective is limited. We use the analysis in very restrictive and simplified cases to arrive at results in Example 6.5 to help us tune PID controllers as in Fig. 6.3b.

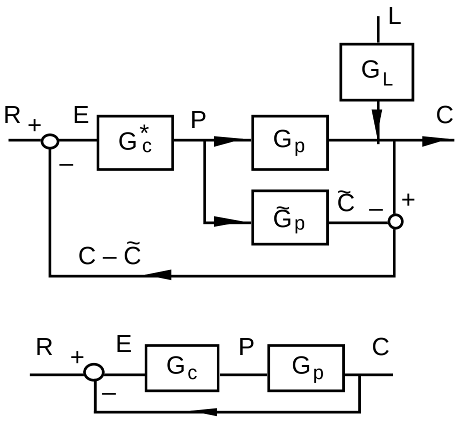


Figure 6.3. A system with IMC (upper panel) as compared with a conventional system in the lower panel.

We first need to derive the closed-loop functions for the system. Based on the block diagram, the error is

$$E = R - (C - \tilde{C})$$

and the model controller output is

$$P = G^*_c E = G^*_c (R - C + \tilde{C})$$

If we substitute $\tilde{C} = \tilde{G}_p P$, we have

$$P = G^*_c (R - C + \tilde{G}_p P) \tag{6-30}$$

from which we can rearrange to obtain

$$P = \frac{G^*_c}{1 - G^*_c \tilde{G}_p} (R - C) \tag{6-28a}$$

The gist of this step is to show the relationship between the conventional controller function G_c and the other functions:

$$G_c = \frac{G^*_c}{1 - G^*_c \tilde{G}_p} \tag{6-31}$$

This is an equation that we will use to retrieve the corresponding PID controller gains. For now, we substitute Eq. (6-28a) in an equation around the process,

$$C = G_L L + G_p P = G_L L + \frac{G_p G^*_c}{1 - G^*_c \tilde{G}_p} (R - C)$$

From this step, we derive the closed-loop equation:

$$C = \left[\frac{(1 - G^*_c \tilde{G}_p) G_L}{1 + G^*_c (G_p - \tilde{G}_p)} \right] L + \left[\frac{G_p G^*_c}{1 + G^*_c (G_p - \tilde{G}_p)} \right] R \tag{6-32}$$

The terms in the brackets are the two closed-loop transfer functions. As always, they have the same denominator—the closed-loop characteristic polynomial.

There is still one unfinished business. We do not know how to choose G^*_c yet. Before we make this decision, we may recall that in direct synthesis, the poles of G_c are "inherited" from the zeros of G_p. If G_p has positive zeros, it will lead to a G_c function with positive poles. To avoid that, we "split" the approximate function as a product of two parts:

$$\tilde{G}_p = \tilde{G}_{p+} \tilde{G}_{p-} \qquad (6\text{-}33)$$

with $\tilde{G}_{p+}$ containing all the positive zeros, if present. The controller will be designed on the basis of $\tilde{G}_{p-}$ only. We now define the model controller function in a way similar to direct synthesis: [1]

$$G^*_c = \frac{1}{\tilde{G}_{p-}} \left[\frac{1}{\tau_c s + 1} \right]^r , \quad \text{where r = 1, 2, etc.} \qquad (6\text{-}34)$$

Like direct synthesis, τ_c is the closed-loop time constant and our *only* tuning parameter. The first order function raised to an integer power of *r* is used to ensure that the controller is physically realizable. [2] Again, we would violate this intention in our simple example just so that we can obtain results that resemble an ideal PID controller.

✎ **Example 6.5**: Repeat the derivation of a controller function for a system with a **first order process with dead time** using IMC.

Say we model our process (read: fitting the open-loop step test data) as a first order function with time delay, and expecting experimental errors or uncertainties, our measured or approximate model function $\tilde{G}_p$ is

$$\tilde{G}_p = \frac{K_p e^{-t_d s}}{\tau_p s + 1}$$

We use the first order Padé approximation for the dead time and isolate the positive zero term as in Eq. (6-33):

$$\tilde{G}_p \approx \frac{K_p}{(\tau_p s + 1)(\frac{t_d}{2} s + 1)} (-\frac{t_d}{2} s + 1) = \tilde{G}_{p-} \tilde{G}_{p+} \qquad (E6\text{-}6)$$

where

$$\tilde{G}_{p+} = (-\frac{t_d}{2} s + 1)$$

If we choose r = 1, Eq. (6-34) gives

$$G^*_c = \frac{(\tau_p s + 1)(\frac{t_d}{2} s + 1)}{K_p} \frac{1}{(\tau_c s + 1)} \qquad (E6\text{-}7)$$

---

[1] If the model is perfect, $G_p = \tilde{G}_p$, and Eq. (6-32) becomes simply $C = G_p G^*_c R$ if we also set L = 0. We choose C/R to be a first order response with unity gain, and we'd arrive at a choice of G^*_c very similar to the definition in (6-34).

[2] The literature refers the term as a first order filter. It only makes sense if you recall your linear circuit analysis or if you wait until the chapter on frequency response analysis.

Substitution of (E6-5) and (E6-6) into Eq. (6-31), and after some algebraic work, will lead to the tuning parameters of an ideal PID controller:

$$K_c = \frac{1}{K_p}\frac{2\frac{\tau_p}{t_d}+1}{2\frac{\tau_c}{t_d}+1} \quad ; \quad \tau_I = \tau_p + \frac{t_d}{2} \quad ; \quad \tau_D = \frac{\tau_p}{2\frac{\tau_p}{t_d}+1} \qquad \text{(E6-8)}$$

✎ **Example 5.7B**: What would be the PID controller settings for the dye mixing problem if we use IMC-based tuning relations?

With the same first order with dead time approximation in Example 5.7A (p. 6-5), and the choice of τ_c being two-thirds the value of dead time, the IMC relations in (E6-8) provide the following PID settings (as computed with our M-file recipe.m):

	K_C	τ_I	τ_D
IMC	3.4	4.5	0.4

Compare this result using other tuning relations in Example 5.7A. The IMC proportional gain falls in between the Cohen-Coon and ITAE settings, but the integral time constant is relatively high. With less integrating action, we expect this IMC tuning to be less oscillatory. Indeed, we shall see that if we do Example 5.7C (or you can cheat and read the plotting result from our *Web Support*).

✎ **Example 5.7C**: How do the different controller settings affect the system time response in the dye mixing problem?

We can use the following MATLAB statements to do time response simulations (explanations are in MATLAB Session 5). Better yet, save them in an M-file. The plotting can be handled differently to suit your personal taste. (Of course, you can use Simulink instead.)

```
alfa=0.1;                      % Real PID
Gc=tf(kc*[taui*taud (taui+taud) 1],[alfa*taui*taud  taui  0]);
td=0.725;
Gm=tf([-td/2 1],[td/2 1]);     %Padé approximation for dead time
Km=2.6;                        %Move Km into the forward path
Gp=tf(0.8,[4 1]);
Ga=tf(0.6,[0.2 1]);
Gcl=feedback(Km*Gc*Ga*Gp,Gm);  % The closed-loop function

step(Gcl)                      % Plotting...
```

We reset the three controller parameters each time we execute the M-file. For example, to use the Cohen-Coon results, we would take from Example 5.7A:

```
kc=4.9; taui=2; taud=0.31;
```

MATLAB calculation details and plots can be found on our *Web Support*. You should observe that Cohen-Coon and Ziegler-Nichols tuning relations lead to roughly 74% and 64% overshoot, respectively, which are more significant than what we expect with a quarter decay ratio criterion.

The ITAE, with about 14% overshoot, is more conservative. Ciancone and Marlin tuning relations are ultra conservative; the system is slow and overdamped.

With the IMC tuning setting in Example 5.7B, the resulting time response plot is (very nicely) slightly underdamped even though the derivation in Example 6.4 predicates on a system response without oscillations. Part of the reason lies in the approximation of the dead time function, and part of the reason is due to how the system time constant was chosen. Generally, it is important to double check our IMC settings with simulations.

At this point, one may be sufficiently confused with respect to all the different controller tuning methods. Use Table 6.3 as a guide to review and compare different techniques this chapter and also Chapters 7 and 8.

Table 6.2. Summary of PID controller settings based on IMC or direct synthesis

Process model	Controller	K_c	τ_I	τ_D
$\dfrac{K_p}{\tau_p s + 1}$	PI	$\dfrac{\tau_p}{K_p \tau_c}$	τ_p	—
$\dfrac{K_p}{(\tau_1 s + 1)(\tau_2 s + 1)}$	PID	$\dfrac{\tau_1 + \tau_2}{K_p \tau_c}$	$\tau_1 + \tau_2$	$\dfrac{\tau_1 \tau_2}{\tau_1 + \tau_2}$
	PID with $\tau_1 > \tau_2$	$\dfrac{\tau_1}{K_p \tau_c}$	τ_1	τ_2
	PI (underdamped)	$\dfrac{\tau_1}{4K_p \zeta^2 \tau_2}$	τ_1	—
$\dfrac{K_p}{\tau^2 s^2 + 2\zeta\tau s + 1}$	PID	$\dfrac{2\zeta\tau}{K_p \tau_c}$	$2\zeta\tau$	$\dfrac{\tau}{2\zeta}$
$\dfrac{K_p}{s(\tau_p s + 1)}$	PD	$\dfrac{1}{K_p \tau_c}$	—	τ_p
$\dfrac{K_p e^{-t_d s}}{\tau_p s + 1}$	PI	$\dfrac{\tau_p}{K_p(\tau_c + t_d)}$	τ_p	—
	PID	$\dfrac{1}{K_p} \dfrac{2\tau_p/t_d + 1}{2\tau_c/t_d + 1}$	$\tau_p + t_d/2$	$\dfrac{\tau_p}{2\tau_p/t_d + 1}$
$\dfrac{K_p}{s}$	P	$\dfrac{1}{K_p \tau_c}$	—	—

❒ **Review Problems**

4. Repeat Example 6.1 when we have $G_p = \dfrac{K_p}{s\,(\tau_p s + 1)}$. What is the offset of the system?

5. What are the limitations to IMC? Especially with respect to the choice of τ_c?

6. What control action increases the order of the system?

7. Refer back to Example 6.4. If we have a third order process

$$G_p = \frac{K_p}{(\tau_1 s + 1)\,(\tau_2 s + 1)\,(\tau_3 s + 1)}$$

what is the controller function if we follow the same strategy as in the example?

8. Complete the time response simulations in Example 5.7C using settings in Example 5.7A.

9. (Optional) How would you implement the PID algorithm in a computer program?

Hints:

1. The result is an ideal PD controller with the choice of $\tau_D = \tau_p$. See that you can obtain the same result with IMC too. Here, take the process function as the approximate model and it has no parts that we need to consider as having positive zeros. There is no offset; the integrating action is provided by G_p.

2. Too small a value of τ_c means too large a K_c and therefore saturation. System response is subject to imperfect pole-zero cancellation.

3. Integration is 1/s.

10. The intermediate step is

$$G_c = \frac{(\tau_1 s + 1)\,(\tau_2 s + 1)\,(\tau_3 s + 1)}{2\zeta\, K_p \tau\, s\,(\tau_c s + 1)}$$

where $\tau_f = \tau/2\zeta$, and now we require $\tau_f = \tau_3$, presuming it is the largest time constant. The final result, after also taking some of the ideas from Example 6.2, is an ideal PID controller with the form:

$$G_c = \frac{(\tau_1 + \tau_2)}{4K_p \zeta^2 \tau_3} \left(1 + \frac{1}{\tau_1 + \tau_2} \frac{1}{s} + \frac{\tau_1 \tau_2}{\tau_1 + \tau_2}\, s \right)$$

The necessary choices of K_c, τ_I, and τ_D are obvious. Again, ζ is the only tuning parameter.

5. See our *Web Support* for the simulations.

6. Use finite difference. The ideal PID in Eq. (5-8a) can be discretized as

$$p_n = p^s + K_c \left[e_n + \frac{\Delta t}{\tau_I} \sum_{k=1}^{n} e_k + \frac{\tau_D}{\Delta t}(e_n - e_{n-1}) \right]$$

where p_n is the controller output at the *n*-th sampling period, p^s is the controller bias, Δt is the sampling period, and e_n is the error. This is referred to as the *position form* algorithm. The alternate approach is to compute the change in the controller output based on the difference between two samplings:

$$\Delta p_n = p_n - p_{n-1} = K_c \left[(e_n - e_{n-1}) + \frac{\Delta t}{\tau_I} e_n + \frac{\tau_D}{\Delta t}(e_n - 2e_{n-1} + e_{n-2}) \right]$$

This is the *velocity form* algorithm which is considered to be more attractive than the position form. The summation of error is not computed explicitly and thus the velocity form is not as susceptible to reset windup.

Table 6.3. Summary of methods to select controller gains

Method	What to do?	What is evaluated?	Comments
❑ **Transient response criteria**			
• Analytical derivation	Derive closed-loop damping ratio from a second order system characteristic polynomial. Relate the damping ratio to the proportional gain of the system.	Usually the proportional gain.	• Limited to second order systems. No unique answer other than a P-controller. • Theoretically can use other transient response criteria. • 1/4 decay ratio provides a 50% overshoot.
❑ **Empirical tuning with open-loop step test**	Measure open-loop step response, the so-called process reaction curve. Fit data to first order with dead-time function.		
• Cohen-Coon • Ziegler-Nichols • Ciacone-Marlin	Apply empirical design relations.	Proportional gain, integral and derivative time constants to PI and PID controllers.	• Cohen-Coon was designed to handle disturbances by preventing a large initial deviation from the set point. The one-quarter decay ratio response is generally too underdamped for set point changes.
• Time integral performance criteria (ISE, IAE, ITAE)	Apply design relations derived from minimization of an error integral of the theoretical time-domain response.	Proportional gain, integral and derivative time constants to PI and PID controllers.	• Different settings for load and set point changes. • Different settings for different definitions of the error integral. • The minimum ITAE criterion provides the least oscillatory response.
❑ **Ziegler-Nichols Continuous Cycling** (empirical tuning with closed loop test)	Increase proportional gain of only a proportional controller until system sustains oscillation. Measure ultimate gain and ultimate period. Apply empirical design relations.	Proportional gain, integral and derivative time constants of PID controllers.	• Experimental analog of the $s = j\omega$ substitution calculation. • Not necessarily feasible with chemical systems in practice. • Tuning relations allow for choices from 1/4 decay ratio to little oscillations.
❑ **Stability analysis methods**			
• Routh-Hurwitz criterion	Apply the Routh test on the closed-loop characteristic polynomial to find if there are closed-loop poles on the right-hand-plane.	Establish limits on the controller gain.	• Usually applies to relatively simple systems with the focus on the proportional gain. • Need be careful on interpretation when the lower limit on proportional gain is negative.
• Direct substitution	Substitute $s = j\omega$ in characteristic polynomial and solve for closed-loop poles on *Im*-axis. The *Im* and *Re* parts of the equation allow the ultimate gain and ultimate frequency to be solved.	Ultimate gain and ultimate period ($P_u = 2\pi/\omega_u$) that can be used in the Ziegler-Nichols continuous cycling relations.	• Result on ultimate gain is consistent with the Routh array analysis. • Limited to relatively simple systems.

Summary (continued)

Method	**What to do?**	**What is evaluated?**	**Comments**
• Root-locus	With each chosen value of proportional gain, plot the closed-loop poles. Generate the loci with either hand-sketching or computer.	The loci of closed-loop poles reveal the effect of controller gain on the probable closed-loop dynamic response. Together with specifications of damping ratio and time constant, the loci can be a basis of selecting proportional gain.	• Rarely used in the final controller design because of difficulty in handling dead-time. • Method is instructive and great pedagogical tool.
❑ **(Model-based) Direct synthesis**	For a given system, synthesize the controller function according to a specified closed-loop response. The system time constant, τ_c, is the only tuning parameter.	Proportional gain, integral and derivative time constants where appropriate.	• The design is not necessarily PID, but where the structure of a PID controller results, this method provides insight into the selection of the controller mode (PI, PD, PID) and settings. • Especially useful with system that has no dead time.
• Internal model control	Extension of direct synthesis. Controller design includes an internal approximation process function.	For a first order function with dead-time, the proportional gain, integral and derivative time constants of an ideal PID controller.	
❑ **Frequency-domain methods**			• Can handle dead-time easily and rigorously. • The Nyquist criterion allows the use of open-loop functions in Nyquist or Bode plots to analyze the closed-loop problem. • The stability criteria have no use for simple first and second order systems with no positive open-loop zeros.
• Nyquist plot • Bode plot	Nyquist plot is a frequency parametric plot of the magnitude and the argument of the open-loop transfer function in polar coordinates. Bode plot is magnitude vs. frequency and phase angle vs. frequency plotted individually.	Calculate proportional gain needed to satisfy the gain or phase margin.	• These plots address the stability problem but need other methods to reveal the probable dynamic response.
• Nichols chart	Nichols chart is a frequency parametric plot of open-loop function magnitude vs. phase angle. The closed-loop magnitude and phase angle are overlaid as contours.	With gain or phase margin, calculate proportional gain. Can also estimate the peak amplitude ratio, and assess the degree of oscillation.	• Nichols chart is usually constructed for unity feedback loops only.
• Maximum closed-loop log modulus	A plot of the magnitude vs. frequency of the closed-loop transfer function.	The peak amplitude ratio for a chosen proportional gain.	

❖ 7. *Stability of Closed-loop Systems*

When we design a closed-loop system, the specifications may dictate features in dynamic response. However, we cannot do that unless the system is stable. Thus the foremost concern in a control system design is to keep the system stable, which in itself can be used as a design tool.

What are we up to?
Analyze stability of a closed-loop system with three techniques:

- Routh-Hurwitz criterion for the stability region
- Substitution of s = jω to find roots at marginal stability
- Root locus plots of the closed-loop poles

7.1 Definition of Stability

Our objective is simple. We want to make sure that the controller settings will not lead to an unstable system. Consider the closed-loop system response that we derived in Section 5.2 (p. 5-7):

$$C = \left[\frac{K_m G_c G_a G_p}{1 + G_m G_c G_a G_p}\right] R + \left[\frac{G_L}{1 + G_m G_c G_a G_p}\right] L \tag{7-1}$$

with the characteristic equation

$$1 + G_m G_c G_a G_p = 0 \tag{7-2}$$

The closed-loop system is **stable** if all the roots of the characteristic polynomial have negative real parts. Or we can say that all the poles of the closed-loop transfer function lie in the left-hand plane (LHP). When we make this statement, the stability of the system is defined entirely on the inherent dynamics of the system, and not on the input functions. In other words, the results apply to both servo and regulating problems.

We also see another common definition—**bounded input bounded output** (BIBO) stability: A system is BIBO stable if the output response is bounded for any bounded input. One illustration of this definition is to consider a hypothetical situation with a closed-loop pole at the origin. In such a case, we know that if we apply an impulse input or a rectangular pulse input, the response remains bounded. However, if we apply a step input, which is bounded, the response is a ramp, which has no upper bound. For this reason, we cannot accept any control system that has closed-loop poles lying on the imaginary axis. They must be in the LHP. [1]

Addition of a feedback control loop can stabilize or destabilize a process. We will see plenty examples of the latter. For now, we use the classic example of trying to stabilize an open-loop unstable process.

[1] Do not be confused by the function of integral control; its pole at the origin is an open-loop pole. This point should clear up when we get to the root locus section.

Furthermore, conjugate poles on the imaginary axis are BIBO stable—a step input leads to a sustained oscillation that is bounded in time. But we do not consider this oscillatory steady state as stable, and hence we exclude the entire imaginary axis. In an advanced class, you should find more mathematical definitions of stability.

✎ **Example 7.1**: Consider the unstable process function $G_p = \frac{K}{s-a}$, which may arise from a linearized model of an exothermic chemical reactor with an improper cooling design. The question is whether we can make a stable control system using simply a proportional controller. For illustration, we consider a unity feedback loop with $G_m = 1$. We also take the actuator transfer function to be unity, $G_a = 1$.

With a proportional controller, $G_c = K_c$, the transfer function of this closed-loop servo system is

$$\frac{Y}{R} = \frac{G_c G_p}{1 + G_c G_p} = \frac{K_c K}{s - a + K_c K}$$

The characteristic equation is

$$s - a + K_c K = 0$$

which means that if we want a stable system, the closed-loop poles must satisfy

$$s = a - K_c K \;<\; 0$$

In other words, the closed-loop system is stable if $K_c > a/K$.

For a more complex problem, the characteristic polynomial will not be as simple, and we need tools to help us. The two techniques that we will learn are the Routh-Hurwitz criterion and root locus. Root locus is, by far, the more important and useful method, especially when we can use a computer. Where circumstances allow (*i.e.*, the algebra is not too ferocious), we can also find the roots on the imaginary axis—the case of marginal stability. In the simple example above, this is where $K_c = a/K$. Of course, we have to be smart enough to pick $K_c > a/K$, and not $K_c < a/K$.

7.2 The Routh-Hurwitz Criterion

We first introduce the time honored (*i.e.*, ancient!) Routh-Hurwitz criterion for stability testing. We will not prove it—hardly any text does anymore. Nonetheless, we use two general polynomials to illustrate some simple properties. First, consider a second order polynomial with the leading coefficient $a_2 = 1$. If the polynomial has two real poles p_1 and p_2, it can be factored as

$$P(s) = s^2 + a_1 s + a_o = (s - p_1)(s - p_2) \qquad (7\text{-}3)$$

We may observe that if a_1 is zero, both roots, $\pm j\sqrt{a_o}$, are on the imaginary axis. If a_o is zero, one of the two roots is at the origin. We can expand the pole form to give

$$P(s) = s^2 - (p_1 + p_2)s + p_1 p_2 \qquad (7\text{-}4)$$

and compare the coefficients with the original polynomial in Eq. (7-3). If both p_1 and p_2 are negative, the coefficients a_1 and a_o must be positive definite, which is the mathematicians' way of saying, $a_1 > 0$ and $a_o > 0$.

Next, consider a third order polynomial with leading coefficient $a_3 = 1$ and the form in terms of the poles:

$$P(s) = s^3 + a_2 s^2 + a_1 s + a_o = (s - p_1)(s - p_2)(s - p_3) \qquad (7\text{-}5)$$

We expand the pole form to

$$P(s) = s^3 - (p_1 + p_2 + p_3)s^2 + (p_1 p_2 + p_1 p_3 + p_2 p_3)s - p_1 p_2 p_3 \qquad (7\text{-}6)$$

Once again, if all three poles are negative, the coefficients a_2, a_1, and a_o must be positive definite.

The idea is that the signs of the pole are related to the coefficients a_n, a_{n-1},... , a_o of an *n*-th order characteristic polynomial. If we require all the poles to have negative real parts, there must be some way that we can tell from the coefficients a_n, a_{n-1},... , etc. without actually having to solve for the roots. The inkling is that all of the coefficients in the characteristic polynomial must be positive definite. One could develop a comprehensive theory, which Routh did. The attractiveness of the Routh criterion is that *without solving* for the closed-loop poles, one can derive inequalities that would provide a bound for stable controller design.

The complete Routh array analysis allows us to find, for example, the number of poles on the imaginary axis. Since BIBO stability requires that all poles lie in the left-hand plane, we will not bother with these details (which are still in many control texts). Consider the fact that we can calculate easily the exact roots of a polynomial with MATLAB, we use the Routh criterion to the extent that it serves its purpose.[1] That would be to derive inequality criteria for proper selection of controller gains of relatively simple systems. The technique loses its attractiveness when the algebra becomes too messy. Now the simplified Routh-Hurwitz recipe without proof follows.

(1) **Hurwitz test for the polynomial coefficients**

For a given *n*-th order polynomial

$$P(s) = a_n s^n + a_{n-1} s^{n-1} + ... + a_2 s^2 + a_1 s + a_o \ , \tag{7-7}$$

all the roots are in the left hand plane if and only if all the coefficients a_o,…, a_n are positive definite.

If any one of the coefficients is negative, at least one root has a positive real part (*i.e.*, in the right hand plane). If any of the coefficients is zero, not all of the roots are in the left hand plane: it is likely that some of them are on the imaginary axis. Either way, stop. This test is a necessary condition for BIBO stability. There is no point in doing more other than to redesign the controller.

(2) **Routh array construction**

If the characteristic polynomial passes the coefficient test, we then construct the Routh array to find the necessary and sufficient conditions for stability. This is one of the few classical techniques that we do not emphasize and the general formula is omitted. The array construction up to a fourth order polynomial is used to illustrate the concept.

Generally, for an *n*-th order polynomial, we need (*n*+1) rows. The first two rows are filled in with the coefficients of the polynomial in a column-wise order. The computation of the array entries is very much like the negative of a normalized determinant anchored by the first column. Even without the general formula, you may pick out the pattern as you read the following three examples.

The **Routh criterion** states that in order to have a stable system, all the coefficients in the first column of the array must be positive definite. If any of the coefficients in the first column is negative, there is at least one root with a positive real part. The number of sign changes is the number of positive poles.

Here is the array for a **second order** polynomial, $p(s) = a_2 s^2 + a_1 s + a_o$:

[1] MATLAB does not even bother wit a Routh function. Such an M-file is provided on our *Web Support* for demonstration purpose.

1:	a_2	a_o
2:	a_1	0
3:	$b_1 = \dfrac{a_1 a_o - (0)\, a_2}{a_1} = a_o$	

In the case of a second order system, the first column of the Routh array reduces to simply the coefficients of the polynomial. The coefficient test is sufficient in this case. Or we can say that both the coefficient test and the Routh array provide the same result.

The array for a **third order** polynomial, $p(s) = a_3 s^3 + a_2 s^2 + a_1 s + a_o$, is

1:	a_3	a_1	0
2:	a_2	a_o	0
3:	$b_1 = \dfrac{a_2 a_1 - a_3 a_o}{a_2}$	$b_2 = \dfrac{(a_2)\, 0 - (a_3)\, 0}{a_2} = 0$	
4:	$c_1 = \dfrac{b_1 a_o - b_2 a_2}{b_1} = a_o$	0	

In this case, we have added one column of zeros; they are needed to show how b_2 is computed. Since $b_2 = 0$ and $c_1 = a_o$, the Routh criterion adds one additional constraint in the case of a third order polynomial:

$$b_1 = \frac{a_2 a_1 - a_3 a_o}{a_2} > 0 \tag{7-8}$$

We follow with the array for a **fourth order** polynomial, $p(s) = a_4 s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_o$,

1:	a_4	a_2	a_o
2:	a_3	a_1	0
3:	$b_1 = \dfrac{a_3 a_2 - a_1 a_4}{a_3}$	$b_2 = \dfrac{a_3 a_o - (0)\, a_4}{a_3} = a_o$	0
4:	$c_1 = \dfrac{b_1 a_1 - b_2 a_3}{b_1}$	$c_2 = \dfrac{b_1\, (0) - (0)\, a_3}{b_1} = 0$	
5:	$d_1 = \dfrac{c_1 b_2 - (0)\, b_1}{c_1} = b_2 = a_o$	0	

The two additional constraints from the Routh array are hence

$$b_1 = \frac{a_3 a_2 - a_1 a_4}{a_3} > 0 \tag{7-9}$$

and

$$c_1 = \frac{b_1 a_1 - b_2 a_3}{b_1} = \frac{b_1 a_1 - a_o a_3}{b_1} > 0 \tag{7-10}$$

✎ **Example 7.2**: If we have only a proportional controller (*i.e.*, one design parameter) and real negative open-loop poles, the Routh-Hurwitz criterion can be applied to a fairly high order system with ease. For example, for the following closed-loop system characteristic equation:

$$1 + K_c \frac{1}{(s+3)(s+2)(s+1)} = 0$$

find the stability criteria.

We expand and rearrange the equation to the polynomial form:

$$s^3 + 6s^2 + 11s + (6 + K_c) = 0$$

The Hurwitz test requires that $K_c > -6$ or simply $K_c > 0$ for positive controller gains.

The Routh array is:

1:	1	11
2:	6	$6 + K_c$
3:	b_1	0

The Routh criterion requires, as in Eq. (7-8), that

$$b_1 = \frac{(6)(11) - (6 + K_c)}{6} > 0 \quad \text{or} \quad 60 > K_c$$

The range of proportional gain to maintain system stability is hence $0 < K_c < 60$.

✎ **Example 7.3**: Consider a second order process function $G_p = \frac{1}{s^2 + 2s + 1}$, which is critically damped. If we synthesize a control system with a PI controller, what are the stability constraints?

For illustration, we take $G_m = G_a = 1$ and the closed-loop transfer function for a servo problem is simply

$$\frac{C}{R} = \frac{G_c G_p}{1 + G_c G_p}$$

In this problem, the closed-loop characteristic equation is

$$1 + G_c G_p = 1 + K_c \left(1 + \frac{1}{\tau_I s}\right) \frac{1}{s^2 + 2s + 1} = 0$$

or

$$\tau_I s^3 + 2\tau_I s^2 + \tau_I (1 + K_c) s + K_c = 0$$

With the Routh-Hurwitz criterion, we need immediately $\tau_I > 0$ and $K_c > 0$. (The *s* term requires $K_c > -1$, which is overridden by the last constant coefficient.) The Routh array for this third order polynomial is

1:	τ_I	$\tau_I (1 + K_c)$
2:	$2\tau_I$	K_c
3:	b_1	0
4:	K_c	

With the use of (7-8), we require

$$b_1 = \frac{2\tau_I^2(1 + K_c) - \tau_I K_c}{2\tau_I} = \tau_I (1 + K_c) - \frac{K_c}{2} > 0$$

The inequality is rearranged to

$$\tau_I > \frac{K_c}{2\,(1 + K_c)} \quad \text{or} \quad \frac{2\tau_I}{1 - 2\tau_I} > K_c$$

which can be interpreted in two ways. For a given K_c, there is a minimum integral time constant. If the proportional gain is sufficiently large such that $K_c \gg 1$, the rough estimate for the integral time constant is $\tau_I > 1/2$. Or if the value of τ_I is less than 0.5, there is an upper limit on how large K_c could be.

If the given value of τ_I is larger than 0.5, the inequality simply infers that K_c must be larger than some negative number. To be more specific, if we pick $\tau_I = 1$,[1] the Routh criterion becomes

$$2 > \frac{K_c}{(1 + K_c)}$$

which of course can be satisfied for all $K_c > 0$. No new stability requirement is imposed in this case. Let us try another choice of $\tau_I = 0.1$. In this case, the requirement for the proportional gain is

$$0.2\,(1 + K_c) > K_c \quad \text{or} \quad K_c < 0.25$$

The entire range of stability for $\tau_I = 0.1$ is $0 < K_c < 0.25$. We will revisit this problem when we cover root locus plots; we can make much better sense without doing any algebraic work!

7.3 Direct Substitution Analysis

The closed-loop poles may lie on the imaginary axis at the moment a system becomes unstable. We can substitute $s = j\omega$ in the closed-loop characteristic equation to find the proportional gain that corresponds to this stability limit (which may be called marginal unstable). The value of this specific proportional gain is called the **critical** or **ultimate gain**. The corresponding frequency is called the **crossover** or **ultimate frequency**.

[1] Note that with this very specific case by choosing $\tau_I = 1$, the open-loop zero introduced by the PI controller cancels one of the open-loop poles of the process function at –1. If we do a root locus plot later, we'd see how the root loci change to that of a purely second order system. With respect to this example, the value is not important as long as $\tau_I > 1/2$.

✎ **Example 7.2A**: Apply direct substitution to the characteristic equation in Example 7.2:

$$s^3 + 6s^2 + 11s + (6 + K_c) = 0$$

Substitution of s = jω leads to

$$-j\omega^3 - 6\omega^2 + 11\omega j + (6 + K_c) = 0$$

The real and imaginary part equations are

$$\text{Re}: \; -6\omega^2 + (6 + K_c) = 0 \quad \text{or} \quad K_c = 6(\omega^2 - 1)$$

$$\text{Im}: \; -\omega^3 + 11\omega = 0 \quad \text{or} \quad \omega(11 - \omega^2) = 0$$

From the imaginary part equation, the ultimate frequency is $\omega_u = \sqrt{11}$. Substituting this value in the real part equation leads to the ultimate gain $K_{c,u} = 60$, which is consistent with the result of the Routh criterion.

If we have chosen the other possibility of $\omega_u = 0$, meaning that the closed-loop poles are on the real axis, the ultimate gain is $K_{c,u} = -6$, which is consistent with the other limit obtained with the Routh criterion.

✎ **Example 7.3A**: Repeat Example 7.3 to find the condition for the ultimate gain.

If we make the s = jω substitution in

$$\tau_I s^3 + 2\tau_I s^2 + \tau_I(1 + K_c)s + K_c = 0$$

it becomes

$$-\tau_I\omega^3 j - 2\tau_I\omega^2 + \tau_I(1 + K_c)\omega j + K_c = 0$$

We have two equations after collecting all the real and imaginary parts, and requiring both to be zero:

$$\text{Re}: \quad K_c - 2\tau_I\omega^2 = 0$$

$$\text{Im}: \quad \tau_I\,\omega\left[-\omega^2 + (1 + K_c)\right] = 0$$

Thus we have either $\omega = 0$ or $-\omega^2 + (1 + K_c) = 0$. Substitution of the real part equation into the nontrivial imaginary part equation leads to

$$-\omega^2 + 1 + 2\tau_I\omega^2 = 0 \quad \text{or} \quad \omega_u^2 = \frac{1}{1 - 2\tau_I}$$

where in the second form, we have added a subscript to denote the ultimate frequency, ω_u. Substitution of the ultimate frequency back in the real part equation gives the relation for the ultimate proportional gain

$$K_{c,u} = \frac{2\tau_I}{1 - 2\tau_I}$$

Note that if we have chosen the other possibility of $\omega_u = 0$, meaning where the closed-loop poles are on the real axis, the ultimate gain is $K_{c,u} = 0$, which is consistent with the other limit obtained using the Routh criterion. The result of direct substitution confirms the inequality derived from the Routh criterion, which should not be a surprise.

One may question whether direct substitution is a better method. There is no clear-cut winner here. By and large, we are less prone to making algebraic errors when we apply the Routh-Hurwitz recipe, and the interpretation of the results is more straightforward. With direct substitution, we do not have to remember any formulas, and we can find the ultimate frequency, which however, can be obtained with a root locus plot or frequency response analysis—techniques that we will cover later.

When the system has dead time, we must make an approximation, such as the Padé approximation, on the exponential dead time function before we can apply the Routh-Hurwitz criterion. The result is hence only an estimate. Direct substitution allows us to solve for the ultimate gain and ultimate frequency exactly. The next example illustrates this point.

✎ **Example 7.4**: Consider a system with a proportional controller and a first order process but with dead time. The closed-loop characteristic equation is given as

$$1 + K_c \frac{0.8e^{-2s}}{5s+1} = 0$$

Find the stability criteria of this system.

Let us first use the first order Padé approximation for the time delay function and apply the Routh-Hurwitz criterion. The approximate equation becomes

$$1 + K_c \frac{0.8}{5s+1}\frac{(-s+1)}{(s+1)} = 0$$

or

$$5s^2 + (6 - 0.8K_c)\,s + (1 + 0.8K_c) = 0$$

The Routh-Hurwitz criterion requires $6 - 0.8K_c > 0$ or $K_c < 7.5$, and $K_c > -1/0.8$. By keeping K_c positive, the approximate range of the proportional gain for system stability is $0 < K_c < 7.5$.

We now repeat the problem with the $s = j\omega$ substitution in the characteristic equation, and rewrite the time delay function with Euler's identity:

$$(5\omega j + 1) + 0.8K_c(\cos 2\omega - j\sin 2\omega) = 0$$

Collecting terms of the real and imaginary parts provides the two equations:

$$\text{Re}: \; 1 + 0.8K_c \cos 2\omega = 0 \quad \text{or} \quad K_c = -1/(0.8\cos 2\omega)$$

$$\text{Im}: \; 5\omega - 0.8K_c \sin 2\omega = 0$$

Substitution of the real part equation into the imaginary part equation gives

$$5\omega + \tan 2\omega = 0$$

The solution of this equation is the ultimate frequency $\omega_u = 0.895$, and from the real part equation, the corresponding ultimate proportional gain is $K_{c,u} = 5.73$. Thus the more accurate range of K_c that provides system stability is $0 < K_c < 5.73$.

Note 1: This result is consistent with the use of frequency response analysis later in Chapter 8.

Note 2: The iterative solution in solving the ultimate frequency is tricky. The equation has poor numerical properties—arising from the fact that $\tan\theta$ "jumps" from infinity at $\theta = (\pi/2)_-$ to negative infinity at $\theta = (\pi/2)_+$. To better see why, use MATLAB to make a plot of the function (LHS of the equation) with $0 < \omega < 1$. With MATLAB, we can solve the equation with the `fzero()` function. Create an M-file named `f.m`, and enter these two statements in it:

```
function y=f(x)
y = 5*x + tan(2*x);
```

After you have saved the file, enter at the MATLAB prompt:

```
fzero('f',0.9)
```

where 0.9 is the initial guess. MATLAB should return 0.8953. If you shift the initial guess just a bit, say using 0.8, you may get a "solution" of 0.7854. Note that (2)(0.7854) is $\pi/2$. If you blindly accept this incorrect value, $K_{c,u}$ will be infinity according to the real part equation. MATLAB is handy, but it is not foolproof!

7.4 Root Locus Analysis

The idea of a root locus plot is simple—if we have a computer. We pick *one* design parameter, say, the proportional gain K_C, and write a small program to calculate the roots of the characteristic polynomial for each chosen value of K_C as in 0, 1, 2, 3,...., 100,..., etc. The results (the values of the roots) can be tabulated or better yet, plotted on the complex plane. Even though the idea of plotting a root locus sounds so simple, it is one of the most powerful techniques in controller design and analysis *when there is no time delay*.

Root locus is *a graphical representation* of the roots of the closed-loop characteristic polynomial (*i.e.*, the closed-loop poles) as a chosen parameter is varied. Only the roots are plotted. The values of the parameter are not shown explicitly. The analysis most commonly uses the proportional gain as the parameter. The value of the proportional gain is varied from 0 to infinity, or in practice, just "large enough." Now, we need a simple example to get this idea across.

✎ **Example 7.5**: Construct the root locus plots of a first and second order system with a proportional controller. See how the loci approach infinity.

(a) Consider the characteristic equation of a simple system with a first order process and a proportional controller:

$$1 + K_c \frac{K_p}{\tau_p s + 1} = 0$$

The solution, meaning the closed-loop poles of the system, is

$$s = \frac{-(1 + K_c K_p)}{\tau_p}$$

The root locus plot (Fig. E7.5) is simply a line on the real axis starting at $s = -1/\tau_p$ when $K_C = 0$, and extends to negative infinity as K_C approaches infinity. As we increase the proportional gain, the system response becomes faster. Would there be an upper limit in reality? (Yes, saturation.)

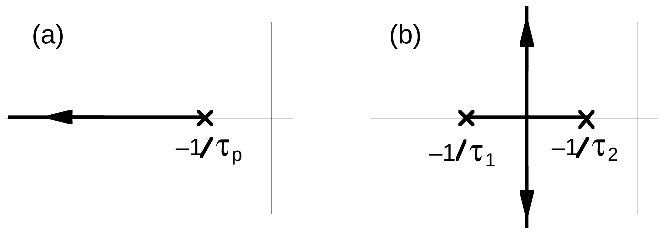


Figure E7.5. Root locus plots of (a) first order, and (b) second order systems.

(b) We repeat the exercise with a second order overdamped process function. The closed-loop characteristic equation of the closed-loop system is

$$1 + K_c \frac{K_p}{(\tau_1 s + 1)(\tau_2 s + 1)} = 0\,, \quad \text{or} \quad \tau_1 \tau_2 s^2 + (\tau_1 + \tau_2)\, s + (1 + K_c K_p) = 0$$

The two closed-loop poles are

$$s = \frac{-(\tau_1 + \tau_2) \pm \sqrt{(\tau_1 + \tau_2)^2 - 4\tau_1 \tau_2 (1 + K_c K_p)}}{2\tau_1 \tau_2}$$

In the mathematical limit of $K_c = 0$, we should find two negative real poles at

$$s = \frac{-(\tau_1 + \tau_2) \pm \sqrt{(\tau_1 - \tau_2)^2}}{2\tau_1 \tau_2} = -\frac{1}{\tau_1} \text{ or } -\frac{1}{\tau_2}$$

which are the locations of the two open-loop poles. (This result is easy to spot if we use the very first step without expanding the terms.) As K_c becomes larger, we should come to a point where we have two repeated roots at

$$s = \frac{-(\tau_1 + \tau_2)}{2\tau_1 \tau_2}$$

If we increase further the value of K_c, the closed-loop poles will branch off (or breakaway) from the real axis and become two complex conjugates (Fig. E7.5). No matter how large K_c becomes, these two complex conjugates always have the same real part as given by the repeated root. Thus what we find are two vertical loci extending toward positive and negative infinity. In this analysis, we also see how as we increase K_c, the *system* changes from overdamped to become underdamped, but it is always stable.

This is the idea behind the plotting of the closed-loop poles—in other words, construction of root locus plots. Of course, we need mathematical or computational tools when we have more complex systems. An important observation from Example 7.5 is that with simple first and second order systems with *no open-loop zeros* in the RHP, the closed-loop system is *always stable*.

We can now state the problem in more general terms. Let us consider a closed-loop characteristic equation $1 + K_c G_o = 0$, where $K_c G_o$ is referred to as the "open-loop" transfer function, G_{OL}. The proportional gain is K_c, and G_o is "everything" else. If we only have a proportional controller, then $G_o = G_m G_a G_p$. If we have other controllers, then G_o would contain parts of the controller function. We further denote G_o as a ratio of two polynomials, Q(s)/P(s), and rewrite the polynomials in the pole-zero form: [1]

$$1 + K_c G_o \;=\; 1 + K_c \frac{Q(s)}{P(s)} \;=\; 1 + K_c \frac{(s - z_1)\,(s - z_2)\ldots\,(s - z_n)}{(s - p_1)\,(s - p_2)\ldots\qquad(s - p_m)} \;=\; 0 \qquad (7\text{-}11)$$

or as

$$\Big[(s - p_1)\,(s - p_2)\ldots\;(s - p_m)\Big] \;+\; K_c\Big[(s - z_1)\,(s - z_2)\ldots\,(s - z_n)\Big] \;=\; 0 \qquad (7\text{-}11a)$$

The roots of the *m*-th order P(s) = 0, p_1, p_2, … , p_m, are the open-loop poles. The roots of the *n*-th order Q(s) = 0, z_1, z_2, … , z_n, are the open-loop zeros. The roots of the entire characteristic equation (7-11) are the closed-loop poles that will constitute the root loci.

There will be *m* root loci, matching the order of the characteristic polynomial. We can easily see that when $K_c = 0$, the poles of the closed-loop system characteristic polynomial $(1 + K_cG_o)$ are essentially the same as the poles of the open-loop. When K_c approaches infinity, the poles of the closed-loop system are the zeros of the open-loop. These are important mathematical features.

In other words, on a root locus plot, we expect the "trace" of the root loci to begin at the open-loop poles and terminate at the open-loop zeros (if there is one). For real systems, m > n, and n≥ 0. In these cases, the (*m* – *n*) root loci will originate from an open-loop pole and extend toward infinity somehow, depending on the specific problem.

Before reading further, it is very important that you go through at least the first half of MATLAB Session 6, and do computations with sample numbers while reading these root locus examples.

✎ **Example 7.2B**: Do the root locus plot and find the ultimate gain of Example 7.2 (p. 7-5).

The closed-loop equation from that example is:

$$1 + K_c \frac{1}{(s + 3)\,(s + 2)\,(s\; + 1)} \;=\; 0$$

We can easily use MATLAB to find that the ultimate gain is roughly 60. The statements to use are:

```
G=zpk([],[-1 -2 -3],1);
k=0:1:100;   % We have to use our own gain vector in this example
rlocus(G,k)  % because the MATLAB default plot does not cross the Im axis
rlocfind(G)
```

After entering the `rlocfind()` command, MATLAB will prompt us to click a point on the root locus plot. In this problem, we select the intersection between the root locus and the imaginary axis for the ultimate gain.

[1] If you cannot follow the fancy generalization, think of a simple problem such as a unity feedback loop with a PD controller and a first order process. The closed-loop characteristic equation is

$$1 + K_c \frac{K_p\,(\tau_D\, s + 1)}{(\tau_p\, s + 1)} = 0$$

The closed-loop pole "runs" from the point $s = –1/\tau_p$ at the mathematical limit of $K_c = 0$ to the point $s = –1/\tau_D$ as K_c approaches infinity.

✎ **Example 7.3B**: Repeat Example 7.3 (p. 7-5) with a root locus analysis.

The closed-loop characteristic equation from Example 7.3 is:

$$1 + K_c \frac{(\tau_I s + 1)}{\tau_I s\,(s^2 + 2s + 1)} = 0$$

Select various values of τ_I and use MATLAB to construct the root locus plots. Sample statements are:

```
taui=0.2;    % Open-loop zero at -5
G=tf([taui 1],conv([taui 0],[1 2 1]));
rlocus(G)
```

We should find that for values of $\tau_I > 0.5$, the system stays stable. For $\tau_I = 0.5$, the system may become unstable, but only at infinitely large K_c. The system may become unstable for $\tau_I < 0.5$ if K_c is too large. Finally, for the choice of $\tau_I = 0.1$, we should find with the MATLAB function `rlocfind` that the ultimate gain is roughly 0.25, the same answer from Example 7.3. How close you get depends on how accurate you can click the axis crossover point.

Even as we rely on MATLAB to generate root locus plots, it is important to appreciate how they are generated. To say the least, we need to know how to identify the open-loop poles and zeros and the direction of the loci. These hints are given in our *Web Support*. The following example illustrates some of the more common ones that we may encounter in control analysis. There are many other possibilities, but this exercise should be a good starting point. MATLAB Session 6 has more suggestions regarding the plots that are associated with the use of controllers.

✎ **Example 7.6**: Construct the root locus plots of some of the more common closed-loop equations with numerical values. Make sure you try them yourself with MATLAB.

(a) A sample first order system, and the MATLAB statement:

$$1 + K \frac{1}{(s + 2)} = 0$$ `rlocus(tf(1,[1 2]))`

(b) A second order system:

$$1 + K \frac{1}{(s + 2)\,(s + 1)} = 0$$ `rlocus(zpk([],[-1 -2],1))`

(c) Second order system with repeated open-loop poles:

$$1 + K \frac{1}{(s + 2)^2} = 0$$ `rlocus(zpk([],[-2 -2],1))`

(d) Second order system with different open-loop zeros:

$$1 + K \frac{(s + 0.5)}{(s + 2)\,(s + 1)} = 0$$ `rlocus(zpk(-0.5,[-1 -2],1))`

$$1 + K \frac{(s + 1.5)}{(s + 2)\,(s + 1)} = 0$$ `rlocus(zpk(-1.5,[-1 -2],1))`

$$1 + K \frac{(s + 4)}{(s + 2)\,(s + 1)} = 0$$ `rlocus(zpk(-4,[-1 -2],1))`

(e) Third order system:

$$1 + K \frac{1}{(s+3)(s+2)(s+1)} = 0$$

```
rlocus(zpk([],[-1 -2 -3],1))
```

(f) Third order system with an open-loop zero:

$$1 + K \frac{(s+1.5)}{(s+3)(s+2)(s+1)} = 0$$

```
rlocus(zpk(-1.5,[-1 -2 -3],1))
```

See also what the plot is like if the open-loop zero is at –0.5, –2.5, and –3.5.

These are rough sketches of what you should obtain with MATLAB. The root locus of the system in (a) is a line on the real axis extending to negative infinity (Fig. E7.6a). The root loci in (b) approach each other (arrows not shown) on the real axis and then branch off toward infinity at 90°. The repeated roots in (c) simply branch off toward infinity.

With only open-loop poles, examples (a) to (c) can only represent systems with a proportional controller. In case (a), the system contains a first orders process, and in (b) and (c) are overdamped and critically damped second order processes.

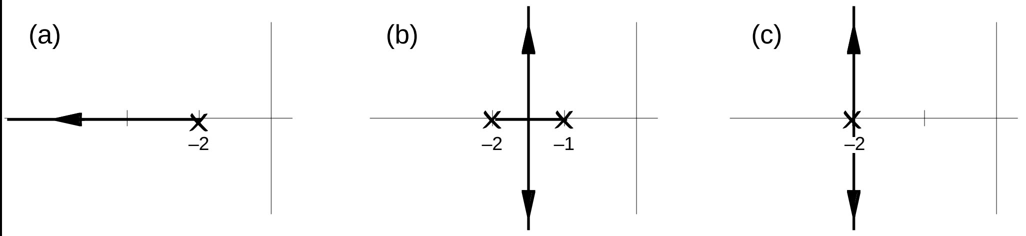


Figure E7.6a.

The sketches for (d) illustrate how an open-loop zero, say, in an ideal PD controller, may affect the root locus plot and dynamics of a system containing an overdamped second order process. Underdamped system behavior is expected only when the open-loop zero is large enough (*i.e.*, τ_D sufficiently small). On the left panel of Fig. E7.6b, one locus goes from the –1 open-loop pole to the open-loop zero at –0.5 (arrow not shown). The second locus goes from –2 to negative infinity on the real axis. In the middle panel, one locus goes from –1 to the open-loop zero at –1.5 (arrow not shown). On the right where the open loop zero is at –4, two root loci move toward each from –1 and –2 (arrows not shown), then branch off. The branches break in later onto the real axis; one locus approaches the zero at –4, the other toward negative infinity.

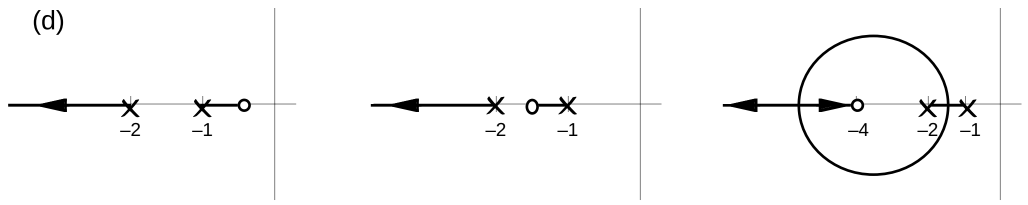


Figure E7.6b.

Systems (e) and (f) would contain a third order process. Of course, we can only have a proportional control in case (e), while (f) represents one probable scenario of using an ideal PD controller.

The system in (e) can become unstable, while a proper addition of an open-loop zero, as in (f), can help stabilize the system (Fig. E7.6c). In (e), the two loci from –1 and –2 approach each other (arrows not shown). They then break away and the closed-loop poles become unstable. The two loci approach positive infinity at ±60°. In (f), the system is always stable. The dominant closed-loop pole is the locus moving from –1 to –1.5 (arrow not shown). This system is faster than if we had put the open-loop zero at, say, –0.5.

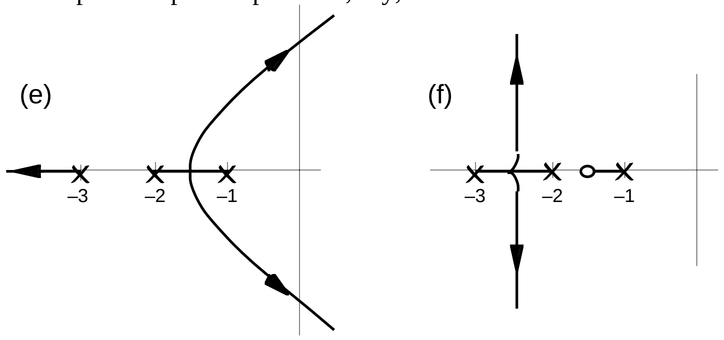


Figure E7.6c.

In most classic control texts, we find **plotting guidelines** to help hand sketching of root locus plots. After going over Example 7.6, you should find that some of them are quite intuitive. These simple guidelines are:

1. The root locus plot is symmetric about the real axis.
2. The number of loci equals the number of open-loop poles (or the order of the system).
3. A locus (closed-loop root path) starts at an open-loop pole and either terminates at an open-loop zero or extends to infinity.
4. On the real axis, a root locus only exists to the *left* of an odd number of real poles and zeros. (The explanation of this point is on our *Web Support*.)
5. The points at which the loci cross the imaginary axis can be found by the Routh-Hurwitz criterion or by substituting s = jω in the characteristic equation. (Of course, we can also use MATLAB to do that.)

To determine the shape of a root locus plot, we need other rules to determine the locations of the so-called breakaway and break-in points, the corresponding angles of departure and arrival, and the angle of the asymptotes if the loci approach infinity. They all arise from the analysis of the characteristic equation. These features, including item 4 above, are explained in our *Web Support* pages. With MATLAB, our need for them is minimal.

Note:

There are two important steps that we must follow. First, make sure you go through the MATLAB tutorial (Session 6) carefully to acquire a feel on the probable shapes of root locus plots. Secondly, test guidelines 3 and 4 listed above for every plot that you make in the tutorial. These guidelines can become your most handy tool to deduce, without doing any algebra, whether a system will exhibit underdamped behavior. Or in other words, whether a system will have complex closed-loop poles.

7.5 Root Locus Design

In terms of controller design, the closed-loop poles (or now the root loci) also tell us about the system dynamics. We can extract much more information from a root locus plot than from a Routh criterion analysis or a $s = j\omega$ substitution. In fact, it is common to impose, say, a time constant or a damping ratio specification on the system when we use root locus plots as a design tool.

✎ **Example 7.5A**: Consider the second order system in Example 7.5 (p. 7-9), what should the proportional gain be if we specify the controlled system to have a damping ratio of 0.7?

The second order closed-loop characteristic equation in Example 7.5 can be rearranged as

$$\frac{\tau_1\tau_2}{(1+K_cK_p)}s^2 + \frac{(\tau_1+\tau_2)}{(1+K_cK_p)}s + 1 = 0$$

We can compare this equation with the general form $\tau^2s^2 + 2\zeta\tau s + 1 = 0$, where now τ is the system time period and ζ is the system damping ratio. From Example 5.2 (p. 5-14), we have derived that

$$\tau = \sqrt{\frac{\tau_1\tau_2}{(1+K_cK_p)}}\ , \quad \text{and} \quad \zeta = \frac{1}{2}\frac{(\tau_1+\tau_2)}{\sqrt{\tau_1\tau_2(1+K_cK_p)}}$$

Thus we can solve for K_c with a given choice of ζ in a specific problem.

However, MATLAB allows us to get the answer with very little work—something that is very useful when we deal with more complex systems. Consider a numerical problem with values of the process gain $K_p = 1$, and process time constants $\tau_1 = 2$ and $\tau_2 = 4$ such that the closed-loop equation is

$$1 + K_c\frac{1}{(2s+1)(4s+1)} = 0$$

We enter the following statements in MATLAB: [1]

```
G=tf(1,conv([2 1],[4 1]));
rlocus(G)
sgrid(0.7,1)            % plot the 0.7 damping ratio lines
[kc,cpole]=rlocfind(G)
```

Where the root locus intersects the 0.7 damping ratio line, we should find, from the result returned by `rlocfind()`, the proportional gain to be 1.29 (1.2944 to be exact), and the closed-loop poles at $-0.375\pm0.382j$. The real and imaginary parts are not identical since $\cos^{-1}0.7$ is not exactly 45°. We can confirm the answer by substituting the values into our analytical equations. We should find that the real part of the closed-loop pole agrees with what we have derived in Example 7.5, and the value of the proportional gain agrees with the expression that we derived in this example.

[1] The technique of using the damp ratio line $\theta = \cos^{-1}\zeta$ in Eq. (2-34) is applied to higher order systems. When we do so, we are implicitly making the *assumption* that we have chosen the dominant closed-loop pole of a system and that this system can be approximated as a second order underdamped function at sufficiently large times. For this reason, root locus is also referred to as *dominant pole design*.

✎ **Example 7.7**: Consider installing a PI controller in a system with a first order process such that we have no offset. The process function has a steady state gain of 0.5 and a time constant of 2 min. Take $G_a = G_m = 1$. The system has the simple closed-loop characteristic equation:

$$1 + K_c \frac{0.5\,(\tau_I s + 1)}{\tau_I s\,(2s + 1)} = 0$$

We also want to have a slightly underdamped system with a reasonably fast response and a damping ratio of 0.7. How should we design the controller? To restrict the scenario for illustration, we consider (a) τ_I = 3 min, and (b) τ_I = 2/3 min.

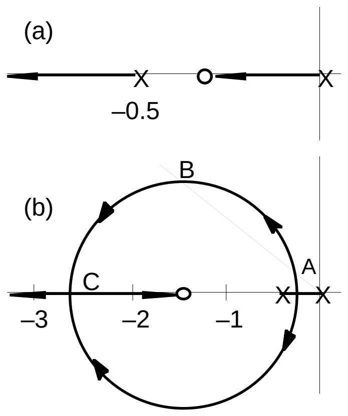


Figure E7.7

To begin with, this is a second order system with no positive zeros and so stability is not an issue. Theoretically speaking, we could have derived and proved all results with the simple second order characteristic equation, but we take the easy way out with root locus plots.

(a) The open-loop poles are at –0.5 and at the origin. If the integral time constant is τ_I = 3 min, the open-loop zero is at –1/3, and all we have are negative and real closed-loop poles (Fig. E7.7a). The system will not become underdamped. The "speed" of the response, as we increase the proportional gain, is limited by the zero position at –1/3. This is where the dominant closed-loop pole will approach.

(b) The situation is more interesting with a smaller integral time constant, τ_I = 2/3 min, with now the open-loop zero at –1.5. The root loci first approach one another (arrows not shown) before breaking away to form the circle (Fig. E7.7b). As K_c increases further, the loci break into the real axis. The dominant closed-loop pole will approach the zero at –1.5.

We use the following MATLAB statements to draw the 0.7 damping ratio line, and it intersects the root loci at two points (Fig. E7.7): at (A), –0.312+0.323j when K_c = 0.55, and at (B), –1.15+1.17j when K_c = 7.17.

```
kc=1; taui=2/3;
Gc=tf(kc*[taui 1], [taui 0]);
Gp=tf(0.5, [2 1]);
rlocus(Gc*Gp)
sgrid(0.7,1)
[kc,cpole]=rlocfind(Gc*Gp)
```

If saturation is not a problem, the proportional gain K_c = 7.17 (point B) is preferred. The corresponding closed-loop pole has a faster time constant. (The calculation of the time period or frequency and confirmation of the damping ratio is left as homework.)

Note 1: Theoretically speaking, the point C on the root locus plot is ideal—the fastest possible response without any oscillation. We rarely can do that in practice; the proportional gain would have been so large that the controller would be saturated.

Note 2: As we reduce the integral time constant from τ_I = 3 min to exactly 2 min, we have the situation of **pole-zero cancellation**. The terms in the closed-loop characteristic equation cancel out, and the response in this example is reduced to that of a first order system. If τ_I is only slightly less than 2 min, we have a very slightly underdamped system; the "circle" in (b) is very small (try this with MATLAB). In reality, it is very difficult to have perfect pole-zero cancellation, and if we design the controller with τ_I too close to τ_p, the system response may "bounce back and forth."

A final remark on root locus plots

For as instructive as root locus plots appear to be, this technique does have its limitations. The most important one is that it cannot handle dead time easily. When we have a system with dead time, we must make an approximation with the Padé formulas. This is the same limitation that applies to the Routh-Hurwitz criterion.

In this computer age, one may question why nobody would write a program that can solve for the roots with dead time accurately? Someone did. There are even refined hand sketching techniques to account for the lag due to dead time. However, these tools are not as easy to apply and are rarely used. Few people use them because frequency response analysis in Chapter 8 can handle dead time accurately and extremely easily.

A second point on root locus plot is that it can only give us the so-called absolute stability, but not relative stability, meaning that there is no easy way to define a good, general "safety margin." We may argue that one can define a certain distance which a closed-loop pole must stay away from the imaginary axis, but such approach is very problem specific. Recall that the proportional gain is an implicit parameter along a locus, and is very difficult to tell what effects one may have with slight changes in the proportional gain (the sensitivity problem). Frequency response analysis once again does better and allows us to define *general* relative stability criteria. Furthermore, frequency response analysis can help us to understand why a certain choice of, for example, an integral time constant may destabilize a system. (Jumping ahead, it has to do with when we bring in phase lead. We shall see that in Chapter 8 and Example 10.1.)

On the other hand, frequency response analysis cannot reveal information on dynamic response easily—something root locus does very well. Hence controller design is always an iterative procedure. There is no one-stop-shopping. There is never a unique answer.

Finally, you may wonder if one can use the integral or the derivative time constant as the parameter. Theoretically, we can. We may even rearrange the characteristic equation in such a way that can take advantage of pre-packaged programs that use the proportional gain as the parameter. In practice, nobody does that. One main reason is that the resulting loci plot will not have this nice interpretation that we have by varying the proportional gain.

❐ Review Problems

1. Repeat Examples 7.2 and 7.4 with the general closed-loop characteristic polynomial:

$$a_3 s^3 + a_2 s^2 + a_1 s + a_o + K_c = 0$$

Derive the ultimate frequency and ultimate gain.

2. No additional reviews are needed as long as you go through each example carefully with MATLAB.

❖ 8. *Frequency Response Analysis*

The frequency response of a given system *at large times* is characterized by its amplitude and phase shift when the input is a sinusoidal wave. These two quantities can be obtained from the transfer function, of course, without inverse transform. The analysis is based on a simple substitution (mapping) $s = j\omega$, and the information is given by the magnitude (modulus) and the phase angle (argument) of the transfer function. Since the analysis begins with Laplace transform, we are still limited to linear or linearized models.

What are we up to?

- Theoretically, we are making the presumption that we can study and understand the dynamic behavior of a process or system by imposing a sinusoidal input and measuring the frequency response. With chemical systems that cannot be subject to frequency response experiments easily, it is very difficult for a beginner to appreciate what we will go through. So until then, take frequency response as a math problem.
- Both the magnitude and the argument are functions of the frequency. The so-named Bode and Nyquist plots are nothing but graphical representations of this functional dependence.
- Frequency response analysis allows us to derive a general relative stability criterion that can easily handle systems with time delay. This property is used in controller design.

8.1 Magnitude and Phase Lag

Our analysis is based on the mathematical property that given a *stable* process (or system) and a sinusoidal input, the response will eventually become a purely sinusoidal function. This output will have the same frequency as the input, but with different amplitude and phase angle. The two latter quantities can be derived from the transfer function.

We first illustrate the idea of frequency response using inverse Laplace transform. Consider our good old familiar first order model equation, [1]

$$\tau_p \frac{dy}{dt} + y = K_p f(t) \tag{8-1}$$

with its transfer function

$$G(s) = \frac{Y(s)}{F(s)} = \frac{K_p}{\tau_p s + 1} \tag{8-2}$$

If the input is a sinusoidal function such that $f(t) = A\sin\omega t$, the output $Y(s)$ is

$$Y(s) = \frac{K_p}{\tau_p s + 1} \frac{A\omega}{s^2 + \omega^2} \tag{8-3}$$

If we do the partial fraction expansion and inverse transform, we should find, after some hard work, the time domain solution:

---

[1] While we retain our usual notation of using a subscript *p* for a first order function in illustrations, the analysis is general; it applies to an open-loop process or a closed-loop system. In other words, we can apply a sinusoidal change to the set point of a system if we need to.

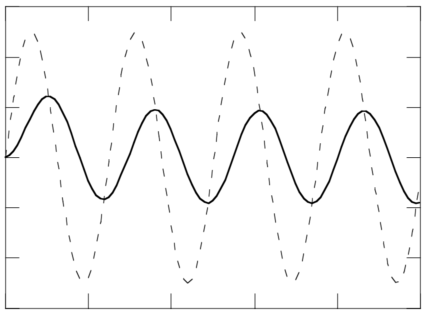


Figure 8.1. Schematic response (solid curve) of a first order function to a sinusoidal input (dashed). The response has a smaller amplitude, a phase lag, and its exponential term decays away quickly to become a pure sinusoidal response.

$$y(t) = \frac{AK_p\tau_p\omega}{\tau_p^2\omega^2 + 1}\, e^{-t/\tau_p} + \left[\frac{AK_p}{\sqrt{\tau_p^2\omega^2 + 1}}\right]\sin(\omega t + \phi) \qquad (8\text{-}4)$$

where

$$\phi = \tan^{-1}(-\omega\tau_p) \qquad (8\text{-}5)$$

is the phase lag.[1] The algebraic details of deriving Eq. (8-4) are not important. The important aspects will be derived via an alternative route just a few steps ahead. For now, the crucial point is to observe that if time is sufficiently large (as relative to τ_p), the exponential term in Eq. (8-4) will decay away, and the time response becomes a purely sinusoidal function (Fig. 8.1).

The time response at sufficiently large times can be normalized with respect to the amplitude of the input sine wave:

$$\frac{y_\infty(t)}{A} = \left[\frac{K_p}{\sqrt{\tau_p^2\omega^2 + 1}}\right]\sin(\omega t + \phi) \qquad (8\text{-}6)$$

where now the amplitude of this normalized large-time response is called the *amplitude ratio* (AR). The response has a different amplitude and phase lag in relation to the input wave. If we further normalize the equation such that the amplitude of the sine wave is bounded by one, we obtain

$$\frac{y_\infty(t)}{AK_p} = \left[\frac{1}{\sqrt{\tau_p^2\omega^2 + 1}}\right]\sin(\omega t + \phi) \qquad (8\text{-}7)$$

The amplitude of this normalized response, y_∞/AK_p, is called the *magnitude ratio.* [2]

Next, let's substitute $s = j\omega$ in the transfer function, *i.e.*, $G_p(s) = G_p(j\omega)$, which makes G_p a complex number:[3]

$$G_p(j\omega) = \frac{K_p}{j\omega\tau_p + 1}\left(\frac{-j\omega\tau_p + 1}{-j\omega\tau_p + 1}\right) = \left(\frac{K_p}{\tau_p^2\omega^2 + 1}\right) - j\left(\frac{K_p\,\omega\tau_p}{\tau_p^2\omega^2 + 1}\right) \qquad (8\text{-}8)$$

If we put (8-8) in polar coordinates, $G_p(j\omega) = |G_p(j\omega)|\, e^{j\phi}$, the magnitude and phase angle are

$$|G_p(j\omega)| = \frac{K_p}{\sqrt{\tau_p^2\omega^2 + 1}}, \text{ and } \phi = \angle G_p(j\omega) = \tan^{-1}(-\omega\tau_p) \qquad (8\text{-}9)$$

A comparison of Eqs. (8-9) with (8-6) shows that the magnitude and the phase angle of $G_p(j\omega)$ are exactly the same as the amplitude and phase lag of the normalized "large time" time domain solution.

---

[1] In these calculations, the units of frequency are radian/time. If the period T is given in s, then frequency $\omega = 1/T$ Hz [Hertz or cycles/s] or $\omega = 2\pi/T$ [radian/s].

[2] This is a major source of confusion in many texts. The magnitude ratio is *not* the magnitude of a transfer function. It is the amplitude ratio that is the same as the magnitude of G(s). To avoid confusion, we'll stick strictly with the mathematical property, *i.e.*, the magnitude of G(s). We will use neither amplitude ratio nor magnitude ratio in our terminology. It is much more sensible to consider the magnitude of a transfer function.

[3] If you need that, a brief summary of complex variable definitions is on our *Web Support*.

We should note that the magnitude and phase angle of $G_p(j\omega)$ are functions of the input frequency. The larger the frequency, the lower the magnitude, and the larger the phase lag. We can make this observation by writing $\tau_p\omega = \omega/\omega_p$. When the imposed frequency is large with respect to the process "natural" frequency, ω_p, the process cannot respond fast enough, resulting in a low magnitude and large phase lag. When the imposed frequency is relatively small, the magnitude approaches the steady state gain and the phase lag approaches zero.

8.1.1 The general analysis

We now generalize our simple illustration. Consider a general transfer function of a *stable* model G(s), which we also denote as the ratio of two polynomials, G(s) = Q(s)/P(s). We impose a sinusoidal input f(t) = A sin ωt such that the output is

$$Y(s) = G(s)\left(\frac{A\omega}{s^2+\omega^2}\right) = \frac{Q(s)}{P(s)}\,\frac{A\omega}{(s+j\omega)(s-j\omega)}$$

Since the model is stable, all the roots of P(s), whether they be real or complex, have negative real parts and their corresponding time domain terms will decay away exponentially. Thus if we are only interested in the time domain response at sufficiently large times, we only need to consider the partial fraction expansion of the two purely sinusoidal terms associated with the input:

$$Y_\infty(s) = \frac{a}{(s+j\omega)} + \frac{a^*}{(s-j\omega)} \qquad (8\text{-}10)$$

We can find their inverse transform easily. Apply the Heaviside expansion,

$$a = (s+j\omega)Y(s)\big|_{s=-j\omega} = G(-j\omega)\,\frac{A\omega}{-2j\omega} = \frac{AG(-j\omega)}{-2j}$$

and its conjugate, redundantly just in case you do not believe the result is correct,

$$a^* = (s-j\omega)Y(s)\big|_{s=+j\omega} = G(j\omega)\,\frac{A\omega}{2j\omega} = \frac{AG(j\omega)}{2j}$$

The time domain solution (at large times) is hence

$$y_\infty(t) = \left[\frac{AG(-j\omega)}{-2j}\right]e^{-j\omega t} + \left[\frac{AG(j\omega)}{2j}\right]e^{j\omega t}$$

Note that $G(j\omega) = |G(j\omega)|\,e^{j\phi}$, and $G(-j\omega) = |G(j\omega)|\,e^{-j\phi}$, and we can write

$$\frac{y_\infty(t)}{A} = |G(j\omega)|\left[\frac{e^{-j\phi}e^{-j\omega t}}{-2j} + \frac{e^{j\phi}e^{j\omega t}}{2j}\right]$$

Apply Euler's identity and the final result for the normalized response is

$$\left[\frac{y_\infty(t)}{A}\right] = |G(j\omega)|\,\sin(\omega t+\phi)\,,\ \text{ where }\ \phi = \angle G(j\omega) \qquad (8\text{-}11)$$

This is a crucial result. It constitutes the basis of frequency response analysis, where in general, all we need are the magnitude and the argument of the transfer function G(s) after the substitution $s = j\omega$.

8.1.2 Some important properties

We need to appreciate some basic properties of transfer functions when viewed as complex variables. They are important in performing frequency response analysis. Consider that any given transfer function can be "broken up" into a *product* of simpler ones:

$$G(s) = G_1(s)G_2(s) \dots G_n(s) \tag{8-12}$$

We do not need to expand the entire function into partial fractions. The functions G_1, G_2, etc., are better viewed as simply first and at the most second order functions. In frequency response analysis, we make the s = jω substitution and further write the function in terms of magnitude and phase angle as:

$$G(j\omega) = G_1(j\omega)\, G_2(j\omega) \dots\ G_n(j\omega) = |G_1(j\omega)|\, e^{j\phi_1} |G_2(j\omega)|\, e^{j\phi_2} \dots\ |G_n(j\omega)|\, e^{j\phi_n}$$

or

$$G(j\omega) = |G_1(j\omega)|\,|G_2(j\omega)| \dots\ |G_n(j\omega)|\, e^{j\,(\phi_1 + \phi_2 + \dots\ + \phi_n)}$$

The magnitude of G(jω) is

$$|G(j\omega)| = |G_1(j\omega)|\,|G_2(j\omega)| \dots\ |G_n(j\omega)| \tag{8-13}$$

or

$$\log|G(j\omega)| = \log|G_1| + \log|G_2| + \dots + \log|G_n|$$

The phase angle is

$$\phi = \angle G(j\omega) = \angle G_1(j\omega) + \angle G_2(j\omega) + \dots\ + \angle G_n(j\omega) \tag{8-14}$$

✎ **Example 8.1**. Derive the magnitude and phase lag of the following transfer function:

$$G(s) = \frac{(\tau_a s + 1)}{(\tau_1 s + 1)\,(\tau_2 s + 1)}$$

We can rewrite the function as a product of

$$G(s) = (\tau_a s + 1)\,\frac{1}{(\tau_1 s + 1)}\,\frac{1}{(\tau_2 s + 1)}$$

The magnitude and phase angle of these terms with the use of Eq. (8-9) are

$$|G(j\omega)| = \sqrt{1 + \tau_a^2 \omega^2}\;\frac{1}{\sqrt{1 + \tau_1^2 \omega^2}}\;\frac{1}{\sqrt{1 + \tau_2^2 \omega^2}}$$

and

$$\phi = \tan^{-1}(\omega\tau_a) + \tan^{-1}(-\,\omega\tau_1) + \tan^{-1}(-\,\omega\tau_2)$$

We have not covered the case of a zero term in the numerator. Here, we are just guessing that its result is the reciprocal of the first order function result in Eq. (8-9). The formal derivation comes later in Example 8.4. Note that τ has units of time, ω is radian/time, and τω is in radian.

We could have put G(s) in a slightly different form:

$$G(j\omega) = \frac{G_a(j\omega)G_b(j\omega) \dots\ G_m(j\omega)}{G_1(j\omega)G_2(j\omega) \dots\ G_n(j\omega)} = \frac{|G_a|\,|G_b| \dots}{|G_1|\,|G_2| \dots}\, e^{j\,(\theta_a + \theta_b + \dots\ - \theta_1 - \theta_2 - \dots)} \tag{8-15}$$

In this case, the equivalent form of Eq. (8-13) is

$$\log|G(j\omega)| = \left[\log|G_a| + \log|G_b| + \ldots + \log|G_m|\right] - \left[\log|G_1| + \log|G_2| + \ldots + \log|G_n|\right] \qquad (8\text{-}16)$$

and the equivalent to Eq. (8-14) is

$$\phi = \left[\angle G_a + \angle G_b + \ldots + \angle G_m\right] - \left[\angle G_1 + \angle G_2 + \ldots + \angle G_n\right] \qquad (8\text{-}17)$$

With these results, we are ready to construct plots used in frequency response analysis. The important message is that we can add up the contributions of individual terms to construct the final curve. The magnitude, of course, would be on the logarithmic scale.

8.2 Graphical analysis tools

We know that both $|G(j\omega)|$ and $\angle(G(j\omega))$ are functions of frequency, ω. We certainly would like to see the relationships graphically. There are three common graphical representations of the frequency dependence. We first describe all three methods briefly. Our introduction relies on the use the so-called Bode plots and more details will follow with respective examples.

8.2.1 Magnitude and Phase Plots — $\log G(j\omega)$ vs. $\log \omega$ and $\angle G(j\omega)$ vs. $\log \omega$

Here, we simply plot the magnitude (modulus) and phase angle (argument) individually against frequency—the so-called **Bode plots**. From Eq. (8-16), we should use a log scale for the magnitude. We also use a log scale for the frequency to cover a larger range. Thus we use a log-log plot for $|G(j\omega)|$ versus ω, and a semi-log plot for $\angle G(j\omega)$ versus ω. The unit of phase angle is commonly converted to degrees, and frequency is in radian per unit time.

In most electrical engineering or industrial control books, the magnitude is plotted in units of decibel (dB) as

$$1 \text{ dB} = 20 \log |G(j\omega)| \qquad (8\text{-}18)$$

Even with MATLAB, we should still know the expected shape of the curves and its “telltale” features. This understanding is crucial in developing our problem solving skills. Thus doing a few simple hand constructions is very instructive. When we sketch the Bode plot, we must identify the corner (break) frequencies, slopes of the magnitude asymptotes and the contributions of phase lags at small and large frequencies. We’ll pick up the details in the examples.

Another advantage of frequency response analysis is that one can “identify” the process transfer function with experimental data. With either a frequency response experiment or a pulse experiment with proper Fourier transform, one can construct the Bode plot using the open-loop transfer functions and use the plot as the basis for controller design.[1]

8.2.2 Polar Coordinate Plots — $G(j\omega)$ in polar coordinates, or $\text{Re}[G(j\omega)]$ vs. $\text{Im}[G(j\omega)]$

We can plot the real and imaginary parts of $G(j\omega)$ on the *s*-plane with ω as the parameter—the so-called **Nyquist plot**. Since a complex number can be put in polar coordinates, the Nyquist plot is also referred to as the polar plot.

$$G(j\omega) = \text{Re}[G(j\omega)] + \text{Im}\,[G(j\omega)] = |G(j\omega)|\,e^{j\phi}$$

This plotting format contains the same information as the Bode plot. The polar plot is more compact, but the information on the frequency is not shown explicitly. If we do not have a computer, we theoretically could read numbers off a Bode plot to construct the Nyquist plot. The use of Nyquist plots is more common in multiloop or multivariable analyses. A Bode plot, on the

[1] The pulse experiment is not crucial for our understanding of frequency response analysis and is provided on our *Web Support*, but we will do the design calculations in Section 8.3.

other hand, is easier to interpret and a good learning tool.

There are hand-plotting techniques, but we'll rely on the computer. Still, we need to know the qualitative features of the plot resulting from several simple transfer functions.

8.2.3 Magnitude vs Phase Plot — log $G(j\omega)$ vs. $\angle G(j\omega)$

In the third rendition of good old $G(j\omega)$, we can plot the logarithmic magnitude against the phase lag—the so-called **Nichols chart**. Generally, this plot is made using the open-loop transfer function of a unity feedback system. The magnitude and argument contours of the closed-loop transfer function are overlaid on it. The plot allows for a frequency response design analysis that better relates to probable closed-loop dynamic response.

For now, we'll take a look at the construction of Bode and Nyquist plots of transfer functions that we have discussed in Chapters 2 and 3. Keep in mind that these plots contain the same information: $G(j\omega)$. It is important that you run MATLAB with sample numerical values while reading the following examples. Yes, you need to go through MATLAB Session 7 first.

✎ **Example 8.2**. What are the Bode and Nyquist plots of a **first order transfer function**?

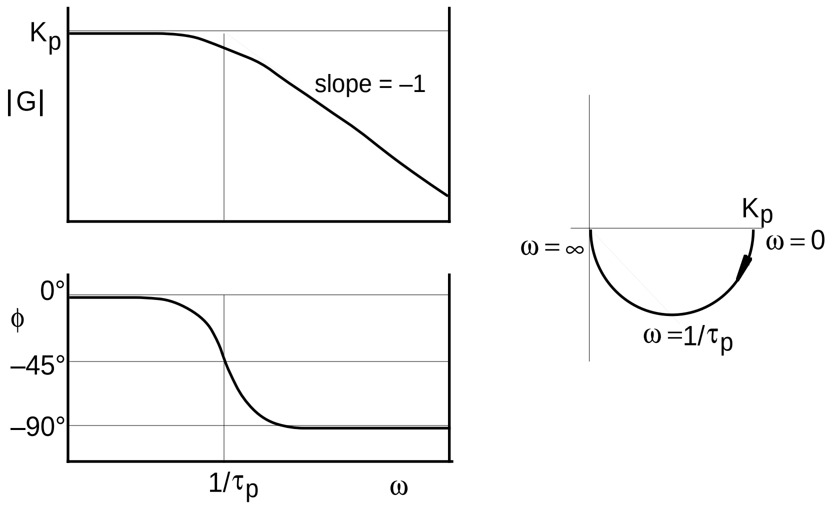


Figure E8.2

We will use the time constant form of transfer functions. The magnitude and phase angle of

$$G(s) = \frac{Y(s)}{F(s)} = \frac{K_p}{\tau_p s + 1}$$

are derived in Eq. (8-9) as

$$\left| G_p(j\omega) \right| = \frac{K_p}{\sqrt{\tau_p^2 \omega^2 + 1}} \qquad \text{and} \qquad \phi = \angle G_p(j\omega) = \tan^{-1}(-\omega\tau_p)$$

We may try the following MATLAB statements to generate Fig. E8.2:

```
kp=1; % Just arbitrary values. Try different ones yourself.
tau=2;
G=tf(kp,[tau 1]);
figure(1), bode(G);
figure(2), nyquist(G);  %If the nyquist default plot is confusing,
                        %follow the instructions in MATLAB Session 7
```

We plot, in theory:

$$\log\left|G_p(j\omega)\right| = \log K_p - \frac{1}{2}\log(1+\tau_p^2\omega^2)$$

To make the phase angle plot, we simply use the definition of $\angle G_p(j\omega)$. As for the polar (Nyquist) plot, we do a frequency parametric calculation of $|G_p(j\omega)|$ and $\angle G_p(j\omega)$, or we can simply plot the real part versus the imaginary part of $G_p(j\omega)$.[1] To check that a computer program is working properly, we only need to use the high and low frequency asymptotes—the same if we had to do the sketch by hand as in the old days. In the limit of low frequencies,

$$\omega \longrightarrow 0,\ |G_p| = K_p, \quad \text{and}\ \ \phi = 0$$

On the magnitude plot, the low frequency (also called zero frequency) asymptote is a horizontal line at K_p. On the phase angle plot, the low frequency asymptote is the 0° line. On the polar plot, the zero frequency limit is represented by the point K_p on the real axis. In the limit of high frequencies,

$$\omega \longrightarrow \infty,\ |G_p| = \frac{K_p}{\tau_p\,\omega}, \quad \text{and}\ \ \phi = \tan^{-1}(-\infty) = -90°$$

With the phase lag, we may see why a first order function is also called a **first order lag**. On the magnitude log-log plot, the high frequency asymptote has a slope of –1. This asymptote also intersects the horizontal K_p line at $\omega = 1/\tau_p$. On the phase angle plot, the high frequency asymptote is the –90° line. On the polar plot, the infinity frequency limit is represented by the origin as the $G_p(j\omega)$ locus approaches it from the –90° angle.

The frequency at which $\omega = 1/\tau_p$ is called the **corner frequency** (also **break** frequency). At this position,

$$\omega = 1/\tau_p,\ |G_p| = K_p/\sqrt{2}, \quad \text{and}\ \ \phi = \tan^{-1}(-1) = -45°$$

One may question the significance of the break frequency, $\omega = 1/\tau$. Let's take the first order transfer function as an illustration. If the time constant is small, the break frequency is large. In other words, a fast process or system can respond to a large range of input frequencies without a diminished magnitude. On the contrary, a slow process or system has a large time constant and a low break frequency. The response magnitude is attenuated quickly as the input frequency increases. Accordingly, the phase lag also decreases quickly to the theoretical high frequency asymptote.

A common term used in control engineering is **bandwidth**, which is defined as the frequency at which any given $|G(j\omega)|$ drops to 70.7% of its low frequency asymptotic value (Fig. 8.2). It

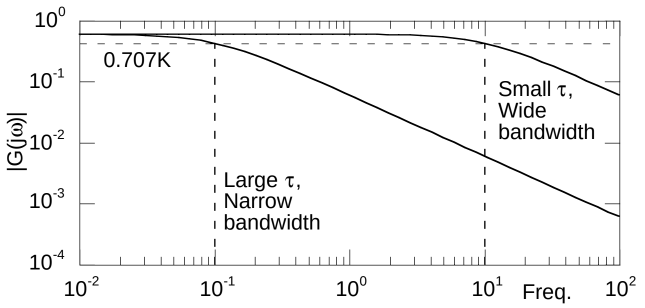


Figure 8.2. Schematic illustration of two systems with wide and narrow bandwidths.

[1] All comments on Nyquist plots are made without the need of formal hand sketching techniques. Strictly speaking, the polar plot is a mapping of the imaginary axis from $\omega = 0_+$ to ∞. You'll see that in texts that provide a more thorough discussion on choosing the so-called Nyquist path.

is obvious that the 70.7% comes from the $1/\sqrt{2}$ of the first order function calculation in Example 8.2.[1] A large bandwidth is related to a fast system with a small time constant. On the other hand, slow responses or large time constants are cases with narrow bandwidth. From another perspective, the first order function is the simplest example of a **low-pass filter**. Signals with frequencies below the bandwidth are not attenuated, while their magnitude diminishes with higher frequencies.

✎ **Example 8.3**. What are the Bode and Nyquist plots of a **second order transfer function**?

We make the s = jω substitution in the transfer function

$$G(s) = \frac{K}{\tau^2 s^2 + 2\zeta\tau s + 1}$$

to obtain

$$G(j\omega) = \frac{K}{(1-\tau^2\omega^2) + j\,2\zeta\tau\omega} = \frac{K\,[(1-\tau^2\omega^2) - j\,2\zeta\tau\omega]}{\left(1-\tau^2\omega^2\right)^2 + \left(2\zeta\tau\omega\right)^2}$$

After a few algebraic steps, the resulting magnitude and phase angle of G(jω) are:

$$|\,G(j\omega)\,| = \frac{K}{\sqrt{\left(1-\tau^2\omega^2\right)^2 + (2\zeta\tau\omega)^2}}\;, \quad \text{and} \quad \phi = \angle G(j\omega) = \tan^{-1}\left(\frac{-2\zeta\tau\omega}{1-\tau^2\omega^2}\right)$$

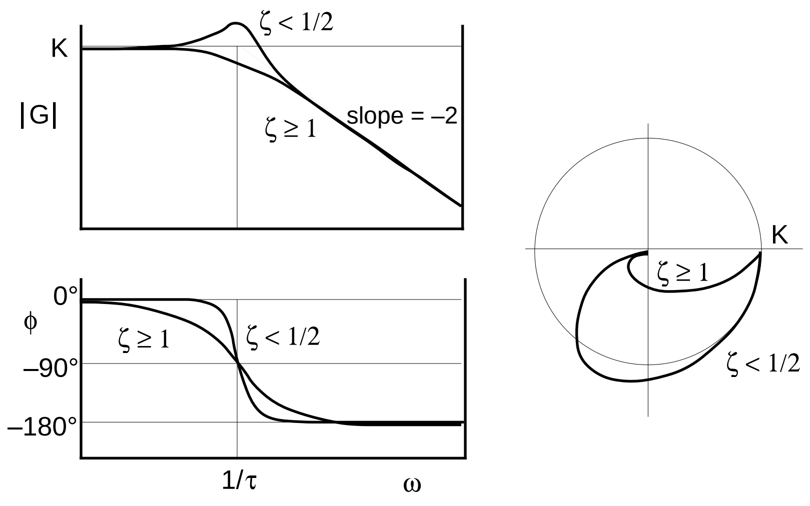


Figure E8.3

These are sample MATLAB statements to plot the magnitude and phase angle as in Fig. E8.3:

```
k=1;   % Just arbitrary values. Try different ones yourself.
tau=2;
zeta=0.2;
G=tf(k,[tau*tau 2*zeta*tau 1]);
damp(G)   % confirm the damping ratio
figure(1), bode(G);
figure(2), nyquist(G);
```

In the limit of low frequencies,

[1] This is also referred to as the 3 dB bandwidth. The term comes from $20\log(1/\sqrt{2}) = -3.01$ dB.

$\omega \longrightarrow 0, \quad |G| = K, \quad \text{and} \quad \phi = 0$

And in the limit of high frequencies,

$$\omega \longrightarrow \infty, \quad |G| = \frac{K}{\tau^2\omega^2}, \quad \text{and} \quad \phi \approx \tan^{-1}\left(\frac{-2\zeta\tau\omega}{-\tau^2\omega^2}\right) = \tan^{-1}\left(\frac{2\zeta}{\tau\omega}\right) = \tan^{-1}(0) = -180°$$

We choose –180° (and not 0°) because we know that there must be a phase lag. On the magnitude log-log plot, the high frequency asymptote has a slope of –2. This asymptote intersects the horizontal K line at $\omega = 1/\tau$.

At the corner frequency,

$$\omega = 1/\tau, \quad |G(j\omega)| = \frac{K}{2\zeta}, \quad \text{and} \quad \phi = \tan^{-1}(-\infty) = -90°$$

For a process or system that is sufficiently underdamped, $\zeta < 1/2$, the magnitude curve will rise above the low frequency asymptote, or the polar plot will extend beyond the K-radius circle.

We can take the derivative of the magnitude equation $|G(j\omega)| = K / \sqrt{\left(1-\tau^2\omega^2\right)^2 + (2\zeta\tau\omega)^2}$ to find the actual maximum and its associated frequency, the so-called **resonant frequency**, ω_r: [1]

$$\omega_r = \frac{\sqrt{1-2\zeta^2}}{\tau} = \omega\sqrt{1-2\zeta^2} \qquad (8\text{-}19)$$

and the maximum magnitude $M_{p\omega}$ is

$$M_{p\omega} = |G(j\omega)|_{\max} = \frac{K}{2\zeta\sqrt{1-\zeta^2}} \qquad (8\text{-}20)$$

From Eq. (8-19), there would only be a maximum if $0 < \zeta < 1/\sqrt{2}$ (or 0.707).

We can design a controller by specifying an upper limit on the value of $M_{p\omega}$. The smaller the system damping ratio ζ, the larger the value of $M_{p\omega}$ is, and the more overshoot, or underdamping we expect in the time domain response. Needless to say, excessive resonance is undesirable.

✎ **Example 8.4**. What are the Bode and Nyquist plots of a **first order lead** $G(s) = (\tau_p s + 1)$?

After the $s = j\omega$ substitution, we have

$G(j\omega) = 1 + j\omega\tau_p$

where

$$|G(j\omega)| = \sqrt{1+\omega^2\tau_p^2} \qquad \text{and} \qquad \angle G(j\omega) = \tan^{-1}(\omega\tau_p)$$

We may try the following MATLAB statements for Fig. E8.4:

```
taud=2;   % Just an arbitrary value
G=tf([taud 1],1);
```

[1] The step just before the result in (8-19) is $2\,(1-\tau^2\omega^2)\,(-2\omega\tau^2) + 2\,(2\zeta\tau\omega)\,(2\zeta\tau) = 0$. In most texts, the derivation is based on unity gain K = 1 and so it will not show up in (8-20). Most texts also plot (8-19) and (8-20) as a function of ζ. But with MATLAB, we can do that ourselves as an exercise.

```
figure(1), bode(G);
figure(2), nyquist(G);
```

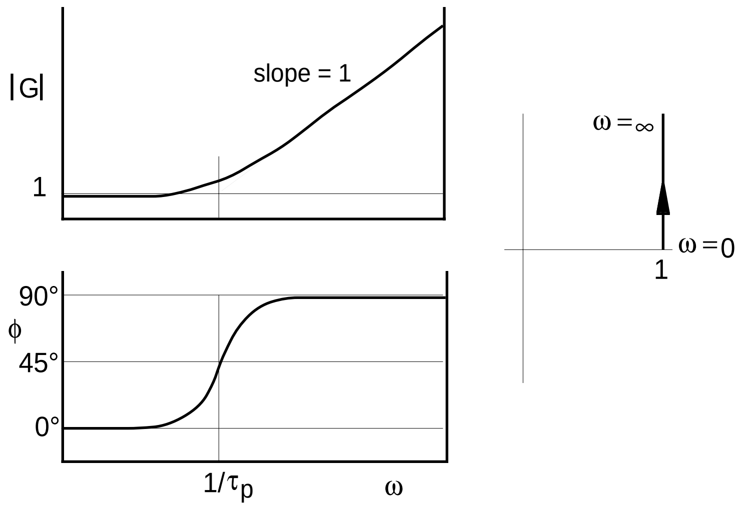


Figure E8.4

The magnitude and phase angle plots are sort of "upside down" versions of first order lag, with the phase angle increasing from 0° to 90° in the high frequency asymptote. The polar plot, on the other hand, is entirely different. The real part of G(jω) is always 1 and not dependent on frequency.

✎ **Example 8.5**. What are the Bode and Nyquist plots of a **dead time** function $G(s) = e^{-\theta s}$?

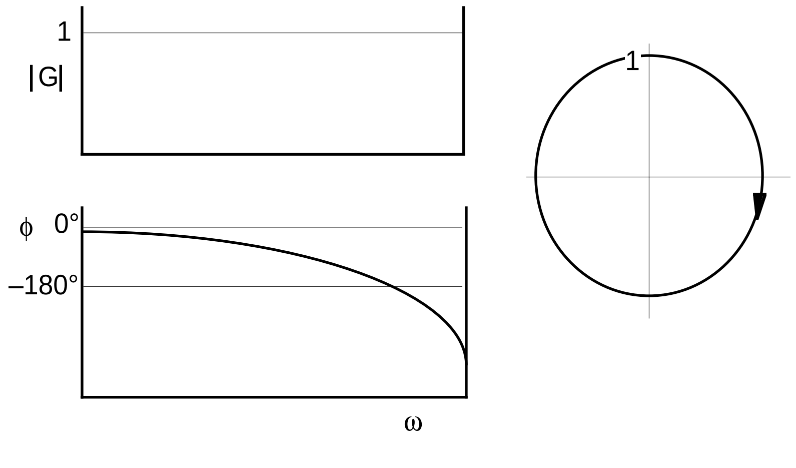


Figure E8.5

Again, we make the s = jω substitution in the transfer function to obtain

$$G(j\omega) = e^{-\theta\omega j}$$

The magnitude is simply unity, $|G(j\omega)| = 1$, and the phase angle is

$$\angle G(j\omega) = -\omega\theta$$

When $\omega = \pi/\theta$, $\angle G(j\omega) = -\pi$. On the polar plot, the dead time function is a unit circle.

We need hints from our MATLAB Session to plot this function in Fig. E8.5. We'll do it together with the next example.

The important point is that the phase lag of the dead time function increases without bound with respect to frequency. This is what is called a **nonminimum phase** system, as opposed to the first and second transfer functions which are minimum phase systems. Formally, a **minimum phase** system is one which has no dead time and has neither poles nor zeros in the RHP. (See Review Problems.)

From here on, we will provide only the important analytical equations or plots of asymptotes in the examples. You should generate plots with sample numerical values using MATLAB as you read them.

✎ **Example 8.6**. What are the Bode and Nyquist plots of a **first order lag with dead time**?

This example shows us the very important reason why and how frequency response analysis can handle dead time so easily. Substitute s = jω in $G(s) = \frac{K_p e^{-t_d s}}{\tau_p s + 1}$, and we have

$$G(j\omega) = \frac{K_p e^{-j\omega t_d}}{j\omega\tau_p + 1}$$

From Example 8.5, we know that the magnitude of the dead time function is 1. Combining also with the results in Example 8.2, the magnitude and phase angle of G(jω) are

$$|\,G(j\omega)\,| = \frac{K_p}{\sqrt{1 + \omega^2\tau_p^2}}\ , \quad \text{and} \quad \angle G(j\omega) = \tan^{-1}(-\omega\tau_p) - \omega\, t_d$$

The results are exact—we do not need to make approximations as we had to with root locus or the Routh array. The magnitude plot is the same as the first order function, but the phase lag increases without bound due to the dead time contribution in the second term. We will see that this is a major contribution to instability. On the Nyquist plot, the G(jω) locus starts at K_p on the real axis and then "spirals" into the origin of the s-plane.

This is how we may do the plots with time delay (details in MATLAB Session 7). Half of the work is taken up by the plotting statements.

```
kp=1; % Some arbitrary values
taup=10;
G=tf(kp,[taup 1]);
tdead=2;
freq=logspace(-1,1);                    %Make a frequency vector
[mag,phase]=bode(G,freq);
mag=mag(1,:); phase=phase(1,:);         %MATLAB specific step

phase = phase - ((180/pi)*tdead*freq);  %Add dead time phase lag

figure(1);
subplot(211), loglog(freq,mag)
              ylabel('Magnitude'),title('Bode Plot'), grid
subplot(212), semilogx(freq,phase)
              ylabel('Phase (degree)'),xlabel('Frequency'), grid

figure(2)            % We have to switch over to the polar plot
phase=phase*pi/180;  % function to do this Nyquist plot
```

```
polar(phase,mag)
```

✎ **Example 8.7**. What are the Bode and Nyquist plots of a **pure integrating function** G(s) = K/s?

The s = jω substitution in the integrating function leads to a pure imaginary number: G(jω) = K/jω = – jK/ω. The magnitude and phase angle are

$$|G(j\omega)| = K/\omega, \quad \text{and} \quad \angle G(j\omega) = \tan^{-1}(-\infty) = -90°$$

Sample MATLAB statements are:

```
G=tf(1,[1 0]);
figure(1), bode(G)
figure(2), nyquist(G)
```

The magnitude log-log plot is a line with slope –1. The phase angle plot is a line at –90°. The polar plot is the negative imaginary axis, approaching from negative infinity with ω = 0 to the origin with ω —> ∞.

✎ **Example 8.8**. What are the Bode and Nyquist plots of a **first order lag with an integrator**?

Our first impulse may be a s = jω substitution in the transfer function

$$G(s) = \frac{K_p}{s\,(\tau_p s + 1)}$$

However, the result is immediately obvious if we consider the function as the product of a first order lag and an integrator. Combining the results from Examples 8.2 and 8.7, the magnitude and phase angle are

$$|\,G(j\omega)\,| = \frac{K_p}{\omega\sqrt{1 + \tau_p^2\,\omega^2}}\ ,$$

and

$$\angle G(j\omega) = \tan^{-1}(-\infty) + \tan^{-1}(-\tau_p\,\omega) = -90° + \tan^{-1}(-\tau_p\,\omega)$$

Sample MATLAB statements are:

```
kp=1;
taup=2;
G=tf(kp,[taup 1 0]);
figure(1), bode(G)
figure(2), nyquist(G)
```

Because of the integrator, the magnitude log-log plot does not have a low or high frequency asymptote. The plot is a curve in which magnitude decreases with frequency. The phase angle plot starts at –90° at the low frequency asymptote and decreases to –180° at the high frequency asymptote. The polar plot curve approaches from negative infinity along the vertical line $-K_p\tau_p$

and approaches the origin as ω—> ∞.

✎ **Example 8.9**. Sketch the Bode plot of the following transfer function:

$$G(s) = \frac{(5s + 1)}{(10s + 1)\,(2s + 1)}$$

The MATLAB statements are:

```
G=tf([5 1],conv([10 1],[2 1]));
bode(G);
```

Formally, we would plot

$$|\,G(j\omega)\,| = \sqrt{1 + 5^2\omega^2}\;\frac{1}{\sqrt{1 + 10^2\omega^2}}\;\frac{1}{\sqrt{1 + 2^2\omega^2}}\;,$$

and

$$\angle G(j\omega) = \tan^{-1}(5\omega) + \tan^{-1}(-\,10\omega) + \tan^{-1}(-\,2\omega)$$

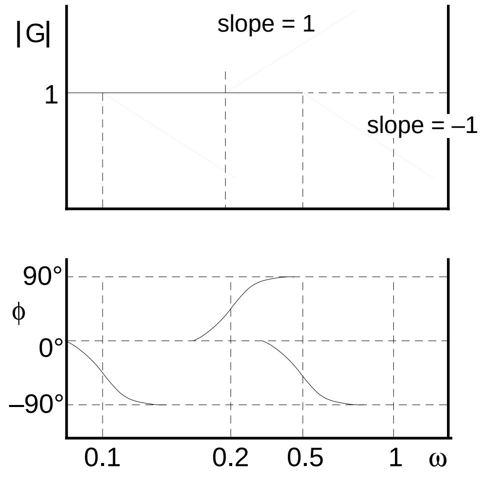


Figure E8.9

With MATLAB, what you find is that the actual curves are very smooth; it is quite different from hand sketching. Nevertheless, understanding the asymptotic features is important to help us check if the results are correct. This is particularly easy (and important) with the phase lag curve.

To help understand MATLAB results, a sketch of the low and high frequency asymptotes is provided in Fig. E8.9. A key step is to identify the corner frequencies. In this case, the corner frequency of the first order lead is at 1/5 or 0.2 rad/s, while the two first order lag terms have their corner frequencies at 1/10, and 1/2 rad/s. The final curve is a superimposition of the contributions from each term in the overall transfer function.

In addition, if you want to better see the little phase lag "hump" that you expect from hand sketching, change the term in the denominator from (2s +1) to (s + 1) so the phase lag of this term will not kick in too soon.

8.3 Stability Analysis

With frequency response analysis, we can derive a general relative stability criterion. The result is applicable to systems with dead time. The analysis of the closed-loop system can be reduced to using only the open-loop transfer functions in the computation.

8.3.1 Nyquist Stability criterion

Consider the characteristic equation of a closed-loop system

$$1 + G_mG_cG_aG_p = 0 \tag{7-2}$$

where often G_{OL} is used to denote the open-loop transfer function: $G_{OL} = G_mG_cG_aG_p$. To "probe" the property on the imaginary axis, we can make a substitution of $s = j\omega$ and rewrite the equation as

$$G_m(j\omega)G_c(j\omega)G_a(j\omega)G_p(j\omega) = -1, \quad \text{or} \quad G_{OL}(j\omega) = -1 \tag{7-2a}$$

This equation, of course, contains information regarding stability, and as it is written, implies that one may match properties on the LHS with the point (–1,0) on the complex plane. The form in (7-2a) also implies that in the process of analyzing the closed-loop stability property, the calculation procedures (or computer programs) only require the open-loop transfer functions. For complex problems, this fact eliminates unnecessary algebra. We just state the Nyquist stability criterion here.[1]

Nyquist stability criterion: Given the closed-loop equation $1 + G_{OL}(j\omega) = 0$, if the function $G_{OL}(j\omega)$ has P open-loop poles and if the polar plot of $G_{OL}(j\omega)$ encircles the (–1,0) point N times as ω is varied from $-\infty$ to ∞, the number of unstable closed-loop poles in the RHP is $Z = N + P$. (Z is named after the number of zeros to $1 + G_{OL}(j\omega) = 0$.)

Do not panic! Without the explanation in our *Web Support*, this statement makes little sense. On the other hand, we do not really need this full definition because we know that just one unstable closed-loop pole is bad enough. Thus the implementation of the Nyquist stability criterion is much simpler than the theory.

Simplified statement of Nyquist stability criterion (Fig. 8.3): Given the closed-loop equation $1 + G_{OL}(j\omega) = 0$, the closed-loop system is stable if the polar plot of $G_{OL}(j\omega)$ does not encircle the (–1,0) point in the G_{OL}-plane.

In this statement, we have used "polar plot of G_{OL}" to replace a mouthful of words. We have added G_{OL}-plane in the wording to emphasize that we are using an analysis based on Eq. (7-2a). The real question lies in what safety margin we should impose on a given system. This question leads to the definitions of gain and phase margins, which constitute the basis of the general relative stability criteria for closed-loop systems.

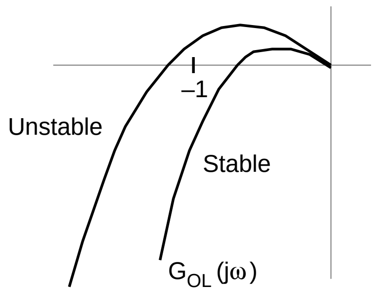


Figure 8.3. Illustration of the stable versus unstable possibilities under the Nyquist stability criterion.

[1] The formal explanation is in our *Web Support*. For a quick idea, our result is based on writing $G_{OL}(j\omega) = -1$. One simple thinking of instability is that if we feed back a sinusoidal wave, it will undergo a –180° phase shift at the summing point of a negative feedback loop. If the amplitude of the wave is less than one after passing through the summing point, it will die out. However, if the amplitude is larger than one, the oscillations will grow.

When we make a Nyquist plot, we usually just map the positive imaginary axis from ω = 0 to infinity, as opposed to the entire axis starting from negative infinity. If a system is unstable, the resulting plot will only contribute π to the (–1,0) point as opposed to 2π—what encirclement really means. However, just mapping the positive imaginary axis is sufficient to observe if the plot may encircle the (–1,0) point.

8.3.2 Gain and Phase Margins

Once we understand the origin of Nyquist stability criterion, putting it to use is easy. Suppose we have a closed-loop system with characteristic equation: 1 + G_cG_p = 0. With the point (–1,0) as a reference and the $G_c(j\omega)G_p(j\omega)$ curve on a Nyquist plot, we can establish a relative measure on how safe we are—that is, how far we are from the (–1,0) point. There are two possibilities. They are shown in Fig. 8.4, together with their interpretations on a Bode plot.

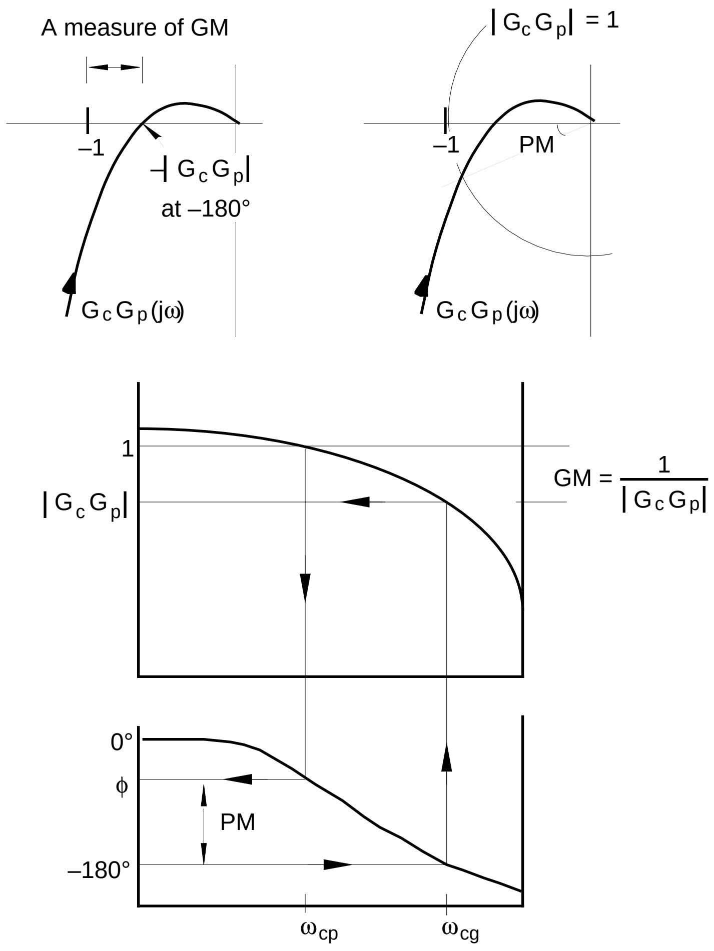


Figure 8.4. Interpretation of the gain and phase margins based on the Nyquist stability criterion using Nyquist and Bode plots.

(1) On the negative real axis (–180°), find the "distance" of $|G_cG_p|$ from (–1,0). This is the **gain margin**, GM. The formal definition is

$$GM = \frac{1}{|G_c(j\omega_{cg})G_p(j\omega_{cg})|} \qquad (8\text{-}21)$$

where $G_c(j\omega)G_p(j\omega)$ is evaluated at where it has a phase lag of –180°. The particular frequency at this point is the **gain crossover frequency**, $\omega = \omega_{cg}$. The smaller the magnitude of G_cG_p at –180°, the larger the gain margin and the "safer we are."

(2) Find the frequency where the magnitude $|G_c(j\omega)G_p(j\omega)|$ is 1. This particular frequency is the **phase crossover frequency**, $\omega = \omega_{cp}$. We then find the angle between G_cG_p and –180°. This is the **phase margin**, PM. The formal definition is

$$PM = \phi - (-180°) = \phi + 180° \qquad (8\text{-}22)$$

where the phase lag ϕ (a negative value) is measured at the point where $G_c(j\omega_{cp})G_p(j\omega_{cp})$ has a magnitude of one. The larger the angle, the larger the phase margin and the "safer we are."

For most control systems, we usually take a gain margin between 1.7 and 2, and a phase margin between 30° and 45° as the design specifications.

The Nyquist stability criterion can be applied to Bode plots. In fact, the calculation using the Bode plot is much easier. To obtain the gain margin, we find the value of $|G_cG_p|$ which corresponds to a phase lag of –180°. To find the phase margin, we look up the phase lag corresponding to when $|G_cG_p|$ is 1.

Once again, recall that simple first and second order systems with no positive zeros are always stable, and no matter what a computer program may return, the gain and phase margins have no meaning. Also, the gain margin crossover frequency is the same as the so-called ultimate frequency when we do the $s = j\omega$ substitution in the closed-loop characteristic equation or when we measure the value with the Ziegler-Nichols ultimate cycle method.

The gain and phase margins are used in the next section for controller design. Before that, let's plot different controller transfer functions and infer their properties in frequency response analysis. Generally speaking, any function that introduces additional phase lag or magnitude tends to be destabilizing, and the effect is frequency dependent.

We'll skip the proportional controller, which is just $G_c = K_c$. Again, do the plots using sample numbers with MATLAB as you read the examples.

✎ **Example 8.10**. Derive the magnitude and phase lag of the transfer function of a **PI** controller.

We could make the substitution $s = j\omega$ in $G_c(s) = K_c\,(1 + \frac{1}{\tau_I s})$. However, we can obtain the result immediately if we see that the function is a product of an integrator and a first order lead:

$$G_c(s) = K_c \frac{1}{\tau_I s} (\tau_I s + 1)$$

Thus

$$|\,G_c(j\omega)\,| = K_c \frac{1}{\omega\tau_I}\sqrt{(1 + \omega^2\tau_I^2)} \quad , \quad \text{and} \quad \angle G_c(j\omega) = -90° + \tan^{-1}\left(\omega\tau_I\right)$$

To do a demonstration plot, we may try the following MATLAB statements:

```
kc=1;   % Just some arbitrary numbers
taui=2;
G=tf(kc*[taui 1],[taui 0]);
figure(1), bode(G);
figure(2), nyquist(G);
```

On the magnitude plot, the low frequency asymptote is a line with slope –1. The high frequency asymptote is a horizontal line at K_c. The phase angle plot starts at –90° at very low frequencies and approaches 0° in the high frequency limit. On the polar plot, the $G_c(j\omega)$ locus is a vertical line that approaches from negative infinity at $\omega = 0$. At infinity frequency, it is at the K_c point on the real axis.

Integral control adds additional phase lag (–90°) at low frequencies below the corner frequency $1/\tau_I$. A larger value of integral time constant will limit the frequency range where the controller introduces phase lag. This is one reason why choosing a large τ_I tends to be more stable than a system with a small τ_I. [1]

[1] Furthermore, we may want to choose τ_I such that $1/\tau_I$ is smaller than the corner frequency associated with the slowest open-loop pole of the process function. This way, we help to stabilize

✎ **Example 8.11**. Derive the magnitude and phase lag of the transfer function of an **ideal PD** controller.

The result is that of a first order lead as in Example 8.4. From $G_c(s) = K_c\,(1 + \tau_D\, s)$, we have, after s = jω substitution,

$$G_c(j\omega) = K_c\,(1 + j\omega\tau_D)$$

and thus

$$|\,G_c(j\omega)\,| = K_c\sqrt{(1 + \omega^2\tau_D^2)}\ ,\quad \text{and} \quad \angle G_c(j\omega) = \tan^{-1}(\omega\tau_D)$$

On the magnitude plot, the low frequency asymptote is a horizontal line at K_C. The high frequency asymptote has a slope of +1. The phase angle plot starts at 0° at very low frequencies and approaches 90° in the high frequency limit. On the polar plot, the $G_C(j\omega)$ locus is a vertical line that starts at the point K_C on the real axis, and approaches infinity. Based on the phase angle plot, the PD controller provides a phase lead and thus stabilizing effect. At the same time, the higher magnitude at higher frequency ranges will amplify noises. There is a practical limit as to how fast a response a PD controller can handle.

The MATLAB statements are essentially the same as the first order lead function:

```
kc=1;           % Just some numbers we pick arbitrarily
taud=2;
G=tf(kc*[taud 1],1);
figure(1), bode(G);
figure(2), nyquist(G);
```

the system by reducing the phase lag due to the integration before the phase lag of the process function “kicks in.” However, integral control will not be effective if τ_I is too large, and there will be a design trade-off when we work with very slow processes. We will test this idea in Homework Problem II.38.

✎ **Example 8.12**. Derive the magnitude and phase lag of the controller transfer function

$$G_c(s) = K_c\left(\frac{1 + \tau_I s}{\tau_I s}\right)(1 + \tau_D s)$$

which is a **PID** controller with ideal derivative action and in the so-called interacting form. We look at this function because it is a product of an integrator and two first order leads, and we can identify the high and low frequency asymptotes easily. It is not identical to the ideal (non-interacting) PID controller, but the numerical results are very similar.

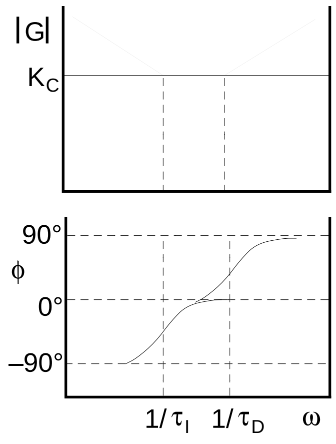


Figure E8.12. Only high and low frequency asymptotes are shown here. Fill in the rest with the help of MATLAB.

First, we need to generate the plots. Use Fig. E8.12 to help interpret the MATLAB generated Bode plot. [1]

```
kc=1;
taui=4;
taud=1;
Gi=tf(kc*[taui 1],[taui 0]);
Gd=tf([taud 1],1);
G=Gi*Gd;
bode(G);
```

By choosing $\tau_D < \tau_I$ (*i.e.*, corner frequencies $1/\tau_D > 1/\tau_I$), the magnitude plot has a notch shape. How sharp it is will depend on the relative values of the corner frequencies. The low frequency asymptote below $1/\tau_I$ has a slope of –1. The high frequency asymptote above $1/\tau_D$ has a slope of +1. The phase angle plot starts at –90°, rises to 0° after the frequency $1/\tau_I$, and finally reaches 90° at the high frequency limit.

Relatively speaking, a PID controller behaves like a PI controller at low frequencies, while it is more like a PD controller at high frequencies. The controller is most desirable in the mid-range where it has the features of both PI and PD controllers. Also in the notch region, the controller function has the lowest magnitude and allows for a larger gain margin for the system.

✎ **Example 8.13**. Derive the magnitude and phase lag of the transfer functions of **phase-lead** and **phase-lag compensators**. In many electromechanical control systems, the controller G_c is built with relatively simple R-C circuits and takes the form of a lead-lag element:

$$G_c(s) = K\frac{(s + z_o)}{(s + p_o)}$$

Here, z_o and p_o are just two positive numbers. There are obviously two possibilities: case (a) $z_o > p_o$, and case (b) $z_o < p_o$. Sketch the magnitude and phase lag plots of G_c for both cases. Identify which case is the phase-lead and which case is the phase-lag compensation. What types of classical controllers may phase-lead and phase-lag compensations resemble?

[1] If you want to see a plot for an ideal PID controller, use

```
Gi=tf(kc*[taui*taud taui 1],[taui 0]);
```

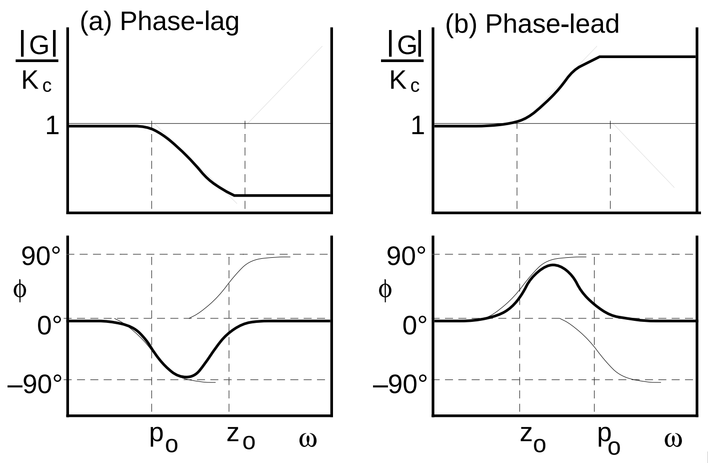


Figure E8.13

We may look at the controller transfer function in the time constant form:

$$G_c(s) = \left(K\frac{z_o}{p_o}\right)\frac{(s/z_o + 1)}{(s/p_o + 1)}$$

where we could further write $K_c = Kz_o/p_o$, but we'll use the pole-zero form. Either way, we should see that the corner frequencies are at $\omega = z_o$ and p_o. To make a Bode plot, we theoretically should do the $s = j\omega$ substitution, but we can write down the magnitude and phase angle immediately if we recognize that the function is a product of a first order lead and a first order lag. Hence, making use of Examples 8.2 and 8.4, we can write

$$\left|G_c(j\omega)\right| = K\sqrt{\omega^2 + z_o^2}\,\frac{1}{\sqrt{\omega^2 + p_o^2}}$$

and

$$\angle G_c(j\omega) = \tan^{-1}(\omega/z_o) + \tan^{-1}(-\,\omega/p_o)$$

Fig. E8.13 is a rough hand sketch with the high and low frequency asymptotes. It is meant to help interpret the MATLAB plots that we will generate next.

(a) With $z_o > p_o$, the phase angle is always negative, and this is the **phase-lag** compensator. The MATLAB statements to get an illustrative plot are:

```
k=1;
zo=4;    % Try repeat with various choices of zo and po
po=1;
G=zpk(-zo,-po,k);
bode(G);
```

The shape of the magnitude plot resembles that of a PI controller, but with an upper limit on the low frequency asymptote. We can infer that the phase-lag compensator could be more stabilizing than a PI controller with very slow systems.[1] The notch-shaped phase angle plot of the phase-lag compensator is quite different from that of a PI controller. The phase lag starts at 0° versus –90°

[1] From the perspective of a root locus plot, a phase-lag compensator adds a large open-loop zero and a relatively small open-loop pole. And for a phase-lead compensator, we are adding a large open-loop pole. When $p_o \gg z_o$ (or $1/p_o \ll 1/z_o$), we can also look at the phase-lead compensator as the real PD controller. How to design their locations, of course, depends on the particular process that we need to control. We'll see that in Example 8.14.

for a PI controller. From a stability point of view, a phase-lag compensator is preferred to a PI controller. On the other hand, without an integrating function, the phase-lag compensator cannot eliminate offset.

(b) With $z_o < p_o$, the phase angle is positive, and this is the **phase-lead** compensator. First, we need an illustrative plot. Sample MATLAB statements to use are:

```
zo=1;
po=4;
G=zpk(-zo,-po,1);
bode(G);
```

The nice feature of the phase-lead compensator, and for that matter a real PD controller, is that it limits the high frequency magnitude. In contrast, an ideal PD controller has no upper limit and would amplify high frequency input noises much more significantly.

✎ **Example 8.14**. Designing **phase-lead** and **phase-lag compensators**. Consider a simple unity feedback loop with characteristic equation $1 + G_cG_p = 0$ and with a first order process function $G_p = \dfrac{K_p}{(\tau_p s + 1)}$. What are the design considerations if one uses either a phase-lead or a phase-lag compensator? Consider the consequences using Bode and root locus plots.

With $K_c = Kz_o/p_o$, the closed-loop transfer function is

$$\frac{C}{R} = \frac{G_cG_p}{1 + G_cG_p} = \frac{K_c K_p (s/z_o + 1)}{(s/p_o + 1)(\tau_p s + 1) + K_c K_p (s/z_o + 1)}$$

and after one more algebraic step, we'll see that the system steady state gain is $K_cK_p/(1 + K_cK_p)$, which means that there will be an offset whether we use a phase-lead or a phase-lag compensator.

From the characteristic polynomial, it is probable that we'll get either overdamped or underdamped system response, depending on how we design the controller. The choice is not clear from the algebra, and this is where the root locus plot comes in handy. From the perspective of a root-locus plot, we can immediately make the decision that no matter what, both z_o and p_o should be larger than the value of $1/\tau_p$ in G_p. That's how we may "steer" the closed-loop poles away from the imaginary axis for better system response. (If we know our root locus, we should know that this system is always stable.)

(a) Let's first consider a phase-lead compensator, $z_o < p_o$. We first construct the Bode and root locus plots that represent a system containing the compensator and a first order process:

```
Kp=1;                       %Arbitrary numbers for the process function
taup=1;
Gp=tf(Kp,[taup 1]);

zo=2;                       %Phase-lead, zo < po
po=4;
Gc=zpk(-zo,-po,1)
figure(1), bode(Gc*Gp)
figure(2), rlocus(Gc*Gp)
```

The root locus plot resembles that of a real PD controller. The system remains overdamped with no complex closed-loop poles. One root locus runs from the "real PD pole" $-p_o$ to negative infinity. The other is from $-\tau_p$ to $-z_o$, which limits the fastest possible system response. How large z_o, and thus K_c, can be depends on the real physical hardware and process.

On the Bode plot, the corner frequencies are, in increasing order, $1/\tau_p$, z_o, and p_o. The frequency asymptotes meeting at $\omega = 1/\tau_p$ and p_o are those of a first-order lag. The frequency asymptotes meeting at $\omega = z_o$ are those of a first-order lead. The largest phase lag of the system is –90° at very high frequencies. The system is always stable as displayed by the root locus plot.

(b) With a phase-lag compensator, $z_o > p_o$, we can use these statements:

```
%Gp remains the same as in part (a)
zo=4;
po=2;
Gc=zpk(-zo,-po,1)
figure(1), bode(Gc*Gp)
figure(2), rlocus(Gc*Gp)
```

The shape of the root locus plot resembles that of a PI controller, except of course we do not have an open-loop pole at the origin anymore. The root loci approach one another from $-\tau_p$ and $-p_o$, then break away from the real axis to form a circle which breaks in to the left of the open-loop zero at $-z_o$. One locus approaches negative infinity and the other toward $-z_o$. One may design the controller with an approach similar to that in Example 7.7 (p. 7-16).

On the Bode plot, the corner frequencies are, in increasing order, $1/\tau_p$, p_o, and z_o. The frequency asymptotes meeting at $\omega = 1/\tau_p$ and p_o are those of a first-order lag. The frequency asymptotes meeting at $\omega = z_o$ are those of a first-order lead. The largest phase lag of the system is larger than –90° just past $\omega = p_o$, but still much less than –180°. This system is always stable.

8.4 Controller Design

The concept of gain and phase margins derived from the Nyquist criterion provides a general relative stability criterion. Frequency response graphical tools such as Bode, Nyquist and Nichols plots can all be used in ensuring that a control system is stable. As in root locus plots, we can only vary one parameter at a time, and the common practice is to vary the proportional gain.

8.4.1 How do we calculate proportional gain without trial-and-error?

This is a big question when we use, for example, a Bode plot. Let's presume that we have a closed-loop system in which we know "everything" but the proportional gain (Fig. 8.5), and we write the closed-loop characteristic equation as

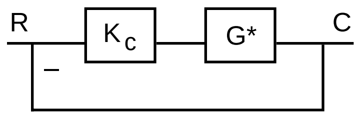


Figure 8.5. A simple unity feedback system.

$$1 + G_{OL} = 1 + K_cG^* = 0$$

where $G_{OL} = G_cG_aG_pG_m$. We further rewrite the function as K_cG^* to indicate that we would like to find K_c. The notation G^* is more than just the product of $G_aG_pG_m$; G^* includes the integral and derivative terms if we use a PID controller.

With the straight textbook definition and explanation, the gain margin and phase margin of the closed-loop system apply only to the magnitude and phase lag plots using the entire open-loop function, $|G_{OL}|$ and $\angle G_{OL}$. It means that we need to know the value for the proportional gain, K_c. Of course, we do not and the whole affair appears to be a trial-and-error calculation. The question is whether we can calculate K_c without guessing. The answer is yes. The next question is whether we can get the answer from the plots of $|G^*|$ and $\angle G^*$. This answer is also yes.

From the definition of gain margin, we have

$$GM = \frac{1}{|G_{OL}(j\omega_{cg})|}, \quad \text{or} \quad GM = \frac{1}{K_c\,|G^*(j\omega_{cg})|} \tag{8-23}$$

where the magnitudes are evaluated at the gain crossover frequency ω_{cg} as defined by the –180° phase lag. We can find $|G^*(j\omega_{cg})|$ simply by a Bode plot of G* itself. The key is that the phase angle of K_c is zero, $\angle G^*$ is identical to $\angle G_{OL}$, and both $|G^*|$ and $|G_{OL}|$ have the same ω_{cg}. [1]

From the definition of gain margin, GM = 1 at marginal stability. Hence, with GM = 1 in Eq. (8-23), we can evaluate $K_{cu} = 1/|G^*(j\omega_{cg})|$. Eq. (8-23) can alternately be stated as

$$GM = \frac{K_{cu}}{K_c} \tag{8-24}$$

Once we know the value of K_{cu}, we can calculate the K_c for a given GM in a problem. Typically, we select GM, say, 1.7, to find the proportional gain.

We can also use the Bode plot of G* to do phase margin calculations. From the textbook definition, we are supposed to find the phase angle $\phi = \angle G_{OL}$ where $|G_{OL}| = 1$. If the phase margin is 45°, ϕ should be –135°. It appears that we need to know K_c beforehand to calculate G_{OL}, but we do not.

We use the fact that $\angle G^*$ is identical to $\angle G_{OL}$, and ω_{cp} is the same in both plots, so we can go backward. On the G* Bode plot, we can find the value of $|G^*(j\omega_{cp})|$ which corresponds to a phase lag of, say, –135°. Now $|G_{OL}| = K_c|G^*|$ and since $|G_{OL}| = 1$ at the phase margin, we can find $K_c = 1/|G^*(j\omega_{cp})|$ that will provide a phase margin of 45° without trial-and-error.

How do I know the answer is correct? Just "plug" K_c back into G_{OL} and repeat the Bode plot using G_{OL}. It does not take that much time to check with MATLAB. Now, we are finally ready for some examples. Again, run MATLAB to confirm the results while you read them.

✎ **Example 7.2C**. Let's revisit Example 7.2 (p. 7-5) with the closed-loop characteristic equation:

$$1 + K_c \frac{1}{(s+3)\,(s+2)\,(s+1)} = 0$$

If we want to design a PI controller, how should we proceed with frequency response methods? Let's presume that the unit of the time constants is in minutes.

The very first step is to find the ultimate gain. With the given third order process transfer function, we use the following MATLAB commands,

```
    p=poly([-1 -2 -3]);
```

[1] We can use MATLAB to do the Bode plot of G* and use the `margin()` function, which will return the "gain margin" of G*, but we now know, is really $1/|G^*(j\omega_{cg})|$, or following Eq. (8-24), also the value for the ultimate gain K_{cu}.

```
     G=tf(1,p);
     margin(G);
```

MATLAB returns K_{cu} as 35.6 dB (at ω_{cg} = 3.3 rad/min),[1] which we easily recalculate as 60 ($10^{35.6/20}$).

Note that the low frequency asymptote of the magnitude plot is not 1 (0 dB). Why? That's because the transfer function is not in the time constant form. If we factor the function accordingly, we should expect a low frequency asymptote of 1/6 (–15.6 dB).

If we take a gain margin of 1.7 in Eq. (8-24), we should use a proportional gain of K_c = 60/1.7 = 35.3. This is the case if we use only a proportional controller as in Example 6.2. We repeat the Bode plot and margin calculation with K_c = 35.3:

```
     G=tf(35.3,p);
     margin(G);
```

Now, we should find that the gain margin is indeed 1.7 (4.6 dB, $10^{4.6/20}$), and the phase margin is 18.8°, which is a bit low according to the design rule of thumb.

We have yet to tackle the PI controller. There are, of course, different ways to find a good integral time constant. With frequency response, we have the handy tool of the Ziegler-Nichols ultimate cycle tuning relations. So with K_{cu} = 60 and ω_{cg} = 3.3 rad/min, we find by referring to the Table of Tuning Relations (Table 6.1) that if we use only a proportional controller, we should use K_c = 30, and if we use a PI controller, we should use K_c = 27.3 and τ_I = 1.58 min.[2]

Using the Ziegler-Nichols tuning parameters, we repeat the proportional controller system Bode plot:

```
     G=tf(30,p);
     margin(G);
```

We should find a gain margin of 2 (6 dB) and a phase margin of 25.4°, which is definitely a bit more conservative than the 1.7 gain margin result.

With the PI controller, we use the following statements:

```
     kc=27.3; taui=1.58;
     Gc=tf(kc*[taui 1],[taui 0]);
     G=tf(1,p);      %p was defined at the beginning of this example
     margin(Gc*G);
```

We should find a gain margin of 1.47 (3.34 dB) and a phase margin of 12.3°. Both margins are a bit small. If we do a root locus plot on each case and with the help of `rlocfind()` in MATLAB , we should find that the corresponding closed-loop poles of these results are indeed quite close to the imaginary axis.

Where do we go from here? We may stay with the design or we may increase the margins. We also can use MATLAB to simulate the closed-loop time domain response and from the

[1] MATLAB always considers time to be in seconds, but this should not concern us as long as we keep our own time units consistent.

[2] All these calculations are done with the M-file `recipe.m` from our *Web Support*.

underdamped time response curve, estimate numerically the effective damping ratio and other quantities such as percent overshoot. If the time domain response does not meet our specification, we will have to tune the values of K_c or τ_I.

If we want to increase the margin, we either have to reduce the value of K_c or increase τ_I. One possibility is to keep τ_I = 1.58 min and repeat the Bode plot calculation to find a new K_c which may provide a gain margin of, say, 2 (6 dB), as in the case of using only the proportional controller. To do so, we first need to find the new ultimate gain using the PI controller:

```
kc=1; taui=1.58;
Gc=tf(kc*[taui 1],[taui 0]);
margin(Gc*G);                %G remains as above
```

MATLAB should return K_{cu} = 40.2 (32.1 dB). Thus following Eq. (8-24), we need to use K_c = 40.2/2 = 20.1 to meet the gain margin specification of 2. You can double check the result yourself with `kc=20.1`, `taui=1.58`. If so, you should find that the phase margin is now 23°—a bit low but we probably can accept that. After this, we may proceed to the time domain response calculations.

The ideas in this example can be applied to a PID controller. Yes, controller design is indeed an iterative process. A computer is a handy tool, but we still need to know what we are doing.

✎ **Example 7.2D**. Back in the last example with a proportional controller, a gain margin of 1.7 created a system with a very small phase margin. What proportional gain should we use to achieve a phase margin of at least 45°?

Following our explanation after Eq. (8-24), we should calculate the phase angle of $G(j\omega) = [(j\omega+3)(j\omega+2)(j\omega+1)]^{-1}$. Of course, we'd rather use `bode()` in MATLAB:

```
p=poly([-1 -2 -3]);
G=tf(1,p);
[mag,phase,w]=bode(G);
mag=mag(1,:);         % MATLAB v5 returns a 3-D array 1
phase=phase(1,:);
tmp=[w';mag;phase]'   % Be careful with the primes here
```

From the `tmp` matrix of frequency, magnitude and phase angle, we find that at ω = 1.74 rad/min, |G| = 0.054, and ∠G = –131.4°, which provides a phase margin of 48.6°. Also, at ω = 2.21 rad/min, |G| = 0.037, and ∠G = –150°, which provides a phase margin of 30°. (We will need to do an interpolation if the problem statement dictates, say, a phase margin of exactly 45°.)

To achieve a phase margin of 48.6°, we use a proportional gain of K_C = 1/0.054 = 18.5. We can repeat the Bode plot calculation with MATLAB using K_C = 18.5, and the statements:

```
G=tf(18.5,p);
margin(G);
```

The result confirms that the system has a phase margin of 48.6°, and a gain margin of 3.2 (10.2 dB), a much more conservative design than a gain margin of 1.7. If we choose to use K_C = 1/0.037

---

1 The MATLAB function `bode()` returns the actual magnitude even though the documentation says dB. This is a detail that we can check with the function `freqresp()` as explained on our *Web Support*, especially with future upgrades of the software.

= 26.96. A MATLAB calculation should confirm a phase margin of 30°, and find a gain margin of 2.2.

✎ **Example 7.4A**. This time, let's revisit Example 7.4 (p. 7-8), which is a system with dead time. We would like to know how to start designing a PI controller. The closed-loop characteristic equation with a proportional controller is (again assuming the time unit is in min)

$$1 + K_c \frac{0.8e^{-2s}}{5s+1} = 0$$

The very first step is to find the ultimate gain. Following Example 8.6 (p. 8-11), we can add easily the extra phase lag due to the dead time function:

```
G=tf(0.8,[5 1]);
tdead=2;
[Mag,Phase,freq]=bode(G);
Mag=Mag(1,:);
Phase=Phase(1,:) - ((180/pi)*tdead*freq');
[Gm,Pm,Wcg,Wcp]=margin(Mag,Phase,freq)
```

We should find $K_{cu} = 5.72$ at $\omega_{cg} = 0.893$ rad/min, which is exactly what we found in Example 7.4, but which takes a lot more work. If we take a gain margin of 1.7, we should use a proportional gain of $K_c = 5.72/1.7 = 3.36$. We use

```
G=tf(3.36*0.8,[5 1]);
```

and repeat the calculation. We should find the corresponding phase margin to be 54.6°, which is plenty.

These are the controller settings if we again use the Ziegler-Nichols tuning relations (or really recipe.m): with $K_{cu} = 5.73$ and $\omega_{cg} = 0.895$ rad/min, we should use a proportional controller with $K_c = 2.87$, and if we use a PI controller, we should use $K_c = 2.61$ and $\tau_I = 5.85$ min. The tasks of checking the gain and phase margins and the time domain response are left as a Review Problem.

✎ **Example 5.7D**: We can now finally wrap up the dye mixing problem that we left in Example 5.7C (p. 6-16).
(a) The control system can be unstable if we place the photodetector too far downstream. To cut down on the algebra, we'll look into the problem with a slight simplification. At this point, we'll use only a proportional controller. Since the regulating valve is so much faster than the mixing process, we'll retain only the mixing first order lag to obtain the approximate closed-loop characteristic equation:

$$1 + \frac{K_c K_v K_p K_m \; e^{-t_d s}}{(\tau_p s + 1)} = 0$$

Again for illustration purpose, we supposedly have chosen K_c such that $K_c K_V K_p K_m = 5$, and τ_p is the mixing process time constant. Find, without trial-and-error and without further approximation, the maximum distance L that the photodetector can be placed downstream such that the system remains stable. (There are two ways to get the answer. The idea of using magnitude and phase angle and the Nyquist criterion is by far the less confusing method and less prone to algebraic mistakes.)

(b) To factor in some safety margin, we install the photodetector at half the maximum distance that we have found in part (a). With the same proportional gain and same approximation used in part (a), what is the gain margin?

(c) Now that we can install the photodetector at a well chosen distance, we can put the dynamics of the regulating valve back into our closed-loop analysis. What is the critical proportional gain when we include the first order lag due to the regulating valve? And what is the proportional gain if we want a gain margin of 1.7?

(d) Finally we come to the controller design. All we know is that customers may be fussy with the color of their jeans. Of course, we also want to avoid the need to dispose of off-spec dye solutions unnecessarily. Despite these concerns, an old plant engineer mentioned that the actual dye-tank downstream is huge and an overshoot as much as 20% to 25% is acceptable as long as the mixing system settles down "quickly." So select your choice of controller, performance specification, and controller gains. Double check your final design with time domain simulation and frequency response analysis to see that we have the proper controller design.

(a) Let's use the abbreviation $G_{OL} = G_cG_vG_pG_m$, and thus the magnitude and phase angle equations, from Example 5.7 (p. 5-17), are:

$$|G_{OL}| = \left|\frac{5}{\tau_p s + 1}\right| |e^{-t_d s}|, \text{ and } \angle G_{OL} = \tan^{-1}(-\omega\tau_p) - t_d\,\omega$$

where now $\tau_p = 4$ s. At crossover ω_{cg}, $\angle G_{OL} = -180°$ and GM = 1, meaning $|G_{OL}| = 1$. Since the magnitude of the dead time transfer function is unity, the magnitude equation is simply

$$1 = \frac{5}{\sqrt{1 + 4^2\omega_{cg}^2}}, \quad \text{or} \quad \omega_{cg} = 1.12 \text{ s}^{-1}$$

With the crossover frequency known, we now use the phase angle equation to find the dead time:

$$-180° = \tan^{-1}(-4 \times 1.12) - t_d\,(1.12)\,(180/\pi), \quad \text{or} \quad t_d = 1.45 \text{ s}$$

Note the unit conversion to angles in the phase equation. With our arbitrary choice of proportional gain such that $K_cK_vK_pK_m = 5$, a dead time of $t_d = 1.45$ s is associated with GM = 1.

(b) Refer back to Example 5.7. The average fluid velocity is 400 cm/s. Thus, the photodetector is located at (1.45)(400) = 580 cm downstream from the mixer. To reduce the distance by half means that we now install the sensor at a location of 580/2 = 290 cm downstream. The reduced transport lag is now 1.45/2 = 0.725 s.

To find the new gain margin, we need to, in theory, reverse the calculation sequence. We first use the phase equation to find the new crossover frequency ω_{cg}. Then we use the magnitude equation to find the new $|G_{OL}|$, and the new GM is of course $1/|G_{OL}|$. However, since we now know the values of t_d, τ_p, and $K_cK_VK_pK_m$, we might as well use MATLAB. These are the statements:

```
k=5;
tdead=0.725;
taup=4;  %The large time constant; dominant pole is at 1/4
G=tf(k,[taup 1]);
```

```
freq=logspace(-1,1)';
[Mag,Phase]=bode(G,freq);
Mag=Mag(1,:);
Phase=Phase(1,:) - ((180/pi)*tdead*freq');
[Gm,Pm,Wcg,Wcp]=margin(Mag,Phase,freq)
```

We should find that the new gain margin is 1.86.

(c) We now include the regulating valve and still use a proportional controller. The closed-loop equation is

$$1 + K_c \left(\frac{0.8}{0.2\,s + 1}\right)\left(\frac{0.6}{4\,s + 1}\right) 2.6\, e^{-0.725\,s} = 0$$

With MATLAB, the statements are:

```
k=0.8*0.6*2.6;
G=tf(k, conv([0.2 1],[4 1]));
[Mag,Phase]=bode(G,freq);
Mag=Mag(1,:);
Phase=Phase(1,:) - ((180/pi)*tdead*freq');
[Gm,Pm,Wcg,Wcp]=margin(Mag,Phase,freq)
```

We should find that the ultimate gain is $K_{c,u} = 6.42$ at the crossover frequency of $\omega_{cg} = 1.86\ s^{-1}$. To keep to a gain margin of 1.7, we need to reduce the proportional gain to $K_c = 6.42/1.7 = 3.77$.

(d) With $K_{c,u} = 6.42$ and $\omega_{cg} = 1.86\ s^{-1}$, we can use the Ziegler-Nichols ultimate-gain tuning relations (with recipe.m) to find, for different objectives:

	K_c	τ_I	τ_D
Quarter decay	3.8	1.7	0.42
Little overshoot	2.1	1.7	1.1
No overshoot	1.3	1.7	1.1

If we repeat the time response simulations as in Example 5.7C, we should find that the settings for the quarter decay leads to a 51% overshoot (a roughly 0.26 decay ratio), the little overshoot settings have a 27% overshoot, and the so-called no overshoot settings still have about 8% overshoot.

Finally, there is no unique solution for the final design. The problem statement leaves us with a lot of latitude. Especially when we have seen from the time response simulations that many combinations of controller settings give us similar closed-loop responses. In process engineering, we do not always have to be fastidious with the exact time response specifications and hence values of the controller settings. Many real life systems can provide acceptable performance within a certain range of response characteristics. As for controller settings, a good chance is that we have to perform field tuning to account for anything ranging from inaccurate process identification to shifting operating conditions of nonlinear processes.

For the present problem, and based on all the settings provided by the different methods, we may select $\tau_I = 3$ s and $\tau_D = 0.5$ s. We next tune the proportional gain to give us the desired response. The closed-loop equation with an ideal PID controller is now:

$$1 + K_c\left(1 + \frac{1}{\tau_I s} + \tau_D s\right) \frac{1.248}{(0.2\,s + 1)(4\,s + 1)}\, e^{-0.725\,s} = 0$$

First, we need MATLAB to find the ultimate gain:

```
taui=3;
taud=0.5;
gc=tf([taui*taud (taui+taud) 1],[taui 0]);  %ideal PID without the Kc
tdead=0.725;
k=0.8*0.6*2.6;
G=tf(k, conv([0.2 1],[4 1]));
[Mag,Phase]=bode(gc*G,freq);
Mag=Mag(1,:);
Phase=Phase(1,:) - ((180/pi)*tdead*freq');
[Gm,Pm,Wcg,Wcp]=margin(Mag,Phase,freq)
```

We should find the ultimate gain to be K_{cu} = 5.87. And at a gain margin of 1.7, we need to use a proportional gain of K_c = 5.87/1.7 = 3.45. A time response simulation shows that the system, with respect to a unit step change in the set point, has an overshoot of 23%. This tuning is slightly less oscillatory than if we had chosen τ_I = 3 s and τ_D = 0.3 s as suggested by ITAE (Example 5.7A). In this case, K_{cu} = 6.79, and K_c = 4, which is closer to the K_c from Ziegler-Nichols tuning. Again, confirm these results in the Review Problems.

8.4.2 A final word: Can frequency response methods replace root locus?

No. These methods complement each other. Very often a design decision is made only after analyses with both methods.

Root locus method gives us a good indication of the transient response of a system and the effect of varying the controller gain. However, we need a relatively accurate model for the analysis, not to mention that root locus does not handle dead time as well.

Frequency methods can give us the relative stability (the gain and phase margins). In addition, we could construct the Bode plot with experimental data using a sinusoidal or pulse input, *i.e.*, the subsequent design does not need a (theoretical) model. If we do have a model, the data can be used to verify the model. However, there are systems which have more than one crossover frequency on the Bode plot (the magnitude and phase lag do not decrease monotonically with frequency), and it would be hard to judge which is the appropriate one with the Bode plot alone.

❐ Review Problems

1. Derive Eqs. (8-19) and (8-20). Use MATLAB to plot the resonant frequency and maximum magnitude as a function of damping ratio with K = 1.
2. What are the low and high frequency asymptotes of the minimum phase function (s + z)/(s + p) versus the simplest nonminimum phase function (s – z)/(s + p) in a Bode plot?
3. What is the bandwidth of a second order function?
4. We used $\tau_D < \tau_I$ in Example 8.12. What if we use $\tau_D > \tau_I$ in our PID controller design? What if we use a real PID controller?
5. Sketch the Bode plots for G(s) = s^n, with n = ±1, ±2, ..., etc.
6. In Example 8.12, we used the interacting form of a PID controller. Derive the magnitude and phase angle equations for the ideal non-interacting PID controller. (It is called non-interacting because the three controller modes are simply added together.) See that this function will have the same frequency asymptotes.
7. Finish the controller calculations in Example 5.7D.

Hint:

1. The plotting statements can be:

```
z = 0.05:0.01:0.7;
wr = sqrt(1 – 2*z.*z);
dum = sqrt(1 – z.*z);
Mp = 1./(2*z.*dum);
plot(z,wr, z,Mp);
```

2. What is the phase angle of the minimum phase function (s + 3)/(s + 6) versus the simplest nonminimum phase function (s - 3)/(s + 6)? Also try plot with MATLAB. The magnitude plots are identical. The phase angle of the nonminimum phase example will go from 0° to –180°, while you'd see a minimum of the phase angle in the minimum phase function. Thus for a transfer function that is minimum phase, one may identify the function from simply the magnitude plot. But we cannot do the same if the function is nonminimum phase.

3. We need to find the frequency ω_b when the magnitude drops from the low frequency asymptote by $1/\sqrt{2}$. From the magnitude equation in Example 8.3, we need to solve

$$\left(1-\tau^2\omega_b^2\right)^2+\left(2\zeta\tau\omega_b\right)^2=2$$

If we now "solve" this equation using $\tau^2\omega^2$ as a variable, we should find

$$\tau^2\omega_b^2=1-2\zeta^2+\sqrt{4\zeta^2(\zeta^2-1)+2}$$

and the final form with ω_b explicitly on the LHS is one small step away.

4. Sketching with MATLAB should help.

5. $G(j\omega) = j^n\omega^n$. This function is real if n is even, imaginary if n is odd. Also, $|G| = \omega^n$, and the phase angle of $G(j\omega)$ is $\tan^{-1}(0)$ when n is even and is $\tan^{-1}(\infty)$ when n is odd.

6. Substitution of $s = j\omega$ in $G_c(s) = K_c\,(1 + \frac{1}{\tau_I s} + \tau_D s)$ gives

$$G_c(j\omega) = K_c\,(1+\frac{1}{j\omega\tau_I}+j\omega\tau_D) = K_c\,(1+j\frac{\tau_I\,\tau_D\,\omega^2-1}{\omega\tau_I})$$

and thus

$$|\,G_c(j\omega)\,| = K_c\sqrt{\;1+\left(\tau_D\,\omega-\frac{1}{\omega\tau_I}\right)^2}$$

and

$$\angle G_c(j\omega)=\tan^{-1}\left(\tau_D\,\omega-\frac{1}{\omega\tau_I}\right)$$

The magnitude equation has slopes of –1 and +1 at very low and very high frequencies. In the phase angle equation, the two limits are –90° and +90° as in Example 8.12. Furthermore, from the phase angle equation of the ideal controller, the "trough" center should be located at the frequency $\omega = (\tau_I\tau_D)^{-1/2}$. The polar plot of the ideal PID controller is like combining the images of a PI and an ideal PD controller—a vertical line at K_c that extends from negative infinity at $\omega = 0$ toward positive infinity at extremely high frequencies.

7. The MATLAB statements and plots are provided on our *Web Support*.

❖ 9. *Design of State Space Systems*

We now return to the use of state space representation that was introduced in Chapter 4. As you may have guessed, we want to design control systems based on state space analysis. State feedback controller is very different from the classical PID controller. Our treatment remains introductory, and we will stay with linear or linearized SISO systems. Nevertheless, the topics here should enlighten(!) us as to what modern control is all about.

What are we up to?

- Evaluate the controllability and observability of a system.
- Pole placement design of state feedback systems. Application of the Ackermann's formula.
- Design with full-state and reduced-order observers (estimators).

9.1 Controllability and Observability

Before we formulate a state space system, we need to raise two important questions. One is whether the choice of inputs (the manipulated variables) may lead to changes in the states, and the second is whether we can evaluate all the states based on the observed output. These are what we call the controllability and observability problems.

9.1.1 Controllability.

A system is said to be completely state controllable if there exists an input u(t) which can drive the system from any given initial state $\mathbf{x_o}(t_o=0)$ to any other desired state $\mathbf{x}(t)$. To derive the controllability criterion, let us restate the linear system and its solution from Eqs. (4-1), (4-2), and (4-10):

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u \tag{9-1}$$

$$y = \mathbf{C}\mathbf{x} \tag{9-2}$$

and

$$\mathbf{x}(t) = e^{\mathbf{A}t}\,\mathbf{x}(0) + \int_0^t e^{-\mathbf{A}(t-\tau)}\,\mathbf{B}u(\tau)\,d\tau \tag{9-3}$$

With our definition of controllability, there is no loss of generality if we choose to have $\mathbf{x}(t) = \mathbf{0}$, *i.e.*, moving the system to the origin. Thus Eq. (9-3) becomes

$$\mathbf{x}(0) = -\int_0^t e^{-\mathbf{A}\tau}\,\mathbf{B}u(\tau)\,d\tau \tag{9-4}$$

We next make use of Eq. (4-15) on page 4-14, *i.e.*, the fact that we can expand the matrix exponential function as a closed-form series:

$$e^{\mathbf{A}t} = \alpha_o(t)\mathbf{I} + \alpha_1(t)\mathbf{A} + \alpha_2(t)\mathbf{A}^2 + \ldots + \alpha_{n-1}(t)\mathbf{A}^{n-1} \tag{9-5}$$

Substitution of Eq. (9-5) into (9-4) gives

$$\mathbf{x}(0) = -\sum_{k=0}^{n-1}\mathbf{A}^k\mathbf{B}\int_0^t \alpha_k(\tau)u(\tau)\,d\tau$$

We now hide the ugly mess by defining the (n x 1) vector β with elements

$$\beta_k(\tau) = \int_0^t \alpha_k(\tau)u(\tau)\, d\tau$$

and Eq. (9-4) appears as

$$\mathbf{x}(0) = -\sum_{k=0}^{n-1} \mathbf{A}^k \mathbf{B}\beta_k = -\left[\mathbf{B} \;\vdots\; \mathbf{AB} \;\vdots\; \mathbf{A}^2\mathbf{B} \;\vdots\; \cdots \;\vdots\; \mathbf{A}^{n-1}\mathbf{B}\right]\begin{bmatrix}\beta_o \\ \beta_1 \\ \vdots \\ \beta_{n-1}\end{bmatrix} \tag{9-6}$$

If Eq. (9-6) is to be satisfied, the (n x n) matrix [**B** **AB** … $\mathbf{A}^{n-1}\mathbf{B}$] must be of rank n. This is a necessary and sufficient condition for controllability. Hence, we can state that a system is completely controllable if and only if the **controllability matrix**

$$\mathbf{C_o} = [\mathbf{B}\;\; \mathbf{AB}\;\; \mathbf{A}^2\mathbf{B}\; \ldots\; \mathbf{A}^{n-1}\mathbf{B}] \tag{9-7}$$

is of rank n.

The controllability condition is the same even when we have multiple inputs, **u**. If we have r inputs, then **u** is (r x 1), **B** is (n x r), each of the β_k is (r x 1), β is (nr x 1), and $\mathbf{C_o}$ is (n x nr).

When we have multiple outputs **y**, we want to control the output rather than the states. Complete state controllability is neither necessary nor sufficient for actual output controllability. With the output **y** = **Cx** and the result in (9-6), we can *infer* that the **output controllability matrix** is

$$\mathbf{C_o} = [\mathbf{CB}\;\; \mathbf{CAB}\;\; \mathbf{CA}^2\mathbf{B}\; \ldots\; \mathbf{CA}^{n-1}\mathbf{B}] \tag{9-8}$$

If we have m outputs, **y** is (m x 1) and **C** is (m x n). If we also have r inputs, then the output controllability matrix is (m x nr). Based on our interpretation of Eq. (9-6), we can also infer that to have complete output controllability, the matrix in (9-8) must have rank m.

9.1.2 Observability

The linear time invariant system in Eqs. (9-1) and (9-2) is completely observable if every initial state **x**(0) can be determined from the output y(t) over a finite time interval. The concept of observability is useful because in a given system, all not of the state variables are accessible for direct measurement. We will need to estimate the unmeasurable state variables from the output in order to construct the control signal.

Since our focus is to establish the link between y and **x**, or observability, it suffices to consider only the unforced problem:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \tag{9-9}$$

and

$$y = \mathbf{C}\mathbf{x} \tag{9-10}$$

Substitution of the solution of (9-9) in (9-10) gives

$$y(t) = \mathbf{C}e^{\mathbf{A}t}\mathbf{x}(0)$$

We again take that we can expand the exponential function as in Eq. (9-5). Thus we have

$$y(t) = \sum_{k=0}^{n-1} \alpha_k(t)\mathbf{CA}^k\mathbf{x}(0) = \left[\alpha_o\; \alpha_1\; \cdots\; \alpha_{n-1}\right]\begin{bmatrix}\mathbf{C} \\ \mathbf{CA} \\ \vdots \\ \mathbf{CA}^{n-1}\end{bmatrix}\mathbf{x}(0) \tag{9-11}$$

With the same reasoning that we applied to Eq. (9-6), we can infer that to have complete observability, the **observability matrix** [1]

$$\mathbf{O_b} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \vdots \\ \mathbf{CA}^{n-1} \end{bmatrix} \tag{9-12}$$

must be of rank *n*. When we have m outputs, **y** is (*m* x 1), **C** is (*m* x *n*), and $\mathbf{O_b}$ is (*mn* x *n*).

✎ **Example 9.1**: Consider a third order model:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

which is the controllable canonical form of the problem in Example 4.9 (p. 4-16). Construct the controllability and observability matrices.

To compute the controllability matrix, we can use the MATLAB function `ctrb()`:

```
A=[0 1 0; 0 0 1; -6 -11 -6];
B=[0; 0; 1];
Co=ctrb(A,B)
```

Or we can use the definition itself:

```
Co=[B A*B A^2*B]
```

Either way, we should obtain

$$\mathbf{C_o} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -6 \\ 1 & -6 & 25 \end{bmatrix}$$

which has a rank of 3 and the model is completely state controllable.

Similarly, we can use the MATLAB function `obsv()` for the observability matrix:

```
C=[1 0 0];
Ob=obsv(A,C)
```

Or we can use the definition:

```
Ob=[C; C*A; C*A^2]
```

We should find that `Ob` is the identity matrix, which of course, is of rank 3.

1 Controllability and observability are dual concepts. With $\mathbf{C} = \mathbf{B}^T$ and $\mathbf{A} = \mathbf{A}^T$, we can see that $\mathbf{O_b} = \mathbf{C_o}^T$.

✎ **Example 4.8A**: We now revisit the fermentor example 4.8 (p. 4-11). Our question is whether we can control the cell mass and glucose concentration by adjusting only D.

From Eq. (E4-38) in Example 4.8, we have

$$\mathbf{A} = \begin{bmatrix} 0 & C_1\mu' \\ -\frac{\mu}{Y} & -\frac{C_1}{Y}\mu' - \mu \end{bmatrix} \quad , \quad \mathbf{B} = \begin{bmatrix} -C_1 \\ \frac{C_1}{Y} \end{bmatrix}$$

First, we evaluate

$$\mathbf{AB} = \begin{bmatrix} 0 & C_1\mu' \\ -\frac{\mu}{Y} & -\frac{C_1}{Y}\mu' - \mu \end{bmatrix} \begin{bmatrix} -C_1 \\ \frac{C_1}{y} \end{bmatrix} = \begin{bmatrix} \frac{C_1^2\mu'}{Y} \\ -\frac{C_1^2\mu}{Y^2} \end{bmatrix}$$

The controllability matrix is

$$\mathbf{C_o} = [\mathbf{B}\ \ \mathbf{AB}] = \begin{bmatrix} -C_1 & \frac{C_1^2\mu'}{Y} \\ \frac{C_1}{Y} & -\frac{C_1^2\mu'}{Y^2} \end{bmatrix}$$

Since the determinant of $\mathbf{C_o}$ is 0, the rank of $\mathbf{C_o}$ is 1, both cell mass and substrate cannot be controlled simultaneously by just varying D. The answer is quite obvious with just a bit of intuition. If we insist on using D as the only input, we can control either C_1 or C_2, but not both quantities. To effectively regulate both C_1 and C_2, we must implement a system with two inputs. An obvious solution is to adjust the glucose feed concentration (C_{2o}) as well as the total flow rate (dilution rate D).

Now, we'll see what happens with two inputs. Compared with Eq. (E4-38), **A** remains the same, while **B** in Eq. (E4-47) is now a (2 x 2) matrix with a rank of 2. Hence the controllability matrix $\mathbf{C_o}$ = [**B** **AB**] is a (2 x 4) matrix and it must have a rank of 2 (since at least **B** is), and both C_1 and C_2 are controllable.

9.2 Pole Placement Design

9.2.1 Pole placement and Ackermann's formula.

When we used root locus for controller design in Chapter 7, we chose a dominant pole (or a conjugate pair if complex). With state space representation, we have the mathematical tool to choose all the closed-loop poles. To begin, we restate the state space model in Eqs. (4-1) and (4-2):

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u \qquad (9\text{-}13)$$

$$y = \mathbf{C}\mathbf{x} \qquad (9\text{-}14)$$

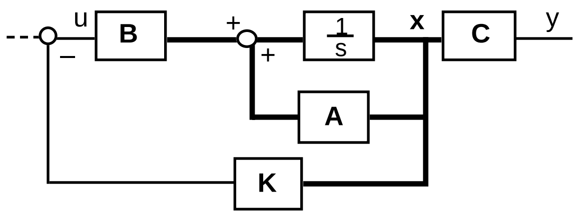


Figure 9.1. Closed-loop system with state feedback

With a control system, the input *u* is now the manipulated variable that is driven by the control signal (Fig. 9.1). For the moment, we consider only the regulator problem and omit changes in the set point. We state the simple control law which depends on full state feedback as

$$u(t) = -\mathbf{K}\mathbf{x} = -K_1x_1(t) - K_2x_2(t) \dots - K_nx_n(t) \qquad (9\text{-}15)$$

where **K** is the **state feedback gain** (1 x *n*) vector. In this formulation, the feedback information requires **x**(t), meaning that we must be able to measure all the state variables.

We now substitute Eq. (9-15) in (9-13) to arrive at the system equation

$$\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x} \qquad (9\text{-}16)$$

The eigenvalues of the system matrix (**A** – **BK**) are called the regulator poles. What we want is to find **K** such that it satisfies how we select all the eigenvalues (or where we put all the closed-loop poles).

To do that easily, we first need to put our model (9-13) in the controllable canonical form as in Eq. (4-19) on page 4-15:

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_o & -a_1 & -a_2 & \dots & -a_{n-1} \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u \qquad (9\text{-}17)$$

After substituting for *u* with Eq. (9-15), the system matrix in (9-16) is

$$\mathbf{A} - \mathbf{B}\mathbf{K} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_o & -a_1 & -a_2 & \dots & -a_{n-1} \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} K_1 \; K_2 \; \cdots \; K_n \end{bmatrix}$$

or

$$\mathbf{A} - \mathbf{BK} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 - K_1 & -a_1 - K_2 & -a_2 - K_3 & \cdots & -a_{n-1} - K_n \end{bmatrix} \qquad (9\text{-}18)$$

As in Eq. (4-21) on page 4-16, the closed-loop characteristic equation $|s\mathbf{I} - \mathbf{A} + \mathbf{BK}| = 0$ will appear as

$$s^n + (a_{n-1} + K_n)s^{n-1} + ... + (a_1 + K_2)s + (a_o + K_1) = 0 \qquad (9\text{-}19)$$

We next return to our assertion that we can choose all our closed-loop poles, or in terms of eigenvalues, λ_1, λ_2, … λ_n. This desired closed-loop characteristic equation is

$$(s - \lambda_1)(s - \lambda_2)\ldots\ (s - \lambda_n) = s^n + \alpha_{n-1}s^{n-1} + ... + \alpha_1 s + \alpha_o = 0 \qquad (9\text{-}20)$$

where the coefficients α_i are computed by expanding the terms in the LHS. By matching the coefficients of like power of *s* in Eqs. (9-19) and (9-20), we obtain

$$\begin{aligned} a_o + K_1 &= \alpha_o \\ a_1 + K_2 &= \alpha_1 \\ &\ldots \\ a_{n-1} + K_n &= \alpha_{n-1} \end{aligned}$$

Thus in general, we can calculate all the state feedback gains in **K** by

$$K_i = \alpha_{i-1} - a_{i-1} \qquad , i = 1, 2, \ldots n \qquad (9\text{-}21)$$

This is the result of full state feedback pole-placement design. If the system is completely state controllable, we can compute the state gain vector **K** to meet our selection of all the closed-loop poles (eigenvalues) through the coefficients α_i.

There are other methods in pole-placement design. One of them is the **Ackermann's formula**. The derivation of Eq. (9-21) predicates that we have put (9-13) in the controllable canonical form. Ackermann's formula only requires that the system (9-13) be completely state controllable. If so, we can evaluate the state feedback gain as [1]

$$\mathbf{K} = [0\ 0\ \ldots\ 1]\ [\mathbf{B}\ \ \mathbf{AB}\ \ \ldots\ \mathbf{A}^{n-1}\mathbf{B}]^{-1}\alpha_c(\mathbf{A}) \qquad (9\text{-}22)$$

where

$$\alpha_c(\mathbf{A}) = \mathbf{A}^n + \alpha_{n-1}\mathbf{A}^{n-1} + ... + \alpha_1\mathbf{A} + \alpha_o\mathbf{I} \qquad (9\text{-}23)$$

is the polynomial derived from the desired eigenvalues as in (9-20), except now $\alpha_c(\mathbf{A})$ is an (n x n) matrix.

9.2.2 Servo systems.

We now re-introduce the change in reference, r(t). We will stay with analyzing a single-input single-output system. By a proper choice in the indexing of the state variables, we select $x_1 = y$. In a feedback loop, the input to the process model may take the form

$$u(t) = K_r r(t) - \mathbf{Kx}(t)$$

[1] Roughly, the Ackermann's formula arises from the application of the Cayley-Hamilton theorem to (9-20). The details of the derivation are in our *Web Support*.

where K_r is some gain associated with the change in the reference, and **K** is the state feedback gain as defined in (9-15). One of the approaches that we can take is to choose $K_r = K_1$, such that u(t) is

$$u(t) = K_1[r(t) - x_1(t)] - K_2x_2(t) - \ldots - K_nx_n(t) \tag{9-24}$$

where we may recognize that $r(t) - x_1(t)$ is the error e(t).

The system equation is now

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}[K_1 r - \mathbf{K}\mathbf{x}]$$

or

$$\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x} + \mathbf{B}K_1 r \tag{9-25}$$

The system matrix and thus design procedures remain the same as in the regulator problem in Eq. (9-16).[1]

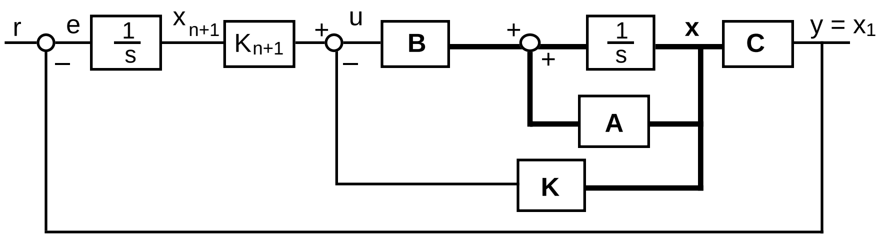


Figure 9.2. State feedback with integral control.

9.2.3 Servo systems with integral control.

You may notice that nothing that we have covered so far does integral control as in a PID controller. To implement integral action, we need to add one state variable as in Fig. 9.2. Here, we integrate the error $[r(t) - x_1(t)]$ to generate the new variable x_{n+1}. This quantity is multiplied by the additional feedback gain K_{n+1} before being added to the rest of the feedback data.

The input to the process model now takes the form

$$u(t) = K_{n+1}x_{n+1}(t) - \mathbf{K}\mathbf{x}(t) \tag{9-26}$$

[1] The system must be asymptotically stable. At the new steady state (as t –> ∞), we have

$$0 = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}(\infty) + \mathbf{B}K_1 r(\infty)$$

and subtracting this equation from (9-25), we have

$$\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{B}\mathbf{K})(\mathbf{x} - \mathbf{x}(\infty)) + \mathbf{B}K_1(r - r(\infty))$$

If we define $\mathbf{e} = \mathbf{x} - \mathbf{x}(\infty)$, and also r(t) as a step function such that r is really a constant for t > 0, the equation is simply

$$\dot{\mathbf{e}} = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{e}$$

Not only is this equation identical to the form in Eq. (9-16), but we also can interpret the analysis as equivalent to a problem where we want to find **K** such that the steady state error **e**(t) approaches zero as quickly as possible.

The differential equation for x_{n+1} is

$$\dot{x}_{n+1} = r(t) - \mathbf{C}\mathbf{x} \tag{9-27}$$

We have written $x_1 = y = \mathbf{Cx}$ just so that we can package this equation in matrix form in the next step. Substitution of Eq. (9-26) in the state model (9-13) and together with (9-27), we can write this (*n* + 1) system as

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{x}_{n+1} \end{bmatrix} = \begin{bmatrix} \mathbf{A}-\mathbf{BK} & \mathbf{B}K_{n+1} \\ -\mathbf{C} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ x_{n+1} \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix} r \tag{9-28}$$

In terms of dimensions, (**A** – **BK**), **B** and **C** remain, respectively, (*n* x *n*), (*n* x 1), and (1 x *n*). We can interpret the system matrix as

$$\begin{bmatrix} \mathbf{A}-\mathbf{BK} & \mathbf{B}K_{n+1} \\ -\mathbf{C} & 0 \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ -\mathbf{C} & 0 \end{bmatrix} - \begin{bmatrix} \mathbf{B} \\ 0 \end{bmatrix} \begin{bmatrix} \mathbf{K} & -K_{n+1} \end{bmatrix} = \hat{\mathbf{A}} - \hat{\mathbf{B}}\hat{\mathbf{K}} \tag{9-29}$$

where now our task is to find the (*n* + 1) state feedback gains

$$\hat{\mathbf{K}} = [\mathbf{K} \;\; -K_{n+1}] \tag{9-30}$$

With Eq. (9-29), we can view the characteristic equation of the system as

$$|s\mathbf{I} - \hat{\mathbf{A}} + \hat{\mathbf{B}}\hat{\mathbf{K}}| = 0 \tag{9-31}$$

which is in the familiar form of the probelm in (9-16). Thus, we can make use of the pole-placement techniques in Section 9.2.1.

✎ **Example 9.2**: Consider the second order model in Example 9.1. What are the state feedback gains if we specify that the closed-loop poles are to be at –3±3j and –6?

With the given model in the controllable canonical form, we can use Eq. (9-21). The MATLAB statements are:

```
A=[0 1 0; 0 0 1; -6 -11 -6];   % Should find
p1=poly(A)                     % [1 6 11 6], coefficients ai in (9-19)
P=[-3+3j -3-3j -6];
p2=poly(P)                     % [1 12 54 108], coefficients αi in (9-20)
p2-p1                          % [0 6 43 102], Ki as in Eq.(9-21)
```

To obtain the state feedback gains with Eq. (9-21), we should subtract the coefficients of the polynomial `p1` from `p2`, starting with the last constant coefficient. The result is, indeed,

$\mathbf{K} = (K_1, K_2, K_3) = (108–6,\ 54–11, 12–6) = (102, 43, 6)$

Check 1. The same result can be obtained with the MATLAB function `acker()` which uses the Ackermann's formula. The statements are:

```
B=[0; 0; 1];
acker(A,B,P)                   %Should return [102 43 6]
```

Check 2. We can do the Ackermann's formula step by step. The statements are:

```
M=[B A*B A^2*B];               %controllability matrix
ac=polyvalm(p2,A);             %Eq.(9-23)
[0 0 1]*inv(M)*ac              %Eq.(9-22)
```

To evaluate the matrix polynomial in Eq. (9-23), we use the MATLAB function `polyvalm()` which applies the coefficients in `p2` to the matrix `A`.

✎ **Example 4.7B**: Let us revisit the two CSTR-in-series problem in Example 4.7 (p. 4-5). Use the inlet concentration as the input variable and check that the system is controllable and observable. Find the state feedback gain such that the reactor system is very slightly underdamped with a damping ratio of 0.8, which is equivalent to about a 1.5% overshoot.

From (E4-27) of Example 4.7, the model is

$$\frac{d}{dt}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -5 & 0 \\ 2 & -4 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} 4 \\ 0 \end{bmatrix} c_o$$

and C_2 is the only output. We can construct the model and check the controllability and observability with

```
A=[-5 0; 2 -4];
B=[4; 0];
C=[0 1];
D=0;
rank(ctrb(A,B))   %should find rank = 2
rank(obsv(A,C))   % for both matrices
```

Both the controllability and observability matrices are of rank two. Hence the system is controllable and observable.

To achieve a damping ratio of 0.8, we can find that the closed-loop poles must be at –4.5±3.38j (using a combination of what we learned in Example 7.5 and Fig. 2.5), but we can cheat with MATLAB and use root locus plots!

```
[q,p]=ss2tf(A,B,C,D); %converts state space to transfer function[1]
Gp=tf(q,p);
rlocus(Gp)
sgrid(0.8,1)
[kc,P]=rlocfind(Gp)  %should find kc = 1.46
```

We now apply the closed-loop poles `P` directly to the Ackermann's formula:

```
K=acker(A,B,P)  %should find K = [0 1.46]
```

The state space state feedback gain (K_2) related to the output variable C_2 is the same as the proportional gain obtained with root locus. Given any set of closed-loop poles, we can find the state feedback gain of a controllable system using state-space pole placement methods. The use of root locus is not necessary, but it is a handy tool that we can take advantage of.

[1] Another way here is to make use of the analytical result in Example 4.7:

```
Gp=zpk([],[-5 -4],8); %transfer function C2/Co taken from (E4-30a)
```

✎ **Example 4.7C**: Add integral action to the system in Example 4.7B so we can eliminate the steady state error.

To find the new state feedback gain is a matter of applying Eq. (9-29) and the Ackermann's formula. The hard part is to make an intelligent decision on the choice of closed-loop poles. Following the lead of Example 4.7B, we use root locus plots to help us. With the understanding that we have two open-loop poles at –4 and –5, a reasonable choice of the integral time constant is 1/3 min. With the open-loop zero at –3, the reactor system is always stable, and the dominant closed-loop pole is real and the reactor system will not suffer from excessive oscillation.

Hence, our first step is to use root locus to find the closed-loop poles of a PI control system with a damping ratio of 0.8. The MATLAB statements to continue with Example 4.7B are:

```
kc=1; taui=1/3;
Gc=tf(kc*[taui 1],[taui 0]);
rlocus(Gc*Gp);                        %Gp is from Example 4.7B
sgrid(0.8,1)
[kc,P]=rlocfind(Gc*Gp) %should find proportional gain kc=1.66
```

The closed-loop poles `P` are roughly at –2.15 and –3.43±2.62j, which we apply immediately to the Ackermann's formula using $\hat{\mathbf{A}}$ and $\hat{\mathbf{B}}$ in Eq. (9-29):

```
Ah=[A zeros(2,1); -C 0];  %Eq. (9-29)
Bh=[B; 0];
Kh=acker(Ah,Bh,P)         %should find Kh = [0 1.66 –4.99]
```

The state feedback gain including integral control $\hat{\mathbf{K}}$ is [0 1.66 –4.99]. Unlike the simple proportional gain, we cannot expect that $K_{n+1} = 4.99$ would resemble the integral time constant in classical PI control. To do the time domain simulation, the task is similar to the hints that we provide for Example 7.5B in the Review Problems. The actual statements will also be provided on our Web Support.

✎ **Example 7.5B**: Consider the second order system in Example 7.5 (p. 7-9). What are the state feedback gains if we specify that the closed-loop poles are to be –0.375±0.382j as determined in Example 7.5A (p. 7-15)?

The problem posed in Examples 7.5 and 7.5A is not in the controllable canonical form (unless we do the transform ourselves). Thus we will make use of the Ackermann's formula. The MATLAB statements are:

```
G=tf(1,conv([2 1],[4 1]));        %Make the state space object from
S=ss(G);                          %  the transfer function
scale=S.c(2);                     %Rescale MATLAB model matrices
S.c=S.c/scale; S.b=S.b*scale;
P=[-0.375+0.382j -0.375-0.382j];  %Define the closed-loop poles
k=acker(S.a,S.b,P)                %Calculate the feedback gains
```

MATLAB will return the vector [0 1.29], meaning that $K_1 = 0$, and $K_2 = 1.29$, which was the proportional gain obtained in Example 7.5A. Since $K_1 = 0$, we only feedback the controlled variable as analogous to proportional control. In this very simple example, the state space system is virtually the classical system with a proportional controller.

A note of caution is necessary when we let MATLAB generate the state space model from a transfer function. The vector **C** (from `S.c`) is [0 0.5], which means that the indexing is reversed such that x_2 is the output variable, and x_1 is the derivative of x_2. Secondly, **C** is not [0 1], and hence we have to rescale the matrices **B** and **C**. These two points are further covered in MALTA Session 4.

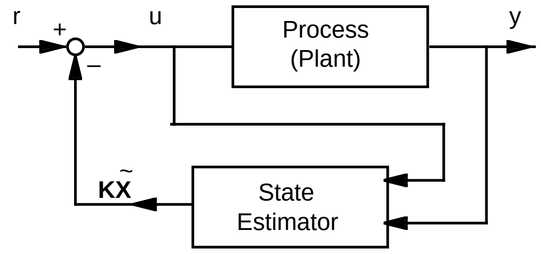


Figure 9.3. Concept of using a state estimator to generate an estimated state feedback signal.

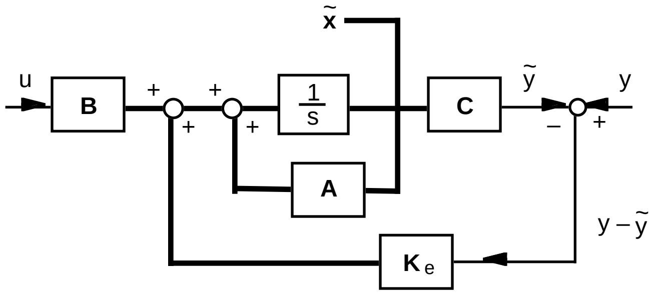


Figure 9.4. A probable model for a state estimator.

9.3 State Estimation Design

9.3.1. State estimator.

The pole placement design predicates on the feedback of *all* the state variables **x** (Fig. 9.1). Under many circumstances, this may not be true. We have to estimate unmeasureable state variables or signals that are too noisy to be measured accurately. One approach to work around this problem is to estimate the state vector with a model. The algorithm that performs this estimation is called the **state observer** or the **state estimator**. The estimated state $\tilde{\mathbf{x}}$ is then used as the feedback signal in a control system (Fig. 9.3). A full-order state observer estimates all the states even when some of them are measured. A reduced-order observer does the smart thing and skip these measurable states.

The next task is to seek a model for the observer. We stay with a single-input single-output system, but the concept can be extended to multiple outputs. The estimate should embody the dynamics of the plant (process). Thus one probable model, as shown in Fig. 9.4, is to assume that the state estimator has the same structure as the plant model, as in Eqs. (9-13) and (9-14), or Fig. 9.1. The estimator also has the *identical* plant matrices **A** and **B**. However, one major difference is the addition of the estimation error, $y - \tilde{y}$, in the computation of the estimated state $\tilde{\mathbf{x}}$.

The estimated state variables based on Fig. 9.4 can be described by (details in Review Problems)

$$\dot{\tilde{\mathbf{x}}} = \mathbf{A}\tilde{\mathbf{x}} + \mathbf{B}u + \mathbf{K}_e(y - \mathbf{C}\tilde{\mathbf{x}})$$

$$= (\mathbf{A} - \mathbf{K}_e\mathbf{C})\tilde{\mathbf{x}} + \mathbf{B}u + \mathbf{K}_e y \qquad (9\text{-}32)$$

Here, $\tilde{y} = \mathbf{C}\tilde{\mathbf{x}}$ has been used in writing the error in the estimation of the output, $(y - \tilde{y})$. The (n x 1) observer gain vector $\mathbf{K}_e$ does a weighting on how the error affects each estimate. In the next two sections, we will apply the state estimator in (9-32) to a state feedback system, and see how we can formulate the problem such that the error $(y - \tilde{y})$ can become zero.

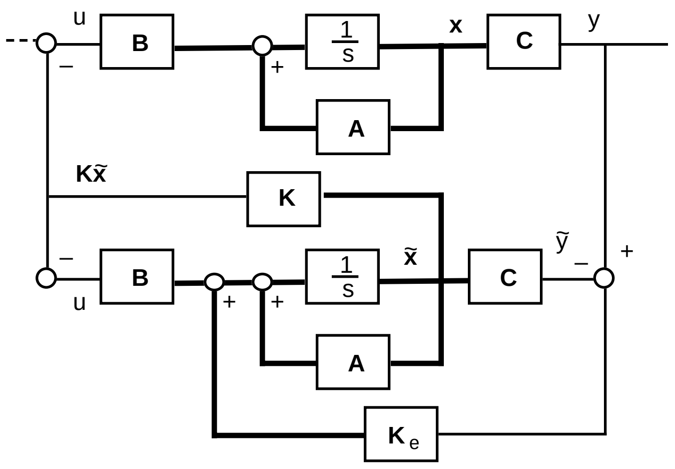


Figure 9.5. A regulator system with controller-estimator

9.3.2. Full-order state estimator system

A system making use of the state estimator is shown in Fig. 9.5, where for the moment, changes in the reference is omitted. What we need is the set of equations that describe this regulator system with state estimation.

By itself, the estimator in Eq. (9-32) has the characteristic equation:

$$|s\mathbf{I} - \mathbf{A} + \mathbf{K}_e\mathbf{C}| = 0 \tag{9-33}$$

Our intention is to use the estimated states to provide feedback information:

$$u = -\mathbf{K}\tilde{\mathbf{x}} \tag{9-34}$$

The state space model Eq. (9-13) now appears as

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u = \mathbf{A}\mathbf{x} - \mathbf{B}\mathbf{K}\tilde{\mathbf{x}} \tag{9-35}$$

If we substitute y = $\mathbf{C}\mathbf{x}$ in (9-32), we can integrate Eqs. (9-32) and (9-35) simultaneously to compute $\mathbf{x}$(t) and $\tilde{\mathbf{x}}$(t). In matrix form, this set of 2*n* equations can be written as

$$\frac{d}{dt}\begin{bmatrix}\mathbf{x}\\ \tilde{\mathbf{x}}\end{bmatrix} = \left[\begin{array}{c:c}\mathbf{A} & -\mathbf{B}\mathbf{K}\\ \hdashline \mathbf{K}_e\mathbf{C} & \mathbf{A}-\mathbf{K}_e\mathbf{C}-\mathbf{B}\mathbf{K}\end{array}\right]\begin{bmatrix}\mathbf{x}\\ \tilde{\mathbf{x}}\end{bmatrix} \tag{9-36}$$

9.3.3. Estimator design

With Eq. (9-36), it is not obvious how $\mathbf{K}_e$ affects the choices of $\mathbf{K}$. We now derive a form of (9-36) that is based on the error of the estimation and is easier for us to make a statement on its properties. We define the state error vector as

$$\mathbf{e}(t) = \mathbf{x}(t) - \tilde{\mathbf{x}}(t) \tag{9-37}$$

Subtract Eq. (9-32) from (9-35), and use y = $\mathbf{C}\mathbf{x}$, we should find

$$(\dot{\mathbf{x}} - \dot{\tilde{\mathbf{x}}}) = (\mathbf{A} - \mathbf{K}_e\mathbf{C})(\mathbf{x} - \tilde{\mathbf{x}}) \quad \text{or} \quad \dot{\mathbf{e}} = (\mathbf{A} - \mathbf{K}_e\mathbf{C})\mathbf{e} \tag{9-38}$$

This error equation has the same characteristic equation as the estimator in Eq. (9-33). The goal is to choose eigenvalues of the estimator such that the error decays away quickly. We may note that the form of (9-38) is the same as that of the regulator problem. Thus we should be able to use the tools of pole-placement for the estimator design. In fact, we can apply, without derivation, a modified form of Ackermann's formula to evaluate

$$\mathbf{K}_e = \alpha_e(\mathbf{A}) \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \vdots \\ \mathbf{CA}^{n-1} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \qquad (9\text{-}39)$$

where as analogous to Eq. (9-20),

$$\alpha_e(s) = s^n + \alpha_{n-1}s^{n-1} + \ldots + \alpha_1 s + \alpha_o \qquad (9\text{-}40)$$

is the polynomial derived from our own chosen estimator eigenvalues. Eq. (9-39) is different from Eq. (9-22) because we are now solving the *dual* problem for the (*n* x 1) vector $\mathbf{K}_e$.

Next, we can replace $\tilde{\mathbf{x}}$ in Eq. (9-35) by the definition of the error vector, and the equation becomes

$$\dot{\mathbf{x}} = \mathbf{Ax} - \mathbf{BK}(\mathbf{x} - \mathbf{e}) \qquad (9\text{-}41)$$

Eqs. (9-38) and (9-41) can be put in matrix form as

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{e}} \end{bmatrix} = \left[\begin{array}{c:c} \mathbf{A} - \mathbf{BK} & \mathbf{BK} \\ \hdashline \mathbf{0} & \mathbf{A} - \mathbf{K}_e\mathbf{C} \end{array}\right] \begin{bmatrix} \mathbf{x} \\ \mathbf{e} \end{bmatrix} \qquad (9\text{-}42)$$

Now, it is clear that the characteristic equation of the controller-estimator system is

$$|s\mathbf{I} - \mathbf{A} + \mathbf{BK}|\;|s\mathbf{I} - \mathbf{A} + \mathbf{K}_e\mathbf{C}| = 0 \qquad (9\text{-}43)$$

We have the very important result that choices for the eigenvalues for the pole-placement design and the observer design can be made independently. Generally, we want the observer response to be two to five times faster than the system response. We should not have to worry about saturation since the entire observer is software-based, but we do have to consider noise and sensitivity problems.

✎ **Example 9.3**: Consider the second order model in Example 9.1, which we have calculated the state feedback gains in Example 9.2. What is the observer gain vector $\mathbf{K}_e$ if we specify that the estimator error should have eigenvalues –9 repeated thrice?

With eigenvalues selected at –9, we have chosen the estimator to be faster than the state feedback, and all the errors are to decay exponentially. We'll make use of the Ackermann's formula in Eq. (9-39) for observer gains. The MATLAB statements are:

```
A=[0 1 0; 0 0 1; -6 -11 -6];   %Define the model
B=[0; 0; 1];
C=[1 0 0];

pe=poly([-9 -9 -9]);           %Make estimator polynomial (9-40)
ae=polyvalm(pe,A);
Ob=[C; C*A; C*A^2];
Ke=ae*inv(Ob)*[0; 0; 1]        %Eq. (9-39)
```

We should find that $\mathbf{K}_e$ = (21, 106, –144). The estimator calculations are purely mathematical, and the values of the observer gains can be negative. Furthermore, we can check that the system of equations in Eq. (9-42) has the correct eigenvalues as suggested by Eq. (4-43).

```
K=[102 43 6];   %Feeback gains calculated from Example 9.2
A11=A-B*K;      %Submatrices in Eq. (9-42)
A12=B*K;
A21=zeros(3,3);
A22=A-Ke*C;
BIGA=[A11 A12; A21 A22];
eig(BIGA)
```

Indeed, we should find that the big matrix `BIGA` has eigenvalues –3±3j, –6, and –9 repeated three times.

9.3.4. Reduced-order estimator

We should not have to estimate variables that we can measure. It is logical to design a reduced-order estimator which estimates only the states that cannot be measured or are too noisy to be measured accurately. Following our introductory practice, we will consider only one measured output. The following development assumes that we have selected x_1 to be the measured variable. Hence, the output is

$$y = \mathbf{C}\mathbf{x} = [1 \;\; 0 \; \ldots \; 0]\, \mathbf{x} \qquad (9\text{-}44)$$

Next, we partition the state vector as

$$\mathbf{x} = \begin{bmatrix} x_1 \\ \mathbf{x}_e \end{bmatrix} \qquad (9\text{-}45)$$

where $\mathbf{x}_e = [x_2 \ldots x_n]$ contains the $(n - 1)$ states that have to be estimated. The state model equation (9-13) is partitioned accordingly as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{\mathbf{x}}_e \end{bmatrix} = \begin{bmatrix} a_{11} & \mathbf{A}_{1e} \\ \mathbf{A}_{e1} & \mathbf{A}_{ee} \end{bmatrix} \begin{bmatrix} x_1 \\ \mathbf{x}_e \end{bmatrix} + \begin{bmatrix} b_1 \\ \mathbf{B}_e \end{bmatrix} u \qquad (9\text{-}46)$$

where the dimensions of $\mathbf{A}_{1e}$, $\mathbf{A}_{e1}$, $\mathbf{A}_{ee}$ are, respectively, $(1 \times n–1)$, $(n–1 \times 1)$, and $(n–1 \times n–1)$, and that of $\mathbf{B}_e$ is $(n–1 \times 1)$.

The next task is to make use of the full state estimator equations. Before that, we have to remold Eq. (9-46) as if it were a full state problem. This exercise requires some careful bookkeeping of notations. Let’s take the first row in Eq. (9-46) and make it to constitute the output equation. Thus we make a slight rearrangement:

$$\dot{x}_1 - a_{11}x_1 - b_1 u = \mathbf{A}_{1e}\mathbf{x}_e$$

such that it takes the form of $y = \mathbf{C}\mathbf{x}$. We repeat with the second row of (9-46) and put it as

$$\dot{\mathbf{x}}_e = \mathbf{A}_{ee}\mathbf{x}_e + (\mathbf{A}_{e1}x_1 + \mathbf{B}_e u)$$

such that it can be compared with $\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{B}u$.

The next step is to take the full state estimator in Eq. (9-32),

$$\dot{\tilde{\mathbf{x}}} = (\mathbf{A} - \mathbf{K}_e\mathbf{C})\tilde{\mathbf{x}} + \mathbf{B}u + \mathbf{K}_e y$$

and substitute term by term using the reduced-order model equations.[1] The result is, finally,

[1] The matching of terms for reduced-order substitution in Eq. (9-31) to derive (9-47) to (9-49):

$$\dot{\tilde{\mathbf{x}}}_e = (\mathbf{A}_{ee} - \mathbf{K}_{er}\mathbf{A}_{1e})\tilde{\mathbf{x}}_e + (\mathbf{A}_{e1}x_1 + \mathbf{B}_e u) + \mathbf{K}_{er}(\dot{x}_1 - a_{11}x_1 - b_1 u) \qquad (9\text{-}47)$$

which is the reduced-order equivalent to (9-32). Note that in this equation, $x_1 = y$.

The computation of the (n – 1) weighting factors in $\mathbf{K}_{er}$ can be based on the equivalent form of Eq. (9-38). Again, doing the substitution for the notations, the error estimate becomes

$$\dot{\mathbf{e}} = (\mathbf{A}_{ee} - \mathbf{K}_{er}\mathbf{A}_{1e})\mathbf{e} \qquad (9\text{-}48)$$

which means that the Ackermann’s formula in Eq. (9-39) now takes the form

$$\mathbf{K}_{er} = \alpha_e(\mathbf{A}_{ee})\begin{bmatrix} \mathbf{A}_{1e} \\ \mathbf{A}_{1e}\mathbf{A}_{ee} \\ \vdots \\ \mathbf{A}_{1e}\mathbf{A}_{ee}^{n-1} \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \qquad (9\text{-}49)$$

We are not quite done yet. If we use Eq. (9-47) to compute $\tilde{\mathbf{x}}_e$, it requires taking the derivative of x_1, an exercise that can easily amplify noise. So we want a modified form that allows us to replace this derivative. To begin, we define a new variable

$$\tilde{\mathbf{x}}_{e1} = \tilde{\mathbf{x}}_e - \mathbf{K}_{er}x_1 \qquad (9\text{-}50)$$

This variable is substituted into (9-47) to give

$$(\dot{\tilde{\mathbf{x}}}_{e1} + \mathbf{K}_{er}\dot{x}_1) = (\mathbf{A}_{ee} - \mathbf{K}_{er}\mathbf{A}_{1e})(\tilde{\mathbf{x}}_{e1} + \mathbf{K}_{er}x_1) + (\mathbf{A}_{e1}x_1 + \mathbf{B}_e u) + \mathbf{K}_{er}(\dot{x}_1 - a_{11}x_1 - b_1 u)$$

After cancellation of the derivative term, we have

$$\begin{aligned}\dot{\tilde{\mathbf{x}}}_{e1} &= (\mathbf{A}_{ee} - \mathbf{K}_{er}\mathbf{A}_{1e})\tilde{\mathbf{x}}_{e1} \\ &\quad + (\mathbf{A}_{ee}\mathbf{K}_{er} - \mathbf{K}_{er}\mathbf{A}_{1e}\mathbf{K}_{er} + \mathbf{A}_{e1} - \mathbf{K}_{er}a_{11})x_1 + (\mathbf{B}_e - \mathbf{K}_{er}b_1)u \qquad (9\text{-}51)\end{aligned}$$

This differential equation is used to compute $\tilde{\mathbf{x}}_{e1}$, which then is used to calculate $\tilde{\mathbf{x}}_e$ with (9-50). With the estimated states, we can compute the feedback to the state space model as

$$u = -\begin{bmatrix} K_1 & \mathbf{K}_{1e}^T \end{bmatrix}\begin{bmatrix} x_1 \\ \tilde{\mathbf{x}}_e \end{bmatrix} \qquad (9\text{-}52)$$

The application of Eqs. (9-50) to (9-52) is a bit involved and best illustrated as shown in Fig. 9.6.

Full-order state estimator	Reduced-order state estimator
$\tilde{\mathbf{x}}$	$\tilde{\mathbf{x}}_e$
y	$\dot{x}_1 - a_{11}x_1 - b_1 u$
$\mathbf{C}$	$\mathbf{A}_{1e}$
$\mathbf{A}$	$\mathbf{A}_{ee}$
$\mathbf{K}_e$, (n x 1)	$\mathbf{K}_{er}$, (n–1 x 1)
$\mathbf{B}u$	$\mathbf{A}_{e1}x_1 + \mathbf{B}_e u$

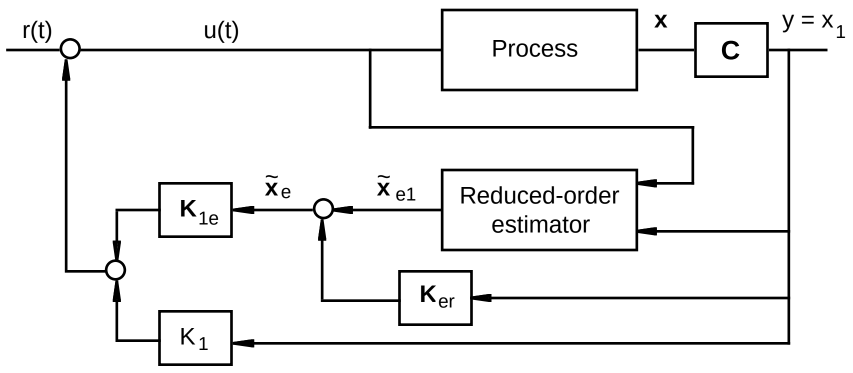


Figure 9.6. State feedback with reduced-order estimator

✎ **Example 9.4**: Consider the estimator in Example 9.3, what is the reduced-order observer gain vector $\mathbf{K}_{er}$ if we specify that the estimator error should have eigenvalues –9 repeated twice?

We can use Eq. (9-49), and the MATLAB statements are:

```
A=[0 1 0; 0 0 1; -6 -11 -6];
N=size(A,1);
a11=A(1,1);       %Extract matrix partitions as in Eq. (9-46)
A1e=A(1,2:N);
Ae1=A(2:N,1);
Aee=A(2:N,2:N);

pe=poly([-9 -9]);              %Make estimator polynomial
ae=polyvalm(pe,Aee);
Ob=[A1e; A1e*Aee];
Ker=ae*inv(Ob)*[0; 1]          %Eq. (9-49) for n=2
```

We should find that $\mathbf{K}_{er} = (12\ \ -2)$.

After all this fancy mathematics, we need a word of caution. It is extremely dangerous to apply the state estimate as presented in this chapter. Why? The first hint is in Eq. (9-32). We have assumed perfect knowledge of the plant matrices. Of course, we rarely do. Furthermore, we have omitted actual terms for disturbances, noises, and errors in measurements. Despite these drawbacks, material in this chapter provides the groundwork to attack serious problems in modern control.

Review Problems

1. For the second order transfer function

$$\frac{Y}{U} = \frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2},$$

derive the controllable canonical form. If the desired poles of a closed-loop system are to be placed at λ_1 and λ_2, what should be the state feedback gains?

2. Presume we do not know what the estimator should be other than that it has the form

$$\dot{\tilde{\mathbf{x}}} = \mathbf{F}\tilde{\mathbf{x}} + \mathbf{G}u + \mathbf{H}y$$

Find Eq. (9-32).

3. Do the time response simulation in Example 7.5B. We found that the state space system has a steady state error. Implement integral control and find the new state feedback gain vector. Perform a time response simulation to confirm the result.

4. With respect to Fig. R9.4, what is the transfer function equivalent to the controller-estimator system in Eq. (9-32)?

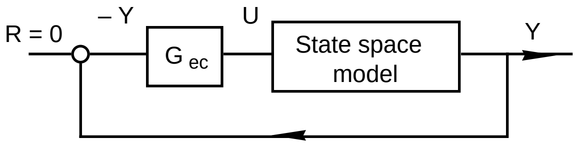


Figure R9.4

Hints:

1. The controllable canonical form was derived in Example 4.1. The characteristic polynomial of (s**I** – **A** + **BK**) should be

$$s^2 + (2\zeta\omega_n + K_2)s + (\omega_n^2 + K_1) = 0$$

The characteristic polynomial of desired poles is

$$s^2 + (\lambda_1 + \lambda_2)s + \lambda_1\lambda_2 = 0$$

Thus

$$K_1 = \lambda_1\lambda_2 - \omega_n^2 \quad \text{and} \quad K_2 = (\lambda_1 + \lambda_2) - 2\zeta\omega_n$$

2. The Laplace transform of the given equation is

$$s\tilde{\mathbf{X}} = \mathbf{F}\tilde{\mathbf{X}} + \mathbf{G}U + \mathbf{H}Y$$

Substituting Y = **CX**, we have

$$\tilde{\mathbf{X}} = (s\mathbf{I} - \mathbf{F})^{-1}[\mathbf{G}U + \mathbf{HCX}]$$

We further substitute for $\mathbf{X} = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}U$ with the simple state space model to give

$$\tilde{\mathbf{X}} = (s\mathbf{I} - \mathbf{F})^{-1}[\mathbf{G} + \mathbf{HC}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}]U$$

What we want is to dictate that the transfer function of this estimator is the same as that of the state space model:

$$(s\mathbf{I} - \mathbf{F})^{-1}[\mathbf{G} + \mathbf{HC}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}] = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$$

Move the second term to the RHS and factor out the $(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$ gives

$$(s\mathbf{I} - \mathbf{F})^{-1}\mathbf{G} = [\mathbf{I} - (s\mathbf{I} - \mathbf{F})^{-1}\mathbf{HC}](s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$$

Thus we can multiply (s**I** –**F**) to both sides to have

$$\mathbf{G} = [(s\mathbf{I} - \mathbf{F}) - \mathbf{HC}](s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$$

And finally,

$$[(s\mathbf{I} - \mathbf{F}) - \mathbf{HC}]^{-1}\mathbf{G} = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$$

Compare term by term, we have

$$\mathbf{F} + \mathbf{HC} = \mathbf{A}\ , \quad \text{or } \mathbf{F} = \mathbf{A} - \mathbf{HC}$$

and

$$\mathbf{G} = \mathbf{B}$$

This result is what we need in (9-32) if we also set $\mathbf{H} = \mathbf{K}_e$.

3. For the time response simulation, we also duplicate the classical control design for comparision. Both classical and state space results have the same damping ratio, but not system steady state gain. The statements are:

```
G=tf(1,conv([2 1],[4 1]));
S=ss(G);                          %MATLAB uses reverse indexing
scale=S.c(2);                     %And need to rescale B and C too
S.c=S.c/scale;
S.b=S.b*scale;
P=[-0.375+0.382j -0.375-0.382j];  %Define the closed-loop poles
K=acker(S.a,S.b,P)
%Compute the system matrices for plotting
A = S.a - S.b*K                   %Makes system matrix, Eq. (9-25)
B = S.b*K(2)
C = S.c
D=0;
step(A,B,C,D)
hold %to add the classical design result
Gcl=feedback(1.29*G,1);
%Kc=1.29 was the proportional gain obtained in Example 7.5A
step(Gcl)
```

To eliminate offset, we need Section 9.2.3. With an added state due to integration, we have to add one more closed-loop poles. We choose it to be –1, sufficiently faster than the real part of the complex poles. The statements are:

```
G=tf(1,conv([2 1],[4 1]));
S=ss(G);  %Generates the matrices S.a, S.b, S.c, S.d
Ah=[S.a  zeros(2,1); -S.c 0]            % Â in (9-29)
Bh=[S.b; 0]                             % B̂
P=[-0.375+0.382j -0.375-0.382j -1];     %Add a faster pole at -1
kh=acker(Ah,Bh,P) %K-head in (9-29)     % K̂
```

We should find $\hat{\mathbf{K}}$ = `[2 3.6 -2.3]`. To do the time response simulation, we can use:

```
Asys=Ah-Bh*kh;              %System matrix (9-29)
Bsys=[0; 0; 1];             %Follows (9-28)
Csys=[S.c 0];
step(Asys, Bsys,Csys,0)
```

4. For the estimator, y is the input and u the output. With $u = -\mathbf{K}\tilde{\mathbf{x}}$, the Laplace transform of Eq. (9-32) is

$$[s\mathbf{I} - \mathbf{A} + \mathbf{K}_e\mathbf{C} + \mathbf{B}\mathbf{K}]\tilde{\mathbf{X}}(s) = \mathbf{K}_e Y(s)$$

or

$$\tilde{\mathbf{X}}(s) = [s\mathbf{I} - \mathbf{A} + \mathbf{K}_e\mathbf{C} + \mathbf{B}\mathbf{K}]^{-1}\mathbf{K}_e Y(s)$$

We now substitute $\tilde{\mathbf{X}}$ back in the Laplace transform of $u = -\mathbf{K}\tilde{\mathbf{x}}$ to obtain

$$U(s) = -\mathbf{K}[s\mathbf{I} - \mathbf{A} + \mathbf{K}_e\mathbf{C} + \mathbf{B}\mathbf{K}]^{-1}\mathbf{K}_e Y(s) = -G_{ec}(s)Y(s)$$

❖ 10. *Multiloop Systems*

There are many advanced strategies in classical control systems. Only a limited selection of examples is presented in this chapter. We start with cascade control, which is a simple introduction to a multiloop, but essentially SISO, system. We continue with feedforward and ratio control. The idea behind ratio control is simple, and it applies quite well to the furnace problem that we use as an illustration. Finally, we address a multiple-input multiple-output system using a simple blending problem as illustration, and use the problem to look into issues of interaction and decoupling. These techniques build on what we have learned in classical control theories.

What are we up to?

- Apply classical controller analysis to cascade control, feedforward control, feedforward-feedback control, ratio control, and the Smith predictor for time delay compensation.
- Analyze a MIMO system with relative gain array , and assess the pairing of manipulated and controlled variables.
- Attempt to decouple and eliminate the interactions in a two-input two-output system.

10.1 Cascade control

A very common design found in process engineering is cascade control. This is a strategy that allows us to handle load changes more effectively with respect to the manipulated variable.

To illustrate the idea, we consider the temperature control of a gas furnace, which is used to heat up a cold process stream. The fuel gas flow rate is the manipulated variable, and its flow is subject to fluctuations due to upstream pressure variations.

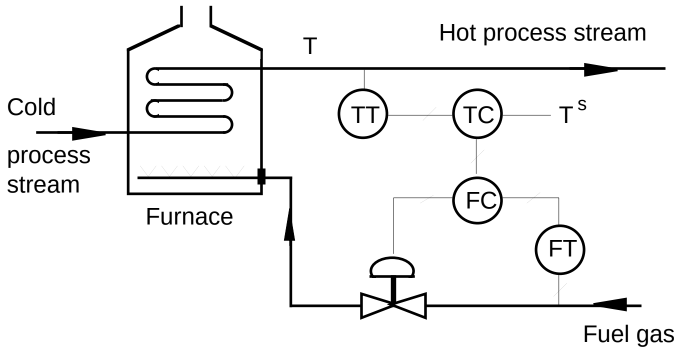


Figure 10.1. Cascade control of the temperature of a furnace, which is taken to be the same as that of the outlet process stream. The temperature controller does not actuate the regulating valve directly; it sends its signal to a secondary flow rate control loop which in turn ensures that the desired fuel gas

In a simple single-loop system, we measure the outlet temperature, and the temperature controller (TC) sends its signal to the regulating valve. If there is fluctuation in the fuel gas flow rate, this simple system will not counter the disturbance until the controller senses that the temperature of the furnace has deviated from the set point (T^s).

A cascade control system can be designed to handle fuel gas disturbance more effectively (Fig. 10.1). In this case, a *secondary loop* (also called the *slave loop*) is used to adjust the regulating valve and thus manipulate the fuel gas flow rate. The temperature controller (the master or primary controller) sends its signal, in terms of the desired flow rate, to the secondary flow control loop—in essence, the signal is the set point of the secondary flow controller (FC).

In the secondary loop, the flow controller compares the desired fuel gas flow rate with the measured flow rate from the flow transducer (FT), and adjusts the regulating valve accordingly. This inner flow control loop can respond immediately to fluctuations in the fuel gas flow to ensure that the proper amount of fuel is delivered.

To be effective, the secondary loop must have a faster response time (smaller time constant) than the outer loop. Generally, we use as high a proportional gain as feasible. In control jargon, we say that the inner loop is tuned very tightly.

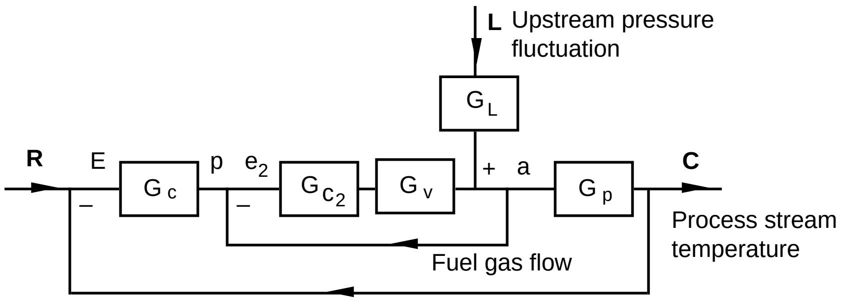


Figure 10.2a. Block diagram of a simple cascade control system with reference to the furnace problem.

We can use a block diagram to describe Fig. 10.1. Cascade control adds an inner control loop with secondary controller function G_{c2} (Fig. 10.2a). This implementation of cascade control requires two controllers and two measured variables (fuel gas flow and furnace temperature). The furnace temperature is the controlled variable, and the fuel gas flow rate remains the only manipulated variable.

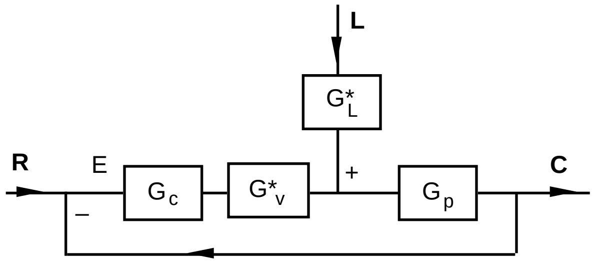


Figure 10.2b. Reduced block diagram of a cascade control system.

For cleaner algebra, we omit the measurement transfer functions, taking $G_{m1} = G_{m2} = 1$. Disturbance, such as upstream pressure, which specifically leads to changes in the fuel gas flow rate is now drawn to be part of the secondary flow control loop. (A disturbance such as change in the process stream inlet temperature, which is not part of the secondary loop, would still be drawn in its usual location as in Section 5.2 on page 5-7.)

We now reduce the block diagram. The first step is to close the inner loop so the system becomes a standard feedback loop (Fig. 10.2b). With hindsight, the result should be intuitively obvious. For now, we take the slow route. Using the lower case letter locations in Fig. 10.2a, we write down the algebraic equations

$$e_2 = p - a$$

and

$$a = G_{c2}G_v e_2 + G_L L$$

Substitution of e_2 leads to

$$a = G_{c2}G_v(p - a) + G_L L$$

and the result after rearrangement is a form that allows us to draw Fig. 10.2b:

$$a = \left[\frac{G_{c_2}G_v}{1 + G_{c_2}G_v}\right] p + \left[\frac{G_L}{1 + G_{c_2}G_v}\right] L = G^*_v\, p + G^*_L\, L$$

where

$$G^*_v = \left[\frac{G_{c_2}G_v}{1 + G_{c_2}G_v}\right] \quad \text{and} \quad G^*_L = \left[\frac{G_L}{1 + G_{c_2}G_v}\right] \tag{10-1}$$

The remaining task to derive the closed-loop transfer functions is routine. Again, slowly, we can write the relation in Fig. 10.2b as

$$C = G^*_L G_p \, L + G_c G^*_v G_p \, E$$

and substituting E = R – C, we have, after rearrangement,

$$C = \left[\frac{G_c G_v^* G_p}{1 + G_c G_v^* G_p}\right] R + \left[\frac{G_p . G_L^*}{1 + G_c G_v^* G_p}\right] L \qquad (10\text{-}2)$$

The closed-loop characteristic polynomial of this cascade system is

$$1 + G_c G_v^* G_p = 0 \qquad (10\text{-}3)$$

If we now substitute G^*_v from (10-1), the characteristic polynomial takes the form [1]

$$1 + G_{c_2} G_v + G_c G_{c_2} G_v G_p = 0 \qquad (10\text{-}3a)$$

So far, we know that the secondary loop helps to reduce disturbance in the manipulated variable. If we design the control loop properly, we should also accomplish a faster response in the actuating element: the regulating valve. To go one step further, cascade control can even help to make the entire system more stable. These points may not be intuitive. We'll use a simple example to illustrate these features.

✎ **Example 10.1**: Consider a simple cascade system as shown in Fig. 10.2a with a PI controller in the primary loop, and a proportional controller in the slave loop. For simplicity, consider first order functions

$$G_p = \frac{0.8}{2s+1},\ G_v = \frac{0.5}{s+1},\ \text{and } G_L = \frac{0.75}{s+1}.$$

(a) How can proper choice of K_{c2} of the controller in the slave loop help to improve the actuator performance and eliminate disturbance in the manipulated variable (e.g., fuel gas flow in the furnace temperature control)?

If we substitute $G_{c2} = K_{c2}$, and $G_v = \dfrac{K_v}{\tau_v s + 1}$ into G^*_v in Eq. (10-1), we should find

$$G_v^* = \left[\frac{K_{c_2} K_v}{(\tau_v s + 1) + K_{c_2} K_v}\right] = \frac{K_v^*}{\tau_v^* s + 1}, \qquad (E10\text{-}1)$$

where

$$K_v^* = \left[\frac{K_{c_2} K_v}{1 + K_{c_2} K_v}\right], \quad \text{and} \quad \tau_v^* = \left[\frac{\tau_v}{1 + K_{c_2} K_v}\right]. \qquad (E10\text{-}2)$$

Similarly, substituting $G_L = \dfrac{K_L}{\tau_v s + 1}$ in G^*_L should give

$$K_L^* = \left[\frac{K_L}{1 + K_{c_2} K_v}\right]. \qquad (E10\text{-}3)$$

Thus as the proportional gain K_{c2} becomes larger, K^*_v approaches unity gain, meaning there

[1] If we remove the secondary loop, this characteristic equation should reduce to that of a conventional feedback system equation. It is not obvious from (10-3) because our derivation has taken the measurement function G_{m2} to be unity. If we had included G_{m2} in a more detailed analysis, we could get the single loop result by setting $G_{c2} = 1$ and $G_{m2} = 0$.

is a more effective change in the manipulated variable, and K^*_L approaches zero, meaning the manipulated variable is becoming less sensitive to changes in the load. Furthermore, the effective actuator time constant τ^*_V will become smaller, meaning a faster response.

(b) The slave loop affords us a faster response with respect to the actuator. What is the proportional gain K_{c2} if we want the slave loop time constant τ^*_V to be only one-tenth of the original time constant τ_V in G_v?

From the problem statement, $K_V = 0.5$ and $\tau_V = 1$ s. Thus $\tau^*_V = 0.1$ s, and substitution of these values in τ^*_V of (E10-2) gives

$$0.1 = \left[\frac{1}{1 + 0.5\,K_{c_2}}\right], \text{ or } K_{c2} = 18.$$

The steady state gain is

$$K_v^* = \frac{(18)\,(0.5)}{1 + (18)\,(0.5)} = 0.9$$

The slave loop will have a 10% offset with respect to desired set point changes in the secondary controller.

(c) So far, we have only used proportional control in the slave loop. We certainly expect offset in this inner loop. Why do we stay with proportional control here?

The modest 10% offset that we have in the slave loop is acceptable under most circumstances. As long as we have integral action in the outer loop, the primary controller can make necessary adjustments in its output and ensure that there is no steady state error in the controlled variable (e.g., the furnace temperature).

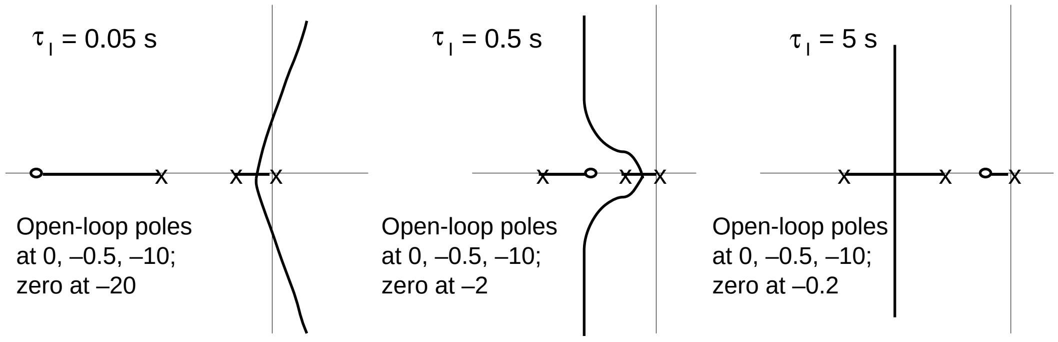
(d) Now, we tackle the entire closed-loop system with the primary PI controller. Our task here is to choose the proper integral time constant among the given values of 0.05, 0.5, and 5 s. We can tolerate underdamped response but absolutely not a system that can become unstable. Of course, we want a system response that is as fast as we can make it, *i.e.*, with a proper choice of proportional gain. Select and explain your choice of the integral time constant.

Among other methods, root locus is the most instructive in this case. With a PI primary controller and numerical values, Eq. (10-3) becomes

$$1 + K_c\left(\frac{\tau_I s + 1}{\tau_I s}\right)\left(\frac{0.9}{0.1\,s + 1}\right)\left(\frac{0.8}{2\,s + 1}\right) = 0$$

With MATLAB, we can easily prepare the root locus plots of this equation for the cases of $\tau_I = 0.05$, 0.5, and 5 s. (You should do it yourself. We'll show only a rough sketch in Fig E10.1. Help can be found in the Review Problems.)

From the root locus plots, it is clear that the system may become unstable when $\tau_I = 0.05$ s. The system is always stable when $\tau_I = 5$ s, but the speed of the system response is limited by the dominant pole between the origin and –0.2. The proper choice is $\tau_I = 0.5$ s in which case the system is always stable but the closed-loop poles can move farther, loosely speaking, away from the origin.



τ_I = 0.05 s
Open-loop poles at 0, –0.5, –10; zero at –20
τ_I = 0.5 s
Open-loop poles at 0, –0.5, –10; zero at –2
τ_I = 5 s
Open-loop poles at 0, –0.5, –10; zero at –0.2

Figure E10.1

(e) Take the case *without* cascade control, and using the integral time constant that you have selected in part (d), determine the range of proportional gain that we can use (without the cascade controller and secondary loop) to maintain a stable system. How is this different from when we use cascade control?

With the choice of τ_I = 0.5 s, but without the inner loop nor the secondary controller, the closed-loop equation is

$$1 + G_c G_v G_p = 1 + K_c \left(\frac{0.5\,s + 1}{0.5\,s}\right)\left(\frac{0.5}{s + 1}\right)\left(\frac{0.8}{2\,s + 1}\right) = 0$$

which can be expanded to

$$s^3 + 1.5\,s^2 + (0.5 + 0.2K_c)\,s + 0.4K_c = 0$$

With the Routh-Hurwitz analysis in Chapter 7, we should find that to have a stable system, we must keep $K_c < 7.5$. (You fill in the intermediate steps in the Review Problems. Other techniques such as root locus, direct substitution or frequency response in Chapter 8 should arrive at the same result.)

With cascade control, we know from part (d) that the system is always stable. Nevertheless, we can write the closed-loop characteristic equation

$$1 + K_c \left(\frac{0.5\,s + 1}{0.5\,s}\right)\left(\frac{0.9}{0.1\,s + 1}\right)\left(\frac{0.8}{2\,s + 1}\right) = 0$$

or

$$0.1\,s^3 + 1.05\,s^2 + (0.5 + 0.36K_c)\,s + 0.72K_c = 0$$

A Routh-Hurwitz analysis can confirm that. The key point is that with cascade control, the system becomes more stable and allows us to use a larger proportional gain in the primary controller. The main reason is the much faster response (smaller time constant) of the actuator in the inner loop.[2]

[2] If you are skeptical of this statement, try do the Bode plots of the systems with and without cascade control and think about the consequence of changing the break frequency (or bandwidth) of the valve function. If you do not pick up the hint, the answer can be found on our *Web Support* on the details of Example 10.1.

10.2 Feedforward control

To counter probable disturbances, we can take an even more proactive approach than cascade control, and use feedforward control. The idea is that if we can make measurements of disturbance changes, we can use this information and our knowledge of the process model to make proper adjustments in the manipulated variable *before* the disturbance has a chance to affect the controlled variable.

We will continue with the gas furnace to illustrate feedforward control. For simplicity, let's make the assumption that changes in the furnace temperature (T) can be effected by changes in the fuel gas flow rate (F_{fuel}) and the cold process stream flow rate (F_s). Other variables such as the process stream temperature are constant.

In Section 10.1, the fuel gas flow rate is the manipulated variable (M) and cascade control is used to handle its fluctuations. Now, we consider also changes in the cold process stream flow rate as another disturbance (L). Let's presume further that we have derived diligently from heat and mass balances the corresponding transfer functions, G_L and G_p, and we have the process model

$$C = G_L L + G_p M \tag{10-4}$$

where we have used the general notation C as the controlled variable in place of furnace temperature T.

We want the controlled variable to track set point changes (R) precisely, so we substitute the ideal scenario C = R, and rearrange Eq. (10-4) to

$$M = \frac{1}{G_p}\, R - \frac{G_L}{G_p}\, L \tag{10-5}$$

This equation provides us with a model-based rule as to how the manipulated variable should be adjusted when we either change the set point or face with a change in the load variable. Eq. (10-5) is the basis of what we call *dynamic* feedforward control because (10-4) has to be derived from a time-domain differential equation (a transient model). [3]

In Eq. (10-5), $1/G_p$ is the *set point tracking* controller. This is what we need if we install only a feedforward controller, which in reality, we seldom do.[4] Under most circumstances, the change in set point is handled by a feedback control loop, and we only need to implement the second term of (10-5). The transfer function $-G_L/G_p$ is the *feedforward* controller (or the *disturbance rejection* controller). In terms of disturbance rejection, we may also see how the feedforward controller arises if we command C = 0 (*i.e.*, no change), and write (10-4) as

$$0 = G_L L + G_p M$$

To see how we implement a feedforward controller, we now turn to a block diagram (Fig. 10.3). [5] For the moment, we omit the feedback path from our general picture. With the

---

[3] In contrast, we could have done the derivation using steady state models. In such a case, we would arrive at the design equation for a *steady state* feedforward controller. We'll skip this analysis. As will be shown later, we can identify this steady state part from the dynamic approach.

[4] The set point tracking controller not only becomes redundant as soon as we add feedback control, but it also unnecessarily ties the feedforward controller into the closed-loop characteristic equation.

[5] If the transfer functions G_L and G_p are based on a simple process model, we know quite well that they should have the same characteristic polynomial. Thus the term $-G_L/G_p$ is nothing but a ratio of the steady state gains, $-K_L/K_p$.

expectation that we'll introduce a feedback loop, we will not implement the set point tracking term in Eq. (10-5). Implementation of feedforword control requires measurement of the load variable.

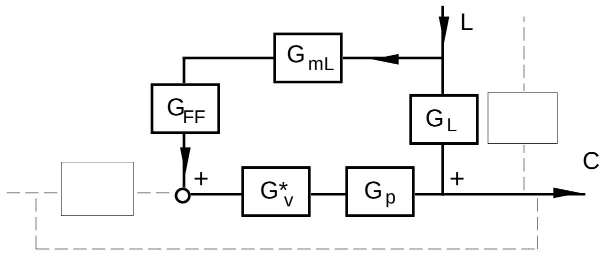


Figure 10.3. A feedforward control system on a major load variable with measurement function G_{ML} and feedforward controller G_{FF}.

If there is more than one load variable, we theoretically could implement a feedforward controller on each one. However, that may not be good engineering. Unless there is a compelling reason, we should select the variable that either undergoes the most severe fluctuation or has the strongest impact on the controlled variable.

Here, we use L to denote the major load variable and its corresponding transfer function is G_L. We measure the load variable with a sensor, G_{mL}, which transmits its signal to the feedforward controller G_{FF}. The feedforward controller then sends its decision to manipulate the actuating element, or valve, G_V. In the block diagram, the actuator transfer function is denoted by G^*_V. The idea is that cascade control may be implemented with the actuator, G_V, as we have derived in Eq. (10-1). We simply use G^*_V to reduce clutter in the diagram.

With the feedforward and load path shown, the corresponding algebraic representation is

$$C = [G_L + G_{mL}G_{FF}G^*_vG_p]\ L \tag{10-6}$$

The ideal feedforward controller should allow us to make proper adjustment in the actuator to achieve perfect rejection of load changes. To have C = 0, the theoretical feedforward controller function is

$$G_{FF} = -\frac{G_L}{G_{mL}G^*_vG_p} \tag{10-7}$$

which is a slightly more complete version of what we have derived in Eq. (10-5).

Before we blindly try to program Eq. (10-7) into a computer, we need to recognize certain pitfalls. If we write out the transfer functions in (10-7), we should find that G_{FF} is not physically realizable—the polynomial in the numerator has a higher order than the one in the denominator.[6]

If we approximate the composite function $G_{mL}G^*_vG_p$ as a first order function with dead time, $Ke^{-\theta s}/(\tau s+1)$, Eq. (10-7) would appear as

[6] If we go by the book, G_L and G_p are the transfer functions to a process and their dynamic terms (characteristic polynomial) in Eq. (10-7) must cancel out. The feedforward transfer function would be reduced to something that looks like $(-K_L/K_{mL}K^*_vK_p)\ (\tau_{mL}s+1)(\tau^*_vs+1)$ while the denominator is just 1.

In the simplest scenario where the responses of the transmitter and the valve are extremely fast such that we can ignore their dynamics, the feedforward function consists of only the steady state gains as in Eq. (10-9).

$$G_{FF} = -\frac{K_L}{K}\frac{\tau s+1}{\tau_p s+1}\; e^{\theta s}$$

Now the dead time appears as a positive exponent or an advance in time. We cannot foresee future and this idea is not probable either.[7]

The consequence is that most simple implementation of a feedforward controller, especially with off-the-shelf hardware, is a lead-lag element with a gain:

$$G_{FF} = K_{FF}\,\frac{\tau_{FLD}s+1}{\tau_{FLG}s+1} \tag{10-8}$$

Based on Eq. (10-7), the gain of this feedforward controller is

$$K_{FF} = -\frac{K_L}{K_{mL}K^*_v K_p} \tag{10-9}$$

This is the *steady state compensator*. The lead-lag element with lead time constant τ_{FLD} and lag time constant τ_{FLG} is the *dynamic compensator*. Any dead time in the transfer functions in (10-7) is omitted in this implementation.

When we tune the feedforward controller, we may take, as a first approximation, τ_{FLD} as the sum of the time constants τ_m and τ^*_v. Analogous to the "real" derivative control function, we can choose the lag time constant to be a tenth smaller, $\tau_{FLG} \approx 0.1\, \tau_{FLD}$. If the dynamics of the measurement device is extremely fast, $G_m = K_{mL}$, and if we have cascade control, the time constant τ^*_v is also small, and we may not need the lead-lag element in the feedforward controller. Just the use of the steady state compensator K_{FF} may suffice. In any event, the feedforward controller must be tuned with computer simulations, and subsequently, field tests.

[7] If the load transfer function in Eq. (10-7) had also been approximated as a first order function with dead time, say, of the form $K_L e^{-t_d s}/(\tau_p s+1)$, the feedforward controller would appear as

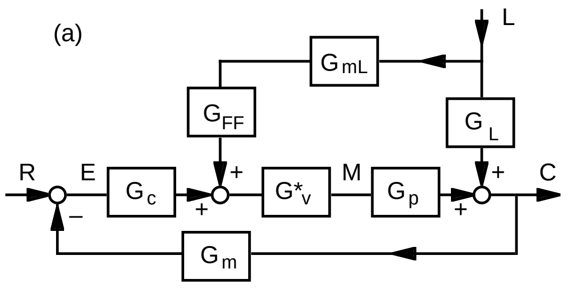
$$G_{FF} = -\frac{K_L}{K}\frac{\tau s+1}{\tau_p s+1}\; e^{-(t_d-\theta)s}.$$

Now, if $t_d > \theta$, it is possible for the feedforward controller to incorporate dead time compensation.

The situation where we may find the load function dead time is larger than that in the feedforward path of $G_m G^*_v G_p$ is not obvious from our simplified block diagram. Such a circumstance arises when we deal with more complex processing equipment consisting of several units (*i.e.*, multicapacity process) and the disturbance enters farther upstream than where the controlled and manipulated variables are located.

10.3 Feedforward-feedback control

Since we do not have the precise model function G_p embedded in the feedforward controller function in Eq. (10-8), we cannot expect perfect rejection of disturbances. In fact, feedforward control is never used by itself; it is implemented in conjunction with a feedback loop to provide the so-called feedback trim (Fig. 10.4a). The feedback loop handles (1) measurement errors, (2) errors in the feedforward function, (3) changes in unmeasured load variables, such as the inlet process stream temperature in the furnace that one single feedforward loop cannot handle, and of course, (4) set point changes.



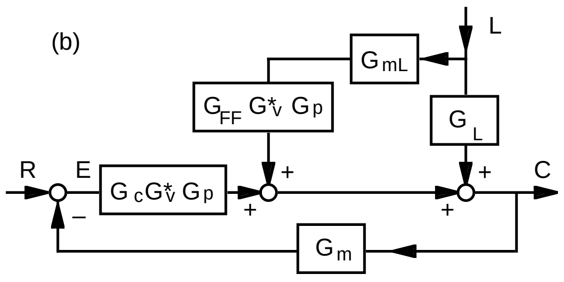


Figure 10.4. (a) A feedforward-feedback control system. (b) The diagram after moving $G^*_vG_p$.

Our next task is to find the closed-loop transfer functions of this feedforward-feedback system. Among other methods, we should see that we can "move" the $G^*_vG_p$ term as shown in Fig. 10.4b. (You can double check with algebra.) After this step, the rest is routine. We can almost write down the final result immediately. Anyway, we should see that

$$C = [G_L + G_{mL}G_{FF}G^*_vG_p]\,L + [G_cG^*_vG_p]\,E$$

and

$$E = R - G_mC$$

After substitution for E and rearrangement, we arrive at

$$C = \frac{G_L + G_{mL}G_{FF}G^*_vG_p}{1 + G_mG_cG^*_vG_p}\,L + \frac{G_cG^*_vG_p}{1 + G_mG_cG^*_vG_p}\,R \qquad (10\text{-}10)$$

If we do not have cascade control, G^*_v is simply G_v. If we are using cascade control, we can substitute for G^*_v with Eq. (10-1), but we'll skip this messy algebraic step. The key point is that the closed-loop characteristic polynomial is

$$1 + G_mG_cG^*_vG_p = 0 \qquad (10\text{-}11)$$

and the feedforward controller G_{FF} does not affect the system stability.

✎ **Example 10.2**: Consider the temperature control of a gas furnace used in heating a process stream. The probable disturbances are in the process stream temperature and flow rate, and the fuel gas flow rate. Draw the schematic diagram of the furnace temperature control system, and show how feedforward, feedback and cascade controls can all be implemented together to handle load changes.

The design in Fig. E10.2 is based on our discussion of cascade control. The fuel gas flow is the manipulated variable, and so we handle disturbance in the fuel gas flow with a flow controller

(FC) in a slave loop. This secondary loop remains the same as the G^*_v function in (10-1), where the secondary transfer function is denoted by G_{C2}.

Of the other two load variables, we choose the process stream flow rate as the major disturbance. The flow transducer sends the signal to the feedforward controller (FFC, transfer function G_{FF}). A summer (Σ) combines the signals from both the feedforward and the feedback controllers, and its output becomes the set point for the secondary fuel gas flow rate controller (FC).

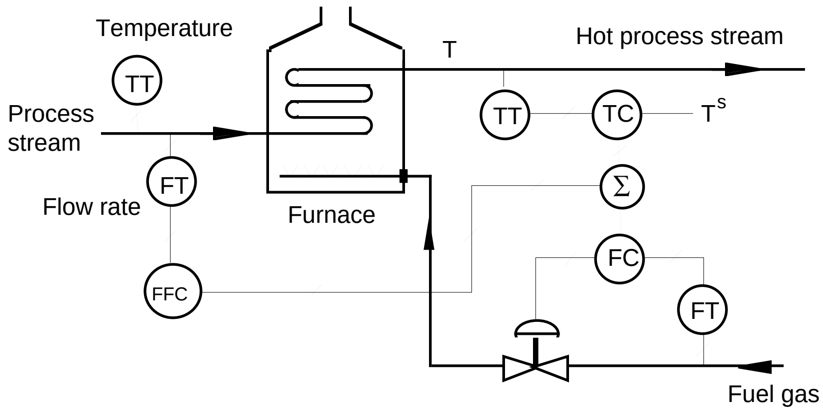


Figure E10.2

Handling of disturbance in the inlet process stream temperature is passive. Any changes in this load variable will affect the furnace temperature. The change in furnace temperature is measured by the outlet temperature transducer (TT) and sent to the feedback temperature controller (TC). The primary controller then acts accordingly to reduce the deviation in the furnace temperature.

10.4 Ratio control

We are not entirely finished with the furnace. There is one more piece missing from the whole picture—the air flow rate. We need to ensure sufficient air flow for efficient combustion. The regulation of air flow is particularly important in the reduction of air pollutant emission.

To regulate the air flow rate with respect to the fuel gas flow rate, we can use ratio control. Fig. 10.5 illustrates one of the simplest implementations of this strategy. Let's say the air to fuel gas flow rates must be kept at some constant ratio

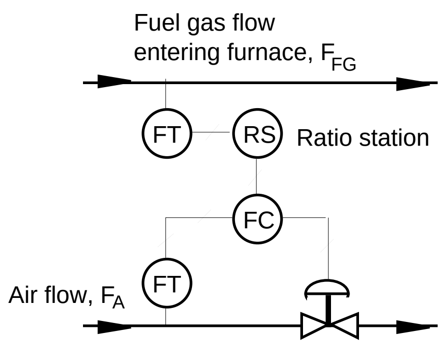


Figure 10.5. Simple ratio control of air flow rate.

$$R = \frac{F_A}{F_{FG}} \tag{10-12}$$

What we can do easily is to measure the fuel gas flow rate, multiply the value by R in the so-called ratio station, and send the signal as the set point to the air flow controller. The calculation can be based on actual flow rates rather than deviation variables.

A more sophisticated implementation is *full metering control* (Fig. 10.6). In this case, we send the signals from the fuel gas controller (FC in the fuel gas loop) and the air flow transmitter (FT) to the ratio controller (RC), which takes the desired flow ratio (R) as the set point. This controller calculates the proper air flow rate, which in turn becomes the set point to the air flow controller (FC in the air flow loop). If we take away the secondary flow control loops on both the fuel gas and air flow rates, what we have is called parallel positioning control. In this simpler case, of course, the performance of the furnace is subject to fluctuations in fuel and air supply lines.

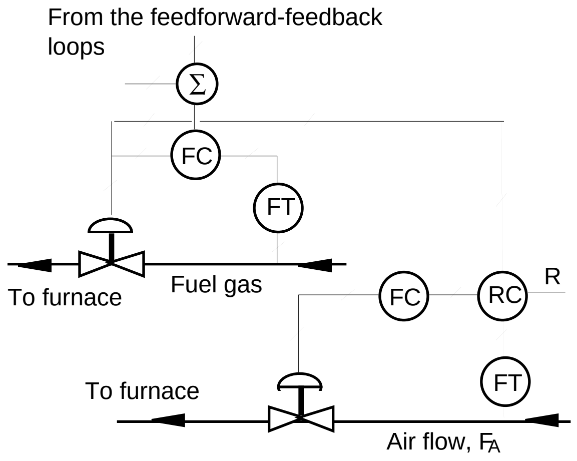


Figure 10.6. Full metering ratio control of fuel and air flows.

We are skipping the equations and details since the air flow regulation should not affect the stability and system analysis of the fuel gas controller, and ratio control is best implemented with Simulink in simulation and design projects.

10.5 Time delay compensation—the Smith predictor

There are different schemes to handle systems with a large dead time. One of them is the Smith predictor. It is not the most effective technique, but it provides a good thought process.

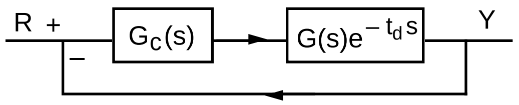


Figure 10.7. System with inherent dead time.

Consider a unit feedback system with a time delay in its process function (Fig. 10.7). The characteristic polynomial is

$$1 + G_c(s)\, G(s)\, e^{-t_d s} = 0 \qquad (10\text{-}13)$$

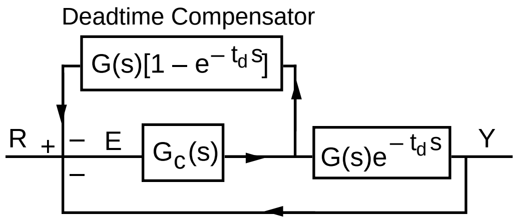


Figure 10.8. Implementation of the Smith predictor.

We know from frequency response analysis that time lag introduces extra phase lag, reduces the gain margin and is a significant source of instability. This is mainly because the feedback information is outdated.

If we have a model for the process, *i.e.*, we know G(s) and t_d, we can predict what may happen and feedback this estimation. The way the dead time compensator (or predictor) is written (Fig. 10.8), we can interpret the transfer function as follows. Assuming that we know the process model, we feedback the "output" calculation based on this model. We also have to subtract out the "actual" calculated time delayed output information.

Now the error E also includes the feedback information from the dead time compensator:

$$E = R - Y - E\left[G_c G\,(1 - e^{-t_d s})\right],$$

and substituting

$$Y = G_c G\, e^{-t_d s}\, E$$

we have

$$E = R - E\left[G_c G\, e^{-t_d s} + G_c G\,(1 - e^{-t_d s})\right]$$

where the exponential terms cancel out and we are left with simply

$$E = R - E\, G_c G \qquad (10\text{-}14)$$

The time delay effect is canceled out, and this equation at the summing point is equivalent to a system without dead time (where the forward path is $C = G_cGE$). With simple block diagram algebra, we can also show that the closed-loop characteristic polynomial with the Smith predictor is simply

$$1 + G_cG = 0 \qquad (10\text{-}15)$$

The time delay is removed. With the delay compensator included, we can now use a larger proportional gain without going unstable. Going back to the fact that the feedback information is G_cGR, we can also interpret the compensator effect as in Fig. 10.9. The Smith predictor is essentially making use of state feedback as opposed to output feedback.

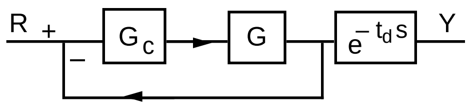


Figure 10.9. An interpretation of the compensator effect.

Just like feedforward control (or any other model-based control), we only have perfect compensation if we know the precise process model. Otherwise, the effectiveness of the compensator (or predictor) is diminished. Assume that we have an imperfect model approximation H(s) and dead time estimation θ ($H \neq G$ and $\theta \neq t_d$), the feedback information is now

$$Y + \left[H\,(1 - e^{-\theta s})\right] G_c R = \left[G_c G\, e^{-t_d s} + G_c H\,(1 - e^{-\theta s})\right] R$$
$$= G_c\left[G\, e^{-t_d s} + H\,(1 - e^{-\theta s})\right] R$$

where the right hand side becomes G_cGR if and only if $H = G$ and $\theta = t_d$. Note that the time delay term is an exponential function. Error in the estimation of the dead time is more detrimental than error in the estimation of the process function *G*.

Since few things are exact in this world, we most likely have errors in the estimation of the process and the dead time. So we only have partial dead time compensation and we must be conservative in picking controller gains based on the characteristic polynomial $1 + G_cG = 0$.

In a chemical plant, time delay is usually a result of transport lag in pipe flow. If the flow rate is fairly constant, the use of the Smith predictor is acceptable. If the flow rate varies for whatever reasons, this compensation method will not be effective.

10.6 Multiple-input Multiple-output Control

In this section, we analyze a multiple input-multiple output (MIMO) system. There are valuable insights that can be gained from using the classical transfer function approach. One decision that we need to appreciate is the proper pairing of manipulated and controlled variables. To do that, we also need to know how strong the interaction is among different variables.

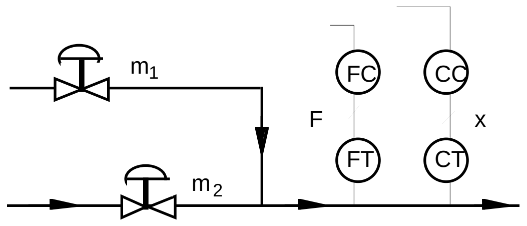
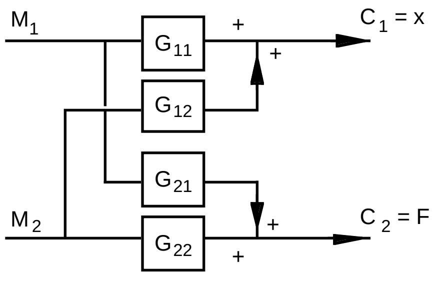


Figure 10.10. A blending system with manipulated and controlled variable pairings yet to be determined.

The key points will be illustrated with a blending process. Here, we mix two streams with mass flow rates m_1 and m_2, and both the total flow rate F and the composition x of a solute *A* are to be controlled (Fig. 10.10). With simple intuition, we know changes in both m_1 and m_2 will affect F and x. We can describe the relations with the block diagram in Fig. 10.11, where interactions are represented by the two, yet to be derived, transfer functions G_{12} and G_{21}.



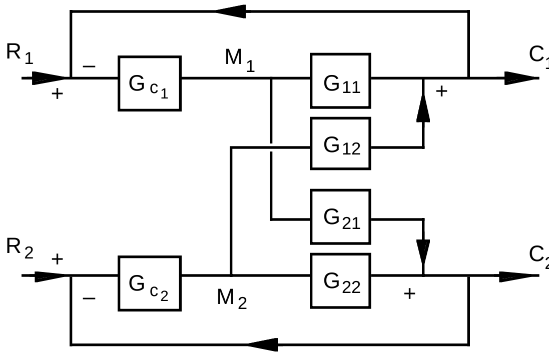


Figure 10.11. Block diagram of an interacting 2 x 2 process, with the output x and F referring to the blending problem.

Figure 10.12. Block diagram of a 2 x 2 servo system. The pairing of the manipulated and controlled variables is not necessarily the same as shown in Fig. 10.11.

Given Fig. 10.11 and classical control theory, we can infer the structure of the control system, which is shown in Fig. 10.12. That is, we use two controllers and two feedback loops, where for simplicity, the measurement and actuator functions have been omitted.

Simple reasoning can illustrate now interactions may arise in Fig. 10.12. If a disturbance (not shown in diagram) moves C_1 away from its reference R_1, the controller G_{c_1} will response to the error and alter M_1 accordingly. The change in M_1, however, will also affect C_2 via G_{21}. Hence C_2 is forced to change and deviate from R_2. Now the second controller G_{c_2} kicks in and adjusts M_2, which in turn also affects C_1 via G_{12}.

With this scenario, the system may eventually settle, but it is just as likely that the system in Fig. 10.12 will spiral out of control. It is clear that loop interactions can destabilize a control system, and tuning controllers in a MIMO system can be difficult. One logical thing that we can do is to reduce loop interactions by proper pairing of manipulated and controlled variables. This is the focus of the analysis in the following sections.

10.6.1 MIMO Transfer functions

We now derive the transfer functions of the MIMO system. This sets the stage for more detailed analysis that follows. The transfer functions in Fig. 10.11 depend on the process that we have to control, and we'll derive them in the next section for the blending process. Here, we consider a general system as shown in Fig.10.12.

With the understanding that the errors are $E_1 = R_1 - C_1$, and $E_2 = R_2 - C_2$ in Fig. 10.12, we can write immediately,

$$M_1 = G_{c_1} (R_1 - C_1) \tag{10-16}$$

$$M_2 = G_{c_2} (R_2 - C_2) \tag{10-17}$$

The process (also in Fig. 10.11) can be written as

$$C_1 = G_{11} M_1 + G_{12} M_2 \tag{10-18}$$

$$C_2 = G_{21} M_1 + G_{22} M_2 \tag{10-19}$$

Substitute for M_1 and M_2 using (10-16, 17), and factor C_1 and C_2 to the left, Eqs. (10-18) and (10-19) becomes

$$(1 + G_{11}G_{c_1}) C_1 + G_{12} G_{c_2} C_2 = G_{11}G_{c_1} R_1 + G_{12} G_{c_2} R_2 \tag{10-20}$$

$$G_{21}G_{c_1} C_1 + (1 + G_{22}G_{c_2}) C_2 = G_{21}G_{c_1} R_1 + G_{22} G_{c_2} R_2 \tag{10-21}$$

Making use of Kramer's rule, we should identify (derive!) the *system characteristic equation*:

$$p(s) = (1 + G_{11}G_{c_1}) (1 + G_{22}G_{c_2}) + G_{12} G_{21} G_{c_1}G_{c_2} = 0 \tag{10-22}$$

which, of course, is what governs the dynamics and stability of the system. We may recognize that when either $G_{12} = 0$ or $G_{21} = 0$, the interaction term is zero.[8] In either case, the system characteristics analysis can be reduced to those of two single loop systems:

$$1 + G_{11}G_{c_1} = 0 , \quad \text{and} \quad 1 + G_{22}G_{c_2} = 0$$

Now back to finding the transfer functions with interaction. To make the algebra appear a bit cleaner, we consider the following two cases. When $R_2 = 0$, we can derive from Eq. (10-20) and (10-21),

$$\frac{C_1}{R_1} = \frac{G_{11}G_{c_1} + G_{c_1}G_{c_2}(G_{11}G_{22} - G_{12}G_{21})}{p(s)} \tag{10-23}$$

And when $R_1 = 0$, we can find

$$\frac{C_1}{R_2} = \frac{G_{12}G_{c_2}}{p(s)} \tag{10-24}$$

[8] When both $G_{12} = G_{21} = 0$, the system is decoupled and behaves identically to two single loops. When either $G_{12} = 0$ or $G_{21} = 0$, the situation is referred to as one-way interaction, which is sufficient to eliminate recursive interactions between the two loops. In such a case, one of the loops is not affected by the second while it becomes a source of disturbance to this second loop.

If both references change simultaneously, we just need to add their effects in (10-23) and (10-24) together. (What about C_2? You'll get to try that in the Review Problems.)

It is apparent from Eq. (10-22) that with interaction, the controller design of the MIMO system is different from a SISO system. One logical question is under what circumstances may we make use of SISO designs as an approximation? Or in other words, can we tell if the interaction may be weak? This takes us to the next two sections.

10.6.2 Process gain matrix

We come back to derive the process transfer functions for the blending problem.[9] The total mass flow balance is

$$F = m_1 + m_2 \tag{10-25}$$

where F is the total flow rate after blending, and m_1 and m_2 are the two inlet flows that we manipulate. The mass balance for a solute *A* (without using the subscript *A* explicitly) is

$$xF = x_1 m_1 + x_2 m_2 \tag{10-26}$$

where *x* is the mass fraction of *A* after blending, and x_1 and x_2 are the mass fractions of *A* in the two inlet streams. We want to find the transfer functions as shown in Fig. 10.11:

$$\begin{bmatrix} X(s) \\ F(s) \end{bmatrix} = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix} \begin{bmatrix} M_1(s) \\ M_2(s) \end{bmatrix} \tag{10-27}$$

We take stream m_1 to be pure solute *A*, and stream m_2 to be pure solvent. In this scenario, $x_1 = 1$ and $x_2 = 0$, and Eq. (10-26) is simplified to

$$x = \frac{m_1}{F} = \frac{m_1}{m_1 + m_2} \tag{10-28}$$

Since x_i and m_i are functions of time, we need to linearize (10-26). A first order Taylor expansion of x is

$$x \approx \left(\frac{m_1}{F}\right)_s + \left[\frac{m_2}{(m_1+m_2)^2}\right]_s (m_1 - m_{1,s}) - \left[\frac{m_1}{(m_1+m_2)^2}\right]_s (m_2 - m_{2,s})$$

where the subscript *s* of the brackets denotes terms evaluated at steady state. The first term on the right is really the value of x at steady state, x_s, which can be moved to the left hand side to make the deviation variable in x. With that, we take the Laplace transform to obtain the transfer functions of the deviation variables:

$$X(s) = G_{11}(s)M_1(s) + G_{12}(s)M_2(s) \tag{10-29}$$

where

$$G_{11}(s) = \left[\frac{m_2}{(m_1+m_2)^2}\right]_s = K_{11} \;, \quad \text{and} \quad G_{12}(s) = -\left[\frac{m_1}{(m_1+m_2)^2}\right]_s = K_{12} \tag{10-30}$$

[9] Since the point is to illustrate the analysis of interactions, we are using only steady state balances and it should not be a surprise that the transfer functions end up being only steady state gains in Eq. (10-32). For a general dynamic problem where have to work with the transfer functions $G_{ij}(s)$, we can still apply the results here by making use of the steady state gains of the transfer functions.

The transfer functions are constants and hence we denote them with the gains K_{11} and K_{12}. If the solvent flow rate m_2 increases, the solute will be diluted. Hence, K_{12} is negative.

The functions G_{21} and G_{22} are much easier. From Eq. (10-25), we can see immediately that

$$G_{21}(s) = K_{21} = 1, \quad \text{and} \quad G_{22}(s) = K_{22} = 1 \tag{10-31}$$

Thus, in this problem, the process transfer function matrix Eq. (10-27) can be written in terms of the steady state gain matrix:

$$\begin{bmatrix} X(s) \\ F(s) \end{bmatrix} = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{bmatrix} M_1(s) \\ M_2(s) \end{bmatrix} \tag{10-32}$$

In more general terms, we replace the LHS of (10-32) with a controlled variable vector:

$$\mathbf{C}(s) = \mathbf{K}\,\mathbf{M}(s) \tag{10-33}$$

where $\mathbf{C} = [\mathbf{X}\ \mathbf{F}]^T$. If there is a policy such that the manipulated variables can regulate the controlled variables, we must be able to find an inverse of the gain matrix such that

$$\mathbf{M}(s) = \mathbf{K}^{-1}\,\mathbf{C}(s) \tag{10-34}$$

Example 10.3. If m_1 = 0.1 g/s, m_2 = 10 g/s, What is the process gain matrix? What are the interpretations?

Making use of (10-30), we can calculate $K_{11} = 9.8 \text{ x } 10^{-2}$, and $K_{12} = -9.8 \text{ x } 10^{-2}$. With (10-31), the process gain matrix is

$$\mathbf{K} = \begin{bmatrix} 9.8 \times 10^{-2} & -9.8 \times 10^{-4} \\ 1 & 1 \end{bmatrix}$$

Under circumstances of the particular set of numbers given, changing either m_1 or m_2 has a stronger effect on the total flow rate F than x. With respect to the composition x, changing the solute flow m_1 has a much stronger effect than changing the solvent flow. The situation resembles very much a one-way interaction.

We may question other obvious scenarios of the process gain matrix. The sweetest is an identity matrix, meaning no interaction among the manipulated and controlled variables. A quick summary of several simple possibilities:[10]

$\mathbf{K} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	No interaction. Controller design is like single-loop systems.
$\mathbf{K} = \begin{bmatrix} 1 & \delta \\ \delta & 1 \end{bmatrix}$	Strong interaction if δ is close to 1; weak interaction if δ « 1.
$\mathbf{K} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$	One-way interaction

[10] There is more to “looking” at **K**. We can, for example, make use of its singular value and condition number, which should be deferred to a second course in control.

10.6.3 Relative gain array

You may not find observing the process gain matrix satisfactory. That takes us to the *relative gain array* (RGA), which can provide for a more quantitative assessment of the effect of changing a manipulated variable on different controlled variables. We start with the blending problem before coming back to the general definition.

For the blending process, the relative gain parameter of the effect of m_1 on x is defined as

$$\lambda_{x,\, m_1} = \frac{\partial x / \partial m_1 \big|_{m_2}}{\partial x / \partial m_1 \big|_F} \tag{10-35}$$

It is the ratio of the partial derivative evaluated under two different circumstances. On top, we look at the effect of m_1 while holding m_2 constant. The calculation represents an open-loop experiment and the value is referred to as an open-loop gain. In the denominator, the total flow rate, the other controlled variable, is held constant. Since we are varying (in theory) m_1, F can only be constant if we have a closed-loop with perfect control involving m_2. The partial derivative in the denominator is referred to as some closed-loop gain.

How do we interpret the relative gain? The idea is that if m_2 does not interfere with m_1, the derivative in the denominator should not be affected by the closed-loop involving m_2, and its value should be the same as the open-loop value in the numerator. In other words, if there is no interaction, $\lambda_{x,m_1} = 1$.

Example 10.4. Evaluate the relative gain array matrix for the blending problem.

The complete relative gain array matrix for the 2 x 2 blending problem is defined as

$$\Lambda = \begin{bmatrix} \lambda_{x,\, m_1} & \lambda_{x,\, m_2} \\ \lambda_{F,\, m_1} & \lambda_{F,\, m_2} \end{bmatrix} \tag{E10-4}$$

For the first element, we use (10-28) to find

$$\frac{\partial x}{\partial m_1}\bigg|_{m_2} = \frac{m_2}{(m_1 + m_2)^2}\,, \quad \text{and} \quad \frac{\partial x}{\partial m_1}\bigg|_F = \frac{1}{F} = \frac{1}{m_1 + m_2}$$

Hence, with the definition in (10-35),

$$\lambda_{x,\, m_1} = \frac{m_2}{m_1 + m_2} = 1 - x \tag{E10-5}$$

Proceed to find the other three elements (see Review Problems) and we have the RGA for the blending problem:

$$\Lambda = \begin{bmatrix} 1 - x & x \\ x & 1 - x \end{bmatrix} \tag{E10-6}$$

There are several notable and general points regarding this problem, *i.e.*, without proving them formally here. The sum of all the entries in each row and each column of the relative gain array Λ is 1. Thus in the case of a 2 x 2 problem, all we need is to evaluate one element. Furthermore, the calculation is based on only open-loop information. In Example 10.4, the derivation is based on (10-25) and (10-26).

We can now state the general definition of the **relative gain array**, Λ. For the element relating the *i*-th controlled variable to the *j*-th manipulated variable,

$$\lambda_{i,\,j} = \frac{\partial c_i/\partial m_j\big|_{m_{k,\,k \neq j}}}{\partial c_i/\partial m_j\big|_{c_{k,\,k \neq i}}} \tag{10-36}$$

where the (open-loop) gain in the numerator is evaluated with all other manipulated variables held constant and all the loops open (no loops!). The (closed-loop) gain in the denominator is evaluated with all the loops—other than the *i*-th loop—closed. The value of this so-called closed-loop gain reflects the effect from other closed-loops *and* the open-loop between m_j and c_i.

The relative gain array can be derived in terms of the process steady state gains. Making use of the gain matrix equation (10-32), we can find (not that hard; see Review Problems)

$$\lambda_{x,\,m_1} = \frac{1}{1 - \dfrac{K_{12}K_{21}}{K_{11}K_{22}}} \tag{10-37}$$

which can be considered a more general form of (E10-5) and hence (E10-6).[11]

The next question comes back to the meaning of the RGA, and how that may influence our decision in pairing manipulated with controlled variables. Here is the simple interpretation making use of (10-36) and (10-37):

$\lambda_{i,j} = 1$	Requires $K_{12}K_{21} = 0$. "Open-loop" gain is the same as the "closed-loop" gain. The controlled variable (or loop) *i* is not subject to interaction from other manipulated variables (or other loops). Of course, we know nothing about whether other manipulated variables may interact and affect other controlled variables. Nevertheless, pairing the *i*-th controlled variable to the *j*-th manipulated variable is desirable.
$\lambda_{i,j} = 0$	The open-loop gain is zero. The manipulated variable *j* has no effect on the controlled variable *i*. Of course m_j may still influence other controlled variables (via one-way interaction). Either way, it makes no sense to pair m_j with c_i in a control loop.
$0 < \lambda_{i,j} < 1$	No doubt there are interactions from other loops, and from (10-37), some of the process gains must have opposite signs (or act in different directions). When $\lambda_{i,j} = 0.5$, we can interpret that the effect of the interactions is identical to the open-loop gain—recall statement after (10-36). When $\lambda_{i,j} > 0.5$, the interaction is less than the main effect of m_j on c_i. However, when $\lambda_{i,j} < 0.5$, the interactive effects predominate and we want to avoid pairing m_j with c_i.
$\lambda_{i,j} > 1$	There are interactions from other loops as well, but now with all the process gains having the same sign. Avoid pairing m_j with c_i if $\lambda_{i,j}$ is much larger than 1.
$\lambda_{i,j} < 0$	We can infer using (10-36) that the open-loop and closed-loop gains have different signs or opposing effects. The overall influence of the

[11] For your information, relative gain array can be computed as the so-called Hadamard product, $\lambda_{ij} = K_{ij}K^{-1}_{ji}$, which is the element-by-element product of the gain matrix **K** and the transpose of its inverse. You can confirm this by repeating the examples with MATLAB calculations.

other loops is in opposition to the main effect of m_j on c_i. Moreover, from (10-37), the interactive product $K_{12}K_{21}$ must be larger than the direct terms $K_{11}K_{22}$. Undesirable interaction is strong. The overall multiloop system may become unstable easily if we open up one of its loops. We definitely should avoid pairing m_j with c_i.

To sum up, the key is to pair the manipulated and controlled variables such that the relative gain parameter is positive and as close to one as possible.

Example 10.5. If $m_1 = 0.1$ g/s, $m_2 = 10$ g/s, what is the proper pairing of manipulated and controlled variables? What if $m_1 = 9$ g/s, $m_2 = 1$ g/s?

In the first case where m_1 is very small, it is like a dosing problem. From (10-28), x = 0.0099. Since $x \ll 1$, λ_{x,m_1} is very close to 1 by (E10-5). Thus interaction is not significant if we pair x with m_1, and F with m_2. Physically, we essentially manipulate the total flow with the large solvent flow m_2 and tune the solute concentration by manipulating m_1.

In the second case, x = 0.9. Now $\lambda_{x,m_1} = 0.1$ by (E10-5). Since $\lambda_{x,m_1} \ll 1$, we do not want to pair x with m_1. Instead, we pair pair x with m_2, and F with m_1. Now we regulate the total flow with the larger solute flow m_1 and tune the concentration with the solvent m_2.

10.7 Decoupling of interacting systems

After proper pairing of manipulated and controlled variables, we still have to design and tune the controllers. The simplest approach is to tune each loop individually and conservatively while the other loop is in manual mode. At a more sophisticated level, we may try to decouple the loops mathematically into two non-interacting SISO systems with which we can apply single loop tuning procedures. Several examples applicable to a 2 x 2 system are offered here.

10.7.1 Alternate definition of manipulated variables

We seek choices of manipulated variables that may decouple the system. A simple possibility is to pick them to be the same as the controlled variables. In the blending problem, the two new manipulated variables can be defined as [12]

$$\mu_1 = F \qquad (10\text{-}38)$$

and

$$\mu_2 = x \qquad (10\text{-}39)$$

Once the controller (a computer) evaluates these two manipulated variables, it also computes on the fly the actual signals necessary for the two mass flow rates m_1 and m_2. The computation

[12] The blending problem can be reduced to one-way interaction if we use m_1 instead of x as the new manipulated variable μ_2. We'll do that in the Review Problems.

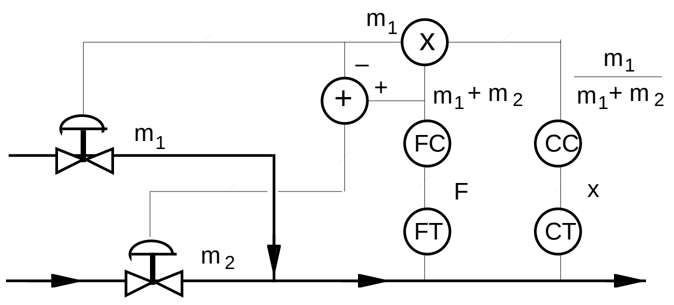


Figure 10.13. A decoupled control scheme. The controller outputs are the manipulated variables in Eqs. (10-38) and (10-39) and they are rewritten based on their definitions in (10-25) and (10-28).

follows directly the balance equations (10-25) and (10-28). Fig. 10.13 is a schematic diagram on how this idea may be implemented.

10.7.2 Decoupler functions

In this section, we add the so-called decoupler functions to a 2 x 2 system. Our starting point is Fig. 10.12. The closed-loop system equations can be written in matrix form, virtually by visual observation of the block diagram, as

$$\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} G_{c_1} & 0 \\ 0 & G_{c_2} \end{bmatrix} \begin{bmatrix} R_1 - C_1 \\ R_2 - C_2 \end{bmatrix} \tag{10-40}$$

In matrix form, this equation looks deceptively simple, but if we expand the algebra, we should arrive at Eqs. (10-20) and (10-21) again.

In a system with interactions, G_{12} and G_{21} are not zero, but we can manipulate the controller signal such that the system appears (mathematically) to be decoupled. So let's try to transform the controller output with a matrix **D**, which will contain our decoupling functions. The manipulated variables are now

$$\begin{bmatrix} M_1 \\ M_2 \end{bmatrix} = \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix} \begin{bmatrix} G_{c_1} & 0 \\ 0 & G_{c_2} \end{bmatrix} \begin{bmatrix} R_1 - C_1 \\ R_2 - C_2 \end{bmatrix}$$

and the system equations become

$$\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix} \begin{bmatrix} G_{c_1} & 0 \\ 0 & G_{c_2} \end{bmatrix} \begin{bmatrix} R_1 - C_1 \\ R_2 - C_2 \end{bmatrix} = \mathbf{GDG}_c \begin{bmatrix} R_1 - C_1 \\ R_2 - C_2 \end{bmatrix} \tag{10-41}$$

To decouple the system equations, we require that $\mathbf{GDG}_C$ be a diagonal matrix. Define $\mathbf{G_o} = \mathbf{GDG}_C$, and the previous step can be solved for **C**:

$$\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = [\mathbf{I} + \mathbf{G_o}]^{-1} \mathbf{G_o} \begin{bmatrix} R_1 \\ R_2 \end{bmatrix} \tag{10-42}$$

Since $\mathbf{G_o}$ is diagonal, the matrix $[I + \mathbf{G_o}]^{-1}\mathbf{G_o}$ is also diagonal, and happily, we have two decoupled equations in (10-42).

Now we have to find **D**. Since $\mathbf{G_C}$ is already diagonal, we require that **GD** be diagonal:

$$\begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix} = \begin{bmatrix} H_1 & 0 \\ 0 & H_2 \end{bmatrix} \tag{10-43}$$

A little bit of algebra to match term by term, we should find (see Review Problems)

$$d_{11} = \frac{G_{22} H_1}{G_{11} G_{22} - G_{12} G_{21}} , \quad d_{22} = \frac{G_{11} H_2}{G_{11} G_{22} - G_{12} G_{21}} \tag{10-44}$$

$$d_{21} = \frac{-G_{21}}{G_{22}} d_{11} , \quad d_{12} = \frac{-G_{12}}{G_{11}} d_{22} \tag{10-45}$$

We have six unknowns (four d_{ij} and two H_i) but only four equations. We have to make two (arbitrary) decisions. One possibility is to choose (or define)

$$H_1 = \frac{G_{11} G_{22} - G_{12} G_{21}}{G_{22}}, \text{ and } \quad H_2 = \frac{G_{11} G_{22} - G_{12} G_{21}}{G_{11}} \tag{10-46}$$

such that d_{11} and d_{22} become 1. (We can also think in terms of choosing both $d_{11} = d_{22} = 1$ and then derive the relations for H_1 and H_2.) It follows that

$$d_{21} = \frac{-G_{21}}{G_{22}}, \quad \text{and} \quad d_{12} = \frac{-G_{12}}{G_{11}} \tag{10-47}$$

Now the closed-loop equations are

$$\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} H_1 & 0 \\ 0 & H_2 \end{bmatrix} \begin{bmatrix} G_{c1} & 0 \\ 0 & G_{c2} \end{bmatrix} \begin{bmatrix} R_1 - C_1 \\ R_2 - C_2 \end{bmatrix} = \begin{bmatrix} H_1 G_{c_1} & 0 \\ 0 & H_2 G_{c_2} \end{bmatrix} \begin{bmatrix} R_1 - C_1 \\ R_2 - C_2 \end{bmatrix} \tag{10-48}$$

from which we can easily write, for each row of the matrices,

$$\frac{C_1}{R_1} = \frac{G_{c_1} H_1}{1 + G_{c_1} H_1}, \quad \text{and} \quad \frac{C_2}{R_2} = \frac{G_{c_2} H_2}{1 + G_{c_2} H_2} \tag{10-49}$$

and the design can be based on the two characteristic equations

$$1 + G_{c1} H_1 = 0 , \text{ and } \quad 1 + G_{c2} H_2 = 0 \tag{10-50}$$

Recall Eq. (10-46) that H_1 and H_2 are defined entirely by the four plant functions G_{ij}. This is another example of model-based control. With the definitions of H_1 and H_2 given in (10-46), the calculations are best performed with a computer.

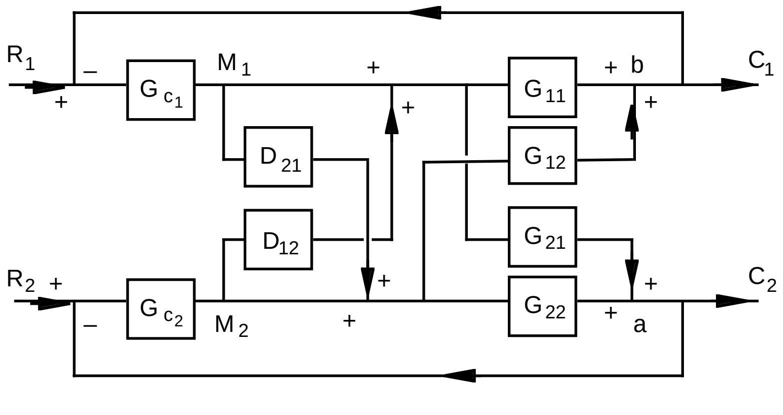


Figure 10.14. A decoupling scheme using two feedforward-like decoupler functions.

10.7.3 Feedforward decoupling functions

A simpler approach is to use only two decoupler functions and implement them as if they were feedforward controllers that may reduce the disturbance arising from loop interaction. As implemented in Fig. 10.14, we use the function D_{12} to “foresee” and reduce the interaction due to G_{12}. Likewise, D_{21} is used to address the interaction due to G_{21}. To find these two decoupling functions, we focus on how to cancel the interaction at the points identified as “a” and “b” in Fig. 10.14.

Let’s pick the point “a” first. If the signal from M_1 through G_{21} can be cancelled by the compensation through D_{21}, we can write

$$G_{21}M_1 + G_{22}D_{21}M_1 = 0$$

Cancel out M_1 and we have

$$D_{21} = -\,G_{21}/G_{22} \tag{10-51}$$

Similarly, if D_{12} can cancel the effect of G_{12} at the point “b,” we have

$$G_{12}M_2 + G_{11}D_{12}M_2 = 0$$

or

$$D_{12} = -\,G_{12}/G_{11} \tag{10-52}$$

We may notice that Eqs. (10-51) and (10-52) are the same as d_{21} and d_{12} in (10-47). The strategy of implementing D_{12} and D_{21} is similar to the discussion of feedforward controllers in Section 10.2, and typically we remove the time delay terms and apply a lead-lag compensator as in Eq. (10-8). If the time constant of the first-order lead is similar to time constant of the first-order lag, then we just need a steady state compensator.

Example 10.6: A classic example of an MIMO problem is a distillation column.[13] From open-loop step tests, the following transfer functions are obtained:

$$\begin{bmatrix} X_D(s) \\ X_B(s) \end{bmatrix} = \begin{bmatrix} \dfrac{0.07\,e^{-3s}}{12s+1} & \dfrac{-0.05\,e^{-s}}{15s+1} \\ \dfrac{0.1\,e^{-4s}}{11s+1} & \dfrac{-0.15\,e^{-2s}}{10s+1} \end{bmatrix} \begin{bmatrix} L(s) \\ V(s) \end{bmatrix}$$

In this model, x_D and x_B are the distillate and bottom compositions, L is the reflux flow rate, and V is the boil-up rate. Design a 2x2 MIMO system with PI controllers and decouplers as in Fig. 10.14.

Before we design the MIMO system, we need to check the paring of variables. The steady state gain matrix is

[13] Pardon us if you have not taken a course in separation processes yet, but you do not need to know what a distillation column is to read the example. In a simple-minded way, we can think of making moonshine. We have to boil a dilute alcohol solution at the bottom and we need a condenser at the top to catch the distillate. This is how we have the V and L manipulated variables. Furthermore, the transfer functions are what we obtain from doing an experiment, not from any theoretical derivation.

$$\mathbf{K} = \begin{bmatrix} 0.07 & -0.05 \\ 0.1 & -0.15 \end{bmatrix}$$

With Eq. (10-37) and (E10-6), the relative gain array is

$$\mathbf{\Lambda} = \begin{bmatrix} 1.91 & -0.91 \\ -0.91 & 1.91 \end{bmatrix}$$

The relative gain parameter $\lambda_{x_D\text{-}L}$ is 1.91. It is not 1 but at least it is not negative. Physically, it also makes sense to manipulate the distillate composition with the more neighboring reflux flow. So we will pair x_D-L and x_B-V. Next, with (10-51) and (10-52), the two decoupling functions are

$$D_{12} = K_{d,\,12}\frac{12\,s + 1}{15\,s + 1}, \quad \text{and} \quad D_{21} = K_{d,\,21}\frac{10\,s + 1}{11\,s + 1} \approx K_{d,\,21}$$

To do the tuning, we can use the initial values $K_{d,12} \approx -0.7$ (–0.05/0.07), and $K_{d,21} \approx -0.7$ (–0.1/0.15).

We will have to skip the details for the remainder of the exercise. You may try to generate a plot similar to Fig. E10.6 in the Review Problems.

This is roughly how we did it. All the simulations are performed with Simulink. First, we use G_{11} and G_{22} as the first order with dead time functions and apply them to the ITAE tuning relations in Table 6.1. With that, we have the PI controller settings of two SISO systems. The single loop response to a unit step change in the set point of x_D is labeled SISO in Fig. E10.6. We retain the ITAE controller settings and apply them to a Simulink block diagram constructed as in Fig. 10.12. The result is labeled MIMO in the figure. Finally, we use Fig. 10.14 and the two decouplers, and the simulation result with the initial setting is labeled "MIMO with decouplers."

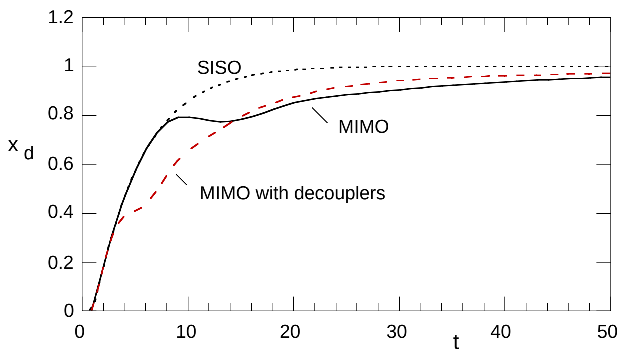


Figure E10.6

In this illustration, we do not have to detune the SISO controller settings. The interaction does not appear to be severely detrimental mainly because we have used the conservative ITAE settings. It would not be the case if we had tried Cohen-Coon relations. The decouplers also do not appear to be particularly effective. They reduce the oscillation, but also slow down the system response. The main reason is that the lead-lag compensators do not factor in the dead times in all the transfer functions.

❒ Review Problems

1. Derive (10-3) and (10-3a) with measurement transfer functions G_{m_1} and G_{m_2} in the primary and secondary loops. Confirm the footnote to (10-3a) that this equation can be reduced to that of a single loop system.

2. Do the root locus plots in Example 10-1(d). Confirm the stability analysis in Example 10-1(e).

3. Draw the block diagram of the system in Example 10-2. Label the diagram with proper variables.

4. Attempt a numerical simulation of a feedforward-feedback system in Fig R10.4. Consider the simplified block diagram with

   $G_v = \frac{0.5}{s+1}$, $G_p = \frac{0.8}{2s+1}$, and $G_L = \frac{-0.4}{2s+1}$.

   (a) The load function has a negative gain. What does it mean?

   (b) Omit for the moment the feedback loop and controller G_c, and consider only G_{FF} as defined in (10-8). Use MATLAB functions (or simulink) to simulate the response in C when we impose a unit step change to L. Experiment with different values of the gain and time constants in the lead-lag element.

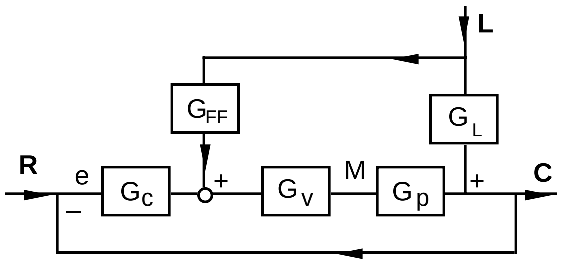


Figure R10.4

   (c) Consider a PI controller for the feedback loop with an integral time of 1.5 s, find the proportional gain such that the system has an underdamped behavior equivalent to a damping ratio of 0.7.

   (d) With the feedback loop and PI controller in part (c), use MATLAB to simulate the response of C to a unit step change in L. Repeat the different values of the feedforward controller as in part (b).

5. Consider the simpler problem in Fig. R10.5 based on Fig. 10.12. If we only implement one feedback loop and one controller, how is the transfer function C_1/M_1 affected by the interaction?

6. Derive the transfer functions C_2/R_1 and C_2/R_2 from Eqs. (10-20) and (10-21).

7. Fill in the details and derive the RGA (E10-6) in Example 10.4.

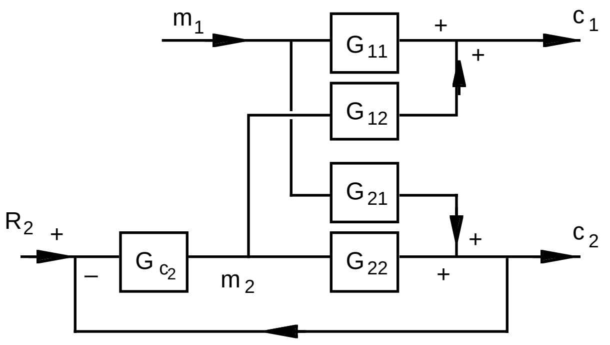


Figure R10.5

8. Derive Eq. (10-37).

9. Show that we also can obtain (E10-6) by applying (10-37) to the blending problem.

(f) Repeat Section 10.7.1 by replacing the second manipulated variable in (10-39) with

$$\mu_2 = m_1$$

Find the gain matrix and show that the relative gain parameter is 1. Show how this partially decoupling scheme can be implemented as analogous to Fig. 10.13.

11. Derive Eqs. (10-44) and (10-45).

12. Try to do the Simulink simulations in Example 10.6. If you need help the Simulink file is on our *Web Support*.

Hint:

2. The MATLAB statements can be:

```
Part (d)
Gp=tf(0.8,[2 1]);
Gv=tf(0.9,[0.1 1]); %With cascade control
taui=0.05;          %Just one example
Gi=tf([taui 1],[taui 0])
rlocus(Gi*Gv*Gp)

Part (e)
Gvo=tf(0.5,[1 1]);
rlocus(Gi*Gvo*Gp)
```

4. (a) If L is the inlet process stream flow rate, how would it affect the furnace temperature?
(b) Use (10-9), and the comments that follow, to select the parameters for the feedforward controller. Compare with the case when we do not have a feedforward controller by setting $K_{FF} = 0$. You should observe that the major compensation to the load change is contributed by the steady-state compensator.
(c) The proportional gain is about 1.4. The feedforward controller does not affect the system stability and we can design the controller G_c with only G_v, G_p, and the feedback loop. We have to use, for example, the root locus method in Chapter 6 to do this part. Root locus can also help us to determine if $\tau_I = 1.5$ s is a good choice.
(d) You should find that the feedback loop takes over much of the burden in load changes. The system response is rather robust even with relatively large errors in the steady-state compensator.

5.
$$C_2 = G_{22}G_{c_2}(R_2 - C_2) + G_{21}M_1$$

$$C_1 = G_{11}M_1 + G_{12}G_{c_2}(R_2 - C_2)$$

Setting $R_2 = 0$,

$$C_2 = \frac{G_{21}}{1 + G_{c_2}G_{22}}\, M_1$$

Substitute C_2 into the C_1 equation, we can find after two algebraic steps,

$$C_1 = \left[G_{11} - \frac{G_{12}G_{21}G_{c_2}}{1 + G_{c_2}G_{22}} \right] M_1$$

The second term in the bracket is due to interaction.

6. We apply Kramer's rule to find C_2 just as we had with C_1. The solution has the same characteristic polynomial in (10-22). The transfer functions:

With $R_1 = 0$,

$$\frac{C_2}{R_2} = \frac{G_{22}G_{c_2} + G_{c_1}G_{c_2}(G_{11}G_{22} - G_{12}G_{21})}{p(s)}$$

With $R_2 = 0$,

$$\frac{C_2}{R_1} = \frac{G_{21}G_{c_1}}{p(s)}$$

7. We still use (10-28) as in Example 10.4. To find λ_{x,m_2}:

$$\left.\frac{\partial x}{\partial m_2}\right|_{m_1} = \frac{-m_1}{(m_1 + m_2)^2} \qquad \left.\frac{\partial x}{\partial m_2}\right|_{F} = \frac{\partial}{\partial m_2}\left(\frac{F - m_2}{F}\right) = -\frac{1}{F}$$

and

$$\lambda_{x,\,m_2} = \frac{m_1}{m_1 + m_2} = x$$

To find λ_{F,m_1}:

$$\left.\frac{\partial F}{\partial m_1}\right|_{m_2} = 1 \text{ , using Eq. (10-25)} \qquad \left.\frac{\partial F}{\partial m_1}\right|_{x} = \frac{1}{x} \text{ , using } F = m_1/x$$

$$\lambda_{F,m_1} = x$$

To find λ_{F,m_2}:

$$\left.\frac{\partial F}{\partial m_2}\right|_{m_1} = 1 \qquad \left.\frac{\partial F}{\partial m_2}\right|_{x} = \frac{1}{1 - x}, \text{ using } xF = F - m_2,\ F = m_2/(1 - x)$$

$$\lambda_{F,m_2} = 1 - x$$

8. We may just as well use Eq. (10-32) in its time-domain form

$$\begin{bmatrix} x \\ F \end{bmatrix} = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}\begin{bmatrix} m_1 \\ m_2 \end{bmatrix}$$

where now x, F, m_1, and m_2 are deviation variables. From the first row, it is immediately obvious that

$$\left.\frac{\partial x}{\partial m_1}\right|_{m_2} = K_{11}$$

We next substitute for m_2 using the second row to get

$$x = K_{11}m_1 + K_{12}\frac{(F - K_{21}m_1)}{K_{22}}$$

Now we can find

$$\left.\frac{\partial x}{\partial m_1}\right|_{F} = K_{11} - \frac{K_{12}K_{21}}{K_{22}}$$

From here on, getting (10-37) is a simple substitution step.

9. To derive E10-6 using K. This is just a matter of substituting the values of the K_{ij}'s from (10-30) and (10-31) into (10-37). We should find once again $\lambda_{x,m_1} = 1 - x$ as in (E10-5), and (E10-6) follows.

10. We need to find how μ_1 and μ_2 affect F and x. With $\mu_1 = F$ and $\mu_2 = m_1$, we can rewrite the definition of $x = m_1/F$ as $x = \mu_1/\mu_2$. This is the form that we use to take a first order Taylor

expansion as we have done with the step after Eq. (10-28). The result in matrix form of the Laplace transform of the deviation variables is

$$\begin{bmatrix} F(s) \\ x(s) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\left.\dfrac{\mu_2}{\mu_1^2}\right|_{s.s.} & \left.\dfrac{1}{\mu_1}\right|_{s.s.} \end{bmatrix} \begin{bmatrix} \mu_1(s) \\ \mu_2(s) \end{bmatrix}$$

By putting F in the first row, it is clear that we have a one-way interaction system. By (10-37), $\lambda = 1$. And with $F = m_1 + m_2$ and m_1 as the output of the controllers, we can implement this scheme as in Fig. R10.10.

11. We'll find d_{11} and d_{21} as an illustration. The first column of the RHS of (10-43) is rewritten as the two equations:

$$G_{11}\, d_{11} + G_{12}\, d_{21} = H_1$$

$$G_{21}\, d_{11} + G_{22}\, d_{21} = 0$$

Solving them simultaneously will lead to d_{11} and d_{21} in (10-44) and (10-45). And choosing $d_{11} = 1$, (10-44) can be rewritten as (10-46).

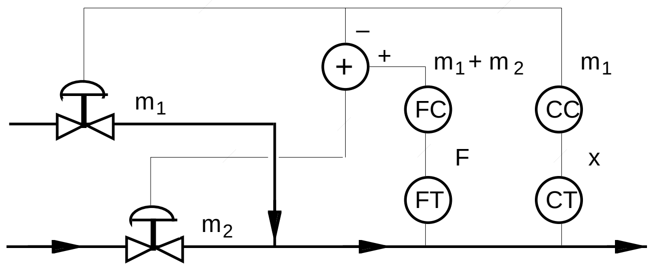


Figure R10.10

MATLAB is a formidable mathematics analysis package. We provide an introduction to the most basic commands that are needed for our use. We make no attempts to be comprehensive. We do not want a big, intimidating manual. The beauty of MATLAB is that we only need to know a tiny bit to get going and be productive. Once we get started, we can pick up new skills quickly with MATLAB's excellent on-line help features. We can only learn with hands-on work; these notes are written as a "walk-through" tutorial—you are expected to enter the commands into MATLAB as you read along.

MATLAB Session 1

For each session, we put the most important functions in a table for easy reference or review. The first one is on the basic commands and plotting. Try the commands as you read. You do not have to enter any text after the "%" sign. Any text behind a "%" is considered a comment and is ignored. We will save some paper and omit the results generated by MATLAB. If you need to see that for help, they are provided on our *Web Support*. There is also where we post any new MATLAB changes and upgrades. Features in our tutorial sessions are based on MATLAB Version 6.1, and Control System Toolbox 5.1.

Important Basic Functions	
General functions:	
`cd`	Change subdirectory
`demo (intro)`	Launch the demo (introduction)
`dir (what)`	List of files in current directory (or only M-files)
`help, helpwin`	Help! Help window
`load`	Load workspace
`lookfor`	Keyword search
`print`	Print graph; can use pull-down menu
`quit`	Quit!
`save`	Save workspace
`who, whos`	List of variables in workspace
Calculation functions:	
`conv`	Convolution function to multiply polynomials
`size, length`	Size of an array, length of a vector
Plotting functions:	
`axis`	Override axis default of plot
`grid`	Add grid to plot
`hold`	Hold a figure to add more plots (curves)
`legend`	Add legend to plot
`plot`	Make plots
`text (gtext)`	Add text (graphical control) to plot
`title`	Add title to plot
`xlabel, ylabel`	Add axis labels to plot

M1.1 Some basic MATLAB commands

Features covered in this session:

- Using help
- Creating vectors, matrices, and polynomials
- Simple matrix operations
- Multiply two polynomials with `conv()`

To begin, we can explore MATLAB using its demonstrations. If you are new to MATLAB, it is highly recommended that you take a look at the introduction.

```
intro       % launch the introduction
demo        % launch the demo program
```

It is important to know that the MATLAB on-line help is excellent, and there are different ways to get that.

```
help        % old-fashioned help inside the Command Window
helpbrowser % launch the help browser window; also available
            % from the Help pull-down menu and toolbar
```

We should make a habit of using the on-line help. The user interface of the help browser, which also works as a Web browser, is extremely intuitive, and it is highly recommended. When we mention the "`help`" command, that is just a general comment; it does not mean that you have to use the old-style help. To use help in the Command Window, turn the page display mode on first. Here's an example of seeking help on the print command with the old-style help:

```
more on                 % turn the page mode on
help print
lookfor print           % general keyword search
which print             % list the pathname of print.m
```

The help features and the Command Window interface tend to evolve quickly. For that reason, we use our *Web Support* to provide additional hints and tidbits so we can keep you update of the latest MATLAB changes. For now, we will introduce a few more basic commands:

```
who         % list the variables that are currently defined
whos        % whos is a more detailed version of who
dir         % list the files in the current subdirectory
what        % list only the M-files
cd          % change the subdirectory
pwd         % list the present working directory
```

For fun, we can try:

```
why
fix(clock)
```

MATLAB is most at home dealing with arrays, which we will refer to as matrices and vectors. They are all created by enclosing a set of numbers in brackets, [ ]. First, we define a row **vector** by entering in the MATLAB Command Window:

```
x = [1 2 3 4 5 6 7 8 9 10]
```

If we add a semicolon at the end of a command, as in

```
x = [1 2 3 4 5 6 7 8 9 10];
```

we can suppress the display of the result. We can check what we have later by entering the name of the variable. To generate a column vector, we insert semicolons between numbers (more specific example below with a matrix). The easier route is to take the transpose of **x**:

```
x = x'
```

Keep in mind that in MATLAB, variables are case sensitive. Small letter x and capital X are two different variables.

We can also generate the row vector with the colon operator:

```
x = 1:10                % same as 1:1:10
y = 0:0.1:2             % just another example
```

The colon operator is very useful when we make longer vectors for plotting or calculations. With this syntax, the increment is squeezed between the beginning and ending values of the vector and they are separated by colons. If the increment value is missing, the default increment is 1. Remember to add a semicolon at the end of each statement to suppress the display of a long string of numbers. We skip that in the illustration just so you may see what is generated. When we do calculations based on vectors, MATLAB will vectorize the computation, which is much faster than if we write a loop construct as in the “for” loop in C or “do” loop in Fortran.

To create a **matrix**, we use a semicolon to separate the rows:

```
a = [1 2 3 ; 4 5 6 ; 7 8 9]
```

In place of the semicolons, we can also simply hit the return key as we generate the matrix.

There are circumstances that we need the size of an array or the length of a vector. They can be found easily:

```
size(y)                 % find the size of an array
length(y)               % find the length of a vector
```

In MATLAB, **polynomials** are stored exactly the same as vectors. Functions in MATLAB will interpret them properly if we follow the convention that a vector stores the coefficients of a polynomial in descending order—it begins with the highest order term and *always* ends with a constant even if it is zero. Some examples:

```
p1=[1 -5 4]      % defines p1(s) = s^2 - 5*s + 4
p2=[1 0 4]       % defines p2(s) = s^2 + 4
p3=[1 -5 0]      % defines p3(s) = s^2 - 5*s
```

We can multiply two polynomials together easily with the convolution function `conv()`. For example, to expand $(s^2 – 5s + 4) (s^2 + 4)$, we can use

```
conv(p1,p2)      % this multiplies p1 to p2
```

or

```
conv([1 -5 4],[1 0 4])
```

MATLAB supports every imaginable way that one can manipulate vectors and matrices. We only need to know a few of them and we will pick up these necessary ones along the way. For now, we'll do a couple of simple operations. With the vector x and matrix a that we've defined above, we can perform simple operations such as

```
y1 = 2*x          % multiplies x by a constant
y2 = sqrt(x)      % takes the square root of each element in x
b  = sqrt(a)      % takes the square root of each element in a
y3 = y1 + y2      % adds the two vectors
c  = a*b          % multiplies the two matrices
```

Note that all functions in MATLAB , such as `sqrt()`, are smart enough that they accept scalars, vectors, and where appropriate, matrices.[1]

When we operate on an *element by element* basis, we need to add a period before the operator. Examples based on the two square matrices a and b:

```
d  = a.^3         % takes the cube of each element
a3 = a^3          %   versus the cube of the matrix
e  = a.*b         % multiplies each element a(i,j)*b(i,j)
f  = a*b          %   versus matrix multiplication a*b
```

[1] In computer science, this is referred to as polymorphism. The fact that mathematical operators can work on different data types is called overloading.

Of course, we can solve the matrix equation Ax = b easily. For example, we can try:

```
A = [ 4   -2   -10;  2  10   -12;  -4  -6  16];
b = [-10;  32;  -16];
x = A\b                          %  Bingo!
```

Let's check the solution by inverting the matrix A:[2]

```
C = inv(A);
x = C*b
```

We can find the eigenvalues and eigenvectors of A easily:

```
[X,D] = eig(A)
```

Finally, we do a simple polynomial fit illustration. Let’s say we have a set of (x,y) data :

```
x = [ 0  1  2  4  6  10];
y = [ 1  7  23  109  307  1231];
```

To make a third-order polynomial fit of y = y(x), all we need is to enter

```
c = polyfit(x,y,3)       % should obtain c = [1 2 3 1]
```

The returned vector c contains the coefficients of the polynomial. In this example, the result should be $y = x^3 + 2x^2 + 3x + 1$. We can check and see how good the fit is. In the following statements, we generate a vector `xfit` so that we can draw a curve. Then, we calculate the corresponding `yfit` values, plot the data with a symbol and the fit as a line.

```
xfit=1:0.5:10;
yfit=xfit.^3 + 2*xfit.^2 + 3*xfit +1;
plot(x,y,'o', xfit,yfit)                % explanation on plotting
title('3rd order polynomial fit')       % is in the next section
legend('data','3rd order fit')
```

Speaking of plotting, this is what we'll get into next.

M1.2 Some simple plotting

Features covered in this session:
- Using the `plot()` function
- Adding titles, labels, and legends

Let's create a few vectors first

```
x = 0:0.5:10;
y1= 2*x;
y2= sqrt(x);
```

Now we plot y1 versus x, and y2 versus x together:

```
plot(x,y1, x,y2)
```

We have a limited selection of line patterns or symbols. For example, we can try [3]

[2] If you have taken a course on numerical methods, you would be pleased to know that MATLAB can do LU decomposition:
```
[L,U] = lu(A);
```
[3] To do multiple plots, we can also use:
```
plot(x,y1,'-.', x,y2,'--')
plot(x,y1,'-.')
hold                    % or use "hold on"
plot(x,y2,'--')
hold                    % or use "hold off"
```

We can find the list of pattern selections with on-line help. The command `hold` allows us to add more plots to the same figure, and `hold` works as a toggle. That is why we do not have to state "on" and "off" explicitly.

We can add a title and axis labels too:

```
title('A boring plot')
xlabel('The x-axis label'), ylabel('The y-axis label')
```

We can issue multiple commands on the same line separated by commas. What makes MATLAB easy to learn is that we can add goodies one after another. We do not have to worry about complex command syntax. We can also do logarithmic plots. Try enter "`help semilogx`, `semilogy`, or `loglog`." We'll skip them because they are not crucial for our immediate needs.

We can add a grid and a legend with

```
grid
legend('y1','y2')
```

A box with the figure legend will appear in the Graph Window. Use the mouse to drag the box to where you want it to be. We can also add text to annotate the plot with:

```
text(1,9,'My two curves')   % starting at the point (1,9)
```

The text entry can be interactive with the use of

```
gtext('My two curves')
```

Now click on the graph window, and a cross-hair will appear. Move it to where you want the legend to begin and click. Presto! Repeat for additional annotations.

In rare cases, we may not like the default axis scaling. To override what MATLAB does, we can define our own minimum and maximum of each axis with

```
axis([0 15 0 30])       % the syntax is [xmin xmax ymin ymax]
```

We need the brackets inside because the argument to the axis function is an array.

Plotting for fun

We do not need to do 3-D plots, but then it's too much fun not to do at least a couple of examples. However, we'll need to use a few functions that we do not need otherwise, so do not worry about the details of these functions that we will not use again. We'll get a pretty looking 3-D picture:

```
[x,y]=meshgrid(-10:0.5:10, -10:0.5:10);
              % meshgrid transforms the specified domain
              $ where -10 < x < 10, and -10 < y < 10
              % into a grid of (x,y) values for evaluating z

r=sqrt(x.^2 + y.^2) + eps;    % We add the machine epsilon eps
z=sin(r)./r;                  % so 1/r won’t blow up
mesh(z)
title('The Sinc Sombrero')
```

So you say wow! But MATLAB can do much more and fancier than that. We try one more example with Bessel functions, which you can come across in heat and mass transfer problems with cylindrical geometry.

```
% Here we do a 3-D mesh plot of Jo(sqrt(x^2+y^2))
% The x and y grids remain the same as in the previous plot

r=sqrt(x.^2+y.^2);
z=bessel(0,r);
mesh(z)
```

M1.3 Making M-files and saving the workspace

Features covered in this session:

- Execute repeated commands in a script, the so-called M-file
- Save a session

For tasks that we have to repeat again and again, it makes sense to save them in some kind of a script and execute them. In MATLAB, these scripts are called M-files. The name came from the use of macros in the old days of computing. We can use M-files to write unstructured scripts or user-defined functions. MATLAB now refers to both as programs. You may want to keep in mind that a script using a scripting interpretive language is not the same as a program written in, say, C.

For our needs, a simple script suffices in most circumstances. To use an M-file: [4]

1. Save all the repetitious MATLAB statements in a text file with the "`.m`" extension.
2. Execute the statements in that file by entering the file name *without* the "`.m`" extension.

Here is one simple example. We need to plot x versus y repeatedly and want to automate the task of generating the plots with an M-file. The necessary statements with comments are:

```
% ______________ M-file script: plotxy.m ________________
% A very simple script to plot x vs y and add the labels
% ...the kind of things we don't want to repeat typing
%     again and again...

plot(x,y)
grid
xlabel('Time [min]')
ylabel('Step Response')
title('PID Controller Simulation')

% End of plotxy.m. An “end” statement is not needed.
```

Save these statements in a file named, say, `plotxy.m`. Anything after the "%" sign is regarded as a comment, which you do not have to enter if you just want to repeat this exercise. After we have defined or updated the values of x and y in the Command Window, all we need is to enter "`plotxy`" at the prompt and MATLAB will do the rest. The key is to note that the M-file has *no* "read" or "input" for x and y. All statements in an M-file are simply executed in the Command Window.

[4] There is another easy way to "cheat." On UNIX/Linux workstations, open up a new text editor and enter your frequently used statements there. On Windows, you can use the really nice MATLAB Editor. You can copy-and-paste multiple commands back and forth between the text editor window and the MATLAB window easily. If you want to save the commands, you certainly can add comments and annotations. You can consider this text file as a "free-format notebook" without having to launch the Microsoft Word Notebook for MATLAB.

If you have an M-file, MATLAB may not find it unless it is located within its search path. Let's see where MATLAB looks first. On UNIX/Linux machines, MATLAB by default looks in the subdirectory from which it is launched. A good habit is to keep all our work in one subdirectory, and change to that specific subdirectory before we launch MATLAB. On Windows machines, MATLAB looks for your files in the WORK folder buried deep inside the Program Files folder. A good chance is that you want to put your files in more convenient locations. To coax MATLAB to find them, we need to change the directory or the search path. So the next question is how to do that and the answer applies to both UNIX/Linux and Windows machines. The formal way is to learn to use the "`cd`" and "`path`" commands. The easy way is to use point-and-click features that can be found under pull-down menus, on tool bars, or in sub-windows. Because these graphical interface features tend to change with each MATLAB upgrade, please refer to our *Web Support*, where we can update quickly to match new changes, for additional help.

If we want to take a coffee break and save all the current variables that we are working with, enter

```
save
```

before we **`quit`** MATLAB. When we launch MATLAB again, type

```
load
```

and everything will be restored. Do not save the workspace if you are going to be away any longer because the old workspace is not very useful if you have all these variables floating around and you forget what they mean.

As a final comment, we can use `load` and `save` to import and export arrays of data. Since we do not really need this feature in what we do here, we defer this explanation to our *Web Support*.

MATLAB Session 2

This tutorial is to complement our development in Chapter 2. You may want to go over the tutorial quickly before you read the text and come back later a second time for the details.

Partial Fraction and Transfer Functions	
`poly`	Construct a polynomial from its roots
`residue`	Partial fraction expansion
`roots`	Find the roots to a polynomial
`tf2zp`	Transfer function to zero-pole form conversion
`zp2tf`	Zero-pole form to transfer function conversion
Object-oriented functions:	
`tf`	Create a transfer function object
`get`	List the object properties
`pole`	Find the poles of a transfer function
`zpk`	Create a transfer function in pole-zero-gain form

M2.1 Partial fractions

Features covered in this session:
- Find the roots of a polynomial with `roots()`
- Generate a polynomial from its roots with `poly()`
- Do partial fractions with `residue()`

Of secondary importance:
- Transfer function to zero-pole form, `tf2zp()`
- Zero-pole form to transfer function, `zp2tf()`

Let's first define a polynomial

```
p = [1 5 4]        % makes p(s) = s^2 + 5*s + 4
```

We can find the roots of p(s) = 0 with the function `roots()`

```
poles = roots(p)
```

MATLAB should return –4 and –1. That means the polynomial can be factored as p(s) = (s + 4)(s + 1). [1]

We can go backwards. Given the roots (or pole positions), we can get the polynomial with:

```
p2 = poly(poles)
```

MATLAB returns the results in a column vector. Most functions in MATLAB take either row or column vectors and we usually do not have to worry about transposing them.

We can do partial fractions with the `residue()` function. Say we have a transfer function

$$G(s) = \frac{q(s)}{p(s)} = \frac{1}{s^2 + 5s + 4}$$

where q(s)=1 and p(s) remains `[1 5 4]` as defined earlier. We can enter

[1] MATLAB has the function **`fzero()`** to find a root of a given function.

```
q = 1;
residue(q,p)
```

MATLAB returns the numbers –0.3333 and 0.3333. That is because the function can be factored as

$$\frac{1}{s^2 + 5s + 4} = \frac{-1/3}{s+4} + \frac{1/3}{s+1}$$

How can we be sure that it is the –0.3333 coefficient that goes with the root at –4? We can use the syntax

```
[a,b,k]=residue(q,p)
```

MATLAB will return the coefficients in a, the corresponding poles in b and whatever is leftover in k, which should be nothing in this case. Note that `[]` denotes an empty matrix or vector.

Let's try another transfer function with poles at 0, –1, –2, and –3:

$$G(s) = \frac{1}{s\,(s+1)\,(s+2)\,(s+3)}$$

To find the partial fractions, this is what we can do:[2]

```
poles=[0 -1 -2 -3];
p=poly(poles);
q=1;
[a,b,k]=residue(q,p)
```

One more example. Find the partial fractions of the nasty looking function:

$$G(s) = \frac{s^2 + 4s + 3}{s^4 - 7s^3 + 11s^2 + 7s - 12}$$

```
q=[1 4 3];
zeros=roots(q)          % should return -3, -1
p=[1 -7 11 7 -12];
poles=roots(p)          % should return 4, 3, 1, -1
[a,b,k]=residue(q,p)
```

See that MATLAB returns the expansion:

$$\frac{s^2 + 4s + 3}{s^4 - 7s^3 + 11s^2 + 7s - 12} = \frac{2.33}{s-4} - \frac{3}{s-3} + \frac{0.67}{s-1}$$

Note that the coefficient associated with the pole at –1 is zero. That is because it is canceled by the zero at –1. In other words, the (s+1) terms cancel out. It is nice to know that the program can do this all by itself. We do not need to know the roots to use `residue()`, but it is a good habit to get a better idea what we are working with.

A transfer function can be written in terms of its poles and zeros. For example,

[2] If we need to write conjugate complex roots, make sure there are no spaces within a complex number. For example, enter: `[-3+4*j  -3-4*j]`. Either `i` or `j` can be used to denote $\sqrt{-1}$.

$$F(s) = \frac{6s^2 - 12}{(s^3 + s^2 - 4s - 4)} = \frac{6\,(s - \sqrt{2})\,(s + \sqrt{2})}{(s + 1)\,(s + 2)\,(s - 2)}$$

The RHS is called the pole-zero form (or zero-pole form). MATLAB provides two functions `tf2zp()` and `zp2tf()` to do the conversion. For instance,

```
q=[6  0  -12];
p=[1  1  -4  -4];
[zeros,poles,k]=tf2zp(q,p)
```

Of course we can go backward with

```
[q,p]=zp2tf(zeros,poles,k)
```

Note: The factor k is 6 here, and in the MATLAB manual, it is referred to as the "gain." This factor is really the ratio of the leading coefficients of the two polynomials q(s) and p(s). Make sure you understand that the "k" here is ***not*** the steady state gain—which is the ratio of the *last* constant coefficients. (In this example, the steady state gain is –12/–4 = 3.) MATLAB actually has a function named `dcgain` to do this.

One more simple example:

```
zero=  -2;                % generate a transfer function
poles=[-4  -3  -1];       % with given poles and zeros
k=1;
[q,p]=zp2tf(zero,poles,k)
```

Double check that we can recover the poles and zeros with

```
[zero,poles,k]=tf2zp(q,p)
```

We can also check with

```
roots(q)
roots(p)
```

Try `zp2tf` or `tf2zp` on your car's license plate!

M2.2 Object-oriented transfer functions

Features covered in this session:

- Define a transfer function object with `tf()` or `zpk()`
- Determine the poles with `pole()`
- Use of overloaded operators

MATLAB is object-oriented. Linear time-invariant (LTI) models are handled as objects. Functions use these objects as arguments. In classical control, LTI objects include transfer functions in polynomial form or in pole-zero form. The LTI-oriented syntax allows us to better organize our problem solving; we no longer have to work with individual polynomials that we can only identify as numerators and denominators.

We will use this syntax extensively starting in Session 3. Here, we see how the object-oriented syntax can make the functions `tf2zp()` and `zp2tf()` redundant and obsolete.

❑ To define a transfer function object, we use `tf()`, which takes the numerator and denominator polynomials as arguments. For example, we define $G(s) = \frac{s}{s^2 - 5s + 4}$ with

```
G1 = tf([1 0], [1 -5 4])
```

We define $G(s) = \frac{6s^2 - 12}{(s^3 + s^2 - 4s - 4)}$ with

```
G2 = tf([6 0 -12], [1 1 -4 -4])
```

We can also use the zero-pole-gain function `zpk()` which takes as arguments the zeros, poles and gain factor of a transfer function. Recall our comments after `zp2tf()`. This gain factor is not the steady state (or dc) gain.

For example, we define $G(s) = \frac{4}{s\,(s + 1)\,(s + 2)\,(s + 3)}$ with

```
G3 = zpk([],[0 -1 -2 -3], 4)  % the [] means there is no zero
```

❑ The `tf()` and `zpk()` functions also serve to perform model conversion from one form to another. We can find the polynomial form of G3 with

```
tf(G3)
```

and the pole-zero form of G2 with

```
zpk(G2)
```

❑ The function `pole()` finds the poles of a transfer function. For example, try:

```
pole(G1)
pole(G2)
```

You can check that the results are identical to the use of `roots()` on the denominator of a transfer function.

❑ We may not need to use them, but it is good to know that there are functions that help us extract the polynomials, or poles and zeros back from an object. For example:

```
[q,p]=tfdata(G1,'v')     % option 'v' for row vectors
[z,p,k]=zpkdata(G3,'v')
```

❑ The addition and multiplication operators are overloaded and we can use them to manipulate or synthesize transfer functions. This capability will come in handy when we analyze control systems. For now, let's consider one simple example. Say we are given

$$G_1 = \frac{1}{s + 1}\ \text{, and}\ G_2 = \frac{2}{s + 2}$$

We can find $G_1 + G_2$ and G_1G_2 easily with

```
G1=tf(1,[1 1]);
G2=tf(2,[1 2]);
G1+G2            % or we can use zpk(G1+G2)
G1*G2            % or we can use zpk(G1*G2)
```

This example is simple enough to see that the answers returned by MATLAB are correct.

❏ With object-oriented programming, an object can hold many properties. We find the associated properties with

```
get(G1)
```

Among the MATLAB result entries, we may find the properties InputName, OutputName, and Notes. We can set them with [3]

```
G1.InputName = 'Flow Rate';
G1.OutputName = 'Level';
G1.Notes = 'My first MATLAB function';
```

You'll see the difference if you enter from now on:

```
G1
get(G1)
```

MATLAB can use **symbolic algebra** to do Laplace transform. Since this skill is not crucial to solve control problems, we skip it here. You can find a brief tutorial on our *Web Support*, and you are encouraged to work through it if you want to know what symbolic algebra means.

[3] We are using the typical structure syntax, but MATLAB also supports the **set()** function to perform the same task.

MATLAB Session 3

This tutorial is to complement our development in Chapter 3. You may want to go over the tutorial quickly before you read the text and come back later a second time for the details.

Time Response Simulation Functions	
damp	Find damping factor and natural frequency
impulse	Impulse response
lsim	Response to arbitrary inputs
step	Unit step response
pade	Time delay Padé approximation
ltiview	Launch the graphics viewer for LTI objects

M3.1 Step and impulse response simulations

Features covered in this session:

- Use of `step()` and `impulse()`
- Time response to any given input, `lsim()`
- Dead time approximation, `pade()`

Instead of spacing out in the Laplace-domain, we can (as we are taught) guess how the process behaves from the pole positions of the transfer function. But wouldn't it be nice if we could actually trace the time profile without having to do the reverse Laplace transform ourselves? Especially the response with respect to step and impulse inputs? Plots of time domain dynamic calculations are extremely instructive and a useful learning tool.[1]

The task of time-domain calculation is easy with MATLAB. Let's say we have

$$\frac{Y(s)}{X(s)} = \frac{1}{s^2 + 0.4\,s + 1}\,,$$

and we want to plot y(t) for a given input x(t). We can easily do

```
q=1;
p=[1 0.4 1];           % poles at -0.2±0.98j
G=tf(q,p)

step(G)                % plots y(t) for unit step input, X(s)=1/s

impulse(G)             % plots y(t) for impulse input, X(s)=1
```

What a piece of cake! Not only does MATLAB perform the calculation, it automatically makes the plot with a properly chosen time axis. Nice! [2] As a habit, find out more about a function with `help` as in

```
help step              % better yet, use helpwin or Help Desk
```

[1] If you are interested, see our *Web Support* for using **Runge-Kutta** integration of differential equations.

[2] How could we guess what the time axis should be? It is not that difficult if we understand how to identify the dominant pole, the significance behind doing partial fractions, and that the time to reach 99% of the final time response is about five time constants.

The functions also handle multiple transfer functions. Let's make a second transfer function in pole-zero form,

$$H(s) = \frac{2}{(s + 2)\,(s^2 + 2\,s + 2)}\ ,$$

```
H=zpk([], [-2 -1+j -1-j], 2)
```

We can compare the unit step responses of the two transfer functions with

```
step(G,H)
```

We can, of course, choose our own axis, or rather, time vector. Putting both the unit step and impulse response plots together may also help us understand their differences.

```
t=0:0.5:40;              % don’t forget the semicolon!
ys=step(G,t);
yi=impulse(G,t);
plot(t,ys,t,yi)
```

Note: In the text, we emphasize the importance of relating pole positions of a transfer function to the actual time-domain response. We should get into the habit of finding what the poles are. The time response plots are teaching tools that reaffirm our confidence in doing analysis in the Laplace-domain. So, we should find the roots of the denominator. We can also use the `damp()` function to find the damping ratio and natural frequency.

```
pole(G)                  % same result with roots(p)
damp(G)                  % same result with damp(p)
```

One more example. Consider the transfer function

$$\frac{Y(s)}{X(s)} = G(s) = \frac{2s + 1}{(4s + 1)\,(s + 1)}$$

We want to plot y(t) if we have a sinusoidal input, x(t) = sin(t). Here we need the function `lsim()`, a general simulation function which takes any given input vector.

```
q=[2 1];                       % a zero at -1/2
p=conv([4 1],[1 1]);           % poles at -1/4 and -1
G=tf(q,p)                      % (can use zpk instead)
t=0:0.5:30;
u=sin(t);
y=lsim(G,u,t);                 % response to a sine function input
plot(t,y,t,u,'-.'), grid
```

Keep this exercise in mind. This result is very useful in understanding what is called frequency response in Chapter 8. You can repeat the simulation with higher frequencies. We can also add what we are familiar with:

```
hold
ys=step(G,t);
yi=impulse(G,t);
plot(t,ys,t,yi)
hold off
```

For fun, try one more calculation with the addition of random noise:

```
u=sin(t)+rand(size(t));
y=lsim(G,u,t);
plot(t,y,'r',t,u,'b'),  grid   % Color lines red and blue
```

For useful applications, `lsim()` is what we need to simulate response to, say, a **rectangular pulse**. This is one simple example using the same transfer function and time vector that we have just defined:

```
t=0:0.5:30;               % t = [0 .5 1 1.5 2 2.5 3 … ]
u=zeros(size(t));         % make a vector with zeros
u(3:7)=1;                 % make a rectangular pulse from t=1 to t=3
y=lsim(G,u,t);
yi=impulse(G,t);          % compare the result with impulse response
plot(t,u,  t,y,  t,yi,'-.');
```

Now, we switch gears and look into the dead time transfer function approximation. To do a Padé approximation, we can use the MATLAB function [3]

```
[q,p]=pade(Td,n)
```

where Td is the dead time, n is the order of the approximation, and the results are returned in q(s)/p(s). For example, with Td=0.2 and n=1, entering

```
[q,p]=pade(0.2,1)         % first order approximation
```

will return

```
q = -1 s + 10
p =  1 s + 10
```

We have expected q(s) = –0.1s + 1, and p(s) = 0.1s + 1. Obviously, MATLAB normalizes the polynomials with the leading coefficients. On second thought, the Padé approximation is so simple that there is no reason why we cannot do it ourselves as in a textbook. For the first order approximation, we have

```
Td=0.2;
q = [-Td/2 1];
p = [ Td/2 1];
```

We can write our own simple-minded M-file to do the approximation. You may now try

```
[q,p]=pade(0.2,2)         % second order approximation
```

and compare the results of this second order approximation with the textbook formula.

[3] When we use `pade()` without the left-hand argument `[q,p]`, the function automatically plots the step and phase responses and compares them with the exact responses of the time delay. Padé approximation has unit gain at all frequencies. These points will not make sense until we get to frequency response analysis in Chapter 8. So for now, keep the `[q,p]` on the left hand side of the command.

M3.2 LTI Viewer

Features covered in this session:

- Graphics viewer for LTI objects, `ltiview` [4]

We can use the LTI Viewer to do all the plots, not only step and impulse responses, but also more general time response and frequency response plots in later chapters. If we know how to execute individual plot statements, it is arguable whether we really need the LTI Viewer. Nonetheless, that would be your personal choice. We will provide here the basic idea and some simple instructions.

To launch the LTI Viewer, enter in the MATLAB command window:

```
ltiview
```

A blank LTI window will pop up. The first task would be to poke into features supported under the *File* and *Tools* pull-down menus and see what we can achieve via point-and-click. There is also a *Help* pull-down menu, which activates the Help Window.

The LTI Viewer runs in its own workspace, which is separate from the MATLAB workspace. The Viewer also works with only LTI objects generated by functions such as `tf()` and `zpk()`, and after Chapter 4, state space objects, `ss()`. So let's generate a couple of objects in the MATLAB command window first:

```
G=tf(1,[1  0.4  1])
H=zpk([],  [-2  -1+j  -1-j],  2)
```

Now, go to the LTI Viewer window and select ***Import*** under the *File* pull-down menu. A dialog box will pop out to help import the transfer function objects. By default, a unit step response will be generated. Click on the axis with the *right mouse button* to retrieve a pop-up menu that will provide options for other plot types, for toggling the object to be plotted, and other features. With a step response plot, the *Characteristics* feature of the pop-up menu can identify the peak time, rise time, and settling time of an underdamped response.

The LTI Viewer was designed to do comparative plots, either comparing different transfer functions, or comparing the time domain and (later in Chapter 8) frequency response properties of a transfer function. So a more likely (and quicker) scenario is to enter, for example,

```
ltiview('step',G,H)
```

The transfer functions G and H will be imported automatically when the LTI Viewer is launched, and the unit step response plots of the two functions will be generated.

Another useful scenario is, for example,

```
ltiview({'step';'bode'},G)
```

In this case, the LTI Viewer will display both the unit step response plot and the Bode plot for the transfer function G. We will learn Bode plot in Chapter 8, so don't panic yet. Just keep this possibility in mind until we get there.

[4] The description is based on Version 4.0 of the MATLAB control toolbox. If changes are introduced in newer versions, they will be presented on our *Web Support*.

MATLAB Session 4

This tutorial is to complement our development in Chapter 4. You may want to go over the tutorial quickly before you read the text and come back later a second time for the details.

State Space Functions	
canon	Canonical state space realization
eig	Eigenvalues and eigenvectors
ss2ss	Transformation of state space systems
ss2tf	Conversion from state space to transfer function
tf2ss	Conversion from transfer function to state space
printsys	Slightly prettier looking display of model equations
ltiview	Launch the graphics viewer for LTI objects
ss	Create state space object

M4.1 Conversion between transfer function and state space

Features covered in this session:

- The use of `ss2tf()` and `tf2ss()`
- Generate object-oriented models with `ss()`

We need to revisit **Example 4.1** with a numerical calculation. Let's use the values $\zeta = 0.5$ and $\omega_n = 1.5$ Hz to establish the transfer function and find the poles.

```
z=0.5;
wn=1.5;                    % Should find
q=wn*wn;                   % q=2.25
p=[1 2*z*wn wn*wn]         % p=[1 1.5 2.25]
roots(p)                   % -0.75±1.3j
```

Based on the results in Example 4.1, we expect to find

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -2.25 & -1.5 \end{bmatrix}; \quad \mathbf{B} = \begin{bmatrix} 0 \\ 2.25 \end{bmatrix}; \quad \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}; \quad \mathbf{D} = 0$$

Now, let's try our hands with MATLAB using its transfer function to state space conversion function:

```
[a,b,c,d]=tf2ss(q,p)
```

and MATLAB returns with

$$\mathbf{a} = \begin{bmatrix} -1.5 & -2.25 \\ 1 & 0 \end{bmatrix}; \quad \mathbf{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}; \quad \mathbf{c} = \begin{bmatrix} 0 & 2.25 \end{bmatrix}; \quad \mathbf{d} = 0$$

which are not the same as Example 4.1 in text. You wonder what's going on? Before you kick the computer, a closer look should reveal that MATLAB probably uses a slightly different convention. Indeed, MATLAB first "split" the transfer function into product form:

$$\frac{Y}{U} = \frac{X_2}{U}\frac{Y}{X_2} = \frac{1}{\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)}\omega_n^2 = \frac{1}{\left(s^2 + 1.5\,s + 2.25\right)}2.25$$

From $X_2/U = 1/(s^2 + 2\zeta\omega_n s + \omega_n^2)$ and with the state variables defined as

$$x_1 = \frac{d\,x_2}{d\,t}\text{, and } x_2 = x_2 \text{ (i.e., same),}$$

we should obtain the matrices **a** and **b** that MATLAB returns. From $Y/X_2 = \omega_n^2$, it should be immediately obvious how MATLAB obtains the array **c**.

In addition, we should beware that the indexing of state variables in MATLAB is in *reverse order* of textbook examples. Despite these differences, the inherent properties of the model remain identical. The most important of all is to check the eigenvalues:

```
eig(a)                          % should be identical to the poles
```

A conversion from state space back to transfer function should recover the transfer function:

```
[q2,p2]=ss2tf(a,b,c,d,1)        % same as q/p as defined earlier
```

The last argument in `ss2tf()` denotes the i-th input, which must be 1 for our single-input single-out model. To make sure we cover all bases, we can set up our own state space model as in Example 4.1:

```
a=[0 1; -2.25 -1.5]; b=[0; 2.25]; c=[1 0]; d=0;
```

and check the results with

```
eig(a)                          % still the same!
[qs,ps]=ss2tf(a,b,c,d,1)
```

The important message is that there is no unique state space representation, but all model matrixes should have the same eigenvalues. In addition, the number of state variables is the same as the order of the process or system.

The fact that the algorithm used by MATLAB does not return a normalized output matrix **C** can create problems when we do feedback calculations in Chapter 9. The easy solution is to rescale the model equations. The output equation can be written as

$$y = [\alpha\ \ 0]\,\mathbf{x} = [1\ \ 0]\,\bar{\mathbf{x}} \quad \text{where } \bar{\mathbf{x}} = \alpha\mathbf{x}$$

Substitution for **x** by $\bar{\mathbf{x}}$ in $d\mathbf{x}/dt = \mathbf{Ax} + \mathbf{B}u$ will lead to

$$\frac{d\bar{\mathbf{x}}}{dt} = \mathbf{A}\bar{\mathbf{x}} + \alpha\mathbf{B}u = \mathbf{A}\bar{\mathbf{x}} + \bar{\mathbf{B}}u\,, \quad \text{where} \quad \bar{\mathbf{B}} = \alpha\mathbf{B}$$

In other words, we just need to change **C** to the normalized vector and multiply **B** with the scaling factor. We can see that this is correct from the numerical results of Example 4.1. (Again, keep in mind that the MATLAB indexing is in reverse order of textbook examples.) We will use this idea in Chapter 9.

We now repeat the same exercise to show how we can create object-oriented state-space LTI models. In later chapters, all control toolbox functions take these objects as arguments. We first repeat the statements above to regenerate the state matrices **a**, **b**, **c**, and **d**. Then we use `ss()` to generate the equivalent LTI object.

```
q=2.25;
```

```
p=[1 1.5 2.25];
[a,b,c,d]=tf2ss(q,p);
sys_obj=ss(a,b,c,d)
```

We should see that the LTI object is identical to the state space model. We can retrieve and operate on individual properties of an object. For example, to find the eigenvalues of the matrix a inside `sys_obj`:

```
eig(sys_obj.a)          % find eigenvalue of state matrix a
```

We can get the transfer function, as analogous to using `ss2tf()`, with

```
tf(sys_obj)
```

Now, you may wonder if we can generate the state space model directly from a transfer function. The answer is, of course, yes. We can use

```
sys2=ss(tf(q,p))
eig(sys2.a)             % should be identical to the poles
```

MATLAB will return with matrices that look different from before:

$$\mathbf{a} = \begin{bmatrix} -1.5 & -1.125 \\ 2 & 0 \end{bmatrix} ; \quad \mathbf{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} ; \quad \mathbf{c} = \begin{bmatrix} 0 & 1.125 \end{bmatrix} ; \quad \mathbf{d} = 0$$

With what we know now, we bet `ss()`uses a different scaling in its algorithm. This time, MATLAB factors the transfer function into this product form:

$$\frac{Y}{U} = \frac{X_2}{U}\frac{Y}{X_2} = \frac{2}{\left(s^2 + 1.5\,s + 2.25\right)}\,1.125$$

From $X_2/U = 2/\left(s^2 + 1.5\,s + 2.25\right)$ and with the state variables defined as

$$x_1 = \frac{1}{2}\frac{d\,x_2}{d\,t} \text{ (i.e., } \frac{d\,x_2}{d\,t} = 2\,x_1\text{), and } x_2 = x_2,$$

we should obtain the new state matrices. Again, the key result is that the state matrix **a** has the same eigenvalue.

This exercise underscores one more time that there is no unique way to define state variables. Since our objective here is to understand the association between transfer function and state space models, we will continue our introduction with the `ss2tf()` and `tf2ss()` functions.

Two minor tidbits before we move on. First, the `printsys()` function displays the model matrices or polynomials in a slightly more readable format. Sample usage:

```
printsys(a,b,c,d)
printsys(q,p,'s')
```

Second, with a second order transfer function, we can generate the textbook state space matrices given a natural frequency wn and damping ratio z:

```
[a,b,c,d]=ord2(wn,z)    % good for only q=1
```

If we examine the values of **b** and **c**, the result is restricted to a unity numerator in the transfer function.

M4.2 Time response simulation

To begin with, we can launch the LTI Viewer with

```
ltiview
```

as we have explained in MATLAB Session 3. The graphics interface is well designed enough that we need no further explanation.

The use of `step()` and `impulse()` on state space models is straightforward as well. We provide here just a simple example. Let's go back to the numbers that we have chosen for **Example 4.1**, and define

```
a=[0 1; -2.25 -1.5]; b=[0; 2.25]; c=[1 0]; d=0;
sys=ss(a,b,c,d);
```

The `step()` function also accepts state space representation, and to generate the unit step response is no more difficult than using a transfer function:

```
step(sys)
```

Now we repeat the calculation in the transfer function form, and overlay the plot on top of the last one:

```
G=tf(2.25,[1 1.5 2.25]);
hold
step(G,'x')
hold off
```

Sure enough, the results are identical. We'd be in big trouble if it were not! In fact, we should get the identical result with other state space representations of the model. (You may try this yourself with the other set of a,b,c,d returned by `tf2ss()` when we first went through Example 4.1.)

Many other MATLAB functions, for example, `impulse()`, `lsim()`, etc., take both transfer function and state space arguments (what you can call polymorphic). There is very little reason to do the conversion back to transfer function once you can live in state space with peace.

M4.3 Transformations

Features covered in this session:

- Similarity and canonical transforms
- Use of functions `canon()` and `ss2ss()`

We first do a demonstration of similarity transform. For a nonsingular matrix **A** with distinct eigenvalues, we can find a nonsingular (modal) matrix **P** such that the matrix **A** can be transformed into a diagonal made up of its eigenvalues. This is one useful technique in decoupling a set of differential equations.

Consider the matrix **A** from **Example 4.6**. We check to see if the rank is indeed 3, and compute the eigenvalues for reference later.

```
A=[0 1 0; 0 -1 -2; 1 0 -10];
rank(A)
eig(A)                  % -0.29, -0.69, -10.02
```

We now enter

```
[P,L] = eig(A)      % L is a diagonal matrix of eigenvalues
                    % P is the modal matrix whose columns are the
                    % corresponding eigenvectors
a = inv(P)*A*P      % Check that the results are correct
```

Indeed, we should find **a** to be the diagonal matrix with the eigenvalues.

The second route is to diagonalize the entire system. With Example 4.6, we further define:

```
B=[0; 2; 0];
C=[1 0 0];
D=[0];
S=ss(A,B,C,D);          % Generates the system object
SD=canon(S)
```

The `canon()` function by default will return the diagonalized system, and in thise case, in the system object `SD`. For example, we should find `SD.a` to be identical to the matrix `L` that we obtained a few steps back.

The third alternative to generate the diagonalized form is to use the state space to state space transformation function. The transformation is based on the modal matrix that we obtained earlier.

```
SD=ss2ss(S,inv(P))
```

To find the observable canonical form of Example 4.6, we use

```
SO=canon(S,'companion')
```

In the returned system `SO`, we should find `SO.a` and `SO.b` to be

$$\mathbf{A}_{ob} = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 0 & -10 \\ 0 & 1 & -11 \end{bmatrix} \quad \text{and} \quad \mathbf{B}_{ob} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Optional reading:
The rest of this section requires material on our *Web Support* and is better read together with Chapter 9. Using the supplementary notes on canonical transformation, we find that the observable canonical form is the transpose of the controllable canonical form. In the observable canonical form, the coefficients of the characteristic polynomial (in reverse sign) are in the last column. The characteristic polynomial is, in this case,

$$P(s) = s^3 + 11s^2 + 10s + 2$$

We can check that with

```
roots([1 11 10 2]) % Check the roots
```

```
poly(A)              % Check the characteristic polynomial of A
```

We can find the canonical forms ourselves. To evaluate the observable canonical form $\mathbf{A}_{ob}$, we define a new transformation matrix based on the controllability matrix:

```
P=[B  A*B  A^2*B];
inv(P)*A*P           % Should be Aob as found by canon()
inv(P)*B             % Shoud be Bob (Bob!)
```

To find the controllable canonical form:

$$\mathbf{A}_{ctr} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -10 & -11 \end{bmatrix} \quad \text{and} \quad \mathbf{B}_{ctr} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

we use the following statements based on the *Web Support* supplementary notes. Be very careful when we construct the matrix **M**.

```
poly(A);   %To confirm that it is [1 11 10 2]
M=[10 11 1; 11 1 0; 1 0 0];
T=P*M;
inv(T)*A*T
inv(T)*B
```

We now repeat the same ideas one more time with **Example 4.9**. We first make the transfer function and the state space objects:

```
G=zpk([],[-1  -2  -3],1);
S=ss(G);
```

As a habit, we check the eigenvalues:

```
eig(S)               % Should be identical to eig(G)
```

To find the modal matrix, we use

```
[P,L]=eig(S.a)
inv(P)*S.a*P         % Just a check of L
```

The observable canonical form is

```
SD=canon(S)
```

The component `SD.a` is, of course, the diagonalized matrix `L` with eigenvalues. We can check that `SD.b` and `SD.c` are, respectively, computed from

```
inv(P)*S.b           % Identical to SD.b
S.c*P                % Identical to SD.c
```

Finally, the observable canonical form is

```
SO=canon(S,'companion')
```

The matrix `SO.a` is

$$\mathbf{A}_{ob} = \begin{bmatrix} 0 & 0 & -6 \\ 1 & 0 & -11 \\ 0 & 1 & -6 \end{bmatrix}$$

meaning,

$$P(s) = s^3 + 6s^2 + 11s + 6$$

which is the characteristic polynomial

```
 poly([-1  -2  -3])
```

as expected from the original transfer function.

MATLAB Session 5

This tutorial is to complement our development in Chapters 5 and 6. You may want to go over the tutorial quickly before you read the text and come back later a second time for the details.

Feedback Simulation Functions	
`feedback`	Generate feedback system transfer function object
`simulink`	Launch Simulink

M5.1 Simulink

Comments with respect to
- Launching Simulink

Simulink is a user-friendly simulation tool with an icon-driven graphics interface that runs within MATLAB. The introduction here is more conceptual than functional for two reasons. One, the Simulink interface design is very intuitive and you may not need help at all! Second, to provide a thorough introduction, we need to reproduce many of the graphics windows. To conserve paper (and trees), we have moved these print-intensive and detailed explanations to our *Web Support*. Furthermore, the HTML-based Help Desk of MATLAB is extremely thorough and should serve as our main guide for further applications.

To launch Simulink, enter in the command window:

```
simulink
```

and MATLAB will launch the Simulink Block Library window with pull-down menus. A few sample block library icons are shown in Fig. M5.1. Each icon represents a toolbox and contains within it a set of models, which will make themselves available if we double-click on the toolbox icons. For example, we can find within the Sources toolbox (Fig. M5.1) a model for generating a step input function, and within the Sinks toolbox a model for graphing results. Within the Continuous toolbox are the important models for generating transfer functions and state space models (Fig. M5.2).

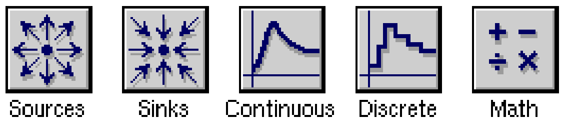


Figure M5.1. Sample icons from the Simulink Block Library Window.

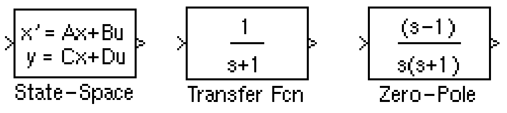


Figure M5.2. Sample icons for model building from the Simulink Continuous Library Window.

All we need is to drag-and-drop the icons that we need from the toolboxes into a blank model window. If this window is not there, open a new one with the *File* pull-down menu. From here on, putting a feedback loop together to do a simulation is largely a point-and-click activity. An example of what Simulink can generate is shown in Fig. M5.3.

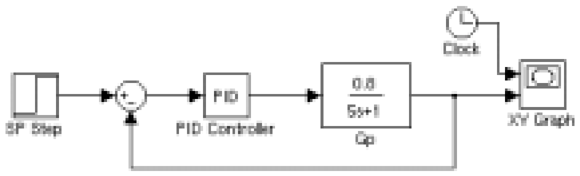


Figure M5.3. A sample negative feedback closed-loop generated within Simulink. This servo system has a first order process function and uses a PID controller. The output is sent to a graphing tool for plotting.

Simulink is easy to learn, fun, and instructive, especially with more complex MIMO systems. For systems with time delays, Simulink can handle the problem much better than the classical control toolbox. Simulink also has ready-made objects to simulate a PID controller.

A few quick pointers:

- Some of the features that we use most often within the Simulink Block Library:

Sources:	Step input; clock for simulation time
Sinks :	Plotting tools; output to MATLAB workspace or a file
Continuous :	Transfer functions in polynomial or pole-zero form; state-space models; transport delay
Math :	Sum; gain or a gain slider
Non-linear :	Saturation; deadzone
Blocksets :	From the Blocksets & Toolboxes, choose "Simulink Extras," and then "Additional Linear." In there are the PID and PID with approximate derivative controllers.

- All Simulink simulation block diagrams are saved as ascii files with the “mdl” extension.

- Before we start a simulation, choose *Parameters* under the *Simulation* pull-down menu to select the time of simulation. If we are using the XY Graph, we need to double-click its icon and edit its parameters to make the time information consistent.

- Simulink shares the main MATLAB workspace. When we enter information into, say, the transfer function block, we can use a variable symbol instead of a number. We then define the variable and assign values to it in the MATLAB command window. This allows for a much quicker route to do parametric studies than changing the numbers within the Simulink icons and dialog boxes.

- We can build our own controllers, but two simple ones are available: an ideal PID and a PID with approximate derivative action.

  For curious minds: The first time you use the PID controllers, drag the icon onto a new simulation window, select the icon and then *Look under mask* under the *Edit* pull-down menu. You will see how the controllers are put together. The simple PID controller is

$$G_c(s) = K_c + \frac{K_I}{s} + K_D s$$

  and the PID with approximate derivative controller is

$$G_c(s) = K_c + \frac{K_I}{s} + \frac{K_D s + 1}{s/N + 1}$$

  We also see the transfer functions used by each icon when we double click on it and open up the parameter entry dialog window. So in terms of notation, we have $K_I = K_c/\tau_I$, $K_D = K_c\tau_D$, and $N = 1/\alpha\tau_D$.

M5.2 Control toolbox functions

Features covered in this session:

- Synthesize a closed-loop transfer function with `feedback()`

The closed-loop transfer function of a servo problem with proper handling of units is Eq. (5-11) in text:

$$\frac{C}{R} = \frac{K_m G_c G_p}{1 + G_m G_c G_p}$$

It can be synthesized with the MATLAB function `feedback()`. As an illustration, we will use a simple first order function for G_p and G_m, and a PI controller for G_c. When all is done, we test the dynamic response with a unit step change in the reference. To make the reading easier, we break the task up into steps. Generally, we would put the transfer function statements inside an M-file and define the values of the gains and time constants outside in the workspace.

Step 1: Define transfer functions in the forward path. The values of all gains and time constants are arbitrarily selected.

```
km=2;        % Gc is a PI controller
kc=10;
taui=100;
Gc=tf(km*kc*[taui 1], [taui 0]);

kp=1;
taup=5;
Gp=tf(kp, [taup 1]);       % Gp is the process function
```

In the definition of the controller G_c, we have included the measurement gain K_m, which usually is in the feedback path and the reference (Fig. 5.4 text). This is a strategy that helps to eliminate the mistake of forgetting about K_m in the reference. One way to spot if you have made a mistake is when the system calculation has an offset when in theory you know that it should not.

Step 2: Define the feedback path function. Let's presume that our measurement function is first order too. The measurement gain has been taken out and implemented in Step 1.

```
taum=1;                    % Gm is the measurement function
Gm=tf(1, [taum 1]);        % Its s.s. gain km is in Gc
```

Step 3: Define the closed-loop function.

```
Gcl=feedback(Gc*Gp,Gm);    % Gcl is the closed-loop function C/R
```

Comments:

- By default, `feedback()` uses negative feedback.
- With unity feedback, i.e., $G_m = K_m = 1$, we would simply use

```
Gcl=feedback(Gc*Gp,1)
```

to generate the closed-loop function.

- One could generate a closed-loop function with, for example, `Gc*Gp/(1 + Gc*Gp)`, but this is not recommended. In this case, MATLAB simply multiplies everything together with no reduction and the resulting function is very unclean.

Step 4: We can now check (if we want to) the closed-loop poles and do the dynamic simulation for a unit step change in R.

```
disp('The closed-loop poles & s.s. gain:')
pole(Gcl)
dcgain(Gcl)

step(Gcl)        % Of course, we can customize the plotting
```

This is the general idea. You can now put it to use by writing M-files for different kinds of processes and controllers.

When we have a really simple problem, we should not even need to use `feedback()`. Yes, we can derive the closed-loop transfer functions ourselves. For example, if we have a proportional controller with $G_c = K_c$, and a first order process, all we need are the following statements, which follow Example 5.1 and Eq. (E5-1) in text:

```
kc=1;
kp=0.8;
taup=10;
Gcl=tf(kc*kp, [taup 1+kc*kp]);
pole(Gcl)
step(Gcl); % Again for unit step change in R
```

Try a proportional controller with a second order process as derived in Example 5.2 in text. This is another simple problem that we do not really need `feedback()`.

We now finish up with what we left behind in Session 4. Let's revisit **Example 4.6**. For checking our results later, we first find the poles of the closed-loop transfer function with

```
q=2*[1 10];
p=[1 11 10 2];
roots(p)               % –0.29, –0.69, and –10.02
```

Next, we define each of the transfer functions in the example:

```
G1=tf(1,[1 0]);
G2=tf(2,[1 1]);
H=tf(1,[1 10]);
```

Note that the numbering and notation areentirely arbitrary. We now generate the closed-loop transfer function, and check that it has the same closed-loop poles:

```
Gcl=feedback(G1*G2,H);
pole(Gcl)
```

And we can easily obtain a state-space representation and see that the eigenvalues of the state matrix are identical to the closed-loop poles:

```
ssm=ss(Gcl);
```

```
eig(ssm.a)
```

For fun, we can recover the closed-loop transfer function `Gcl` with:

```
tf(ssm)
```

One final check with our own derivation. We define the coefficient matrices with Eqs. (E4-23) and (E4-24) and then do the conversion:

```
a=[0 1 0; 0 -1 -2; 1 0 -10];
b=[0; 2; 0];
c=[1 0 0];
d=0;
eig(a)                        % should return the same
[q3,p3]=ss2tf(a,b,c,d,1)      % eigenvalues and transfer function
```

If this is not enough to convince you that everything is consistent, try `step()` on the transfer function and different forms of the state space model. You should see the same unit step response.

MATLAB Session 6

This tutorial is to complement our development in Chapter 7. You may want to go over the tutorial quickly before you read the text and come back later a second time for the details.

Root Locus Functions	
`rlocus`	Root locus plot
`rlocfind`	Find the closed-loop gain graphically
`sgrid`	Draw the damping and natural frequency lines
`rltool`	Launch the root locus design graphics interface

M6.1 Root locus plots

Features covered in this session:

- Root locus calculation and plots, `rlocus()`
- Frequency and damping factor grid, `sgrid()`
- Get gain of chosen closed-loop pole, `rlocfind()`

In simple terms, we want to solve for *s* in the closed-loop equation

$$1 + G_o(s) = 1 + kG(s) = 0$$

where we further write $G_o = kG(s)$, and G(s) is the ratio of two polynomials, G(s) = q(s)/p(s).

In the simplest scenario, we can think of the equation as a unity feedback system with only a proportional controller (i.e., $k = K_c$) and G(s) as the process function. We are interested in finding the roots for different values of the parameter k. We can either tabulate the results or we can plot the solutions *s* in the complex plane—the result is the root-locus plot.

Let's pick an arbitrary function such that q(s) = 1, and $p(s) = s^3 + 6s^2 + 11s + 6$. We can generate the root locus plot of the system with:

```
p=[1 6 11 6];
roots(p)                    % Check the poles
G=tf(1,p);
rlocus(G)                   % Bingo!
```

For the case where q(s) = s +1, we use

```
G=tf([1 1],p);              % Try an open-loop zero at -1
rlocus(G)                   % to cancel the open-loop pole at -1
```

MATLAB automatically selects a reasonable vector for k, calculates the roots, and plots them. The function `rlocus()` also adds the open-loop zeros and poles of G(s) to the plot.

Let's try two more examples with the following two closed-loop characteristic equations:

$$1 + K\frac{1}{(s+1)(s+3)} = 0 \quad \text{and} \quad 1 + K\frac{1}{(s+1)(s+2)(s+3)} = 0$$

```
G=zpk([],[-1 -3],1)             % The second order example
rlocus(G)

G=zpk([],[-1 -2 -3],1)          % The third order example
rlocus(G)
```

The point of the last two calculations is that a simple second order system may become extremely underdamped, but it never goes unstable.

Reminder: We supply the polynomials q(s) and p(s) in G(s), but do not lose sight that MATLAB really solves for s in the equation 1 + kq(s)/p(s) = 0.

Optional reading: In the initial learning stage, it can be a bad habit to rely on MATLAB too much. Hence the following tutorial goes the slow way in making root locus plots, which hopefully may make us more aware of how the loci relate to pole and zero positions. The first thing, of course, is to identify the open-loop poles.

```
q=[2/3 1];          % Redefine q(s) and p(s)
p=[1 6 11 6];
poles=roots(p)'     % display poles and zeros as row vectors
zeros=roots(q)'
G=tf(q,p);
k=0:0.5:100;        % define our own gains; may need
                    % 0:0.1:10 to better see the break off point
rlocus(G,k);        % MATLAB will plot the roots with '+'
```

Until we have more experience, it will take some trial-and-error to pick a good range and increment for k, but then that is the whole idea of trying it ourselves. This manual approach makes us better appreciate the placements and changes of closed-loop poles as we vary the proportional gain.[1]

We may also want to override the MATLAB default format and use little dots, like how many textbooks like to do:

```
r=rlocus(G,k);      % Save loci to array "r" first
plot(r,'.')         % Now use plot() to do the dots
hold                % hold the plot to add goodies
pzmap(G)            % pzmap() draws the open-loop poles
hold off            %         and zeros
```

Be careful to read where the loci are on the real axis because `pzmap()` traces the axis with also little dots that can be confusing.

We may want to find the ultimate gain when the loci cross the imaginary axis. Again there are many ways to do it. The easiest method is to estimate with the MATLAB function `rlocfind()`, which we will introduce next.

There are two very useful MATLAB features. First, we can overlay onto the root locus plot lines of constant damping factor and natural frequency. These lines help us pick the controller gain if the design specification is in terms of the frequency or the damping factor.

```
sgrid               % use the default grid
```

or better yet,

[1] The gain vector generated automatically by MATLAB is not always instructive if we want to observe the region close to the imaginary axis. We can use "tricks" like making two gain vectors with different increments, concatenating them and using the result in `rlocus()`. However, we should not get bogged down with fine details here. Certainly, for day-to-day routine calculations, we can omit the gain vector and let MATLAB generate it for us.

```
 sgrid(zeta,wn)        % plot only lines with given damping ratio
                       % and natural frequency
```

Example:

```
 sgrid(0.7,1)          % add the approx. 45° line for zeta=0.7 and
                       % the unit circle (frequency=1)
```

The second feature is the function `rlocfind()`, which allows us to find the gain associated with a closed-loop pole. We can enter

```
   [ck,cpole]=rlocfind(G)
```

or

```
   rlocfind(G)             % this simple form will not return
                           % the values of the closed-loop poles
```

MATLAB will wait for us to click on a point (the chosen closed-loop pole) in the root locus plot and then returns the closed-loop gain (`ck`) and the corresponding closed-loop poles (`cpole`). MATLAB does the calculation with the root locus magnitude rule, which is explained on our *Web Support*.

What if we click a point not exactly on a root locus? When we select a point s*, MATLAB calculates the value k* = –p(s*)/q(s*), which will only be a real positive number if s* satisfies the closed-loop equation. Otherwise, k* is either complex, or negative if the pole is a real number. In this case, MATLAB calculates the magnitude of k*, uses it as the gain and computes the corresponding closed-loop poles. Thus we find the chosen points are always right on the root loci no matter where we click.

We may also want to use the **`zoom`** feature of MATLAB to zoom in and out of a plot to get a better picture of, say, the break-off point of two loci. Make sure you enter "zoom off" when you are done.

M6.2 Root locus design graphics interface

Features covered in this session:

- Graphics user interface for doing root locus plots, `rltool`[2]

The control toolbox supports an extremely nice root locus design graphics design tool that is ideal for experimentation. The interface is even more intuitive and self-explanatory than that of Simulink. We take the same approach as our introduction to Simulink and have moved the not so necessary and print-intensive window display and instructions to our *Web Support*. A very brief conceptual introduction is provided in this section.

To launch the root locus design tool, enter within the MATLAB command window:

```
 rltool
```

Or if we know beforehand the open-loop transfer function to be used:

```
 rltool(G)
```

[2] The description is based on Version 4.0 of the MATLAB control toolbox. If changes are introduced in newer versions, they will be presented on our *Web Support*.

A graphics window with pull-down menus and tool buttons will pop out. Here are some pointers on the usage of the tool.

- Similar to the LTI Viewer, the root locus tool runs in its own functional space. We have to import the transfer functions under the *File* pull-down menu.
- The tool supports a very flexible block diagram as shown in Fig. M6.1. The import dialog box handles all F, G, and H. The feedback can be either positive or negative.

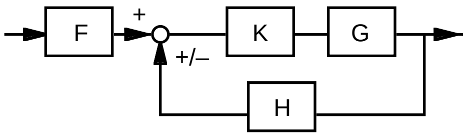


Figure M6.1. Block diagram supported by the root locus design tool.

- The default is a proportional controller, but the K block in Fig. M6.1 can easily be changed to become a PI, PD, or PID controller. The change can be accomplished in different ways. One is to retrieve the compensator-editing window by clicking on the K block or by using the *Tools* pull-down menu. The other is to use the set of arrow tools in the root locus window to add or move open-loop poles and zeros associated with the compensator.
- Once a root locus plot is generated, we can interactively change the locations of the closed-loop poles and the tool will compute the closed-loop gain K for us.
- For a given system and closed-loop gain displayed in the root locus plot, we can generate its corresponding time response (step and impulse) and frequency response (Bode, Nyquist, and Nichols) plots.

In the next section, you can use the root locus tool if you prefer, but we will do the explanation using commands. It is easier for us to get the message across with commands, and at the beginner’s learning stage, we believe entering your own command can give you a better mental imprint on the purpose of the exercise.

M6.3 Root locus plots of PID control systems

Features covered in this session:

- Make root locus plots that model situations of PID control systems

Here are some useful suggestions regarding root locus plots of control systems. In the following exercises, we consider only the simple unity feedback closed-loop characteristic equation:

$$1 + G_cG_p = 0$$

We will ignore the values of any gains. We focus only on the probable open-loop pole and zero positions introduced by a process or by a controller, or in other words, the shape of the root locus plots.

Let's begin with a **first order process** $G_p = 1/(s + 1)$. The root locus plot of a system with this simple process and a **proportional** controller, $G_c = K_c$, is generated as follows:

```
Gp=tf(1,[1  1]);                    % open-loop pole at -1
subplot(221),  rlocus(Gp)           % Gc = Kc
```

To implement an ideal **PD** controller, we'll have an additional open-loop zero. Two (of infinite) possibilities are

```
taud=2;                          % open-loop zero at -1/2
Gc=tf([taud 1],1);
subplot(222), rlocus(Gc*Gp)
```

and

```
taud=1/2;                        % open-loop zero at -2
Gc=tf([taud 1],1);
subplot(223), rlocus(Gc*Gp)
```

What are the corresponding derivative time constants? Which one would you prefer?

We next turn to a **PI** controlle. We first make a new figure and repeat proportional control for comparison:

```
figure(2)
subplot(221), rlocus(Gp)         % Gc = Kc
```

Integral control will add an open-loop pole at the origin. Again, we have two regions where we can put the open-loop zero:

```
taui=2;                          % open-loop zero at -1/2
Gc=tf([taui 1],[taui 0]);
subplot(222), rlocus(Gc*Gp)
```

and

```
taui=1/2;                        % open-loop zero at -2
Gc=tf([taui 1],[taui 0]);
subplot(223), rlocus(Gc*Gp)
```

Once again, what are the corresponding integral time constants? Which one would you prefer?

Finally, let's take a look at the probable root loci of a system with an ideal **PID** controller, which introduces one open-loop pole at the origin and two open-loop zeros. For illustration, we will not use the integral and derivative time constants explicitly, but only refer to the two zeros that the controller may introduce. We will also use `zpk()` to generate the transfer functions.

```
figure(3)
subplot(221), rlocus(Gp)         % re-do Gc = Kc

op_pole=[0];                     % open-loop pole at 0

op_zero=[-0.3 -0.8];             % both zeros less than -1
Gc=zpk(op_zero,op_pole,1);
subplot(222),rlocus(Gc*Gp)

op_zero=[-1.5 -3];               % both zeros larger than -1
Gc=zpk(op_zero,op_pole,1);
subplot(223),rlocus(Gc*Gp)
```

```
op_zero=[-0.5  -1.8];             % one zero in each region
Gc=zpk(op_zero,op_pole,1);
subplot(224),rlocus(Gc*Gp)
```

Yes, you know the question is coming. Which case would you prefer? We can use the rule of thumb that the derivative time constant is usually around one-fourth the value of the integral time constant, meaning that the zero farther away from the origin is the one associated with the derivative time constant.

Note that the system remains stable in all cases, as it should for a first or second order system. One final question: Based on the design guidelines by which the system should respond faster than the process and the system should be slightly underdamped, what are the ranges of derivative and integral time constants that you would select for the PD, PI, and PID controllers? And in what region are the desired closed-loop poles?

We'll finish with implementing the P, PI and PD controllers on a **second order overdamped process**. As in the exercise above, try to calculate the derivative or integral time constants, and take a minute to observe the plots and see what may lead to better controller designs.

Let's consider an overdamped process with two open-loop poles at –1 and –2 (time constants at 1 and 0.5 time units). A system with a proportional controller would have a root locus plot as follows. We stay with `tf()`, but you can always use `zpk()`.

```
figure(1)
p=poly([-1  -2]);                    % open-loop poles -1, -2
Gp=tf(1,p);
subplot(221),rlocus(Gp)              % proportional control
```

To implement an ideal **PD** controller, we now have three possible regions to put the zero.

```
taud=2;                              % open-loop zero at -1/2
Gc=tf([taud  1],1);
subplot(222),  rlocus(Gc*Gp)

taud=2/3;                            % open-loop zero at -1.5
Gc=tf([taud  1],1);
subplot(223),  rlocus(Gc*Gp)

taud=1/3;                            % open-loop zero at -3
Gc=tf([taud  1],1);
subplot(224),  rlocus(Gc*Gp)
```

We will put the **PI** controller plots on a new figure.

```
figure(2)
subplot(221),rlocus(Gp)              % re-do proportional control
```

The major regions for placing the zero are the same, but the interpretation as to the choice of the integral time constant is very different. We now repeat adding the open-loop zeros:

```
taui=2;                              % open-loop zero at -1/2
Gc=tf([taui  1],[taui  0]);
```

```
subplot(222), rlocus(Gc*Gp)

taui=2/3;                               % open-loop zero at -1.5
Gc=tf([taui 1],[taui 0]);
subplot(223), rlocus(Gc*Gp)

taui=1/3;                               % open-loop zero at -3
Gc=tf([taui 1],[taui 0]);
subplot(224), rlocus(Gc*Gp)
```

You may want to try some sample calculations using a PID controller. One way of thinking: we need to add a second open-loop zero. We can limit the number of cases if we assume that the value of the derivative time constant is usually smaller than the integral time constant.

MATLAB Session 7

This tutorial is to complement our development in Chapter 8. You may want to go over the tutorial quickly before you read the text and come back later a second time for the details.

Frequency Response Functions	
bode	Bode plots
`freqresp`	Frequency response of a transfer function
`logspace`	Logarithmically spaced vector
`margin`	Gain margin and crossover frequency interpolation
`nichols, ngrid`	Nichols plots
`nyquist`	Nyquist plots
`sisotool`	Launch the SISO system design graphics interface

M7.1 Nyquist and Nichols Plots

Features covered in this session:

- Nyquist plots, `nyquist()`

We will simply state that the SISO system design tool `sisotool`, as explained in Session 6, can be used to do frequency response plots. Now, we want to use the default view, so we just need to enter:

```
sisotool
```

Hints to make better use of the design tool are on our *Web Support*. We use commands here because they give us a better idea behind the calculations. We shall keep this section brief since our main tool will be Bode plots, which will be explained in the next section.

Let say we have a simple open-loop transfer function G_o of the closed-loop characteristic equation

$$1 + G_o = 0,$$

and we want to find the proportional gain which will give us an unstable system. For this simple exercise, we take $G_o(s) = KG(s)$.

```
p=poly([-1; -2; -3]);      % Open-loop poles at -1, -2, -3
G=tf(10,p);                % Arbitrary value K=10
nyquist(G);                % Bingo!
```

We'll see two curves. By default, MATLAB maps and plots also the image of the negative Im-axis. That can make the plot too busy and confusing, at least for a beginner. So we'll stay away from the default in the following exercises.

```
[re,im]=nyquist(G);
plot(re(1,:),im(1,:))      % Only the positive Im-axis image
```

Of course, we can define our own frequency vector.

```
w=logspace(-1,1);        % Generate numbers between [10^-1, 10^1]
[re,im]=nyquist(G,w);
plot(re(1,:),im(1,:))
```

The function `logspace()` generates a vector with numbers nicely spaced on the logarithmic scale. Its use is optional. The default of the function gives 50 points and is usually adequate. For a smoother curve, use more points. For example, this command will use 150 points: `logspace(-1,1,150)`.

```
hold   % to add the (-1,0) point and the axes on the plot
x=-1; y=0;
xh=[-2 2]; yh=[0 0];       % the axes
xv=[0 0];    yv=[-2 1];
plot(x,y,'o',xh,yh,'-',xv,yv,'-')
```

We can increase the gain K and repeat the calculation with, for example, two more trials:[1]

```
G=tf(50,p);                % try again
[re,im]=nyquist(G,w);
plot(re(1,:),im(1,:))

G=tf(60,p);                % and again
[re,im]=nyquist(G,w);
plot(re(1,:),im(1,:))
hold off
```

We do not use **Nichols plot** (log magnitude versus phase) much anymore, but it is nice to know that we can do it just as easily:

```
p=poly([-1; -2; -3]);
G=tf(10,p);
nichols(G)
ngrid
zoom        % need to zoom into the meaningful region
```

The plot with default settings is quite useless unless we use **`ngrid`** to superimpose the closed-loop gain and phase grid lines. Instead of zooming in, we can reset the axes with:

```
axis([-360 0 -40 20])
```

M7.2 Magnitude and Phase Angle (Bode) Plots

Features covered in this session:

- Bode plot calculation, `bode()`
- Find the gain and phase margins, `margin()`
- Bode plots for transfer functions with dead time

We begin with one simple example. Let's say we want to analyze the closed-loop characteristic equation

$$1+\frac{1}{s^2 + 0.4s + 1} = 0$$

We generate the Bode plot with:

[1] All functions like `nyquist()`, `bode()`, etc., can take on multiple LTI objects, as in
`nyquist(G1,G2,G3)`
but only when we do not use left-hand side arguments.

```
G=tf(1,[1 0.4 1]);
bode(G)                    % Done!
```

The MATLAB default plot is perfect! That is except when we may not want dB as the unit for the magnitude. We have two options. One, learn to live with dB, the convention in the control industry. Or two, we do our own plots. This is a task that we need to know when we analyze systems with dead time. This is how we can generate our own plots:

```
w=logspace(-1,1);
[mag,phase]=bode(G,w);
mag=mag(1,:);          % required since MATLAB v.5
phase=phase(1,:);

subplot(211),  loglog(w,mag)
               ylabel('Magnitude'), grid
subplot(212),  semilogx(w,phase)
               ylabel('Phase, deg'), grid
               xlabel('Frequency (rad/time)')
```

As an option, we can omit the subplot command and put the magnitude and phase plots in individual figures.

This is how we can make a Bode plot with dB as the scale for the magnitude.

```
dB=20*log10(mag);          % converts mag to dB units
```

Now we do the plotting. Note that dB is already a logarithmic scale.

```
subplot(211),  semilogx(w,dB)    % Use semilogx for dB
               ylabel('Magnitude (dB)')
               grid
subplot(212),  semilogx(w,phase)
               ylabel('Phase angle (degree)')
               xlabel('Frequency, w')
               grid
```

We most often use radian/s as the unit for frequency. In case cycle/s or Hz is needed, the conversion is

```
f=w/(2*pi);              % Converts w [rad/s] to [Hz]
```

After using the `subplot()` command and before doing any other plots, we should have a habit to reset the window with

```
clf                      % clear figure
```

We now find the **gain margin** with its crossover frequency (`Gm, Wcg`), and **phase margin** with its crossover frequency (`Pm, Wcp`) with either one of the following options:

```
[Gm,Pm, Wcg,Wcp]=margin(mag,phase,w)     % option 1
```

where `mag` and `phase` are calculated with the function `bode()` beforehand. We can skip the `bode()` step and use the transfer function directly as the argument:

```
[Gm,Pm, Wcg,Wcp]=margin(G)               % option 2
```

or simply,

```
margin(G)                                % option 3, Gm in dB
```

In the last option without any left-hand side arguments, MATLAB will do the Bode plot, mark the gain margins with vertical lines, and display the margin calculations on the plot.

Two important comments:

1. With `G=tf(1,[1 0.4 1])`, *i.e.*, a simple second order system, it is always stable. The gain margin calculation is meaningless. Nevertheless, MATLAB returns a set of results anyway. Again, a computer is not foolproof. All `margin()` does is an interpolation calculation.

2. If you use option 1 or 2 above, `margin()` returns the "linear scale" gain margin in the variable `Gm`. With option 3, however, the gain margin displayed in the plot is in the unit of dB. You need to convert it back with $10^{dB/20}$.

To handle **dead time**, all we need is a simple modification using the fact that the time delay transfer function has magnitude 1 and phase angle $-t_d\omega$. We need one single statement to "tag on" the lag due to dead time, and do it after the bode`()` function call.

So let's start with the second order function which is always stable:

```
G=tf(1,[1  0.4  1]);
freq=logspace(-1,1);          % freq is in radian/time
[mag,phase]=bode(G,freq);
mag=mag(1,:);
phase=phase(1,:);
```

Now let's say we also have dead time:

```
tdead=0.2;    % [time unit]
```

The following statement is the only addition needed to introduce the phase lag due to dead time:

```
phase = phase - ((180/pi)*tdead*freq);    % phase is in degrees
```

We can now proceed with the plotting and phase/gain margin interpolation:

```
subplot(211),   loglog(freq,mag)
                ylabel('Magnitude'),title('Bode  Plot')
                grid
subplot(212),   semilogx(freq,phase)
                ylabel('Phase  (degree)'),xlabel('Frequency')
                grid

% now using new phase variable that includes dead time phase lag
[Gm,Pm,Wcg,Wcp]=margin(mag,phase,freq)
```

The whole idea of handling dead time applies to other types of frequency-domain plots, but the Bode plot is the easiest to learn from.

There is no magic in the functions `nyquist()` or `bode()`. We could have done all our calculations using the more basic **`freqresp()`** function. What it does is essentially making the s=jω substitution numerically in a given transfer function G(s). A sample usage is

```
w=logspace(-1,1);
gjw=freqresp(G,w); %does the s=jw calculation for each value in w
```

After that, we can use the result to do frequency response analysis. If you are interested in the details, they are provided in the Session 7 Supplement on our *Web Support*.