EE 407
Homework 3 Solutions

1 Answers

1. **Non-interacting Tanks** Consider the non-interacting tanks system depicted in Fig. 1. Let $A_k(m^2)$ denote the cross-sectional area of tank k, $q_k(m^3/min)$ and $q_{k-1}(m^3/min)$ its infow and outflow, respectively, and $R_k(min/m^2)$ its outlow valve resistance. Assuming that the outflow is linearly dependent to tank water height, $q_k = h_k/R_k$. Assume also that the actuating valve and the level transmitter are ideal; that is, level transmitter can give instantaneous measurements and actuator can supply the desired water flow without any limits or delay.

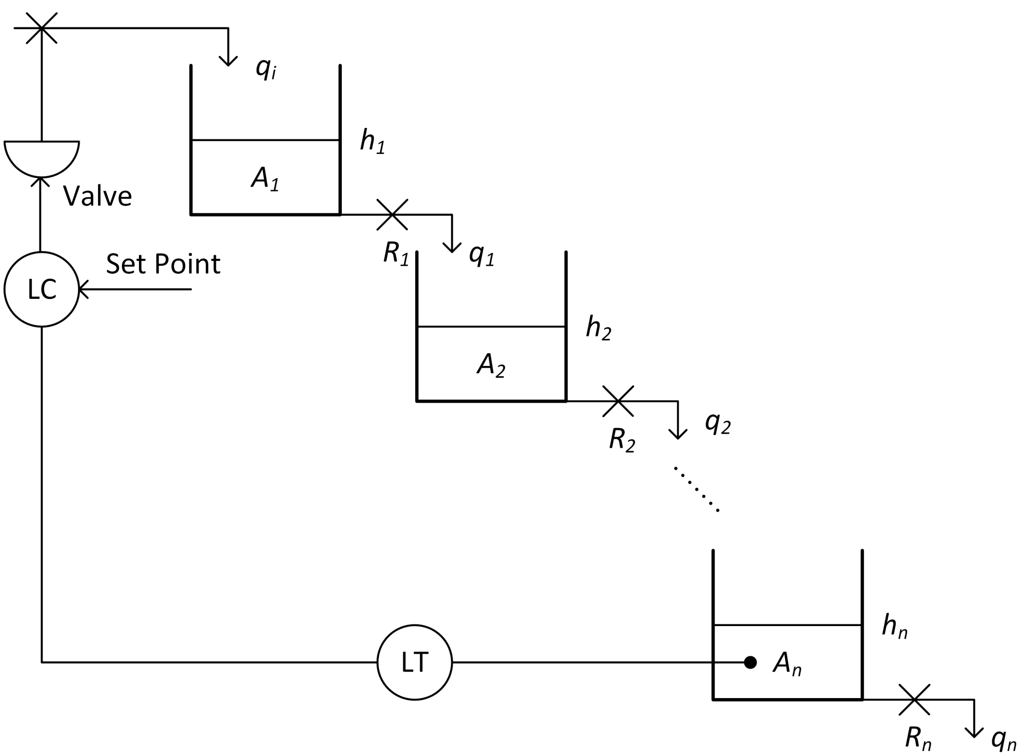


Figure 1

(a) Write down the differential equations satisfied by water heights of all tanks and obtain the state space representation of this system in the following form $\dot{\mathbf{h}} = \mathbf{A}\mathbf{h} + \mathbf{B}q_i$, where $\mathbf{h} = [h_1, h_2, \ldots, h_n]^T$.

Starting with *Volumetric change = inflow - outflow*, the differential equations governing this system can be found as follows.

$$
\begin{aligned}
A_1\dot{h_1} &= q_i - q_1 = q_i - \frac{h_1}{R_1} \implies \dot{h_1} = \frac{1}{A_1}(-\frac{h_1}{R_1} + q_i)\\
A_2\dot{h_2} &= q_1 - q_2 = \frac{h_1}{R_1} - \frac{h_2}{R_2} \implies \dot{h_2} = \frac{1}{A_2}(\frac{h_1}{R_1} - \frac{h_2}{R_2})\\
&\vdots\\
A_n\dot{h_n} &= q_{n-1} - q_n = \frac{h_{n-1}}{R_{n-1}} - \frac{h_n}{R_n} \implies \dot{h_n} = \frac{1}{A_n}(\frac{h_{n-1}}{R_{n-1}} - \frac{h_n}{R_n})
\end{aligned} \tag{1}
$$

From this, open loop SS representation can easily be obtained as

$$
\begin{aligned}
\dot{\mathbf{h}} &= \begin{bmatrix} \dot{h_1} \\ h_2 \\ \vdots \\ h_n \end{bmatrix} = \begin{bmatrix} -\frac{1}{R_1A_1} & 0 & 0 & \dots & 0 & 0 \\ \frac{1}{R_1A_2} & -\frac{1}{R_2A_2} & 0 & \dots & 0 & 0 \\ 0 & \frac{1}{R_2A_3} & -\frac{1}{R_3A_3} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \frac{1}{A_nR_{n-1}} & -\frac{1}{A_nR_n} \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \\ h_3 \\ \vdots \\ h_n \end{bmatrix} + \begin{bmatrix} \frac{1}{A_1} \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} q_i \\
y &= \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix} \mathbf{h}
\end{aligned} \tag{2}
$$

(b) Find the transfer function, $G(s) = H_n(s)/Q_i(s)$, of the system. If you observe the stucture of state space representation (or the governing differential equations), each state is dependent only on itself and its previous state. Thus, starting from $G_1(s) = H_1(s)/Q_i(s)$ and proceeding sequentially may ease up the calculations.

Let's start according to the recommendation and introduce intermediate transfer function $G_1'(s) = Q_1(s)/Q_i(s)$.

$$
\begin{aligned}
G_1(s) &= \frac{H_1(s)}{Q_i(s)} = \frac{R_1}{R_1C_1s+1} \implies G_1'(s) = \frac{1}{R_1A_1s+1}\\
G_2(s) &= \frac{H_2(s)}{Q_i(s)} = \frac{H_2(s)}{Q_1(s)}\frac{Q_1(s)}{Q_i(s)} = \frac{R_2}{R_2C_2s+1}\frac{1}{R_1C_1s+1} \implies G_2'(s) = \frac{1}{(R_1A_1s+1)(R_2A_2s+1)}\\
&\vdots\\
G_n(s) &= \frac{H_2(s)}{Q_i(s)} = \frac{H_n(s)}{Q_{n-1}(s)}\prod_{i=1}^{n-1} G_i'(s) \implies G(s) = R_n \prod_{i=1}^{n} \frac{1}{R_iA_is+1}
\end{aligned} \tag{3}
$$

SS representation of this system has a nice structure, allowing us to easily use the following alternative as well. In the adjoint matrix, entries marked as x are not important as they would be canceled at the multiplication step. The last step while finding $G(s)$ involves plugging the adjoint and determinant values into the first line, and performing simple algebraic manipulations.

$$G(s) = C(s\mathcal{I} - A)^{-1}B = \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \end{bmatrix} \frac{\operatorname{adj}(s\mathcal{I} - A)}{\det(s\mathcal{I} - A)} \begin{bmatrix} \frac{1}{A_1} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$s\mathcal{I} - A = \begin{bmatrix} s + \frac{1}{R_1A_1} & 0 & 0 & \dots & 0 & 0 \\ -\frac{1}{R_1A_2} & s + \frac{1}{R_2A_2} & 0 & \dots & 0 & 0 \\ 0 & -\frac{1}{R_2A_3} & s + \frac{1}{R_3A_3} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -\frac{1}{A_nR_{n-1}} & s + \frac{1}{A_nR_n} \end{bmatrix}$$

$$\implies \det(sI - A) = \prod_{i=1}^{n}(s + \frac{1}{R_iA_i}) = \prod_{i=1}^{n}(\frac{sR_iA_i + 1}{R_iA_i})$$

$$\& \operatorname{adj}(s\mathcal{I} - A) = \begin{bmatrix} x & x & \dots & x \\ \vdots & \vdots & \ddots & \dots \\ \operatorname{adj}(s\mathcal{I} - A)_{n,1} & x & \dots & x \end{bmatrix} \tag{4}$$

$$\text{where} \operatorname{adj}(s\mathcal{I} - A)_{n,1} = \operatorname{cof}(s\mathcal{I} - A)_{1,n}$$

$$= (-1)^{(n+1)} \begin{vmatrix} -\frac{1}{R_1A_2} & s + \frac{1}{R_2A_2} & 0 & \dots & 0 \\ 0 & -\frac{1}{R_2A_3} & s + \frac{1}{R_3A_3} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & -\frac{1}{A_nR_{n-1}} \end{vmatrix}$$

$$= (-1)^{(n+1)} \prod_{i=1}^{n-1}(-\frac{1}{R_iA_{i+1}}) = \prod_{i=1}^{n-1} \frac{1}{R_iA_{i+1}}$$

$$\implies G(s) = R_n \prod_{i=1}^{n} \frac{1}{R_iA_is + 1}$$

(c) If you were to modify the previous transfer function as $G'(s) = Q_n(s)/Q_i(s)$, what would be its DC gain? Does its DC gain make sense on physical grounds and why (not)?

With that modification, we have the transfer function given in equation 5. Now, knowing that the DC gain of a plant is its gain at $s = 0$, that gain is found to be equal to 1 (note that each tank's modified transfer function also has this gain). This is as expected on physical grounds as well. Each tank is a self regulating process; that is, when its input is a unit step, it should settle down at a certain operating point. After that point is reached there is no accumulation in the process by the definition of being in a steady state, which, in turn, implies that inflow must be equal to the outflow.

$$G'(s) = \prod_{i=1}^{n} \frac{1}{R_iA_is + 1} \tag{5}$$

(d) In Ziegler-Nichols method, the idea is to control the system under proportional-only control and increase its gain slowly until sustaining oscillations are observed at the output. The system has to be open loop stable (as it is the case in our self regulating process) and one has to be able to push the system to critical stability. Now, suppose that $n = 2$ and note that all of the constants are positive. Show that continuous cycling method cannot be applied to this system (with transfer function $G(s)$) using:

- an approximate root locus plot and
- an approximate bode plot. (**Hint:** At critical stability, a system has zero phase margin.)

Open loop poles of the system are at $s = -\frac{1}{\tau_1}$ and $s = -\frac{1}{\tau_2}$ and the real axis root location is in between these two. Also, as the # of poles - # of zeros is equal to 2, there will be two asymptotes at the angles of off multiples of $90°/2$: $90°$ and $270°$. The intersection point of these asymptotes is $\frac{\sum \text{open loop poles} - \sum \text{open loop zeros}}{\text{\# of asymptotes}} = -\frac{1}{2}(\frac{1}{\tau_1} + \frac{1}{\tau_2})$. There is no need to calculate arrival or departure angles as all the poles (and, if any, zeros) are completely real. Break-away point is calculated below, which is coincident with the intersection point of the asymptotes. An approximate plot for this case is provided in Fig. 2. Process gain is assumed to be unity; this is a valid assumption as if it were not, it would only change the controller gain at which the corresponding closed loop root loci would be obtained without changing the behavior of the pole locations.

$$\begin{aligned} 1 + K\frac{n(s)}{d(s)} = 0 &\implies K = -\frac{d(s)}{n(s)} = -(\tau_1 s + 1)(\tau_2 s + 1) \\ \frac{dK}{ds} = 0 &\implies -\tau_1(\tau_2 s + 1) - \tau_2(\tau_1 s + 1) = 0 \implies s = -\frac{1}{2}(\frac{1}{\tau_1} + \frac{1}{\tau_2}) \end{aligned} \tag{6}$$

Regarding the bode plot part, magnitude response of this system would show a DC gain of $20\log(K_p)$ around $\omega = 0$. As you increase the angular frequency, it would start a decrease of $20dB/dec$ as each pole is reached by ω. Similarly, phase response would show an approximate drop of $45°/dec$ between $\omega = \frac{1}{10\tau_1}$ and $\omega = \frac{10}{\tau_1}$ (and the same argument goes for τ_2). However, the phase response would never reach the $-180°$ line (it reaches there at $\omega = \infty$ as it is shown below).That is why, no matter what the controller gain is, you cannot make your system critically stable. Therefore, the gain margin is $K = \infty$. An approximate plot is provided in Fig. 3.

$$\begin{aligned} \angle(H(j\omega)) &= \angle\frac{1}{j\tau_1\omega + 1} + \angle\frac{1}{j\tau_2\omega + 1} = -(\tan^{-1}(\tau_1\omega) + \tan^{-1}(\tau_1\omega)) = \pi \\ &\text{Having } -\pi/2 \leq \tan^{-1}(.) \leq \pi/2 \\ \tan^{-1}\tau_1\omega &= \pi/2 \implies \omega = \frac{\tan(\pi/2)}{\tau_1} = \infty \end{aligned} \tag{7}$$

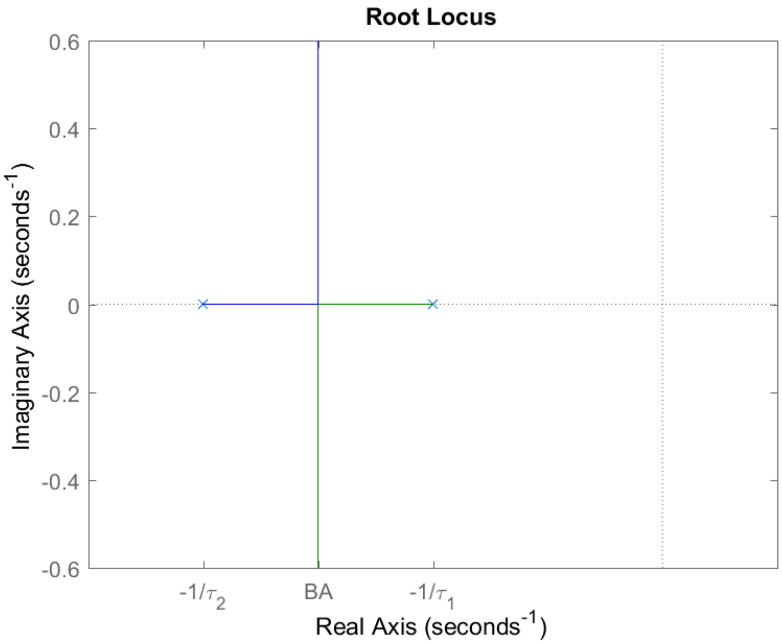


Figure 2

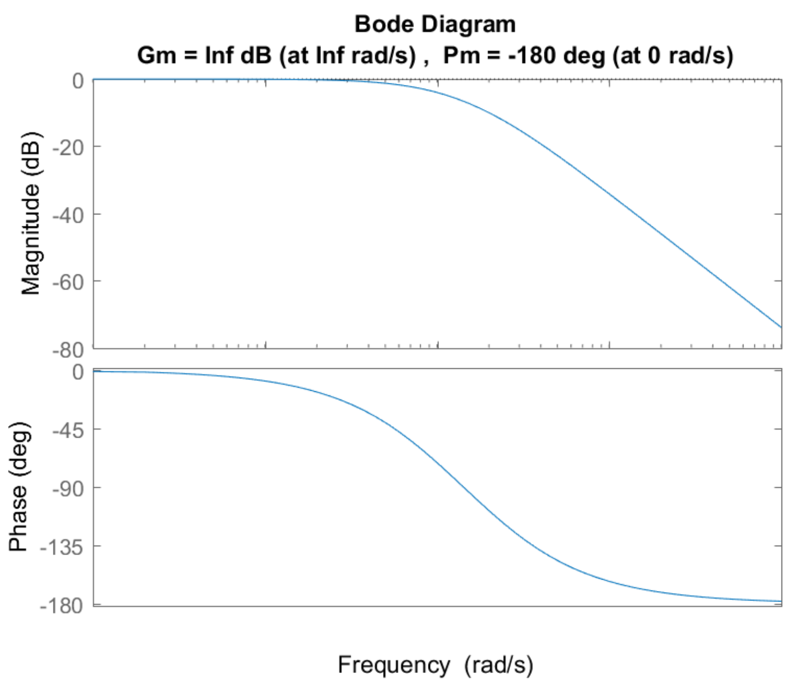


Figure 3

(e) Now, let $n = 3$. This time you have a system which can be made unstable/critically stable. Also let $\tau_i = R_i A_i = 1(min)$ and $R_i = 1(min/m^2)$ for $i = 1, 2, 3$. Find the value of gain K_c (critical/ultimate gain), using both of the methods of step 1d, to make the system critically stable. Also find the angular frequencies of oscillation, $\omega_{cu}(rad/sec)$. Simulate the system to verify and report your result. Note that slight numerical inaccuracy may make the system unstable; you may have to decrease it slightly to have a bounded response. In practical systems, this is of no concern as the actuator output is limited unlike your simulation, and you do not even need a model of the system to find these parameters.

Three asymptotes would exist for root locus at the odd multiples of $180°/(3-0) = 60°$: $60°$, $180°$ and $270°$. The two poles are already at the break-away point and one of them would follow go towards $-\infty$. Critical gain can be obtained either using Routh-Hurwitz or by finding the purely imaginary roots of the denominator.

Before proceeding any further, note that this problem has two equivalent solutions. One of them is obtained by letting $G(s) = \frac{1}{(s+1)^3}$ and treating every result of Simulink/MATLAB as well as your calculations with the unit of *min* instead of *sec*. Although you can change the default time unit in MATLAB, it is not possible to change the units in Simulink models. Equivalently, you can let $G(s) = \frac{1}{(60s+1)^3}$ and proceed without the need of the aforementioned treatment. Both solutions are acceptable and are presented on the next page; both root loci are provided in Fig. 4.

From Routh-Hurwitz, after rearranging the closed loop denominator as $d_{cl}(s) = s^3 + 3s^2 + 3s + 1 + K$,

$$\begin{array}{c|cc} s^3 & 1 & 3 \\ s^2 & 3 & 1+K \\ s^1 & \frac{9-1-K}{3} & \\ s^0 & 1+K & \end{array}$$

Noting that s^1 line would be a zero line for $K = 8(m^2/min)$ (and sign change would occur if K is increased further), the critical gain is $K_c = 8(m^2/min)$. At that gain, solving the above line (s^2 line, which becomes a factor of the closed loop denominator polynomial as the line below it is a zero line), $3s^2 + 9s^0 = 0 \implies \omega_{cu} = \sqrt{3}(rad/min) = \frac{\sqrt{3}}{60}(rad/sec)$, from which the period of oscillations is found as $P_{cu} = \frac{2\pi}{\sqrt{3}}(min) = \frac{120\pi}{\sqrt{3}}(sec)$.

An alternative approach would be solving the purely imaginary roots of the closed loop denominator polynomial. By letting $s = j\omega$, we have the following. Note that $\omega = 0$ is not on the root locus for $K > 0$; hence, it is ignored in the solution.

$$\begin{aligned} & -j\omega^3 - 3\omega^2 + 3j\omega + 1 + K = 0 \\ & \implies j\omega(3 - \omega^2) = 0 \ \&\ 1 + K - 3\omega^2 = 0 \\ & \omega = \omega_{cu} = \sqrt{3}(rad/min) \text{ and } K_c = 8(m^2/min) \end{aligned} \tag{8}$$

From Routh-Hurwitz, after rearranging the closed loop denominator as $d_{cl}(s) = (60s)^3 + 3(60s)^2 + 3(60s) + 1 + K$,

$$\begin{array}{c|cc} s^3 & 60^3 & 3 \cdot 60 \\ s^2 & 3 \cdot 60^2 & 1+K \\ s^1 & 60^3 \frac{9-1-K}{3 \cdot 60^2} & \\ s^0 & 1+K & \end{array}$$

Noting that s^1 line would be a zero line for $K = 8(m^2/min)$ (and sign change would occur if K is increased further), the critical gain is $K_c = 8(m^2/min)$. At that gain, solving the above line (s^2 line, which becomes a factor of the closed loop denominator polynomial as the line below it is a zero line), $3(60s)^2 + 9s^0 = 0 \implies \omega_{cu} = \frac{\sqrt{3}}{60}(rad/sec)$, from which the period of oscillations is found as $P_{cu} = \frac{120\pi}{\sqrt{3}}(sec)$.

An alternative approach would be solving the purely imaginary roots of the closed loop denominator polynomial. By letting $s = j\omega$, we have the following. Note that $\omega = 0$ is not on the root locus for $K > 0$; hence, it is ignored in the solution.

$$\begin{aligned} & -j(60\omega)^3 - 3(60\omega)^2 + 3j(60\omega) + 1 + K = 0 \\ & \implies j\omega(3 - (60\omega)^2) = 0 \ \&\ 1 + K - 3(60\omega)^2 = 0 \\ & \omega = \omega_{cu} = \frac{\sqrt{3}}{60}(rad/sec) \text{ and } K_c = 8(m^2/min) \end{aligned} \tag{9}$$

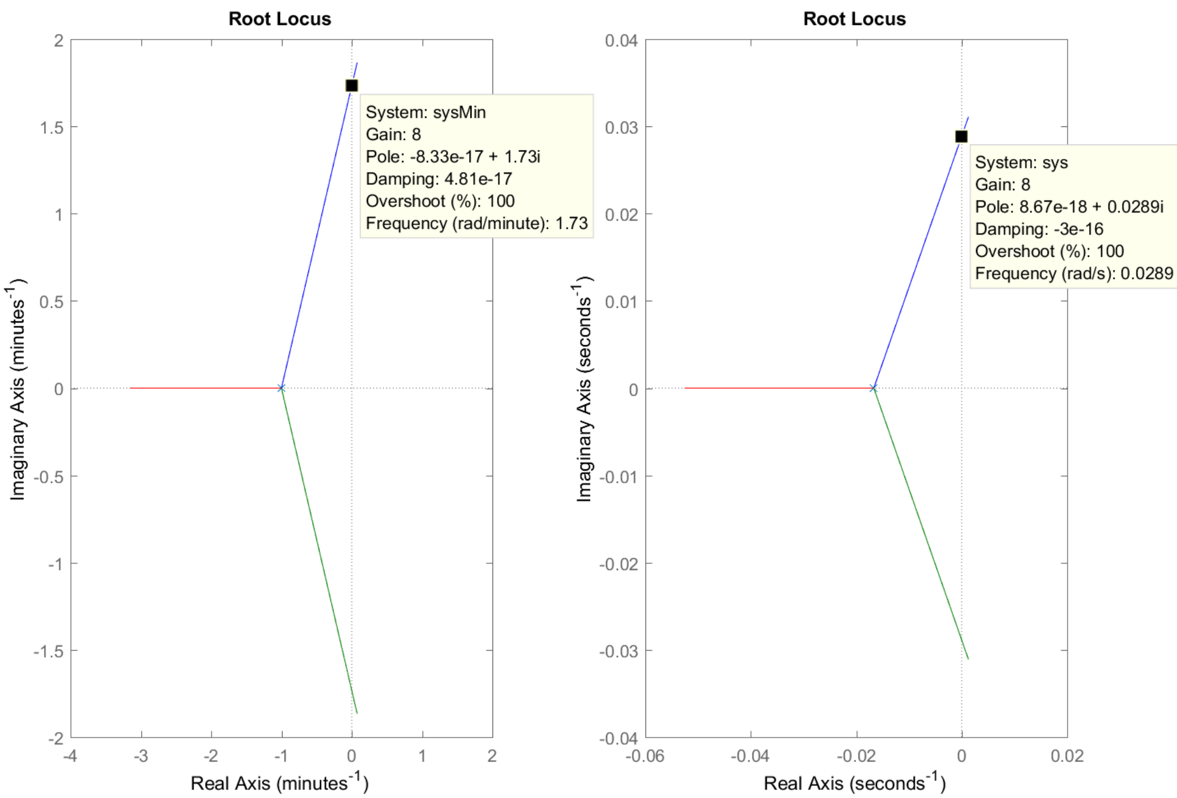


Figure 4

Bode plot of the system is shown in Fig. 4. The frequency of oscillations is equal to the angular frequency where $\angle(H(j\omega)) = -180°$ if the controller provides proportional gain of $-20\log|H(j\omega_{cu})|$. Reading from the plots, $\omega_{cu} = 1.73(rad/min) = 0.0289(rad/sec)$. Controller gain is then 18.1 dB, or equivalently, $10^{\frac{18.1}{20}} \cong 8.03(m^2/min)$ which verifies the previous results.

(f) From your simulation result, determine the so-called critical/ultimate period P_u (period of oscillations under critical gain). Using K_{cu} and P_u, find the controller parameters for a suitable PID controller.

The step response of the critically stable system is given in Fig. 6. You can read the period of oscillations approximately as $3.7min \cong 217sec$, which verifies the results from the previous part. Using the following formula, and the exact values from the previous part, K, T_i and T_d can be found as:

$$
\begin{aligned}
K &= 0.6 \cdot K_c = 4.8(m^2/min) \\
T_i &= P_{cu}/2 = 108.82(sec) = 1.814(min) \qquad (10) \\
T_d &= P_{cu}/8 = 27.21(sec) = 0.4534(min)
\end{aligned}
$$

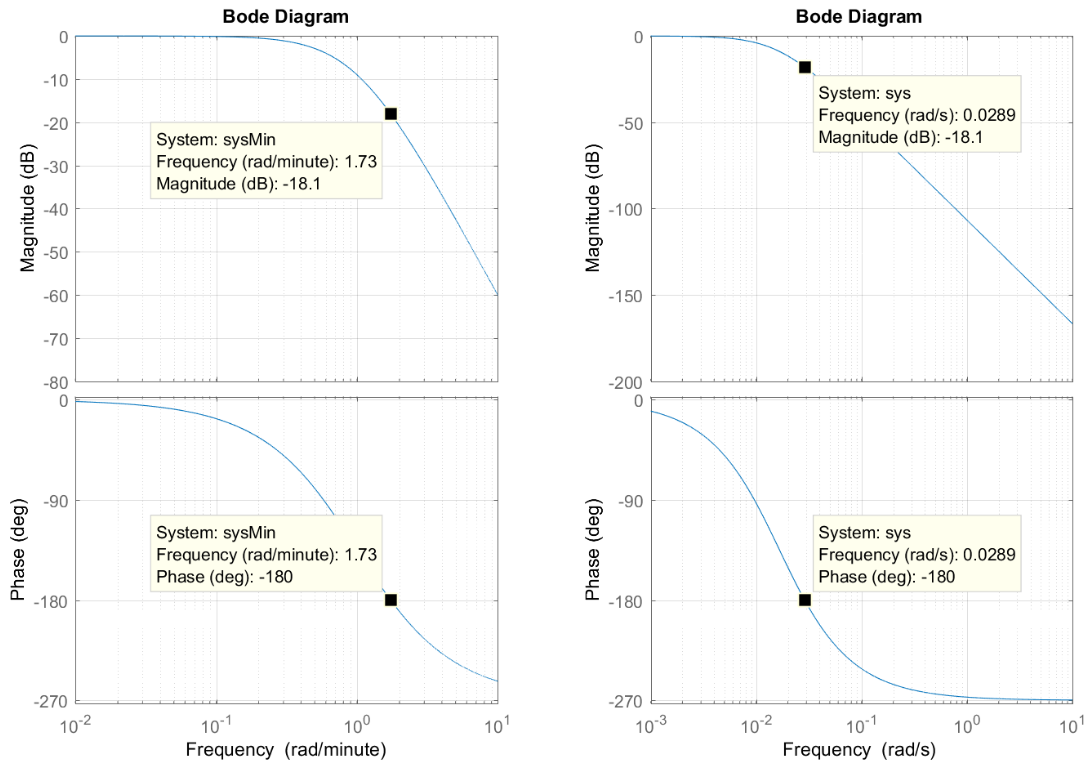


Figure 5

(g) Simulate the system and find the decay ratio (decay ratio is explained in the video Introduction to Proportional Integral Control).

Simulation result can be found in Fig. 7. Decay ratio is $\frac{0.07}{0.41} \cong 0.171$, which is an approximate result to quarter decay ratio (0.25) of Ziegler-Nichols method that was originally designed to be used with FOPDT plants (although you don't need this to be the case).

(h) Commit your simulation model. In this model, you are required to show both your simulation results (of steps 1e and 1g).

Model results are provided at the previous steps.

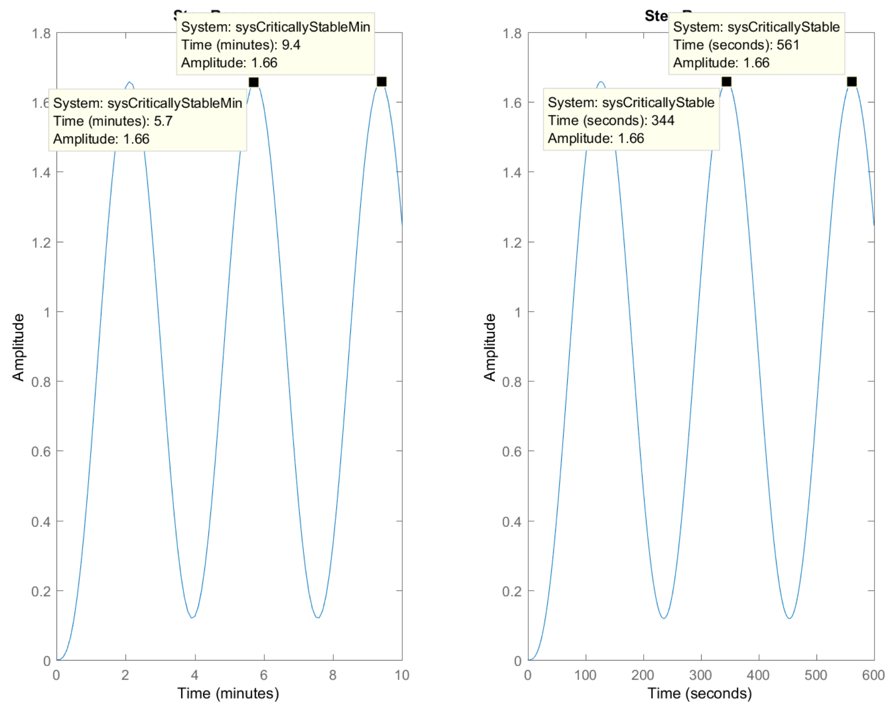


Figure 6

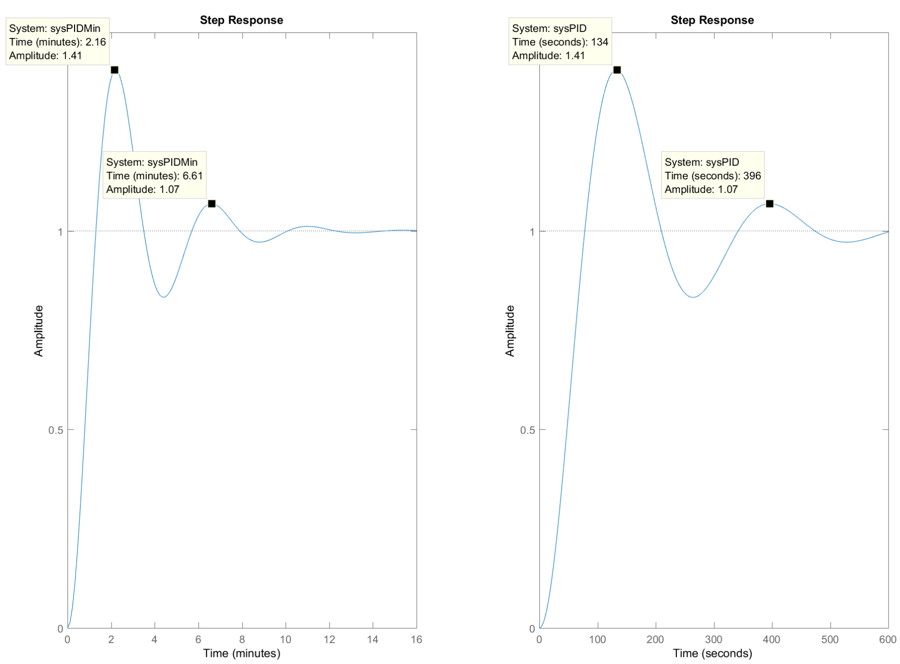


Figure 7

2. **Heating of Interacting Masses**

Consider the heating process of the masses shown in Fig. 8 (which is the setup of experiment #2). Let m_1 and c_1 denote the mass and heat capacity of the wirewound resistor (outer mass), and m_2 and c_2 be those of the thermocouple (inner mass). h_i and A_i are the heat transfer coefficient and the area of inner contact, and h_o and A_o are those of the outer one. Assume that each body is at a uniform temperature and heat is transferred only by convection. q_{in} is the power supplied to the resistor (by the means of Joule heating, ohmic heating or resistive heating) and $q_{out} = h_o A_o (T_1 - T_e)$ is the power lost to the environment. A similar equation of heat transfer governs the heat transfer between the masses, which is left for you to find at the first step.

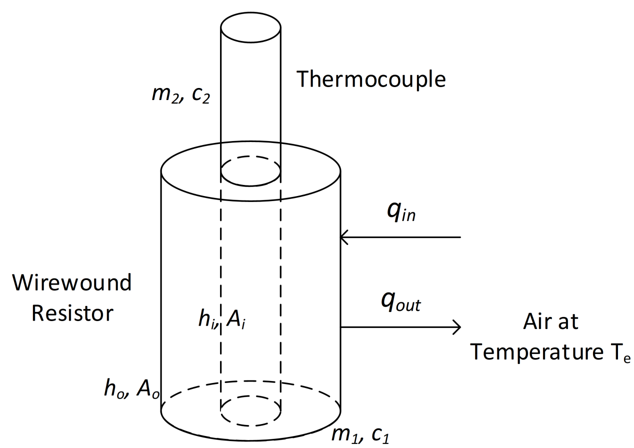


Figure 8

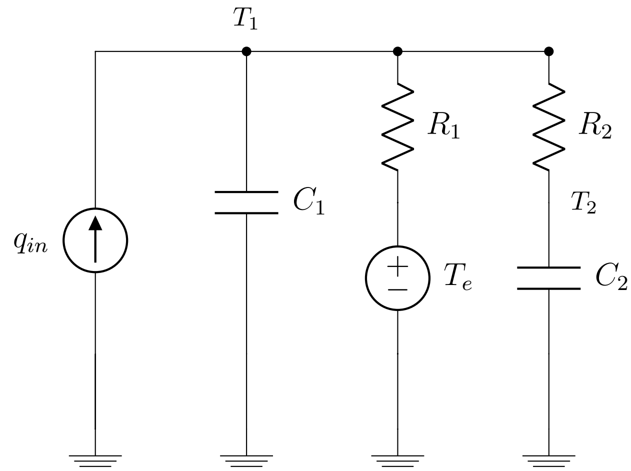
(a) Find the differential equations governing the temperatures of each mass.

Before writing the differential equations, let's consider how each of these parameters would affect the speed of the process. By doing so, the goal is to get you some insights instead of pure memorization of the formulas. As m_1 and c_1 increase, we would need to input more power into mass 1 before seeing a temperature rise, indicating more heat capacity. Also, as h_o (heat transfer coefficient, or electrical equivalent of conductance) and A_o (consider a heat sink where we intentionally increase the area to allow more heat flow) increase, we permit the system to have more heat flow between mass 1 and the environment. With these in mind as well as the basic principle of conservation as accumulation=inflow-outflow, the governing differential equation can be written as:

$$\begin{aligned} m_1 c_1 \frac{\partial T_1}{\partial t} &= h_i A_i (T_2 - T_1) + h_o A_o (T_e - T_1) + q_{in} \\ m_2 c_2 \frac{\partial T_2}{\partial t} &= h_i A_i (T_1 - T_2) \end{aligned} \tag{11}$$

(b) Find the electrical equivalent circuit representing this system. Find the values of circuit elements $R_1, \ldots, C_1, \ldots, V_1, \ldots$ in terms of the given quantities.

After the discussion at the previous step, and after observing the DEs, the electrical equivalent can easily be obtained as:



where $C_1 = m_1c_1$, $R_1 = \frac{1}{h_oA_o}$, $C_2 = m_2c_2$ and $R_2 = \frac{1}{h_iA_i}$. Also, $i_{in} = q_{in}$, $v_1 = T_1$ and $v_e = T_e$ if you have decided to label them so. The reason why this is called an interacting process is that a raise in T_2 should cause a raise in T_1, which is the obvious result of the main system and its electrical equivalent.

(c) Knowing that you can superpose the effect of independent sources in an electrical circuit, kill the active circuit element corresponding to the environmental temperature (set its value to 0). Find the transfer function $G(s) = T_1(s)/Q(s)$ in terms of the circuit variables you introduced.

Now let $I_1(s)$, $I_2(s)$ and $I_3(s)$ be the Laplace transforms of currents passing through C_1, R_1 and R_2 pointing downwards. Then,

$$
\begin{aligned}
&I_1 \propto sC_1, I_2 \propto \frac{1}{R_1}, I_3 \propto \frac{sC_2}{sC_2R_2+1} \text{ and } I_1 + I_2 + I_3 = Q_{in} \\
&a(sC_1 + \frac{1}{R_1} + \frac{sC_2}{sR_2C_2+1}) = Q_{in} \\
&\implies a = \frac{R_1(sR_2C_2+1)}{s^2R_1C_1R_2C_2 + s(R_1C_1 + R_2C_2 + R_1C_2) + 1}Q_{in} = \frac{R_1(s\tau_2+1)}{s^2\tau_1\tau_2 + s(\tau_1+\tau_2+\tau_{12}) + 1}Q_{in} \\
&\implies T_1 = \frac{I_1}{sC_1} = \frac{a \cdot sC_1}{sC_1} = a \\
&\implies G(s) = \frac{R_1(s\tau_2+1)}{s^2\tau_1\tau_2 + s(\tau_1+\tau_2+\tau_{12}) + 1}
\end{aligned} \tag{12}
$$

This is a meaningful result on physical grounds as well. This is a self-regulating process, meaning that any step input should result in a finite output value. When the system reaches that steady state, the final temperature is limited by the power loss to the environment and that is controlled only by the heat resistance R_1 (so is the DC gain).

(d) The transfer function you found in the previous step should be of the following form. Suppose that the system parameters are as follows. Approximate the system as a FOPDT system.

$$G(s) = \frac{K_p(\tau_1 s + 1)}{(\tau^2 s^2 + 2\xi\tau s + 1)} = \frac{0.5(10s+1)}{2000s^2 + 410s + 1}(°\mathrm{C/W})$$

Open loop step response is given in 9. Reading the following values as $t_{1/3} = 160(sec)$ and $t_{2/3} = 438(sec)$, and $K_p = 0.5(°\mathrm{C/W})$, we have

$$\begin{aligned} T_p &= \frac{1}{0.7}(t_{2/3} - t_{1/3}) = 397.14(sec) \\ \theta_p &= 1.1429(sec) \end{aligned} \tag{13}$$

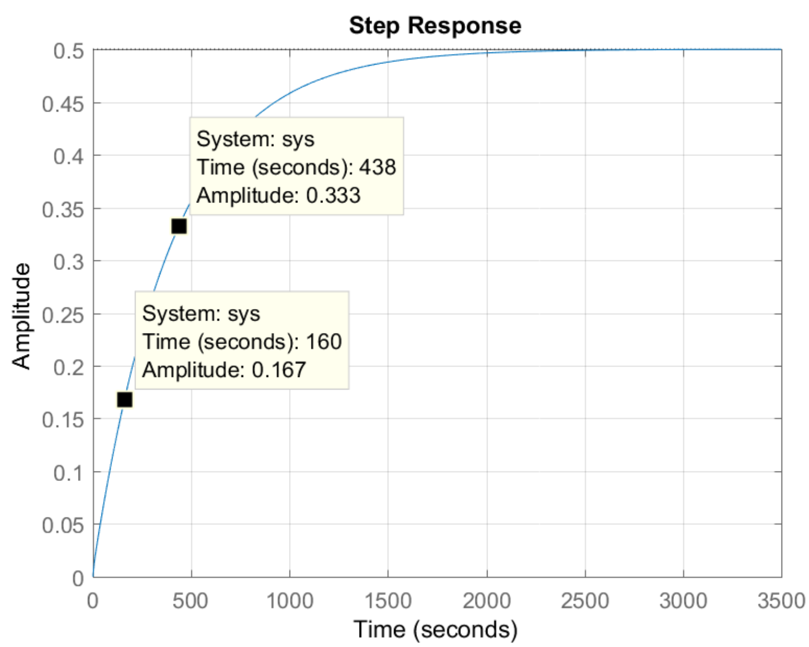


Figure 9

(e) Find the Internal Model Control PID parameters using this approximation. As the desired closed loop time constants, use the following two $\tau_c = T_p/2$ and $\tau_c = T_p/8$. Simulate the actual plant with these controllers and comment on your results.

Controller parameters can be obtained as

$$
\begin{aligned}
T_i &= T_p + 0.5 \cdot \theta_p \cong 397.71(sec) \\
T_d &= \frac{T_p \theta_p}{2 \cdot T_p + \theta_p} \cong 0.571(sec) \\
K_{c1} &= \frac{1}{K_p}\left(\frac{T_p + 0.5 \cdot \theta_p}{\tau_c + 0.5 \cdot \theta_p}\right)\Bigg|_{\tau_c = T_p/2} \cong 3.994(\mathrm{W/°C}) \\
K_{c2} &= \frac{1}{K_p}\left(\frac{T_p + 0.5 \cdot \theta_p}{\tau_c + 0.5 \cdot \theta_p}\right)\Bigg|_{\tau_c = T_p/8} \cong 15.841(\mathrm{W/°C})
\end{aligned} \tag{14}
$$

Step responses of both systems can be seen in Fig. 10 (the left one for the slower closed loop time constant, right one for the faster). Note that controlled plant is the actual one whose transfer function is provided above. Although the system is not with a FOPDT plant, IMC modeling with such an assumption in mind provided useful controller parameters. The closed loop time constants were approximately reduced to 0.5 and 0.125 times the open loop time constant (0.5590 and 0.1410 to be more exact). One last note is that you cannot make the system as fast as you desire (even the plant with $K_{c2} has a small overshoot$). Although such a second order plant cannot be made unstable (consider its root locus!), magnitude of the overshoots may not be acceptable.

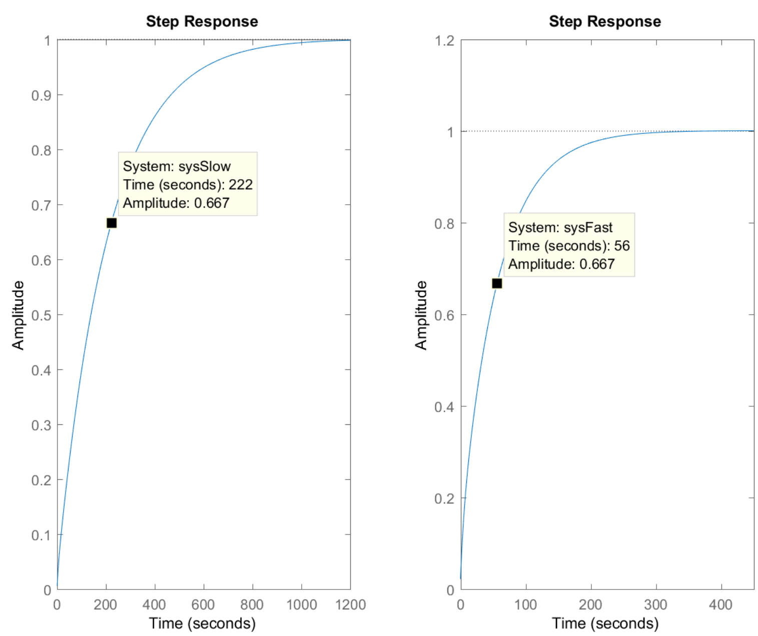


Figure 10

(f) Commit your simulation model of step 1e. Both controlled outputs are expected to be observed in this model.

NA

3. **Miscellaneous Questions**

(a) Consider the step response of a bump test shown in Fig. 11. One way that you can obtain the FOPDT parameters would be to draw a tangent line at the inflection point (that is, the point where the step response has the highest scope), and intersect it with the y_0 line to find θ_p. Then, find the time of $0.632\Delta y = (1 - e^{-1})\Delta y$, and, from the results of the two, find T_p. Interested readers are encouraged to have a look at **this video** giving the explanations of these steps and **here** to see how to achieve this programmatically. However, this would either require you to draw the tangent line rudimentarily or be prone to errors due to spikes caused by noise. Instead, finding a mathematical expression for the time constant and the dead time may be more beneficial. To achieve this, let $\Delta u = u_1 - u_0$ and $\Delta y = y_1 - y_0$. Also let $y_{1/3} = y_0 + \Delta y/3$ and $y_{2/3} = y_0 + 2\Delta y/3$. You are also given that the response of $y(t)$ is of the following form, where t_0 is the input step time and $u'(.)$ is the unit step function. Rearrange this equation to find an expression for t where $t > t_0 + \theta_p$, and then find $t_{1/3}$ & $t_{2/3}$ (corresponding to $y_{1/3}$ & $y_{2/3}$, respectively). Then, rearrange these results to find T_p and θ_p. Using these results and shifting the plots as $t_0 = 0$, show that the formulae given for them in your lectures are valid.

$$y(t) = (1 - e^{-\frac{t-t_0-\theta_p}{T_p}})u'(t - t_0 - \theta_p)\Delta y + y_0$$

For $t > t_0 + \theta_p$, the following equation is satisfied at every point. Arrange it to obtain t in terms of the remaining quantities. Evaluate this expression for $y_{1/3}$ and $y_{2/3}$ to find the result. Letting $t_0 = 0$ shows the validity of the formulae given in lectures.

$$\begin{aligned}
&\frac{y(t) - y_0}{\Delta y} = 1 - e^{-\frac{t-t_0-\theta_p}{T_p}} \\
&t = \ln\left((1 - \frac{y(t) - y_0}{\Delta y})^{-1}\right)T_p + t_0 + \theta_p \\
&t_{1/3} = \ln\left((1 - \frac{y_0 + \Delta y/3 - y_0}{\Delta y})^{-1}\right)T_p + t_0 + \theta_p = 0.4055T_p + t_0 + \theta_p \cong 0.4T_p + t_0 + \theta_p \\
&t_{2/3} = \ln\left((1 - \frac{y_0 + 2\Delta y/3 - y_0}{\Delta y})^{-1}\right)T_p + t_0 + \theta_p = 1.0986T_p + t_0 + \theta_p \cong 1.1T_p + t_0 + \theta_p \\
&\implies T_p = \frac{1}{0.7}(t_{2/3} - t_{1/3}) \ \&\ \theta_p = t_{1/3} - t_0 - 0.4T_p
\end{aligned} \tag{15}$$

(b) Answer the following questions after you have watched the videos on ODTUClass regarding P, PI and PID controllers.

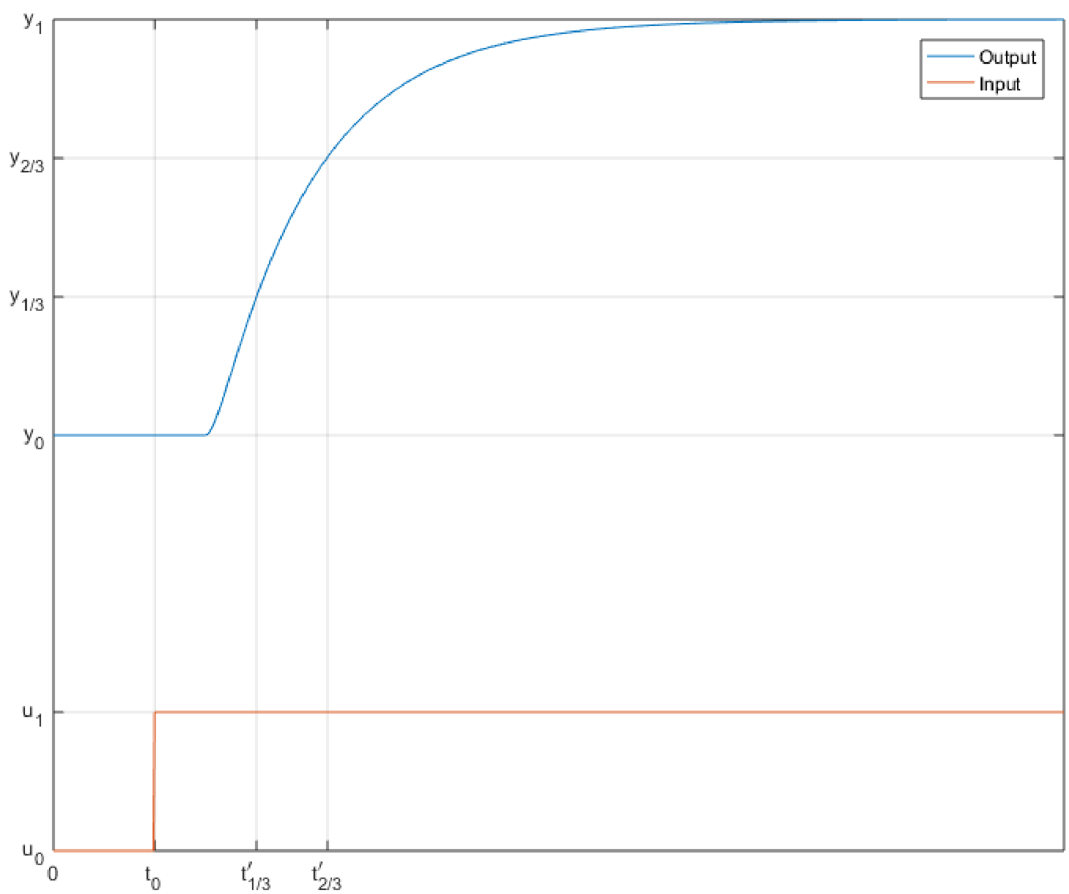


Figure 11

i. Why is it a good idea to have a dead band in an on-off controller? Find an additional example of a practical on-off controller apart from those stated in the video.
Without a dead band, the controller may change its state too frequently (for a first order plant this frequency would be quite high). In order to protect your actuator from being worn off, it is a good idea to introduce a dead band to reduce the frequency of state changes. Additional example of a practical on-off controller (from Wikipedia): a vacuum pump, where you pump out air with full power until the pressure inside a chamber decreases below your desired set point & turn off the pump when the pressure is at an acceptable level.

ii. Which one of the following systems may observe non-zero e_{ss} with a properly chosen P-only control for different set points, and why? A self-regulating process or an integrating one.
The answer is "a self-regulating process". A self-regulating process can observe

zero e_{ss} under P control only when the set point is equal to the value for which CO_{bias} (or u_{bias}) is adjusted to. For different set points (assuming them to change step-wise), an integrating system is of type-1 and $e_{ss} = 0$ provided that the closed loop system is stable.

iii. Consider the problem of driving a car. What are the optimal (in the sense of ITAE) P-control parameters for the cases of (a) you intend to change the speed of the car regularly and (b) slope of the road changes when you intend to drive the car at a constant speed?

The cases (a) & (b) correspond to servo control & regulatory control, respectively. Their, optimal (in the sense of ITAE for a FOPDT plant) P control parameters are:

$$\begin{aligned} \text{(a)}:K_c &= \frac{0.202}{K_p}(\theta_p/\tau_p)^{-1.219} \\ \text{(b)}:K_c &= \frac{0.490}{K_p}(\theta_p/\tau_p)^{-1.084} \end{aligned} \tag{16}$$

iv. ITAE, IAE and ISE controller parameters are found for a certain type of a system by the minimization of $\int_0^\infty t|e(t)|dt$, $\int_0^\infty |e(t)|dt$ and $\int_0^\infty e^2(t)dt$, respectively. State which parameter tuning method (a) penalizes all errors equally, (b) penalizes large errors more and (c) penalizes initial errors less than those at a later time.

When a (P, PI or PID) controller is designed by the minimization of ITAE, errors that occur when t is large can affect the integration result more compared to those that occur when t is low. That is why ITAE tuning method is expected to penalize initial errors less than those at a later time. IAE, on the other hand, treats and is expected to penalize all errors equally. ISE tuning method, on the other hand, is expected to penalize large errors more as large errors increase the integral value more than the small ones.

v. Consider the water level control of a tank where your actuator is a pump either supplying water to the tank or draining water from it. Show the configurations of the pump in order to have a reverse acting and direct acting controllers. Describe what would happen if the pump is in reverse acting configuration and the controller is in direct acting mode.

The two configurations are shown in Fig. 12, (a) for reverse acting & (b) for direct acting. If the pump is in reverse acting configuration as in (a) and the controller is in direct acting mode, the case would be equivalent to a positive feedback controller: either you will completely drain the tank or overfill it. Suppose, $h_{set} > h$ where you would have to increase the amount of inflow. However, as $e > 0$ & $K_c < 0$, the controller output is negative and the actuator output is 0 (or negative if it can drain water from it), resulting in an empty tank at the end. If $h_{set} > h$, on the other hand, the controller output is positive and you are supplying more water when you have to decrease it, resulting in a tank with overflow.

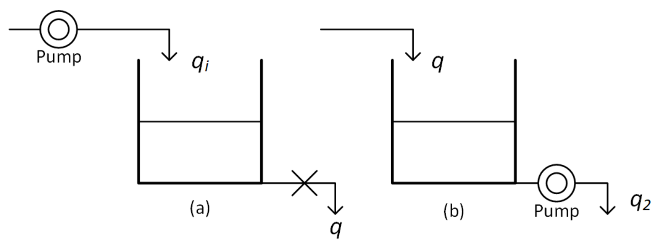


Figure 12

vi. Suppose that $SP(t) > PV(t_{ss})$ where $PV(t_{ss})$ is the process variable when the controller output is $CO = CO_{bias}$. Draw approximately the integral of error and discuss how this achieves moving the bias term.

Such a plot of $\int_0^t e(t)$ is provided in Fig. **??** (obtained from the video on PI control). Integral of error does not have to show oscillatory behavior, or it may settle at negative values (depending on whether your controller is direct or reverse acting mode). However, its final value cannot be equal to 0 because the process we are trying to control is a self-regulating one; otherwise, $PV(t_{ss})$ would not exist. When the controlled variable reaches its steady state value, $e(t) = 0$ and its integral would remain constant from that point on. This residual term will be added to CO_{bias}, and the integrating action will effectively move this bias term to adjust to new operating conditions.

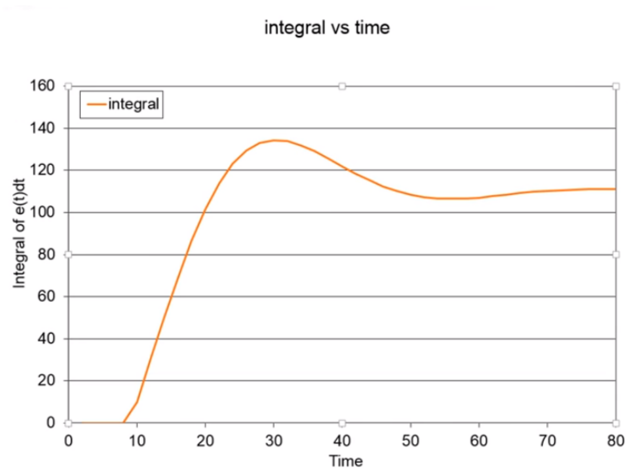


Figure 13

vii. What is reset or integral windup? Suggest a method to eliminate this.

Integral windup occurs when positive errors persist over a long period of time after a large set point change (a similar argument can be made for negative errors). When integral windup occurs, we would need the error to change sign to let the controller unwind during which we observe overshoots. Accumulating action of integration causes the controller output to be unachievable (out of the bounds of the actuator); hence, integrating error at that point is of no use. One solution would be "preventing the integral term from accumulating above or below pre-determined bounds" (from Wikipedia). Several other methods can also be found on Wikipedia.

viii. How do decay ratio and settling time generally correlate? How about rise time and peak time?

Generally speaking, higher decay ratio implies faster settling as the initial overshoots are damped more. As the peak time increases, the time the controlled variable first reaches its set point value is likely to increase as well.

ix. What is the derivative kick that occurs when the set point changes stepwise? Why should you and how can you eliminate it?

Knowing that $e(t) = SP - PV$ (SP for set point, PV for the controlled process value), if you change the set point stepwise, the output of the controller would have an impulsive term due to differentiation. Even if you approximate differentiation with a finite difference equation, a large initial CO value would exist for a single sampling time. This is called a derivative kick. This kick should be eliminated as it may cause mechanical final control elements to be worn off. You can eliminate this by performing differentiation only on PV. By doing so, the initial kick is avoided. Also, derivative of SP after the step time is zero; hence, this has the same effect as a traditional derivative controller except for the step time.

x. When is the ideal derivative action detrimental to controller performance? What should you do if you want to use derivative action in such cases?

Differentiation is detrimental to controller performance when the measurements are noisy as differentiation amplifies this noise and reflects this into CO. When noise level increases, alternating CO actions (chatters) would also increase, deteriorating the controller performance. (Low pass) Filtering the measurements or (low pass) filtering the CO values before sending them to the FCE are the two methods that are suggested in the video.