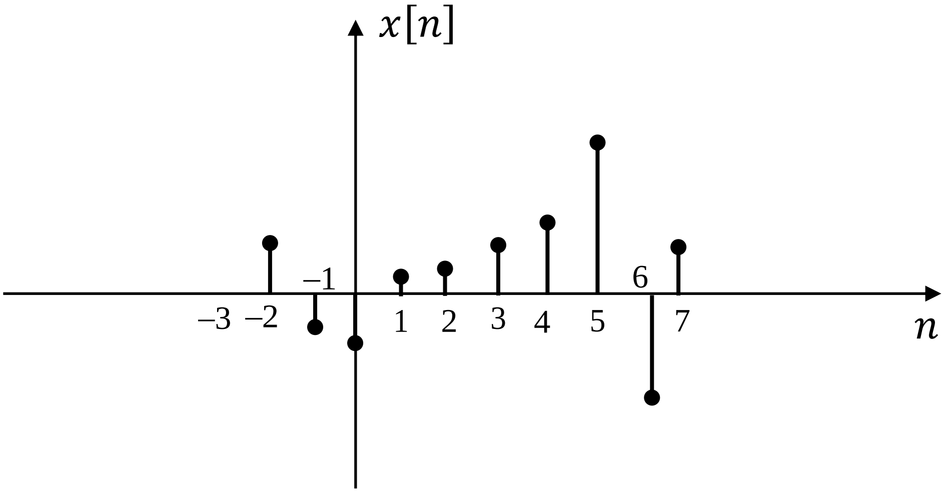
Discrete-Time Signals

- Mathematical Representation of DT Signals
- Unit Sample Sequence
- Unit Step Sequence
- Exponential Sequences
- Sinusoidal Sequences
- Two Fundamental Properties of DT Sinusoidal Sequences
    - $\cos(\omega_0 n) = \cos\big((\omega_0 + k2\pi)\, n\big), \quad k \in Z$
    - A DT sinusodal is not necessarily periodic.

Discrete-Time Signals

A discrete-time signal is a sequence of real or complex numbers.

Its n^{th} element is $x[n]$, $n \in Z$.



n is sample index.
It is integer valued.
Unitless.

A discrete-time signal $x[n]$ may have been obtained by sampling a continuous-time signal, i.e.,

$$x[n] = x_C(t)|_{t=nT} \,, \quad n \in Z$$

<u>Ex</u>: Let T = 0.001 sec = 1 msec.

We do NOT write

$$\ldots, x[-0.001], x[0], x[0.001], x[0.002], \ldots$$

We write

$$\ldots, x[-1], x[0], x[1], x[2\ ], \ldots$$

Exercise:

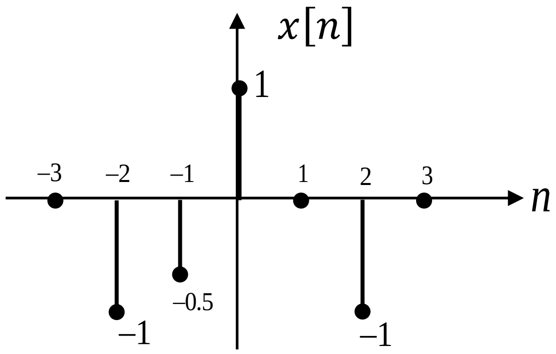
Using MATLAB, write a code to record sound via the microphone of a computer.

Plot some signals of your choice.
Try both ‘plot’ and ‘stem’ commands of MATLAB to plot the signals.
Describe the difference between the results of these two commands.

Inspect close views of different segments of the signals and describe the characteristics and the differences between of these segments.

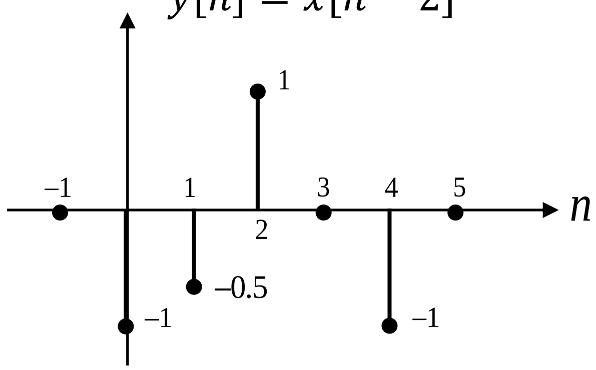
Time Shift of a Signal

<u>Ex</u>: Delay



$$y[n] = x[n - n_0]$$

$$y[n] = x[n - 2]$$



n_0 is always an integer!

We do NOT have sth. like $x[n - 2.15]$

Comment on continuous-time shift discrete-time shift relationship:

Let

$$x[n] = x_C(t)|_{t=nT}$$

and

$$y[n] = x_C(t - t_o)|_{t=nT}$$

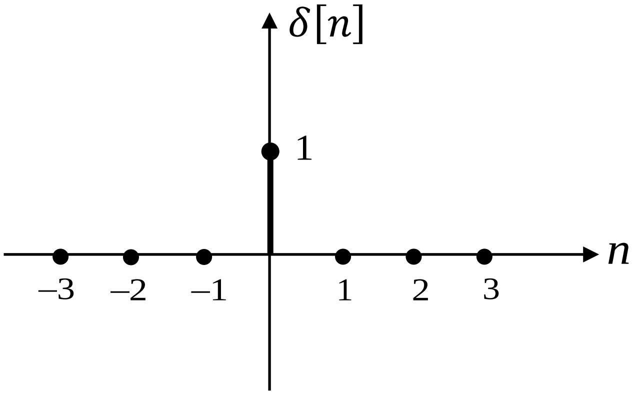
where $t_0 = rT, \qquad 0 < r < 1.$

Then, $x[n]$ and $y[n]$ cannot be related by a discrete-time shift.

However, we will see that it may be possible to obtain $y[n]$ from $x[n]$ by a discrete-time operation, i.e. without resampling the shifted continuous-time signal obtained from $x[n]$.

(related keywords: filtering, interpolation, sampling rate change)

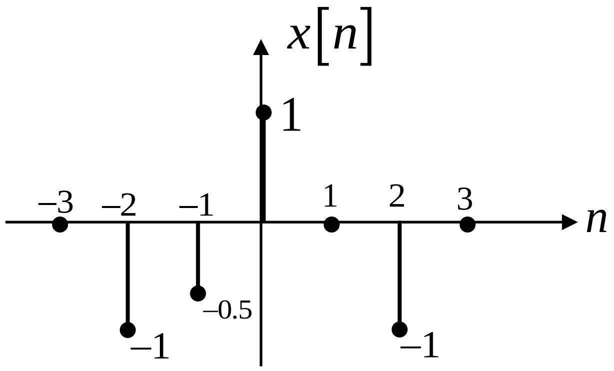
UNIT SAMPLE SEQUENCE



$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

Expressing an arbitrary sequence $x[n]$ in terms of Unit Sample sequences

Ex: Let $x[n]$ be



It can be written as:

$$x[n] = -\delta[n+2] - 0.5\delta[n+1] + \delta[n] - \delta[n-2]$$

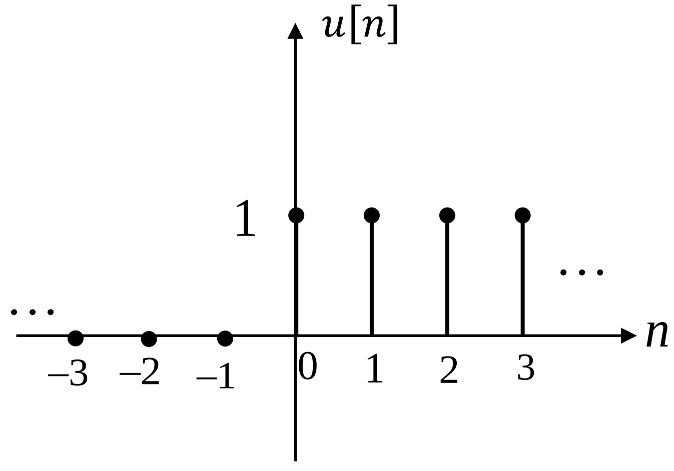
In general, any seq. can be written as

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

This is the fundamental expression in the derivation of "convolution"[1].

[1] the output of a LTI system is the convolution of the input and the system's impulse response.

UNIT STEP SEQUENCE



$$u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$

Relationship between $u[n]$ and $\delta[n]$

$$u[n] = \sum_{k=0}^{\infty} \delta[n-k]$$

or

$$u[n] = \sum_{k=-\infty}^{n} \delta[k]$$

(like integration in cont. time)

on the other hand

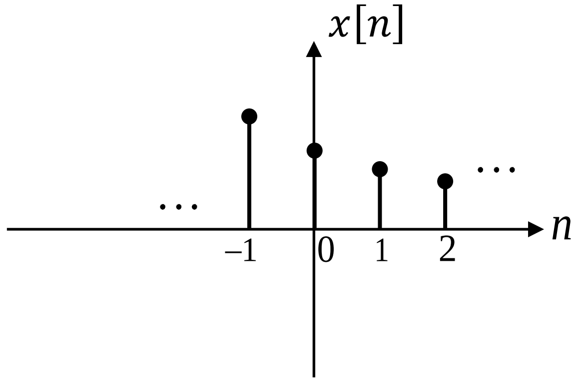
$$\delta[n] = u[n] - u[n-1]$$

(like differentation in cont. time)

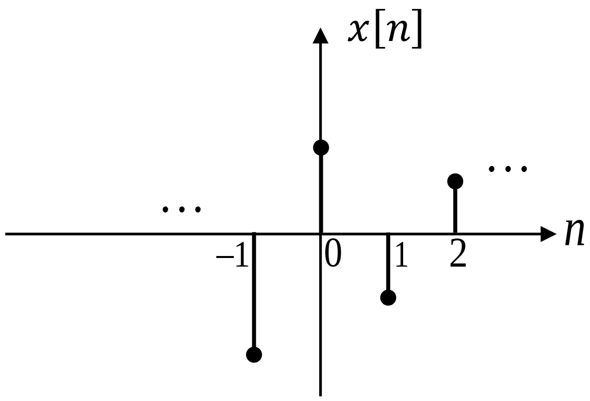
EXPONENTIAL SEQUENCES (real valued)

They appear in the solution and analysis of LTI systems.

$$x[n] = A\alpha^n$$



$0 < \alpha < 1$



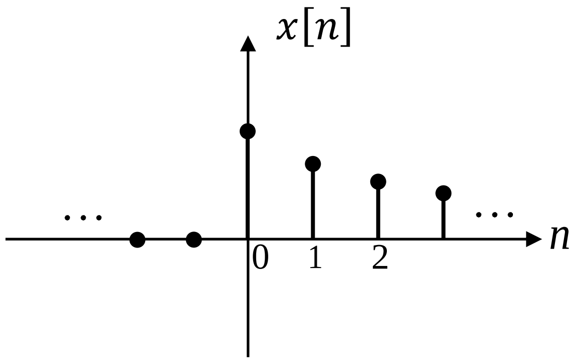
$-1 < \alpha < 0$

if $|\alpha| > 1$ then $|x[n]|$ grows as $n \to \infty$

TRUNCATED EXPONENTIAL SEQUENCE

$$x[n] = \begin{cases} A\alpha^n & n \geq 0 \\ 0 & n < 0 \end{cases}$$

$$= A\alpha^n u[n]$$

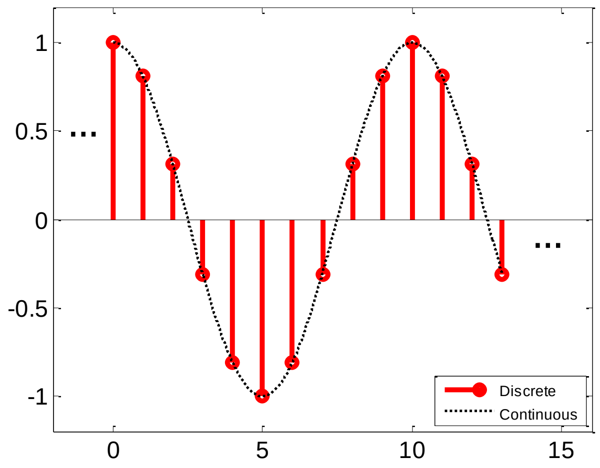


$0 < \alpha < 1$

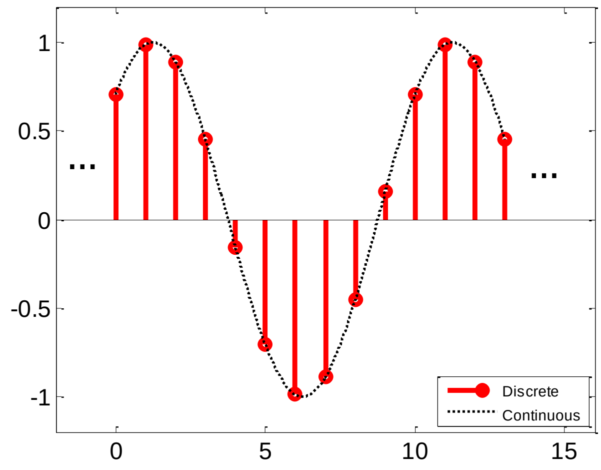
SINUSOIDAL SEQUENCES

$$x[n] = A\cos(\omega_0 n + \phi)$$

$\omega_0 = \frac{2\pi}{10} \quad \phi = 0$



$\omega_0 = \frac{2\pi}{10} \quad \phi = \frac{\pi}{4}$



Note that, $x_1[n]$ and $x_2[n]$cannot be related by a simple time shift.

$$\cancel{x_2[n] = x_1\left[n - \frac{5}{4}\right]} \qquad \cancel{x_2[n] \neq x_1\left[n - \frac{5}{4}\right]}$$

code_1.m

```
clear all
close all

N = 10;
w0 = (2*pi / N);
alfa = 1;
phase = 0;  %pi/4;

M = N + 3;
n =  [0:0.01:M];
y = alfa.^n.*cos(w0*n-phase);

nn = [0:M];
x = alfa.^nn.*cos(w0*nn-phase);

stem(nn,x,'r','LineWidth',3)
hold
plot(n,y,'k:','LineWidth',2)
v = axis;
dV = v(4)-v(3);
v = [v(1)-2 v(2)+2 v(3)-0.1*dV v(4)+0.1*dV];
axis(v)

set(gca,'fontsize',14)
hleg =
legend('Discrete','Continuous','location','southe
ast');
set(hleg,'fontsize', 9)
```

EXPONENTIAL SEQUENCES (complex valued)

$$x[n] = A\,\alpha^n \qquad A, \alpha \in C$$

$$A = |A|e^{j\phi} \qquad \alpha = |\alpha|e^{j\omega_0}$$

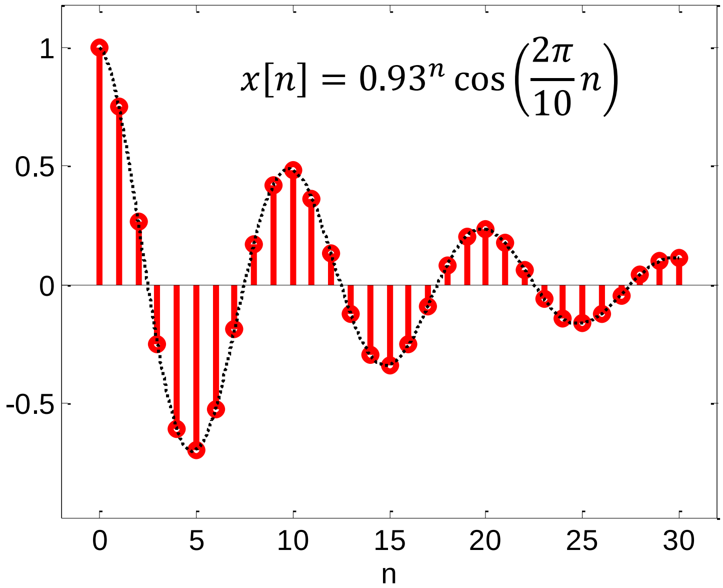
$$\Rightarrow x[n] = |A||\alpha|^n \cos(\omega_0 n + \phi) + j|A||\alpha|^n \sin(\omega_0 n + \phi)$$

Euler’s formula:

$$e^{jx} = \cos x + j \sin x$$

Ex:

$$A = 1 \qquad |\alpha| = 0.93 \qquad \phi = 0 \qquad \omega_0 = \frac{2\pi}{10}$$



Plotted for $0 \leq n \leq 30$

Ex:

$$A = e^{j\frac{\pi}{3}} \qquad \alpha = 0.9e^{j\frac{2\pi}{7}}$$

$$\phi = \frac{\pi}{3} \qquad \omega_0 = \frac{2\pi}{7}$$

$$x[n] = e^{j\frac{\pi}{3}}\, 0.9^n e^{j\frac{2\pi}{7}n}$$

$$= 0.9^n \left(\cos\left(\frac{2\pi}{7}n + \frac{\pi}{3}\right) + j\sin\left(\frac{2\pi}{7}n + \frac{\pi}{3}\right)\right)$$

COMPLEX EXPONENTIAL SEQUENCES

Let

$$|\alpha| = 1$$

in

$$x[n] = A\,\alpha^n \qquad A, \alpha \in C$$

Then,

$$|A|e^{j(\omega_0 n+\phi)}$$

is called a complex exponential sequence.

$$\Rightarrow\ Ae^{j(\omega_0 n)} = |A|\cos(\omega_0 n+\phi) + j|A|\sin(\omega_0 n+\phi)$$

A sinusoidal sequence can be expressed in terms of a complex exponential sequence.

$$
\begin{aligned}
M\cos(\omega_0 n + \phi) &= Re\{Ae^{j\omega_0 n}\} \\
&= \frac{1}{2}\left(Ae^{j\omega_0 n} + A^* e^{-j\omega_0 n}\right)
\end{aligned}
$$

$$
\begin{aligned}
M\sin(\omega_0 n + \phi) &= Im\{Ae^{j\omega_0 n}\} \\
&= \frac{1}{2j}\left(Ae^{j\omega_0 n} - A^* e^{-j\omega_0 n}\right)
\end{aligned}
$$

$$A = Me^{j\phi}, \qquad M \in R$$

ω_0 : frequency (radians/sample or, shortly, radians, NOT rad/sec!)

ϕ : phase shift (radians)

TWO FUNDAMENTAL PROPERTIES OF COMPLEX EXPONENTIAL (SINUSOIDAL) DISCRETE-TIME SEQUENCES

These properties do not hold for continuous-time signals!

FIRST PROPERTY

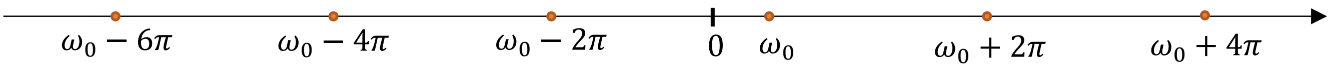
For any frequency value ω_0, $\omega_0 + k2\pi$ (k: $integer$) is an equivalent frequency value, i.e.,

if $\quad x[n] = Ae^{j\omega_0 n}$ and $x[n] = Ae^{j(\omega_0 + k2\pi)\, n}$

then $\quad x[n] = y[n]\ \forall n \in Z$

$$\cos(\omega_0 n) = \cos\big((\omega_0 + k2\pi)\, n\big)$$

$$\sin(\omega_0 n) = \sin\big((\omega_0 + k2\pi)\, n\big)$$



In other words, the elements of the set $\{\omega | \omega = \omega_0 + k2\pi, \omega_0 \in R, k \in Z\}$ are equivalent if they are considered as the frequencies of discrete-time complex exponentials/sinusoids.

$$\ldots = \cos\left(-\frac{9\pi}{5}n\right) = \cos\left(\frac{\pi}{5}n\right) = \cos\left(\frac{11\pi}{5}n\right) = \cos\left(\frac{21\pi}{5}n\right) = \cdots$$

$$\ldots = e^{-j\frac{9\pi}{5}n} = e^{j\frac{\pi}{5}n} = e^{j\frac{11\pi}{5}n} = e^{j\frac{21\pi}{5}n} = \cdots$$

Indeed an interval of 2π covers all distinct frequencies.

Effectively an interval of π!   Why?

$$e^{j(2\pi-\omega)n} = e^{-j\omega n}e^{j2\pi n}$$

$$= e^{-j\omega n}$$

$$\cos(2\pi-\omega)n = \cos\omega n$$

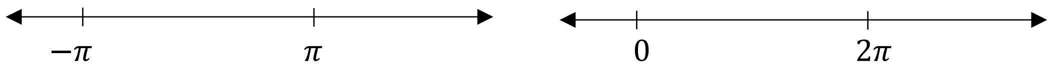
$$\sin(2\pi-\omega)n = -\sin\omega n$$

CONVENTION FOR THE REFERENCE FREQUENCY INTERVAL

Since frequencies are equivalent when multiples of 2π is added/subtracted, convention is to use either of the following as the basic interval

$$-\pi \leq \omega < \pi$$

$$0 \leq \omega < 2\pi$$



Ex:

$$\cos(416.31\pi n) = \cos(0.31\pi n)$$

$$\sin(416.31\pi n) = \sin(0.31\pi n)$$

Ex:

$$\cos(417.31\pi n) = \cos(1.31\pi n)$$

$$= \cos(0.69\pi n)$$

$$\sin(417.31\pi n) = \sin(1.31\pi n)$$

$$= -\sin(0.69\pi n)$$

Remark: For a given sampling period, T_s, (frequency, $f_s = \frac{1}{T_s}$) one can find an infinite set of sinusoids which yield the same discrete-time (sinusoidal) signal.

$$\cos(\Omega t) \xrightarrow{uniform\ sampling\ by\ T_s} \cos(\Omega n T_s)$$

$$\Omega = \Omega_0 + k\frac{2\pi}{T_s}\ , \qquad k \in Z$$

$$\cos(\Omega_0 n T_s) \overset{?}{=} \cos\big((\Omega_0 + \Delta\Omega) n T_s\big)$$

$$\cos(\Omega_0 n T_s + \Delta\Omega n T_s)$$

$$\Delta\Omega T_s = k2\pi \qquad\qquad k \in Z$$

Therefore

$$\Delta\Omega = k\frac{2\pi}{T_s}$$

$$\Delta f = k\frac{1}{T_s} = kf_s$$

Ex:

A 100 MHz signal $x_C(t) = \cos(2 \times 10^8 \pi t)$ is sampled at a rate of 250 MHz

(i.e. sampling period is $T_S = \frac{1}{250\ 000\ 000} = 4 \text{ pico sec.}$)

$$x[n] = \cos(0.8\pi n)$$

Find another continuous-time (CT) sinusoid that would yield the same discrete-time sinusoid (i.e., $x[n]$) at this sampling frequency.

What other CT sinusoids would yield the same DT sequence?

Answer: 100 MHz, 350 MHz, 600 MHz, …

Ex: When we sample

$$\cos(2\pi 4000t)\,,\cos(2\pi 26000t)\,,\cos(2\pi 48000t)\,,\ldots$$

by a sampling frequency of $f_s = 22$ kHz,

they all yield the same discrete-time signal.

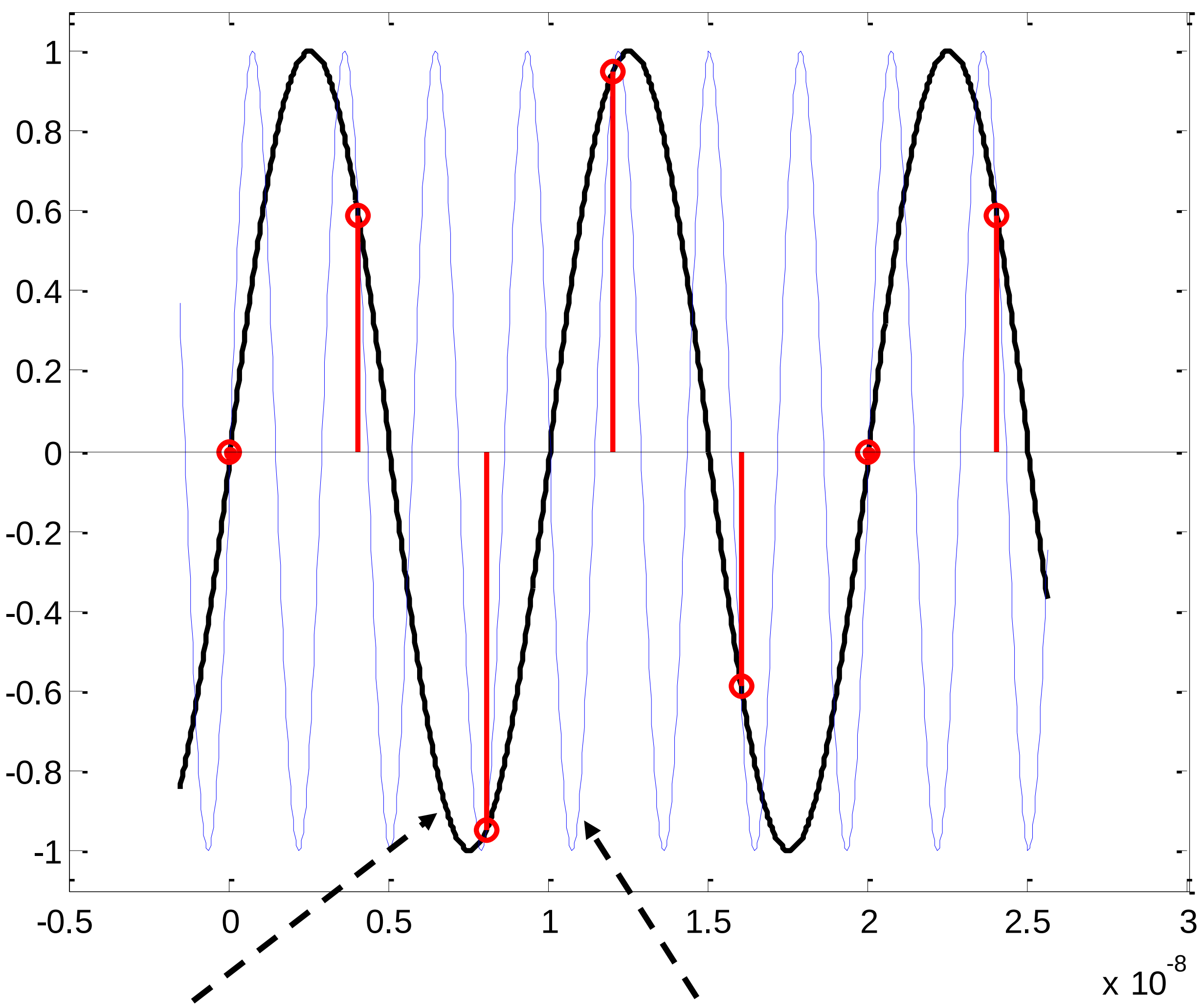
Ex: When we sample

$$\cos(2\pi 4000t)\,,\cos(2\pi 17000t)\,,\cos(2\pi 30000t)\,,\ldots$$

by a sampling frequency of $f_s = 13$ kHz,

they all yield the same discrete-time signal.

Ex:



$\cos(2\pi 10^8 t)$

$\cos(2\pi(f_S + 10^8)t)$

To create the above figure

```
clear all
close all

fS = 250*10^6;
f0 = 100*10^6;
f1 = f0 + fS;
f2 = f1 + fS;

t = [-0.4:0.01:6.4];
nn = [0:6];
x0 = sin(t*2*pi*f0/fS);
x1 = sin(t*2*pi*f1/fS);
x2 = sin(t*2*pi*f2/fS);
z = sin(nn*2*pi*f0/fS);
plot(t/fS,x0,'k','LineWidth',2);
hold
plot(t/fS,x1);
% plot(t/fS,x2);

stem(nn/fS,z,'r','LineWidth',2)
v=axis;
v=[v(1) v(2) -1.1 1.1];
axis(v)
```

Try code_2.m

Exercise: Is it possible to obtain the same discrete-time sinusoidal signal by sampling a continuous-time sinusoidal signal at different rates? If yes, describe those set of frequencies.

$$\cos \Omega_0 n(T_s + \Delta T) = \cos(\Omega_0 n T_s + \Omega_0 n \Delta T)$$

$$\Omega_0 \Delta T = k2\pi\ , \quad k \in Z$$

$$\Delta T = \frac{k2\pi}{\Omega_0}$$

$$\Delta T = \frac{k2\pi}{2\pi f_0}$$

$$\Delta T = \frac{k}{f_0} = kT_0$$

$$f_s' = \frac{1}{T_s + \Delta T}$$

$$= \frac{1}{\frac{1}{f_s} + \frac{k}{f_0}}$$

$$= \frac{f_s f_0}{f_0 + k f_s}$$

Ex: $\Omega_0 = 2\pi 1200\, rad/sec$

$$f_s = 10^4 \;\; samples/sec \qquad\qquad T_s = 10^{-4}\, sec$$

The set of sampling periods are

$$T_s + \Delta T\!:\left\{10^{-4},\quad 10^{-4} + \frac{1}{1200},\quad 10^{-4} + \frac{2}{1200}, \dots\right\}$$

$$T_s + \Delta T\!:\left\{10^{-4},\;\; \frac{112}{12} 10^{-4},\;\; \frac{212}{12}\; 10^{-4}, \dots\right\}$$

$$f_0 = 1200\, cyc/sec$$

$$T_0 = \frac{1}{1200}\; sec$$

$$f_s'\!:\; \left\{\frac{120000}{100k + 12};\;\; k \in Z\right\} = \{10^4, 1.0714 \times 10^3, 566.0377, 384.6154, \dots\}$$

AN IMPORTANT CONSEQUENCE OF THE FIRST PROPERTY

Remember (from EE301) that

“Discrete-Time Fourier Transform is Periodic with 2π”.

A sequence $x[n]$ with DTFT $X(e^{j\omega})$ can be written as

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega})\, e^{j\omega n} d\omega$$

SECOND PROPERTY

A DT sinusodal ($\cos(\omega_0 n + \phi)$) or complex exponential signal $e^{j(\omega_0 n+\phi)}$ is not necessarily periodic!

To be periodic,

ω_0 must be a *rational* multiple of π,

i.e.,

$$\omega_0 = \frac{p}{q}\pi, \qquad p, q \in Z$$

$$A\cos(\omega_0 n + \phi) \overset{?}{=} A\cos(\omega_0(n+N) + \phi)$$

Proof:

$$A\cos(\omega_0(n+N) + \phi) = A\cos(\omega_0 n + \omega_0 N + \phi)$$

For periodicity

$$\omega_0 N = k2\pi$$

has to be satisfied for integer N and k.

$$\Rightarrow \quad \omega_0 = k\frac{2\pi}{N}$$

or

$$\frac{\omega_0}{\pi} = \frac{2k}{N} \qquad k \in Z$$

has to be satisfied.

Ex:

$$\cos(5n) \qquad\qquad \omega_0 = 5$$

$$\frac{\omega_0}{\pi} = \frac{5}{\pi}$$

is not rational so $\cos(5n)$ is not periodic.

```
clear all
close all

n=0:144;

x = cos(0.5*n);

stem(n,x)
```

FUNDAMENTAL PERIOD, N, IS NOT NECESSARILY EQUAL TO $\frac{2\pi}{\omega_0}$

Since, for periodic sinusoids,

$$\omega_0 N = k2\pi \quad k \in Z$$

i.e,

$$N = \frac{k2\pi}{\omega_0},$$

fundamental period, *N*, is not necessarily equal to $\frac{2\pi}{\omega_0}$.

FINDING THE FUNDAMENTAL PERIOD OF A SINUSOID

Find the smallest k, k_{min} , so that $k_{min}\frac{2\pi}{\omega_0}$ is an integer.

Then, the fundamental period is

$$N = k_{min}\frac{2\pi}{\omega_0} \quad .$$

Ex:

$$\cos\left(\frac{\pi}{5}n\right) \qquad \omega_0 = \frac{\pi}{5} \qquad \frac{\omega_0}{2\pi} = \frac{1}{10} \qquad N = k\frac{2\pi}{\omega_0} = k\frac{2\pi}{\frac{\pi}{5}} = 10 \qquad (k = 1)$$

Ex:

$$\cos\left(\frac{5\pi}{17}n\right) \qquad \omega_0 = \frac{5\pi}{17} \qquad \frac{\omega_0}{2\pi} = \frac{5}{34} \qquad N = k\frac{34}{5} = 34 \qquad (k = 5)$$

Ex:

$$\cos\left(\frac{6\pi}{5}n\right) \qquad \omega_0 = \frac{6\pi}{5} \qquad \frac{\omega_0}{2\pi} = \frac{3}{5} \qquad N = k\frac{5}{3} = 5 \qquad (k = 3)$$

Ex:

Let

$$x_1[n] = \cos(\omega_1 n) \qquad\qquad and \qquad\qquad x_2[n] = \cos(\omega_2 n)$$

Find two "frequencies" ω_1 and $\omega_2$such that $\omega_1 \neq \omega_2 + k2\pi$ for any integer k, and $x_1[n]$ and $x_2[n]$ are both periodic with fundamental period $N = 13$.

$$N = 13 = k\frac{2\pi}{\omega} \qquad\qquad k \in Z$$

$$\Rightarrow \quad \omega = k\frac{2\pi}{13}$$

Choose, for example, $k = 1$ and $k = 2$

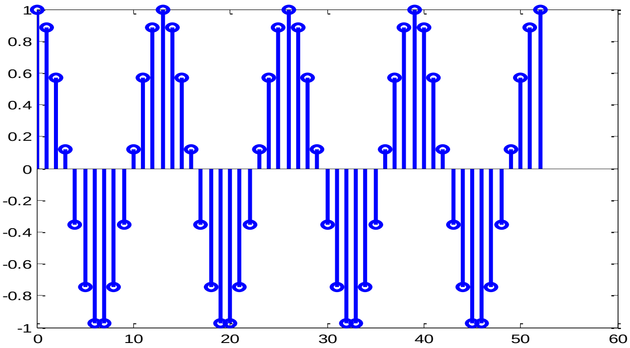
$$\omega_1 = \frac{2\pi}{13} \qquad\qquad \omega_2 = \frac{4\pi}{13}$$

Therefore,
DT sinusoids may have different "frequencies"
although
their fundamental periods are the same!

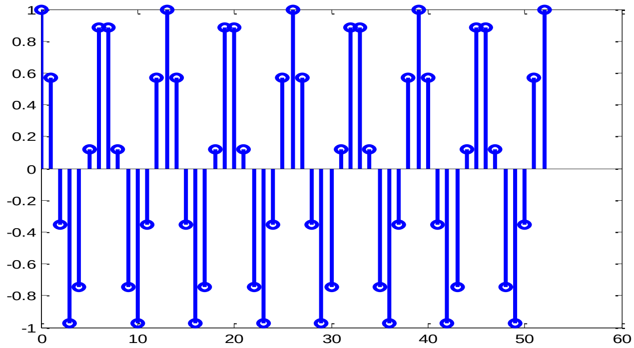
What do the discrete-time sinusoids look like?

Some frequencies between 0 and π

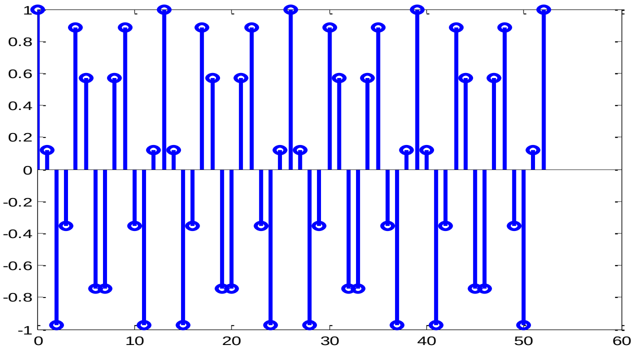
$\cos\left(\frac{2\pi}{13}n\right)$ *N*=13



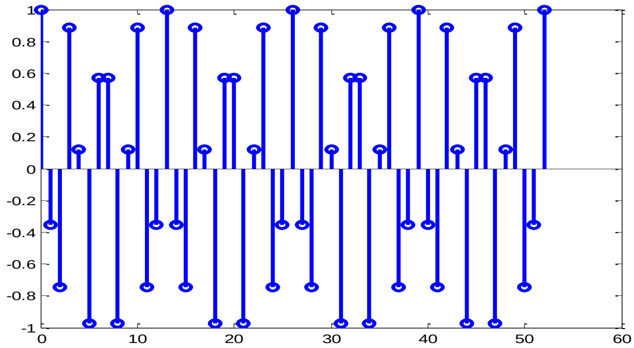
$\cos\left(\frac{4\pi}{13}n\right)$ *N*=13



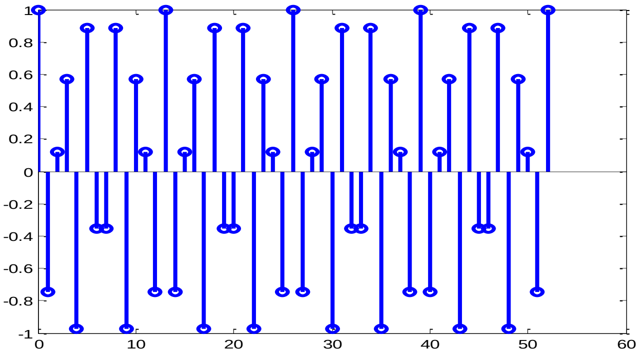
$\cos\left(\frac{6\pi}{13}n\right)$ *N*= 13



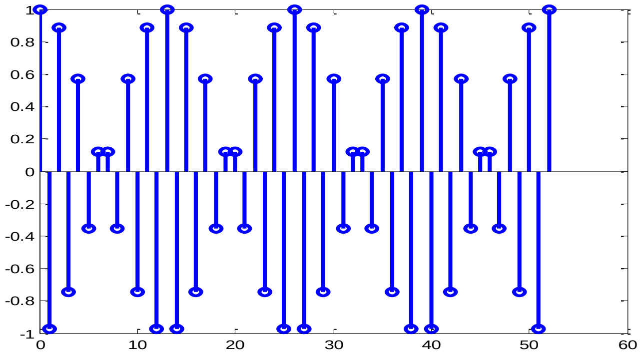
$\cos\left(\frac{8\pi}{13}n\right)$ *N*=13



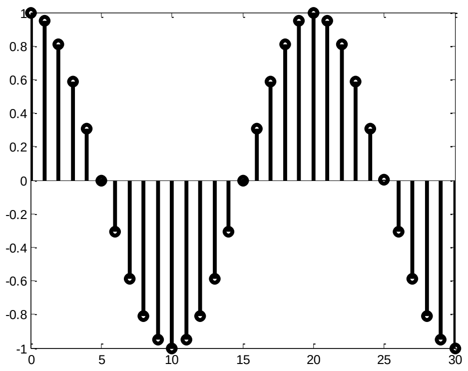
$\cos\left(\frac{10\pi}{13}n\right)$ *N* = 13



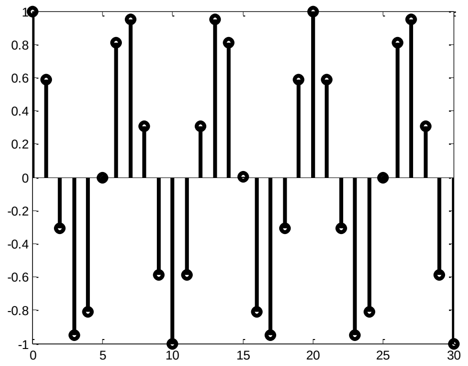
$\cos\left(\frac{12\pi}{13}n\right)$ *N*=13



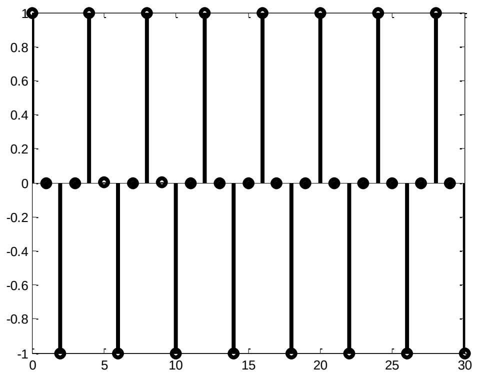
$\cos(0.1\pi n)$ N=20



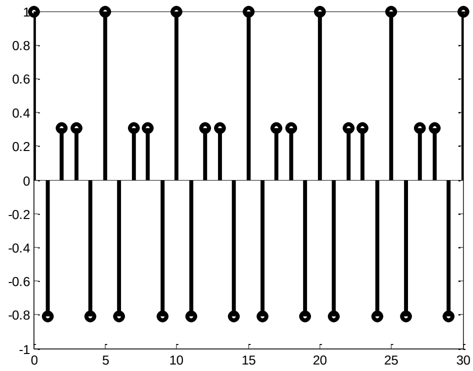
$\cos(0.3\pi n)$ N=20



$\cos(0.5\pi n)$ N=4

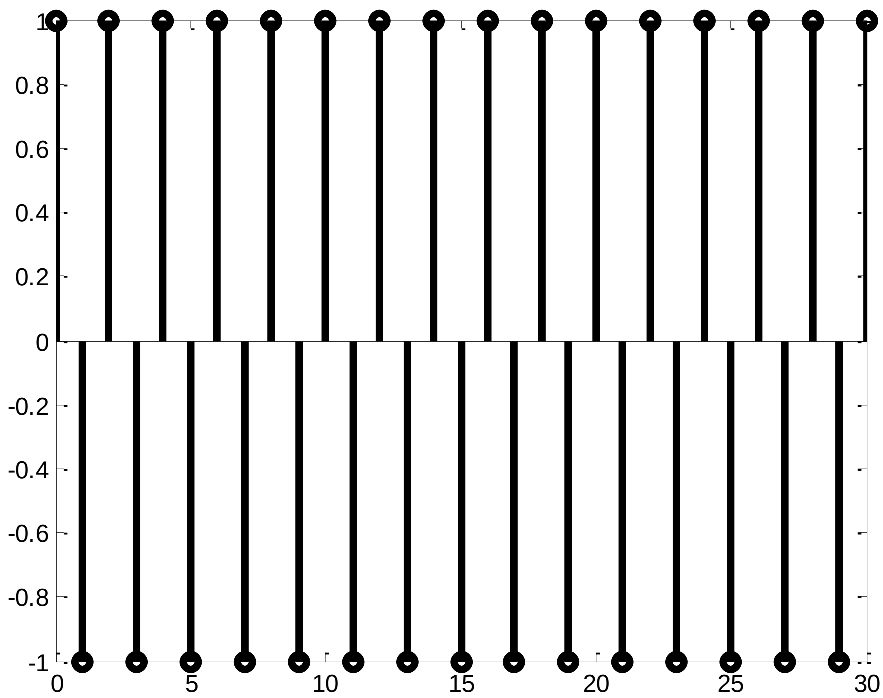


$\cos(0.8\pi n)$ N=5



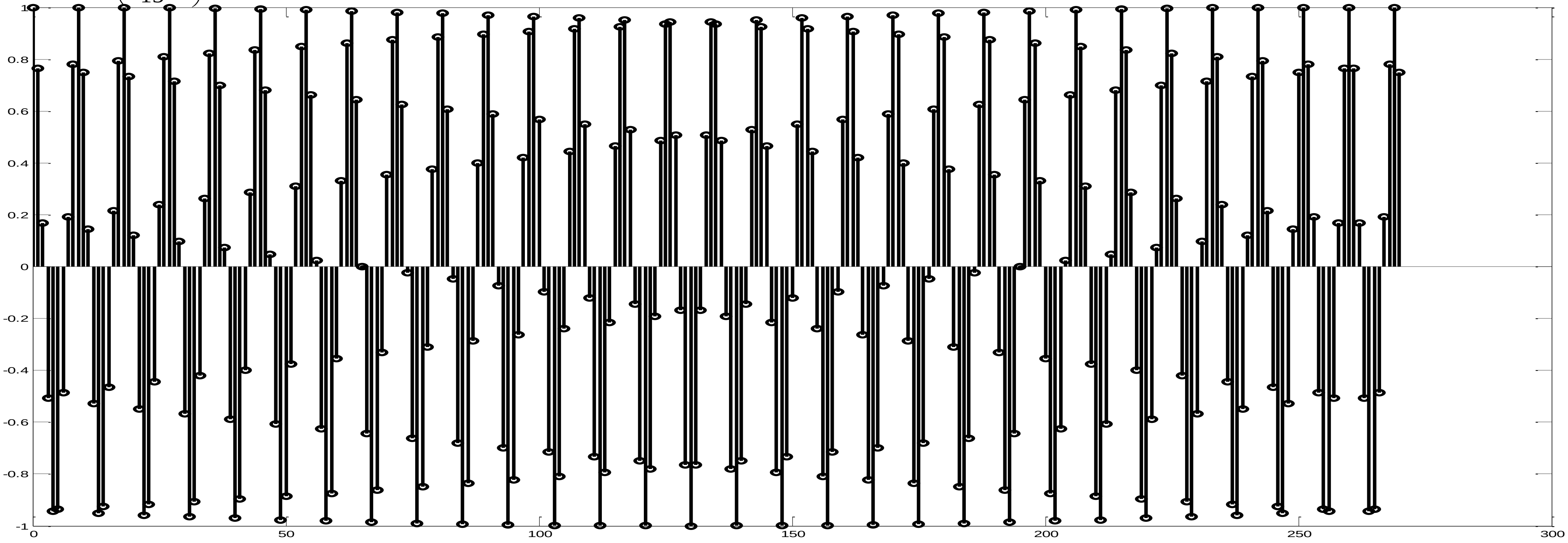
THE HIGHEST FREQUENCY SIGNAL IN DISCRETE-TIME

$$\cos(\pi n)$$

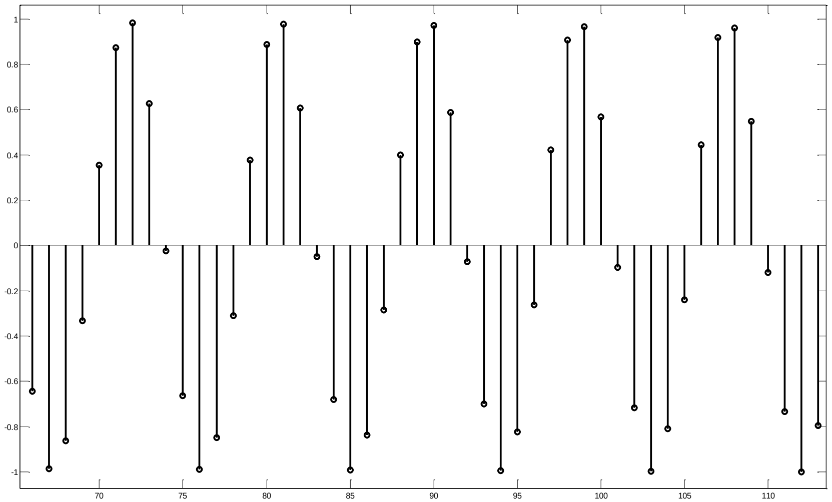


$$\cos\left(\frac{2.9\pi}{13}n\right)$$

Fundamental period is N=260

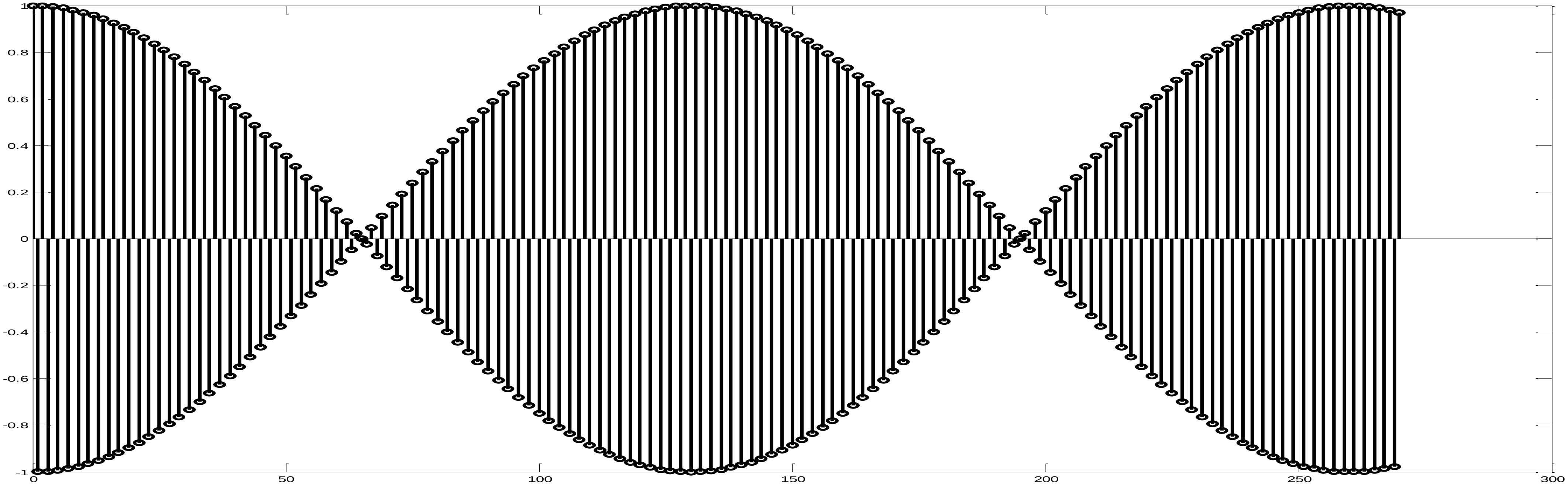


detail



$$\cos\left(\frac{12.9\pi}{13}n\right)$$

Fundamental period is N=260



detail