Date: 29.11.2016 Time: 15:40 Duration: **110 minutes** Attempt all questions Closed books and notes

Middle East Technical University
Electrical-Electronics Engineering Department

EE 430 Digital Signal Processing

Midterm Examination II

CLOSED BOOKS
110 MINUTES

LASTNAME	
NAME	
STUDENT ID:	

Question	Grade
Q1 (25pts)	
Q2 (25pts)	
Q3 (25pts)	
Q4 (25pts)	
TOTAL	

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I have read and fully understood the warning, and I pledge to comply with the exam rules.
SIGNATURE :

Q1)

a) Assume the z-transform for the complex sequence, $x[n] = x_R[n] + jx_I[n]$, is equal to $X(z)$. Find the z-transforms for the following sequences in terms of $X(z)$.

(i) $x^*[-n]$

(ii) $x_R[n]$

b) The autocorrelation function of a real sequence, $x[n]$, is given as

$$r_{xx}[k] = \sum_{n=-\infty}^{\infty} x[n]x[n-k]$$

Find the z-transform of the autocorrelation function, $R_{xx}(z)$, in terms of $X(z)$.

c) If $x[n] = a^n\, u[n]$, $(|a| < 1)$, determine $R_{xx}(z)$ and $r_{xx}[n]$.

a) (i) $X(z) = \sum_{n=-\infty}^{+\infty} x(n) z^{-n} \Rightarrow X^*(z) = \sum_{n=-\infty}^{+\infty} x^*(n)(z^*)^{-n} \Rightarrow X^*(z) = \sum_{m=+\infty}^{-\infty} x^*[-m](z^*)^{+m}$

$\Rightarrow X^*(1/z) = \sum_{n=-\infty}^{+\infty} x^*[-n](z^*)^{-n} \Rightarrow \boxed{x^*[-n] \longleftrightarrow X^*(1/z^*)}$

(ii) $x_R(n) = \dfrac{x(n) + x^*(n)}{2} \Rightarrow x^*(n) \longleftrightarrow X^*(z^*)$

$\Rightarrow \boxed{x_r(n) \longleftrightarrow (X(z) + X^*(z^*))/2}$

b) $r_{xx}(k) = \sum_{n=-\infty}^{+\infty} x(n)x(n-k) = \sum_{m=-\infty}^{+\infty} x[k-m]x[-m] = \boxed{x(k] * x(-k]}$

$n-k \triangleq -m$

Using convolution property : $\boxed{R_{xx}(z) = X(z).X(z^{-1})}$ $\left(x(-n] \longleftrightarrow X(z^{-1})\right)$

c) $x(n) = a^n u(n) \Rightarrow X(z) = \dfrac{1}{1-az^{-1}} \Rightarrow R_{xx}(z) = \dfrac{1}{1-az^{-1}} \cdot \dfrac{1}{1-az}$

ROC: $|z| > |a|$ ROC: $\frac{1}{|a|} > |z| > |a|$

$\Rightarrow \boxed{R_{xx}(z) = \dfrac{1}{1+a^2-az^{-1}-az}} = \dfrac{1}{1-az^{-1}} \cdot \dfrac{z^{-1}}{z^{-1}-a} = \dfrac{A}{1-az^{-1}} + \dfrac{B}{z^{-1}-a} = \dfrac{A}{1-az^{-1}} - \dfrac{B/a}{1-a^{-1}z^{-1}}$

$(|z| > |a|)$ $(|z| < \frac{1}{|a|})$

$\Rightarrow Az^{-1} - aA + B - Baz^{-1} = z^{-1} \Rightarrow \left.\begin{array}{l} -aA + B = 0 \\ A - Ba = 1 \end{array}\right\} \Rightarrow \begin{array}{l} A = 1/1-a^2 \\ B = a/1-a^2 \end{array}$

$\Rightarrow r_{xx}(k) = \left(\dfrac{1}{1-a^2}\right) a^k u[k] + \left(\dfrac{1}{1-a^2}\right) a^{-k} u(-k-1] \boxed{= \dfrac{1}{(1-a^2)} a^{|k|} \quad -\infty < k < \infty}$

Q2) Let $v[n]$ be a real sequence of length $2N$ with $V[k]$ denoting its 2N-point DFT. Define two real sequences, $g[n]$ and $h[n]$ of length N each as

$$g[n]= v[2n], \qquad h[n]= v[2n+1], \qquad 0 \le n \le N-1.$$

while $G[k]$ and $H[k]$ denote their N-point DFTs, respectively.

a) Determine $V[k]$ in terms of $g[n]$ and $h[n]$.

b) Determine $V[k]$ ($2N$-point DFT) in terms of $G[k]$ and $H[k]$ (two N-point DFTs) for $0 \le k \le 2N-1$

c) Assume $v[n]=\{1\ 2\ 2\ 1\}$. Calculate $G[k]$, $H[k]$ and $V[k]$ by using the method in part-b.

a) $V(k] = \sum_{n=0}^{2N-1} v(n] W_{2N}^{nk} = \sum_{m=0}^{N-1} v[2m] W_{2N}^{2m} + \sum_{m=0}^{N-1} v[2m+1] W_{2N}^{2m+1}$

$$\boxed{V[k] = \sum_{m=0}^{N-1} g(m] W_{2N}^{2mk} + \sum_{m=0}^{N-1} h(m] W_{2N}^{(2m+1)k}}$$

$= -j\frac{2\pi}{2N}(2m)k$

b) Note that $G[k] = \sum_{m=0}^{N-1} g(m] W_N^{mk} = \sum_{m=0}^{N-1} g(m] e^{-j\frac{2\pi}{N}mk}$

$\Rightarrow \sum_{m=0}^{N-1} g(m] W_{2N}^{2mk} = \sum_{m=0}^{N-1} g(m] W_N^{mk} = G(k]$

Similarly

$\Rightarrow \sum_{m=0}^{N-1} h(m] W_{2N}^{(2m+1)k} = W_{2N}^{k} \sum_{m=0}^{N-1} h(m] W_N^{mk} = W_{2N}^{k} H[k]$.

$$\Rightarrow \boxed{V(k] = G[k] + W_{2N}^{k} H[k], \quad 0 \le k \le 2N-1}$$

c) If $v(n] = \{1\ 2\ 2\ 1\} \Rightarrow g(n] = \{1\ 2\}, \quad h(n] = \{2\ 1\}$

$G[k] = g(0] + e^{-j\pi k} g(1] \quad \Rightarrow \quad G[0] = 1+2 = 3$
$\qquad\qquad\qquad\qquad\qquad\qquad\quad G(1] = 1-2 = -1$

$H[k] = h(0] + e^{-j\pi k} h(1] \quad \Rightarrow \quad H(0] = 2+1 = 3$
$\qquad\qquad\qquad\qquad\qquad\qquad\quad H(1] = 2-1 = 1$

$V[0] = 3+1.3 = 6, \quad V[1] = -1 + e^{-j\frac{\pi}{2}}.1, \quad V[2] = 3 + e^{-j\pi}.3, \quad V(3] = -1 + e^{-j\frac{3\pi}{2}}$

Q3) The impulse response of an LTI system is given as $h[n] = \left(\frac{1}{4}\right)^n u[n]$

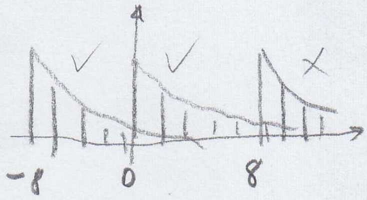
a) Find DTFT of $h[n]$, as $H(e^{jw})$.

b) Take 8 samples from $H(e^{jw})$ in part-a for $w = \frac{2\pi k}{8}, k = 0, \ldots, 7$ to obtain $G[k]$. Find $g[n]$ in its simplest form which is equal to the 8-point inverse DFT of $G[k]$. Compare $g[n]$ with $h[n]$ for $0 \leq n < 8$.

c) Assume a 3-point sequence, $x[n]$, is input to this system, whose impulse response is $h[n]$, to obtain the output, $y[n]$. If only 9-point DFT is available, explain how to obtain the output, $y[n]$, by using any technique based on DFT.

a) $H(e^{jw}) = \sum_{n=-\infty}^{+\infty} h(n) e^{-jwn} = \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n e^{-jwn} = \boxed{\frac{1}{1-\left(\frac{1}{4}\right)e^{-jw}}}$

b) $G[k] = H(e^{jw})\Big|_{w=\frac{2\pi}{8}k} = \boxed{\frac{1}{1-\left(\frac{1}{4}\right)e^{-jw}}\Big|_{w=\frac{2\pi}{8}k}}$



$\boxed{g(n) = \sum_{k=-\infty}^{+\infty} h(n-8k] \quad 0 \leq n \leq 7} = \sum_{k=-\infty}^{+\infty} \left(\frac{1}{4}\right)^{n-8k} u(n-8k] \quad 0 \leq n \leq 7$

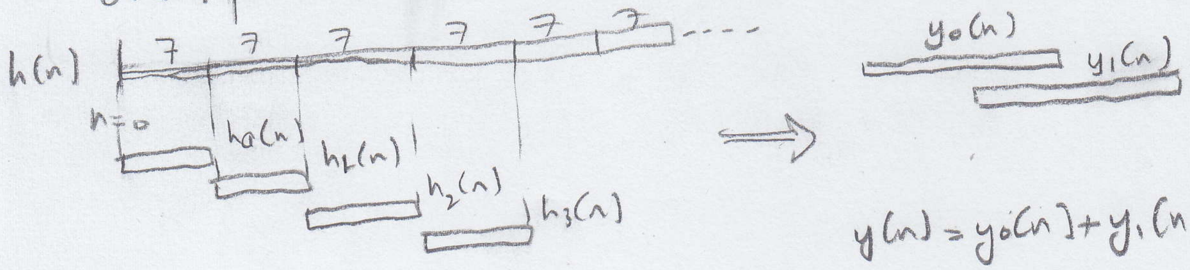
$= \sum_{k=-\infty}^{0} \left(\frac{1}{4}\right)^{n-8k} = \sum_{m=0}^{\infty} \left(\frac{1}{4}\right)^{n+8m} = \boxed{\left(\frac{1}{4}\right)^n \cdot \frac{1}{1-\left(\frac{1}{4}\right)^8} \quad 0 \leq n \leq 7.}$

Since $h[n] = \left(\frac{1}{4}\right)^n u(n]$, $h(n]$ & $g(n]$ have very similar values for $0 \leq n < 8$.

c) Due to commutation property: $x(n] * h(n] = h(n] * x(n] = y(n)$

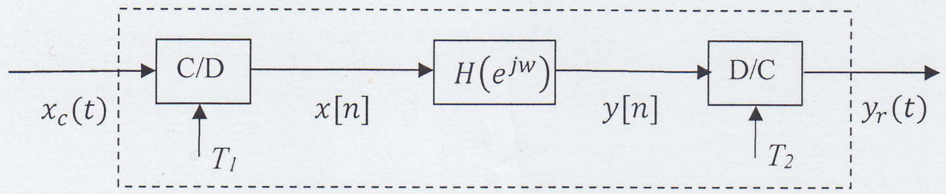
To use 9-point DFT, divide $h(n]$ into 7-point chunks.

Take 9-point DFT of $h_i(n]$ & $x(n)$ and use overlap-and-add.



$y(n) = y_0(n] + y_1(n-N] + y_2(n-2N]$

Q4) Consider the following signal processing system with ideal C/D and D/C components:



a) Write $Y_r(j\Omega)$ in terms of $X_c(j\Omega)$ for $T_1 \neq T_2$.

b) Assume $x_c(t) = \cos(250\pi t)$ and the frequency response of the DT system in the figure is equal to, $H(e^{jw}) = |\frac{6w}{\pi}|\, e^{-jw}$, $|w| \leq \pi$,

i. Find maximum sampling period T_1 such that given $x_c(t)$ can be recovered from x[n].

ii. Let T_1 = 1/300. Find T_2, α, β so that output is equal to $y_r(t) = \alpha\cos(750\pi t + \beta)$.

iii. Let T_2 = 1/200. Determine T_1 that results with the output $y_r(t) = 3\sin(100\pi t)$, while this value of T_1 causes aliasing at the input.

a) $X(e^{jw}) = \frac{1}{T_1}\sum_{k=-\infty}^{+\infty} X_c\left(j\left(\frac{w}{T_1} - \frac{2\pi}{T_1}k\right)\right)$; $Y_r(j\Omega) = H_r(j\Omega)Y_s(j\Omega) = H_r(j\Omega)\,Y(e^{j\Omega T_2})$

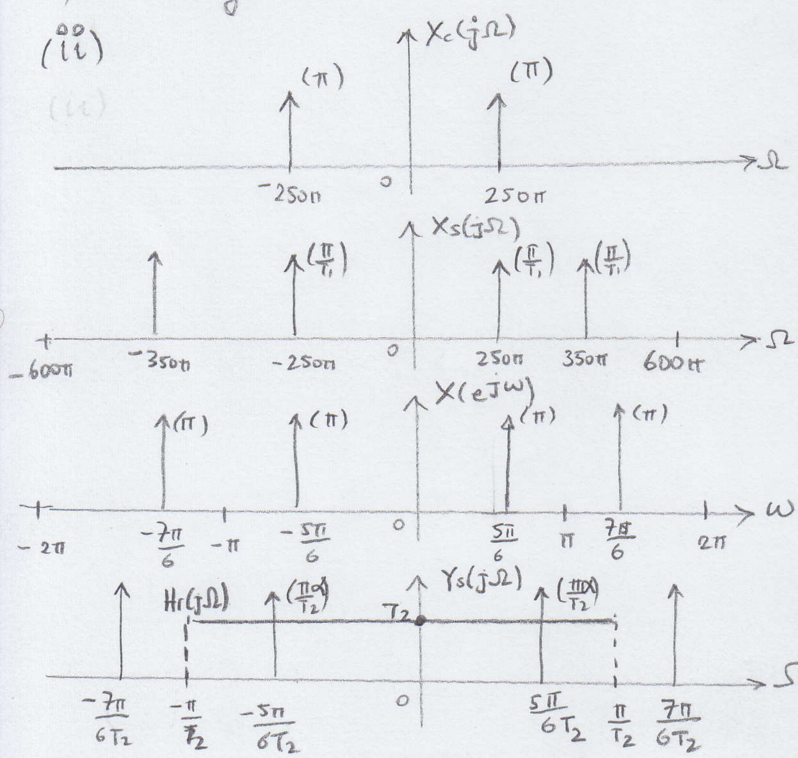
Since $Y(e^{jw}) = H(e^{jw})X(e^{jw}) \Rightarrow Y_r(j\Omega) = H_r(j\Omega)H(e^{j\Omega T_1})X(e^{j\Omega T_1})$

$\Rightarrow Y_r(j\Omega) = H_r(j\Omega)H(e^{j\Omega T_1})\left(\frac{1}{T_1}\sum_{k=-\infty}^{+\infty} X_c\left(j\left(\Omega - \frac{2\pi}{T_1}k\right)\right)\right) = \begin{cases} \left(\frac{T_2}{T_1}\right) H(e^{j\Omega T_1})\left(\sum X_c\left(j\left(\Omega - \frac{2\pi k}{T_1}\right)\right)\right) & |\Omega| < \pi/T_2 \\ 0 & \text{e.w.} \end{cases}$

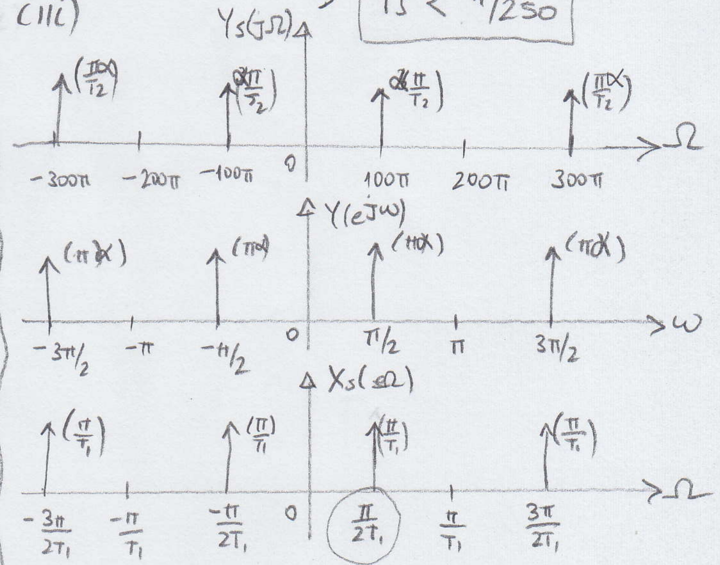
b) (i) Using Nyquist Theorem, $x_c(t) = \cos(\Omega_0 t) \Rightarrow \Omega_0 = 250\pi \Rightarrow \Omega_s > 2.250\pi$

$\Rightarrow \boxed{T_s < 1/250}$

(ii)



(iii)



No aliasing, since $T_1 = \frac{1}{300} < \frac{1}{250}$.

Since $y_r(t) = \alpha\cos(750\pi t + \beta) \Rightarrow \frac{5\pi}{6T_2} = 750\pi \Rightarrow \boxed{T_2 = \frac{1}{900}}$

$H(e^{j5\pi/6}) = 5.e^{-j5\pi/6} \Rightarrow \boxed{\alpha = 5}$ $\boxed{\beta = -5\pi/6}$

For $X_s(j\Omega)$; impulses due to $\cos(250\pi t)$ could be located at the following freq:

$\Omega = \left\{+250\pi,\ \frac{2\pi}{T_1} - 250\pi,\ -\frac{2\pi}{T_1} + 250\pi, \ldots\right\}$

If $250\pi = \frac{\pi}{2T_1} \Rightarrow T_1 = 1/500$ no aliasing

If $\frac{2\pi}{T_1} - 250\pi = \frac{\pi}{2T_1} \Rightarrow \boxed{T_1 = 3/500}$ ✓

If $-\frac{2\pi}{T_1} + 250\pi = \frac{\pi}{2T_1} \Rightarrow \boxed{T_1 = 1/100}$ ✓