EE430 - Digital Signal Processing

Homework 3

Due Date: 24.12.2017 until 23:55

Submit your answers for the questions 3, 8, 14, 15, 17, 21, M1, M2, M4

1. $x(t)$ is a complex valued signal whose CTFT is zero for $\Omega \leq 100\pi$ and $\Omega \geq 1000\pi$. Find the minimum sampling frequency in Hz so that $x(t)$ can be recovered from its samples, $x[n]$.

2. a) Are all-pass systems invertible?
   b) Are minimum phase systems invertible?
   c) What is a linear phase system? What is a generalized linear-phase system? Clarify their difference.
   d) May a linear-phase system have an arbitrary impulse response? May a generalized linear-phase system have an arbitrary impulse response? Explain.

3. Consider the two systems in Fig. 1.

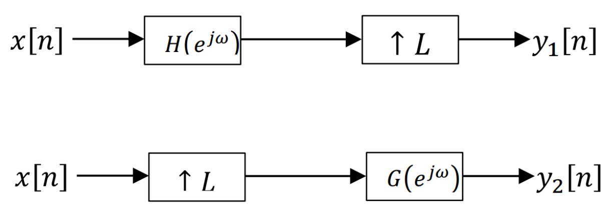

Figure 1

Can you find a relationship between $H(e^{jw})$ and $G(e^{jw})$ so that the two systems are equivalent, i.e., $y_1[n] = y_2[n]$? Prove your claim.

4. Consider the two systems in Fig. 2.

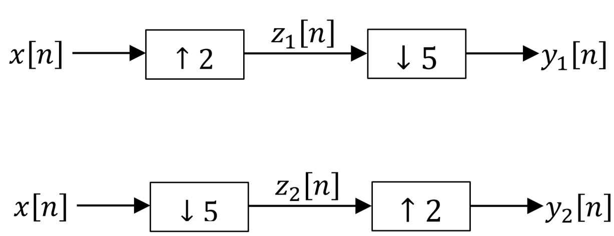

Figure 2

   a) Express $Z_1(e^{jw})$ and $Y_1(e^{jw})$ in terms of $X(e^{jw})$.

b) Express $Z_2(e^{jw})$ and $Y_2(e^{jw})$ in terms of $X(e^{jw})$.

c) Are $y_1[n]$ and $y_2[n]$ the same? Prove your claim.

d) Does your answer to part (c) change if the downsampling factor is 4?

5. It is desired to have an ideal bandpass filter with the system below.

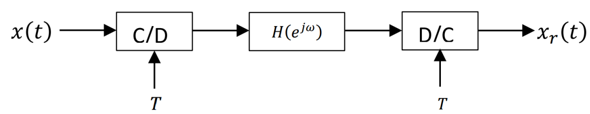

Figure 3

In particular, it is desired to have

$$H_{eff(j\Omega)} = \frac{X_r(j\Omega)}{X(j\Omega)} = \begin{cases} 1, & 9000\pi \le |\Omega| \le 10000\pi \\ 0, & \text{otherwise} \end{cases}$$

It is known that $H(e^{jw})$ is a highpass filter.

a) Find T, specify $H(e^{jw})$. Are there any constraints that must also be stated so that the system acts as the specified bandpass filter without any reservation?

b) Answer part (a) to have

$$H_{eff(j\Omega)} = \frac{X_r(j\Omega)}{X(j\Omega)} = \begin{cases} 1, & 900\pi \le |\Omega| \le 1000\pi \\ 0, & \text{otherwise} \end{cases}$$

6. a) Suppose that an audio signal is sampled at $fs = 48$KHz and a block of 500 samples are collected for analysis, forming a signal, $x[n]$. The 512-point DFT of $x[n]$ (after zero-padding) is $X[k]$ and the magnitude of $X[k]$ contains peaks (showing dominant frequencies) corresponding at the indices, $k = \{20, 45, 162, 350, 467, 492\}$.

   i) How many dominant frequency components of this signal will be heard?

   ii) What are the values of the frequencies, in Hz, in part (a) for this analog audio signal?

   b) Given the system in Fig. 4, the frequency response of the digital filter is equal to $H(e^{jw}) = 1 - e^{-j5w}$. Assume the input is given to the system, as $x_c(t) = 2 + 3\cos(\Omega_1 t) + 4\sin(\Omega_2 t)$ for $-\infty < t < \infty$.

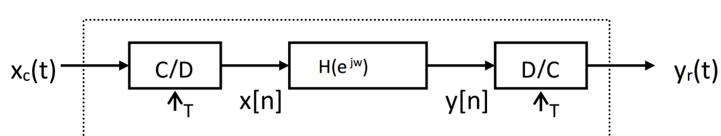

Figure 4

i) Given $\Omega_1 = 200\pi$ and $\Omega_2 = 440\pi$, state and explain the Nyquist rate of this signal.

ii) Let the sampling rate be 1000Hz. Given $\Omega_1 = 500\pi$ and $\Omega_2 = 1500\pi$, sketch $|X(e^{jw})|$, $|Y(e^{jw})|$, and $|Y_r(e^{jw})|$.

iii) For the same sampling rate 1000 Hz, now, let Ω_1 and Ω_2 be unknown and not equal to each other. Determine non-zero analog frequencies for Ω_1 and Ω_2 which causes output signal to be zero for the input $x_c(t)$. Pick Ω_1 and Ω_2, so that there is no aliasing.

7. A discrete-time signal, $x[n]$, is obtained by sampling a continuous-time signal, $x_c(t)$, bandlimited to $\frac{\pi}{T}$, where T is the sampling period. $x[n]$ is applied to a causal LTI system whose impulse response is

$$h_D[n] = \begin{cases} \dfrac{\sin\left(\frac{\pi}{T}\left(\frac{T}{5} + (n-D)T\right)\right)}{\frac{\pi}{T}\left(\frac{T}{5} + (n-D)T\right)}, & n = 0, 1, \ldots, 2D \\ 0, & \text{otherwise} \end{cases}$$

Discuss the relationship of the output, $y[n]$, of this system to $x_c(t)$. May your discussion have any practical implications?

8. Increasing sampling rate over the Nyquist-rate (i.e. oversampling) is usually performed in order to minimize distortions. In a typical DSP system, oversampling can be utilized in A/D conversion stage. For A/D stage, continuous-time frequency response of a non-ideal anti-aliasing filter, $H(j\Omega)$, is given below.

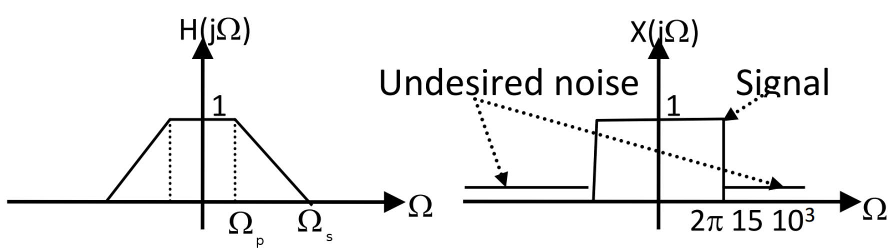

Figure 5

Assume input signal $x(t)$ with a frequency response $X(j\Omega)$ is first processed with this filter in order to remove the undesired noise before sampling for the following rates :

a) Nyquist-rate,

b) 35 KHz,

c) 44 KHz.

If Ω_p is chosen as the cutoff frequency ($2\pi 15 \times 10^3$rad/sec) of $X(j\Omega)$, then for each sampling case, find value of Ω_s, yielding maximum filter transition band, i.e., $\Omega_s - \Omega_p$ while avoiding any aliasing on the signal. For which case, the corresponding filter is easiest to implement?

9. Plot the phase functions of $\dfrac{1}{1-0.9z^{-1}}$ and $\dfrac{1}{1-\frac{10}{9}z^{-1}}$, and describe their difference.

10. Can you make a minimum-phase and all-pass decomposition of $(z) = \dfrac{(1-2z^{-1})(z+2)(z-\frac{1}{2})}{1-\frac{1}{2}z^{-1}+\frac{1}{2}z^{-2}}$?

    If yes, find the minimum-phase and all-pass functions.

11. Can you make a minimum-phase and all-pass decomposition of $(z) = \dfrac{(1-2z^{-1})(z+2)}{(1-\frac{1}{2}z^{-1}+\frac{1}{2}z^{-2})(z+\frac{1}{2})}$?

    If yes, find the minimum-phase and all-pass functions.

12. a) The squared magnitude response of $H(e^{jw})$ is given as,

$$\left|H(e^{jw})\right|^2 = \frac{\frac{5}{4}-\cos(w)}{\frac{10}{9}-\frac{2}{3}\cos(w)}.$$

    Find the minimum-phase system $H_{min}(z)$ which has the same magnitude response as $H(e^{jw})$.

    b) Assume $H(z)$ is given as,

$$H(z) = \frac{(1-0.5z^{-1})(1+4z^{-2})}{1-0.64z^{-2}}.$$

    Find the minimum-phase $H_{min}(z)$ and linear-phase $H_{lin}(z)$ filters such that $H(z) = H_{min}(z)H_{lin}(z)$. Plot the approximate magnitude response of $H_{min}(z)$ indicating only the critical values.

    c) Assume that $h_{lp}[n]$ is the impulse response of an FIR generalized linear-phase lowpass filter of length N. A generalized linear-phase highpass FIR filter $h_{hp}[n]$ will be obtained from $h_{lp}[n]$ by multiplication with a cosine (cosine modulation).

       i) Determine the requirement for ω_0 of the frequency of the cosine.

       ii) Write the expression for $h_{hp}[n]$. Also write the expression for $H_{hp}(e^jw)$.

13. Prove that minimum-phase systems has the minimum energy delay property.

14. Consider a causal and stable LTI system with a system function $H(z)$, whose pole-zero diagram is shown in Fig. 6, where $p_1 = p_2^* = 0.8e^{j\frac{\pi}{6}}$, $p_3 = p_4^* = 0.8e^{j\frac{5\pi}{6}}$, $z_1 = z_2^* = 1.2e^{j\frac{\pi}{4}}$, and $z_3 = z_4^* = 1.2e^{j\frac{3\pi}{4}}$.

    a) Express $H(z)$ as $H(z) = H_{min}(z)H_{ap}(z)$ where $H_{min}(z)$ and $H_{ap}(z)$ are minimum-phase and all-pass system functions, respectively. Justify your answer.

    b) Find $H_1(z)$ and $H_2(z)$ such that $H_{min}(z) = H_1(z)H_2(z)$ and $H_1(z)$ and $H_2(z)$ have, roughly, low-pass and high-pass frequency responses, respectively. Justify your answer.

    c) Find $\left|H_1(e^{j\frac{\pi}{3}})\right|$ using a purely geometric approach on the pole-zero plot of $H_1(z)$. Justify your answer.

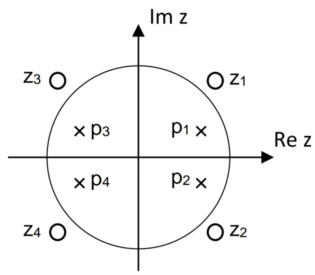

Figure 6: Pole-zero diagram of $H(z)$

15. a) Assume a real casual FIR filter, $g[n]$, with the system function $\mathrm{G}(z) = 1 + z^{-1}$. Find pole(s) and zero(s) of $G(z)$ and plot only the principal value of the phase for this system function, namely $\angle G(e^{jw})$.

    b) Another real casual FIR filter, $h[n]$, has the following principal value of its phase, $\angle H(e^{jw})$, presented in the Fig. 7.

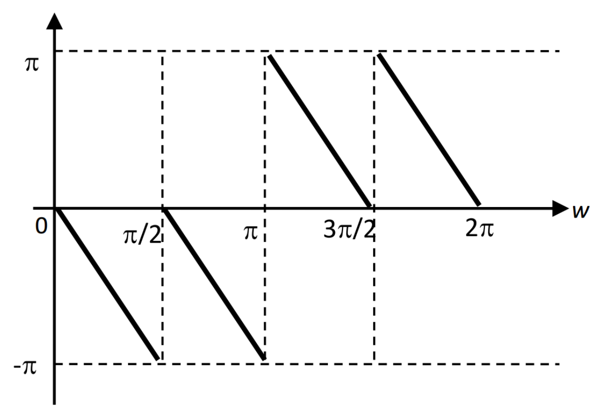

Figure 7

i) Explain reasons of discontinuities in $\angle H(e^{jw})$ for $\pi/2$, π, and $3\pi/2$.

ii) Determine group delay of this filter. State whether this filter has a generalized linear phase; if so, try to determine its type, when $h[n] = 0$, $n > 4$.

c) Apart from the properties of $h[n]$ given in part (b), $H(z)$ also contains a zero at $z = 1/2$, and $H(1) = 1$. Find $H(z)$ and indicate its poles and zeros on the z-plane.

d) Determine $\left|H(e^{jw})\right|$ as a closed form expression and approximately plot $\left|H(e^{jw})\right|$.

16. Given a real causal generalized linear-phase FIR system, $H(z)$, of Type III, with some of its zeros at the following locations$z = -j\frac{1}{2}$ and $z = 2$,

    a) Plot the complete pole-zero diagram of $h(z)$, indicating its pole and zero locations.

    b) Find the impulse response of this system, if $\lim_{z\to\infty} H(z) = 0.5$ Check its symmetry in comparison to Type III systems.

c) Using $H(z)$obtain a cascaded linear phase system, consisting of $H_4(z)$ and $H_2(z)$, such that their overall response is equal to $H(z)$. $H_4(z)$ has a zero at $z = 0.5$. $H_4(z)$ and $H_2(z)$ must also be of Type IV and Type II, respectively. Plot their respective pole-zero diagrams for both $H_4(z)$ and $H_2(z)$.

d) Among the following types, *low-pass*, *high-pass* and *band-pass*, which ones can be obtained by using $H_4(z)$, and $H_2(z)$, individually.

M1. a) Write a MATLAB script that generates a chirp signal, which is 0 Hz and 200 Hz at time instants 0 and 100 with time step of 0.001. Listen to it by using *sound* command. Describe your experience in terms of frequency components you hear.

b) Compute the DFT of this signal. Plot its magnitude and phase.

c) Using *downsample* command, reduce the sampling rate of the signal by a factor of $M = 2$. Listen to the downsampled signal. Repeat this procedure for $M = \{5, 10, 20\}$. Compare what you hear for different rates by justifying your answer.

d) Plot the magnitudes of DFT's of the signals that you downsampled in part (c). Comment on the results.

M2. Consider the DFT of a bandlimited signal given in Fig. 8.

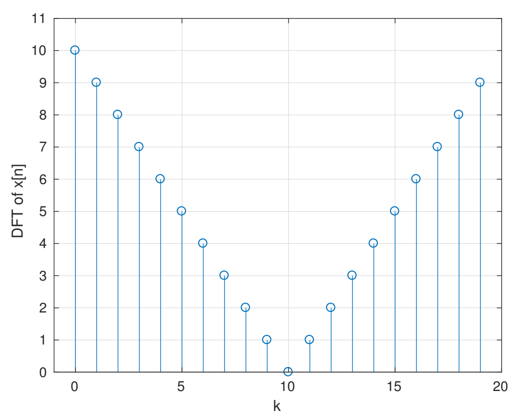

Figure 8: DFT of $x[n]$

a) Compute the inverse DFT of this signal, $x[n]$, and plot its magnitude.

b) Increase the sampling rate of this signal by $N = 5$ using *upsample* command of MATLAB and plot the the resultant sequence, $x_u[n]$.

c) Compute DFT of the upsampled signal and plot its magnitude. Compare it with the one given in Fig. 8 without referring to the derivation of DFT of an upsampled signal, that is covered in lectures, but considering the effect of zero-padding between the samples.

d) Write a small script that applies zero-order hold instead of zero-padding to interpolate samples between the original samples. Plot your result for $x[n]$, denoted by $x_{ZOH}[n]$, when $N = 5$. Plot the magnitude of the DFT of $x_{ZOH}[n]$.

e) Repeat part (d) when linear interpolation is performed in between the original samples instead of zero-order hold. (You can designate the upsampled signal by $x_{FOH}[n]$ referring that is also known as first-order hold.)

f) Compare the DFTs you obtained in part (c), (d), and (e). Which one would you prefer for which reason? Can you think of an ideal interpolator that fills the missing samples of an properly sampled continuous-time signal perfectly? Justify your answer.

M3. Given the impulse response $h[n]$ of a length-N FIR system, develop a code to check whether or not the system has minimum phase. Further, if the system is non-minimum-phase, we want to construct a minimum-phase system with the same magnitude $\left|H(e^{jw})\right|$. Write a MATLAB program such that it takes $h[n]$ as its input and returns $h_{min}[n]$ and plot it. You can generate $h[n]$, for example, by $h = randn(1, 10)$.

M4. Consider

$$H(z) = \frac{(1 - z_1 z^{-1})(1 - z_1^* z^{-1})}{(1 - p_1 z^{-1})(1 - p_1^* z^{-1})}$$

where $z_1 = 1.2e^{j\frac{3\pi}{4}}$ and $p_1 = 0.5e^{j\frac{pi}{2}}$.

a) Find the impulse response, $h[n]$.

b) Find the impulse response of the minimum phase *counterpart*, $h_{min}[n]$.

c) Plot the phase-lag functions of $H(z)$ and $H_{min}(z)$, the minimum-phase system function associated with $H(z)$.

d) Plot the group-delay functions related to $H(z)$ and $H_{min}(z)$.

e) Verify the *minimum phase-lag* property of minimum-phase systems.

f) Verify the *minimum group-delay* property of minimum-phase systems.

g) Verify the *minimum energy-delay* property of minimum-phase systems.

M5. In this problem, you will write your own group delay function by using a simple property. Express $X(e^{jw})$ in polar form $X(e^{jw}) = \left|X(e^{jw})\right| e^{j\theta(w)}$. Prove that

$$-\frac{d\theta(w)}{dw} = \mathrm{Re}\left\{ \frac{f\frac{dX(e^{jw})}{dw}}{X(e^{jw})} \right\}$$

.

Now write a function $gd = gdel(x, n, L)$ to find the group delay of a signal $x[n]$, where n is the time index (may start from 0), L is the dft length. Use the above property and *fft* command to write your function.

a) Find the group delay for $x = [\ 1\ \ 2\ \ 3\ \ 4\ \ 4\ \ 3\ \ 2\ \ 1\ ]$ and plot it.

b) Use the MATLAB function filter to generate the impulse response of the causal filter.

$$H(z) = \frac{1}{1 - 0.7z^{-1}}$$

Plot impulse response for $0 \leq n \leq 128$. Find the frequency response magnitude and group delay and plot them.

M6. In this problem, you are going to investigate the non-causal filtering as well as zero-phase IIR filtering. Consider

$$H(z) = \frac{1}{1 - 0.7z}.$$

This is a not a causal and stable filter. In order to filter an input in a stable manner with this filter, we have to apply a non-causal processing. Prove that $Y(z) = H(z)X(z)$ can be obtained by $Y_1(z) = X(z^{-1})H(z^{-1})$ and $Y(z) = Y_1(z^{-1})$.

a) Write a function, which takes the filter coefficients and input and outputs the non-causal response, i.e., function $out = noncas(a, x)$, where a is the vector of $H(z)$ denominator polynomial, x is the vector of input signal.

b) Take $x = [\ 1 \ \ 0 \ \ 0 \ \ \ldots \ \ 0\ ]$ of length 64, and find and plot the output with its frequency magnitude as well.

c) Now assume that $H(z)$ is

$$H(z) = \frac{1}{(1 - 0.7z^{-1})(1 - 0.7z)}$$

Use the above function and generate the impulse 128 points of the impulse response of $H(z)$ and plot it. Find the frequency magnitude and group delay and plot them.

17. Problem 4.23 from the third edition of the textbook given in Fig. 9.

18. Problem 4.29 from the third edition of the textbook given in Fig. 10.

19. Problem 4.32 from the third edition of the textbook given in Fig. 11.

20. Problem 5.32 from the third edition of the textbook given in Fig. 12.

21. Problem 5.45 from the third edition of the textbook given in Fig. 13.

22. Problem 5.47 from the third edition of the textbook given in Fig. 14.

23. Problem 5.54 from the third edition of the textbook given in Fig. 15.

Figure P23-1 shows a continuous-time filter that is implemented using an LTI discrete-time filter ideal lowpass filter with frequency response over $-\pi \le \omega \le \pi$ as

$$H(e^{j\omega}) = \begin{cases} 1 & |\omega| < \omega_c \\ 0 & \omega_c < |\omega| \le \pi. \end{cases}$$

(a) If the continuous-time Fourier transform of $x_c(t)$, namely $X_c(j\Omega)$, is as shown in Figure P23-2 and $\omega_c = \frac{\pi}{5}$, sketch and label $X(e^{j\omega})$, $Y(e^{j\omega})$ and $Y_c(j\Omega)$ for each of the following cases:

(i) $1/T_1 = 1/T_2 = 2 \times 10^4$
(ii) $1/T_1 = 4 \times 10^4,\ \ 1/T_2 = 10^4$
(iii) $1/T_1 = 10^4,\ \ 1/T_2 = 3 \times 10^4$.

(b) For $1/T_1 = 1/T_2 = 6 \times 10^3$, and for input signals $x_c(t)$ whose spectra are bandlimited to $|\Omega| < 2\pi \times 5 \times 10^3$ (but otherwise unconstrained), what is the maximum choice of the cutoff frequency ω_c of the filter $H(e^{j\omega})$ for which the overall system is LTI? For this maximum choice of ω_c, specify $H_c(j\Omega)$.

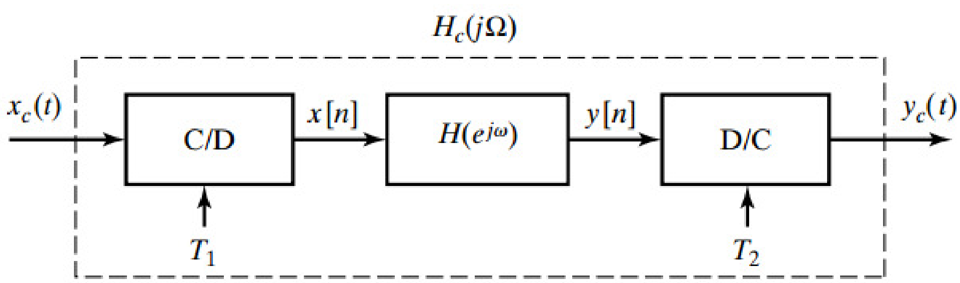

Figure P23-1

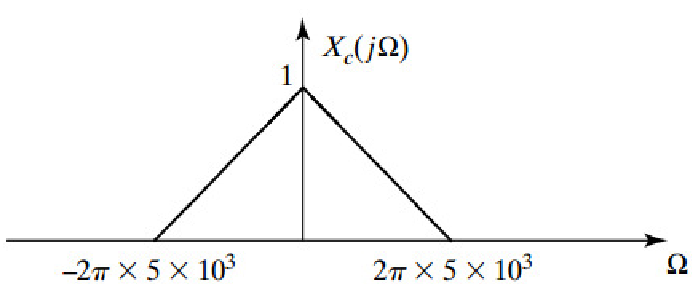

Figure P23-2

Figure 9: Problem 4.23 from the third edition of the textbook.

In Figure P29, assume that $X_c(j\Omega) = 0$, $|\Omega| \geq \pi/T_1$. For the general case in which $T_1 \neq T_2$ in the system, express $y_c(t)$ in terms of $x_c(t)$. Is the basic relationship different for $T_1 > T_2$ and $T_1 < T_2$?

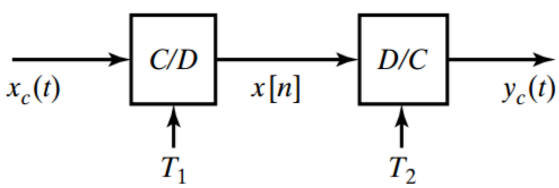

Figure P29

Figure 10: Problem 4.29 from the third edition of the textbook.

Consider the discrete-time system shown in Figure P32-1

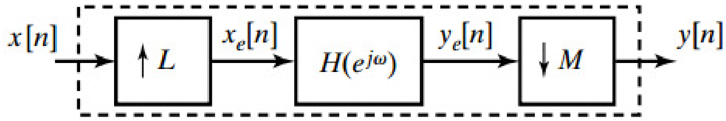

Figure P32-1

where

(i) L and M are positive integers.

(ii) $x_e[n] = \begin{cases} x[n/L] & n = kL, \quad k \text{ is any integer} \\ 0 & \text{otherwise.} \end{cases}$

(iii) $y[n] = y_e[nM]$.

(iv) $H(e^{j\omega}) = \begin{cases} M & |\omega| \leq \frac{\pi}{4} \\ 0 & \frac{\pi}{4} < |\omega| \leq \pi \end{cases}$.

(a) Assume that $L = 2$ and $M = 4$, and that $X(e^{j\omega})$, the DTFT of $x[n]$, is real and is as shown in Figure P32-2. Make an appropriately labeled sketch of $X_e(e^{j\omega})$, $Y_e(e^{j\omega})$, and $Y(e^{j\omega})$, the DTFTs of $x_e[n]$, $y_e[n]$, and $y[n]$, respectively. Be sure to clearly label salient amplitudes and frequencies.

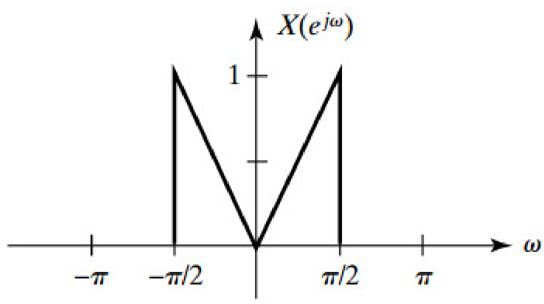

Figure P32-2

(b) Now assume $L = 2$ and $M = 8$. Determine $y[n]$ in this case.
Hint: See which diagrams in your answer to part (a) change.

Figure 11: Problem 4.32 from the third edition of the textbook.

Suppose that a causal LTI system has an impulse response of length 6 as shown in Figure P32, where c is a real-valued constant (positive or negative).

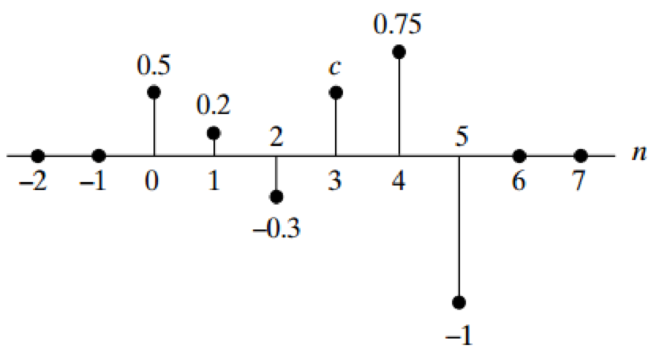

Figure P32

Which of the following statements is true:

(a) This system must be minimum phase.
(b) This system cannot be minimum phase.
(c) This system may or may not be minimum phase, depending on the value of c.

Justify your answer.

Figure 12: Problem 5.32 from the third edition of the textbook.

The pole–zero plots in Figure P45 describe six different causal LTI systems.

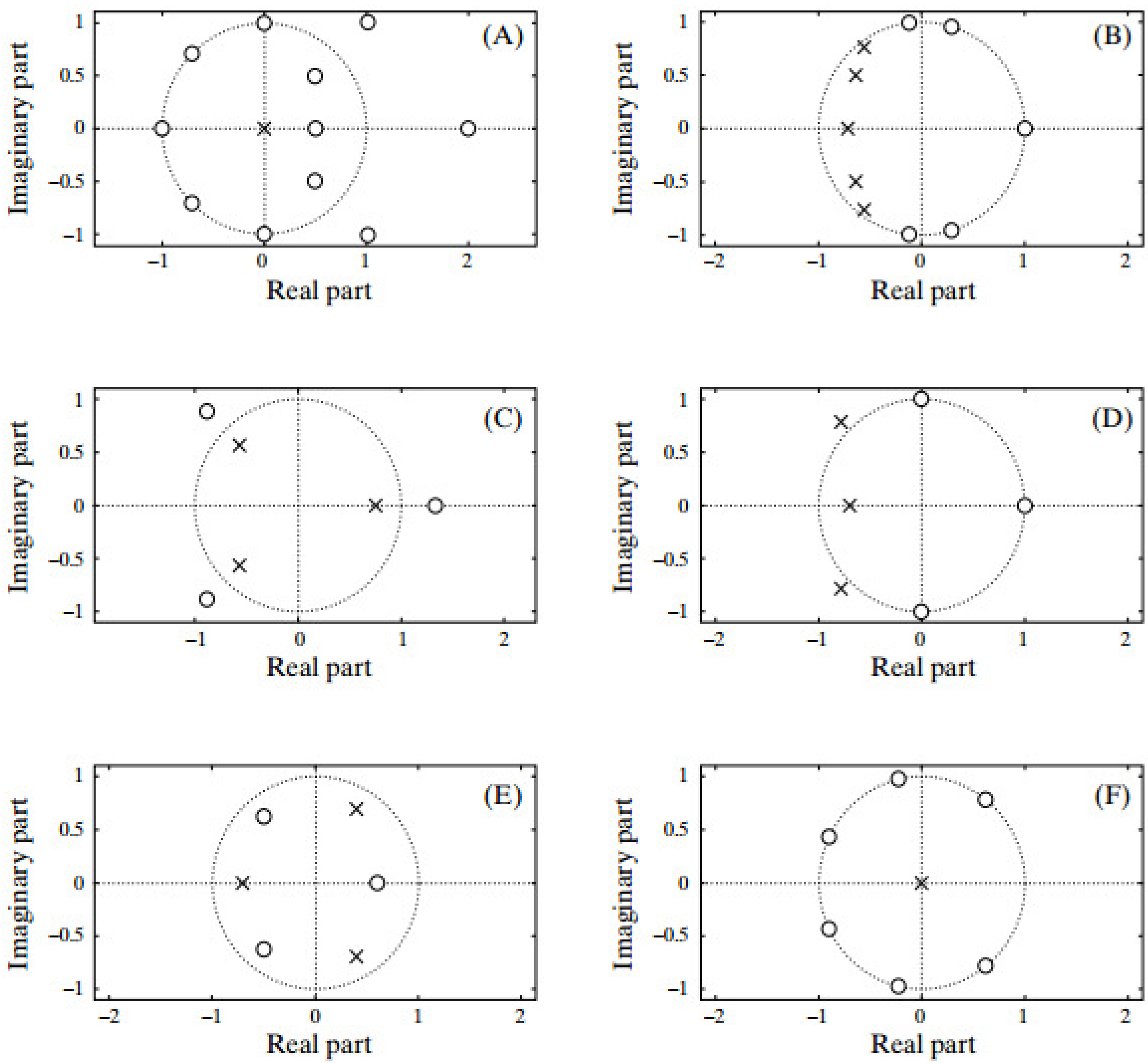

Figure P45

Answer the following questions about the systems having the above pole–zero plots. In each case, an acceptable answer could be *none* or *all*.

(a) Which systems are IIR systems?
(b) Which systems are FIR systems?
(c) Which systems are stable systems?
(d) Which systems are minimum-phase systems?
(e) Which systems are generalized linear-phase systems?
(f) Which systems have $|H(e^{j\omega})|$=constant for all ω?
(g) Which systems have corresponding stable and causal inverse systems?
(h) Which system has the shortest (least number of nonzero samples) impulse response?
(i) Which systems have lowpass frequency responses?
(j) Which systems have minimum group delay?

Figure 13: Problem 5.45 from the third edition of the textbook.

A linear-phase FIR system has a real impulse response $h[n]$ whose z-transform is known to have the form

$$H(z) = (1 - az^{-1})(1 - e^{j\pi/2}z^{-1})(1 - bz^{-1})(1 - 0.5z^{-1})(1 - cz^{-1})$$

where a, b, and c are zeros of $H(z)$ that you are to find. It is also known that $H(e^{j\omega}) = 0$ for $\omega = 0$. This information and knowledge of the properties of linear-phase systems are sufficient to completely determine the system function (and therefore the impulse response) and to answer the following questions:

(a) Determine the length of the impulse response (i.e., the number of nonzero samples).
(b) Is this a Type I, Type II, Type III, or Type IV system?
(c) Determine the group delay of the system in samples.
(d) Determine the unknown zeros a, b, and c. (The labels are arbitrary, but there are three more zeros to find.)
(e) Determine the values of the impulse response and sketch it as a stem plot.

Figure 14: Problem 5.47 from the third edition of the textbook.

54. Shown in Figure P54 are eight different finite-duration sequences. Each sequence is four points long. The magnitude of the Fourier transform is the same for all sequences. Which of the sequences has all the zeros of its z-transform *inside* the unit circle?

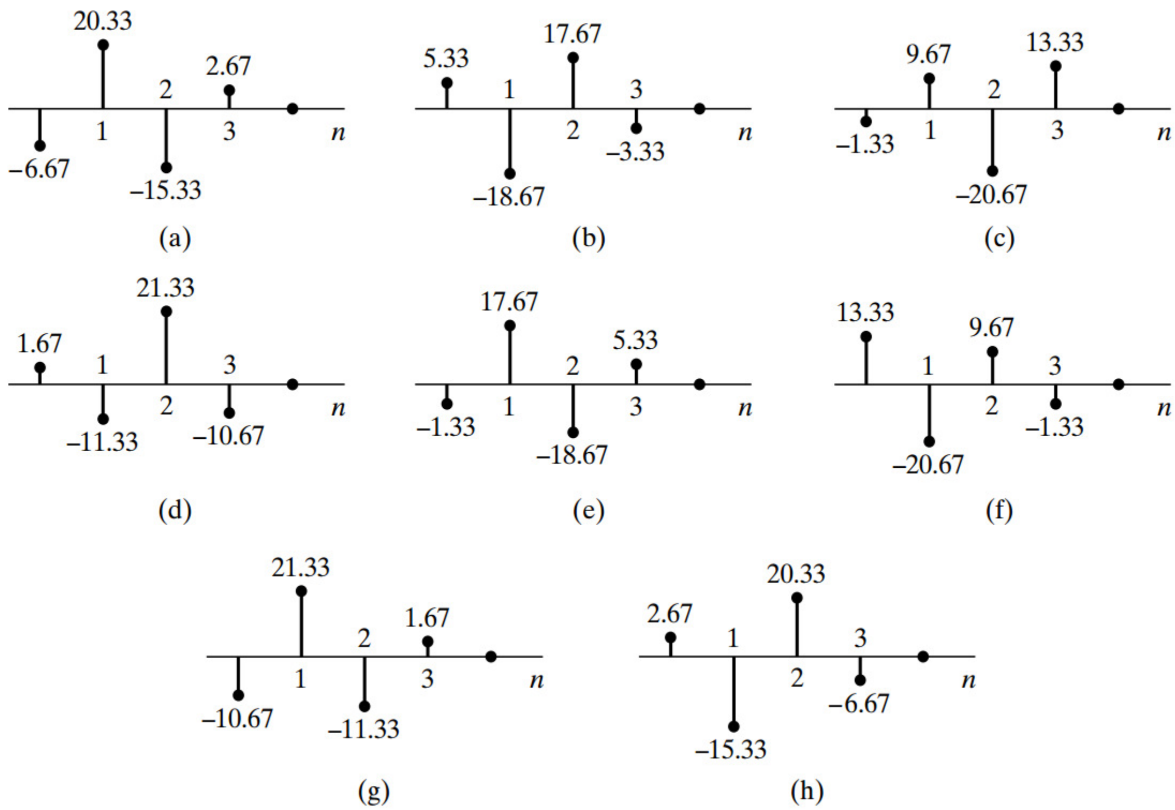

Figure P54

Figure 15: Problem 5.54 from the third edition of the textbook.