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FREQUENCY DOMAIN REPRESENTATION OF LTI SYSTEMS

In the context of LTI systems, “Frequency Domain” refers to the representation of signals and analysis of LTI systems using sinusoidal signals.

EULER'S FORMULA

Sinusoidal expressions are closely related to complex exponential expressions (Euler's formula).

$$e^{j\omega n} = \cos(\omega n) + j\sin(\omega n)$$

$$\cos(\omega n) = \frac{1}{2}\left(e^{j\omega n} + e^{-j\omega n}\right)$$

$$\sin(\omega n) = \frac{1}{2}\left(e^{j\omega n} - e^{-j\omega n}\right)$$

Note:

Euler's identity: $e^{j\pi} + 1 = 0$

Let the input signal be $e^{j\omega_0}$, i.e. a complex exponential with frequency ω_0

$$x[n]=e^{j\omega_0 n} \longrightarrow \boxed{h[n]} \longrightarrow y[n]=?$$

The output will be

$$y[n] = e^{j\omega_0 n} * h[n]$$

$$= \sum_{k=-\infty}^{\infty} h[k] e^{j\omega_0 (n-k)}$$

$$= e^{j\omega_0 n} \underbrace{\sum_{k=-\infty}^{\infty} h[k] e^{j\omega_0 k}}_{H\left(e^{j\omega_0}\right)} \quad = e^{j\omega_0 n} \; H\left(e^{j\omega_0}\right)$$

Assumed that $\sum_{k=-\infty}^{\infty} h[k] e^{j\omega_0 k}$ is finite.

Therefore, the output is the same (<u>except a complex scaling factor</u>) as the input.

The complex scale factor, $H\left(e^{j\omega_0}\right)$, is called the frequency response.

Complex exponentials or real sinusoids are called as the *eigenfunctions* of LTI systems.

Ex: $\left[1 \quad -1 \quad -\frac{1}{2}\right] \quad and \quad e^{j\frac{\pi}{3}n}$

$$H\left(e^{j\omega}\right)=\sum_{k=-\infty}^{\infty} h[k]e^{j\omega k}$$

is called the "frequency response".

It is the complex valued gain of a LTI system to a complex exponential of particular "frequency".

$$\ldots + h[-2]e^{j2\omega} + h[-1]e^{j\omega} + h[0] + h[1]e^{-j\omega} + h[2]e^{j2\omega} + \cdots$$

FREQUENCY RESPONSE IS PERIODIC WITH 2π.

$$H\left(e^{j\omega}\right)=\sum_{n=-\infty}^{\infty}h[n]e^{-j\omega n}=\sum_{n=-\infty}^{\infty}h[n]e^{-j(\omega+k2\pi)n}=\sum_{n=-\infty}^{\infty}h[n]e^{-j\omega n}\underbrace{e^{-jk2\pi n}}_{1}=H\left(e^{j(\omega+k2\pi)}\right)$$

MAGNITUDE AND PHASE

$$H\left(e^{j\omega}\right) = \left|H\left(e^{j\omega}\right)\right| e^{j\measuredangle H\left(e^{j\omega}\right)}$$

$$= \left|H\left(e^{j\omega}\right)\right| e^{j\,\theta(\omega)}$$

$$\Rightarrow$$

$$H\left(e^{j\omega_0}\right)e^{j\,\omega_0 n} = \left|H\left(e^{j\omega_0}\right)\right| e^{j\,\theta(\omega_0)} e^{j\,\omega_0 n}$$

$$= \left|H\left(e^{j\omega_0}\right)\right| e^{j\left(\omega_0 n + \theta(\omega_0)\right)}$$

$$H\left(e^{j\omega}\right) = H_R\left(e^{j\omega}\right) + jH_I\left(e^{j\omega}\right)$$

$$\left|H\left(e^{j\omega}\right)\right| : \text{magnitude of the frequency response} \left(\sqrt{H_R^2 + H_I^2}\right)$$

$$\angle H\left(e^{j\omega}\right) : \text{phase of the frequency response} \left(\tan^{-1}\frac{H_I}{H_R}\right)$$

FOR A REAL LTI SYSTEM, FREQUENCY RESPONSE IS CONJUGATE SYMMETRIC

Real LTI system means impulse response is real valued.

$$H^*\left(e^{j\omega}\right) = \left(\sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n}\right)^*$$

$$= \sum_{n=-\infty}^{\infty} h[n]e^{j\omega n}$$

$$= H\left(e^{-j\omega}\right)$$

$$H^*\left(e^{j\omega}\right) = H\left(e^{-j\omega}\right)$$

Ex: [1 1]

$$H^*\left(e^{j\omega}\right) = \left(\left|H\left(e^{j\omega}\right)\right|e^{j\theta(\omega)}\right)^*$$

$$= \left|H\left(e^{j\omega}\right)\right|e^{-j\theta(\omega)}$$

$$H\left(e^{-j\omega}\right) = \left|H\left(e^{-j\omega}\right)\right|e^{j\theta(-\omega)}$$

Therefore

Magnitude is even symmetric
Phase is odd symmetric

THE RESPONSE TO A SINUSOID

When the impulse response is real,

$$H\left(e^{j\omega}\right) = H^{*}\left(e^{-j\omega}\right)$$

i.e., $H(e^{j\omega})$ is conjugate symmetric.

$$x[n]=\cos(\omega n) \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

$$x[n] = \cos(\omega n) = \frac{1}{2}\left(e^{j\omega n} + e^{-j\omega n}\right)$$

since LTI

$$y[n] = \frac{1}{2}\left(H\left(e^{j\omega}\right)e^{j\omega n} + H\left(e^{-j\omega}\right)e^{-j\omega n}\right)$$

Hence

$$y[n] = \frac{1}{2}\left|H\left(e^{j\omega}\right)\right|\left(e^{j\theta(\omega)}e^{j\omega n} + e^{-j\theta(\omega)}e^{-j\omega n}\right)$$

$$= \frac{1}{2}\left|H\left(e^{j\omega}\right)\right| \cos\left(\omega n + \theta(\omega)\right)$$

In summary,

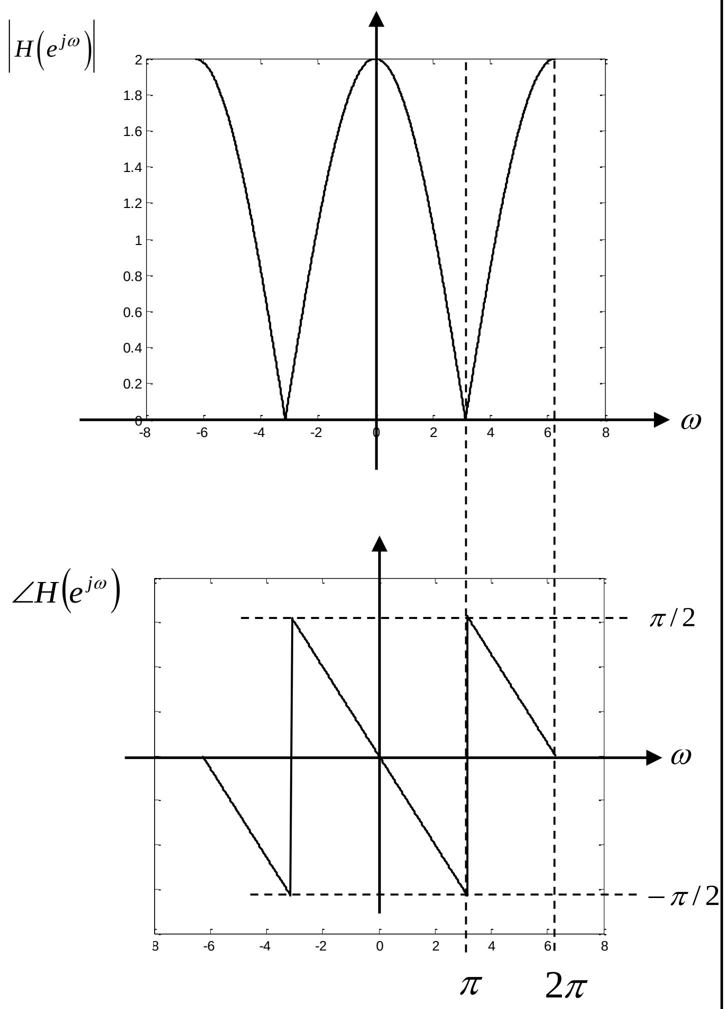
$$\cos(\omega n) \xrightarrow{\text{LTI system}} \left|H\left(e^{j\omega}\right)\right| \cos\left(\omega n + \theta(\omega)\right)$$

Ex: Let $h[n]=\delta[n]+\delta[n-1]$

$$H(e^{j\omega})=1+e^{-j\omega}$$

$$=e^{-j\frac{\omega}{2}}\left(e^{j\frac{\omega}{2}}+e^{-j\frac{\omega}{2}}\right)$$

$$=2\cos\left(\frac{\omega}{2}\right)e^{-j\frac{\omega}{2}}$$



The magnitude and phase are

$$|H(e^{j\omega})|=\left|2\cos\left(\frac{\omega}{2}\right)\right|$$

$$\angle H(e^{j\omega})=\begin{cases}-\dfrac{\omega}{2} & \cos\left(\dfrac{\omega}{2}\right)\geq 0\\ -\dfrac{\omega}{2}\pm\pi & \cos\left(\dfrac{\omega}{2}\right)<0\end{cases}$$

```
h = [1 1];
[H,w] = freqz(h,1,1000,'whole');  (to plot in [0,2π])
[H,w] = freqz(h,1,1000);  (to plot in [0, π])
plot(w,abs(H));
figure
plot(w,angle(H));
```

Let the input be $x[n] = \cos\left(\frac{\pi}{5}n\right)$

The output is $y[n] = 1.9021 \ \cos\left(\frac{\pi}{5}n - \frac{\pi}{10}\right)$

Since $H\left(e^{j\frac{\pi}{5}}\right) = 2\cos\left(\frac{\pi}{10}\right) e^{-j\frac{\pi}{10}} = 1.9021 \ e^{-j\frac{\pi}{10}}$

or

If the input is $x[n] = \cos\left(\frac{6\pi}{5}n\right)$

The output is $y[n] = 0.6180 \cos\left(\frac{6\pi}{5}n - \frac{6\pi}{10} + \pi\right)$

Since

$$H\left(e^{j\frac{6\pi}{5}}\right) = 2\cos\left(\frac{6\pi}{10}\right) e^{-j\frac{6\pi}{10}}$$

$$= -0.6180 \ e^{-j\frac{6\pi}{10}}$$

$$= 0.6180 \ e^{-j\frac{6\pi}{10}+\pi}$$

FOR A REAL LTI SYSTEM, FREQUENCY RESPONSE IS ALSO CONJUGATE SYMMETRIC WRT π

$$H(\omega) = H(\omega + 2\pi) \text{ and } H(\omega) = H^*(-\omega)$$

$$\Rightarrow H^*(-\omega) = H(\omega + 2\pi)$$

$$\Rightarrow H^*(\pi - \omega) = H(\omega + \pi)$$

USEFUL TIPS

$$e^{jx}+e^{jy}=e^{j\frac{x-y}{2}}\left(e^{j\frac{x+y}{2}}+e^{-j\frac{x+y}{2}}\right)=e^{j\frac{x-y}{2}}2\cos\left(\frac{x+y}{2}\right)$$

$$e^{jx}+e^{jy}=e^{j\frac{x+y}{2}}\left(e^{j\frac{x-y}{2}}+e^{-j\frac{x-y}{2}}\right)=e^{j\frac{x+y}{2}}2\cos\left(\frac{x-y}{2}\right)$$

$$a+be^{-j\omega}+be^{-j2\omega}+ae^{-j3\omega}$$

$$=e^{-j\frac{3\omega}{2}}\left(ae^{j\frac{3\omega}{2}}+be^{j\frac{\omega}{2}}+be^{-j\frac{\omega}{2}}+ae^{-j\frac{3\omega}{2}}\right)$$

$$=e^{-j\frac{3\omega}{2}}2\left(b\cos\left(\frac{\omega}{2}\right)+a\cos\left(\frac{3\omega}{2}\right)\right)$$

in general, let $a_k=a_{N-k}\quad k=0,1,\ldots,N$

$$\sum_{k=0}^{N}a_k e^{-jk\omega}=\begin{cases} e^{-j\frac{N}{2}}\;2\displaystyle\sum_{k=1}^{\frac{N+1}{2}}a_k\cos\left(\frac{k}{2}\omega\right) & \text{if } N \text{ is odd}\\ e^{-j\frac{N}{2}}\left(a_{\frac{N}{2}}+2\displaystyle\sum_{k=1}^{\frac{N}{2}}a_k\cos(k\omega)\right) & \text{if } N \text{ is even}\end{cases}$$

or, if $a_k = -a_{N-k} \quad k = 0,1,\ldots,N$

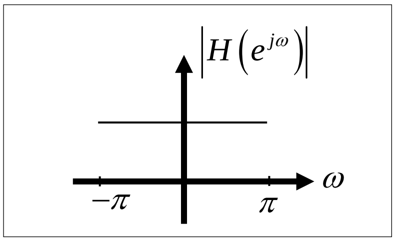
$$\sum_{k=0}^{N} a_k e^{-jk\omega} = \begin{cases} j\, e^{-j\frac{N}{2}}\; 2 \sum_{k=1}^{\frac{N+1}{2}} a_k \sin\left(\frac{k}{2}\,\omega\right) & \text{if } N \text{ is odd} \\ e^{-j\frac{N}{2}} \left(a_{\frac{N}{2}} + j2 \sum_{k=1}^{\frac{N}{2}} a_k \sin(k\omega) \right) & \text{if } N \text{ is even} \end{cases}$$

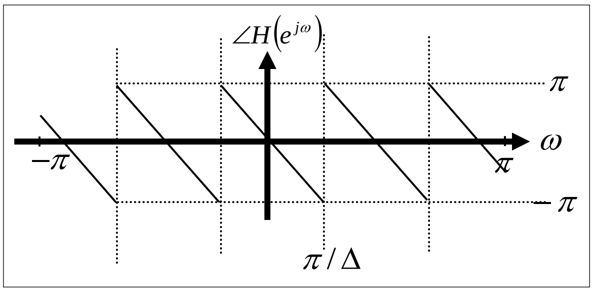
Ex: A pure delay.

$$y[n] = x[n - \Delta]$$

Let $x[n] = e^{j\omega n}$ → $y[n] = e^{j\omega(n-\Delta)} = e^{-j\omega\Delta}\, e^{j\omega n}$

$$\Rightarrow H\left(e^{j\omega}\right) = e^{-j\omega\Delta}$$





It can also be computed as,

$$h[n] = \delta[n - \Delta]$$

$$\Rightarrow H\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} \delta[n - \Delta] e^{-j\omega n}$$

$$= e^{-j\omega\Delta}$$

$$\left|H\left(e^{j\omega}\right)\right| = 1 \quad , \qquad \angle H\left(e^{j\omega}\right) = -\Delta\omega$$

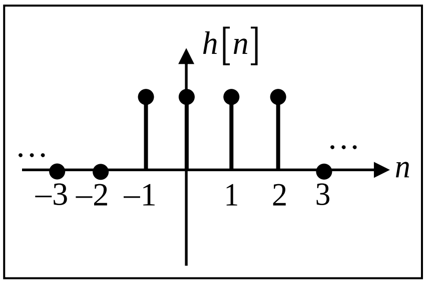
LINEAR PHASE SYSTEMS

The phase of the frequency response of a pure delay system is a “**linear”** function.

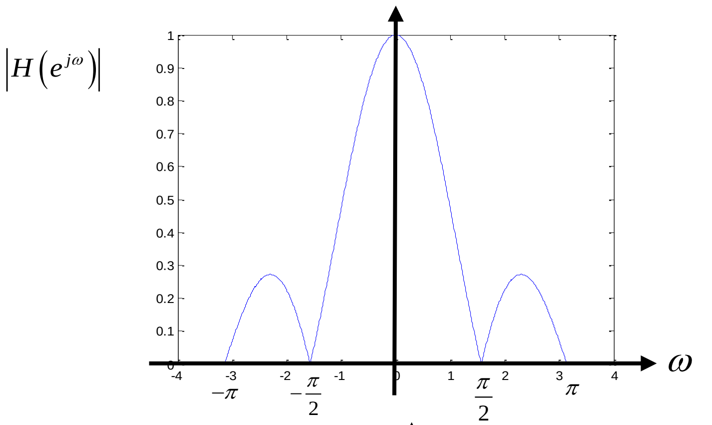
Such systems are called “**linear phase**” systems.

Ex: Frequency response of a moving average system.

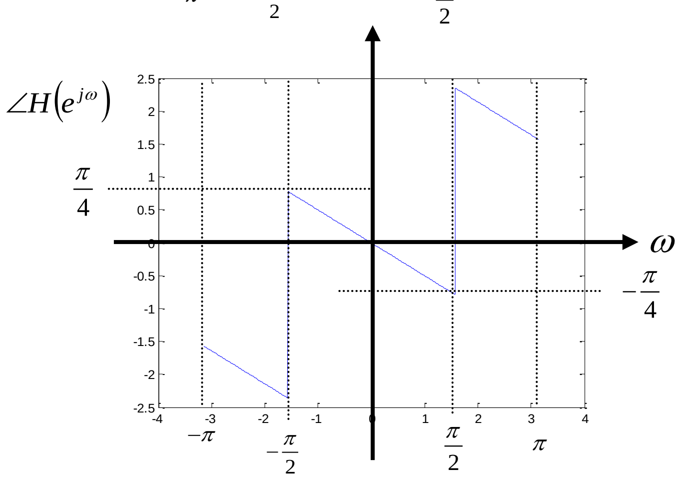
$$y[n]=\frac{1}{4}\left(x[n+1]+x[n]+x[n-1]+x[n-2]\right)$$
$$=\frac{1}{4}\sum_{k=-1}^{2}x[n-k]$$



$$h[n]=\frac{1}{4}\sum_{k=-1}^{2}\delta[n-k]$$



$$H\left(e^{j\omega}\right)=\sum_{n=-\infty}^{\infty}h[n]e^{-j\omega n}$$
$$=\frac{1}{4}\left(e^{j\omega}+1+e^{-j\omega}+e^{-j2\omega}\right)$$
$$=\frac{1}{4}\sum_{n=-1}^{2}e^{-j\omega n}$$
$$=\frac{1}{4}e^{j\omega}\sum_{n=0}^{3}e^{-j\omega n}$$
$$=\frac{1}{4}e^{j\omega}\frac{1-e^{-j4\omega}}{1-e^{-j\omega}}$$
$$=\frac{1}{4}e^{j\omega}\frac{e^{-j2\omega}}{e^{-j\frac{\omega}{2}}}\frac{e^{j2\omega}-e^{-j2\omega}}{e^{j\frac{\omega}{2}}-e^{-j\frac{\omega}{2}}}$$
$$=\frac{1}{4}e^{-j\frac{\omega}{2}}\frac{\sin(2\omega)}{\sin\left(\frac{\omega}{2}\right)}$$



Is it a linear phase system?

Note that, according to the “useful tip”, the frequency response above can also be written as

$$
\begin{aligned}
H\left(e^{j\omega}\right) &= \frac{1}{4}\left(e^{j\omega} + 1 + e^{-j\omega} + e^{-j2\omega}\right) \\
&= \frac{1}{4}e^{j\omega}\left(1 + e^{-j\omega} + e^{-j2\omega} + e^{-j3\omega}\right) \\
&= \frac{1}{4}e^{j\omega}e^{-j\frac{3\omega}{2}}\left(e^{j\frac{3\omega}{2}} + e^{j\frac{\omega}{2}} + e^{-j\frac{\omega}{2}} + e^{-j\frac{3\omega}{2}}\right) \\
&= \frac{1}{2}e^{-j\frac{\omega}{2}}\left(\cos\left(\frac{\omega}{2}\right) + \cos\left(\frac{3\omega}{2}\right)\right)
\end{aligned}
$$

Therefore we have

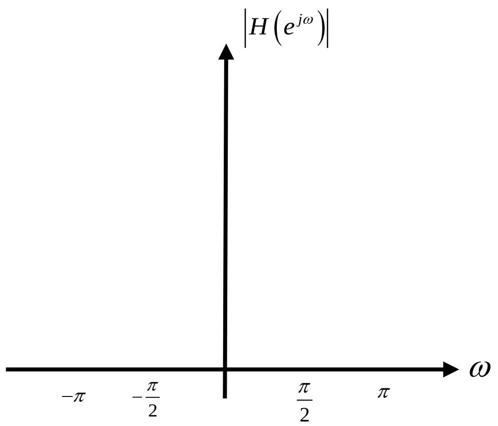
$$
\frac{1}{4}\frac{\sin(2\omega)}{\sin\left(\frac{\omega}{2}\right)} = \frac{1}{2}\left(\cos\left(\frac{\omega}{2}\right) + \cos\left(\frac{3\omega}{2}\right)\right)
$$

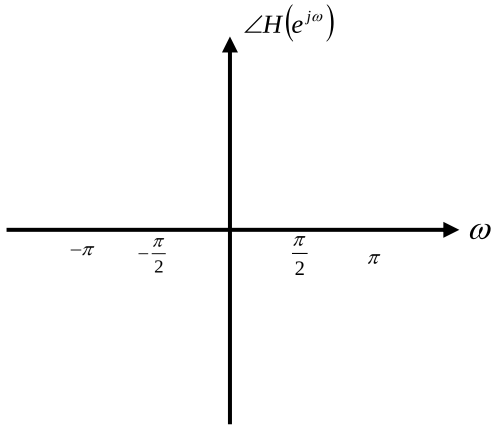
To check:

```
clear all; close all;
w = linspace(0,pi,1000);
x = 0.25 * (sin(2*w)) ./ (sin(w/2));  y = 0.5 * (cos(w/2) + cos(3 * w/2));
plot(w,x,'r'); hold; plot(w,y,'k');
```

Exercise: Using the results of last two examples, find and plot the frequency response of a LTI system whose impulse response is

$$h[n] = \frac{1}{4}\sum_{k=3}^{6}\delta[n-k]$$





“SUDDENLY” APPLIED COMPLEX EXPONENTIAL INPUTS

Study! Textbook Section 2.6.2, “Suddenly” Applied Complex Exponential Inputs

$$x[n] = e^{j\omega n}u[n]$$

LTI Sytem

$\Downarrow$

$$y[n] = \underbrace{e^{j\omega n}H\left(e^{j\omega}\right)}_{y_{ss}[n]} - \underbrace{e^{j\omega n}\sum_{k=n+1}^{\infty} h[k]e^{-j\omega k}}_{y_t[n]}$$