EE430 Digital Signal Processing
HW 2

1. The impulse response of the system is given as $h[n] = \delta[n] - \sqrt{2}\delta[n-1] + \delta[n-2]$

   (a) The system function $H(z)$ can be found as follows;

$$\mathcal{Z}\{\delta[n]\} = 1 \quad \forall z$$

$$\mathcal{Z}\{x[n-n_0]\} = z^{-n_0}X(z)$$

Thus,

$$\boxed{H(z) = 1 - \sqrt{2}z^{-1} + z^{-2}}$$

$H(z)$ can be put in more classical transfer function form to find poles and zeros.

$$H(z) = \frac{z^2 - \sqrt{2}z + 1}{z^2} = \frac{\left(z - (\frac{1}{2} + \frac{j}{2})\right)\left(z - (\frac{1}{2} - \frac{j}{2})\right)}{z^2}$$

Therefore, poles and zeros of the system function can be easily found to be as,

$$\boxed{z_{1,2} = \frac{1}{2} \mp \frac{j}{2}}, \quad \boxed{p_{1,2} = 0}$$

Since the impulse response of the system is finite-length , the region of convergence of system function is **All Z Plane** except $z = 0$ due to poles.Pole-zero plot can be seen at *Figure 2*.

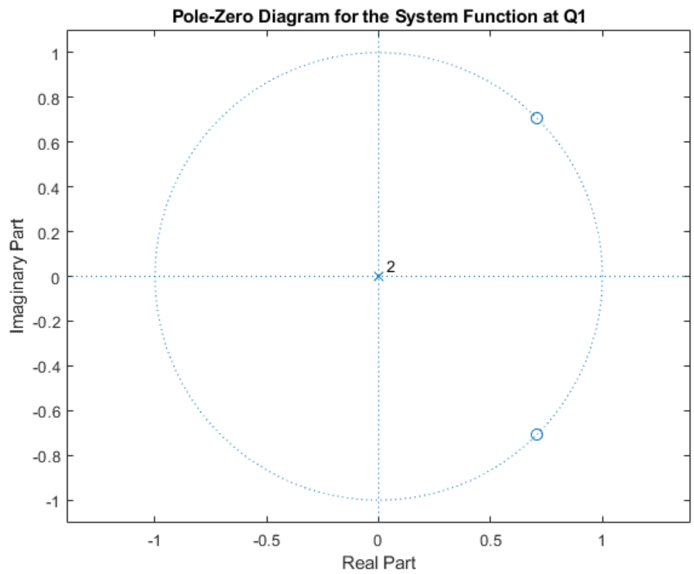
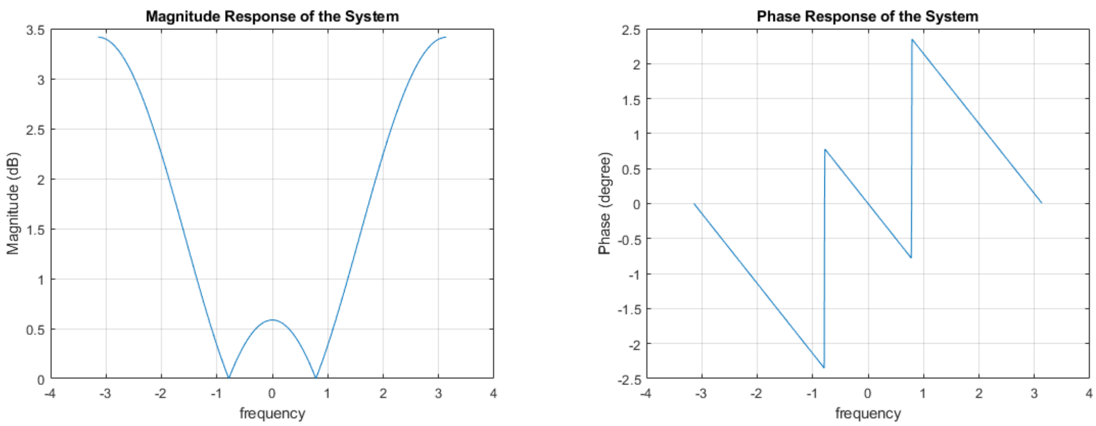


Figure 1: Pole-Zero Diagram for the System Function

(b) Since the ROC contains the unit circle, the system has a frequency response and it can be found by simply replacing z with e^{jw}.

$$H(e^{jw}) = H(z)|_{z=e^{jw}} = 1 - \sqrt{2}e^{-jw} + e^{-j2w}$$
$$H(e^{jw}) = e^{-jw}[e^{jw} + e^{-jw} - \sqrt{2}]$$
$$\boxed{H(e^{jw}) = e^{-jw}\left[\cos(w) - \sqrt{2}\right]}$$

Magnitude and phase response of the frequency response can be seen at *Figure 2a* and *Figure 2b* respectively.



(a) Magnitude Response (b) Phase Response

Figure 2: Magnitude and Phase Response of the System

(c) The output of the system for the different inputs can be found using Z-transform of given inputs.

i. $x_1[n] = \sin(\frac{\pi}{4}n + \frac{\pi}{4}) = x_1[n]\,(u[-n-1] + u[n])$, since the ROC of the z transform of this input would be empty set, let us use another property of DTFT since we already have frequency response.

$$y[n] = \sum_{k=-\inf}^{\inf} = a_k H(e^{jw})e^{jwn}$$

$$x_1[n] = \sin(\frac{\pi}{4}n)\cos(\frac{\pi}{4}) + \sin(\frac{\pi}{4})\cos(\frac{\pi}{4}n) = \frac{1}{\sqrt{2}}\left(\sin(\frac{\pi}{4}n) + \cos(\frac{\pi}{4}n)\right)$$

$$x_1[n] = \frac{1}{\sqrt{2}}\left(\frac{1}{2j}(e^{j\frac{\pi}{4}n - e^{-j\frac{\pi}{4}n}}) + \frac{1}{2}(e^{j\frac{\pi}{4}n} + e^{-j\frac{\pi}{4}n})\right)$$

$$x_1[n] = \frac{1}{2\sqrt{2}}\left(e^{j\frac{\pi}{4}n}(1-j) + e^{-j\frac{\pi}{4}n}(1+j)\right)$$

Thus,

$$y_1[n] = \frac{1}{2\sqrt{2}} \left(e^{j\frac{\pi}{4}n}(1-j)H(e^{j\frac{\pi}{4}}) + e^{-j\frac{\pi}{4}n}(1+j)H(e^{\frac{-j\pi}{4}}) \right)$$

Since $H(e^{j\frac{\pi}{4}}) = H(e^{-j\frac{\pi}{4}}) = 0$

$$\boxed{y_1[n] = 0}$$

ii. $x_2[n] = \sin(\frac{\pi}{4}n + \frac{\pi}{4})u[n]$ can be simplifeid further as follows;

$$x_2[n] = \frac{1}{\sqrt{2}} \left(\sin(\frac{\pi}{4}n) + \cos(\frac{\pi}{4}n) \right)$$

using

$$\mathcal{Z}\{u[n]\} = \frac{1}{1-z^{-1}} \ , \ ROC : |z| > |a|$$

$$\mathcal{Z}\{z_0^n x[n]\} = X(z/z_0) \ , \ ROC : R_x|z_0|$$

$$\mathcal{Z}\{\cos(w_0)n\}u[n] = \frac{1-\cos(w_0)z^{-1}}{1-2\cos(w_0)z^{-1}+z^{-2}} \ , \ |z| > 1$$

$$\mathcal{Z}\{\sin(w_0)n\}u[n] = \frac{\sin(w_0)z^{-1}}{1-2\cos(w_0)z^{-1}+z^{-2}} \ , \ |z| > 1$$

the z-transform $X_2(z)$ can be found as follow;

$$X_2(z) = \frac{1}{\sqrt{2}} \left(\frac{1-\cos(\frac{\pi}{4})z^{-1}}{1-2\cos(\frac{\pi}{4})z^{-1}+z^{-2}} + \frac{\sin(\frac{\pi}{4})z^{-1}}{1-2\cos(\frac{\pi}{4})z^{-1}+z^{-2}} \right)$$

$$X_2(z) = \frac{1}{\sqrt{2}} \left(\frac{1-\frac{1}{\sqrt{2}}}{1-\sqrt{2}z^{-1}+z^{-2}} + \frac{\frac{1}{\sqrt{2}}z^{-1}}{1-\sqrt{2}z^{-1}+z^{-2}} \right)$$

$$X_2(z) = \frac{1}{\sqrt{2}} \left(\frac{1}{1-\sqrt{2}z^{-1}+z^{-2}} \right)$$

$$Y_2(z) = X_2(z)H(z) = \frac{1}{\sqrt{2}}$$

$$\boxed{y[n] = \frac{1}{\sqrt{2}}\delta[n]}$$

iii. $x_3[n] = \sin(\frac{\pi}{4}n + \frac{\pi}{4}) + \sin(\frac{3\pi}{4}n)$ Using linearity property of Z-Transform, we can safely say that the output for the first term is zero, let us find the output for the second term using the eigenfunction property again,

$$\sin(\frac{3\pi}{4}n) = \frac{1}{2j} \left(e^{j\frac{3\pi}{4}n} - e^{-j\frac{3\pi}{4}n} \right)$$

$$y_3[n] = \frac{1}{2j}\left(e^{j\frac{3\pi}{4}n}H(e^{j\frac{3\pi}{4}}) - e^{-j\frac{3\pi}{4}n}H(e^{-j\frac{3\pi}{4}})\right)$$

$$H(e^{j\frac{3\pi}{4}}) = e^{-j\frac{3\pi}{4}}\left[\cos(\frac{3\pi}{4}) - \sqrt{2}\right] = 2\sqrt{2}e^{j\frac{\pi}{4}}$$

$$H(e^{-j\frac{3\pi}{4}}) = e^{j\frac{3\pi}{4}}\left[\cos(\frac{3\pi}{4}) - \sqrt{2}\right] = 2\sqrt{2}e^{-j\frac{\pi}{4}}$$

$$y_3[n] = \frac{2\sqrt{2}}{2j}\left(e^{j(\frac{3\pi}{4}n+\frac{\pi}{4})} - e^{-j(\frac{3\pi}{4}n+\frac{\pi}{4})}\right)$$

$$\boxed{y_3[n] = 2\sqrt{2}\sin(\frac{3\pi}{4}n + \frac{\pi}{4})}$$

(d) Since on the unit circle, the Z-Transform is equivalent to the DTFT, the zeros of the Z-Transform at unit circle is also the zeros of the DTFT.

2. The system function is given as $H(z) = \frac{1-\sqrt{2}z^{-1}+z^{-2}}{1-2z^{-1}}$

(a) a

(b) b

(c)

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1-\sqrt{2}z^{-1}+z^{-2}}{1-2z^{-1}}$$

Thus,

$$\boxed{y[n] - 2y[n-1] = x[n] - \sqrt{2}x[n-1] + x[n-2]}$$

3. Since $x[n]$ is a right-sided sequence, the ROC of the $H(z)$ would be outward. It is implied that the ROC of the $H(z)$ includes $|z| = 4$ circle. This means that the ROC can be something close to a ROC at *Figure 3*. The circle can be smaller but can not be larger that $|z| = 4$ since it is given that the Z Transform exist for that circle.

Thus, in any circumstances, $X(z)$ exists for $z = 4.1e^{jw}$ since the ROC includes $|z| = 4.1$ circle in any case. However, it is possible that $X(z)$ may not exist for $z = 3.9e^{jw}$ since the ROC may not include $|z| = 3.9$ circle as in *Figure 3*.

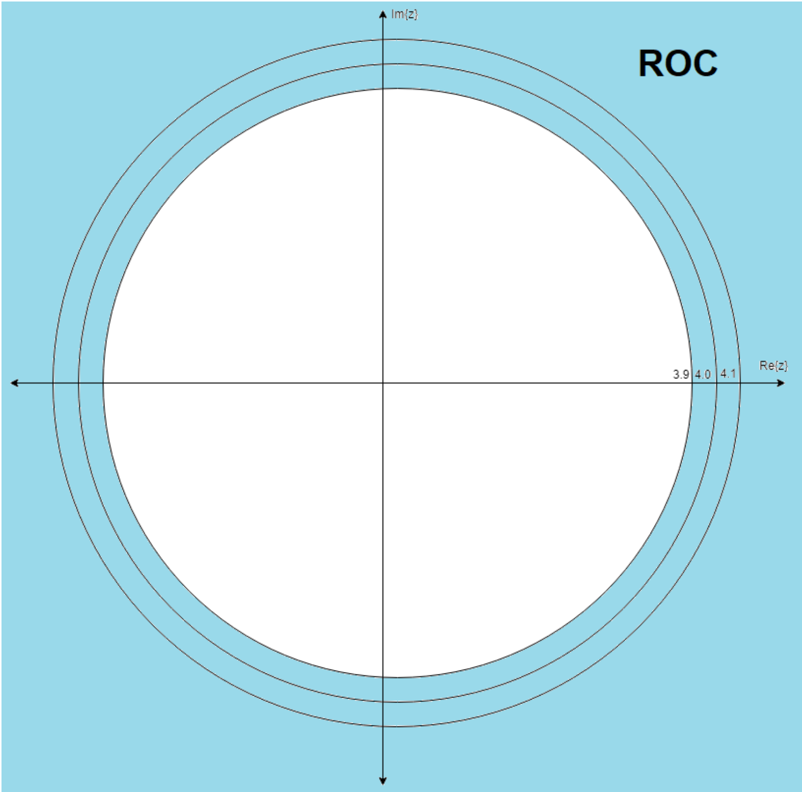


Figure 3: Possible ROC for the Given System

4. Given that $x[n] = \delta[n+1] + (\frac{1}{2})^n u[n]$

   (a)

$$\boxed{X(z) = z - \frac{1}{1-\frac{1}{2}z^{-1}} = \frac{z(z+\frac{1}{2})}{z-\frac{1}{2}} \ , \ |z| > \frac{1}{2}}$$

$$\boxed{p_{a-1} = \frac{1}{2}} \ , \ \boxed{p_{a-2} = \text{inf}} \ , \ \boxed{z_{a-1} = 0} \ , \ \boxed{z_{a-2} = \frac{-1}{2}}$$

   (b)

$$\boxed{\mathcal{Z}\{x[n-5]\} = z^{-5}X(z) = \frac{z+\frac{1}{2}}{z^4(z-\frac{1}{2})} \ , \ |z| > \frac{1}{2}}$$

$$\boxed{p_{b-1} = \frac{1}{2}} \ , \ \boxed{p_{b-2,3,4,5} = 0} \ , \ \boxed{z_{b-1} = \frac{-1}{2}} \ , \ \boxed{z_{b-2,3,4,5} = \text{inf}}$$

   (c)

$$\mathcal{Z}\{nx[n]\} = -z\frac{d}{dz}X(z) = -z\frac{d}{dz}(\frac{z+\frac{1}{2}}{z^4(z-\frac{1}{2})}) \ , \ |z| > \frac{1}{2}$$

$$\mathcal{Z}\{nx[n]\} = -z\left(-\frac{1+z^2}{z^5(z-0.5^2)}\right)$$

$$\boxed{\mathcal{Z}\{nx[n]\} = \frac{1+z^2}{z^4(z-0.5^2)}}$$

$$\boxed{p_{c-1,2} = \frac{1}{2}}\ ,\ \boxed{p_{c-3,4,5,6} = 0}\ ,\ \boxed{z_{c-1,2} = \mp j}\ ,\ \boxed{p_{c-3,4,5,6} = \text{inf}}$$

(d)

$$\cos(\frac{\pi}{2}n)x[n] = \frac{1}{2}\left(e^{j\frac{\pi}{2}n}x[n] + e^{-j\frac{\pi}{2}n}x[n]\right)$$

$$\mathcal{Z}\{\cos(\frac{\pi}{2}n)x[n]\} = \frac{1}{2}\left(X(\frac{z}{e^{j\frac{\pi}{2}}}) + X(\frac{z}{e^{-j\frac{\pi}{2}}})\right)$$

$$\boxed{\mathcal{Z}\{\cos(\frac{\pi}{2}n)x[n]\} = \frac{1}{2}\left(\frac{(\frac{z}{e^{j\frac{\pi}{2}}})(\frac{z}{e^{j\frac{\pi}{2}}}+\frac{1}{2})}{\frac{z}{e^{j\frac{\pi}{2}}}-\frac{1}{2}} + \frac{(\frac{z}{e^{-j\frac{\pi}{2}}})(\frac{z}{e^{-j\frac{\pi}{2}}}+\frac{1}{2})}{\frac{z}{e^{-j\frac{\pi}{2}}}-\frac{1}{2}}\right)}$$

5. $H(z)$ will be in the form of

$$\frac{\alpha(z+\frac{1}{2})}{(z-3)(z-\frac{1}{2})}$$

$$H(1) = \frac{\alpha(3/2)}{(-2)(1/2)} = 1$$

$$\boxed{\alpha = \frac{-2}{3}}$$

$$\boxed{H(z) = -\frac{2}{3}\frac{(z+\frac{1}{2})}{(z-3)(z-\frac{1}{2})}}$$

(a) System to be stable, the ROC must include unit circle. The ROC can not also include any pole and should be one piece. Thus the ROC can only be

$$\boxed{ROC : \frac{1}{2} < |z| < 3}$$

The pole-zero diagram for this system and its ROC can be seen at *Figure 4*.

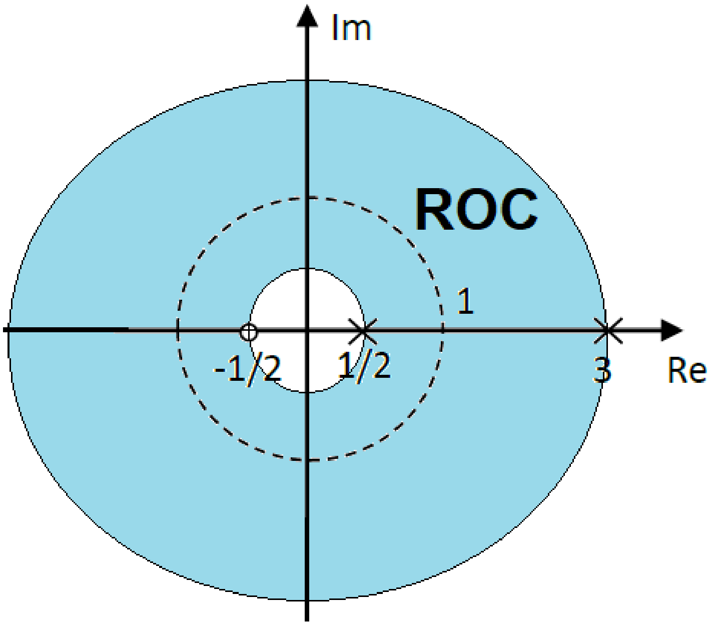


Figure 4: ROC for the Given System

To find $h[n]$, partial fraction method can be used;

$$H(z) = \left[\frac{a}{z-3} + \frac{b}{z-\frac{1}{2}}\right]$$

where a and b can be found easily as

$$\boxed{a = -\frac{14}{15}}, \boxed{b = \frac{4}{15}}$$

Thus, considering the ROC, $h[n]$ becomes;

$$\boxed{h[n] = \frac{4}{5}(\frac{1}{2})^n u[n] - \frac{14}{15}(3)^n u[-n-1]}$$

(b) Given that,

$$h_2[n] = h[-n+2]$$

Z-Transform of $h_2[n]$ can be calculated as follows,

$$H_2(z) = z^2 X(1/z) = H(z) = -\frac{2}{3}\frac{z^2(z^{-1}+\frac{1}{2})}{(z^{-1}-3)(z^{-1}-\frac{1}{2})}$$

$$\boxed{H_2(z) = -\frac{2}{9}\frac{z^3(z+2)}{(z-\frac{1}{3})(z-2)}\ ,\ \frac{1}{3} < |z| < 2}$$

Pole-zero diagram for this transform and its ROC can be seen at *Figure 5*.

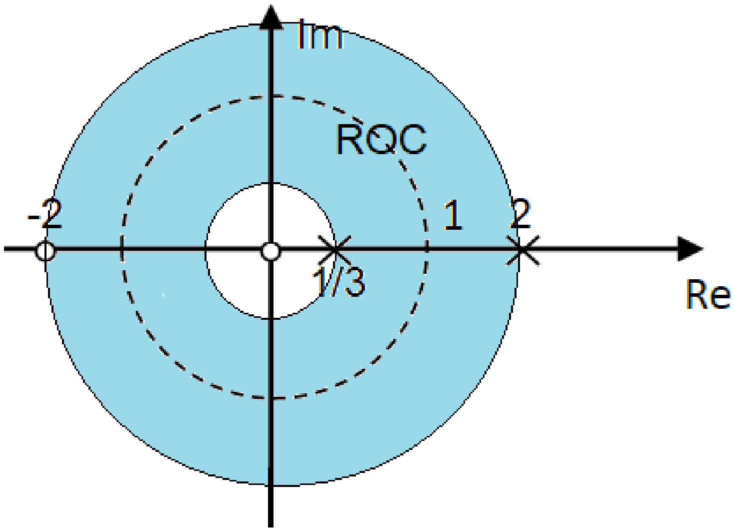


Figure 5: ROC for the Updated System

6. Given that

$$H(z) = \frac{1-z^{-1}}{1-0.25z^{-2}} = \frac{1-z^{-1}}{(1-0.5z^{-1})(1+0.5z^{-1})}$$

(a)

$$X(z) = \frac{1}{1-z^{-1}}$$

$$Y_1(z) = X_1(z)H(z) = \frac{1}{(1-0.5z^{-1})(1+0.5z^{-1})} = \frac{a}{1-0.5z^{-1}} + \frac{b}{1+0.5z^{-1}}$$

$$\boxed{a = b = \frac{1}{2}}$$

$$Y_1(z) = \frac{1}{2}(\frac{1}{1-0.5z^{-1}}) + \frac{1}{2}(\frac{1}{1+0.5z^{-1}})$$

$$\boxed{y_1[n] = \frac{1}{2}\left[(-0.5)^n + (0.5)^n\right]u[n]}$$

(b)

$$Y_2(z) = 1 - z^{-1} = X_2(z)H(z)$$
$$X_2(z) = 1 - 0.25z^{-2}$$

$$\boxed{x_2[n] = \delta[n] - 0.25\delta[n-2]}$$

(c) Since the system is casual

$$h[n] = 0 \ for \ n < 0$$

Thus, the input can behave as $cos(0.5\pi n)u[n] = x_3[n]$.

$$X_3(z) = \frac{1 - \cos(0.5\pi)z^{-1}}{1 - 2\cos(0.5\pi)z^{-1} + z^{-2}} \ , \ |z| > 1$$

$$X_3(z) = \frac{1}{1 + z^{-2}}$$

$$Y_3(z) = X_3(z)H(z) = \frac{1 - z^{-1}}{(1 + z^{-2})(1 - 0.25z^{-2})}$$

$$\boxed{y_3[n] = \mathcal{Z}^{-1}\{Y_3(z)\}}$$

7. Given that

$$\hat{X}(z) = logX(z)$$

and

$$x[n] = \delta[n] + a\delta[n-N]$$

$\hat{x}[n]$ can be found as follows,

$$X(z) = 1 + az^{-N}$$

$$\hat{X}(z) = \Updownarrow\wr\}\{1 + az^{-N}\}$$

Using Taylor Series Expansion, more specifically Mercator Series expansion,

$$log(1 + x) = \sum_{n=1}^{\text{inf}} (-1)^{n+1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} \dots..$$

assume, $x = az^{-N}$ for our case,

$$\hat{X}(z) = \sum_{n=1}^{\text{inf}} (-1)^{n+1} \frac{az^{-Nn}}{n}$$

Thus,

$$\boxed{\hat{x}[n] = \sum_{n=1}^{\text{inf}} \frac{(-1)^{n+1}}{n} \delta[n - Nn]}$$

8. Given that

$$c_{xx}[n] = \sum_{k=-\inf}^{\inf} x[k]x[k+1]$$

(a)

$$C_{xx}(z) = \sum_{n=-\inf}^{\inf} \sum_{k=-\inf}^{\inf} x[k]x[k+n]z^{-n}$$

$$C_{xx}(z) = \sum_{k=-\inf}^{\inf} x[k] \sum_{n=-\inf}^{\inf} x[k+n]z^{-n}$$

Let $m = k + n$

$$C_{xx}(z) = \sum_{k=-\inf}^{\inf} x[k]z^{k} \sum_{m=-\inf}^{\inf} x[m]z^{-m}$$

$$C_{xx}(z) = \sum_{k=-\inf}^{\inf} x[k]z^{k}X(z)$$

Let $\hat{k} = -k$

$$C_{xx}(z) = \sum_{\hat{k}=-\inf}^{\inf} x[-\hat{k}]z^{-\hat{k}}X(z)$$

$$C_{xx}(z) = \mathcal{Z}\{x[-k]\}X(z)$$

$$\boxed{C_{xx}(z) = X(\frac{1}{z})X(z)}$$

(b)

$$X(z) = \frac{1}{1 - az^{-1}}$$

$$X(z^{-1}) = \frac{1}{1 - az}$$

$$C_{xx} = X(z)X(z^{-1}) = \frac{1}{(1 - az^{-1})(1 - az)} = \frac{-az^{-1}}{(1 - az^{-1})(1 - a^{-1}z^{-1})}$$

$$C_{xx} = \frac{a_1}{1 - az^{-1}} + \frac{a_2}{1 - a^{-1}z^{-1}}$$

$$\boxed{a_1 = -\frac{1}{1 - a^{-2}}}, \boxed{a_2 = -\frac{a^2}{1 - a^2}}$$

Since it is stable, the ROC includes unit circle. Thus, ROC would be in a ring form. Assuming $a < 1$ and $1/a > 1$, the ROC becomes $\frac{1}{a} < |z| < a$. Considering the ROC, c_{xx} becomes,

$$\boxed{c_{xx}[n] = -\frac{1}{1-a^{-2}}a^n u[n] - -\frac{a^2}{1-a^2}a^{-n}u[-n-1]}$$

The pole-zero diagram and ROC foe the given system can be seen at *Figure 6*.

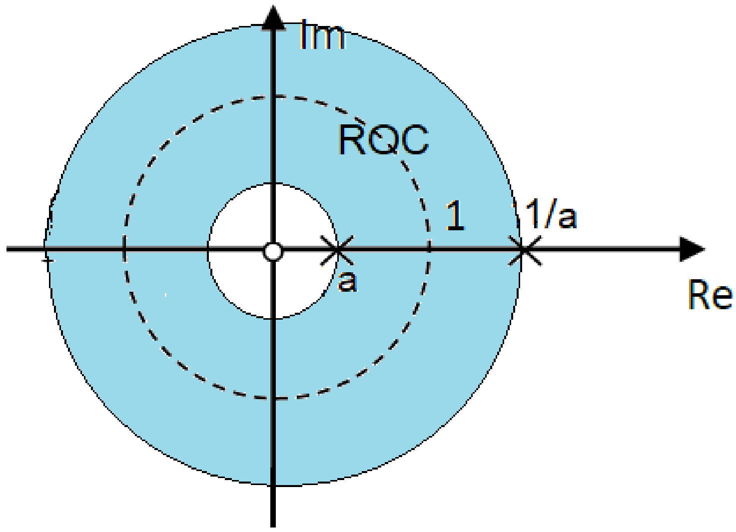


Figure 6: ROC for Given System

(c) It is clear that $c_{xx}(z) = c_{xx}(z^{-1})$. Thus, $x[-n]$ would have same autocorrelation function with $x[n]$.

(d) Also notice that $C_{xx} = X(z)X(z^{-1}) = \frac{1}{(1-az^{-1})(1-az)}$, if we could have only extra polynomial at the dominator like z^{-l}, we could potentially get rid of it as we multiply with z^l term of $X(z^{-1})$. This can be satisfied with any shifted version of $x[n]$, i.e., $x[n-l]$

9. 9th Question

(a) Z-Transfer found using conv command can be seen at *Figure 7*. The source code for this part can be found in **Appendix B**.

```
x_z =

-z^-1 - z^-2 + 7 z^-3 - 16 z^-4 + 22 z^-5 + 17 z^-6 - 28 z^-7
```

Figure 7: Z-Transfer using conv command

(b) Inverse Z-Transform of the given system function using residuez can be seen at *Figure 8*. The source code for this part can be found in **Appendix B**.

```
The inverse z-transform of given function is:
x[1]= 1.000000 (1.000000)^(n)
x[2]= 0.000000 (0.000000)^(n)
x[3]= 0.000000 (0.000000)^(n)
```

Figure 8: Inverse Z-Transform of the Given System Function

(c) Pole-Zero diagram for the system function given can be seen at *Figure 9*. The source code for this part can be found in **Appendix B**.

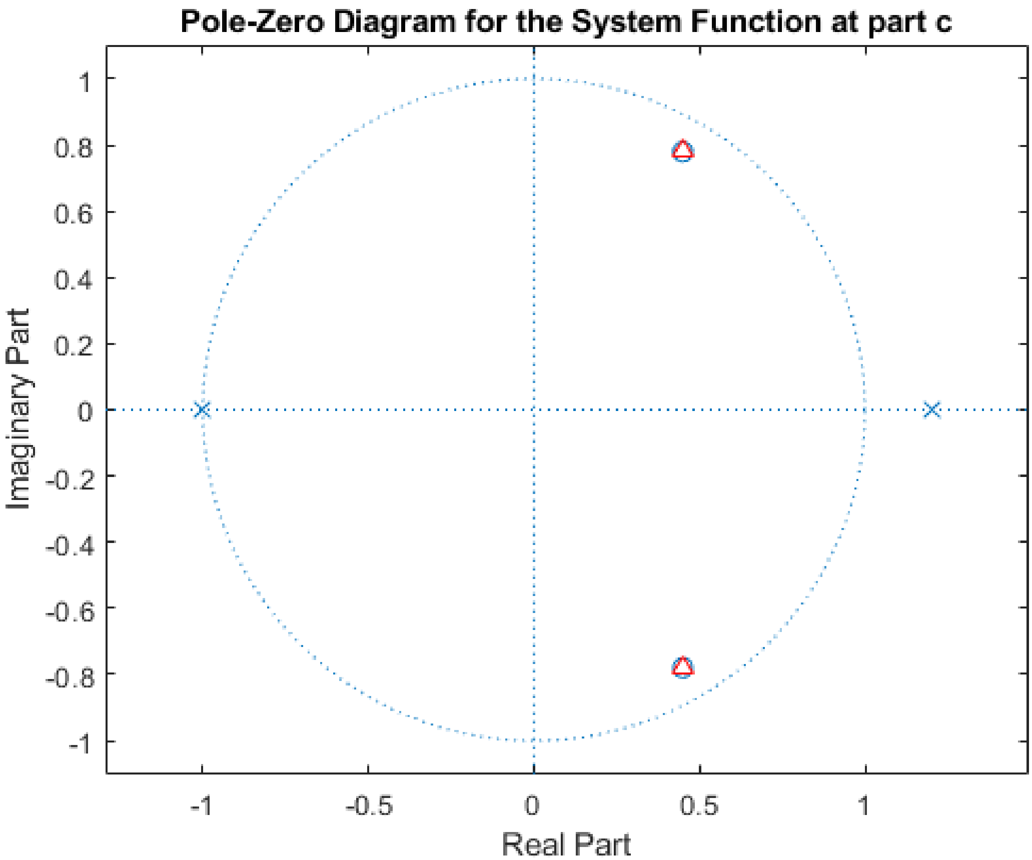


Figure 9: Pole-Zero Diagram for the System Function Given at part c

Appendices

A Source Code for Question 1

```
%% Q1a
num =[1 -sqrt(2) 1]
den=[1 0 0]
x = tf (num,den)
figure(1)
zplane(num,den)
title("Pole-Zero Diagram for the System Function at Q1")


%% Q1b
L=1000;
dw=2*pi/L;
w = -pi:dw:pi-dw;

HH=freqz(num,den,w);

figure(2)
mag=abs(HH)
plot(w,mag)
title('Magnitude Response of the System')
grid on
xlabel("frequency")
ylabel("Magnitude (dB)")

figure(3)
phase=angle(HH)
plot(w,phase)
title('Phase Response of the System')
grid on
xlabel("frequency")
ylabel("Phase (degree)")
```

B Source Code for Question 9

```
%% Q9a
x_1=[ 1 3 -4]
x_2=[-1 2 -3 1 7]
x_k=conv(x_1,x_2)

x_z=filt([0 x_k] , [1])

%% Q9b

a= [1 -3 4];
b= [1 -1 1 -1];
[r,p,k]=residuez(a , b);
[rsize1,rsize2]=size(r);

i=1;
fprintf('The inverse z-transform of given function is:\n')
while i<(rsize1+1)
    fprintf('x[%i]= %f (%f)^(n)\n',i, r(i),p(i));
    i=i+1;
end

%% Q9c

a_c= [1 -0.2 -1.2 ]
b_c= [1 -0.9 0.81]
[r_c,p_c,k_c]=residuez(a_c , b_c)


zplane(b_c,a_c)
hold on
plot(p_c,'^r')
title("Pole-Zero Diagram for the System Function at part c")
hold off
```