FREQUENCY DOMAIN REPRESENTATION OF LTI SYSTEMS

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LCCDES AND FREQUENCY RESPONSE

DICRETE TIME FOURIER TRANSFORM (DTFT)

The Fourier transform of a sequence $x[n]$ is defined as

$$X\left(e^{j\omega}\right)=\sum_{n=-\infty}^{\infty}x[n]e^{-j\omega n}$$

If the FT exists (summation converges) the sequence can be obtained from its FT as

$$x[n]=\frac{1}{2\pi}\int_{-\pi}^{\pi}X\left(e^{j\omega}\right)e^{j\omega n}d\omega$$

Fourier Transform is periodic with 2π.

The frequency response function,

$$H\left(e^{j\omega}\right)$$

is the FT of the impulse response

$$h[n]$$

EXISTENCE

FT of a sequence $x[n]$ exists, i.e.,

$$\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

converges to a continuous function of ω,

if $x[n]$ is absolutely summable.

(sufficient condition)

Proof: Exercise

$$\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = \sum_{n=-\infty}^{\infty} x[n](\cos(\omega n) - j\sin(\omega n))$$

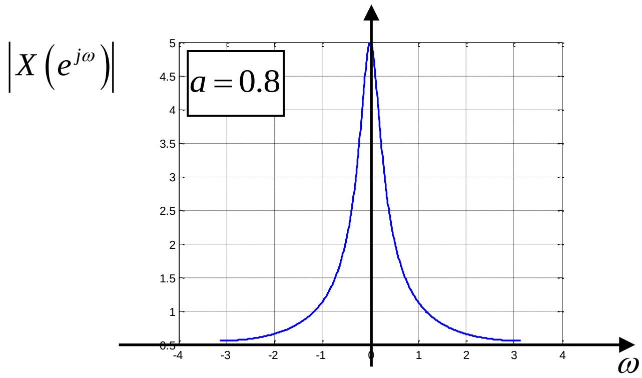
$$= \sum_{n=-\infty}^{\infty} x[n]\cos(\omega n) - j\sum_{n=-\infty}^{\infty} x[n]\sin(\omega n)$$

Both sums have to converge

→ All stable LTI systems have frequency response functions.

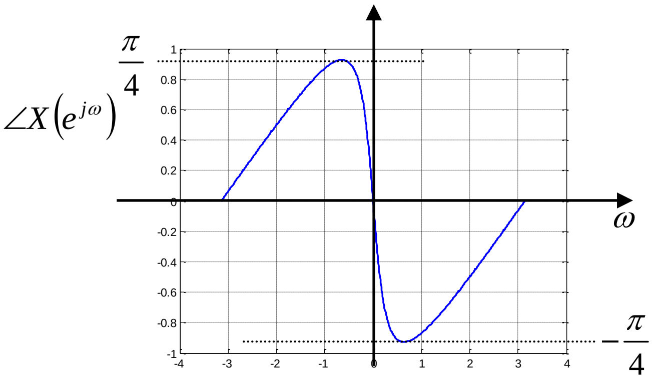
Ex: $x[n] = a^n \, u[n]$

$$X\left(e^{j\omega}\right) = \sum_{n=0}^{\infty} a^n e^{-j\omega n} = \sum_{n=0}^{\infty} \left(a e^{-j\omega}\right)^n = \frac{1}{1 - a e^{-j\omega}} \qquad \text{if} \quad \left|a e^{-j\omega}\right| < 1 \quad \text{or} \quad |a| < 1$$



$$\begin{aligned}
\left|X\left(e^{j\omega}\right)\right|^2 &= \frac{1}{\left|1 - a e^{-j\omega}\right|^2} \\
&= \frac{1}{\left|1 - a\cos\omega + j\sin\omega\right|^2} \\
&= \frac{1}{1 + a^2 - 2a\cos\omega}
\end{aligned}$$

$$\begin{aligned}
\angle X\left(e^{j\omega}\right) &= \angle 1 - \angle\left(1 - a e^{-j\omega}\right) \\
&= 0 - \angle\left(1 - a\cos\omega + ja\sin\omega\right) \\
&= -\tan^{-1}\left(\frac{a\sin\omega}{1 - a\cos\omega}\right)
\end{aligned}$$



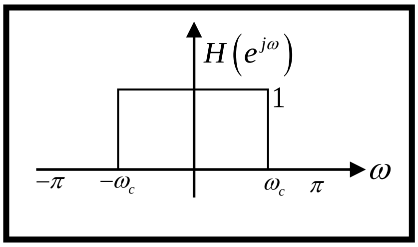
MEAN SQUARE CONVERGENCE

Some sequences, which are not absolutely summable but square summable

$$\sum_{n=-\infty}^{\infty} \left|x[n]\right|^2 < \infty$$

can still be represented by Fourier Transform, but…

Ex: Ideal lowpass filter.



$$H(e^{j\omega}) = \begin{cases} 1 & |\omega| < \omega_c \\ 0 & \omega_c < |\omega| \le \pi \end{cases}$$

Let's find $h[n]$!

$$h[n] = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} e^{j\omega n} d\omega$$

$$= \frac{1}{j2\pi n}\left(e^{j\omega_c n} - e^{-j\omega_c n}\right)$$

$$= \frac{\sin(\omega_c n)}{\pi n}$$

Note that,

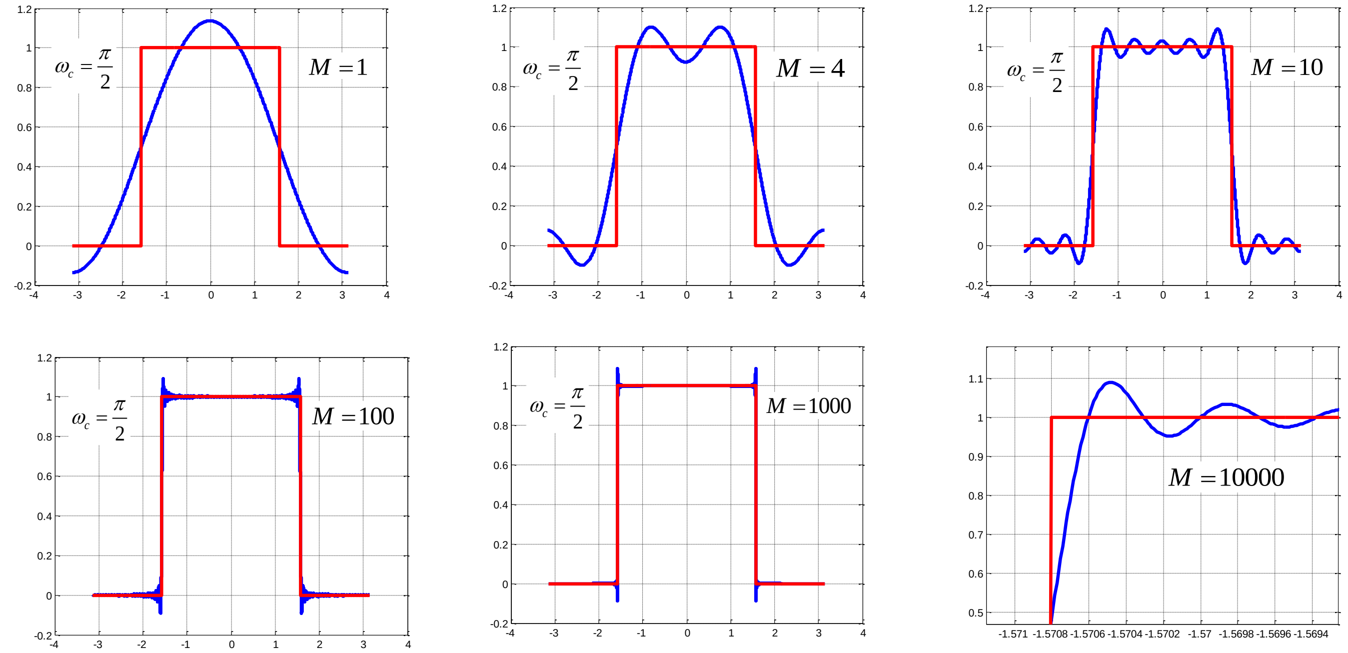
$$h[n] = \frac{\sin(\omega_c n)}{\pi n}$$

is not absolutely summable!

Then, one may question the Fourier transform of $h[n]$,

$$\sum_{n=-\infty}^{\infty} \frac{\sin(\omega_c n)}{\pi n} e^{-j\omega n} =?$$

Define $H_M = \sum_{n=-M}^{M} \frac{\sin(\omega_c n)}{\pi n} e^{-j\omega n}$



detail

Even if you take $M \to \infty$ oscillations do not die to zero.

However $\lim_{M\to\infty} \int_{-\pi}^{\pi} \left| H\left(e^{j\omega}\right) - H_M\left(e^{j\omega}\right) \right|^2 d\omega = 0$. This is called "mean square" convergence.

The oscillatory behavior around $\omega = \omega_c$ is called the Gibbs phenomenon.

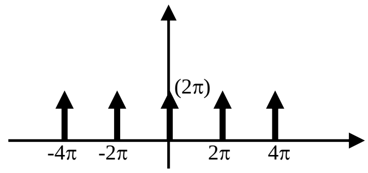
MATLAB code

```
clear all;   close all;
precision =0.0001
w = [-pi:precision:pi];
idealL = zeros(1,length(w));
wc = pi/2;
orta = round(length(w)/2);
idealL((orta-round(wc/precision)):(orta+round(wc/precision)))=1;
M = 10000;
H = 0;
for n = -M:-1
   H = H+(sin(wc*n)/(pi*n))*exp(-i*w*n);
end
for n = 1:M
   H = H+(sin(wc*n)/(pi*n))*exp(-i*w*n);
end
   H = H+(wc/pi);
plot(w,H); hold on;
plot(w,idea,'r')
grid
```

FOURIER TRANSFORM OF A CONSTANT SEQUENCE

$$x[n] = 1 \quad \leftrightarrow \quad X\left(e^{j\omega}\right) = 2\pi \sum_{r=-\infty}^{\infty} \delta\left(\omega + 2\pi r\right)$$

not absolutely summable



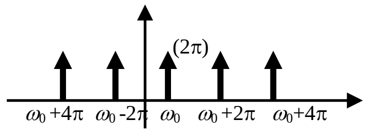
or we can write as

$$X\left(e^{j\omega}\right) = 2\pi\,\delta(\omega) \qquad 0 \le \omega < 2\pi$$

keeping in mind that FT is periodic with 2π.

FOURIER TRANSFORM OF A COMPLEX EXPONENTIAL SEQUENCE

$$x[n] = e^{j\omega_0 n} \quad \leftrightarrow \quad X\left(e^{j\omega}\right) = 2\pi \sum_{r=-\infty}^{\infty} \delta\left(\omega - \omega_0 + 2\pi r\right)$$



not absolutely summable

or we can write as

$$X\left(e^{j\omega}\right) = 2\pi\delta\left(\omega - \omega_0\right) \qquad 0 \le \omega < 2\pi$$

keeping in mind that FT is periodic with 2π

FOURIER TRANSFORM OF A SINUSOIDAL SEQUENCE

$$x[n] = \cos(\omega_0 n) = \frac{1}{2}\left(e^{j\omega_0 n} + e^{-j\omega_0 n}\right) \quad \leftrightarrow \quad X\left(e^{j\omega}\right) = \pi\left(\sum_{r=-\infty}^{\infty} \delta(\omega + \omega_0 + 2r\pi) + \delta(\omega - \omega_0 + 2r\pi)\right)$$

not absolutely summable

or $X\left(e^{j\omega}\right) = \pi\left(\delta(\omega + \omega_0) + \delta(\omega - \omega_0)\right)$ $\quad 0 \le \omega < 2\pi$ since FT is periodic with 2π

FOURIER TRANSFORM OF UNIT STEP SEQUENCE

$$x[n] = u[n] \quad \leftrightarrow \quad X\left(e^{j\omega}\right) = \frac{1}{1-e^{-j\omega}} + \pi \sum_{r=-\infty}^{\infty} \delta(\omega + 2\pi r)$$

not absolutely summable

SYMMETRY PROPERTIES OF FOURIER TRANSFORM

Definitions:

Conjugate symmetric (CS) sequence. $x[n] = x^*[-n]$

Conjugate antisymmetric (CaS) sequence. $x[n] = -x^*[-n]$

Using the above definitions, any sequence can be written as

$$x[n] = x_e[n] + x_o[n]$$

where

$$x_e[n] = \frac{1}{2}\left(x[n] + x^*[-n]\right)$$ is the CS part

and

$$x_o[n] = \frac{1}{2}\left(x[n] - x^*[-n]\right)$$ is the CaS part.

SYMMETRY PROPERTIES

Fundamental relations

Let $x[n] \leftrightarrow X\left(e^{j\omega}\right)$ be a FT pair. Then, the following hold:

$x^*[n] \leftrightarrow X^*\left(e^{-j\omega}\right)$ since $X^*\left(e^{-j\omega}\right) = \sum_{n=-\infty}^{\infty} x^*[n] e^{-j\omega n}$

$x[-n] \leftrightarrow X\left(e^{-j\omega}\right)$ since $X\left(e^{-j\omega}\right) = \sum_{n=-\infty}^{\infty} x[-n] e^{-j\omega n}$

Above yields

$$x^*[-n] \leftrightarrow X^*\left(e^{j\omega}\right)$$

The two relations above also yield:

1) $\mathrm{Re}\{x[n]\} = \dfrac{x[n]+x^*[n]}{2} \;\leftrightarrow\; X_e\left(e^{j\omega}\right) = \dfrac{X\left(e^{j\omega}\right)+X^*\left(e^{-j\omega}\right)}{2}$ (CS part of $X\left(e^{j\omega}\right)$)

2) $j\,\mathrm{Im}[x[n]] = \dfrac{x[n]-x^*[n]}{2} \;\leftrightarrow\; X_o\left(e^{j\omega}\right) = \dfrac{X\left(e^{j\omega}\right)-X^*\left(e^{-j\omega}\right)}{2}$ (CaS part of $X\left(e^{j\omega}\right)$)

3) $x_e[n] = \dfrac{x[n]+x^*[-n]}{2} \;\leftrightarrow\; \mathrm{Re}\left\{X\left(e^{j\omega}\right)\right\} = \dfrac{X\left(e^{j\omega}\right)+X^*\left(e^{j\omega}\right)}{2}$ (real part of $X\left(e^{j\omega}\right)$)

Therefore FT of an even seq. is real!

4) $x_o[n] = \dfrac{x[n]-x^*[-n]}{2} \;\leftrightarrow\; j\,\mathrm{Im}\left\{X\left(e^{j\omega}\right)\right\} = \dfrac{X\left(e^{j\omega}\right)-X^*\left(e^{j\omega}\right)}{2}$ (imag. part of $X\left(e^{j\omega}\right)$)

Therefore FT of an odd seq. is purely imaginary!

Ex: Let $a[n]$ and $b[n]$ be two real sequences with their DTFTs $A\left(e^{j\omega}\right)$ and $B\left(e^{j\omega}\right)$, respectively.

Let

$$x[n] = a[n] + jb[n]$$

Then,

$$X\left(e^{j\omega}\right) = A\left(e^{j\omega}\right) + jB\left(e^{j\omega}\right)$$

Note that $A\left(e^{j\omega}\right)$ is NOT the real part of $X\left(e^{j\omega}\right)$.

However,

$$A\left(e^{j\omega}\right) = \frac{X\left(e^{j\omega}\right) + X^*\left(e^{-j\omega}\right)}{2}$$

Since

$$X^*\left(e^{-j\omega}\right) = \underbrace{A^*\left(e^{-j\omega}\right)}_{A\left(e^{j\omega}\right)} - j\,\underbrace{B^*\left(e^{-j\omega}\right)}_{B\left(e^{j\omega}\right)}$$

Similarly,

$$jB\left(e^{j\omega}\right) = \frac{X\left(e^{j\omega}\right) - X^*\left(e^{-j\omega}\right)}{2}$$

Ex: (cont'd)

$$a[n]:\ [-1\ \ 1]$$

$$b[n]:\ [1\ \ 1]$$

$$x[n]:\ [-1+j\quad 1+j]$$

$$A\left(e^{j\omega}\right) = -1 + e^{-j\omega}$$

$$B\left(e^{j\omega}\right) = 1 + e^{-j\omega}$$

$$X\left(e^{j\omega}\right) = -1 + e^{-j\omega} + j + je^{-j\omega}$$

$$X^*\left(e^{-j\omega}\right) = -1 + e^{-j\omega} - j - je^{-j\omega}$$

$$\frac{X\left(e^{j\omega}\right) + X^*\left(e^{-j\omega}\right)}{2} = -1 + e^{-j\omega}$$

$$\frac{X\left(e^{j\omega}\right) - X^*\left(e^{-j\omega}\right)}{2} = j + je^{-j\omega}$$

REAL SEQUENCES

Based on the above relations, for real sequences ($x[n] = x^*[n]$):

$$X\left(e^{j\omega}\right) = X^*\left(e^{-j\omega}\right), \quad \text{conjugate symmetry}$$

which implies

Magnitude is even.............................. $\left|X\left(e^{j\omega}\right)\right| = \left|X\left(e^{-j\omega}\right)\right|$

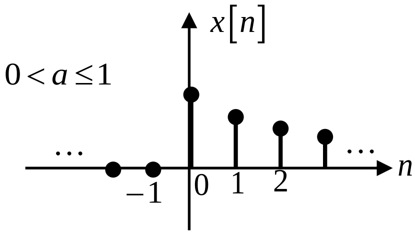
Phase is odd... $\angle X\left(e^{j\omega}\right) = -\angle X\left(e^{-j\omega}\right)$

Real part is even.................................. $X_R\left(e^{j\omega}\right) = X_R\left(e^{-j\omega}\right)$

Imaginary part is odd......................... $X_I\left(e^{j\omega}\right) = -X_I\left(e^{-j\omega}\right)$

Verification by an example

Ex: $x[n] = a^n \ u[n] \quad \leftrightarrow \quad X\left(e^{j\omega}\right) = \dfrac{1}{1 - ae^{-j\omega}} \qquad |a| < 1$



a) FT is conjugate symmetric:

$$X\left(e^{j\omega}\right) = X^*\left(e^{-j\omega}\right) = \frac{1}{1 - ae^{-j\omega}} \qquad |a| < 1$$

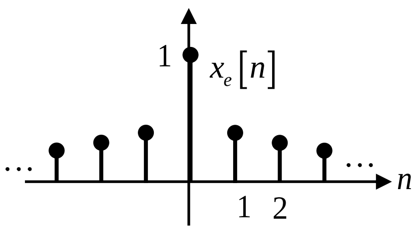
b) Real part of FT is an even function:

$$\mathrm{Re}\left\{X\left(e^{j\omega}\right)\right\} = X_R\left(e^{j\omega}\right) = \frac{1 - a\cos\omega}{1 + a^2 - 2a\cos\omega} = X_R\left(e^{-j\omega}\right)$$

c) $X_R\left(e^{j\omega}\right)$ is the FT of $x_e[n]$:

$$X_R\left(e^{j\omega}\right) = \frac{1 - a\cos\omega}{1 + a^2 - 2a\cos\omega}$$

$$x_e[n] = \frac{1}{2}\left(a^n \ u[n] + a^{-n} \ u[-n]\right)$$



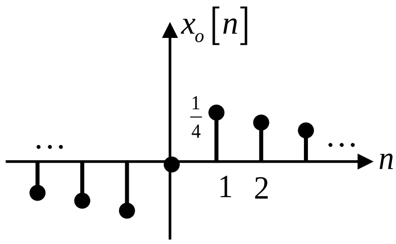
d) Imaginary part of Ft is an odd function.

$$\mathrm{Im}\left\{X\left(e^{j\omega}\right)\right\} = X_I\left(e^{j\omega}\right) = \frac{-a\sin\omega}{1 + a^2 - 2a\cos\omega} = -X_I\left(e^{-j\omega}\right)$$

e) $X_I(e^{j\omega})$ is the FT of $x_o[n]$:

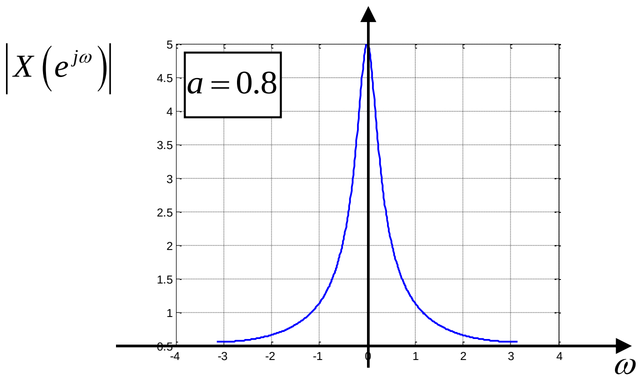
$$X_I\left(e^{j\omega}\right)=\frac{-a\sin\omega}{1+a^2-2a\cos\omega}$$

$$x_o[n]=\frac{1}{2}\left(a^n\,u[n]-a^{-n}\,u[-n]\right)$$



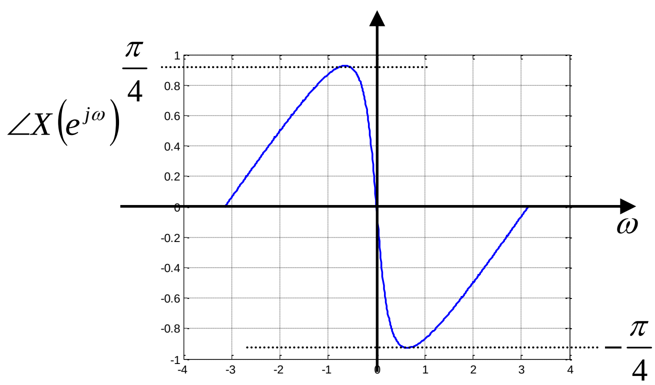
f) Magnitude of FT is an even function:

$$\left|X\left(e^{j\omega}\right)\right|=\frac{1}{\sqrt{1+a^2-2a\cos\omega}}=\left|X\left(e^{-j\omega}\right)\right|$$



g) Phase of FT is an odd function

$$\angle X\left(e^{j\omega}\right)=-\tan^{-1}\left(\frac{a\sin\omega}{1-a\cos\omega}\right)=-\angle X\left(e^{-j\omega}\right)$$



$$x[n] \overset{F}{\leftrightarrow} X\left(e^{j\omega}\right) \qquad \left(x[n] = F\left\{X\left(e^{j\omega}\right)\right\}, \quad X\left(e^{j\omega}\right) = F^{-1}\left\{x[n]\right\}\right)$$

1) $ax[n]+by[n] \overset{F}{\leftrightarrow} aX\left(e^{j\omega}\right)+bY\left(e^{j\omega}\right)$ Linearity

2) $x[n-n_0] \overset{F}{\leftrightarrow} e^{-j\omega n_0}X\left(e^{j\omega}\right)$ Time-shift

3) $e^{j\omega_0 n}x[n] \overset{F}{\leftrightarrow} X\left(e^{j(\omega-\omega_0)}\right)$ Freq. shift

4) $x[-n] \overset{F}{\leftrightarrow} X\left(e^{-j\omega}\right)$ Time reversal

5) $n\,x[n] \overset{F}{\leftrightarrow} j\dfrac{dX\left(e^{j\omega}\right)}{d\omega}$ Differentiation in frequency domain

6) $x[n]*y[n] \overset{F}{\leftrightarrow} X\left(e^{j\omega}\right)Y\left(e^{j\omega}\right)$ Convolution

7) $y[n]=x[n]w[n] \overset{F}{\leftrightarrow} Y\left(e^{j\omega}\right)=\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} X\left(e^{j\phi}\right)W\left(e^{j\omega-\phi}\right)d\phi$ Modulation, windowing

Parseval’s theorem (prove as an exercise)

8) $\displaystyle\sum_{n=-\infty}^{\infty}\left|x[n]\right|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|X\left(e^{j\omega}\right)\right|^2 d\omega$ energy of the signal

$\left|X\left(e^{j\omega}\right)\right|^2$ is called the “**energy density spectrum**”.

9) $\displaystyle\sum_{n=-\infty}^{\infty} x[n]y^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X\left(e^{j\omega}\right)Y^*\left(e^{j\omega}\right)d\omega$

Proof of (6):

Let

$$w[n] \stackrel{\text{def}}{=} \sum_{k=-\infty}^{\infty} x[k]y[n-k] = x[n] * y[n]$$

$$W(e^{j\omega}) = ?$$

$$W(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} x[k]y[n-k] \right) e^{-j\omega n}$$

$$= \sum_{k=-\infty}^{\infty} x[k] \left(\sum_{n=-\infty}^{\infty} y[n-k]e^{-j\omega n} \right)$$

$$= \sum_{k=-\infty}^{\infty} x[k]e^{-j\omega k} \left(\sum_{m=-\infty}^{\infty} y[m]e^{-j\omega m} \right)$$

$$= X(e^{j\omega})Y(e^{j\omega})$$

Proof of (8) using (6):

Let

$$w[n] \stackrel{\text{def}}{=} \sum_{k=-\infty}^{\infty} x[k]x^*[k-n] = x[n] * x^*[-n]$$

$$W\left(e^{j\omega}\right) = X\left(e^{j\omega}\right)X^*\left(e^{j\omega}\right) = \left|X\left(e^{j\omega}\right)\right|^2$$

$$w[0] = \sum_{k=-\infty}^{\infty} x[k]x^*[k] = \sum_{k=-\infty}^{\infty} |x[k]|^2$$

$$w[0] = \frac{1}{2\pi}\int_{-\pi}^{\pi} W\left(e^{j\omega}\right) d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|X\left(e^{j\omega}\right)\right|^2 d\omega$$

Proof of (9):

$$\sum_{n=-\infty}^{\infty} x[n]y^*[n]$$

$$= \sum_{n=-\infty}^{\infty} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n}\, d\omega \right) y^*[n]$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) \underbrace{\left(\sum_{n=-\infty}^{\infty} y^*[n] e^{j\omega n} \right)}_{Y^*(e^{j\omega})} d\omega$$

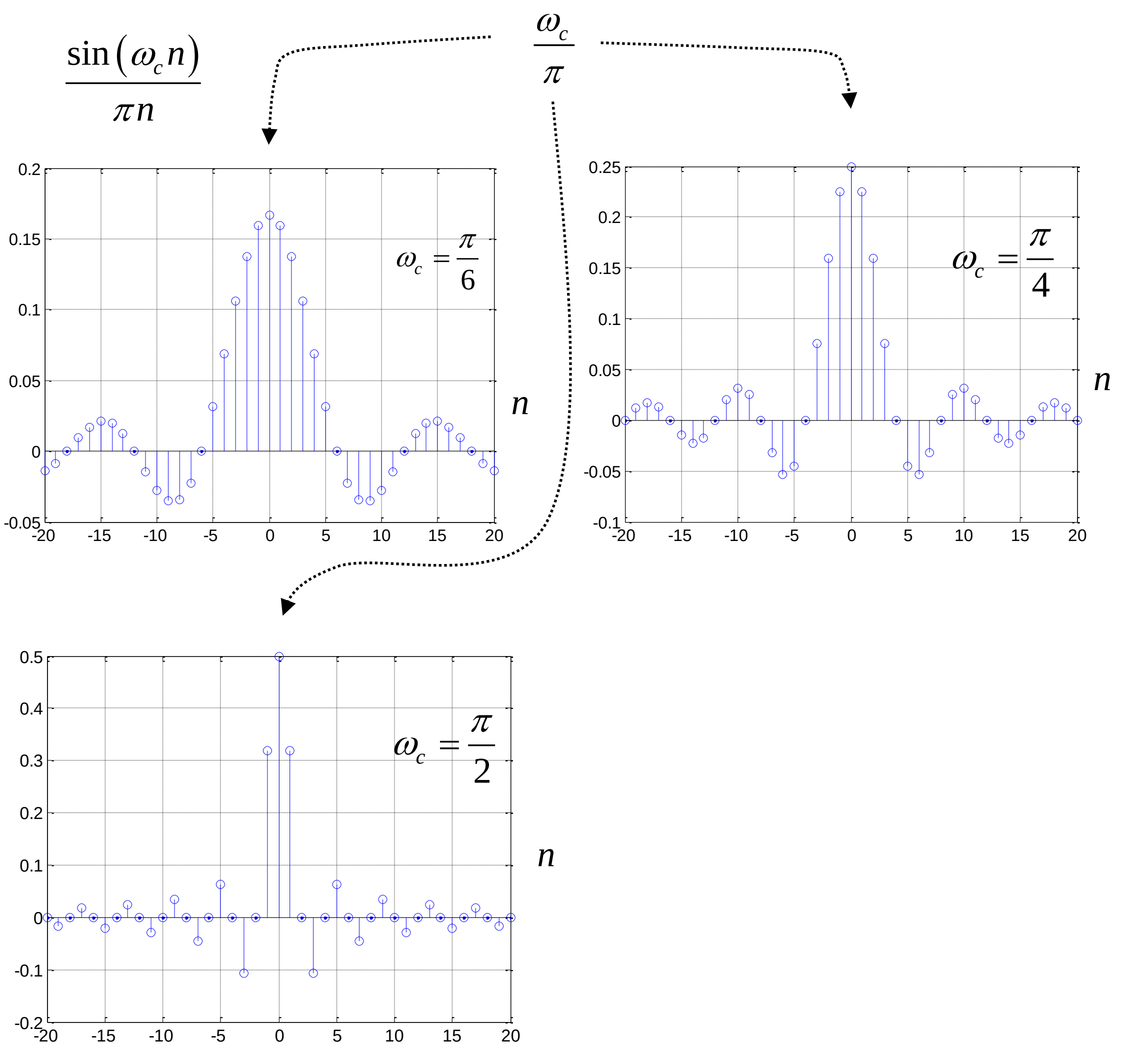
$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) Y^*(e^{j\omega}) d\omega$$

FOURIER TRANSFORM PAIRS

$\delta[n]$	1
$\delta[n-n_0]$	$e^{-j\omega n_0}$
1 $\quad -\infty < n < \infty$	$X\left(e^{j\omega}\right) = 2\pi \sum_{r=-\infty}^{\infty} \delta(\omega + 2\pi r)$
$x[n] = a^n\ u[n] \qquad \lvert a\rvert < 1$	$\dfrac{1}{1-ae^{-j\omega}}$
$x[n] = u[n]$	$\dfrac{1}{1-e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\ \delta(\omega + 2\pi k)$
$n a^n\ u[n] \qquad \lvert a\rvert < 1$	$\dfrac{ae^{-j\omega}}{\left(1-ae^{-j\omega}\right)^2}$
$(n+1)a^n\ u[n] \quad \lvert a\rvert < 1$ $= \left(a^n\ u[n]\right) * \left(a^n\ u[n]\right)$	$\dfrac{1}{\left(1-ae^{-j\omega}\right)^2}$
$\dfrac{(n+2)(n+1)a^n\ u[n]}{2}$	$\dfrac{1}{\left(1-ae^{-j\omega}\right)^3}$
$\dfrac{1}{(k-1)!}\ \dfrac{(n+k-1)!}{n!}\ a^n u[n]$ $\dfrac{1}{(k-1)!}\ (n+k-1)(n+k-2)\ldots(n+1)\ a^n u[n]$	$\dfrac{1}{\left(1-ae^{-j\omega}\right)^k}$
$\dfrac{1}{\sin\omega_p}\ r^n \sin\left(\omega_p (n+1)\right) u[n] \qquad \lvert r\rvert < 1$	$\dfrac{1}{1-2r\cos\left(\omega_p\right)e^{-j\omega} + r^2 e^{-j2\omega}}$ show using $x[n] = a^n\ u[n]$
$\dfrac{\sin(\omega_c n)}{\pi n}$	$X\left(e^{j\omega}\right) = \begin{cases} 1, & \lvert\omega\rvert < \omega_c \\ 0, & \omega_c < \lvert\omega\rvert \le \pi \end{cases}$
$x[n] = \begin{cases} 1, & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$	$\dfrac{\sin\left(\omega(M+1)/2\right)}{\sin(\omega/2)}\ e^{-j\omega M/2}$
$e^{j\omega_0 n}$	$2\pi \sum_{r=-\infty}^{\infty} \delta(\omega - \omega_0 + r2\pi)$
$\cos(\omega_0 n + \phi)$	$\pi e^{j\phi} \sum_{r=-\infty}^{\infty} \delta(\omega - \omega_0 + r2\pi) + \pi e^{-j\phi} \sum_{r=-\infty}^{\infty} \delta(\omega + \omega_0 + r2\pi)$
$\sin(\omega_0 n + \phi)$	$-j\pi e^{j\phi} \sum_{r=-\infty}^{\infty} \delta(\omega - \omega_0 + r2\pi) + j\pi e^{-j\phi} \sum_{r=-\infty}^{\infty} \delta(\omega + \omega_0 + r2\pi)$

Ex: IDEAL LOWPASS FILTER (IMPULSE RESPONSE)

Plots of $\sin \frac{(\omega_c n)}{\pi n}$ sequence

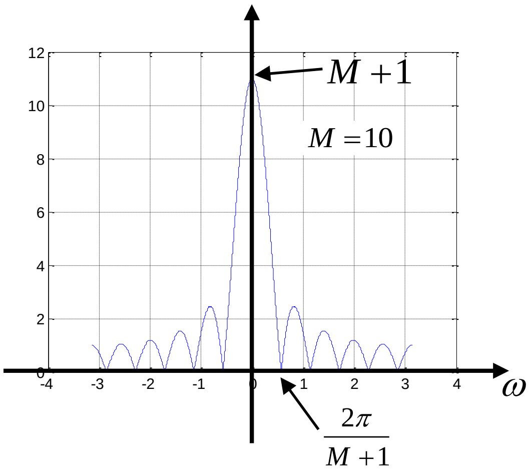


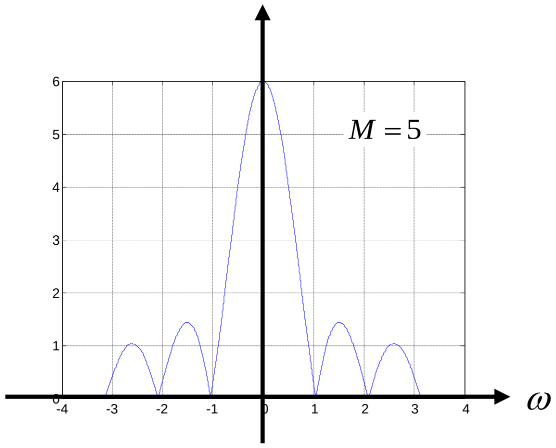
They are infinitely long sequences in $-\infty < n < \infty$

Plots are arbitrarily in $-20 < n < 20$

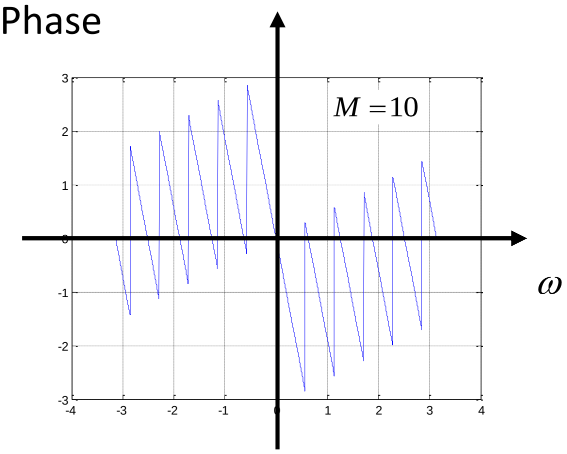
Ex: MOVING AVERAGE FILTER

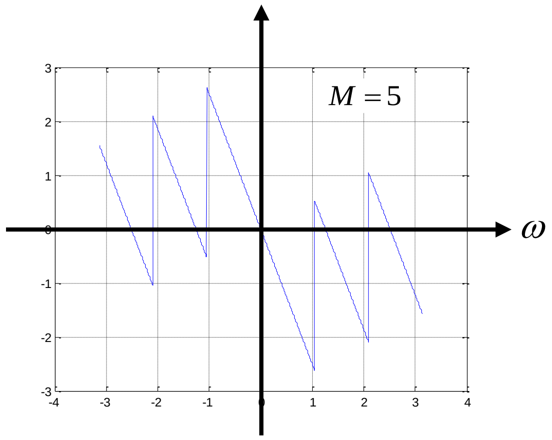
Magnitude $$\left| \frac{\sin\left(\omega\left(M+1\right)/2\right)}{\sin\left(\omega/2\right)} e^{-j\omega M/2} \right|$$





Phase



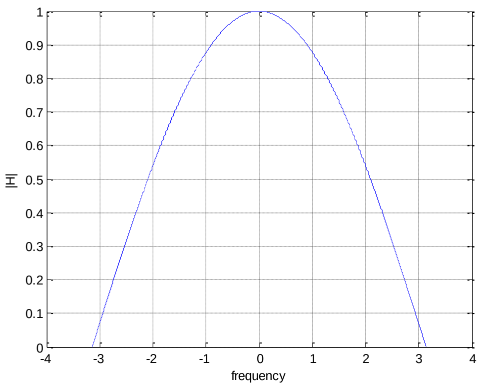


DTFT functions are plotted in $-\pi \le \omega \le \pi$ or in $0 \le \omega \le 2\pi$

M = 1

$$h[n]=\frac{1}{2}\left(\delta[n]+\delta[n-1]\right) \rightarrow H\left(e^{j\omega}\right)=\frac{1}{2}\left(1+e^{-j\omega}\right)=e^{-j\frac{\omega}{2}}\cos\frac{\omega}{2}$$

$$y[n]=\frac{1}{2}\left(x[n]+x[n-1]\right)$$



M = 4

$$y[n]=\frac{1}{5}\left(x[n]+x[n-1]+x[n-2]+x[n-3]+x[n-4]\right)$$

$$h[n]=\frac{1}{5}\left(\delta[n]+\delta[n-1]+\delta[n-2]+\delta[n-3]+\delta[n-4]\right)$$

$$\rightarrow \quad H\left(e^{j\omega}\right)=\frac{1}{5}\left(1+e^{-j\omega}+e^{-j2\omega}+e^{-j3\omega}+e^{-j4\omega}\right)$$

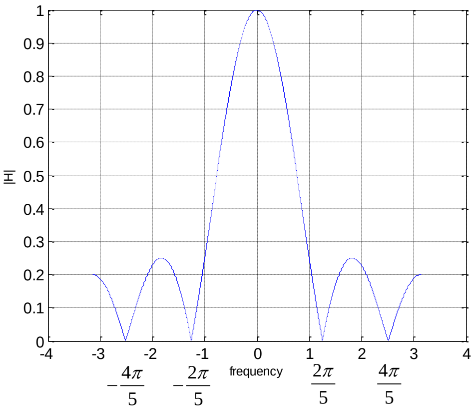
$$=\frac{1}{5}e^{-j2\omega}\left(1+2\cos\omega+2\cos 2\omega\right)$$

or

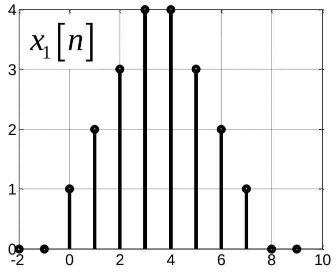
$$H\left(e^{j\omega}\right)=\frac{1}{5}\left(1+e^{-j\omega}+e^{-j2\omega}+e^{-j3\omega}+e^{-j4\omega}\right)$$

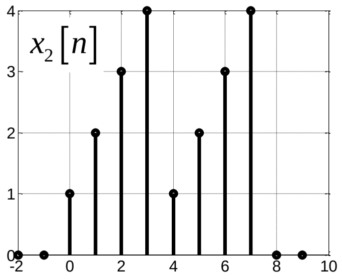
$$=\frac{1}{5}\frac{1-e^{-j5\omega}}{1-e^{-j\omega}}=\frac{1}{5}\frac{e^{-j\frac{5}{2}\omega}\left(e^{j\frac{5}{2}\omega}-e^{-j\frac{5}{2}\omega}\right)}{e^{-j\frac{\omega}{2}}\left(e^{j\frac{\omega}{2}}-e^{-j\frac{\omega}{2}}\right)}$$

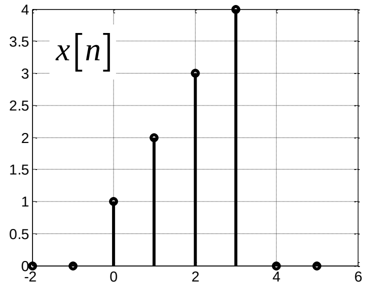
$$=\frac{1}{5}e^{-j2\omega}\frac{\sin\left(\frac{5}{2}\omega\right)}{\sin\left(\frac{\omega}{2}\right)}$$



Ex: Express $X_1\left(e^{j\omega}\right)$, $X_2\left(e^{j\omega}\right)$, in terms of $X\left(e^{j\omega}\right)$, the DTFT of $x[n]$.







$x_1[n]=x[n]+x[-(n-7)]=x[n]+x[-n+7]$ → $X_1\left(e^{j\omega}\right)=X\left(e^{j\omega}\right)+e^{-j7\omega}X\left(e^{-j\omega}\right)$

$x_2[n]=x[n]+x[n-4]$ → $X_2\left(e^{j\omega}\right)=X\left(e^{j\omega}\right)+e^{-j4\omega}X\left(e^{j\omega}\right)$

One can also write as

$$X_1\left(e^{j\omega}\right) = X\left(e^{j\omega}\right) + e^{-j7\omega}X\left(e^{-j\omega}\right)$$

$$= X\left(e^{j\omega}\right) + e^{-j7\omega}X^*\left(e^{j\omega}\right) \qquad \text{since } x[n] \text{ is real}$$

$$= \left|X\left(e^{j\omega}\right)\right| \left(e^{j\measuredangle X\left(e^{j\omega}\right)} + e^{-j7\omega}e^{-j\measuredangle X\left(e^{j\omega}\right)}\right)$$

$$= \left|X\left(e^{j\omega}\right)\right| e^{-j\frac{7}{2}\omega} \left(e^{j\frac{7}{2}\omega}e^{j\measuredangle X\left(e^{j\omega}\right)} + e^{-j\frac{7}{2}\omega}e^{-j\measuredangle X\left(e^{j\omega}\right)}\right)$$

$$= 2\,\mathrm{Re}\left\{e^{j\left(\measuredangle X\left(e^{j\omega}\right)+\frac{7}{2}\omega\right)}\right\} \left|X\left(e^{j\omega}\right)\right| e^{-j\frac{7}{2}\omega}$$

$$X_2\left(e^{j\omega}\right) = X\left(e^{j\omega}\right)\left(1 + e^{-j4\omega}\right)$$

$$= X\left(e^{j\omega}\right)e^{-j2\omega}\left(e^{j2\omega} + e^{-j2\omega}\right)$$

$$= X\left(e^{j\omega}\right)e^{-j2\omega}2\cos(2\omega)$$

$$= 2\cos(2\omega)\left|X\left(e^{j\omega}\right)\right| e^{-j2\omega}e^{j\angle X\left(e^{j\omega}\right)}$$

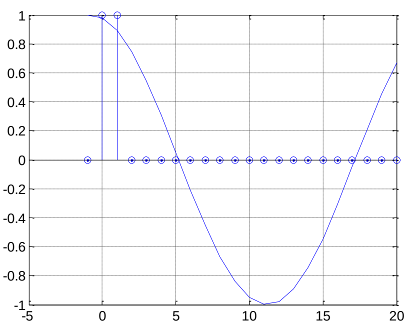
Ex: What is the inverse DTFT, $y[n]$, of $Y\left(e^{j\omega}\right) = \frac{2e^{-j3\omega}}{\left(1-\frac{1}{8}e^{-j\omega}\right)^2}$?

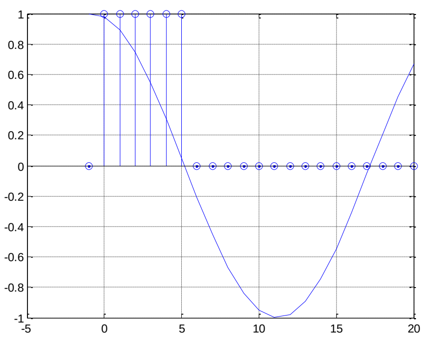
From the table $\quad (n+1)a^n\, u[n] \quad \leftrightarrow \quad \frac{1}{\left(1-ae^{-j\omega}\right)^2} \quad$ for $\quad |a|<1$

$$2(n+1)\frac{1}{8}^{n}\, u[n] \quad \leftrightarrow \quad \frac{2}{\left(1-\frac{1}{8}e^{-j\omega}\right)^2}$$

$$2(n-2)\frac{1}{8}^{(n-3)}\, u[n-3] \quad \leftrightarrow \quad \frac{2e^{-j3\omega}}{\left(1-\frac{1}{8}e^{-j\omega}\right)^2}$$

Why does the high frequency gain of MA filter "<u>decrease</u>" as M increases?





Comment on the above illustrations.

$$y[n] = x[n] * h[n] \xleftrightarrow{FT} Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$$

$$\Rightarrow H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})}$$

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k] \xleftrightarrow{FT} \sum_{k=0}^{N} a_k e^{-jk\omega} Y(e^{j\omega})$$
$$= \sum_{k=0}^{M} b_k e^{-jk\omega} X(e^{j\omega})$$

$$\Rightarrow \quad Y(e^{j\omega}) \sum_{k=0}^{N} a_k e^{-jk\omega} = X(e^{j\omega}) \sum_{k=0}^{M} b_k e^{-jk\omega}$$

$$\Rightarrow \quad \frac{Y(e^{j\omega})}{X(e^{j\omega})} = H(e^{j\omega}) = \frac{\sum_{k=0}^{M} b_k e^{-jk\omega}}{\sum_{k=0}^{N} a_k e^{-jk\omega}}$$

$$H(e^{j\omega}) = \frac{b_0 + b_1 e^{-j\omega} + b_2 e^{-j2\omega} + \cdots + b_M e^{-jM\omega}}{a_0 + a e^{-j\omega} + a_2 e^{-j2\omega} + \cdots + a_N e^{-jN\omega}}$$