# COMPUTATION OF DFT

## DIRECT COMPUTATION OF DFT

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn} \qquad k = 0,1,\dots,N-1$$

For each $k$,

$N$ complex multiplications (multiplication of two complex numbers)

$N-1$ complex additions (addition of two complex numbers)

For all

$N^2$ complex multiplications

$N(N-1)$ complex additions

Each complex multiplication = $4$ real multiplications + $2$ real additions

Each complex addition = $2$ real additions

Therefore, for each $k$

$4N$ real multiplications

$4N-2$ real additions $\quad (2(N-1)+2N)$

In total

$4N^2$ real multiplications

$N(4N-2)$ real additions $\quad (N(4N-2) \cong 4N^2 \quad for \quad N \gg 1)$

If $x[n]$ is real, for each $k$

$2N$ real multiplications

$2(N-1)$ real additions

In total

$2N^2$ real multiplications

$2N(N-1)$ real additions $\quad (2N(N-1) \cong 2N^2 \quad for \quad N \gg 1)$

FAST FOURIER TRANSFORM (FFT)

Periodicity and symmetry of $W_N^q$.

1) $W_N^q = W_N^{q+N}$

$$e^{-j\frac{2\pi}{N}(q+N)} = e^{-j\frac{2\pi}{N}q}\underbrace{e^{-j\frac{2\pi}{N}N}}_{=1}$$

Ex: $W_{16}^4 = W_{16}^{20}$

2) $W_N^q = -W_N^{q+\frac{N}{2}}$

$$e^{-j\frac{2\pi}{N}\left(q+\frac{N}{2}\right)} = e^{-j\frac{2\pi}{N}q}\underbrace{e^{-j\pi}}_{=-1}$$

Ex: $W_{16}^3 = -W_{16}^{11}$

Let's have a close look at an 8-point DFT.

$X[0] = x[0] + x[1] + x[2] + x[3] + x[4] + x[5] + x[6] + x[7]$

$X[1] = x[0] + x[1]W_8 + x[2]W_8^2 + x[3]W_8^3 + x[4]W_8^4 + x[5]W_8^5 + x[6]W_8^6 + x[7]W_8^7$

$X[2] = x[0] + x[1]W_8^2 + x[2]W_8^4 + x[3]W_8^6 + x[4]\underbrace{W_8^8}_{=1} + x[5]\underbrace{W_8^{10}}_{=W_8^2} + x[6]\underbrace{W_8^{12}}_{=W_8^4} + x[7]\underbrace{W_8^{14}}_{=W_8^6}$

$X[3] = x[0] + x[1]W_8^3 + x[2]W_8^6 + x[3]\underbrace{W_8^9}_{=W_8^1} + x[4]\underbrace{W_8^{12}}_{=W_8^4} + x[5]\underbrace{W_8^{15}}_{=W_8^7} + x[6]\underbrace{W_8^{18}}_{=W_8^2} + x[7]\underbrace{W_8^{21}}_{=W_8^5}$

$X[4] = x[0] + x[1]\underbrace{W_8^4}_{=-1} + x[2]\underbrace{W_8^8}_{=1} + x[3]\underbrace{W_8^{12}}_{=-1} + x[4]\underbrace{W_8^{16}}_{=1} + x[5]\underbrace{W_8^{20}}_{=-1} + x[6]\underbrace{W_8^{24}}_{=1} + x[7]\underbrace{W_8^{28}}_{=-1}$

$X[5] = x[0] + x[1]W_8^5 + x[2]\underbrace{W_8^{10}}_{=W_8^2} + x[3]\underbrace{W_8^{15}}_{=W_8^7} + x[4]\underbrace{W_8^{20}}_{=-1} + x[5]\underbrace{W_8^{25}}_{=W_8^1} + x[6]\underbrace{W_8^{30}}_{=W_8^6} + x[7]\underbrace{W_8^{35}}_{=W_8^3}$

$X[6] = x[0] + x[1]W_8^6 + x[2]\underbrace{W_8^{12}}_{=-1} + x[3]\underbrace{W_8^{18}}_{=W_8^2} + x[4]\underbrace{W_8^{24}}_{=1} + x[5]\underbrace{W_8^{30}}_{=W_8^6} + x[6]\underbrace{W_8^{36}}_{=-1} + x[7]\underbrace{W_8^{42}}_{=W_8^2}$

$X[7] = x[0] + x[1]W_8^7 + x[2]\underbrace{W_8^{14}}_{=W_8^6} + x[3]\underbrace{W_8^{21}}_{=W_8^5} + x[4]\underbrace{W_8^{28}}_{=-1} + x[5]\underbrace{W_8^{35}}_{=W_8^3} + x[6]\underbrace{W_8^{42}}_{=W_8^2} + x[7]\underbrace{W_8^{49}}_{=W_8^1}$

Note that, since

$$W_8^4 = -W_8^0 \qquad W_8^5 = -W_8^1 \qquad W_8^6 = -W_8^2 \qquad W_8^7 = -W_8^3$$

i.e.

$$W_N^q = -W_N^{q+\frac{N}{2}}$$

we effectively need only $W_8^1, W_8^2, W_8^3$ instead of $W_8^0, W_8^1, W_8^2, \dots, W_8^{49}$.

Rewriting

$$X[1]=(x[0]-x[4])+(x[1]-x[5])W_8^1+(x[2]-x[6])W_8^2+(x[3]-x[7])W_8^3$$

$$X[2]=(x[0]-x[2]+x[4]-x[6])+(x[1]-x[3]+x[5]-x[7])W_8^2$$

$$X[3]=(x[0]-x[4])+(x[3]-x[7])W_8^1+(-x[2]+x[6])W_8^2+(x[1]-x[5])W_8^3$$

$$X[4]=x[0]-x[1]+x[2]-x[3]+x[4]-x[5]+x[6]-x[7]$$

$$X[5]=(x[0]-x[4])+(-x[1]+x[5])W_8^1+(x[2]-x[6])W_8^2+(-x[3]+x[7])W_8^3$$

$$X[6]=x[0]-x[2]+x[4]-x[6]+(-x[1]+x[3]-x[5]+x[7])W_8^2$$

$$X[7]=(x[0]-x[4])+(-x[3]+x[7])W_8^1+(-x[2]+x[6])W_8^2+(-x[1]+x[5])W_8^3$$

Therefore, the computation of the following expressions is sufficient to get all 8-point DFT values…

| | |
|---|---|
| 1) $x[0]+x[4],\ \ x[0]-x[4]$ | 2 adds |
| 2) $x[1]+x[5],\ \ x[1]-x[5]$ | 2 adds |
| 3) $x[2]+x[6],\ \ x[2]-x[6]$ | 2 adds |
| 4) $x[3]+x[7],\ \ x[3]-x[7]$ | 2 adds |
| 5) $(x[1]-x[5])W_8^1,\ \ (-x[1]+x[5])W_8^3$ | 4 multiplies |
| 6) $(x[3]-x[7])W_8^3,\ \ (-x[3]+x[7])W_8^1$ | 4 multiplies |
| 7) $(x[2]-x[6])W_8^2,\ \ (x[1]-x[3]+x[5]-x[7])W_8^2$ | 1 add, 4 multiplies |

plus “some” additions in the previous page.

We will study the two systematic methods

1) Decimation-in-time

2) Decimation-in-frequency

# Decimation-in-Time

<u>Assume that $N$ is a power of two, i.e. $N = 2^m$</u>.

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn} \qquad k = 0,1,\dots,N-1$$

Decompose for even and odd indexed terms of $x[n]$:

$$X[k] = \sum_{n:even}^{N-1} x[n]W_N^{kn} + \sum_{n:odd}^{N-1} x[n]W_N^{kn}$$

$$= \sum_{r=0}^{\frac{N}{2}-1} x[2r]W_N^{2kr} + \sum_{r=0}^{\frac{N}{2}-1} x[2r+1]W_N^{k(2r+1)}$$

$$= \underbrace{\sum_{r=0}^{\frac{N}{2}-1} x[2r]W_{\frac{N}{2}}^{kr}}_{G[k]} + W_N^k \underbrace{\sum_{r=0}^{\frac{N}{2}-1} x[2r+1]W_{\frac{N}{2}}^{kr}}_{H[k]}$$

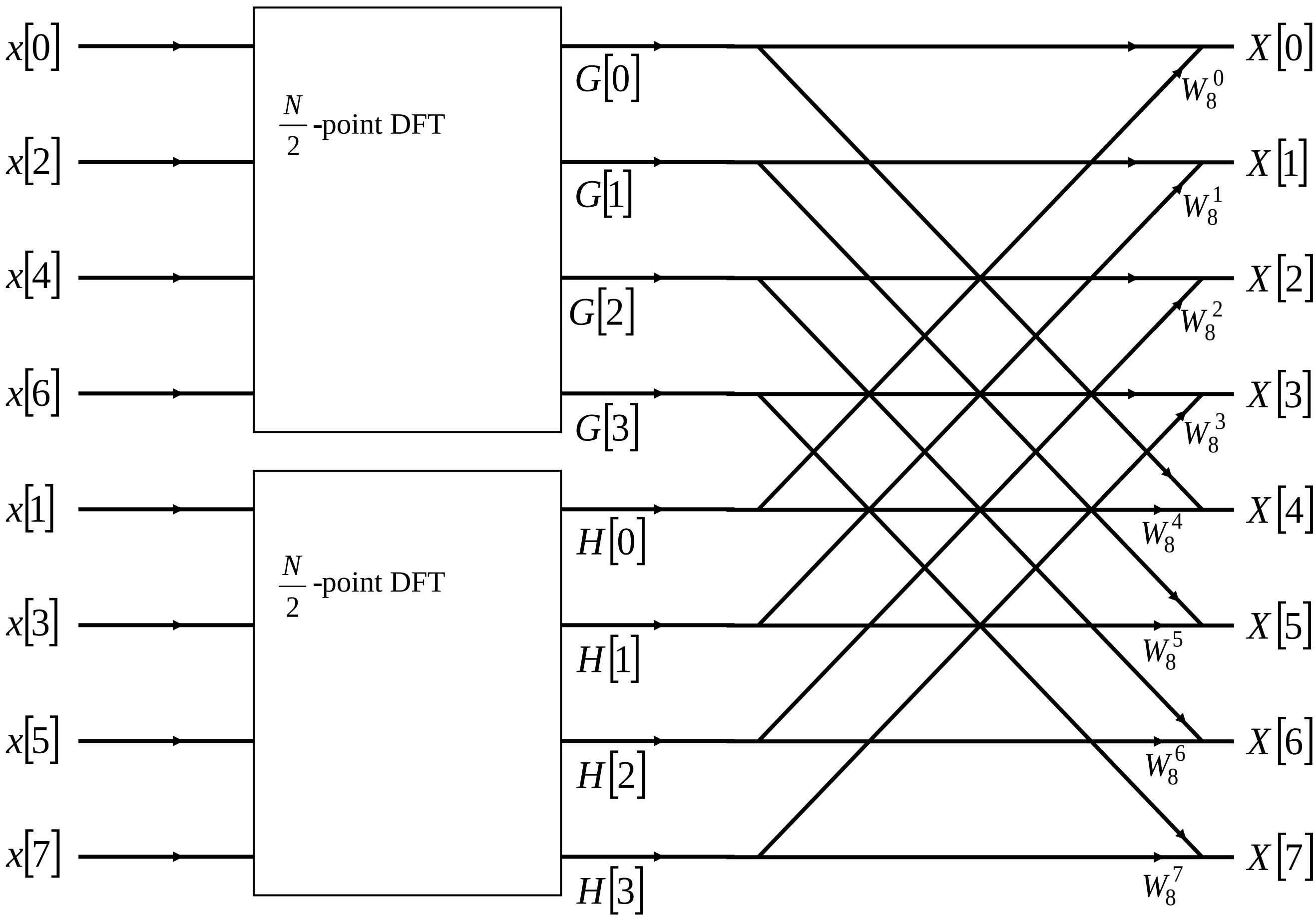
$$G[k]\colon \frac{N}{2} - point\ DFT\ of\ even\ indexed\ samples$$

$$H[k]\colon \frac{N}{2} - point\ DFT\ of\ odd\ indexed\ samples$$

Therefore

$$X[k] = G[k] + W_N^k H[k] \qquad k = 0,1,\dots,\underbrace{N-1}_{!!!}$$

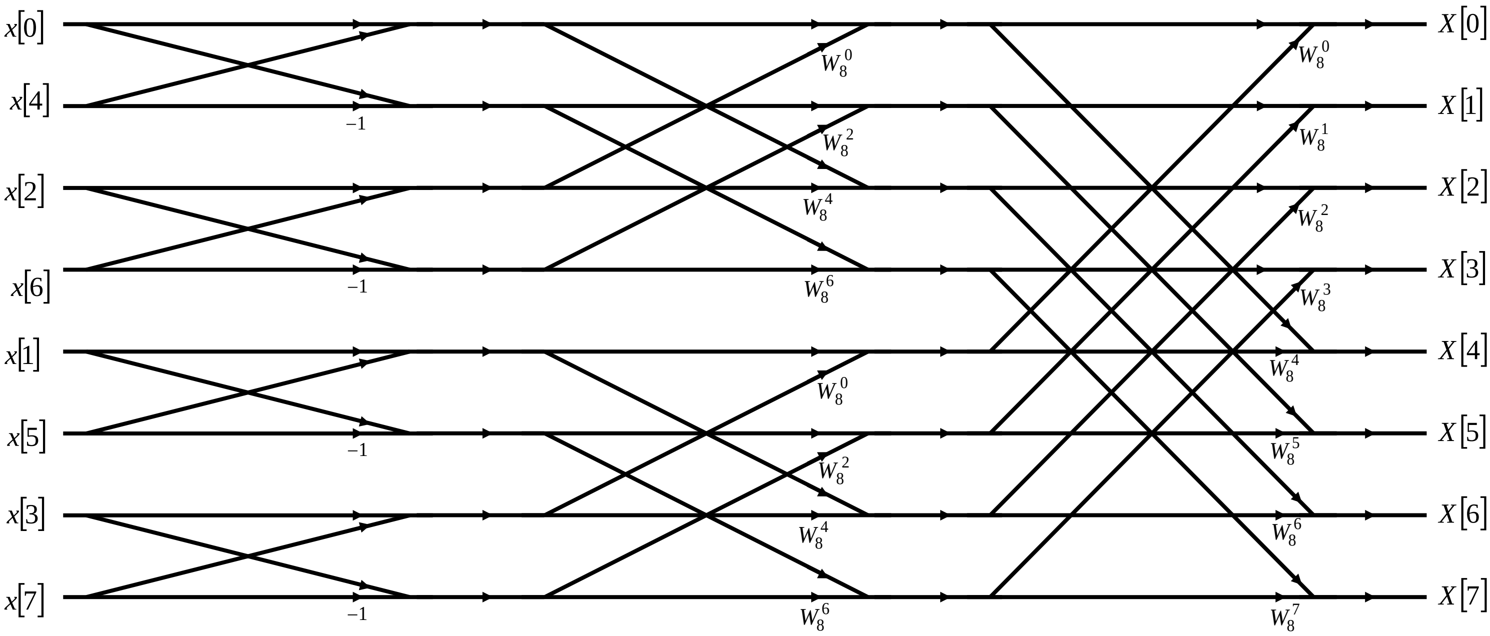
Attention: $G[k]$ and $H[k]$ are $\frac{N}{2}$−point DFTs.




$$2\left(\frac{N}{2}\right)^2 + N = \frac{N^2}{2} + N$$ complex multiplications required

$\frac{N}{2}$ −point DFTs can be decomposed into $\frac{N}{4}$ −point DFTs.

Then the following computational flow arises.




$$4\left(\frac{N}{4}\right)^2 + N + N = \frac{N^2}{4} + 2N$$

complex multiplications required

In general, the number of sections will be

$$\log_2 N = v$$

Therefore

$$\underbrace{\frac{N}{2}\left(\frac{N}{\frac{N}{2}}\right)^2}_{2N} + N + N + \cdots + N = (v+1)N$$

complex multiplications required

Considering the multiplications by -1 and 1, this value is approximately referred to as

$$N \log_2 N$$

This is also the approximate number of additions.

Ex:

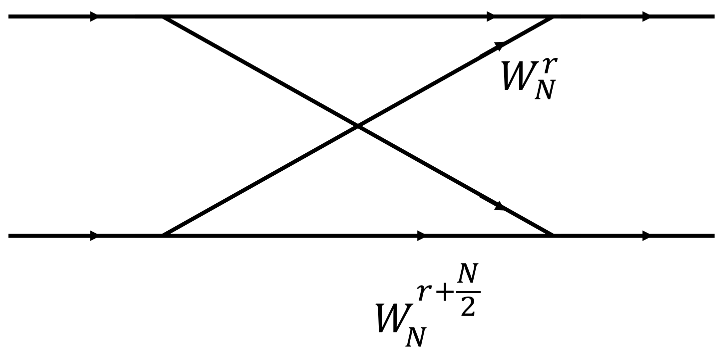
$$N = 2^{10} = 1024$$

$$N^2 = 1\,048\,576 \approx 10^6$$

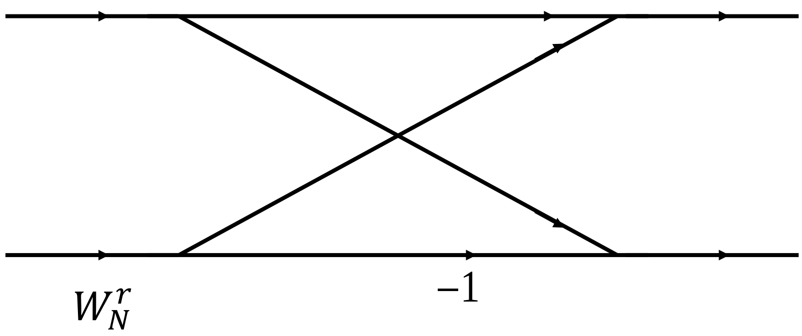
$$N \log_2 N = 10\,240 \approx 10^4$$

A reduction by a factor of 100 approximately.

Indeed, above diagrams contain many "butterfly" structures,




Such butterflies can be simplified as



# DECIMATION-IN-FREQUENCY

Assume that $N$ is a power of two, $N = 2^m$.

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn} \qquad k = 0,1,\ldots,N-1$$

Consider "even" and "odd" indexed elements of the DFT:

$$X[k] \begin{array}{l} \nearrow \; X[2r] \\ \searrow \; X[2r+1] \end{array} \qquad r = 0,1,\ldots,\frac{N}{2}-1$$

Even indexed elements:

$$X[2r] = \sum_{n=0}^{N-1} x[n] W_N^{2rn} \qquad r = 0,1,\dots,\frac{N}{2} - 1$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x[n] W_N^{2rn} + \sum_{n=\frac{N}{2}}^{N-1} x[n] W_N^{2rn}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x[n] W_N^{2rn} + \sum_{n=0}^{\frac{N}{2}-1} x\left[n + \frac{N}{2}\right] W_N^{2r\left(n+\frac{N}{2}\right)}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} \left(x[n] + x\left[n + \frac{N}{2}\right]\right) W_{\frac{N}{2}}^{rn}$$

This is an $\frac{N}{2}$ -point DFT of

$$\left(x[n] + x\left[n + \frac{N}{2}\right]\right) \qquad n = 0,1,\dots,\frac{N}{2} - 1.$$

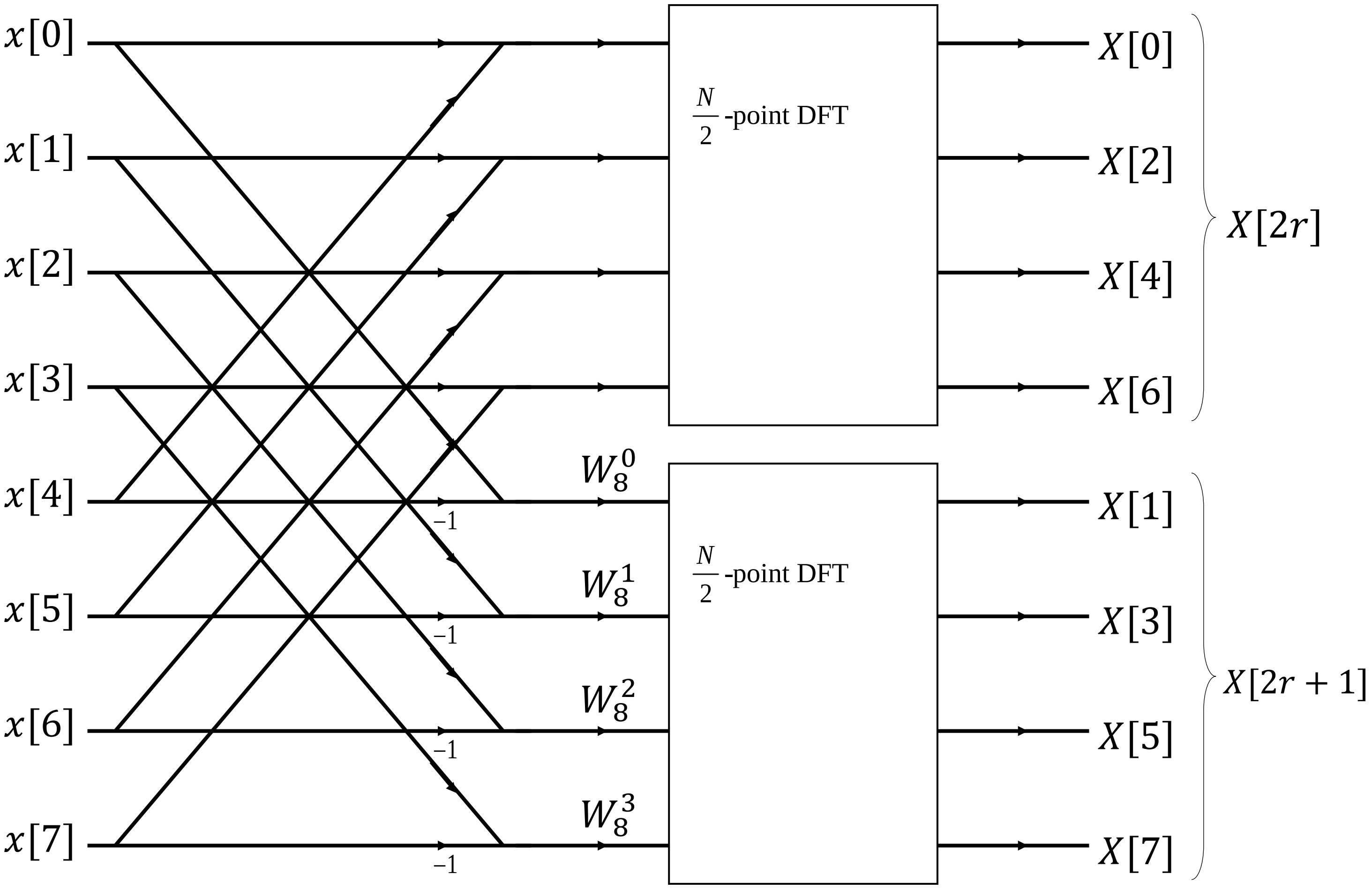
Odd indexed elements: Similarly,

$$X[2r+1] = \sum_{n=0}^{\frac{N}{2}-1} \left( x[n] - x\left[n + \frac{N}{2}\right] \right) W_N^n \, W_{\frac{N}{2}}^{rn}$$

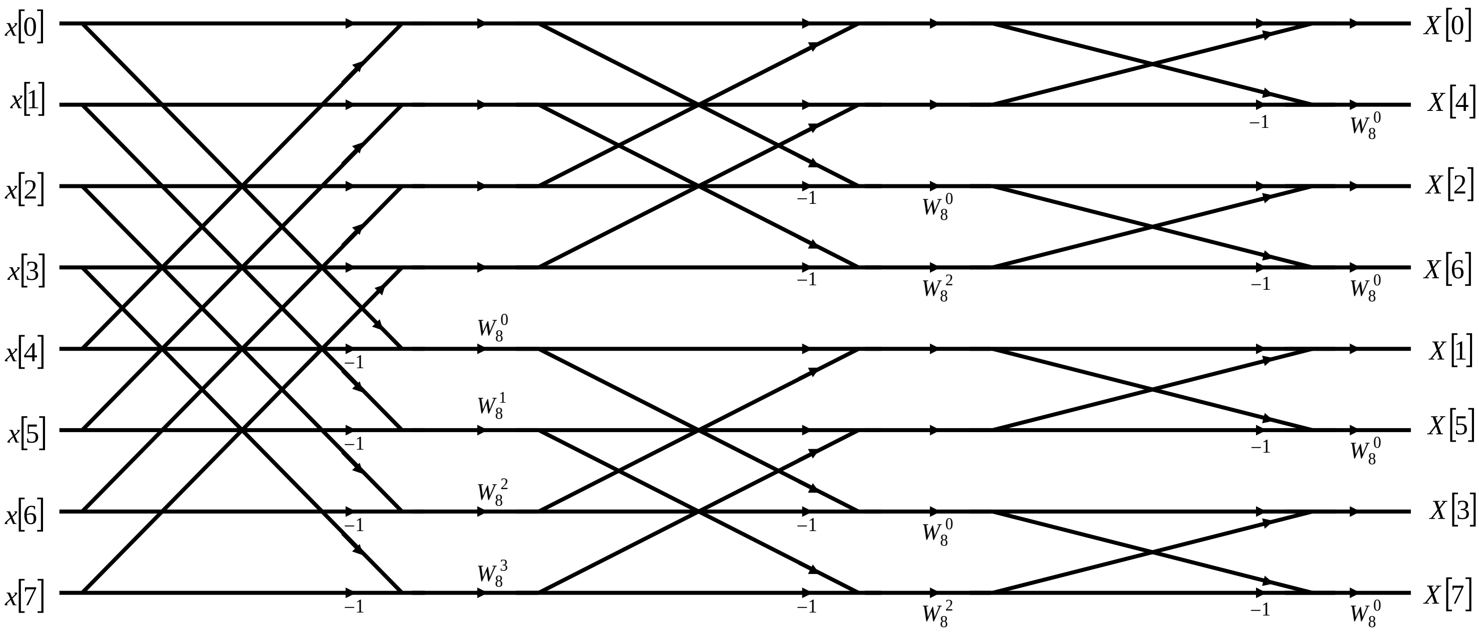
This is a $\frac{N}{2}$-point DFT of

$$\left( x[n] - x\left[n + \frac{N}{2}\right] \right) W_N^n \qquad n = 0,1,\dots,\frac{N}{2} - 1.$$

For $N = 8$


x[0]
x[1]
x[2]
x[3]
x[4]
x[5]
x[6]
x[7]
N/2 -point DFT
N/2 -point DFT
W_8^0
W_8^1
W_8^2
W_8^3
−1
−1
−1
−1
X[0]
X[2]
X[4]
X[6]
X[2r]
X[1]
X[3]
X[5]
X[7]
X[2r + 1]

Decomposing $\frac{N}{2}$ -point DFT blocks down to 2-point DFTs we get the following flow diagram



# THE GOERTZEL ALGORITHM

The algorithm reduces the storage requirement

DFT expression:

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn} \qquad k = 0,1,\ldots,N-1$$

can be written in a recursive form.

For example, let $N = 8$

$$X[k] = x[0] + x[1]W_N^k + x[2]W_N^{2k} + x[3]W_N^{3k} + x[4]W_N^{4k} + x[5]W_N^{5k} + x[6]W_N^{6k} + x[7]W_N^{7k}$$

$$= x[0] + W_N^k\left(x[1] + x[2]W_N^k + x[3]W_N^{2k} + x[4]W_N^{3k} + x[5]W_N^{4k} + x[6]W_N^{5k} + x[7]W_N^{6k}\right)$$

$$\vdots$$

$$= x[0] + W_N^k\left(x[1] + W_N^k\left(x[2] + W_N^k\left(x[3] + W_N^k\left(x[4] + W_N^k\left(x[5] + W_N^k\left(x[6] + x[7]W_N^k\right)\right)\right)\right)\right)\right)$$

OR

Since $W_N^{-kN} = 1$

$$X[k] = W_N^{-kN} \sum_{n=0}^{N-1} x[n] W_N^{kn}$$

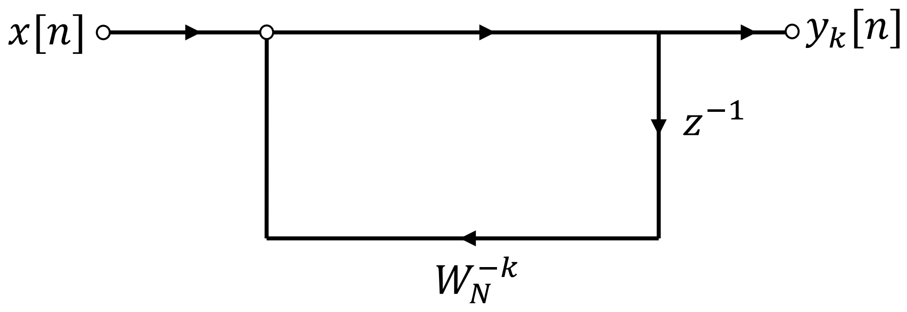
$$= \sum_{n=0}^{N-1} x[n] W_N^{-k(N-n)}$$

For $N = 8$

$$W_8^{-8k} X[k] =$$

$$\left(\left(\left(\left(\left(\left(\left(x[0]W_8^{-k} + x[1]\right)W_8^{-k} + x[2]\right)W_8^{-k} + x[3]\right)W_8^{-k} + \right.\right.\right.\right.$$

$$\left.\left.\left.\left. x[4]\right)W_8^{-k} + x[5]\right)W_8^{-k} + x[6]\right)W_8^{-k} + x[7]\right)$$




$$y_k[n] = W_N^{-k} y_k[n-1] + x[n] \qquad n = 0,1,\dots,N$$

$$y_k(-1) = 0 \qquad x[N] = 0$$

Take $X[k] = y_k[N]$

Note that $X[k]$'s are computed sequentially.

Computational demand:

For each $k$, $y_k[1], y_k[2], \ldots, y_k[N-1]$ must be computed to find $y_k[N] = X[k]$.

This requires, (assuming a complex input) $4N$ multiplications and $4N$ additions.

This amount is almost the same as that of direct computation.

However, the storage requirement of $W_N^{kn}$ values vanishes.

# DOWNSAMPLING AND DECIMATION-IN-TIME

For the $N$-point DFT of $x[n]$ we obtained

$$X[k] = G[k] + W_N^k H[k]$$

where $G[k]$ and $H[k]$ are the $\frac{N}{2}$-point DFTs, respectively, of even and odd indexed elements of the sequence $x[n]$, i.e.,

$$g[n] = x[2n]$$

$$h[n] = x_{-1}[2n] \qquad\qquad x_{-1}[n] = x[n+1]$$

$$G\left(e^{j\omega}\right) = \frac{1}{2}\left(X\left(e^{j\frac{\omega}{2}}\right) + X\left(e^{j\frac{1}{2}(\omega-2\pi)}\right)\right)$$

$$H\left(e^{j\omega}\right) = \frac{1}{2}\left(e^{j\frac{\omega}{2}}X\left(e^{j\frac{\omega}{2}}\right) + e^{j\frac{1}{2}(\omega-2\pi)}X\left(e^{j\frac{1}{2}(\omega-2\pi)}\right)\right)$$

Therefore

$$X\left(e^{j\omega}\right) = G\left(e^{j2\omega}\right) + e^{-j\omega}H\left(e^{j2\omega}\right)$$

$$X[k] = X\left(e^{j\omega}\right)\Big|_{\omega=k\frac{2\pi}{N}}$$

$$G\left(e^{j2\omega}\right) = \sum_{n=0}^{\frac{N}{2}-1} x[2n]e^{-j2\omega n}$$

$$G\left(e^{j2\omega}\right)\Big|_{\omega=k\frac{2\pi}{N}} = \sum_{n=0}^{\frac{N}{2}-1} x[2n]e^{-j2k\frac{2\pi}{N}n}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x[2n]W_{\frac{N}{2}}^{kn}$$

Likewise

$$H\left(e^{j2\omega}\right)\Big|_{\omega=k\frac{2\pi}{N}} = \sum_{n=0}^{\frac{N}{2}-1} x[2n+1]W_{\frac{N}{2}}^{kn}$$

RADIX 3 vs, RADIX 2

In place computations

Chirp transform 9.6