COMPUTATION OF DFT

FAST FOURIER TRANSFORM (FFT)

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn} \qquad k = 0,1,\dots,N-1$$

For each k,

N	complex multiplications
$N-1$	complex additions

For all

N^2	complex multiplications
$N(N-1)$	complex additions

Each complex multiplication = 4 real multiplications + 2 real additions

Each complex addition = 2 real additions

Therefore, for each k

$4N$ real multiplications

$4N-2$ real additions $\qquad (2(N-1)+2N)$

In total

$4N^2$ real multiplications

$N(4N-2)$ real additions

Let's have a close look at an 8-point DFT.

$$X[0]=x[0]+x[1]+x[2]+x[3]+x[4]+x[5]+x[6]+x[7]$$

$$X[1]=x[0]+x[1]\,W_8+x[2]\,W_8^2+x[3]\,W_8^3+x[4]\,W_8^4+x[5]\,W_8^5+x[6]\,W_8^6+x[7]\,W_8^7$$

$$X[2]=x[0]+x[1]\,W_8^2+x[2]\,W_8^4+x[3]\,W_8^6+x[4]\,\underbrace{W_8^8}_{1}+x[5]\,\underbrace{W_8^{10}}_{W_8^2}+x[6]\,\underbrace{W_8^{12}}_{W_8^4}+x[7]\,\underbrace{W_8^{14}}_{W_8^6}$$

$$X[3]=x[0]+x[1]\,W_8^3+x[2]\,W_8^6+x[3]\,\underbrace{W_8^9}_{W_8^1}+x[4]\,\underbrace{W_8^{12}}_{W_8^4}+x[5]\,\underbrace{W_8^{15}}_{W_8^7}+x[6]\,\underbrace{W_8^{18}}_{W_8^2}+x[7]\,\underbrace{W_8^{21}}_{W_8^5}$$

$$X[4]=x[0]+x[1]\,\underbrace{W_8^4}_{-1}+x[2]\,\underbrace{W_8^8}_{1}+x[3]\,\underbrace{W_8^{12}}_{W_8^4=-1}+x[4]\,\underbrace{W_8^{16}}_{1}+x[5]\,\underbrace{W_8^{20}}_{W_8^4=-1}+x[6]\,\underbrace{W_8^{24}}_{1}+x[7]\,\underbrace{W_8^{28}}_{W_8^4=-1}$$

$$X[5]=x[0]+x[1]\,W_8^5+x[2]\,\underbrace{W_8^{10}}_{W_8^2}+x[3]\,\underbrace{W_8^{15}}_{W_8^7}+x[4]\,\underbrace{W_8^{20}}_{W_8^4=-1}+x[5]\,\underbrace{W_8^{25}}_{W_8^1}+x[6]\,\underbrace{W_8^{30}}_{W_8^6}+x[7]\,\underbrace{W_8^{35}}_{W_8^3}$$

$$X[6]=x[0]+x[1]\,W_8^6+x[2]\,\underbrace{W_8^{12}}_{W_8^4}+x[3]\,\underbrace{W_8^{18}}_{W_8^2}+x[4]\,\underbrace{W_8^{24}}_{1}+x[5]\,\underbrace{W_8^{30}}_{W_8^6}+x[6]\,\underbrace{W_8^{36}}_{W_8^4}+x[7]\,\underbrace{W_8^{42}}_{W_8^2}$$

$$X[7]=x[0]+x[1]\,W_8^7+x[2]\,\underbrace{W_8^{14}}_{W_8^6}+x[3]\,\underbrace{W_8^{21}}_{W_8^5}+x[4]\,\underbrace{W_8^{28}}_{W_8^4=-1}+x[5]\,\underbrace{W_8^{35}}_{W_8^3}+x[6]\,\underbrace{W_8^{42}}_{W_8^2}+x[7]\,\underbrace{W_8^{49}}_{W_8^1}$$

Note that

$$W_8^4 = -W_8^0 \qquad W_8^5 = -W_8^1 \qquad W_8^6 = -W_8^2 \qquad W_8^7 = -W_8^3$$

i.e.

$$W_N^a = -W_N^{a+\frac{N}{2}}$$

Rewriting

$$X[1]=(x[0]-x[4])+(x[1]-x[5])W_8^1+(x[2]-x[6])W_8^2+(x[3]-x[7])W_8^3$$

$$X[2]=(x[0]-x[2]+x[4]-x[6])+(x[1]-x[3]+x[5]-x[7])W_8^2$$

$$X[3]=(x[0]-x[4])+(x[3]-x[7])W_8^1+(-x[2]+x[6])W_8^2+(x[1]-x[5])W_8^3$$

$$X[4]=x[0]-x[1]+x[2]-x[3]+x[4]-x[5]+x[6]-x[7]$$

$$X[5]=(x[0]-x[4])+(-x[1]+x[5])W_8^1+(x[2]-x[6])W_8^2+(-x[3]+x[7])W_8^3$$

$$X[6]=x[0]-x[2]+x[4]-x[6]+(-x[1]+x[3]-x[5]+x[7])W_8^2$$

$$X[7]=(x[0]-x[4])+(-x[3]+x[7])W_8^1+(-x[2]+x[6])W_8^2+(-x[1]+x[5])W_8^3$$

Therefore, the computation of the following is sufficient to get all 8-point DFT values…

$x[0]+x[4]$	$x[0]-x[4]$
$x[1]+x[5]$	$x[1]-x[5]$
$x[2]+x[6]$	$x[2]-x[6]$
$x[3]+x[7]$	$x[3]-x[7]$
$(x[1]-x[5])W_8^1$	$(-x[1]+x[5])W_8^3$
$(x[3]-x[7])W_8^3$	$(-x[3]+x[7])W_8^1$
$(x[2]-x[6])W_8^2$	$(x[1]-x[3]+x[5]-x[7])W_8^2$

We will study the two systematic methods

1) Decimation-in-time

2) Decimation-in-frequency

Decimation-in-Time

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn} \qquad k = 0,1, \dots, N-1$$

Assume that *N* is a power of two, $N = 2^m$.

$$X[k] = \sum_{n:even}^{N-1} x[n]W_N^{kn} + \sum_{n:odd}^{N-1} x[n]W_N^{kn}$$

$$= \sum_{r=0}^{\frac{N}{2}-1} x[2r]W_N^{2kr} + \sum_{r=0}^{\frac{N}{2}-1} x[2r+1]W_N^{k(2r+1)}$$

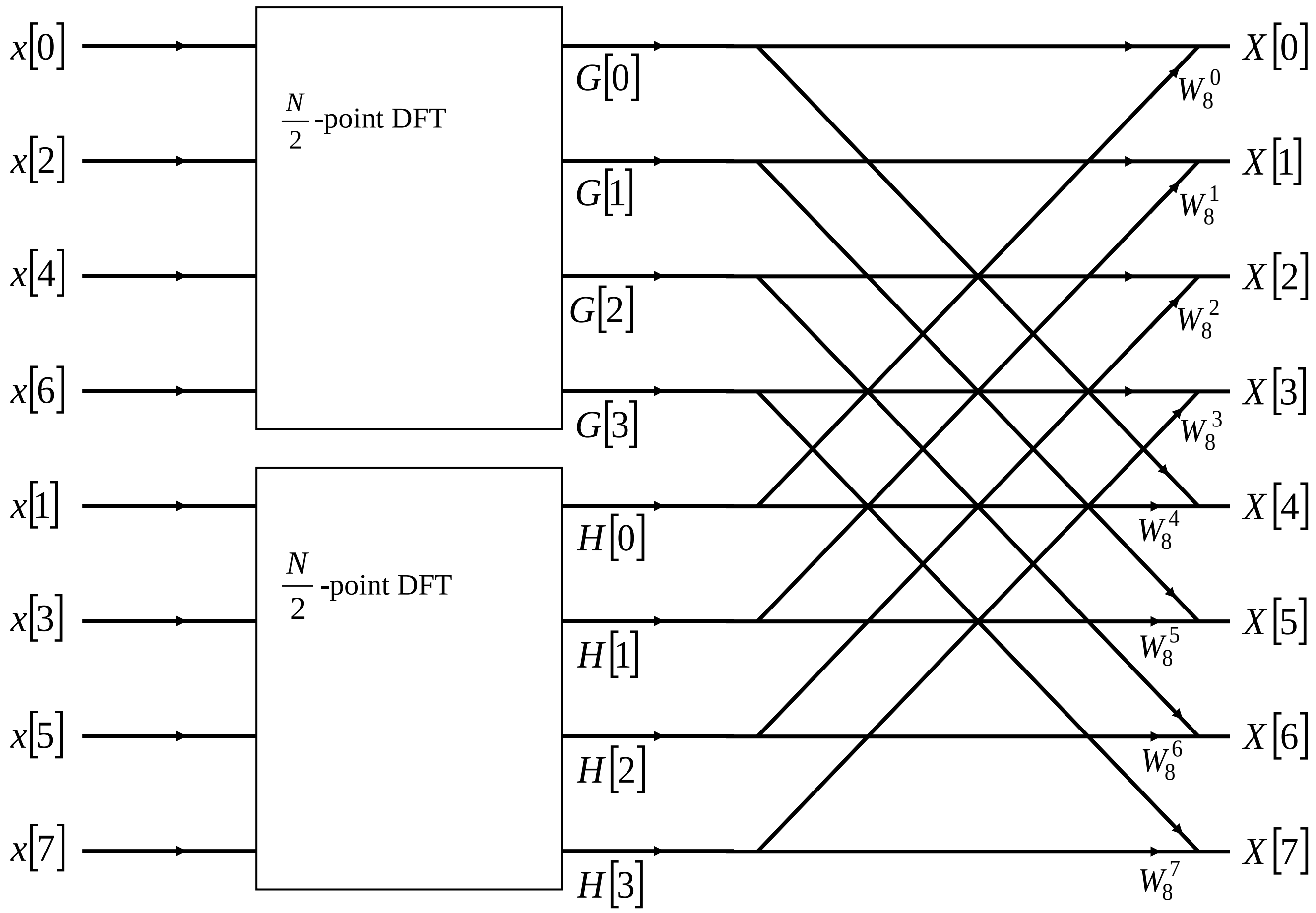
$$= \underbrace{\sum_{r=0}^{\frac{N}{2}-1} x[2r]W_{\frac{N}{2}}^{kr}}_{G[k]} + W_N^k \underbrace{\sum_{r=0}^{\frac{N}{2}-1} x[2r+1]W_{\frac{N}{2}}^{kr}}_{H[k]}$$

$$G[k]\colon \frac{N}{2} - point\ DFT\ of\ even\ indexed\ samples$$

$$H[k]\colon \frac{N}{2} - point\ DFT\ of\ odd\ indexed\ samples$$

Therefore

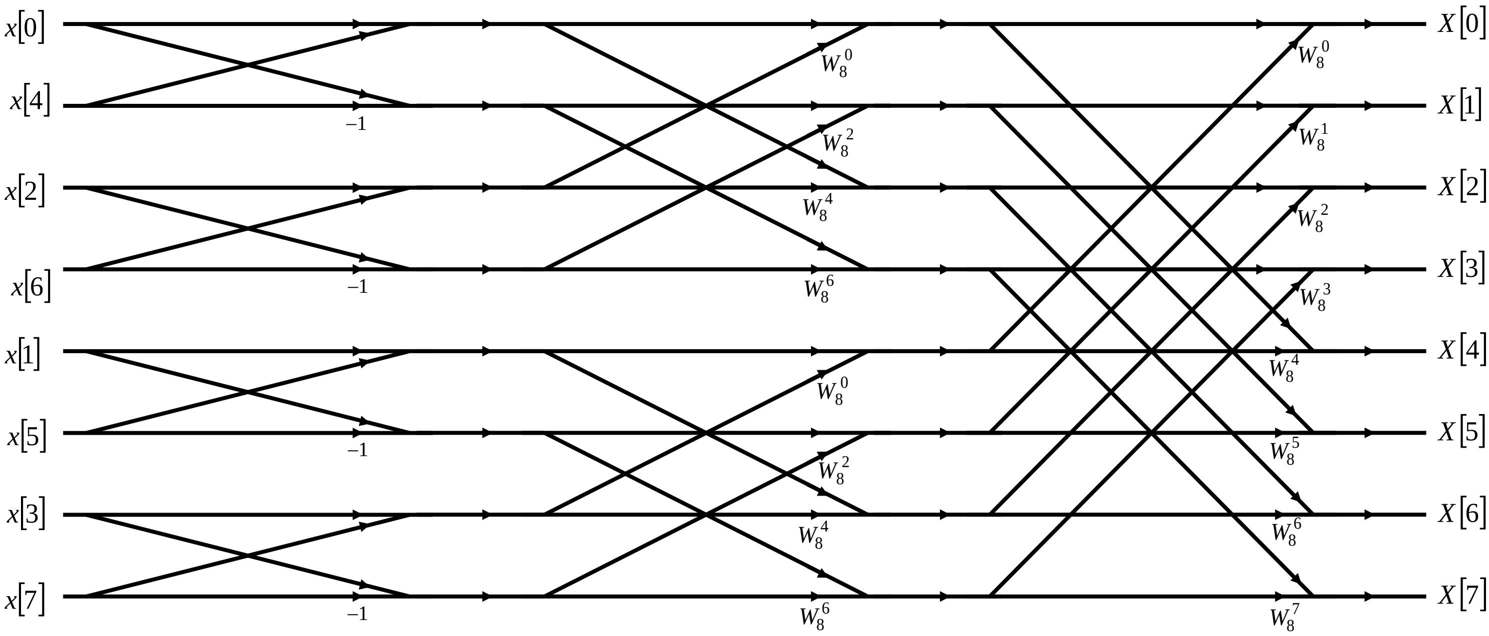
$$X[k] = G[k] + W_N^k H[k] \qquad k = ,1, \dots, N-1$$



$2\left(\frac{N}{2}\right)^2 + N = \frac{N^2}{2} + N$ complex multiplications required

$\frac{N}{2}$ −point DFTs can be decomposed into $\frac{N}{4}$ −point DFTs.

Then the following computation diagram arises.



$$4\left(\frac{N}{4}\right)^2 + N + N = \frac{N^2}{4} + 2N$$

complex multiplications required

In general, the number of sections will be

$$\log_2 N = v$$

Therefore

$$\underbrace{\frac{N}{2}\left(\frac{N}{\frac{N}{2}}\right)^2}_{2N} + N + N + \cdots + N = (v+1)N$$

complex multiplications required

Considering the multiplications by -1 and 1, this value is approximately referred to as

$$N \log_2 N$$

This is also the approximate number of additions.

Ex:

$$N = 2^{10} = 1024$$

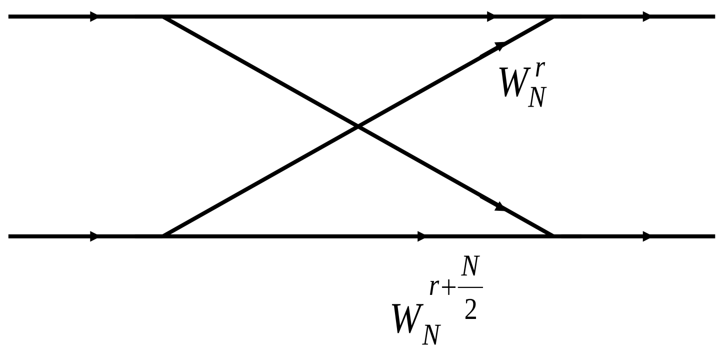
$$N^2 = 1\,048\,576 \approx 10^6$$

$$N \log_2 N = 10\,240 \approx 10^4$$

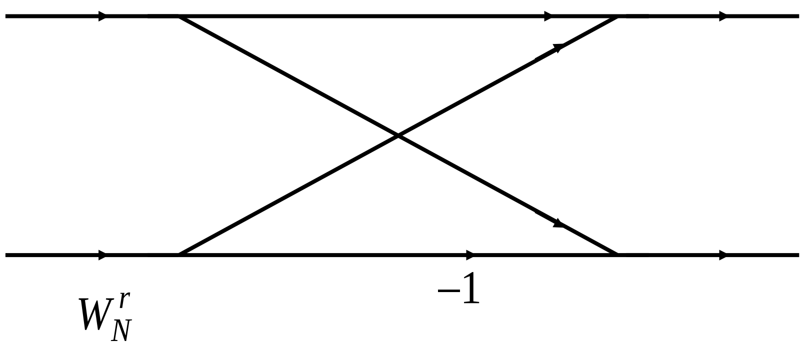
A reduction by a factor of 100 approximately.

“BUTTERFLY”s

Indeed, above diagrams contain many “butterfly” structures,



Such butterflies can be simplified as



DECIMATION-IN-FREQUENCY

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn} \qquad k = 0,1,\ldots,N-1$$

Assume that *N* is a power of two, $N = 2^m$.

Consider "even" and "odd" indexed elements of the DFT:

$$X[k] \begin{cases} \nearrow X[2r] \\ \searrow X[2r+1] \end{cases} \qquad r = 0,1,\ldots,\frac{N}{2}-1$$

Even indexed elements:

$$X[2r] = \sum_{n=0}^{N-1} x[n] W_N^{2rn} \qquad r = 0,1,\dots,\frac{N}{2}-1$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x[n] W_N^{2rn} + \sum_{n=\frac{N}{2}}^{N-1} x[n] W_N^{2rn}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x[n] W_N^{2rn} + \sum_{n=0}^{\frac{N}{2}-1} x\left[n+\frac{N}{2}\right] W_N^{2r\left(n+\frac{N}{2}\right)}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} \left(x[n] + x\left[n+\frac{N}{2}\right]\right) W_{\frac{N}{2}}^{rn}$$

This is a $\frac{N}{2}$-point DFT of

$$\left(x[n] - x\left[n+\frac{N}{2}\right]\right) \qquad n = 0,1,\dots,\frac{N}{2}-1.$$

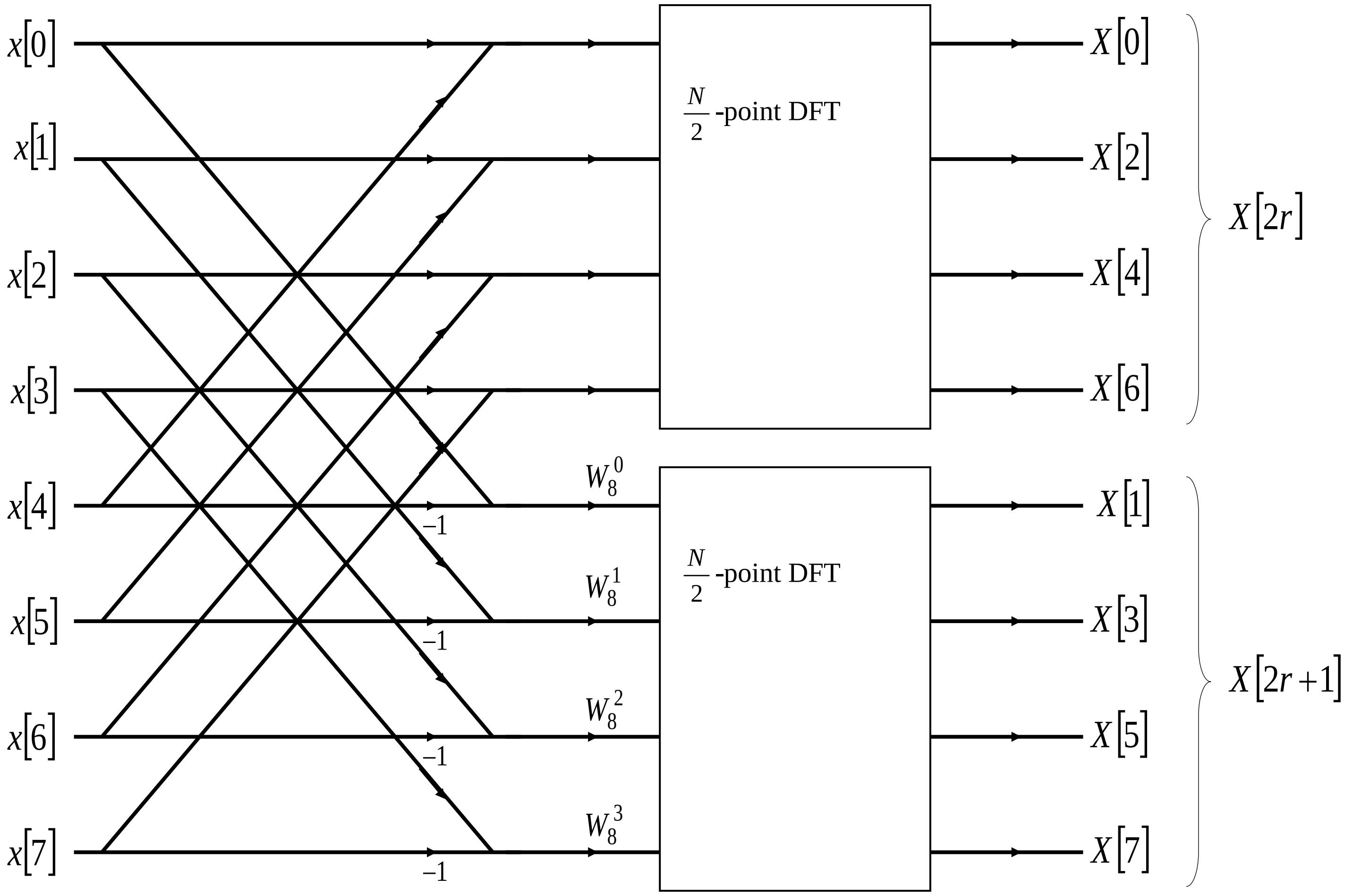
Odd indexed elements: Similarly,

$$X[2r+1] = \sum_{n=0}^{\frac{N}{2}-1} \left(x[n] - x\left[n+\frac{N}{2}\right]\right) W_N^n \, W_{\frac{N}{2}}^{rn}$$

This is a $\frac{N}{2}$-point DFT of

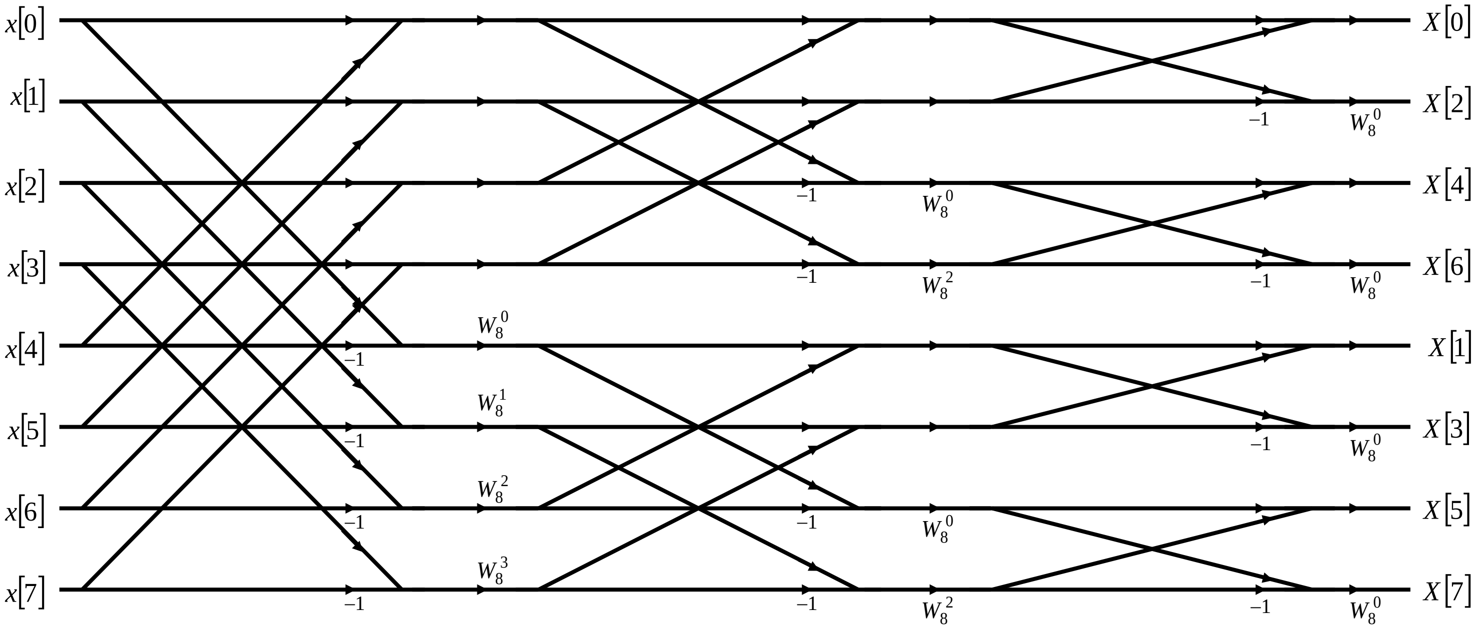
$$\left(x[n] - x\left[n+\frac{N}{2}\right]\right) W_N^n \qquad n = 0,1,\dots,\frac{N}{2}-1.$$

For N=8



$x[0]$
$x[1]$
$x[2]$
$x[3]$
$x[4]$
$x[5]$
$x[6]$
$x[7]$
-1
-1
-1
-1
W_8^0
W_8^1
W_8^2
W_8^3
$\frac{N}{2}$-point DFT
$\frac{N}{2}$-point DFT
$X[0]$
$X[2]$
$X[4]$
$X[6]$
$X[1]$
$X[3]$
$X[5]$
$X[7]$
$X[2r]$
$X[2r+1]$

Decomposing $\frac{N}{2}$-point DFT blocks down to 2-point DFTs we get the following flow diagram



THE GOERTZEL ALGORITHM

The algorithm reduces the storage requirement

DFT expression:

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn} \qquad k = 0,1, \dots , N-1$$

can be written in a recursive form.

For example, let N=8

$$X[k] = x[0] + x[1]W_N^k + x[2]W_N^{2k} + x[3]W_N^{3k} + x[4]W_N^{4k} + x[5]W_N^{5k} + x[6]W_N^{6k} + x[7]W_N^{7k}$$

$$= x[0] + W_N^k\left(x[1] + x[2]W_N^k + x[3]W_N^{2k} + x[4]W_N^{3k} + x[5]W_N^{4k} + x[6]W_N^{5k} + x[7]W_N^{6k}\right)$$

$$\vdots$$

$$= x[0] + W_N^k\left(x[1] + W_N^k\left(x[2] + W_N^k\left(x[3] + W_N^k\left(x[4] + W_N^k\left(x[5] + W_N^k\left(x[6] + x[7]W_N^k\right)\right)\right)\right)\right)\right)$$

OR

Since $W_N^{-kN}=1$

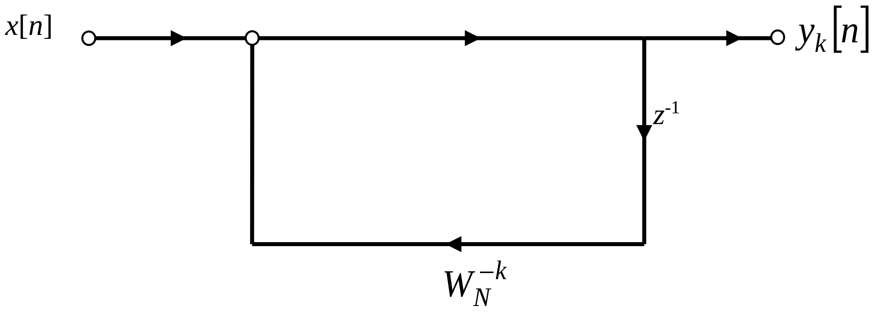
$$X[k]=W_N^{-kN}\sum_{n=0}^{N-1}x[n]\,W_N^{kn}$$

$$=\sum_{n=0}^{N-1}x[n]\,W_N^{-k(N-n)}$$

For N=8

$W_8^{-8k}X[k]=$

$$\left(\left(\left(\left(\left(\left(\left(x[0]\,W_8^{-7k}+x[1]\right)W_8^{-k}+x[2]\right)W_8^{-k}+x[3]\right)W_8^{-k}+x[4]\right)W_8^{-k}+x[5]\right)W_8^{-k}+x[6]\right)W_8^{-k}+x[7]\right)W_8^{-k}$$



$$y_k[n]=W_N^{-k}y_k[n-1]+x[n] \qquad n=0,1,\dots,N$$

$$y_k(-1)=0 \qquad x[N]=0$$

Take $X[k]=y_k[N]$

Note that $X[k]$'s are computed sequentially.

Computational demand:

For each k, $y_k[1], y_k[2], \ldots, y_k[N-1]$ must be computed to find $y_k[N] = X[k]$.

This requires, (assuming a complex input) $4N$ multiplications and $4N$ additions.

This amount is almost the same as that of direct computation.

However, the storage requirement of W_N^{kn} values vanishes.

For the N-point DFT of $x[n]$ we obtained

$$X[k] = G[k] + W_N^k H[k]$$

where $G[k]$ and $H[k]$ are the $\frac{N}{2}$-point DFTs, respectively, of even and odd indexed elements of the sequence $x[n]$, i.e.,

$$g[n] = x[2n]$$

$$h[n] = x_{-1}[2n] \qquad\qquad x_{-1}[n] = x[n+1]$$

$$G\left(e^{j\omega}\right) = \frac{1}{2}\left(X\left(e^{j\frac{\omega}{2}}\right) + X\left(e^{j\frac{1}{2}(\omega-2\pi)}\right)\right)$$

$$H\left(e^{j\omega}\right) = \frac{1}{2}\left(e^{j\frac{\omega}{2}} X\left(e^{j\frac{\omega}{2}}\right) + e^{j\frac{1}{2}(\omega-2\pi)} X\left(e^{j\frac{1}{2}(\omega-2\pi)}\right)\right)$$

Therefore

$$X\left(e^{j\omega}\right) = G\left(e^{j2\omega}\right) + e^{-j\omega} H\left(e^{j2\omega}\right)$$

$$X[k] = X\left(e^{j\omega}\right)\Big|_{\omega = k\frac{2\pi}{N}}$$

$$G\left(e^{j2\omega}\right) = \sum_{n=0}^{\frac{N}{2}-1} x[2n]e^{-j2\omega n}$$

$$G\left(e^{j2\omega}\right)\Big|_{\omega = k\frac{2\pi}{N}} = \sum_{n=0}^{\frac{N}{2}-1} x[2n]e^{-j2k\frac{2\pi}{N}n}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x[2n]W_{\frac{N}{2}}^{kn}$$

Likewise

$$H\left(e^{j2\omega}\right)\Big|_{\omega = k\frac{2\pi}{N}} = \sum_{n=0}^{\frac{N}{2}-1} x[2n+1]W_{\frac{N}{2}}^{kn}$$

RADIX 3 vs, RADIX 2

In place computations

Chirp transform 9.6