EE 430

Fall 2014

Sec. 1 & 3

HW 4

(1) a)

$$y[n] = \begin{cases} x[n/L], & n = 0, \pm L, \pm 2L, \dots \\ 0 & \text{otherwise} \end{cases}$$

i) Consider two arbitrary inputs $x_1[n]$ and $x_2[n]$. Suppose that $y_1[n]$ and $y_2[n]$ are the outputs of the system for these inputs, i.e.,

$$y_1[n] = \begin{cases} x_1[n/L], & n = 0, \pm L, \pm 2L, \dots \\ 0 & \text{o.w.} \end{cases}$$

$$y_2[n] = \begin{cases} x_2[n/L], & n = 0, \pm L, \pm 2L, \\ 0 & \text{o.w} \end{cases}$$

Consider any two arbitrary scalars α and β:

$$x_3[n] = \alpha x_1[n] + \beta x_2[n]$$

$$\Downarrow$$

$$y_3[n] = \begin{cases} x_3[n/L] & n = 0, \pm L, \pm 2L \dots \\ 0 & \text{o.w} \end{cases}$$

$$= \begin{cases} \alpha x_1[n/L] + \beta x_2[n/L], & n = 0, \pm L, \pm 2L \dots \\ 0 & \text{o.w} \end{cases}$$

$$= \alpha y_1[n] + \beta y_2[n] \implies \text{Linear system}$$

ii) Consider $x_2[n] = x_1[n - m_0]$ where m_0 is any constant integer

$$y_2[n] = \begin{cases} x_2[n/L], & n = 0, \pm L, \pm 2L, \dots \\ 0 & \text{o.w} \end{cases}$$

$$= \begin{cases} x_1[n/L - m_0] & n = 0, \pm L, \pm 2L \\ 0 & \text{o.w} \end{cases}$$

$$y_2[n] \neq y_1[n-m_0] = \begin{cases} x_1\left[\frac{(n-m_0)}{L}\right], & n-m_0 = 0, \pm L, \pm 2L, \dots \\ 0 & \text{o.w} \end{cases}$$

$\Rightarrow$ Not time-invariant $\Rightarrow$ time varying system

b) $y[n] = x[nM]$

i) $x_1[n] \rightarrow \boxed{\downarrow M} \rightarrow y_1[n] = x_1[nM]$

$x_2[n] \rightarrow \boxed{\downarrow M} \rightarrow y_2[n] = x_2[nM]$

Consider $x_3[n] = \alpha x_1[n] + \beta x_2[n]$ where α, β are arbitrary scalars.

$$y_3[n] = x_3[nM] = \alpha x_1[nM] + \beta x_2[nM]$$
$$= \alpha y_1[n] + \beta y_2[n]$$

$\Rightarrow$ Linear system

ii)

$x_1[n] \rightarrow \boxed{\downarrow M} \rightarrow y_1[n] = x_1[nM]$

Consider $x_2[n] = x_1[n-m_0]$ where m_0 is any constant integer

$$y_2[n] = x_2[nM] = x_1[nM - m_0] \neq y_1[n-m_0]$$
$$= x_1[M(n-m_0)]$$

$\Rightarrow$ Time varying system

c) $x[n] \rightarrow \boxed{\downarrow 2} \xrightarrow{g_1[n]} \boxed{\uparrow 3} \rightarrow y_1[n]$

$g_1[n] = x[2n]$

$$G_1(e^{jw}) = \frac{1}{2} X\left(e^{j\frac{w}{2}}\right) + \frac{1}{2} X\left(e^{j\left(\frac{w}{2} - \pi\right)}\right)$$

$$y_1[n] = \begin{cases} g_1[n/3], & n = 0, \pm 3, \pm 6, \dots \\ 0 & \text{o.w} \end{cases}$$

$$Y_1(e^{j\omega}) = G_1(e^{j\omega 3})$$
$$= \frac{1}{2} X(e^{j\frac{3\omega}{2}}) + \frac{1}{2} X(e^{j(\frac{3\omega}{2} - \pi)})$$

d) $$G_2(e^{j\omega}) = X(e^{j\omega 3})$$
$$Y_2(e^{j\omega}) = \frac{1}{2} G_2(e^{j\frac{\omega}{2}}) + \frac{1}{2} G_2(e^{j(\frac{\omega}{2} - \pi)})$$
$$= \frac{1}{2} X(e^{j\frac{3\omega}{2}}) + \frac{1}{2} X(e^{j3(\frac{\omega}{2} - \pi)})$$
$$= \frac{1}{2} X(e^{j\frac{3\omega}{2}}) + \frac{1}{2} X(e^{j(\frac{3\omega}{2} - 3\pi)})$$

e) $$Y_2(e^{j\omega}) = \frac{1}{2} X(e^{j\frac{3\omega}{2}}) + \frac{1}{2} X(e^{j(\frac{3\omega}{2} - \pi)})$$
$$= Y_1(e^{j\omega}) \Rightarrow y_1[n] = y_2[n].$$

Consider the general case.

<u>First System</u>

$$x[n] \rightarrow \boxed{\downarrow M} \xrightarrow{g_1[n]} \boxed{\uparrow L} \rightarrow y_1[n]$$

$$Y_1(e^{j\omega}) = \frac{1}{M} \sum_{i=0}^{M-1} X(e^{j(\frac{\omega L}{M} - \frac{2\pi i}{M})})$$

<u>Second System</u>

$$x[n] \rightarrow \boxed{\uparrow L} \xrightarrow{g_2[n]} \boxed{\downarrow M} \rightarrow y_2[n]$$

$$Y_2(e^{j\omega}) = \frac{1}{M} \sum_{i=0}^{M-1} X(e^{j(\frac{\omega L}{M} - \frac{2\pi L i}{M})})$$

For $Y_1(e^{j\omega}) = Y_2(e^{j\omega})$,

any integer

$$\frac{2\pi L i}{M} = \frac{2\pi i}{M} + 2\pi k \qquad i = 1, 2, \ldots M-1$$

is a sufficient condition.

$$\frac{Li}{M} = \frac{i}{M} + k \Rightarrow \boxed{(L-1)i = Mk \quad i = 1, 2, \ldots M-1}$$

If $\frac{L-1}{M}$ is an integer then the condition above is satisfied. In this problem, $L=3$, $M=2$. $\frac{L-1}{M} = \frac{2}{2} = 1 \rightarrow$ an integer ✓

f) $x[n] = x_c(nT)$

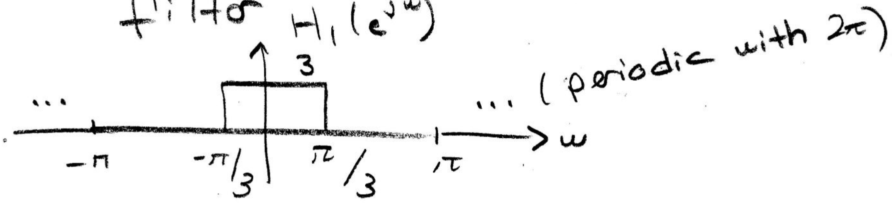
$$x_1[n] = x_c\left(T\left(n - \frac{2}{3}\right)\right) = x_c\left(nT - \frac{2}{3}T\right)$$

$$x[n] \rightarrow \boxed{\uparrow 3} \xrightarrow{x_e[n]} \boxed{H_1(e^{jw})} \rightarrow x_i[n] = x_c\left(\frac{nT}{3}\right)$$

$x[n]$: Sampling period : T

$H_1(e^{jw})$: lowpass filter

$x_i[n]$: Sampling period : $\frac{T}{3}$



$$x_i[n] \rightarrow \boxed{z^{-2}} \rightarrow x_{i_2}[n] = x_i[n-2] = x_c\left(\frac{T(n-2)}{3}\right)$$

$$= x_c\left(\frac{nT}{3} - \frac{2T}{3}\right)$$

z^{-2}: 2 samples delay, $h[n] = \delta[n-2]$

Now we need to decrease sampling rate (increase sampling period)

Since $x_{i_2}[n]$ is bandlimited to $\frac{\pi}{3}$ we don't need any antialiasing lowpass filter in decimator.

$$x_{i_2}[n] \rightarrow \boxed{\downarrow 3} \rightarrow x_1[n] = x_c\left(nT - \frac{2T}{3}\right) \checkmark$$

$x_{i_2}[n]$: Sampling period : $\frac{T}{3}$

$x_1[n]$: Sampling period : T

The overall DT structure is;

$$x[n] \rightarrow \boxed{\uparrow 3} \rightarrow \boxed{H_1(e^{j\omega})} \rightarrow \boxed{z^{-2}} \rightarrow \boxed{\downarrow 3} \rightarrow x_1[n]$$

(2)

$$x_1[n] \rightarrow \boxed{\uparrow 2} \xrightarrow{g_1[n]} \boxed{H(e^{j\omega})} \xrightarrow{g_2[n]} \boxed{\downarrow 2} \rightarrow y_1[n]$$

$$G_1(e^{j\omega}) = X_1(e^{j2\omega})$$

$$G_2(e^{j\omega}) = X_1(e^{j2\omega})\, H(e^{j\omega})$$

$$Y_1(e^{j\omega}) = \frac{1}{2} G_2(e^{j\frac{\omega}{2}}) + \frac{1}{2} G_2(e^{j(\frac{\omega}{2}-\pi)})$$

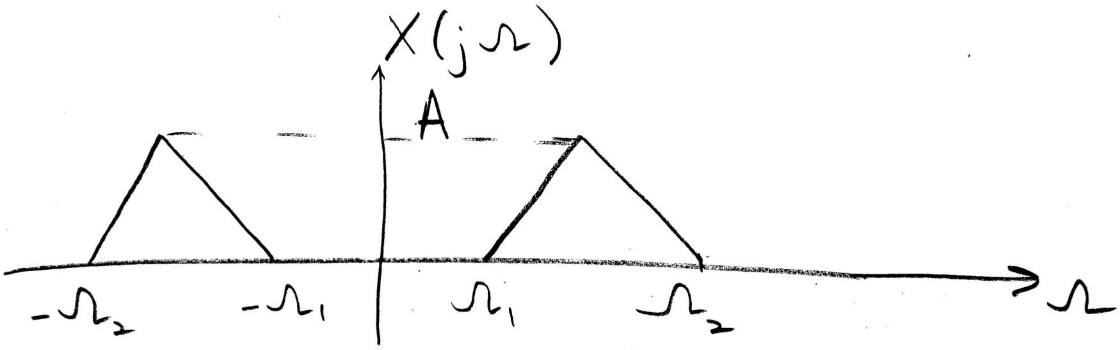
$$= \frac{1}{2} X_1(e^{j\omega}) H(e^{j\frac{\omega}{2}}) + \frac{1}{2} \underbrace{X_1(e^{j2(\frac{\omega}{2}-\pi)})}_{\substack{= X_1(e^{j(\omega-2\pi)}) \\ = X_1(e^{j\omega})}} H(e^{j(\frac{\omega}{2}-\pi)})$$

$$= X_1(e^{j\omega}) \underbrace{\left[\frac{1}{2} H(e^{j\frac{\omega}{2}}) + \frac{1}{2} H(e^{j(\frac{\omega}{2}-\pi)}) \right]}_{= G(e^{j\omega})}$$

$$Y_2(e^{j\omega}) = X_1(e^{j\omega})\, G(e^{j\omega}) = Y_1(e^{j\omega})$$

where $G(e^{j\omega}) = \frac{1}{2} H(e^{j\frac{\omega}{2}}) + \frac{1}{2} H(e^{j(\frac{\omega}{2}-\pi)})$

3

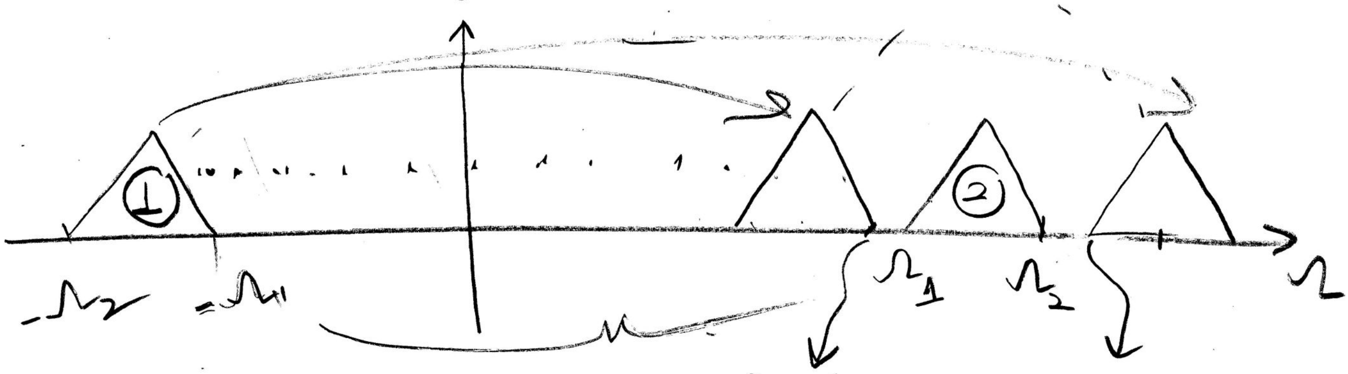


$\Omega_2 > \Omega_1 > 0$

Nyquist rate $= 2\Omega_2$

$\Omega_s \leq \Omega_2$

replicas of 1



$-\Omega_1 + (k-1)\Omega_s$ $\quad$ $-\Omega_2 + k\Omega_s$

$$-\Omega_1 + (k-1)\Omega_s \leq \Omega_1 \qquad (1)$$

$$\Omega_2 \leq -\Omega_2 + k\Omega_s \qquad (2)$$

$k > 1$ integer

$$\Omega_s \leq \frac{2\Omega_1}{k-1}$$

$$\frac{2\Omega_2}{k} \leq \Omega_s$$

$\Downarrow$

$$\frac{2\Omega_2}{k} \leq \frac{2\Omega_1}{k-1}$$

$$2\Omega_2 k - 2\Omega_2 \leq 2\Omega_1 k$$

$$k \leq \frac{\Omega_2}{\Omega_2 - \Omega_1}$$

We want to select k as large as possible to obtain minimum sampling rate since

$$\frac{2\Omega_2}{k} \leq \Omega_s$$

$\Downarrow$

$$(*) \quad \Omega_s = \frac{2\Omega_2}{\left\lfloor \frac{\Omega_2}{\Omega_2 - \Omega_1} \right\rfloor} \leq \Omega_2$$

Furthermore, we should consider replicas of (2).

$\Omega_1 + \Omega_s \geq \Omega_2$

$\Omega_2 - \Omega_s \leq \Omega_1 \Rightarrow \Omega_s \geq \Omega_2 - \Omega_1$

This condition is already satisfied by (*).

If the $X(j\Omega)$ is not zero at Ω_2 and Ω_1, select Ω_s such that $\frac{2\Omega_2}{k} < \Omega_s < \frac{2\Omega_1}{k-1}$

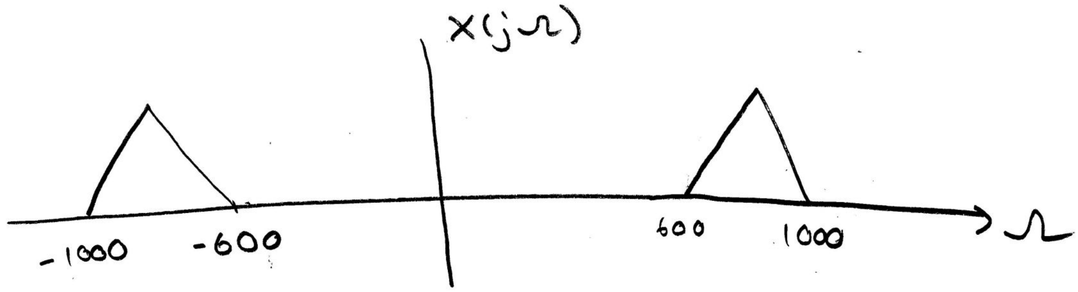
Now let's construct our problem. Let's choose $\Omega_2 = 1000$ Hz. Ω_s should be less than or equal to $\Omega_2 = 1000$ Hz

$$\frac{2 \times 1000}{\left\lfloor \frac{1000}{1000 - \Omega_1} \right\rfloor} \leq 1000$$

$$\Rightarrow \left\lfloor \frac{1000}{1000 - \Omega_1} \right\rfloor \geq 2 \Rightarrow \frac{1000}{1000 - \Omega_1} \geq 2$$

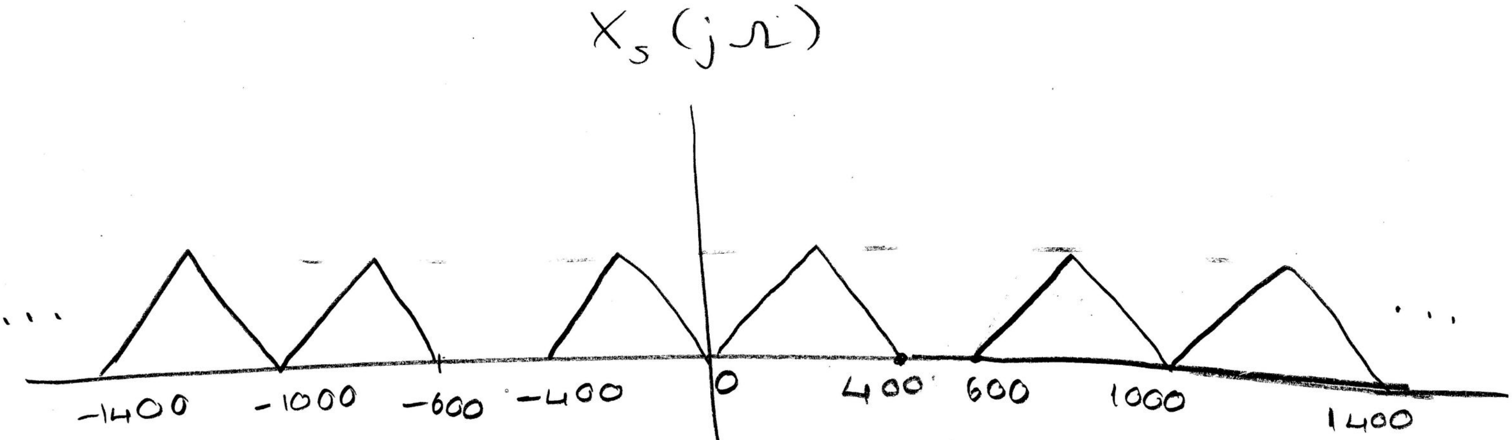
$$\boxed{\Omega_1 \geq 500 \text{ Hz}}$$

Let's choose $\Omega_1 = 600$ Hz.



minimum sampling rate is;

$$\Omega_s = \frac{2 \times 1000}{\left\lfloor \dfrac{1000}{1000-600} \right\rfloor} = \frac{2000}{2} = 1000 \text{ Hz}$$



↳ no aliasing

(4) i) • C/D converter converts a continuous time signal into a discrete time signal, where each sample is known with infinite precision. However A/D converter converts a continuous time signal into a digital signal, i.e., a sequence of finite precision or quantized samples.

• C/D converters are idealized converters. A/D converters are practical devices

• C/D converters are ideal. They are used to understand mathematical concepts about the sampling. So they are assumed to convert continuous time signal instantaneously. However

A/D converters use a sample and hold device. They complete an A/D conversion every T seconds.

ii) • D/C converter is an idealized system.
A physically realizable counterpart to the ideal D/C converter is a D/A converter

• D/A converter uses sample and hold device while D/C not. It uses different reconstruction filter which is not the ideal interpolating filter $H_r(j\Omega)$ as in D/C converters.

iii) $SNR_Q \approx 6B - 1.25\,dB > 48\,dB$

$$6B > 49.25$$

$$B > 8.208$$

$B = 9. \Rightarrow$ Minimum $B+1 = 10$ bits are needed

iv) Consider ideal D/C converter.

$$X_r(j\Omega) = X(e^{j\Omega T}) H_r(j\Omega)$$

ideal reconstruction filter

$$H_r(j\Omega) = \begin{cases} T, & |\Omega| < \pi/T \\ 0, & |\Omega| \geq \pi/T \end{cases}$$

However D/A converter uses zero order hold filter $h_0(t)$ and produces,

$$x_{DA}(t) = \sum_{n=-\infty}^{\infty} \hat{x}[n] h_0(t-nT)$$

If we use the additive-noise model to represent the effects of quantization,

$$x_{DA}(t) = \underbrace{\sum_{n=-\infty}^{\infty} x[n] h_0(t-nT)}_{x_0(t)} + \sum_{n=-\infty}^{\infty} e[n] h_0(t-nT)$$

$x_0(t)$: signal part (related to the input signal $x_a(t)$)

$x[n] = x_a(nT)$

Consider $x_0(t)$.

$$X_0(j\Omega) = \sum_{n=-\infty}^{\infty} x[n] H_0(j\Omega) e^{-j\Omega nT}$$

$$= \left(\sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega nT} \right) H_0(j\Omega)$$

$$= X(e^{j\Omega T}) H_0(j\Omega)$$

$$X(e^{j\Omega T}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_a\left(j\left(\Omega - \frac{2\pi k}{T}\right)\right)$$

$$X_o(j\Omega) = \left[\frac{1}{T} \sum_{k=-\infty}^{\infty} X_a\left(j\left(\Omega - \frac{2\pi k}{T}\right)\right)\right] H_o(j\Omega)$$

If $X_a(j\Omega)$ is bandlimited to frequencies below $\frac{\pi}{T}$ and if we define a compensated reconstruction filter as

$$\tilde{H}_r(j\Omega) = \frac{H_r(j\Omega)}{H_o(j\Omega)}$$

$\Longrightarrow$ output of the filter will be $x_a(t)$.

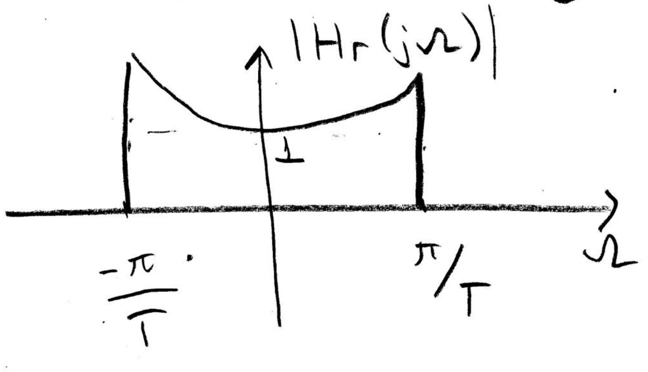
if the input is $x_o(t)$.

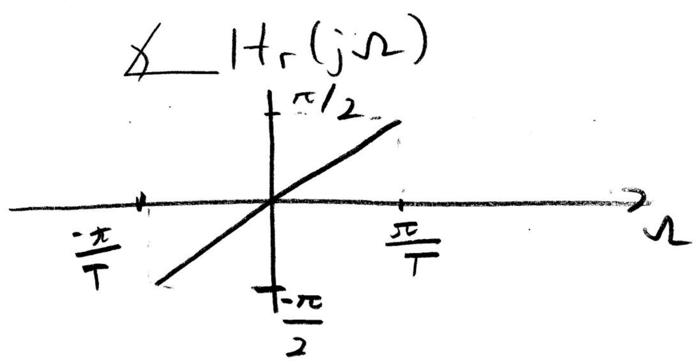
$$H_o(j\Omega) = \int_0^T e^{-j\Omega t}\,dt = \left.\frac{e^{-j\Omega t}}{-j\Omega}\right|_0^T$$

$$= \frac{e^{-j\Omega T} - 1}{-j\Omega} = e^{-j\Omega T/2}\,\frac{-j\,2\sin(\Omega T/2)}{-j\Omega}$$

$$= \frac{2\sin(\Omega T/2)}{\Omega}\, e^{-j\Omega\frac{T}{2}}$$

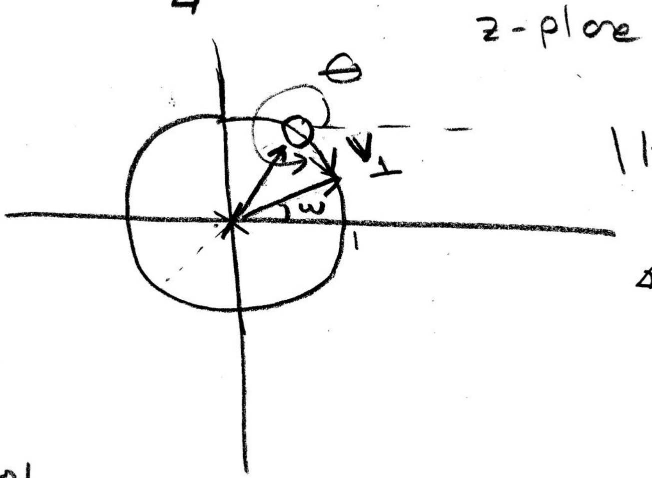
$$\tilde{H}_r(j\Omega) = \begin{cases} \dfrac{\Omega T/2}{\sin(\Omega T/2)}\, e^{j\Omega T/2} & |\Omega| < \pi/T \\ 0 & |\Omega| \geq \pi/T \end{cases}$$





5) $H(z) = 1 - re^{j\theta} z^{-1}$

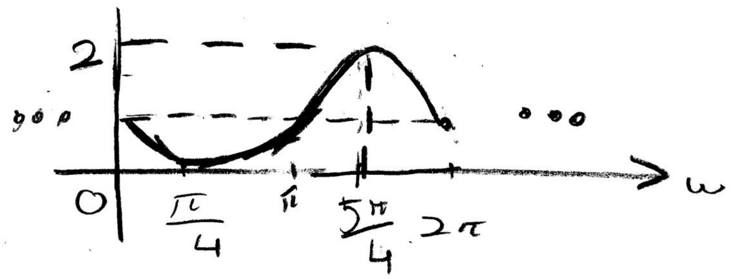
a) $r = 1,\ \theta = \frac{\pi}{4}$



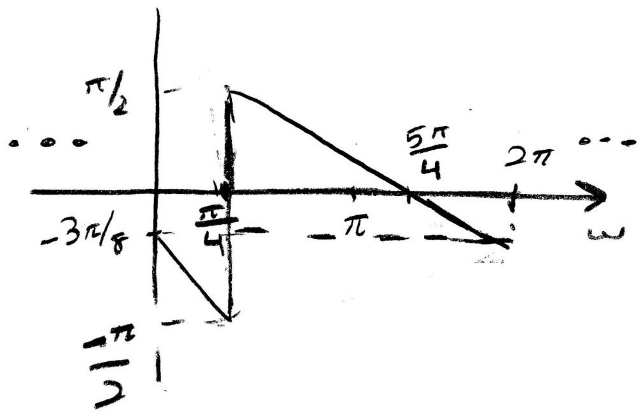
$$|H(e^{j\omega})| = \frac{|V_1|}{1} = |V_1|$$

$$\angle H(e^{j\omega}) = \theta - \omega$$

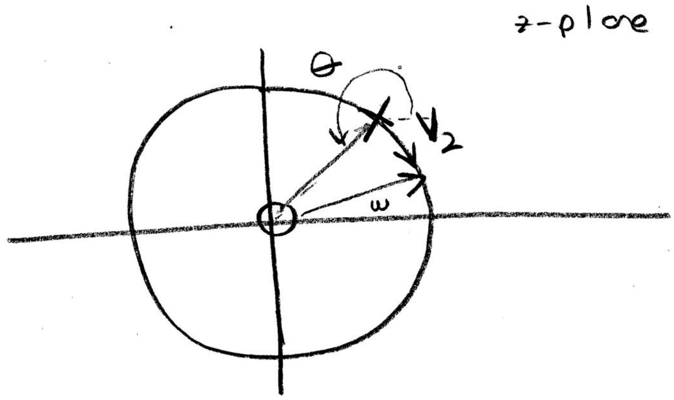
$|H(e^{j\omega})|$



$\angle H(e^{j\omega})$



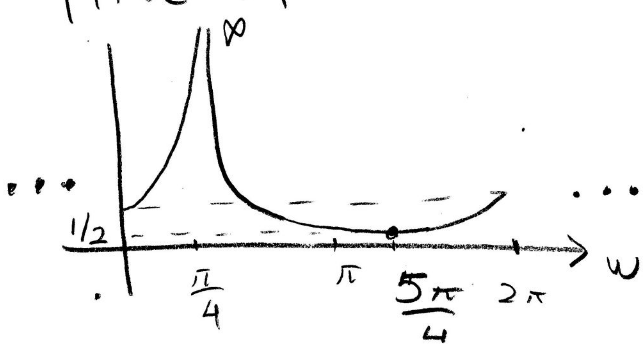
b)



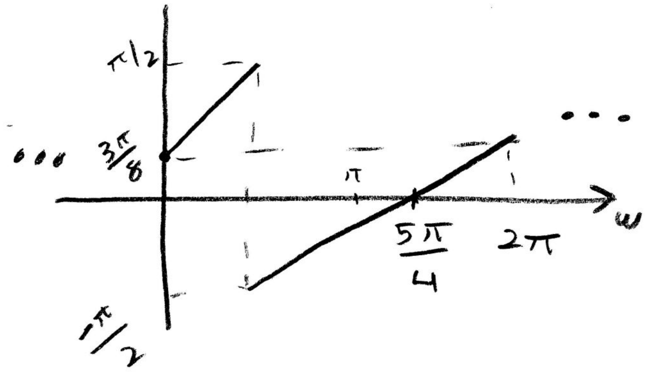
$$|H(e^{j\omega})| = \frac{1}{|V_2|}$$

$$\angle H(e^{j\omega}) = \omega - \theta$$

$|H(e^{j\omega})|$



$\angle H(e^{j\omega})$



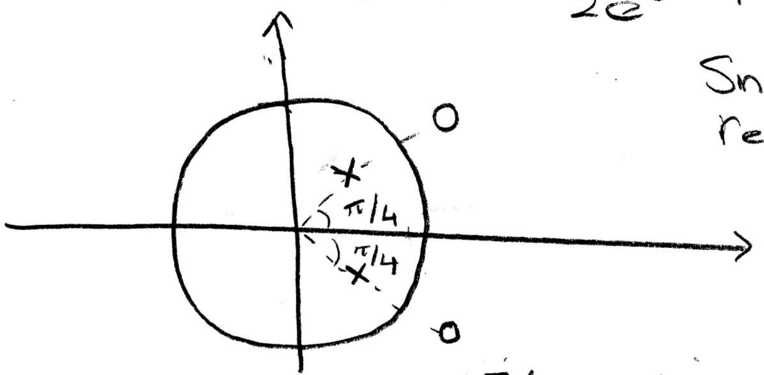
c)

$$H_{ap}(z) = \frac{z^{-1} - a^*}{1 - a z^{-1}} \Rightarrow \text{pole} : z = a$$
$$\text{zero} : z = \frac{1}{a^*}$$

first order all pass filter

$$\frac{1}{a^*} = 2e^{j\pi/4}$$

$$a^* = \frac{1}{2e^{j\pi/4}} \Rightarrow a = \frac{1}{2} e^{j\pi/4}$$



Since it is a real impulse response filter, include also complex conjugates of pole and zero

$$H(z) = \frac{z^{-1} - 0.5e^{-j\pi/4}}{1 - 0.5e^{j\pi/4}z^{-1}} \cdot \frac{z^{-1} - 0.5e^{j\pi/4}}{1 - 0.5e^{-j\pi/4}z^{-1}}, \ |z| > \frac{1}{2}$$

6) $$|H(e^{jw})|^2 = \frac{1}{\frac{5}{4} - \cos(w)} = \frac{1}{\frac{5}{4} - \frac{1}{2}e^{jw} - \frac{1}{2}e^{-jw}}$$

$$= \frac{1}{1 - \frac{1}{2}e^{jw}} \times \frac{1}{1 - \frac{1}{2}e^{-jw}}$$

$$H(e^{jw}) = \frac{1}{1 - \frac{1}{2}e^{jw}} \quad \text{or} \quad \frac{1}{1 - \frac{1}{2}e^{-jw}}$$

$\Downarrow$

$$H(z) = \frac{1}{1 - \frac{1}{2}z} \quad \text{or} \quad \frac{1}{1 - \frac{1}{2}z^{-1}}$$

Take $H(z) = \dfrac{1}{1-\frac{1}{2}z^{-1}}$ since the pole $z_1 = \frac{1}{2}$ is inside the unit circle.

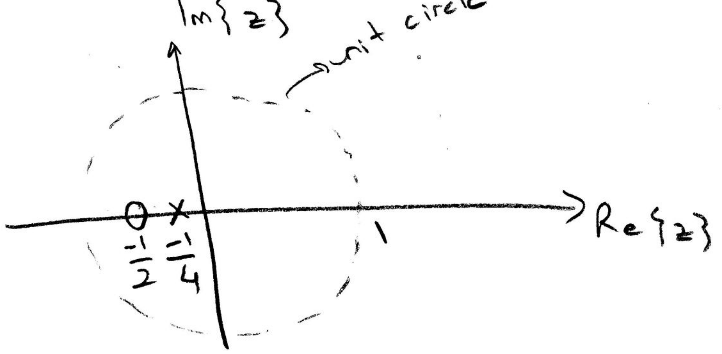
$$H(z) = \frac{z}{z-\frac{1}{2}}$$

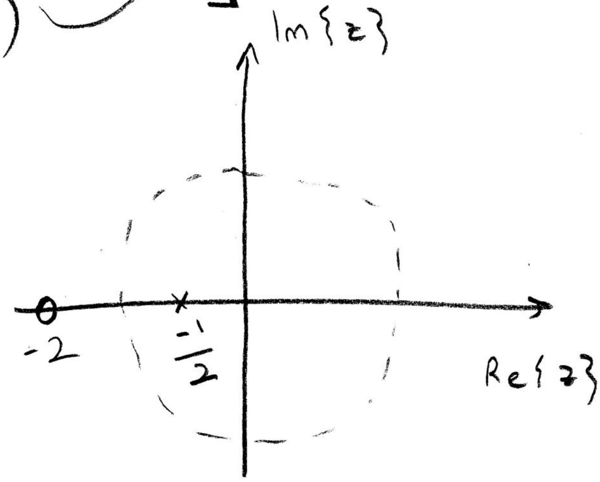
zero at $z = 0$
pole at $z = \frac{1}{2}$
inside the unit circle.

(7) $H(z) = \dfrac{1+2z^{-1}}{1+\frac{1}{4}z^{-1}}$

zero at $z=-2$
pole at $z = \frac{-1}{4}$ include it into $H_{min}(z)$

$$H(z) = \underbrace{\frac{1+\frac{1}{2}z^{-1}}{1+\frac{1}{4}z^{-1}}}_{H_{min}(z)} \; \underbrace{\frac{2\left(z^{-1}+\frac{1}{2}\right)}{1+\frac{1}{2}z^{-1}}}_{H_{ap}(z)}$$





(8) a) <u>Type I</u>

$h[n] = h[M-n]$ $\quad 0 \le n \le M$, M: even integer

$$H(z) = \sum_{n=0}^{M} h[n]z^{-n} = \sum_{n=0}^{M} h[\underbrace{M-n}_{\triangleq k}]z^{-n} = \sum_{k=0}^{M} h[k]z^{k}z^{-M}$$

$$H(z) = z^{-M}H(z^{-1})$$

Type II

$h[n] = h[M-n] \quad 0 \le n \le M$, M: odd integer

$H(z) = z^{-M} H(z^{-1})$

Type III

$h[n] = -h[M-n] \quad 0 \le n \le M$, M: even integer

$H(z) = -z^{-M} H(z^{-1})$

Type IV

$h[n] = -h[M-n] \quad 0 \le n \le M$, M: odd integer

$H(z) = -z^{-M} H(z^{-1})$

b) M = even integer . $M = 2$

$h[n] = -h[2-n] \quad 0 \le n \le 2$

$h[0] = -h[2]$

$h[1] = -h[1] = 0$

Let $h[0] = 1 \Rightarrow h[2] = -1$

$h[n] = \delta[n] - \delta[n-2]$

c) $H(z) = 1 + 2z^{-1} + 4z^{-2} + 2z^{-3} + z^{-4}$

$$= \frac{z^4 + 2z^3 + 4z^2 + 2z + 1}{z^4}$$

You can use MATLAB to find zeros.

$$\cong \frac{(z - 0.59e^{j0.65\pi})(z - 0.59e^{-j0.65\pi})(z - 1.7e^{j0.65\pi})(z - 1.7e^{-j0.65\pi})}{z^4}$$

$$H_{min}(z) = \frac{\left(z - 0.59 e^{j0.65\pi}\right)^2 \left(z - 0.59 e^{-j0.65\pi}\right)^2}{z^4}$$

$$H_{ap}(z) = \frac{\left(z - 1.7 e^{j0.65\pi}\right)\left(z - 1.7 e^{-j0.65\pi}\right)}{\left(z - 0.59 e^{j0.65\pi}\right)\left(z - 1.7 e^{-j0.65\pi}\right)}$$

$$= \frac{(1.7)^2 \left(z^{-1} - 0.59 e^{-j0.65\pi}\right)\left(z^{-1} - 0.59 e^{j0.65\pi}\right)}{\left(1 - 0.59 e^{j0.65\pi} z^{-1}\right)\left(1 - 0.59 e^{-j0.65\pi} z^{-1}\right)}$$

The gain of this all pass filter $(1.7)^2$. If you want to obtain a unity gain allpass filter take $(1.7)^2$ term into $H_{min}(z)$