TRANSFORM DOMAIN ANALYSIS

- RATIONAL SYSTEM FUNCTIONS
- STABLE, CAUSAL SYSTEMS
- INVERSE SYSTEMS
- PHASE DELAY AND GROUP DELAY
- FREQUENCY RESPONSE FUNCTIONS IN TERMS OF FIRST ORDER FACTORS
  - FREQUENCY RESPONSE OF A SINGLE ZERO OR A SINGLE POLE
- APPROXIMATE PLOTS OF FREQUENCY RESPONSE FROM POLE ZERO DIAGRAMS
- FREQUENCY RESPONSES OF CONJUGATE RECIPROCAL ZEROS (POLES)
  - RELATIONSHIP BETWEEN $A(z)$ AND $B(z)$
  - PHASES AND GROUP DELAYS OF $A(\omega)$ AND $B(\omega)$
  - FREQUENCY RESPONSES OF $\frac{1}{A(\omega)}$ AND $\frac{1}{B(\omega)}$
- RELATIONSHIP BETWEEN THE MAGNITUDE AND THE PHASE OF A FREQUENCY RESPONSE FUNCTION
  - $H(z)$ AND $H^*\left(\frac{1}{z^*}\right)$ YIELD THE SAME MAGNITUDE RESPONSE
  - THE POLES AND ZEROS OF $H^*\left(\frac{1}{z^*}\right)$ ARE THE *CONJUGATE-RECIPROCAL*'S OF THOSE OF $H(z)$
- ALLPASS SYSTEMS
  - PHASE AND GROUP DELAY FUNCTIONS OF ALLPASS SYSTEMS
- MINIMUM PHASE SYSTEMS
  - FREQUENCY RESPONSE MAGNITUDE COMPENSATION
  - MINIMUM GROUP-DELAY PROPERTY
  - MINIMUM PHASE-LAG PROPERTY
  - MINIMUM ENERGY-DELAY PROPERTY
- GENERALIZED LINEAR PHASE SYSTEMS

RATIONAL SYSTEM FUNCTIONS

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

$$H(z) = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \cdots + a_N z^{-N}}$$

$$= \frac{b_0}{a_0} \frac{(1 - c_1 z^{-1})(1 - c_2 z^{-1}) \ldots (1 - c_M z^{-1})}{(1 - d_1 z^{-1})(1 - d_2 z^{-1}) \ldots (1 - d_N z^{-1})}$$

STABLE AND CAUSAL SYSTEMS

Stable Systems

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

ROC of $H(z)$ includes unit circle

Causal Systems

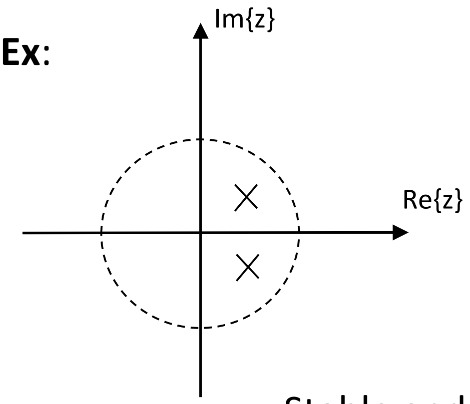
$$h[n] = 0 \qquad\qquad n < 0$$

ROC of $H(z)$ is out of the outermost pole circle

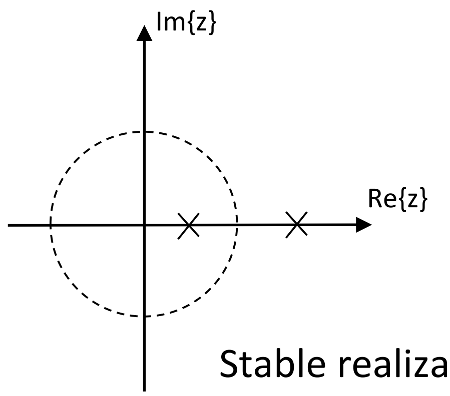
Stable and Causal Systems

Poles are inside the unit circle

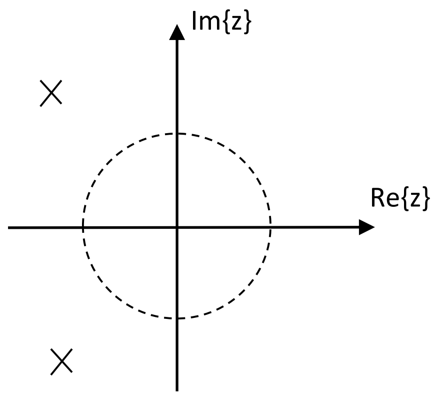
Ex:



Stable and causal realization (impulse response) is possible



Stable realization is noncausal (two-sided impulse response).
Causal realization is unstable.



Stable realization is noncausal (left-sided impulse response)
Causal realization is unstable.

INVERSE SYSTEMS

FREQUENCY RESPONSE

$$H\left(e^{j\omega}\right)$$

$$e^{j\omega_0 n} \xrightarrow{\text{LTI system}} e^{j\omega_0 n}\, H\left(e^{j\omega_0}\right)$$

$$Y\left(e^{j\omega}\right) = X\left(e^{j\omega}\right) H\left(e^{j\omega}\right)$$

$$\left|Y\left(e^{j\omega}\right)\right| = \left|X\left(e^{j\omega}\right)\right| \left|H\left(e^{j\omega}\right)\right|$$

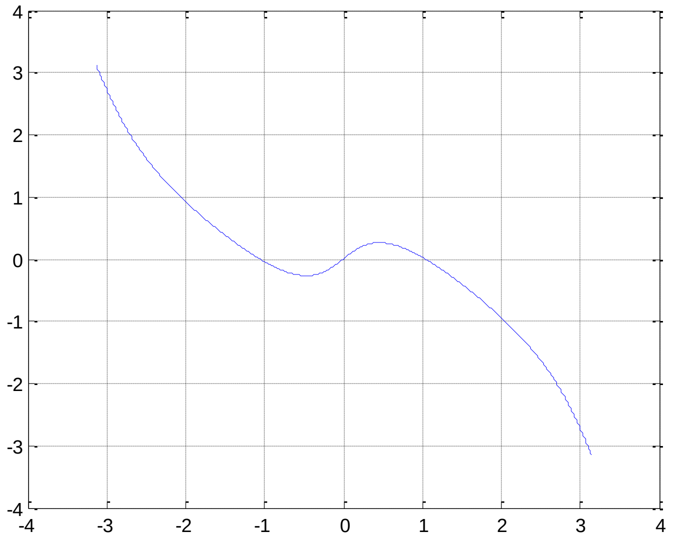
$$\angle Y\left(e^{j\omega}\right) = \underbrace{\angle X\left(e^{j\omega}\right)}_{\substack{\text{phase(response)} \\ \text{phase(shift)}}} + \angle H\left(e^{j\omega}\right)$$

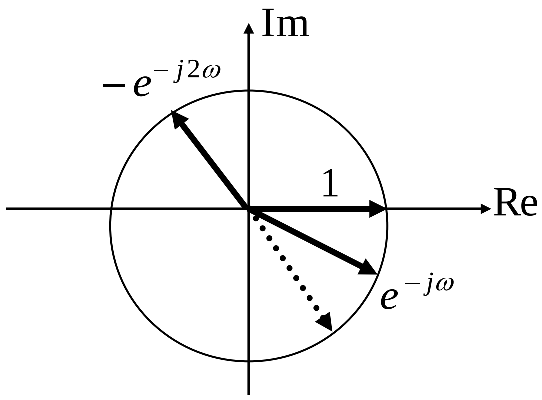
Ex:

$$h[n] = [\ldots 0 \;\; \underset{\substack{\uparrow \\ n=0}}{1} \;\; 1 \;\; -1 \;\; 0 \ldots]$$

$$\begin{aligned} H\left(e^{j\omega}\right) &= 1 + e^{-j\omega} - e^{-j2\omega} \\ &= 1 + \cos\omega - \cos 2\omega - j(\sin\omega + \sin 2\omega) \end{aligned}$$

$$\angle H\left(e^{j\omega}\right) = \arctan\left(\frac{-\sin\omega - \sin 2\omega}{1 + \cos\omega - \cos 2\omega}\right)$$

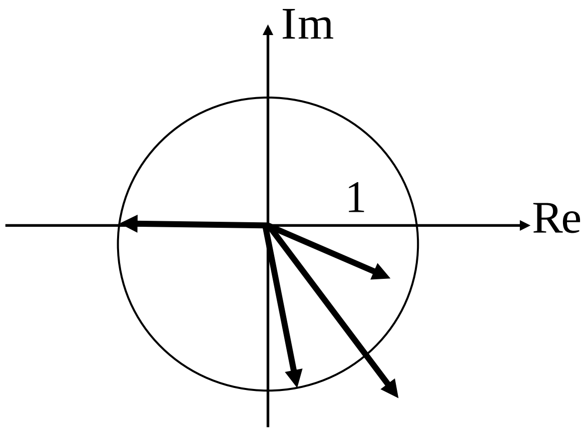


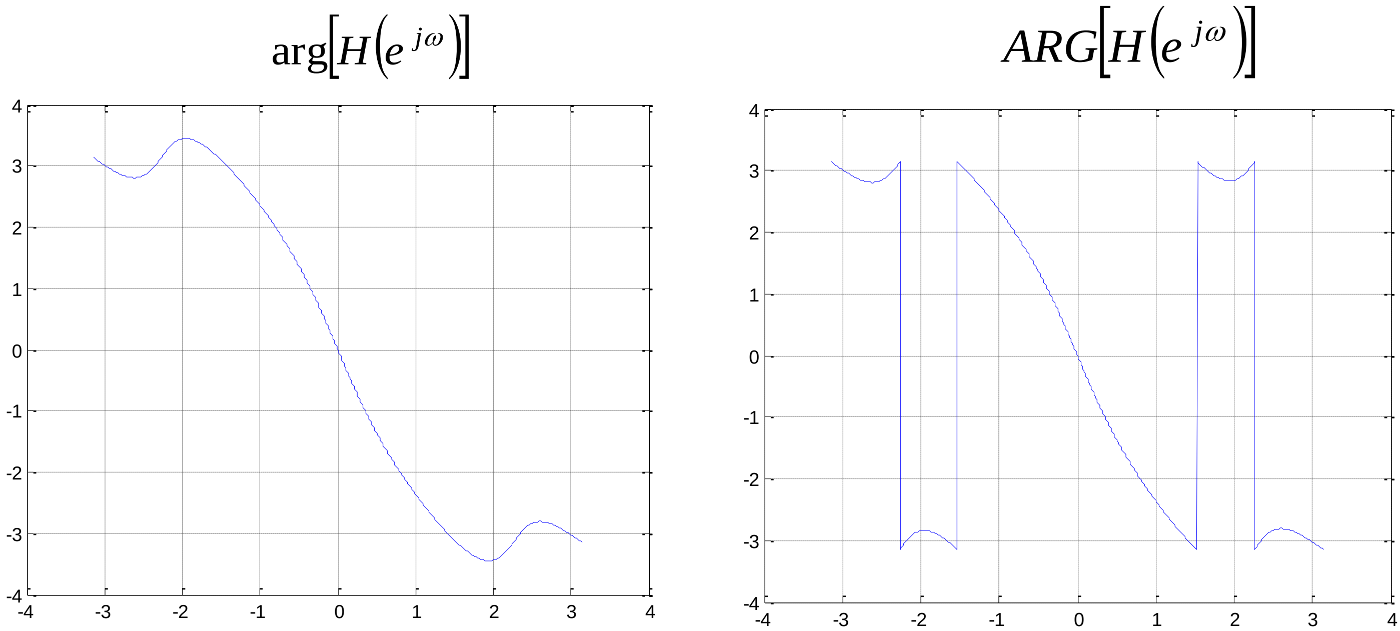


Ex:

$$h[n]=[\ldots 0 \;\; \underset{\substack{\uparrow \\ n=0}}{-1} \;\; 0.9 \;\; 1.2 \;\; 1 \;\; 0\ldots]$$

$$H(e^{j\omega})=-1+0.9e^{-j\omega}+1.2e^{-j2\omega}+e^{-j3\omega}$$





PHASE DELAY AND GROUP DELAY

$$\tau_{ph}(\omega) = -\frac{\angle H\left(e^{j\omega}\right)}{\omega}$$

$$\tau_{gr}(\omega) = -\frac{d}{d\omega}\angle H\left(e^{j\omega}\right)$$

Phase delay represents the amount of delay experienced by a sinusoidal signal through a LTI system.

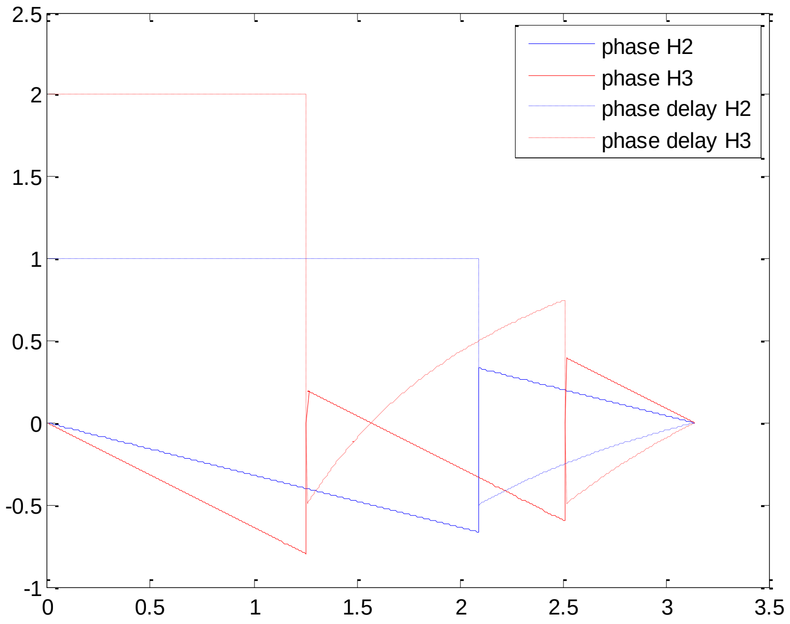
$$\cos(\omega_0 n) \longrightarrow \boxed{\begin{array}{c}\text{LTI system}\\ H\left(e^{j\omega}\right)\end{array}} \longrightarrow \left|H\left(e^{j\omega_0}\right)\right|\cos\left(\omega_0 n + \angle H\left(e^{j\omega_0}\right)\right)$$

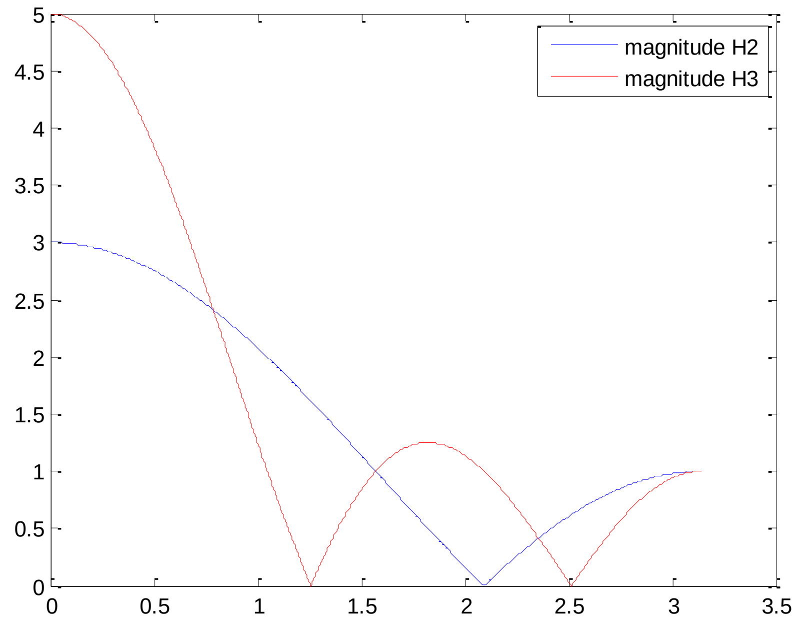
$$= \left|H\left(e^{j\omega_0}\right)\right|\cos\left(\omega_0\left(n + \angle\frac{H\left(e^{j\omega_0}\right)}{\omega_0}\right)\right)$$

Which one has “larger phase delay” ?

h2 = [1 1 1];

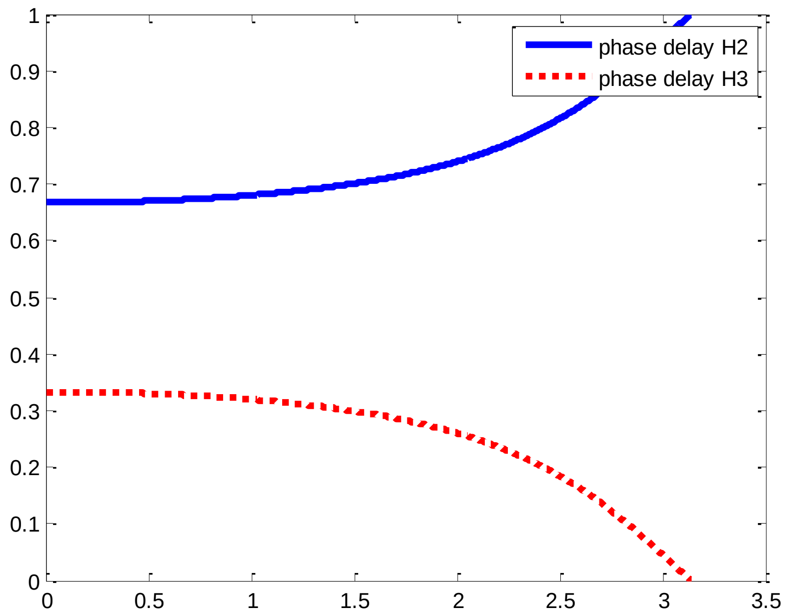
h3 = [1 1 1 1 1];

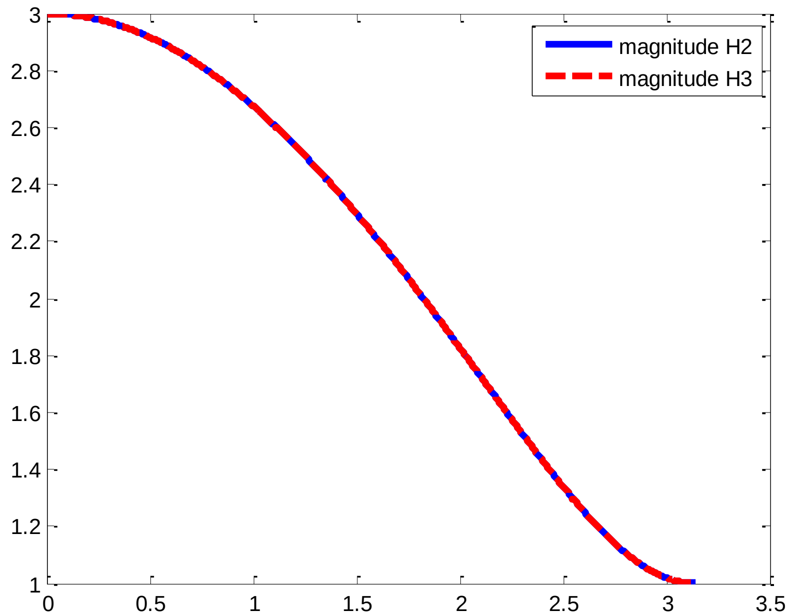





```
clear all
close all

h2 = [1 1 1];
h3 = [1 1 1 1 1];

[H2,w] = freqz(h2,1,1000);
[H3,w] = freqz(h3,1,1000);
pd2 = -unwrap(angle(H2)./w);
pd3 = -unwrap(angle(H3)./w);

plot(w,angle(H2)/pi)
hold on
plot(w,angle(H3)/pi,'r')
plot(w,pd2,'b:')
plot(w,pd3,'r:')
legend('phase H2','phase H3','phase delay H2','phase delay H3')
figure
plot(w,abs(H2))
hold
plot(w,abs(H3),'r')
```

Which one has “larger phase delay” ?





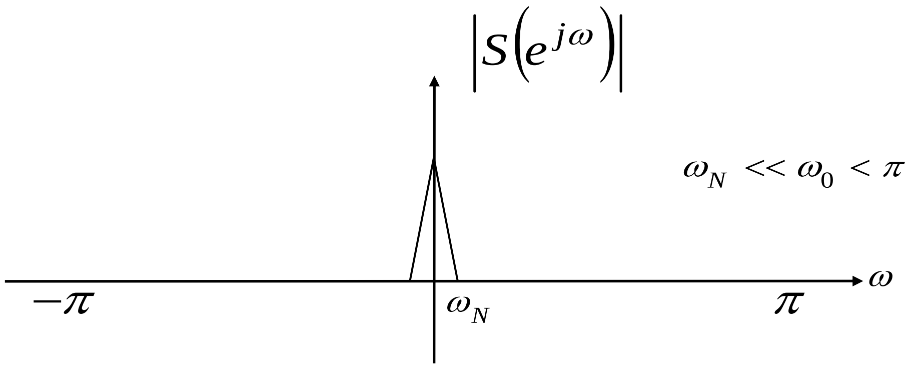

```
clear all
close all

h2 = [1 2];

h3 = [2 1];

[H2,w] = freqz(h2,1,1000);
[H3,w] = freqz(h3,1,1000);
plot(w,-angle(H2)./w,'linewidth', 3)
hold on
plot(w,-angle(H3)./w,'r:','linewidth', 3)
legend('phase delay H2','phase delay H3')

figure
plot(w,abs(H2),'linewidth', 3)
hold on
plot(w,abs(H3),'r--','linewidth', 3)
legend('magnitude H2','magnitude H3')
```

What does the group delay represent?

Ex:

$$s[n]\cos(\omega_0 n) \longrightarrow \boxed{\text{LTI system } H(e^{j\omega})} \longrightarrow y[n] = ?$$

$s[n]$: "narrowband" signal



Let $\angle H(e^{j\omega_0})$ be modeled, to a first order approximation, around ω_0 as

$$\angle H(e^{j\omega_0}) \cong -\phi_0 - \omega n_d$$

Then,

$$\tau_{gr}(\omega) = n_d$$

and

$$y[n] \cong |H(e^{j\omega_0})| s[n - n_d] \cos(\omega_0 n - \phi_0 - \omega n_d)$$

$$= |H(e^{j\omega_0})| s[n - \tau_{gr}(\omega_0)] \cos\left(\omega_0 (n - \tau_{ph}(\omega_0)\right)$$

```
clear all
close all

zz1 = 0.98*exp(j*0.8*pi);
zz2 = conj(zz1);
pp1 = 0.8*exp(j*0.4*pi);
pp2 = conj(pp1);

for k=1:4
   pp(k) = 0.95*exp(j*(0.15*pi+0.02*pi*k));
end
for k=5:8
   pp(k) = conj(pp(k-4));
end

zz = 1./pp;

p = [pp pp pp1 pp2];
z = [zz zz zz1 zz2];
K = 0.95^16;

plot(p,'rx','linewidth',2,'markersize',10)
hold
plot(z,'bo','linewidth',2,'markersize',10)
om = 0:999;
plot(exp(j*om*2*pi/1000));

[Num, Den] = zp2tf(z',p',K);

[H,w] = freqz(Num,Den,2048);
figure
plot(w/pi,abs(H))
title('magnitude')
figure
plot(w/pi,20*log10(abs(H)))
title('magnitude, dB')
figure
plot(w/pi,angle(H)/pi)
title('phase, principal value, (ARG),      X pi rads')
figure
plot(w/pi,unwrap(angle(H))/pi)
title('phase, (arg),      X pi rads')
figure
[Phi,w] = phasedelay(Num,Den,2048);
plot(w/pi,Phi)
title('phase delay, samples')
figure
[Grp,w] = grpdelay(Num,Den,2048);
plot(w/pi,Grp)
title('group delay, samples')

M = 60;
window = hamming(M+1);
n = 0:M;
x1 = window' .* cos(0.2*pi*n);
x2 = window' .* cos(0.4*pi*n - pi/2);
x3 = window' .* cos(0.8*pi*n + pi/5);
figure
plot(x1)
figure
```

```
plot(x2)
figure
plot(x3)

x = [x3 x1 x2 zeros(1,130)];
figure
plot(x)
title('input signal')
figure
[X,w] = freqz(x,1,2048)
plot(w/pi,abs(X))
title('DTFT of input signal')
y = filter(Num,Den,x);
figure
plot(y)
title('output signal')
```

FREQUENCY RESPONSE FUNCTIONS IN TERMS OF FIRST ORDER FACTORS

The magnitude function

$$\left|H\left(e^{j\omega}\right)\right| = \left|\frac{b_0}{a_0}\right| \frac{\left|1 - c_1 e^{-j\omega}\right|\left|1 - c_2 e^{-j\omega}\right| \ldots \left|1 - c_M e^{-j\omega}\right|}{\left|1 - d_1 e^{-j\omega}\right|\left|1 - d_2 e^{-j\omega}\right| \ldots \left|1 - d_N e^{-j\omega}\right|}$$

$$= \left|\frac{b_0}{a_0}\right| \frac{\prod_{k=1}^{M}\left|1 - c_k e^{-j\omega}\right|}{\prod_{k=1}^{N}\left|1 - d_k e^{-j\omega}\right|}$$

The magnitude-squared function

$$\left|H\left(e^{j\omega}\right)\right|^2 = H\left(e^{j\omega}\right)H^*\left(e^{j\omega}\right) = \left(\frac{b_0}{a_0}\right)^2 \frac{\prod_{k=1}^{M}\left(1 - c_k e^{-j\omega}\right)\left(1 - c_k^* e^{j\omega}\right)}{\prod_{k=1}^{N}\left(1 - d_k e^{-j\omega}\right)\left(1 - d_k^* e^{j\omega}\right)}$$

dB Magnitude

$$\text{Gain in dB} = 20\log_{10}\left|H\left(e^{j\omega}\right)\right|$$

$$= 20\log_{10}\left|\frac{b_0}{a_0}\right| + \sum_{k=1}^{M} 20\log_{10}\left(1 - c_k e^{-j\omega}\right) - \sum_{k=1}^{N} 20\log_{10}\left(1 - d_k e^{-j\omega}\right)$$

Sum of 1st order terms

Phase

$$\arg\left[H\left(e^{j\omega}\right)\right] = \arg\left[\frac{b_0}{a_0}\right] + \sum_{k=1}^{M} \arg\left[\left(1 - c_k e^{-j\omega}\right)\right] - \sum_{k=1}^{N} \arg\left[\left(1 - d_k e^{-j\omega}\right)\right]$$

Sum of 1st order terms

Group Delay

$$\mathrm{grd}\left[H\left(e^{j\omega}\right)\right] = \sum_{k=1}^{N} \frac{d}{d\omega} \arg\left[\left(1 - d_k e^{-j\omega}\right)\right] - \sum_{k=1}^{M} \frac{d}{d\omega} \arg\left[\left(1 - c_k e^{-j\omega}\right)\right]$$

Sum of 1st order terms

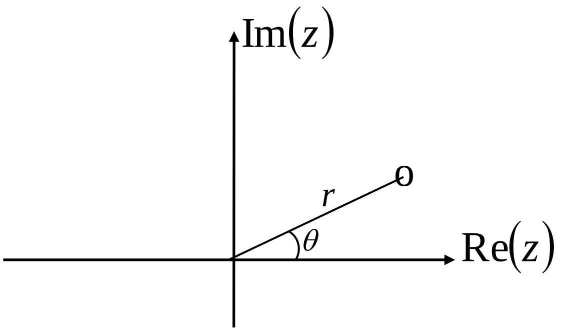
FREQUENCY RESPONSE OF A SINGLE ZERO OR A SINGLE POLE

$$1-c_k e^{-j\omega} \quad \text{or} \quad \frac{1}{1-d_k e^{-j\omega}}$$

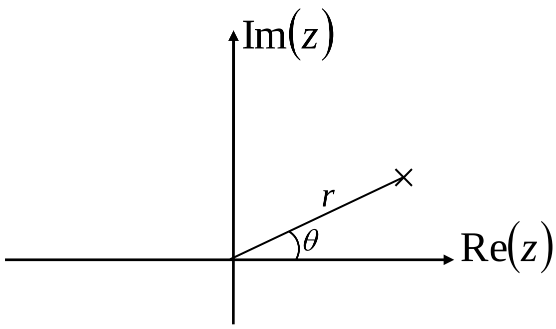
where

$$c_k, d_k = re^{j\theta}$$

A zero at $re^{j\theta}$



A pole at $re^{j\theta}$

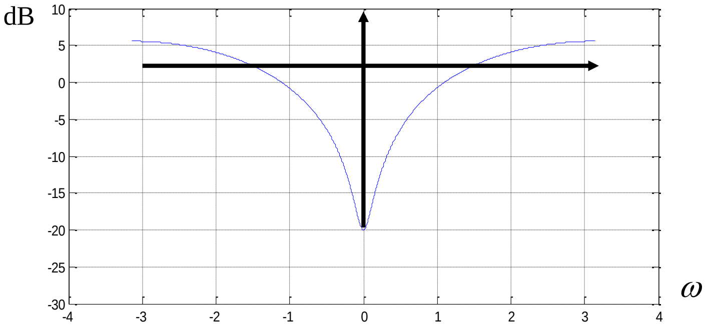


Magnitude response

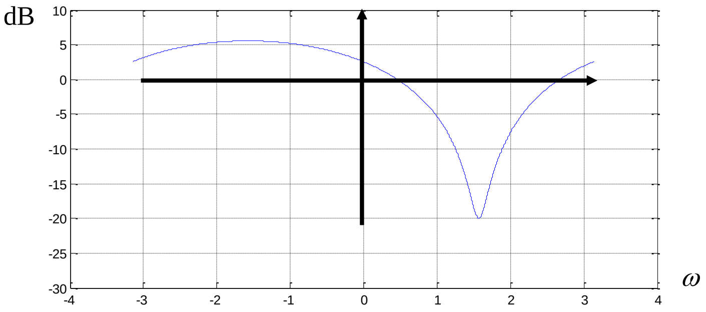
$$\left|1 - re^{j\theta}e^{-j\omega}\right|^2 = 1 + r^2 - 2r\cos(\omega - \theta)$$

$$20\log_{10}\left|1 - re^{j\theta}e^{-j\omega}\right| = 10\log_{10}\left(1 + r^2 - 2r\cos(\omega - \theta)\right)$$

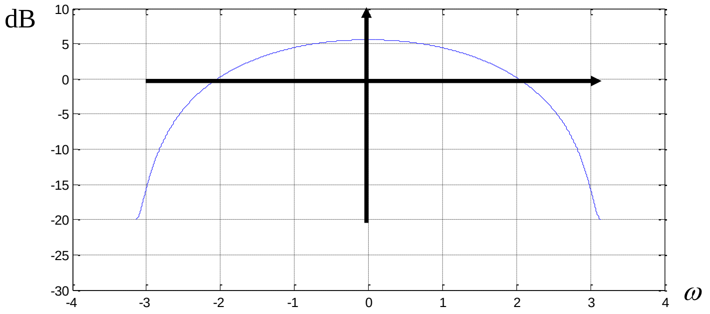
r=0.9, θ=0



r=0.9, θ=π/2



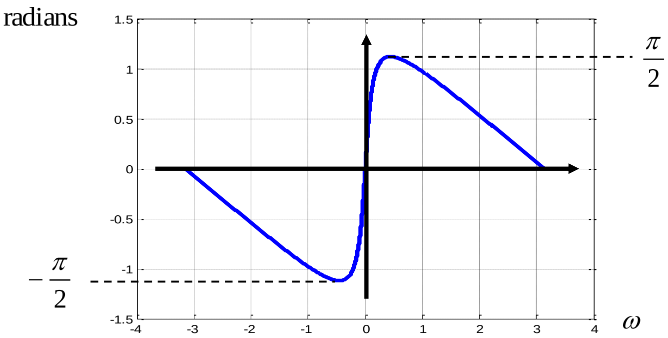
r=0.9, θ=π



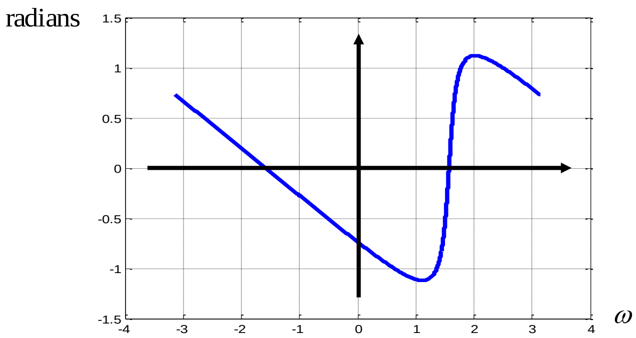
Phase response:

$$\angle\left(1 - re^{j\theta}e^{-j\omega}\right) = \tan^{-1}\frac{r\sin(\omega-\theta)}{1 - r\cos(\omega-\theta)}$$

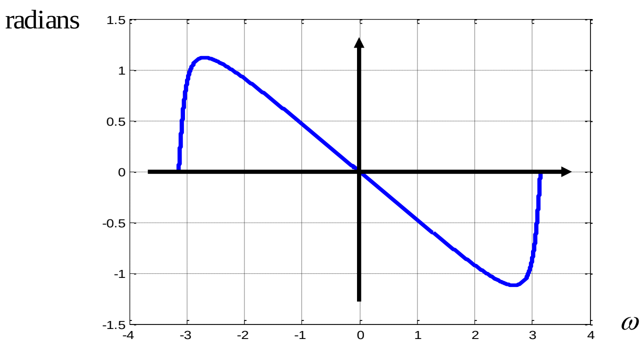
r=0.9, θ=0



r=0.9, θ= π/2



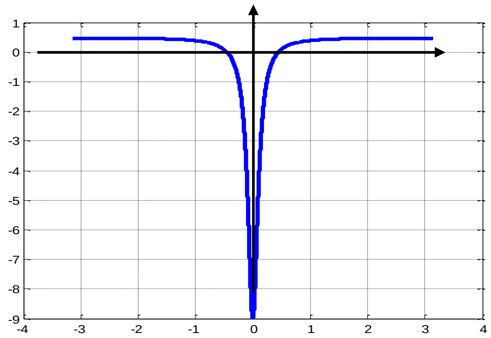
r=0.9, θ= π/2



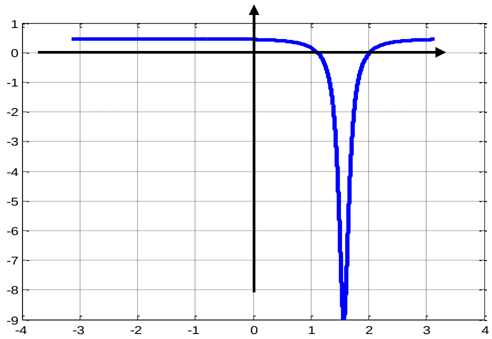
Group delay:

$$\tau_{grd} = -\frac{d}{d\omega}\angle\left(1 - re^{j\theta}e^{-j\omega}\right) = \tan^{-1}\frac{r^2 - r\cos(\omega-\theta)}{1 + r^2 - 2r\cos(\omega-\theta)}$$

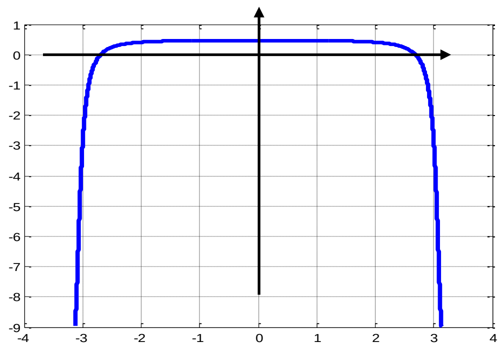
r=0.9, θ= 0



r=0.9, θ= π/2



r=0.9, θ= π



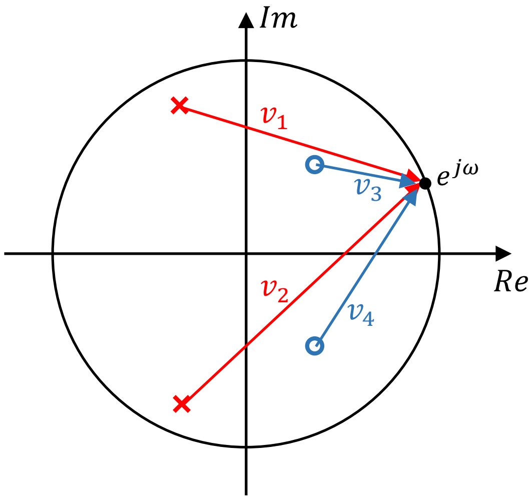
```
close all
clear all
w=-pi:0.001:pi;
r=0.9;
teta=pi;
% mag=10*log10(1+r^2-2*r*cos(w-teta));
% plot(w,mag,'linewidth',3)
% grid

c=r*exp(j*teta);
H = 1-c*exp(-j*w);
% plot(w,20*log10(abs(H)),'linewidth',3);
plot(w,angle(H),'linewidth',3);
grid
% v=[-4 -4 -30 10];
% axes([-4 -4 -30 10])
```

APPROXIMATE PLOTS OF FREQUENCY RESPONSE FROM POLE ZERO DIAGRAMS

Definition:

Zero vector: a vector drawn from a zero to a point, $e^{j\omega}$, on the unit circle.

Pole vector: a vector drawn from a pole to a point, $e^{j\omega}$, on the unit circle.



$$H(z) = \frac{(1 - z_1 z^{-1})(1 - z_2 z^{-1})}{(1 - p_1 z^{-1})(1 - p_2 z^{-1})}$$

$$H(e^{j\omega}) = \frac{\left(1 - z_1 e^{-j\omega}\right)\left(1 - z_2 e^{-j\omega}\right)}{\left(1 - p_1 e^{-j\omega}\right)\left(1 - p_2 e^{-j\omega}\right)}$$

$$= \frac{\underbrace{\left(e^{j\omega} - z_1\right)}_{v_3}\underbrace{\left(e^{j\omega} - z_2\right)}_{v_4}}{\underbrace{\left(e^{j\omega} - p_1\right)}_{v_1}\underbrace{\left(e^{j\omega} - p_2\right)}_{v_2}}$$

1) The magnitude of a frequency response function, $\left|H\left(e^{j\omega}\right)\right|$, is proportional to

$$\frac{\text{product of the magnitudes of zero vectors}}{\text{product of the magnitudes of pole vectors}}$$

$$\left|H\left(e^{j\omega}\right)\right| = \frac{\left|e^{j\omega} - z_1\right|\left|e^{j\omega} - z_2\right|}{\left|e^{j\omega} - p_1\right|\left|e^{j\omega} - p_2\right|}$$

2) The phase of a frequency response function, $\measuredangle H\left(e^{j\omega}\right)$, is (except possible addition of $\pm\pi$)

$$\begin{aligned}&(\text{sum of the angles of zero vectors})\\ &\qquad - (\text{sum of the angles of pole vectors})\end{aligned}$$

Observations (on first order examples above):

A zero at $re^{j\theta}$ $(r < 1) \Rightarrow$

1) The magnitude has a ***minimum*** at (around, for higher orders) $\omega = \theta$.
2) The absolute rate of change of the phase is maximum around $\omega = \theta$.
3) The phase tends to ***increase*** towards $\omega = \theta$.
4) (After (2)) The absolute value of group delay has a maximum around $\omega = \theta$.
5) These four effects become stronger as $|r| \rightarrow 1$, i.e., as the zero approaches the unit circle.

A pole at $re^{j\theta}$ $(r < 1) \Rightarrow$

1) The magnitude has a ***maximum*** at (around, for higher orders) $\omega = \theta$.
2) The absolute rate of change of the phase is maximum around $\omega = \theta$.
3) The phase tends to ***decrease*** towards $\omega = \theta$.
4) (After (2)) The absolute value of group delay has a maximum at $\omega = \theta$.
5) These four effects become stronger as $|r| \rightarrow 1$, i.e., as the zero approaches the unit circle.

In general, when there are multiple poles and zeros, the above principles can be used to make a rough sketch of frequency response magnitude, phase and group delay.

Note that with multiple poles and zeros, the absolute rate of change of the phase is not necessarily maximum at the pole/zero angle, i.e., $\omega = \theta$.

Investigate the following examples and judge on the reliability of the above principles in inferring the frequency response magnitude, phase and group delay.

The following examples are for 2nd order systems of the form

$$H(z) = \frac{1}{(1 - dz^{-1})(1 - d^{*}z^{-1})}$$

$$= \frac{1}{(1 - re^{j\theta}z^{-1})(1 - re^{-j\theta}z^{-1})}$$

$$= \frac{1}{1 - 2r\cos\theta\, z^{-1} + r^2 z^{-2}}$$

Poles at $re^{j\theta}$, $re^{-j\theta}$

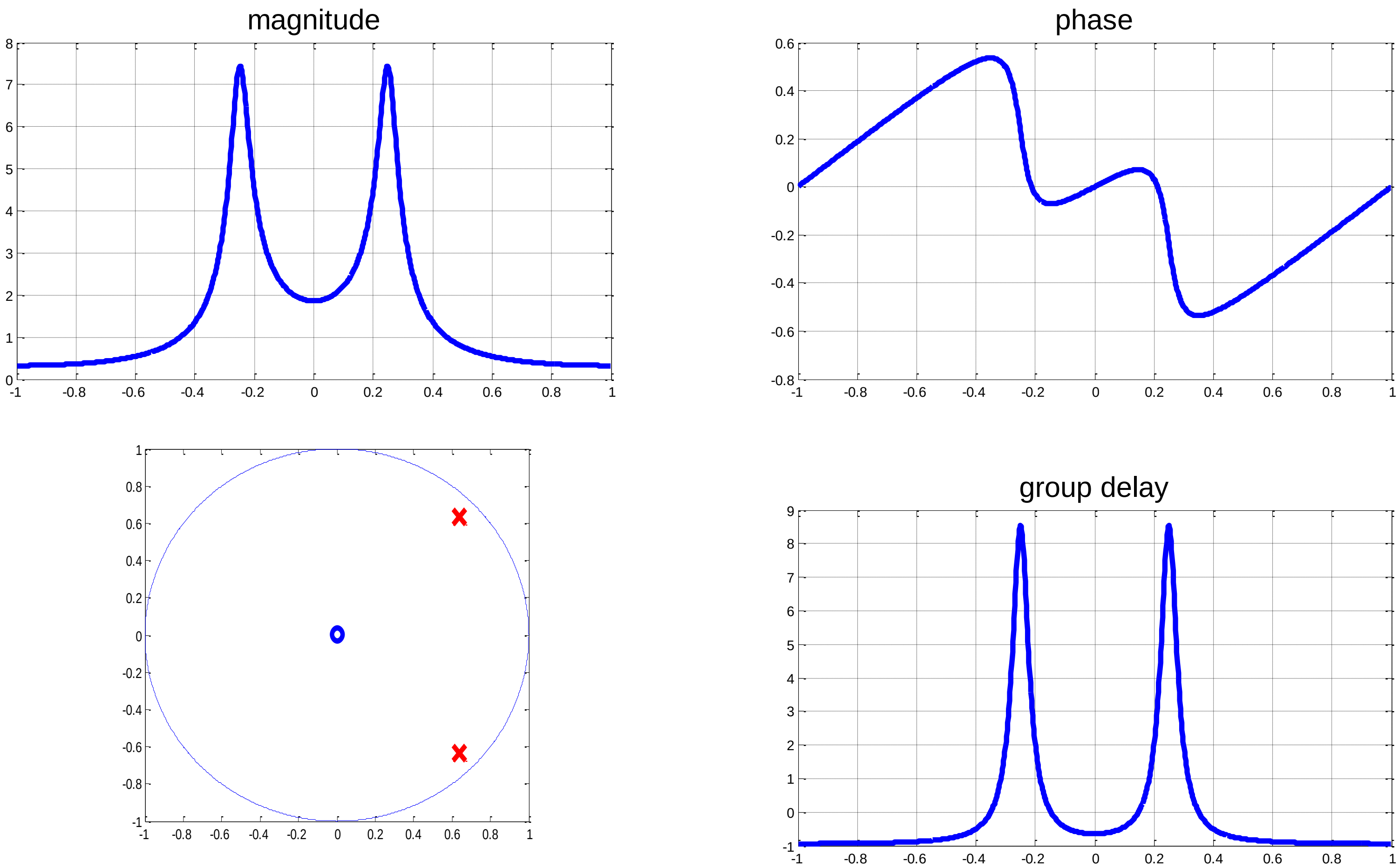
Double zero at 0

Impulse response,

$$h[n] = \frac{1}{\sin\theta} r^n \sin\big(\theta(n+1)\big)\, u[n]$$

Ex:

Let $r = 0.9$ and $\theta = \frac{\pi}{4}$

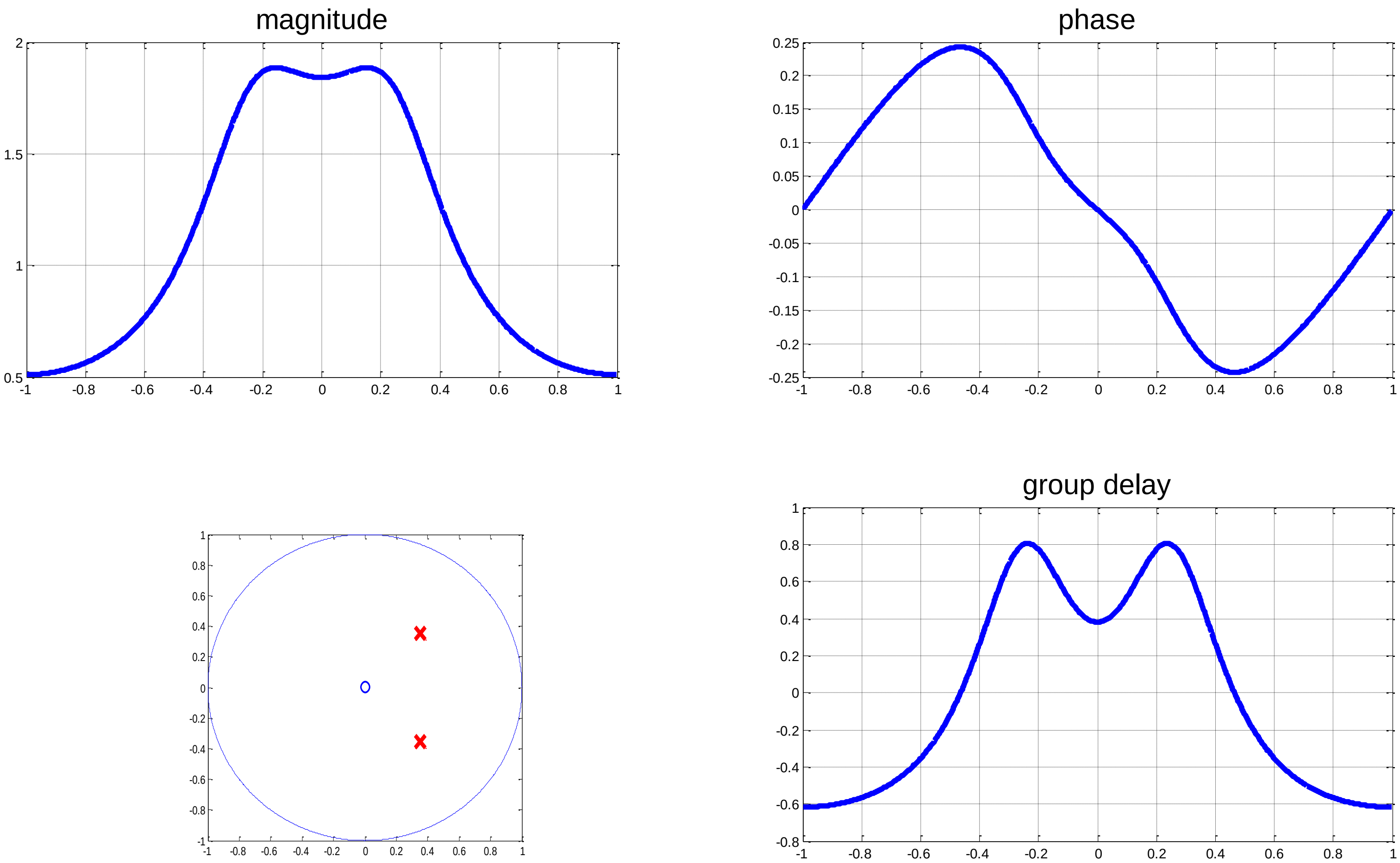


```
clear all
close all

phaseshift = 0; % 0, 1, -1 degerlerini alır
rp = 0.9;
thtp = pi/4;
p1 = rp*exp(j*thtp);
p2 = conj(p1);

rz = 0 ;
thtz = pi/4 ;
z1 = rz*exp(j*thtz);
z2 = conj(z1);

poles = [p1; p2];
zeros = [z1; z2];
gain = 1;

plot(poles+(0.0001+j*0.0001),'rx','linewidth',4,'markersize',15)
hold
 plot(zeros+(0.0001+j*0.0001),'bo','linewidth',4,'markersize',10)
% plot(0,0,'bo','linewidth',2,'markersize',10)
om = 0:999;
plot(exp(j*om*2*pi/1000));


[num den] = zp2tf(zeros,poles,gain)

[HH w] = freqz(num,den,1024,'whole');
H = fftshift(HH);
figure
subplot(2,2,1); plot(w/pi-1, abs(H),'linewidth',4); grid;
title('magnitude','fontsize',20)
% v = axis;
% v(3) = 0;
% v(4) =5;
% axis(v);
% figure
subplot(2,2,2); plot(w/pi-1,(2*pi*phaseshift+unwrap(angle(H)))/pi,'linewidth',4); grid;
% subplot(2,2,2); plot(w/pi-1,fftshift(unwrap(angle(HH)))/pi,'linewidth',4); grid;
title('phase','fontsize',20)

% figure
subplot(2,2,4);plot(w/pi-1,fftshift(grpdelay(num,den,1024,'whole')),'linewidth',4)
title('group delay','fontsize',20)
grid
```

Ex: 2nd order system $H(z) = \frac{1}{(1-re^{j\theta}z^{-1})(1-re^{-j\theta}z^{-1})}$

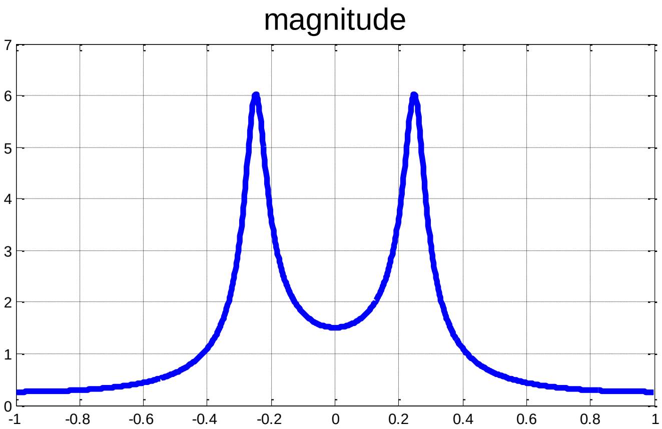
Poles at $0.5e^{j\frac{\pi}{4}}$, $0.5e^{-j\frac{\pi}{4}}$,

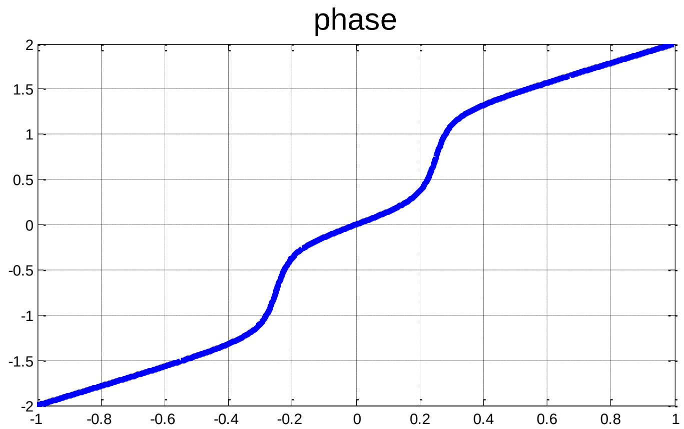
Double zero at 0

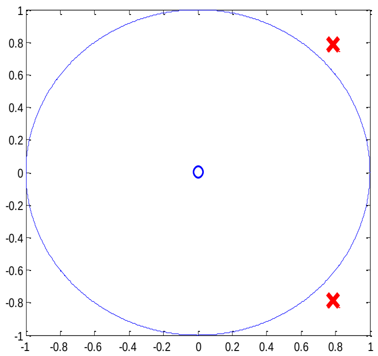


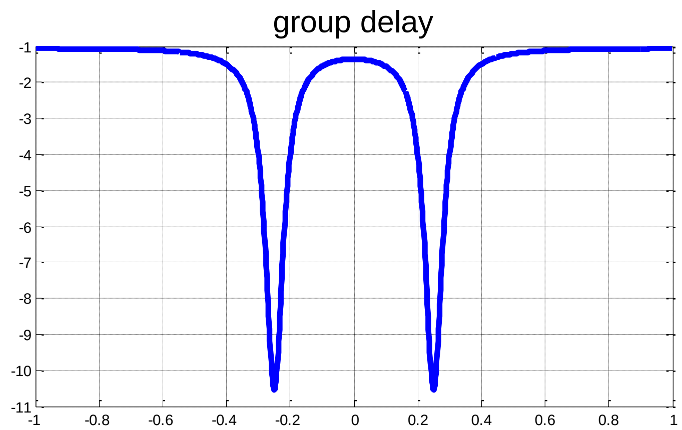
Ex: 2nd order system $H(z) = \frac{1}{(1-re^{j\theta}z^{-1})(1-re^{-j\theta}z^{-1})}$

Poles at $\frac{10}{9}e^{j\frac{\pi}{4}}$, $\frac{10}{9}e^{-j\frac{\pi}{4}}$,
double zero at 0





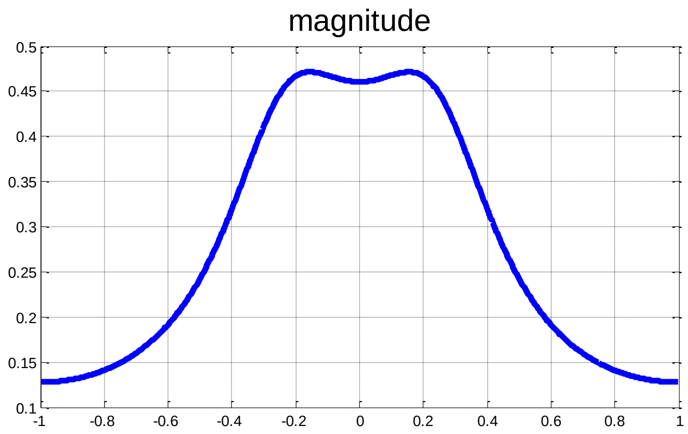


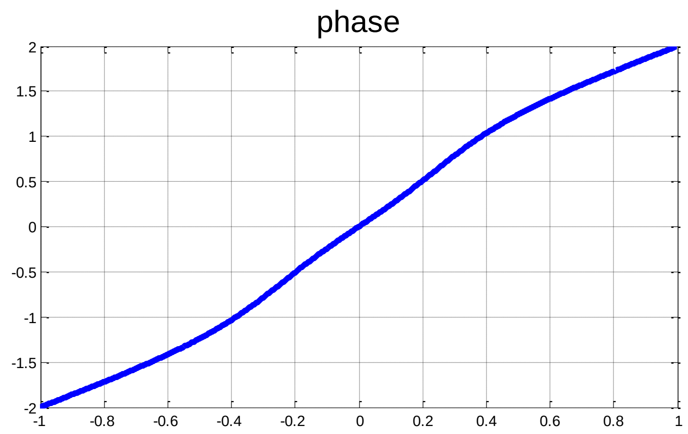


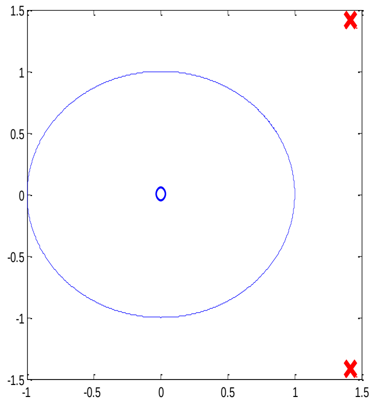
Ex: 2^{nd} order system $\quad H(z) = \dfrac{1}{(1-re^{j\theta}z^{-1})(1-re^{-j\theta}z^{-1})}$

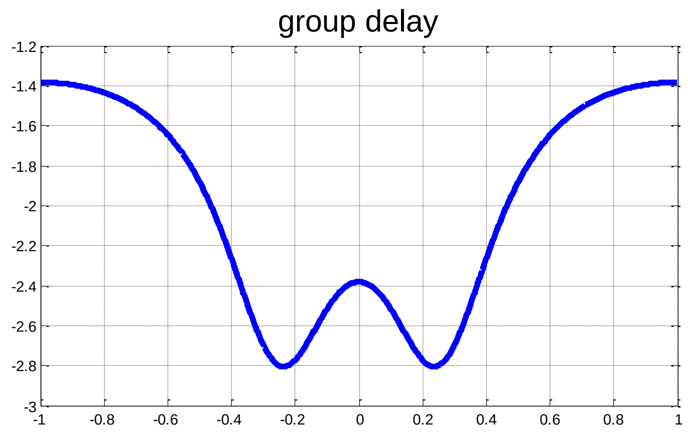
Poles at $2e^{j\frac{\pi}{4}},\ 2e^{-j\frac{\pi}{4}}$

Double zero at 0





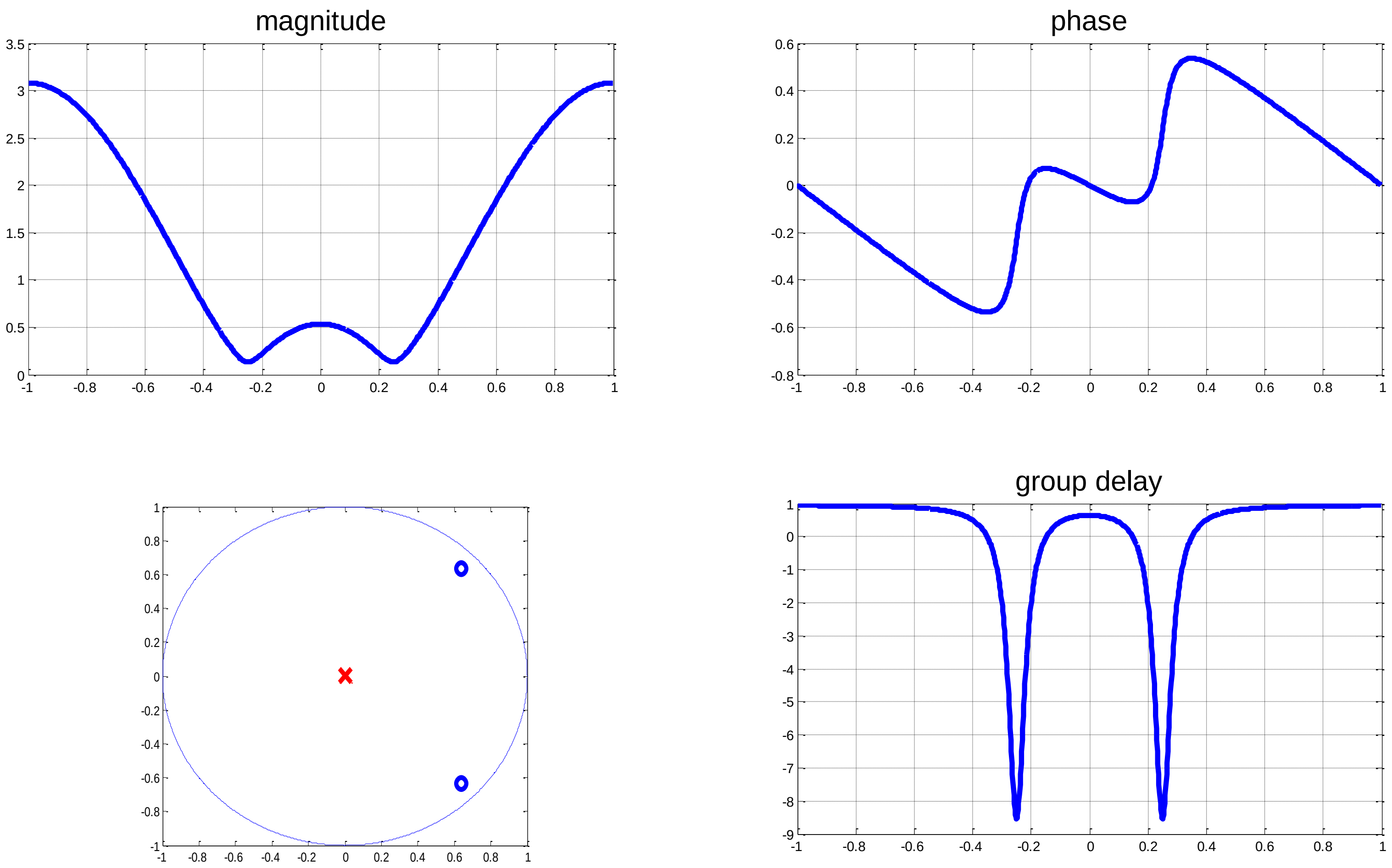




Ex: 2^{nd} order system $H(z) = \left(1 - re^{j\theta}z^{-1}\right)\left(1 - re^{-j\theta}z^{-1}\right)$

Double pole at 0

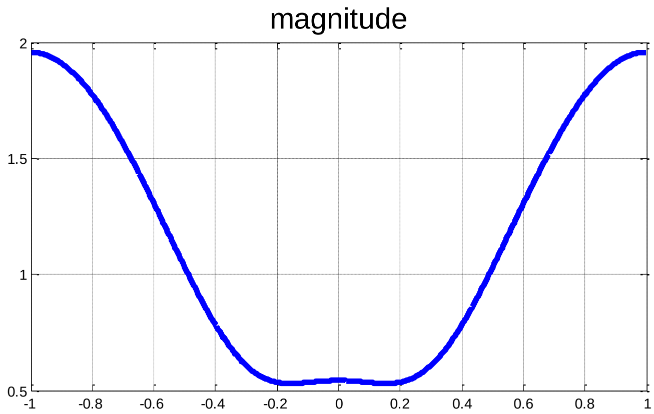
Zeros at $0.9e^{j\frac{\pi}{4}}$, $0.9e^{-j\frac{\pi}{4}}$,

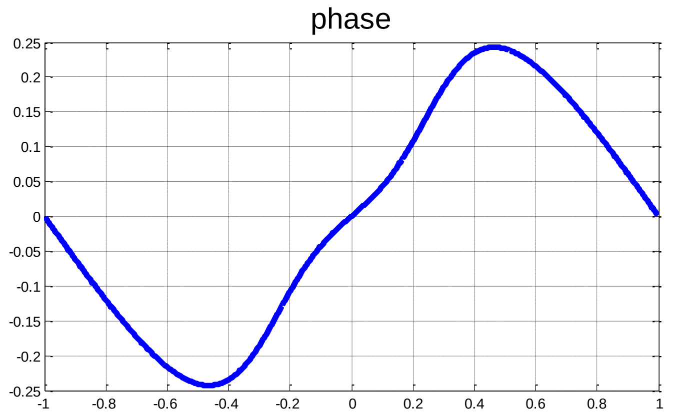


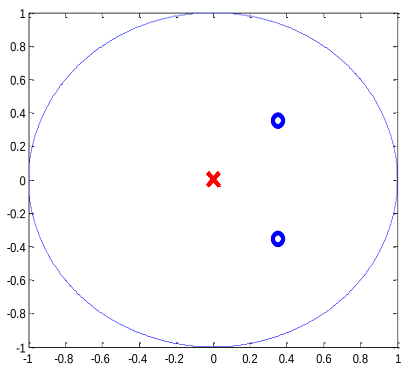
Ex: 2nd order system $\quad H(z) = \left(1 - re^{j\theta}z^{-1}\right)\left(1 - re^{-j\theta}z^{-1}\right)$

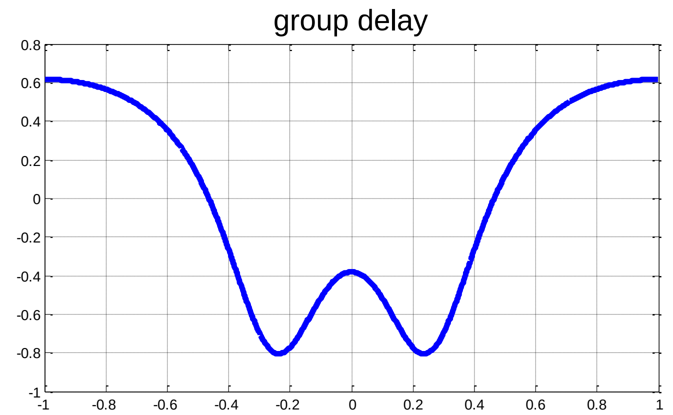
Double pole at 0

Zeros at $0.5e^{j\frac{\pi}{4}},\ 0.5e^{-j\frac{\pi}{4}}$





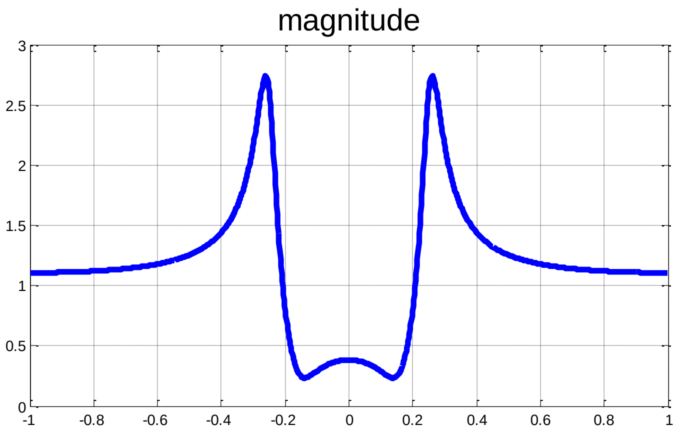


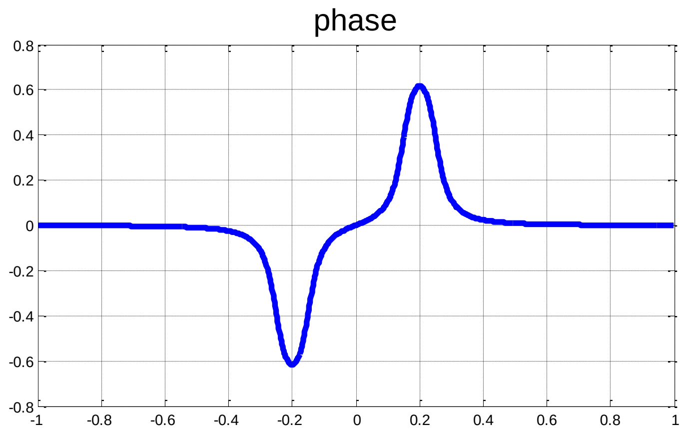


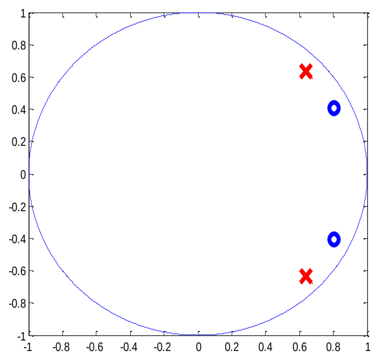
Ex: 2nd order, $H(z) = \dfrac{(1-z_1 z^{-1})(1-z_1^* z^{-1})}{(1-p_1 z^{-1})(1-p_1^* z^{-1})}$,

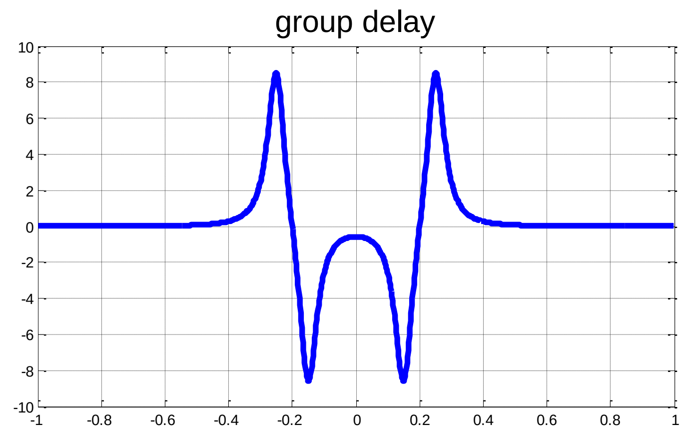
Poles: $0.9e^{j\frac{\pi}{4}},\ 0.9e^{-j\frac{\pi}{4}}$

Zeros: $0.9e^{j\left(\frac{\pi}{4}-\frac{\pi}{10}\right)},\ 0.9e^{-j\left(\frac{\pi}{4}-\frac{\pi}{10}\right)}$





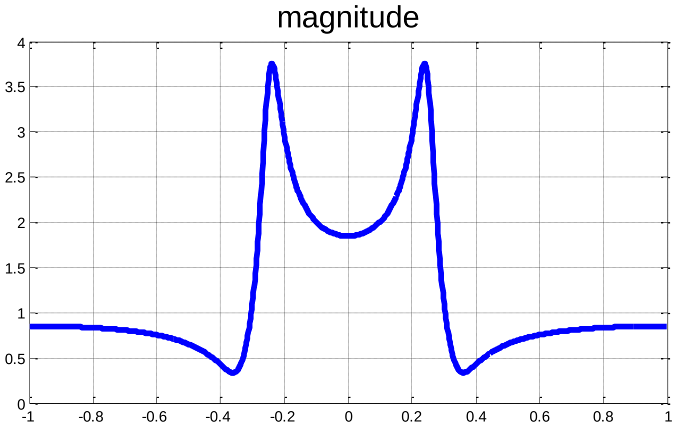


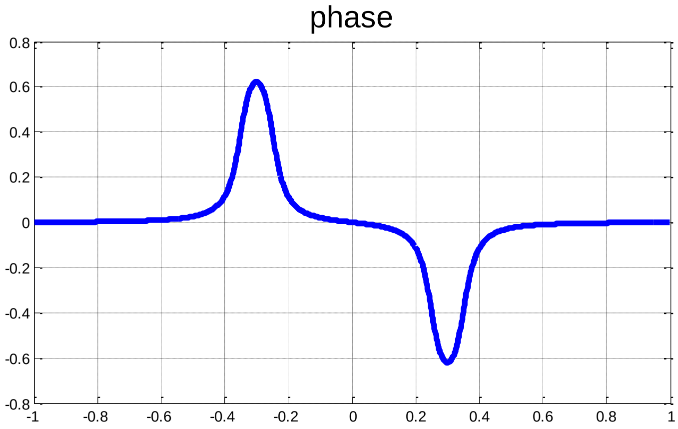


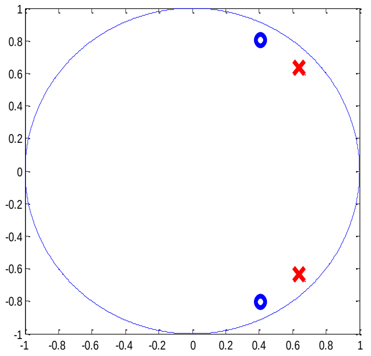
Ex: 2nd order, $H(z) = \frac{(1-z_1 z^{-1})(1-z_1^* z^{-1})}{(1-p_1 z^{-1})(1-p_1^* z^{-1})}$

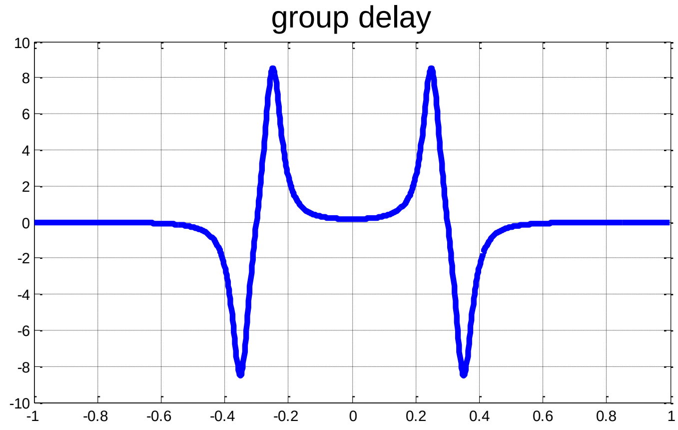
Poles: $0.9e^{j\frac{\pi}{4}},\ 0.9e^{-j\frac{\pi}{4}}$

Zeros: $0.9e^{j\left(\frac{\pi}{4}+\frac{\pi}{10}\right)},\ 0.9e^{-j\left(\frac{\pi}{4}+\frac{\pi}{10}\right)}$





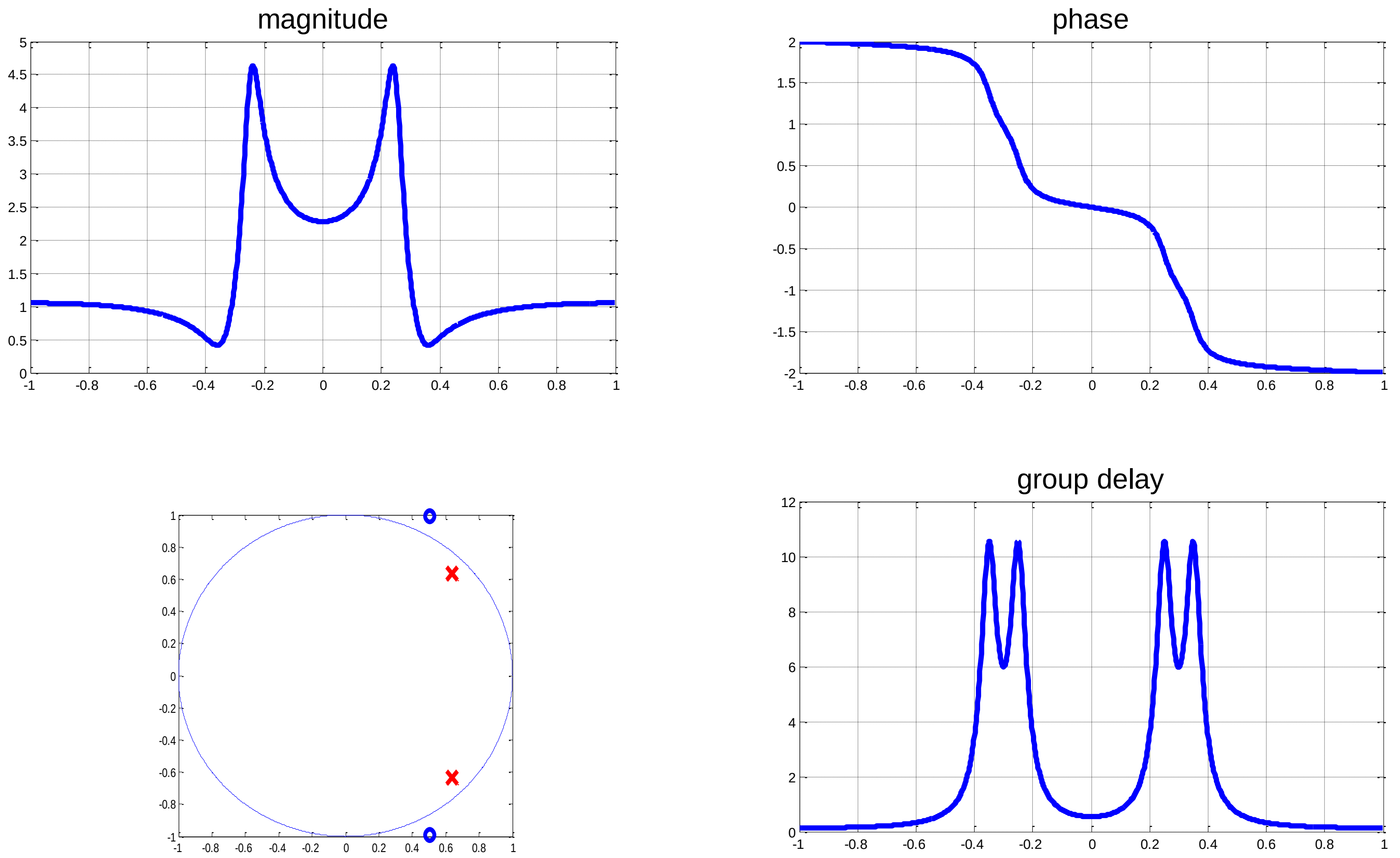




Ex: 2nd order, $H(z) = \frac{(1-z_1 z^{-1})(1-z_1^* z^{-1})}{(1-p_1 z^{-1})(1-p_1^* z^{-1})}$

Poles: $0.9e^{j\frac{\pi}{4}},\ 0.9e^{-j\frac{\pi}{4}}$,

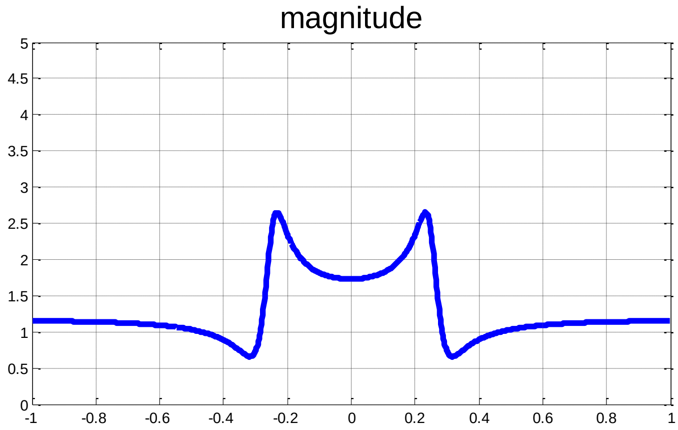
Zeros: $\frac{10}{9}e^{j\left(\frac{\pi}{4}+\frac{\pi}{10}\right)},\ \frac{10}{9}e^{-j\left(\frac{\pi}{4}+\frac{\pi}{10}\right)}$

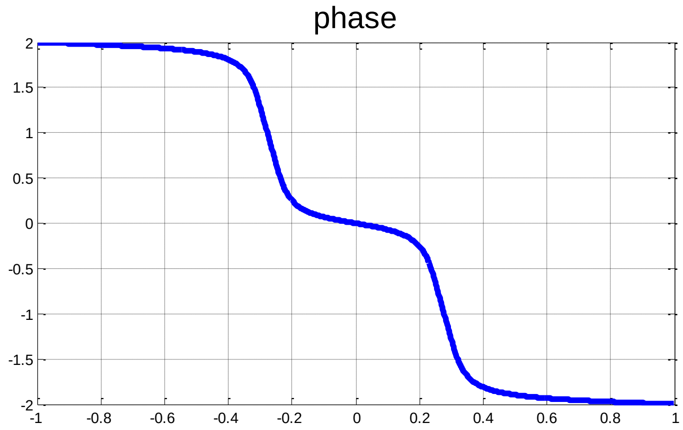


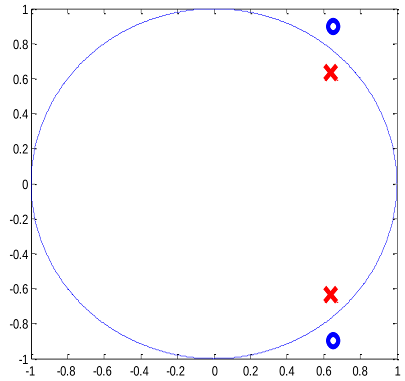
Ex: 2nd order, $H(z)=\frac{(1-z_1z^{-1})(1-z_1^*z^{-1})}{(1-p_1z^{-1})(1-p_1^*z^{-1})}$

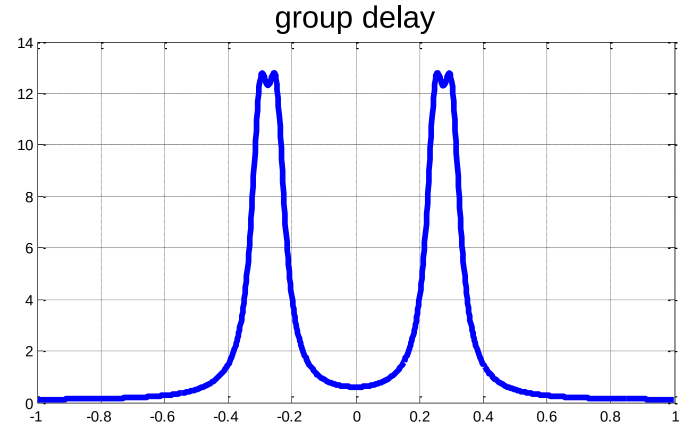
Poles: $0.9e^{j\frac{\pi}{4}},\ 0.9e^{-j\frac{\pi}{4}}$

Zeros: $\frac{10}{9}e^{j\left(\frac{\pi}{4}+\frac{\pi}{20}\right)},\ \frac{10}{9}e^{-j\left(\frac{\pi}{4}+\frac{\pi}{20}\right)}$





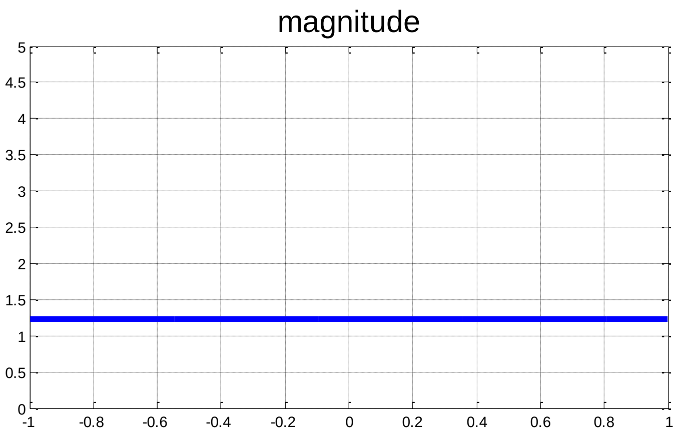


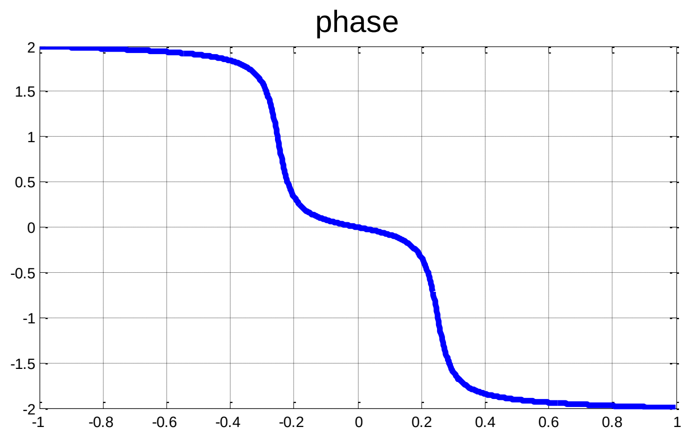


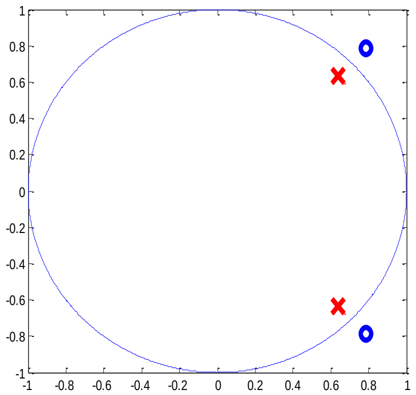
Ex: 2nd order, $H(z) = \frac{(1-z_1 z^{-1})(1-z_1^* z^{-1})}{(1-p_1 z^{-1})(1-p_1^* z^{-1})}$

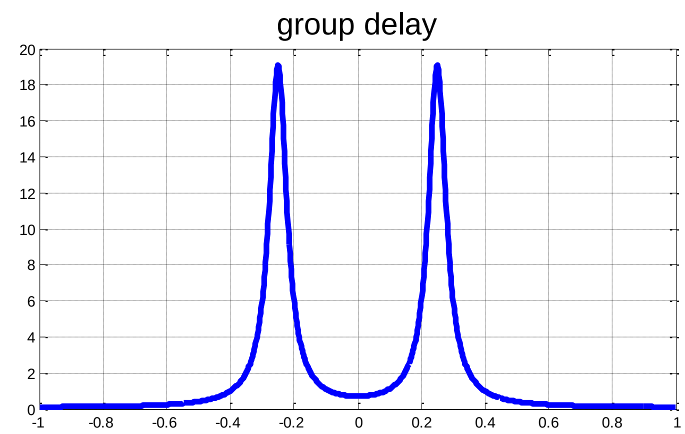
Poles: $0.9e^{j\frac{\pi}{4}},\ 0.9e^{-j\frac{\pi}{4}}$

Zeros: $\frac{10}{9}e^{j\frac{\pi}{4}},\ \frac{10}{9}e^{-j\frac{\pi}{4}}$





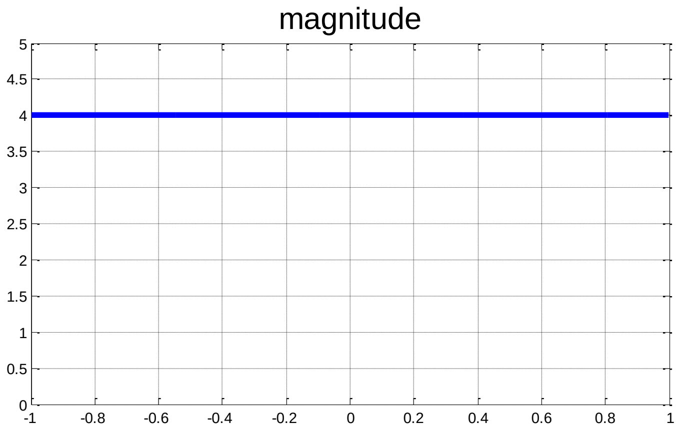


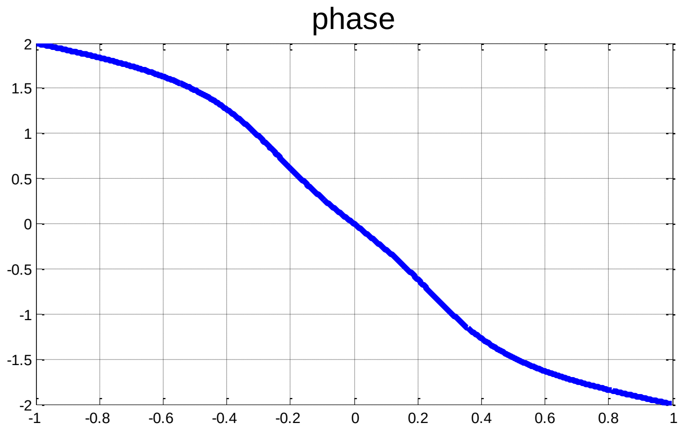


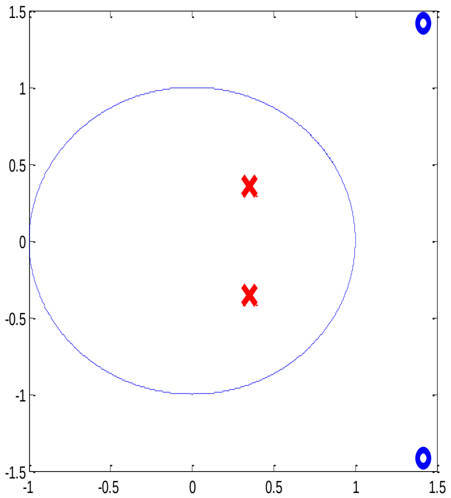
Ex: 2nd order, $H(z) = \frac{(1-z_1 z^{-1})(1-z_1^* z^{-1})}{(1-p_1 z^{-1})(1-p_1^* z^{-1})}$

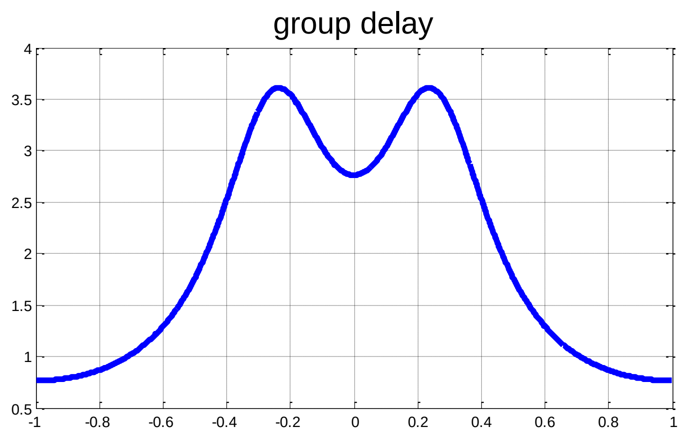
Poles: $0.5e^{j\frac{\pi}{4}},\ 0.5e^{-j\frac{\pi}{4}}$

Zeros: $2e^{j\frac{\pi}{4}},\ 2e^{-j\frac{\pi}{4}}$





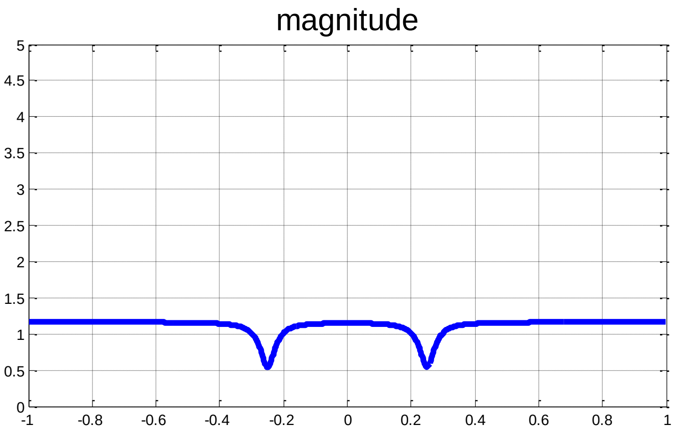


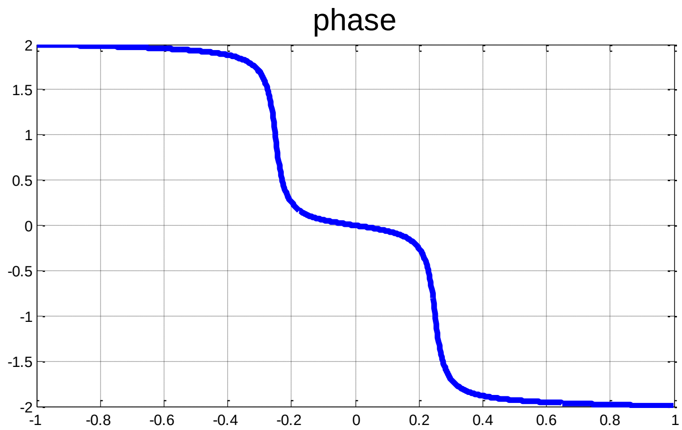


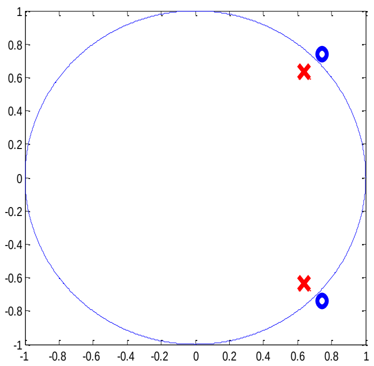
Ex: 2nd order, $H(z) = \frac{(1-z_1 z^{-1})(1-z_1^* z^{-1})}{(1-p_1 z^{-1})(1-p_1^* z^{-1})}$

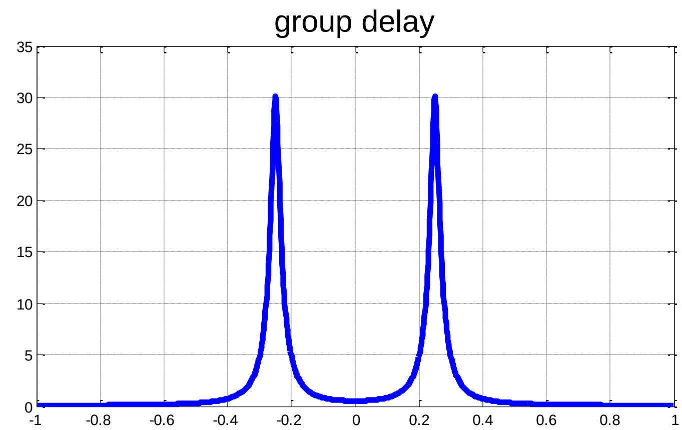
Poles: $0.9e^{j\frac{\pi}{4}},\ 0.9e^{-j\frac{\pi}{4}}$

Zeros: $1.05e^{j\frac{\pi}{4}},\ 1.05e^{-j\frac{\pi}{4}}$





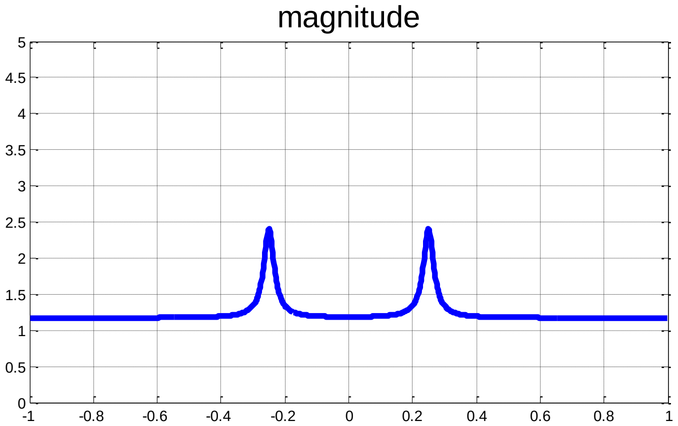


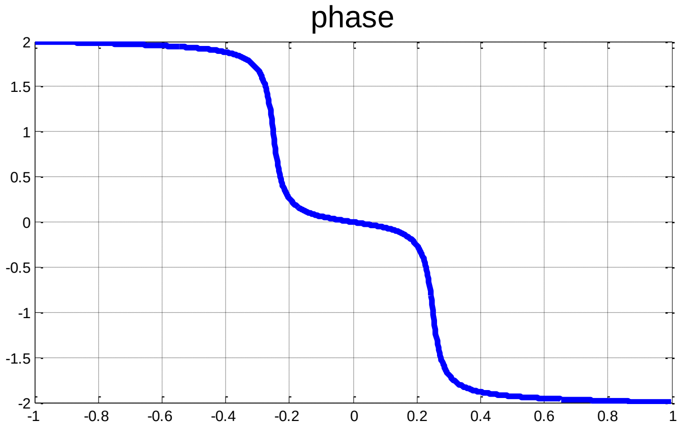


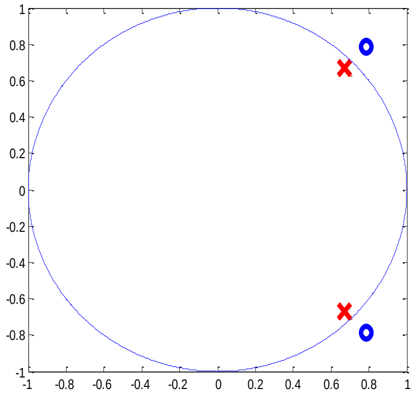
Ex: 2nd order, $H(z) = \frac{(1-z_1 z^{-1})(1-z_1^* z^{-1})}{(1-p_1 z^{-1})(1-p_1^* z^{-1})}$

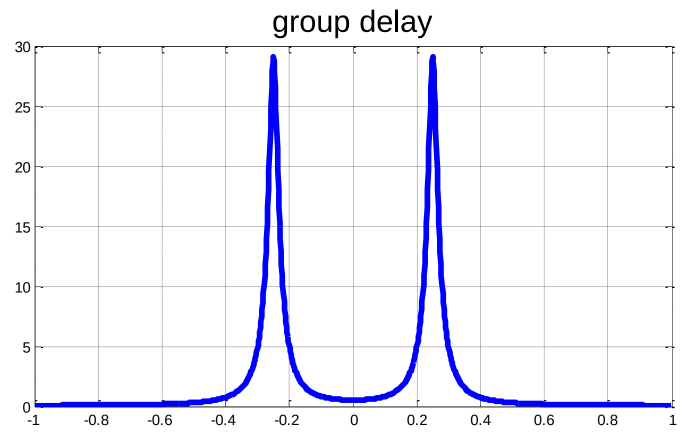
Poles: $0.95e^{j\frac{\pi}{4}},\ 0.95e^{-j\frac{\pi}{4}}$

Zeros: $\frac{10}{9}e^{j\frac{\pi}{4}},\ \frac{10}{9}e^{-j\frac{\pi}{4}}$

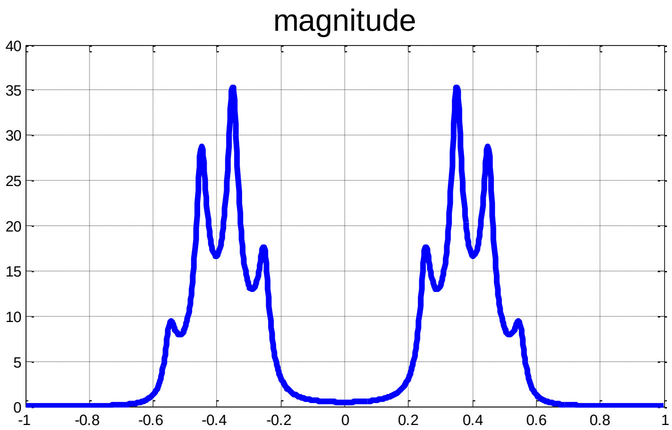


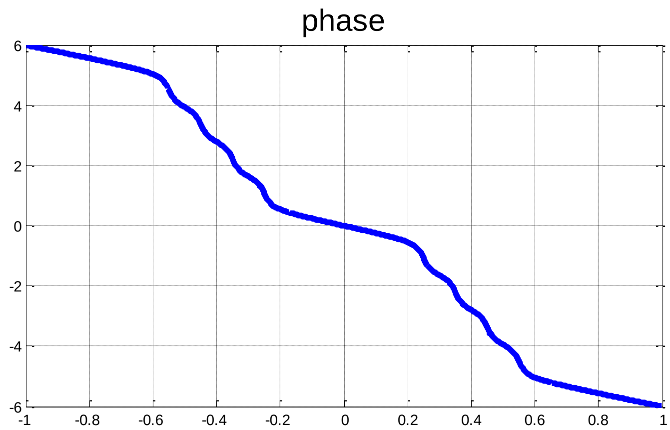


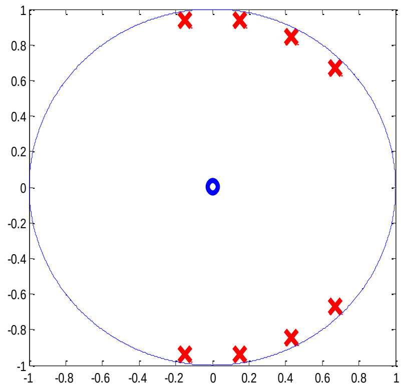


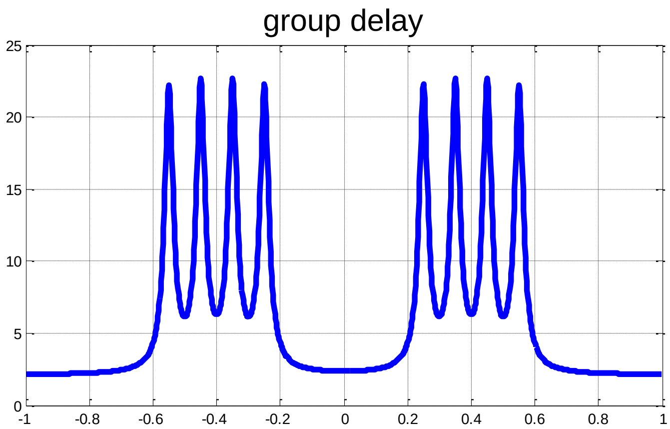


Ex: 8 poles 2 zeros

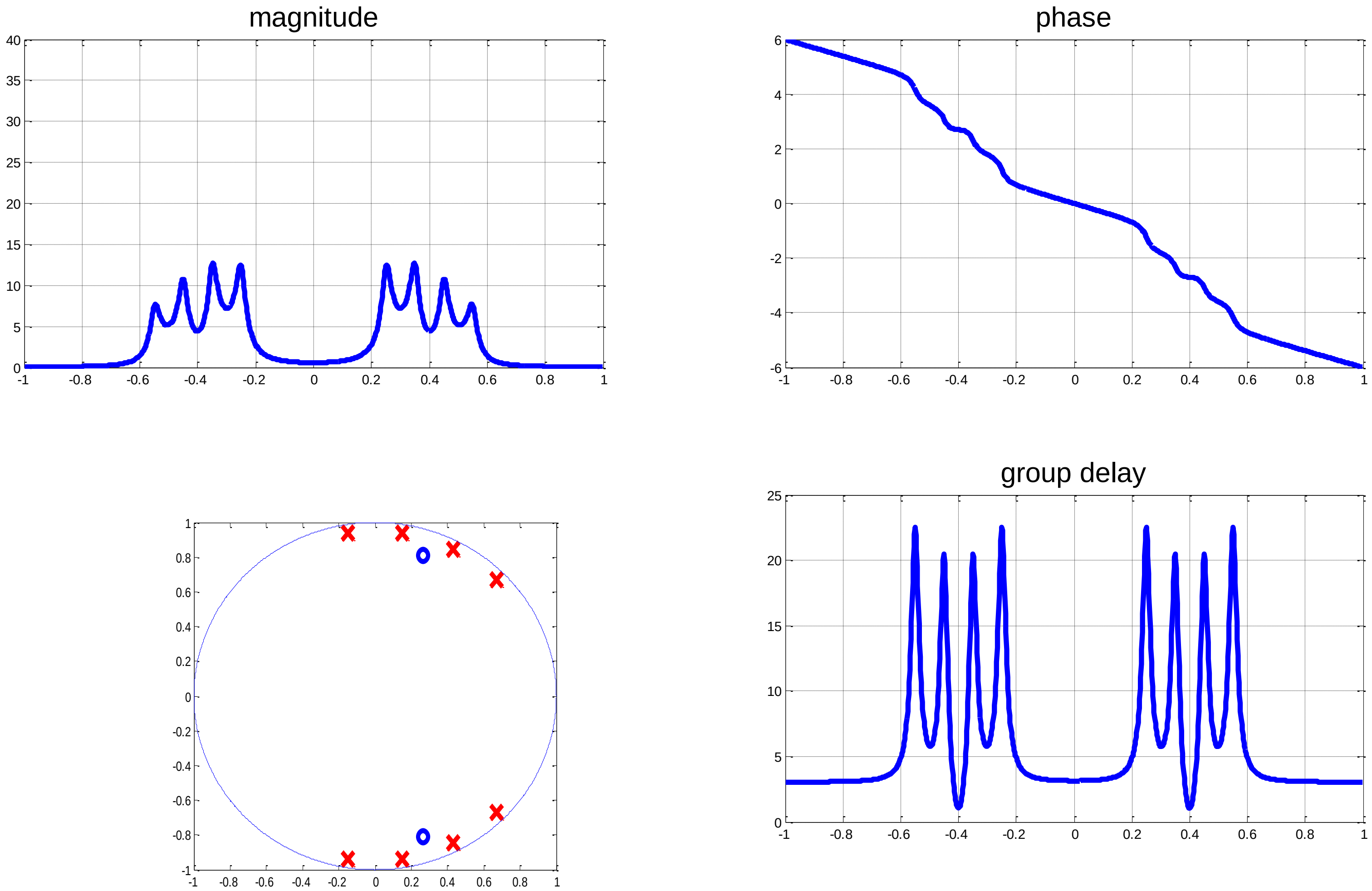








Ex: 8 poles 2 zeros



FREQUENCY RESPONSES OF CONJUGATE RECIPROCAL ZEROS (POLES)

CONJUGATE RECIPROCAL ZEROS

$$A(\omega) = \left(1 - ce^{-j\omega}\right)$$

$$B(\omega) = \left(1 - \frac{1}{c^*}e^{-j\omega}\right)$$

Notice that

$$B(\omega) = \left(1 - \frac{1}{c^*}e^{-j\omega}\right)$$

$$= -\frac{1}{c^*}e^{-j\omega}\left(1 - c^*e^{j\omega}\right)$$

Now, compare the **MAGNITUDES** of $A(\omega)$ and $B(\omega)$:

Since $\left(1 - c^*e^{j\omega}\right)$ and $\left(1 - ce^{-j\omega}\right)$ are conjugates of each other

$$|B(\omega)| = \left|-\frac{1}{c^*}\right| \left|e^{-j\omega}\right| |A(\omega)|$$

$$= \left|\frac{1}{c}\right| |A(\omega)|$$

RELATIONSHIP BETWEEN $A(z)$ AND $B(z)$

$$A^*\left(\frac{1}{z^*}\right) = 1 - c^* z$$

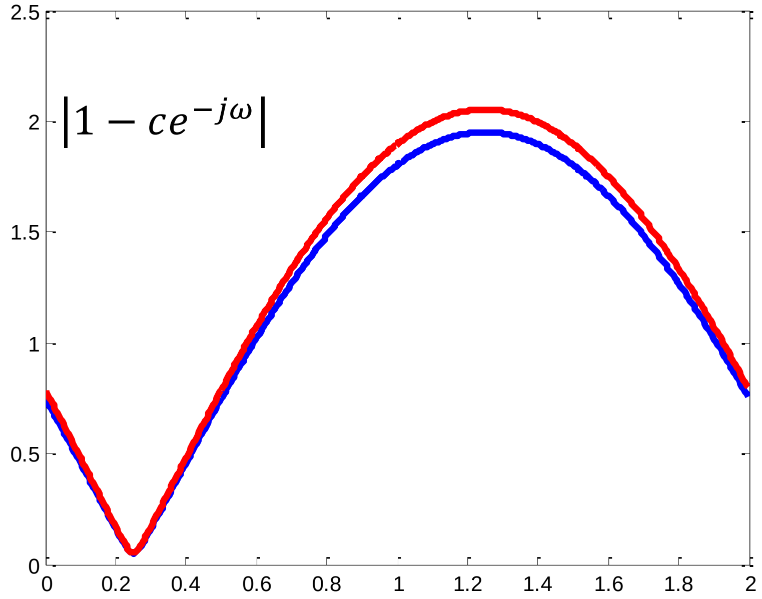
$$= -c^* z \left(1 - \frac{1}{c^*} z^{-1}\right)$$

$$= -c^* z \, B(z)$$

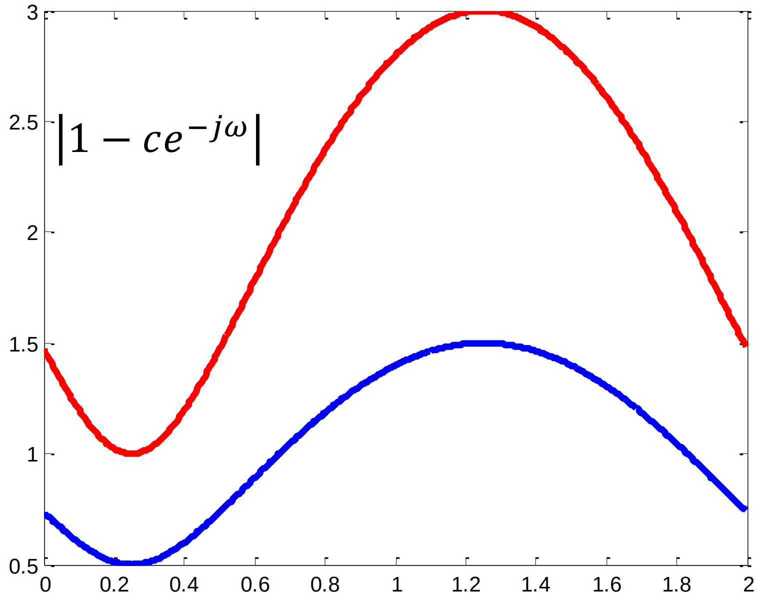
$$\Rightarrow \; B(z) = \frac{-1}{c^* z} A^*\left(\frac{1}{z^*}\right)$$

```
clear all
close all

c = 0.5 * exp(j*pi/4);
cij = 1/conj(c);

a = [1 -c];
b = [1 -cij];

[A, w] = freqz(a, 1, 4096, 'whole');

[B, w] = freqz(b, 1, 4096, 'whole');

plot(w/pi,abs(A), 'linewidth', 3);
hold
plot(w/pi,abs(B),'r', 'linewidth', 3)


figure
plot(w/pi,angle(A)/pi, 'linewidth', 3);
hold
plot(w/pi,angle(B)/pi,'r', 'linewidth', 3)

figure
[Grp_a,w] = grpdelay(a, 1, 4096, 'whole');
[Grp_b,w] = grpdelay(b, 1, 4096, 'whole');
plot(w/pi,Grp_a,'linewidth', 3)
hold
plot(w/pi,Grp_b,'r','linewidth', 3)
title('group delay, samples')
```

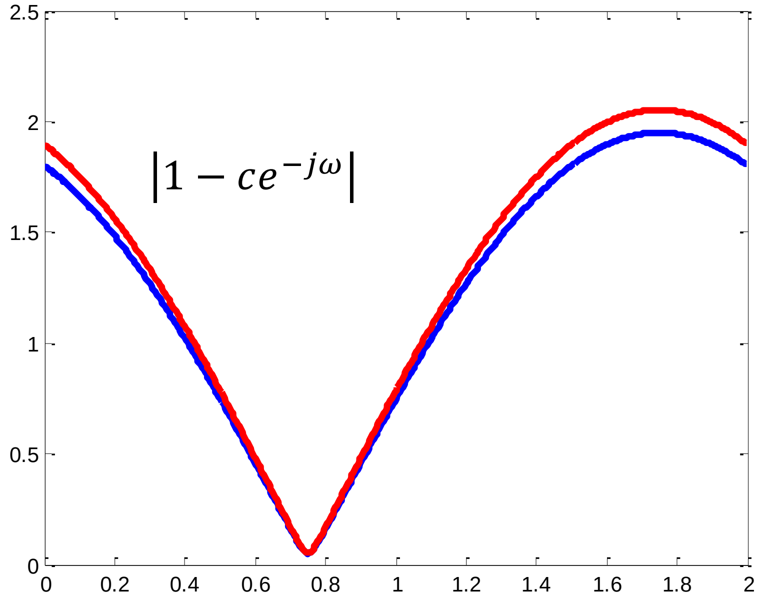
Ex: $c = 0.95e^{j\frac{\pi}{4}}$



Ex: $c = 0.5e^{j\frac{\pi}{4}}$



Ex: $c = 0.95e^{j\frac{3\pi}{4}}$



We can state that

$$\left(1 - ce^{-j\omega}\right) \text{ and } \left(e^{-j\omega} - c^*\right)$$

have the same magnitudes.

Hence,

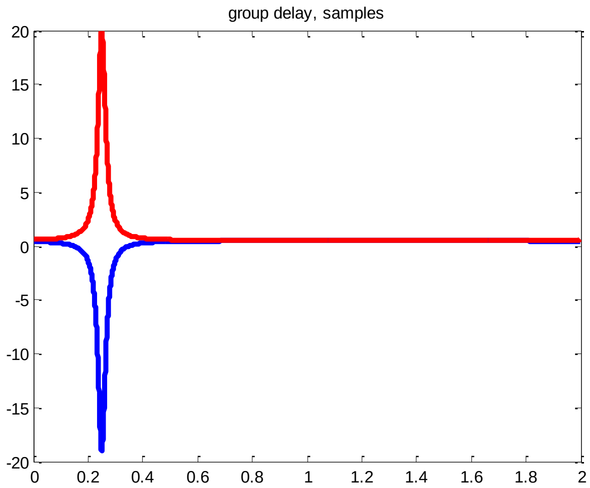
$$H\left(e^{j\omega}\right) = \frac{\left(e^{-j\omega} - c^*\right)}{\left(1 - ce^{-j\omega}\right)}$$

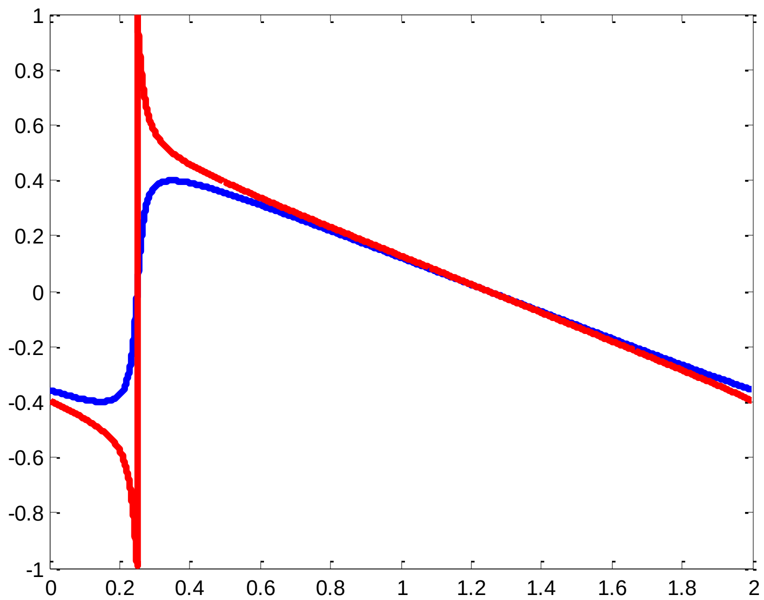
has an allpass characteristics.

PHASES AND GROUP DELAYS OF $A(\omega)$ AND $B(\omega)$

Ex:

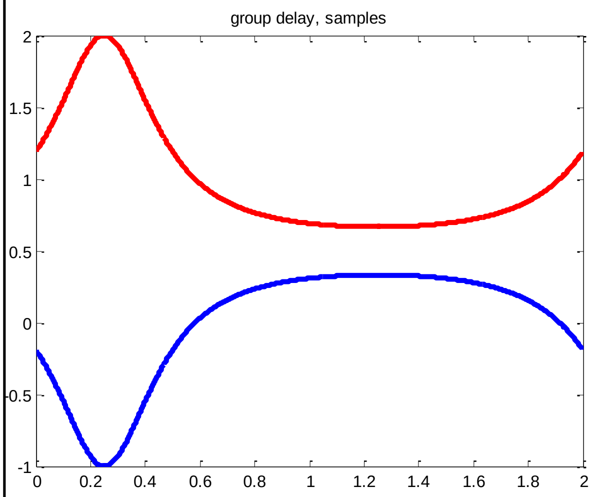
$$c = 0.95e^{j\frac{\pi}{4}}$$

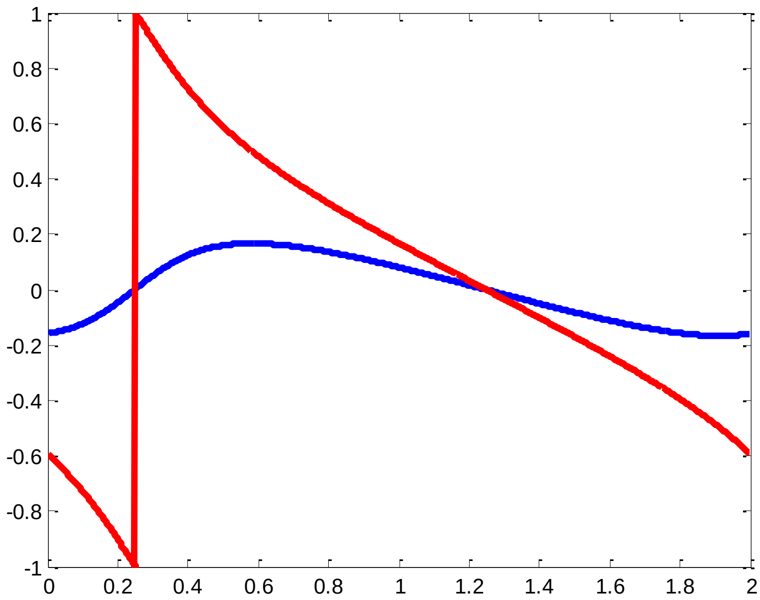




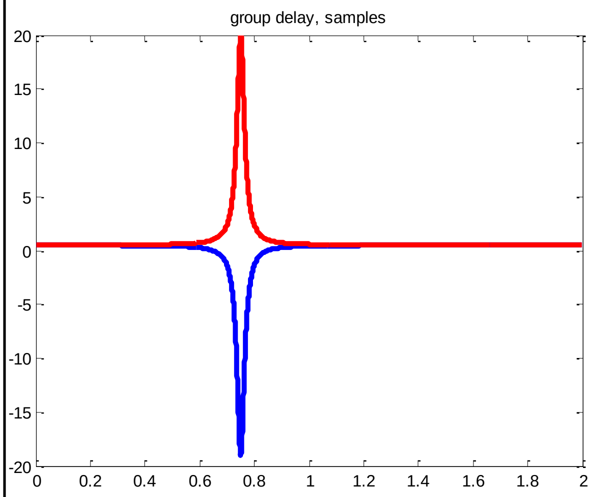
Ex:

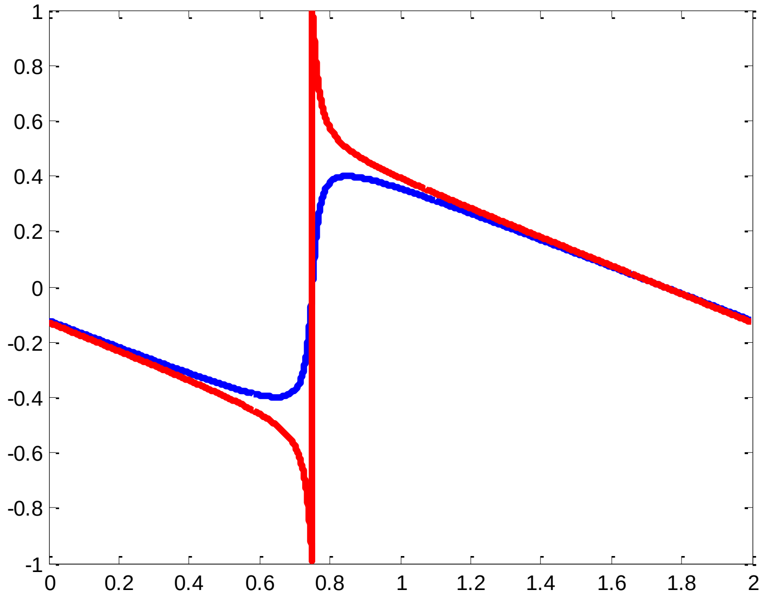
$$c = 0.5e^{j\frac{\pi}{4}}$$





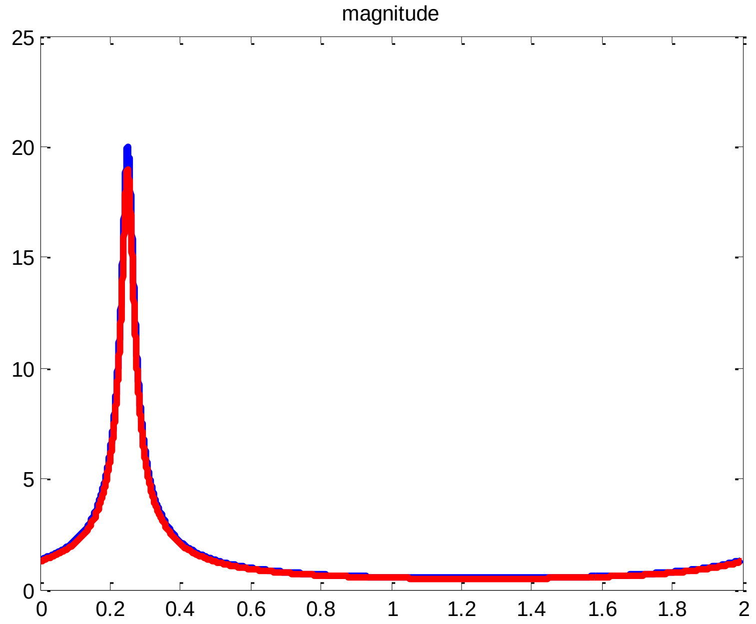
Ex: $c = 0.95e^{j\frac{3\pi}{4}}$

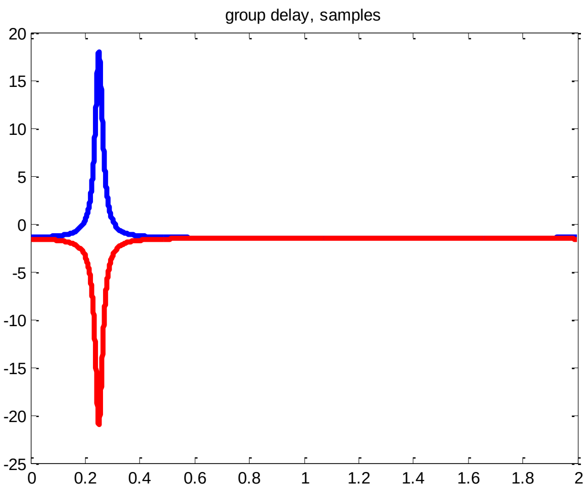


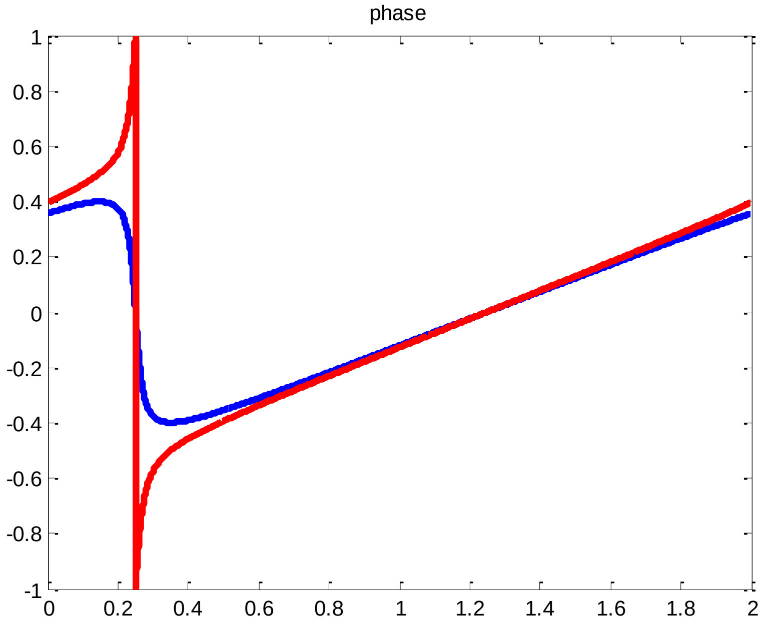


FREQUENCY RESPONSES OF $\frac{1}{A(\omega)}$ AND $\frac{1}{B(\omega)}$

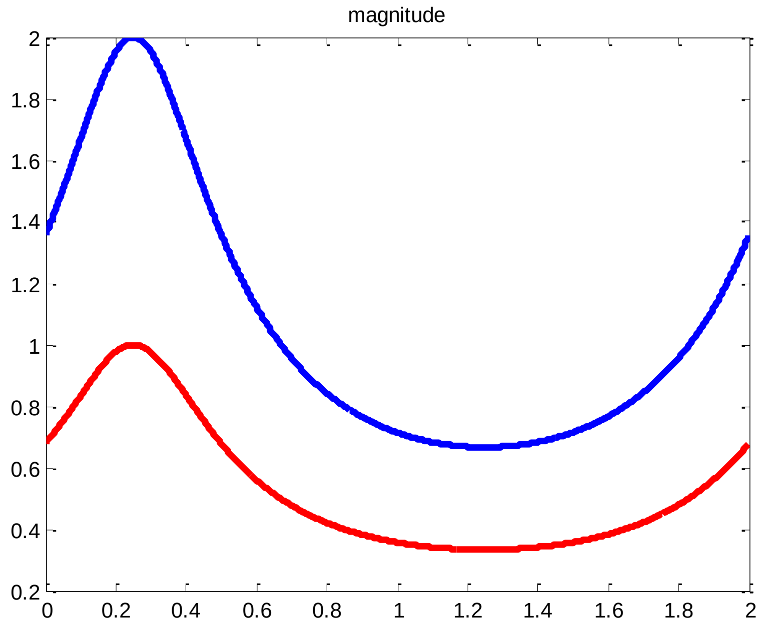
Ex: $c = 0.95e^{j\frac{\pi}{4}}$

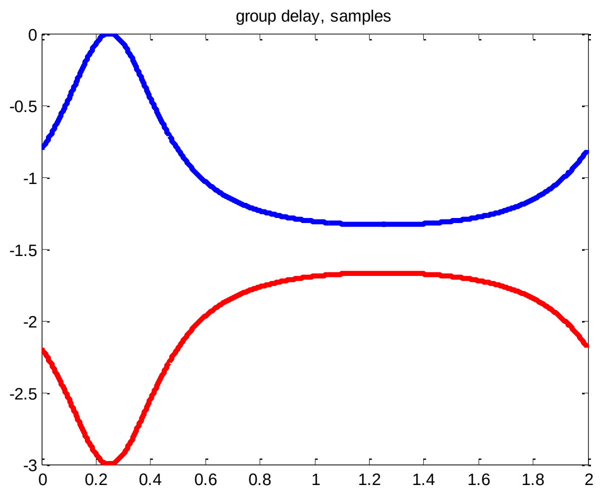


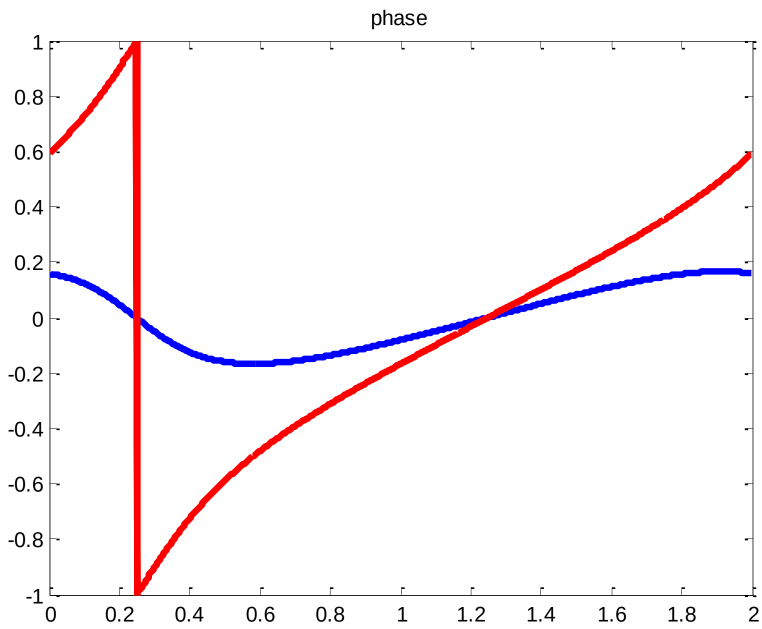




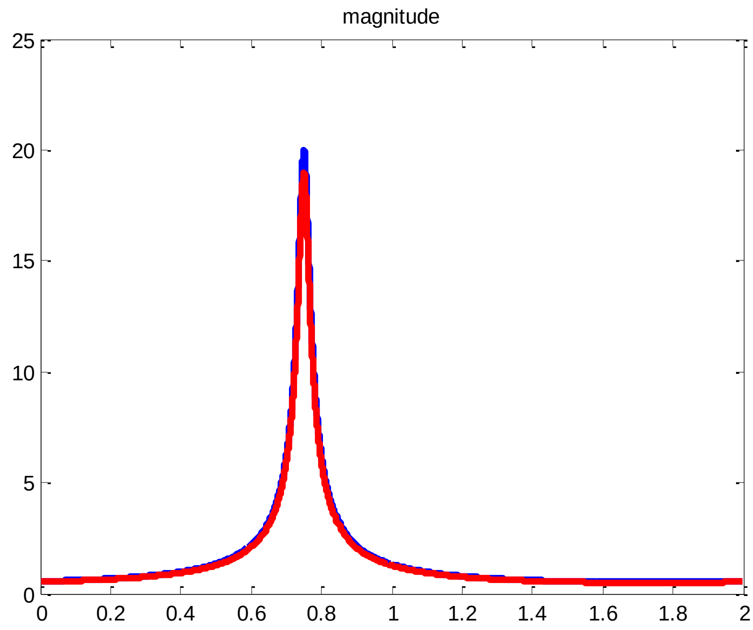
Ex: $c = 0.5e^{j\frac{\pi}{4}}$

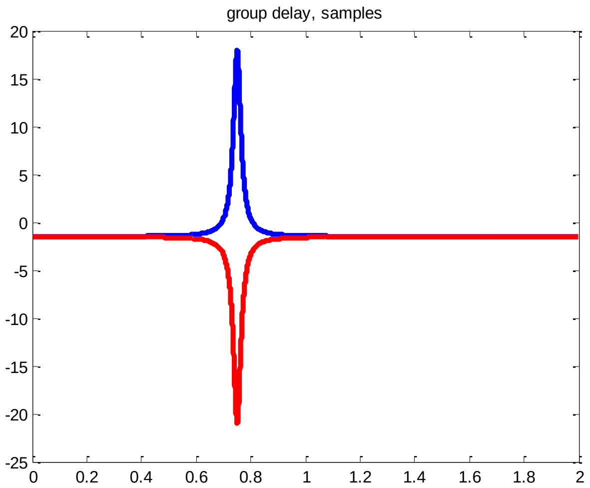


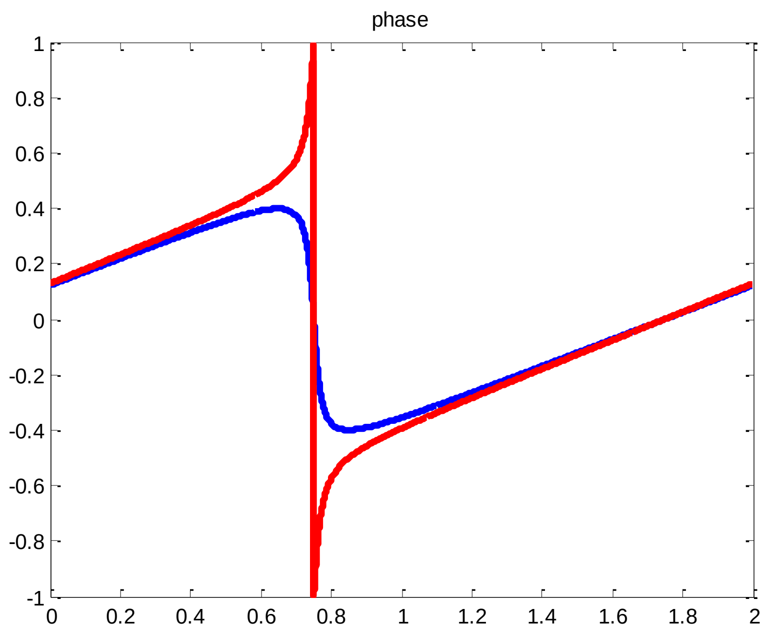




Ex: $c = 0.95e^{j\frac{3\pi}{4}}$







RELATIONSHIP BETWEEN THE MAGNITUDE AND THE PHASE OF A FREQUENCY RESPONSE FUNCTION

For LTI systems with a rational $H(z)$

If the magnitude (phase) of the frequency response and the number of poles and zeros are known there are a limited number of choices for the phase (magnitude) of the frequency response

$H(z)$ AND $H^*\left(\frac{1}{z^*}\right)$ YIELD THE SAME MAGNITUDE RESPONSE

since

$$H^*\left(\frac{1}{z^*}\right)\bigg|_{z=e^{j\omega}} = H^*\left(e^{j\omega}\right)$$

and

$$\left|H^*\left(e^{j\omega}\right)\right| = \left|H\left(e^{j\omega}\right)\right|$$

THE POLES AND ZEROS OF $H^*\left(\frac{1}{z^*}\right)$
ARE THE *CONJUGATE-RECIPROCAL*'S
OF THOSE OF $H(z)$

If z_0 is a zero of $H(z)$, then $\frac{1}{z_0^*}$ is a zero of $H^*\left(\frac{1}{z^*}\right)$.

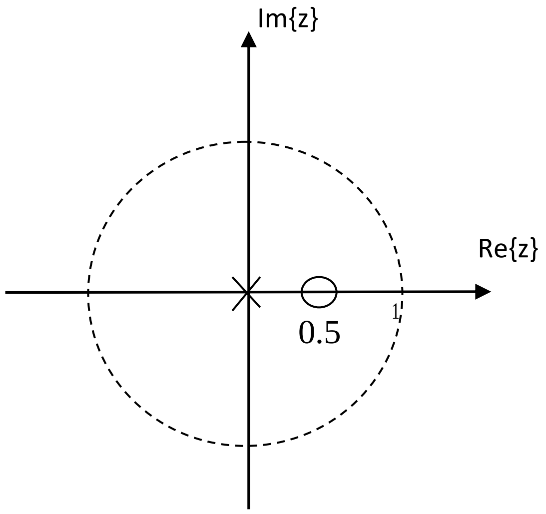
If p_0 is a pole of $H(z)$, then $\frac{1}{p_0^*}$ is a pole of $H^*\left(\frac{1}{z^*}\right)$.

Ex:

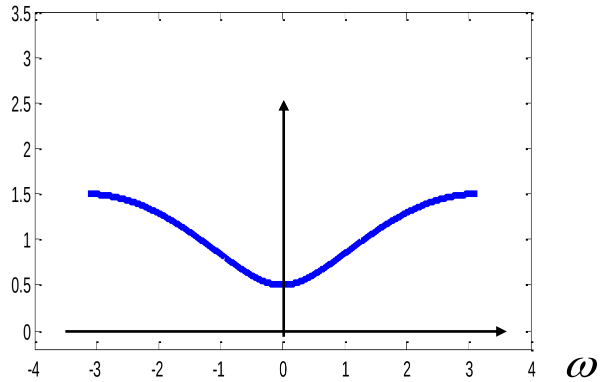
$$H(z) = 1 - 0.5z^{-1}$$

zero at $z = 0.5$ pole at $z = 0$

$$H(e^{j\omega}) = 1 - 0.5e^{-j\omega}$$



$$\left|1 - 0.5e^{-j\omega}\right|$$

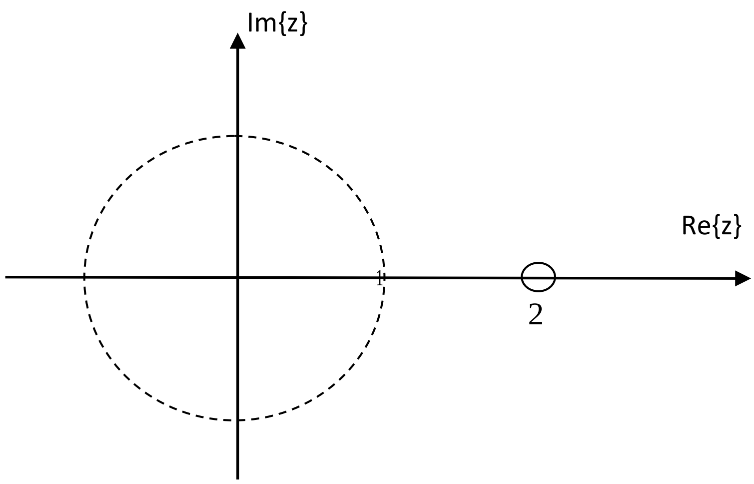


On the other hand,

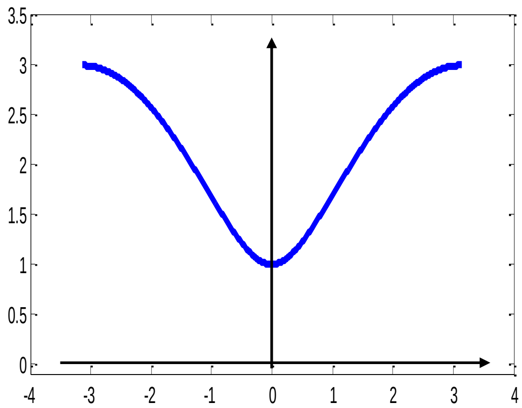
$$H^*\left(\frac{1}{z^*}\right) = 1 - 0.5z \qquad = -0.5z(1 - 2z^{-1})$$

zero at $z = 2$ pole at $z \to \infty$

$$H^*(e^{j\omega}) = 1 - 0.5e^{j\omega} \qquad = -0.5e^{j\omega}(1 - 2e^{-j\omega})$$



$$\left|1 - 2e^{-j\omega}\right|$$



In general,

$$H(z) = \frac{b_0}{a_0} \frac{\prod_{k=1}^{M}(1 - c_k z^{-1})}{\prod_{k=1}^{N}(1 - d_k z^{-1})}$$

$$H^*\left(\frac{1}{z^*}\right) = \frac{b_0}{a_0} \frac{\prod_{k=1}^{M}(1 - c_k^* z)}{\prod_{k=1}^{N}(1 - d_k^* z)}$$

$$C(z) = H(z)\, H^*\left(\frac{1}{z^*}\right)$$

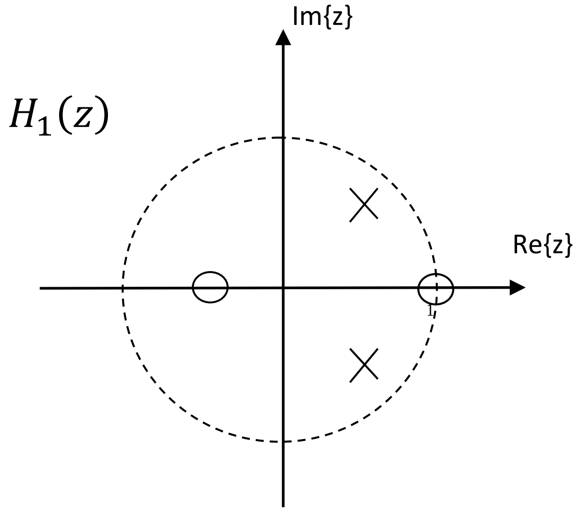
$$C\left(e^{j\omega}\right) = \left|H\left(e^{j\omega}\right)\right|^2$$

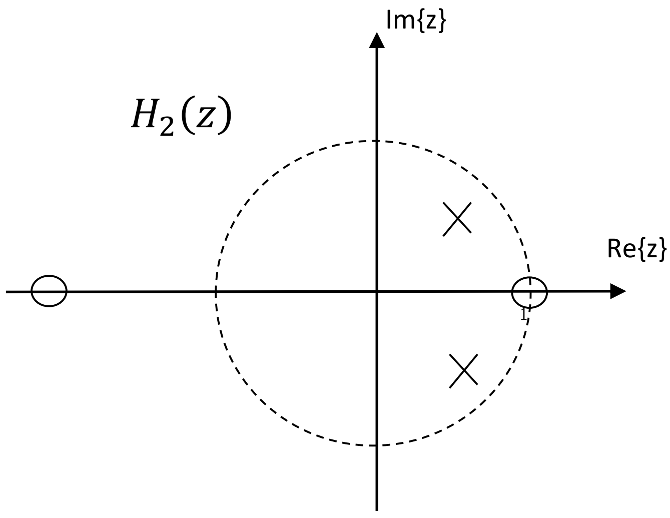
- Given $\left|H\left(e^{j\omega}\right)\right|^2$, one can construct $C(z)$.
  - Express $\left|H\left(e^{j\omega}\right)\right|^2$ in terms of $e^{j\omega}$
  - Replace $e^{j\omega} by\ z$

- Then, allocate the poles and zeros of $C(z)$ to its factors, $H(z)$ and $H^*\left(\frac{1}{z^*}\right)$.

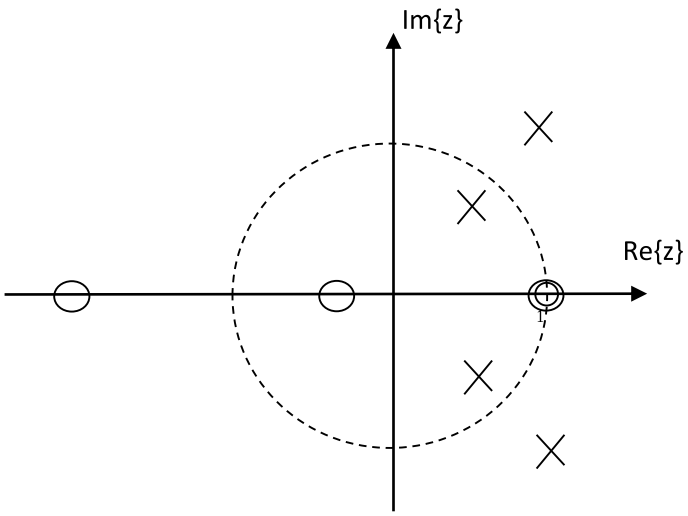
- There will be a finite number of choices for $H(z)$ and $H^*\left(\frac{1}{z^*}\right)$.

Ex:





$$C(z) = H_1(z)H_1^*\left(\frac{1}{z^*}\right) = H_2(z)H_2^*\left(\frac{1}{z^*}\right)$$



Ex: Let

$$H(z) = \frac{\left(1-\frac{1}{2}z^{-1}\right)\left(1+\frac{3}{2}z^{-1}\right)\left(1-\frac{7}{5}e^{j\frac{3\pi}{4}}\right)\left(1-\frac{7}{5}e^{-j\frac{3\pi}{4}}\right)}{\left(1+\frac{1}{3}z^{-1}\right)\left(1+\frac{2}{5}z^{-1}\right)\left(1-\frac{4}{5}e^{j\frac{3\pi}{4}}\right)\left(1-\frac{4}{5}e^{-j\frac{3\pi}{4}}\right)}$$

a) How many systems with 4 poles-zeros (real or complex, stable or unstable, causal or noncausal) have the same magnitude?
b) Write one of them.
c) How many of them are real, and causal and stable? Write one of them.

ALLPASS SYSTEMS

An LTI system is called allpass if

$$\left|H\left(e^{j\omega}\right)\right| = 1$$

OBTAINING A FIRST ORDER ALLPASS SYSTEM

Let

$$A(z) = 1 - az^{-1} \qquad |a| < 1$$

One can obtain a first order n allpass system function as

$$\frac{A^*\left(\frac{1}{z^*}\right)}{A(z)} = \frac{1 - a^*z}{1 - az^{-1}}$$

IMPULSE RESPONSE

Taking the ROC as $|z| > |a|$

$$h[n] = a^n u[n] - a^* a^{n+1} u[n+1]$$

$$= -a^* \delta[n+1\,] + (1 - |a|^2) a^n u[n]$$

which is stable but noncausal.

CAUSAL AND CANONICAL FORM

To have it causal, a first order allpass is defined as

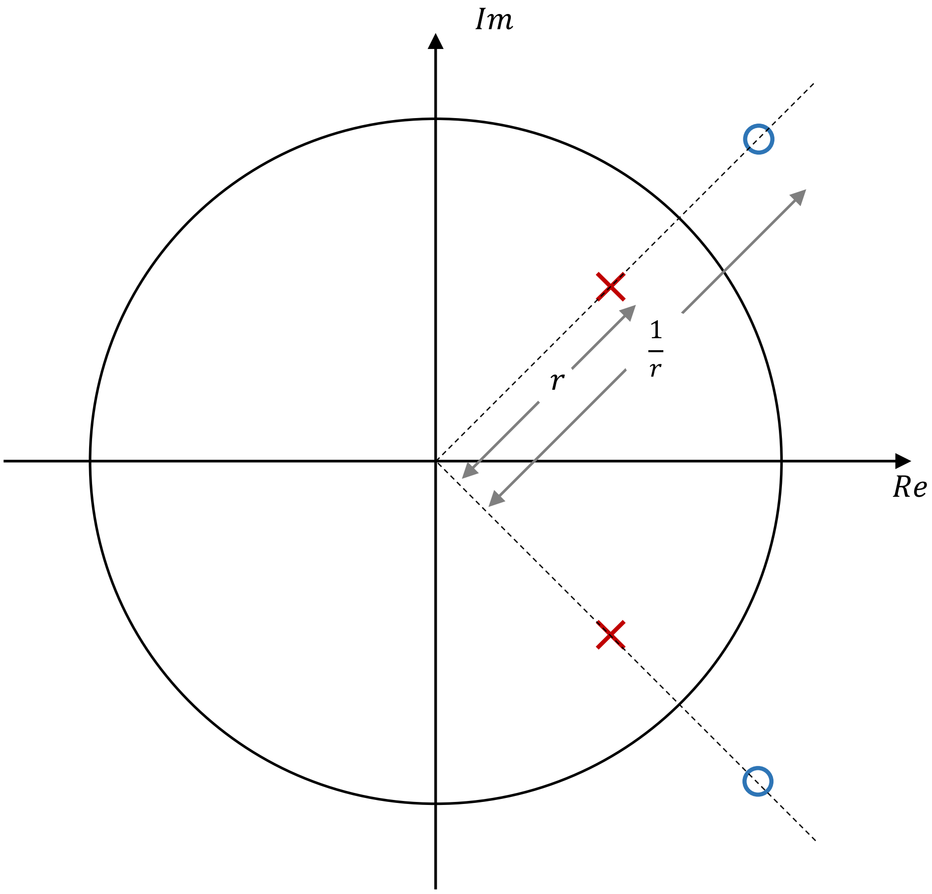
$$H_{ap}\,(z) = z^{-1}\,\frac{1 - a^{*}z}{1 - az^{-1}}$$

$$= \frac{z^{-1} - a^{*}}{1 - az^{-1}}$$

In general

$$H_{ap}(z) = A\prod_{k=1}^{M_r}\frac{z^{-1}-d_k}{1-d_k z^{-1}}\prod_{k=1}^{M_c}\frac{(z^{-1}-e_k^*)(z^{-1}-e_k)}{(1-e_k^* z^{-1})(1-e_k z^{-1})}$$

The poles and zeros of an allpass system are at conjugate reciprocal locations.



Therefore, causal, stable allpass systems have all zeros outside the unit circle.

Ex:

a) Two LTI systems have the transfer functions $H(z)$ and $H^*\left(\frac{1}{z^*}\right)$. Assume that the ROCs of both system functions include the unit circle. Show that the magnitudes of the frequency responses of these systems are the same.

b) Let $H(z) = 1 - \frac{1}{2}z^{-1}$.

Consider the all-pass system function $G(z) = \frac{H^*\left(\frac{1}{z^*}\right)}{H(z)}$.

i) Find the poles and zeros of $G(z)$.

ii) Find the impulse response of the stable system represented by $G(z)$.

iii) Write the difference equation for $G(z)$. Is a causal and stable implementation possible?

If yes, why?

If no, find another first-order, all-pass transfer function $M(z)$ having a pole at $z = \frac{1}{2}$ by modifying $G(z)$, so that causal and stable implementation is possible.

PHASE AND GROUP DELAY FUNCTIONS OF ALLPASS SYSTEMS

1st order

$$H_{ap}(z) = \frac{z^{-1} - a^*}{1 - az^{-1}}$$

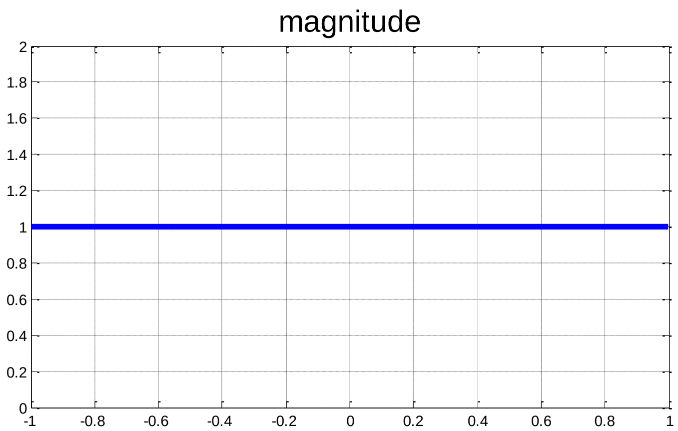
$$a = re^{j\theta}$$

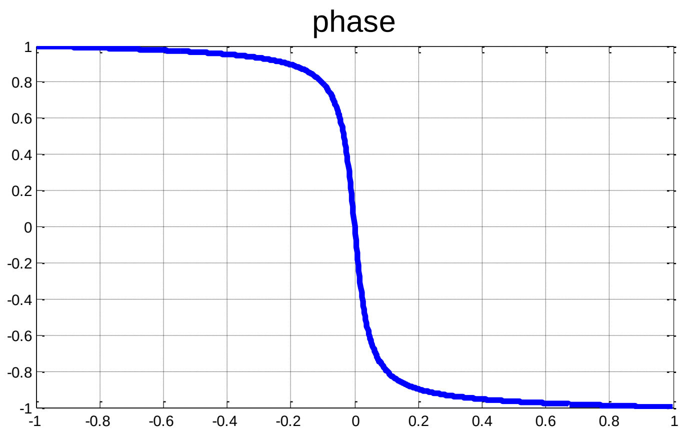
$$\begin{aligned} H_{ap}(e^{j\omega}) &= \frac{e^{-j\omega} - a^*}{1 - ae^{-j\omega}} \\ &= e^{-j\omega} \frac{1 - a^* e^{j\omega}}{1 - ae^{-j\omega}} \end{aligned}$$

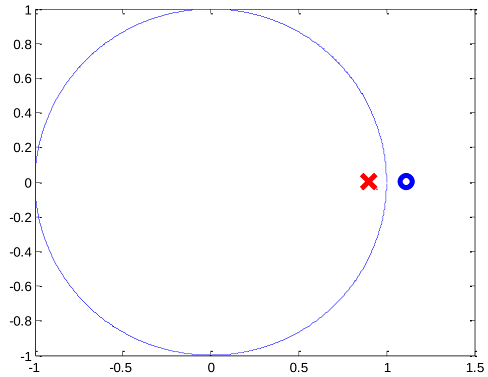
Phase

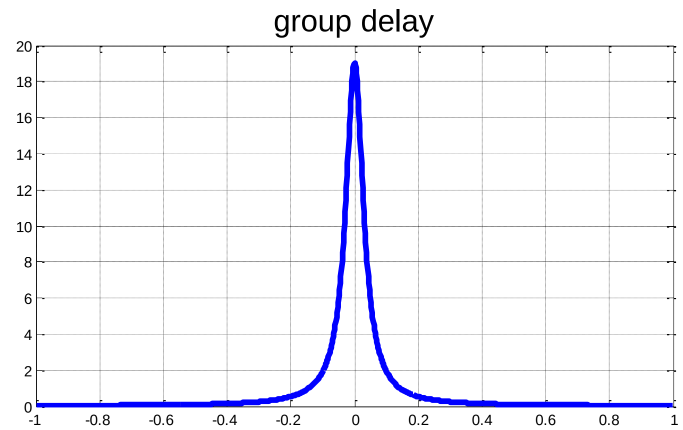
$$\begin{aligned} \measuredangle H(e^{j\omega}) &= -\omega + \measuredangle(1 - a^* e^{j\omega}) - \measuredangle(1 - ae^{-j\omega}) \\ &= -\omega + 2\measuredangle(1 - a^* e^{j\omega}) \\ &= -\omega - 2\tan^{-1} \frac{r\sin(\omega - \theta\ )}{1 - r\cos(\omega - \theta\ )} \end{aligned}$$

Ex: $r = 0.9 \quad \theta = 0$

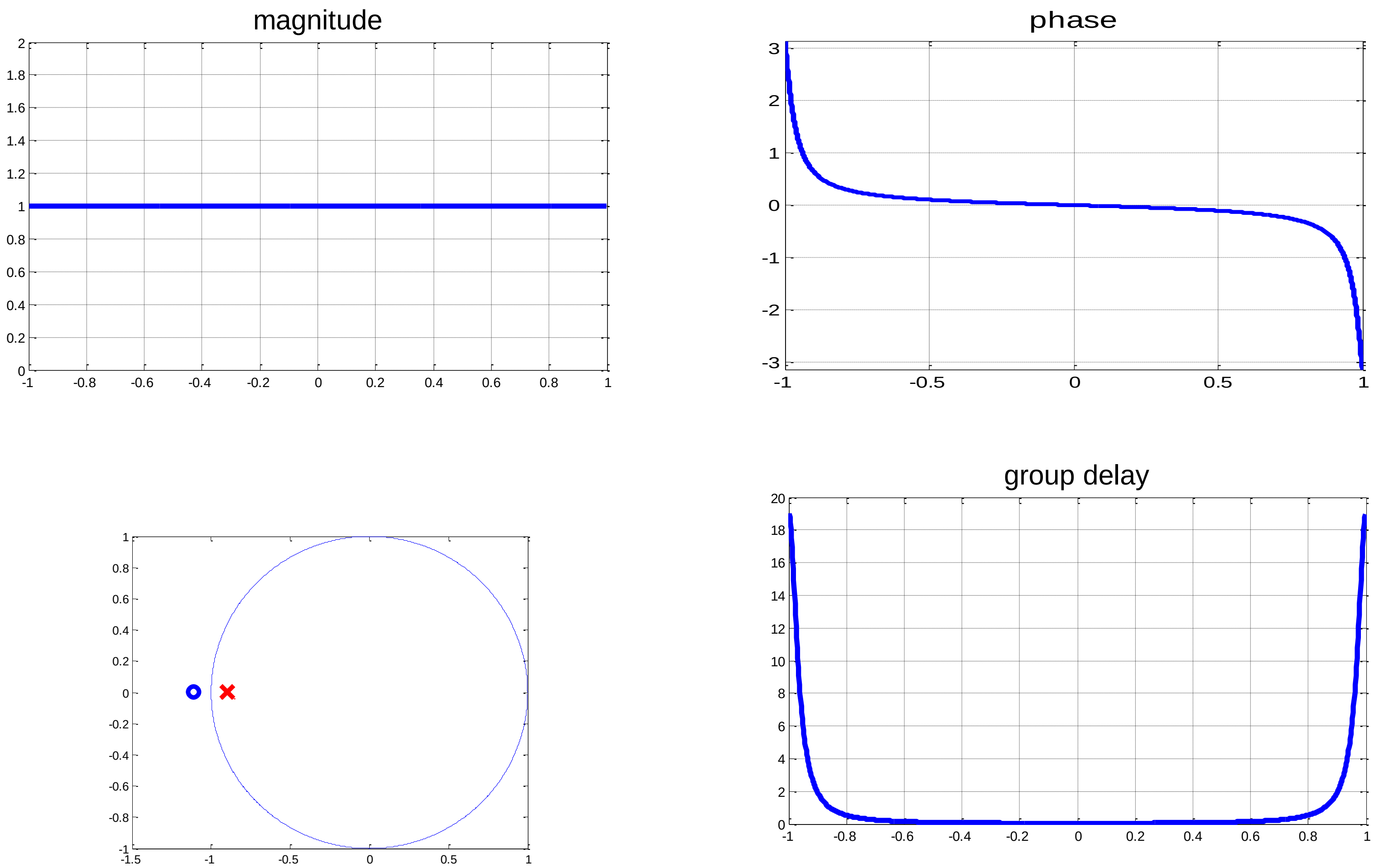




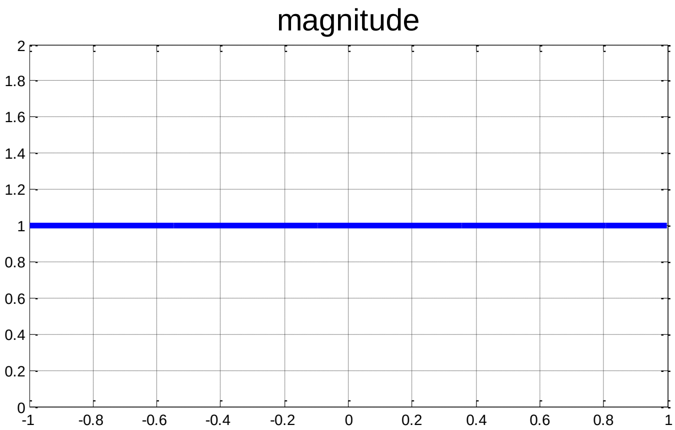


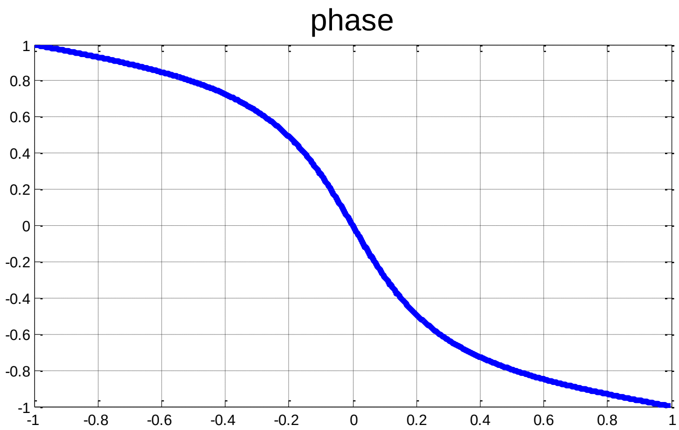


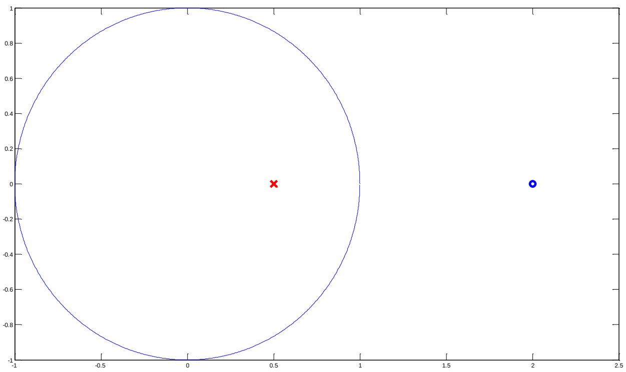
Ex: $r = 0.9 \quad \theta = \pi$

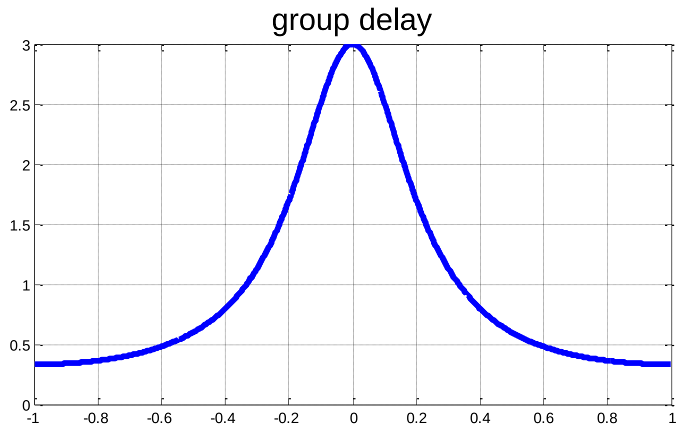


Ex: $r = 0.5 \quad \theta = 0$





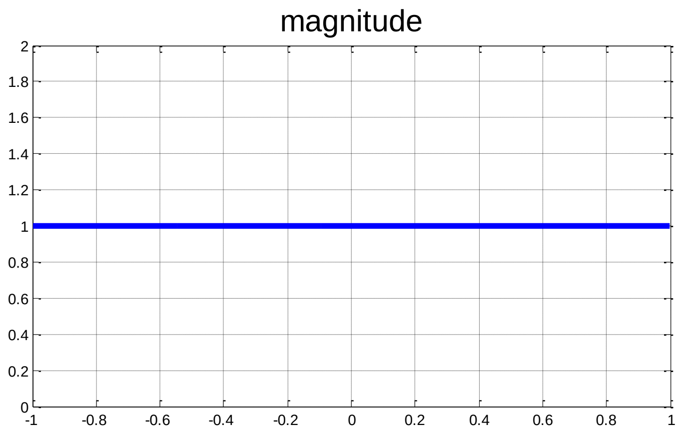


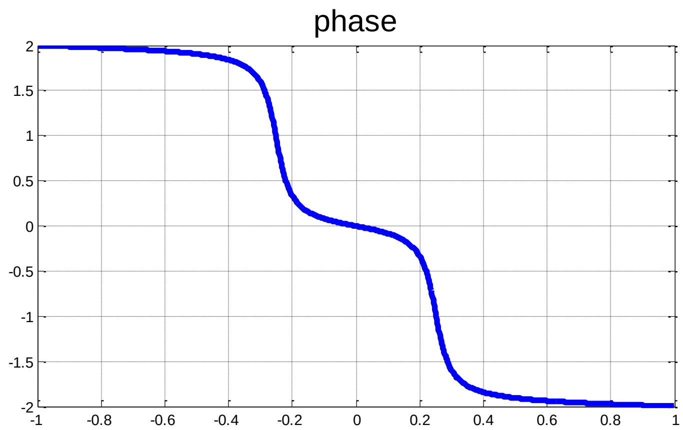


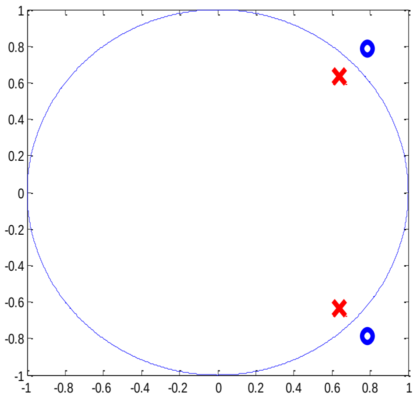
2nd order

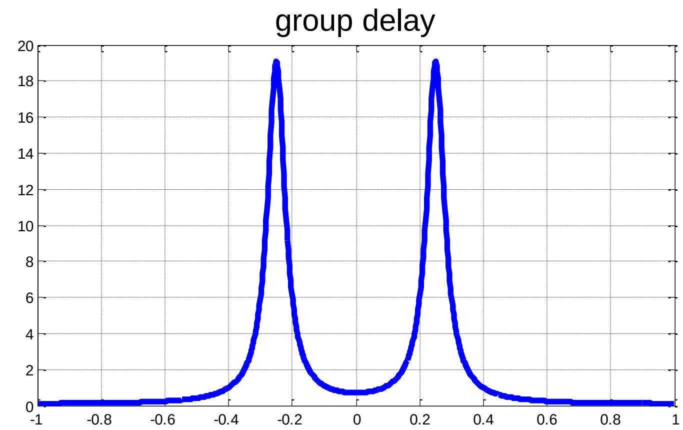
$$H_{ap}(z) = \frac{z^{-1} - a^*}{1 - az^{-1}} \frac{z^{-1} - a}{1 - a^* z^{-1}} \qquad\qquad a = re^{j\theta}$$

$$r = 0.9 \quad \theta = \frac{\pi}{4}$$









GROUP DELAY OF ALLPASS SYSTEMS

$$\tau_{gr}(\omega) = -\frac{d\measuredangle H(e^{j\omega})}{d\omega}$$
$$= \frac{1-r^2}{|1-re^{j\theta}e^{-j\omega}|^2} \qquad (see\ end\ of\ notes)$$

Group delay is positive since $|r| < 1$. (This is true for any order!)

Positivity of group delay implies that

$$\measuredangle H(e^{j\omega}) \leq 0 \qquad 0 \leq \omega \leq \pi$$

since

$$\measuredangle H(e^{j\omega}) = -\int_0^\omega \tau_{gr}(\alpha)d\alpha + \underbrace{\measuredangle H(e^{j0})}_{\leq 0} \qquad 0 \leq \omega \leq \pi$$

Therefore, phase response of allpass systems is negative and monotonically decreasing over $0 \leq \omega \leq \pi$.

(This is true for any order!)

MINIMUM PHASE SYSTEMS

Definition: A causal and stable LTI system is said to be minimum-phase (maximum-phase) if all of its zeros are inside (outside) the unit circle.

(Note the restriction to CAUSAL and STABLE systems)

(MAXIMUM PHASE CLARIFICATION $H_{max}H_{ap} = \widehat{H}_{max}$)

Inverse of a minimum-phase system has causal and stable implementation.

Why?

Any (causal and stable) system function can be decomposed as

$$H(z) = H_{min}(z)H_{ap}(z)$$

where $H_{min}(z)$ is a minimum phase system and $H_{ap}(z)$ is a causal-stable allpass system.

To see that the last claim holds, consider a causal and stable system, $H(z)$ with one of its zeros outside the unit circle.

Let $\frac{1}{z_0^*}$ be the zero outside.

Then, $H(z)$ can be expressed as

$$H(z) = \widehat{H}(z)(z^{-1} - z_0^*)$$

Then,

$$H(z) = \widehat{H}(z)(z^{-1} - z_0^*)\frac{(1 - z_0^* z^{-1})}{(1 - z_0^* z^{-1})}$$

$$= \underbrace{\widehat{H}(z)(1 - z_0^* z^{-1})}_{H_{min}(z)} \underbrace{\frac{(z^{-1} - z_0^*)}{(1 - z_0^* z^{-1})}}_{H_{ap}(z)}$$

Above development can be generalized to any number of zeros outside the unit circle.

Verbally,

1) for every zero outside the unit circle, add a pole and a zero which are conjugate reciprocals of that zero.
2) Then separate the minimum phase and allpass parts.

Ex: Find the minimum-phase and allpass components of

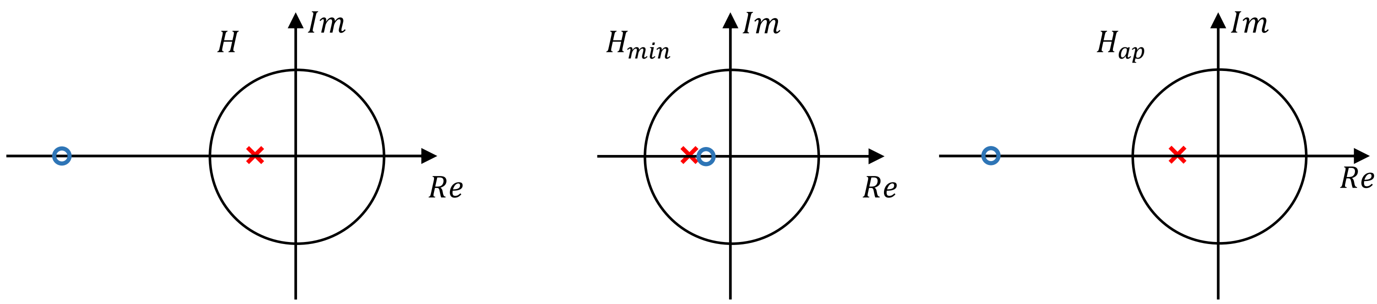
$$H(z) = \frac{1 + 3z^{-1}}{1 + \frac{1}{2}z^{-1}} \; .$$

$$H(z) = \frac{1 + 3z^{-1}}{1 + \frac{1}{2}z^{-1}}$$

$$= \frac{1}{1 + \frac{1}{2}z^{-1}} 3\left(\frac{1}{3} + z^{-1}\right)$$

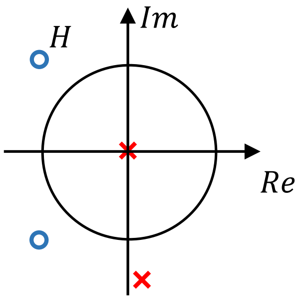
$$= \frac{1}{1 + \frac{1}{2}z^{-1}} 3\left(z^{-1} + \frac{1}{3}\right) \frac{\left(1 + \frac{1}{3}z^{-1}\right)}{\left(1 + \frac{1}{3}z^{-1}\right)}$$

$$= 3 \underbrace{\frac{\left(1 + \frac{1}{3}z^{-1}\right)}{1 + \frac{1}{2}z^{-1}}}_{H_{min}(z)} \; \underbrace{\frac{\left(z^{-1} + \frac{1}{3}\right)}{\left(1 + \frac{1}{3}z^{-1}\right)}}_{H_{ap}(z)}$$



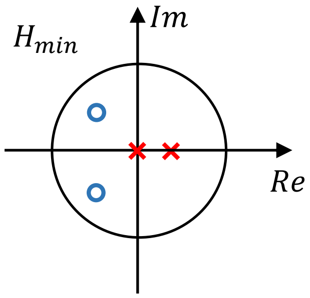
Ex: Find the minimum-phase and allpass components of

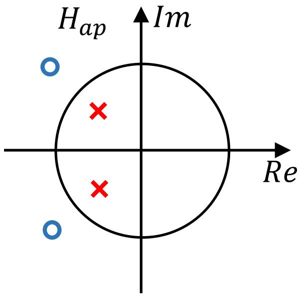
$$H(z)=\frac{\left(1-\frac{3}{2}e^{j\frac{3\pi}{4}}z^{-1}\right)\left(1-\frac{3}{2}e^{-j\frac{3\pi}{4}}z^{-1}\right)}{1-\frac{1}{3}z^{-1}}.$$



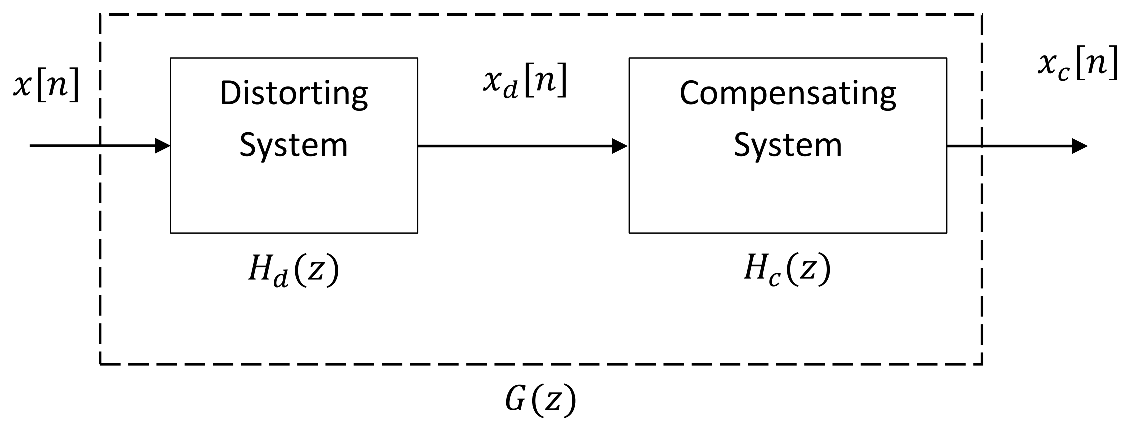
$$H(z)=\frac{\frac{9}{4}\left(z^{-1}-\frac{2}{3}e^{-j\frac{3\pi}{4}}\right)\left(z^{-1}-\frac{2}{3}e^{j\frac{3\pi}{4}}\right)}{1-\frac{1}{3}z^{-1}}$$

$$=\underbrace{\frac{\frac{9}{4}\left(1-\frac{2}{3}e^{-j\frac{3\pi}{4}}z^{-1}\right)\left(1-\frac{2}{3}e^{j\frac{3\pi}{4}}z^{-1}\right)}{1-\frac{1}{3}z^{-1}}}_{H_{min}}\ \underbrace{\frac{\left(z^{-1}-\frac{2}{3}e^{j\frac{3\pi}{4}}\right)}{\left(1-\frac{2}{3}e^{-j\frac{3\pi}{4}}z^{-1}\right)}\ \frac{\left(z^{-1}-\frac{2}{3}e^{-j\frac{3\pi}{4}}\right)}{\left(1-\frac{2}{3}e^{j\frac{3\pi}{4}}z^{-1}\right)}}_{H_{ap}}$$





FREQUENCY RESPONSE MAGNITUDE COMPENSATION



If $H_d(z)$ is a minimum phase system, then $H_c(z)$ can be taken as

$$H_c(z) = \frac{1}{H_d(z)}$$

since $\frac{1}{H_d(z)}$ has a causal and stable realization.

Otherwise, one possibility is to set

$$H_c(z) = \frac{1}{H_{d,min}(z)}$$

where $H_{d,min}(z)$ satisfies

$$H_d(z) = H_{d,min}(z) H_{d,ap}(z) .$$

Such a choice yields

$$\left|G\left(e^{j\omega}\right)\right| = 1 ,$$

i.e., magnitude distortion is compensated.

Ex: Let

$$H_d(z) = \left(1 - \frac{9}{10}e^{j\frac{3\pi}{5}}z^{-1}\right)\left(1 - \frac{9}{10}e^{-j\frac{3\pi}{5}}z^{-1}\right)\left(1 - \frac{5}{4}e^{j\frac{4\pi}{5}}z^{-1}\right)\left(1 - \frac{5}{4}e^{-j\frac{4\pi}{5}}z^{-1}\right)$$

$$H_{d,min}(z) = \frac{25}{16}\left(1 - \frac{9}{10}e^{j\frac{3\pi}{5}}z^{-1}\right)\left(1 - \frac{9}{10}e^{-j\frac{3\pi}{5}}z^{-1}\right)\left(1 - \frac{4}{5}e^{j\frac{4\pi}{5}}z^{-1}\right)\left(1 - \frac{4}{5}e^{-j\frac{4\pi}{5}}z^{-1}\right)$$

$$H_{d,ap}(z) = \frac{\left(z^{-1} - \frac{4}{5}e^{-j\frac{4\pi}{5}}\right)\left(z^{-1} - \frac{4}{5}e^{j\frac{4\pi}{5}}\right)}{\left(1 - \frac{4}{5}e^{j\frac{4\pi}{5}}z^{-1}\right)\left(1 - \frac{4}{5}e^{-j\frac{4\pi}{5}}z^{-1}\right)}$$

$$H_c(z) = \frac{16}{25}\frac{1}{\left(1 - \frac{9}{10}e^{j\frac{3\pi}{5}}z^{-1}\right)\left(1 - \frac{9}{10}e^{-j\frac{3\pi}{5}}z^{-1}\right)\left(1 - \frac{4}{5}e^{j\frac{4\pi}{5}}z^{-1}\right)\left(1 - \frac{4}{5}e^{-j\frac{4\pi}{5}}z^{-1}\right)}$$

PROPERTIES OF MINIMUM PHASE SYSTEMS

MINIMUM PHASE-LAG PROPERTY

Definition

Phase-lag: $-\measuredangle H\left(e^{j\omega}\right)$, i.e. negative of the phase of the frequency response.

Let $H_{min}(z)$ be a minimum-phase system.

Let $H(z)$ be one of those systems that have the same frequency response magnitude function with $H_{min}(z)$.

Then, the phase-lag function of $H_{min}(z)$ is smaller than that of $H(z)$.

$$H(z) = H_{min}(z)H_{ap}(z)\ ,$$

$$H\left(e^{j\omega}\right) = H_{min}\left(e^{j\omega}\right)H_{ap}\left(e^{j\omega}\right)$$

$$-\measuredangle H\left(e^{j\omega}\right) = -\measuredangle H_{min}\left(e^{j\omega}\right) \underbrace{-\measuredangle H_{ap}\left(e^{j\omega}\right)}_{=0 \quad for\ 0\leq\omega<\pi}$$

$$\Rightarrow \qquad -\measuredangle H_{min}\left(e^{j\omega}\right) \leq -\measuredangle H\left(e^{j\omega}\right)$$

Note that $H(e^{j0}) = \sum_{n=-\infty}^{\infty} h[n]$ is required in defining a minimum phase system since $h[n]$ and $-h[n]$ have the same poles and zeros but they have a phase difference of π.

Remember that group-delay of allpass systems are always positive.

$$H(z) = H_{min}(z)H_{ap}(z)$$

$$\Rightarrow \quad grd\left(H(z)\right) = grd\left(H_{min}(z)\right) + \underbrace{grd\left(H_{ap}(z)\right)}_{>0}$$

$$\Rightarrow \quad grd\left(H_{min}(z)\right) \leq grd\left(H(z)\right)$$

Let

$$H(z) = H_{min}(z)H_{ap}(z)$$

Then, the following holds

a) $$\sum_{n=0}^{\infty} |h[n]|^2 = \sum_{n=0}^{\infty} |h_{min}[n]|^2$$

Proof: Using Parseval's theorem and $\left|H\left(e^{j\omega}\right)\right| = \left|H_{min}\left(e^{j\omega}\right)\right|$

b) $$\sum_{n=0}^{m} |h[n]|^2 \leq \sum_{n=0}^{m} |h_{min}[n]|^2$$

Proof: Let $h_{min}[n]$ be a minimum-phase sequence and $H_{min}(z)$ be its z-transform. $H_{min}(z)$ can be written as

$$H_{min}(z) = Q(z)(1 - z_k z^{-1}) \qquad\qquad |z_k| < 1$$

where z_k is one of its zeros. Note that $Q(z)$ is also minimum-phase.

Let $h[n]$ be another sequence such that

$$\left|H\left(e^{j\omega}\right)\right| = \left|H_{min}\left(e^{j\omega}\right)\right|$$

and $H(z)$ has a zero at $\frac{1}{z_k^*}$.

Then, since

$$Q(z)(1 - z_k^* z) = Q(z)\, z\, (z^{-1} - z_k^*)$$

let

$$H(z) = Q(z)(z^{-1} - z_k^*)\,.$$

Hence,

$$h[n] = -z_k^* q[n] + q[n-1]$$

$$h_{min}[n] = q[n] - z_k q[n-1]$$

Now, let's evaluate

$$\sum_{m=0}^{n} |h_{min}[m]|^2 - \sum_{m=0}^{n} |h[m]|^2$$

$$\begin{aligned}
|h_{min}[m]|^2 &= h_{min}[m] h_{min}^*[m] \\
&= (q[m] - z_k q[m-1])(q^*[m] - z_k^* q^*[m-1]) \\
&= |q[m]|^2 - z_k^* q[m] q^*[m-1] - z_k q[m-1] q^*[m] \\
&\quad + |z_k|^2 |q[m-1]|^2
\end{aligned}$$

$$\begin{aligned}
|h[m]|^2 &= h[m] h^*[m] \\
&= (-z_k^* q[m] + q[m-1])(-z_k q^*[m] + q^*[m-1])
\end{aligned}$$

$$= |z_k|^2|q[m]|^2 - z_k^* q[m]q^*[m-1] - z_k q[m-1]q^*[m] + |q[m-1]|^2$$

$$\sum_{m=0}^{n} |h_{min}[m]|^2 - \sum_{m=0}^{n} |h[m]|^2 = \sum_{m=0}^{n} (1-|z_k|^2)(|q[m]|^2 - |q[m-1]|^2)$$

$$= \underbrace{(1-|z_k|^2)|q[n]|^2}_{>0}$$

Ex: Let

$$H(z) = 1 + 2z^{-1} + 5z^{-2}$$

$$= (1 - az^{-1})(1 - a^* z^{-1})$$

$$a = -1 + j2 = re^{j\theta} = \sqrt{5}e^{j\theta}$$

$$H_{min}(z) = |a|^2 \left(1 - \frac{1}{a} z^{-1}\right)\left(1 - \frac{1}{a^*} z^{-1}\right)$$

$$= r^2 \left(1 - 2\frac{1}{r}\cos\theta\, z^{-1} + \frac{1}{r^2} z^{-2}\right)$$

$$= 5 + 2z^{-1} + z^{-2}$$

Ex: Let

$$H(z) = 1 + z^{-1} - 6z^{-2}$$

$$H_{min}(z) = -6 + z^{-1} + z^{-2}$$

Ex: The squared magnitude of the frequency response of a *minimum phase* system is

$$\left|H\left(e^{j\omega}\right)\right|^2 = \frac{1}{2.5 + 2\cos(\omega)}$$

Find $H(z)$; plot its poles and zeros.

$$\left|H\left(e^{j\omega}\right)\right|^2 = \frac{1}{\frac{5}{2} + e^{j\omega} + e^{-j\omega}}$$

$$H(z)H^*\left(\frac{1}{z^*}\right) = \frac{1}{\frac{5}{2} + z + z^{-1}}$$

$$= \frac{1}{\left(1 + \frac{1}{2}z^{-1}\right)(2 + z)}$$

$$= \frac{\frac{1}{2}}{\left(1 + \frac{1}{2}z^{-1}\right)\left(1 + \frac{1}{2}z\right)}$$

Therefore,

$$H(z) = \frac{\frac{1}{\sqrt{2}}}{\left(1 + \frac{1}{2}z^{-1}\right)}$$

GENERALIZED LINEAR PHASE SYSTEMS

Linear Phase Systems

$$\angle H\left(e^{j\omega}\right) = -\alpha\omega$$

Ex:

Δ: integer

$$h[n] = \delta(n - \Delta)$$

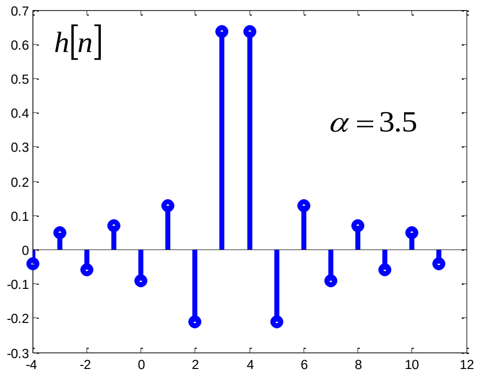
$$H\left(e^{j\omega}\right) = e^{-j\Delta\omega}$$

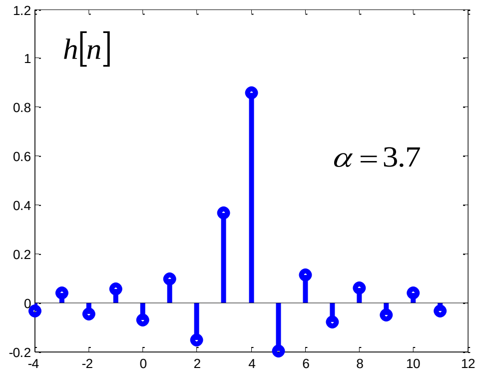
Ex:

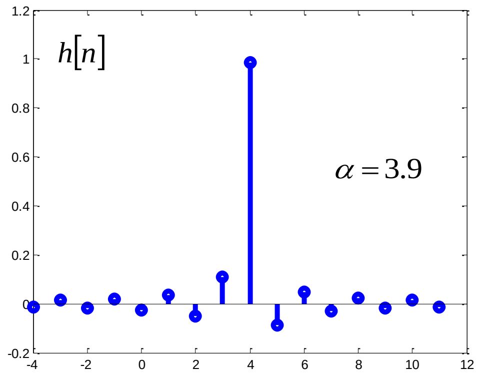
α: real

$$H(e^{j\omega}) = e^{-j\alpha\omega}$$

$$h[n] = \sin\frac{(\pi(n-\alpha))}{\pi(n-\alpha)}$$





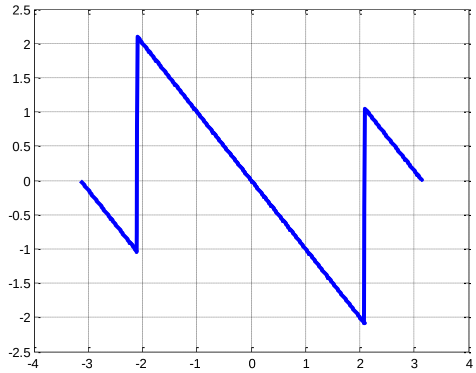


There are systems having linear-like phase responses and constant group delay.

Ex:

$$h[n]: \quad \ldots 0 \;\; 1 \;\; 1 \;\; 1 \; 0 \ldots$$

$$H\left(e^{j\omega}\right) = e^{-j\omega}(1 + 2\cos(\omega))$$

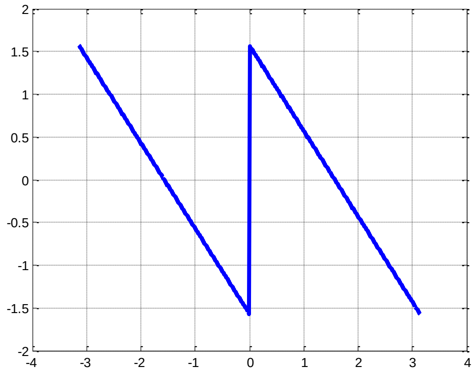


Ex:

$$h[n]: \quad \ldots 0 \;\; 1 \;\; 0 \quad -1 \quad 0 \ldots$$

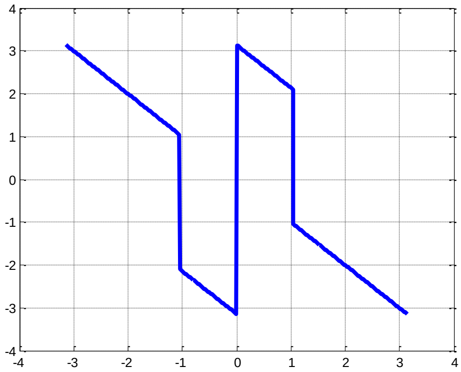
$$H\left(e^{j\omega}\right) = je^{-j\omega}(2\sin(\omega))$$

$$= e^{-j(\omega-\pi)}(2\sin(\omega))$$



$$h[n]: \; [\ldots 0 \quad -1 \;\; 1 \quad -1 \quad 0 \ldots]$$

$$H\left(e^{j\omega}\right) = e^{-j\omega}(1 - 2\cos(\omega))$$



ZERO LOCATIONS

$$
\begin{aligned}
H(z) &= \sum_{n=0}^{M} h[n] z^{-n} \\
&= \sum_{n=0}^{M} h[M-n] z^{-n} \qquad \text{even symmetric } h[n] \text{assumed} \\
&= \sum_{n=0}^{M} h[n] z^{-(M-n)} \\
&= z^{-M} \sum_{n=0}^{M} h[n] z^{n} \\
&= z^{-M} H\left(\frac{1}{z}\right)
\end{aligned}
$$

Therefore, for even symmetric filters (Type I and Type II)

$$H(z) = z^{-M} H\left(\frac{1}{z}\right)$$

Similarly, for odd symmetric filters (Type III and Type IV)

$$H(z) = -z^{-M} H\left(\frac{1}{z}\right)$$

Both equalities indicate that, zeros are in pairs, i.e., a zero at $z = z_0$ is always accompanied by a zero at $z = \frac{1}{z_0}$

GENERALIZED LINEAR PHASE SYSTEMS

$$H\left(e^{j\omega}\right) = A(\omega)e^{-j(\alpha\omega-\beta)}$$

$A(\omega) \in R$ (bipolar)

Linear phase if

$$A(\omega) \geq 0 \quad \text{and} \quad \beta = 0$$

	Lowpass	Highpass
Type-I		
Type-II		No
Type-III	No	No
Type-IV	No	

	zero at z =1 (ω = 0)	zero at z =-1 (ω = π)
Type-I		
Type-II		always
Type-III	always	always
Type-IV	always	

GROUP DELAY OF $\frac{z^{-1}-a^*}{1-az^{-1}}$

$$-\frac{d}{d\omega}\measuredangle\left(e^{-j\omega}\frac{1-re^{-j\theta}e^{j\omega}}{1-re^{j\theta}e^{-j\omega}}\right)=-\frac{d}{d\omega}\left(-\omega+\measuredangle\left(\frac{1-re^{-j\theta}e^{j\omega}}{1-re^{j\theta}e^{-j\omega}}\right)\right)$$

$$=-\frac{d}{d\omega}\left(-\omega+2\measuredangle\left(1-re^{-j\theta}e^{j\omega}\right)\right)$$

$$=1-2\frac{d}{d\omega}\arctan\left(\frac{-r\sin(\omega-\theta)}{1-r\cos(\omega-\theta)}\right)$$

$$=1-2\left(\frac{1}{1+\left(\frac{-r\sin(\omega-\theta)}{1-r\cos(\omega-\theta)}\right)^2}\right)\left(\frac{-r\cos(\omega-\theta)\,(1-r\cos(\omega-\theta))+r^2\sin^2(\omega-\theta)}{1-2r\cos(\omega-\theta)+r^2\cos^2(\omega-\theta)}\right)$$

$$=1-2\left(\frac{1-2r\cos(\omega-\theta)+r^2\cos^2(\omega-\theta)}{1+r^2-2r\cos(\omega-\theta)}\right)\left(\frac{r^2-r\cos(\omega-\theta)}{1-2r\cos(\omega-\theta)+r^2\cos^2(\omega-\theta)}\right)$$

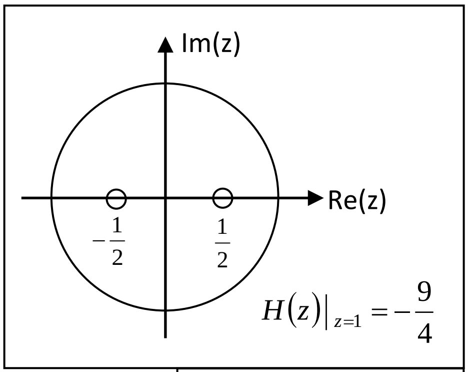
$$=1-2\left(\frac{r^2-r\cos(\omega-\theta)}{1+r^2-2r\cos(\omega-\theta)}\right)$$

$$=\frac{1+r^2-2r\cos(\omega-\theta)-2r^2+2r\cos(\omega-\theta)}{1+r^2-2r\cos(\omega-\theta)}$$

$$=\frac{1-r^2}{1+r^2-2r\cos(\omega-\theta)}$$

$$=\frac{1-r^2}{|1-re^{j\theta}e^{-j\omega}|^2}$$

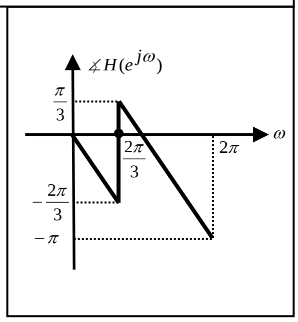
Ex: Consider a causal, generalized linear phase system. The length of the impulse response is 5. Some of the zeros of the transfer function, $H(z)$, of this system are shown in the *upper* panel.



a) Find and <u>plot</u> the impulse response of this system.

b) Find the frequency response and <u>plot</u> its magnitude and phase.

c) Find and <u>plot</u> the impulse response of the minimum phase system that have the same magnitude response.

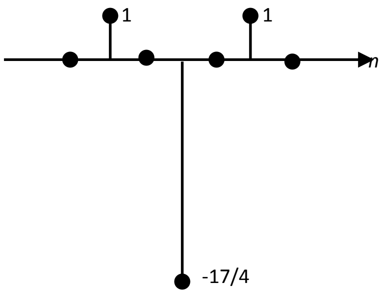


d) (**This part is independent of the previous parts.**) The phase of the frequecy response of an *even symmetric* generalized linear phase system is shown in the *lower* panel.

i) What is the length of the impulse response? Why?

ii) Let $h[0]=1$. Write the frequency response in terms of $h[0]$ and the other elements of the impulse response. Plot the magnitude of the frequency response.

iii) Find the whole impulse response.



a) The other zeros are at z = 2 and z = -2.

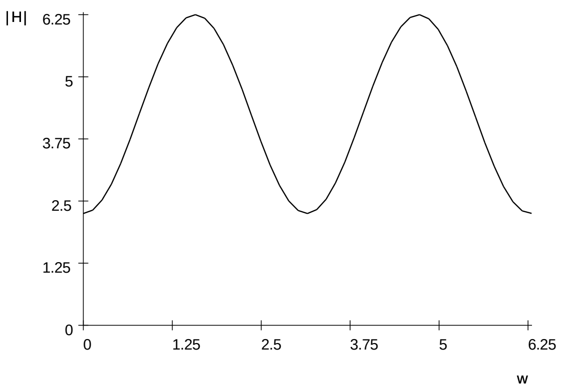
$$\Rightarrow H(z)=A\left(1-\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{2}z^{-1}\right)\left(1-2z^{-1}\right)\left(1+2z^{-1}\right)$$

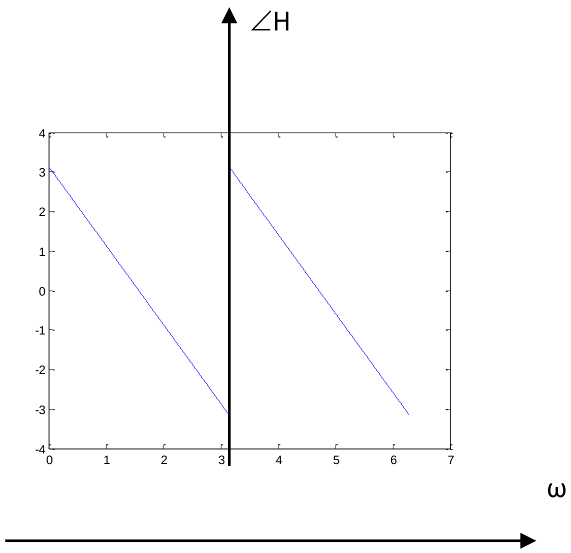
$$H(z)=A\left(1-\frac{17}{4}z^{-2}+z^{-4}\right),\qquad H(1)=A\ -\frac{9}{4}\ \Rightarrow A=1.$$

$$\Rightarrow h[n]=\delta[n]-\frac{17}{4}\delta[n-2]+\delta[n-4]$$

b) $H\left(e^{j\omega}\right)=1-\frac{17}{4}e^{-j2\omega}+e^{-j4\omega}=e^{-j2\omega}\left(-\frac{17}{4}+e^{j2\omega}+e^{-j2\omega}\right)=e^{-j2\omega}\left(-\frac{17}{4}+2\cos(2\omega)\right)$

$$H\left(e^{j\omega}\right)=1-\frac{17}{4}e^{-j2\omega}+e^{-j4\omega}=e^{-j2\omega}\left(-\frac{17}{4}+e^{j2\omega}+e^{-j2\omega}\right)=e^{-j2\omega}\left(-\frac{17}{4}+2\cos(2\omega)\right)$$





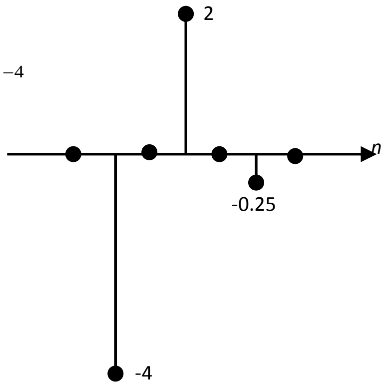
c) $H(z)=\left(1-\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{2}z^{-1}\right)\left(1-2z^{-1}\right)\left(1+2z^{-1}\right)=\left(1-\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{2}z^{-1}\right)(-2)\left(z^{-1}-\frac{1}{2}\right)(2)\left(z^{-1}+\frac{1}{2}\right)$

$$=\left(1-\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{2}z^{-1}\right)(-2)\left(z^{-1}-\frac{1}{2}\right)(2)\left(z^{-1}+\frac{1}{2}\right)\frac{\left(1-\frac{1}{2}z^{-1}\right)}{\left(1-\frac{1}{2}z^{-1}\right)}\frac{\left(1+\frac{1}{2}z^{-1}\right)}{\left(1+\frac{1}{2}z^{-1}\right)}$$

$$=\underbrace{(-4)\left(1-\frac{1}{2}z^{-1}\right)^2\left(1+\frac{1}{2}z^{-1}\right)^2}_{H_{\min}(z)}\underbrace{\left(\frac{\left(z^{-1}-\frac{1}{2}\right)}{\left(1-\frac{1}{2}z^{-1}\right)}\frac{\left(z^{-1}+\frac{1}{2}\right)}{\left(1+\frac{1}{2}z^{-1}\right)}\right)}_{H_{ap}(z)}$$

$$H_{\min}(z)=(-4)\left(1-\frac{1}{2}z^{-1}\right)^2\left(1+\frac{1}{2}z^{-1}\right)^2=(-4)\left(1-0.5z^{-1}+\frac{1}{16}z^{-4}\right)=-4+2z^{-1}-\frac{1}{4}z^{-4}$$

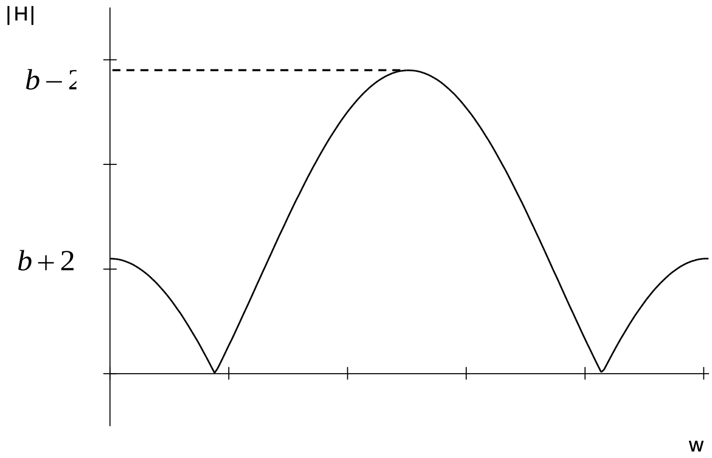
$$h_{\min}[n]=-4\delta[n]+2\delta[n-2]-0.25\delta[n-4]$$



d) **i**)The slope of the phase is -1 (-1x group delay) so the filter length is 3.

$$h[n] = a\delta[n] + b\delta[n-1] + a\delta[n-2]$$

$$H\left(e^{j\omega}\right) = a + be^{-j\omega} + ae^{-j2\omega} = e^{-j\omega}\left(b + 2a\cos(\omega)\right)$$



ii) $h[0] = 1 \Rightarrow$a=1

$$H\left(e^{j\omega}\right) = e^{-j\omega}\left(b + 2\cos(\omega)\right)$$

$$\measuredangle H\left(e^{j0}\right) = 0 \Rightarrow b > -2$$

iii) The first zero crossing must be at 3

$$b + 2\cos\left(\frac{2\pi}{3}\right) = 0 \Rightarrow b = 1$$