

# FREQUENCY DOMAIN REPRESENTATION OF LTI SYSTEMS

- EIGENFUNCTIONS OF LTI SYSTEMS
- FREQUENCY RESPONSE
  - MAGNITUDE AND PHASE
- THE RESPONSE TO A SINUSOID
- FOR A REAL LTI SYSTEM, FREQUENCY RESPONSE IS ALSO CONJUGATE SYMMETRIC WRT  $\pi$
- USEFUL TIPS
- FREQUENCY RESPONSE OF PURE DELAY
- LINEAR PHASE SYSTEMS
- “SUDDENLY” APPLIED COMPLEX EXPONENTIAL INPUTS

## FREQUENCY DOMAIN REPRESENTATION OF LTI SYSTEMS

In the context of LTI systems, “Frequency Domain” refers to the representation of signals and analysis of LTI systems using sinusoidal signals.

Sinusoidal expressions are closely related to complex exponential expressions (Euler’s formula).

$$e^{j\omega n} = \cos(\omega n) + j \sin(\omega n)$$

## EIGENFUNCTIONS OF LTI SYSTEMS

Let the input signal be  $e^{j\omega_0 n}$ , i.e. a complex exponential with frequency  $\omega_0$

$$x[n] = e^{j\omega_0 n} \longrightarrow \boxed{h[n]} \longrightarrow y[n] = ?$$

The output will be

$$\begin{aligned} y[n] &= e^{j\omega_0 n} * h[n] \\ &= \sum_{k=-\infty}^{\infty} h[k] e^{j\omega_0 (n-k)} \\ &= e^{j\omega_0 n} \underbrace{\sum_{k=-\infty}^{\infty} h[k] e^{j\omega_0 k}}_{H(e^{j\omega_0})} = e^{j\omega_0 n} H(e^{j\omega_0}) \end{aligned}$$

Assumed that  $\sum_{k=-\infty}^{\infty} h[k] e^{j\omega_0 k}$  is finite.

Therefore, the output is the same (except a complex scaling factor) as the input.

The complex scale factor,  $H(e^{j\omega_0})$ , is called the frequency response.

Complex exponentials or real sinusoids are called as the *eigenfunctions* of LTI systems.

## FREQUENCY RESPONSE

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k] e^{j\omega k}$$

is called the “frequency response”.

It is the complex valued gain of a LTI system to a complex exponential of particular “frequency”.

## MAGNITUDE AND PHASE

Note that,

$$H(e^{j\omega}) = |H(e^{j\omega})| e^{j\angle H(e^{j\omega})}$$

$$= |H(e^{j\omega_0})| e^{j\theta(\omega_0)}$$

$\Rightarrow$

$$H(e^{j\omega_0}) e^{j\omega_0 n} = |H(e^{j\omega_0})| e^{j\theta(\omega_0)} e^{j\omega_0 n}$$

$$= |H(e^{j\omega_0})| e^{j(\omega_0 n + \theta(\omega_0))}$$

Frequency response is a periodic function; period is  $2\pi$ .

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n} = \sum_{n=-\infty}^{\infty} h[n]e^{-j(\omega+k2\pi)n} = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n} \underbrace{e^{-jk2\pi n}}_1 = H(e^{j(\omega+k2\pi)})$$

$$H(e^{j\omega}) = H_R(e^{j\omega}) + jH_I(e^{j\omega})$$

$$|H(e^{j\omega})|: \text{magnitude of the frequency response } \left( \sqrt{H_R^2 + H_I^2} \right)$$

$$\angle H(e^{j\omega}): \text{phase of the frequency response } \left( \tan^{-1} \frac{H_I}{H_R} \right)$$

## THE RESPONSE TO A SINUSOID

When the impulse response is real,

$$H(e^{j\omega}) = H^*(e^{-j\omega})$$

i.e.,  $H(e^{j\omega})$  is conjugate symmetric.

$$x[n] = \cos(\omega n) \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

$$x[n] = \cos(\omega n) = \frac{1}{2}(e^{j\omega n} + e^{-j\omega n})$$

since LTI

$$y[n] = \frac{1}{2}(H(e^{j\omega})e^{j\omega n} + H(e^{-j\omega})e^{-j\omega n})$$

Hence

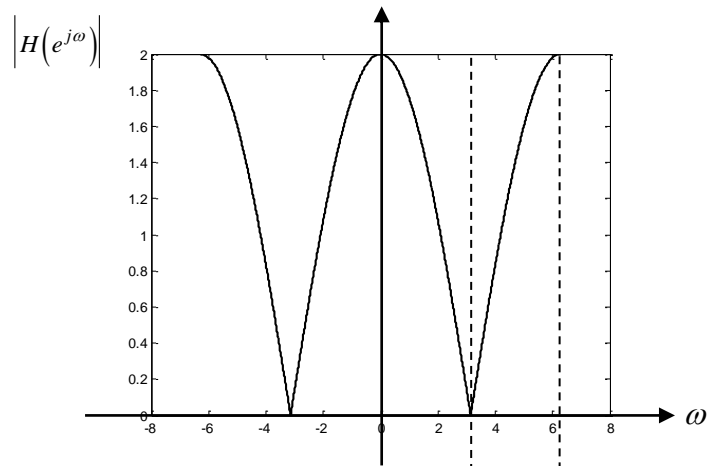
$$\begin{aligned} y[n] &= \frac{1}{2}|H(e^{j\omega})|(e^{j\theta(\omega)}e^{j\omega n} + e^{-j\theta(\omega)}e^{-j\omega n}) \\ &= \frac{1}{2}|H(e^{j\omega})|\cos(\omega n + \theta(\omega)) \end{aligned}$$

In summary,

$$\cos(\omega n) \xrightarrow{\text{LTI system}} |H(e^{j\omega})| \cos(\omega n + \theta(\omega))$$

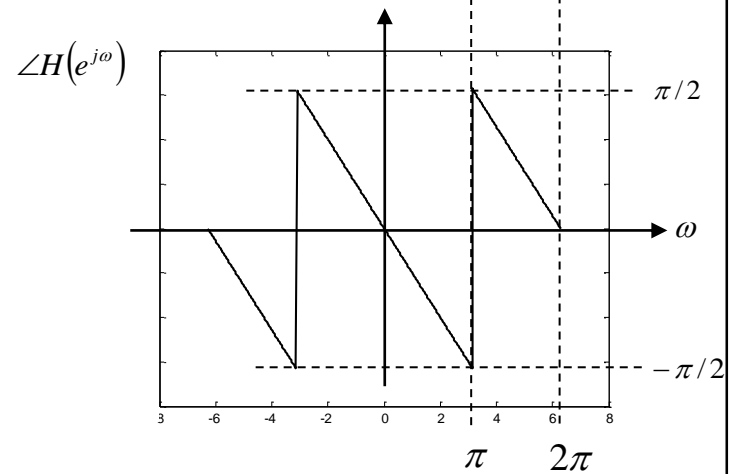
**Ex:** Let  $h[n] = \delta[n] + \delta[n-1]$

$$\begin{aligned} H(e^{j\omega}) &= 1 + e^{-j\omega} \\ &= e^{-j\frac{\omega}{2}} \left( e^{j\frac{\omega}{2}} + e^{-j\frac{\omega}{2}} \right) \\ &= 2 \cos\left(\frac{\omega}{2}\right) e^{-j\frac{\omega}{2}} \end{aligned}$$



The magnitude and phase are

$$|H(e^{j\omega})| = \left| 2 \cos\left(\frac{\omega}{2}\right) \right|$$



$$\angle H(e^{j\omega}) = \begin{cases} -\frac{\omega}{2} & \cos\left(\frac{\omega}{2}\right) \geq 0 \\ -\frac{\omega}{2} \pm \pi & \cos\left(\frac{\omega}{2}\right) < 0 \end{cases}$$

```
h = [1 1];
[H,w] = freqz(h,1,1000,'whole'); (to plot in [0,2\pi])
[H,w] = freqz(h,1,1000); (to plot in [0, \pi])
plot(w,abs(H));
figure
plot(w,angle(H));
```



Let the input be

$$x[n] = \cos\left(\frac{\pi}{5}n\right)$$

The output is

$$y[n] = 1.9021 \cos\left(\frac{\pi}{5}n - \frac{\pi}{10}\right)$$

Since

$$H\left(e^{j\frac{\pi}{5}}\right) = 2 \cos\left(\frac{\pi}{10}\right) e^{-j\frac{\pi}{10}} = 1.9021 e^{-j\frac{\pi}{10}}$$

or

If the input is

$$x[n] = \cos\left(\frac{6\pi}{5}n\right)$$

The output is

$$y[n] = 0.6180 \cos\left(\frac{6\pi}{5}n - \frac{6\pi}{10} + \pi\right)$$

Since

$$H\left(e^{j\frac{6\pi}{5}}\right) = 2 \cos\left(\frac{6\pi}{10}\right) e^{-j\frac{6\pi}{10}}$$

$$= -0.6180 e^{-j\frac{6\pi}{10}}$$

$$= 0.6180 e^{-j\frac{6\pi}{10} + \pi}$$

FOR A REAL LTI SYSTEM, FREQUENCY RESPONSE IS ALSO CONJUGATE  
SYMMETRIC WRT  $\pi$

$$H(\omega) = H(\omega + 2\pi) \text{ and } H(\omega) = H^*(-\omega)$$

$$\Rightarrow H^*(-\omega) = H(\omega + 2\pi)$$

$$\Rightarrow H^*(\pi - \omega) = H(\omega + \pi)$$

## USEFUL TIPS

$$e^{jx} + e^{jy} = e^{j\frac{x-y}{2}} \left( e^{j\frac{x+y}{2}} + e^{-j\frac{x+y}{2}} \right) = e^{j\frac{x-y}{2}} 2 \cos\left(\frac{x+y}{2}\right)$$

$$e^{jx} + e^{jy} = e^{j\frac{x+y}{2}} \left( e^{j\frac{x-y}{2}} + e^{-j\frac{x-y}{2}} \right) = e^{j\frac{x+y}{2}} 2 \cos\left(\frac{x-y}{2}\right)$$

$$a + be^{-j\omega} + be^{-j2\omega} + ae^{-j3\omega}$$

$$= e^{-j\frac{3\omega}{2}} \left( ae^{j\frac{3\omega}{2}} + be^{j\frac{\omega}{2}} + be^{-j\frac{\omega}{2}} + ae^{-j\frac{3\omega}{2}} \right)$$

$$= e^{-j\frac{3\omega}{2}} 2 \left( b \cos\left(\frac{\omega}{2}\right) + a \cos\left(\frac{3\omega}{2}\right) \right)$$

in general, let  $a_k = a_{N-k} \quad k = 0, 1, \dots, N$

$$\sum_{k=0}^N a_k e^{-jk\omega} = \begin{cases} e^{-j\frac{N}{2}} 2 \sum_{k=1}^{\frac{N+1}{2}} a_k \cos\left(\frac{k}{2}\omega\right) & \text{if } N \text{ is odd} \\ e^{-j\frac{N}{2}} \left( a_{\frac{N}{2}} + 2 \sum_{k=1}^{\frac{N}{2}} a_k \cos(k\omega) \right) & \text{if } N \text{ is even} \end{cases}$$

or, if  $a_k = -a_{N-k} \quad k = 0, 1, \dots, N$

$$\sum_{k=0}^N a_k e^{-jk\omega} = \begin{cases} j e^{-j\frac{N}{2}} 2 \sum_{k=1}^{\frac{N+1}{2}} a_k \sin\left(\frac{k}{2}\omega\right) & \text{if } N \text{ is odd} \\ e^{-j\frac{N}{2}} \left( a_{\frac{N}{2}} + j 2 \sum_{k=1}^{\frac{N}{2}} a_k \sin(k\omega) \right) & \text{if } N \text{ is even} \end{cases}$$

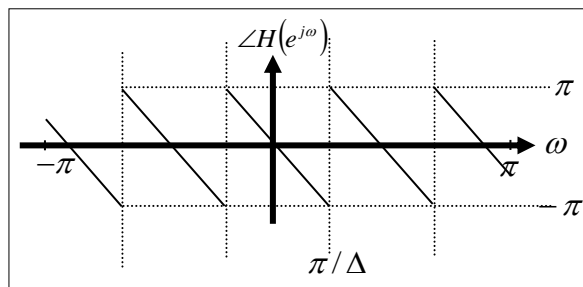
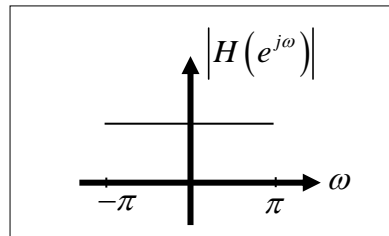
## FREQUENCY RESPONSE OF PURE DELAY

**Ex:** A pure delay.

$$y[n] = x[n - \Delta]$$

$$\text{Let } x[n] = e^{j\omega n} \rightarrow y[n] = e^{j\omega(n-\Delta)} = e^{-j\omega\Delta} e^{j\omega n}$$

$$\Rightarrow H(e^{j\omega}) = e^{-j\omega\Delta}$$



It can also be computed as,

$$\begin{aligned}h[n] &= \delta[n - \Delta] \\ \Rightarrow H(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} \delta[n - \Delta] e^{-j\omega n} \\ &= e^{-j\omega\Delta} \\ |H(e^{j\omega})| &= 1 \quad , \quad \angle H(e^{j\omega}) = -\Delta\omega\end{aligned}$$

## LINEAR PHASE SYSTEMS

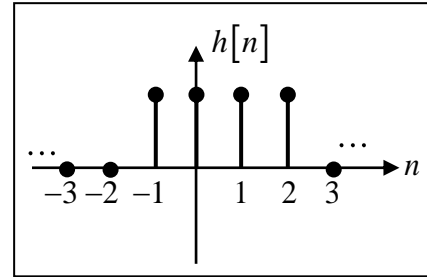
The phase of the frequency response of a pure delay system is a “**linear**” function.

Such systems are called “**linear phase**” systems.

**Ex:** Frequency response of a moving average system.

$$y[n] = \frac{1}{4}(x[n+1] + x[n] + x[n-1] + x[n-2])$$

$$= \frac{1}{4} \sum_{k=-1}^2 x[n-k]$$



$$h[n] = \frac{1}{4} \sum_{k=-1}^2 \delta[n-k]$$

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n] e^{-j\omega n}$$

$$= \frac{1}{4} (e^{j\omega} + 1 + e^{-j\omega} + e^{-j2\omega})$$

$$= \frac{1}{4} \sum_{n=-1}^2 e^{-j\omega n}$$

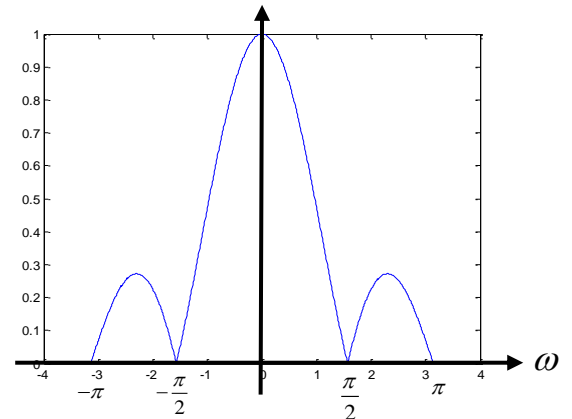
$$= \frac{1}{4} e^{j\omega} \sum_{n=0}^3 e^{-j\omega n}$$

$$= \frac{1}{4} e^{j\omega} \frac{1 - e^{-j4\omega}}{1 - e^{-j\omega}}$$

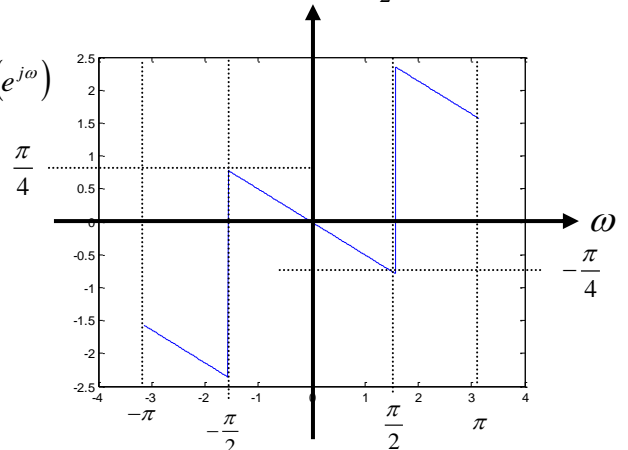
$$= \frac{1}{4} e^{j\omega} \frac{e^{-j2\omega}}{e^{-j\frac{\omega}{2}}} \frac{e^{j2\omega} - e^{-j2\omega}}{e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}}}$$

$$= \frac{1}{4} e^{-j\frac{\omega}{2}} \frac{\sin(2\omega)}{\sin\left(\frac{\omega}{2}\right)}$$

$$|H(e^{j\omega})|$$



$$\angle H(e^{j\omega})$$



Is it a linear phase system?



Note that, according to the “useful tip”, the frequency response above can also be written as

$$\begin{aligned} H(e^{j\omega}) &= \frac{1}{4} (e^{j\omega} + 1 + e^{-j\omega} + e^{-j2\omega}) \\ &= \frac{1}{4} e^{j\omega} (1 + e^{-j\omega} + e^{-j2\omega} + e^{-j3\omega}) \\ &= \frac{1}{4} e^{j\omega} e^{-j\frac{3\omega}{2}} \left( e^{j\frac{3\omega}{2}} + e^{j\frac{\omega}{2}} + e^{-j\frac{\omega}{2}} + e^{-j\frac{3\omega}{2}} \right) \\ &= \frac{1}{2} e^{-j\frac{\omega}{2}} \left( \cos\left(\frac{\omega}{2}\right) + \cos\left(\frac{3\omega}{2}\right) \right) \end{aligned}$$

Therefore we have

$$\frac{1}{4} \frac{\sin(2\omega)}{\sin\left(\frac{\omega}{2}\right)} = \frac{1}{2} \left( \cos\left(\frac{\omega}{2}\right) + \cos\left(\frac{3\omega}{2}\right) \right)$$

To check:

```
clear all; close all;
```

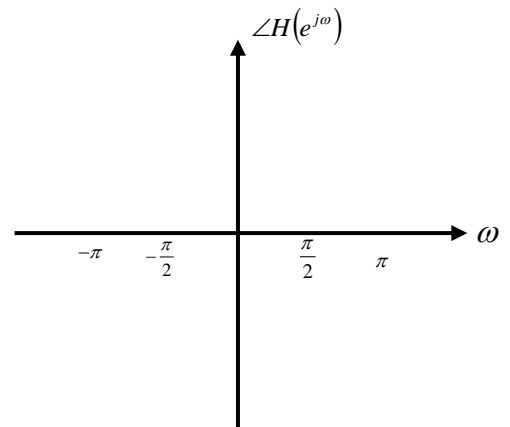
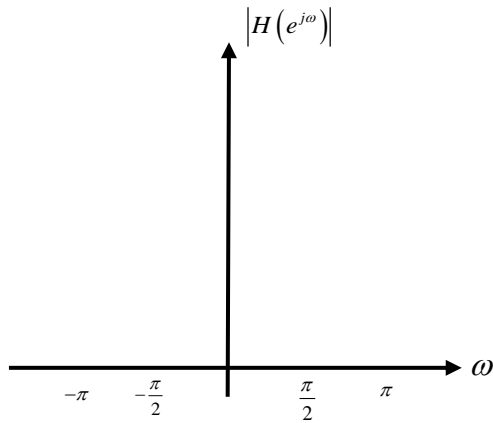
```
w = linspace(0,pi,1000);
```

```
x = 0.25 * (sin(2*w)) ./ (sin(w/2)); y = 0.5 * (cos(w/2) + cos(3 * w/2));
```

```
plot(w,x,'r'); hold; plot(w,y,'k');
```

**Exercise:** Using the results of last two examples, find and plot the frequency response of a LTI system whose impulse response is

$$h[n] = \frac{1}{4} \sum_{k=3}^6 \delta[n-k]$$



## “SUDDENLY” APPLIED COMPLEX EXPONENTIAL INPUTS

**Study!** Textbook Section 2.6.2, “Suddenly” Applied Complex Exponential Inputs

$$x[n] = e^{j\omega n}u[n]$$

LTI Sytem

↓

$$y[n] = \underbrace{e^{j\omega n}H(e^{j\omega})}_{y_{ss}[n]} - \underbrace{e^{j\omega n} \sum_{k=n+1}^{\infty} h[k]e^{-j\omega k}}_{y_t[n]}$$