FREQUENCY DOMAIN REPRESENTATION OF LTI SYSTEMS

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DICRETE TIME FOURIER TRANSFORM (DTFT)

The Fourier transform of a sequence $x[n]$ is defined as

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

If the FT exists (summation converges) the sequence can be obtained from its FT as

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega$$

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X\left(e^{j\omega}\right) e^{j\omega n}\, d\omega$$

What is this?

Fourier Transform is periodic with 2π.

The frequency response function of a LTI system,

$$H\left(e^{j\omega}\right),$$

is the FT of its impulse response

$$h[n].$$

EXISTENCE

FT of a sequence $x[n]$ exists, i.e.,

$$\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

converges to a continuous function of ω, if $x[n]$ is absolutely summable.

(sufficient condition)

Proof: Exercise

$$\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = \sum_{n=-\infty}^{\infty} x[n](\cos(\omega n) - j\sin(\omega n))$$

$$= \sum_{n=-\infty}^{\infty} x[n]\cos(\omega n) - j\sum_{n=-\infty}^{\infty} x[n]\sin(\omega n)$$

Both sums have to converge

→ All stable LTI systems have frequency response functions.

Ex: DTFT of

$$x[n] = a^n u[n]$$

$$X(e^{j\omega}) = \sum_{n=0}^{\infty} a^n e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} \left(a e^{-j\omega}\right)^n$$

$$= \frac{1}{1 - a e^{-j\omega}} \qquad if \ \ \left|a e^{-j\omega}\right| < 1 \quad or \quad |a| < 1$$

Magnitude of

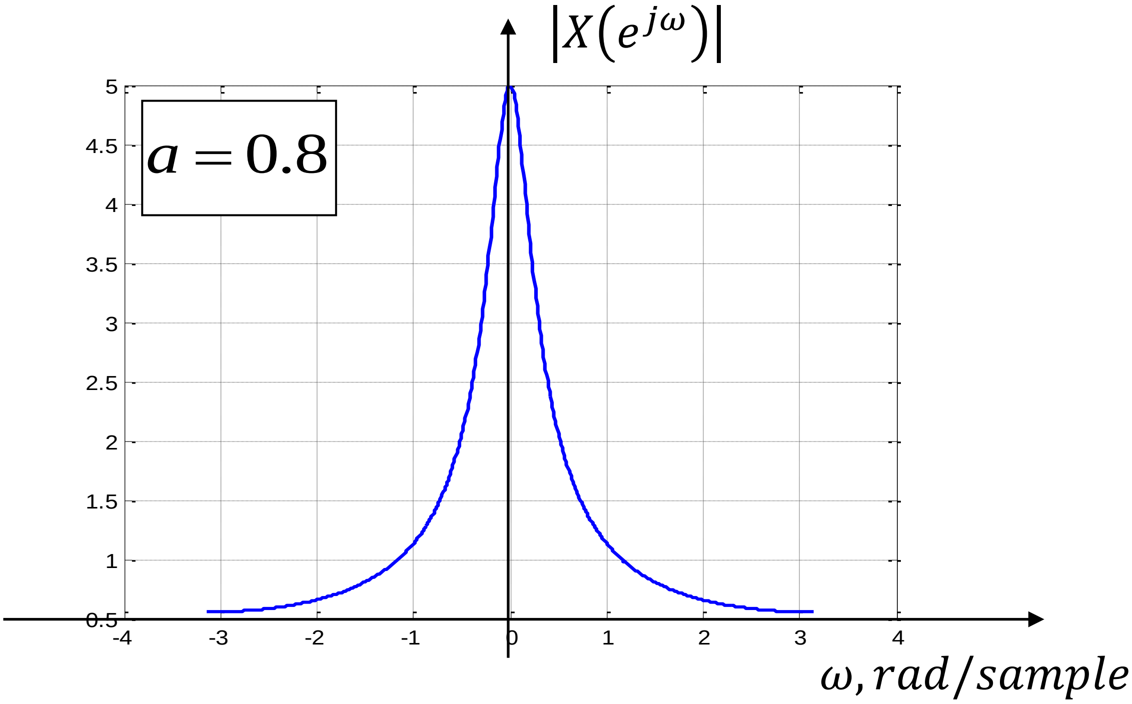
$$X(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}, \qquad |a| < 1$$

$$\left|X(e^{j\omega})\right|^2 = \frac{1}{|1 - ae^{-j\omega}|^2}$$

$$= \frac{1}{|1 - a\cos(\omega) + ja\sin(\omega)|^2}$$

$$= \frac{1}{1 + a^2 - 2a\cos(\omega)}$$

$$\left|X(e^{j\omega})\right| = \frac{1}{\sqrt{1 + a^2 - 2a\cos(\omega)}}$$



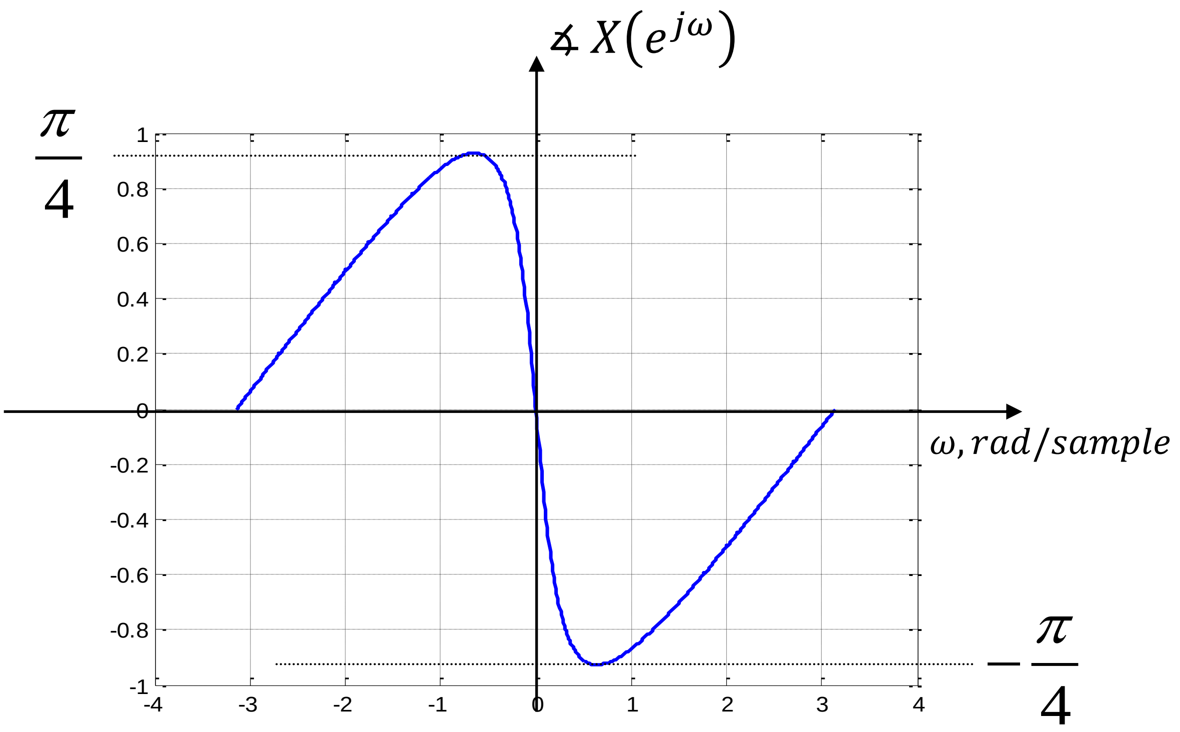
Phase of

$$X(e^{j\omega}) = \frac{1}{1-ae^{-j\omega}}, \qquad |a|<1$$

$$\measuredangle X(e^{j\omega}) = \measuredangle 1 - \measuredangle 1 - ae^{-j\omega}$$

$$= 0 - \measuredangle(1 - a\cos(\omega) + ja\sin(\omega))$$

$$= -\tan^{-1}\left(\frac{a\sin(\omega)}{1-a\cos(\omega)}\right)$$



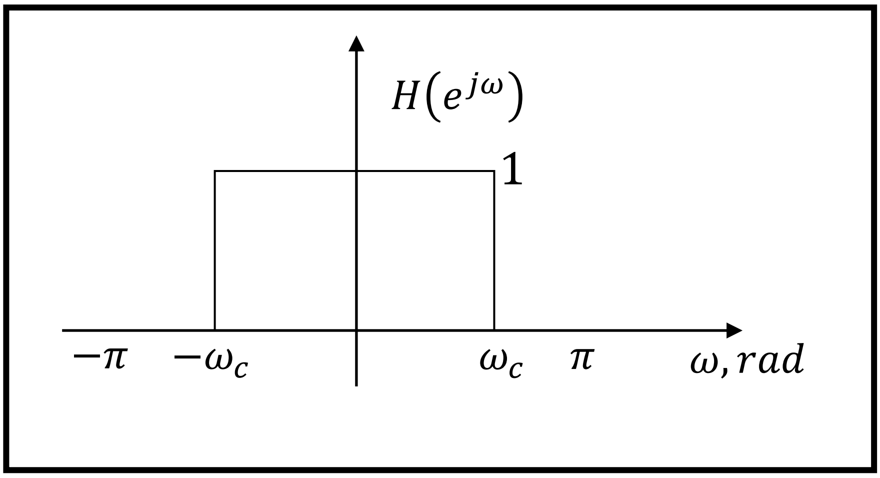
MEAN SQUARE CONVERGENCE

Some sequences, which are not absolutely summable but square summable

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 < \text{finite constant}$$

can still be represented by Fourier Transform, but…

Ex: Ideal lowpass filter.



$$H(e^{j\omega}) = \begin{cases} 1 & |\omega| < \omega_c \\ 0 & \omega_c < |\omega| \leq \pi \end{cases}$$

Let's find $h[n]$

$$h[n] = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} e^{j\omega n} d\omega$$

$$= \frac{1}{j2\pi n}\left(e^{j\omega_c n} - e^{-j\omega_c n}\right)$$

$$= \frac{\sin(\omega_c n)}{\pi n}$$

Note that,

$$h[n] = \frac{\sin(\omega_c n)}{\pi n}$$

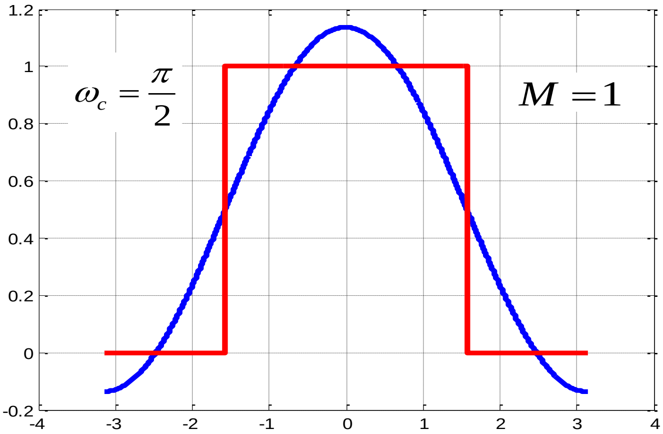
is not absolutely summable!

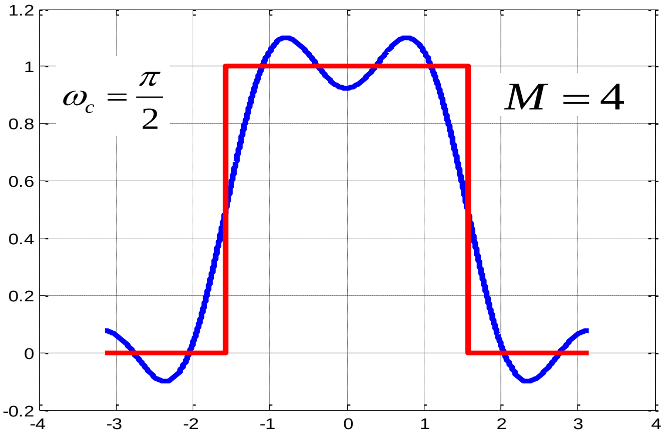
Then, one may question the Fourier transform of $h[n]$,

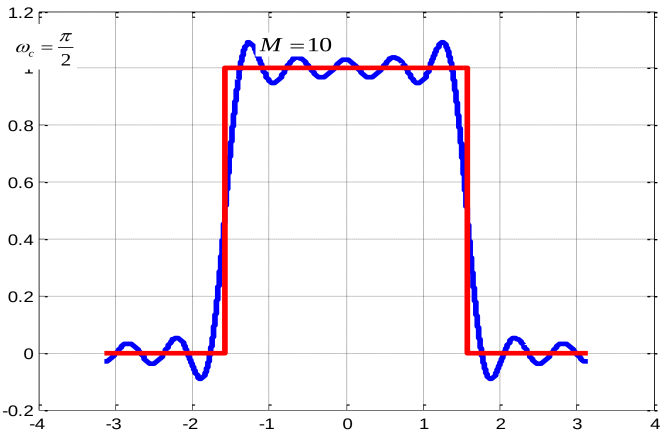
$$\sum_{n=-\infty}^{\infty} \frac{\sin(\omega_c n)}{\pi n} e^{-j\omega n} = ?$$

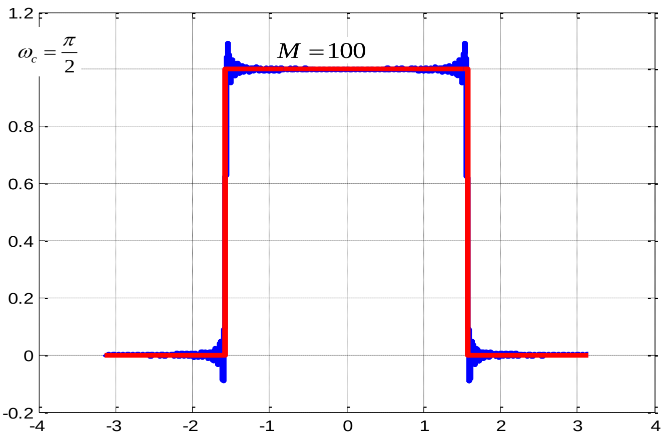
Define

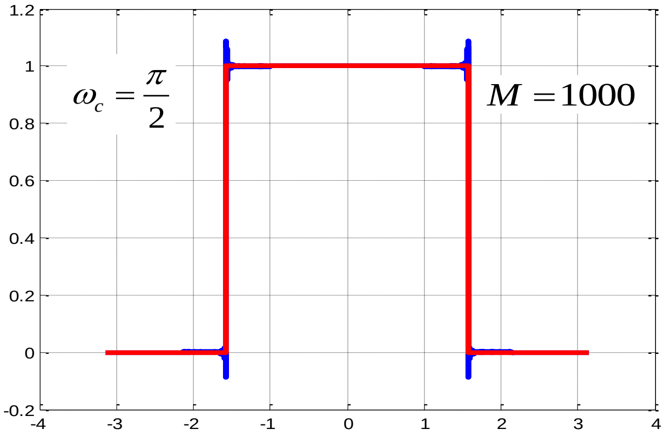
$$H_M = \sum_{n=-M}^{M} \frac{\sin(\omega_c n)}{\pi n} e^{-j\omega n}$$

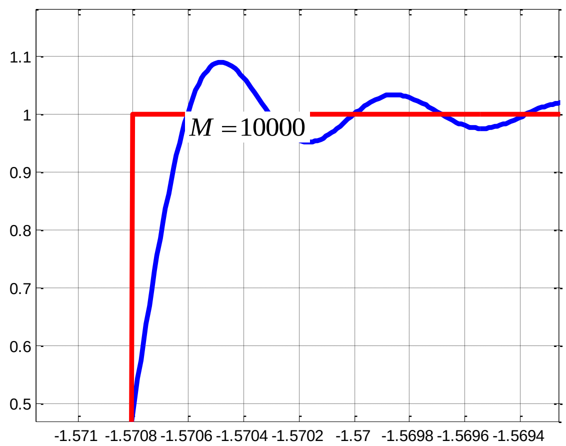

$\omega_c = \frac{\pi}{2}$
$M = 1$
1.2 1 0.8 0.6 0.4 0.2 0 -0.2
-4 -3 -2 -1 0 1 2 3 4


$\omega_c = \frac{\pi}{2}$
$M = 4$
1.2 1 0.8 0.6 0.4 0.2 0 -0.2
-4 -3 -2 -1 0 1 2 3 4


$\omega_c = \frac{\pi}{2}$
$M = 10$
1.2 0.8 0.6 0.4 0.2 0 -0.2
-4 -3 -2 -1 0 1 2 3 4


$\omega_c = \frac{\pi}{2}$
$M = 100$
1.2 0.8 0.6 0.4 0.2 0 -0.2
-4 -3 -2 -1 0 1 2 3 4


$\omega_c = \frac{\pi}{2}$
$M = 1000$
1.2 1 0.8 0.6 0.4 0.2 0 -0.2
-4 -3 -2 -1 0 1 2 3 4


$M = 10000$
1.1 1 0.9 0.8 0.7 0.6 0.5
-1.571 -1.5708 -1.5706 -1.5704 -1.5702 -1.57 -1.5698 -1.5696 -1.5694

detail

Even if you take $M \to \infty$ oscillations do not die to zero.

However

$$\lim_{M\to\infty} \int_{-\pi}^{\pi} \left|H\left(e^{j\omega}\right) - H_M\left(e^{j\omega}\right)\right|^2 d\omega = 0.$$

This is called “mean square” convergence.

The oscillatory behavior around $\omega = \omega_c$ is called the Gibbs phenomenon.

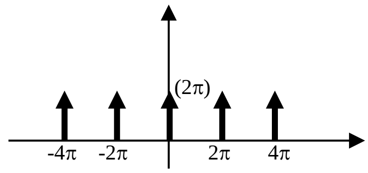
MATLAB code

```
clear all;   close all;
precision =0.0001
w = [-pi:precision:pi];
idealL = zeros(1,length(w));
wc = pi/2;
orta = round(length(w)/2);
idealL((orta-round(wc/precision)):(orta+round(wc/precision)))=1;
M = 10000;
H = 0;
for n = -M:-1
   H = H+(sin(wc*n)/(pi*n))*exp(-i*w*n);
end
for n = 1:M
   H = H+(sin(wc*n)/(pi*n))*exp(-i*w*n);
end
   H = H+(wc/pi);
plot(w,H); hold on;
plot(w,idea,'r')
grid
```

FOURIER TRANSFORM OF A CONSTANT SEQUENCE

$$x[n] = 1 \qquad \leftrightarrow \qquad X\left(e^{j\omega}\right) = 2\pi \sum_{r=-\infty}^{\infty} \delta(\omega + 2\pi r)$$

not absolutely summable



or we can write as

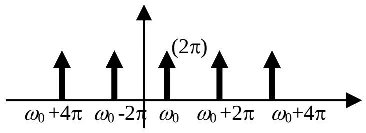
$$X\left(e^{j\omega}\right) = 2\pi\delta(\omega) \qquad 0 \leq \omega < 2\pi$$

keeping in mind that FT is periodic with 2π.

FOURIER TRANSFORM OF A COMPLEX EXPONENTIAL SEQUENCE

$$x[n] = e^{j\omega_0 n} \qquad \leftrightarrow \qquad X(e^{j\omega}) = 2\pi \sum_{r=-\infty}^{\infty} \delta(\omega - \omega_0 + 2\pi r)$$

not absolutely summable



or we can write as

$$X(e^{j\omega}) = 2\pi\delta(\omega - \omega_0) \qquad 0 \leq \omega \leq 2\pi$$

keeping in mind that FT is periodic with 2π.

FOURIER TRANSFORM OF A SINUSOIDAL SEQUENCE

$$x[n] = \cos(\omega_0 n)$$

$$= \frac{1}{2}\left(e^{j\omega_0 n} + e^{-j\omega_0 n}\right)$$

Note that it is not absolutely summable.

$$X\left(e^{j\omega}\right) = \pi\left(\sum_{r=-\infty}^{\infty} \delta(\omega + \omega_0 + 2r\pi) + \delta(\omega - \omega_0 + 2r\pi)\right)$$

FOURIER TRANSFORM OF UNIT STEP SEQUENCE

$$x[n] = u[n] \qquad \leftrightarrow \qquad X\left(e^{j\omega}\right) = \frac{1}{1 - e^{-j\omega}} + \pi \sum_{r=-\infty}^{\infty} \delta(\omega + 2\pi r)$$

not absolutely summable

SYMMETRY PROPERTIES OF FOURIER TRANSFORM

Definitions:

Conjugate symmetric (CS) sequence.	$x[n] = x^*[-n]$
Conjugate antisymmetric (CaS) sequence.	$x[n] = -x^*[-n]$

Using the above definitions, any sequence can be written as

$$x[n] = x_e[n] + x_o[n]$$

where

$$x_e[n] = \frac{1}{2}(x[n] + x^*[-n]) \text{ is the CS part}$$

and

$$x_o[n] = \frac{1}{2}(x[n] - x^*[-n]) \quad \text{is the CaS part.}$$

SYMMETRY PROPERTIES

Fundamental relations

Let $x[n] \leftrightarrow X\left(e^{j\omega}\right)$ be a FT pair. Then, the following hold:

$x^*[n] \leftrightarrow X^*\left(e^{-j\omega}\right)$ since $X^*\left(e^{-j\omega}\right) = \sum_{n=-\infty}^{\infty} x^*[n] e^{-j\omega n}$

and

$x[-n] \leftrightarrow X\left(e^{-j\omega}\right)$ since $X\left(e^{-j\omega}\right) = \sum_{n=-\infty}^{\infty} x[-n] e^{-j\omega n}$

Above yields

$$x^*[-n] \leftrightarrow X^*\left(e^{j\omega}\right)$$

The two relations above also yield:

1)

$$Re\{x[n]\} = \frac{x[n] + x^*[n]}{2} \quad \leftrightarrow \quad \underbrace{X_e(e^{j\omega}) = \frac{X(e^{j\omega}) + X^*(e^{-j\omega})}{2}}_{\text{CS part of } X(e^{j\omega})}$$

2)

$$jIm\{x[n]\} = \frac{x[n] - x^*[n]}{2} \quad \leftrightarrow \quad \underbrace{X_o(e^{j\omega}) = \frac{X(e^{j\omega}) - X^*(e^{-j\omega})}{2}}_{\text{CaS part of } X(e^{j\omega})}$$

3)

$$x_e[n] = \frac{x[n] + x^*[-n]}{2} \quad \leftrightarrow \quad Re\{X(e^{j\omega})\} = \frac{X(e^{j\omega}) + X^*(e^{j\omega})}{2}$$

Therefore FT of an even sequence is real!

4)

$$x_o[n] = \frac{x[n] - x^*[-n]}{2} \quad \leftrightarrow \quad jIm\{X(e^{j\omega})\} = \frac{X(e^{j\omega}) - X^*(e^{j\omega})}{2}$$

Therefore FT of an odd sequence is purely imaginary!

Ex: Let $a[n]$ and $b[n]$ be two real sequences with their DTFTs $A(e^{j\omega})$ and $B(e^{j\omega})$, respectively.

Let

$$x[n] = a[n] + jb[n]$$

Then,

$$X(e^{j\omega}) = A(e^{j\omega}) + jB(e^{j\omega})$$

Note that $A(e^{j\omega})$ is NOT the real part of $X(e^{j\omega})$.

However,

$$A(e^{j\omega}) = \frac{X(e^{j\omega}) + X^*(e^{-j\omega})}{2}$$

i.e. conjugate symmetric part of $X(e^{j\omega})$,

since

$$X^*(e^{-j\omega}) = \underbrace{A^*(e^{-j\omega})}_{A(e^{j\omega})} - j\,\underbrace{B^*(e^{-j\omega})}_{B(e^{j\omega})} .$$

Similarly,

$$jB(e^{j\omega}) = \frac{X(e^{j\omega}) - X^*(e^{-j\omega})}{2}$$

i.e. conjugate antisymmetric part of $X(e^{j\omega})$.

Ex: (cont'd)

$$a[n]:\ [-1\ \ 1]$$

$$b[n]:\ [1\ \ 1]$$

$$x[n]:\ [-1+j \quad 1+j]$$

$$A\left(e^{j\omega}\right) = -1 + e^{-j\omega}$$

$$B\left(e^{j\omega}\right) = 1 + e^{-j\omega}$$

$$X\left(e^{j\omega}\right) = -1 + e^{-j\omega} + j + je^{-j\omega}$$

$$X^*\left(e^{-j\omega}\right) = -1 + e^{-j\omega} - j - je^{-j\omega}$$

$$\frac{X\left(e^{j\omega}\right) + X^*\left(e^{-j\omega}\right)}{2} = -1 + e^{-j\omega}$$

$$\frac{X\left(e^{j\omega}\right) - X^*\left(e^{-j\omega}\right)}{2} = j + je^{-j\omega}$$

Based on the above relations, for real sequences ($x[n] = x^*[n]$):

$$X\left(e^{j\omega}\right) = X^*\left(e^{-j\omega}\right) \quad \text{(conjugate symmetry)}$$

which implies

Magnitude is even………………………… $\left|X\left(e^{j\omega}\right)\right| = \left|X\left(e^{-j\omega}\right)\right|$

Phase is odd………………………………. $\measuredangle X\left(e^{j\omega}\right) = -\measuredangle X\left(e^{-j\omega}\right)$

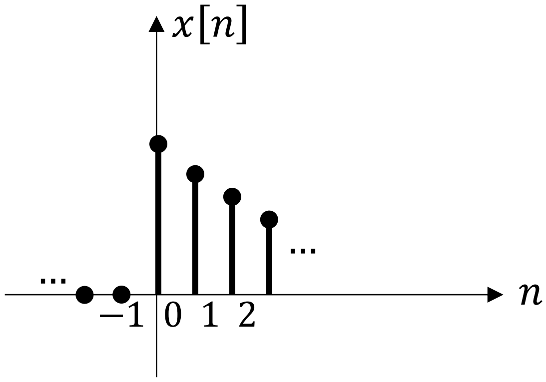
Real part is even……………………………. $Re\left\{X\left(e^{j\omega}\right)\right\} = Re\left\{X\left(e^{-j\omega}\right)\right\}$

Imaginary part is odd......................... $Im\left\{X\left(e^{j\omega}\right)\right\} = -Im\left\{X\left(e^{-j\omega}\right)\right\}$

Verification by an example

$$x[n] = a^n u[n] \quad \leftrightarrow \quad X(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}} \quad |a| < 1$$

$0 < a < 1$



a) FT is conjugate symmetric:

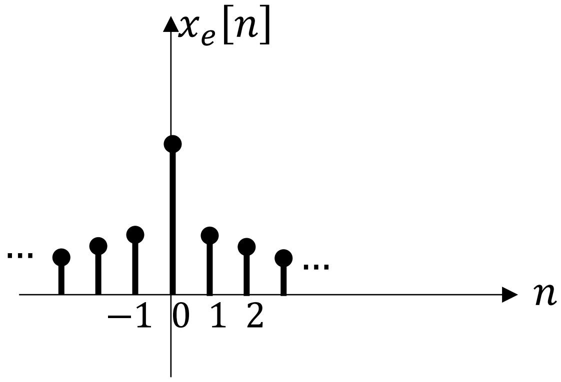
$$X(e^{j\omega}) = X^*(e^{-j\omega}) = \frac{1}{1 - ae^{-j\omega}}$$

b) Real part of FT is an even function:

$$Re\{X(e^{j\omega})\} = Re\{X(e^{-j\omega})\} = \frac{1 - a\cos(\omega)}{1 + a^2 - 2a\cos(\omega)}$$

c) $Re\{X(e^{j\omega})\}$ is the FT of $x_e[n]$:

$0 < a < 1$



$$Re\{X(e^{j\omega})\} = \frac{1 - a\cos(\omega)}{1 + a^2 - 2a\cos(\omega)}$$

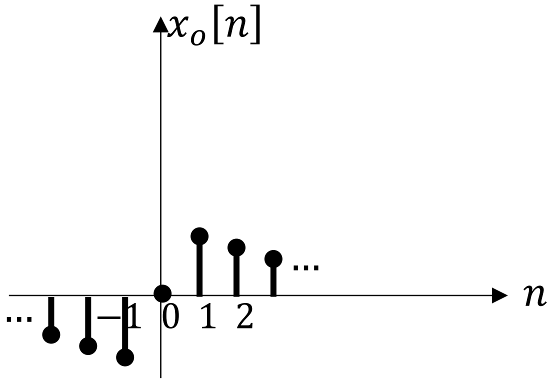
$$x_e[n] = \frac{1}{2}(a^n u[n] + a^{-n}u[-n])$$

d) Imaginary part of FT is an odd function.

$$Im\{X(e^{j\omega})\} = -Im\{X(e^{-j\omega})\} = \frac{-a\sin(\omega)}{1 + a^2 - 2a\cos(\omega)}$$

e) $Im\{X(e^{j\omega})\}$ is the FT of $x_o[n]$:

$0 < a < 1$

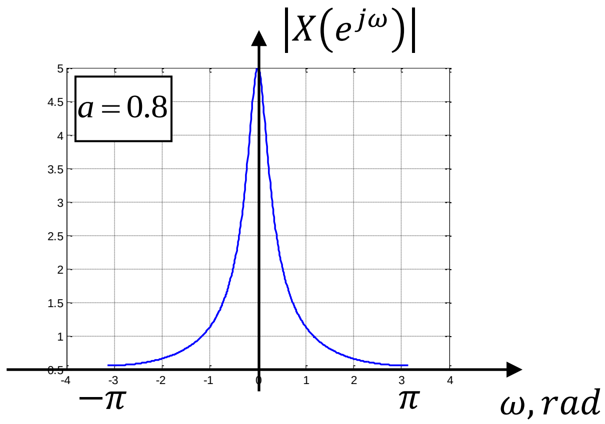


$$Im\{X(e^{j\omega})\} = \frac{-a\sin(\omega)}{1 + a^2 - 2a\cos(\omega)}$$

$$x_o[n] = \frac{1}{2}(a^n u[n] - a^{-n}u[-n])$$

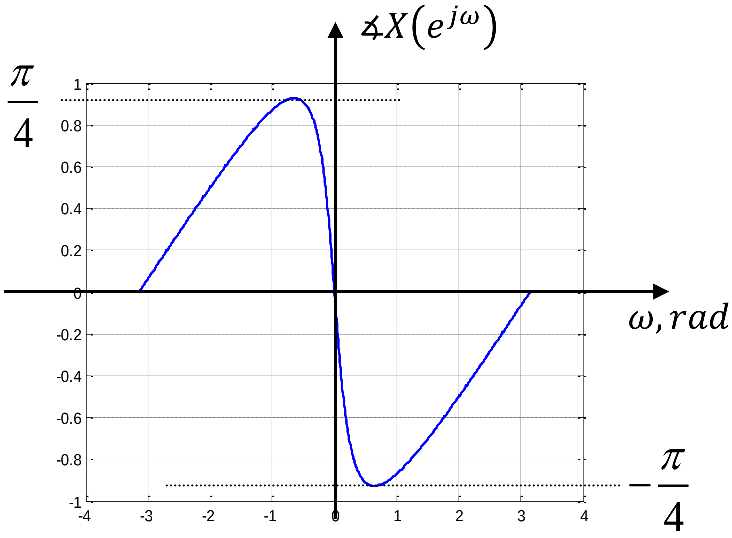
f) Magnitude of FT is an even function:

$$\left|X\left(e^{j\omega}\right)\right| = \left|X\left(e^{-j\omega}\right)\right| = \frac{1}{\sqrt{1 + a^2 - 2a\cos(\omega)}}$$



g) Phase of FT is an odd function

$$\measuredangle X(e^{j\omega}) = -\measuredangle X(e^{j\omega}) = -\tan^{-1}\left(\frac{a\sin(\omega)}{1-a\cos(\omega)}\right)$$



FOURIER TRANSFORM THEOREMS

$$x[n] \leftrightarrow X(e^{j\omega})$$

1) $ax[n] + by[n] \leftrightarrow aX(e^{j\omega}) + bY(e^{j\omega})$ linearity

2) $x[n - n_0] \leftrightarrow e^{-j\omega n_0} X(e^{j\omega})$ time-shift

3) $e^{j\omega_0 n} x[n] \leftrightarrow X(e^{j(\omega-\omega_0)})$ freq.-shift

4) $x[-n] \leftrightarrow X(e^{-j\omega})$ time reversal

5) $nx[n] \leftrightarrow j\frac{dX(e^{j\omega})}{d\omega}$ differentiation in freq. domain

6) $x[n] * y[n] \leftrightarrow X(e^{j\omega})Y(e^{j\omega})$ convolution (proof below)

7)

$$x[n]y[n] \leftrightarrow \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$$

modulation, windowing

8) Parseval's theorem (prove as an exercise)

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left|X\left(e^{j\omega}\right)\right|^2 d\omega$$

Note that $\left|X\left(e^{j\omega}\right)\right|^2$ is called the "**energy density spectrum**".

9)

$$\sum_{n=-\infty}^{\infty} x[n]y^*[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X\left(e^{j\omega}\right)Y^*\left(e^{j\omega}\right)d\omega$$

Proof of (6):

Let

$$w[n] \stackrel{\text{def}}{=} \sum_{k=-\infty}^{\infty} x[k]y[n-k] = x[n] * y[n]$$

$$W(e^{j\omega}) = ?$$

$$W(e^{j\omega}) = \sum_{n=-\infty}^{\infty}\left(\sum_{k=-\infty}^{\infty} x[k]y[n-k]\right)e^{-j\omega n}$$

$$= \sum_{k=-\infty}^{\infty} x[k]\left(\sum_{n=-\infty}^{\infty} y[n-k]e^{-j\omega n}\right)$$

$$= \sum_{k=-\infty}^{\infty} x[k]e^{-j\omega k}\left(\sum_{m=-\infty}^{\infty} y[m]e^{-j\omega m}\right)$$

$$= X(e^{j\omega})Y(e^{j\omega})$$

Proof of (8) using (6):

Let

$$w[n] \stackrel{\text{def}}{=} \sum_{k=-\infty}^{\infty} x[k]x^*[k-n] = x[n] * x^*[-n]$$

$$W(e^{j\omega}) = X(e^{j\omega})X^*(e^{j\omega}) = \left|X(e^{j\omega})\right|^2$$

$$w[0] = \sum_{k=-\infty}^{\infty} x[k]x^*[k] = \sum_{k=-\infty}^{\infty} |x[k]|^2$$

$$w[0] = \frac{1}{2\pi}\int_{-\pi}^{\pi} W(e^{j\omega})\, d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|X(e^{j\omega})\right|^2 d\omega$$

Proof of (9):

$$\sum_{n=-\infty}^{\infty} x[n]y^*[n]$$

$$= \sum_{n=-\infty}^{\infty} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n}\, d\omega \right) y^*[n]$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) \underbrace{\left(\sum_{n=-\infty}^{\infty} y^*[n] e^{j\omega n} \right)}_{Y^*(e^{j\omega})} d\omega$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) Y^*(e^{j\omega}) d\omega$$

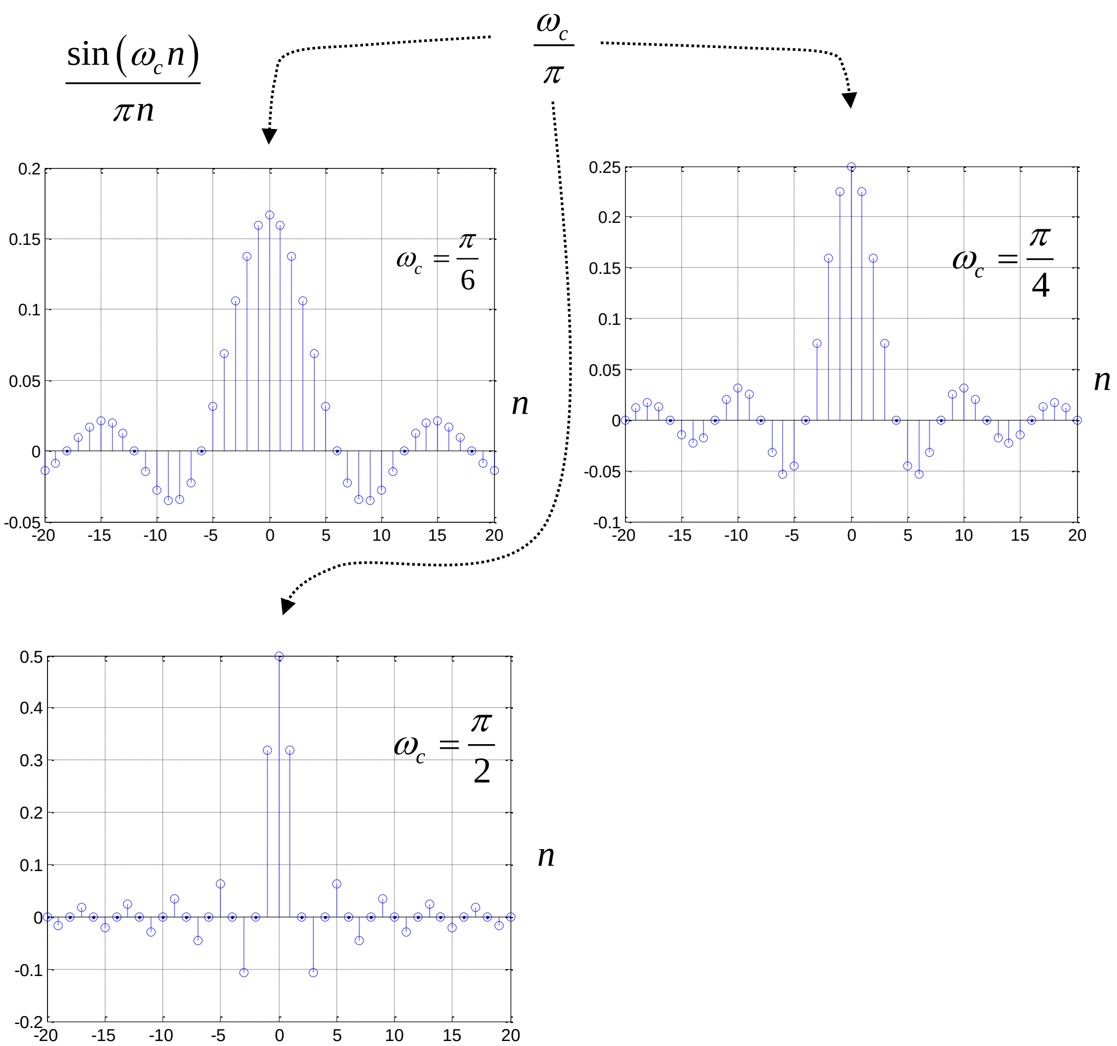
FOURIER TRANSFORM PAIRS

$\delta[n]$	1
$\delta[n-n_0]$	$e^{-j\omega n_0}$
$1 \qquad -\infty < n < \infty$	$X(e^{j\omega}) = 2\pi \sum_{k=-\infty}^{\infty} \delta[\omega + 2\pi k]$
$a^n u[n] \qquad \|a\| < 1$	$\frac{1}{1 - ae^{-j\omega}}$
$u[n]$	$\frac{1}{1 - ae^{-j\omega}} + \pi \sum_{k=-\infty}^{\infty} \delta(\omega + 2\pi k)$
$na^n u[n] \qquad \|a\| < 1$	$\frac{ae^{-j\omega}}{(1 - ae^{-j\omega})^2}$
$(n+1)a^n u[n]$ $= (a^n u[n]) * (a^n u[n])$ $\|a\| < 1$	$\frac{1}{(1 - ae^{-j\omega})^2}$
$\frac{1}{2}(n+2)(n+1)a^n u[n]$	$\frac{1}{(1 - ae^{-j\omega})^3}$
$\frac{(n+k-1)!}{(k-1)!\,n!} a^n u[n]$	$\frac{1}{(1 - ae^{-j\omega})^k}$
$\frac{1}{\sin(\omega_0)} r^n \sin\big(\omega_0(n+1)\big)\, u[n]$ $\|r\| < 1$	$\frac{1}{1 - 2r\cos(\omega_0)\, e^{-j\omega} + r^2 e^{-j2\omega}}$ show using the DTFT of $a^n u[n]$

$\dfrac{\sin(\omega_c n)}{\pi n}$	$X(e^{j\omega}) = \begin{cases} 1 & \lvert\omega\rvert < \omega_c \\ 0 & \omega_c < \lvert\omega\rvert \le \pi \end{cases}$
$x[n] = \begin{cases} 1 & 0 \le n \le M \\ 0 & otherwise \end{cases}$	$\dfrac{\sin\left(\omega \dfrac{(M+1)}{2}\right)}{\sin\left(\dfrac{\omega}{2}\right)} e^{-j\omega\frac{M}{2}}$
$e^{j\omega_0 n}$	$2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - \omega_0 + 2\pi k)$
$\cos(\omega_0 n + \phi)$	$\pi e^{j\phi} \sum_{k=-\infty}^{\infty} \delta(\omega - \omega_0 + 2\pi k)$ $+$ $\pi e^{-j\phi} \sum_{k=-\infty}^{\infty} \delta(\omega + \omega_0 + 2\pi k)$
$\sin(\omega_0 n + \phi)$	$-j\pi e^{j\phi} \sum_{k=-\infty}^{\infty} \delta(\omega - \omega_0 + 2\pi k)$ $+$ $j\pi e^{-j\phi} \sum_{k=-\infty}^{\infty} \delta(\omega + \omega_0 + 2\pi k)$

Ex: IDEAL LOWPASS FILTER (IMPULSE RESPONSE)

Plots of sin $\frac{(\omega_c n)}{\pi n}$ sequence

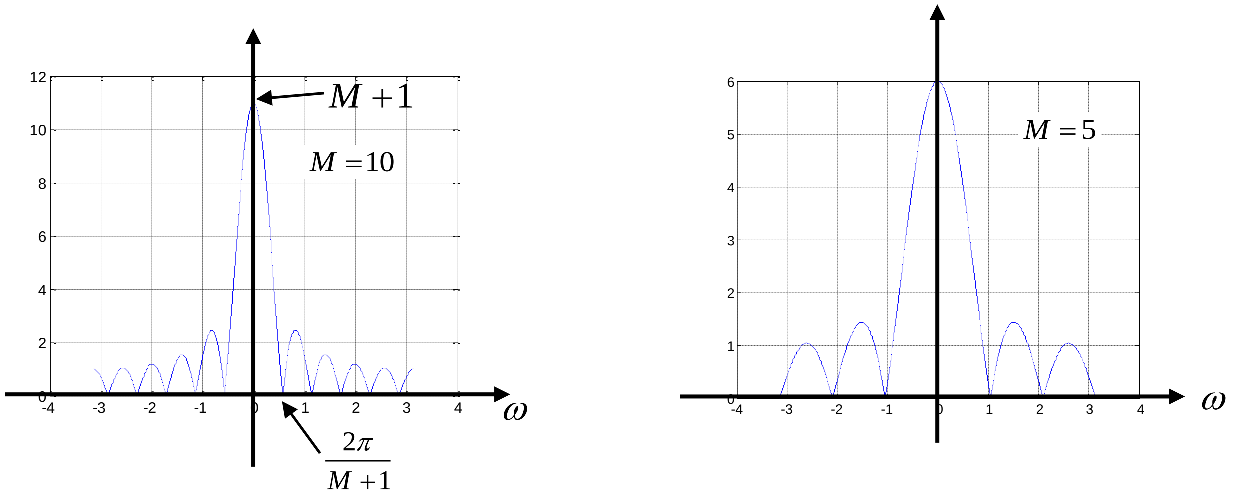


They are infinitely long sequences in $-\infty < n < \infty$
Plots are arbitrarily in $-20 < n < 20$

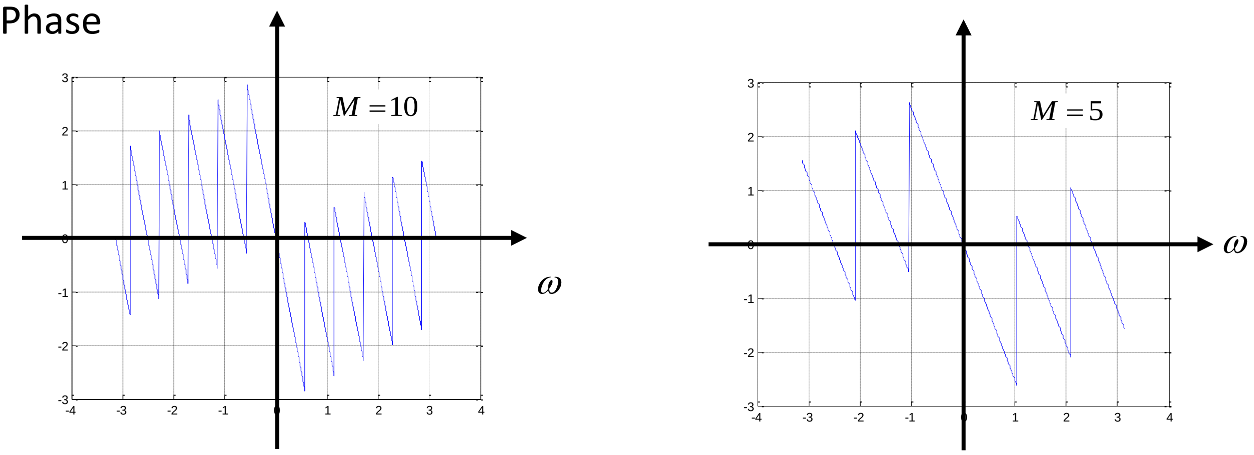
Ex: MOVING AVERAGE FILTER

Magnitude

$$\frac{\sin\left(\omega\frac{(M+1)}{2}\right)}{\sin\left(\frac{\omega}{2}\right)}e^{-j\omega\frac{M}{2}}$$



Phase



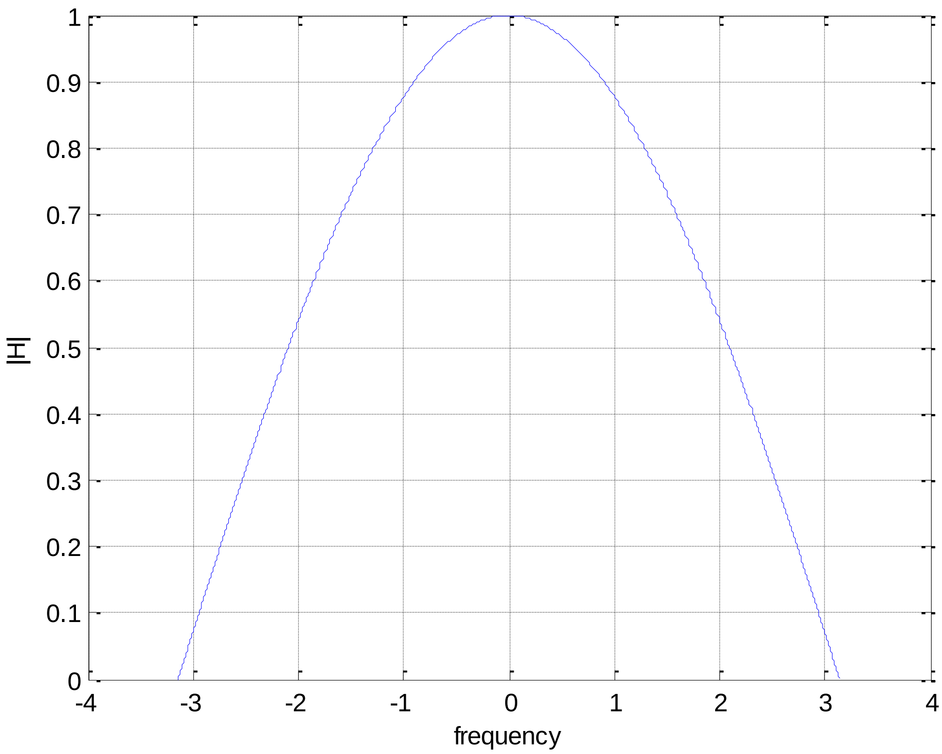
DTFT functions are plotted in $-\pi \leq \omega < \pi$ or in $0 \leq \omega < 2\pi$

$M = 1$

$$h[n] = \frac{1}{2}(\delta[n] + \delta[n-1]) \leftrightarrow H(e^{j\omega}) = \frac{1}{2}(1 + e^{-j\omega})$$

$$= e^{-j\frac{\omega}{2}} \cos\left(\frac{\omega}{2}\right)$$

$$y[n] = \frac{1}{2}(x[n] + x[n-1])$$



$M = 4$

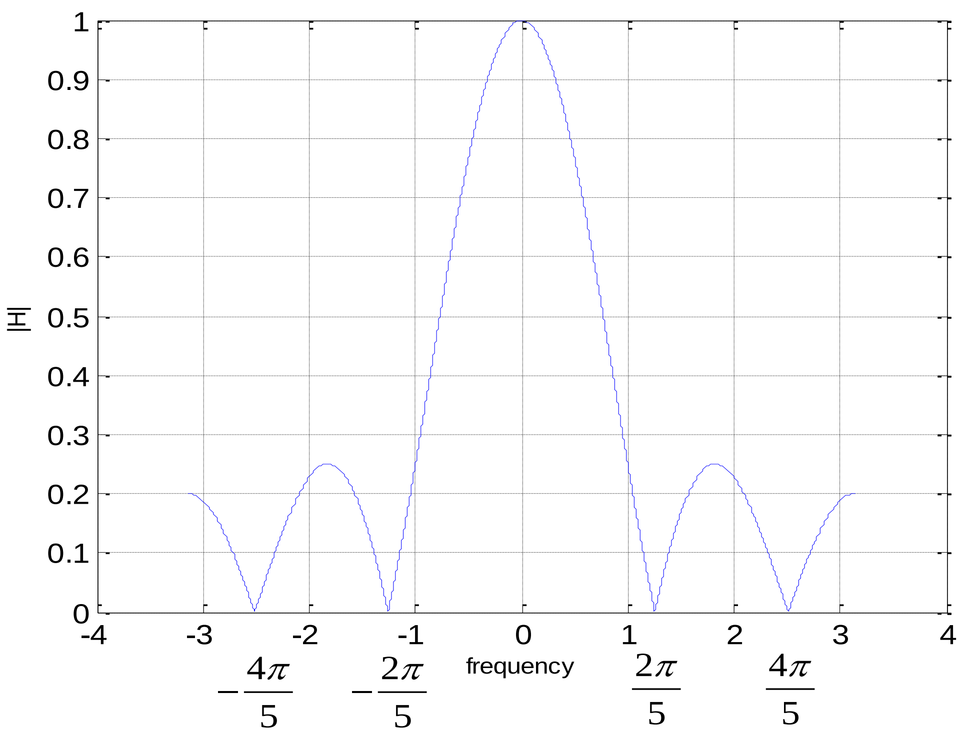
$$h[n] = \frac{1}{5}(\delta[n] + \delta[n-1] + \delta[n-2] + \delta[n-3] + \delta[n-4]) \leftrightarrow H(e^{j\omega})y[n]$$

$$y[n] = \frac{1}{2}(x[n] + x[n-1] + x[n-2] + x[n-3] + x[n-4])$$

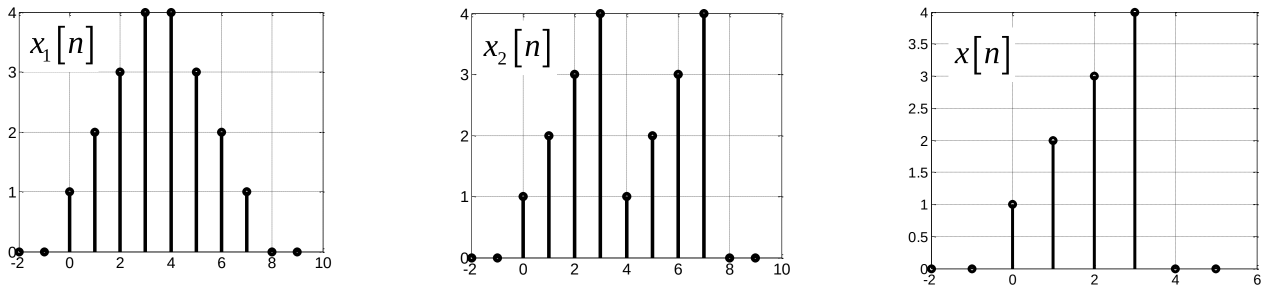
$$H(e^{j\omega}) = \frac{1}{5}\left(1 + e^{-j\omega} + e^{-j2\omega} + e^{-j3\omega} + e^{-j4\omega}\right)$$

$$= \frac{1}{5}e^{-j2\omega}(1 + 2\cos(\omega) + 2\cos(2\omega))$$

$$= \frac{1}{5}e^{-j2\omega}\frac{\sin\left(\frac{5}{2}\omega\right)}{\sin\left(\frac{\omega}{2}\right)}$$



Ex: Express $X_1(e^{j\omega})$, $X_2(e^{j\omega})$, in terms of $X(e^{j\omega})$, the DTFT of $x[n]$.



$$x_1[n] = x[n] + x[-(n-7)]$$

$$X_1(e^{j\omega}) = X(e^{j\omega}) + e^{-j7\omega}X(e^{-j\omega})$$

$$x_2[n] = x[n] + x[n-4]$$

$$X_2(e^{j\omega}) = X(e^{j\omega}) + e^{-j4\omega}X(e^{j\omega})$$

One can also write as

$$
\begin{aligned}
X_1(e^{j\omega}) &= X(e^{j\omega}) + e^{-j7\omega}X(e^{-j\omega}) \\
&= X(e^{j\omega}) + e^{-j7\omega}X^*(e^{j\omega}) \qquad \text{since } x[n] \text{ is real} \\
&= \left|X(e^{j\omega})\right|\left(e^{j\measuredangle X(e^{j\omega})} + e^{-j7\omega}e^{-j\measuredangle X(e^{j\omega})}\right) \\
&= \left|X(e^{j\omega})\right|e^{-j\frac{7}{2}\omega}\left(e^{j\frac{7}{2}\omega}e^{j\measuredangle X(e^{j\omega})} + e^{-j\frac{7}{2}\omega}e^{-j\measuredangle X(e^{j\omega})}\right) \\
&= 2\,\mathrm{Re}\left\{e^{j\left(\measuredangle X(e^{j\omega})+\frac{7}{2}\omega\right)}\right\}\left|X(e^{j\omega})\right|e^{-j\frac{7}{2}\omega}
\end{aligned}
$$

$$
\begin{aligned}
X_2(e^{j\omega}) &= X(e^{j\omega})(1 + e^{-j4\omega}) \\
&= X(e^{j\omega})e^{-j2\omega}(e^{j2\omega} + e^{-j2\omega}) \\
&= X(e^{j\omega})e^{-j2\omega}2\cos(2\omega) \\
&= 2\cos(2\omega)\left|X(e^{j\omega})\right|e^{-j2\omega}e^{j\angle X(e^{j\omega})}
\end{aligned}
$$

Ex: What is the inverse DTFT, $y[n]$, of

$$Y\left(e^{j\omega}\right) = \frac{2e^{-j3\omega}}{\left(1 - \frac{1}{8}e^{-j\omega}\right)^2} \text{ ?}$$

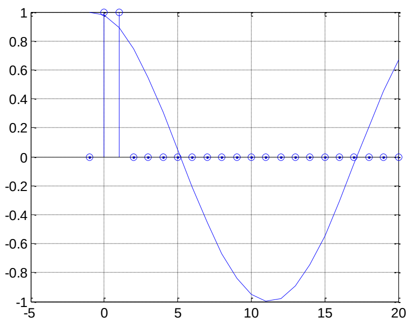
From the table

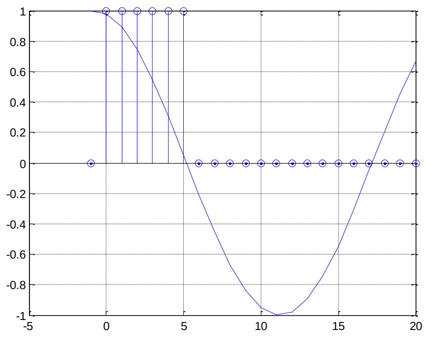
$$(n+1)a^n u[n] \quad \leftrightarrow \quad \frac{1}{(1 - ae^{-j\omega})^2} \quad |a| < 1$$

$$2(n+1)\left(\frac{1}{8}\right)^n u[n] \quad \leftrightarrow \quad \frac{2}{\left(1 - \frac{1}{8}e^{-j\omega}\right)^2}$$

$$2(n-2)\left(\frac{1}{8}\right)^{n-3} u[n-3] \quad \leftrightarrow \quad \frac{2e^{-j3\omega}}{\left(1 - \frac{1}{8}e^{-j\omega}\right)^2}$$

Why does the high frequency gain of MA filter “<u>decrease</u>” as M increases?





Comment on the above illustrations.

LCCDEs AND FREQUENCY RESPONSE

$$y[n] = x[n] * h[n] \xleftrightarrow{FT} Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$$

$$\Rightarrow H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})}$$

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k] \xleftrightarrow{FT} \sum_{k=0}^{N} a_k e^{-jk\omega} Y(e^{j\omega})$$
$$= \sum_{k=0}^{M} b_k e^{-jk\omega} X(e^{j\omega})$$

$$\Rightarrow \quad Y(e^{j\omega}) \sum_{k=0}^{N} a_k e^{-jk\omega} = X(e^{j\omega}) \sum_{k=0}^{M} b_k e^{-jk\omega}$$

$$\Rightarrow \quad \frac{Y(e^{j\omega})}{X(e^{j\omega})} = H(e^{j\omega}) = \frac{\sum_{k=0}^{M} b_k e^{-jk\omega}}{\sum_{k=0}^{N} a_k e^{-jk\omega}}$$

$$H(e^{j\omega}) = \frac{b_0 + b_1 e^{-j\omega} + b_2 e^{-j2\omega} + \cdots + b_M e^{-jM\omega}}{a_0 + a e^{-j\omega} + a_2 e^{-j2\omega} + \cdots + a_N e^{-jN\omega}}$$