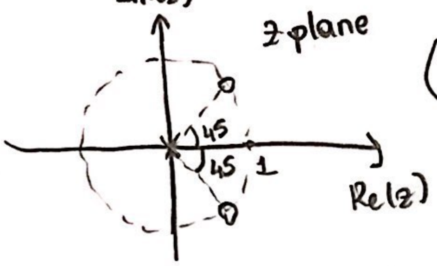
HW-2 Solutions

3) $H(z) = 1 - \sqrt{2} \cdot z^{-1} + z^{-2}$ ROC = entire z plane $- \{0\}$

a)

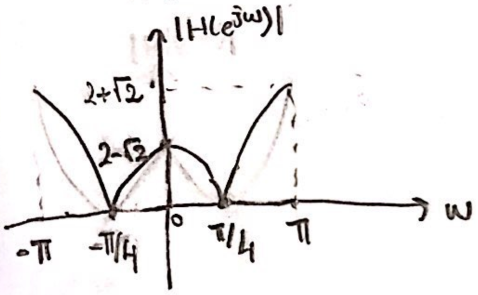
$$= \frac{z^2 - \sqrt{2}z + 1}{z^2} = \frac{(z - e^{+j\frac{\pi}{4}})(z - e^{-j\frac{\pi}{4}})}{z^2}$$

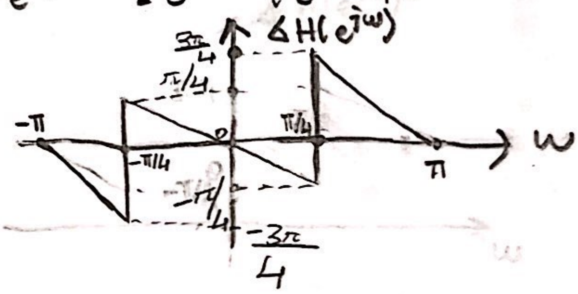


(two poles at 0)

b) Yes, because $\boxed{|z| = 1 \in ROC}$

$$H(e^{j\omega}) = 1 - \sqrt{2}\, e^{-j\omega} + e^{-j2\omega} = e^{-j\omega}\left(e^{+j\omega} + e^{-j\omega} - \sqrt{2}\right) = e^{-j\omega}\left(2\cos(\omega) - \sqrt{2}\right)$$





c) $H(e^{j\frac{\pi}{4}}) = 1 - \sqrt{2} \cdot e^{-j\frac{\pi}{4}} + e^{-j\frac{\pi}{2}} = 1 - (1 - j) - j = 0 \rightarrow$ as we expected

$x_1[n] * h[n] = 0$

$H(e^{j\frac{3\pi}{4}}) = e^{-j\frac{3\pi}{4}}(-2\sqrt{2}) = 2\sqrt{2} \cdot e^{j\frac{\pi}{4}}$

$x_3[n] * h[n] = 2\sqrt{2} \cdot \sin\left(\frac{3\pi}{4}n + \frac{\pi}{4}\right)$

$x_2[n] \xrightarrow{z} X_2[z] = \dfrac{\sin\frac{\pi}{4} + z^{-1} \cdot \sin(\omega_0 - \frac{\pi}{4})}{(1 - z^{-1}e^{j\omega_0})(1 - z^{-1} \cdot e^{-j\omega_0})}$

$(\omega_0 = \pi/4)$

$= \dfrac{\sin \pi/4}{(1 - z^{-1} \cdot e^{j\pi/4})(1 - z^{-1} \cdot e^{-j\pi/4})}$

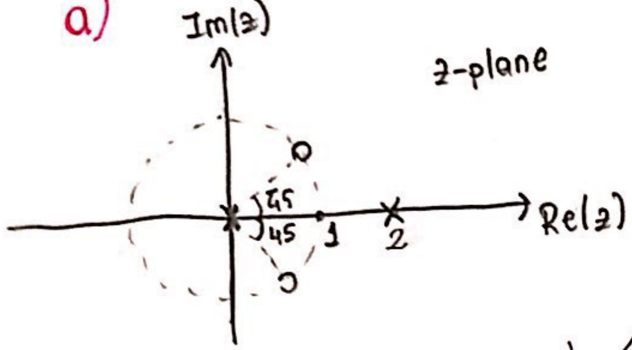
$X_2[z] \cdot H[z] = \sin \pi/4$

$\boxed{x_2[n] * h[n] = \sin\pi/4 \cdot \delta[n]}$

d) If there is a zero on the unit circle, the frequ. response become zero at corrosponding frequency.

4) $H(z) = \dfrac{1-\sqrt{2}\,z^{-1}+z^{-2}}{1-2z^{-1}} = \dfrac{(z-e^{+j\frac{\pi}{4}})(z-e^{-j\frac{\pi}{4}})}{z.\,(z-2)}$

a)



$H(e^{j\frac{\pi}{2}}) = \sqrt{\frac{2}{5}}\,.\,e^{j(\tan^{-1}\frac{1}{2})}$

ROC should contain $z = e^{j\frac{\pi}{2}}$

$\downarrow$

$(|z|<2$ and $|z| \neq 0)$

(output $y[n]$ is bounded.)

$$H(z) = 1 + \frac{(2-\sqrt{2})z+1}{z(z-2)}$$

$$H(z)-1 = \frac{A}{z} + \frac{B}{z-2} \Rightarrow A = -1/2 \qquad B = \frac{5-2\sqrt{2}}{2}$$

$$H(z) = 1 - \frac{1}{2}\,.\,z^{-1} + \frac{5-2\sqrt{2}}{2(z-2)}$$

$$h[n] = \delta[n] - \frac{1}{2}\,\delta[n-1] + \underbrace{(2.5-\sqrt{2})(-2^{n-1}u[-n])}_{}$$

b) non-causal part

(system is non-causal)

$|z| = 1 \in ROC$

(stable system)

c) $H(z) = \dfrac{1-\sqrt{2}\,z^{-1}+z^{-2}}{1-2z^{-1}} = \dfrac{Y(z)}{X(z)}$

$$y[n] - 2y[n-1] = x[n] - \sqrt{2}\,x[n-1] + x[n-2]$$

6) Right sided means the region of

$$\sum_{n=-\infty}^{+\infty} x[n]\, z^{-n} \xrightarrow{\text{right sided}} \sum_{n=a}^{\infty} x[n] \cdot z^{-n}$$

$$\left| \sum_{n=a}^{\infty} x[n] \cdot (4e^{jw})^{-n} \right| < \underset{\substack{\uparrow \\ \text{bounded}}}{M}$$

$$y = \sum_{n=a}^{\infty} x[n] (4{,}1 \cdot e^{jw})^{-n} = \sum_{n=a}^{\infty} x[n] (4 \cdot e^{jw})^{-n} \left(\frac{4}{4.1}\right)^{n}$$

$$|y| < \left| \sum_{n=a}^{\infty} | x[n] \cdot (4 \cdot e^{jw})^{-n} | \left| \left(\frac{4}{4.1}\right)^{n} \right| \right| < M \cdot \underbrace{\sum_{n=a}^{\infty} \left(\frac{4}{4.1}\right)^{n}}_{\text{also bounded}} \Big\} \text{ finite result}$$

for $z = 4{,}1e^{jw}$, we have $|X(z)| < \underset{\substack{\downarrow \\ \text{finite}}}{K}$. $\rightarrow$ for all $x[n]$.

$z = 4{,}1e^{jw} \in ROC$

For $z = 3{,}9\, e^{jw} \rightarrow \sum_{n=a}^{\infty} \left(\frac{4}{3.9}\right)^{n} = \text{infinite}$

For example, $x[n] = (1)^{n} u[n-a]$, $X(z)$ exists for $z = 3.9e^{jw}$, but not for all $x[n]$.

$\downarrow$

for example

$x[n] = (3.95)^{n} u[n-a]$

ROC

8)

- $X(z) = \sum_{n=-\infty}^{+\infty} \left(\delta[n+1] + \left(\frac{1}{2}\right)^n u[n]\right) z^{-n} = z + \frac{1}{1-\frac{1}{2}z^{-1}} \quad \left(\frac{1}{2} < |z| < \infty\right)$

$= \frac{z + 1/2}{1-\frac{1}{2}z^{-1}} = \frac{z(z+1/2)}{(z-\frac{1}{2})}$

poles $\rightarrow z = 1/2, \infty$

zeros $\rightarrow z = -1/2, 0$

- $x[n-5] \xrightarrow{z} \frac{z+1/2}{z^4(z-1/2)} \quad \left(\frac{1}{2} < |z|\right)$

poles $\rightarrow z = 1/2, 0, 0, 0, 0$

zeros $\rightarrow z = -1/2, \infty, \infty, \infty, \infty$

- $n \cdot x[n] \xrightarrow{z} -z \frac{d}{dz} X(z)$ ROC $\rightarrow |z| > \frac{1}{2}$

$-z \cdot \frac{d}{dz} X(z) = -z \cdot \frac{z^2 - z - 1/4}{(z-1/2)^2} = \frac{-z \cdot (z - \frac{1}{2} - \frac{1}{\sqrt{2}})(z - \frac{1}{2} + \frac{1}{\sqrt{2}})}{(z-1/2)^2}$

poles $\rightarrow 1/2, 1/2, \infty$

zeros $\rightarrow \frac{1}{2} + \frac{1}{\sqrt{2}}, \frac{1}{2} - \frac{1}{\sqrt{2}}, 0$

ROC : $\frac{1}{2} < |z| < \infty$

- $\sum_{n=-\infty}^{+\infty} \cos\left(\frac{\pi}{2} n\right) x[n] z^{-n} = \frac{1 - \frac{1}{2}\cos\left(\frac{\pi}{2}\right) z^{-1}}{1 - \cos\left(\frac{\pi}{2}\right) z^{-1} + \frac{1}{4} z^{-2}} = \frac{1}{1 + \frac{1}{4} z^{-2}} = \frac{z^2}{z^2 + 1/4}$

$\downarrow$

$(z - \frac{1}{2}j)(z + \frac{1}{2}j)$

$\left(\delta[n+1] \cos\left(\frac{\pi}{2} n\right) = 0\right)$

ROC $\rightarrow |z| > \frac{1}{2}$

poles $\rightarrow z = \frac{1}{2}j, -\frac{1}{2}j$

zeros $\rightarrow z = 0, 0$

g) Stable

a) ROC $\rightarrow \frac{1}{2} < |z| < 3$

$$H(z) = \frac{(z+1/2)}{(z-3)(z-1/2)} \cdot A$$

$$H(1) = \frac{3/2 \cdot A}{(-2)(1/2)} = 1$$

$$\boxed{A = -2/3}$$

$$H(z) = \frac{-2/3\,(z+1/2)}{(z-3)(z-1/2)}$$

$$= \frac{B}{z-3} + \frac{C}{z-1/2} \Rightarrow B = \frac{-\frac{2}{3} \cdot \frac{7}{2}}{\frac{5}{2}} = \boxed{-\frac{14}{15} = B}$$

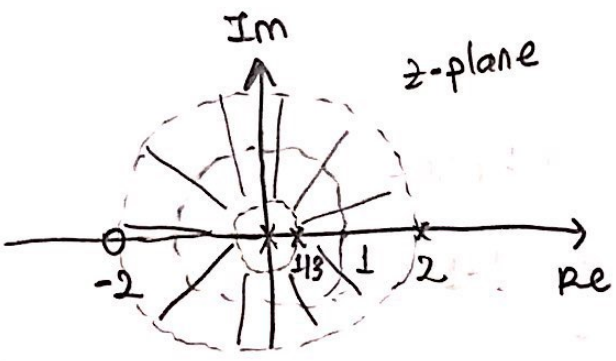
$$C = \frac{-2/3 \cdot 1}{-5/2} = \boxed{\frac{4}{15} = C}$$

$$h[n] = \frac{4}{15}\left(\frac{1}{2}\right)^{n-1} u[n-1] + \frac{14}{15}\, 3^{(n-1)}\, u[-n]$$

b) $$H_1(z) = \sum_{n=-\infty}^{+\infty} h[\underbrace{-n+2}_{-n+2}] \cdot z^{-n}$$

$$= \sum_{n=-\infty}^{+\infty} h[n]\, z^{+n-2} = z^{-2} H(z^{-1}) = \frac{z^{-2}(z^{-1}+1/2)}{(z^{-1}-3)(z^{-1}-1/2)} \cdot -\frac{2}{3}$$

$$= \frac{-2 z^{-2}(z^{-1}+1/2)}{3(z^{-1}-1/2)(z^{-1}-3)}$$



$$= \frac{-2(z+2)}{9z(z-2)(z-1/3)}$$

ROC: $\frac{1}{3} < |z| < 2$

11) a) $X(z) = \frac{1}{1-z^{-1}}$ $|z|>1$

(3.30)

$H(z) \rightarrow$ ROC: $0{,}5<|z|$
$\downarrow$
causal

$$Y(z) = \frac{1}{(1-0{,}5z^{-1})(1+0{,}5z^{-1})} = \frac{A}{\underset{(1+0{,}5z^{-1})}{(1-0{,}5z^{-1})}} + \frac{B}{\underset{(1-0{,}5z^{-1})}{(1+0{,}5z^{-1})}}$$

$$A+\frac{A}{2}z^{-1}+B-\frac{B}{2}z^{-1} = 1$$

$\boxed{A=B=1/2}$

$$\boxed{y[n] = (0{,}5)^n u[n] + (-0{,}5)^n u[n]}$$

b) $Y(z) = 1-z^{-1}$ $X(z).H(z)=Y(z)$

$$X(z) = (1-0{,}5z^{-1})(1+0{,}5z^{-1}) \Rightarrow x[n] = \delta[n] - 0{,}25\,\delta[n-2]$$

$$= 1-0{,}25z^{-2}$$

c) ROC contains $|z|=1 \Rightarrow$ stable system

$$H(e^{j\omega}) = \frac{1-e^{-j\omega}}{1-0.25.e^{-j2\omega}}$$

$$H(e^{j0{,}5\pi}) = \frac{1-e^{-j\frac{\pi}{2}}}{1-0.25.e^{-j\pi}} = \frac{1+j}{1+0{,}25} = \frac{\sqrt{2}}{1{,}25}.e^{j\frac{\pi}{4}}$$

$$x[n] * h[n] = \boxed{\frac{\sqrt{2}}{1{,}25} \cdot \cos\left(\frac{\pi}{2}n+\frac{\pi}{4}\right) = y[n]}$$

12) $x[n] = \delta[n] + a\,\delta[n-N] \quad |a|<1$

(3.52) $X(z) = 1 + a \cdot z^{-N}$

$\boxed{\log(1+x) = +x - \frac{x^2}{2} + \frac{x^3}{3} \cdots}$

$$\hat{X}(z) = \log X(z) = a \cdot z^{-N} - \frac{a^2 \cdot z^{-2N}}{2} + \frac{a^3 z^{-3N}}{3} - \cdots$$

$$\boxed{\hat{x}[n] = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \cdot a^k \cdot \delta[n-kN]}$$

13) a) $C_{xx}[n] = \sum_{k=-\infty}^{+\infty} x[k]\,x[n+k] = \sum_{k=-\infty}^{+\infty} x[-k]\,x[n-k] = x[-n] * x[n]$

(3.58)

$C_{xx}[z] = X(z^{-1})\,X(z)$

$\underline{X(z)}$
ROC: $a<|z|<b$

$\underline{X(z^{-1})}$
ROC: $1/b < |z| < 1/a$

$\underline{C_{xx}(z)}$
$\max(a, 1/b) < |z| < \min(1/a, b)$

b) $X(z) = \frac{1}{1-az^{-1}} \quad |a|<|z|$

$(|a|<1)$
stable sequence

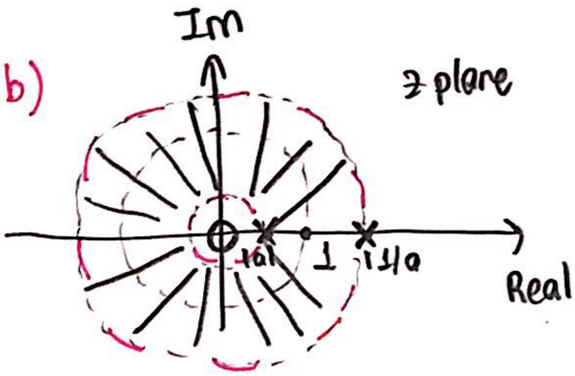
$X(z^{-1}) = \frac{1}{1-az} \quad |z| < |1/a|$

$$C_{xx}[z] = \frac{1}{1-az^{-1}} \cdot \frac{1}{1-az} = \frac{-a^{-1} \cdot z^{-1}}{(1-az^{-1})(1-a^{-1}z^{-1})} = \frac{\frac{1}{1-a^2}}{1-az^{-1}} - \frac{\frac{1}{1-a^2}}{1-a^{-1}z^{-1}}$$

$|a|<|z|<|a^{-1}|$

$$C_{xx}[n] = \frac{1}{1-a^2}\left[a^n u[n] + a^{-n} u[-n-1]\right]$$

13) b)



$|a| < |z| < |a^{-1}|$

(1 zero at infinity)

c) $x_1[n] = x[-n]$ ROC : $|z| < |1/a|$

$$X_1(z) = \frac{1}{1-az}$$

$x_1[-n] = x[n]$ ROC : $|a| < |z|$

$$X_1(z^{-1}) = \frac{1}{1-az^{-1}}$$

$$C_{x_1x_1}(z) = \frac{1}{1-az} \cdot \frac{1}{1-az^{-1}} \qquad |a| < |z| < |1/a|$$

$$= C_{xx}(z)$$

d) $x_2[n] = x[n-m]$ ROC: $|a| < |z|$

$$X_2(z) = \frac{z^{-m}}{1-az^{-1}} \qquad |a| < |z|$$

$$C_{x_2x_2}(z) = \frac{z^{-m}}{1-az^{-1}} \cdot \frac{z^{m}}{1-az} = C_{xx}(z)$$