TRANSFORM DOMAIN ANALYSIS

- RATIONAL SYSTEM FUNCTIONS
- STABLE, CAUSAL SYSTEMS
- INVERSE SYSTEMS
- PHASE LAG, PHASE DELAY AND GROUP DELAY
- FREQUENCY RESPONSE FUNCTIONS IN TERMS OF FIRST ORDER FACTORS
    - FREQUENCY RESPONSE OF A SINGLE ZERO OR A SINGLE POLE
- APPROXIMATE PLOTS OF FREQUENCY RESPONSE FROM POLE ZERO DIAGRAMS
- ALLPASS SYSTEMS
    - RELATIONSHIP BETWEEN THE MAGNITUDE AND THE PHASE OF A FREQUENCY RESPONSE FUNCTION
        - $H(z)$ AND $H^*\left(\frac{1}{z^*}\right)$ YIELD THE SAME MAGNITUDE RESPONSE
        - THE POLES AND ZEROS OF $H^*\left(\frac{1}{z^*}\right)$ ARE THE *CONJUGATE-RECIPROCALS* OF THOSE OF $H(z)$
    - PHASE AND GROUP DELAY FUNCTIONS OF ALLPASS SYSTEMS
- MINIMUM PHASE SYSTEMS
    - FREQUENCY RESPONSE MAGNITUDE COMPENSATION
    - MINIMUM GROUP-DELAY PROPERTY
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    - MINIMUM ENERGY-DELAY PROPERTY
- GENERALIZED LINEAR PHASE SYSTEMS
- Appendix A: FREQUENCY RESPONSES OF CONJUGATE RECIPROCAL ZEROS (POLES)

    RELATIONSHIP BETWEEN $A(z)$ AND $B(z)$

    PHASES AND GROUP DELAYS OF $A(\omega)$ AND $B(\omega)$

    FREQUENCY RESPONSES OF $\frac{1}{A(\omega)}$ AND $\frac{1}{B(\omega)}$
- Appendix B: GROUP DELAY OF $\frac{z^{-1}-a^*}{1-az^{-1}}$

RATIONAL SYSTEM FUNCTIONS

LTI systems can be implemented by using

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

$$H(z) = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \cdots + a_N z^{-N}}$$

$$= \frac{b_0}{a_0} \frac{(1 - c_1 z^{-1})(1 - c_2 z^{-1}) \ldots (1 - c_M z^{-1})}{(1 - d_1 z^{-1})(1 - d_2 z^{-1}) \ldots (1 - d_N z^{-1})}$$

Poles and zeros…

STABLE AND CAUSAL SYSTEMS

Stable Systems

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

ROC of $H(z)$ includes unit circle

Causal Systems

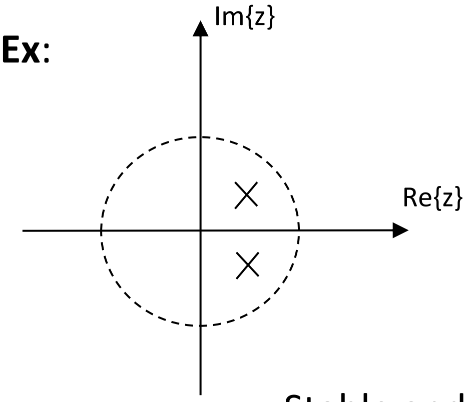
$$h[n] = 0 \qquad n < 0$$

ROC of $H(z)$ is out of the outermost pole circle

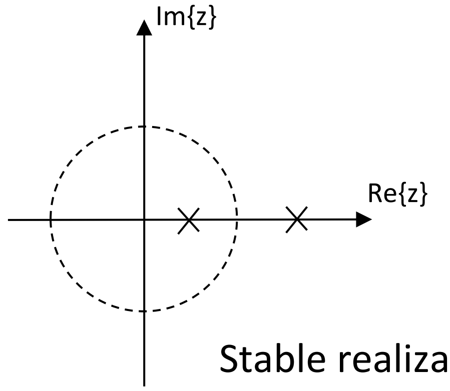
Stable and Causal Systems

Poles are inside the unit circle

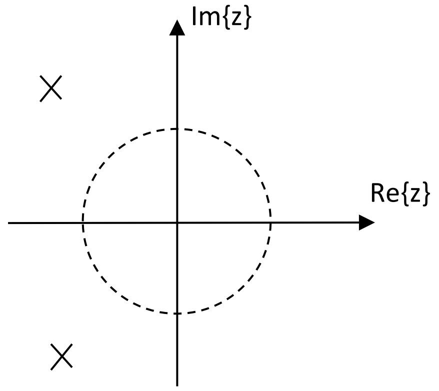
Ex:



Stable and causal realization (impulse response) is possible

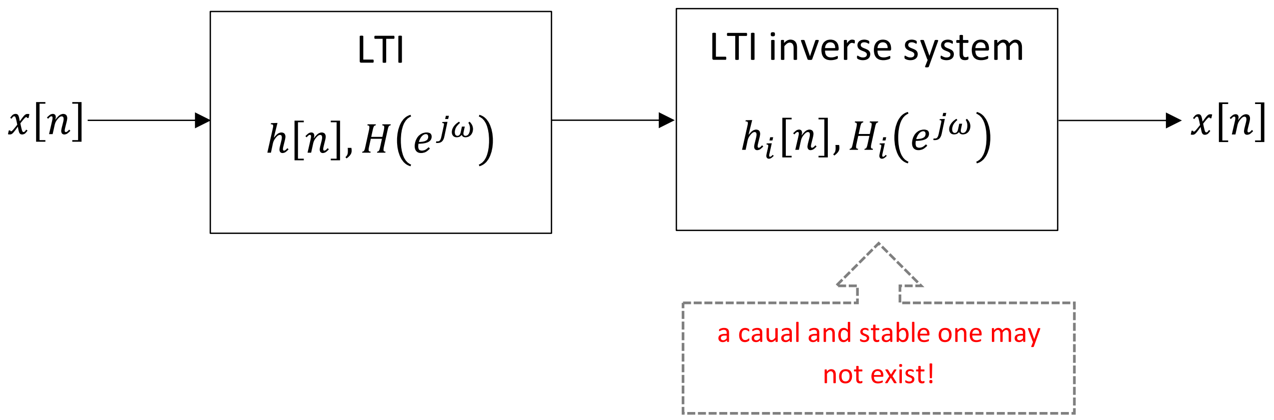


Stable realization is noncausal (two-sided impulse response).
Causal realization is unstable.



Stable realization is noncausal (left-sided impulse response)
Causal realization is unstable.

INVERSE SYSTEMS



For example, ideal (lowpass) filter do not have an inverse!

Inverse system, if exists, has to satisfy

$$h[n] * h_i[n] = \delta[n]$$

$$H(z)H_i(z) = 1$$

$$H_i(z) = \frac{1}{H(z)}$$

poles of $H(z) \rightarrow \ zeros\ of\ H_i(z)$
zeros of $H(z) \rightarrow \ poles\ of\ H_i(z)$

If it also has a frequency response (if $H_i(z)$ has a stable realization)

$$H(e^{j\omega})H_i(e^{j\omega}) = 1$$

$$H_i(e^{j\omega}) = \frac{1}{H(e^{j\omega})}$$

ROC of $H_i(z)$

$$H(z)H_i(z) = 1$$

implies that ROCs of $H(z)$ and $H_i(z)$ must have an overlap.

Ex:

$$H(z) = \frac{1 - 0.5z^{-1}}{1 - 0.9z^{-1}} \qquad |z| > 0.9$$

$$H_i(z) = \frac{1 - 0.9z^{-1}}{1 - 0.5z^{-1}}$$

ROC may be $|z| > 0.5$ or $|z| < 0.5$.

We have to choose $|z| > 0.5$.

In this case the inverse system has a stable and causal implementation:

$$h[n] = (0.5)^n u[n] - 0.9\,(0.5)^{n-1} u[n-1]$$

Ex: Inverse for a system with a zero in the ROC

$$H(z) = \frac{1 - 2z^{-1}}{1 - 0.9z^{-1}} \qquad |z| > 0.9$$

$$H_i(z) = \frac{1 - 0.9z^{-1}}{1 - 2z^{-1}}$$

ROC may be $|z| > 2$ or $|z| < 2$.

Both overlap with $|z| > 0.9$.

Choosing $|z| > 2$ yields

$$h_{i1}[n] = 2^n u[n] - 0.9\, 2^{n-1} u[n-1]$$

which is causal but unstable.

Choosing $|z| < 2$ yields

$$h_{i2}[n] = -2^n u[-n-1] + 0.9\, 2^{n-1} u[-n]$$

which is stable but noncausal.

To have a stable and causal inverse, the zeros of the system must be inside the unit circle.

Such systems are called “minimum phase” systems.

They have

- MINIMUM GROUP-DELAY PROPERTY
- MINIMUM PHASE-LAG PROPERTY
- MINIMUM ENERGY-DELAY PROPERTY

We will talk about them later...

FREQUENCY RESPONSE

$$e^{j\omega_0 n} \xrightarrow[H(e^{j\omega})]{LTI\ system} e^{j\omega_0 n}H\left(e^{j\omega_0}\right)$$

Multiplication of a signal by a complex constant.

$$Y\left(e^{j\omega_0}\right) = X\left(e^{j\omega_0}\right)H\left(e^{j\omega_0}\right)$$

Multiplication of two complex constants.

Magnitude and Phase

$$\left|Y\left(e^{j\omega}\right)\right| = \left|X\left(e^{j\omega}\right)\right|\left|H\left(e^{j\omega}\right)\right|$$

$$\angle Y\left(e^{j\omega}\right) = \angle X\left(e^{j\omega}\right) + \angle H\left(e^{j\omega}\right)$$

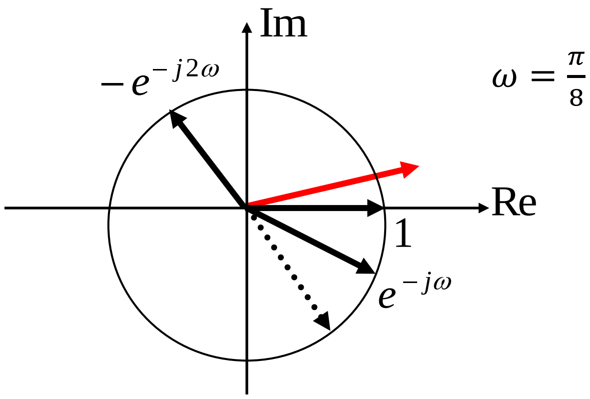
Phase Response

Ex:

$$h[n] = [\dots 0 \quad \underset{\substack{\uparrow \\ n=0}}{1} \quad 1 \quad -1 \quad 0 \dots]$$

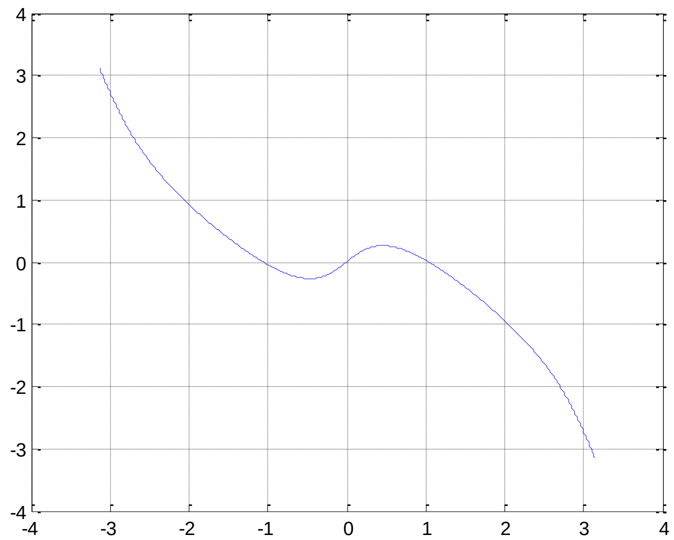
$$H(e^{j\omega}) = 1 + e^{-j\omega} - e^{-j2\omega}$$

$$= (1 + \cos\omega - \cos 2\omega) - j(\sin\omega + \sin 2\omega)$$



Phase:

$$\angle H(e^{j\omega}) = \arctan\left(\frac{-\sin\omega - \sin 2\omega}{1 + \cos\omega - \cos 2\omega}\right)$$



Phase Plots: “arg” vs. “ARG”

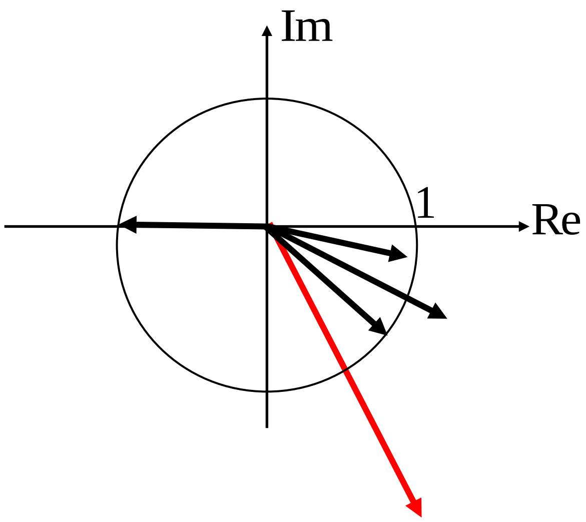
Ex:

$$h[n] = [\dots 0 \quad \underset{\substack{\uparrow \\ n=0}}{-1} \quad 0.9 \quad 1.2 \quad 1 \quad 0 \dots]$$

$$H(e^{j\omega}) = -1 + 0.9e^{-j\omega} + 1.2e^{-j2\omega} + e^{-j3\omega}$$

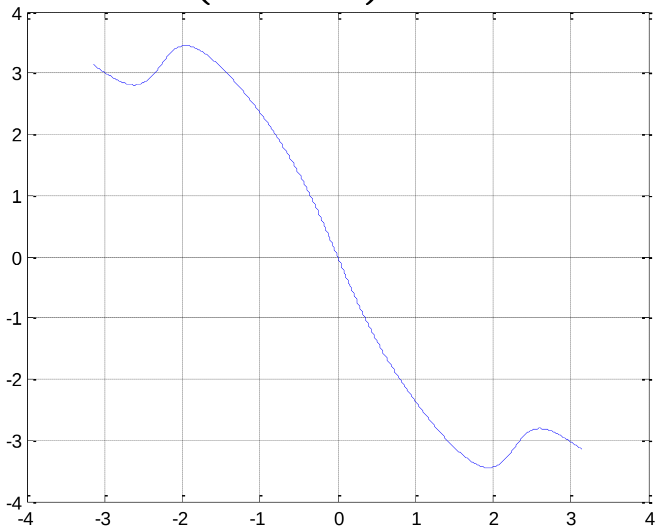
$$= (-1 + 0.9\cos\omega + 1.2\cos 2\omega + \cos 3\omega) - j(0.9\sin\omega + 1.2\sin 2\omega + \sin 3\omega)$$

$\omega = \frac{\pi}{8}$



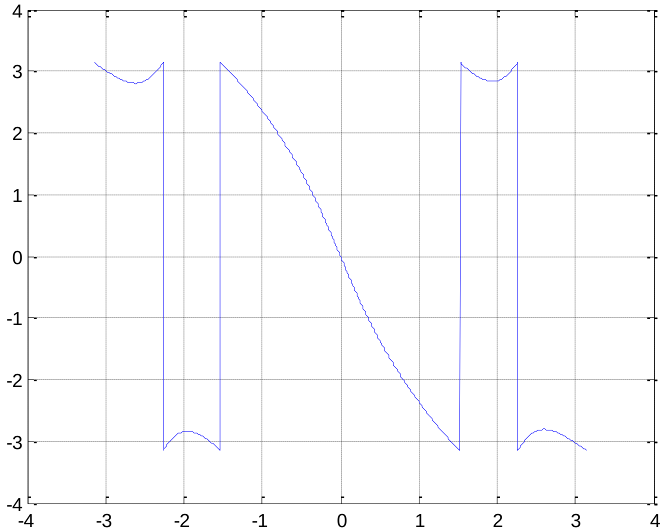
$$\angle H\left(e^{j\omega}\right) = \arctan\left(\frac{-(0.9\sin\omega + 1.2\sin 2\omega + \sin 3\omega)}{-1 + 0.9\cos\omega + 1.2\cos 2\omega + \cos 3\omega}\right)$$

$\arg\left(H\left(e^{j\omega}\right)\right)\ \ (rad.)$



$\omega\ (rad.)$

$\mathrm{ARG}\left(H\left(e^{j\omega}\right)\right)\ \ (rad.)$

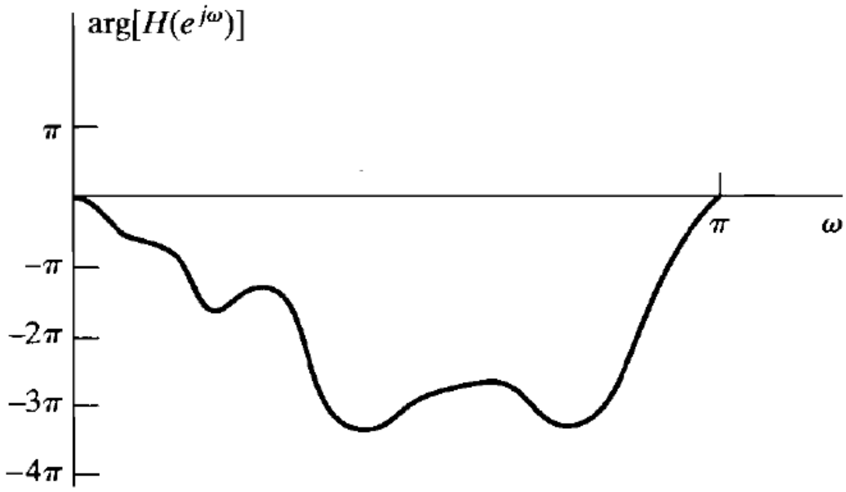


$\omega\ (rad.)$

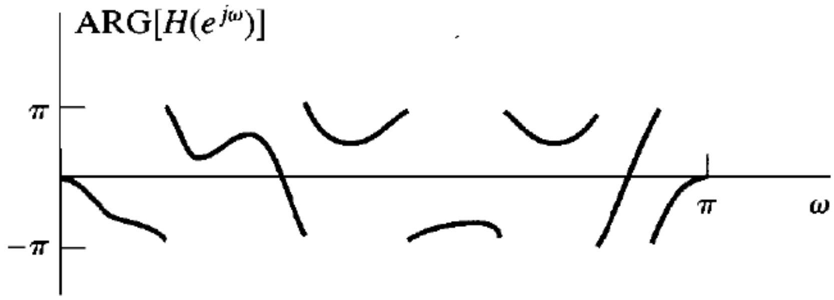
$\arg\left(H\left(e^{j\omega}\right)\right)$: unwrapped (continuous) phase

$\mathrm{ARG}\left(H\left(e^{j\omega}\right)\right)$: principal value of the phase

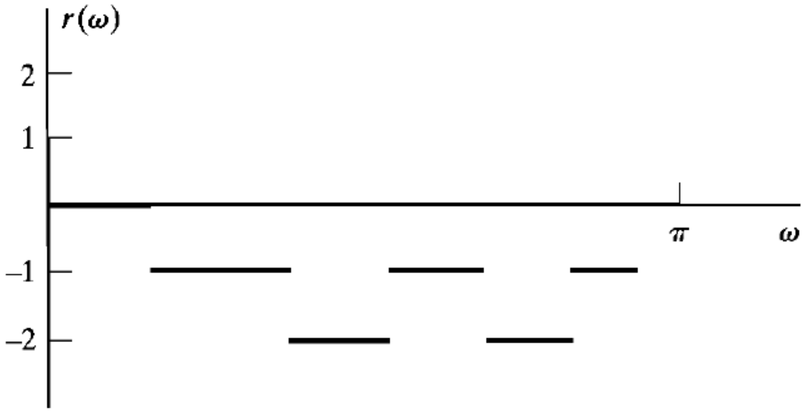
$$\arg\left(H\left(e^{j\omega}\right)\right) = \underbrace{\mathrm{ARG}\left(H\left(e^{j\omega}\right)\right)}_{\in(-\pi,\pi]} \quad + \quad 2\pi\ \underbrace{r(\omega)}_{integer\ valued}$$



(a)



(b)



(c)

From Oppenheim, Schafer, 3rd ed., p.276.

PHASE LAG

$$phase\ lag = -\angle H\left(e^{j\omega}\right) \qquad (radians)$$

$$\cos(\omega n) \xrightarrow{LTI\ system} \left|H\left(e^{j\omega}\right)\right| \cos\left(\omega n + \angle H\left(e^{j\omega}\right)\right)$$

PHASE DELAY

Phase Delay:

$$\tau_{ph}(\omega) = -\frac{\angle H\left(e^{j\omega}\right)}{\omega} \quad (samples)$$

$$\cos(\omega n) \xrightarrow{LTI\ system} \left|H\left(e^{j\omega}\right)\right| \cos\left(\omega\left(n + \frac{\angle H\left(e^{j\omega}\right)}{\omega}\right)\right)$$

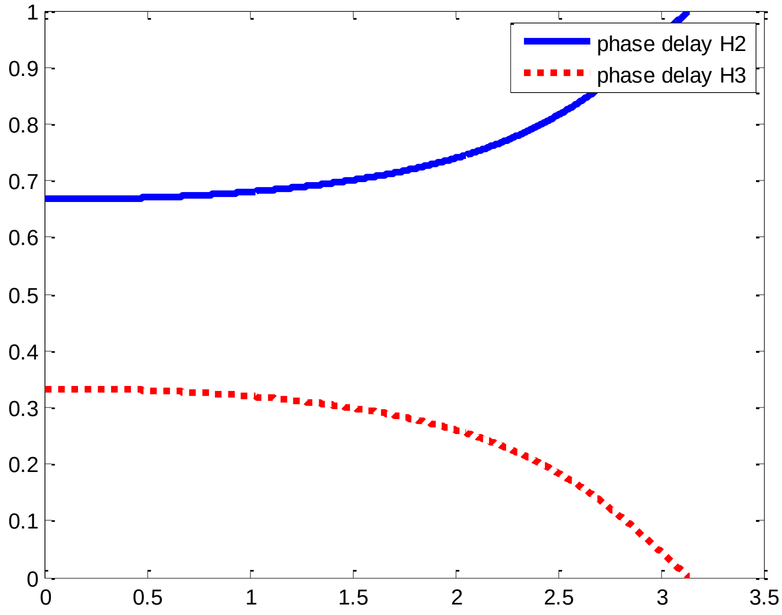
The significance of **phase delay** can be demonstrated by considering the output of an LTI system when its input is a sinusoid.

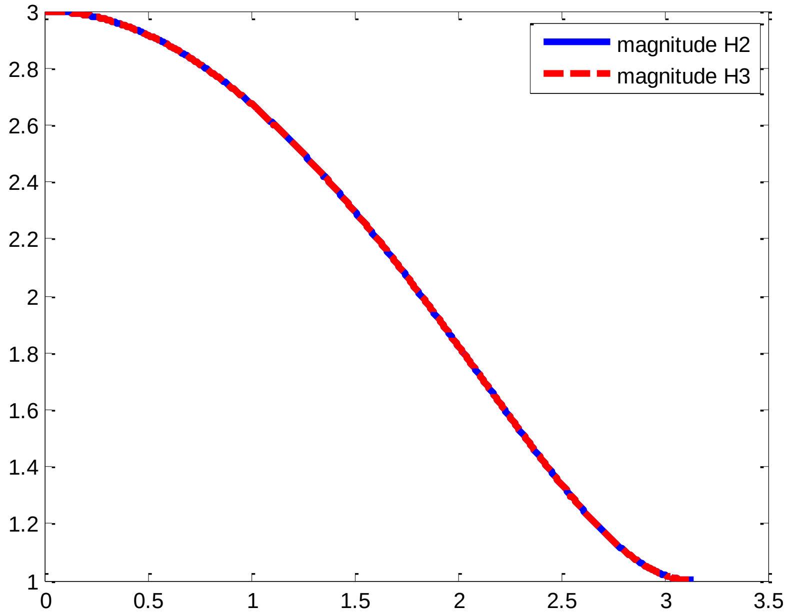
$$\cos(\omega_0 n) \longrightarrow \boxed{\begin{array}{c}\text{LTI system}\\ H\left(e^{j\omega}\right)\end{array}} \longrightarrow \left|H\left(e^{j\omega_0}\right)\right|\cos\left(\omega_0 n + \angle H\left(e^{j\omega_0}\right)\right)$$

$$= \left|H\left(e^{j\omega_0}\right)\right|\cos\left(\omega_0\left(n + \frac{\angle H\left(e^{j\omega_0}\right)}{\omega_0}\right)\right)$$

Ex: Phase delay

$h_2 = [1 \ \ 2]$
$h_3 = [2 \ \ 1]$



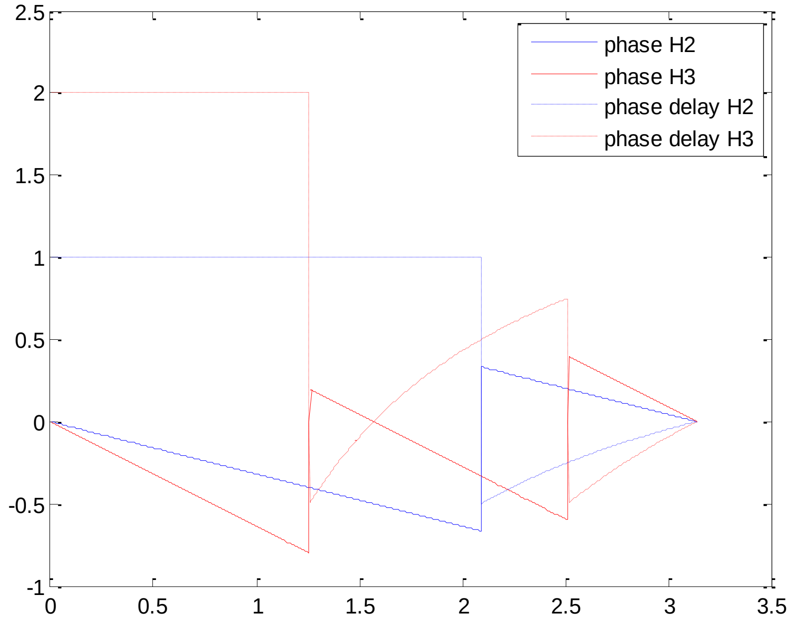


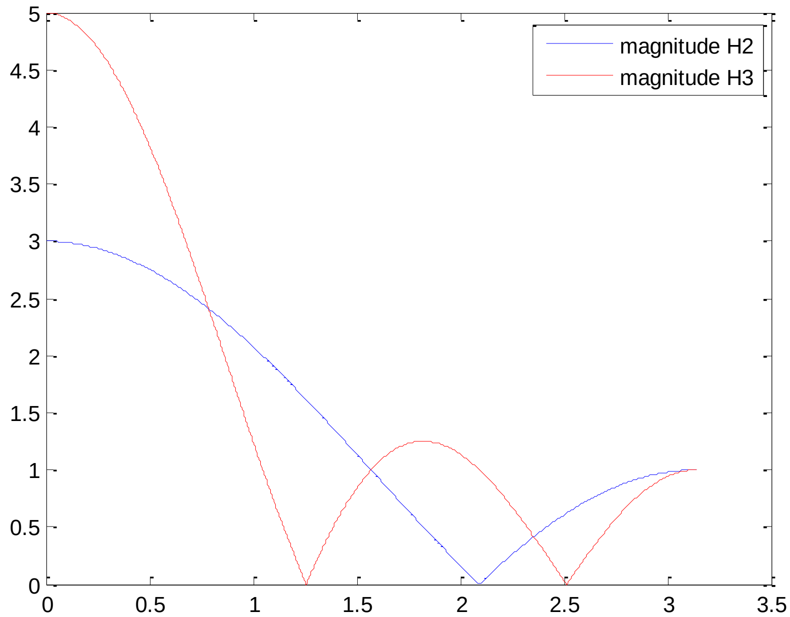

```
clear all
close all

h2 = [1 2];

h3 = [2 1];

[H2,w] = freqz(h2,1,1000);
[H3,w] = freqz(h3,1,1000);
plot(w,-angle(H2)./w,'linewidth', 3)
hold on
plot(w,-angle(H3)./w,'r:','linewidth', 3)
legend('phase delay H2','phase delay H3')

figure
plot(w,abs(H2),'linewidth', 3)
hold on
plot(w,abs(H3),'r--','linewidth', 3)
legend('magnitude H2','magnitude H3')
```

Ex: Phase delay

$h_2 = [1 \;\; 1 \;\; 1]$

$h_3 = [1 \;\; 1 \;\; 1 \;\; 1 \;\; 1]$






```
clear all
close all

h2 = [1 1 1];
h3 = [1 1 1 1 1];

[H2,w] = freqz(h2,1,1000);
[H3,w] = freqz(h3,1,1000);
pd2 = -unwrap(angle(H2)./w);
pd3 = -unwrap(angle(H3)./w);

plot(w,angle(H2)/pi)
hold on
plot(w,angle(H3)/pi,'r')
plot(w,pd2,'b:')
plot(w,pd3,'r:')
legend('phase H2','phase H3','phase delay H2','phase delay H3')
figure
plot(w,abs(H2))
hold
plot(w,abs(H3),'r')
```

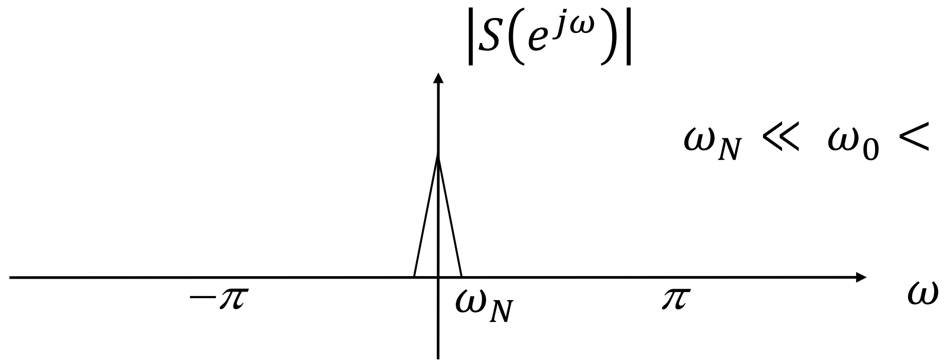
GROUP DELAY

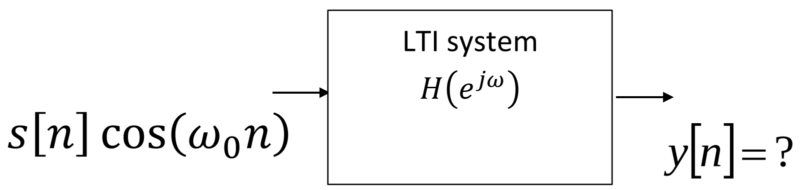
Group Delay:

$$\tau_{gr}(\omega) = -\frac{d}{d\omega}\angle H\left(e^{j\omega}\right) \quad (samples)$$

What does the **group delay** represent?

Ex: Let $s[n]$ be a "narrowband" signal.





Let $\angle H(e^{j\omega_0})$ be modeled, to a first order approximation, around ω_0 as

$$\angle H(e^{j\omega_0}) \cong -\phi_0 - \omega n_d$$

Then,

$$\tau_{gr}(\omega_0) = n_d$$

and

$$y[n] \cong |H(e^{j\omega_0})| s[n - n_d] \cos(\omega_0 n - \phi_0 - \omega_0 n_d)$$

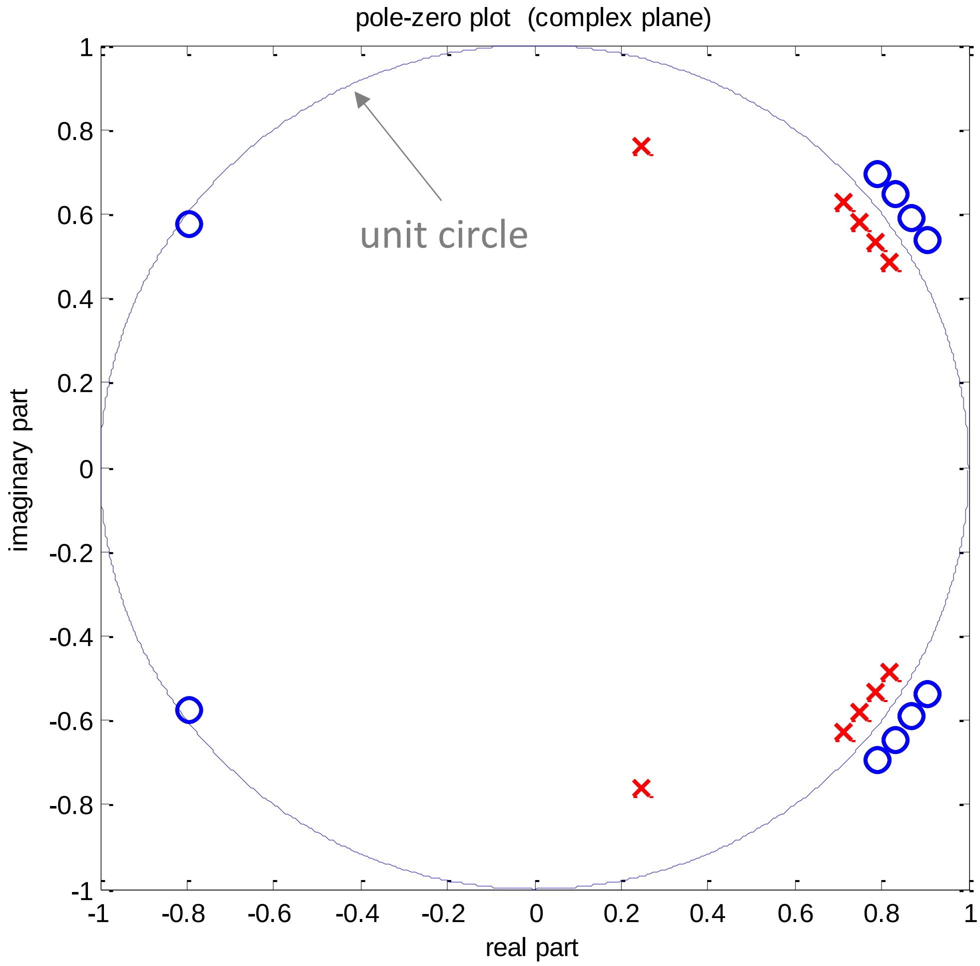
$$= |H(e^{j\omega_0})| s[n - \tau_{gr}(\omega_0)] \cos\left(\omega_0 (n - \tau_{ph}(\omega_0)\right)$$

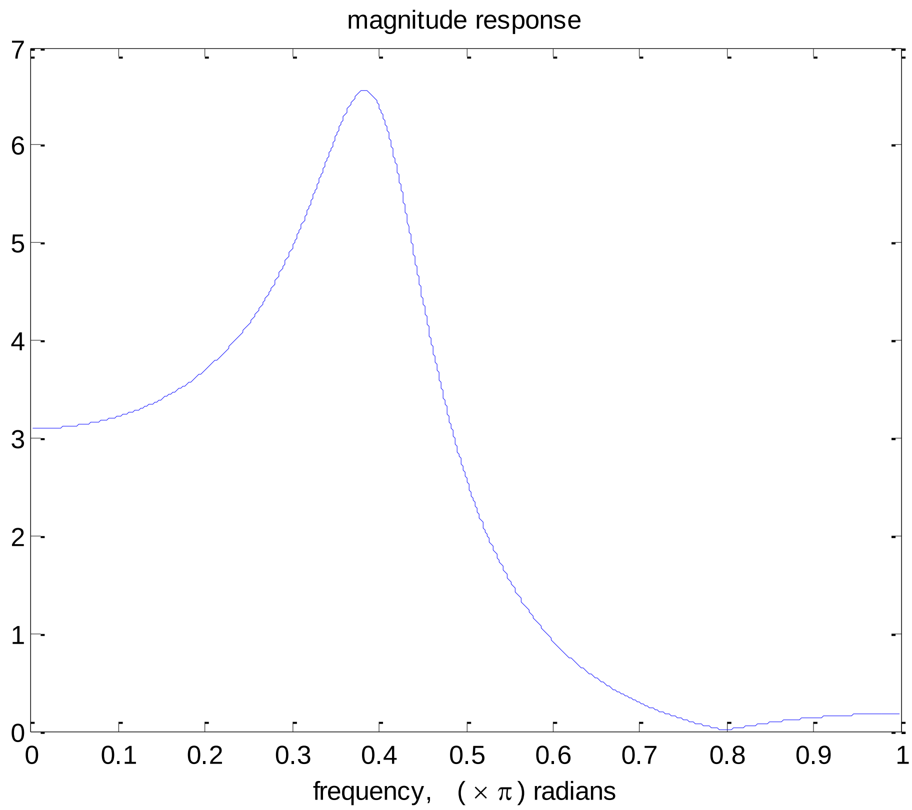
MATLAB Implementation of the System Described in Section 5.1.2 of the Textbook

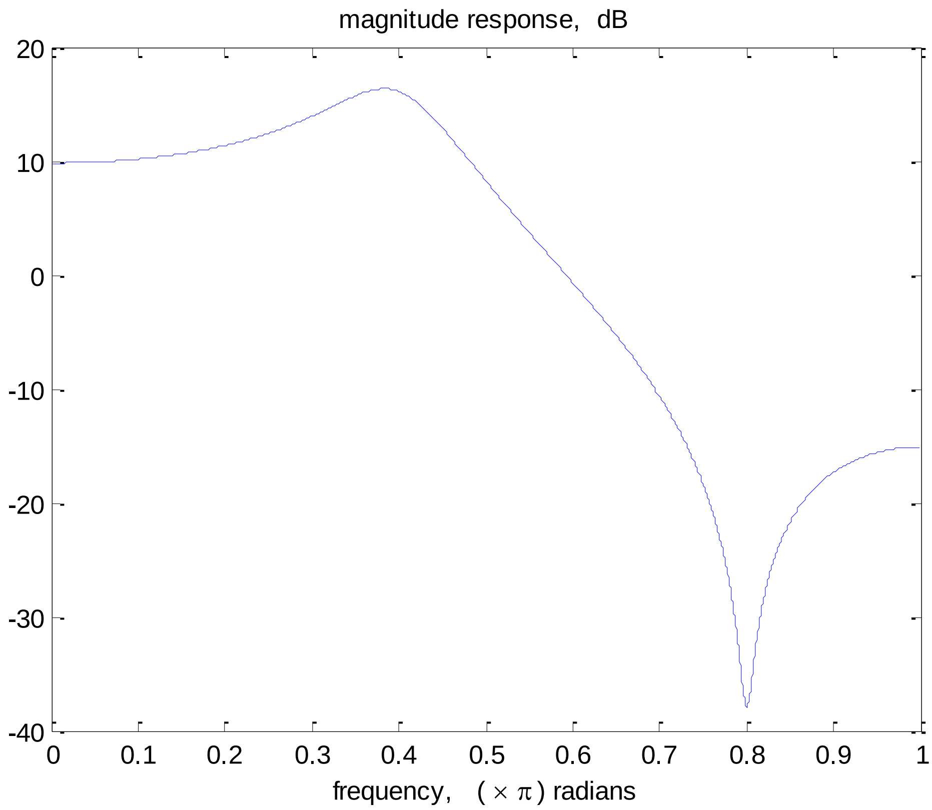
```
clear all
close all

zz1 = 0.98*exp(j*0.8*pi);
zz2 = conj(zz1);
pp1 = 0.8*exp(j*0.4*pi);
pp2 = conj(pp1);

for k=1:4
   pp(k) = 0.95*exp(j*(0.15*pi+0.02*pi*k));
end
for k=5:8
   pp(k) = conj(pp(k-4));
end

zz = 1./pp;

p = [pp pp pp1 pp2];
z = [zz zz zz1 zz2];
K = 0.95^16;

plot(p,'rx','linewidth',2,'markersize',10)
hold
plot(z,'bo','linewidth',2,'markersize',10)
om = 0:999;
plot(exp(j*om*2*pi/1000));

[Num, Den] = zp2tf(z',p',K);

[H,w] = freqz(Num,Den,2048);
figure
plot(w/pi,abs(H))
title('magnitude')
figure
plot(w/pi,20*log10(abs(H)))
title('magnitude, dB')
figure
plot(w/pi,angle(H)/pi)
```

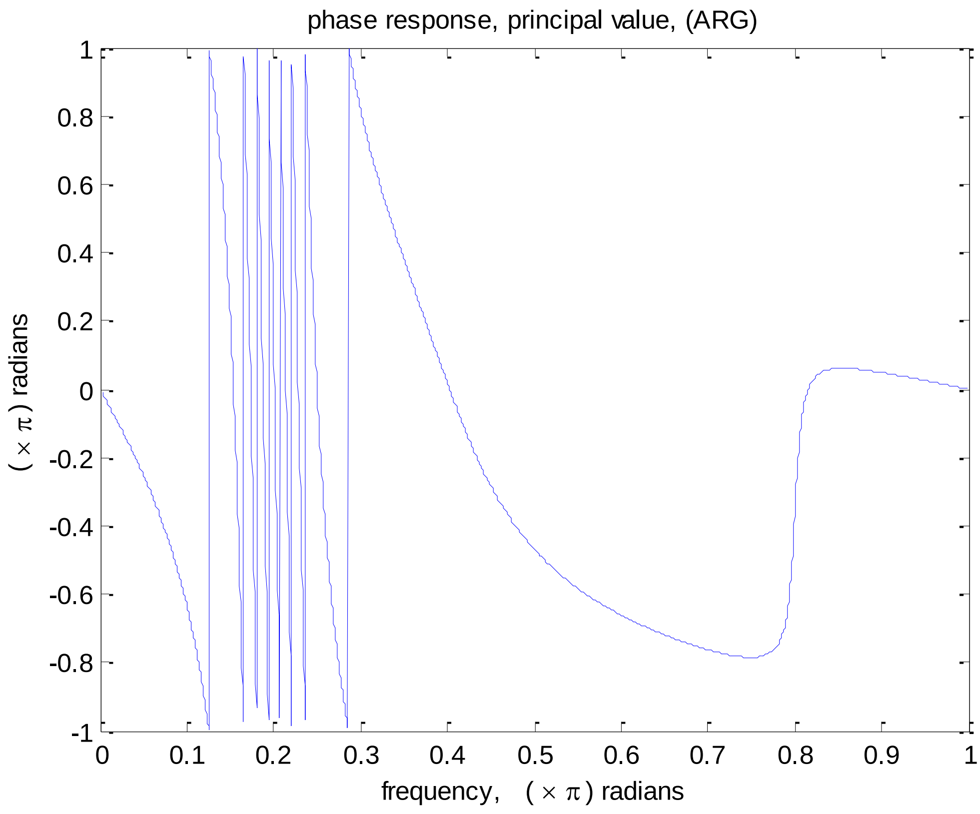
```
title('phase, principal value, (ARG),      X pi rads')
figure
plot(w/pi,unwrap(angle(H))/pi)
title('phase, (arg),       X pi rads')
figure
[Phi,w] = phasedelay(Num,Den,2048);
plot(w/pi,Phi)
title('phase delay, samples')
figure
[Grp,w] = grpdelay(Num,Den,2048);
plot(w/pi,Grp)
title('group delay, samples')

M = 60;
window = hamming(M+1);
n = 0:M;
x1 = window' .* cos(0.2*pi*n);
x2 = window' .* cos(0.4*pi*n - pi/2);
x3 = window' .* cos(0.8*pi*n + pi/5);
figure
plot(x1)
figure
plot(x2)
figure
plot(x3)

x = [x3 x1 x2 zeros(1,130)];
figure
plot(x)
title('input signal')
figure
[X,w] = freqz(x,1,2048)
plot(w/pi,abs(X))
title('DTFT of input signal')
y = filter(Num,Den,x);
figure
plot(y)
title('output signal')
```

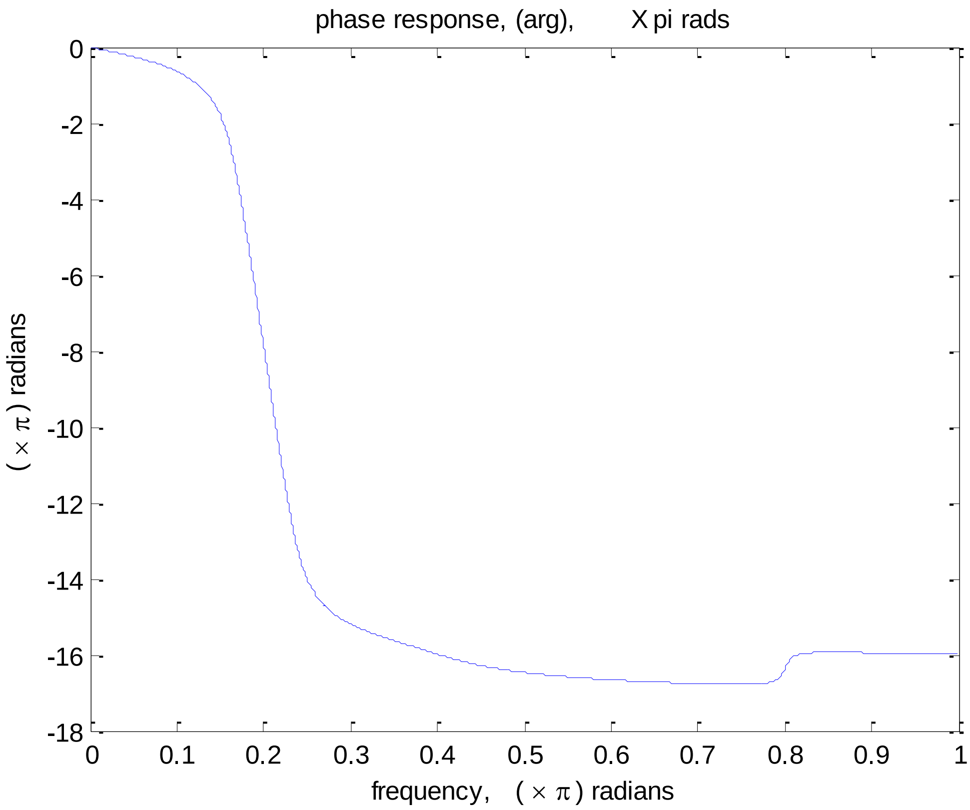
The Plots Produced by the Code Above

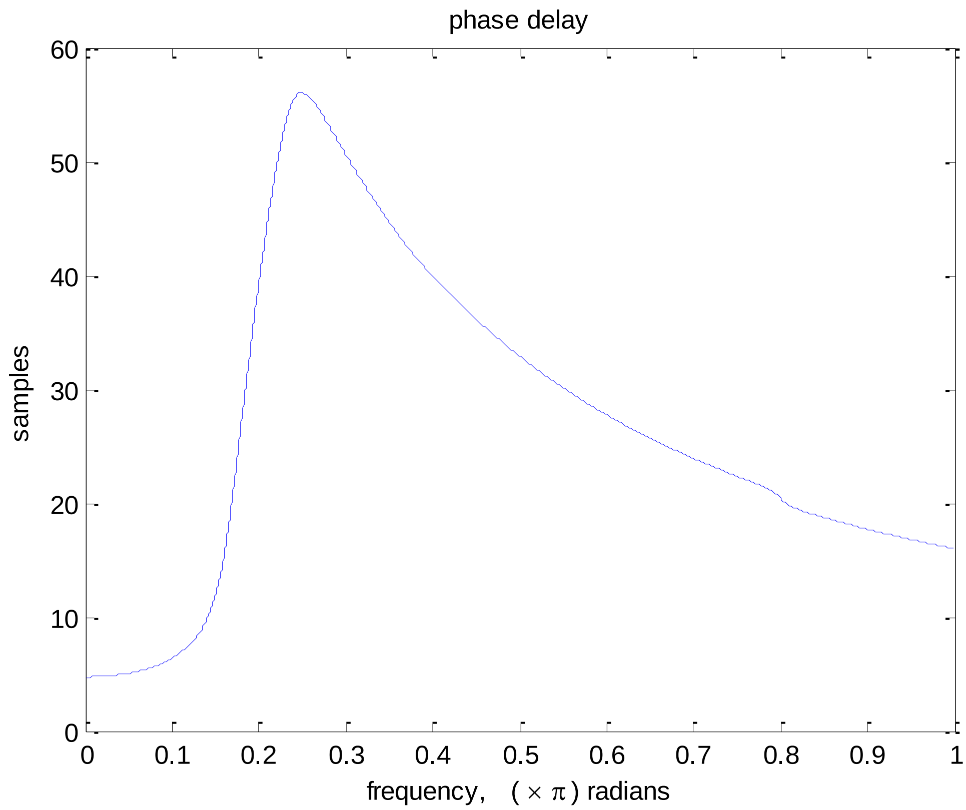


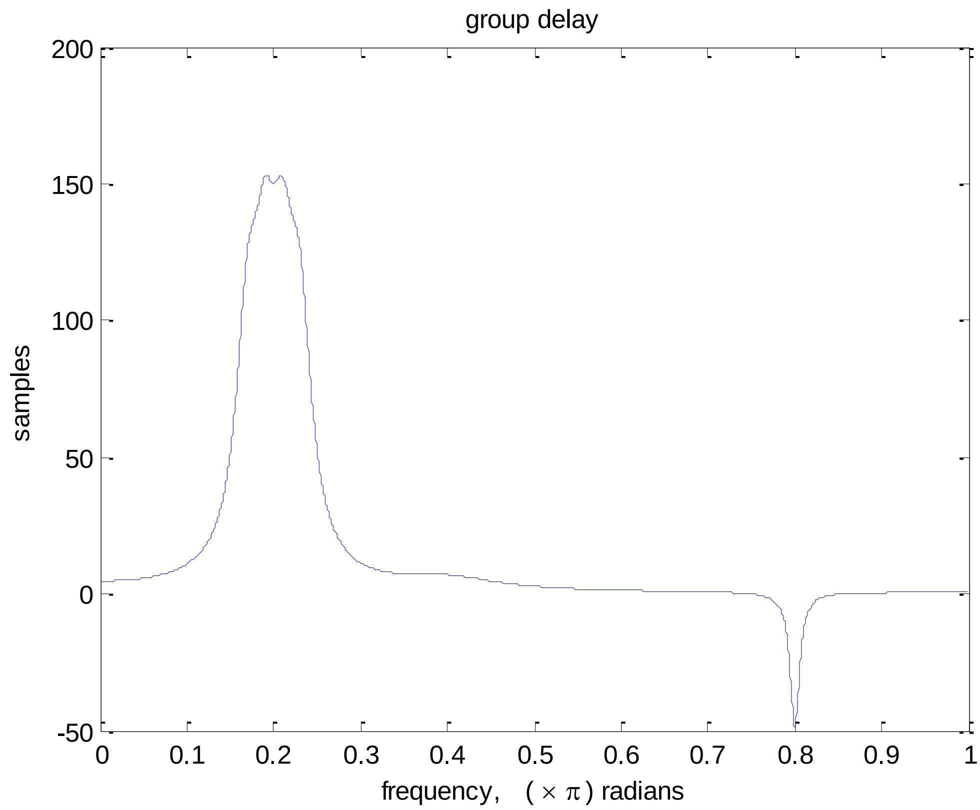


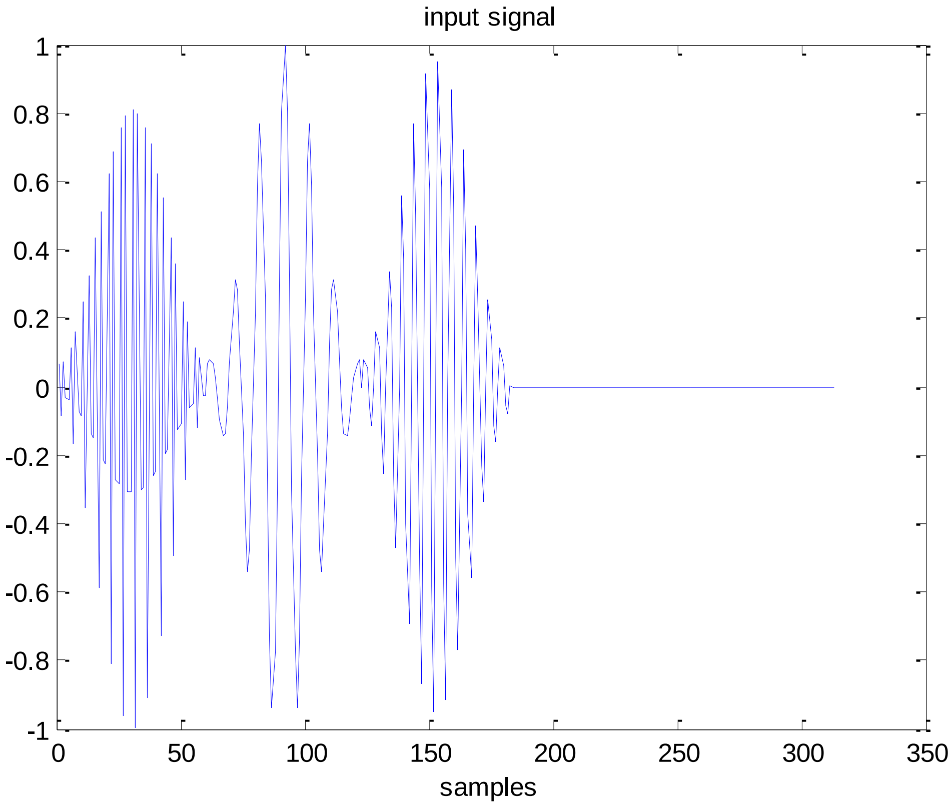

magnitude response, dB
20
10
0
-10
-20
-30
-40
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
frequency, (× π) radians

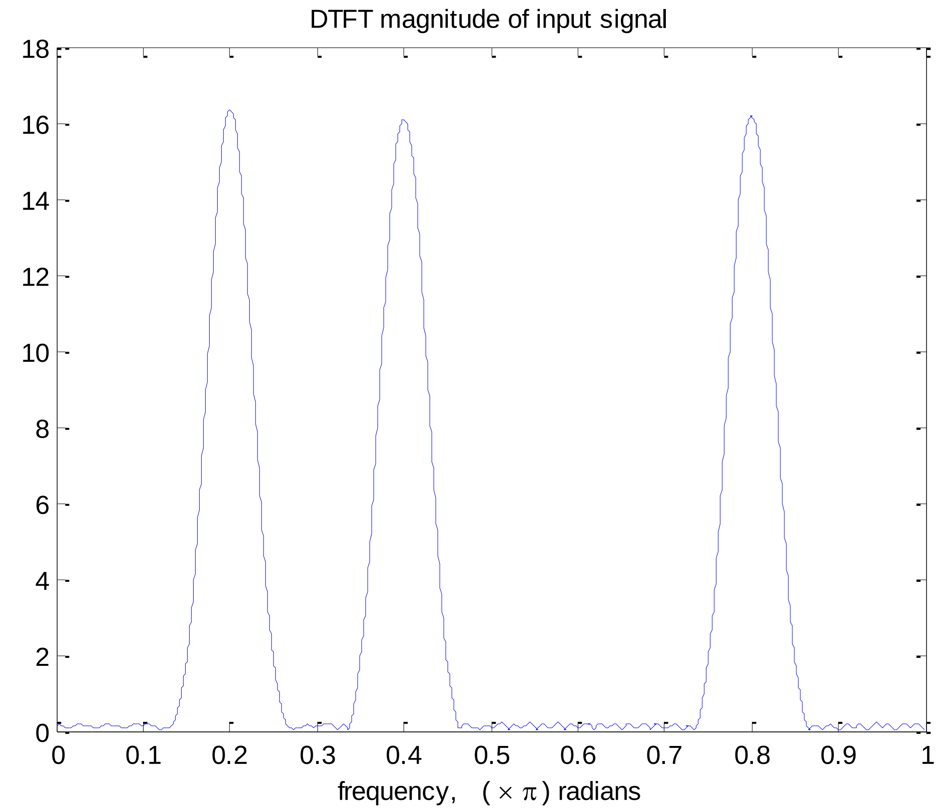

phase response, principal value, (ARG)
(× π) radians
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
frequency, (× π) radians

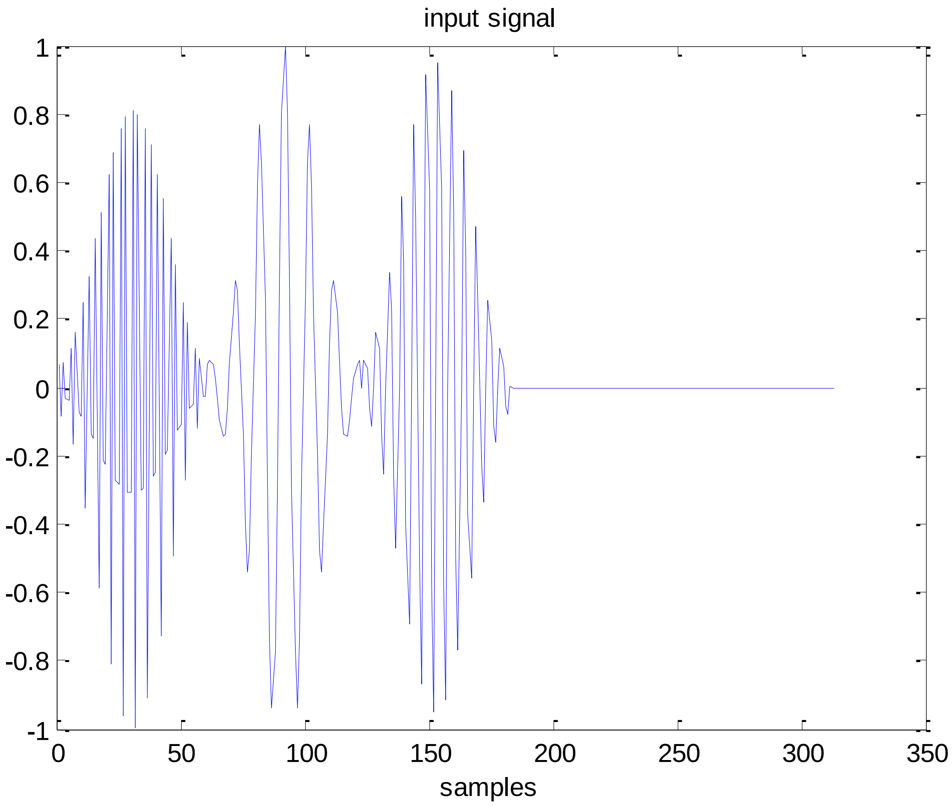

phase response, (arg), X pi rads
0
-2
-4
-6
-8
-10
-12
-14
-16
-18
(× π) radians
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
frequency, (× π) radians

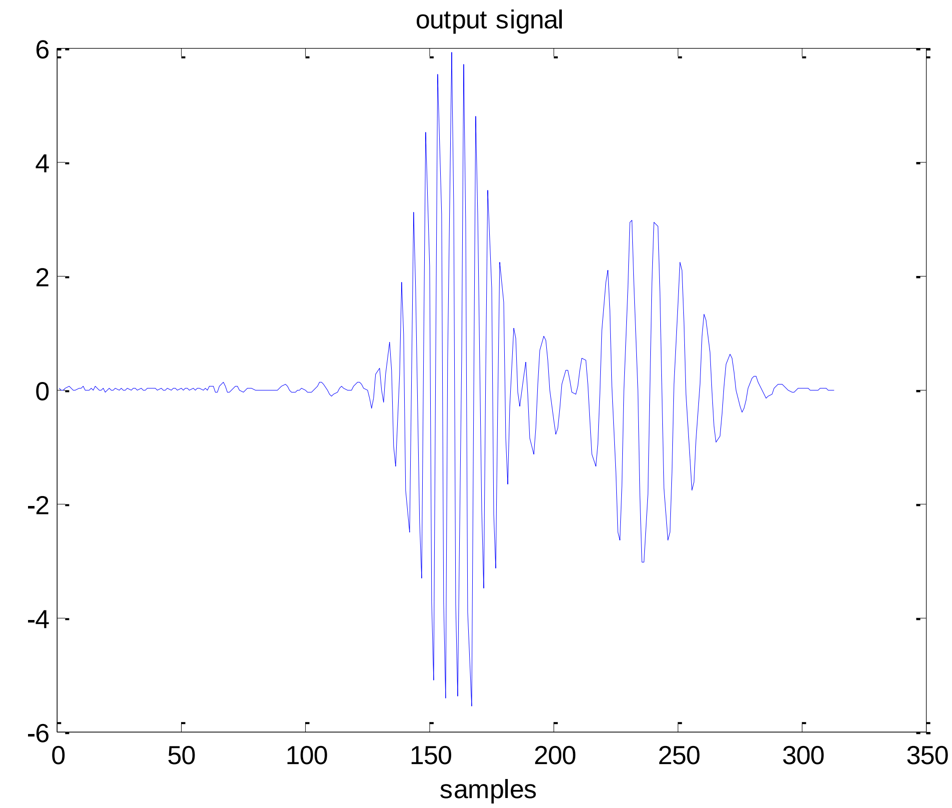

phase delay
60
50
40
30
20
10
0
samples
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
frequency, (× π) radians


group delay
200
150
100
50
0
-50
samples
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
frequency, (× π) radians


input signal
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
0
50
100
150
200
250
300
350
samples


DTFT magnitude of input signal
18
16
14
12
10
8
6
4
2
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
frequency, (× π) radians


input signal
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
0
50
100
150
200
250
300
350
samples


output signal
6
4
2
0
-2
-4
-6
0
50
100
150
200
250
300
350
samples

FREQUENCY RESPONSE FUNCTIONS IN TERMS OF FIRST ORDER FACTORS

The magnitude function

$$\left|H\left(e^{j\omega}\right)\right| = \left|\frac{b_0}{a_0}\right| \frac{\left|1 - c_1 e^{-j\omega}\right|\left|1 - c_2 e^{-j\omega}\right| \ldots \left|1 - c_M e^{-j\omega}\right|}{\left|1 - d_1 e^{-j\omega}\right|\left|1 - d_2 e^{-j\omega}\right| \ldots \left|1 - d_N e^{-j\omega}\right|}$$

$$= \left|\frac{b_0}{a_0}\right| \frac{\prod_{k=1}^{M}\left|1 - c_k e^{-j\omega}\right|}{\prod_{k=1}^{N}\left|1 - d_k e^{-j\omega}\right|}$$

The magnitude-squared function

$$\left|H\left(e^{j\omega}\right)\right|^2 = H\left(e^{j\omega}\right)H^*\left(e^{j\omega}\right) = \left(\frac{b_0}{a_0}\right)^2 \frac{\prod_{k=1}^{M}\left(1 - c_k e^{-j\omega}\right)\left(1 - c_k^* e^{j\omega}\right)}{\prod_{k=1}^{N}\left(1 - d_k e^{-j\omega}\right)\left(1 - d_k^* e^{j\omega}\right)}$$

dB Magnitude

$$\text{Gain in dB} = 20\log_{10}\left|H\left(e^{j\omega}\right)\right|$$

$$= 20\log_{10}\left|\frac{b_0}{a_0}\right| + \sum_{k=1}^{M} 20\log_{10}\left|1 - c_k e^{-j\omega}\right| - \sum_{k=1}^{N} 20\log_{10}\left|1 - d_k e^{-j\omega}\right|$$

This is a sum of the dB magnitudes of 1st order terms.

Phase

$$\arg[H(e^{j\omega})] = \arg\left[\frac{b_0}{a_0}\right] + \sum_{k=1}^{M} \arg[(1 - c_k e^{-j\omega})] - \sum_{k=1}^{N} \arg[(1 - d_k e^{-j\omega})]$$

This is a sum of the phases of 1st order terms.

Group Delay

$$\mathrm{grd}[H(e^{j\omega})] = \sum_{k=1}^{N} \frac{d}{d\omega} \arg[(1 - d_k e^{-j\omega})] - \sum_{k=1}^{M} \frac{d}{d\omega} \arg[(1 - c_k e^{-j\omega})]$$

This is a sum of the group delays of 1st order terms.

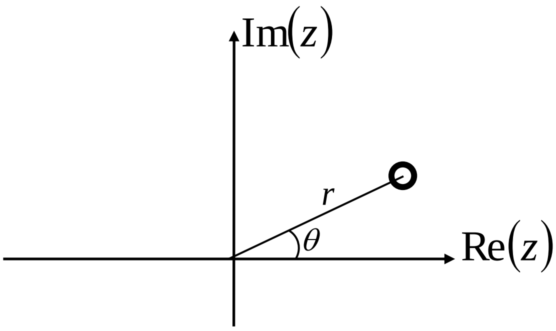
FREQUENCY RESPONSE OF A SINGLE ZERO OR A SINGLE POLE

$$1 - c_k e^{-j\omega} \qquad or \qquad \frac{1}{1 - d_k e^{-j\omega}}$$

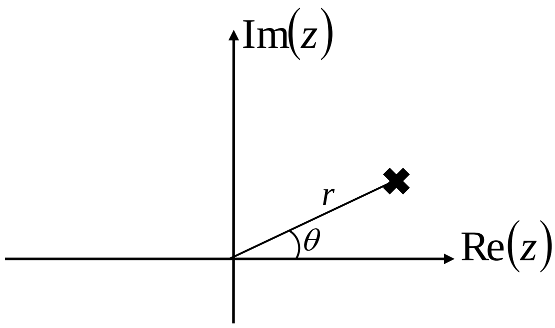
where

$$c_k, d_k \quad : re^{j\theta}$$

A zero at $re^{j\theta}$



A pole at $re^{j\theta}$

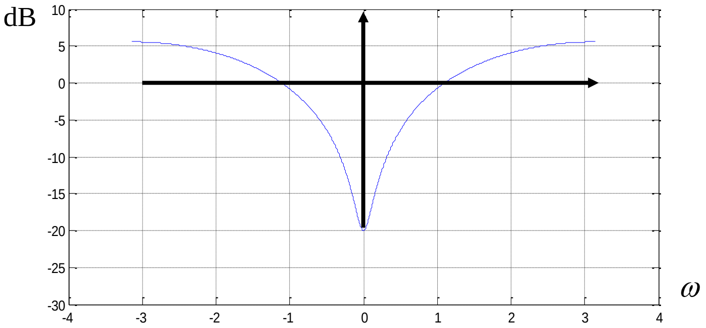


Magnitude response

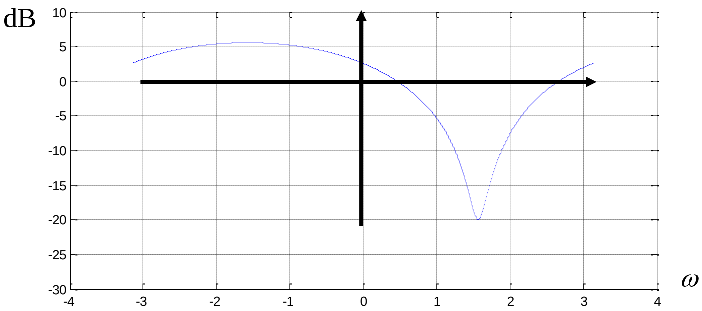
$$\left|1 - re^{j\theta}e^{-j\omega}\right|^2 = 1 + r^2 - 2r\cos(\omega - \theta)$$

$$20\log_{10}\left|1 - re^{j\theta}e^{-j\omega}\right| = 10\log_{10} 1 + r^2 - 2r\cos(\omega - \theta)$$

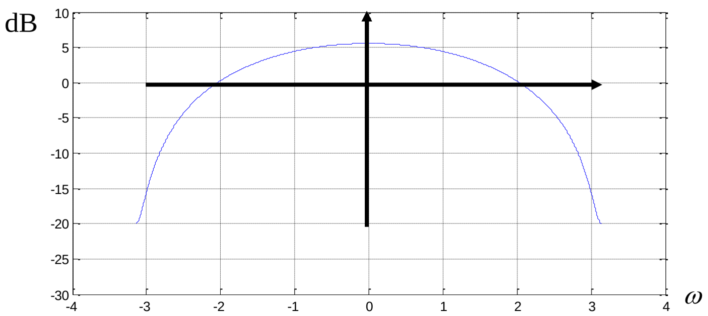
$$r = 0.9, \theta = 0$$



$$r = 0.9, \theta = \frac{\pi}{2}$$



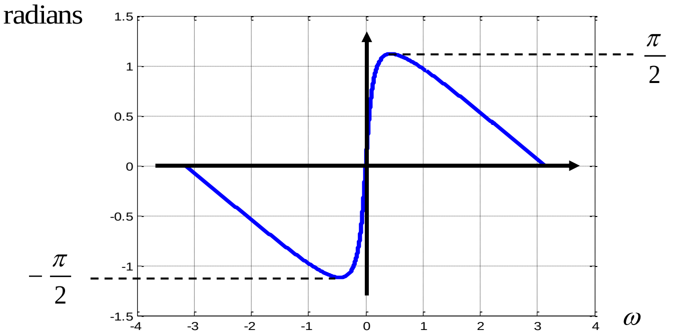
$$r = 0.9, \theta = \pi$$



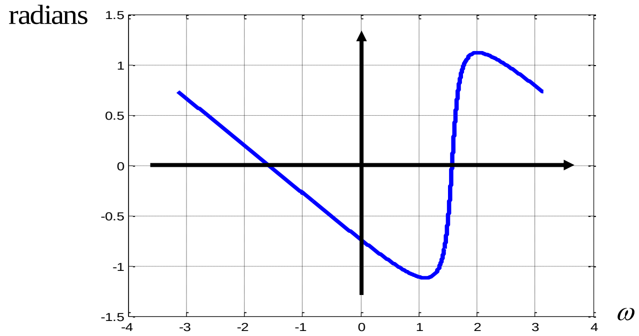
Phase response:

$$\angle\left(1 - re^{j\theta}e^{-j\omega}\right) = \tan^{-1}\frac{r\sin(\omega-\theta)}{1 - r\cos(\omega-\theta)}$$

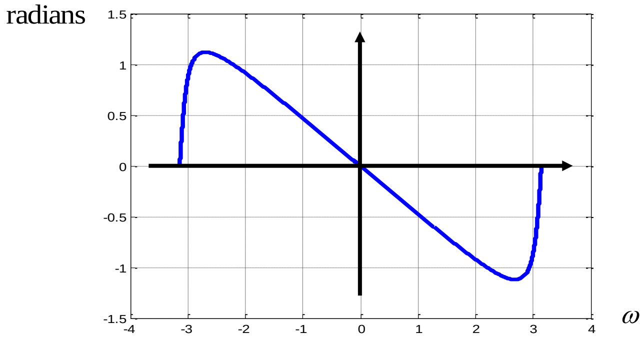
$r = 0.9, \theta = 0$



$r = 0.9, \theta = \pi/2$



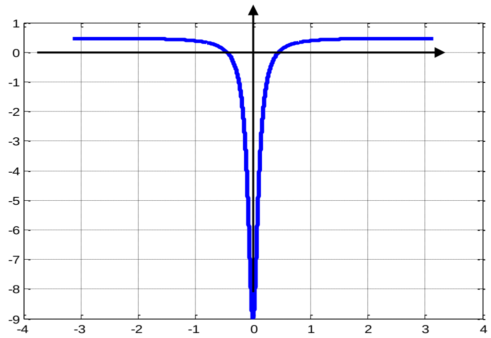
$r = 0.9, \theta = \pi/2$



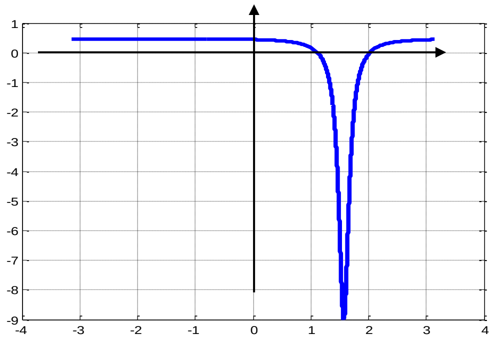
Group delay:

$$\tau_{grd} = -\frac{d}{d\omega}\angle\left(1 - re^{j\theta}e^{-j\omega}\right) = \frac{r^2 - r\cos(\omega-\theta)}{1 + r^2 - 2r\cos(\omega-\theta)}$$

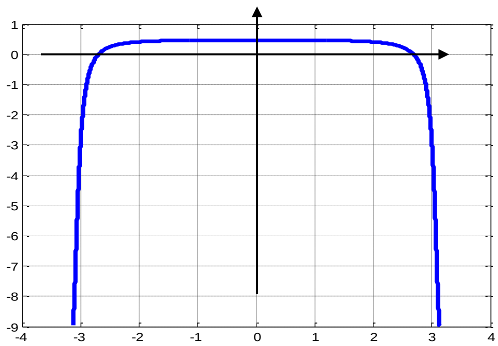
$r = 0.9, \theta = \ 0$



$r = 0.9, \theta = \ \pi/2$



$r = 0.9, \theta = \ \pi$



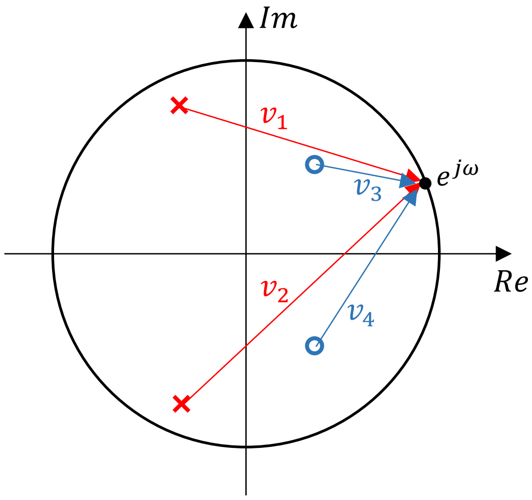
```
close all
clear all
w=-pi:0.001:pi;
r=0.9;
teta=0;
% mag=10*log10(1+r^2-2*r*cos(w-teta));
% plot(w,mag,'linewidth',3)
% grid

c=r*exp(j*teta);
H = 1./(1-c*exp(-j*w));
plot(w,20*log10(abs(H)),'linewidth',3);
grid
figure
plot(w,angle(H),'linewidth',3);
grid
% v=[-4 -4 -30 10];
% axes([-4 -4 -30 10])
B = 1;
A = [1 -c];
[Gd,w] = grpdelay(B,A,1000,'whole');
Gd = fftshift(Gd);
w = w-pi;
figure
plot(w,Gd,'linewidth',3)
grid
```

APPROXIMATE PLOTS OF FREQUENCY RESPONSE FROM POLE ZERO DIAGRAMS

Definition:

Zero vector: a vector drawn from a zero to a point, $e^{j\omega}$, on the unit circle.

Pole vector: a vector drawn from a pole to a point, $e^{j\omega}$, on the unit circle.



$$H(z) = \frac{(1 - z_1 z^{-1})(1 - z_2 z^{-1})}{(1 - p_1 z^{-1})(1 - p_2 z^{-1})}$$

$$= \frac{(z - z_1)(z - z_2)}{(z - p_1)(z - p_2)}$$

$$H\left(e^{j\omega}\right) = \frac{\left(1 - z_1 e^{-j\omega}\right)\left(1 - z_2 e^{-j\omega}\right)}{(1 - p_1 e^{-j\omega})(1 - p_2 e^{-j\omega})}$$

$$= \frac{\underbrace{\left(e^{j\omega} - z_1\right)}_{v_3}\underbrace{\left(e^{j\omega} - z_2\right)}_{v_4}}{\underbrace{\left(e^{j\omega} - p_1\right)}_{v_1}\underbrace{\left(e^{j\omega} - p_2\right)}_{v_2}}$$

1) The magnitude of a frequency response function, $\left|H\left(e^{j\omega}\right)\right|$, is proportional to

$$\frac{\text{product of the magnitudes of zero vectors}}{\text{product of the magnitudes of pole vectors}}$$

$$\left|H\left(e^{j\omega}\right)\right| = \frac{\left|e^{j\omega} - z_1\right|\left|e^{j\omega} - z_2\right|}{\left|e^{j\omega} - p_1\right|\left|e^{j\omega} - p_2\right|}$$

2) The phase of a frequency response function, $\measuredangle H\left(e^{j\omega}\right)$, is (except possible addition of $\pm\pi$)

$$(\text{sum of the angles of zero vectors}) \\ - (\text{sum of the angles of pole vectors})$$

Animation:
https://engineering.purdue.edu/VISE/ee438/demos/flash/pole_zero.html

Observations (on first order examples above):

A zero at $re^{j\theta}$ $(r < 1, r > 1) \Rightarrow$

1) The magnitude has a ***minimum*** at (around for higher orders) $\omega = \theta$.
2) The absolute rate of change of the phase is maximum around $\omega = \theta$.
3) The phase tends to *increase*/decrease towards $\omega = \theta$.
4) (After (2)) The absolute value of group delay has a maximum around $\omega = \theta$.
5) These four effects become stronger as $|r| \to 1$, i.e., as the zero approaches the unit circle.

A pole at $re^{j\theta}$ $(r < 1, r > 1) \Rightarrow$

1) The magnitude has a ***maximum*** at (around, for higher orders) $\omega = \theta$.
2) The absolute rate of change of the phase is maximum around $\omega = \theta$.
3) The phase tends to *decrease*/*increase* towards $\omega = \theta$.
4) (After (2)) The absolute value of group delay has a maximum at $\omega = \theta$.
5) These four effects become stronger as $|r| \to 1$, i.e., as the zero approaches the unit circle.

In general, when there are multiple poles and zeros, the above principles can be used to make a rough sketch of frequency response magnitude, phase and group delay.

Note that with multiple poles and zeros, the absolute rate of change of the phase is not necessarily maximum at the exact pole/zero angle, i.e., $\omega = \theta$.

Investigate the following examples and judge on the reliability of the above principles in inferring the frequency response magnitude, phase and group delay.

The following 4 examples are for 2nd order systems of the form

$$H(z) = \frac{1}{(1 - dz^{-1})(1 - d^{*}z^{-1})}$$

$$= \frac{1}{(1 - re^{j\theta}z^{-1})(1 - re^{-j\theta}z^{-1})}$$

$$= \frac{1}{1 - 2r\cos\theta\, z^{-1} + r^{2}z^{-2}}$$

Poles at $re^{j\theta},\ re^{-j\theta}$

Double zero at 0

Impulse response,

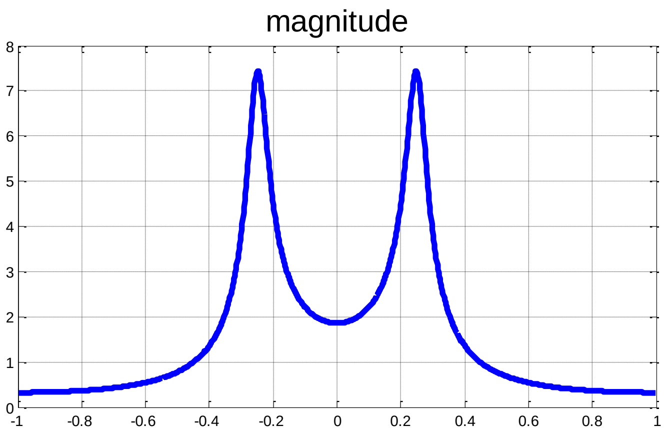
$$h[n] = \frac{1}{\sin\theta} r^{n} \sin\big(\theta(n+1)\big)\, u[n]$$

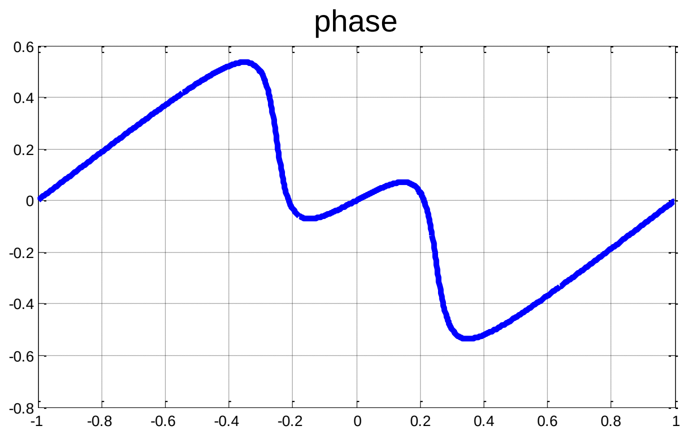
or

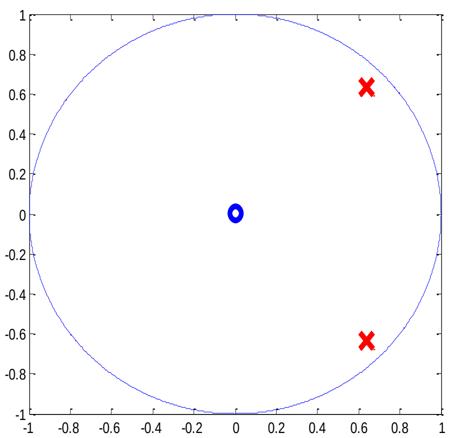
$$h[n] = -\frac{1}{\sin\theta} r^{n} \sin\big(\theta(n+1)\big)\, u[-n-1]$$

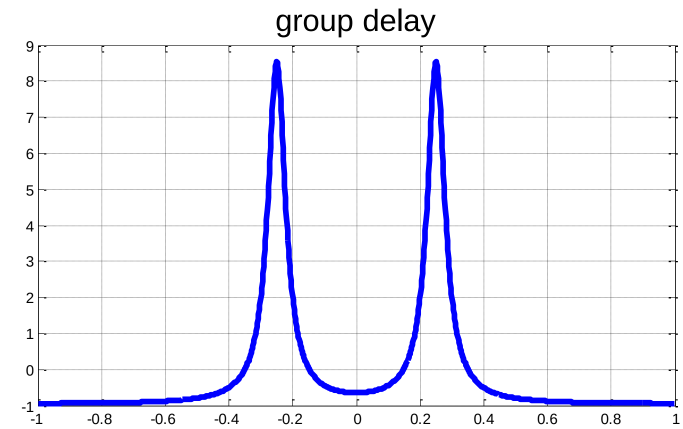
Ex (1):

Let $r = 0.9$ and $\theta = \frac{\pi}{4}$







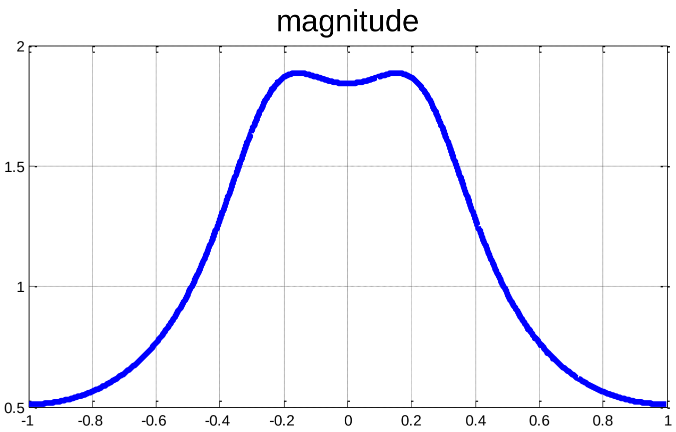


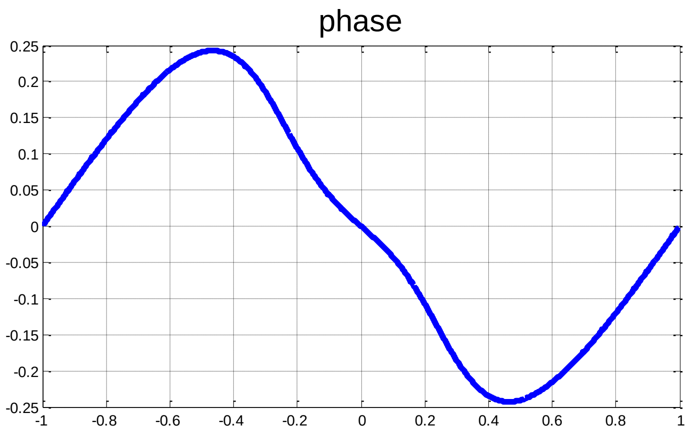
```
clear all
close all

phaseshift = 0; % 0, 1, -1 degerlerini alır
rp = 0.9;
thtp = pi/4;
p1 = rp*exp(j*thtp);
p2 = conj(p1);

rz = 0 ;
thtz = pi/4 ;
z1 = rz*exp(j*thtz);
z2 = conj(z1);

poles = [p1; p2];
zeros = [z1; z2];
gain = 1;

plot(poles+(0.0001+j*0.0001),'rx','linewidth',4,'markersize',15)
hold
 plot(zeros+(0.0001+j*0.0001),'bo','linewidth',4,'markersize',10)
% plot(0,0,'bo','linewidth',2,'markersize',10)
om = 0:999;
plot(exp(j*om*2*pi/1000));


[num den] = zp2tf(zeros,poles,gain)

[HH w] = freqz(num,den,1024,'whole');
H = fftshift(HH);
figure
subplot(2,2,1); plot(w/pi-1, abs(H),'linewidth',4); grid;
title('magnitude','fontsize',20)
% v = axis;
% v(3) = 0;
% v(4) =5;
% axis(v);
% figure
subplot(2,2,2); plot(w/pi-1,(2*pi*phaseshift+unwrap(angle(H)))/pi,'linewidth',4); grid;
% subplot(2,2,2); plot(w/pi-1,fftshift(unwrap(angle(HH)))/pi,'linewidth',4); grid;
title('phase','fontsize',20)

% figure
subplot(2,2,4);plot(w/pi-1,fftshift(grpdelay(num,den,1024,'whole')),'linewidth',4)
title('group delay','fontsize',20)
grid
```

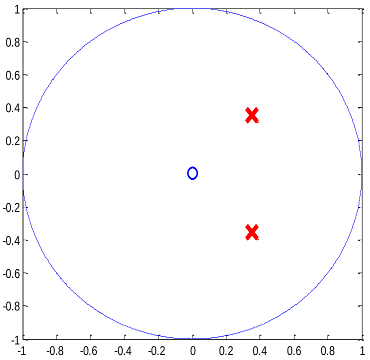
Ex (2): 2nd order system $H(z) = \frac{1}{\left(1 - re^{j\theta} z^{-1}\right)\left(1 - re^{-j\theta} z^{-1}\right)}$

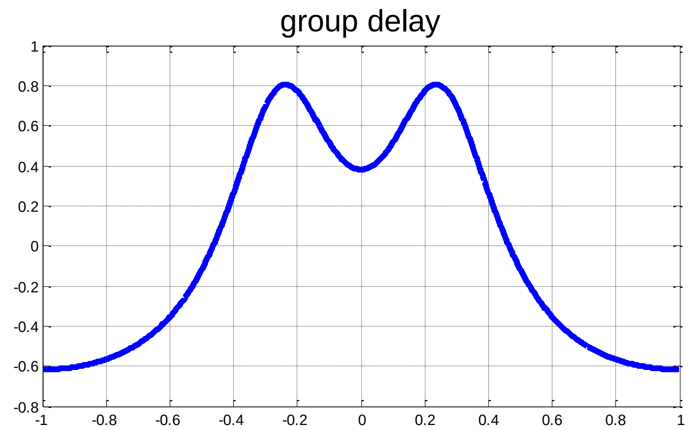
Poles at $0.5e^{j\frac{\pi}{4}}$, $0.5e^{-j\frac{\pi}{4}}$,

Double zero at 0



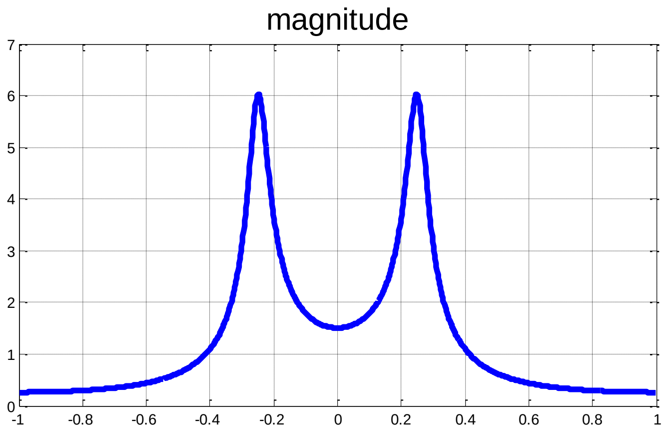


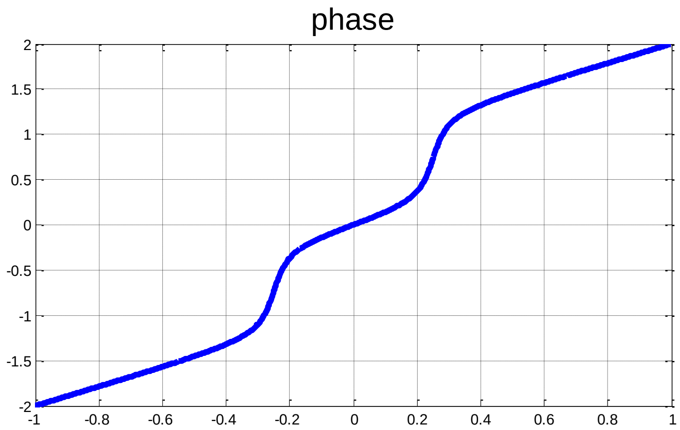


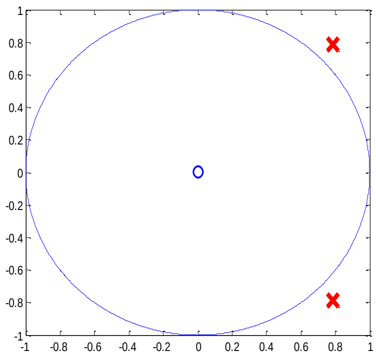


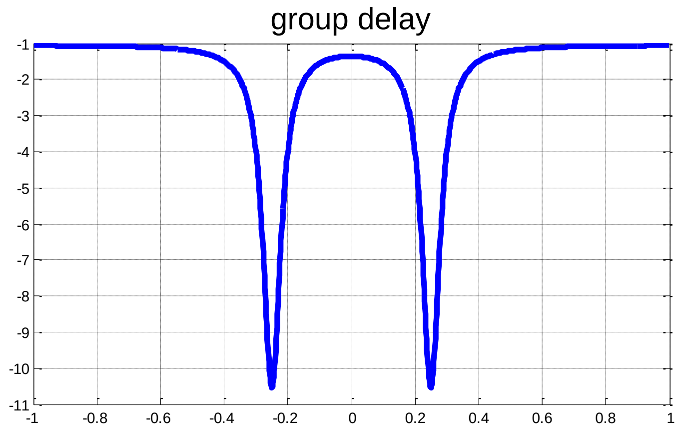
Ex (3): 2nd order system $H(z) = \frac{1}{(1-re^{j\theta}z^{-1})(1-re^{-j\theta}z^{-1})}$

Poles at $\frac{10}{9}e^{j\frac{\pi}{4}}$, $\frac{10}{9}e^{-j\frac{\pi}{4}}$,
double zero at 0



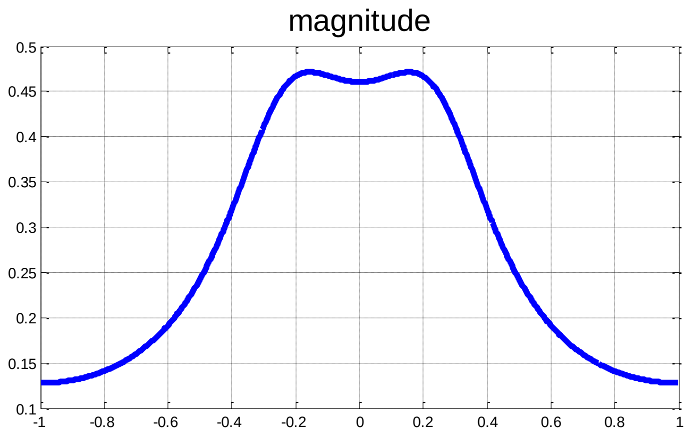


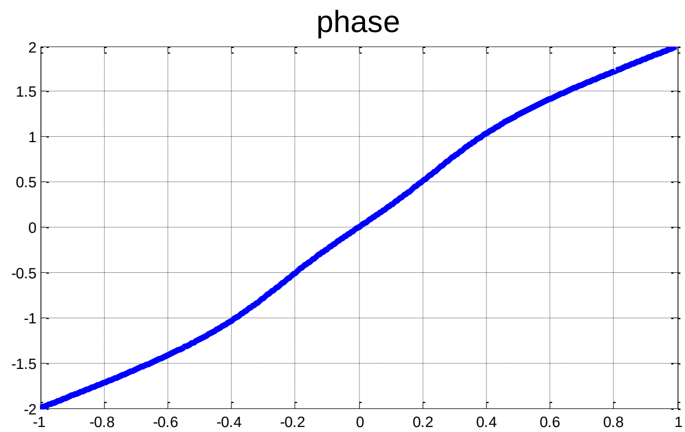


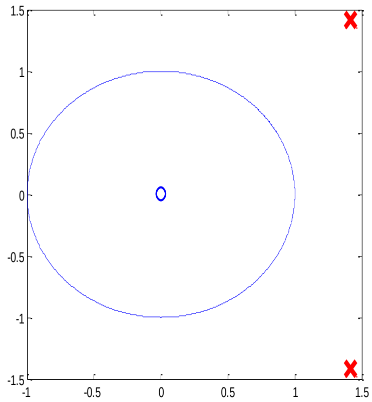


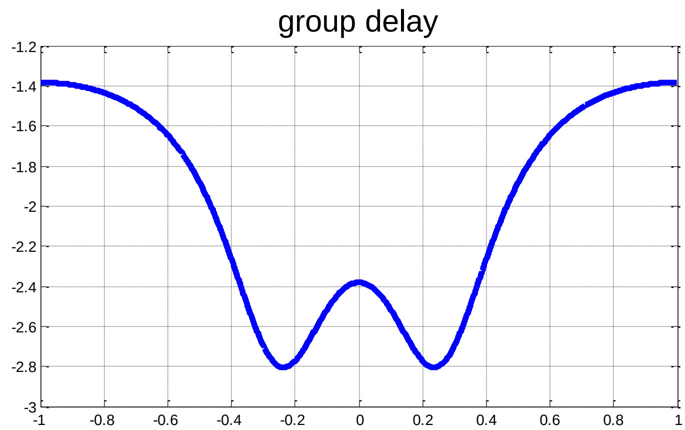
Ex (4): 2nd order system $H(z) = \frac{1}{(1-re^{j\theta}z^{-1})(1-re^{-j\theta}z^{-1})}$

Poles at $2e^{j\frac{\pi}{4}}$, $2e^{-j\frac{\pi}{4}}$
Double zero at 0





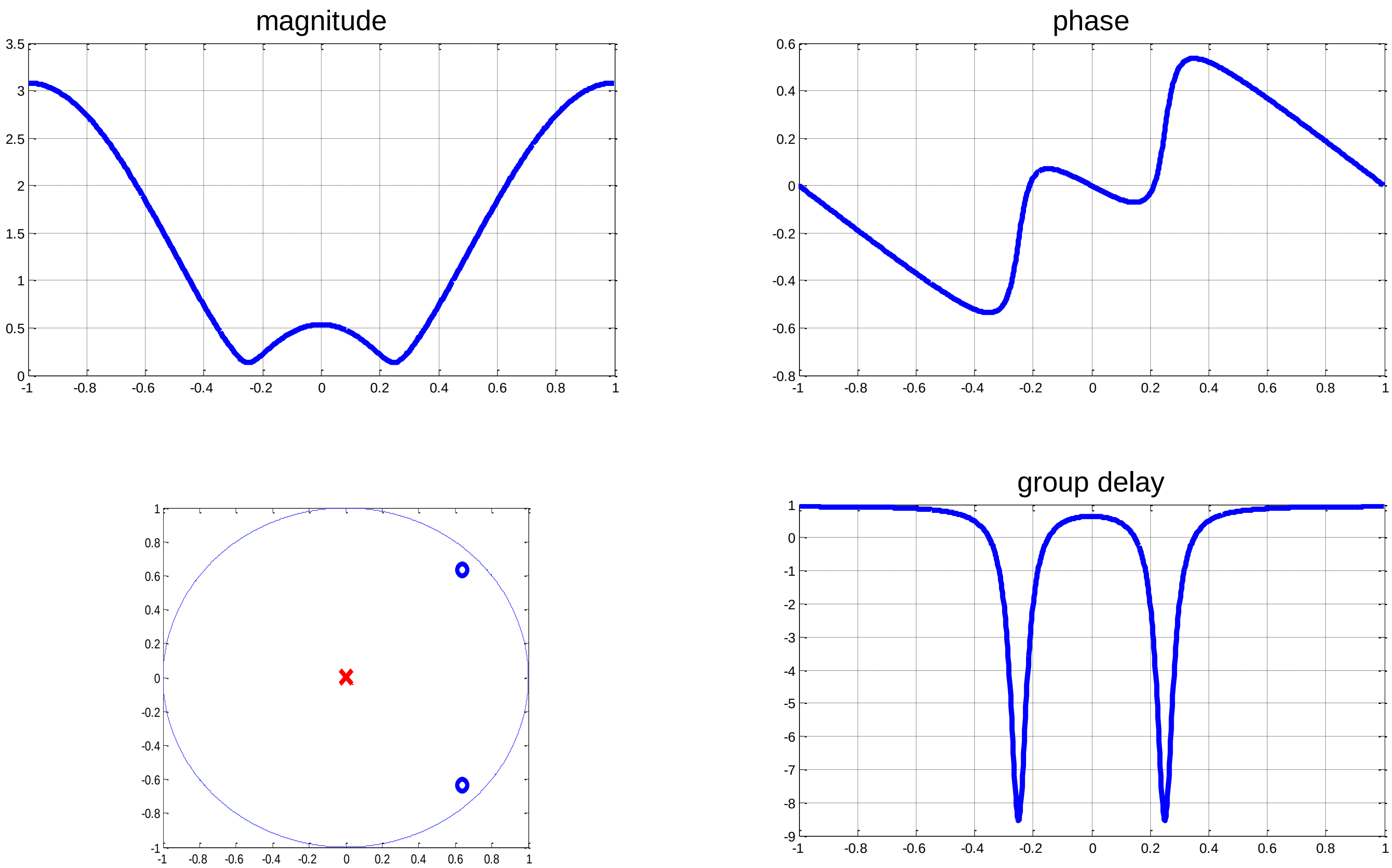




Ex: 2nd order system $H(z) = \left(1 - re^{j\theta}z^{-1}\right)\left(1 - re^{-j\theta}z^{-1}\right)$

Double pole at 0

Zeros at $0.9e^{j\frac{\pi}{4}},\ 0.9e^{-j\frac{\pi}{4}}$,

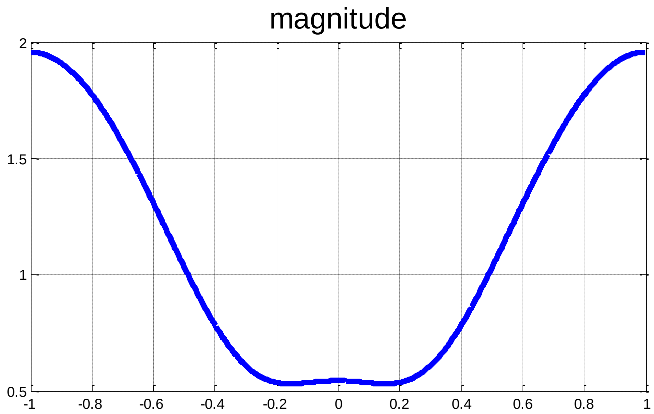


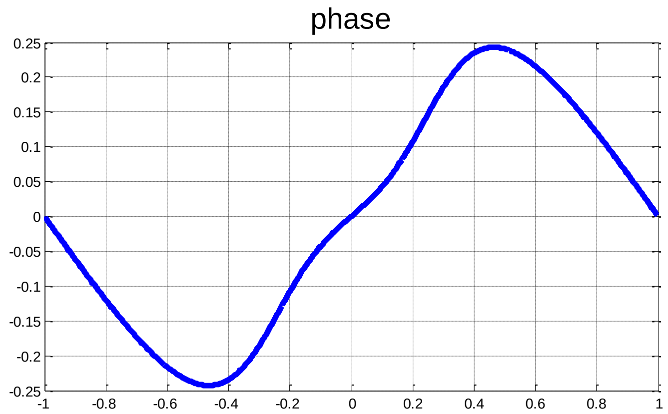
$$h[n] = \delta[n] - 2r\cos(\theta)\,\delta[n-1] + r^2\delta[n-2]$$

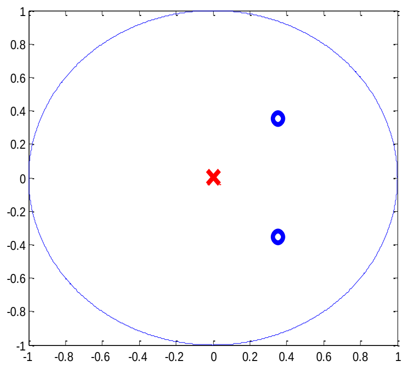
Ex: 2nd order system $H(z) = \left(1 - re^{j\theta}z^{-1}\right)\left(1 - re^{-j\theta}z^{-1}\right)$

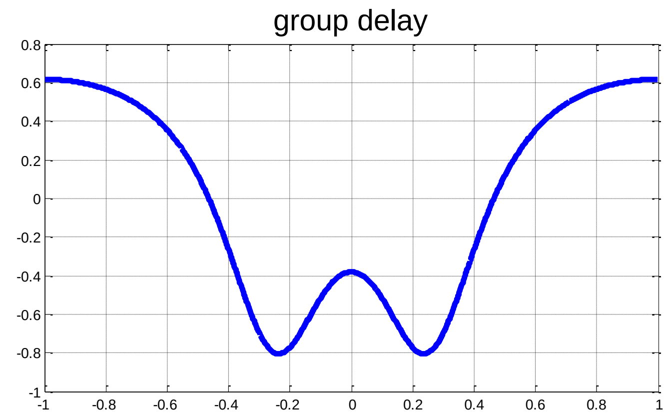
Double pole at 0

Zeros at $0.5e^{j\frac{\pi}{4}},\ 0.5e^{-j\frac{\pi}{4}}$





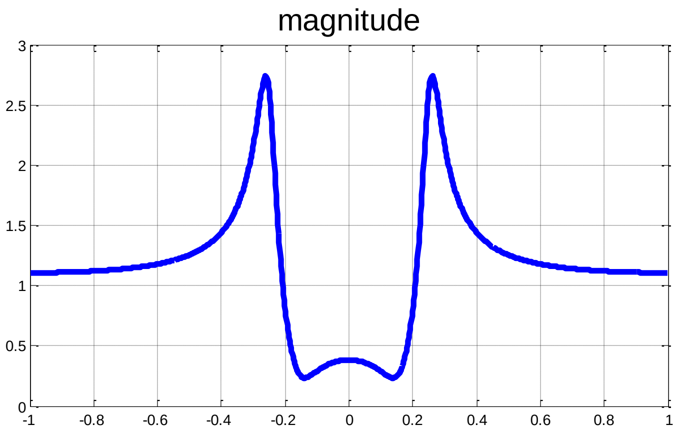


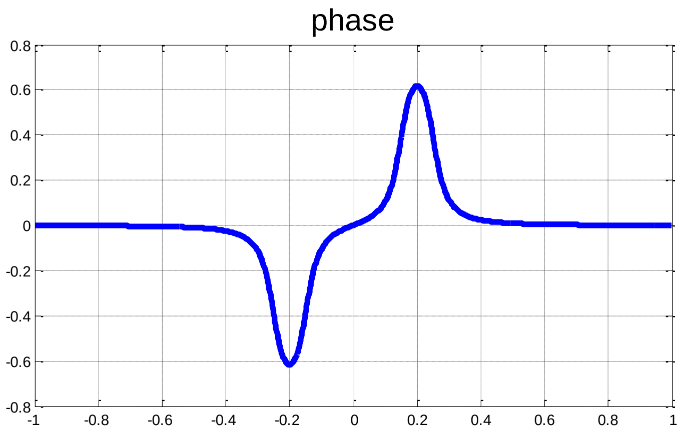


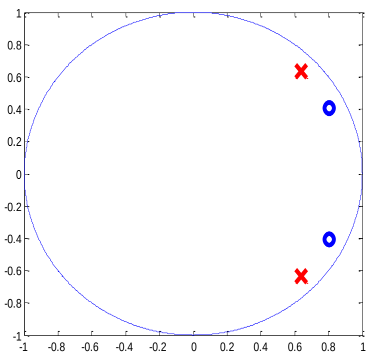
Ex: 2nd order, $H(z) = \frac{(1-z_1 z^{-1})(1-z_1^* z^{-1})}{(1-p_1 z^{-1})(1-p_1^* z^{-1})}$,

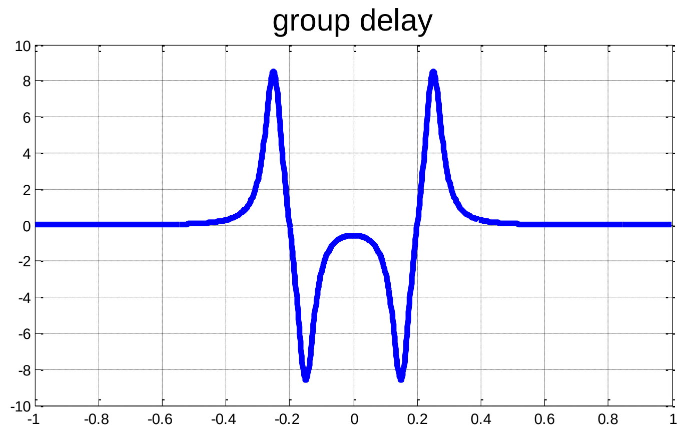
Poles: $0.9e^{j\frac{\pi}{4}},\ 0.9e^{-j\frac{\pi}{4}}$

Zeros: $0.9e^{j\left(\frac{\pi}{4}-\frac{\pi}{10}\right)},\ 0.9e^{-j\left(\frac{\pi}{4}-\frac{\pi}{10}\right)}$





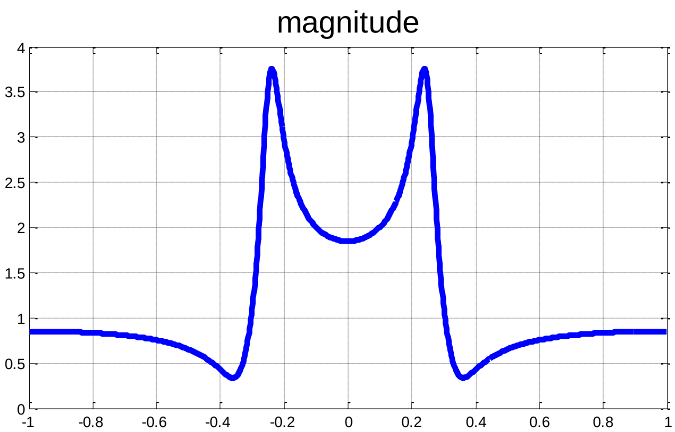


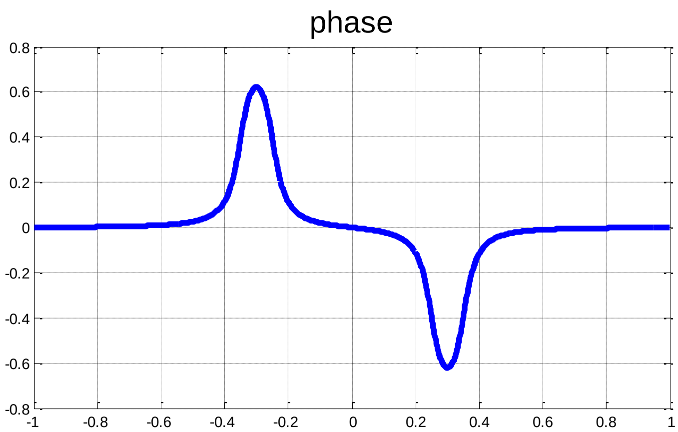


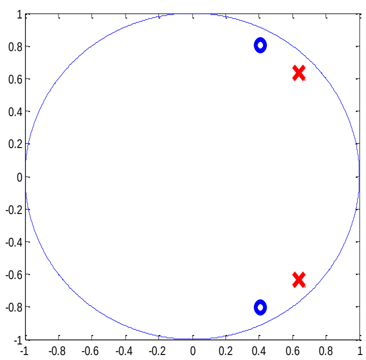
Ex: 2nd order, $H(z) = \frac{(1-z_1 z^{-1})(1-z_1^* z^{-1})}{(1-p_1 z^{-1})(1-p_1^* z^{-1})}$

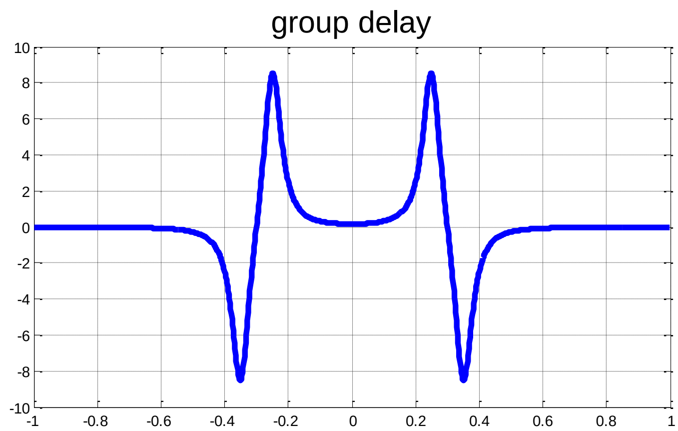
Poles: $0.9e^{j\frac{\pi}{4}},\ 0.9e^{-j\frac{\pi}{4}}$

Zeros: $0.9e^{j\left(\frac{\pi}{4}+\frac{\pi}{10}\right)},\ 0.9e^{-j\left(\frac{\pi}{4}+\frac{\pi}{10}\right)}$





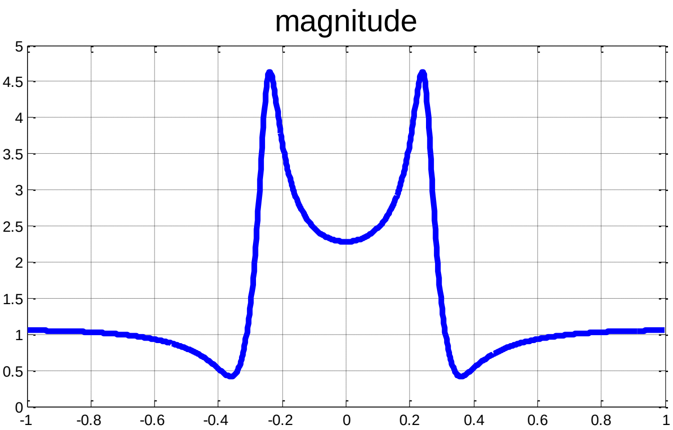


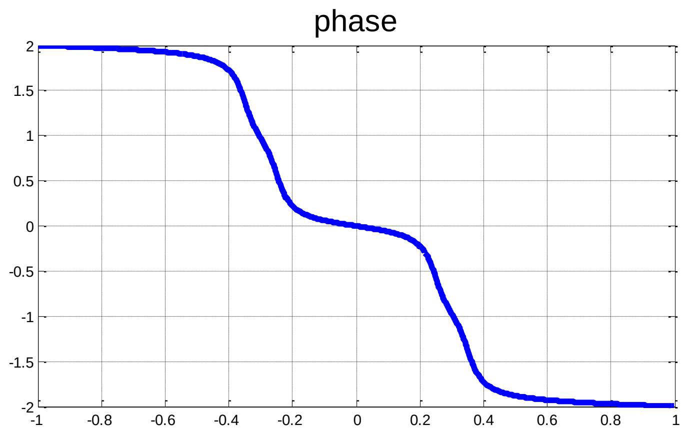


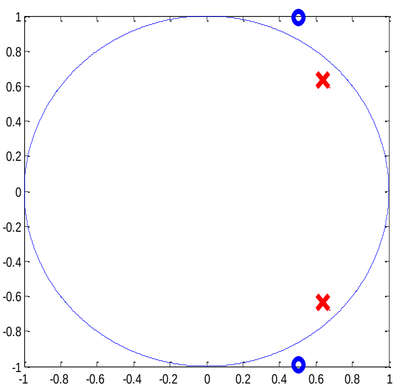
Ex: 2nd order, $H(z)=\frac{(1-z_1z^{-1})(1-z_1^*z^{-1})}{(1-p_1z^{-1})(1-p_1^*z^{-1})}$

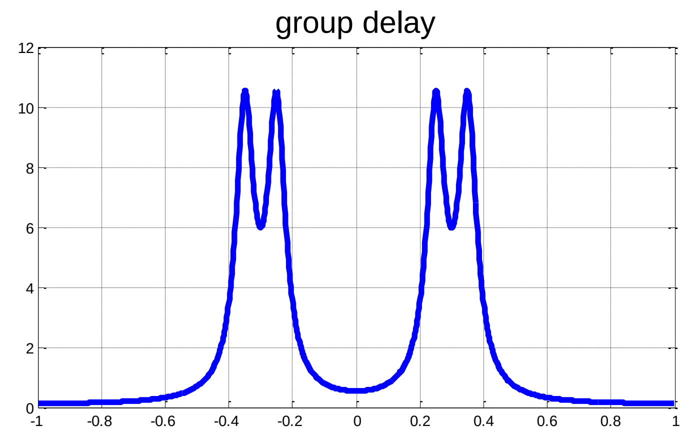
Poles: $0.9e^{j\frac{\pi}{4}},\ 0.9e^{-j\frac{\pi}{4}}$,

Zeros: $\frac{10}{9}e^{j\left(\frac{\pi}{4}+\frac{\pi}{10}\right)},\ \frac{10}{9}e^{-j\left(\frac{\pi}{4}+\frac{\pi}{10}\right)}$





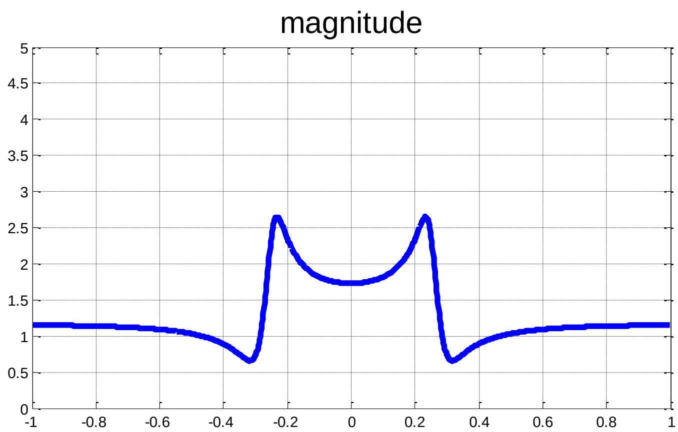


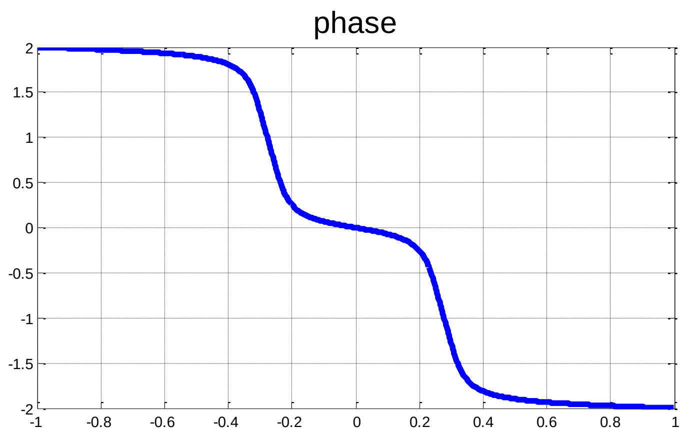


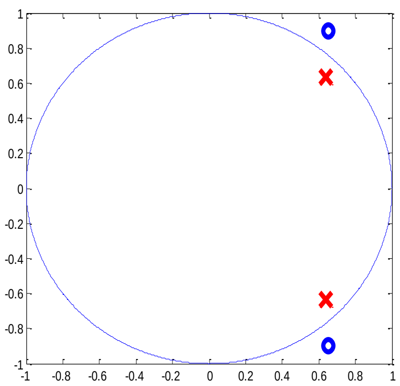
Ex: 2nd order, $H(z)=\frac{(1-z_1 z^{-1})(1-z_1^* z^{-1})}{(1-p_1 z^{-1})(1-p_1^* z^{-1})}$

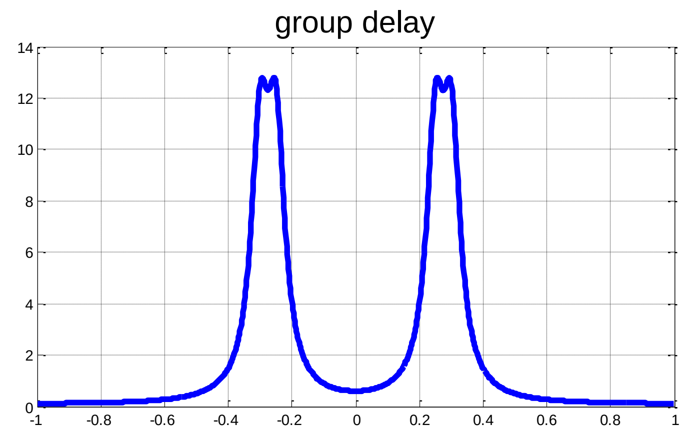
Poles: $0.9e^{j\frac{\pi}{4}},\ 0.9e^{-j\frac{\pi}{4}}$

Zeros: $\frac{10}{9}e^{j\left(\frac{\pi}{4}+\frac{\pi}{20}\right)},\ \frac{10}{9}e^{-j\left(\frac{\pi}{4}+\frac{\pi}{20}\right)}$





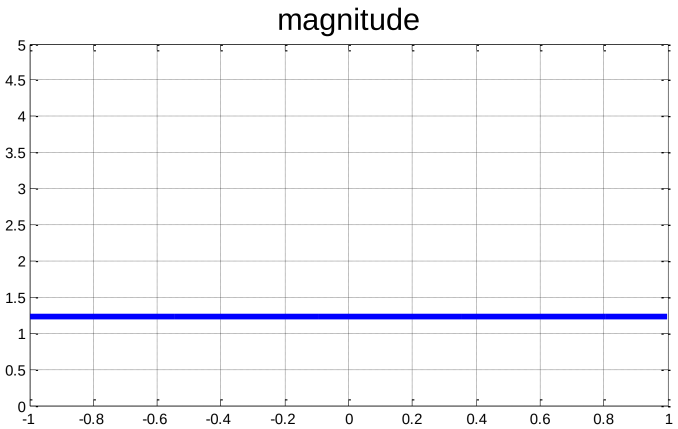


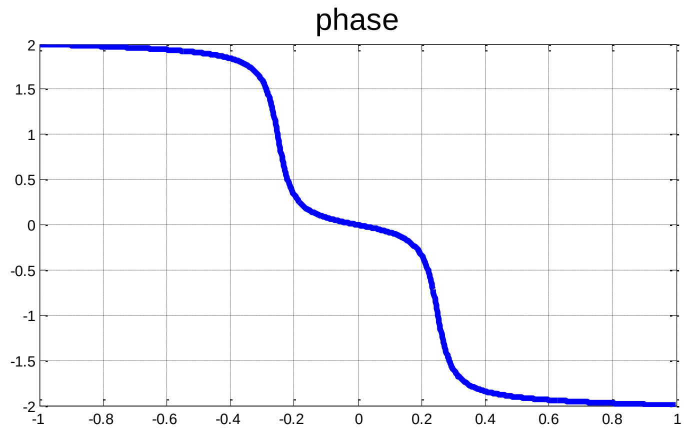


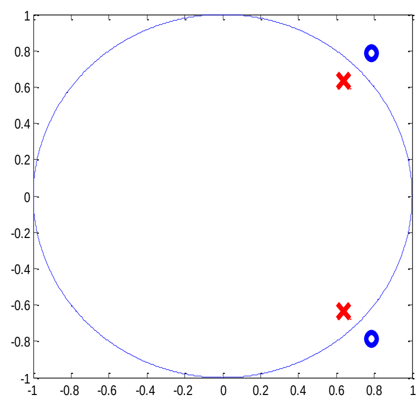
Ex: 2nd order, $H(z) = \dfrac{(1-z_1 z^{-1})(1-z_1^* z^{-1})}{(1-p_1 z^{-1})(1-p_1^* z^{-1})}$

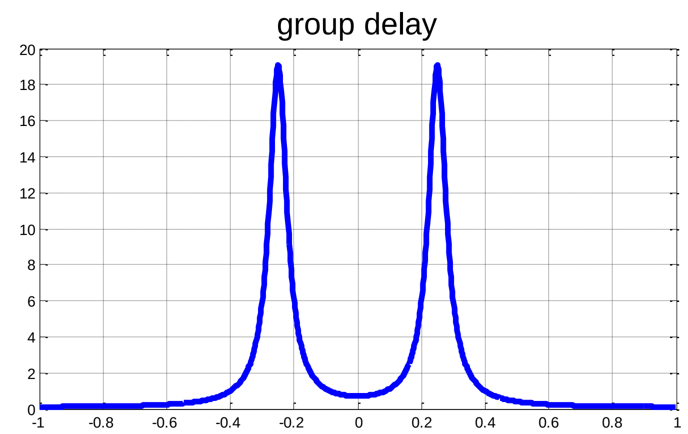
Poles: $0.9e^{j\frac{\pi}{4}},\ 0.9e^{-j\frac{\pi}{4}}$

Zeros: $\frac{10}{9}e^{j\frac{\pi}{4}},\ \frac{10}{9}e^{-j\frac{\pi}{4}}$





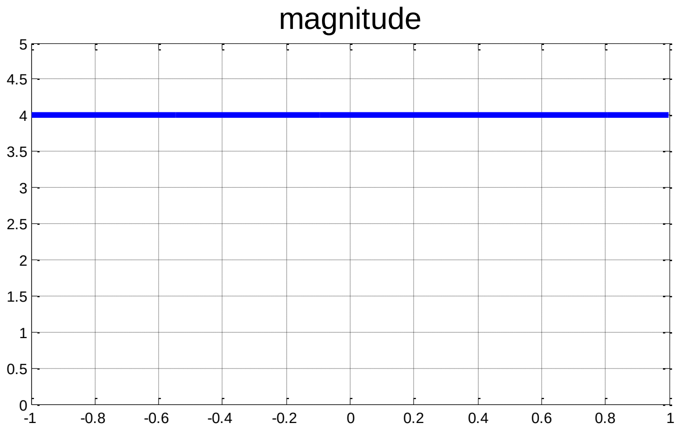


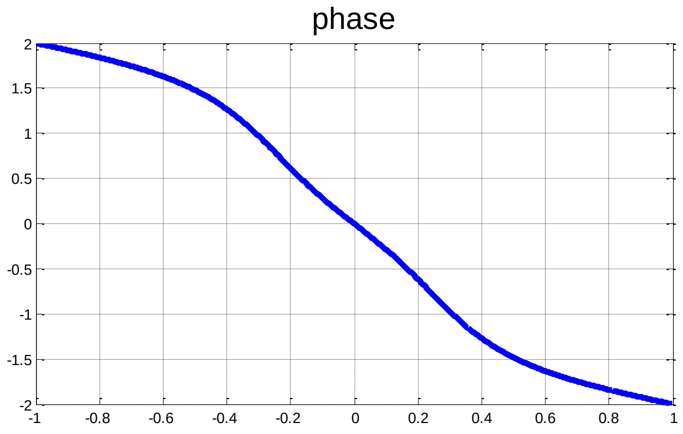


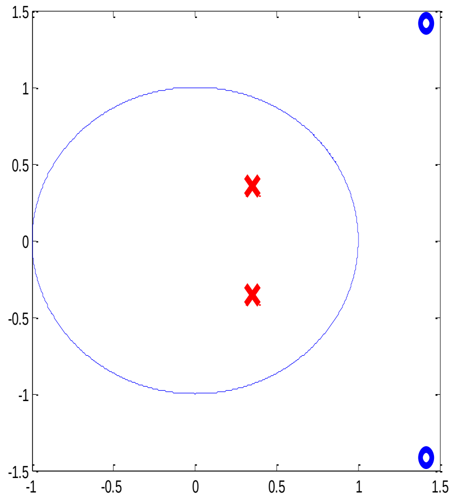
Ex: 2nd order, $H(z) = \frac{(1-z_1 z^{-1})(1-z_1^* z^{-1})}{(1-p_1 z^{-1})(1-p_1^* z^{-1})}$

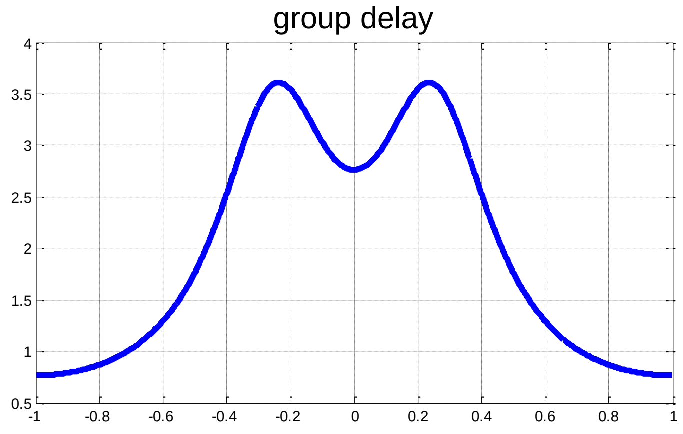
Poles: $0.5e^{j\frac{\pi}{4}},\ 0.5e^{-j\frac{\pi}{4}}$

Zeros: $2e^{j\frac{\pi}{4}},\ 2e^{-j\frac{\pi}{4}}$





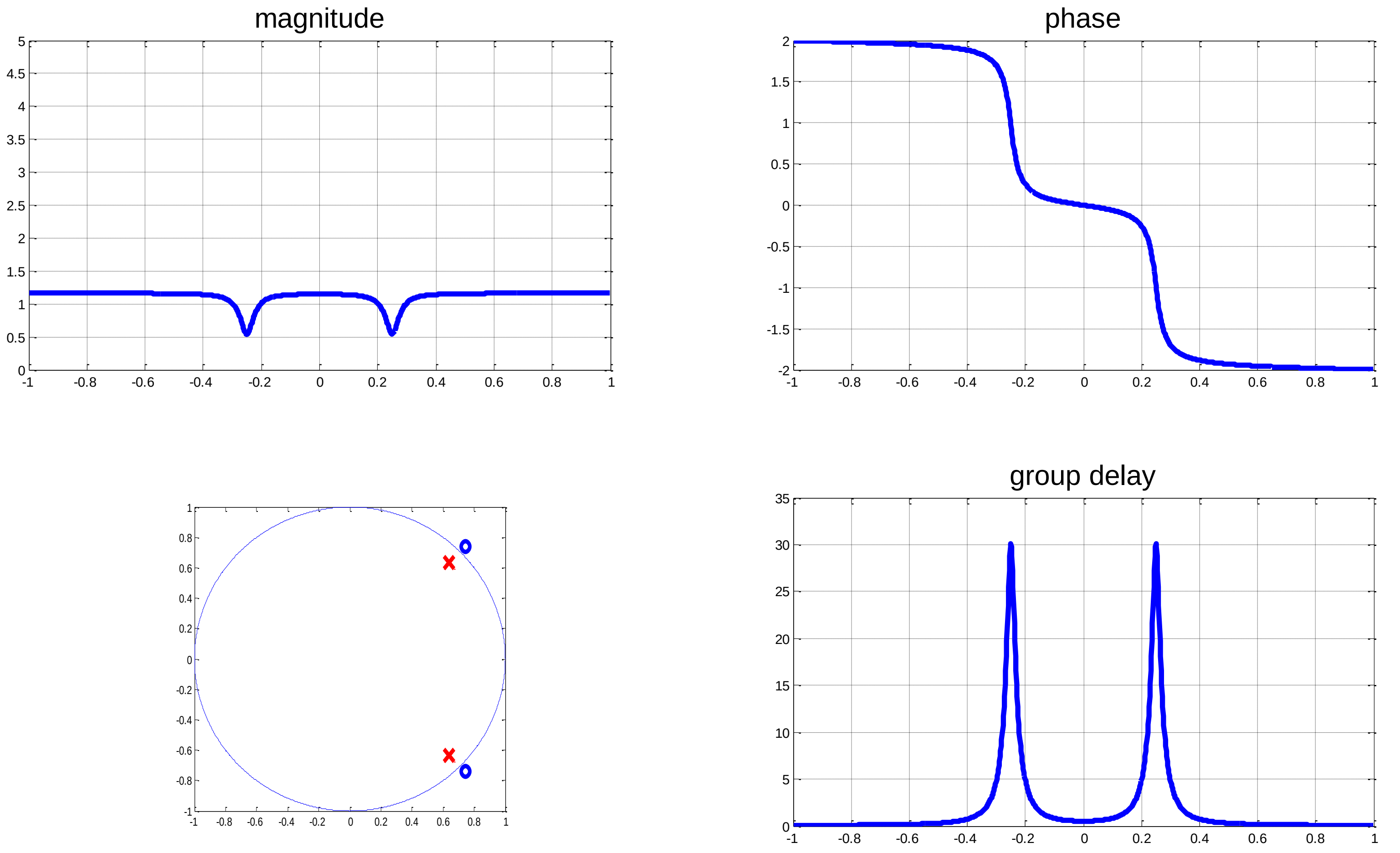




Ex: 2nd order, $H(z) = \frac{(1-z_1 z^{-1})(1-z_1^* z^{-1})}{(1-p_1 z^{-1})(1-p_1^* z^{-1})}$

Poles: $0.9e^{j\frac{\pi}{4}},\ 0.9e^{-j\frac{\pi}{4}}$

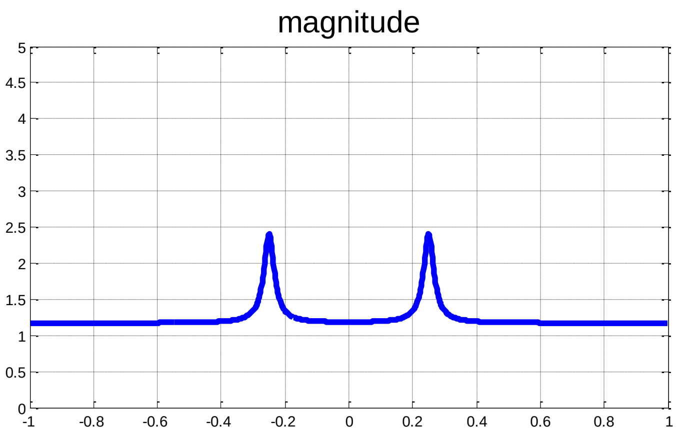
Zeros: $1.05e^{j\frac{\pi}{4}},\ 1.05e^{-j\frac{\pi}{4}}$

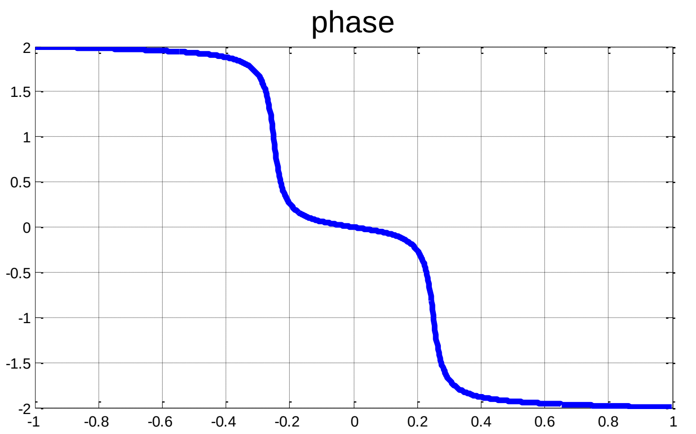


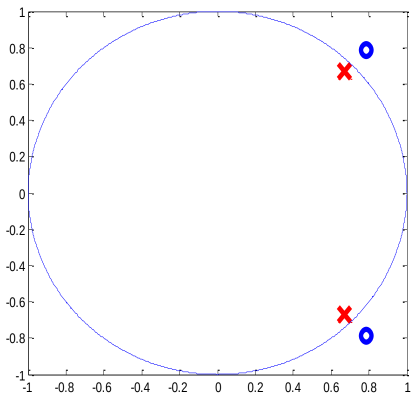
Ex: 2[nd] order, $H(z) = \frac{(1-z_1 z^{-1})(1-z_1^* z^{-1})}{(1-p_1 z^{-1})(1-p_1^* z^{-1})}$

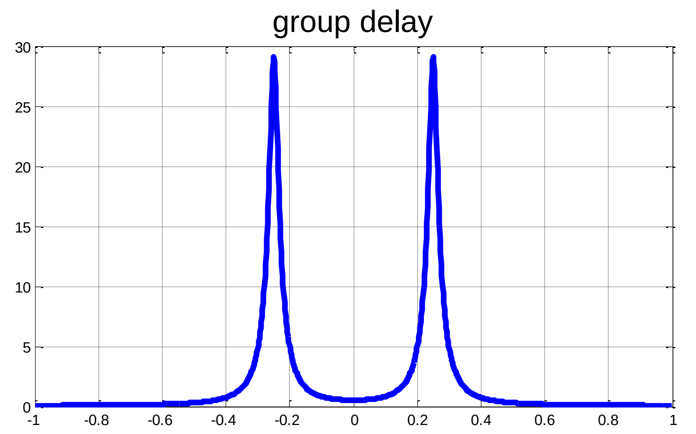
Poles: $0.95e^{j\frac{\pi}{4}},\ 0.95e^{-j\frac{\pi}{4}}$

Zeros: $\frac{10}{9}e^{j\frac{\pi}{4}},\ \frac{10}{9}e^{-j\frac{\pi}{4}}$

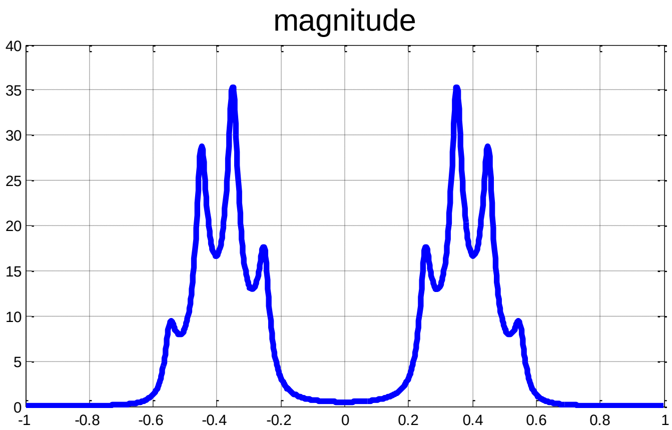


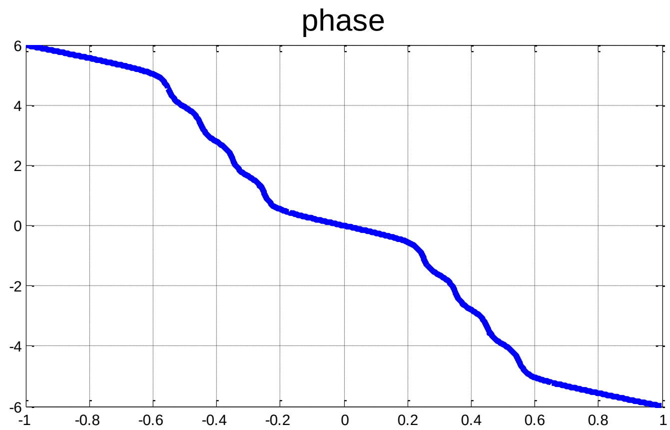


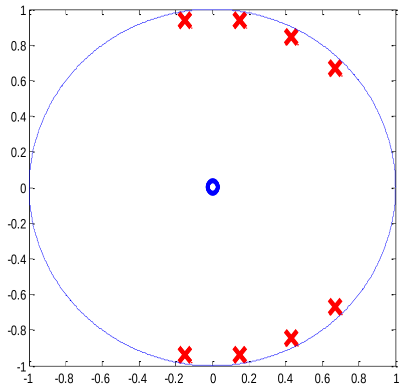


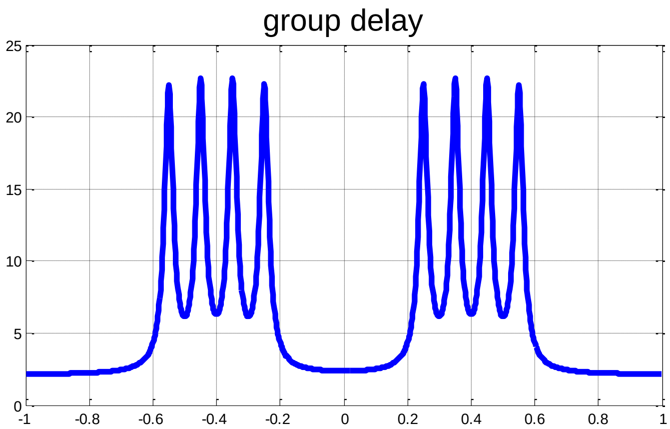


Ex: 8 poles   2 zeros

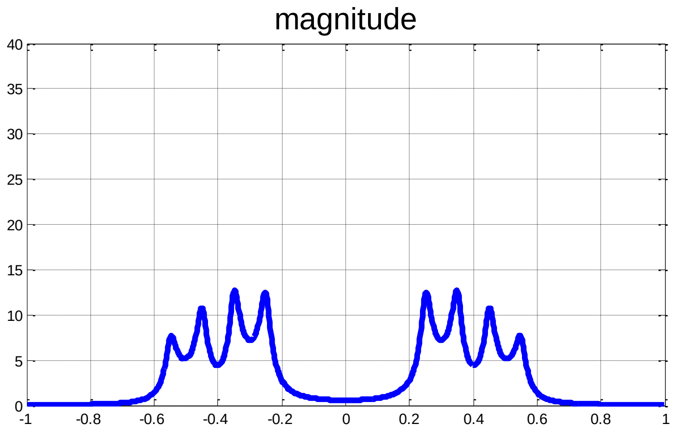


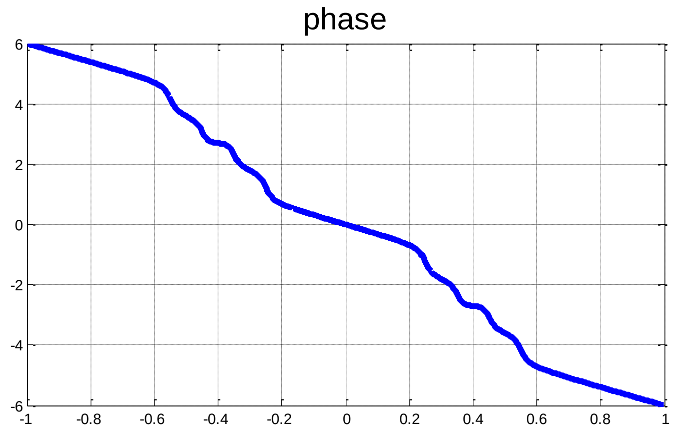


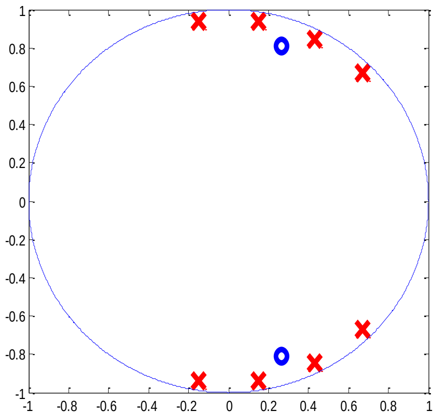


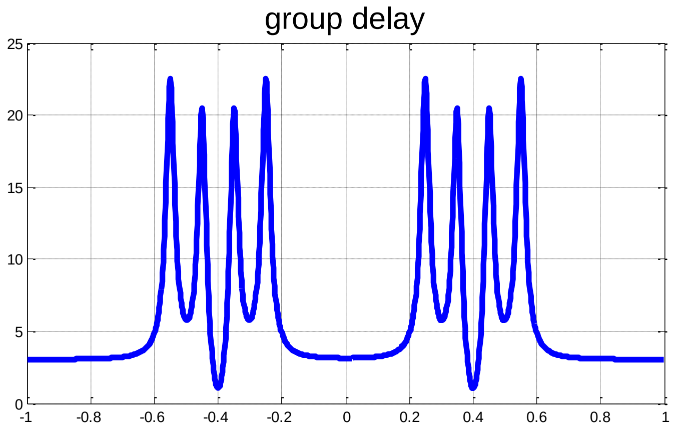


Ex: 8 poles 2 zeros









ALLPASS SYSTEMS

An LTI system is called allpass if

$$\left|H\left(e^{j\omega}\right)\right| = 1$$

PROPERTIES OF ALLPASS SYSTEMS

1) Poles and zeros are in <u>conjugate reciprocal</u> pairs.

$$H_{ap}(z_0) \to \infty \qquad \Leftrightarrow \qquad H_{ap}\left(\frac{1}{z_0^*}\right) = 0$$

2) Group delays of allpass systems are positive.

3) Phase responses of allpass systems are monotonically decreasing.

4) Phase responses of allpass systems are negative over $0 \leq \omega \leq \pi$.

RELATIONSHIP BETWEEN THE MAGNITUDE AND THE PHASE OF A FREQUENCY RESPONSE FUNCTION

For LTI systems with a rational $H(z)$

If the magnitude (phase) of the frequency response and the number of poles and zeros are known there are a limited number of choices for the phase (magnitude) of the frequency response

$H(z)$ AND $H^*\left(\frac{1}{z^*}\right)$ YIELD THE SAME MAGNITUDE RESPONSE

since

$$H^*\left(\frac{1}{z^*}\right)\bigg|_{z=e^{j\omega}} = H^*\left(e^{j\omega}\right)$$

and

$$\left|H^*\left(e^{j\omega}\right)\right| = \left|H\left(e^{j\omega}\right)\right|$$

Reading:

Appendix A: Frequency Responses Of Conjugate Reciprocal Zeros (Poles)

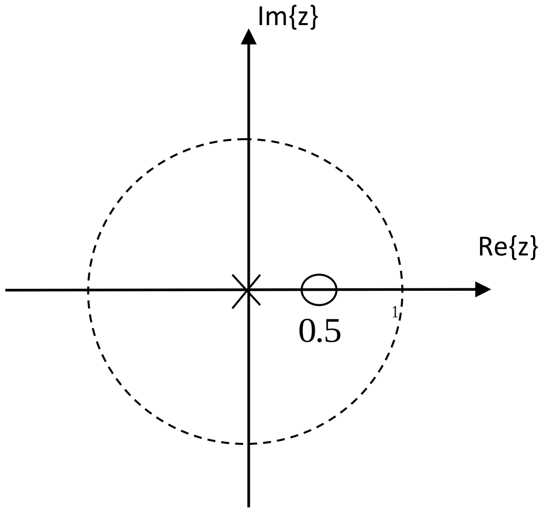
THE POLES AND ZEROS OF $H^*\left(\frac{1}{z^*}\right)$
ARE THE *CONJUGATE-RECIPROCALS*
OF THOSE OF $H(z)$

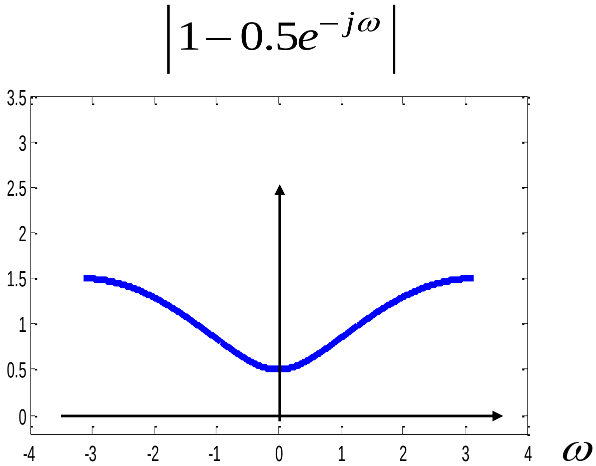
z_0 is a zero of $H(z)$ $\Rightarrow$ $\frac{1}{z_0^*}$ is a zero of $H^*\left(\frac{1}{z^*}\right)$.

p_0 is a pole of $H(z)$ $\Rightarrow$ $\frac{1}{p_0^*}$ is a pole of $H^*\left(\frac{1}{z^*}\right)$.

Ex:

$$H(z) = 1 - 0.5z^{-1}$$
$$H(e^{j\omega}) = 1 - 0.5e^{-j\omega}$$

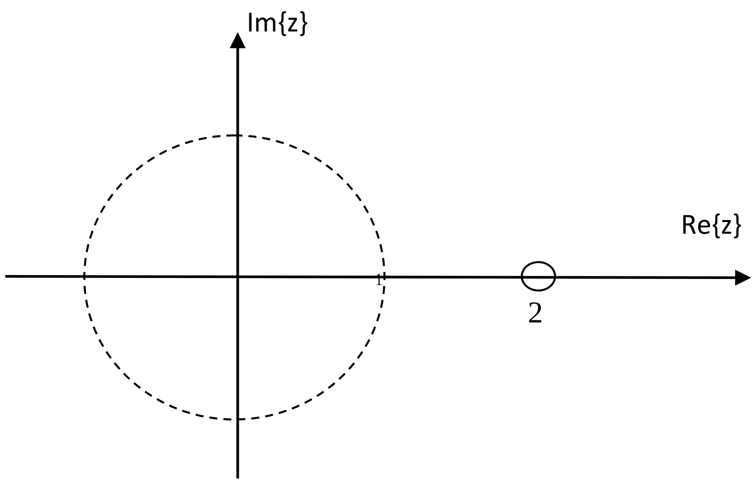


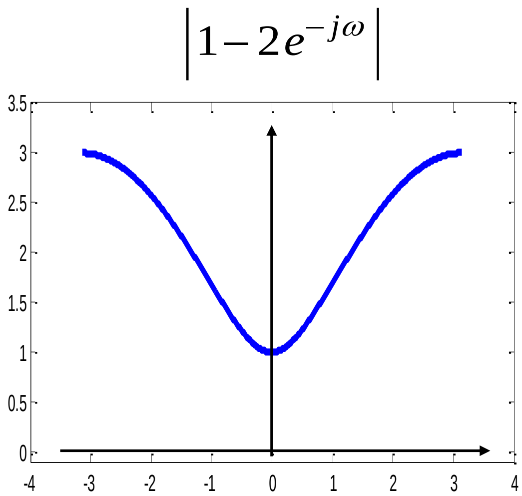


On the other hand,

$$H^*\left(\frac{1}{z^*}\right) = 1 - 0.5z \qquad = -0.5z(1 - 2z^{-1})$$

$$H^*(e^{j\omega}) = 1 - 0.5e^{j\omega} \qquad = -0.5e^{j\omega}(1 - 2e^{-j\omega})$$





In general,

$$H(z) = \frac{b_0}{a_0} \frac{\prod_{k=1}^{M}(1 - c_k z^{-1})}{\prod_{k=1}^{N}(1 - d_k z^{-1})}$$

$$H^*\left(\frac{1}{z^*}\right) = \frac{b_0}{a_0} \frac{\prod_{k=1}^{M}(1 - c_k^* z)}{\prod_{k=1}^{N}(1 - d_k^* z)}$$

$$C(z) = H(z)\, H^*\left(\frac{1}{z^*}\right)$$

$$C\left(e^{j\omega}\right) = \left|H\left(e^{j\omega}\right)\right|^2$$

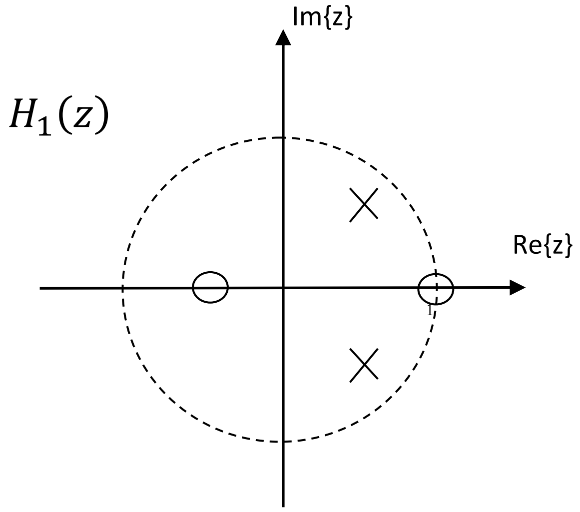
Therefore poles of $C(z)$ are in conjugate-reciprocal pairs and zeros of $C(z)$ are also in conjugate-reciprocal pairs.

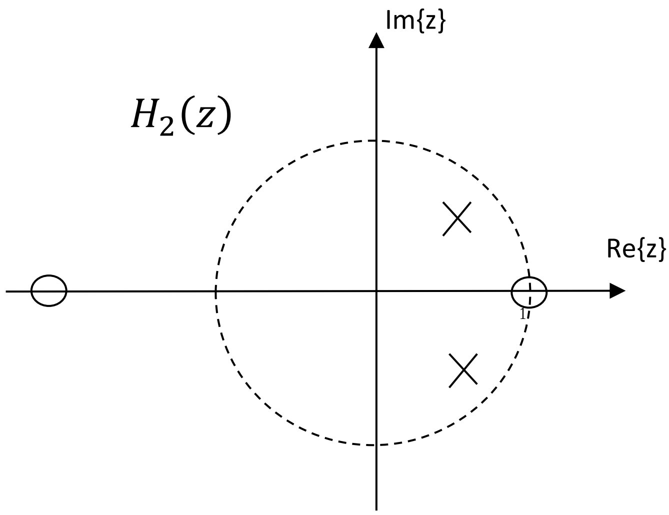
- Given $\left|H(e^{j\omega})\right|^2$, one can construct $C(z)$.
  - Express $\left|H(e^{j\omega})\right|^2$ in terms of $e^{j\omega}$
  - Replace $e^{j\omega} by\ z$

- Then, allocate the poles and zeros of $C(z)$ to its factors, $H(z)$ and $H^*\left(\frac{1}{z^*}\right)$.

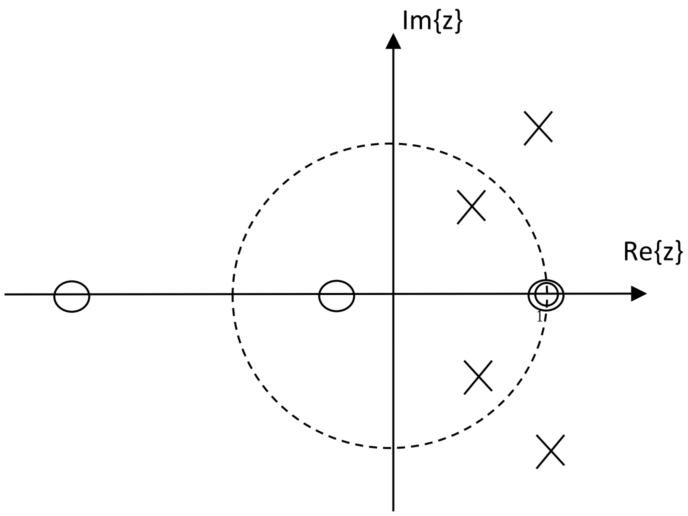
- There will be a finite number of choices for $H(z)$ and $H^*\left(\frac{1}{z^*}\right)$.

Ex:





$$C(z) = H_1(z)H_1^*\left(\frac{1}{z^*}\right) = H_2(z)H_2^*\left(\frac{1}{z^*}\right)$$



Ex: Let

$$H(z) = \frac{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 + \frac{3}{2}z^{-1}\right)\left(1 - \frac{7}{5}e^{j\frac{3\pi}{4}}\right)\left(1 - \frac{7}{5}e^{-j\frac{3\pi}{4}}\right)}{\left(1 + \frac{1}{3}z^{-1}\right)\left(1 + \frac{2}{5}z^{-1}\right)\left(1 - \frac{4}{5}e^{j\frac{3\pi}{4}}\right)\left(1 - \frac{4}{5}e^{-j\frac{3\pi}{4}}\right)}$$

a) How many other systems with 4 poles, 4 zeros (no matter real or complex, stable or unstable, causal or noncausal) have the same magnitude?

b) Write one of them.

c) How many of them are real, and causal and stable? Write one of them.

Ex: The squared magnitude of the frequency response of a *minimum phase (a system with all poles and zeros inside the unit circle)* system is

$$\left|H\left(e^{j\omega}\right)\right|^2 = \frac{1}{2.5 + 2\cos(\omega)}$$

Find $H(z)$; plot its poles and zeros.

$$\left|H\left(e^{j\omega}\right)\right|^2 = \frac{1}{\frac{5}{2} + e^{j\omega} + e^{-j\omega}}$$

$$H(z)H^*\left(\frac{1}{z^*}\right) = \frac{1}{\frac{5}{2} + z + z^{-1}}$$

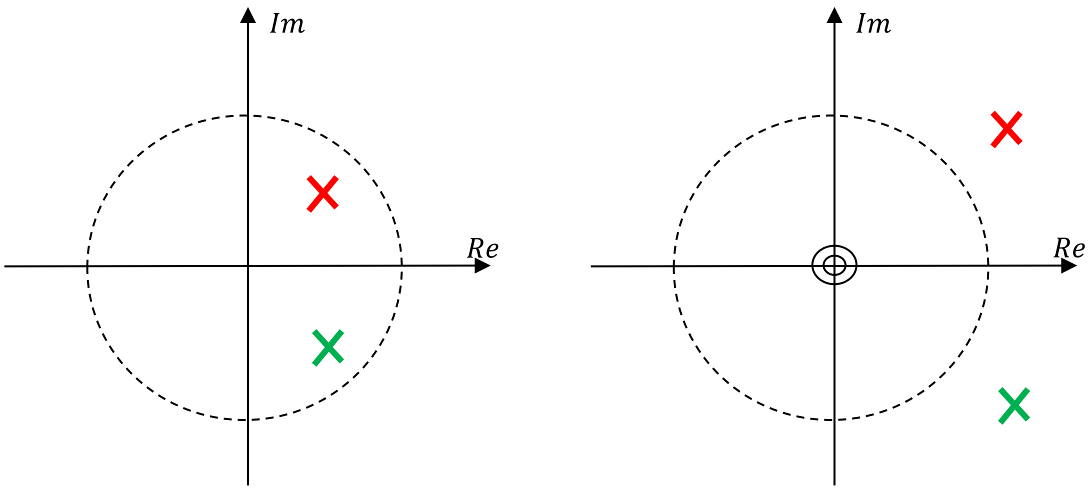
$$= \frac{1}{\left(1 + \frac{1}{2}z^{-1}\right)(2 + z)}$$

$$= \frac{\frac{1}{2}}{\left(1 + \frac{1}{2}z^{-1}\right)\left(1 + \frac{1}{2}z\right)}$$

Therefore,

$$H(z) = \frac{\frac{1}{\sqrt{2}}}{\left(1 + \frac{1}{2}z^{-1}\right)}$$

Ex:



$$H_a(z) = \frac{1}{(z-p_1)(z-p_1^*)}, \qquad H_b(z) = H^*\left(\frac{1}{z^*}\right) = \frac{1}{(z^{-1}-p_1^*)(z^{-1}-p_1)}$$

$$= \frac{1}{|p|^2} \frac{z^2}{\left(z-\frac{1}{p_1^*}\right)\left(z-\frac{1}{p_1}\right)}$$

$$\left|H_b(e^{j\omega})\right| = \left|H_b(e^{j\omega})\right|$$

ALLPASS SYSTEMS

OBTAINING A FIRST ORDER ALLPASS SYSTEM

Let

$$A(z) = 1 - az^{-1} \qquad |a| < 1$$

One can obtain a first order allpass system function as

$$\frac{A^*\left(\frac{1}{z^*}\right)}{A(z)} = \frac{1 - a^*z}{1 - az^{-1}}$$

Taking the ROC as $|z| > |a|$

$$h[n] = a^n u[n] - a^* a^{n+1} u[n+1]$$

$$= -a^* \delta[n+1\,] + (1 - |a|^2) a^n u[n]$$

which is stable but noncausal.

CAUSAL AND CANONICAL FORM

To have it causal, a first order allpass is canonically defined as

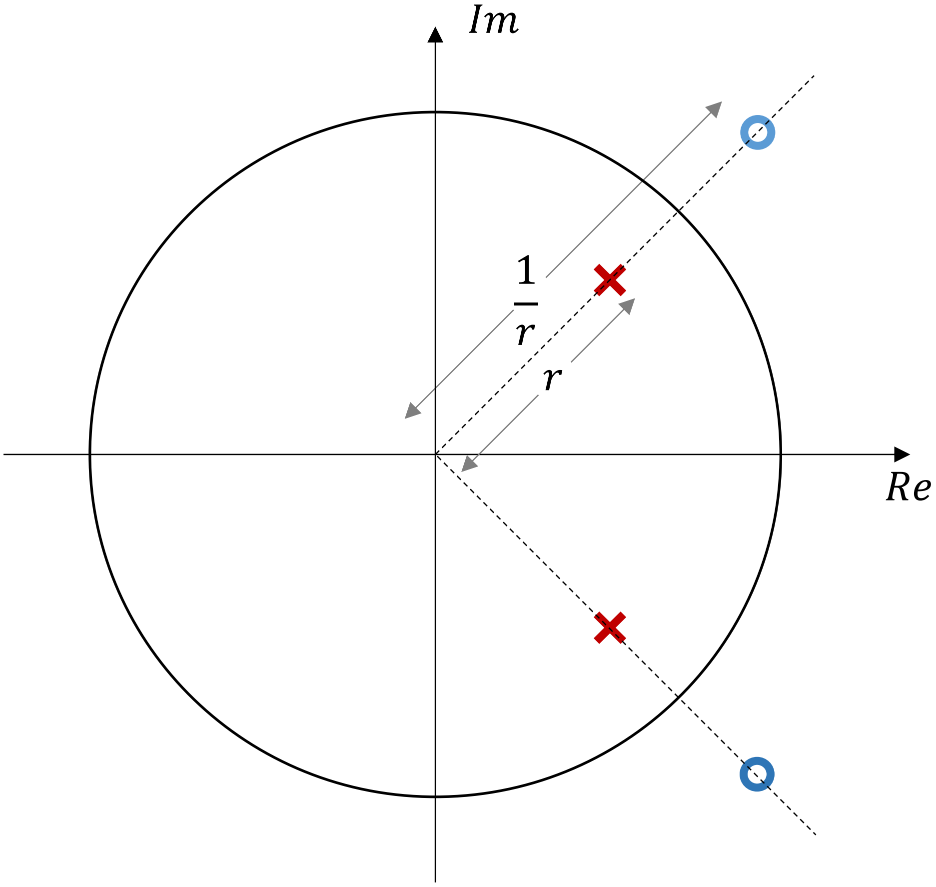
$$H_{ap}(z) = z^{-1} \frac{1 - a^* z}{1 - a z^{-1}}$$

$$= \frac{z^{-1} - a^*}{1 - a z^{-1}}$$

In general

$$H_{ap}(z) = A \prod_{k=1}^{M_r} \frac{z^{-1} - d_k}{1 - d_k z^{-1}} \prod_{k=1}^{M_c} \frac{(z^{-1} - e_k^*)(z^{-1} - e_k)}{(1 - e_k^* z^{-1})(1 - e_k z^{-1})}$$

The poles and zeros of an allpass system are at conjugate reciprocal locations.



Therefore, causal, stable allpass systems have all zeros outside the unit circle.

Ex:

a) Two LTI systems have the transfer functions $H(z)$ and $H^*\left(\frac{1}{z^*}\right)$. Assume that the ROCs of both system functions include the unit circle. Show that the magnitudes of the frequency responses of these systems are the same.

b) Let $H(z) = 1 - \frac{1}{2}z^{-1}$.

Consider the all-pass system function $G(z) = \frac{H^*\left(\frac{1}{z^*}\right)}{H(z)}$.

i) Find the poles and zeros of $G(z)$.

ii) Find the impulse response of the stable system represented by $G(z)$.

iii) Write the difference equation for $G(z)$. Is a causal and stable implementation possible?

If yes, why?

If no, find another first-order, all-pass transfer function $M(z)$ having a pole at $z = \frac{1}{2}$ by modifying $G(z)$, so that causal and stable implementation is possible.

1st order

$$H_{ap}(z) = \frac{z^{-1} - a^*}{1 - az^{-1}}$$

$$a = re^{j\theta}$$

$$H_{ap}\left(e^{j\omega}\right) = \frac{e^{-j\omega} - a^*}{1 - ae^{-j\omega}}$$

$$= e^{-j\omega}\,\frac{1 - a^*e^{j\omega}}{1 - ae^{-j\omega}}$$

Phase

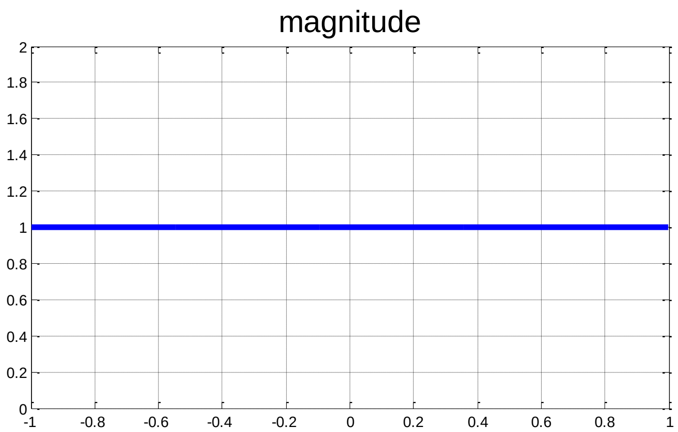
$$\measuredangle H\left(e^{j\omega}\right) = -\omega + \measuredangle\left(1 - a^*e^{j\omega}\right) - \measuredangle\left(1 - ae^{-j\omega}\right)$$

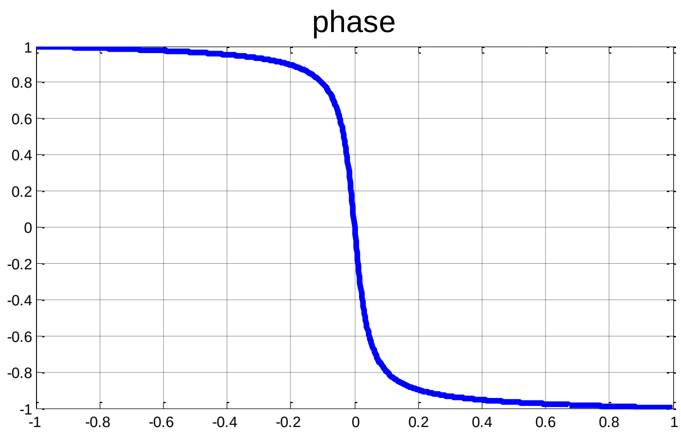
$$= -\omega + 2\measuredangle\left(1 - a^*e^{j\omega}\right)$$

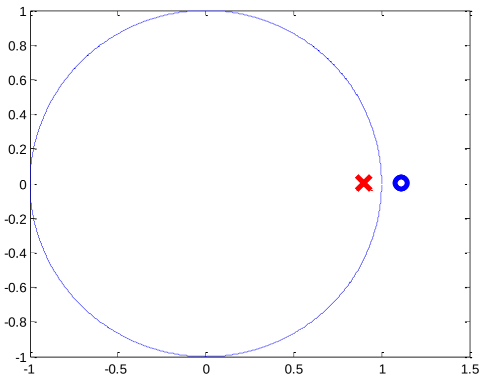
$$= -\omega - 2\tan^{-1}\frac{r\sin(\omega - \theta\ )}{1 - r\cos(\omega - \theta\ )}$$

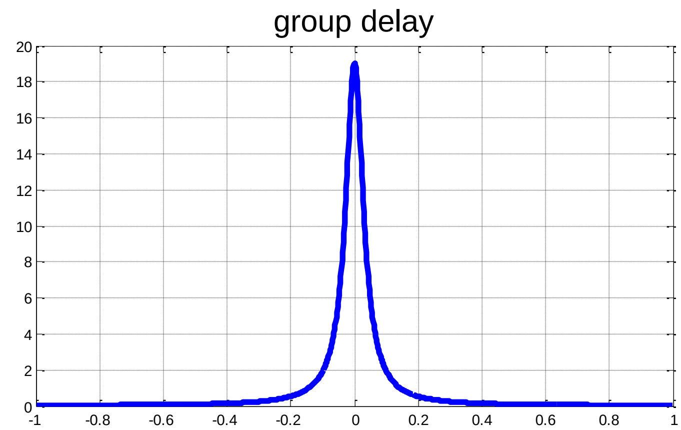
Ex: $r = 0.9 \quad \theta = 0$

$$H(z) = \frac{z^{-1} - 0.9}{1 - 0.9z^{-1}}$$



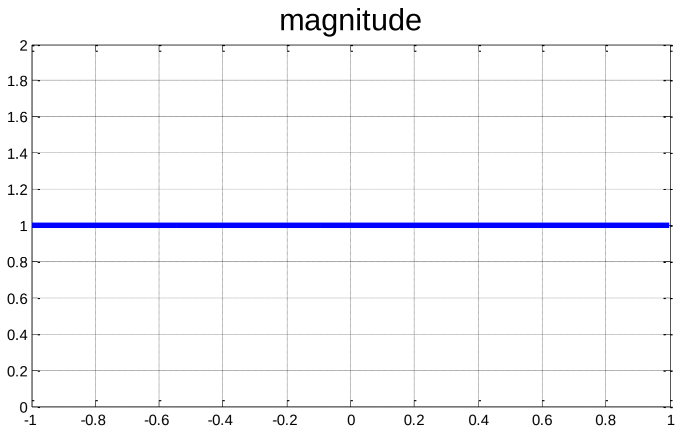


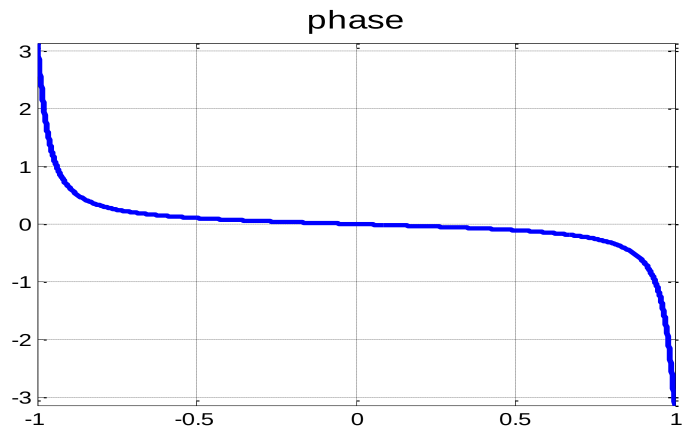


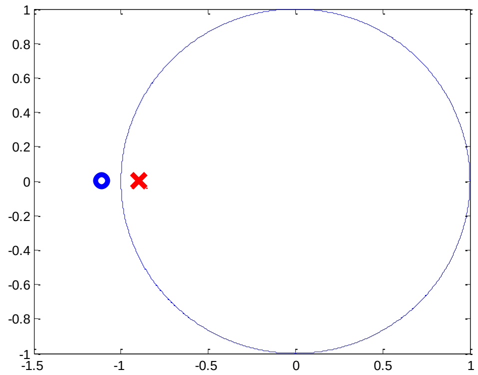


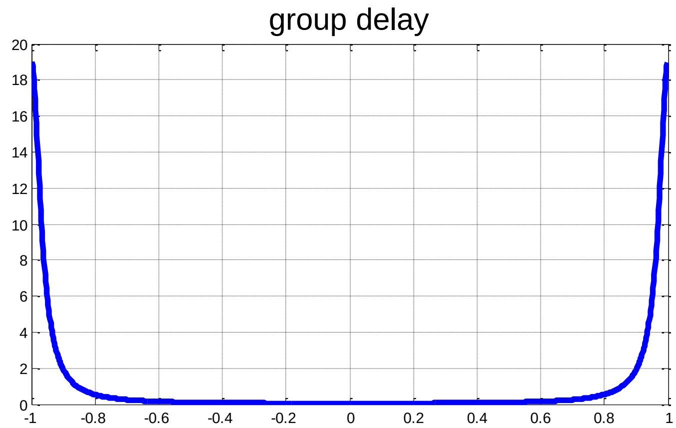
Ex: $r = 0.9 \quad \theta = \pi$

$$H(z) = \frac{z^{-1} + 0.9}{1 + 0.9z^{-1}}$$



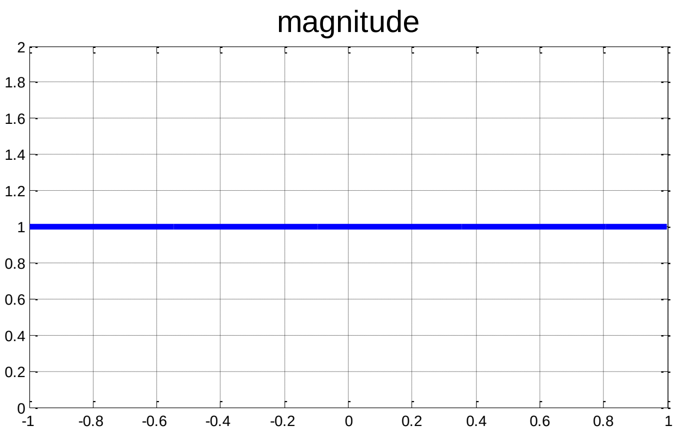


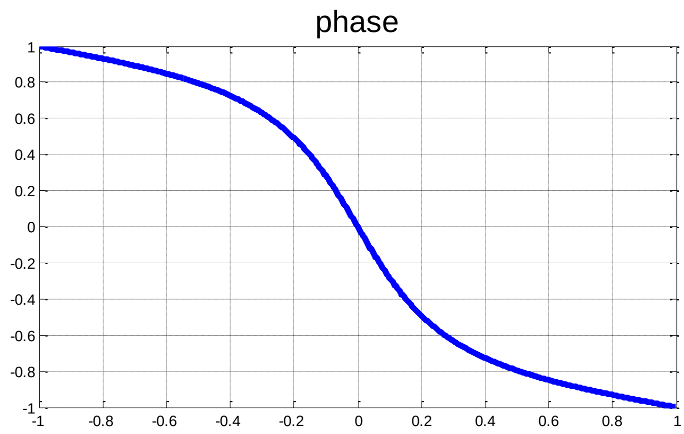


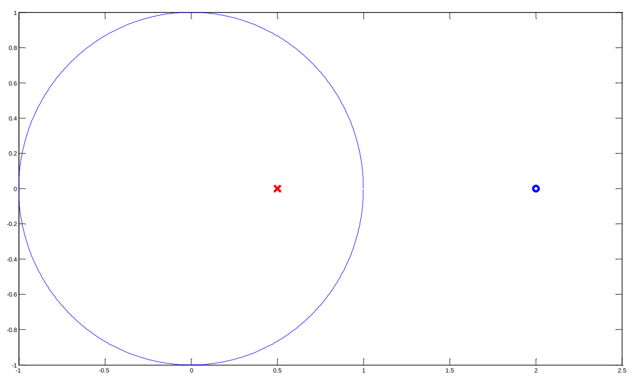


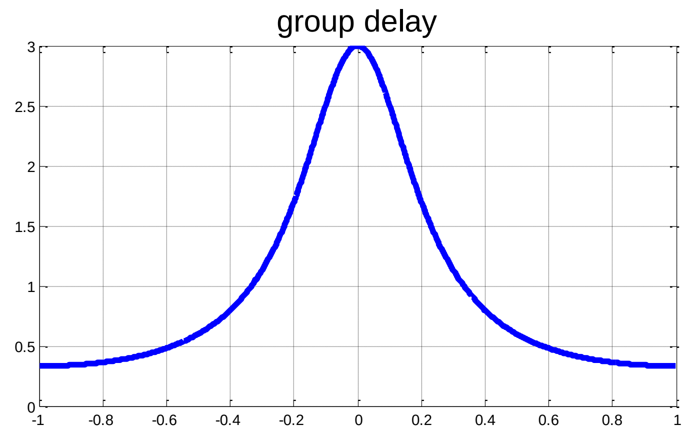
Ex: $r = 0.5 \quad \theta = 0$

$$H(z) = \frac{z^{-1} - 0.5}{1 - 0.5z^{-1}}$$









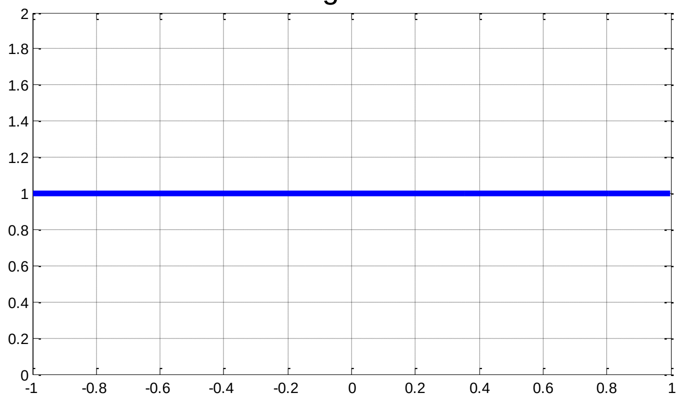
2nd order

$$r = 0.9 \quad \theta = \frac{\pi}{4}$$

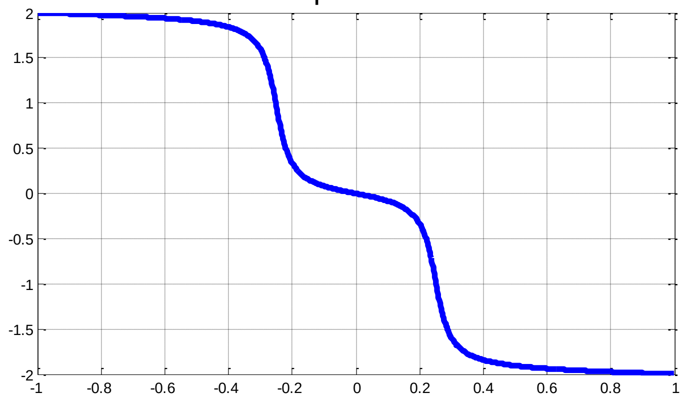
$$H_{ap}(z) = \frac{z^{-1} - a^*}{1 - az^{-1}} \frac{z^{-1} - a}{1 - a^* z^{-1}} \qquad a = re^{j\theta}$$

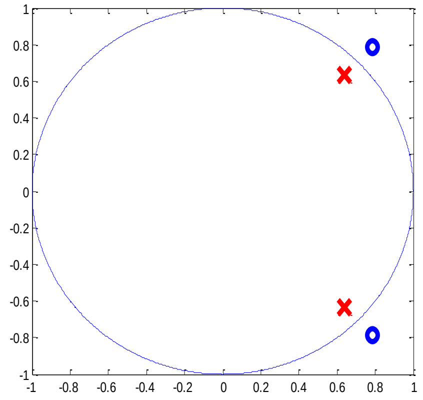
$$= \frac{z^{-1} - 0.9e^{-j\frac{\pi}{4}}}{1 - 0.9e^{j\frac{\pi}{4}}z^{-1}} \frac{z^{-1} - 0.9e^{j\frac{\pi}{4}}}{1 - 0.9e^{-j\frac{\pi}{4}}\, z^{-1}}$$

magnitude

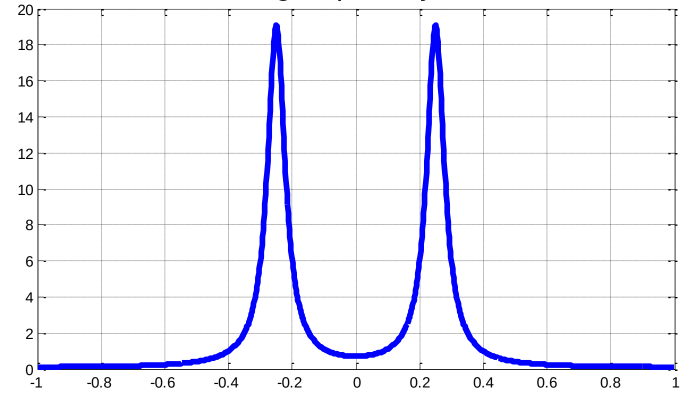


phase





group delay



NOTE THAT

1) All phase responses above are monotonically decreasing.
2) The frequency response of an allpass system has a total phase variation of "$order \times 2\pi$" in $[-\pi, \pi)$ interval.

PHASE OF REAL ALLPASS SYSTEMS (IN CANONICAL FORM)
IS ZERO AT $z = 1 \ \ (\omega = 0)$

Ex:

$$H(z) = \frac{z^{-1} - a^*}{1 - az^{-1}} \frac{z^{-1} - a}{1 - a^* z^{-1}}$$

$$H(1) = \frac{1 - a^*}{1 - a} \frac{1 - a}{1 - a^*}$$

$$= 1$$

$$\Rightarrow \quad \measuredangle H\left(e^{j0}\right) = 0$$

GROUP DELAYS OF ALLPASS SYSTEMS (IN CANONICAL FORM)

$$\tau_{gr}(\omega) = -\frac{d\measuredangle H(e^{j\omega})}{d\omega}$$

For a first order allpass system, i.e., $H(z) = \frac{z^{-1}-a^*}{1-az^{-1}}$

$$-\frac{d\measuredangle H(e^{j\omega})}{d\omega} = \frac{1-r^2}{|1-re^{j\theta}e^{-j\omega}|^2}$$ (read Appendix B)

Group delay is positive since $|r| < 1$. (This is true for any order!)

Positivity of group delay implies that

$$\measuredangle H(e^{j\omega}) \leq 0 \qquad\qquad 0 \leq \omega \leq \pi$$

since

$$\measuredangle H(e^{j\omega}) = -\int_0^{\omega} \tau_{gr}(\alpha)d\alpha + \underbrace{\measuredangle H(e^{j0})}_{=0} \qquad\qquad 0 \leq \omega \leq \pi$$

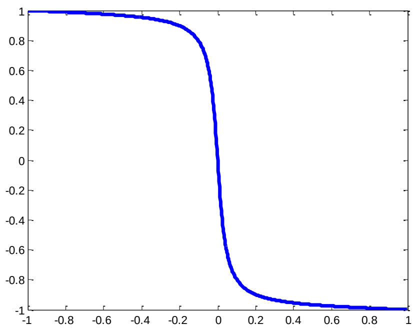
Therefore, phase responses of allpass systems (<u>in canonical form</u>) are negative and monotonically decreasing over $0 \leq \omega \leq \pi$.

(This is true for any order!)

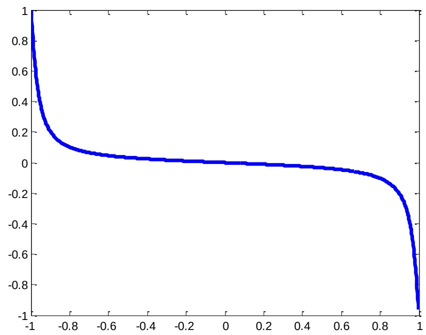
Ex:

Plot the pole-zero diagrams and phase responses of the following allpass systems.

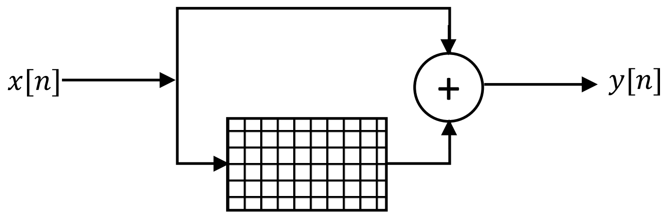
a) $H_1(z) = \frac{z^{-1}-0.9}{1-0.9z^{-1}}$



b) $H_2(z) = \frac{z^{-1}+0.9}{1+0.9z^{-1}}$



c) Which of $H_1(z)$ and $H_2(z)$ has to be placed into the grid box so that $H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})}$ is a highpass frequency response? $X(e^{j\omega})$ and $Y(e^{j\omega})$ are the DTFTs of $x[n]$ and $y[n]$, respectively. Explain clearly and briefly.



None of them. (For the other exam booklet, both of them.)

$$x[n] = x_U[n] = e^{j\omega_0 n}$$

$$x_L[n] = e^{j\left(\omega_0 n + \angle H_{ap}(e^{j\omega_0})\right)} = e^{j\omega_0 n} e^{j\angle H_{ap}(e^{j\omega_0})}$$

$$y[n] = e^{j\omega_0 n} \underbrace{\left(1 + e^{j\angle H_{ap}(e^{j\omega_0})}\right)}_{H(e^{j\omega_0})}$$

$$0 \leq \left|H(e^{j\omega_0})\right| \leq 2$$

$$\angle H_{ap}(e^{j\omega_0}) \cong k2\pi \quad \Rightarrow \quad \left|H(e^{j\omega_0})\right| \cong 2$$

$$\angle H_{ap}(e^{j\omega_0}) \cong (2k+1)\pi \quad \Rightarrow \quad \left|H(e^{j\omega_0})\right| \cong 0$$

Ex: In what way do the frequency responses of the following systems differ in terms of the properties of allpass systems considered so far?

$$H_1(z) = \frac{z^{-1} - a^*}{1 - az^{-1}} \qquad H_2(z) = \frac{1 - \dfrac{1}{a^*} z^{-1}}{1 - az^{-1}} \qquad |a| < 1$$

MINIMUM PHASE SYSTEMS (definition)

Definition: A <u>causal and stable</u> LTI system is said to be minimum-phase (maximum-phase) if all of its zeros are inside (outside) the unit circle.

(Note the restriction to CAUSAL and STABLE systems)

(MAXIMUM PHASE CLARIFICATION $H_{max}H_{ap} = \widehat{H}_{max}$)

Inverse of a minimum-phase system has a causal and stable implementation.

Why?

MINIMUM PHASE ALLPASS DECOMPOSITION

Any (causal and stable) system function, $H(z)$, can be decomposed as

$$H(z) = H_{min}(z)H_{ap}(z)$$

where $H_{min}(z)$ is a minimum phase system and $H_{ap}(z)$ is a causal-stable allpass system in canonical form.

To see that the above claim holds, consider a causal and stable system, $H(z)$ with one of its zeros outside the unit circle.

Let $\frac{1}{z_0^*}$ be the zero outside the unit circle.

Then, $H(z)$ can be expressed as

$$H(z) = \widehat{H}(z)(z^{-1} - z_0^*)$$

Then,

$$H(z) = \widehat{H}(z)(z^{-1} - z_0^*)\frac{(1 - z_0^* z^{-1})}{(1 - z_0^* z^{-1})}$$

$$= \underbrace{\widehat{H}(z)(1 - z_0^* z^{-1})}_{H_{min}(z)} \underbrace{\frac{(z^{-1} - z_0^*)}{(1 - z_0^* z^{-1})}}_{H_{ap}(z)}$$

Above development can be generalized to any number of zeros outside the unit circle.

Roughly speaking,

1) For every zero outside the unit circle, “add” a pole and a zero which are conjugate reciprocals of that zero.
2) Then “separate” the minimum phase and allpass parts.

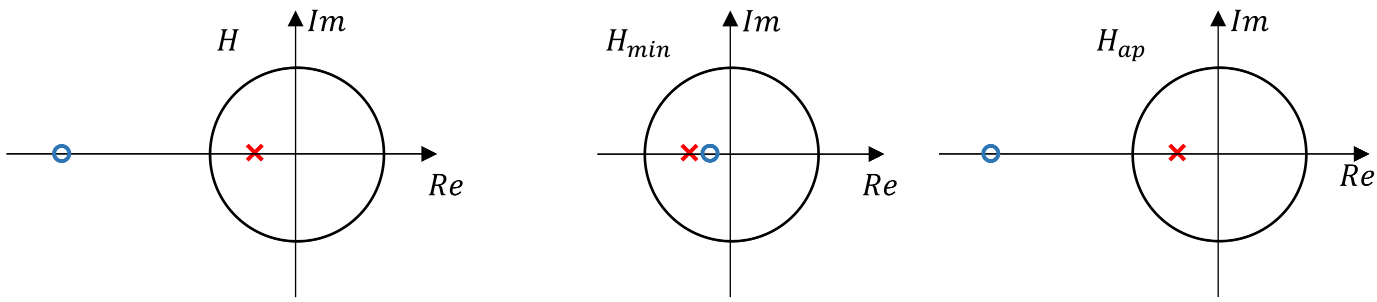
Ex: Find the minimum-phase and allpass components of $H(z) = \frac{1+3z^{-1}}{1+\frac{1}{2}z^{-1}}$.

$$H(z) = \frac{1+3z^{-1}}{1+\frac{1}{2}z^{-1}}$$

$$= \frac{1}{1+\frac{1}{2}z^{-1}} 3\left(z^{-1}+\frac{1}{3}\right)$$

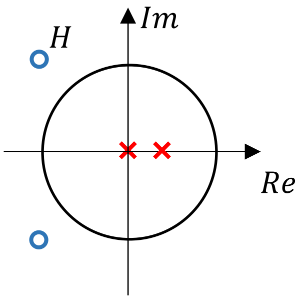
$$= \frac{1}{1+\frac{1}{2}z^{-1}} 3\left(z^{-1}+\frac{1}{3}\right)\frac{\left(1+\frac{1}{3}z^{-1}\right)}{\left(1+\frac{1}{3}z^{-1}\right)}$$

$$= \underbrace{3\frac{\left(1+\frac{1}{3}z^{-1}\right)}{1+\frac{1}{2}z^{-1}}}_{H_{min}(z)} \underbrace{\frac{\left(z^{-1}+\frac{1}{3}\right)}{\left(1+\frac{1}{3}z^{-1}\right)}}_{H_{ap}(z)}$$



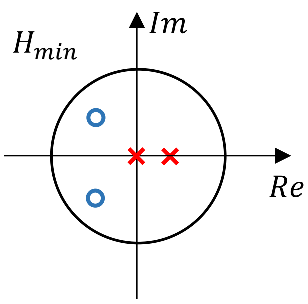
Ex: Find the minimum-phase and allpass components of

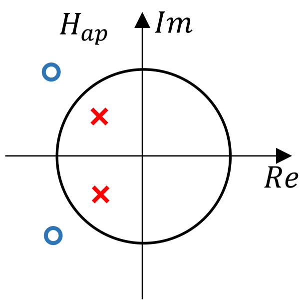
$$H(z)=\frac{\left(1-\frac{3}{2}e^{j\frac{3\pi}{4}}z^{-1}\right)\left(1-\frac{3}{2}e^{-j\frac{3\pi}{4}}z^{-1}\right)}{1-\frac{1}{3}z^{-1}}\,.$$



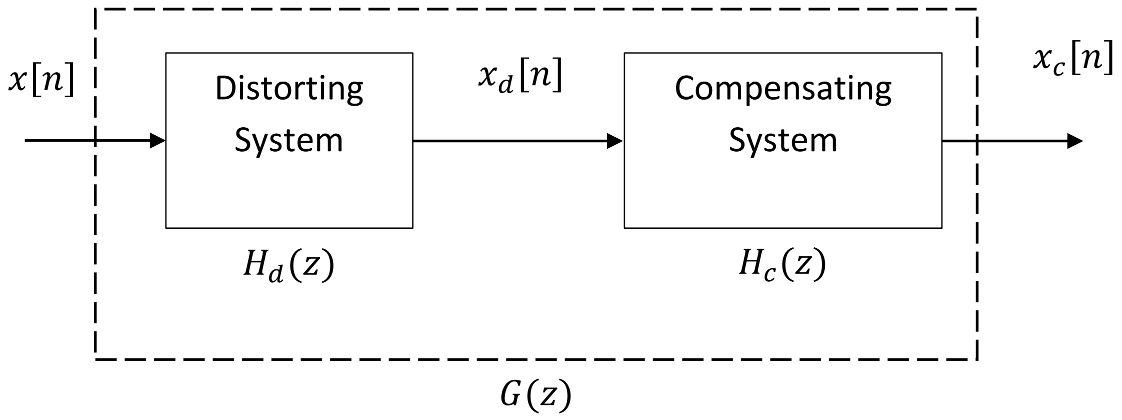
$$H(z)=\frac{\frac{9}{4}\left(z^{-1}-\frac{2}{3}e^{-j\frac{3\pi}{4}}\right)\left(z^{-1}-\frac{2}{3}e^{j\frac{3\pi}{4}}\right)}{1-\frac{1}{3}z^{-1}}$$

$$=\underbrace{\frac{\frac{9}{4}\left(1-\frac{2}{3}e^{-j\frac{3\pi}{4}}z^{-1}\right)\left(1-\frac{2}{3}e^{j\frac{3\pi}{4}}z^{-1}\right)}{1-\frac{1}{3}z^{-1}}}_{H_{min}}\;\underbrace{\frac{\left(z^{-1}-\frac{2}{3}e^{j\frac{3\pi}{4}}\right)}{\left(1-\frac{2}{3}e^{-j\frac{3\pi}{4}}z^{-1}\right)}\,\frac{\left(z^{-1}-\frac{2}{3}e^{-j\frac{3\pi}{4}}\right)}{\left(1-\frac{2}{3}e^{j\frac{3\pi}{4}}z^{-1}\right)}}_{H_{ap}}$$





FREQUENCY RESPONSE MAGNITUDE COMPENSATION



If $H_d(z)$ is a minimum phase system, then $H_c(z)$ can be taken as

$$H_c(z) = \frac{1}{H_d(z)}$$

since $\frac{1}{H_d(z)}$ has a causal and stable realization.

Otherwise, one possibility is to set

$$H_c(z) = \frac{1}{H_{d,min}(z)}$$

where $H_{d,min}(z)$ satisfies

$$H_d(z) = H_{d,min}(z) H_{d,ap}(z) \,.$$

Such a choice yields

$$\left| G\left(e^{j\omega}\right) \right| = 1 \,,$$

i.e., only magnitude distortion is compensated.

Ex: Let

$$H_d(z) = \left(1 - \frac{9}{10}e^{j\frac{3\pi}{5}}z^{-1}\right)\left(1 - \frac{9}{10}e^{-j\frac{3\pi}{5}}z^{-1}\right)\left(1 - \frac{5}{4}e^{j\frac{4\pi}{5}}z^{-1}\right)\left(1 - \frac{5}{4}e^{-j\frac{4\pi}{5}}z^{-1}\right)$$

$$H_{d,min}(z) = \frac{25}{16}\left(1 - \frac{9}{10}e^{j\frac{3\pi}{5}}z^{-1}\right)\left(1 - \frac{9}{10}e^{-j\frac{3\pi}{5}}z^{-1}\right)\left(1 - \frac{4}{5}e^{j\frac{4\pi}{5}}z^{-1}\right)\left(1 - \frac{4}{5}e^{-j\frac{4\pi}{5}}z^{-1}\right)$$

$$H_{d,ap}(z) = \frac{\left(z^{-1} - \frac{4}{5}e^{-j\frac{4\pi}{5}}\right)\left(z^{-1} - \frac{4}{5}e^{j\frac{4\pi}{5}}\right)}{\left(1 - \frac{4}{5}e^{j\frac{4\pi}{5}}z^{-1}\right)\left(1 - \frac{4}{5}e^{-j\frac{4\pi}{5}}z^{-1}\right)}$$

$$H_c(z) = \frac{1}{H_{d,min}(z)}$$

$$= \frac{16}{25}\frac{1}{\left(1 - \frac{9}{10}e^{j\frac{3\pi}{5}}z^{-1}\right)\left(1 - \frac{9}{10}e^{-j\frac{3\pi}{5}}z^{-1}\right)\left(1 - \frac{4}{5}e^{j\frac{4\pi}{5}}z^{-1}\right)\left(1 - \frac{4}{5}e^{-j\frac{4\pi}{5}}z^{-1}\right)}$$

PROPERTIES OF MINIMUM PHASE SYSTEMS

MINIMUM PHASE-LAG PROPERTY

Remember

Phase-lag: $-\measuredangle H(e^{j\omega})$

Let $H_{min}(z)$ be a minimum-phase system.

Let $H(z)$ be one of those systems so that $|H(z)| = |H_{min}(z)|$.

Then, the phase-lag function of $H_{min}(z)$ is smaller than that of $H(z)$.

$$-\measuredangle H_{min}(e^{j\omega}) < -\measuredangle H(e^{j\omega}) ,$$

Proof:

$$H(z) = H_{min}(z)H_{ap}(z) ,$$

$$H(e^{j\omega}) = H_{min}(e^{j\omega})H_{ap}(e^{j\omega})$$

$$-\measuredangle H(e^{j\omega}) = -\measuredangle H_{min}(e^{j\omega}) \underbrace{- \underbrace{\measuredangle H_{ap}(e^{j\omega})}_{\leq 0 \;\; for\ 0\leq\omega<\pi}}_{\geq 0 \;\; for\ 0\leq\omega<\pi}$$

$$\Rightarrow \qquad -\measuredangle H_{min}(e^{j\omega}) \leq -\measuredangle H(e^{j\omega})$$

MINIMUM GROUP-DELAY PROPERTY

Let $H_{min}(z)$ be a minimum-phase system.

Let $H(z)$ be one of those systems so that $|H(z)| = |H_{min}(z)|$.

Then, $grd\left(H_{min}(z)\right) \leq grd\left(H(z)\right)$.

Proof:

$$H(z) = H_{min}(z)H_{ap}(z)$$

$$\Rightarrow \quad \measuredangle H\left(e^{j\omega}\right) = \measuredangle H_{min}\left(e^{j\omega}\right) + \measuredangle H_{ap}\left(e^{j\omega}\right)$$

Remember that group-delays of allpass systems are always positive.

$$\Rightarrow \quad grd\left(H(z)\right) = grd\left(H_{min}(z)\right) + \underbrace{grd\left(H_{ap}(z)\right)}_{\geq 0}$$

$$\Rightarrow \quad grd\left(H_{min}(z)\right) \leq grd\left(H(z)\right)$$

MINIMUM ENERGY-DELAY PROPERTY

Let $H_{min}(z)$ be a minimum-phase system.

Let $H(z)$ be one of those systems so that $|H(z)| = |H_{min}(z)|$.

Then, the following holds

a) $$\sum_{n=0}^{\infty} |h[n]|^2 = \sum_{n=0}^{\infty} |h_{min}[n]|^2$$

Proof: Using $H(z) = H_{min}(z)H_{ap}(z)$, $\left|H\left(e^{j\omega}\right)\right| = \left|H_{min}\left(e^{j\omega}\right)\right|$ and Parseval's theorem.

b) $$\sum_{n=0}^{m} |h[n]|^2 \leq \sum_{n=0}^{m} |h_{min}[n]|^2 \qquad (integer)\ m \geq 0$$

Proof: (one choice) We will evaluate

$$\sum_{m=0}^{n} |h_{min}[m]|^2 - \sum_{m=0}^{n} |h[m]|^2.$$

Let $h_{min}[n]$ be a minimum-phase sequence and $H_{min}(z)$ be its z-transform. $H_{min}(z)$ can be written as

$$H_{min}(z) = Q(z)(1 - z_k z^{-1}) \qquad\qquad |z_k| < 1$$

where z_k is one of its zeros. Note that $Q(z)$ is also minimum-phase.

Let $h[n]$ be another sequence such that

$$\left|H\left(e^{j\omega}\right)\right| = \left|H_{min}\left(e^{j\omega}\right)\right|$$

and $H(z)$ has a zero at $\frac{1}{z_k^*}$.

Then, $H(z)$ can be expressed as

$$H(z) = Q(z)(z^{-1} - z_k^*) \,.$$

Hence,

$$h[n] = -z_k^* q[n] + q[n-1]$$

$$h_{min}[n] = q[n] - z_k q[n-1]$$

Now, let’s evaluate

$$\sum_{m=0}^{n} |h_{min}[m]|^2 - \sum_{m=0}^{n} |h[m]|^2$$

$$|h_{min}[m]|^2 = h_{min}[m]h^*_{min}[m]$$

$$= (q[m] - z_k q[m-1])(q^*[m] - z_k^* q^*[m-1])$$

$$= |q[m]|^2 - z_k^* q[m]q^*[m-1] - z_k q[m-1]q^*[m]$$

$$+ |z_k|^2 |q[m-1]|^2$$

$$|h[m]|^2 = h[m]h^*[m]$$

$$= (-z_k^* q[m] + q[m-1])(-z_k q^*[m] + q^*[m-1])$$

$$= |z_k|^2 |q[m]|^2 - z_k^* q[m]q^*[m-1] - z_k q[m-1]q^*[m]$$

$$+ |q[m-1]|^2$$

$$\sum_{m=0}^{n} |h_{min}[m]|^2 - \sum_{m=0}^{n} |h[m]|^2 = \sum_{m=0}^{n} (1 - |z_k|^2)(|q[m]|^2 - |q[m-1]|^2)$$

$$= \underbrace{(1 - |z_k|^2)|q[n]|^2}_{>0}$$

Demonstration of the minimum energy-delay property.

Ex: Let

$$H(z) = 1 + 2z^{-1} + 5z^{-2}$$

$$= (1 - az^{-1})(1 - a^* z^{-1})$$

$$a = -1 + j2 = re^{j\theta} = \sqrt{5}e^{j\theta}$$

$$H_{min}(z) = |a|^2 \left(1 - \frac{1}{a} z^{-1}\right)\left(1 - \frac{1}{a^*} z^{-1}\right)$$

$$= r^2 \left(1 - 2\frac{1}{r}\cos\theta\, z^{-1} + \frac{1}{r^2} z^{-2}\right)$$

$$= 5 + 2z^{-1} + z^{-2}$$

Demonstration of the minimum energy-delay property.

Ex: Let

$$H(z) = 1 + z^{-1} - 6z^{-2}$$

$$H_{min}(z) = -6 + z^{-1} + z^{-2}$$

Appendix A

FREQUENCY RESPONSES OF CONJUGATE RECIPROCAL ZEROS (POLES)

CONJUGATE RECIPROCAL ZEROS

$$A(\omega) = \left(1 - ce^{-j\omega}\right)$$

$$B(\omega) = \left(1 - \frac{1}{c^*}e^{-j\omega}\right)$$

Notice that

$$B(\omega) = \left(1 - \frac{1}{c^*}e^{-j\omega}\right)$$

$$= -\frac{1}{c^*}e^{-j\omega}\left(1 - c^*e^{j\omega}\right)$$

Now, compare the **MAGNITUDES** of $A(\omega)$ and $B(\omega)$:

Since $\left(1 - c^*e^{j\omega}\right)$ and $\left(1 - ce^{-j\omega}\right)$ are conjugates of each other

$$|B(\omega)| = \left|-\frac{1}{c^*}\right| \left|e^{-j\omega}\right| |A(\omega)|$$

$$= \frac{1}{|c|}|A(\omega)|$$

RELATIONSHIP BETWEEN $A(z)$ AND $B(z)$

$$A^*\left(\frac{1}{z^*}\right) = 1 - c^* z$$

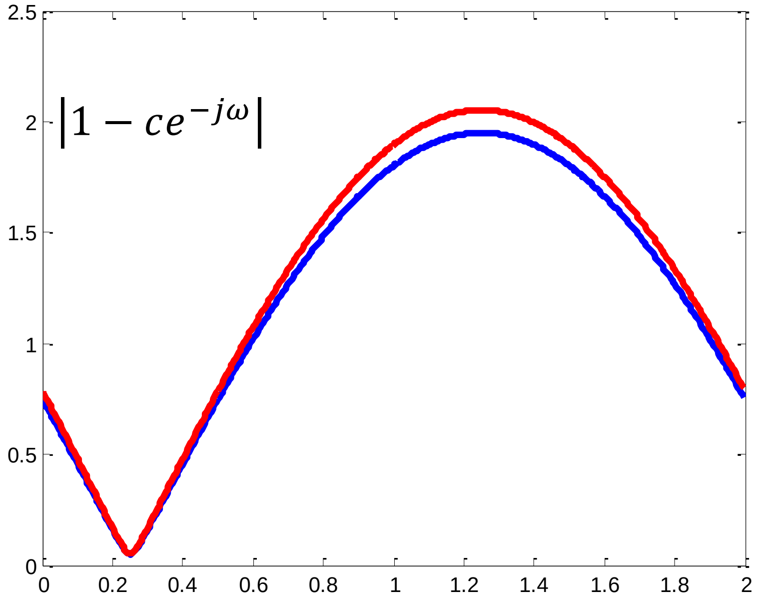
$$= -c^* z\left(1 - \frac{1}{c^*} z^{-1}\right)$$

$$= -c^* z \, B(z)$$

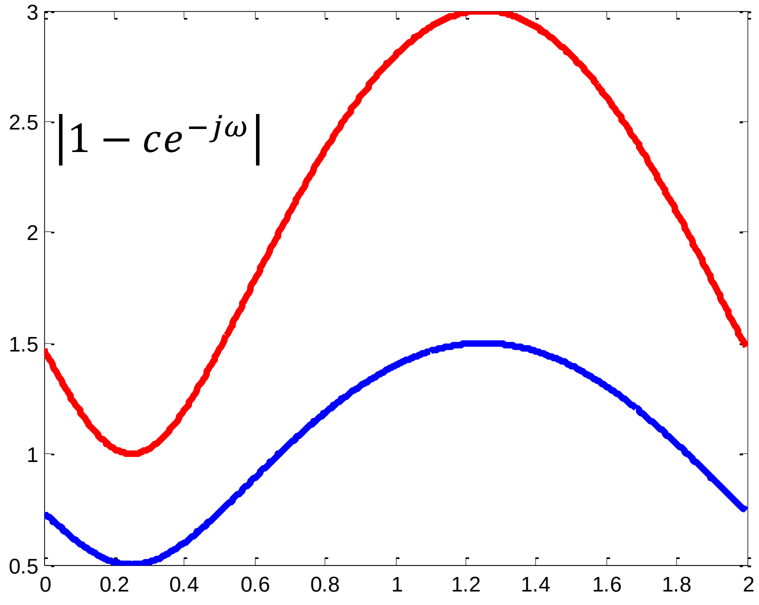
$$\Rightarrow \; B(z) = \frac{-1}{c^* z} A^*\left(\frac{1}{z^*}\right)$$

```
clear all
close all

c = 0.5 * exp(j*pi/4);
cij = 1/conj(c);

a = [1 -c];
b = [1 -cij];

[A, w] = freqz(a, 1, 4096, 'whole');

[B, w] = freqz(b, 1, 4096, 'whole');

plot(w/pi,abs(A), 'linewidth', 3);
hold
plot(w/pi,abs(B),'r', 'linewidth', 3)


figure
plot(w/pi,angle(A)/pi, 'linewidth', 3);
hold
plot(w/pi,angle(B)/pi,'r', 'linewidth', 3)

figure
[Grp_a,w] = grpdelay(a, 1, 4096, 'whole');
[Grp_b,w] = grpdelay(b, 1, 4096, 'whole');
plot(w/pi,Grp_a,'linewidth', 3)
hold
plot(w/pi,Grp_b,'r','linewidth', 3)
title('group delay, samples')
```

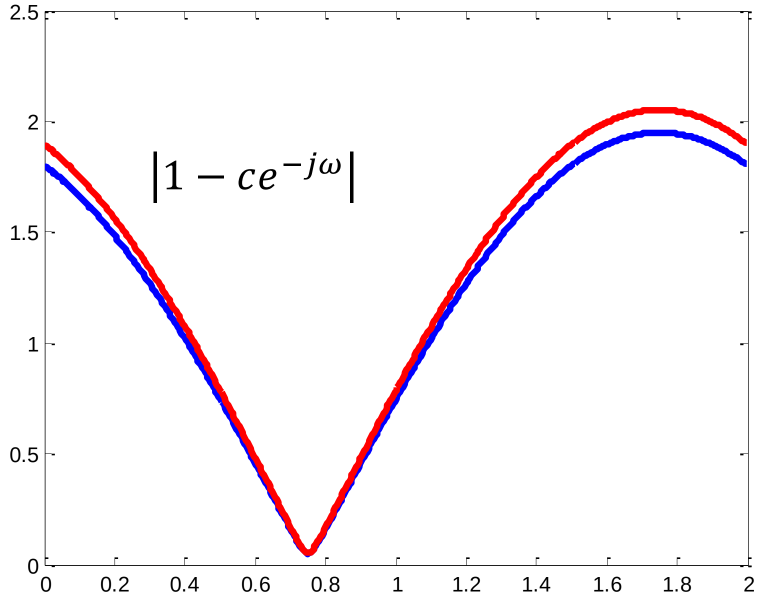
Ex: $c = 0.95e^{j\frac{\pi}{4}}$



Ex: $c = 0.5e^{j\frac{\pi}{4}}$



Ex: $c = 0.95e^{j\frac{3\pi}{4}}$



Note that

$$\left|1 - ce^{-j\omega}\right| = \left|e^{-j\omega} - c^*\right|$$

Therefore,

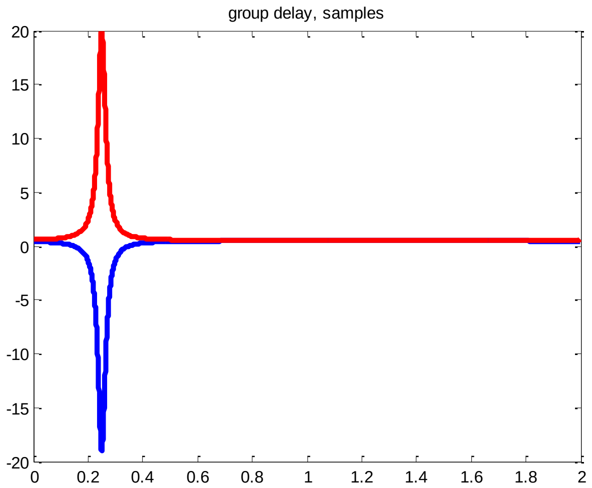
$$H\left(e^{j\omega}\right) = \frac{\left(e^{-j\omega} - c^*\right)}{\left(1 - ce^{-j\omega}\right)}$$

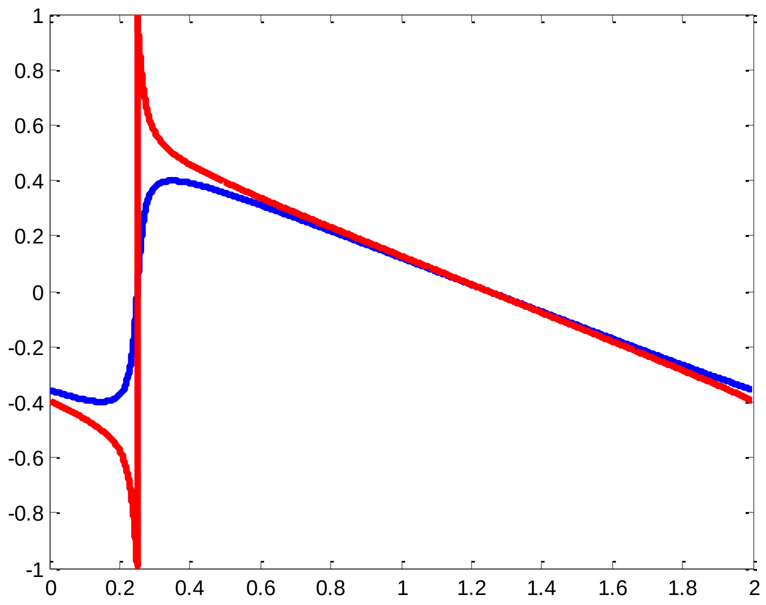
has an allpass characteristics.

PHASES AND GROUP DELAYS OF $A(\omega)$ AND $B(\omega)$

Ex:

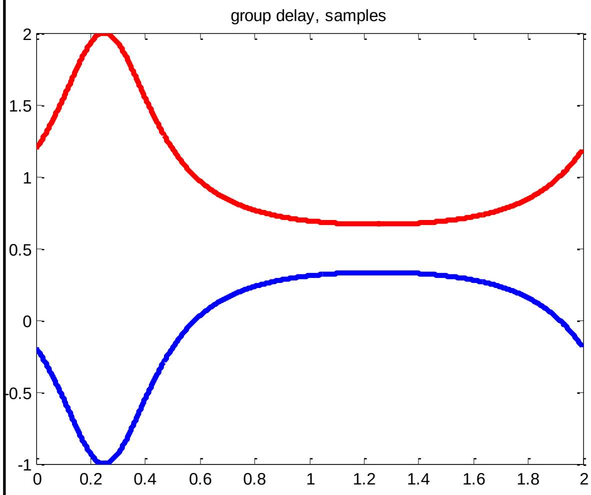
$$c = 0.95e^{j\frac{\pi}{4}}$$

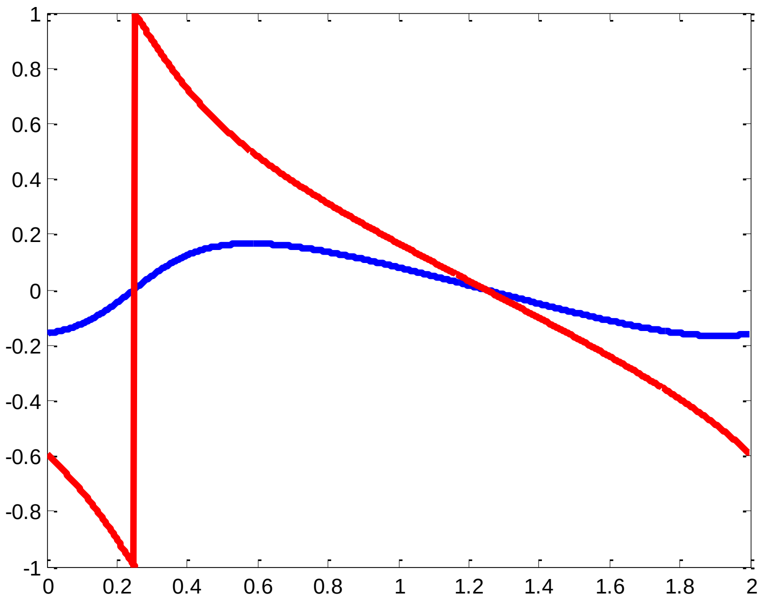




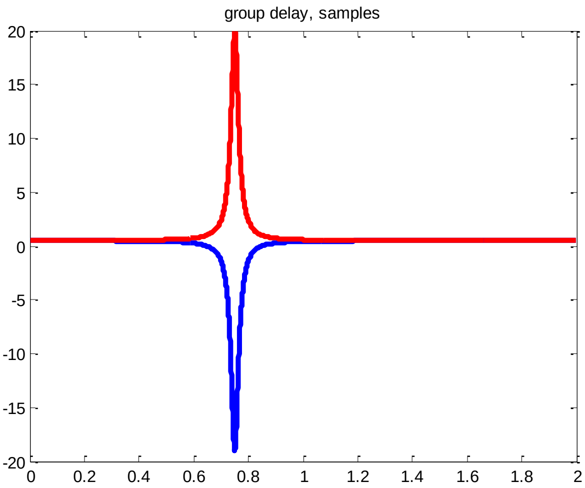
Ex:

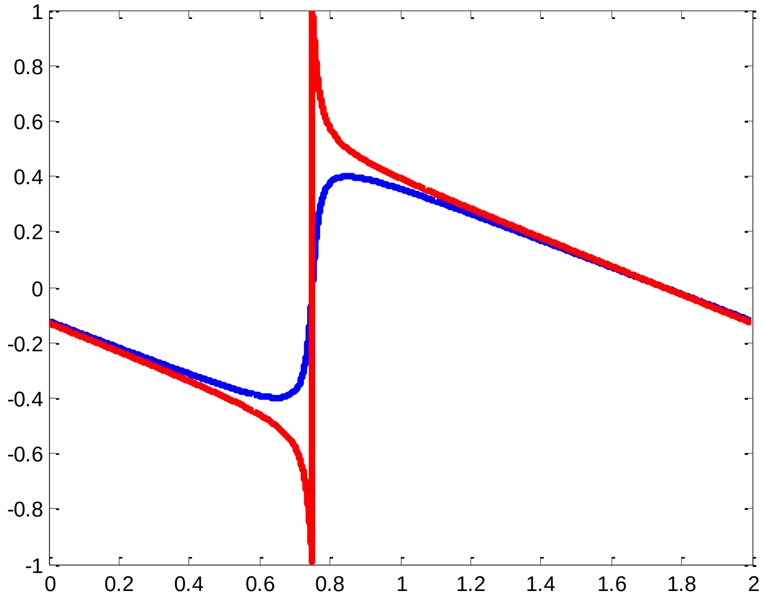
$$c = 0.5e^{j\frac{\pi}{4}}$$





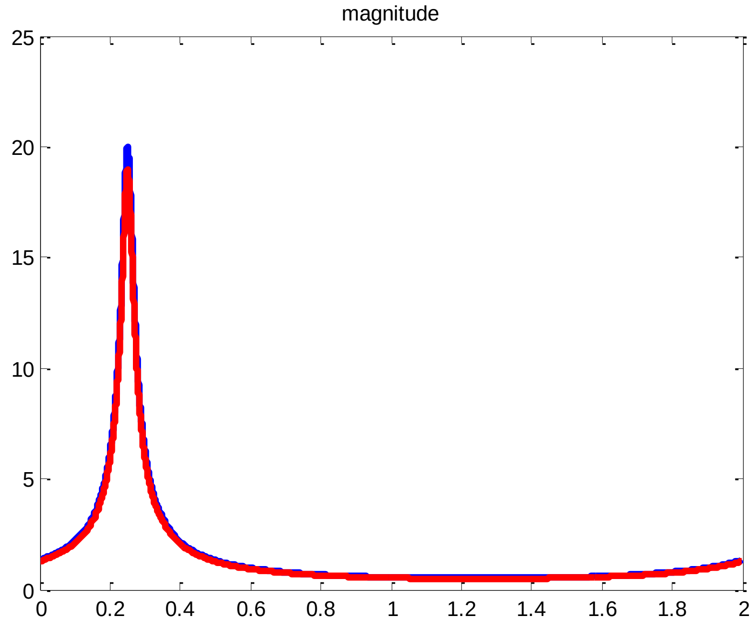
Ex: $c = 0.95e^{j\frac{3\pi}{4}}$

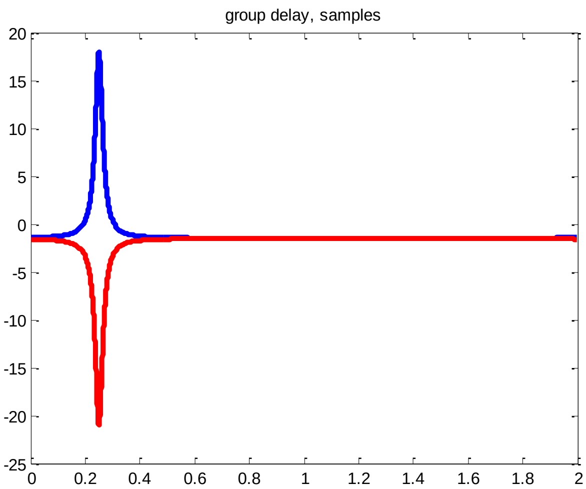


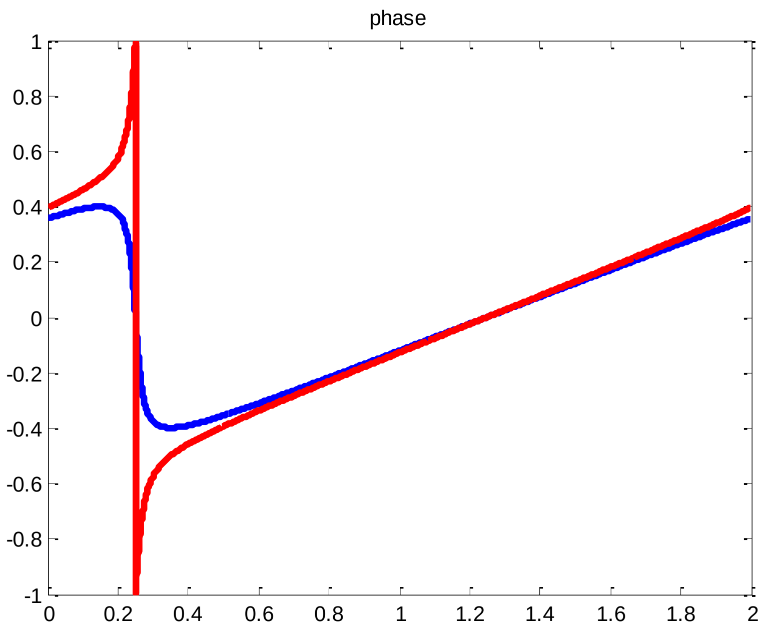


FREQUENCY RESPONSES OF $\frac{1}{A(\omega)}$ AND $\frac{1}{B(\omega)}$

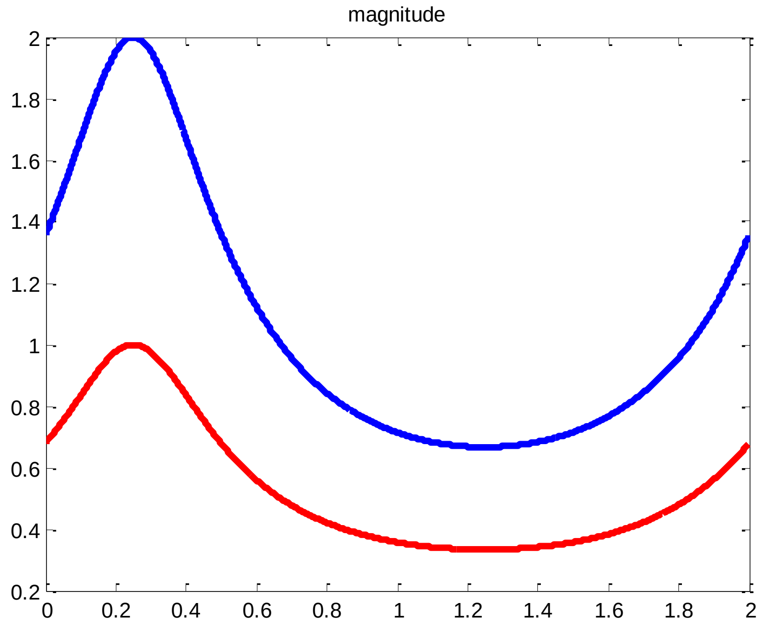
Ex: $c = 0.95e^{j\frac{\pi}{4}}$

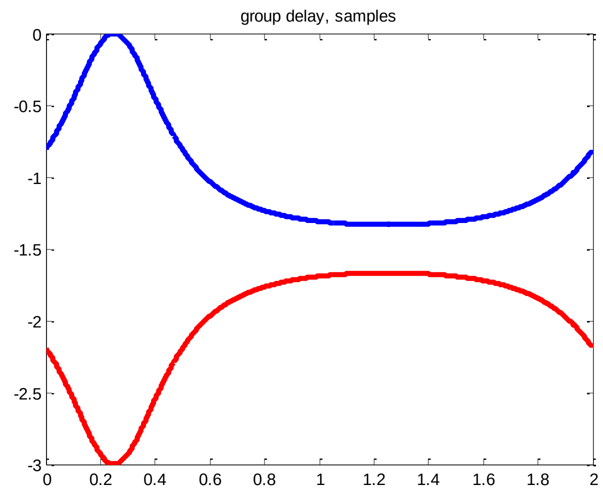


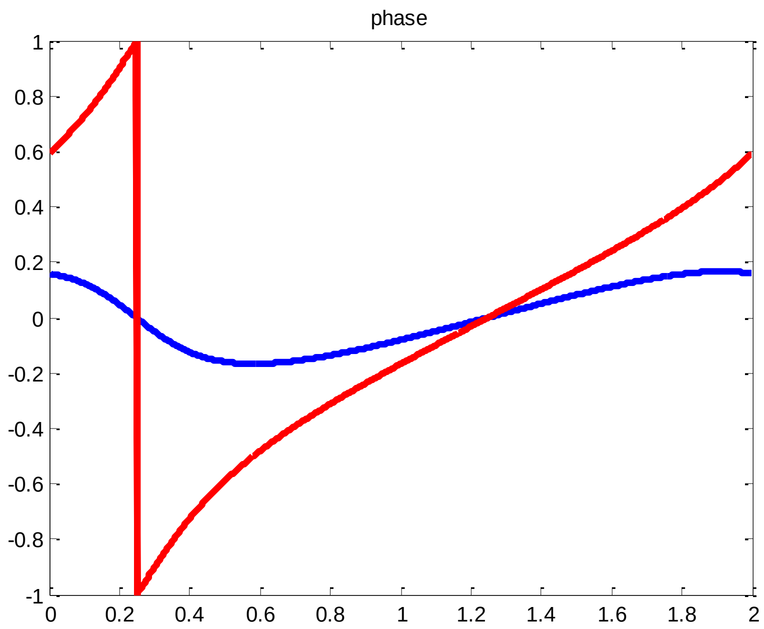




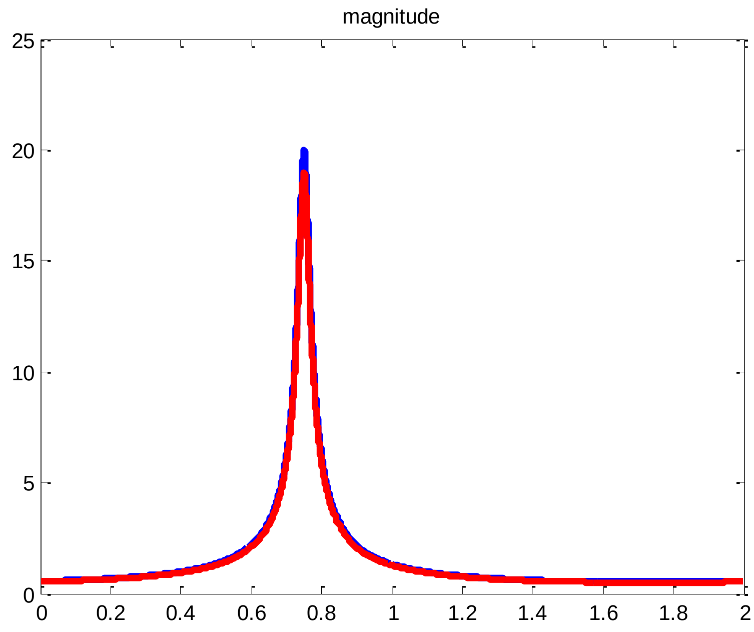
Ex: $c = 0.5e^{j\frac{\pi}{4}}$

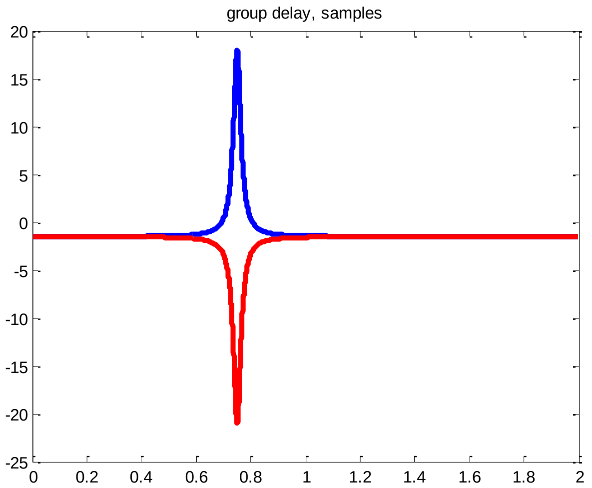


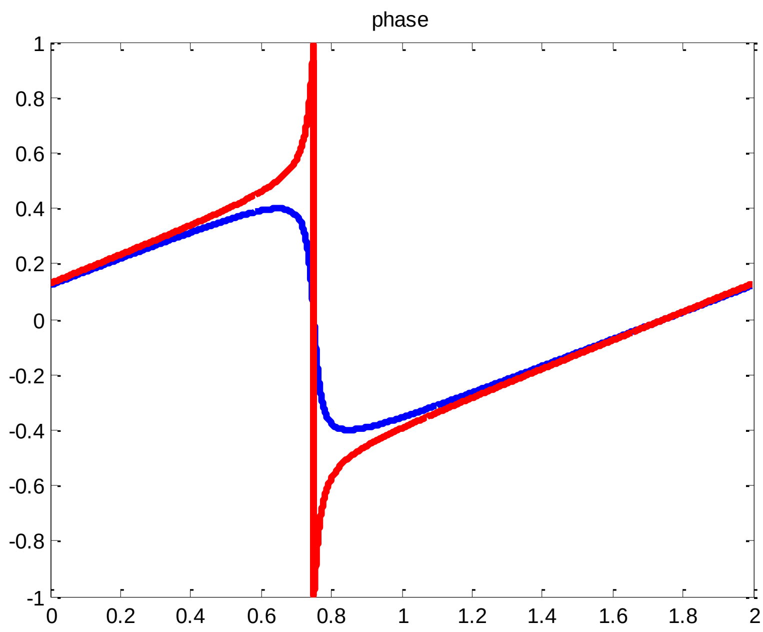




Ex: $c = 0.95e^{j\frac{3\pi}{4}}$







Appendix B

GROUP DELAY OF $H(z) = \frac{z^{-1}-a^*}{1-az^{-1}}$

$$-\frac{d}{d\omega}\measuredangle\left(e^{-j\omega}\frac{1-re^{-j\theta}e^{j\omega}}{1-re^{j\theta}e^{-j\omega}}\right) = -\frac{d}{d\omega}\left(-\omega+\measuredangle\left(\frac{1-re^{-j\theta}e^{j\omega}}{1-re^{j\theta}e^{-j\omega}}\right)\right)$$

$$= -\frac{d}{d\omega}\left(-\omega+2\measuredangle\left(1-re^{-j\theta}e^{j\omega}\right)\right)$$

$$= 1-2\frac{d}{d\omega}\arctan\left(\frac{-r\sin(\omega-\theta)}{1-r\cos(\omega-\theta)}\right)$$

$$= 1-2\left(\frac{1}{1+\left(\frac{-r\sin(\omega-\theta)}{1-r\cos(\omega-\theta)}\right)^2}\right)\left(\frac{-r\cos(\omega-\theta)\,(1-r\cos(\omega-\theta))+r^2\sin^2(\omega-\theta)}{1-2r\cos(\omega-\theta)+r^2\cos^2(\omega-\theta)}\right)$$

$$= 1-2\left(\frac{1-2r\cos(\omega-\theta)+r^2\cos^2(\omega-\theta)}{1+r^2-2r\cos(\omega-\theta)}\right)\left(\frac{r^2-r\cos(\omega-\theta)}{1-2r\cos(\omega-\theta)+r^2\cos^2(\omega-\theta)}\right)$$

$$= 1-2\left(\frac{r^2-r\cos(\omega-\theta)}{1+r^2-2r\cos(\omega-\theta)}\right)$$

$$= \frac{1+r^2-2r\cos(\omega-\theta)-2r^2+2r\cos(\omega-\theta)}{1+r^2-2r\cos(\omega-\theta)}$$

$$= \frac{1-r^2}{1+r^2-2r\cos(\omega-\theta)}$$

$$= \frac{1-r^2}{|1-re^{j\theta}e^{-j\omega}|^2}$$

$$\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}$$