EE430 DIGITAL SIGNAL PROCESSING

HOMEWORK 2

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1)

a) $y[n]-\frac{1}{2}y[n-1]=0$ (homogenous equation)

$y[n]=Ar^n$

$Ar^n-\frac{1}{2}Ar^{n-1}=0$

$Ar^{n-1}(r-\frac{1}{2})=0 \Rightarrow r=\frac{1}{2}$

$y[n]=A(\frac{1}{2})^n$

Assume $y[n]-\frac{1}{2}y[n-1]=x[n]$ where $x[n]=\delta[n]$

$y[n]=\frac{1}{2}y[n-1]+\delta[n]$

$y[0]=\frac{1}{2}y[-1]+\delta[0]=1$

$y[0]=A=1 \Rightarrow y[n]=(\frac{1}{2})^n u[n]$

We evaluate that for x[n]. However, we have a linear combination of shifted inputs. Thus in order to find h[n] we must use linearity and time-invariance properties.

$h[n]=y[n]-y[n-1]+y[n-2]=(\frac{1}{2})^n u[n]-(\frac{1}{2})^{n-1} u[n-1]+(\frac{1}{2})^{n-2} u[n-2]=(\frac{1}{2})^n(\delta[n]-\delta[n-1]+3u[n-2])$

Hence;

$h[n]=(\frac{1}{2})^n(\delta[n]-\delta[n-1]+3u[n-2])$

b) $H(e^{j\omega})=\sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n}=$

$=\sum_{n=-\infty}^{\infty}(\frac{1}{2})^n\delta[n]e^{-j\omega n}-\sum_{n=-\infty}^{\infty}(\frac{1}{2})^n\delta[n-1]e^{-j\omega n}+3\sum_{n=-\infty}^{\infty}(\frac{1}{2})^n u[n-2]e^{-j\omega n}=$

$=1-\frac{1}{2}e^{-j\omega}+3\sum_{n=2}^{\infty}(\frac{e^{-j\omega}}{2})^n = 1-\frac{1}{2}e^{-j\omega}+3(\sum_{n=0}^{\infty}(\frac{e^{-j\omega}}{2})^n-1-\frac{1}{2}e^{-j\omega})=$

$=-2-2e^{-j\omega}+\frac{6}{2-e^{-j\omega}}$ for $-\pi<\omega<\pi$

c) "freqz" command operates by the coefficients of the z-transform of a sequence. So, first transform should be evaluated.

$$Y(z) - \frac{1}{2} Y(z) z^{-1} = X(z) - X(z) z^{-1} + X(z) z^{-2}$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 - z^{-1} + z^{-2}}{1 - \frac{1}{2} z^{-1}}$$

Corresponding MATLAB code and plots are given below.

```
b=[1 -1 1];
a=[1 -0.5];

[h,w] = freqz(b,a,'whole',2001);
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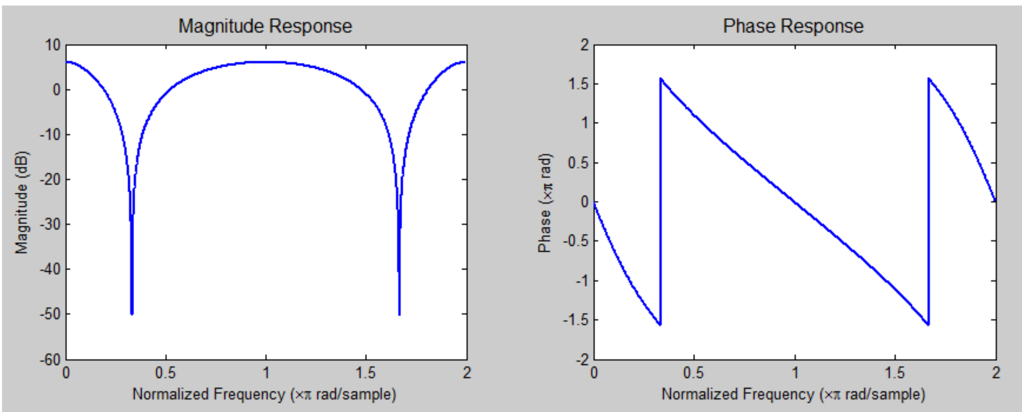


Magnitude response is even symmetric and phase response is odd symmetric as we expect from a real h[n] sequence.

d) $x[n]=\cos(\frac{\pi}{3}n)+\sin(\frac{\pi}{2}n+\frac{\pi}{4})= \cos(\frac{\pi}{3}n)+\sin(\frac{\pi}{2}n)\cos(\frac{\pi}{4})+ \cos(\frac{\pi}{2}n)\sin(\frac{\pi}{4})=$

$= \cos(\frac{\pi}{3}n)+\frac{1}{\sqrt{2}}\sin(\frac{\pi}{2}n) + \frac{1}{\sqrt{2}}\cos(\frac{\pi}{2}n)$

$X(e^{j\omega})= \pi[\delta(\omega-\frac{\pi}{3})+ \delta(\omega+\frac{\pi}{3})]- \frac{j\pi}{\sqrt{2}}[\delta(\omega-\frac{\pi}{2})- \delta(\omega+\frac{\pi}{2})]+ \frac{\pi}{\sqrt{2}}[\delta(\omega-\frac{\pi}{2})+ \delta(\omega+\frac{\pi}{2})]=$

$= \pi\, \delta(\omega-\frac{\pi}{3})+ \pi\, \delta(\omega+\frac{\pi}{3})+ \frac{\pi}{\sqrt{2}}(1-j)\, \delta(\omega-\frac{\pi}{2})+ \frac{\pi}{\sqrt{2}}(1+j)\, \delta(\omega+\frac{\pi}{2})$

$Y(e^{j\omega})= X(e^{j\omega})\, H(e^{j\omega})= \pi\, (-2-2e^{-j\frac{\pi}{3}}+\frac{6}{2-e^{-j\frac{\pi}{3}}})\, \delta(\omega-\frac{\pi}{3}) + \pi\, (-2-2e^{j\frac{\pi}{3}}+\frac{6}{2-e^{j\frac{\pi}{3}}})\, \delta(\omega+\frac{\pi}{3})$

$+ \frac{\pi}{\sqrt{2}}(1-j)(-2-2e^{-j\frac{\pi}{2}}+\frac{6}{2-e^{-j\frac{\pi}{2}}})\, \delta(\omega-\frac{\pi}{2}) + \frac{\pi}{\sqrt{2}}(1+j)(-2-2e^{j\frac{\pi}{2}}+\frac{6}{2-e^{j\frac{\pi}{2}}})\, \delta(\omega+\frac{\pi}{2})$

Here $-2-2e^{-j\frac{\pi}{3}}+\frac{6}{2-e^{-j\frac{\pi}{3}}} = -2-2e^{j\frac{\pi}{3}}+\frac{6}{2-e^{j\frac{\pi}{3}}} =0.$

Thus two terms with delta functions vanish. Also;

$-2-2e^{-j\frac{\pi}{2}}+\frac{6}{2-e^{-j\frac{\pi}{2}}} = \frac{2}{5} + j\frac{4}{5}$ $\qquad$ $-2-2e^{j\frac{\pi}{2}}+\frac{6}{2-e^{j\frac{\pi}{2}}} = \frac{2}{5} - j\frac{4}{5}$

$Y(e^{j\omega}) = \frac{\pi}{\sqrt{2}}\left(\frac{6}{5} + j\,\frac{2}{5}\right)\delta(\omega-\frac{\pi}{2}) + \frac{\pi}{\sqrt{2}}\left(\frac{6}{5} - j\,\frac{2}{5}\right)\delta(\omega+\frac{\pi}{2})$

$y[n]= \frac{\pi}{\sqrt{2}}\left(\frac{6}{5} + j\,\frac{2}{5}\right)\frac{1}{2\pi}\,e^{j\frac{\pi}{2}n} + \frac{\pi}{\sqrt{2}}\left(\frac{6}{5} - j\,\frac{2}{5}\right)\frac{1}{2\pi}\,e^{-j\frac{\pi}{2}n} = \frac{1}{\sqrt{2}}\left(\frac{3}{5} + j\,\frac{1}{5}\right)e^{j\frac{\pi}{2}n} + \frac{1}{\sqrt{2}}\left(\frac{3}{5} - j\,\frac{1}{5}\right)e^{-j\frac{\pi}{2}n}$

Hence;

$y[n]= \frac{1}{\sqrt{2}}\left(\frac{3}{5} + j\,\frac{1}{5}\right)e^{j\frac{\pi}{2}n} + \frac{1}{\sqrt{2}}\left(\frac{3}{5} - j\,\frac{1}{5}\right)e^{-j\frac{\pi}{2}n}$

e) $H(e^{j\omega}) = -2-2e^{-j\omega}+\frac{6}{2-e^{-j\omega}}$

$H^*(e^{j(2\pi-\omega)}) = \left(-2-2e^{-j(2\pi-\omega)}+\frac{6}{2-e^{-j(2\pi-\omega)}}\right)^* = \left(-2-2e^{-j2\pi}\,e^{j\omega}+\frac{6}{2-e^{-j2\pi}e^{j\omega}}\right)^* =$

$= \left(-2-2e^{j\omega}+\frac{6}{2-e^{j\omega}}\right)^* = -2-2e^{-j\omega}+\frac{6}{2-e^{-j\omega}}$

Hence; $H(e^{j\omega})=H^*(e^{j(2\pi-\omega)})$ for this question. This is not always the case. This equation is only valid when h[n] is a real sequence.

2) We must assume $|a|<1$ for convergence of the DTFT sum. Let's start with DTFT of $a^n u[n]$ and we can reach the desired result by applying properties to it.

$a^n u[n] \leftrightarrow \frac{1}{1-ae^{-j\omega}}$

By applying time-shift property;

$a^{n-2}u[n-2] \leftrightarrow \frac{e^{-j\omega 2}}{1-ae^{-j\omega}}$

Then applying the differentiation to frequency domain;

$-j\,n\,a^{n-2}u[n-2] \leftrightarrow \frac{d}{d\omega}\left(\frac{e^{-j\omega 2}}{1-ae^{-j\omega}}\right) = \frac{-j\,2\,e^{-j\omega 2}\,(1-ae^{-j\omega})-(-a)(-j)e^{-j\omega}\,e^{-j\omega 2}}{(1-ae^{-j\omega})^2}$

$n\,a^{n-2}u[n-2] \leftrightarrow \frac{2\,e^{-j\omega 2}-ae^{-j\omega 3}}{(1-ae^{-j\omega})^2}$

Hence the DTFT of x[n] is equal to $X(e^{j\omega}) = \frac{2\,e^{-j\omega 2}-ae^{-j\omega 3}}{(1-ae^{-j\omega})^2}$.

3)

a) $y_1[n] = x_1[n] * h[n] = \sum_{k=-\infty}^{\infty} x_1[n-k]h[k] = \frac{1}{2}\sum_{k=0}^{1} \sin\left(\frac{\pi}{7}(n-k)\right) + \sin\left(\frac{\pi}{3}(n-k)\right) =$

$= \frac{1}{2}\left(\sin\left(\frac{\pi}{7}n\right) + \sin\left(\frac{\pi}{3}n\right) + \sin\left(\frac{\pi}{7}(n-1)\right) + \sin\left(\frac{\pi}{3}(n-1)\right)\right)$

$y_2[n] = x_2[n] * h[n] = \sum_{k=-\infty}^{\infty} x_2[n-k]h[k] = \frac{1}{2}\sum_{k=0}^{1} \left[\sin\left(\frac{\pi}{7}(n-k)\right) + \sin\left(\frac{\pi}{3}(n-k)\right)\right] u[n-k] =$

$= \frac{1}{2}\left[\sin\left(\frac{\pi}{7}n\right) + \sin\left(\frac{\pi}{3}n\right)\right] u[n] + \frac{1}{2}\left[\sin\left(\frac{\pi}{7}(n-1)\right) + \sin\left(\frac{\pi}{3}(n-1)\right)\right] u[n-1]$

$y_1[n] = y_2[n]$ for $n \geq 1$.

b) $y_1[n] = x_1[n] * h[n] = \sum_{k=-\infty}^{\infty} x_1[n-k]h[k] = \frac{1}{2}\sum_{k=2}^{3} \sin\left(\frac{\pi}{7}(n-k)\right) + \sin\left(\frac{\pi}{3}(n-k)\right) =$

$= \frac{1}{2}\left(\sin\left(\frac{\pi}{7}(n-2)\right) + \sin\left(\frac{\pi}{3}(n-2)\right) + \sin\left(\frac{\pi}{7}(n-3)\right) + \sin\left(\frac{\pi}{3}(n-3)\right)\right)$

$y_2[n] = x_2[n] * h[n] = \sum_{k=-\infty}^{\infty} x_2[n-k]h[k] = \frac{1}{2}\sum_{k=2}^{3} \left[\sin\left(\frac{\pi}{7}(n-k)\right) + \sin\left(\frac{\pi}{3}(n-k)\right)\right] u[n-k] =$

$= \frac{1}{2}\left[\sin\left(\frac{\pi}{7}(n-2)\right) + \sin\left(\frac{\pi}{3}(n-3)\right)\right] u[n-2] + \frac{1}{2}\left[\sin\left(\frac{\pi}{7}(n-3)\right) + \sin\left(\frac{\pi}{3}(n-3)\right)\right] u[n-3]$

$y_1[n] = y_2[n]$ for $n \geq 3$.

c) $y_1[n] = x_1[n] * h[n] = \sum_{k=-\infty}^{\infty} x_1[n-k]h[k] = \frac{1}{6}\sum_{k=0}^{5} \sin\left(\frac{\pi}{7}(n-k)\right) + \sin\left(\frac{\pi}{3}(n-k)\right)$

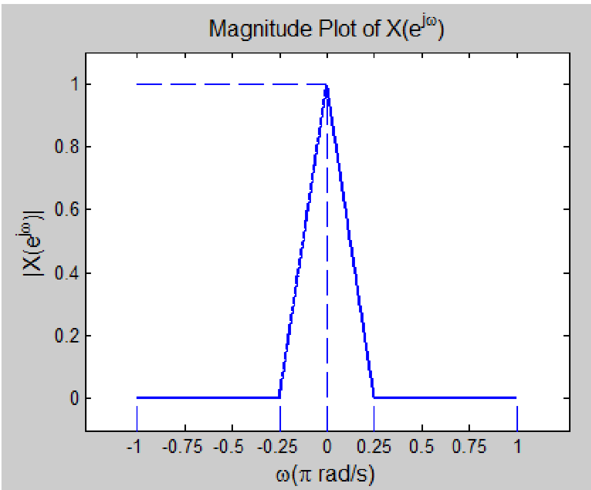
$y_2[n] = x_2[n] * h[n] = \sum_{k=-\infty}^{\infty} x_2[n-k]h[k] = \frac{1}{6}\sum_{k=0}^{5} \left[\sin\left(\frac{\pi}{7}(n-k)\right) + \sin\left(\frac{\pi}{3}(n-k)\right)\right] u[n-k]$

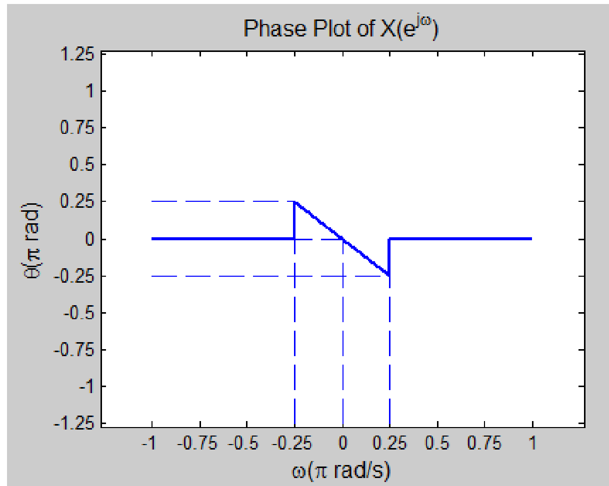
There are 6 terms in each of these summations. I did not write them in the open form since they are too long but it is clearly seen that $y_1[n] = y_2[n]$ for $n \geq 5$.

The intervals that $y_1[n] = y_2[n]$ can be defined as $n = [a, \infty)$. Here "a" is the number that for values greater than a, h[n] becomes zero.

4)

a) Since x[n] is a real sequence its magnitude response must be even symmetric and phase response must be odd symmetric.





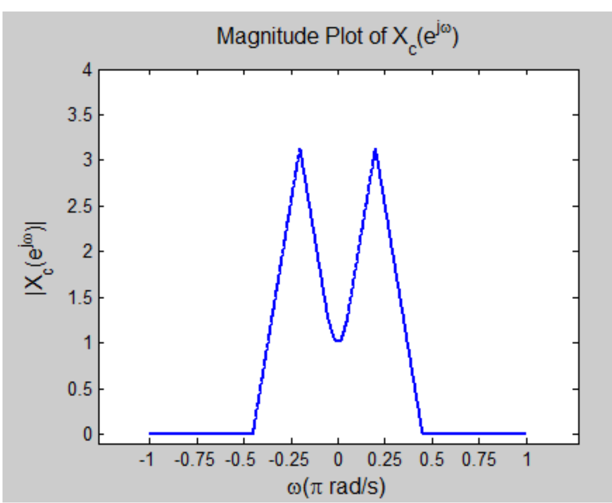
b) $F\{\cos(\frac{\pi}{5}n)\} = \pi\,[\delta(\omega-\frac{\pi}{5})+\delta(\omega+\frac{\pi}{5})]$

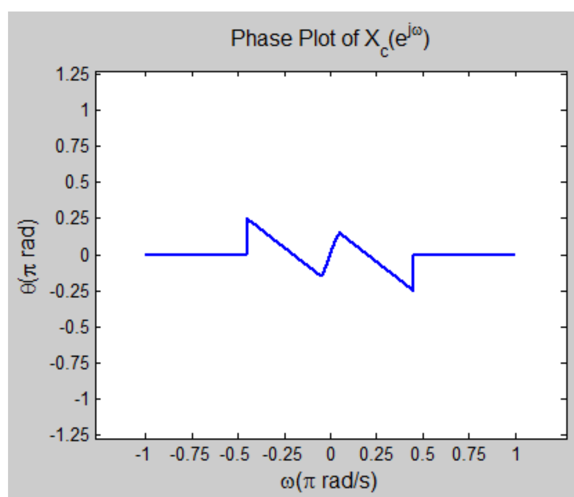
$F\{\sin(\frac{\pi}{5}n)\} = -j\pi\,[\delta(\omega-\frac{\pi}{5})-\delta(\omega+\frac{\pi}{5})]$

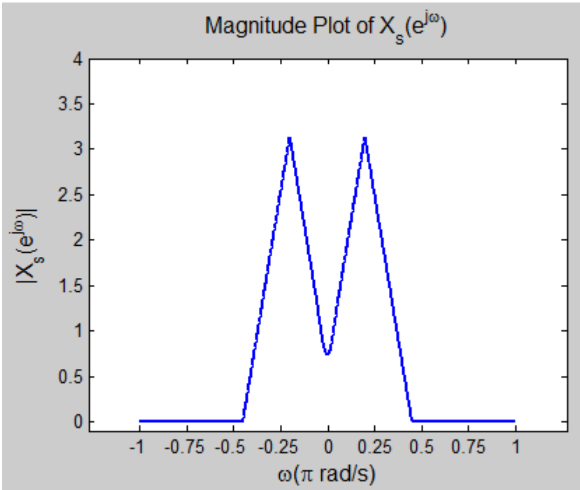
$X_c(e^{j\omega})= X(e^{j\omega}) * F\{\cos(\frac{\pi}{5}n)\}= \pi\,[X(e^{j(\omega+\pi/5)})+ X(e^{j(\omega-\pi/5)})]$

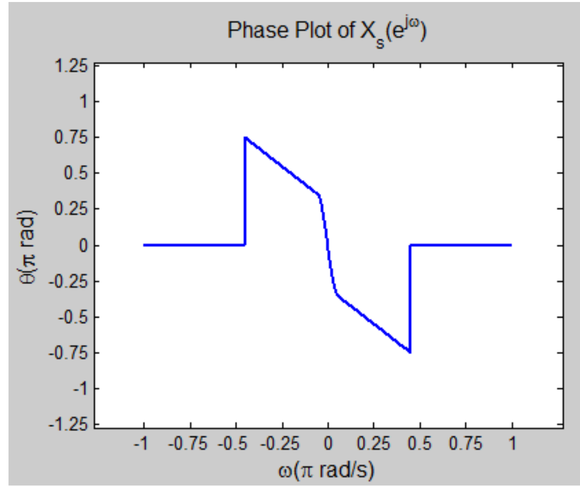
$X_s(e^{j\omega})= X(e^{j\omega}) * F\{\sin(\frac{\pi}{5}n)\}= -j\pi\,[X(e^{j(\omega+\pi/5)}) - X(e^{j(\omega-\pi/5)})]$

c)









5) First of all since the integration variable frequency ω is a real parameter instead of taking the square of absolute value, we can directly take the square of the function.

$$I = \int_{-\pi}^{0} \left(\frac{\sin(\frac{5}{2}\omega)}{\sin(\frac{\omega}{2})}\right)^2 d\omega = \int_{-\pi}^{0} \left(\frac{\sin\left(\frac{\omega}{2}\right)\cos(2\omega) + \cos(\frac{\omega}{2})\sin(2\omega)}{\sin(\frac{\omega}{2})}\right)^2 d\omega =$$

$$= \int_{-\pi}^{0} \left(\cos(2\omega) + \frac{\cos\left(\frac{\omega}{2}\right) 4\cos(\omega)\cos(\frac{\omega}{2})\sin(\frac{\omega}{2})}{\sin(\frac{\omega}{2})}\right)^2 d\omega = \int_{-\pi}^{0} (\cos(2\omega) + 2\,(\cos(\omega) + 1)\cos\omega)^2\, d\omega =$$

$$= \int_{-\pi}^{0} (\cos(2\omega) + 2\cos^2\omega + 2\cos\omega)^2\, d\omega = \int_{-\pi}^{0} (2\cos(2\omega) + 1 + 2\cos\omega)^2\, d\omega =$$

$$= 4\int_{-\pi}^{0} \cos^2(2\omega)\ d\omega + 4\int_{-\pi}^{0} \cos(2\omega)\, d\omega + 8\int_{-\pi}^{0} \cos(2\omega)\cos\omega\, d\omega + 4\int_{-\pi}^{0} \cos\omega\, d\omega +$$

$$+ 4\int_{-\pi}^{0} \cos^2\omega\, d\omega + \int_{-\pi}^{0} 1\, d\omega$$

Lots of the integrals in the last expression vanish after substituting the integral bounds. After that we are left with;

$$I = 4\int_{-\pi}^{0} \cos^2(2\omega)\ d\omega + 4\int_{-\pi}^{0} \cos^2\omega\, d\omega + \int_{-\pi}^{0} 1\, d\omega = 5\pi$$

Hence;

$$\int_{-\pi}^{0} \left|\frac{\sin\left(\frac{5}{2}\omega\right)}{\sin\left(\frac{\omega}{2}\right)}\right|^2 d\omega = 5\pi$$

6) $X(e^{j\omega}) = 2\pi\, \delta(\omega - \omega_0)$

$$y[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} Y(e^{j\omega})\ e^{j\omega n}\, d\omega = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{j\omega})\ 2\pi\, \delta(\omega - \omega_0)\, e^{j\omega n}\, d\omega = H(e^{j\omega_0})\, e^{j\omega_0 n}$$

7)

a) $y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[n-k]h[k] = \sum_{k=-\infty}^{\infty} 3^{n-k}[\left(\frac{1}{2}\right)^k u[k] + 2^k u[k]] =$

$= 3^n \left(\sum_{k=0}^{\infty}\left(\frac{1}{6}\right)^k + \sum_{k=0}^{\infty}\left(\frac{2}{3}\right)^k\right) = 3^n \left(\frac{1}{1-\frac{1}{6}} + \frac{1}{1-\frac{2}{3}}\right) = \frac{21}{5}\,3^n$

$C=\frac{21}{5}$

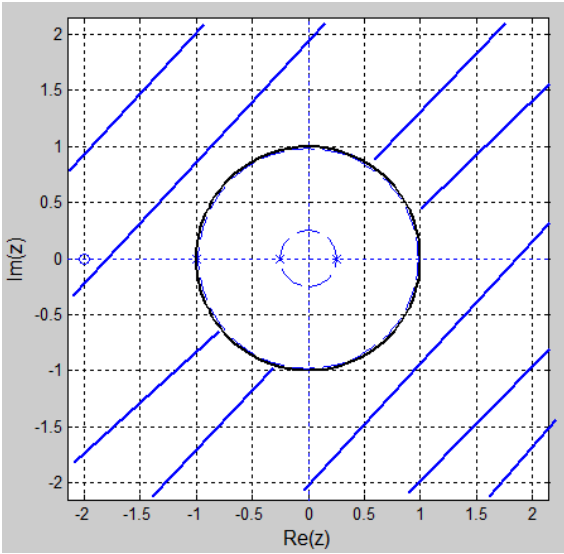
b) $y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[n-k]h[k] = \sum_{k=-\infty}^{\infty} 3^{n-k}[\left(\frac{1}{2}\right)^k u[k] + 2^k u[-k]] =$

$= 3^n \left(\sum_{k=0}^{\infty}\left(\frac{1}{6}\right)^k + \sum_{k=-\infty}^{0}\left(\frac{2}{3}\right)^k\right) = 3^n \left(\frac{6}{5} + \infty\right) \to \infty$

This output diverges to infinity. Thus we cannot write it in the form of $C\,3^n$.

8)

a) This transfer function has a zero at z=2 and poles at z=1/4, -1/4, -1. The problem about that function is that ROC of a stable system must include the unit circle in the pole-zero plot. However, since we have a pole on that circle our ROC cannot contain it and this system cannot be stable. For causality ROC must include the area which is out of the outermost circle. The corresponding plot is given below.



Let's find the time domain expression.

$$H(z) = \frac{1+2z^{-1}}{(1-\frac{1}{4}z^{-1})(1+z^{-1})(1+\frac{1}{4}z^{-1})} = \frac{A_1}{1-\frac{1}{4}z^{-1}} + \frac{A_2}{1+z^{-1}} + \frac{A_3}{1+\frac{1}{4}z^{-1}}$$

$$1+2z^{-1} = (1+z^{-1})(1+\frac{1}{4}z^{-1})A_1 + (1-\frac{1}{16}z^{-2})A_2 + (1-\frac{1}{4}z^{-1})(1+z^{-1})A_3$$

$1+2z^{-1} = (1+\frac{5}{4}z^{-1}+\frac{1}{4}z^{-2})A_1 + (1-\frac{1}{16}z^{-2})A_2 + (1+\frac{3}{4}z^{-1}-\frac{1}{4}z^{-2})A_3$

$A_1+A_2+A_3=1 \qquad 5A_1+3A_3=8 \qquad 4A_1-A_2-4A_3=0$

$A_1=\frac{1}{9} \qquad A_2= -\frac{43}{27} \qquad A_3= \frac{67}{27}$

$H(z) = \frac{1}{9}\frac{1}{1-\frac{1}{4}z^{-1}} - \frac{43}{27}\frac{1}{1+z^{-1}} + \frac{67}{27}\frac{1}{1+\frac{1}{4}z^{-1}}$

In order to convert that into time domain we must know the system properties. We have found that it cannot be stable but if we assume it to be causal;

$h[n] = \frac{1}{9}(\frac{1}{4})^n u[n] - \frac{43}{27}(-1)^n u[n] + \frac{67}{27}(-\frac{1}{4})^n u[n]$

9)

c) $X(e^{j\omega})= 2\pi(3+j5)\delta(\omega) - j\pi[\delta(\omega - \frac{\pi}{4}) - \delta(\omega + \frac{\pi}{4})]$

$H(e^{j\omega}) = |H(e^{j\omega})|\, e^{j\angle H(e^{j\omega})}$ where

$$|H(e^{j\omega})| = \begin{cases} \frac{2}{\pi}\omega + 1 & \text{for } -\frac{\pi}{2} < \omega < 0 \\ -\frac{2}{\pi}\omega + 1 & \text{for } 0 < \omega < \frac{\pi}{2} \\ 0 & \text{otherwise} \end{cases} \qquad \angle H(e^{j\omega}) = -\frac{\omega}{2} \qquad -\pi < \omega < \pi$$

$Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega}) = 2\pi(3+j5)H(e^{j0})\delta(\omega) - j\pi H(e^{j\frac{\pi}{4}})\delta(\omega - \frac{\pi}{4}) + j\pi H(e^{-j\frac{\pi}{4}})\delta(\omega + \frac{\pi}{4}) =$

$= 2\pi(3+j5)\delta(\omega) - j\frac{\pi}{2}e^{j(-\frac{\pi}{8})}\delta(\omega - \frac{\pi}{4}) + j\frac{\pi}{2}e^{j(\frac{\pi}{8})}\delta(\omega + \frac{\pi}{4})$

$y[n] = 3+j5 - j\frac{1}{4}e^{-j\frac{\pi}{8}}e^{j\frac{\pi}{4}n} + j\frac{1}{4}e^{j\frac{\pi}{8}}e^{-j\frac{\pi}{4}n} =$

$= 3+j5 + j\frac{1}{4}(-\cos(\frac{\pi}{4}n - \frac{\pi}{8}) - j\sin(\frac{\pi}{4}n - \frac{\pi}{8}) + \cos(\frac{\pi}{4}n - \frac{\pi}{8}) - j\sin(\frac{\pi}{4}n - \frac{\pi}{8})) =$

$= 3+j5 + \frac{1}{2}\sin(\frac{\pi}{4}n - \frac{\pi}{8})$

Hence; $y[n] = 3+j5 + \frac{1}{2}\sin(\frac{\pi}{4}n - \frac{\pi}{8})$

Instead of making analytical calculations we can write the answer directly also. x[n] has a DC term and a sinusoidal term with 0.25π rad/sample frequency. Magnitude plot is 1 and phase plot is 0 for 0 rad/sample so this term passes the system without any change. 0.25π rad/sample is $\frac{1}{2}$ for magnitude and $-\frac{\pi}{8}$ rad for phase plots. These are also the results of our calculation.