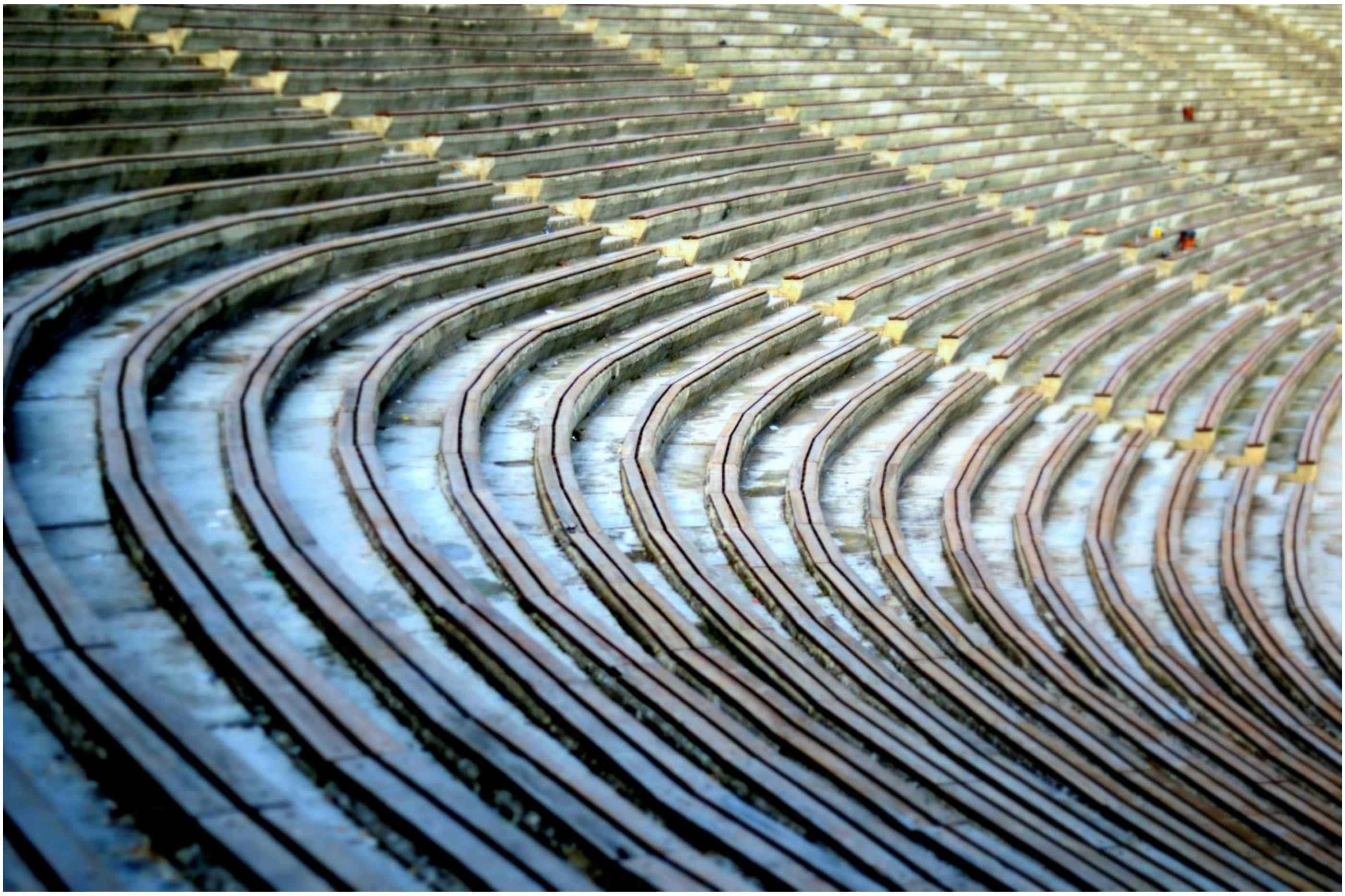


SAMPLING

UNIFORM SAMPLING

C/D, D/C (A/D, D/A)

A MATHEMATICAL MODEL OF SAMPLING

IMPULSE SAMPLING

ALIASING

EXPRESSING $X(e^{j\omega})$ IN TERMS OF $X_c(\Omega)$

NYQUIST-SHANNON SAMPLING THEOREM

RECONSTRUCTION OF A CT SIGNAL FROM A DT SIGNAL

DISCRETE-TIME PROCESSING OF CONTINUOUS-TIME SIGNALS

IMPULSE RESPONSES OF EQUIVALENT CT AND DT SYSTEMS

CHANGING THE SAMPLING RATE IN DISCRETE-TIME

RATE REDUCTION BY AN INTEGER FACTOR

RATE INCREASE BY AN INTEGER FACTOR

CHANGING THE SAMPLING RATE BY A NONINTEGER (RATIONAL) FACTOR

DIGITAL PROCESSING OF ANALOG SIGNALS

ANALOG TO DIGITAL CONVERSION

QUANTIZATION

DIGITAL TO ANALOG CONVERSION

Ex:
Consider sampling of a sinusoidal signal,

$$x_c(t) = \cos(2\pi f_0 t)$$

$$x[n] = x_c(nT) = x_c\left(\frac{n}{f_s}\right) = \cos(2\pi \frac{f_0}{f_s} n)$$

$$x[n] = \cos\left(2\pi \frac{f_0}{f_s} n\right) = \cos(\omega_0 n)$$

If

$$\frac{f_0}{f_s} \geq 1$$

Let

$$\hat{f}_0 = f_0 \; modulo \; f_s$$

$$x[n] = \cos\left(2\pi \frac{f_0}{f_s} n\right) = \cos\left(2\pi \frac{\hat{f}_0}{f_s} n\right) = \cos(\omega_0 n)$$

Furthermore, if

$$\frac{\hat{f}_0}{f_s} > \frac{1}{2}$$

$$x[n] = \cos\left(2\pi \frac{f_0}{f_s} n\right) = \cos\left(2\pi \frac{\hat{f}_0}{f_s} n\right) = \cos(\omega_0 n)$$

Ex:

$$f_s = 2000 \text{ Hz}$$

$f_0 = 3400 \text{ Hz}$ $\omega_0 = 3.4\pi$ $x[n] = \cos(3.4\pi n) = \cos(1.4\pi n) = \cos(0.6\pi n)$

$f = 3000 \text{ Hz}$ $\omega_0 = 3\pi$ $x[n] = \cos(3\pi n) = \cos(\pi n)$

$f = 2800 \text{ Hz}$ $\omega_0 = 2.8\pi$ $x[n] = \cos(2.8\pi n) = \cos(0.8\pi n)$

$f = 2000 \text{ Hz}$ $\omega_0 = 2\pi$ $x[n] = \cos(2\pi n) = \cos(0) = 1$

$f = 1400 \text{ Hz}$ $\omega_0 = 1.4\pi$ $x[n] = \cos(1.4\pi n) = \cos(0.6\pi n)$

$f = 1000 \text{ Hz}$ $\omega_0 = \pi$ $x[n] = \cos(\pi n)$

$f = 800 \text{ Hz}$ $\omega_0 = 0.8\pi$ $x[n] = \cos(0.8\pi n)$

$f = 600 \text{ Hz}$ $\omega_0 = 0.6\pi$ $x[n] = \cos(0.6\pi n)$

Same color --> same DT frequency

Continuous-time signals are commonly processed by using digital systems.

Mobile devices, TV receivers and displays, radar, sonar,…

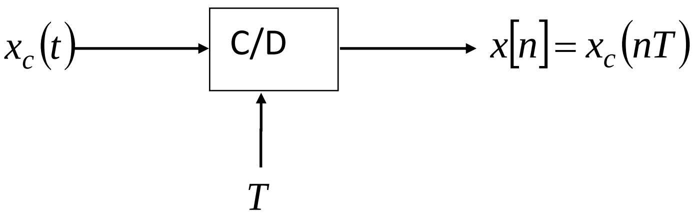
“**analog to digital**” conversion: A/D

“**continuous-time to discrete-time**” conversion: C/D

A/D = C/D and quantization of sample values

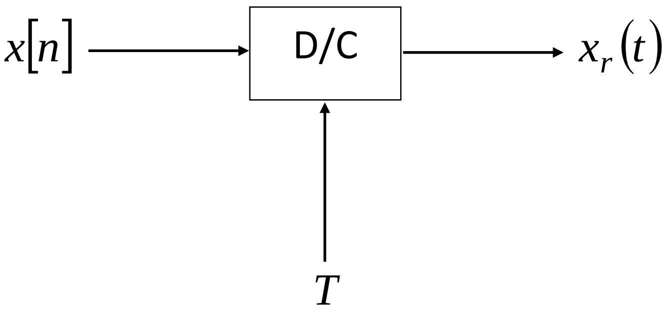
UNIFORM SAMPLING

Samples are spaced uniformly in time.



T: sampling period

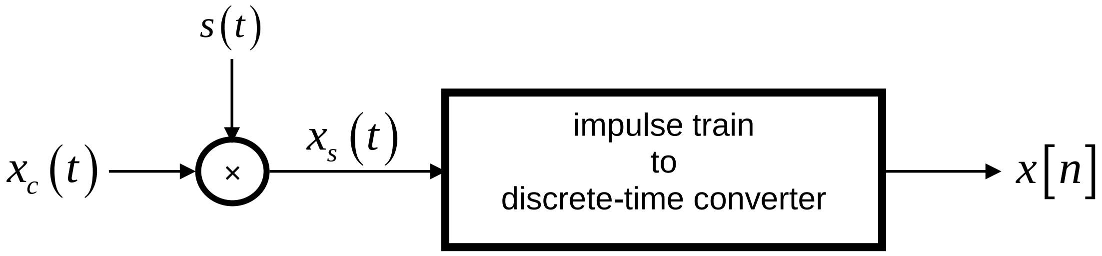
“discrete-time to continuous-time conversion”, D/C.



In practice, we have the term “digital to analog” (D/A) conversion.

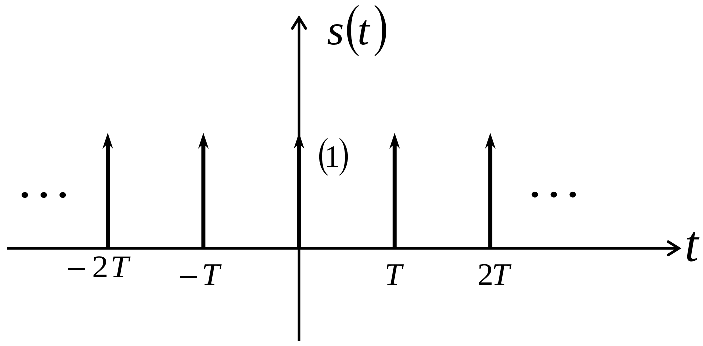
A central question is about determining, if exist, the conditions required to recover a continuous-time signal from its uniformly acquired samples.

A MATHEMATICAL MODEL OF SAMPLING



$$s(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$$

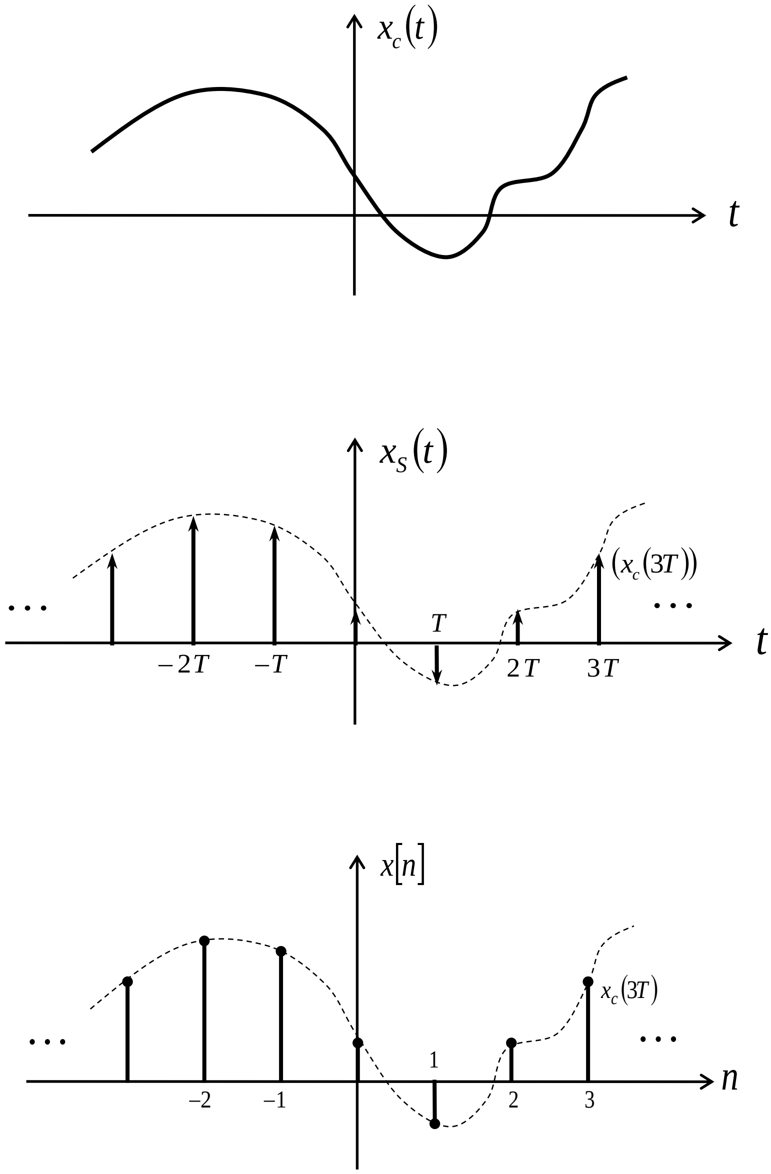
impulse train



T : sampling period

$f_s = \frac{1}{T}$: sampling frequency (Hz)

$\Omega_s = \frac{2\pi}{T}$: sampling frequency (rad/sec)



$$x_s(t) = \sum_{n=-\infty}^{\infty} x_c(nT)\delta(t - nT)$$

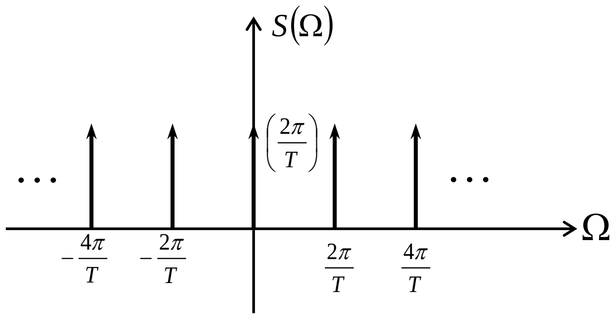
$$x[n] = x_c(nT)$$

IMPULSE TRAIN IN FREQUENCY DOMAIN

Using Fourier series representation of $s(t)$ with coefficients $a_k = \frac{1}{T}$ and

$$e^{-jk\frac{2\pi}{T}t} \quad \overset{CTFT}{\longleftrightarrow} \quad 2\pi\delta\left(\Omega - k\frac{2\pi}{T}\right)$$

$$S(\Omega) = \frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta\left(\Omega - k\frac{2\pi}{T}\right)$$



$$a_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} s(t)e^{-jk\frac{2\pi}{T}t}dt$$

Note that, an alternative expression is

$$S(\Omega) = \sum_{k=-\infty}^{\infty} e^{-jkT\Omega}$$

since

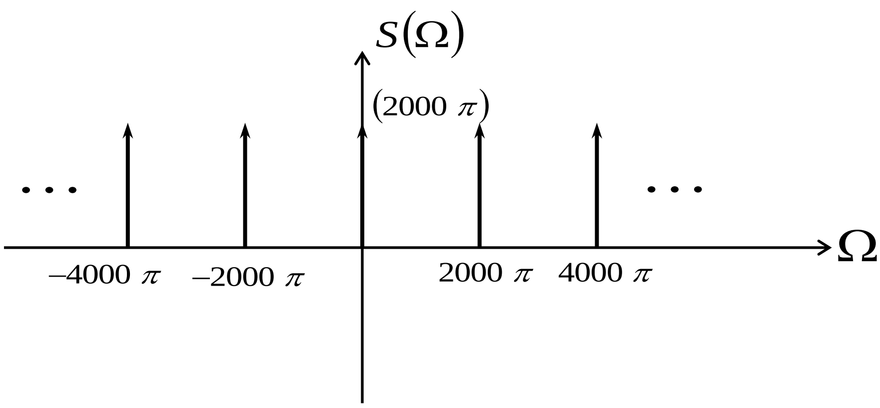
$$\delta(t-kT) \quad \overset{CTFT}{\longleftrightarrow} \quad e^{-jkT\Omega}$$

Ex:

$T = 1$ ms,

$f = 1000$ Hz,

$\Omega = \frac{2\pi}{T} = 2000\pi$ rad/sec

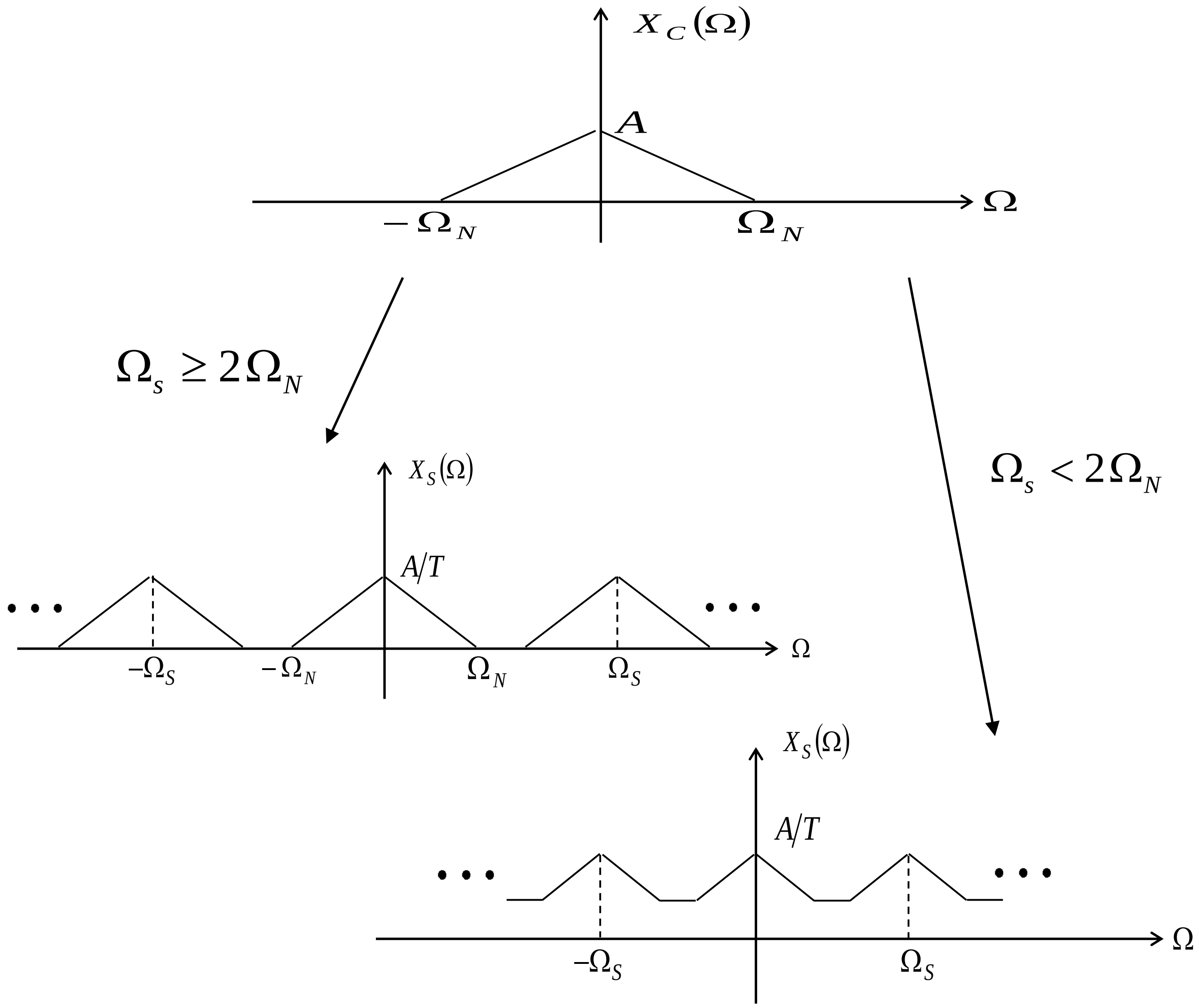


After multiplication with Impulse Train

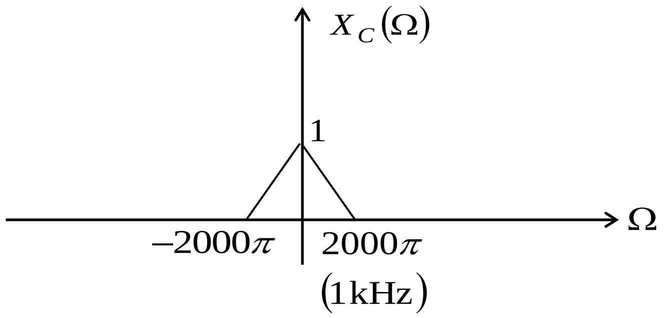
$$X_s(\Omega) = \frac{1}{2\pi} X_c(\Omega) * S(\Omega)$$

Since $S(\Omega)$ is an impulse train

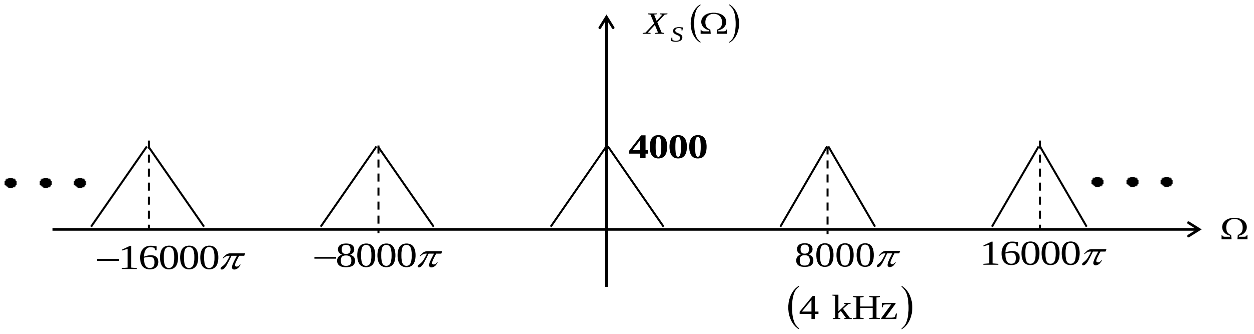
$$X_s(\Omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c\left(\Omega - k\frac{2\pi}{T}\right)$$



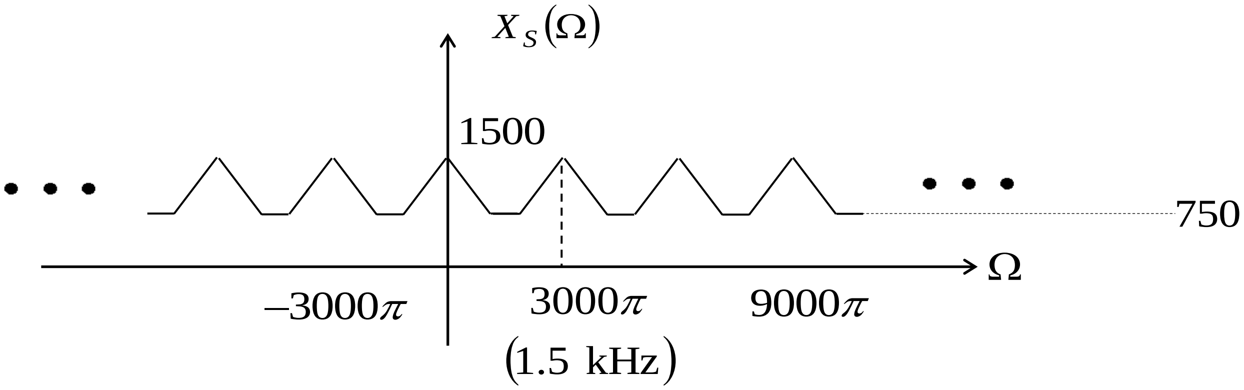
Ex: Let $x_c(t)$ be "*bandlimited*" to 1 kHz or equivalently to 2000π rad/sec., i.e. $\Omega_N = 2000\pi$



Let $T = 0.25$ ms ($f_s = 4$ kHz) $\Omega_s = 8000\pi$



However, if $T = 2/3$ ms $f_s = 1.5$ kHz) $\Omega_s = 3000\pi$



Definition

The overlap (distortion) of the spectrum when $\Omega_S < 2\,\Omega_N$ is called "**aliasing**".

EXPRESSING $X\left(e^{j\omega}\right)$ IN TERMS OF $X_c(\Omega)$

$$x_s(t) = \sum_{n=-\infty}^{\infty} x_c(nT)\delta(t - nT)$$

$$= \sum_{n=-\infty}^{\infty} x[n]\delta(t - nT)$$

Time-shift property of CTFT,

$$X_s(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega T n}$$

Compare this expression to

$$X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

Conclude that

$$X\left(e^{j\omega}\right) = X_s(\Omega)\big|_{\Omega=\frac{\omega}{T}}$$

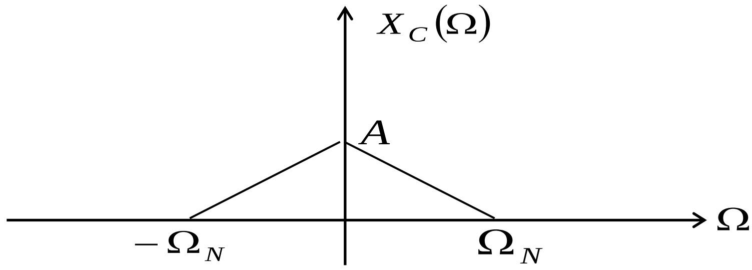
Hence

$$X\left(e^{j\omega}\right) = \frac{1}{T}\sum_{n=-\infty}^{\infty} X_c\left(\frac{\omega}{T} - k\frac{2\pi}{T}\right)$$

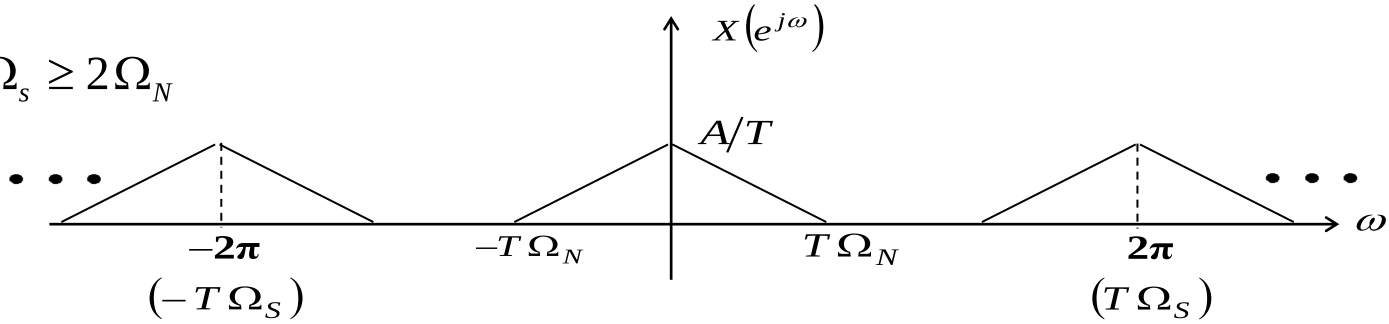
A “linear warping” of the frequency scale and its periodic extension.

$$
\begin{aligned}
X\left(e^{j\omega}\right) = \cdots &+ \frac{1}{T} X_c\left(\frac{1}{T}(\omega + 4\pi)\right) \\
&+ \frac{1}{T} X_c\left(\frac{1}{T}(\omega + 2\pi)\right) \\
&+ \frac{1}{T} X_c\left(\frac{\omega}{T}\right) \\
&+ \frac{1}{T} X_c\left(\frac{1}{T}(\omega - 2\pi)\right) \\
&+ \frac{1}{T} X_c\left(\frac{1}{T}(\omega - 4\pi)\right) \\
&+ \cdots
\end{aligned}
$$

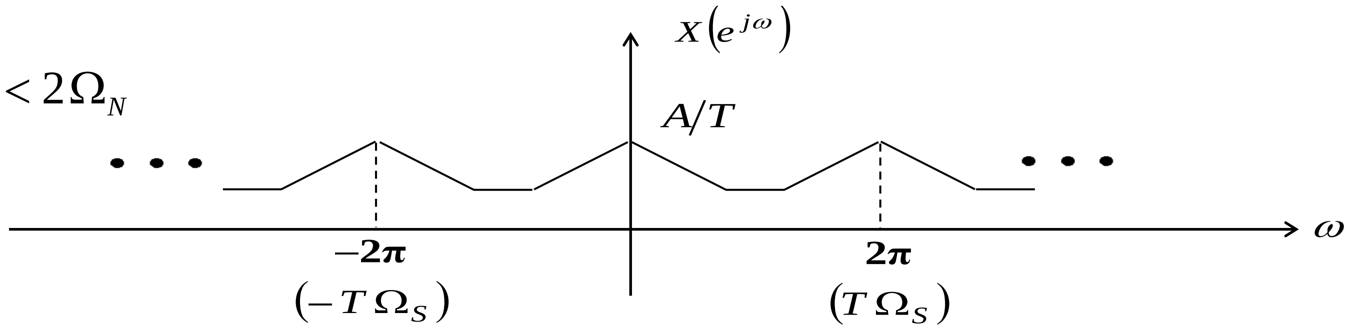
Therefore



$\Omega_s \geq 2\Omega_N$

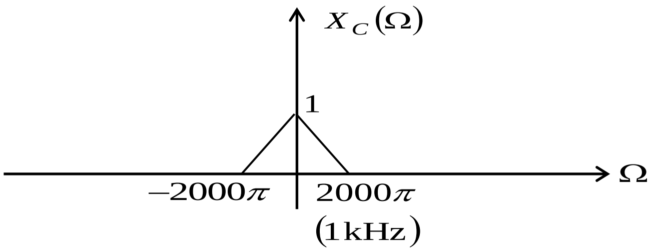


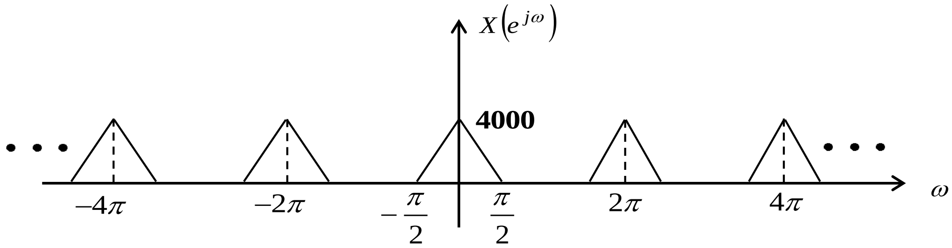
$\Omega_s < 2\Omega_N$



Ex: Let $x_c(t)$ be "*bandlimited*" to 1 kHz or equivalently to 2000π rad/sec., i.e., $\Omega_N = 2000\pi$

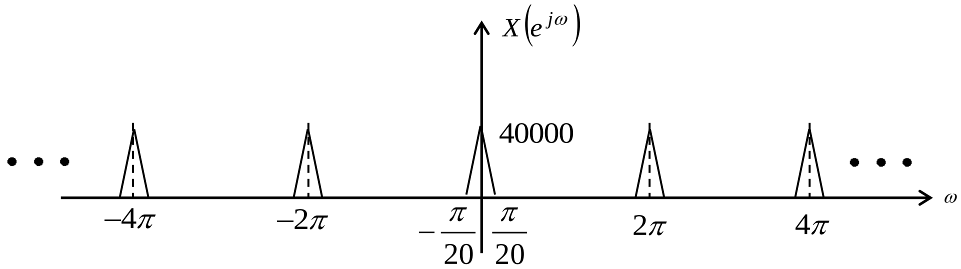
Let $T = 0.25$ ms ($f_s = 4$ kHz) $\Omega_s = 8000\pi$





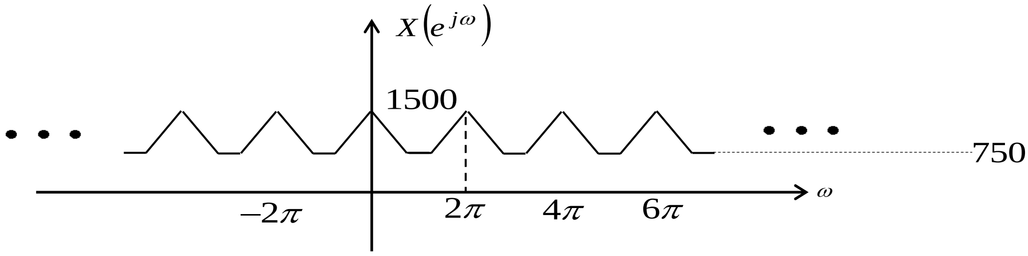
If sampling frequency is increased ten times:

$T = 0.025$ ms ($f_s = 40$ kHz) $\Omega_s = 80000\pi$



If sampling frequency is decreased,

$T = \frac{2}{3}$ ms ($f_s = 1.5$ kHz) $\Omega_s = 3000\pi$



Aliasing!

Notes

- The discrete-time and continuous-time frequency scales are related by $\omega = \Omega T$

- Sampling frequency, Ω_s, is "always" mapped to 2π in discrete-time frequency scale.

- Discrete-time frequency "equivalent", ω_a, of any continuous-time frequency, Ω_a, can be found using the ratio $\frac{\Omega_a}{\Omega_s}$:

  "In the last example, the band edge frequency, 1 kHz, is one fourth of the sampling frequency, 4 kHz. Therefore, the band edge of $X\left(e^{j\omega}\right)$ is at $\frac{\pi}{2}$, one fourth of 2π."

NYQUIST-SHANNON SAMPLING THEOREM

Let $x_c(t)$ be bandlimited to Ω_N, i.e.,

$$X_c(\Omega) = 0 \quad |\Omega| \geq \Omega_N$$

$x_c(t)$ can be determined uniquely from its samples

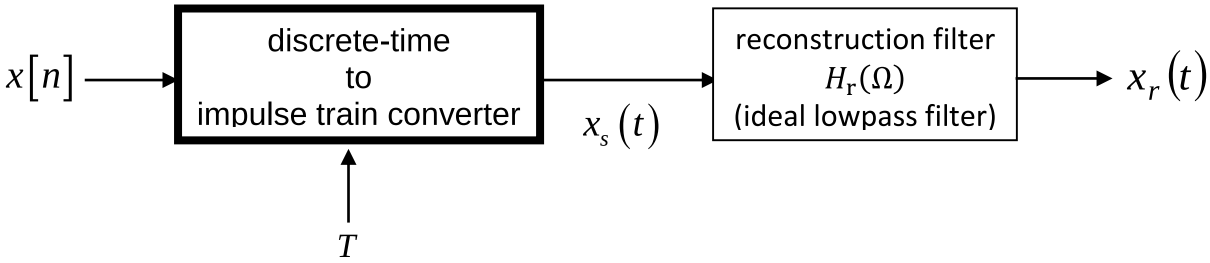
$$x[n] = x_c(nT)$$

if

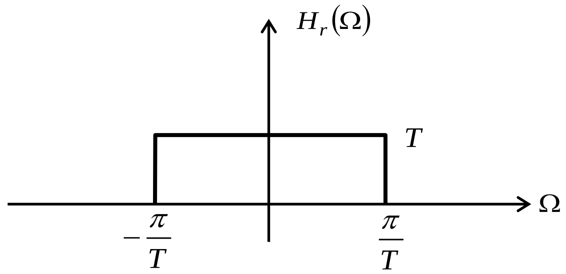
$$\Omega_s \geq 2\Omega_N$$

Pay attention to the equality!

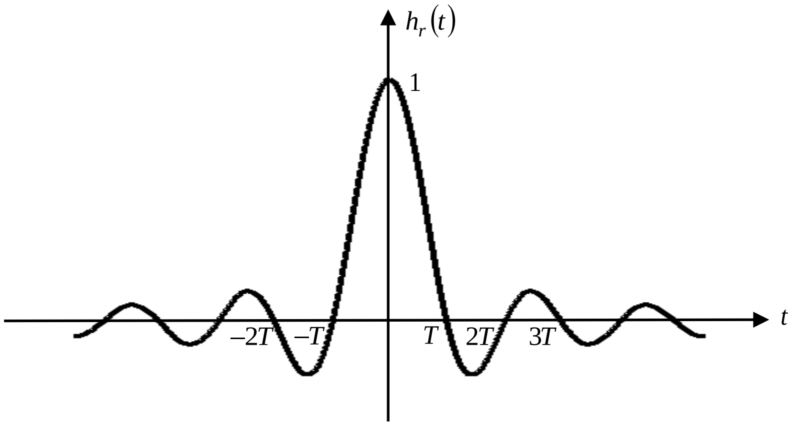
RECONSTRUCTION OF A CT SIGNAL FROM A DT SIGNAL



$$x_s(t) = \sum_{n=-\infty}^{\infty} x_c(nT)\,\delta(t - nT)$$



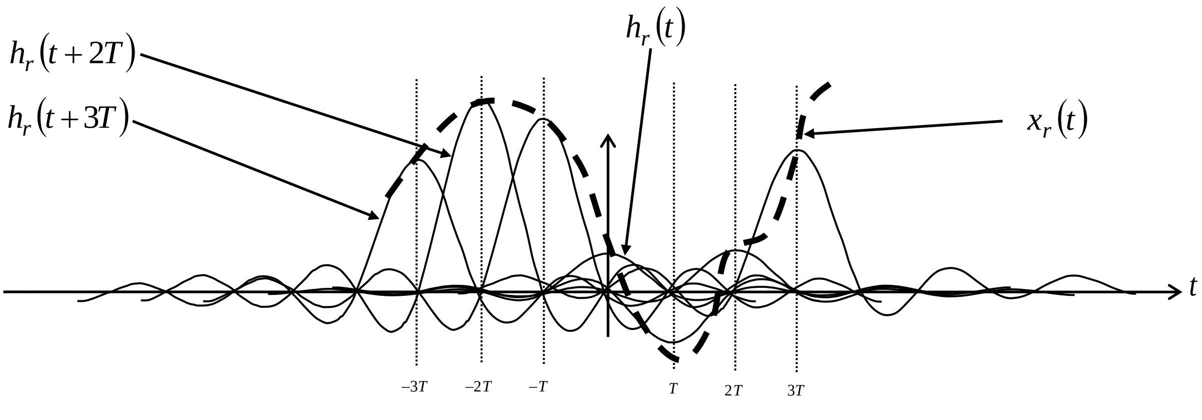
$$h_r(t) = \frac{\sin\left(\frac{\pi}{T}t\right)}{\frac{\pi}{T}t}$$



$$
\begin{aligned}
x_r(t) &= x_S(t) * h_r(t) \\
&= \left(\sum_{n=-\infty}^{\infty} x[n]\delta(t-nT) \right) * h_r(t) \\
&= \sum_{n=-\infty}^{\infty} x[n] h_r(t-nT)
\end{aligned}
$$

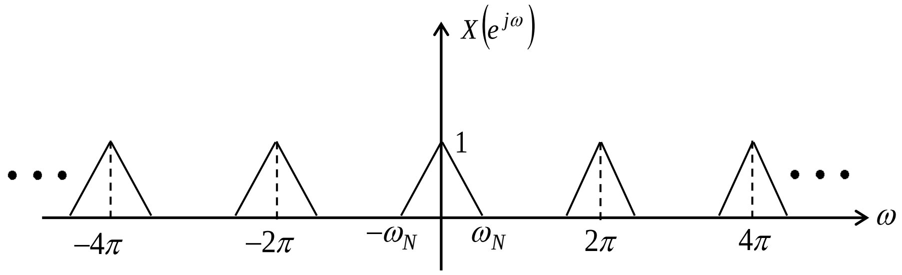
$$
\begin{aligned}
X_r(\Omega) &= \sum_{n=-\infty}^{\infty} x[n] e^{-jnT\Omega} H_r(\Omega) \\
&= \left(\sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega T n} \right) H_r(\Omega) \\
&= X\left(e^{j\Omega T}\right) H_r(\Omega)
\end{aligned}
$$

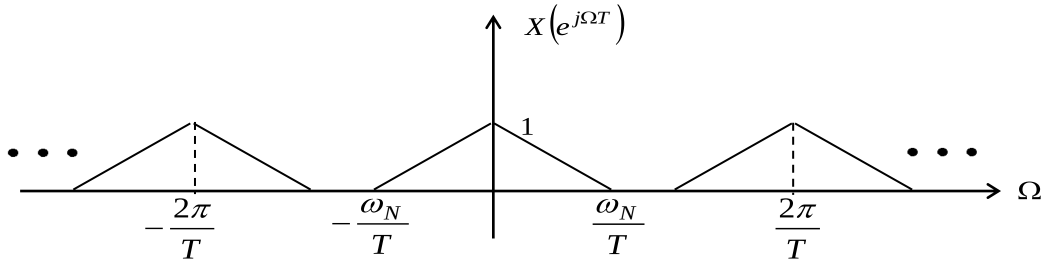
$$
x_r(t) = \sum_{n=-\infty}^{\infty} x[n] h_r(t-nT)
$$

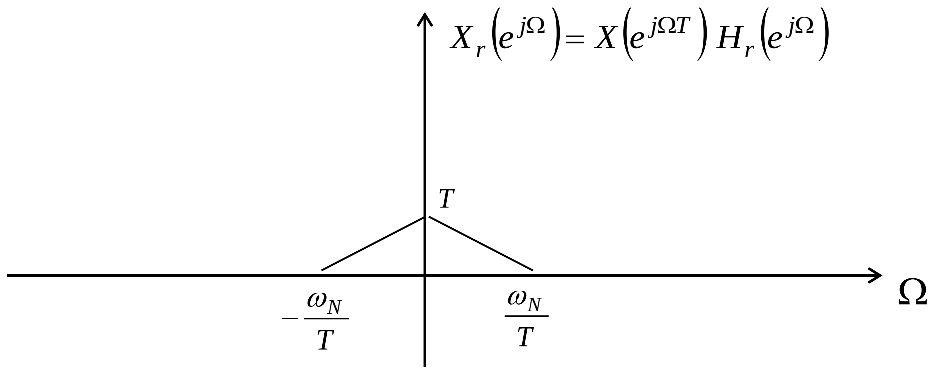


```
% bandlimited interpolation
% using 2N+1 terms in the summation

clear all
close all

N = 25  ;
f = 1   ;              % frequency of continuous time sinusoid to be sampled
Omega = 2 * pi * f ;
f_s   = 7          ;    % sampling frequency
T_s   = 1/f_s      ;
k     = -N:N       ;
x     = sin(Omega*k*T_s)   ; % samples of the continuous time sinusoid

delta = 4;
t = -delta:0.01:delta  ; % time interval in which we plot our results

% computing the sinc signals in the summation
for n = -N:N
   h_r(n+N+1,:) = x(n+N+1) * sin(f_s*pi*(t-n*T_s))./(f_s*pi*(t-n*T_s));
   h_r(n+N+1,find(isnan(h_r(n+N+1,:)))) = x(n+N+1) ;
end

 x_r = sum(h_r);   % reconstructed cont-time signal

plot(t,sin(Omega*t),'k','LineWidth',3)
hold
pause
plot(t,h_r(N+1-3,:),'g','LineWidth',3)
pause
plot(t,h_r(N+1-2,:),'r','LineWidth',3)
pause
plot(t,h_r(N+1-1,:),'c','LineWidth',3)
pause
plot(t,h_r(N+1,:),'m','LineWidth',3)
pause
plot(t,h_r(N+1+1,:),'y','LineWidth',3)
pause
plot(t,h_r(N+1+2,:),'b','LineWidth',3)
pause
plot(t,x_r,'r--','LineWidth',1.5)
```

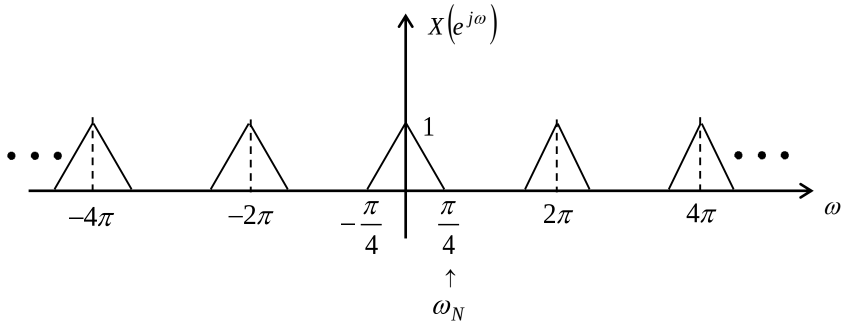
Ex: $$X_r(\Omega) = X\left(e^{j\Omega T}\right) H_r(\Omega)$$



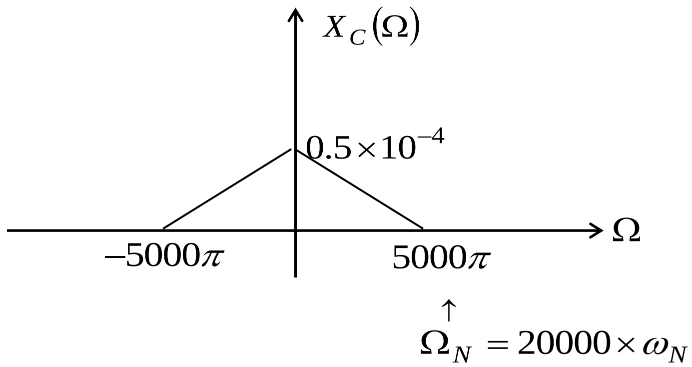




Ex: Given

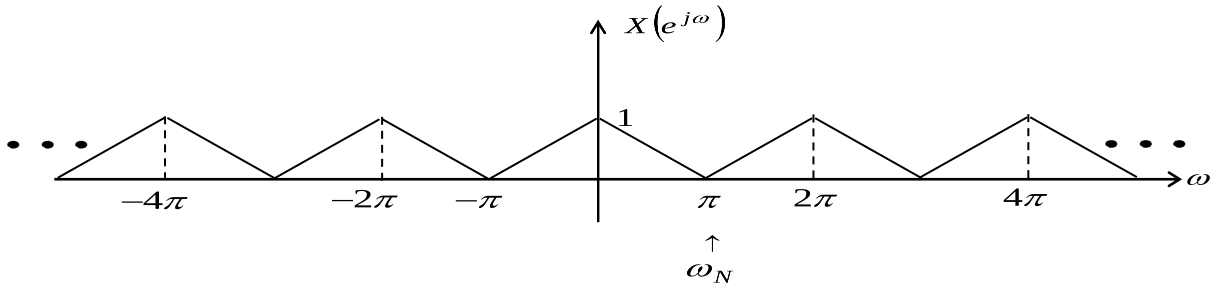


If $f_s = 20$ kHz, i.e., $\Omega_s = 40\pi$ krad/s, $T = \frac{1}{20}$ ms

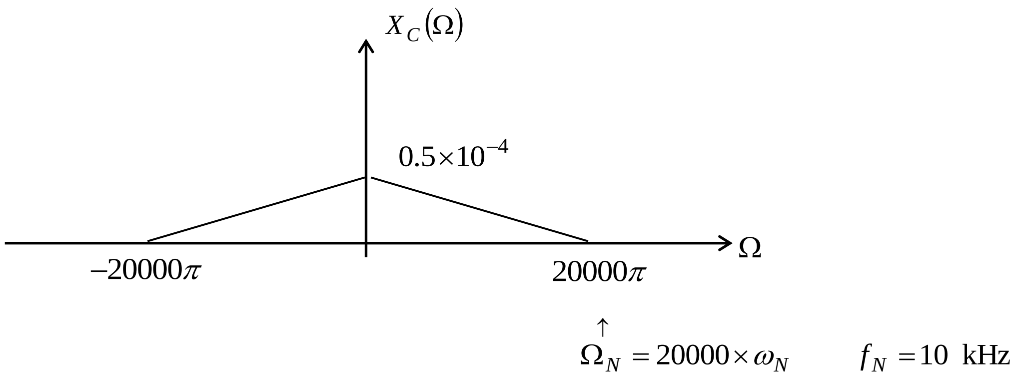


$(f_N = 2.5$ kHz$)$

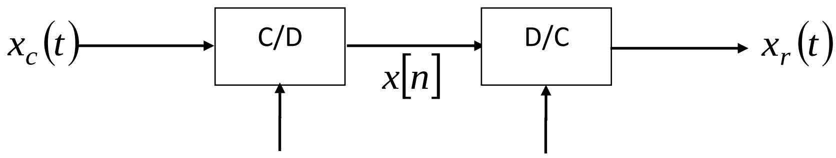
Ex: Given



If $f_s = 20$ kHz, i.e., $\Omega_s = 40\pi$ krad/s, $T = \frac{1}{20}$ ms

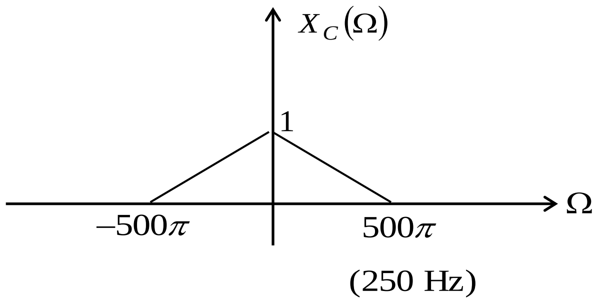


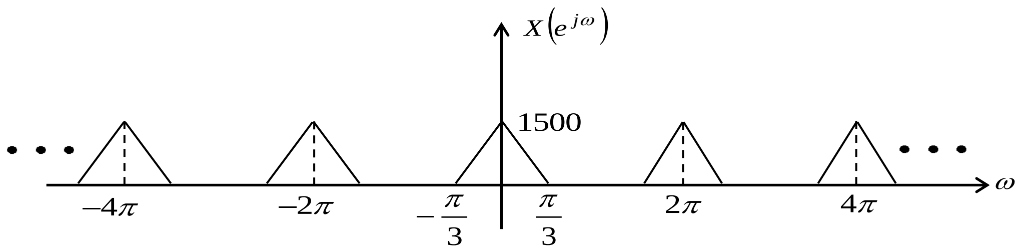
Ex:

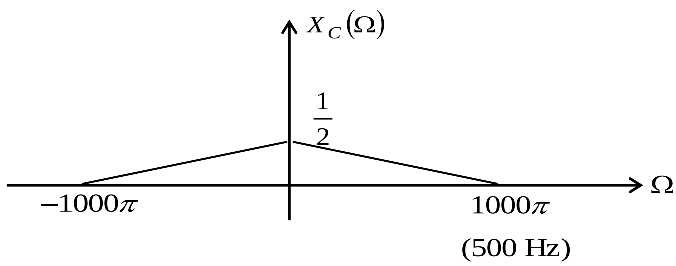


$$T_1 = \frac{1}{1500} \qquad T_2 = \frac{1}{3000}$$

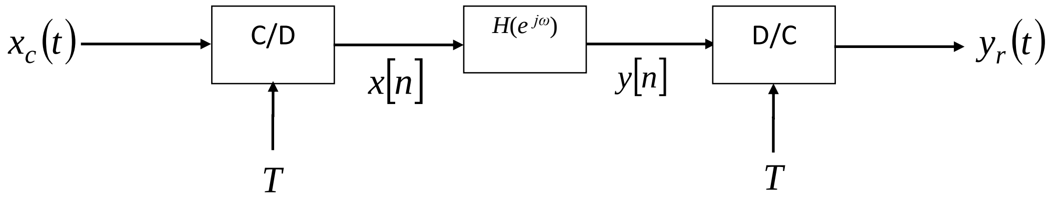
$$f_1 = 1500 \text{ Hz} \qquad f_2 = 3000 \text{ Hz}$$







DISCRETE-TIME PROCESSING OF CONTINUOUS-TIME SIGNALS



$$
\begin{aligned}
Y_r(\Omega) &= H_r(\Omega)\, Y\left(e^{j\Omega T}\right) \\
&= H_r(\Omega) H\left(e^{j\Omega T}\right) X\left(e^{j\Omega T}\right) \\
&= H_r(\Omega) H\left(e^{j\Omega T}\right)\left(\frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(\Omega - k\frac{2\pi}{T}\right)\right)
\end{aligned}
$$

Assuming that $x_c(t)$ is badlimited to $\frac{\pi}{T}$

$$
Y_r(\Omega) = \begin{cases} H\left(e^{j\Omega T}\right) X_c(\Omega) & |\Omega| < \dfrac{\pi}{T} \\ 0 & o.w. \end{cases}
$$

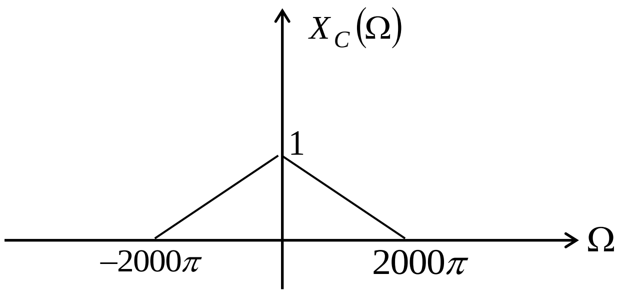
Hence

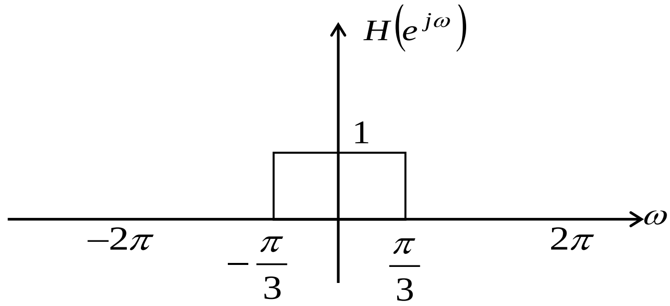
$$Y_r(\Omega) = H_{eff}(\Omega) X_c(\Omega)$$

where

$$H_{eff}(\Omega) = \begin{cases} H\left(e^{j\Omega T}\right) & |\Omega| < \dfrac{\pi}{T} \\ 0 & o.w. \end{cases}$$

Ex: Let T = 0.25 ms (4 kHz) and

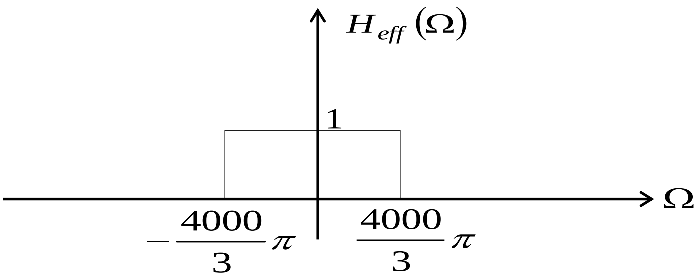




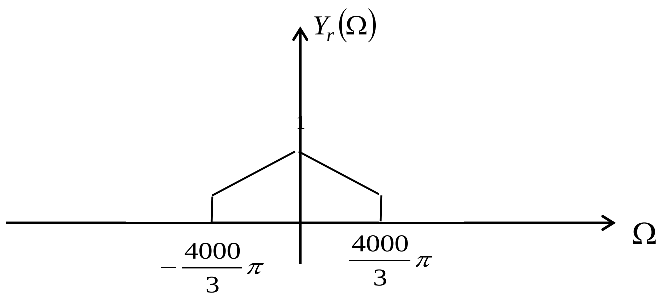
Therefore

$$H_{eff}(\Omega) = \begin{cases} H\left(e^{j\Omega T}\right) & |\Omega| < 4000\pi \\ 0 & o.w. \end{cases}$$

$$= \begin{cases} 1 & |\Omega| < \frac{4000}{3}\pi \\ 0 & o.w. \end{cases}$$



and



Ex: What should the cut-off frequency of the discrete-time ideal lowpass filter be so that the continuous-time signal is lowpass filtered with a cut-off frequency of 3 kHz when the sampling frequency 8 kHz?

$$\omega_C = \frac{3}{8} 2\pi = \frac{3}{4} \pi$$