EE 430 HOMEWORK III SOLUTIONS

Q1 ⓐ

$$X_1(e^{jw}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X\left(\frac{\Omega - 2\pi k}{T}\right)$$

$$\boxed{X_1(e^{jw}) = 2\times10^4 \sum_{k=-\infty} X(2\times10^4 w - 4\times10^4\pi.k) \text{ for } |w|<\pi}$$

$$X_2(e^{jw}) = X_1(e^{j10w}) \quad \text{due to upsampling}$$

$$\boxed{X_2(e^{jw}) = 2\times10^4 \sum_{k=-\infty} X(2\times10^5 w - 4\times10^4\pi k) \text{ for } |w|<\pi}$$

ⓑ

$$\boxed{Y_2(e^{jw}) = H(e^{jw}).X_2(e^{jw}) = e^{-jwN} X_1(e^{j10w}) \text{ for } |w|<\frac{\pi}{10}}$$

$$Y_1(e^{jw}) = \frac{1}{10}\cdot\sum_{k=-\infty}^{\infty} Y_2\left(e^{j\left(\frac{w-2\pi k}{10}\right)}\right)$$

$$\boxed{Y_1(e^{jw}) = e^{-j\frac{wN}{10}}\,\frac{1}{10}\sum_{k=-\infty}^{\infty} X_1\left(e^{j(w-2\pi k)}\right)}$$

ⓒ

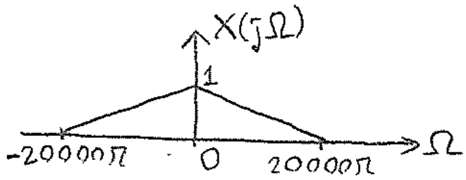
$$Y(\Omega) = \frac{1}{2\times10^4}\, Y_1\left(e^{\frac{j\Omega}{2\times10^4}}\right) \text{ for } |w|<\pi\times2\times10^4$$

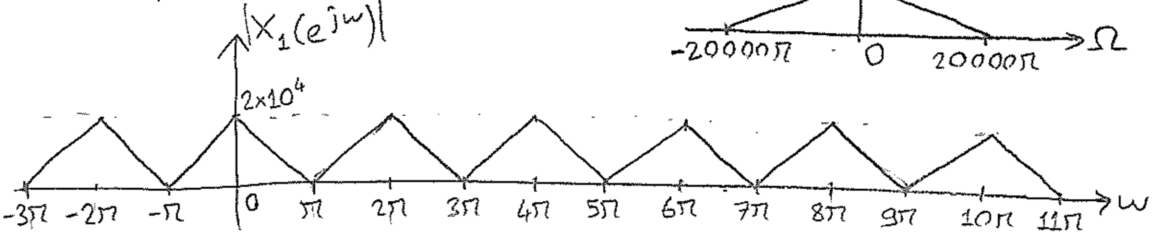
$$Y(\Omega) = \frac{1}{2\times10^4}\left(\frac{e^{-j\frac{N\Omega}{2\times10^5}}}{10}\cdot 2\times10^4\cdot X(\Omega)\right)$$

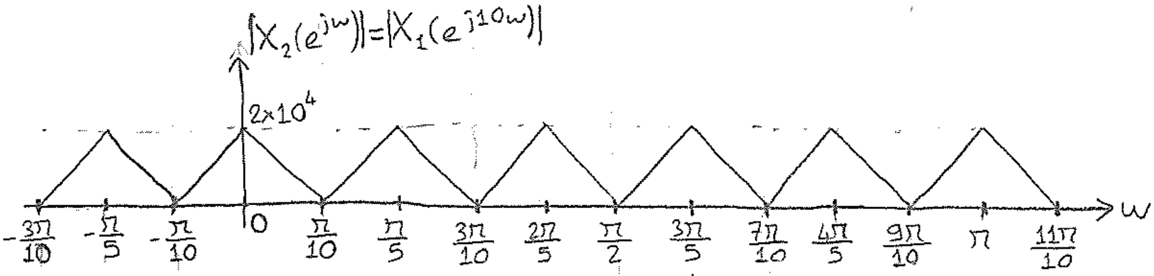
$$Y(\Omega) = \frac{e^{-j\frac{N\Omega}{2\times10^5}}\, X(\Omega)}{10}$$

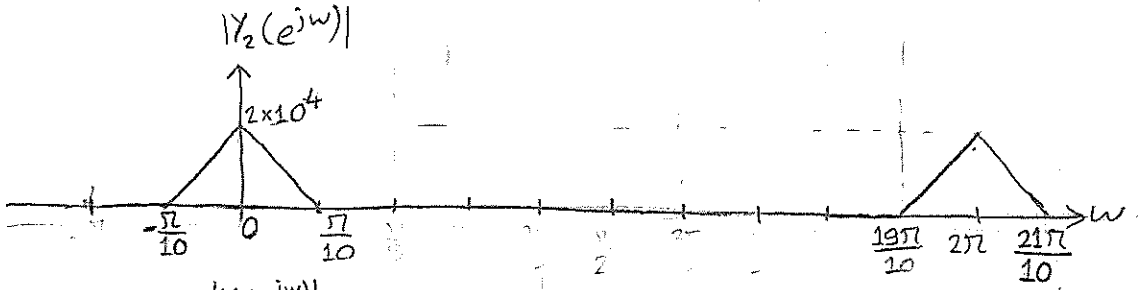
$$\boxed{y(t) = \frac{1}{10}.x\left(t - \frac{N}{2\times10^5}\right)}$$

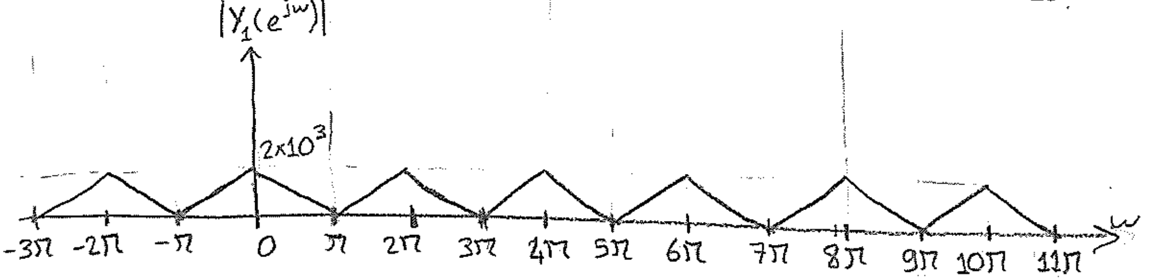
(d) The plots are obtained for

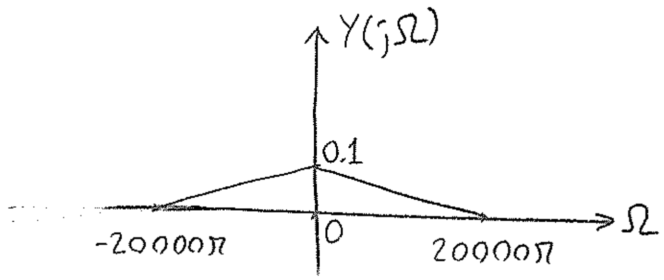












2) a) $x[n] \rightarrow \boxed{H(e^{j\omega})} \rightarrow \boxed{\uparrow L} \rightarrow y_1[n]$

$x[n] \rightarrow \boxed{\uparrow L} \rightarrow \boxed{G(e^{j\omega})} \rightarrow y_2[n]$

$y_1[n] \rightarrow Y_1(e^{j\omega})$ and $y_2[n] \rightarrow Y_2(e^{j\omega})$

$$Y_1(e^{j\omega}) = X(e^{j\omega L})\, H(e^{j\omega L})$$

$$Y_2(e^{j\omega}) = X(e^{j\omega L})\, G(e^{j\omega})$$

$\Rightarrow Y_1(e^{j\omega}) = Y_2(e^{j\omega})$ if $H(e^{j\omega L}) = G(e^{j\omega})$
$(y_1[n] = y_2[n])$

b) $x[n] \rightarrow \boxed{H(e^{j\omega})} \rightarrow \boxed{\downarrow M} \rightarrow y_1[n]$

$x[n] \rightarrow \boxed{\downarrow M} \rightarrow \boxed{G(e^{j\omega})} \rightarrow y_2[n]$

$$Y_1(e^{j\omega}) = \frac{1}{M} \sum_{k=0}^{M-1} X(e^{j(\frac{\omega-2\pi k}{M})})\, H(e^{j(\frac{\omega-2\pi k}{M})})$$

$$Y_2(e^{j\omega}) = \frac{1}{M} \sum_{k=0}^{M-1} X(e^{j(\frac{\omega-2\pi k}{M})})\, G(e^{j\omega})$$

If $X(e^{j\omega})$ is not bandlimited to $\frac{\pi}{M}$, aliasing occurs. Assuming it is bandlimited to $\frac{\pi}{M}$;

$$Y_1(e^{j\omega}) = \frac{1}{M} \sum_{k=0}^{M-1} X(e^{j(\frac{\omega-2\pi k}{M})}) \sum_{k=0}^{M-1} H'(e^{j(\frac{\omega-2\pi k}{M})})$$

where $H'(e^{j\omega}) = \begin{cases} H(e^{j\omega}) & \text{for } |\omega| \leq \frac{\pi}{M} \\ 0, & \text{otherwise} \end{cases}$

Thus, if $G(e^{j\omega}) = \sum_{k=0}^{M-1} H'(e^{j(\frac{\omega - 2\pi k}{M})})$

$Y_1(e^{j\omega}) = Y_2(e^{j\omega}) \Rightarrow y_1[n] = y_2[n]$

3) $$x_c(t) \longrightarrow \underset{\substack{\uparrow \\ T_1}}{\boxed{C/D}} \xrightarrow{x[n]} \underset{\substack{\uparrow \\ T_2}}{\boxed{D/C}} \longrightarrow y_c(t)$$

$X_c(j\Omega) = 0$ for $|\Omega| > \frac{\pi}{T_1}$, so after sampling

$x_c(t)$ with T_1, aliasing does not occur.

a) If $T_1 = T_2$ $\quad X_c(j\Omega) = Y_c(j\Omega) \Rightarrow x_c(t) = y_c(t)$

No, it does not hold for $X_c(j\Omega) \neq 0$ when $|\Omega| > \frac{\pi}{T_1}$ (due to aliasing)

b) For $T_1 \neq T_2$;

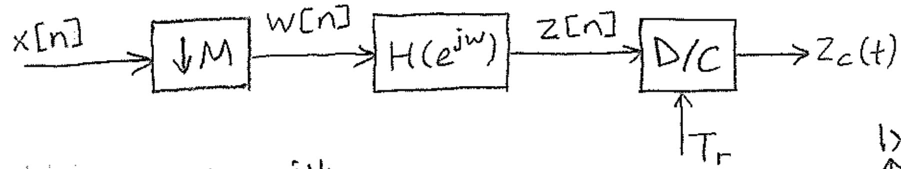
$$X(e^{j\omega}) = \frac{1}{T_1} \sum_{k=-\infty}^{\infty} X_c\left(\frac{\omega - k2\pi}{T_1}\right)$$

and $Y_c(\Omega) = T_2\, X(e^{j\Omega T_2})$ for $|\Omega| < \frac{\pi}{T_2}$

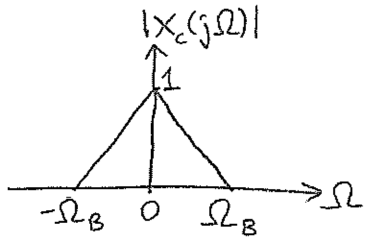
$$\Rightarrow Y_c(\Omega) = \frac{T_2}{T_1} \sum_{k=-\infty}^{\infty} X_c\left(\frac{\Omega T_2 - k2\pi}{T_1}\right) \qquad |\Omega| < \frac{\pi}{T_2}$$

$$\Rightarrow Y_c(\Omega) = \frac{T_2}{T_1} X_c\left(\frac{\Omega T_2}{T_1}\right) \Rightarrow \boxed{y_c(t) = x_c\left(t\frac{T_1}{T_2}\right)}$$

Q4



$$H(e^{jw})=\begin{cases} A\cdot e^{-jNw}, & |w|<w_c \text{ rad.} \\ 0, & \text{otherwise} \end{cases}$$



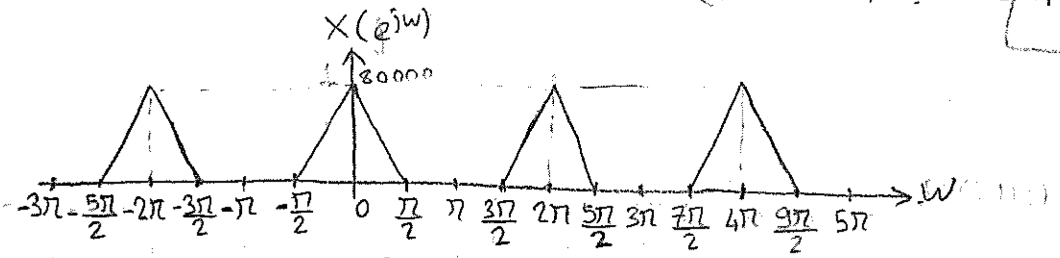
$\Omega_B = 40000\pi$

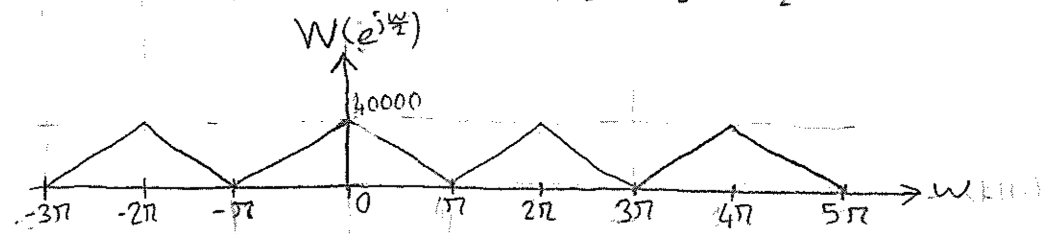
As $z_c(t) = x_c(\alpha \cdot t - \beta)$ where $\alpha = 2$, $x[n]$ should be downsampled with $\boxed{M = \alpha = 2.}$

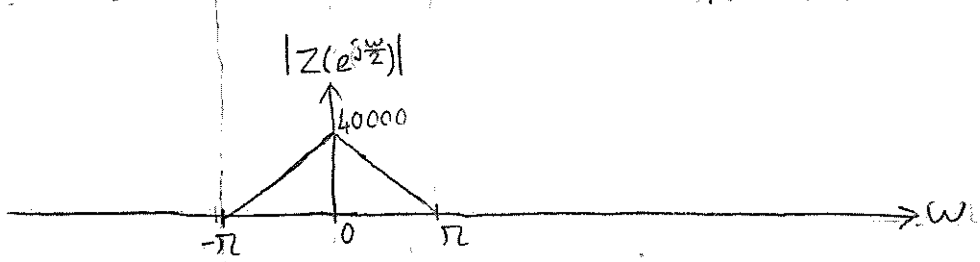
The gain of the low-pass filter is $\boxed{A=1.}$ Beside this, $\boxed{w_c = \frac{2\pi}{M} = \pi.}$ Finally, $N \cdot T_r = \beta = 2$ so $\boxed{N = \frac{2}{T_r} = 2 f_s = 160000}$

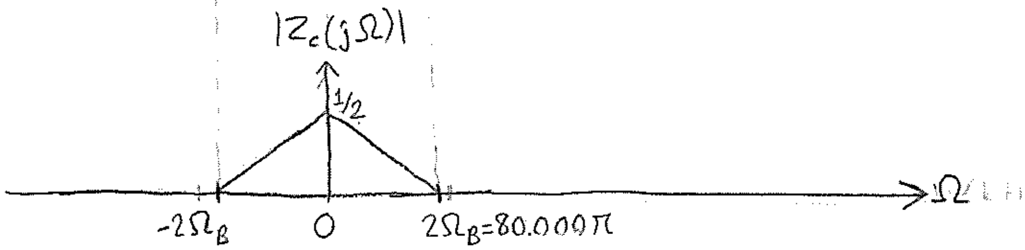
$$H(e^{jw})=\begin{cases} e^{-j160000w}, & |w|<w_c \text{ rad} \\ 0, & \text{otherwise} \end{cases}$$

Notice that $f_s = 80\ \text{KHz} \geq 2\times(\underbrace{2}_{\text{since } M=2}\times 20\ \text{KHz})$. Choose $\boxed{T_r = \frac{1}{f_s} = 12.5\,\mu s.}$



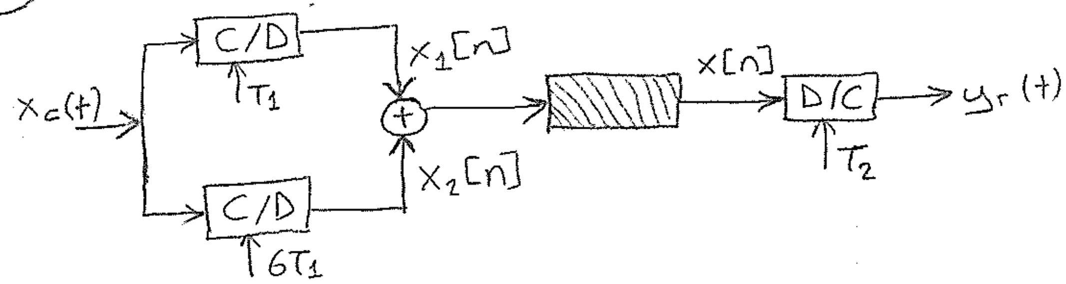




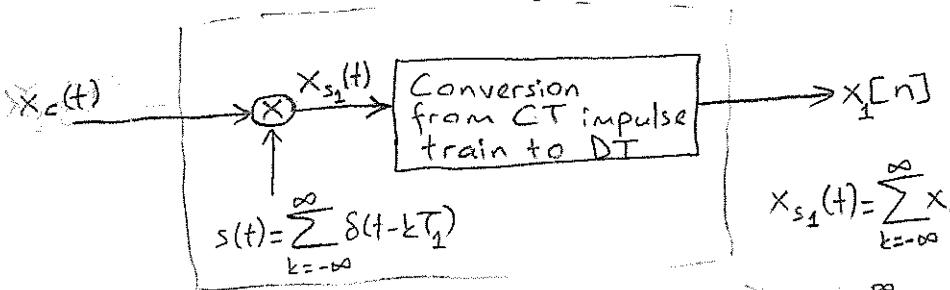


Q5

$x_c(t) = \cos(200\pi t)$ and $T_1 = 10^{-3}$



a



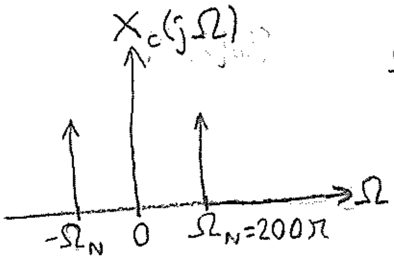
$$x_{s_1}(t)=\sum_{k=-\infty}^{\infty}x_c(kT_1)\,\delta(t-kT_1)$$

$$x_1[n]=\sum_{k=-\infty}^{\infty}\underbrace{x_c(kT_1)}\,\delta[n-k] = \sum_{k=-\infty}^{\infty}x[k]\,\delta[n-k]$$

Similarly, $x_2[n]=\sum_{k=-\infty}^{\infty}x_c(6kT_1)\,\delta[n-k]$

$$\boxed{x_2[n]=\sum_{k=-\infty}^{\infty}\cos(1.2\pi k)\cdot\delta[n-k]} \qquad \boxed{x_1[n]=\sum_{k=-\infty}^{\infty}\cos(0.2\pi k)\,\delta[n-k]}$$

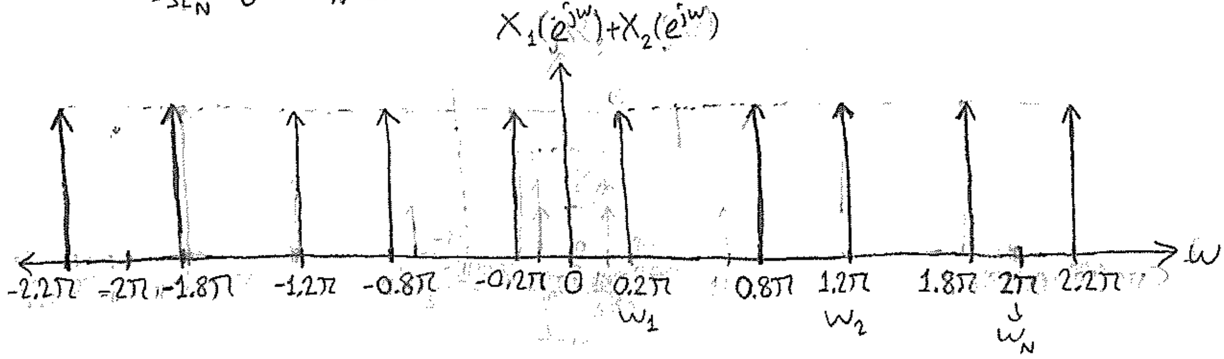
b



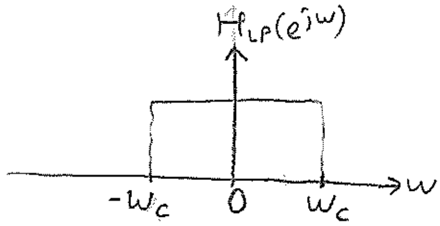
$$\Omega_{s_1}=\frac{2\pi}{T_1}=2000\pi \Rightarrow w_1=0.2\pi$$

$$\Omega_{s_2}=\frac{2\pi}{6T_1}=\frac{1000}{3}\pi \Rightarrow w_2=1.2\pi$$

(Notice that $\Omega_N=200\pi$, $w_N=2\pi$)



The striped box should include an ideal low pass filter as below and choose $T_2 = 10^{-3}$, $y_R(t)=x_c(t)$.

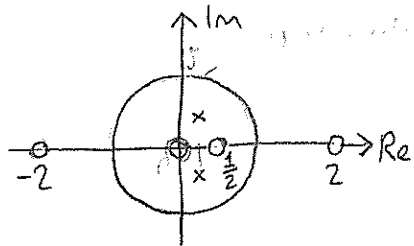


$$0.2\pi < w_c < 0.8\pi$$

Q7 (a) $H(z) = \dfrac{(1-2z^{-1})(z+2)(z-\frac{1}{2})}{\left(1-\frac{1}{2}z^{-1}+\frac{1}{2}z^{-2}\right)}$. Find $H_{min}(z)$ and $H_{ap}(z)$.

$$H(z) = \frac{\cancel{(-2)}\left(z^{-1}-\frac{1}{2}\right)2z\left(z^{-1}+\frac{1}{2}\right)\cancel{\left(-\frac{1}{2}\right)}z\left(z^{-1}-2\right)}{\Big[1-\underbrace{\left(\frac{1}{4}+j\frac{\sqrt{3}}{4}\right)}_{e^{j\frac{\pi}{3}}}z^{-1}\Big]\Big[1-\underbrace{\left(\frac{1}{4}-j\frac{\sqrt{3}}{4}\right)}_{e^{-j\frac{\pi}{3}}}z^{-1}\Big]}$$

Pole-zero plot of $H(z)$



From the pole-zero plot, it can be observed that two zeros, namely -2 and 2, are out of unit circle. This makes $H(z)$ not a min-phase system in its current form. Therefore, we design an all-pass subsystem by adding conjugate reciprocal poles for these zeros -2 and 2 as follows:

$$\boxed{H_{ap}(z) = \frac{\left(z^{-1}-\frac{1}{2}\right)\left(z^{-1}+\frac{1}{2}\right)}{\left(1-\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{2}z^{-1}\right)}}$$

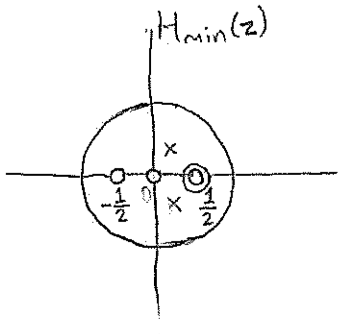
$\Longrightarrow$ zeros at -2 and 2

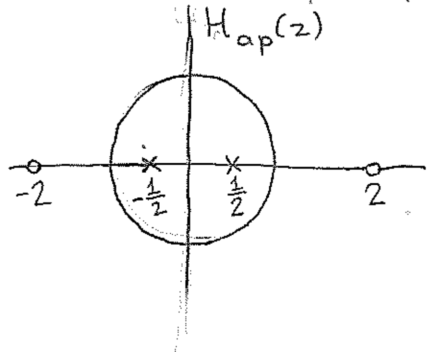
$\Longrightarrow$ The conjugate reciprocal poles.

To satisfy $H(z) = H_{min}(z) . H_{ap}(z)$, we multiply the remaining part with two zeros which can cancel the conjugate reciprocal poles. Hence, minimum phase subsystem, $H_{min}(z)$, can be obtained as follows:

$$\boxed{H_{min}(z) = \underbrace{\frac{2z^2 . (z^{-1}-2)}{\left(1-e^{j\frac{\pi}{3}}z^{-1}\right)\left(1-e^{-j\frac{\pi}{3}}z^{-1}\right)}}_{\text{The remaining part}} \cdot \underbrace{\left(1-\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{2}z^{-1}\right)}_{\text{The zeros to satisfy } H(z)=H_{min}(z).H_{ap}(z)}}$$

The pole-zero plots of min-phase and all pass systems





(b) $|H(e^{jw})|^2 = \dfrac{\frac{5}{4} - \cos(w)}{\frac{10}{9} - \frac{2}{3}\cos(w)}$

$$|H(e^{jw})|^2 = H(e^{jw})H^*(e^{jw})$$

$$|H(e^{jw})|^2 = \frac{\frac{5}{4} - \cos(w)}{\frac{10}{9} - \frac{2}{3}\cos(w)} = \frac{\overbrace{\frac{1}{2}\cdot\frac{1}{2}}^{\frac{1}{4}} + \overbrace{(e^{jw})(-e^{-jw})}^{1} + \overbrace{\left(-\frac{1}{2}e^{jw} - \frac{1}{2}e^{-jw}\right)}^{-\cos(w)}}{\underbrace{\frac{1}{3}\cdot\frac{1}{3}}_{\frac{1}{9}} + \underbrace{(e^{jw})(-e^{-jw})}_{1} + \underbrace{\left(-\frac{1}{3}e^{jw} - \frac{1}{3}e^{-jw}\right)}_{-\frac{2}{3}\cos(w)}}$$

$$H(e^{jw}) = \frac{\frac{1}{2} - e^{jw}}{\frac{1}{3} - e^{jw}} \quad \text{and} \quad H^*(e^{jw}) = \frac{\frac{1}{2} - e^{-jw}}{\frac{1}{3} - e^{-jw}}$$

By taking $z = e^{jw}$,

$$H(z) = \frac{\left(\frac{1}{2} - z\right)}{\left(\frac{1}{3} - z\right)}$$

$$H(z) \cdot H^*\left(\frac{1}{z^*}\right) = \frac{\frac{1}{2} - z}{\frac{1}{3} - z} \cdot \frac{\frac{1}{2} - z^{-1}}{\frac{1}{3} - z^{-1}}$$

$$= \underbrace{\left(\frac{1 - \frac{1}{2}z^{-1}}{1 - \frac{1}{3}z^{-1}}\right)}_{H(z)} \cdot \frac{\frac{1}{2} - z^{-1}}{\frac{1}{3} - z^{-1}}$$

Both pole and zero of $H(z)$ is inside of the unit circle. (It has a pole at $\frac{1}{3}$ and a zero at $\frac{1}{2}$). Therefore, $H(z) = H_{min}(z)$

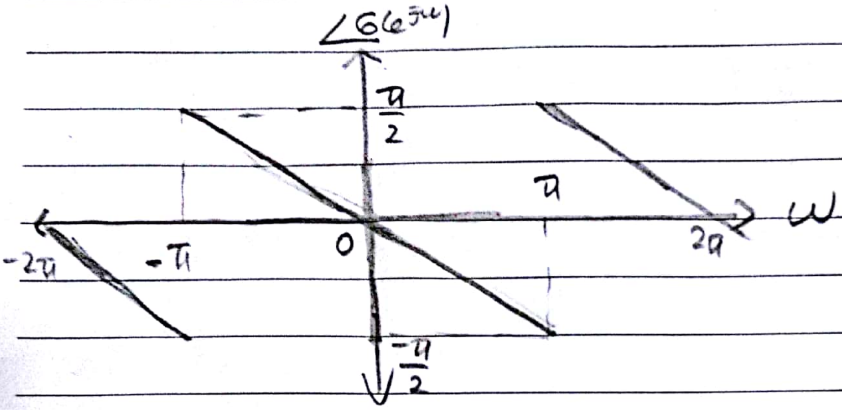
$$\boxed{H_{min}(z) = \frac{1 - \frac{1}{2}z^{-1}}{1 - \frac{1}{3}z^{-1}} \qquad H_{min}(e^{jw}) = \frac{1 - \frac{1}{2}e^{-jw}}{1 - \frac{1}{3}e^{-jw}}}$$

8) a) $G(z) = 1 + z^{-1}$ → one pole at $z=0$

→ one zero at $z=-1$

$G(e^{jw}) = 1 + e^{-jw} = e^{-\frac{jw}{2}}\left(e^{\frac{jw}{2}} + e^{-\frac{jw}{2}}\right) = 2\cos\left(\frac{w}{2}\right) e^{-\frac{jw}{2}}$

$\Rightarrow \angle G(e^{jw}) = \frac{-w}{2}$ for $-\pi < w < \pi$



b) i. At $w = \frac{\pi}{2}$ and $w = \frac{3\pi}{2}$; the discontinuities are resulted from zeros at $z=j$ and $z=-j$.

Since, only the principal value of the phase is plotted, we observe a discontinuity at $w=\pi$, as well.

ii. Group delay: $\tau(w) = \frac{-d}{dw} \angle H(e^{jw}) = 2$

$H(e^{jw}) = A(w)\, e^{-j2w}$ and $A(w) \in \mathbb{R})$

$\Rightarrow$ Generalized Linear Phase with Type-1
($h[n]$ is odd length and even symmetric)

c) It is given that there is a zero at $z=\frac{1}{2}$

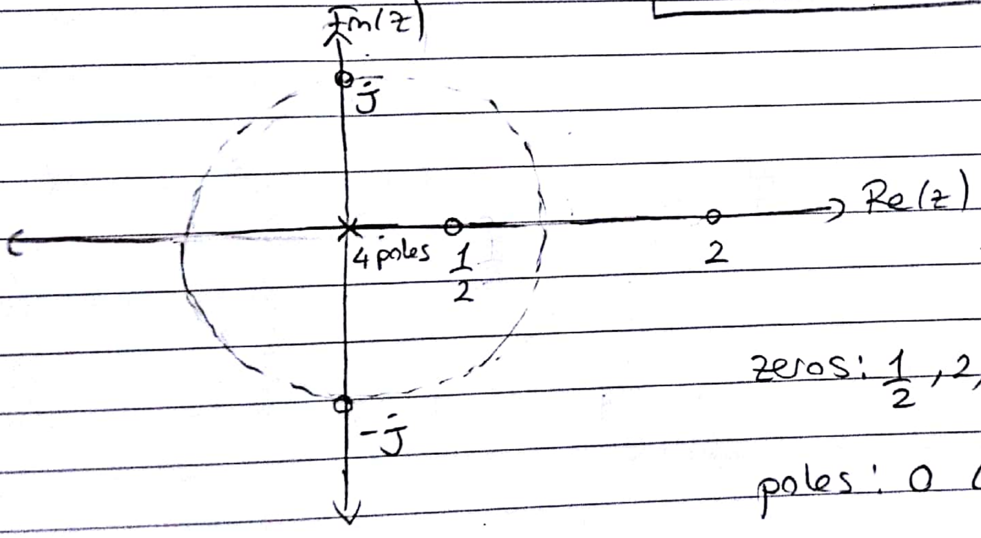
$\Rightarrow$ there is also a zero at $z=2$
(zeros are in reciprocal pairs)

We also know that $h[n]$ is causal and $h[n]=0$ for $n>4$

Then, the form of $H(z) = c\left(1-z^{-1}\frac{1}{2}\right)\left(1-z^{-1}2\right)\left(1-z^{-1}j\right)\left(1+z^{-1}j\right)$

$$\Rightarrow H(z) = c\,\frac{\left(z-\frac{1}{2}\right)(z-2)(z-j)(z+j)}{z^4} \qquad c: \text{a constant}$$

$$H(1)=1 \Rightarrow c=-1 \Rightarrow \boxed{H(z) = -\frac{\left(z-\frac{1}{2}\right)(z-2)(z-j)(z+j)}{z^4}}$$



zeros: $\frac{1}{2}, 2, j, -j$

poles: 0 (4 poles)

d) $H(z) = -1 + \frac{5}{2}z^{-1} - 2z^{-2} + \frac{5}{2}z^{-3} - z^{-4}$

$$\Rightarrow H(e^{jw}) = -1 + \frac{5}{2}e^{-jw} - 2e^{-j2w} + \frac{5}{2}e^{-j3w} - e^{-j4w}$$

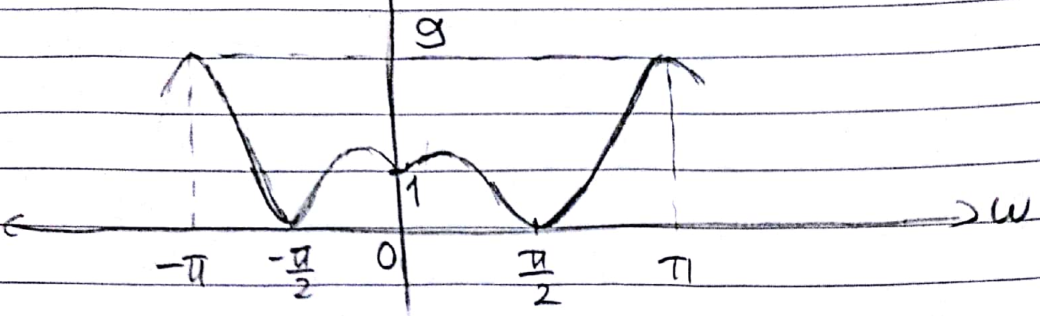
$$H(e^{jw}) = e^{-j2w}\left(-2\cos 2w + 5\cos w - 2\right)$$

$$\cos 2w = 2\cos^2 w - 1 \Rightarrow H(e^{jw}) = e^{-j2w}\cos w\,(5-4\cos w)$$

$$|H(e^{jw})| = |\cos w|\,\underbrace{(5-4\cos w)}$$

alway positive since $|\cos w| \leq 1$

$|H(e^{j\omega})| = (5-4\cos\omega)|\cos\omega|$



9) a. IIR systems: B, C, D, E

b. FIR systems: A, F

c. Stable systems: A, B, C, E, F

d. Minimum-phase systems: E

e. Linear-phase systems: A, F

f. $|H(e^{j\omega})|$ = constant for all ω : C

g. The systems with stable&causal inverse systems: E

h. Having the shortest impulse response: F

i. The systems with LP frequency response: F

j. The systems having minimum group delay: E