Lecture Notes

EE430 Digital Signal Processing

Department of Electrical and Electronics Engineering

Middle East Technical University (METU)

Contents

1 Discrete-time Signals and Systems **2**

1.1 Discrete-time (DT) signals . . . 3

1.1.1 Basic sequences and sequence operations . . . 3

1.2 Discrete-time (DT) systems . . . 7

1.2.1 Memoryless systems: . . . 7

1.2.2 Linear systems: . . . 7

1.2.3 Time-invariant systems: . . . 8

1.2.4 Causality: . . . 8

1.2.5 Stability: . . . 8

1.3 Linear time-invariant (LTI) systems . . . 9

1.3.1 Computation of convolution sum: . . . 10

1.4 Properties of convolution and LTI systems . . . 11

1.4.1 Properties of convolution . . . 11

1.4.2 Properties of LTI systems . . . 11

1.4.3 FIR and IIR systems . . . 13

1.5 Linear constant-coefficient difference equations (LCCDE) . . . 13

1.6 Frequency domain representation of DT signals and systems . . . 15

1.7 Representation of sequences by Fourier transforms . . . 17

1.8 Symmetry properties of DT Fourier transforms . . . 17

1.9 DT Fourier transform theorems . . . 17

Chapter 1

Discrete-time Signals and Systems

Contents

1.1 Discrete-time (DT) signals . . . 3
- 1.1.1 Basic sequences and sequence operations . . . 3

1.2 Discrete-time (DT) systems . . . 7
- 1.2.1 Memoryless systems: . . . 7
- 1.2.2 Linear systems: . . . 7
- 1.2.3 Time-invariant systems: . . . 8
- 1.2.4 Causality: . . . 8
- 1.2.5 Stability: . . . 8

1.3 Linear time-invariant (LTI) systems . . . 9
- 1.3.1 Computation of convolution sum: . . . 10

1.4 Properties of convolution and LTI systems . . . 11
- 1.4.1 Properties of convolution . . . 11
- 1.4.2 Properties of LTI systems . . . 11
- 1.4.3 FIR and IIR systems . . . 13

1.5 Linear constant-coefficient difference equations (LCCDE) . . . 13

1.6 Frequency domain representation of DT signals and systems . . . 15

1.7 Representation of sequences by Fourier transforms . . . 17

1.8 Symmetry properties of DT Fourier transforms . . . 17

1.9 DT Fourier transform theorems . . . 17

This chapter covers the fundamental concepts of discrete-time (DT) signals and systems. In particular, we cover Sections 2.1 to 2.9 from our textbook.

Reading assignment for this chapter :

- Sections 2.1 to 2.9 from our textbook.

1.1 Discrete-time (DT) signals

A DT signal is simply a sequence of numbers indexed by integer n.

Our notation to show a DT signal is :

A DT signal can be obtained from

- an inherently discrete event (e.g. number of students attending lecture n)
- sampling of a continuous-time (CT) signal :

1.1.1 Basic sequences and sequence operations

Unit sample sequence

Any sequence $x[n]$ can be written in terms of delayed and scaled $\delta[n]$.

For an arbitrary $x[n]$, we have :

Unit step sequence

Relations between unit step and sample sequences:

Exponential sequences

General form :

If A and α are real, $x[n]$ is real.

- $A > 0,\ 0 < \alpha < 1$: $x[n]$ decreases in time
- $A > 0,\ -1 < \alpha < 0$: $x[n]$ increases in time with alternating sign
- $|\alpha| > 1$: $x[n]$ grows in magnitude as n increases

If A and α are complex :

Complex exponentials

$$x[n] =$$

Properties :

1. Complex exponentials $Ae^{j(\omega_0+2\pi r)n}$ with frequencies $(\omega_0+2\pi r)$, $r \in \mathbb{Z}$ (e.g. ω_0, $\omega_0+2\pi$, $\omega_0+4\pi$, ...) are equivalent to each other:

2. Based on above property, when discussing complex exponentials $Ae^{j\omega_0 n}$ (or sinusoids $\cos(\omega_0 n+\phi)$), we only need to consider and interval of length 2π for frequency ω_0:

3. Complex exponentials $Ae^{j\omega_0 n}$ (or sinusoids $\cos(\omega_0 n+\phi)$) are <u>periodic only if</u> $\frac{2\pi}{\omega_0}$ is a ratio of integers, i.e.

   Remember periodicity requirement for any sequence $x[n]$:

   Ex: Are following sequences periodic ? If so, find the periods.

   $x_1[n] = \cos(n)$ $\qquad$ $x_2[n] = \cos(\frac{2\pi}{8}n)$ $\qquad$ $x_3[n] = \cos(\frac{3\pi}{8}n+\phi)$

4. (Prop.1 + Prop. 3) There are only N distinguishable frequencies for which the complex exponentials $Ae^{j\omega_0 n}$ (or sinusoids $\cos(\omega_0 n+\phi)$) are periodic with N:

5. For complex exponentials $Ae^{j\omega_0 n}$ (or sinusoids $\cos(\omega_0 n + \phi)$),
   - low frequencies are in the vicinity of $\omega_0 =$
   - high frequencies are in the vicinity of $\omega_0 =$

Rate of oscillation of complex exp. (or sinusoid) determines whether frequency is high or low:

<u>Note :</u> For CT complex exponential $x(t) = Ae^{j\phi_0 t}$, <u>none</u> of the above 5 properties hold :

1.
2.
3.
4.
5.

Transformation of independent variable n

Time shift:

Time reversal:

Note : First time shift, then time reversal $\neq$ first time reversal, then time shift

1.2 Discrete-time (DT) systems

Notation:

1.2.1 Memoryless systems:

Output $y[n]$ does not depend on past or future values of input $x[n]$.
Ex:

1.2.2 Linear systems:

The systems satisfies the following relation for any $a, b, x_1[n], x_2[n]$:

In a linear system, if input

Ex: Are these systems linear ?

1.2.3 Time-invariant systems:

Any time shift at the input causes a time shift at the output by the same amount.

Ex: Are these systems time-invariant ? (Accumulator)

Ex: Are these systems time-invariant ? (Compressor)

1.2.4 Causality:

Current output sample $y[n]$ depends only on current and past input samples $x[n], x[n-1], x[n-2], ...$
Ex:

1.2.5 Stability:

A system is stable if and only if (iff) every bounded input (i.e.) produces a bounded output (i.e.).
Ex:

1.3 Linear time-invariant (LTI) systems

LTI systems have both linearity and time-invariance properties.
LTI systems are a very important class of systems.
The output of a LTI system to an arbitrary input can be calculated by the famous convolution sum:

Hence, for any input $x[n]$ to an LTI system, output

An LTI system is completely characterized by its impulse response $h[n] = T\{\delta[n]\}$.

1.3.1 Computation of convolution sum:

Ex: $x[n] = \delta[n+2] + 2\delta[n] - \delta[n-3]$ is input to LTI system with impulse response $h[n] = 3\delta[n] + 2\delta[n-1] + \delta[n-2]$. Find output $y[n]$ using two methods.

<u>Echo method</u> : Add outputs to each weighted and delayed delta function in the input. (Useful when input has few samples.)

<u>Sliding average method</u> : Apply definition of convolution sum.

Ex: Impulse response of LTI system is $h[n] = u[n] - u[n-N]$ and input $x[n] = a^n u[n], \quad 0 < a < 1$. Find output $y[n]$.

1.4 Properties of convolution and LTI systems

1.4.1 Properties of convolution

- **Distribution** over addition:

- **Commutative** property:

- **Associative** property:

Figure 1.1:
(Figure 2.11 in textbook) (a) Parallel combination of LTI systems (b) an equivalent system.

Figure 1.2:
(Figure 2.12 in textbook) (a) Cascade combination of LTI systems (b) equivalent cascade system (c) single equivalent system.

1.4.2 Properties of LTI systems

- **Impulse response property:** An LTI system is completely characterized/specified/determined by its impulse response $h[n]$.
  $\Longrightarrow$

- **Memory property:** LTI system is memoriless $\Longleftrightarrow$

- **Causality property:** LTI system is causal $\Longleftrightarrow$

- **Stability property:** LTI system is stable $\Longleftrightarrow$
  Proof given in two steps.
  Step-1 : Sufficiency, i.e. if $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$, then LTI system is stable.

  Step-2 : Necessity, i.e. for LTI system to be stable, we must have $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$.

- **Invertibility property:** LTI system $(h[n])$ is invertible $\Longleftrightarrow$ There is another LTI system $(g[n])$ such that $h[n] * g[n] = \delta[n]$.

1.4.3 FIR and IIR systems

FIR : **F**inite (-duration) **I**mpulse **R**esponse ($h[n]$ has finite number of nonzero samples)
Ex:

IIR : **I**nfinite (-duration) **I**mpulse **R**esponse ($h[n]$ has infinite number of nonzero samples)
Ex:

1.5 Linear constant-coefficient difference equations (LCCDE)

An important subclass of LTI systems exist, where the input $x[n]$ and output $y[n]$ satisfy an LCCDE.

Ex: Accumulator :

Ex: LTI system with impulse response $h[n] = \frac{1}{n^2}u[n-1]$. System is LTI, but cannot be represented with and LCCDE.

Initial rest conditions (IRC)

An LCCDE alone does not uniquely specify a system. (i.e. there may be multiple systems satisfying the same LCCDE)

- Auxiliary conditions are required together with LCCDE to uniquely specify the system.
- Some auxiliary conditions may result in non-LTI system.
- The so-called **"initial rest" auxiliary conditions** lead to a unique LTI and causal system.
  Initial rest conditions (IRC) :

In this course, we are mostly interested in finding impulse response $h[n]$ of LTI systems (and not necessarily the output $y[n]$ for an arbitrary $x[n]$, because we can then use convolution..).
$\Longrightarrow$

We can find the desired $h[n]$ in two steps :

1. Find **homogeneous solution** from

2. Find **initial conditions** for the homogeneous solution using

Ex: System satisfies : $y[n] - 3y[n-1] - 4y[n-2] = x[n] + 2x[n]$. Find $h[n]$ under IRC.

Recursive calculation from LCCDE

The **recursive nature of LCCDE** are very powerful/useful and can also be used to calculate the output $y[n]$ recursively, under many auxiliary conditions.
Ex: (Same LCCDE) $y[n] - 3y[n-1] - 4y[n-2] = x[n] + 2x[n]$

LCCDE of FIR and IIR systems

<u>If $N = 0$</u>, in the LCCDE equation, we have

- no recursion and thus no initial/auxiliary conditions are required to compute output

- actually, the LCCDE is in the form of a convolution where

If $N > 1$, in the LCCDE equation, we have

- recursion is required to compute output
- if IRC used, system is LTI and causal and IIR due to recursion

Transform domain approaches

Transform domain approaches are best/useful for LCCDE describing LTI systems.
Ex: LCCDE : $y[n] - 3y[n-1] - 4y[n-2] = x[n] + 2x[n]$

1.6 Frequency domain representation of DT signals and systems

Consider an LTI system with impulse response $h[n]$ and input $x[n]$. The output $y[n]$ is

If $x[n] = e^{j\omega n}$ for $-\infty < n < \infty$ (i.e. complex exponential with frequency ω)

$\Longrightarrow$
- $e^{j\omega n}$ is the **eigenfunction** for all LTI systems.
- The corresponding **eigenvalue** is $H(e^{j\omega})$, also called the **frequency response** of the system.

Ex: What is the frequency response of and ideal delay system ?

It will be shown that a broad class of signals can be represented by a sum of complex exponentials

Hence, for an LTI system, the output can be easily calculated

Note that the frequency response is periodic with 2π:

Therefore, frequency response can be defined only over a range of 2π :

Note that the signals $e^{j\omega n}$ and $e^{j(\omega+2\pi)n}$ are equal and hence the system cannot differentiate between these eigenfunctions.

Ex: Input to an LTI systems is $x[n] = A\cos(\omega_0 n + \phi)$. Find output in terms of $H(e^{j\omega})$.

An important class of LTI systems, called **frequency selective filters**, have frequency response $H(e^{j\omega})$ that is unity (i.e. 1) over a range of frequencies and 0 for the remaining frequencies.

Figure 1.3:
(Figure 2.17 in textbook) Ideal lowpass filter showing (a) periodicity of frequency response and (b) one period of frequncy response.

Figure 1.4:
(Figure 2.18 in textbook) Ideal frequency selective filters (a) Highpass filter (b) Bandstop filter (c) Bandpass filter.

Ex: Moving average system: $y[n] = \frac{1}{M_1+M_2+1}\sum_{k=-M_1}^{M2} x[n-k]$. LTI ? If so, find $h[n]$ and $H(e^{j\omega})$.

Suddenly applied complex exponential inputs

This subsection (Sec. 2.6.2 in textbook) is a reading assignment. It discusses LTI system when inputs are of the form $x[n] = e^{j\omega_0 n}u[n]$ instead of $x[n] = e^{j\omega_0 n}$.

1.7 Representation of sequences by Fourier transforms

1.8 Symmetry properties of DT Fourier transforms

1.9 DT Fourier transform theorems

(WRITE YOUR PART HERE)