FILTER DESIGN

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WINDOW FUNCTIONS IN MATLAB

PARKS-MCCLELLAN OPTIMAL EQUIRIPPLE FIR FILTER DESIGN IN MATLAB (firpm)

Filter Design Problem

May be stated as “given the specifications, find $H(z)/h[n]$”.

What does finding $H(z)$ or $h[n]$ imply?

“Specifications” may be for the frequency response (magnitude, phase) and/or for the impulse response.

TRANSFORMING CT FILTERS TO DT FILTERS

1) Impulse Invariance 2) Bilinear Transformation

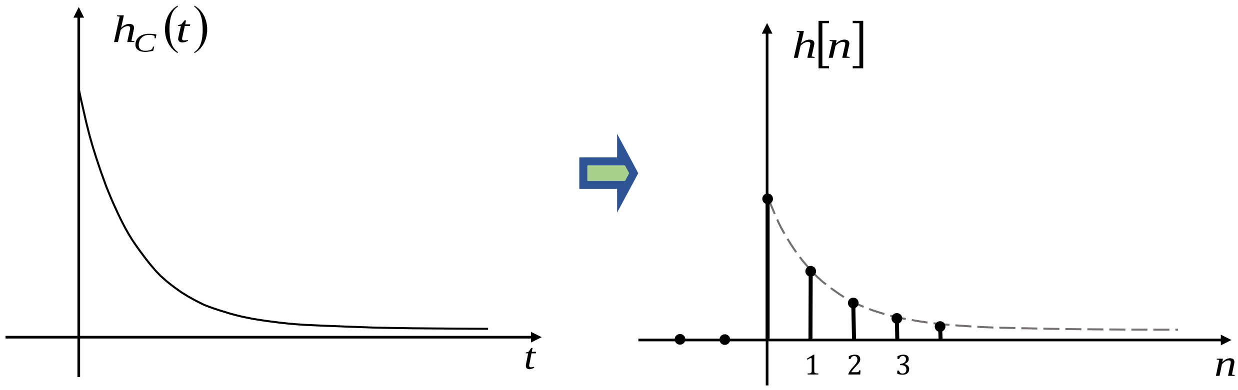
Given a CT filter with

system function : $H_c(s)$

impulse response : $h_c(t)$

Transformed filter (i.e. the DT filter) is obtained as

$$h[n] = Th_c(nT)$$



The frequency response of the DT filter so obtained is

$$H(e^{j\omega}) = T\left(\frac{1}{T}\sum_{k=-\infty}^{\infty} H_c\left(\frac{1}{T}(\omega - k2\pi)\right)\right)$$

$$= \sum_{k=-\infty}^{\infty} H_c\left(\frac{1}{T}(\omega - k2\pi)\right)$$

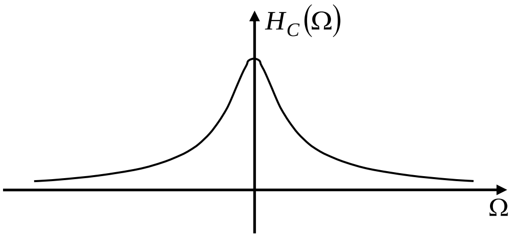
If $H_c(\Omega)$ is bandlimited, i.e.,

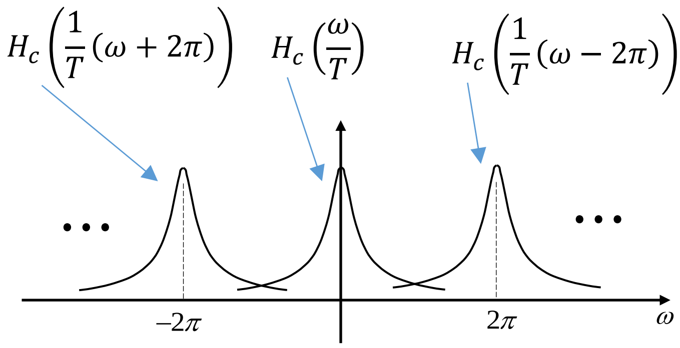
$$H_c(\Omega) = 0 \qquad\qquad |\Omega| \geq \frac{\pi}{T}$$

Then

$$H\left(e^{j\omega}\right) = H_c\left(\frac{\omega}{T}\right) \qquad\qquad |\omega| < \pi$$

In practice $H_c(\Omega)$ will not be bandlimited and $H\left(e^{j\omega}\right)$ will contain aliasing.





Therefore, once you obtain $H(e^{j\omega})$ by using *impulse invariance* method, you have to check for whether it satisfies the specifications!

If not, the design has to be reworked with modified parameter modifications.

OBTAINING $H\left(e^{j\omega}\right)$ FROM $H_C(\Omega)$ IN IMPULSE INVARIANCE

Express $H_C(s)$ in partial fractions

$$H_c(s) = \sum_{k=1}^{N} \frac{A_k}{s - s_k} \qquad (1)$$

which means

$$h_c(t) = \sum_{k=1}^{N} A_k e^{s_k t} u(t)$$

According to impulse invariance method

$$h[n] = T h_c(nT)$$

$$= T \sum_{k=1}^{N} A_k (e^{s_k T})^n u[n]$$

which yields

$$H(z) = \sum_{k=1}^{N} \frac{T A_k}{1 - e^{s_k T} z^{-1}} \qquad (2)$$

Asuming that $H_C(s)$ and T are known, the procedure is as follows:

1) Determine A_k and s_k in the partial fraction expansion of $H_c(s)$; expression (1) above.

2) Obtain $H(z)$ using expression (2) above.

3) Then the (IIR) filter can be implemented using the corresponding LCCDE.

If a CT filter has a causal-stable implementation, when it is transformed to a DT filter by using Impulse Invariance, DT filter also has a causal-stable implementation since:

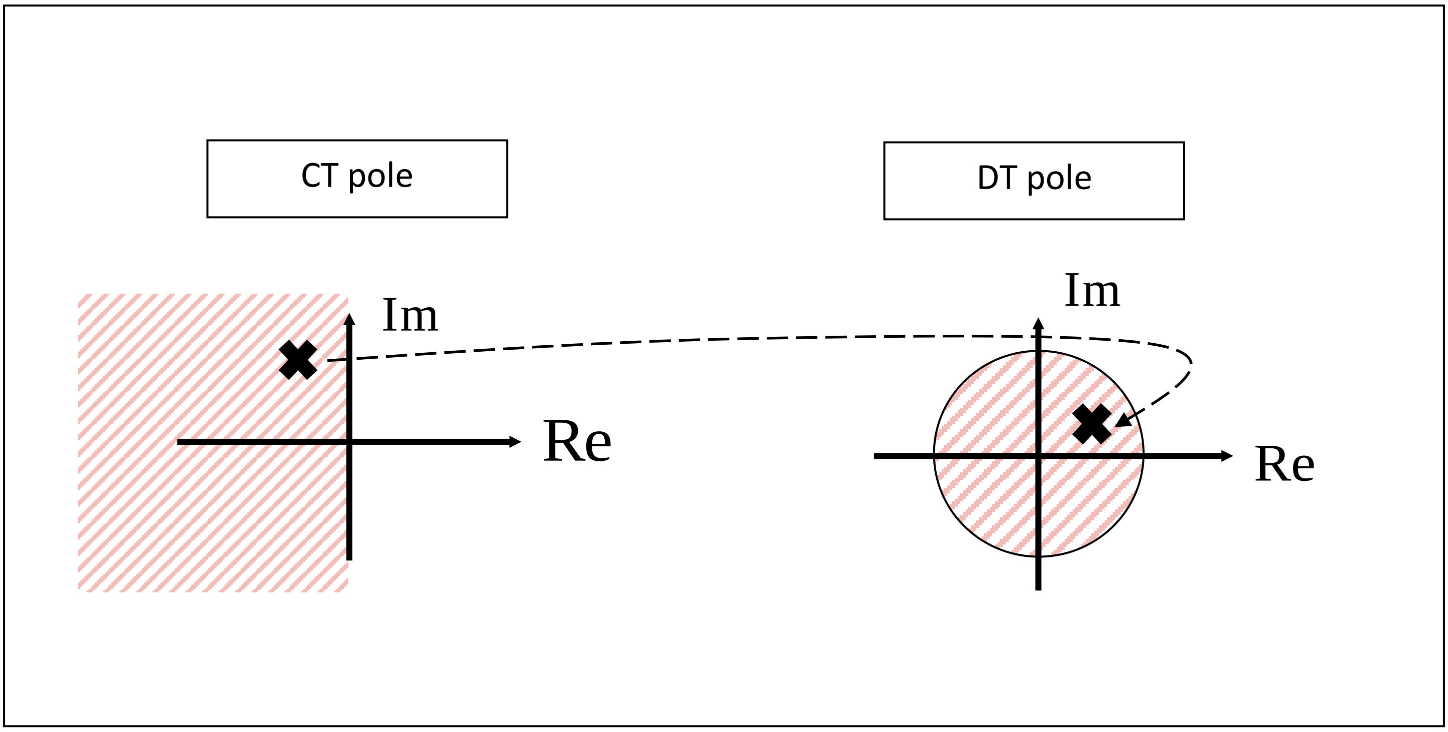
A pole of $H_C(s)$ at s_k is transformed to a pole of $H(z)$ at $e^{s_k T}$.

If $s_k = r + j\omega$,

$$\Rightarrow \quad e^{rT+j\omega T} = e^{rT} e^{j\omega T}$$

Therefore CT poles on the left half plane are transformed to DT poles inside the unit circle since

$$Re\{s_k\} = r < 0 \quad \Rightarrow \quad |e^{s_k T}| < 1$$



Poles of $H_C(s)$ on a vertical line are transformed as poles of $H(z)$ onto a circle.

Poles of $H_C(s)$ with a common real part, and imaginary parts satisfying

$$\omega_k T = \omega_l T + m2\pi, \qquad m \in \mathbb{Z}$$

become multiple poles of $H(z)$.

Note: If the design starts with the specifications stated in DT then the value of T is not of concern and it can be taken as $T = 1$ for simplicity.

The reason is due to the fact that the relationship $\Omega = \frac{\omega}{T}$ is used in twice in opposite "directions":

Once $\omega \rightarrow \Omega$ to get the CT frequency domain specifications and then (after the CT filter, $H_c(s)$, is obtained) $\Omega \rightarrow \omega$ to get $H(z)$ from $H_c(s)$.

Ex: Filter Design by *Impulse Invariance* technique.

1) Let the design specifications be given in DT as

$$0.89125 \leq \left|H\left(e^{j\omega}\right)\right| \leq 1 \qquad\qquad 0 \leq |\omega| \leq 0.2\pi$$

$$\left|H\left(e^{j\omega}\right)\right| \leq 0.17783 \qquad\qquad 0.3\pi \leq |\omega| \leq \pi$$

2) The specifications will be expressed in CT frequency domain .

For this purpose we need the value of T (sampling period) so that we can determine the CT frequency values using $\Omega = \frac{\omega}{T}$.

Therefore it is reasonable to have $T = 1$ for the sake of simplicity.

Accordingly, $\Omega = \omega$ and CT frequency domain specifications become

$$0.89125 \leq |H_c(\Omega)| \leq 1 \qquad\qquad 0 \leq |\Omega| \leq 0.2\pi$$

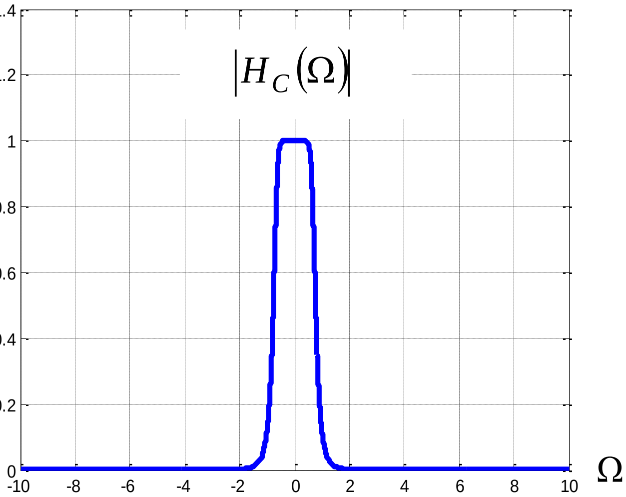
$$|H_c(\Omega)| \leq 0.17783 \qquad\qquad 0.3\pi \leq |\Omega| \leq \pi$$

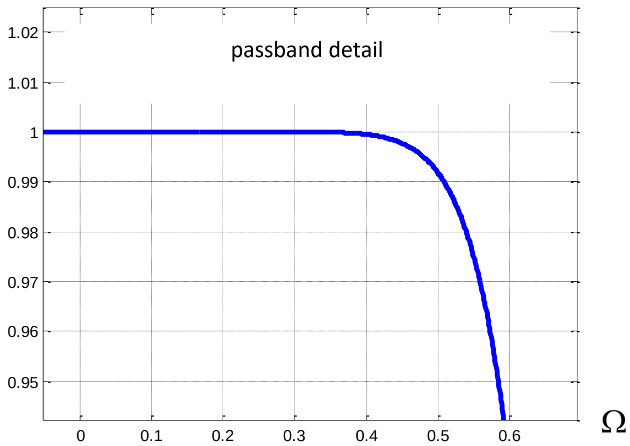
3) Now, suppose a CT Butterworth filter that satisfies the above specifications is available:

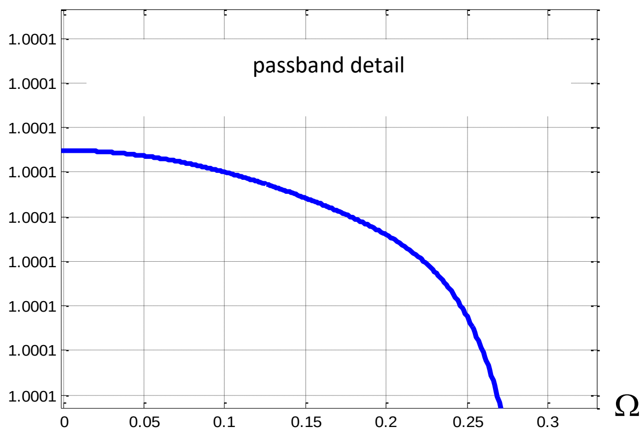
$$H_C(s)=\frac{0.12093}{\left(s^2+0.3640s+0.4945\right)\left(s^2+0.9945s+0.4945\right)\left(s^2+1.3585s+0.4945\right)}$$

Poles

$$p_{1,2}=-0.182\pm j0.679 \qquad p_{3,4}=-0.497\pm j0.497 \qquad p_{5,6}=-0.679\pm j0.182$$







4) Now, using the *impulse invariance method* obtain the DT filter

$$H_c(s) = \frac{\overbrace{0.1435 + j0.2486}^{r_1}}{s - p_1} + \frac{r_1^*}{s - p_1^*} + \frac{\overbrace{-1.0715}^{r_2}}{s - p_3} + \frac{r_2^*}{s - p_3^*}$$

$$+ \frac{\overbrace{0.9280 + j1.6074}^{r_3}}{s - p_5} + \frac{r_3^*}{s - p_5^*}$$

$$H(z) = \frac{r_1}{s - e^{p_1} z^{-1}} + \frac{r_1^*}{s - e^{p_1^*} z^{-1}} + \frac{r_2}{s - e^{p_3} z^{-1}} + \frac{r_2^*}{s - e^{p_3^*} z^{-1}}$$

$$+ \frac{r_3}{s - e^{p_5} z^{-1}} + \frac{r_3^*}{s - e^{p_5^*} z^{-1}}$$

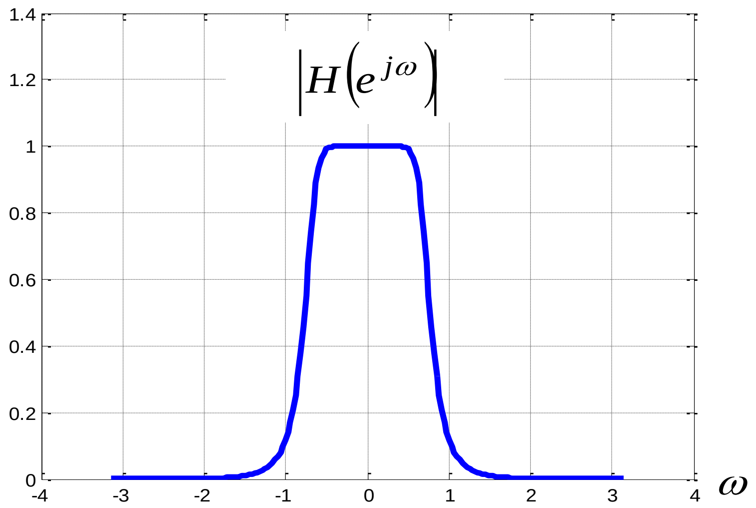
$$H(z) = \frac{(r_1 + r_1^*) - \left(r_1 e^{p_1^*} + r_1^* e^{p_1}\right) z^{-1}}{1 - \left(e^{p_1} + e^{p_1^*}\right) z^{-1} + e^{(p_1 + p_1^*)} z^{-2}}$$

$$+ \frac{(r_2 + r_2^*) - \left(r_2 e^{p_3^*} + r_2^* e^{p_3}\right) z^{-1}}{1 - \left(e^{p_3} + e^{p_3^*}\right) z^{-1} + e^{(p_3 + p_3^*)} z^{-2}}$$

$$+ \frac{(r_3 + r_3^*) - \left(r_3 e^{p_5^*} + r_3^* e^{p_5}\right) z^{-1}}{1 - \left(e^{p_5} + e^{p_5^*}\right) z^{-1} + e^{(p_5 + p_5^*)} z^{-2}}$$

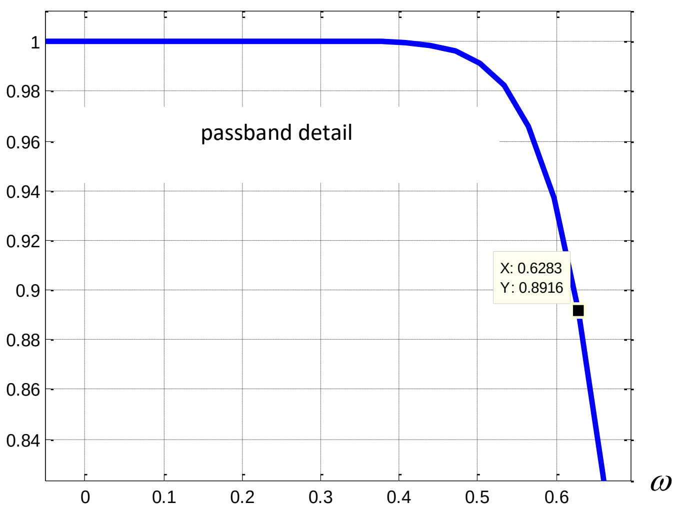
$$H(z) = \frac{0.2871 - 0.4466 z^{-1}}{1 - 1.2971 z^{-1} + 0.6949 z^{-2}} + \frac{-2.1428 + 1.1455 z^{-1}}{1 - 1.0691 z^{-1} + 0.3699 z^{-2}} + \frac{1.8557 - 0.6303 z^{-1}}{1 - 0.9972 z^{-1} + 0.2570 z^{-2}}$$

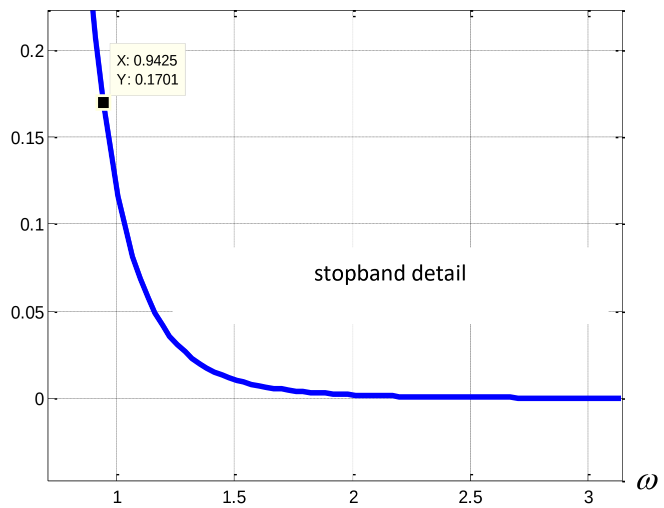
5) Since impulse invariance technique yields aliasing in the formation of $H(e^{j\omega})$ from $H_c(\Omega)$, the design has to be verified.



$$0.89125 \leq \left|H\left(e^{j\omega}\right)\right| \leq 1 \qquad 0 \leq |\omega| \leq 0.2\pi = 0.6283$$

$$\left|H\left(e^{j\omega}\right)\right| \leq 0.17783 \qquad 0.9425 = 0.3\pi \leq |\omega| \leq \pi$$





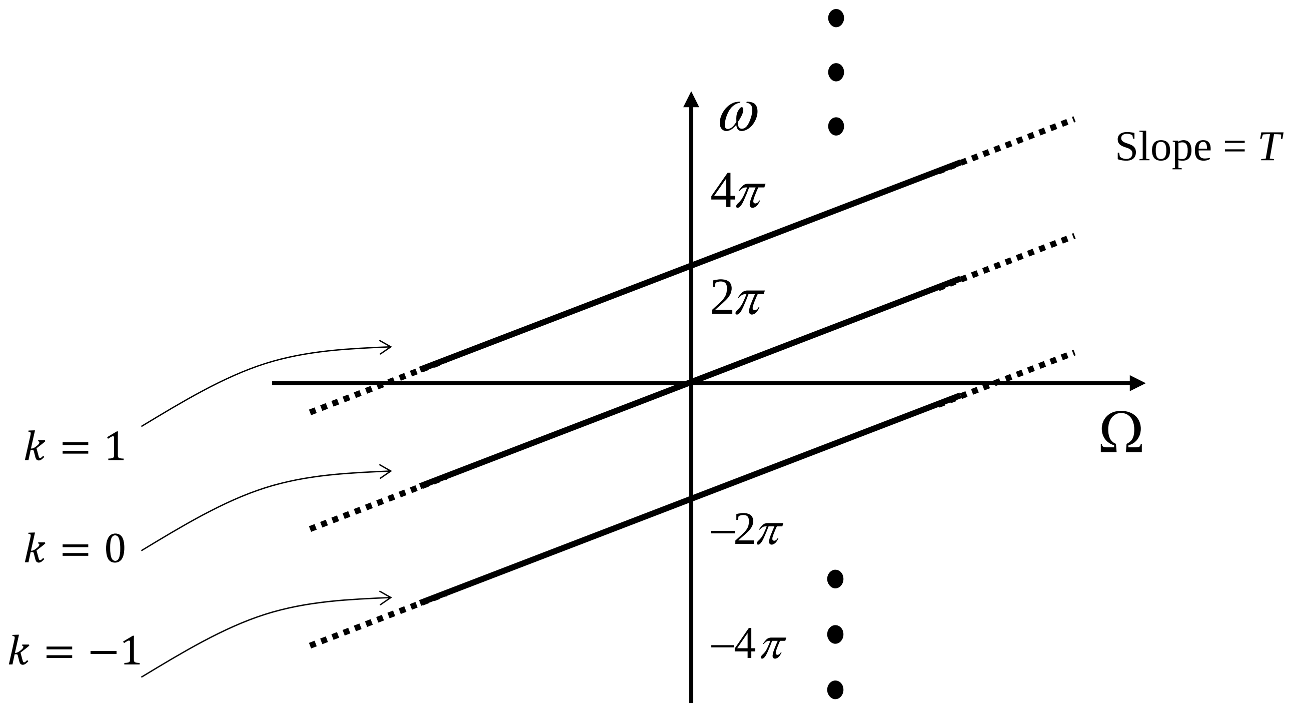
Satisfied ☺

BILINEAR TRANSFORMATION

Remember that *impulse invariance* method transforms the frequency response as,

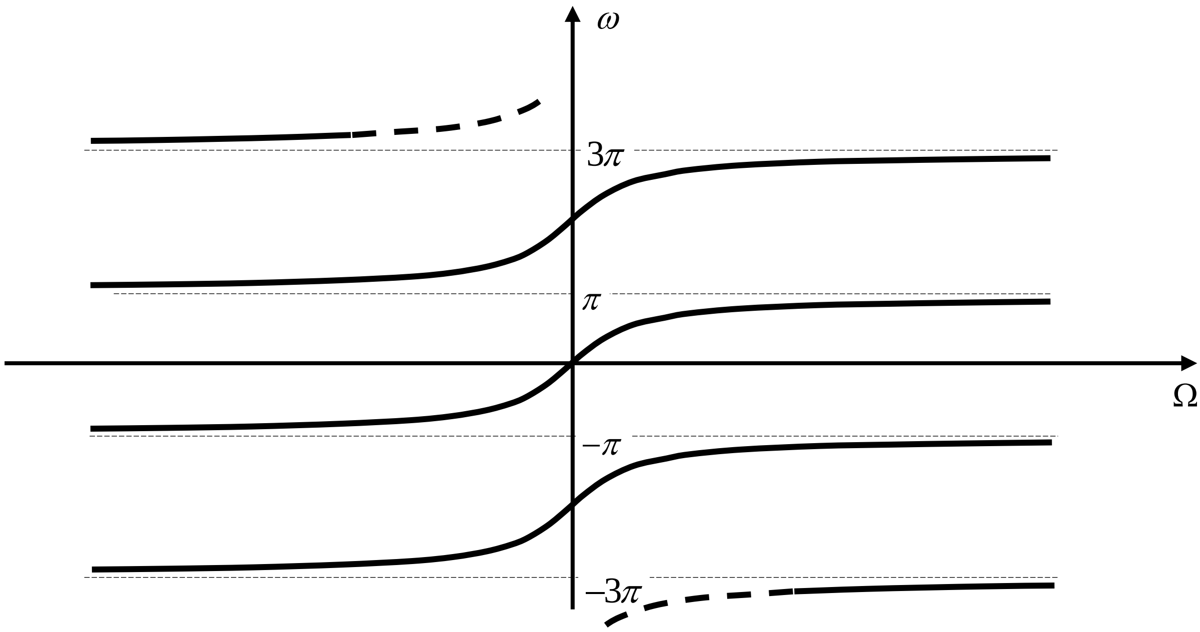
$$H\left(e^{j\omega}\right) = \sum_{k=-\infty}^{\infty} H_c\left(\frac{1}{T}(\omega - k2\pi)\right)$$

Accordingly, Ω is mapped to ω as



which causes *aliasing*.

Bilinear transformation yields a mapping from CT frequency domain to DT frequency domain as



Since $\Omega \in (-\infty, \infty)$ is mapped to finite intervals of size 2π in ω domain, bilinear transformation does not yield aliasing. However, it is a *nonlinear* mapping.

BILINEAR TRANSFORMATION

Given a CT filter, $H_C(s)$, the DT filter is obtained as

$$H(z) = H_C(s)\Big|_{s=\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}}$$

PROPERTIES OF BILINEAR TRANSFORMATION

1) “CT poles on the left half plane are transformed to DT poles inside the unit circle”

If a CT filter has a causal-stable implementation, when it is transformed to a DT filter by using Bilinear Transformation, DT filter also has a causal-stable implementation.

2) $j\Omega$ axis maps onto unit circle

To verify the properties:

Solve $s = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}$ for z,

$$z = \frac{1 + \frac{T}{2}s}{1 - \frac{T}{2}s}.$$

Substitute $s = \sigma + j\Omega$

$$z = \frac{1 + \frac{T}{2}\sigma + j\frac{T}{2}\Omega}{1 - \frac{T}{2}\sigma - j\frac{T}{2}\Omega} \qquad (***)$$

which implies *the first property*

$$|z| < 1 \text{ whenever } \sigma < 0$$

and

$$|z| > 1 \text{ whenever } \sigma > 0$$

Furthermore if $\sigma = 0$ in $(***)$

$$z = \frac{1 + j\frac{T}{2}\Omega}{1 - j\frac{T}{2}\Omega}$$

it becomes the ratio of a complex number to its conjugate and therefore $|z| = 1$ when $s = j\Omega$, which is *the second property*.

Furthermore ω and Ω can be related as

$$s = \frac{2}{T}\frac{1 - z^{-1}}{1 + z^{-1}}$$

Let $z = e^{j\omega}$

$$s = \frac{2}{T}\frac{1 - e^{-j\omega}}{1 + e^{-j\omega}}$$

$$= \frac{2}{T}\,\frac{e^{-j\frac{\omega}{2}}}{e^{-j\frac{\omega}{2}}}\,\frac{e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}}}{e^{j\frac{\omega}{2}} + e^{-j\frac{\omega}{2}}}$$

$$= j\frac{2}{T}\tan\left(\frac{\omega}{2}\right)$$

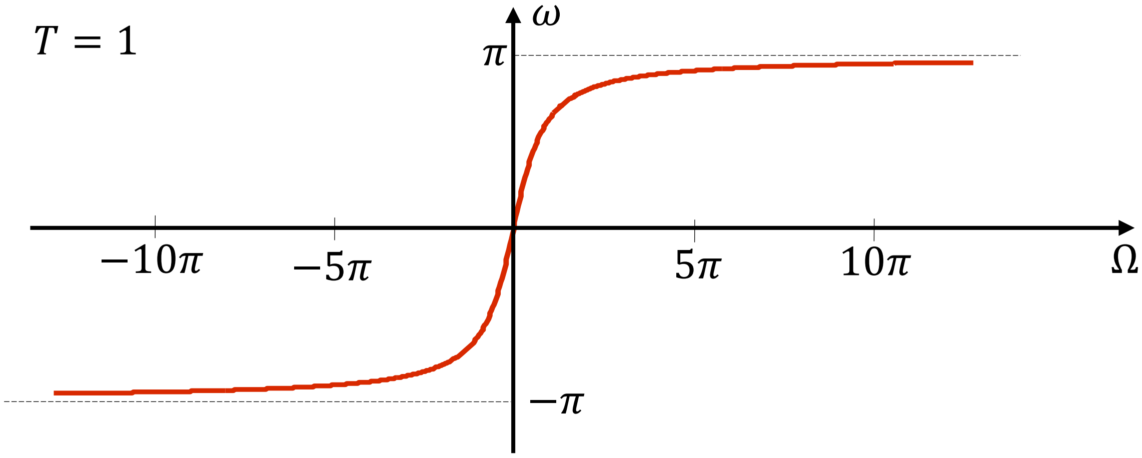
$$\Rightarrow$$

$$\Omega = \frac{2}{T}\tan\left(\frac{\omega}{2}\right)$$

or

$$\omega = 2\arctan\left(\frac{\Omega T}{2}\right)$$

$$\omega = 2\arctan\left(\frac{\Omega T}{2}\right)$$



BILINEAR TRANSFORMATION PROCEDURE

Assuming that the filter specifications are given in DT, (in such a case the value of T is not of concern, it can be taken as $T = 1$)

1) Use $\Omega = \frac{2}{T}\tan\left(\frac{\omega}{2}\right)$ to get the specifications in CT.

2) "Design the CT filter, i.e., find $H_c(s)$.

3) Obtain $H(z)$ from $H_c(s)$ using $s = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}$, i.e.,

$$H(z) = H_c\left(\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}\right)$$

Note: You do not need to verify the specifications for $H(e^{j\omega})$ since bilinear transformation does not cause aliasing.

However, doing so is a right engineering attitude!

Ex: Filter Design by *Bilinear Transformation*.

1) Let the design specifications be given in DT as (the specifications are the same as those in the previous impulse invariance example)

$$0.89125 \le \left|H\left(e^{j\omega}\right)\right| \le 1 \qquad 0 \le |\omega| \le 0.2\pi$$

$$\left|H\left(e^{j\omega}\right)\right| \le 0.17783 \qquad 0.3\pi \le |\omega| \le \pi$$

2) The specifications will be expressed in CT frequency domain using

$$\Omega = \frac{2}{T}\tan\left(\frac{\omega}{2}\right)$$

$$0.89125 \le |H_c(\Omega)| \le 1 \qquad 0 \le |\Omega| \le \frac{2}{T}\tan\left(\frac{0.2\pi}{2}\right)$$

$$|H_c(\Omega)| \le 0.17783 \qquad \frac{2}{T}\tan\left(\frac{0.3\pi}{2}\right) \le |\Omega| \le \infty$$

Here ,we take $T = 1$ as explained before.

3) Now, a CT Butterworth filter that satisfies the above specifications is available:

$$H_C(s) = \frac{0.20238}{\left(s^2 + 0.3996s + 0.5871\right)\left(s^2 + 1.0836s + 0.5871\right)\left(s^2 + 1.4802s + 0.5871\right)}$$

4) Now, using the *bilinear transformation* obtain the DT filter

$$H(z) = H_C\left(2\frac{1 - z^{-1}}{1 + z^{-1}}\right)$$

$$= \frac{0.0007378\left(1 + z^{-1}\right)^6}{\left(1 - 1.2686z^{-1} + 0.7051z^{-2}\right)\left(1 - 1.0106z^{-1} + 0.3583z^{-2}\right)\left(1 - 0.9044z^{-1} + 0.2155z^{-2}\right)}$$

BUTTERWORTH FILTERS

By Stephen Butterworth (1885–1958), a British physicist.

- Maximally flat in the passband, i.e., for an N^{th} order filter, the first $2N-1$ derivatives of squared magnitude response at $\Omega = 0$ are zero.

- Magnitude response is monotonic everywhere.

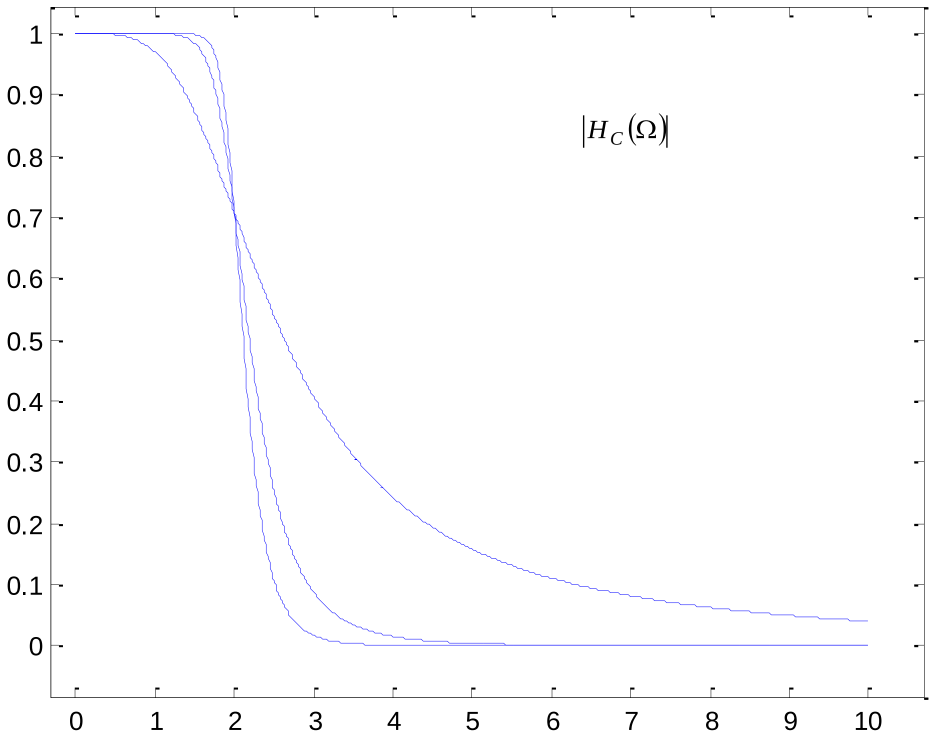
- Squared magnitude is

$$|H_c(\Omega)|^2 = \frac{1}{1 + \left(\frac{j\Omega}{j\Omega_c}\right)^{2N}}$$

Magnitude plots

$$\Omega_C = 2$$

2^{nd} , 6^{th}, 10^{th} orders



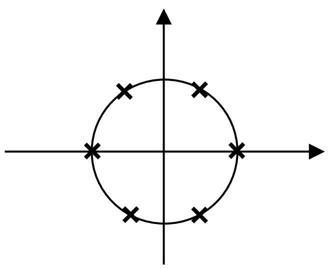
POLE LOCATIONS

$$H_c(s)H_c(-s)|_{s=j\Omega} = |H_c(\Omega)|^2$$

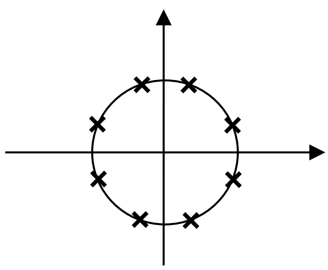
Poles of $H_c(s)H_c(-s)$

- Located on a circle of radius Ω_c
- Uniformly spaced
- Symmetrical wrt real/imaginary axis.
- No poles on imaginary axis.

Odd N



Even N



Proof (pole locations):

Note that

$$H_c(s)H_c(-s)|_{s=j\Omega} = |H_c(\Omega)|^2$$

$$= \frac{1}{1+\left(\frac{j\Omega}{j\Omega_c}\right)^{2N}}$$

Therefore

$$H_c(s)H_c(-s) = \frac{1}{1+\left(\frac{s}{j\Omega_c}\right)^{2N}}$$

Then, set

$$1+\left(\frac{s}{j\Omega_c}\right)^{2N} = 0$$

$$\left(\frac{s}{j\Omega_c}\right)^{2N} = e^{j(2k-1)\pi}$$

$$\Rightarrow \qquad s = j\Omega_c e^{j\frac{(2k-1)\pi}{2N}}$$

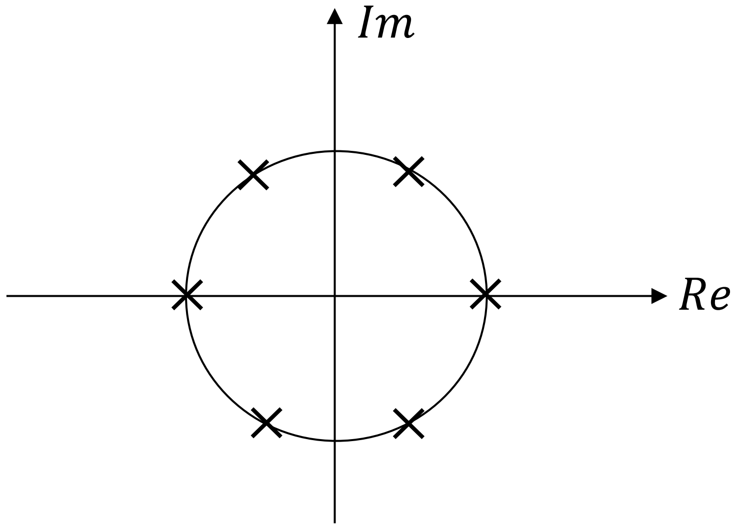
$$= \Omega_c\, e^{j\frac{\pi}{2}}\, e^{j\frac{(2k-1)\pi}{2N}}$$

$$= \Omega_c\ e^{j\frac{(2k-1+N)\pi}{2N}} \qquad k = 1,2,\ldots,2N$$

$N = 3$

$$\left\{\Omega_c e^{j\frac{4\pi}{6}}, \Omega_c e^{j\frac{6\pi}{6}}, \Omega_c e^{j\frac{8\pi}{6}}, \Omega_c e^{j\frac{10\pi}{6}}, \Omega_c e^{j\frac{12\pi}{6}}, \Omega_c e^{j\frac{14\pi}{6}}\right\} =$$

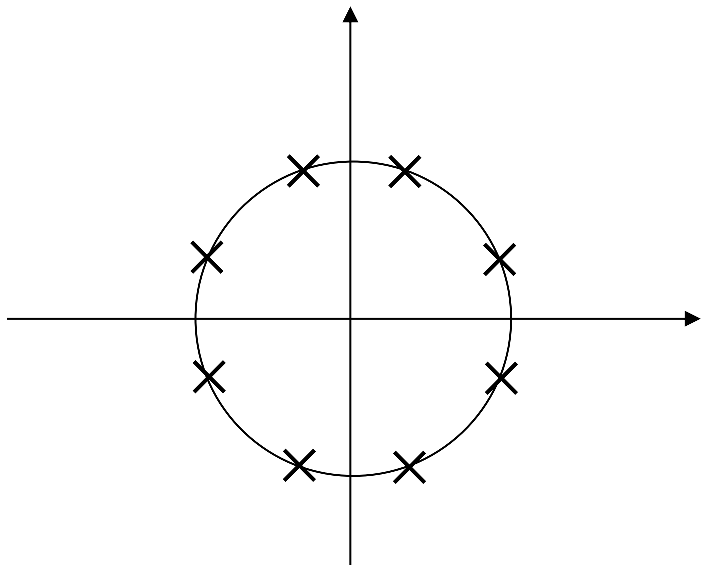
$$\left\{\Omega_c e^{j\frac{\pi}{3}}, \Omega_c e^{j\frac{2\pi}{3}}, \Omega_c e^{j\pi}, \Omega_c e^{j\frac{4\pi}{3}}, \Omega_c e^{j\frac{5\pi}{3}}, \Omega_c e^{j2\pi}\right\}$$



$N = 4$

$$\left\{\Omega_c e^{j\frac{5\pi}{8}}, \Omega_c e^{j\frac{7\pi}{8}}, \Omega_c e^{j\frac{9\pi}{8}}, \Omega_c e^{j\frac{11\pi}{8}}, \Omega_c e^{j\frac{13\pi}{8}}, \Omega_c e^{j\frac{15\pi}{8}}, \Omega_c e^{j\frac{17\pi}{8}}, \Omega_c e^{j\frac{19\pi}{8}}\right\} =$$

$$\left\{\Omega_c e^{j\frac{\pi}{8}}, \Omega_c e^{j\frac{3\pi}{8}}, \Omega_c e^{j\frac{5\pi}{8}}, \Omega_c e^{j\frac{7\pi}{8}}, \Omega_c e^{j\frac{9\pi}{8}}, \Omega_c e^{j\frac{11\pi}{8}}, \Omega_c e^{j\frac{13\pi}{8}}, \Omega_c e^{j\frac{15\pi}{8}}\right\}$$



BUTTERWORTH DESIGN

BY IMPULSE INVARIANCE

Let the specifications be given in discrete-time frequency domain

$$0.89125 \leq \left|H\left(e^{j\omega}\right)\right| \leq 1 \qquad\qquad 0 \leq |\omega| \leq 0.2\pi$$

$$\left|H\left(e^{j\omega}\right)\right| \leq 0.17783 \qquad\qquad 0.3\pi \leq |\omega| \leq \pi$$

Take $T = 1$ since specifications are given in DT frequency domain

$\Omega = \omega$ (This step is impulse invariance specific)

$$0.89125 \leq |H_c(\Omega)| \leq 1 \qquad\qquad 0 \leq |\Omega| \leq 0.2\pi$$

$$|H_c(\Omega)| \leq 0.17783 \qquad\qquad 0.3\pi \leq |\Omega| \leq \pi$$

1) Find N and Ω_c

$$\left.\begin{aligned} 1+\left(\frac{0.2\pi}{\Omega_C}\right)^{2N} &= \left(\frac{1}{0.89125}\right)^2 \\ 1+\left(\frac{0.3\pi}{\Omega_C}\right)^{2N} &= \left(\frac{1}{0.1783}\right)^2 \end{aligned}\right\} \Rightarrow N = 5.8858 \qquad \Omega_C = 0.70474$$

N must be integer $\quad \Rightarrow N = 6$

Specifications will be exceeded at the stopband and passband edges.

Recalculate Ω_C so as to provide maximum gap at the stopband edge against aliasing (*since impulse invariance is used*)

$$1+\left(\frac{0.2\pi}{\Omega_C}\right)^{12} = \left(\frac{1}{0.89125}\right)^2 \;\Rightarrow\; \Omega_C = 0.7032$$

2) Select the poles and form $H_c(s)$.

The set of poles of $H_c(s)H_c(-s)$;

$$\left\{\begin{matrix} 0.7032e^{j\frac{\pi}{12}}, 0.7032e^{j\frac{3\pi}{12}}, 0.7032e^{j\frac{5\pi}{12}}, 0.7032e^{j\frac{7\pi}{12}}, 0.7032e^{j\frac{9\pi}{12}}, 0.7032e^{j\frac{11\pi}{12}}, \\ 0.7032e^{j\frac{13\pi}{12}}, 0.7032e^{j\frac{15\pi}{12}}, 0.7032e^{j\frac{17\pi}{12}}, 0.7032e^{j\frac{19\pi}{12}}, 0.7032e^{j\frac{21\pi}{12}}, 0.7032e^{j\frac{23\pi}{12}} \end{matrix}\right\}$$

Select those in the left half plane, i.e.,

$$s_1 = 0.7032e^{j\frac{7\pi}{12}}, \; s_1^* = 0.7032e^{j\frac{17\pi}{12}}$$

$$s_2 = 0.7032e^{j\frac{9\pi}{12}}, \; s_2^* = 0.7032e^{j\frac{15\pi}{12}}$$

$$s_3 = 0.7032e^{j\frac{11\pi}{12}}, \; s_3^* = 0.7032e^{j\frac{13\pi}{12}}$$

Then

$$H_C(s) = \frac{A}{(s-s_1)(s-s_1^*)\ (s-s_2)(s-s_2^*)(s-s_3)(s-s_3^*)}$$

$$H_C(s) = \frac{A}{(s^2+0.3640s+04945)(s^2+0.9945s+04945)(s^2+1.3585s+04945)}$$

and A is calculated to make $H_c(0) = 1$ $\Rightarrow A = 0.12093.$

Butterworth Design

BY BILINEAR TRANSFORMATION

Given the specifications in discrete-time frequency domain

$$0.89125 \le \left|H\left(e^{j\omega}\right)\right| \le 1 \qquad 0 \le |\omega| \le 0.2\pi$$

$$\left|H\left(e^{j\omega}\right)\right| \le 0.17783 \qquad 0.3\pi \le |\omega| \le \pi$$

Let $T = 1$ since specifications are given in DT frequqency domain

$$\Omega = \frac{2}{T}\tan\left(\frac{\omega}{2}\right)$$

(This step is specific to bilinear transformation)

$$0.89125 \le |H_c(\Omega)| \le 1 \qquad 0 \le |\Omega| \le \frac{2}{T}\tan\left(\frac{0.2\pi}{2}\right)$$

$$|H_c(\Omega)| \le 0.17783 \qquad \frac{2}{T}\tan\left(\frac{0.3\pi}{2}\right) \le |\Omega| \le \infty$$

1) Find N and Ω_C

$$\left.\begin{aligned} 1+\left(\frac{2\tan(0.1\pi)}{\Omega_C}\right)^{2N} &= \left(\frac{1}{0.89125}\right)^2 \\ 1+\left(\frac{2\tan(0.15\pi)}{\Omega_C}\right)^{2N} &= \left(\frac{1}{0.1783}\right)^2 \end{aligned}\right\} \Rightarrow N = 5.2871 \qquad \Omega_C = 0.7375$$

N must be integer $\Rightarrow N = 6$

Specifications will be exceeded at the stopband and passband edges.

Recalculate Ω_C according to your specific needs, for example

$$1+\left(\frac{2\tan(0.15\pi)}{\Omega_C}\right)^{12} = \left(\frac{1}{0.1783}\right)^2 \;\Rightarrow\; \Omega_C = 0.766$$

2) Select the poles and form $H_c(s)$.

From the complete set

$$0.766e^{j\frac{\pi}{12}}, 0.766e^{j\frac{3\pi}{12}}, 0.766e^{j\frac{5\pi}{12}}, 0.766e^{j\frac{7\pi}{12}}, 0.766e^{j\frac{9\pi}{12}}, 0.766e^{j\frac{11\pi}{12}},$$

$$0.766e^{j\frac{13\pi}{12}}, 0.766e^{j\frac{15\pi}{12}}, 0.766e^{j\frac{17\pi}{12}}, 0.766e^{j\frac{19\pi}{12}}, 0.766e^{j\frac{21\pi}{12}}, 0.766e^{j\frac{23\pi}{12}}$$

We select those in the left half plane, i.e.

$$s_1 = 0.766e^{j\frac{7\pi}{12}},\ s_1^* = 0.766e^{j\frac{17\pi}{12}}$$

$$s_2 = 0.766e^{j\frac{9\pi}{12}},\ s_2^* = 0.766e^{j\frac{15\pi}{12}}$$

$$s_3 = 0.766e^{j\frac{11\pi}{12}},\ s_3^* = 0.766e^{j\frac{13\pi}{12}}$$

Then

$$H_C(s) = \frac{A}{(s-s_1)(s-s_1^*)\ (s-s_2)(s-s_2^*)(s-s_3)(s-s_3^*)}$$

$$H_C(s) = \frac{A}{(s^2+0.3996s+0.5871)(s^2+1.0836s+0.5871)(s^2+1.4802s+0.5871)}$$

and A is found to make $H_c(0) = 1$, $A = 0.20238$.

CHEBYCHEV FILTERS

Pafnuty Lvovich Chebyshev (Russian: Пафну́тий Льво́вич Чебышёв, (May 16 1821 – December 8 1894) is a Russian mathematician. His name can be alternatively transliterated as Chebychev, Chebysheff, Chebyshov, Tchebychev or Tchebycheff, or Tschebyschev or Tschebyscheff.

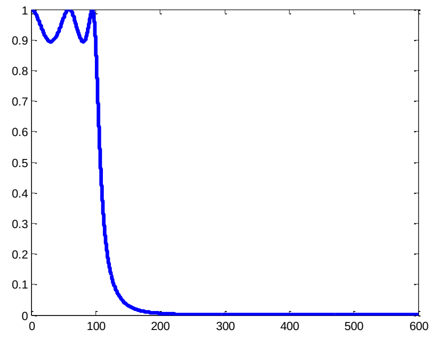
Chebychev filters are described by three parameters:

$$\left.\begin{array}{l} N \\ \Omega_c \end{array}\right\} \textit{roles similar to those in Butterworth}$$

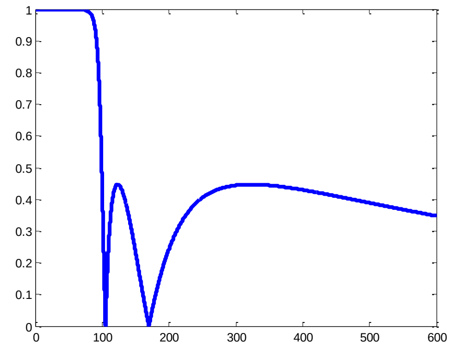
$$\varepsilon \qquad \textit{ripple size}$$

$N = 5 \qquad \varepsilon = 0.5$

Type-I

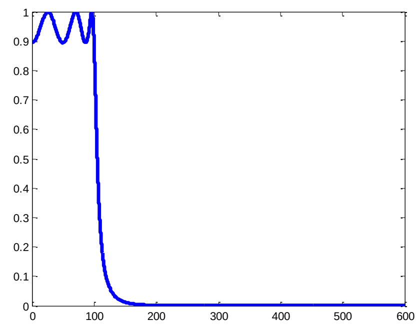


Type-II

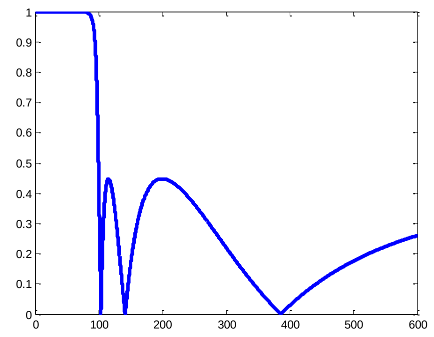


$N = 6 \qquad \varepsilon = 0.5$

Type-I



Type-II



Type-I

$$\left|H\left(j\Omega\right)\right|^2 = \frac{1}{1+\varepsilon^2 V_N^2\left(\dfrac{\Omega}{\Omega_C}\right)}$$

Type-II

$$\left|H\left(j\Omega\right)\right|^2 = \frac{1}{1+\dfrac{1}{\varepsilon^2 V_N^2\left(\dfrac{\Omega_C}{\Omega}\right)}}$$

$V_N(x) = \cos(N\cos^{-1}(x))$: N^{th} order Chebychev polynomial

$$V_N(x) = 2xV_{N-1}(x) - V_{N-2}(x)$$

$$\begin{aligned}
V_0(x) &= 1 \\
V_1(x) &= x \\
V_2(x) &= \cos\left(2\cos^{-1} x\right) \\
&= \cos^2\left(\cos^{-1} x\right) - \sin^2\left(\cos^{-1} x\right) \\
&= 2\cos^2\left(\cos^{-1} x\right) - 1 \\
&= 2x^2 - 1 \\
V_3(x) &= 4x^3 - 3x \\
V_4(x) &= 8x^4 - 8x^2 - 1
\end{aligned}$$

POLES OF TYPE I

Poles are located on an ellipse.

Minor axis length : $2a\Omega_c$

Major axis length : $2b\Omega_c$

$$a = \frac{1}{2}\left(\alpha^{\frac{1}{N}} - \alpha^{-\frac{1}{N}}\right)$$

$$b = \frac{1}{2}\left(\alpha^{\frac{1}{N}} + \alpha^{-\frac{1}{N}}\right)$$

$$\alpha = \varepsilon^{-1} + \sqrt{1 + \varepsilon^{-2}}$$

To locate the poles:

Points equally spaced by $\frac{\pi}{N}$ radians on the major and minor circles.

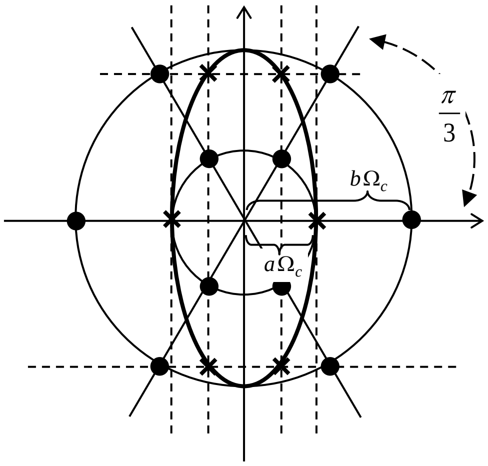
Horizontal lines through major-circle points.

Vertical lines through minor-circle points.

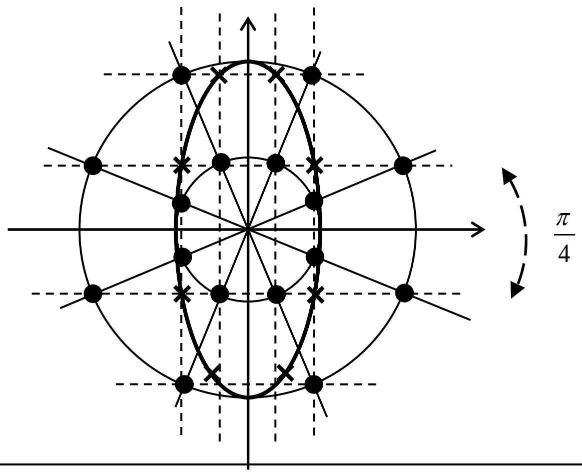
Poles are located at the intersections.

Poles never on imaginary axis.

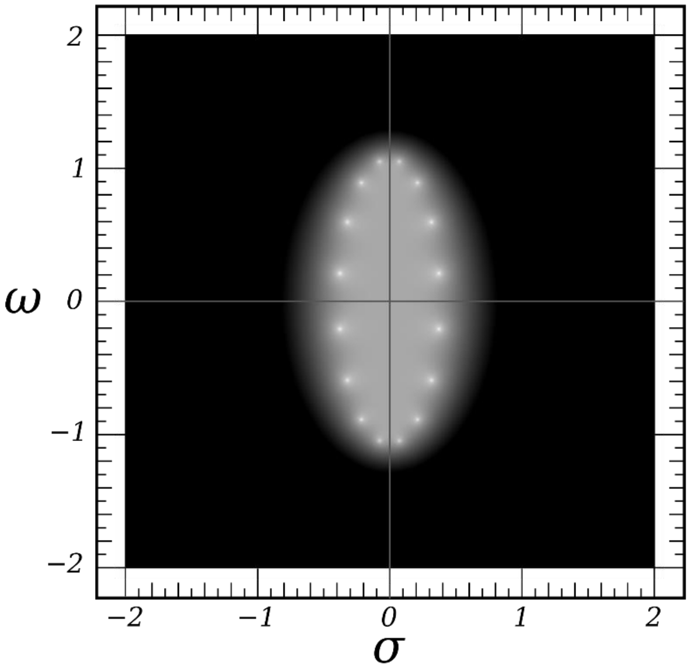
Ex: $N = 3$



Ex: $N = 4$

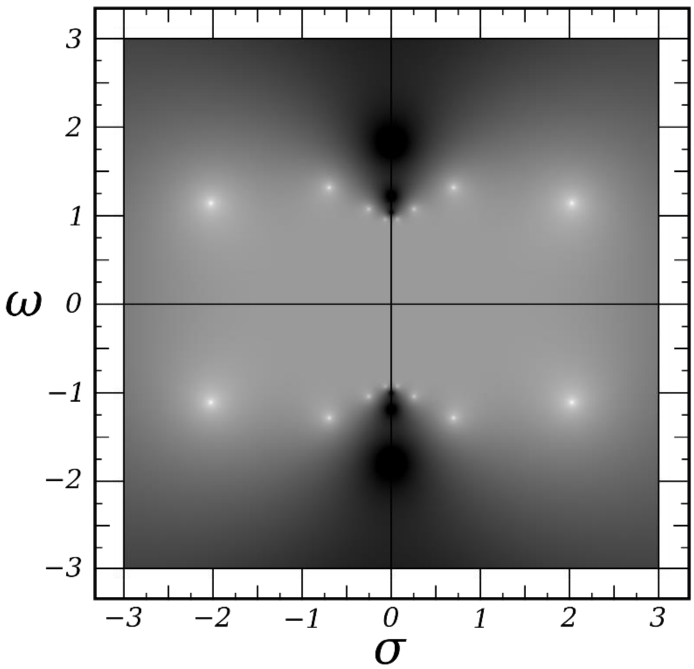


Type I



Picture: Wikipedia

Type II



Picture: Wikipedia

1) Find ε.

Passband response varies between $\frac{1}{\sqrt{1+\varepsilon^2}}$ and 1.

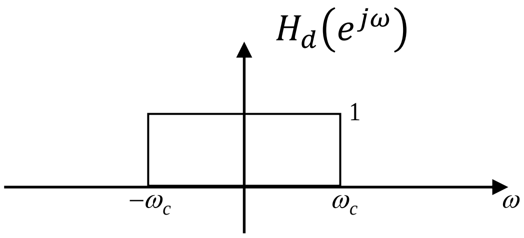
2) Find Ω_c .

Passband edge is Ω_c.

3) Find N to satisfy the stopband edge.

FIR FILTER DESIGN BY WINDOWING

Consider the magnitude response of an ideal lowpass filter

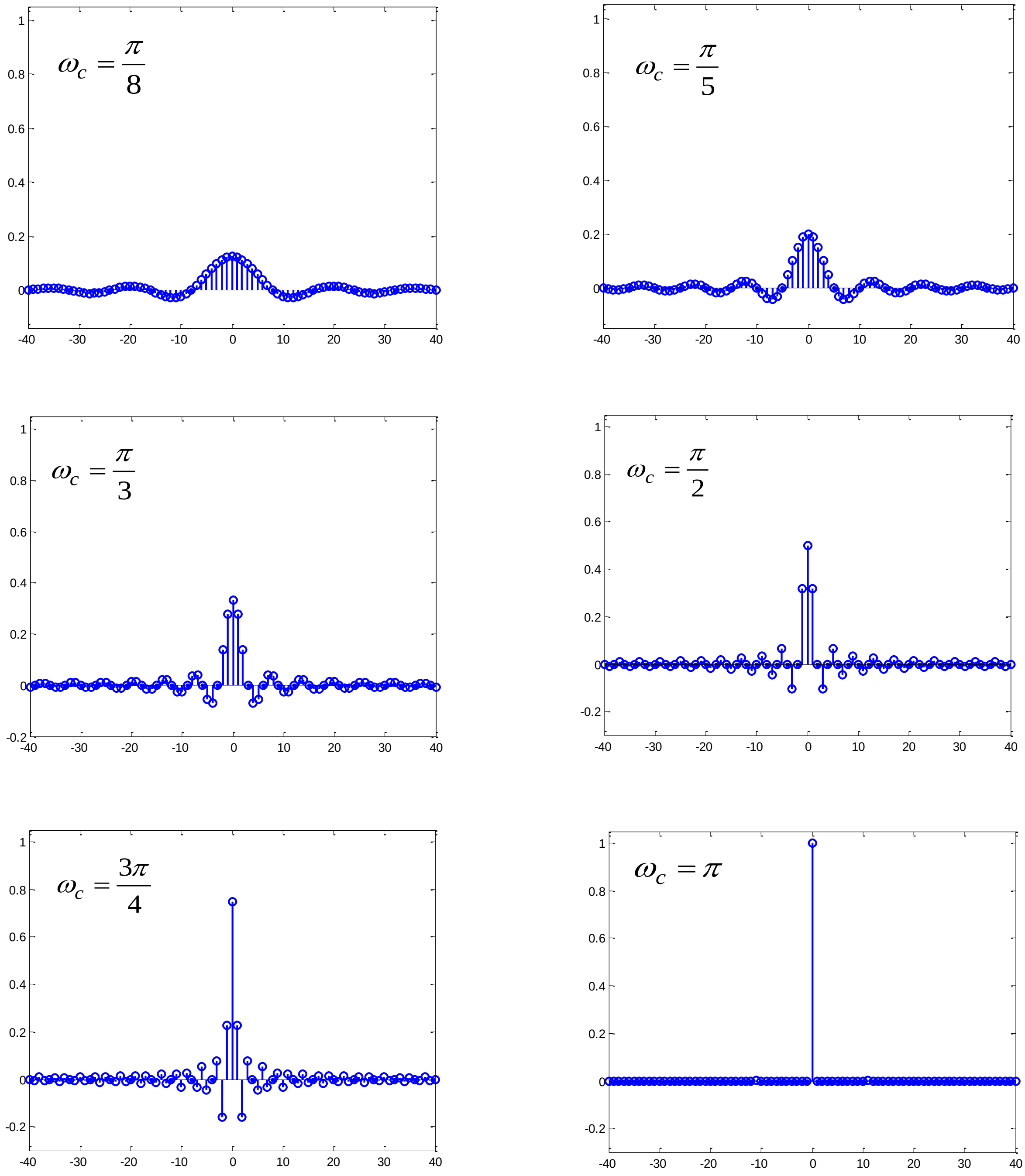


Its impulse response is

$$h_d[n] = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} e^{-j\omega n}\, d\omega$$

$$= \frac{1}{-jn2\pi}\left(e^{-j\omega_c n} - e^{j\omega_c n}\right)$$

$$= \frac{\sin(\omega_c n)}{\pi n}$$

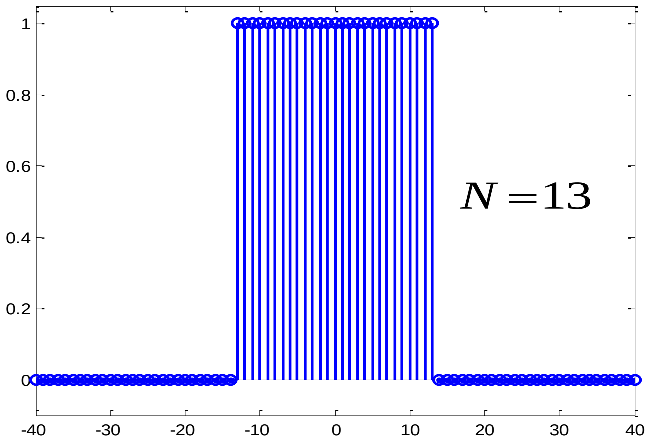


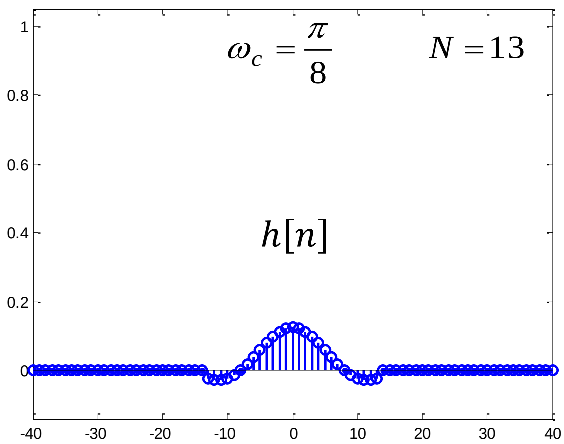
- $h_d[n]$ covers $(-\infty, \infty)$
- $h_d[n]$ is noncausal

Suppose you truncate $h_d[n]$ and call it $h[n]$

$$h[n] = h_d[n] \times w[n]$$
$$= \begin{cases} h_d[n] & -N \le n \le N \\ 0 & \text{o.w.} \end{cases}$$

where $w[n]$ is a rectangular "*window*" function $w[n] = \begin{cases} 1 & -N \le n \le N \\ 0 & o.w. \end{cases}$

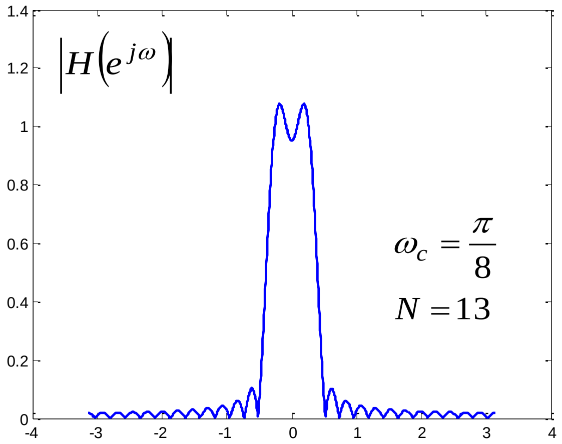


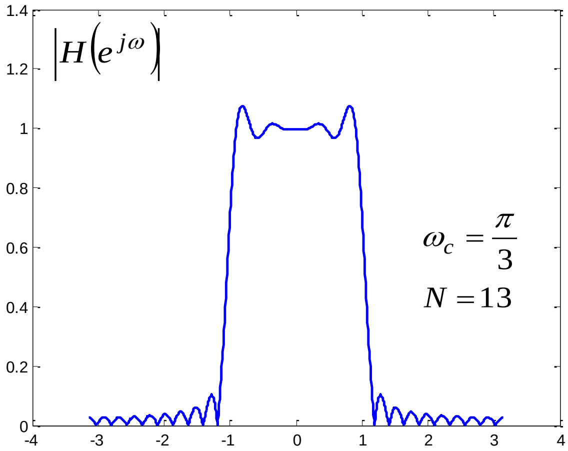


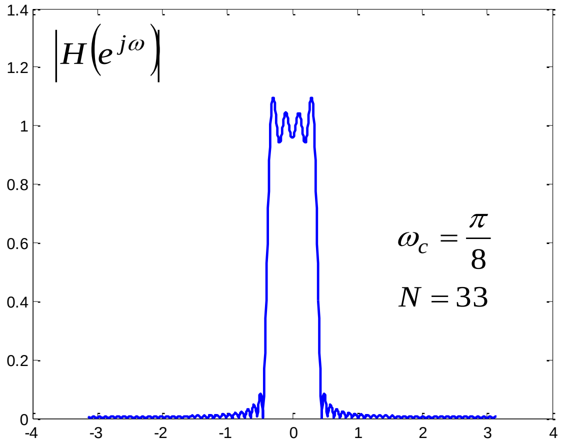
How do $H_d(e^{j\omega})$ and $H(e^{j\omega})$ differ?

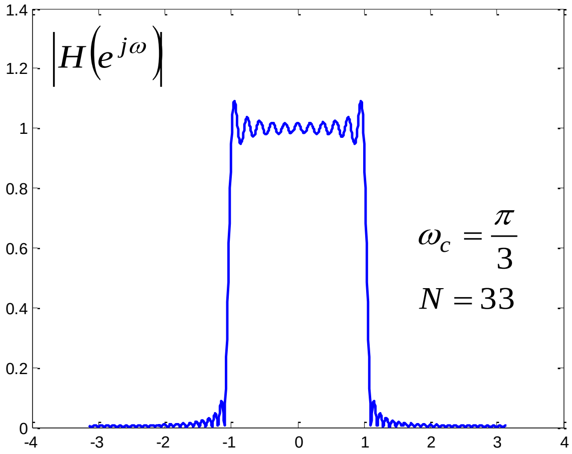
$$h[n] = h_d[n] \times w[n]$$

$$H(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\theta})W(e^{j(\omega-\theta)})d\theta$$









FOURIER TRANSFORM OF RECTANGULAR WINDOW FUNCTION

$$W\left(e^{j\omega}\right) = \sum_{n=-N}^{N} e^{-j\omega n}$$

$$= e^{j\omega N} \sum_{n=0}^{2N} e^{-j\omega n}$$

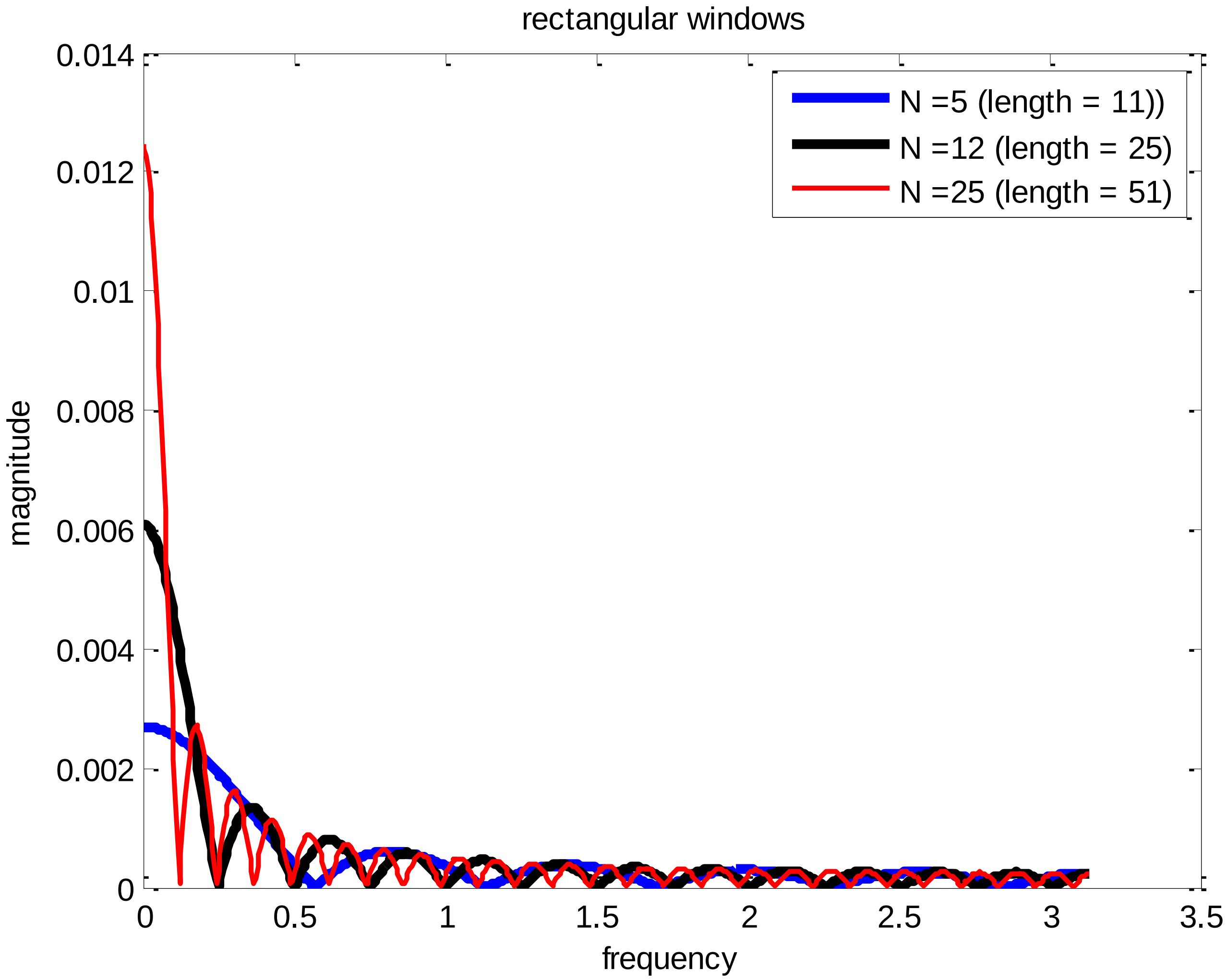
$$= e^{j\omega N} \frac{1 - e^{-j\omega(2N+1)}}{1 - e^{-j\omega}}$$

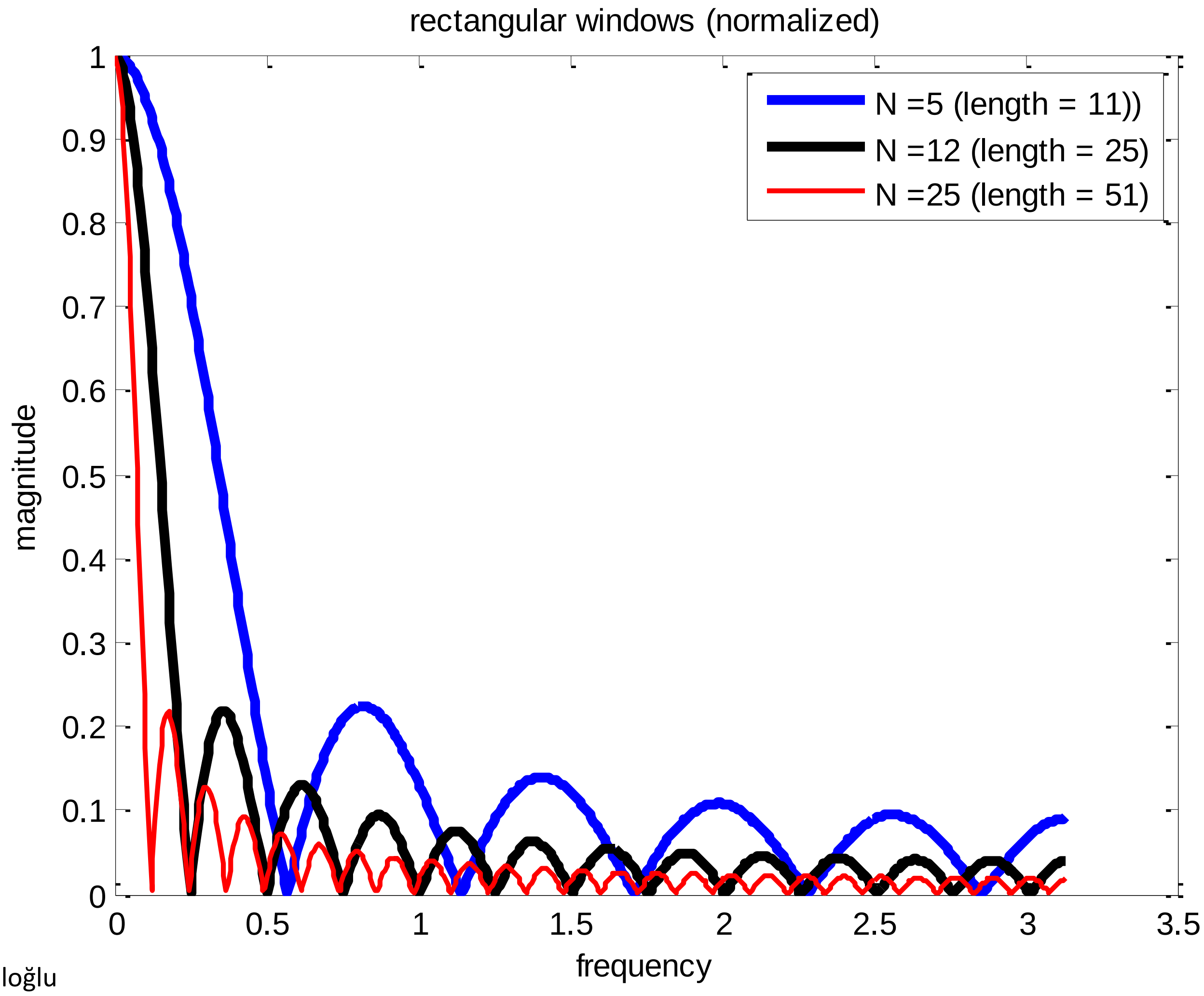
$$= \frac{e^{j\omega N} e^{-j\omega \frac{2N+1}{2}}}{e^{-j\frac{\omega}{2}}} \frac{e^{j\omega \frac{2N+1}{2}} - e^{-j\omega \frac{2N+1}{2}}}{e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}}}$$

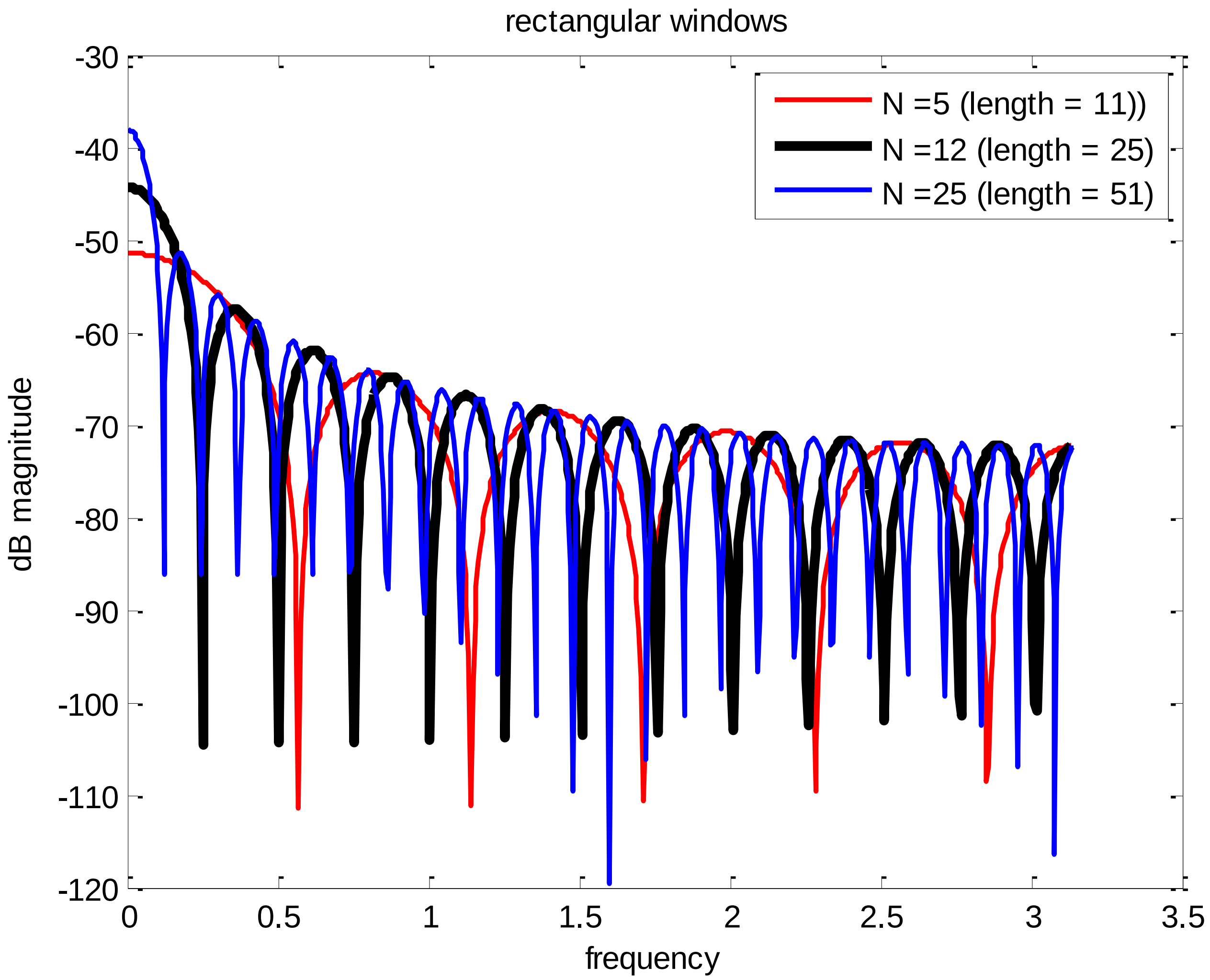
$$= \frac{e^{j\omega N} e^{-j\omega \frac{2N+1}{2}}}{e^{-j\frac{\omega}{2}}} \frac{\sin\left(\omega \frac{2N+1}{2}\right)}{\sin\left(\frac{\omega}{2}\right)}$$

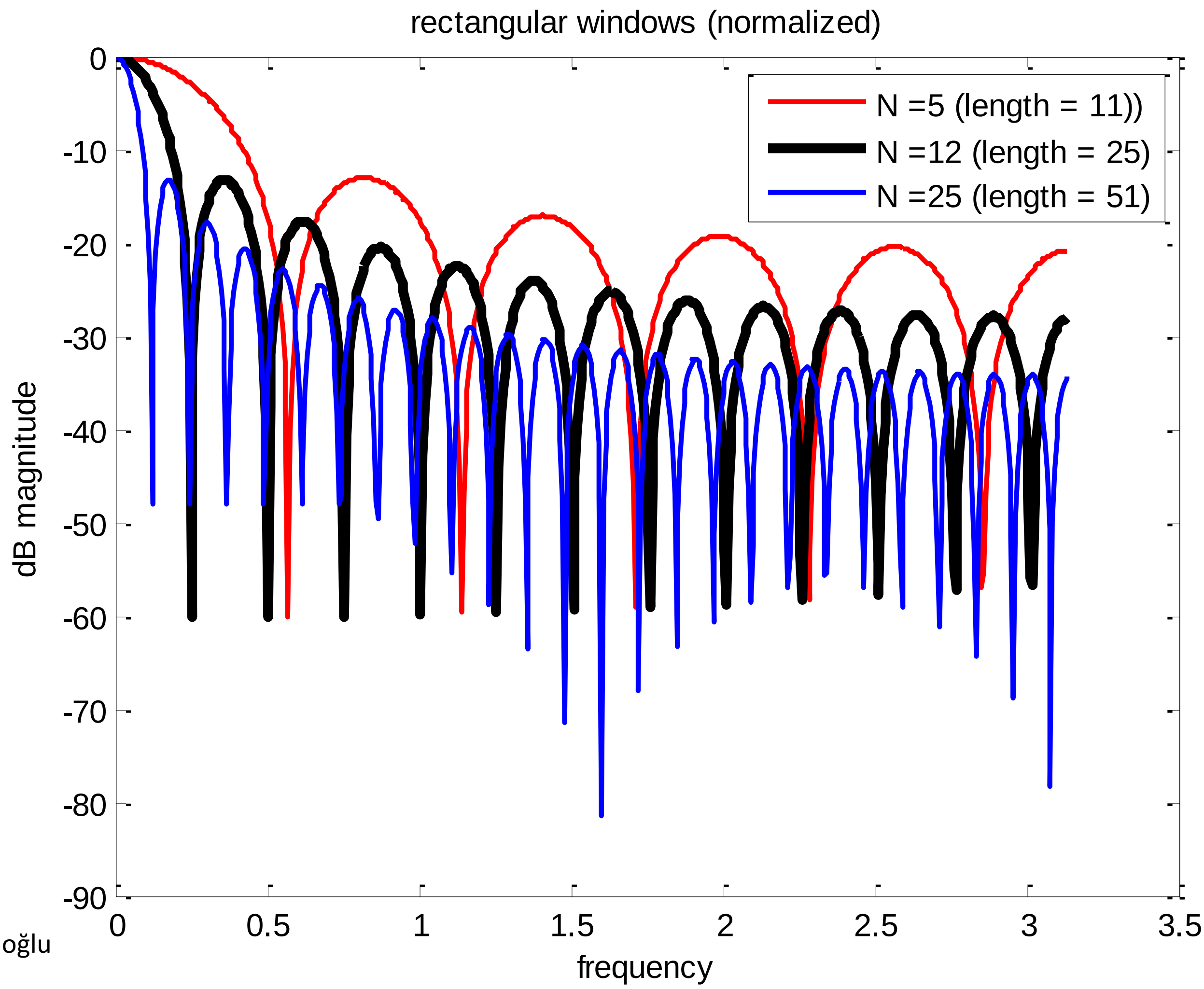
$$= \frac{\sin\left(\omega \frac{2N+1}{2}\right)}{\sin\left(\frac{\omega}{2}\right)}$$

The only parameter is the window length, 2*N*+1.


rectangular windows
0.014
0.012
0.01
0.008
0.006
0.004
0.002
0
magnitude
N =5 (length = 11))
N =12 (length = 25)
N =25 (length = 51)
0
0.5
1
1.5
2
2.5
3
3.5
frequency


rectangular windows (normalized)
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
magnitude
N =5 (length = 11))
N =12 (length = 25)
N =25 (length = 51)
0
0.5
1
1.5
2
2.5
3
3.5
frequency

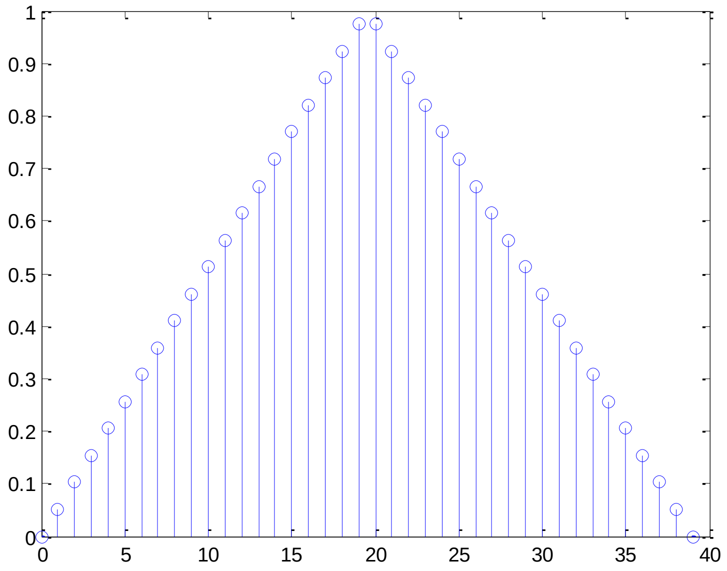

rectangular windows
N =5 (length = 11))
N =12 (length = 25)
N =25 (length = 51)
dB magnitude
frequency
-30
-40
-50
-60
-70
-80
-90
-100
-110
-120
0
0.5
1
1.5
2
2.5
3
3.5

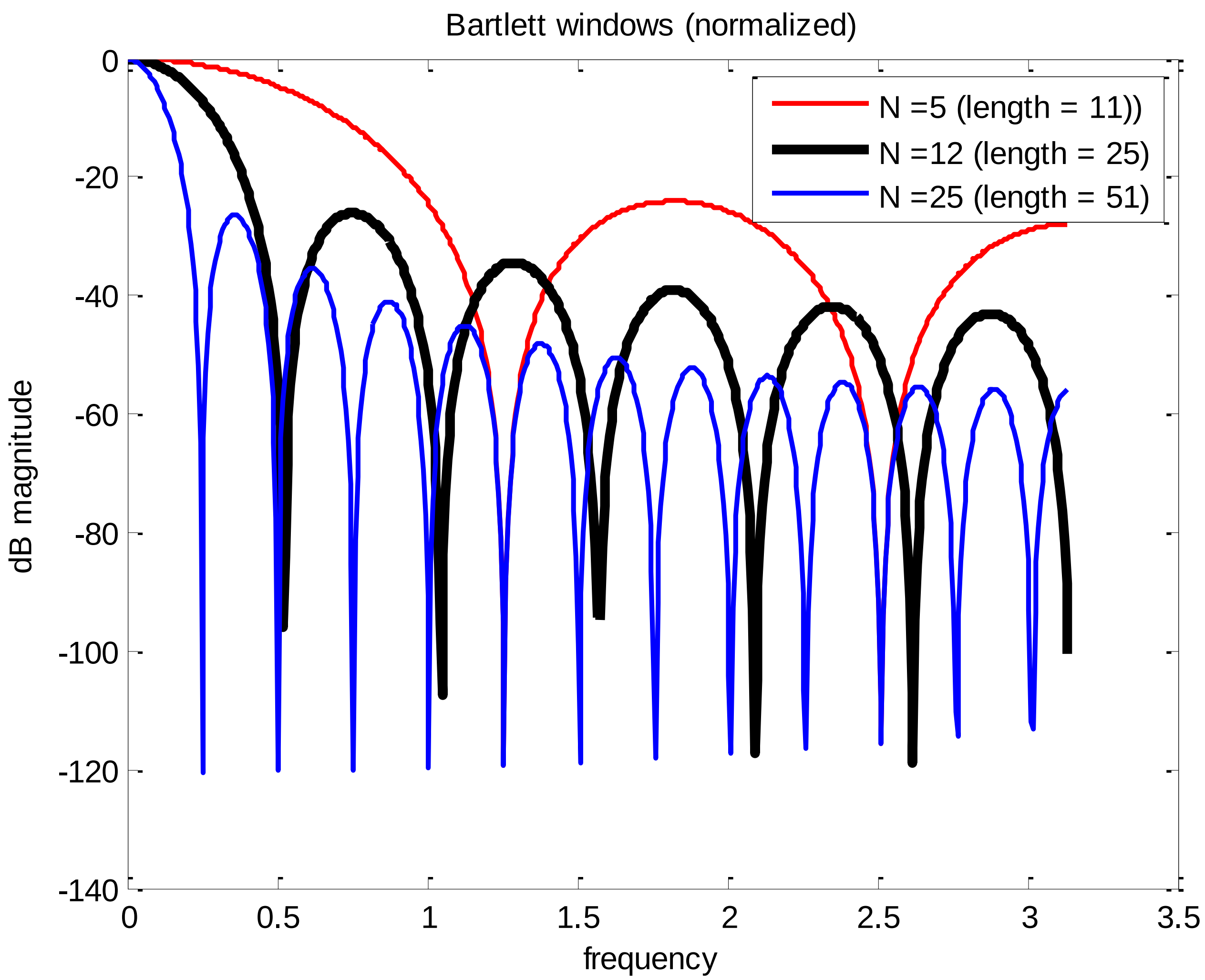

rectangular windows (normalized)
N =5 (length = 11))
N =12 (length = 25)
N =25 (length = 51)
dB magnitude
frequency
0
-10
-20
-30
-40
-50
-60
-70
-80
-90
0
0.5
1
1.5
2
2.5
3
3.5

```
%rectangular windows
clear all
close all

N1 = 11;
N2 = 25;
N3 = 51;

w1 = rectwin(N1);
w2 = rectwin(N2);
w3 = rectwin(N3);

[W1,f] = freqz(w1,4096);
[W2,f] = freqz(w2,4096);
[W3,f] = freqz(w3,4096);

W1 = W1 / abs(W1(1));
W2 = W2 / abs(W2(1));
W3 = W3 / abs(W3(1));

figure
plot(f,20*log10(abs(W1)),'r', 'linewidth', 2)
hold on
plot(f,20*log10(abs(W2)),'k', 'linewidth', 3)
plot(f,20*log10(abs(W3)),'b', 'linewidth', 2)

% plot(f,abs(W1), 'linewidth', 3)
% hold on
% plot(f,abs(W2),'k', 'linewidth', 3)
% plot(f,abs(W3),'r', 'linewidth', 2)

legend('N =5 (length = 11))', 'N =12 (length = 25)', 'N =25 (length = 51)');
title('rectangular windows (normalized)');
xlabel('frequency');
ylabel('dB magnitude');
```

Some Other Window Functions

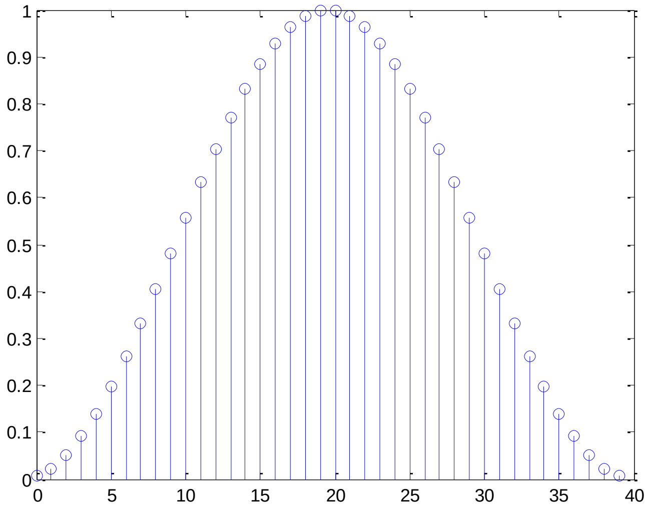
Bartlett (triangular)



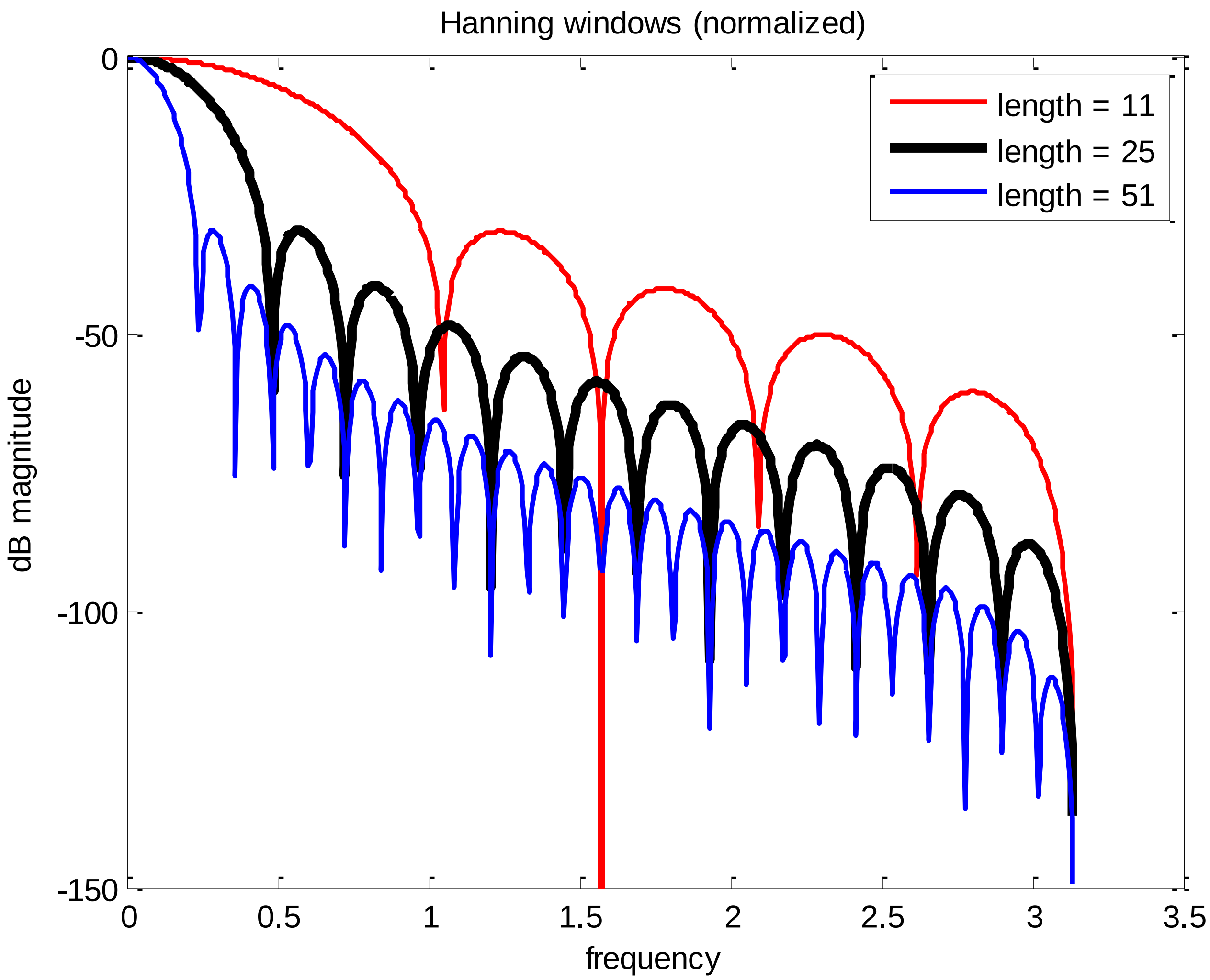


Hanning

(raised cosine)

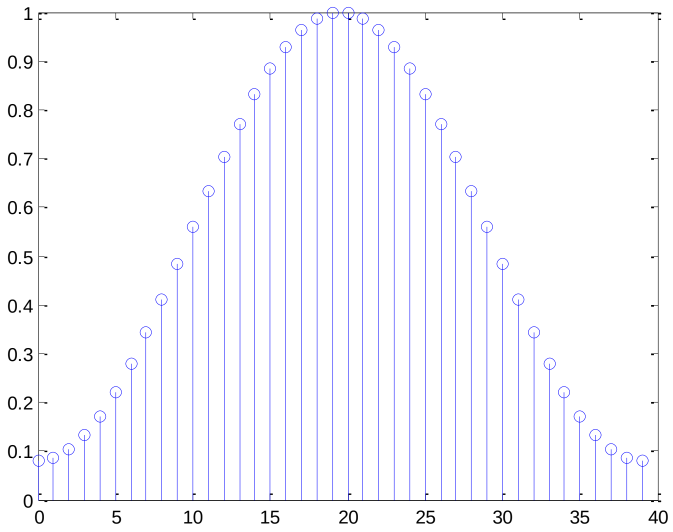


$$w[n]=\begin{cases}0.5-0.5\cos\left(\dfrac{2\pi}{M}n\right) & 0\le n\le M\\ 0 & \text{o.w.}\end{cases}$$

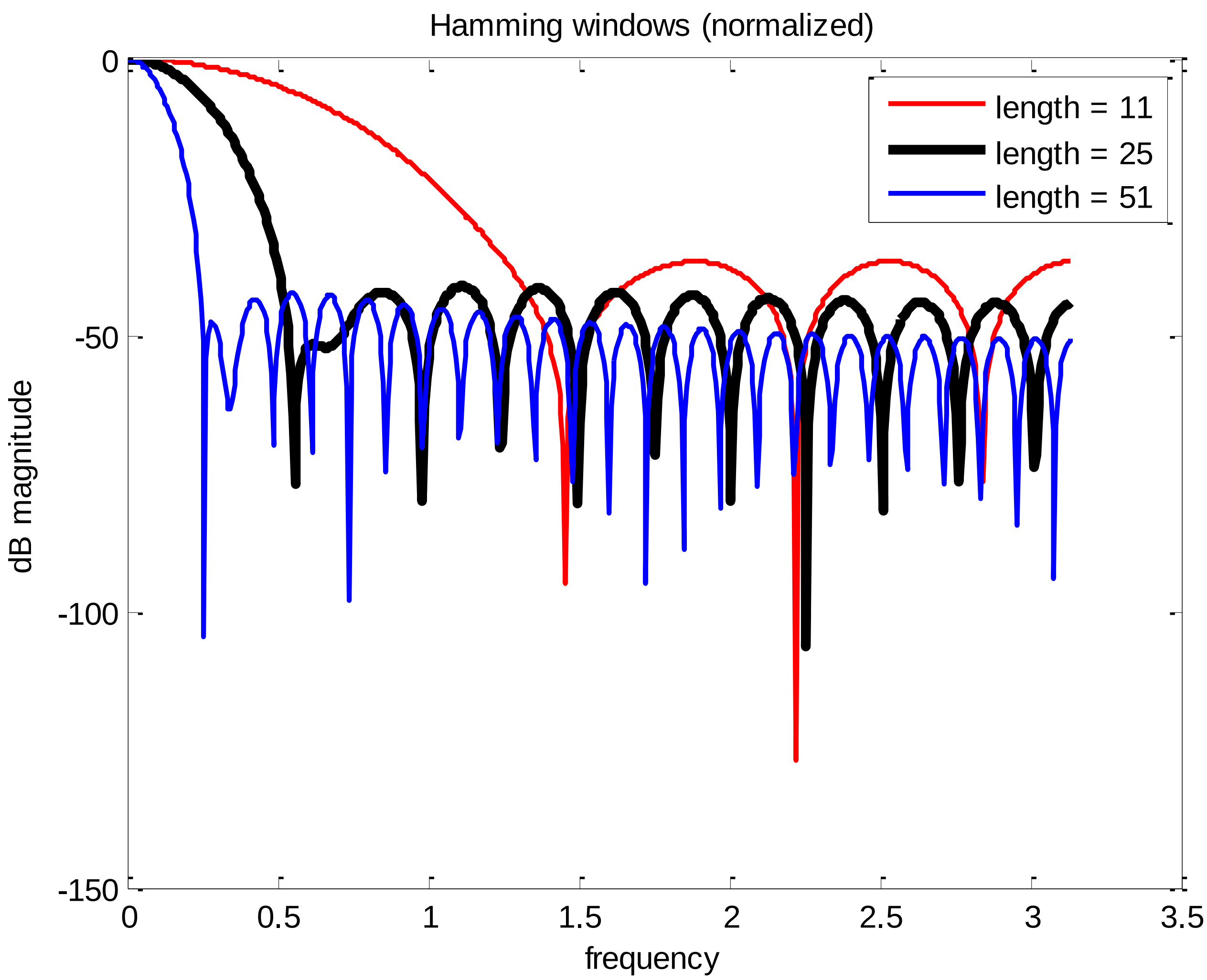


Hamming

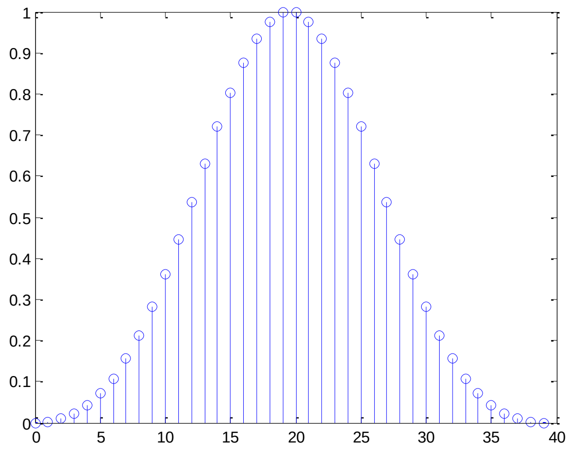
(raised cosine)



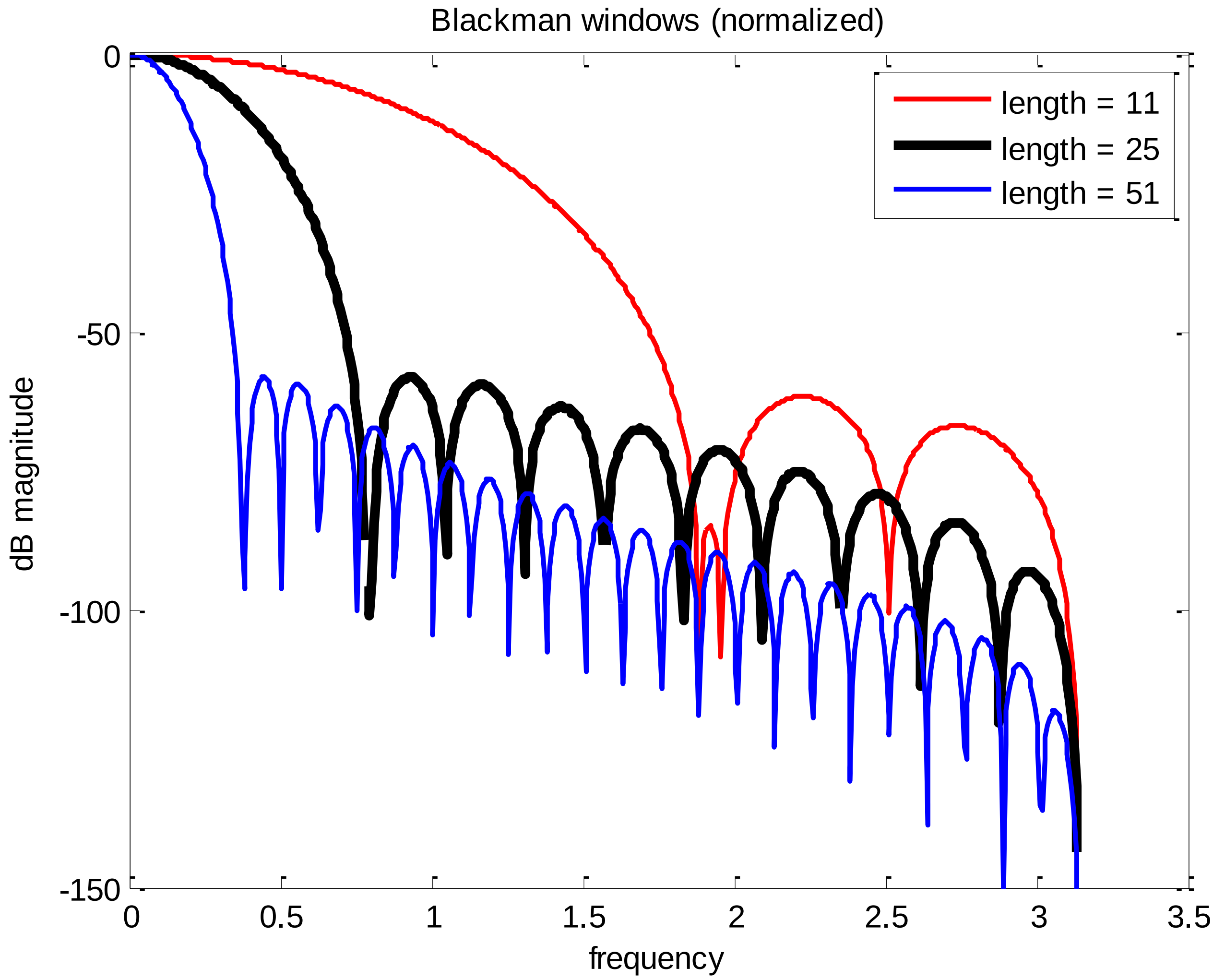
$$w[n]=\begin{cases}0.54-0.46\cos\left(\dfrac{2\pi}{M}n\right) & 0\le n\le M\\ 0 & \text{o.w.}\end{cases}$$

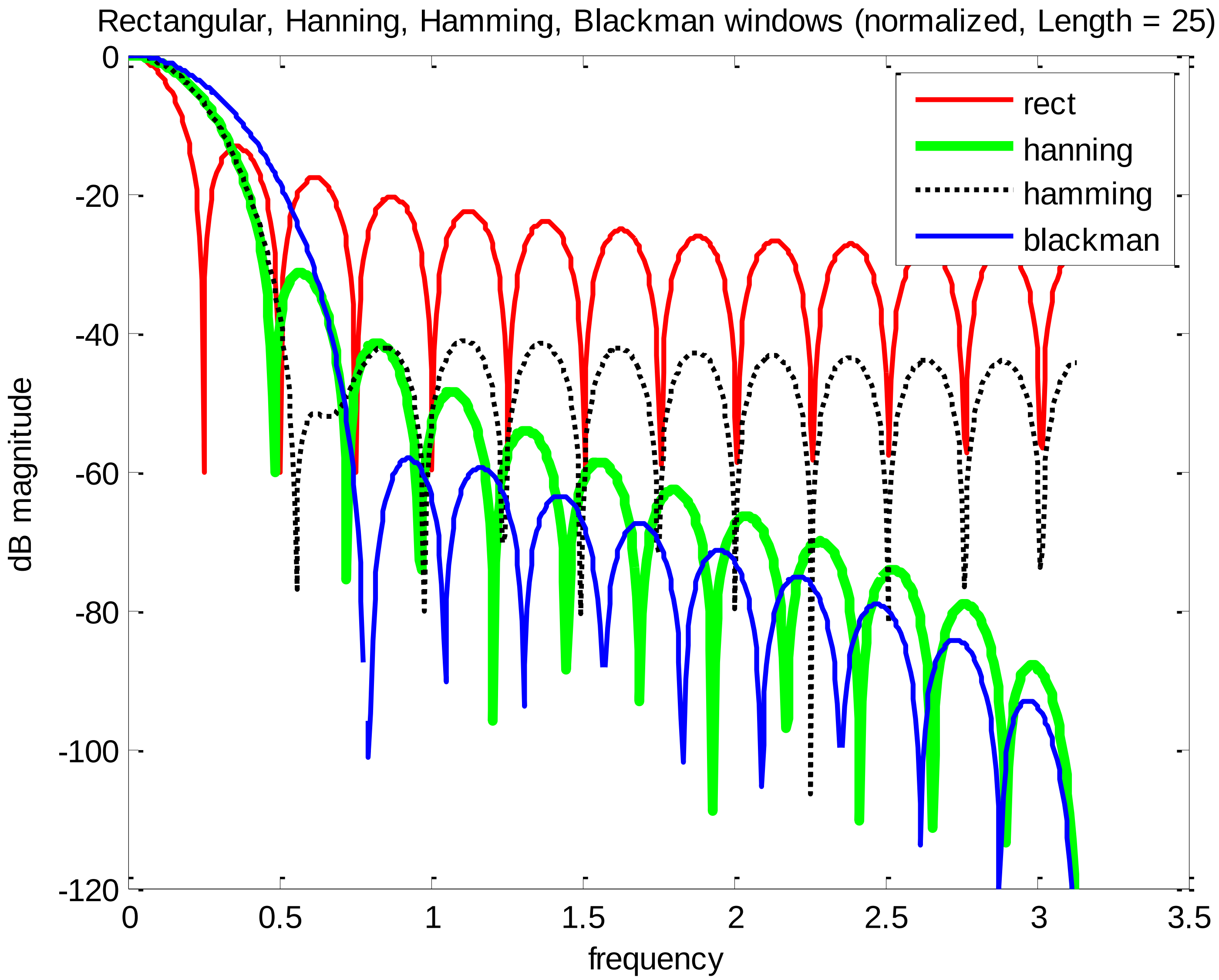


Blackman



$$w[n]=\begin{cases}0.42-0.5\cos\left(\dfrac{2\pi}{M}n\right)+0.08\cos\left(\dfrac{4\pi}{M}n\right) & 0\le n\le M\\ 0 & \text{o.w.}\end{cases}$$

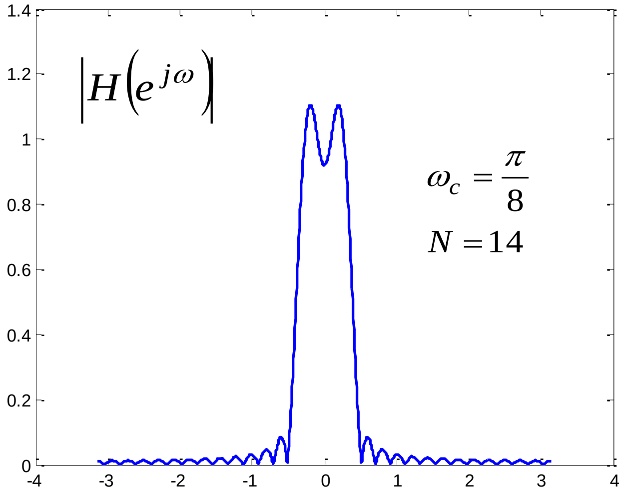




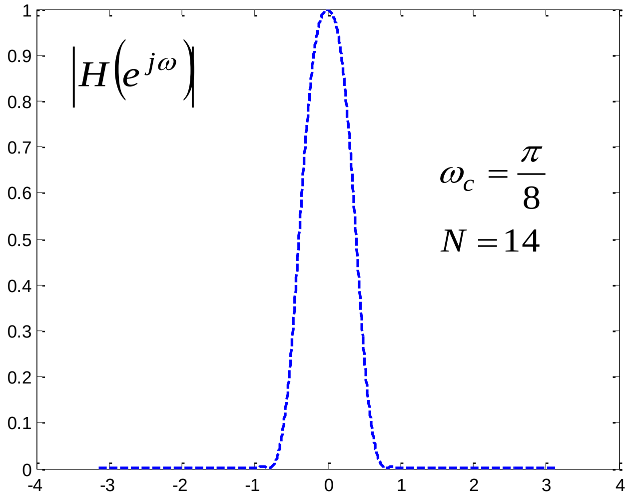

```
clear all
close all

N = 25;

w1 = rectwin(N);
w2 = hanning(N);
w3 = hamming(N);
w4 = blackman(N);

[W1,f] = freqz(w1,4096);
[W2,f] = freqz(w2,4096);
[W3,f] = freqz(w3,4096);
[W4,f] = freqz(w4,4096);

W1 = W1 / abs(W1(1));
W2 = W2 / abs(W2(1));
W3 = W3 / abs(W3(1));
W4 = W4 / abs(W4(1));

figure
plot(f,20*log10(abs(W1)),'r', 'linewidth', 2)
hold on
plot(f,20*log10(abs(W2)),'g', 'linewidth', 3)
plot(f,20*log10(abs(W3)),'k:', 'linewidth', 2)
plot(f,20*log10(abs(W4)),'b', 'linewidth', 2)

v = axis;
v(3) = -150;
axis(v)
legend('rect', 'hanning', 'hamming','blackman');
title('Rectangular, Hanning, Hamming, Blackman windows (normalized, Length = 25)');
xlabel('frequency');
ylabel('dB magnitude');
```

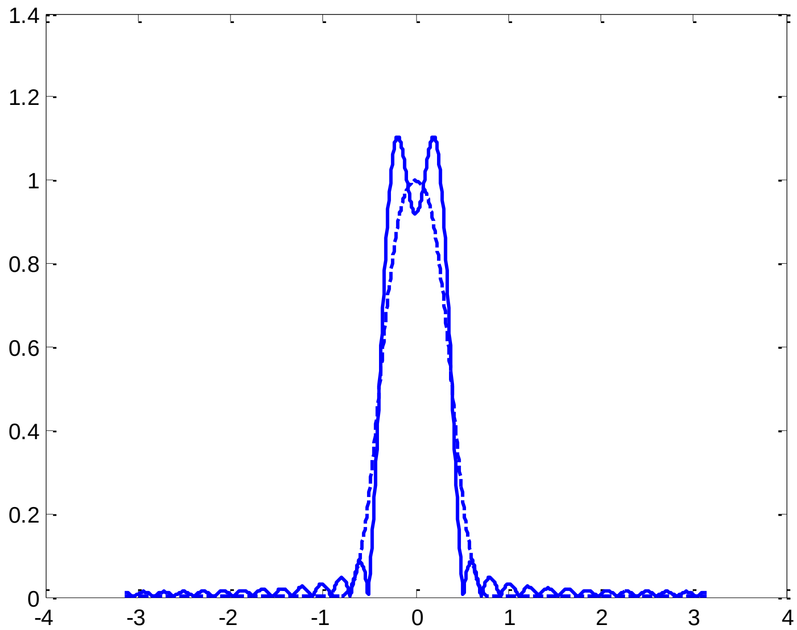
Rectangular vs Hamming (Window) Designs

using rectangular window

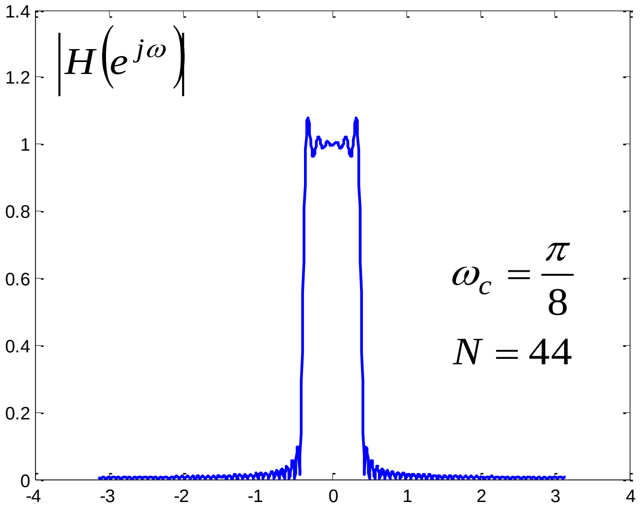


using Hamming window

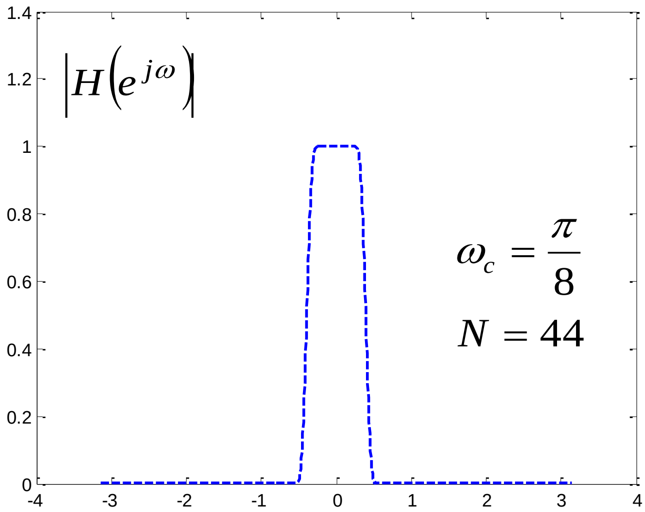


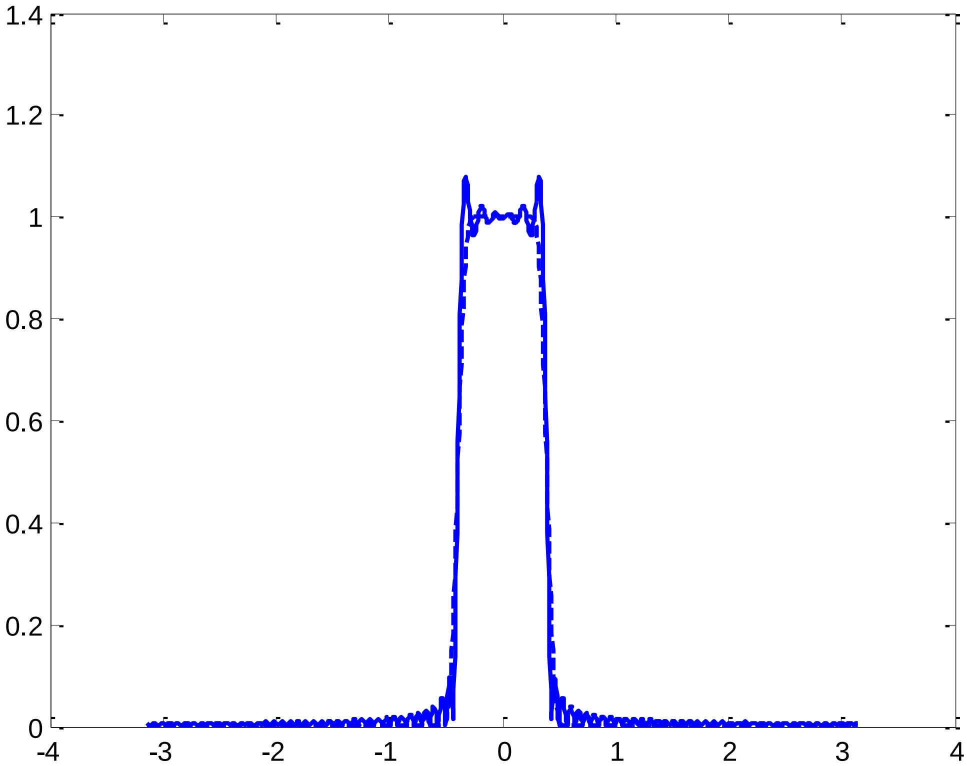


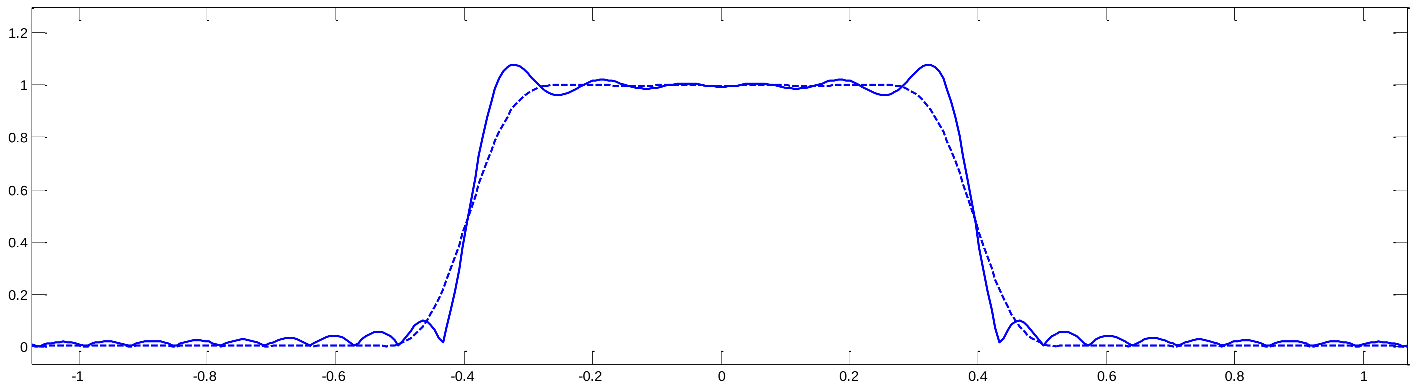
using rectangular window



using Hamming window







KAISER WINDOW

$$w[n] = \begin{cases} \dfrac{I_0\left(\beta\sqrt{1-\left(\dfrac{n-\alpha}{\alpha}\right)^2}\right)}{I_0(\beta)} & 0 \le n \le M \\ 0 & o.w. \end{cases}$$

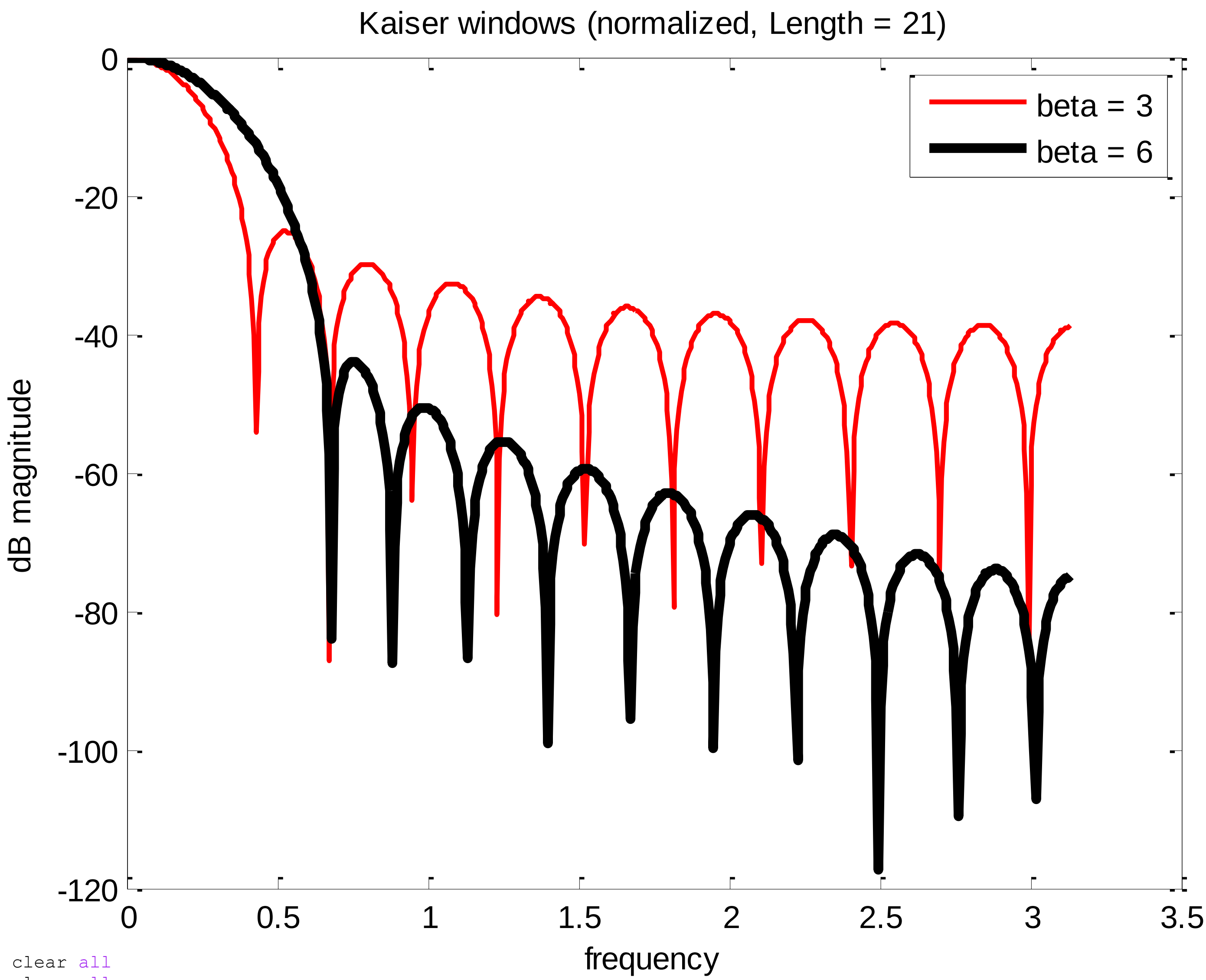
$$\alpha = \frac{M}{2}$$

$I_0(\cdot)$: $zero^{th}$ $order\ Bessel\ function\ of\ the\ first\ kind$

Two parameters $M+1$ and β.

By varying $M+1$ and β, main lobe width and side lobe level can be traded.

Increasing M, keeping β constant → Main lobe narrows, side lobe levels do not change




```
clear all
close all

N = 21;

beta1 = 3;
beta2 = 6;

w1 = kaiser(N,beta1);
w2 = kaiser(N,beta2);

[W1,f] = freqz(w1,4096);
[W2,f] = freqz(w2,4096);

W1 = W1 / abs(W1(1));
W2 = W2 / abs(W2(1));

figure
plot(f,20*log10(abs(W1)),'r', 'linewidth', 2)
hold on
plot(f,20*log10(abs(W2)),'k', 'linewidth', 3)

v = axis;
v(3) = -120;
axis(v)
legend('beta = 3', 'beta = 6');
title('Kaiser windows (normalized, Length = 21)');
xlabel('frequency');
ylabel('dB magnitude');
```

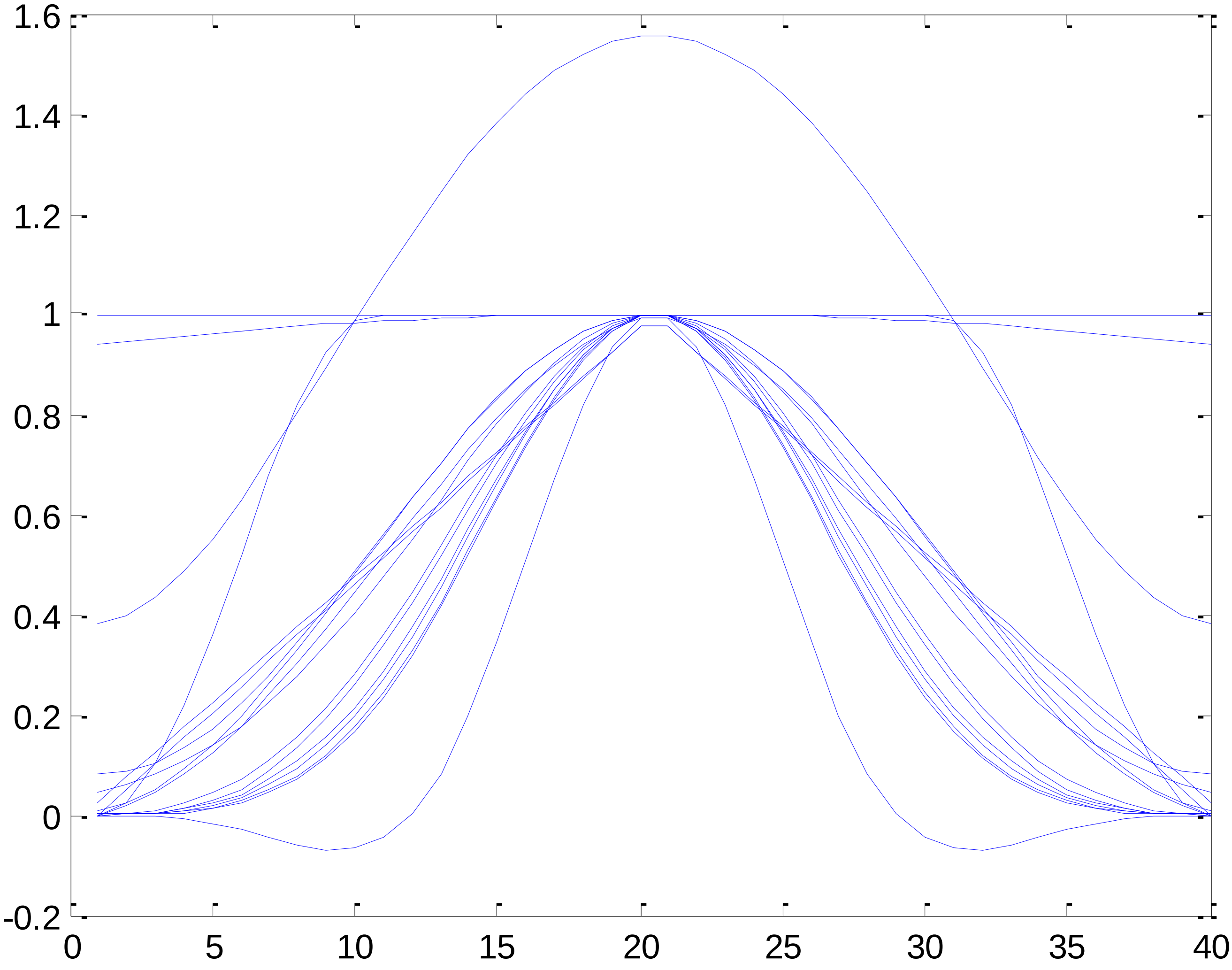
MATLAB Window Functions

```
WINDOW Window function gateway.
   WINDOW(@WNAME,N) returns an N-point window of type specified
   by the function handle @WNAME in a column vector.  @WNAME can
   be any valid window function name, for example:

   @bartlett       - Bartlett window.
   @barthannwin    - Modified Bartlett-Hanning window.
   @blackman       - Blackman window.
   @blackmanharris - Minimum 4-term Blackman-Harris window.
   @bohmanwin      - Bohman window.
   @chebwin        - Chebyshev window.
   @flattopwin     - Flat Top window.
   @gausswin       - Gaussian window.
   @hamming        - Hamming window.
   @hann           - Hann window.
   @kaiser         - Kaiser window.
   @nuttallwin     - Nuttall defined minimum 4-term Blackman-Harris window.
   @parzenwin      - Parzen (de la Valle-Poussin) window.
   @rectwin        - Rectangular window.
   @taylorwin      - Taylor window.
   @tukeywin       - Tukey window.
   @triang         - Triangular window.

   WINDOW(@WNAME,N,OPT1,OPT2) designs the window with the optional input
   arguments specified in OPT1 and OPT2. To see what the optional input
   arguments are, see the help for the individual windows, for example,
   KAISER or CHEBWIN.
```

WINDOW launches the Window Design & Analysis Tool (WinTool).

```
EXAMPLE:
  N  = 65;
  w  = window(@blackmanharris,N);
  w1 = window(@gausswin,N,2.5);
  w2 = window(@taylorwin,N,5,-35);
  plot(1:N,[w,w1,w2]); axis([1 N 0 2]);
  legend('Blackman-Harris','Gaussian','Taylor');
```

ALL WINDOWS ABOVE ….



FIR filter design

cfirpm - Complex and nonlinear phase equiripple FIR filter design

fir1 - Window based FIR filter design - low, high, band, stop, multi

fir2 - FIR arbitrary shape filter design using the frequency sampling method

fircls - Constrained Least Squares filter design - arbitrary response

fircls1 - Constrained Least Squares FIR filter design - low and highpass

firls - Optimal least-squares FIR filter design

firpm - Parks-McClellan optimal equiripple FIR filter design (generalized linear phase)

firpmord - Parks-McClellan optimal equiripple FIR order estimator

intfilt - Interpolation FIR filter design

kaiserord - Kaiser window design based filter order estimation

sgolay - Savitzky-Golay FIR smoothing filter design

Communications filters

firrcos - Raised cosine FIR filter design

gaussfir - Gaussian FIR Pulse-Shaping Filter Design

IIR digital filter design

butter - Butterworth filter design

cheby1 - Chebyshev Type I filter design (passband ripple)

cheby2 - Chebyshev Type II filter design (stopband ripple)

ellip - Elliptic filter design

maxflat - Generalized Butterworth lowpass filter design

yulewalk - Yule-Walker filter design

IIR filter order estimation

buttord - Butterworth filter order estimation

cheb1ord - Chebyshev Type I filter order estimation

cheb2ord - Chebyshev Type II filter order estimation

ellipord - Elliptic filter order estimation

Filter analysis

abs - Magnitude

angle - Phase angle

filternorm - Compute the 2-norm or inf-norm of a digital filter

freqz - Z-transform frequency response

fvtool - Filter Visualization Tool

grpdelay - Group delay

impz - Discrete impulse response

phasedelay - Phase delay of a digital filter

phasez - Digital filter phase response (unwrapped)

stepz - Digital filter step response

unwrap - Unwrap phase angle

zerophase - Zero-phase response of a real filter

zplane - Discrete pole-zero plot

Filter implementation

conv - Convolution

conv2 - 2-D convolution

convmtx - Convolution matrix

deconv - Deconvolution

fftfilt - Overlap-add filter implementation

filter - Filter implementation

filter2 - Two-dimensional digital filtering

filtfilt - Zero-phase version of filter

filtic - Determine filter initial conditions

latcfilt - Lattice filter implementation

medfilt1 - 1-Dimensional median filtering

sgolayfilt - Savitzky-Golay filter implementation

sosfilt - Second-order sections (biquad) filter implementation

upfirdn - Upsample, FIR filter, downsample

PARKS-MCCLELLAN OPTIMAL EQUIRIPPLE FIR FILTER DESIGN IN MATLAB (firpm)

B = firpm(N,F,A)

- FIR filter
- length N+1
- generalized linear phase (real, symmetric coefficients)
- best approximation to the desired frequency response described by F and A in the MINIMAX SENSE

F: It is a vector of frequency band edges in pairs, in ascending order between 0 and 1. 1 corresponds to the Nyquist frequency or half the sampling frequency.

A: It is a real vector the same size as F which specifies the desired amplitude of the frequency response of the resultant filter B.

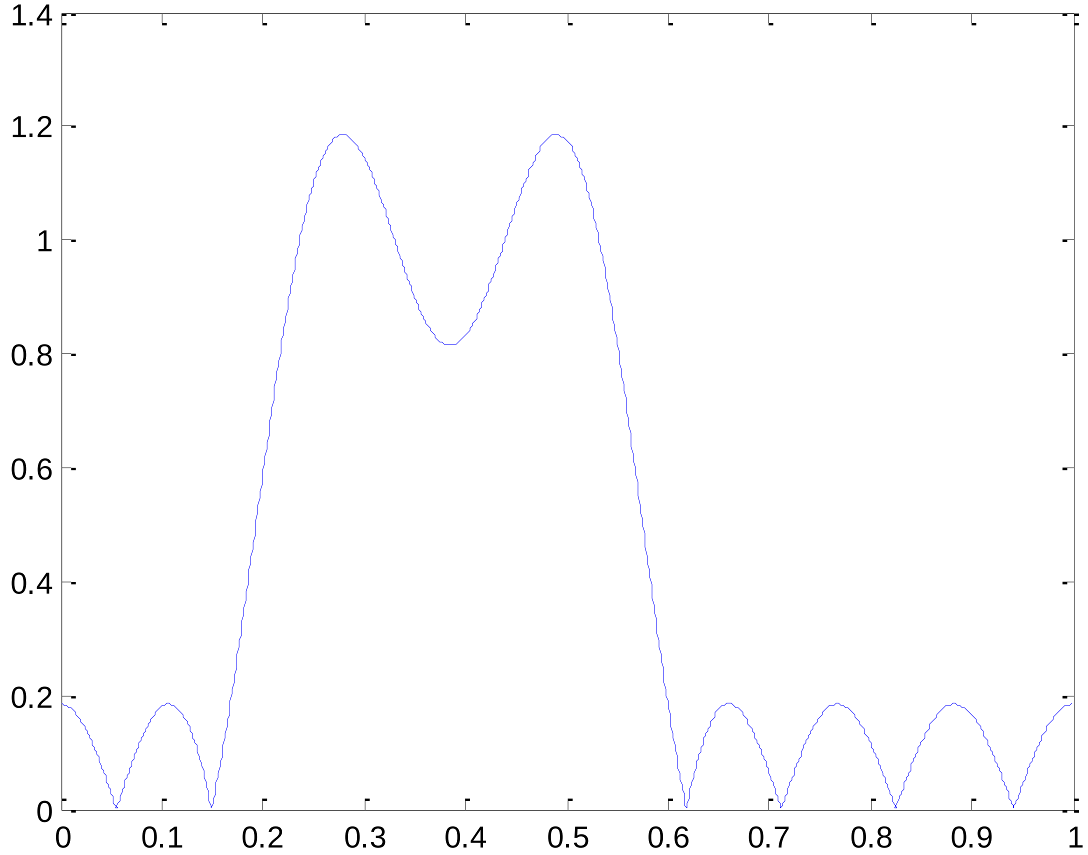
For filters with a gain other than zero at Fs/2, e.g., highpass and bandstop filters, N must be even. Otherwise, N will be incremented by one. Alternatively, you can use a trailing 'h' flag to design a type 4 linear phase filter and avoid incrementing N.

B=firpm(N,F,A,W)

uses the weights in W to weight the error. W has one entry per band (so it is half the length of F and A) which tells firpm how much emphasis to put on minimizing the error in each band relative to the other bands.

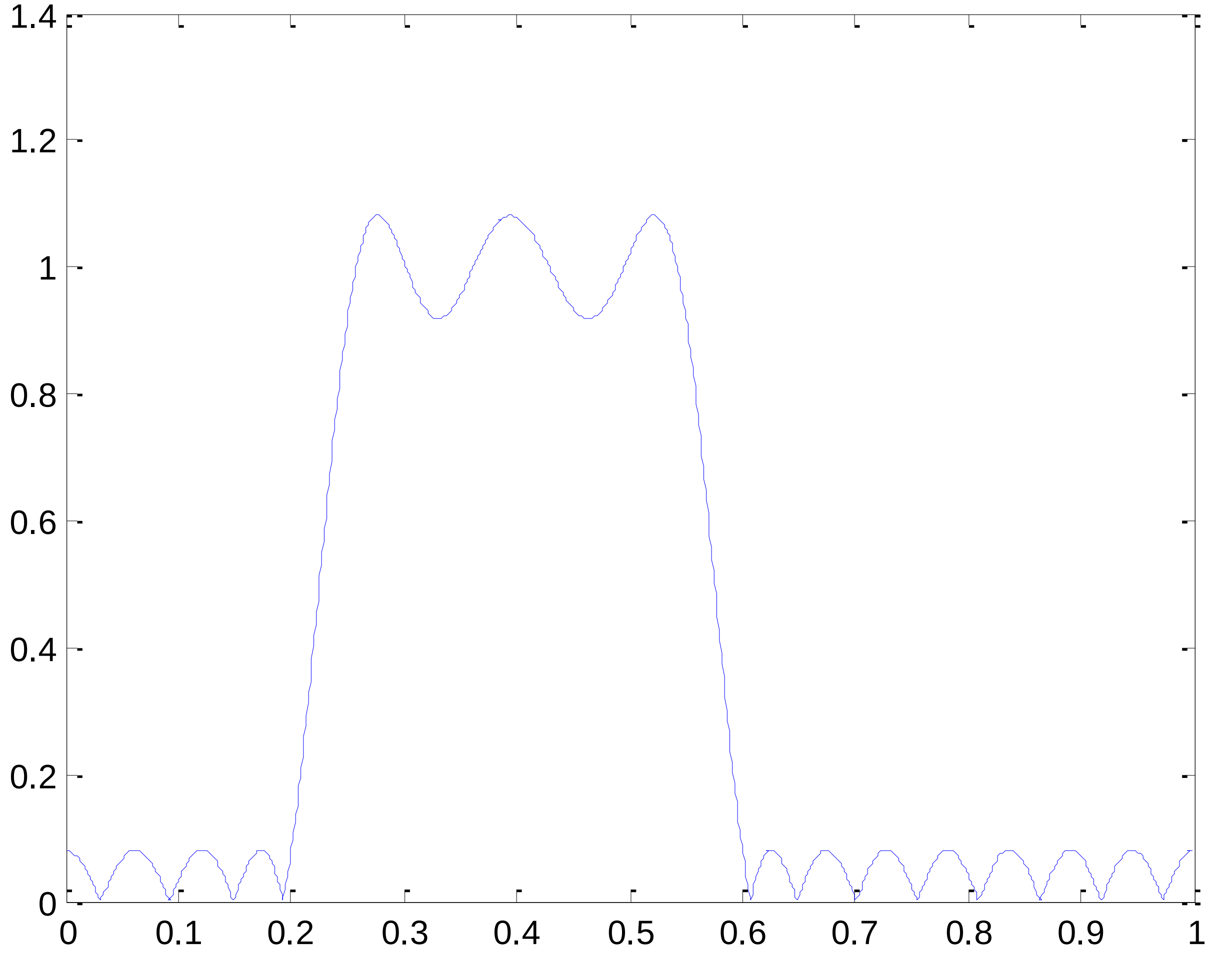
```
clear all
close all

N = 18;
F = [0 0.15 0.25 0.55 0.6 1];
A = [0 0 1 1 0 0];
h = firpm(N,F,A);
[H,w] =freqz(h,1,1024);
plot(w/pi,abs(H))
```

```
N = 18;
```



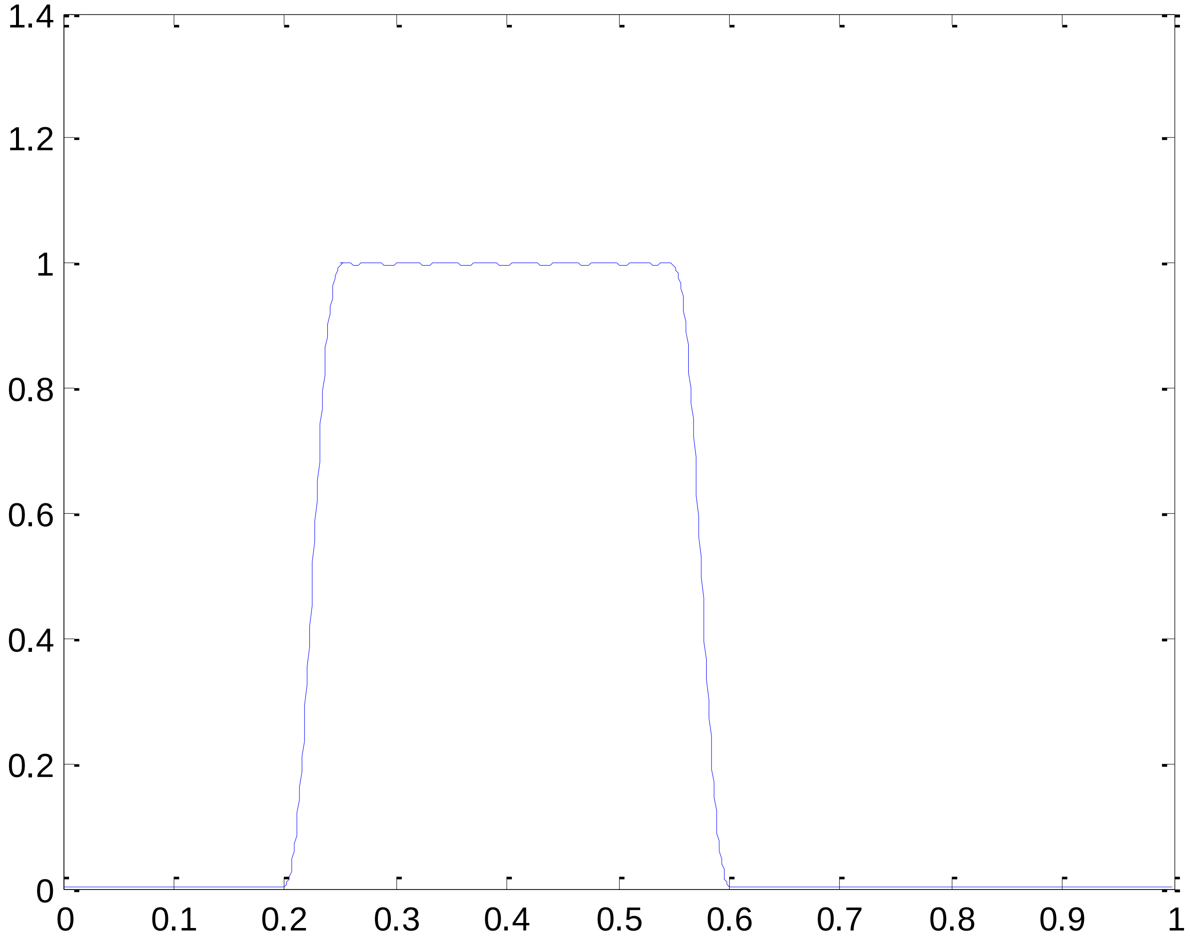
```
clear all
close all

N = 38;
F = [0 0.2 0.25 0.55 0.6 1];
A = [0 0 1 1 0 0];
h = firpm(N,F,A);
[H,w] =freqz(h,1,1024);
plot(w/pi,abs(H))
```

```
N = 38;
```



```
clear all
close all

N = 118;
F = [0 0.2 0.25 0.55 0.6 1];
A = [0 0 1 1 0 0];
h = firpm(N,F,A);
[H,w] =freqz(h,1,1024);
plot(w/pi,abs(H))
```

N = 118;



zplane(B,A) where B and A are row vectors containing transfer function
polynomial coefficients plots the poles and zeros of B(z)/A(z). Note that if B and A are both scalars they will be interpreted as Z and P.

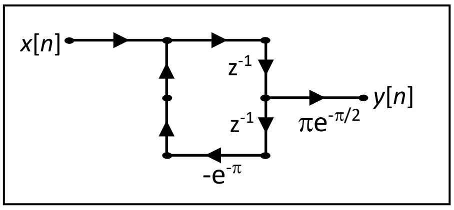
[HZ,HP,Hl] = zplane(Z,P) returns vectors of handles to the lines and text objects generated. HZ is a vector of handles to the zeros lines, HP is a vector of handles to the poles lines, and Hl is a vector of handles to the axes / unit circle line and to text objects which are present when there are multiple zeros or poles. In case there are no zeros or no poles, HZ or HP is set to the empty matrix [].

zplane(Z,P,AX) puts the plot into the axes specified by the handle AX.

```
% Example 1:
%   Design a lowpass FIR filter with normalized cut-off frequency at
%   0.3 and show its zero-pole plot.

b=fircls1(54,0.3,0.02,0.008);
zplane(b)                   % zero-pole plot for filter

% Example 2:
%   Design a 5th order lowpass elliptic IIR filter and show its
%   zero-pole plot.

[b,a] = ellip(5,0.5,20,0.4);
zplane(b,a)                 % zero-pole plot for filter

% Example 3:
%   Design a lowpass Butterworth IIR filter and show its Zero-Pole
%   plot.

[z,p] = butter(4,0.15);
zplane(z,p)                 % zero-pole plot for filter
```

See also freqz, grpdelay, impz, fvtool.

Q6) The signal flow graph representation of a digital filter is given in the figure.



a) Determine the transfer function, $H(z)$, and the poles of this filter.

b) Assume that this filter has been designed by using *impulse invariance* method from a *Butterworth* filter. Find the order (N) of the Butterworth filter, as well as its parameter Ω_c. Mark the poles of the Butterworth filter system function, $H(s)$, on the complex plane. Take the sampling period as $T = 1$ sec.

c) Now, assume that the Butterworth filter ($H(s)$) of part-b is used to design another digital filter ($G(z)$) via *bilinear transformation* method (by taking the sampling period as $T = 1$ sec.). Determine the values of the passband and stopband (edge) frequencies (ω_p and ω_s, respectively) of this digital filter, if the minimum value of $\left|G\left(e^{jw}\right)\right|^2$ in the passband is allowed to be 64/65, whereas the maximum value of $\left|G\left(e^{jw}\right)\right|^2$ in the stopband is allowed to be 1/1025 (writing only the necessary equations is sufficient).

Bilinear transformation: $\Omega = \frac{2}{T}\tan\left(\frac{\omega}{2}\right)$

Butterworth filter: $|H(j\Omega)|^2 = \frac{1}{1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}}$