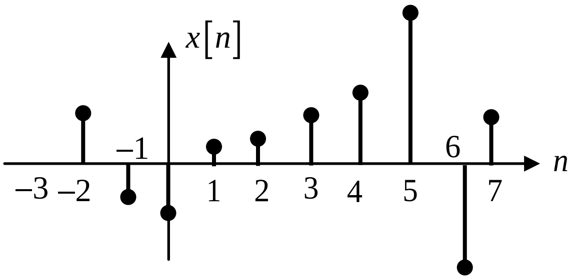
Discrete-Time Signals

A sequence (ordering) of (real, complex) numbers, n^{th} element is $x[n]$, $n \in Z$.



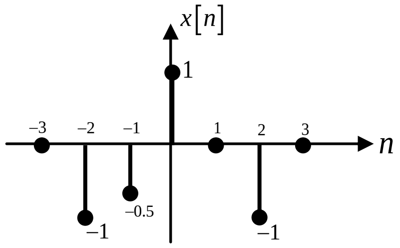
May have been obtained by sampling a continuous-time signal, i.e., $x[n] = x_C(t)|_{t=nT}$, $n \in Z$

<u>Ex</u>: Let T = 0.001 sec = 1 msec.

We do NOT write $\ldots, x[-0.001], x[0], x[0.001], x[0.002]\ldots$!

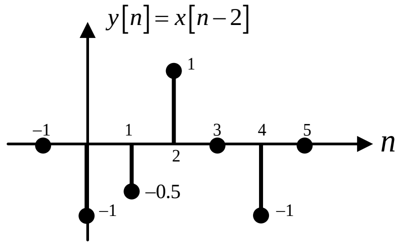
We write $\ldots, x[-1], x[0], x[1], x[2]\ldots$

<u>Ex</u>: Delay



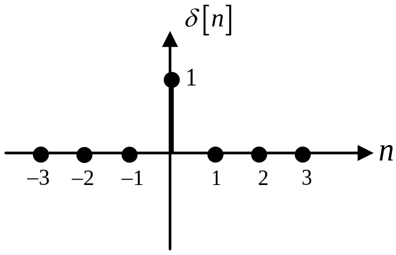
$$y[n] = x[n - n_0]$$

n_0 is always an integer!



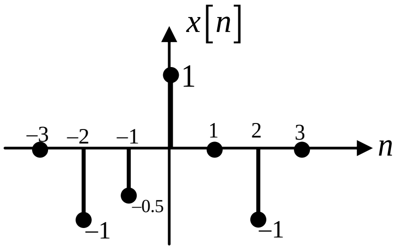
We do NOT have sth. like $x[n - 2.15]$

UNIT SAMPLE SEQUENCE:



$$\delta[n]=\begin{cases}1 & n=0\\ 0 & n\neq 0\end{cases}$$

<u>Ex</u>: Let $x[n]$ be



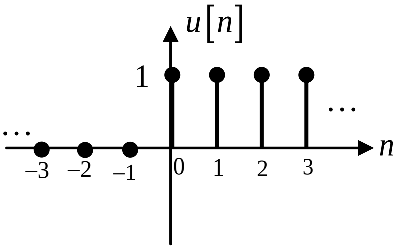
Can be written as:

$$x[n]=-\delta[n+2]-0.5\delta[n+1]+\delta[n]-\delta[n-2]$$

In general, any seq. can be written as $x[n]=\sum_{k=-\infty}^{\infty} x[k]\;\delta[n-k]$

This is the fundamental expression in the derivation of the fact that the output of a LTI system is the convolution of the input and the system's impulse response.

UNIT STEP SEQUENCE:



$$u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$

Ex: $u[n] = \sum_{k=0}^{\infty} \delta[n-k]$ (convolution of $u[n]$ and $\delta[n]$)

or

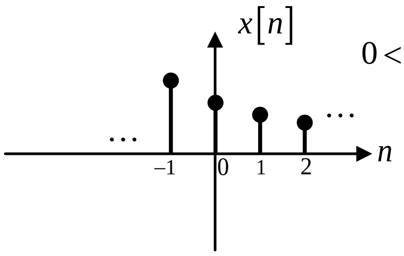
Ex: $u[n] = \sum_{k=-\infty}^{n} \delta[k]$ (like integration in cont. time)

on the other hand

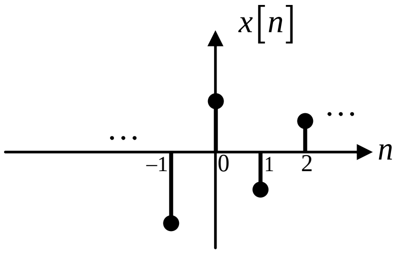
Ex: $\delta[n] = u[n] - u[n-1]$ (like differentation in cont. time)

EXPONENTIAL SEQUENCES (real valued): They appear in the solution and analysis of LTI systems.

$$x[n] = A\alpha^n$$



$0 < \alpha < 1$

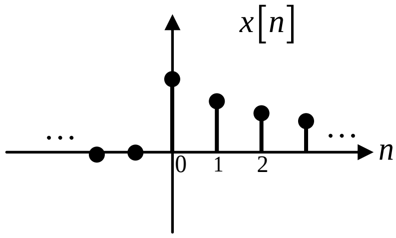


$-1 < \alpha < 0$

if $|\alpha| > 1$ then $|x[n]|$ grows as $n \to \infty$

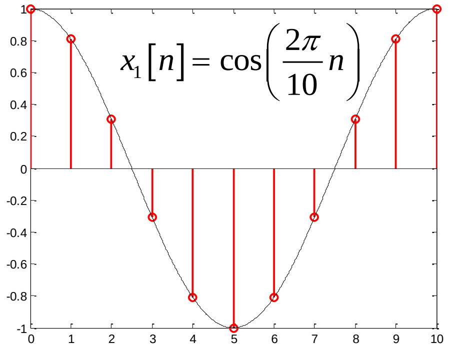
TRUNCATED EXPONENTIAL SEQUENCE:

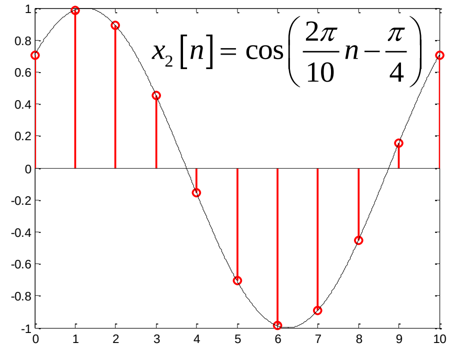
$$x[n] = \begin{cases} A\alpha^n & n \geq 0 \\ 0 & n < 0 \end{cases} = A\alpha^n \, u[n]$$



$0 < \alpha < 1$

SINUSOIDAL SEQUENCES: $x[n] = A\cos(\omega_0 n + \phi)$





Note that, $x_1[n]$ and $x_2[n]$ cannot be related by a simple time shift.

$\cancel{x_2[n] = x_1\left[n - \frac{5}{4}\right]}$ $\cancel{x_2[n] \neq x_1\left[n - \frac{5}{4}\right]}$

EXPONENTIAL SEQUENCES (complex valued):

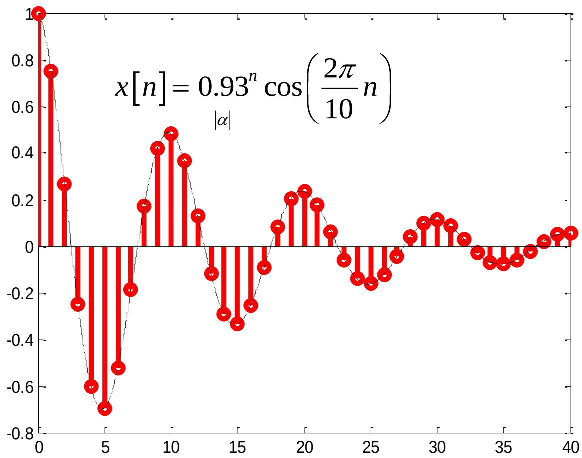
$$x[n] = A\alpha^n \qquad A, \alpha \in C$$

$$A = |A| e^{j\phi} \qquad \alpha = |\alpha| e^{j\omega_0}$$

$$\Rightarrow \; A\alpha^n = |A||\alpha|^n \; e^{j\phi} e^{j\omega_0 n} = |A||\alpha|^n \; e^{j(\omega_0 n + \phi)}$$

$$\Rightarrow \; A\alpha^n = |A||\alpha|^n \; \left(\cos(\omega_0 n + \phi) + j \sin(\omega_0 n + \phi)\right)$$

$$\Rightarrow \; A\alpha^n = |A||\alpha|^n \cos(\omega_0 n + \phi) + \; j |A||\alpha|^n \sin(\omega_0 n + \phi)$$



COMPLEX EXPONENTIAL SEQUENCES:

Let $|\alpha| = 1, \quad A\alpha^n = Ae^{j\omega_0 n} = |A| e^{j(\omega_0 n + \phi)}$ is called a complex exponential sequence.

$$\Rightarrow \; Ae^{j\omega_0 n} = |A| \cos(\omega_0 n + \phi) + \; j |A| \sin(\omega_0 n + \phi)$$

A sinusoidal sequence can be expressed in terms of a complex exponential sequence.

$$M \cos(\omega_0 n + \phi) = Re\{Ae^{j\omega_0 n}\} = \frac{1}{2}\left(Ae^{j\omega_0 n} + A^* e^{-j\omega_0 n}\right); \quad A = Me^{j\phi}, \qquad M \in R$$

$$M \sin(\omega_0 n + \phi) = Im\{Ae^{j\omega_0 n}\} = \frac{1}{2j}\left(Ae^{j\omega_0 n} - A^* e^{-j\omega_0 n}\right); \quad A = Me^{j\phi}, \qquad M \in R$$

ω_0	: frequency	(radians/sample or radians)
ϕ	: phase shift	(radians)

TWO FUNDAMENTAL PROPERTIES OF COMPLEX EXPONENTIAL (SINUSOIDAL) DISCRETE-TIME SEQUENCES

FIRST: For any frequency value ω_0, $\omega_0 + k2\pi$ (k: $integer$) is an equivalent frequency value, i.e.,

if $\quad x[n] = Ae^{j\omega_0 n}$ and $y[n] = Ae^{j(\omega_0 + k2\pi)n}$

then $\quad x[n] = y[n] \quad \forall n \in Z$

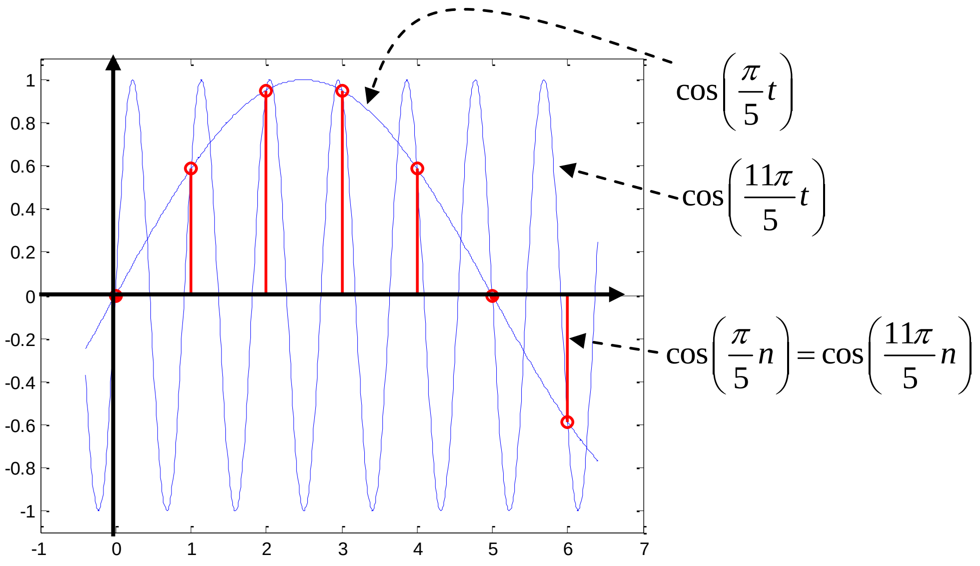
In other words, the elements of the set $\{\omega | \omega = \omega_0 + k2\pi, \omega_0 \in R, k \in Z\}$ are equivalent if they are considered as the frequencies of discrete-time complex exponentials/sinusoids.

Note that $\quad \cos(\omega_0 n) = \cos(\omega_0 n + k2\pi n)$ and $\sin(\omega_0 n) = \sin(\omega_0 n + k2\pi n)$

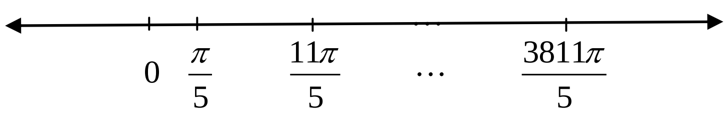
Ex: $\quad \ldots = \cos\left(-\frac{9\pi}{5}n\right) = \cos\left(\frac{\pi}{5}n\right) = \cos\left(\frac{11\pi}{5}n\right) = \cos\left(\frac{21\pi}{5}n\right) = \cdots$

Ex: $\quad \ldots = e^{-j\frac{9\pi}{5}n} = e^{j\frac{\pi}{5}n} = e^{j\frac{11\pi}{5}n} = e^{j\frac{21\pi}{5}n} = \cdots$

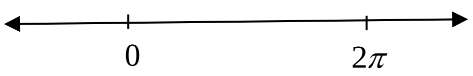
Ex:

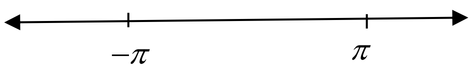



```
clear all
close all
n = [-0.4:0.01:6.4];
nn = [0:6];
x = sin(n*pi/5);
y = sin(n*11*pi/5);
z = sin(nn*11*pi/5);
plot(n,x);
hold
plot(n,y)
stem(nn,z,'r','LineWidth',2
)
v=axis;
v=[v(1) v(2) -1.1 1.1];
axis(v)
```



Therefore an interval of 2π (indeed an interval of π! Why?) covers all distinct frequencies.





Ex:

i) cos(416.31πn) = cos(208.155(2πn)) = cos(0.155(2πn)) = cos(0.31πn)

ii) sin(416.31πn) = sin(208.155(2πn)) = sin(0.155(2πn)) = sin(0.31πn)

Ex:

i) cos(417.31πn) = cos(208.655(2πn)) = cos(0.655(2πn)) = cos(1.31πn) = cos(0.69πn)

ii) sin(417.31πn) = sin(208.655(2πn)) = sin(0.655(2πn)) = sin(1.31πn) = -sin(0.69πn) (minus sign !!!)

Practically, it is sufficient consider

$$\cos(f\pi n), \sin(f\pi n) \quad \text{for} \quad 0 \le f \le 1$$

since

$$\cos(f\pi n) = \cos\big((2-f)\pi n\big)$$

and

$$\sin(f\pi n) = -\sin\big((2-f)\pi n\big)$$

Ex: Consider $x[n] = \cos(0.8\pi n)$ obtained by sampling a 100 MHz signal $x_C(t) = \cos(2 \times 10^8 \pi t)$ at a sampling rate of 250 MHz (i.e. sampling period is $T = \frac{1}{250\,000\,000} = 4$ pico sec.). Find another continuous-time (CT) sinusoid that would yield the same discrete-time sinusoid (i.e., $x[n]$) at this sampling frequency. How many other CT sinusoids would yield the same DT sequence?

Solution:
We know that

$$x[n] = \cos(0.8\pi n) = x_C(t)|_{t=nT} = \cos(2 \times 10^8 \pi T n).$$

Note that

$$x[n] = \cos(0.8\pi n) = \cos(0.4 \times 2\pi n) = \cos(f_0 T 2\pi n)$$

i.e., $f_0 T = 0.4$ where $f_0 = 10^8\ Hz$.

Remember that

$$\cos(0.4 \times 2\pi n) = \cos(1.4 \times 2\pi n) = \cos(2.4 \times 2\pi n) = \cdots$$

Therefore, for example selecting $\cos(1.4 \times 2\pi n)$ yields

$$\cos(1.4 \times 2\pi n) = \cos(f_0' T 2\pi n)$$

$$f_0' = \frac{1.4}{T} = 350 \text{ MHz}$$

(indeed all CT sinusoids at frequencies 350 MHz, 600 MHz, 850 MHz, … yield the same DT sinusoid for this sampling frequency. Their frequencies can be expressed as $\frac{(k+0.4)}{T} = (k+0.4)250$ MHz .)

SECOND:

A DT sinusodal ($\cos(\omega_0 n + \phi)$) or complex exponential signal $e^{j(\omega_0 n+\phi)}$ is not necessarily periodic!

To be periodic,

$$\omega_0 \text{ must be a } \textit{rational} \text{ multiple of } \pi,$$

i.e.,

$$\omega_0 = \frac{p}{q}\pi, \qquad p, q \in Z$$

Proof:

$$A\cos(\omega_0 n+\phi) \overset{?}{=} A\cos(\omega_0(n+N)+\phi)$$

$$A\cos(\omega_0(n+N)+\phi) = A\cos(\omega_0 n+\omega_0 N+\phi)$$

For periodicity $\omega_0 N = k2\pi \quad \Rightarrow \quad \omega_0 = \frac{k}{N}2\pi$ or $\frac{\omega_0}{2\pi} = \frac{k}{N} \qquad k \in \mathrm{Z}$

has to be satisfied.

<u>Ex</u>: $\cos(5n) \qquad \omega_0 = 5 \qquad \frac{\omega_0}{2\pi} = \frac{5}{2\pi}$ is not rational so it is not periodic.

FUNDAMENTAL PERIOD, *N*, IS NOT NECESSARILY EQUAL TO $\frac{2\pi}{\omega_0}$

Corollary: Since, for periodic sinusoids,

$$\omega_0 N = k2\pi$$

i.e,

$$N = \frac{k2\pi}{\omega_0},$$

fundamental period, *N*, is not necessarily equal to $\frac{2\pi}{\omega_0}$.

Finding the Fundamental Period of a Sinusoid

Find the smallest k, k_{min} , so that $k_{min}\frac{2\pi}{\omega_0}$ is an integer.

Then, the fundamental period is

$$N = k_{min}\frac{2\pi}{\omega_0} \quad .$$

Ex: $\cos\left(\frac{\pi}{5}n\right) \qquad \omega_0 = \frac{\pi}{5} \qquad \frac{\omega_0}{2\pi} = \frac{1}{10} \qquad N = k\frac{2\pi}{\omega_0} = k\frac{2\pi}{\frac{\pi}{5}} = 10 \quad (k=1)$

Ex: $\cos\left(\frac{5\pi}{17}n\right) \qquad \omega_0 = \frac{5\pi}{17} \qquad \frac{\omega_0}{2\pi} = \frac{5}{34} \qquad N = k\frac{34}{5} = 34 \quad (k=5)$

Ex: $\cos\left(\frac{6\pi}{5}n\right) \qquad \omega_0 = \frac{6\pi}{5} \qquad \frac{\omega_0}{2\pi} = \frac{3}{5} \qquad N = k\frac{5}{3} = 5 \quad (k=3)$

<u>Ex</u>: Let $x_1[n]=\cos(\omega_1 n)$ and $x_2[n]=\cos(\omega_2 n)$. Find two "frequencies" ω_1 and ω_2 such that $\omega_1 \neq \omega_2 + k2\pi$ for any integer *k*, and $x_1[n]$ and $x_2[n]$ are both periodic with fundamental period *N*=13.

$$N = 13 = k\frac{2\pi}{\omega}, \; k\text{: integer}$$

$$\Rightarrow \omega = k\frac{2\pi}{13}$$

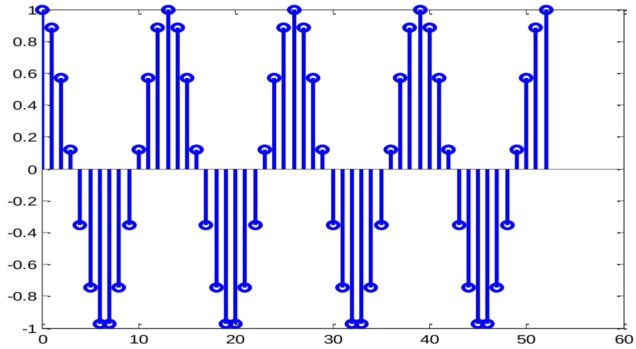
Choose, for example, *k* =1 and *k* =2 $\Rightarrow \omega_1 = \frac{2\pi}{13}, \;\; \omega_2 = \frac{4\pi}{13}$

Therefore,
DT sinusoids may have different "frequencies"
although
their fundamental periods are the same!

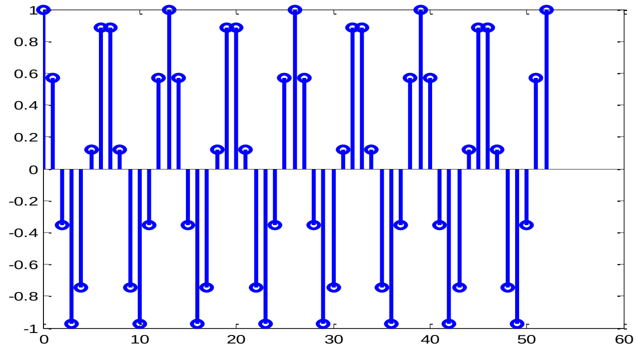
What do the discrete-time sinusoids look like?

Some frequencies between 0 and π

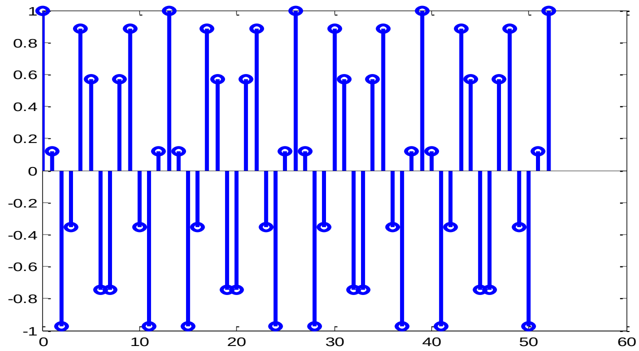
$\cos\left(\frac{2\pi}{13}n\right)$ N=13



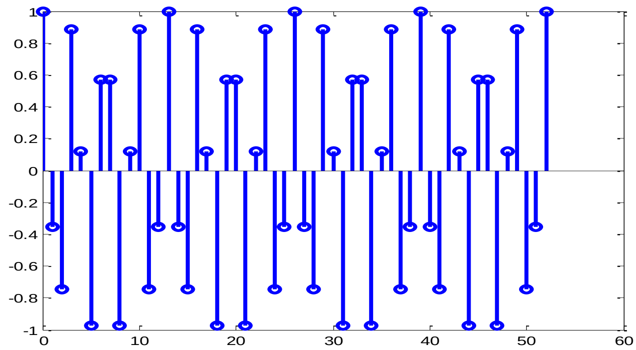
$\cos\left(\frac{4\pi}{13}n\right)$ N=13



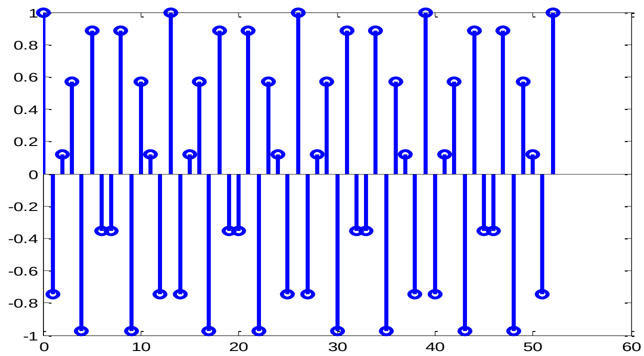
$\cos\left(\frac{6\pi}{13}n\right)$ N= 13



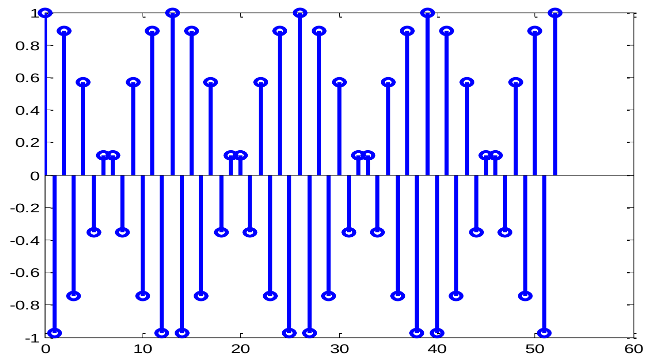
$\cos\left(\frac{8\pi}{13}n\right)$ N=13



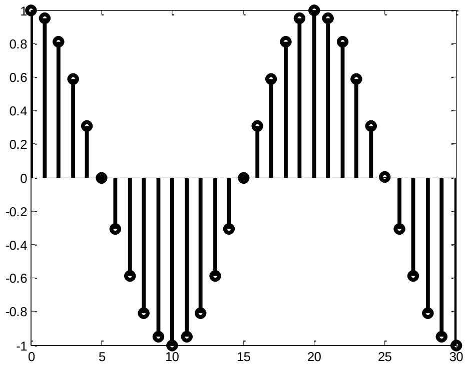
$\cos\left(\frac{10\pi}{13}n\right)$ N = 13



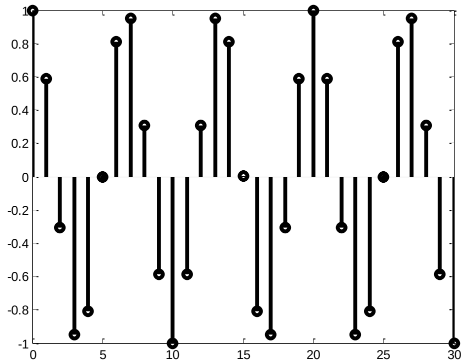
$\cos\left(\frac{12\pi}{13}n\right)$ N=13



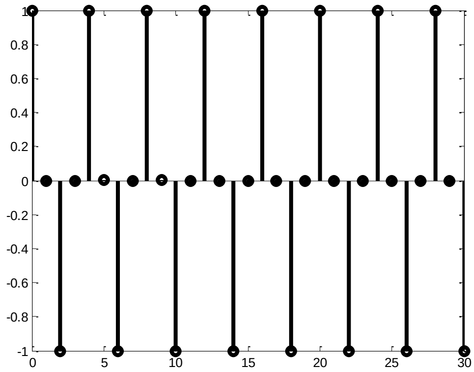
$\cos(0.1\pi n)$ N=20



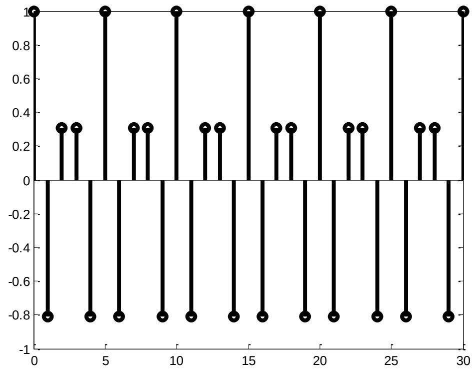
$\cos(0.3\pi n)$ N=20



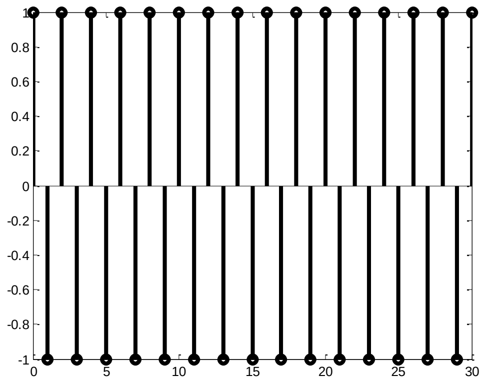
$\cos(0.5\pi n)$ N=4



$\cos(0.8\pi n)$ N=5

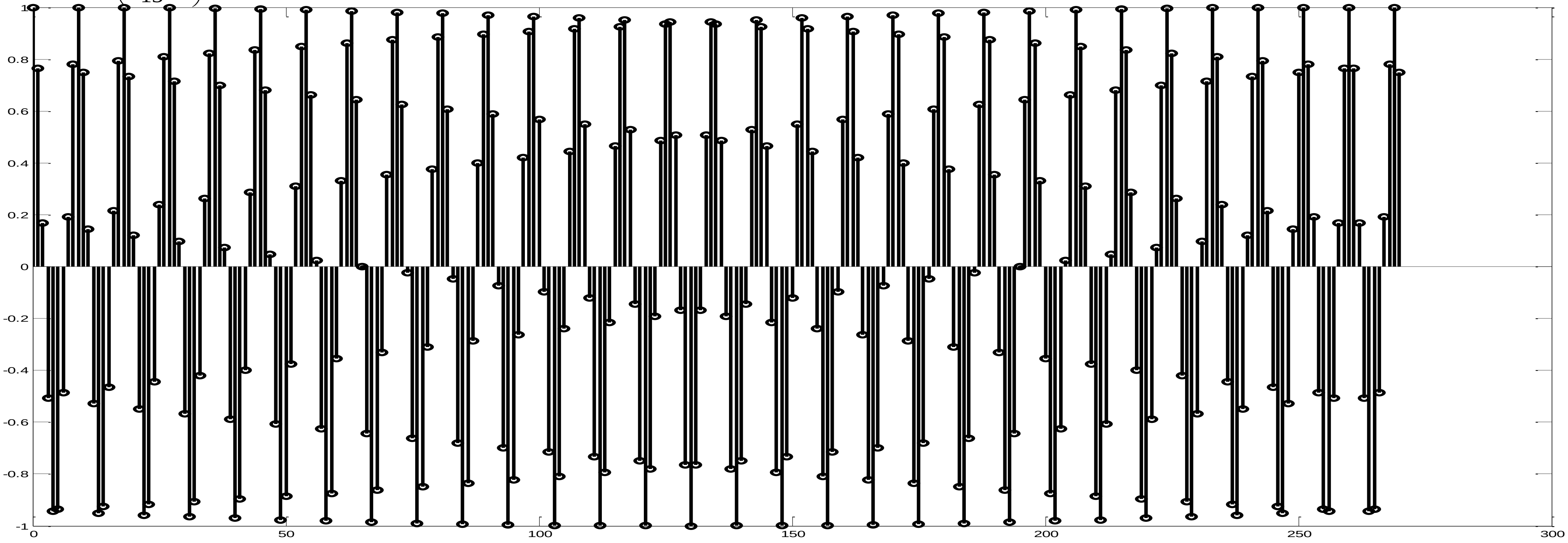


$\cos(\pi n)$ N= 2

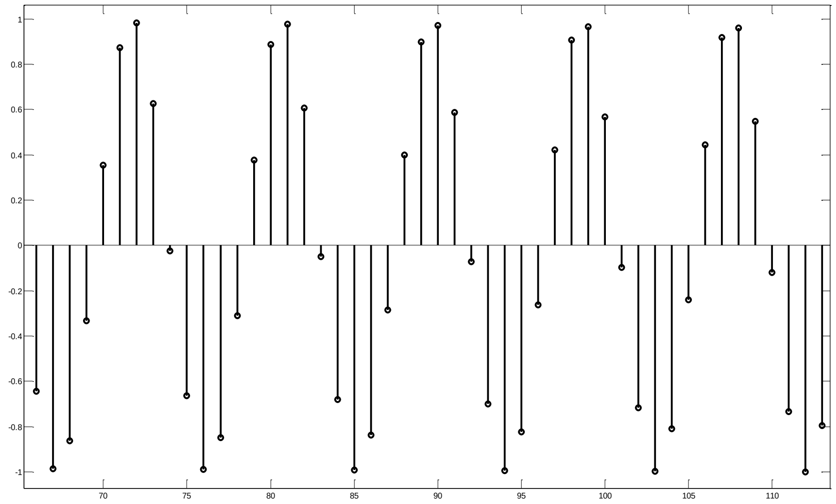


$$\cos\left(\frac{2.9\pi}{13}n\right)$$

Fundamental period is N=260

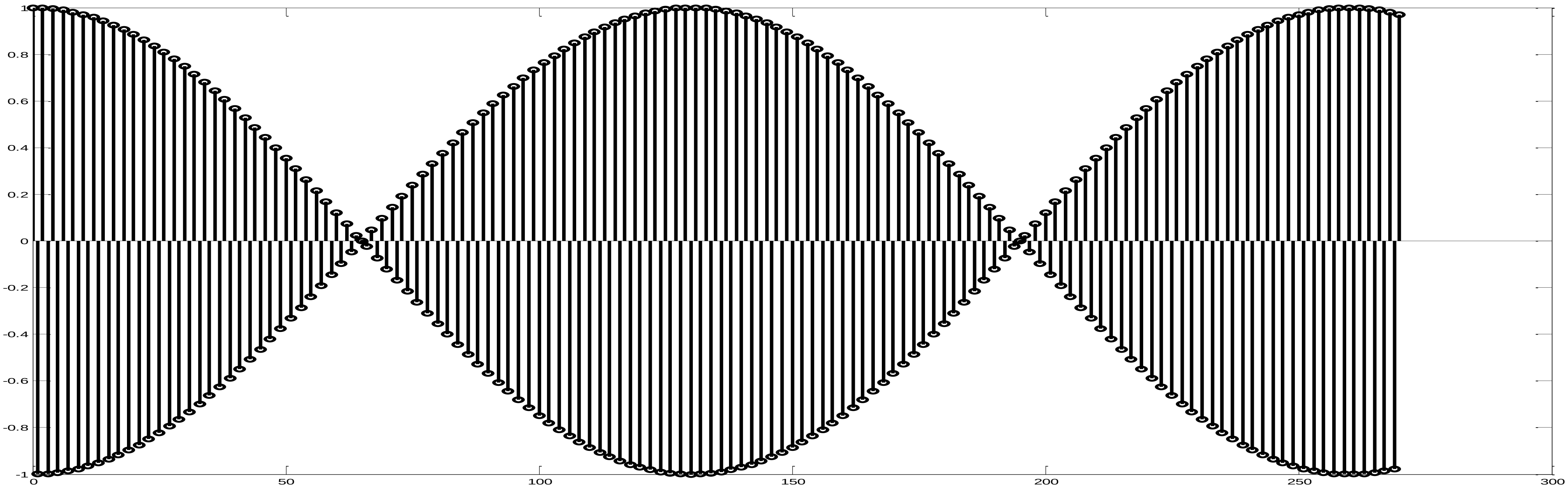


detail



$$\cos\left(\frac{12.9\pi}{13}n\right)$$

Fundamental period is N=260



detail