EE 430

Fall 2014

HW 2 (Section 2)

Solutions for 10-29

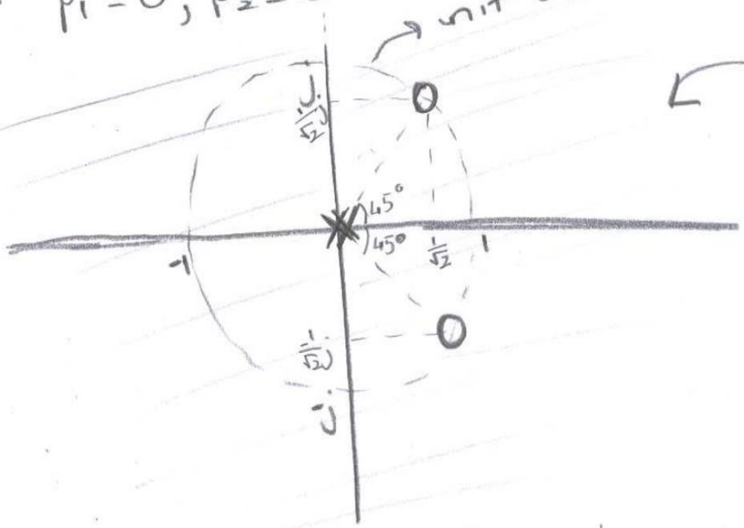
10)

$$h[n] = \delta[n] - \sqrt{2}\,\delta[n-1] + \delta[n-2]$$

a)
$$H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n}$$
$$= \sum_{n=0}^{2} h[n] z^{-n} = 1 - \sqrt{2} z^{-1} + z^{-2}$$
$$= \left(1 - \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}j\right) z^{-1}\right)\left(1 - \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}j\right) z^{-1}\right)$$
$$= \frac{\left(z - \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}j\right)\right)\left(z - \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}j\right)\right)}{z^2}$$

zeros at $z_1 = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}j$, $z_2 = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}j$

poles at $p_1 = 0$, $p_2 = 0$



ROC: $|z| > 0$

(since the only poles at $p_1 = p_2 = 0$)

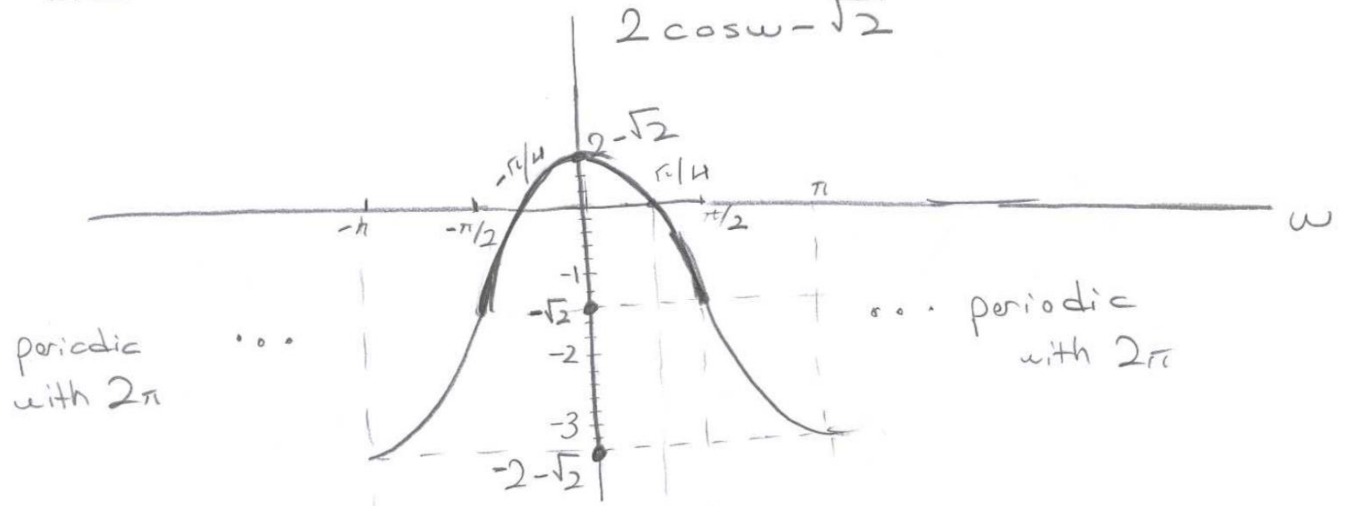
b) Since ROC contains the unit circle, this system has a frequency response.

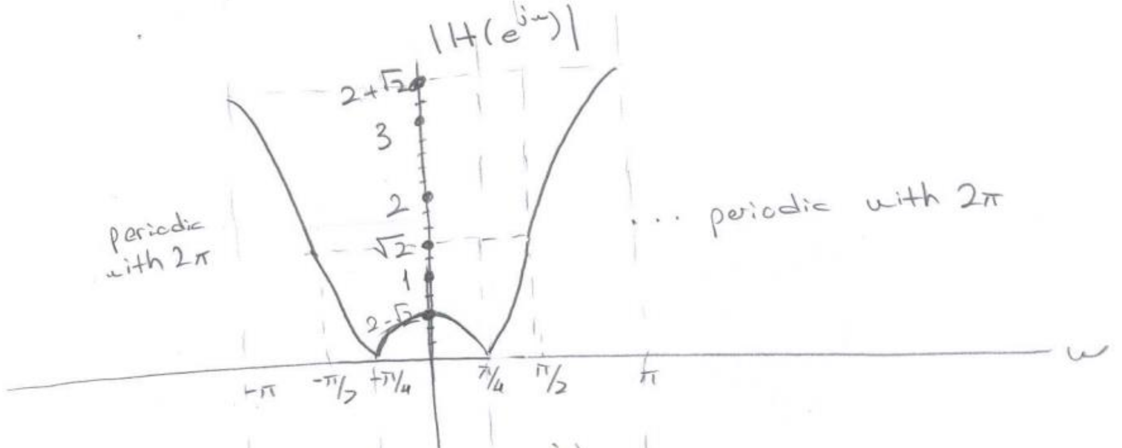
$$H(e^{j\omega}) = H(z)\Big|_{z=e^{j\omega}} = 1 - \sqrt{2}e^{-j\omega} + 1e^{-j2\omega}$$

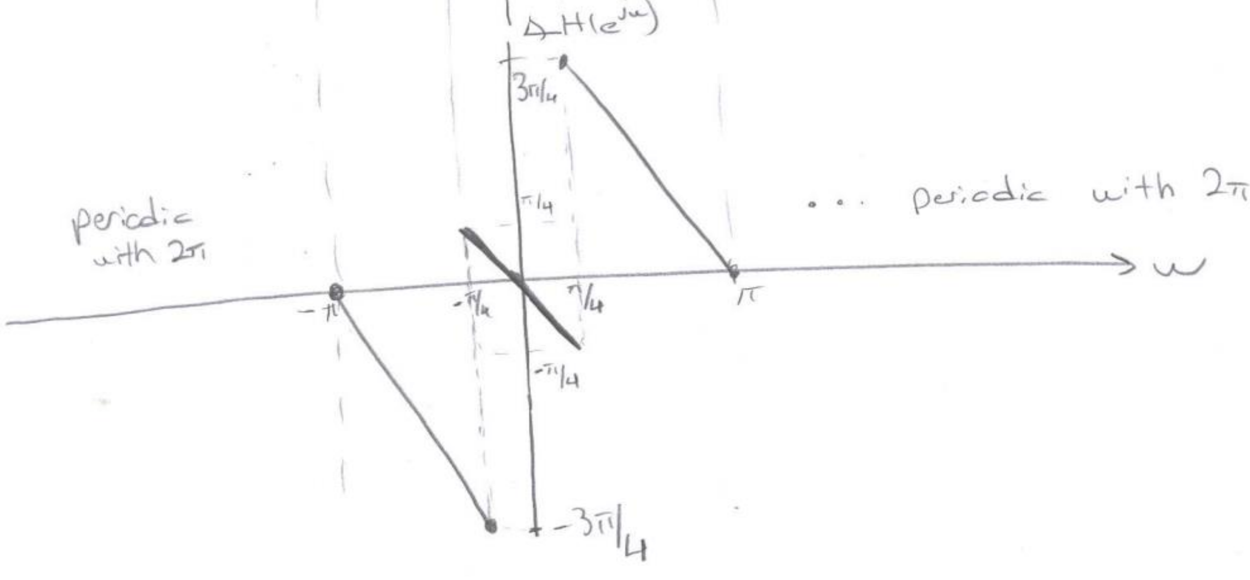
$$= e^{-j\omega}\left(e^{j\omega} - \sqrt{2} + e^{-j\omega}\right) = e^{-j\omega}\left(2\cos\omega - \sqrt{2}\right)$$

$$|H(e^{j\omega})| = |2\cos\omega - \sqrt{2}|$$

$$\measuredangle H(e^{j\omega}) = -\omega + \measuredangle\left(2\cos\omega - \sqrt{2}\right)$$







c) Using the eigenfunction property;

$$x[n] = \sum_k a_k e^{j\omega_k n} \longrightarrow \boxed{h[n]} \longrightarrow y[n] = \sum_k a_k H(e^{j\omega_k}) e^{j\omega_k n}$$

i) $x_1[n] = \sin\left(\frac{\pi}{4} n + \frac{\pi}{4}\right) = \dfrac{e^{j\frac{\pi}{4}n + \frac{\pi}{4}} - e^{-j\frac{\pi}{4}n - \frac{\pi}{4}}}{2j}$

$$y_1[n] = \frac{e^{j\frac{\pi}{4}}}{2j} \underbrace{H(e^{j\frac{\pi}{4}})}_{0} e^{j\frac{\pi}{4}n} - \frac{e^{-j\frac{\pi}{4}}}{2j} \underbrace{H(e^{-j\frac{\pi}{4}})}_{0} e^{-j\frac{\pi}{4}n}$$

$y_1[n] = 0$

ii) $x[n] = e^{j\omega n} u[n]$

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k] = \sum_{k=-\infty}^{n} h[k]\, e^{j\omega(n-k)}$$

Since $h[n]$ is causal;

$$y[n] = \begin{cases} 0 & n < 0 \\ \left(\sum_{k=0}^{n} h[k] e^{-j\omega k} \right) e^{j\omega n} & n \geq 0 \end{cases}$$

For $n \geq 0$

$$y[n] = \left(\sum_{k=0}^{\infty} h[k] e^{-j\omega k} \right) e^{j\omega n} - \left(\sum_{k=n+1}^{\infty} h[k] e^{-j\omega k} \right) e^{j\omega n}$$

$$= H(e^{j\omega}) e^{j\omega n} - \left(\sum_{k=n+1}^{\infty} h[k] e^{-j\omega k} \right) e^{j\omega n}$$

$$x_2[n] = \sin\left(\frac{\pi}{4}n + \frac{\pi}{4}\right)u[n] = \left(\frac{e^{j\pi/4}}{2j} e^{j\frac{\pi}{4}n} - \frac{e^{-j\pi/4}}{2j} e^{-j\frac{\pi}{4}n}\right)u[n]$$

$$y_2[n] = \frac{e^{j\pi/4}}{2j} e^{j\frac{\pi}{4}n} \underbrace{H(e^{j\pi/4})}_{0} - \frac{e^{-j\pi/4}}{2j} e^{-j\pi/4 n} \underbrace{H(e^{-j\pi/4})}_{0}$$

$$-\left(\sum_{k=n+1}^{\infty} h[k] e^{-j\frac{\pi}{4}k}\right) \frac{e^{j\pi/4}}{2j} e^{j\pi/4 n} + \left(\sum_{k=n+1}^{\infty} h[k] e^{+j\frac{\pi}{4}k}\right) \frac{e^{-j\pi/4}}{2j} e^{-j\pi/4 n}$$

$n \geq 0$

$y_2[n] = 0 \qquad n < 0$

$y_2[n] = 0 \qquad n < 0$

$$y_2[0] = -\left(\sum_{k=1}^{2} h[k] e^{-j\pi/4 k}\right) \frac{e^{j\pi/4}}{2j} e^{j\pi/4 \cdot 0} + \left(\sum_{k=1}^{2} h[k] e^{+j\frac{\pi}{4}k}\right) \frac{e^{-j\pi/4}}{2j} e^{-j\pi/4 \cdot 0}$$

$$= -\left(-\sqrt{2}\, e^{-j\pi/4} + 1 e^{-j\pi/2}\right) \frac{e^{j\pi/4}}{2j} + \left(-\sqrt{2}\, e^{j\pi/4} + 1 e^{j\pi/2}\right) \frac{e^{-j\pi/4}}{2j}$$

$$= \cancel{-j\frac{\sqrt{2}}{2}} + j\frac{e^{-j\pi/4}}{2} \cancel{+j\frac{\sqrt{2}}{2}} - j\frac{e^{j\pi/4}}{2} = \frac{j}{2}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}j\right) - \frac{j}{2}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}j\right)$$

$$= \frac{1}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$y_2[1] = -\left(\sum_{k=2}^{2} h[k]\, e^{-j\pi/4 k}\right) \frac{e^{j\pi/4}}{2j} e^{j\pi/4} + \left(\sum_{k=2}^{2} h[k] e^{j\pi/4 k}\right) \frac{e^{-j\pi/4}}{2j} e^{-j\pi/4}$$

$$= -1\, e^{-j\pi/2} \frac{e^{j\pi/2}}{2j} + e^{j\pi/2} \frac{e^{-j\pi/2}}{2j} = 0$$

$$y_2[n] = 0 \quad n \geq 2$$

$$y_2[n] = \frac{1}{\sqrt{2}} \delta[n]$$

iii) $x_3[n] = \underbrace{\sin\left(\frac{\pi}{4}n + \frac{\pi}{4}\right)}_{x_4[n]} + \underbrace{\sin\left(\frac{3\pi}{4}n\right)}_{x_5[n]}$

$y_3[n] = y_4[n] + y_5[n]$

$y_4[n]$: $x_4[n] * h[n] = 0$ (from i))

$y_5[n]$: $x_5[n] * h[n]$

$$y_3[n] = \frac{e^{j3\pi/4 n}}{2j} H(e^{j3\pi/4}) - \frac{e^{-j3\pi/4 n}}{2j} H(e^{-j3\pi/4})$$

$$H(e^{j\frac{3\pi}{4}}) = e^{-j\frac{3\pi}{4}}\Big(2\underbrace{\cos\frac{3\pi}{4}}_{\frac{-1}{\sqrt{2}}} - \sqrt{2}\Big) = 2\sqrt{2}\, e^{j\pi/4}$$

$$H(e^{-j3\pi/4}) = e^{j3\pi/4}\Big(2\cos\Big(\frac{-3\pi}{4}\Big) - \sqrt{2}\Big) = 2\sqrt{2}\, e^{-j\pi/4}$$

$$y_3[n] = 2\sqrt{2}\,\frac{e^{j\left(\frac{3\pi}{4}n + \frac{\pi}{4}\right)}}{2j} - 2\sqrt{2}\,\frac{e^{j\left(-\frac{3\pi}{4}n - \frac{\pi}{4}\right)}}{2j}$$

$$\boxed{y_3[n] = 2\sqrt{2}\sin\Big(\frac{3\pi}{4}n + \frac{\pi}{4}\Big)}$$

d) When the zeros of $H(z)$ are on the unit circle, at the frequencies where these zeros are located $H(e^{j\omega})$ (frequency response) is 0. For example when $H(z)$ has a zero where $z_0 = e^{j\omega_0}$; $H(e^{j\omega_0})$ is 0.

(11)

a) $h[n] = -2^{n} u[-n-1] + \sqrt{2} \times 2^{n-1} u[-n] - 2^{n-2} u[-n+1]$.

b) Since $h[n] \neq 0$ for $n<0$, the system is not causal.

c) $y[n] - 2y[n-1] = x[n] - \sqrt{2}\, x[n-1] + x[n-2]$

(12)

$x[n] \xleftrightarrow{F} X(e^{j\omega})$

$y[n] \xleftrightarrow{F} Y(e^{j\omega})$

$z[n] = x[n]y[n] \xleftrightarrow{F} Z(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})\, Y(e^{j(\omega-\theta)})\, d\theta$

Proof:

$$Z(e^{j\omega}) = \sum_{n=-\infty}^{\infty} z[n] e^{-j\omega n} = \sum_{n=-\infty}^{\infty} \underbrace{x[n]}_{\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta}) e^{j\theta n} d\theta} y[n] e^{-j\omega n}$$

$$Z(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta}) e^{j\theta n} d\theta \; y[n] e^{-j\omega n}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta}) \underbrace{\sum_{n=-\infty}^{\infty} y[n] e^{-jn(\omega-\theta)}}_{Y(e^{j(\omega-\theta)})} d\theta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})\, Y(e^{j(\omega-\theta)})$$

(13)

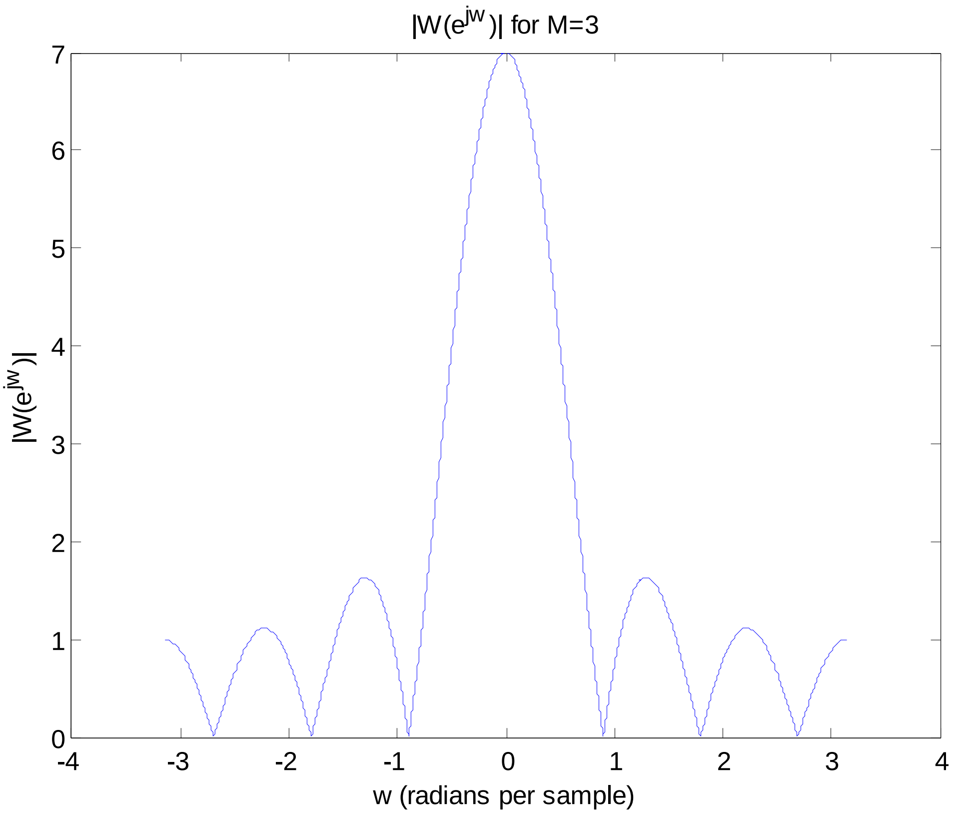
a) $Y(e^{jw}) = \frac{1}{2} W(e^{j(w-\frac{\pi}{4})}) + \frac{1}{2} W(e^{j(w+\frac{\pi}{4})})$

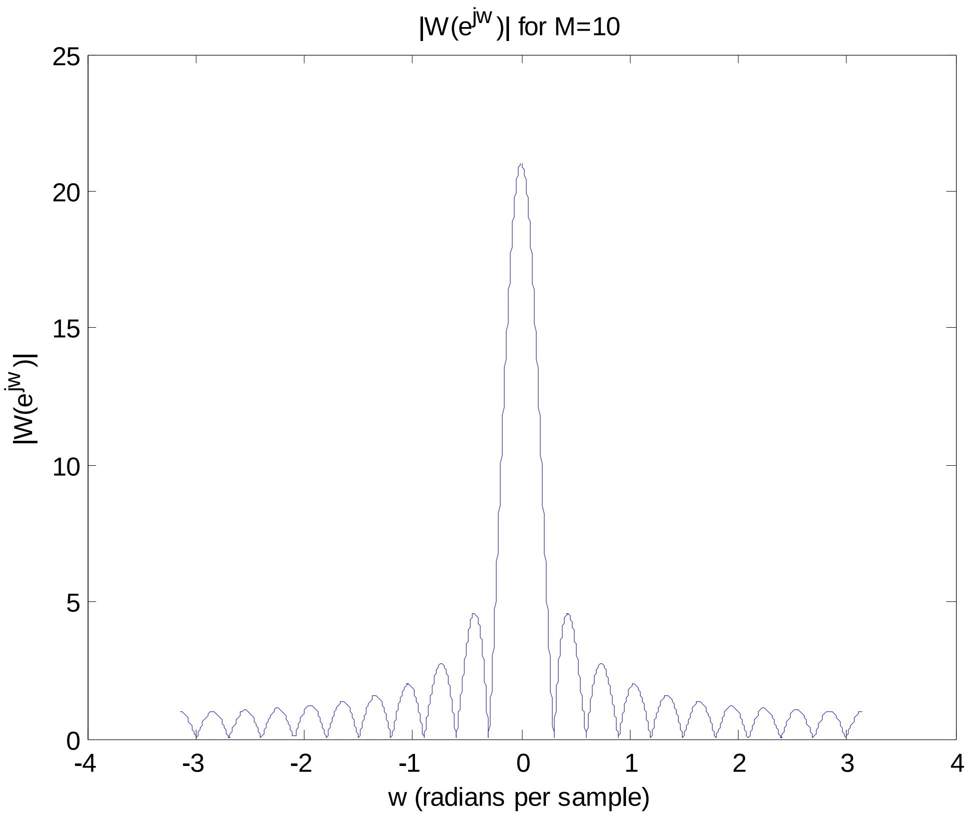
b) $W(e^{jw}) = \dfrac{\sin\left(\left(M+\frac{1}{2}\right)w\right)}{\sin\left(\frac{w}{2}\right)}$

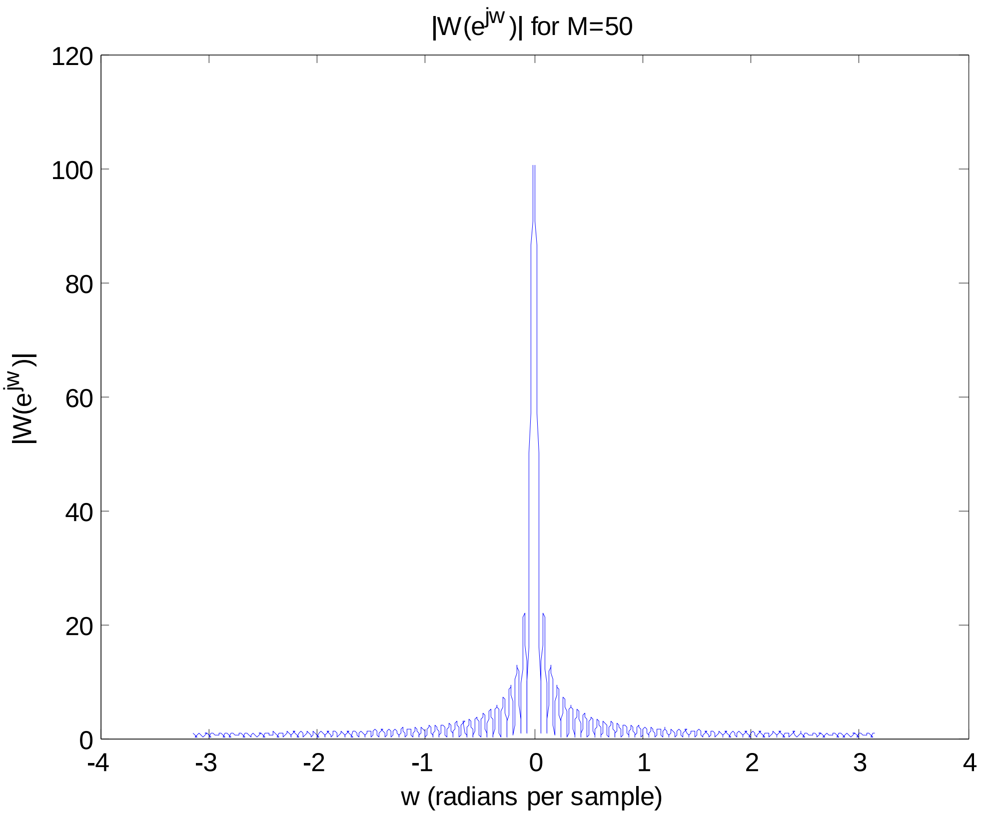
You can use the following code for plotting $|W(e^{jw})|$ in MATLAB:

```
M=3
w=-pi: pi/1000:pi
W= (sin((M+0.5)*w))./(sin(w/2));
plot(w, abs(W))
```

* Note that at $w=0$. W is indefinite in MATLAB
You can insert the actual value "2M+1" at $w=0$
as `W(w==0) =2M+1;`.


|W(e^jw)| for M=3
|W(e^jw)|
w (radians per sample)
7
6
5
4
3
2
1
0
-4
-3
-2
-1
0
1
2
3
4


|W(e^jw)| for M=10
|W(e^jw)|
w (radians per sample)
25
20
15
10
5
0
-4
-3
-2
-1
0
1
2
3
4


$|W(e^{jw})|$ for M=50
$|W(e^{jw})|$
w (radians per sample)
120
100
80
60
40
20
0
-4
-3
-2
-1
0
1
2
3
4

c) $Y(e^{j\omega}) = \frac{1}{2} W(e^{j(\omega-\frac{\pi}{4})}) + \frac{1}{2} W(e^{j(\omega+\frac{\pi}{4})})$

i) MATLAB Command:

```
M=3;
w=-pi:pi/1000:pi
Y = 1/2 (sin((M+0.5)*(w-pi/4))./(sin((w-pi/4)/2)...
+ 1/2 (sin((M+0.5)*(w+pi/4))./(sin((w+pi/4)/2));
plot(w, abs(Y))
```

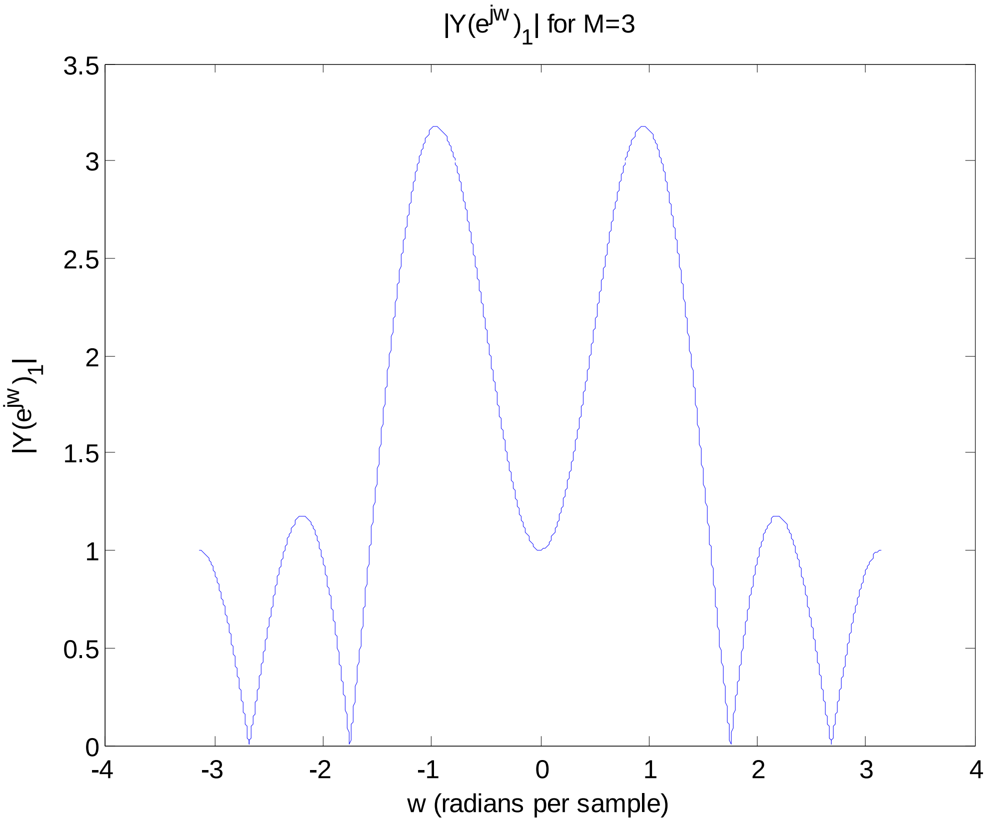
ii) The frequencies at which the peak of $|Y(e^{j\omega})|$ is observed:

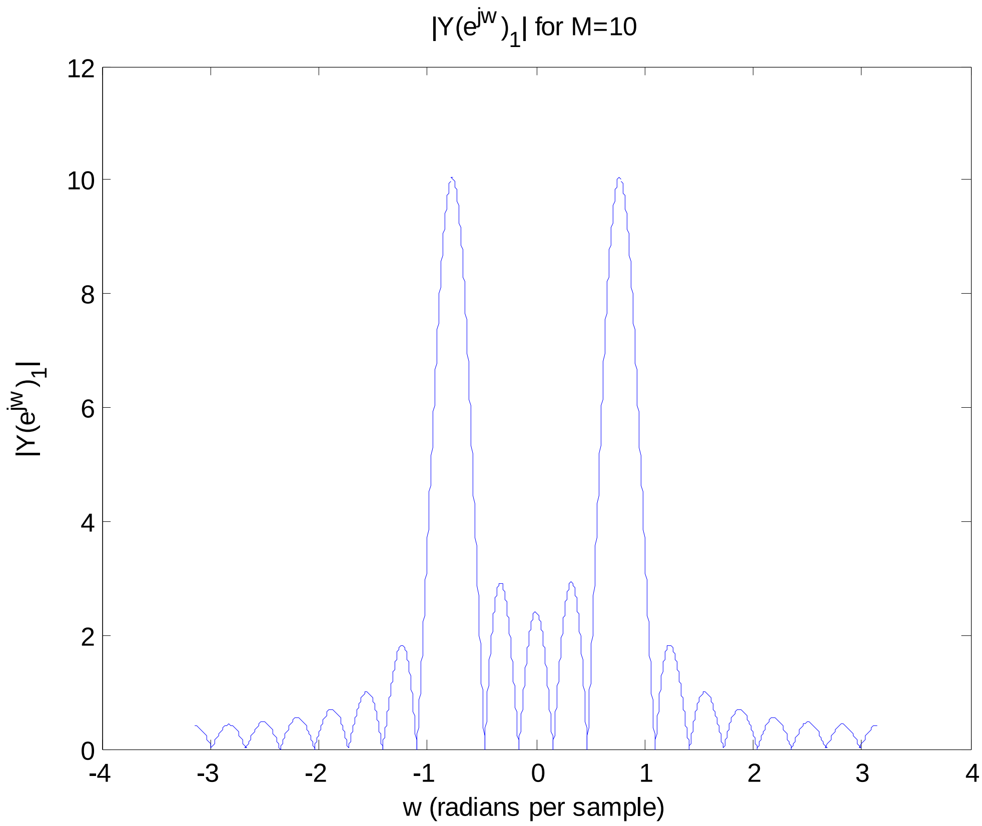
$M=3 \Longrightarrow \omega \approx 0.302\pi, -0.302\pi$

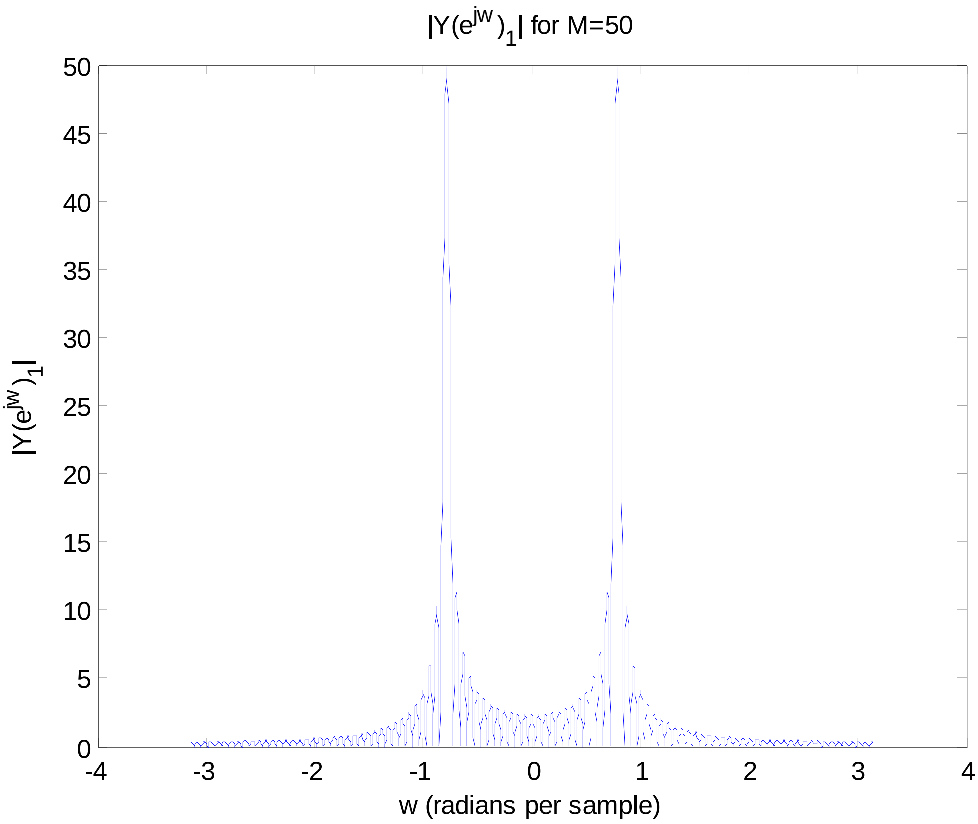
$M=10 \Longrightarrow \omega \approx 0.245\pi, -0.245\pi$

$M=50 \Longrightarrow \omega \approx 0.249\pi, -0.249\pi$

If $M=\infty$ we expect that these frequencies are $\omega = 0.25\pi, -0.25\pi$ since $W(e^{j\omega}) = \sum_{k=-\infty}^{\infty} 2\pi(\delta(\omega+2\pi k))$ and there is no interperetration between $W(e^{j(\omega-\frac{\pi}{4})})$ and $W(e^{j(\omega+\frac{\pi}{4})})$. However for finite values of M, $W(e^{j\omega})$ has not an impulse shape but sinc characteristics. The side lobes of the sinc functions result interperetration and the maxima of $Y(e^{j\omega})$ occurs at some frequency other than $\omega = 0.25\pi, -0.25\pi$. As M increases, the peak values of side lobes decrease and ω_{max} approaches $0.25\pi, -0.25\pi$.


|Y(e^jw)_1| for M=3
3.5
3
2.5
2
1.5
1
0.5
0
-4
-3
-2
-1
0
1
2
3
4
w (radians per sample)
|Y(e^jw)_1|


|Y(e^jw)_1| for M=10
12
10
8
6
4
2
0
-4
-3
-2
-1
0
1
2
3
4
w (radians per sample)
|Y(e^jw)_1|



|Y(e^jw)_1| for M=50
|Y(e^jw)_1|
50
45
40
35
30
25
20
15
10
5
0
-4
-3
-2
-1
0
1
2
3
4
w (radians per sample)

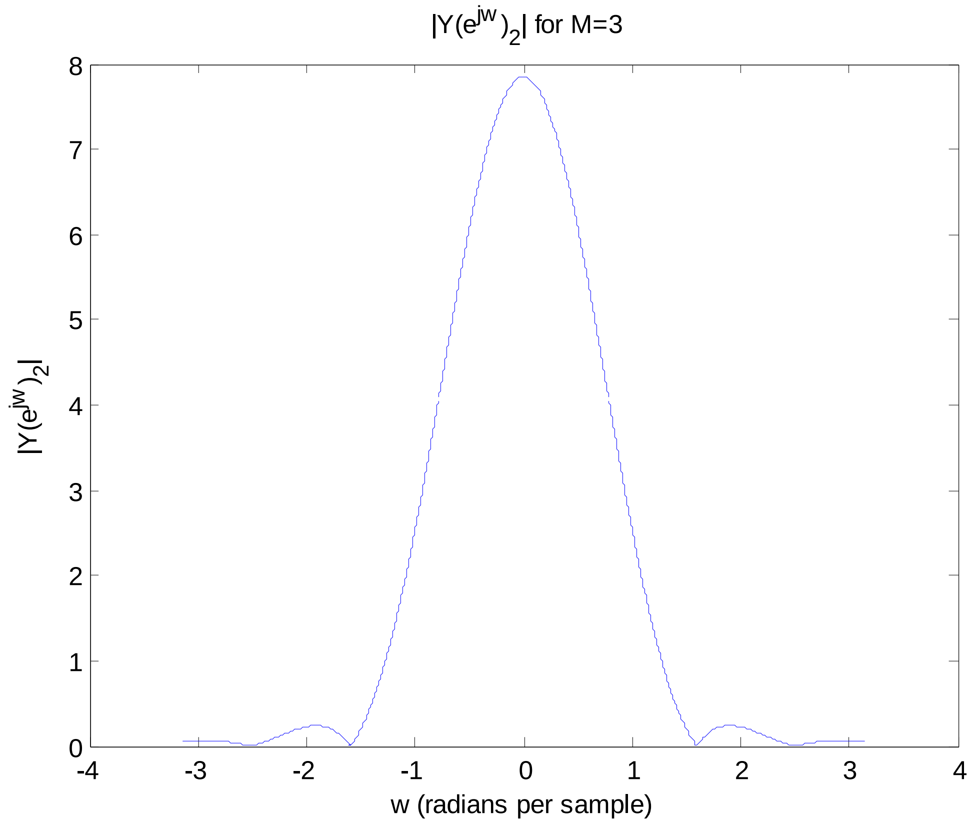
d) $x[n] = \cos\left(\frac{\pi}{4}n\right) + \cos\left(\frac{\pi}{30}n\right)$

i) $$X(e^{jw}) = \sum_{k=-\infty}^{\infty} \pi\delta(w - \frac{\pi}{4} + 2\pi k) + \pi\delta(w + \frac{\pi}{4} + 2\pi k) + \pi\delta(w - \frac{\pi}{30} + 2\pi k) + \pi\delta(w + \frac{\pi}{30} + 2\pi k)$$

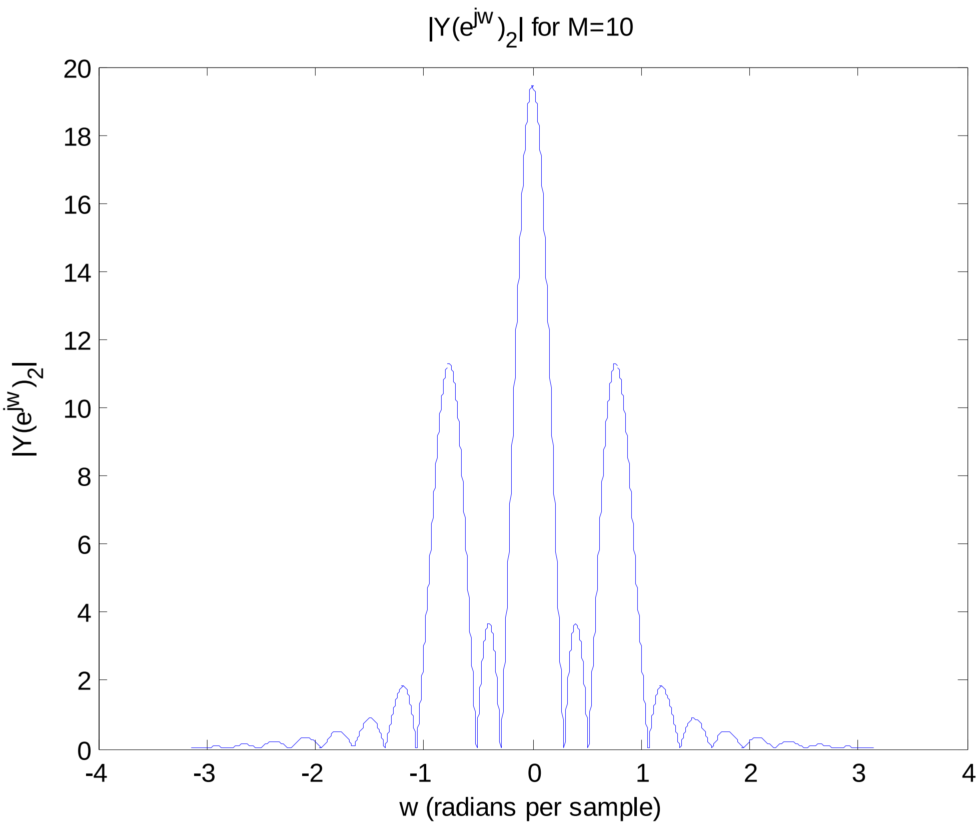
From modulation property,

$$Y(e^{jw}) = \frac{1}{2} W(e^{j(w-\frac{\pi}{4})}) + \frac{1}{2} W(e^{j(w+\frac{\pi}{4})}) + \frac{1}{2} W(e^{j(w-\frac{\pi}{30})}) + \frac{1}{2} W(e^{j(w+\frac{\pi}{30})})$$

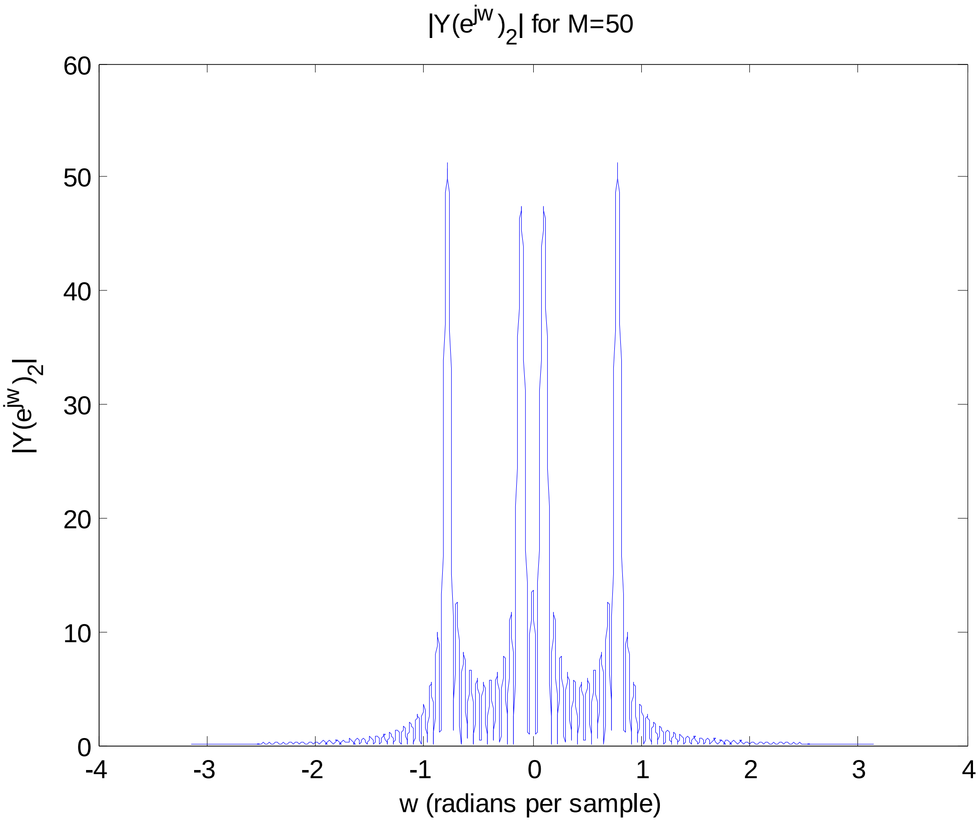
ii) We again see the same effect as in part c). This time, the effect of interpenetration is more visible. In fact for $M=3$, there is only one peak.



|Y(e^jw)_2| for M=3
|Y(e^jw)_2|
w (radians per sample)



|Y(e^jw)_2| for M=10
|Y(e^jw)_2|
w (radians per sample)


$|Y(e^{jw})_2|$ for M=50
$|Y(e^{jw})_2|$
w (radians per sample)
60
50
40
30
20
10
0
-4
-3
-2
-1
0
1
2
3
4