$$\sum_{r=-\infty}^{\infty} \delta(n-rN) = \frac{1}{N}\sum_{p=0}^{N-1} e^{\pm jp\frac{2\pi}{N}n}$$

DFT: DISCRETE FOURIER TRANSFORM

DFS: DISCRETE FOURIER SERIES

DFT: DISCRETE FOURIER TRANSFORM

IDFT: INVERSE DFT

THE SIGNIFICANCE OF DFT AND DFS: SAMPLING THE DTFT

DFT OF WINDOWED SINUSOID (ESTIMATING THE FREQUENCY, …)

PROPERTIES OF DFS AND DFT

1) Linearity
2) Time Shift Property (DFS) – Circular Time Shift Property (DFT)
3) Multiplication by a complex exponential
4) Duality
5) Symmetry Properties Real Sequences
6) Convolution Property
    - Circular Convolution
    - Getting the Result of Linear Convolution Using DFT
7) Sampling the DTFT
8) Multiplication in Time Domain

IMPLEMENTING LTI SYSTEMS USING DFT

- Overlap-Add
- Overlap-Save

LINEAR CONVOLUTION AND CIRCULAR CONVOLUTION

$$\sum_{r=-\infty}^{\infty} \delta(n-rN) = \frac{1}{N}\sum_{p=0}^{N-1} e^{\pm jp\frac{2\pi}{N}n}$$

DFT: DISCRETE FOURIER TRANSFORM

Remember DTFT representation

$$X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X\left(e^{j\omega}\right) e^{j\omega n}\, d\omega$$

$$\sum_{r=-\infty}^{\infty} \delta(n - rN) = \frac{1}{N} \sum_{p=0}^{N-1} e^{\pm jp\frac{2\pi}{N}n}$$

Disadvantage of DTFT from a computational point of view is that DTFT, $X\left(e^{j\omega}\right)$, is a function of a continuous variable, ω.

Therefore it requires infinite storage.

$$\sum_{r=-\infty}^{\infty} \delta(n - rN) = \frac{1}{N}\sum_{p=0}^{N-1} e^{\pm jp\frac{2\pi}{N}n}$$

However, if $x[n]$ has finite length "N", i.e.,

$$x[n] = 0 \quad for \qquad \begin{matrix} n < n_0 \\ n > n_0 + N - 1 \end{matrix} \;,$$

it can be considered as one period of its periodic extension, $\tilde{x}[n]$, defined as

$$\tilde{x}[n] = \sum_{r=-\infty}^{\infty} x[n - rN]$$

$$= x[(n)\ \ modulo\ N]$$

$$\triangleq x\left[\left((n)\right)_N\right]$$

Then, the Fourier series representation of $\tilde{x}[n]$ can be used to represent $x[n]$ as well since

$$x[n] = \begin{cases} \tilde{x}[n] & n = n_0, n_0 + 1, \dots, n_0 + N - 1 \\ 0 & otherwise \end{cases}$$

for some n_0.

$$\sum_{r=-\infty}^{\infty} \delta(n - rN) = \frac{1}{N} \sum_{p=0}^{N-1} e^{\pm jp\frac{2\pi}{N}n}$$

Note that "N" has a lower limit but not upper limit.

Ex: $x[n] = \delta[n] + 2\delta[n-1] - 4\delta[n-2]$

$\Rightarrow N \in \{3,4,5, \dots\}$

Plot a few $\tilde{x}[n]$ sequences for different values of N.

$$\sum_{r=-\infty}^{\infty} \delta(n - rN) = \frac{1}{N} \sum_{p=0}^{N-1} e^{\pm jp\frac{2\pi}{N}n}$$

Note that the DFS representation of a periodic sequence of period N has N coefficients.

Therefore, instead of an infinite set of numbers as required by DTFT, a finite length sequence of length N can be represented by N complex values (Fourier series coefficients).

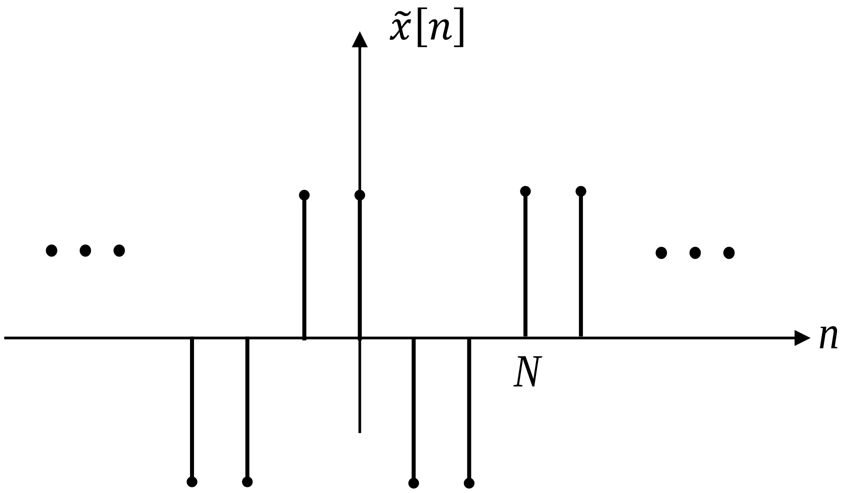
$$\sum_{r=-\infty}^{\infty} \delta(n-rN) = \frac{1}{N}\sum_{p=0}^{N-1} e^{\pm jp\frac{2\pi}{N}n}$$

Now, let’s review Fourier series representation of periodic sequences.

$$\sum_{r=-\infty}^{\infty} \delta(n - rN) = \frac{1}{N} \sum_{p=0}^{N-1} e^{\pm jp\frac{2\pi}{N}n}$$

DFS: DISCRETE FOURIER SERIES

Let $\tilde{x}[n]$ be an arbitrary periodic sequence with fundamental period N;



Its DFS representation is

$$\tilde{x}[n] = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k]\, e^{jk\frac{2\pi}{N}n}$$

$$= \frac{1}{N}\left(\tilde{X}[0] + \tilde{X}[1]\, e^{j\frac{2\pi}{N}n} + \cdots + \tilde{X}[N-1]\, e^{j(N-1)\frac{2\pi}{N}n}\right)$$

$$\sum_{r=-\infty}^{\infty} \delta(n - rN) = \frac{1}{N} \sum_{p=0}^{N-1} e^{\pm jp\frac{2\pi}{N}n}$$

This is a representation in terms of sinusoidal sequences at a fundamental frequency, $\frac{2\pi}{N}$, and its multiples (harmonic components), and a DC component.

$$\left\{1, e^{j\frac{2\pi}{N}n}, e^{j2\frac{2\pi}{N}n}, \dots, e^{j(N-1)\frac{2\pi}{N}n}\right\} = \left\{1, e^{j\omega_0 n}, e^{j2\omega_0 n}, \dots, e^{j(N-1)\omega_0 n}\right\}\Big|_{\omega_0=\frac{2\pi}{N}}$$

$$\sum_{r=-\infty}^{\infty} \delta(n-rN) = \frac{1}{N}\sum_{p=0}^{N-1} e^{\pm jp\frac{2\pi}{N}n}$$

Ex: Let

$$x[n] = \delta[n] + 2\delta[n-1] - \delta[n-2]$$

and

$$\tilde{x}[n] = \sum_{r=-\infty}^{\infty} x[n-3r]$$

$$= x\left[\big((n)\big)_3\right].$$

$$\tilde{x}[n] = \frac{1}{3}\left(2 + \left(\frac{1}{2} - j\,\frac{3\sqrt{3}}{2}\right)e^{j\frac{2\pi}{3}n} + \left(\frac{1}{2} + j\,\frac{3\sqrt{3}}{2}\right)e^{j\frac{4\pi}{3}n}\right)$$

$$\cos\left(\frac{2\pi}{3}n\right) = \cdots, 1, -\frac{1}{2}, -\frac{1}{2}, 1, -\frac{1}{2}, -\frac{1}{2}, \cdots$$
$$\sin\left(\frac{2\pi}{3}n\right) = \cdots, 0, \frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}, 0, \frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}, \cdots$$

$$\cos\left(\frac{4\pi}{3}n\right) = \cdots, 1, -\frac{1}{2}, -\frac{1}{2}, 1, -\frac{1}{2}, -\frac{1}{2}, \cdots$$
$$\sin\left(\frac{4\pi}{3}n\right) = \cdots, 0, -\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}, 0, -\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}, \cdots$$

$$X[1] = 1 + 2e^{-j\frac{2\pi}{3}} - e^{-j\frac{4\pi}{3}} = 1 + 2e^{-j\frac{2\pi}{3}} - e^{j\frac{2\pi}{3}} = 1 + 2\left(-\frac{1}{2} - j\frac{\sqrt{3}}{2}\right) - \left(-\frac{1}{2} + j\frac{\sqrt{3}}{2}\right)$$

$$\sum_{r=-\infty}^{\infty} \delta(n - rN) = \frac{1}{N} \sum_{p=0}^{N-1} e^{\pm jp\frac{2\pi}{N}n}$$

REMARKS:

1) The number of frequency components depends on signal period, N.

$$\sum_{r=-\infty}^{\infty} \delta(n-rN) = \frac{1}{N}\sum_{p=0}^{N-1} e^{\pm jp\frac{2\pi}{N}n}$$

2) In the set

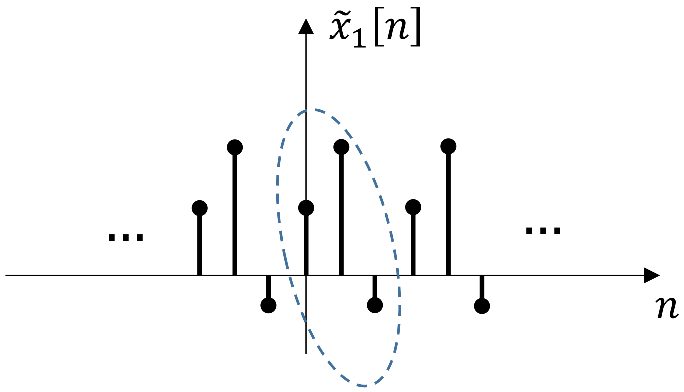
$$\left\{1, e^{j\frac{2\pi}{N}n}, e^{j2\frac{2\pi}{N}n}, \dots, e^{j(N-1)\frac{2\pi}{N}n}\right\} = \left\{1, e^{j\omega_0 n}, e^{j2\omega_0 n}, \dots, e^{j(N-1)\omega_0 n}\right\}\Big|_{\omega_0=\frac{2\pi}{N}}$$

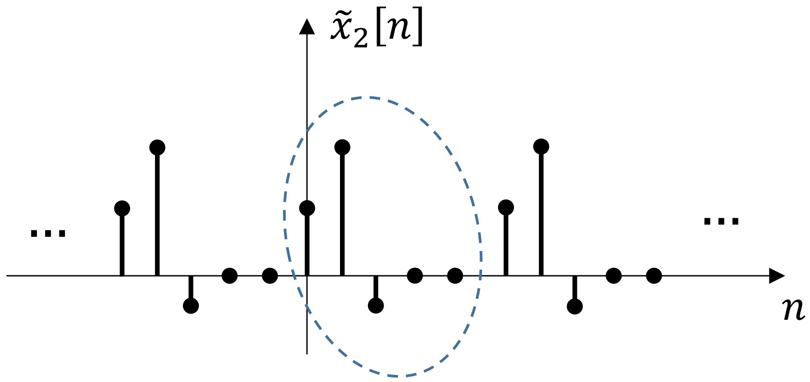
$e^{jk\frac{2\pi}{N}n}$ and $e^{j(N-k)\frac{2\pi}{N}n}$ are complex conjugates of each other, i.e.,

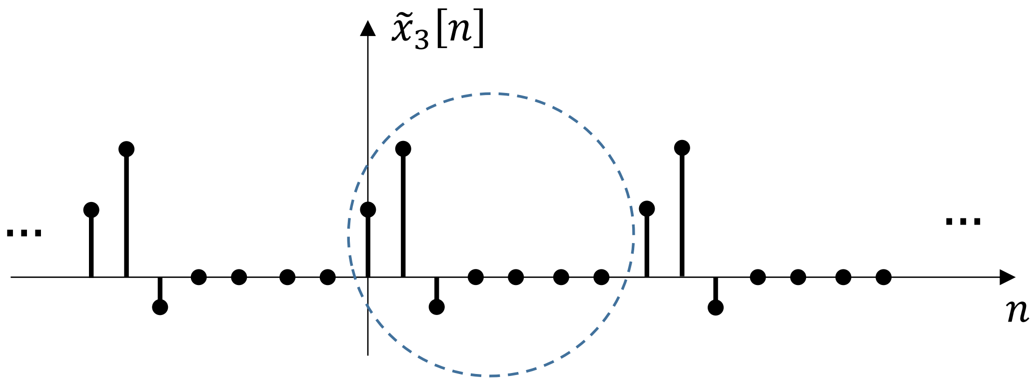
$$e^{jk\frac{2\pi}{N}n} = e^{-j(N-k)\frac{2\pi}{N}n}$$

$$\sum_{r=-\infty}^{\infty} \delta(n - rN) = \frac{1}{N}\sum_{p=0}^{N-1} e^{\pm jp\frac{2\pi}{N}n}$$

3) Sequences of the following nature are equivalent for practical purposes (to handle finite length sequences)







$$\sum_{r=-\infty}^{\infty} \delta(n-rN) = \frac{1}{N}\sum_{p=0}^{N-1} e^{\pm jp\frac{2\pi}{N}n}$$

The DFS coefficients $\tilde{X}[k],\ k = 0,1,\dots,N-1$, can be obtained as

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n]\, e^{-jk\frac{2\pi}{N}n}$$

$$= \frac{1}{N}\left(\tilde{x}[0] + \tilde{x}[1]\, e^{-jk\frac{2\pi}{N}} + \cdots + \tilde{x}[N-1]\, e^{-jk\frac{2\pi}{N}(N-1)}\right)$$

$$\sum_{r=-\infty}^{\infty} \delta(n - rN) = \frac{1}{N} \sum_{p=0}^{N-1} e^{\pm jp\frac{2\pi}{N}n}$$

To obtain this expression multiply

$$\tilde{x}[n] = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k]\, e^{jk\frac{2\pi}{N}n}$$

by

$$e^{-jm\frac{2\pi}{N}n}$$

and sum over $n = 0,1,2,\ldots,N-1$.

$$\sum_{r=-\infty}^{\infty} \delta(n - rN) = \frac{1}{N}\sum_{p=0}^{N-1} e^{\pm jp\frac{2\pi}{N}n}$$

$$\sum_{n=0}^{N-1} \tilde{x}[n]\, e^{-jm\frac{2\pi}{N}n} = \frac{1}{N}\sum_{n=0}^{N-1}\sum_{k=0}^{N-1} \tilde{X}[k]\, e^{jk\frac{2\pi}{N}n}\, e^{-jm\frac{2\pi}{N}n}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} \tilde{X}[k] \sum_{n=0}^{N-1} e^{j(k-m)\frac{2\pi}{N}n}$$

Here

$$\sum_{n=0}^{N-1} e^{j(k-m)\frac{2\pi}{N}n} = \begin{cases} N & if \quad k - m = qN \quad q \in Z \\ 0 & otherwise \end{cases}$$

Therefore

$$\sum_{n=0}^{N-1} \tilde{x}[n]\, e^{-jm\frac{2\pi}{N}n} = \tilde{X}[m]$$

$$\sum_{r=-\infty}^{\infty} \delta(n - rN) = \frac{1}{N} \sum_{p=0}^{N-1} e^{\pm jp\frac{2\pi}{N}n}$$

Note that $\tilde{X}[k]$ is periodic with N since

$$\tilde{X}[k + rN] = \sum_{n=0}^{N-1} \tilde{x}[n]\, e^{-jk\frac{2\pi}{N}n}\, \underbrace{e^{-jr\frac{2\pi}{N}Nn}}_{=1}$$

$$= \tilde{X}[k]\,.$$

So, it is sufficient to know N values.

$$\sum_{r=-\infty}^{\infty} \delta(n - rN) = \frac{1}{N} \sum_{p=0}^{N-1} e^{\pm jp\frac{2\pi}{N}n}$$

<u>Notation</u>: For convenience define $W_N \triangleq e^{-j\frac{2\pi}{N}}$

$$\tilde{x}[n] = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k]\, W_N^{-kn}$$

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n]\, W_N^{kn}$$

$W_N^{-kn} = e^{jk\frac{2\pi}{N}n}$ is the k^{th} harmonic.

$$\sum_{r=-\infty}^{\infty} \delta(n - rN) = \frac{1}{N} \sum_{p=0}^{N-1} e^{\pm jp\frac{2\pi}{N}n}$$

Ex:

$$\tilde{x}[n] = \sum_{r=-\infty}^{\infty} \delta[n - rN]$$

plot

DFS coefficients of $\tilde{x}[n]$

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \delta[n]\, W_N^{kn} = 1$$

plot

(independent of N)

$$\sum_{r=-\infty}^{\infty} \delta(n - rN) = \frac{1}{N} \sum_{p=0}^{N-1} e^{\pm jp\frac{2\pi}{N}n}$$

DFS representation of $\tilde{x}[n]$ in the above example

$$\tilde{x}[n] = \frac{1}{N} \sum_{k=0}^{N-1} W_N^{-kn}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} e^{jk\frac{2\pi}{N}n}$$

$$= \sum_{k=-\infty}^{\infty} \delta[n - kN]$$

$$\sum_{r=-\infty}^{\infty} \delta(n - rN) = \frac{1}{N} \sum_{p=0}^{N-1} e^{\pm jp\frac{2\pi}{N}n}$$

For $N = 2$

$$\tilde{x}[n] = \frac{1}{2} \sum_{k=0}^{1} W_2^{-kn}$$

$$= \frac{1}{2}\left(1 + e^{j\pi n}\right)$$

$$= \sum_{k=\infty}^{\infty} \delta[n - k2]$$

$$\sum_{r=-\infty}^{\infty} \delta(n - rN) = \frac{1}{N} \sum_{p=0}^{N-1} e^{\pm jp\frac{2\pi}{N}n}$$

For $N = 3$

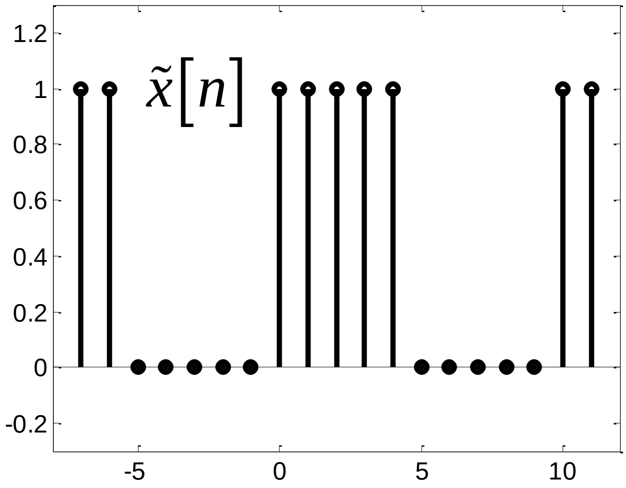
$$\tilde{x}[n] = \frac{1}{3} \sum_{k=0}^{2} W_3^{-kn}$$

$$= \frac{1}{2}\left(1 + e^{j\frac{2\pi}{3}n} + e^{j2\frac{2\pi}{3}n}\right)$$

$$= \sum_{k=-\infty}^{\infty} \delta[n - k3]$$

$$\sum_{r=-\infty}^{\infty} \delta(n - rN) = \frac{1}{N} \sum_{p=0}^{N-1} e^{\pm jp\frac{2\pi}{N}n}$$

Ex:



$N = 10$

DFS coefficients of $\tilde{x}[k]$

$$\tilde{X}[k] = \sum_{n=0}^{4} W_{10}^{kn}$$

$$= \frac{1 - W_{10}^{k5}}{1 - W_{10}^{k}}$$

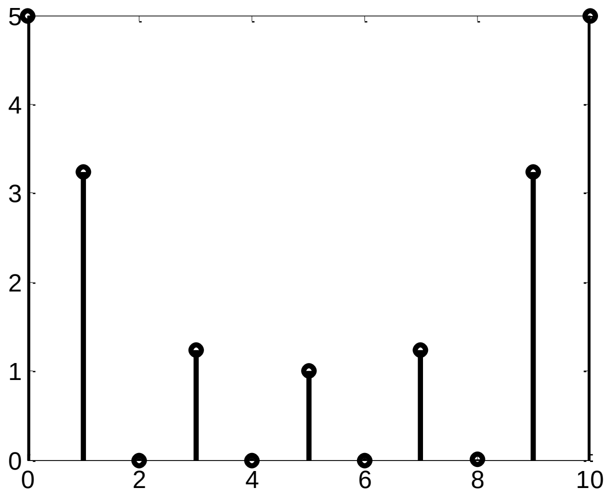
$$= \frac{1 - e^{-j\pi k}}{1 - e^{-j\frac{\pi}{5}k}}$$

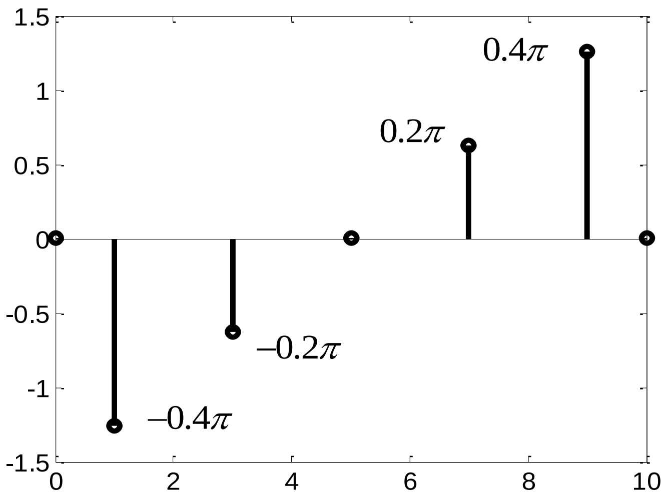
$$= e^{-j\frac{4\pi}{10}k} \frac{\sin\left(\frac{\pi}{2}k\right)}{\sin\left(\frac{\pi}{10}k\right)}$$

$$\sum_{r=-\infty}^{\infty} \delta(n - rN) = \frac{1}{N} \sum_{p=0}^{N-1} e^{\pm jp\frac{2\pi}{N}n}$$

Verify that these are the samples of the DTFT of one period ($n = 0,1,2,3,4$) of $\tilde{x}[n]$.

$\tilde{X}[k]$ is periodic with $N = 10$.






```
close all
clear all
w=[0:0.01:2*pi];
k=[0:2*pi/10:2*pi];

Xw = exp(-j*2*w).*sin(5*w/2)./sin(w/2);
Xk = exp(-j*2*k).*sin(5*k/2)./sin(k/2);
plot(w,abs(Xw))
hold
stem(k,abs(Xk),'r')

figure
plot(w,angle(Xw))
hold
stem(k,angle(Xk),'r')
```

$$\sum_{r=-\infty}^{\infty} \delta(n - rN) = \frac{1}{N}\sum_{p=0}^{N-1} e^{\pm jp\frac{2\pi}{N}n}$$

DFT: DISCRETE FOURIER TRANSFORM

$$\sum_{r=-\infty}^{\infty} \delta(n - rN) = \frac{1}{N} \sum_{p=0}^{N-1} e^{\pm jp\frac{2\pi}{N}n}$$

The Length and the 'Minimum Length' of a Sequence

In the context of DFT,

- Finite length sequences are defined to have a support over $n \geq 0$.

- The length, N, of a finite sequence can be assigned arbitrarily as $N \geq N_x$, where N_x is the minimum integer that can be assigned as the length of the sequence.

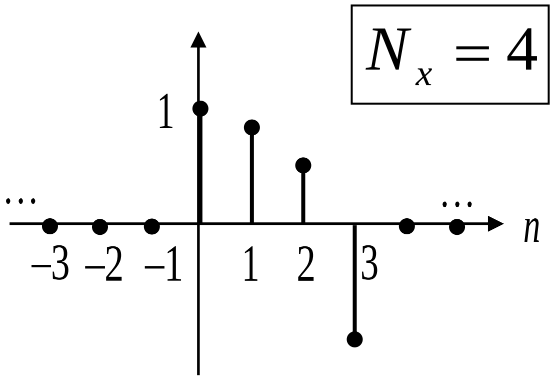
- 'Minimum length', N_x of a sequence can be defined as follows:

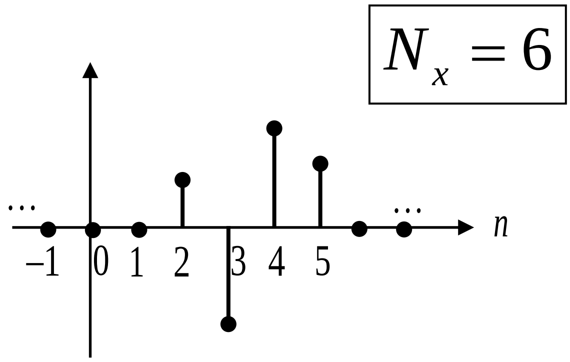
  Let $x[n]$ be a finite length sequence of minimum length $= N_x$ then,

$$x[n] \begin{cases} = 0 & n < 0 \quad and \quad n > N_x - 1 \\ \neq 0 & n = N_x - 1 \end{cases}.$$

$$\sum_{r=-\infty}^{\infty} \delta(n - rN) = \frac{1}{N}\sum_{p=0}^{N-1} e^{\pm jp\frac{2\pi}{N}n}$$

Ex:





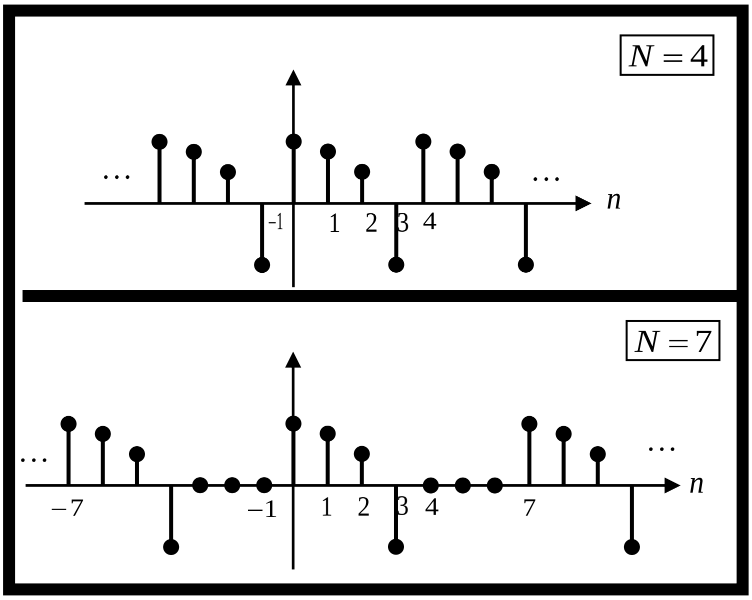
$$\sum_{r=-\infty}^{\infty} \delta(n-rN) = \frac{1}{N}\sum_{p=0}^{N-1} e^{\pm jp\frac{2\pi}{N}n}$$

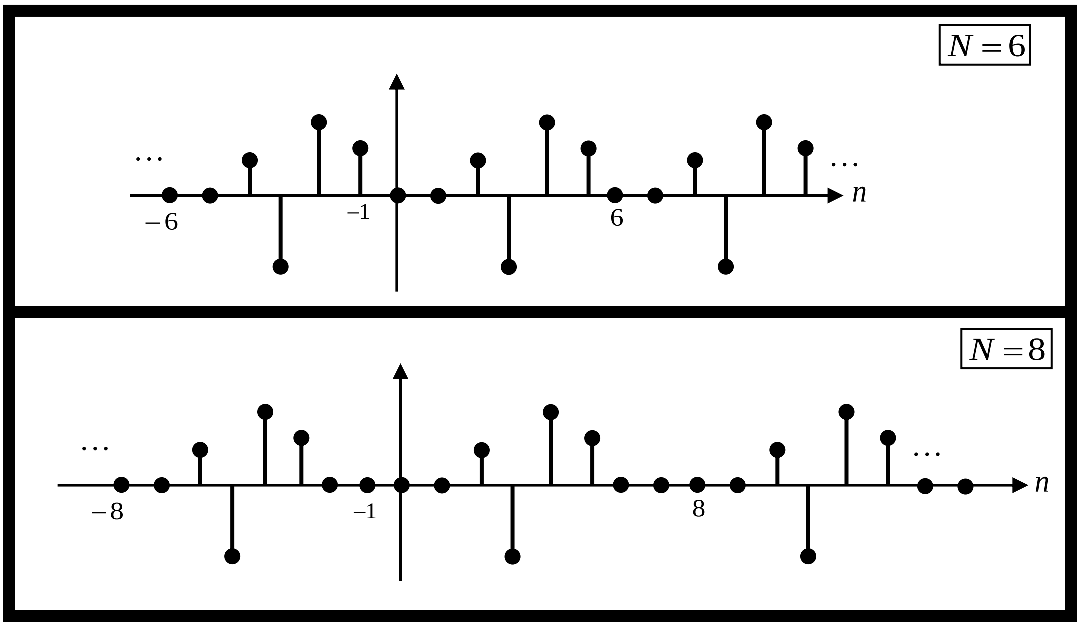
Let $x[n]$ be a finite length sequence and let

$$\tilde{x}[n] = \sum_{r=-\infty}^{\infty} x[n-rN] \quad N \geq N_x$$

be its periodic extension with period N:

$$\sum_{r=-\infty}^{\infty} \delta(n - rN) = \frac{1}{N} \sum_{p=0}^{N-1} e^{\pm jp\frac{2\pi}{N}n}$$





$$\sum_{r=-\infty}^{\infty} \delta(n - rN) = \frac{1}{N} \sum_{p=0}^{N-1} e^{\pm jp\frac{2\pi}{N}n}$$

Let

$$\tilde{X}[k]; \quad k \in Z$$

be the DFS coefficients of

$$\tilde{x}[n].$$

Then, $\boldsymbol{N}$**-point DFT** of $x[n]$ is defined as

$$X[k] = \begin{cases} \sum_{n=0}^{N-1} x[n]\, e^{-jk\frac{2\pi}{N}n} & k = 0,1, \dots, N-1 \\ \\ 0 & otherwise \end{cases}$$

$$= \begin{cases} \tilde{X}[k] & k = 0,1, \dots, N-1 \\ \\ 0 & otherwise \end{cases}$$

$$\sum_{r=-\infty}^{\infty} \delta(n-rN) = \frac{1}{N}\sum_{p=0}^{N-1} e^{\pm jp\frac{2\pi}{N}n}$$

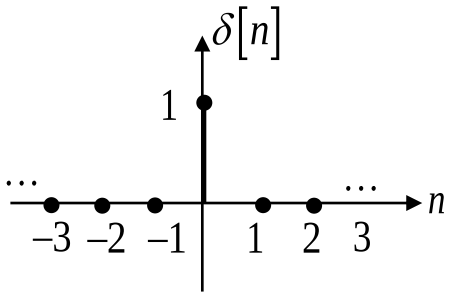
IDFT: INVERSE DFT

$$x[n] = \begin{cases} \dfrac{1}{N}\displaystyle\sum_{k=0}^{N-1} X[k]\, e^{jk\frac{2\pi}{N}n} & n = 0,1,\dots,N-1 \\ 0 & otherwise \end{cases}$$

$$\sum_{r=-\infty}^{\infty} \delta(n - rN) = \frac{1}{N} \sum_{p=0}^{N-1} e^{\pm jp\frac{2\pi}{N}n}$$

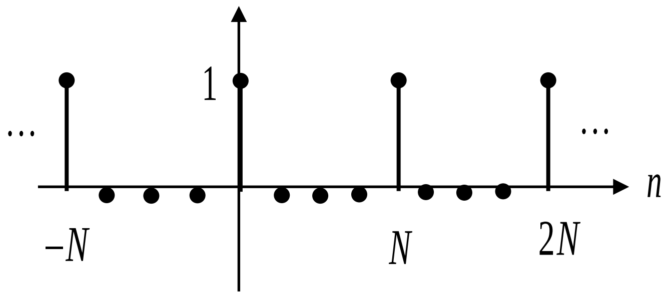
Ex:

$$x[n] = \delta[n]$$



$N_x = 1$

$$\tilde{x}[n] = \sum_{p=-\infty}^{\infty} \delta[n - pN]$$



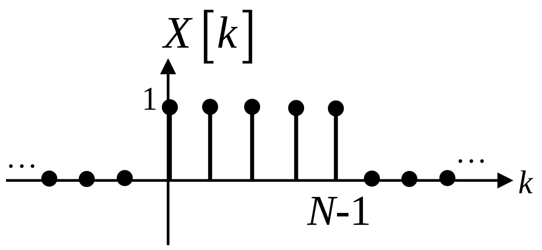
$$\sum_{r=-\infty}^{\infty} \delta(n - rN) = \frac{1}{N} \sum_{p=0}^{N-1} e^{\pm jp\frac{2\pi}{N}n}$$

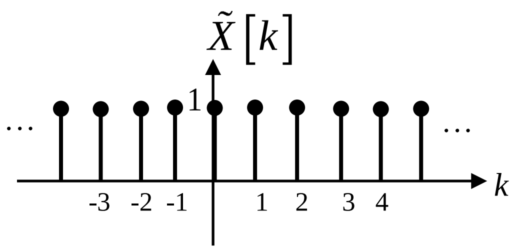
Ex cont'd

$$X[k] = \begin{cases} \tilde{X}[k] & k = 0,1,\dots,N-1 \\ 0 & otherwise \end{cases}$$

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \delta[n]\, e^{-jk\frac{2\pi}{N}n}$$

$$= 1 \qquad k \in Z$$





$$\sum_{r=-\infty}^{\infty} \delta(n-rN) = \frac{1}{N}\sum_{p=0}^{N-1} e^{\pm jp\frac{2\pi}{N}n}$$

Ex cont'd

Using 1-point DFT, $x[n] = \delta[n]$ can be obtained by IDFT as,

$$x[n] = \begin{cases} \frac{1}{1}\left(X[0]W_1^{-0n}\right) & n = 0 \\ \\ 0 & otherwise \end{cases}$$

$$= \begin{cases} 1 & n = 0 \\ \\ 0 & otherwise \end{cases}$$

$$\sum_{r=-\infty}^{\infty} \delta(n - rN) = \frac{1}{N} \sum_{p=0}^{N-1} e^{\pm jp\frac{2\pi}{N}n}$$

Ex cont'd

Using 2-point DFT, $x[n] = \delta[n]$ can be obtained by IDFT as,

$$x[n] = \begin{cases} \frac{1}{2}\left(X[0]W_2^{-0n} + X[1]W_2^{-1n}\right) & n = 0,1 \\ 0 & otherwise \end{cases}$$

$$= \begin{cases} 1 & n = 0 \\ 0 & n = 1 \\ 0 & otherwise \end{cases}$$

$$\sum_{r=-\infty}^{\infty} \delta(n - rN) = \frac{1}{N}\sum_{p=0}^{N-1} e^{\pm jp\frac{2\pi}{N}n}$$

Ex cont'd

Using 3-point DFT, $x[n] = \delta[n]$ can be obtained by IDFT as,

$$x[n] = \begin{cases} \frac{1}{3}\left(X[0]W_3^{-0n} + X[1]W_3^{-1n} + X[2]W_3^{-2n}\right) & n = 0,1,2 \\ \\ 0 & otherwise \end{cases}$$

$$= \begin{cases} \frac{1}{3}\left(X[0] + X[1]e^{j\frac{2\pi}{3}n} + X[2]e^{j2\frac{2\pi}{3}n}\right) & n = 0,1,2 \\ \\ 0 & otherwise \end{cases}$$

$$= \begin{cases} 1 & n = 0 \\ 0 & n = 1 \\ 0 & n = 2 \\ \\ 0 & otherwise \end{cases}$$

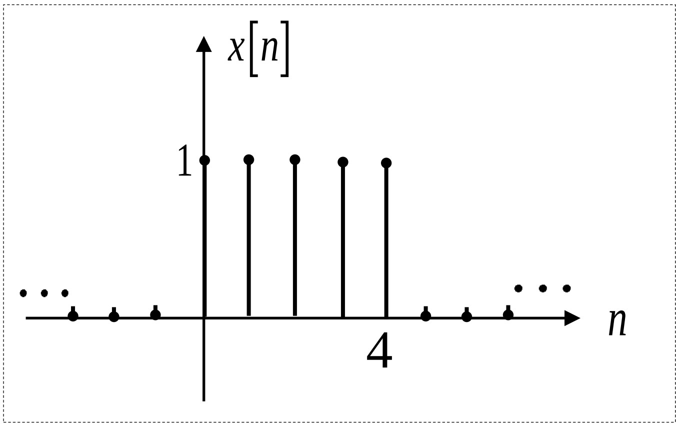
$$\sum_{r=-\infty}^{\infty} \delta(n - rN) = \frac{1}{N}\sum_{p=0}^{N-1} e^{\pm jp\frac{2\pi}{N}n}$$

<u>Ex</u>:

$x[n]$: … 0 0 1 1 1 1 1 0 0 …

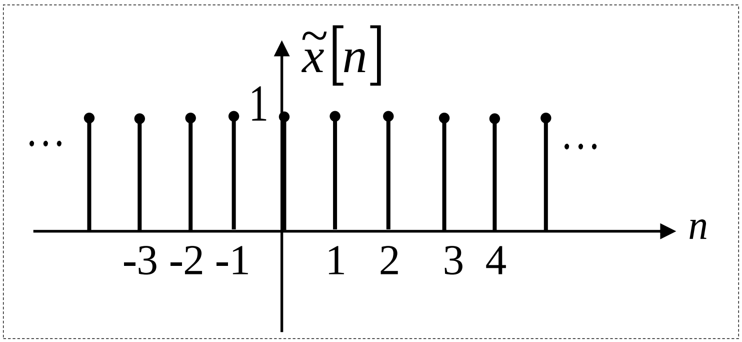
$\uparrow$

$n = 0$



Length of $x[n]$ is 5.

$$\tilde{x}[n] = \sum_{p=-\infty}^{\infty} x[n-5p]$$

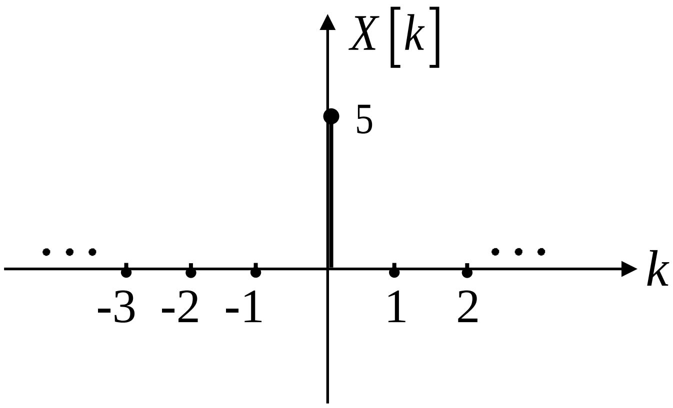


$$\sum_{r=-\infty}^{\infty} \delta(n - rN) = \frac{1}{N}\sum_{p=0}^{N-1} e^{\pm jp\frac{2\pi}{N}n}$$

Let's consider 5-point DFT ($N = 5$), so

$$X[k] = \begin{cases} \sum_{n=0}^{4} x[n]\, W_5^{kn} & k = 0,1,\dots,4 \\ 0 & otherwise \end{cases}$$

$$\begin{aligned}\sum_{n=0}^{4} x[n]\, W_5^{kn} &= \frac{1 - e^{-j2\pi k}}{1 - e^{-j\frac{2\pi}{5}k}} \\ &= e^{-j\frac{4\pi}{5}k}\,\frac{\sin(\pi k)}{\sin\left(\frac{\pi}{5}k\right)} \\ &= \begin{cases} 5 & k = 0 \\ 0 & k = 1,2,3,4 \end{cases} \\ &= 5\delta[k]\end{aligned}$$

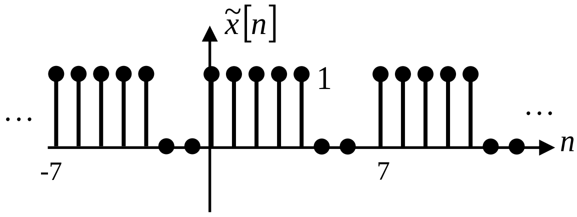


$$\sum_{r=-\infty}^{\infty} \delta(n - rN) = \frac{1}{N}\sum_{p=0}^{N-1} e^{\pm jp\frac{2\pi}{N}n}$$

Cont'd

Now consider 7-point DFT (N =7), so

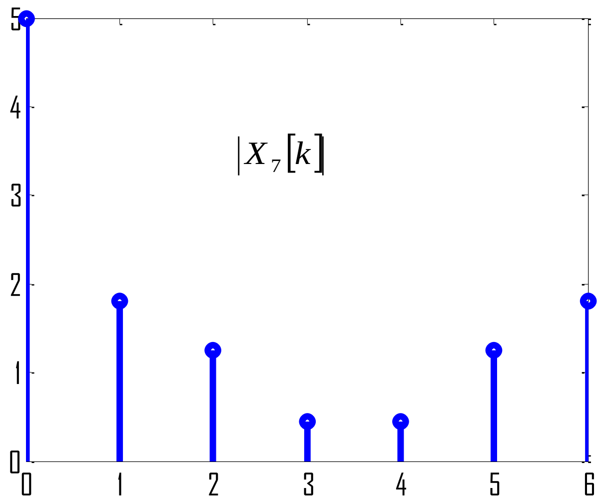
$$\tilde{x}[n] = \sum_{r=-\infty}^{\infty} x[n - r7]$$

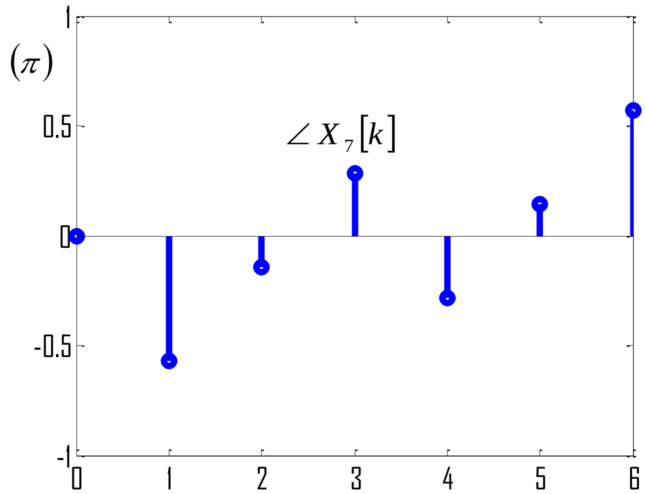


$$X[k] = \begin{cases} \sum_{n=0}^{4} x[n]\, W_7^{kn} & k = 0,1,\ldots,6 \\ 0 & otherwise \end{cases}$$

$$\sum_{n=0}^{4} x[n]\, W_7^{kn} = \frac{1 - e^{-j\frac{10\pi}{7}k}}{1 - e^{-j\frac{2\pi}{7}k}}$$

$$= e^{-j\frac{4\pi}{7}k} \frac{\sin\left(\frac{5\pi}{7}k\right)}{\sin\left(\frac{\pi}{7}k\right)}$$



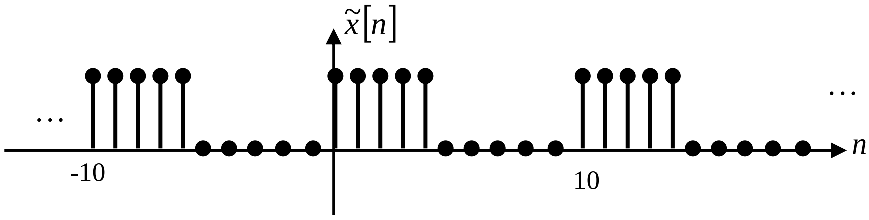


$$\sum_{r=-\infty}^{\infty} \delta(n-rN) = \frac{1}{N}\sum_{p=0}^{N-1} e^{\pm jp\frac{2\pi}{N}n}$$

Cont'd

Now consider 10-point DFT ($N = 10$)

$$\tilde{x}[n] = \sum_{r=-\infty}^{\infty} x[n-r10]$$



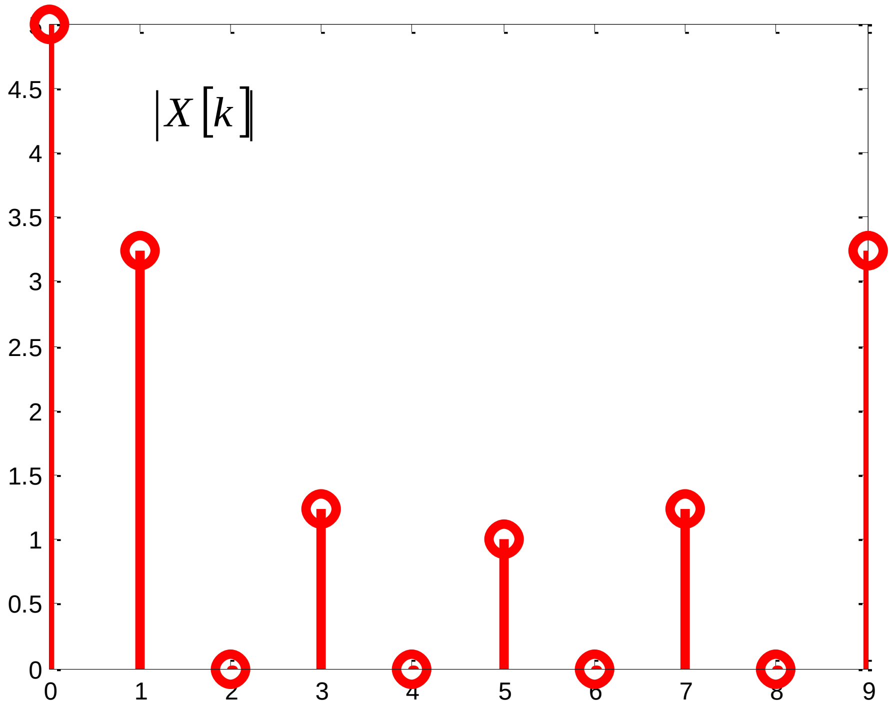
So

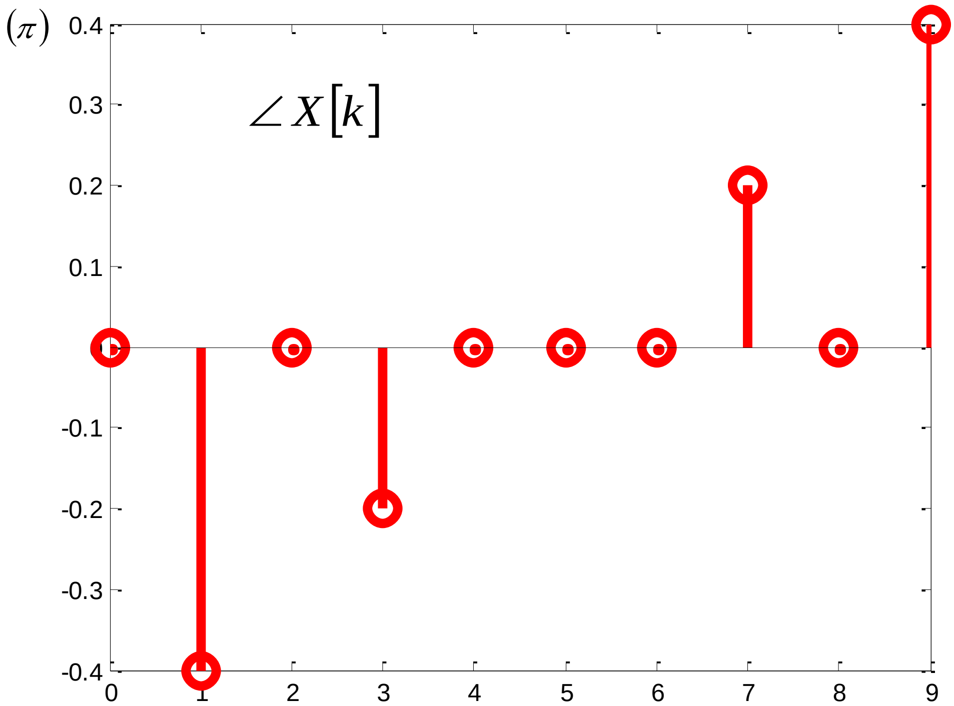
$$X[k] = \begin{cases} \sum_{n=0}^{4} x[n]\, W_{10}^{kn} & k = 0,1,\dots,9 \\ 0 & otherwise \end{cases}$$

$$\sum_{n=0}^{4} x[n]\, W_{10}^{kn} = \frac{1-e^{-j\frac{10\pi}{7}k}}{1-e^{-j\frac{2\pi}{7}k}}$$

$$= e^{-j\frac{4\pi}{10}k}\,\frac{\sin\left(\frac{\pi}{2}k\right)}{\sin\left(\frac{\pi}{10}k\right)}$$

$$\sum_{r=-\infty}^{\infty} \delta(n - rN) = \frac{1}{N} \sum_{p=0}^{N-1} e^{\pm jp\frac{2\pi}{N}n}$$





$$\sum_{r=-\infty}^{\infty} \delta(n-rN) = \frac{1}{N}\sum_{p=0}^{N-1} e^{\pm jp\frac{2\pi}{N}n}$$

Note that all of the above DFTs can be used to get $x[n]$ back!

$$\sum_{r=-\infty}^{\infty} \delta(n-rN) = \frac{1}{N}\sum_{p=0}^{N-1} e^{\pm jp\frac{2\pi}{N}n}$$

THE SIGNIFICANCE OF DFT AND DFS: SAMPLING THE DTFT

Given a finite length $x[n]$ of length N_x, and $N \geq N_x$.

Let

$$\tilde{x}[n] = \sum_{r=-\infty}^{\infty} x[n-rN].$$

Then,

$\tilde{X}[k]$ are N uniformly spaced samples of $X(e^{j\omega})$, $\omega \in [0,2\pi)$.

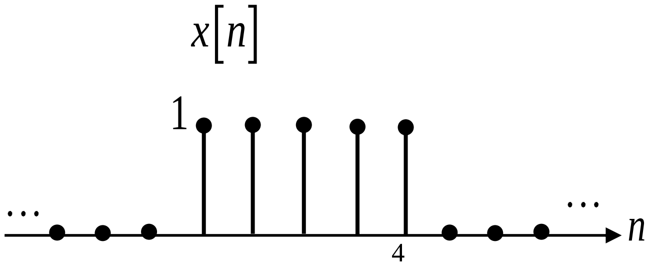
Therefore, also are $X[k]$.

$$X[k] = X(e^{j\omega})\Big|_{\omega=\frac{2\pi}{N}k}$$

$$= \sum_{n=0}^{N-1} x[n]e^{-j\left(\frac{2\pi}{N}k\right)n} \qquad\qquad k = 0,1,\dots,N-1$$

$$\sum_{r=-\infty}^{\infty} \delta(n - rN) = \frac{1}{N} \sum_{p=0}^{N-1} e^{\pm jp\frac{2\pi}{N}n}$$

Ex: Continued

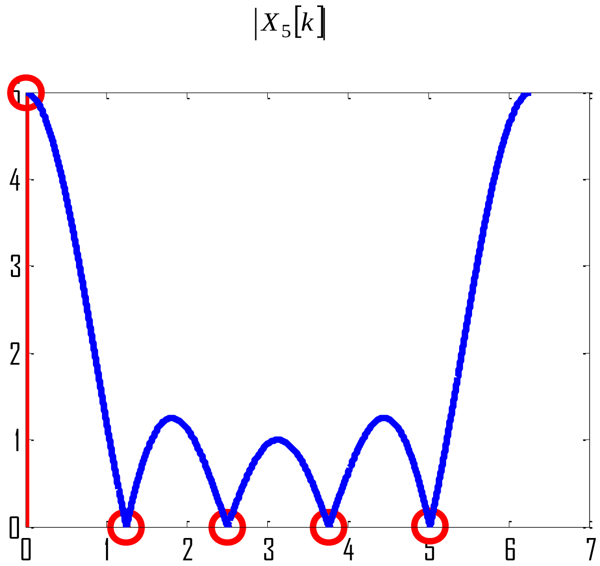


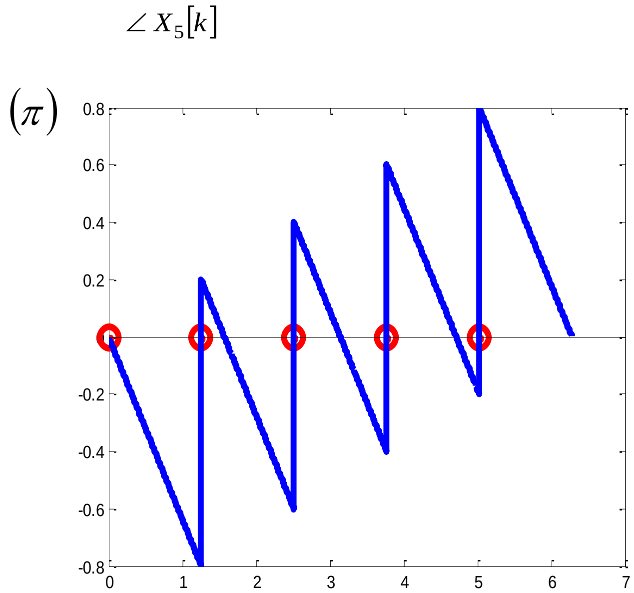
$$X\left(e^{j\omega}\right) = \sum_{n=0}^{4} x[n]\, e^{-j\omega n}$$

$$= e^{-j2\omega} \frac{\sin\left(\frac{5}{2}\omega\right)}{\sin\left(\frac{\omega}{2}\right)}$$

$$\sum_{r=-\infty}^{\infty} \delta(n - rN) = \frac{1}{N} \sum_{p=0}^{N-1} e^{\pm jp\frac{2\pi}{N}n}$$

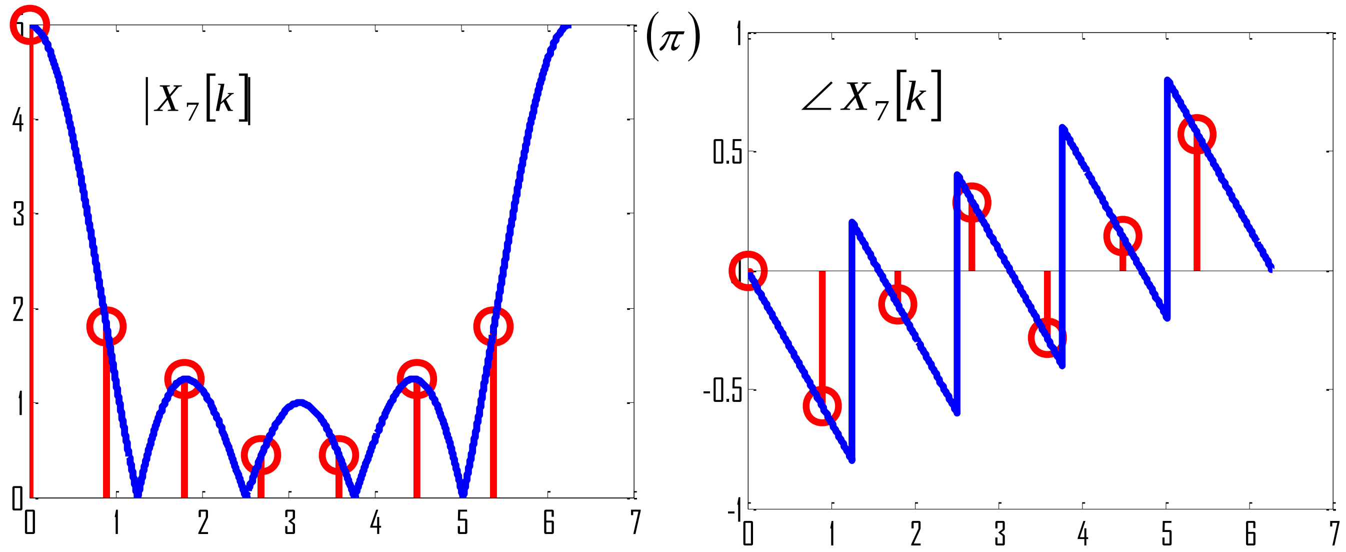
5-point DFT



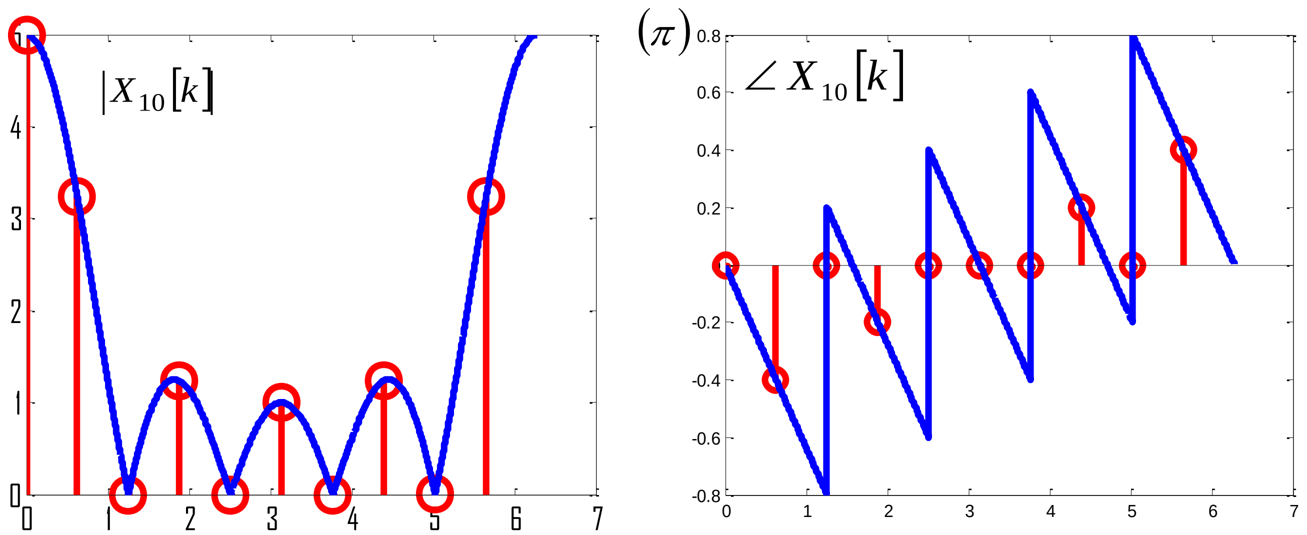


$$\sum_{r=-\infty}^{\infty} \delta(n - rN) = \frac{1}{N} \sum_{p=0}^{N-1} e^{\pm jp\frac{2\pi}{N}n}$$

7-point DFT



10-point DFT



HORIZONTAL SCALE !

$$\sum_{r=-\infty}^{\infty}\delta(n-rN)=\frac{1}{N}\sum_{p=0}^{N-1}e^{\pm jp\frac{2\pi}{N}n}$$

In matrix notation

$$x[n]=\sum_{n=0}^{N-1}X[n]e^{j\left(\frac{2\pi}{N}k\right)n}\qquad\qquad n=0,1,\dots,N-1$$

$$X[k]=\sum_{n=0}^{N-1}x[n]e^{-j\left(\frac{2\pi}{N}k\right)n}\qquad\qquad k=0,1,\dots,N-1$$

$$\underbrace{\begin{bmatrix}x[0]\\x[1]\\\vdots\\x[N-1]\end{bmatrix}}_{\bar{x}}=\frac{1}{N}\underbrace{\begin{bmatrix}1&1&1&1&\cdots&1\\1&e^{j\frac{2\pi}{N}}&e^{j\frac{4\pi}{N}}&e^{j\frac{6\pi}{N}}&\cdots&e^{j(N-1)\frac{2\pi}{N}}\\1&e^{j\frac{4\pi}{N}}&e^{j\frac{8\pi}{N}}&e^{j\frac{12\pi}{N}}&\cdots&e^{j(N-1)\frac{4\pi}{N}}\\1&e^{j\frac{6\pi}{N}}&e^{j\frac{12\pi}{N}}&e^{j\frac{18\pi}{N}}&\cdots&e^{j(N-1)\frac{6\pi}{N}}\\\vdots&\vdots&\vdots&\vdots&\ddots&\vdots\\1&e^{j\frac{(N-1)2\pi}{N}}&e^{j\frac{(N-1)4\pi}{N}}&e^{j\frac{(N-1)6\pi}{N}}&\cdots&e^{j\frac{(N-1)^2 2\pi}{N}}\end{bmatrix}}_{\boldsymbol{W}^*}\underbrace{\begin{bmatrix}X[0]\\X[1]\\X[2]\\X[3]\\\vdots\\X[N-1]\end{bmatrix}}_{\bar{X}}$$

$$\begin{bmatrix}X[0]\\X[1]\\X[2]\\X[3]\\\vdots\\X[N-1]\end{bmatrix}=\underbrace{\begin{bmatrix}1&1&1&1&\cdots&1\\1&e^{-j\frac{2\pi}{N}}&e^{-j\frac{4\pi}{N}}&e^{-j\frac{6\pi}{N}}&\cdots&e^{-j(N-1)\frac{2\pi}{N}}\\1&e^{-j\frac{4\pi}{N}}&e^{-j\frac{8\pi}{N}}&e^{-j\frac{12\pi}{N}}&\cdots&e^{-j(N-1)\frac{4\pi}{N}}\\1&e^{-j\frac{6\pi}{N}}&e^{-j\frac{12\pi}{N}}&e^{-j\frac{18\pi}{N}}&\cdots&e^{-j(N-1)\frac{6\pi}{N}}\\\vdots&\vdots&\vdots&\vdots&\ddots&\vdots\\1&e^{-j\frac{(N-1)2\pi}{N}}&e^{-j\frac{(N-1)4\pi}{N}}&e^{-j\frac{(N-1)6\pi}{N}}&\cdots&e^{-j\frac{(N-1)^2 2\pi}{N}}\end{bmatrix}}_{\boldsymbol{W}}\begin{bmatrix}x[0]\\x[1]\\\vdots\\x[N-1]\end{bmatrix}$$

$$\sum_{r=-\infty}^{\infty} \delta(n - rN) = \frac{1}{N}\sum_{p=0}^{N-1} e^{\pm jp\frac{2\pi}{N}n}$$

$$\bar{x} = \frac{1}{N}\boldsymbol{W}^*\bar{X}$$

$$\bar{X} = \boldsymbol{W}\bar{x}$$

$$\begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix} = \frac{1}{N}\begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N^{-1} & W_N^{-2} & W_N^{-3} & \cdots & W_N^{-(N-1)} \\ 1 & W_N^{-2} & W_N^{-4} & W_N^{-6} & \cdots & W_N^{-2(N-1)} \\ 1 & W_N^{-3} & W_N^{-6} & W_N^{-9} & \cdots & W_N^{-3(N-1)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & W_N^{-(N-1)} & W_N^{-2(N-1)} & W_N^{-3(N-1)} & \cdots & W_N^{-(N-1)^2} \end{bmatrix}\begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ X[3] \\ \vdots \\ X[N-1] \end{bmatrix}$$

$$\begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ X[3] \\ \vdots \\ X[N-1] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N & W_N^2 & W_N^3 & \cdots & W_N^{(N-1)} \\ 1 & W_N^2 & W_N^4 & W_N^6 & \cdots & W_N^{2(N-1)} \\ 1 & W_N^3 & W_N^6 & W_N^9 & \cdots & W_N^{3(N-1)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & W_N^{(N-1)} & W_N^{2(N-1)} & W_N^{3(N-1)} & \cdots & W_N^{(N-1)^2} \end{bmatrix}\begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix}$$

You can show that

$$\boldsymbol{WW}^* = \begin{bmatrix} N & 0 & 0 & 0 & \cdots & 0 \\ 0 & N & 0 & 0 & \cdots & 0 \\ 0 & 0 & N & 0 & \cdots & 0 \\ 0 & 0 & 0 & N & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & N \end{bmatrix}$$

$$= N\boldsymbol{I}_{N\times N}$$

where $\boldsymbol{I}_{N\times N}$ is the identity matrix.

Therefore $\frac{1}{\sqrt{N}}\boldsymbol{W}$ is a unitary matrix.

$$\sum_{r=-\infty}^{\infty} \delta(n - rN) = \frac{1}{N} \sum_{p=0}^{N-1} e^{\pm jp\frac{2\pi}{N}n}$$

PROPERTIES OF DFS AND DFT

Properties of DFT are borrowed from the properties of DFS.

We just need to fit the notation!

Pay attention to the fact that, in the context of DFT, signals are finite length and DFTs are also finite length.

These finite length portions are considered as one period of their periodic extensions…

$$\sum_{r=-\infty}^{\infty} \delta(n - rN) = \frac{1}{N}\sum_{p=0}^{N-1} e^{\pm jp\frac{2\pi}{N}n}$$

1) Linearity

DFS

Let $\tilde{x}_1[n]$ and $\tilde{x}_2[n]$ be periodic sequences with the same period N.

$$\tilde{x}_1[n] \overset{DFS}{\longleftrightarrow} \tilde{X}_1[k]$$

$$\tilde{x}_2[n] \overset{DFS}{\longleftrightarrow} \tilde{X}_2[k]$$

Then

$$\tilde{x}_3[n] = a\tilde{x}_1[n] + b\tilde{x}_2[n] \overset{DFS}{\longleftrightarrow} \tilde{X}_3[k] = a\tilde{X}_1[k] + b\tilde{X}_2[k]$$

$$\sum_{r=-\infty}^{\infty} \delta(n-rN) = \frac{1}{N}\sum_{p=0}^{N-1} e^{\pm jp\frac{2\pi}{N}n}$$

DFT

Let $x_1[n]$ and $x_2[n]$ be finite length sequences.

Let $N \geq \max\{N_1, N_2\}$ where N_1 and N_2 are the lengths of $x_1[n]$ and $x_2[n]$.

$$x_1[n] \xleftrightarrow{N-point\ DFT} X_1[k]$$

$$x_2[n] \xleftrightarrow{N-point\ DFT} X_2[k]$$

Then

$$x_3[n] = ax_1[n] + bx_2[n] \xleftrightarrow{N-point\ DFT} X_3[k] = aX_1[k] + bX_2[k]$$

$$\sum_{r=-\infty}^{\infty} \delta(n-rN) = \frac{1}{N}\sum_{p=0}^{N-1} e^{\pm jp\frac{2\pi}{N}n}$$

<u>**Ex**</u>: Let

$$x_1[n] = \delta[n] + \delta[n-1]$$

and

$$x_2[n] = \delta[n] + \delta[n-1] + \delta[n-2] + \delta[n-3].$$

Note that $x_1[n]$ is a 2-point sequence, $x_2[n]$ is a 4-point sequence.

4-point DFTs are

$$X_1[k] = 1 + e^{-j\frac{2\pi}{4}k},$$

$$X_2[k] = 1 + e^{-j\frac{2\pi}{4}k} + e^{-j2\frac{2\pi}{4}k} + e^{-j3\frac{2\pi}{4}k}.$$

Then, if

$$\begin{aligned} x_3[n] &= x_1[n] + x_2[n] \\ &= 2\delta[n] + 2\delta[n-1] + \delta[n-2] + \delta[n-3] \end{aligned}$$

4-point DFT of $x_3[n]$ is

$$\begin{aligned} X_3[k] &= X_1[k] + X_2[k] \\ &= 2 + 2e^{-j\frac{2\pi}{4}k} + e^{-j2\frac{2\pi}{4}k} + e^{-j3\frac{2\pi}{4}k} \end{aligned}$$

$$\sum_{r=-\infty}^{\infty} \delta(n - rN) = \frac{1}{N} \sum_{p=0}^{N-1} e^{\pm jp\frac{2\pi}{N}n}$$

Notation: Modulo

Define

$$\big((n)\big)_N \triangleq (n)\ \ mod\ \ N$$

then

$$\tilde{X}[k] = X\left[\big((k)\big)_N\right]$$

and

$$\tilde{x}[n] = x\left[\big((n)\big)_N\right]$$

$$\sum_{r=-\infty}^{\infty} \delta(n - rN) = \frac{1}{N} \sum_{p=0}^{N-1} e^{\pm jp\frac{2\pi}{N}n}$$

Modulo Table, $N = 5$

$\boldsymbol{n}$	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7
$((\boldsymbol{n}))_{\mathbf{5}}$	3	4	0	1	2	3	4	0	1	2	3	4	0	1	2
$((\boldsymbol{n}-\mathbf{2}))_{\mathbf{5}}$	1	2	3	4	0	1	2	3	4	0	1	2	3	4	5
$((\boldsymbol{n}+\mathbf{3}))_{\mathbf{5}}$	1	2	3	4	0	1	2	3	4	0	1	2	3	4	5
$((-\boldsymbol{n}))_{\mathbf{5}}$	2	1	0	4	3	2	1	0	4	3	2	1	0	4	3
$((-\boldsymbol{n}-\mathbf{1}))_{\mathbf{5}}$	1	0	4	3	2	1	0	4	3	2	1	0	4	3	2
$((-\boldsymbol{n}+\mathbf{1}))_{\mathbf{5}}$	3	2	1	0	4	3	2	1	0	4	3	2	1	0	4

$$\sum_{r=-\infty}^{\infty} \delta(n-rN) = \frac{1}{N}\sum_{p=0}^{N-1} e^{\pm jp\frac{2\pi}{N}n}$$

MATLAB Example

$$\left((-\boldsymbol{n})\right)_{\mathbf{5}}$$

```
>> x=1:5          x =   1   2   3   4   5
>>   for i=1:5
          y(i) = x(mod(-(i-1),5)+1);          (since vector indices start from 1 in MATLAB)
     end
>> y  y =   1   5   4   3   2
```

$$\sum_{r=-\infty}^{\infty} \delta(n - rN) = \frac{1}{N}\sum_{p=0}^{N-1} e^{\pm jp\frac{2\pi}{N}n}$$

2) Time Shift Property (DFS) – Circular Time Shift Property (DFT)

Shift Property (DFS)

$$\tilde{x}[n] \overset{DFS}{\longleftrightarrow} \tilde{X}[k] \quad \Rightarrow \quad \tilde{x}[n-\Delta] \overset{DFS}{\longleftrightarrow} e^{-j\Delta\frac{2\pi}{N}k}\tilde{X}[k]$$

Proof:

$$\tilde{y}[n] = \tilde{x}[n-\Delta]$$

$$\tilde{Y}[k] = \sum_{n=0}^{N-1} \tilde{x}[n-\Delta] e^{-jk\frac{2\pi}{N}n}$$

$$= e^{-jk\frac{2\pi}{N}\Delta} \sum_{m=-\Delta}^{N-1-\Delta} \tilde{x}[m] e^{-jk\frac{2\pi}{N}m}$$

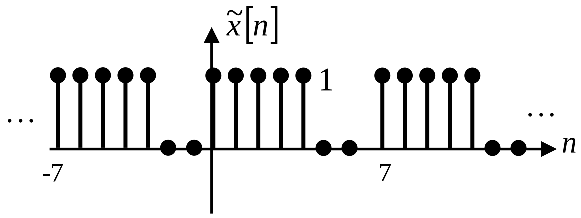
Since $\tilde{x}[m]$ and $e^{-jk\frac{2\pi}{N}m}$ are periodic with N, and the summation is over N consecutive values

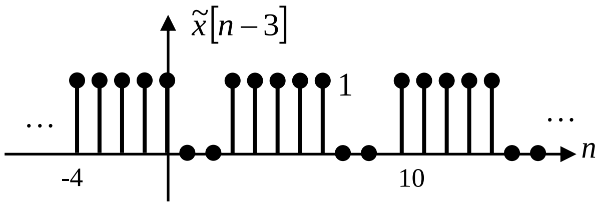
$$\tilde{Y}[k] = e^{-jk\frac{2\pi}{N}\Delta} \sum_{m=0}^{N-1} \tilde{x}[m] e^{-jk\frac{2\pi}{N}m}$$

$$= e^{-j\frac{2\pi}{N}\Delta}\ \tilde{X}[k]$$

$$\sum_{r=-\infty}^{\infty} \delta(n - rN) = \frac{1}{N} \sum_{p=0}^{N-1} e^{\pm jp\frac{2\pi}{N}n}$$

Ex:





$$\tilde{x}[n-3] \overset{DFS}{\longleftrightarrow} W_7^{3k}\tilde{X}[k]$$

$$\tilde{X}[k] = e^{-j\frac{4\pi}{7}k} \frac{\sin\left(\frac{5\pi}{7}k\right)}{\sin\left(\frac{\pi}{7}k\right)}$$

$$W_7^{3k}\tilde{X}[k] = e^{-j5\frac{2\pi}{7}k} \frac{\sin\left(\frac{5\pi}{7}k\right)}{\sin\left(\frac{\pi}{7}k\right)}$$

$$\sum_{r=-\infty}^{\infty} \delta(n-rN) = \frac{1}{N}\sum_{p=0}^{N-1} e^{\pm jp\frac{2\pi}{N}n}$$

Circular Shift Property (DFT)

$$x[n] \overset{DFT}{\longleftrightarrow} X[k]$$

$$a[n] =? \overset{DFT}{\longleftrightarrow} W_N^{k\Delta} X[k]$$

$$a[n] = \begin{cases} x\left[\left((n-\Delta)\right)_N\right] & n = 0,1,\dots,N-1 \\ \\ 0 & otherwise \end{cases}$$

Since

$$x[n] \triangleq \begin{cases} \tilde{x}[n] & n = 0,1,\dots,N-1 \\ \\ 0 & otherwise \end{cases}$$

$$\Rightarrow$$

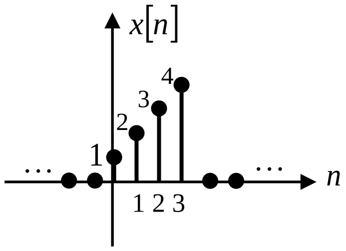
$$a[n] \triangleq \begin{cases} \tilde{x}[n-\Delta] & n = 0,1,\dots,N-1 \\ \\ 0 & otherwise \end{cases}$$

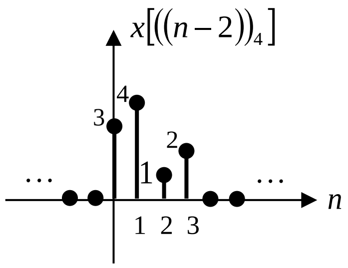
$$\sum_{r=-\infty}^{\infty} \delta(n-rN) = \frac{1}{N}\sum_{p=0}^{N-1} e^{\pm jp\frac{2\pi}{N}n}$$

<u>Ex</u>: If "signal length = DFT length".

$$x[n] \xleftrightarrow{4-point\ DFT} X[k]$$

$$x\left[\left((n-2)\right)_4\right] \xleftrightarrow{4-point\ DFT} W_4^{2k}\, X[k]$$





"distorted" compared to $x[n-2]$.

$$\sum_{r=-\infty}^{\infty} \delta(n-rN) = \frac{1}{N}\sum_{p=0}^{N-1} e^{\pm jp\frac{2\pi}{N}n}$$

Cont'd

"signal length < DFT length".

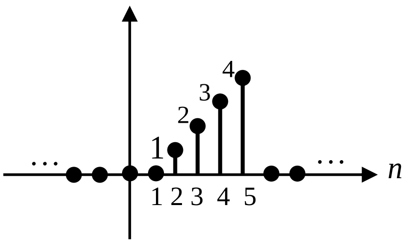
If 6-point DFT is used

$$x[n] \xleftrightarrow{6-point\ DFT} X[k]$$

$$x\left[\left((n-2)\right)_6\right] \xleftrightarrow{6-point\ DFT} W_6^{2k}\, X[k]$$

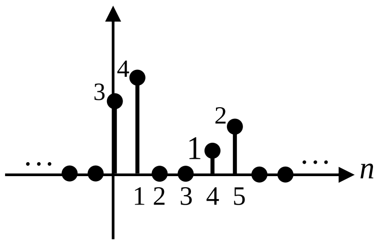
$$x\left[\left((n+2)\right)_6\right] \xleftrightarrow{6-point\ DFT} W_6^{-2k}\, X[k]$$

$$x\left[\left((n-2)\right)_6\right]$$



"undistorted"

$$x\left[\left((n-4)\right)_6\right] = x\left[\left((n+2)\right)_6\right]$$



"distorted"

$$\sum_{r=-\infty}^{\infty} \delta(n-rN) = \frac{1}{N}\sum_{p=0}^{N-1} e^{\pm jp\frac{2\pi}{N}n}$$

3) Multiplication by a complex exponential

DFS

For a periodic sequence with period N

$$e^{j\Delta\frac{2\pi}{N}n}\,\tilde{x}[n] \xleftrightarrow{DFS} \tilde{X}[k-\Delta]$$

Note that, in the context of the above property, complex exponentials being multiplied are restricted to have frequency as an integer multiple of $\frac{2\pi}{N}$.

$$\sum_{r=-\infty}^{\infty} \delta(n - rN) = \frac{1}{N}\sum_{p=0}^{N-1} e^{\pm jp\frac{2\pi}{N}n}$$

DFT

$$e^{j\Delta\frac{2\pi}{N}n}\, x[n] \xleftrightarrow{N-point\ DFT} X\left[\left((k-\Delta)\right)_N\right] \quad k = 0,1,\dots,N-1$$

$$\sum_{r=-\infty}^{\infty} \delta(n-rN) = \frac{1}{N}\sum_{p=0}^{N-1} e^{\pm jp\frac{2\pi}{N}n}$$

Ex: Let $x[n]$ be of length 7.

Find N and Δ so that

$$e^{j\frac{2\pi}{3}n}x[n] = W_N^{-\Delta n}x[n] \xleftrightarrow{N-point\ DFT} X\left[\left((k-\Delta)\right)_N\right]$$

$$k = 0,1,\dots,N-1$$

$$\frac{2\pi}{3} = \Delta\frac{2\pi}{N}$$

$$N = 3\Delta \qquad (N\ is\ a\ multiple\ of\ 3)$$

$$N > 7 \qquad (since\ it\ is\ given\ that\ x[n]\ is\ of\ length\ 7)$$

$$\Delta = \frac{N}{3}$$

Solution: Many possibilities,

$$N = 9 \quad \Delta = 3$$

$$N = 12 \quad \Delta = 4$$

$$N = 15 \quad \Delta = 5$$

$$\vdots$$

$$\sum_{r=-\infty}^{\infty} \delta(n - rN) = \frac{1}{N} \sum_{p=0}^{N-1} e^{\pm jp\frac{2\pi}{N}n}$$

4) Duality

DFS

$$\tilde{x}[n] \xleftrightarrow{DFS} \tilde{X}[k] \quad \Leftrightarrow \quad \tilde{X}[n] \xleftrightarrow{DFS} N\tilde{x}[-k]$$

DFT

$$x[n] \xleftrightarrow{DFT} X[k] \quad \Leftrightarrow \quad X[n] \xleftrightarrow{DFT} Nx\left[\left((-k)\right)_N\right]$$

or easier to remember

$$x[n] \xleftrightarrow{N-point\ DFT} X[k] \xleftrightarrow{N-point\ DFT} Nx\left[\left((-k)\right)_N\right]$$

$$\sum_{r=-\infty}^{\infty} \delta(n-rN) = \frac{1}{N}\sum_{p=0}^{N-1} e^{\pm jp\frac{2\pi}{N}n}$$

Proof:

$$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1} \tilde{X}[k] W_N^{-kn} \qquad \leftrightarrow \qquad \tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n] W_N^{kn}$$

$$\Downarrow$$

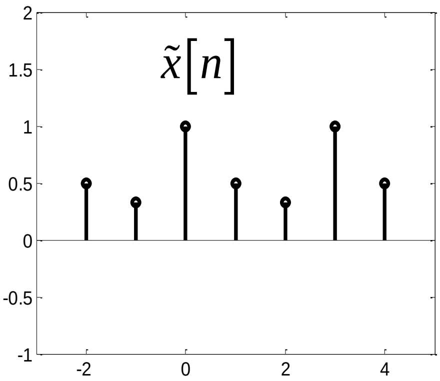
compare

$$\widetilde{Nx}[-n] = \sum_{k=0}^{N-1} \tilde{X}[k] W_N^{kn} \qquad \Longrightarrow \qquad \tilde{X}[n] \xleftrightarrow{DFS} N\tilde{x}[-k]$$

$$\sum_{r=-\infty}^{\infty} \delta(n - rN) = \frac{1}{N} \sum_{p=0}^{N-1} e^{\pm jp\frac{2\pi}{N}n}$$

Ex: Let $\tilde{x}[n]: \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \end{bmatrix}$ be periodic with period 3.

$$\begin{matrix} \uparrow \\ n = 0 \end{matrix}$$



$$\tilde{X}[k] = 1 + \frac{1}{2} e^{-j\frac{2\pi}{3}k} + \frac{1}{3} e^{-j\frac{2\pi}{3}2k}$$

$$\tilde{X}[k]: \begin{bmatrix} \frac{11}{6} & \frac{7 - j\sqrt{3}}{12} & \frac{7 + j\sqrt{3}}{12} \end{bmatrix} \qquad \textit{periodic with period } 3$$

$$\begin{matrix} \uparrow \\ k = 0 \end{matrix}$$

Then the DFS coefficients of the periodic sequence

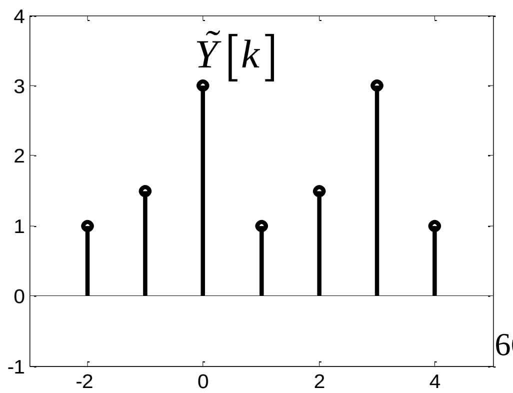
$$\tilde{y}[n] = \tilde{X}[n]: \begin{bmatrix} \frac{11}{6} & \frac{7 - j\sqrt{3}}{12} & \frac{7 + j\sqrt{3}}{12} \end{bmatrix} \qquad \textit{periodic with period } 3$$

$$\begin{matrix} \uparrow \\ k = 0 \end{matrix}$$

are

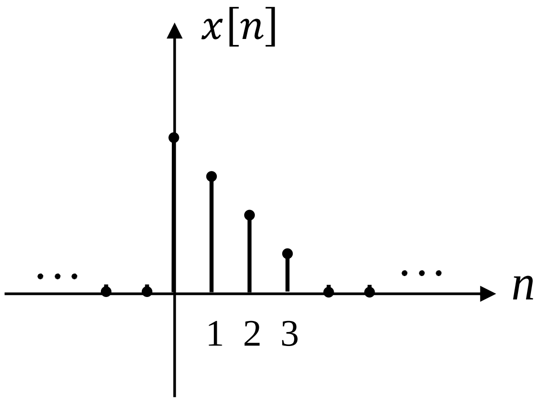
$$\tilde{Y}[k] = 3\tilde{x}[-k]: \begin{bmatrix} 1 & \frac{1}{3} & \frac{1}{2} \end{bmatrix} \qquad \textit{periodic with period } 3$$

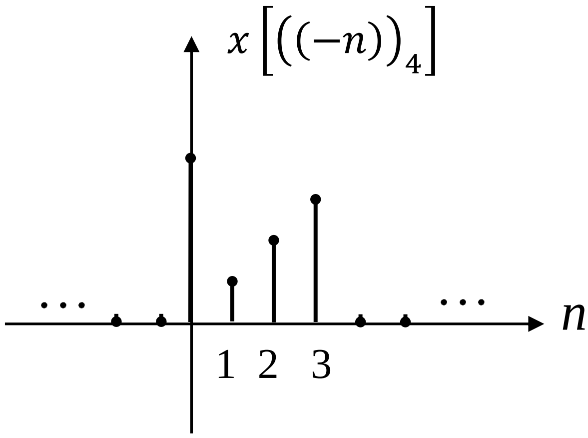
$$\begin{matrix} \uparrow \\ k = 0 \end{matrix}$$

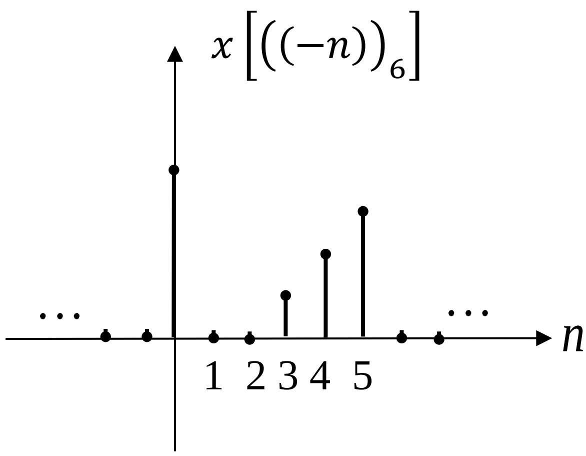


$$\sum_{r=-\infty}^{\infty} \delta(n - rN) = \frac{1}{N} \sum_{p=0}^{N-1} e^{\pm jp\frac{2\pi}{N}n}$$

Ex: What is $x\left[\left((-n)\right)_N\right]$?

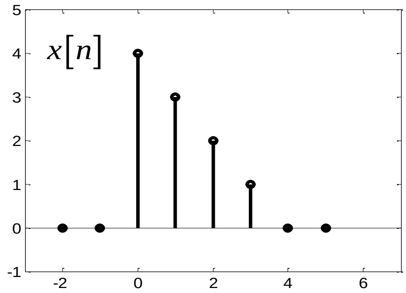






$$\sum_{r=-\infty}^{\infty} \delta(n-rN) = \frac{1}{N}\sum_{p=0}^{N-1} e^{\pm jp\frac{2\pi}{N}n}$$

Ex: Let $x[n]$ be

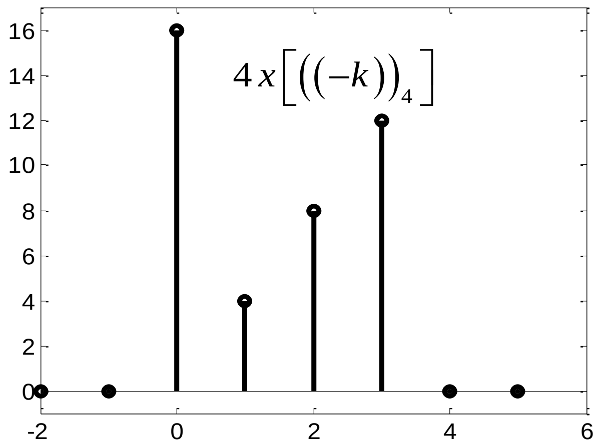


Using 4-point DFT,

$$4x\left[\left((-k)\right)_4\right]$$

is the DFT of

$$X[n] = \begin{cases} 4 + 3e^{-j\frac{2\pi}{4}n} + 2e^{-j\frac{2\pi}{4}2n} + e^{-j\frac{2\pi}{4}3n} & n = 0,1,\ldots,3 \\ 0 & o.w. \end{cases}$$



$$\sum_{r=-\infty}^{\infty} \delta(n-rN) = \frac{1}{N}\sum_{p=0}^{N-1} e^{\pm jp\frac{2\pi}{N}n}$$

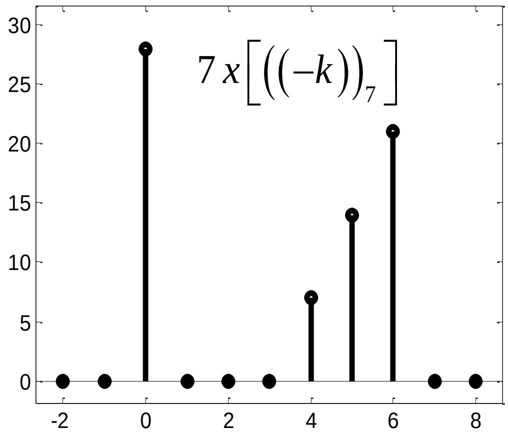
However, if, for example, 7-point DFT is used

Then

$$7x\left[((-k))_7\right]$$

is the 7-point DFT of

$$X[n] = \begin{cases} 4 + 3e^{-j\frac{2\pi}{7}n} + 2e^{-j\frac{2\pi}{7}2n} + e^{-j\frac{2\pi}{7}3n} & n = 0,1,\dots,6 \\ 0 & o.w. \end{cases}$$



$$\sum_{r=-\infty}^{\infty} \delta(n-rN) = \frac{1}{N}\sum_{p=0}^{N-1} e^{\pm jp\frac{2\pi}{N}n}$$

5) Symmetry Properties

Symmetry Properties of DFT are strictly related to those of DTFT since DFT is obtained by sampling DTFT

$$X[k] = X\left(e^{j\omega}\right)\Big|_{\omega=k\frac{2\pi}{N}}$$

$$\sum_{r=-\infty}^{\infty} \delta(n - rN) = \frac{1}{N} \sum_{p=0}^{N-1} e^{\pm jp\frac{2\pi}{N}n}$$

Definition:

Periodic Conjugate Symmetric sequence

$$x[n] = x^{*}\left[\left((-n)\right)_{N}\right]$$

Periodic Conjugate Antisymmetric sequence

$$x[n] = -x^{*}\left[\left((-n)\right)_{N}\right]$$

$$\sum_{r=-\infty}^{\infty} \delta(n-rN) = \frac{1}{N}\sum_{p=0}^{N-1} e^{\pm jp\frac{2\pi}{N}n}$$

DFS	**DFT** $n = 0,1,\dots,N-1$ $\quad$ $k = 0,1,\dots,N-1$
$\tilde{x}[n] \leftrightarrow \tilde{X}[k]$	$x[n] \leftrightarrow X[k]$
$\tilde{x}^*[n] \leftrightarrow \tilde{X}^*[-k]$	$x^*[n] \leftrightarrow X^*\left[((-k))_N\right]$
$\tilde{x}[-n] \leftrightarrow \tilde{X}[-k]$	$x^*\left[((-n))_N\right] \leftrightarrow X\left[((-k))_N\right]$
$Re\{\tilde{x}[n]\}$ $\leftrightarrow \underbrace{\tilde{X}_e[k] = \frac{1}{2}\left(\tilde{X}[k] + \tilde{X}^*[-k]\right)}_{conjugate\ symmetric\ part}$	$Re\{x[n]\}$ $\leftrightarrow \underbrace{X_{pe}[k] = \frac{1}{2}\left(X[k] + X^*\left[((-k))_N\right]\right)}_{periodic\ conjugate\ symmetric\ part}$
$jIm\{\tilde{x}[n]\}$ $\leftrightarrow \underbrace{\tilde{X}_o[k] = \frac{1}{2}\left(\tilde{X}[k] - \tilde{X}^*[-k]\right)}_{conjugate\ antisymmetric\ part}$	$jIm\{x[n]\}$ $\leftrightarrow \underbrace{X_{po}[k] = \frac{1}{2}\left(X[k] - X^*\left[((-k))_N\right]\right)}_{periodic\ conjugate\ antisymmetric\ part}$
$\overbrace{\tilde{x}_e[n] = \frac{1}{2}\left(\tilde{x}[n] + \tilde{x}^*[-n]\right)}^{conjugate\ symmetric\ part}$ $\leftrightarrow Re\{\tilde{X}[k]\}$	$\overbrace{x_{pe}[n] = \frac{1}{2}\left(x[n] + x^*\left[((-n))_N\right]\right)}^{periodic\ conjugate\ symmetric\ part}$ $\leftrightarrow Re\{X[k]\}$
$\overbrace{\tilde{x}_o[n] = \frac{1}{2}\left(\tilde{x}[n] - \tilde{x}^*[-n]\right)}^{conjugate\ antisymmetric\ part}$ $\leftrightarrow jIm\{\tilde{X}[k]\}$	$\overbrace{x_o[n] = \frac{1}{2}\left(x[n] - x^*\left[((-n))_N\right]\right)}^{periodic\ conjugate\ antisymmetric\ part}$ $\leftrightarrow jIm\{X[k]\}$

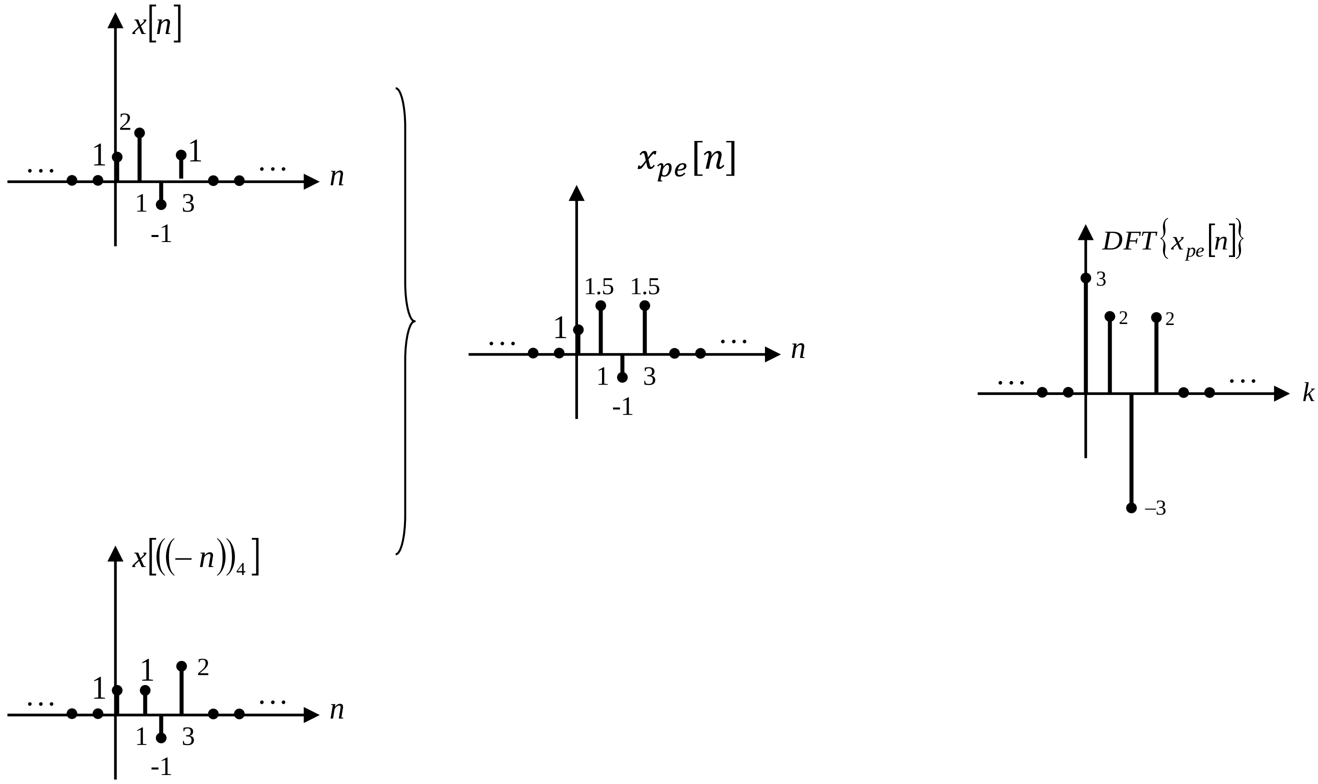
$$\sum_{r=-\infty}^{\infty} \delta(n-rN) = \frac{1}{N}\sum_{p=0}^{N-1} e^{\pm jp\frac{2\pi}{N}n}$$

Ex: The DFT of a periodic conjugate symmetric sequence is real valued.

Let

$$x[n] = x^*\left[\left((-n)\right)_N\right]$$

$$\Rightarrow X[k] = X^*[k]$$



$$\sum_{r=-\infty}^{\infty} \delta(n-rN) = \frac{1}{N}\sum_{p=0}^{N-1} e^{\pm jp\frac{2\pi}{N}n}$$

REAL SEQUENCES

$$x[n] = x^*[n] \qquad \Rightarrow \qquad X\left(e^{j\omega}\right) = X^*\left(e^{-j\omega}\right)$$

We know that DTFT is also conjugate symmetric wrt. π.

Then, since DFT is obtained by uniformly sampling DTFT,

$X[k]$ is conjugate symmetric over $k = 1,2,\ldots,N-1$.

$$X[k] = X^*\left[\left((-k)\right)_N\right] \qquad k = 0,1,\ldots,N-1$$

$$= X^*[N-k] \qquad k = 1,2,\ldots,N-1$$

$$\sum_{r=-\infty}^{\infty} \delta(n-rN) = \frac{1}{N}\sum_{p=0}^{N-1} e^{\pm jp\frac{2\pi}{N}n}$$

DFS	DFT
$\tilde{X}[k] = \tilde{X}^*[-k]$	$X[k] = X^*\left[((-k))_N\right]$ or $X[k] = X[N-k] \quad k = 1, 2, \ldots, N-1$
$\mathrm{Re}\{\tilde{X}[k]\} = \mathrm{Re}\{\tilde{X}[-k]\}$	$\mathrm{Re}\{X[k]\} = \mathrm{Re}\{X[N-k]\}$ $k = 1,2,\ldots,N-1$
$\mathrm{Im}\{\tilde{X}[k]\} = -\mathrm{Im}\{\tilde{X}[-k]\}$	$\mathrm{Im}\{X[k]\} = -\mathrm{Im}\{X[N-k]\}$ $k = 1,2,\ldots,N-1$
$\lvert\tilde{X}[k]\rvert = \lvert\tilde{X}[-k]\rvert$	$\lvert X[k]\rvert = \lvert X[N-k]\rvert$ $k = 1,2,\ldots,N-1$
$\tilde{X}[k] = -\angle\tilde{X}[-k]$	$\angle X[k] = -\angle X[N-k]$ $k = 1,2,\ldots,N-1$

$$\sum_{r=-\infty}^{\infty} \delta(n - rN) = \frac{1}{N} \sum_{p=0}^{N-1} e^{\pm jp\frac{2\pi}{N}n}$$

Ex: $x[n] = \delta[n] + \delta[n-1] \qquad \Rightarrow \qquad X\left(e^{j\omega}\right) = 1 + e^{-j\omega} = e^{-j\frac{\omega}{2}} \cos\frac{\omega}{2}$

For 8-point DFT

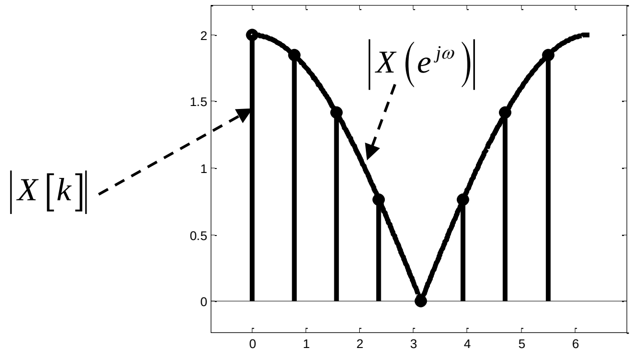
$$X[k] = 1 + e^{-jk\frac{2\pi}{8}}$$

$|X[k]| = \left|X\left[\left((-k)\right)_8\right]\right|$

$|X[1]| = |X[7]|$

$|X[2]| = |X[6]|$

$|X[3]| = |X[5]|$

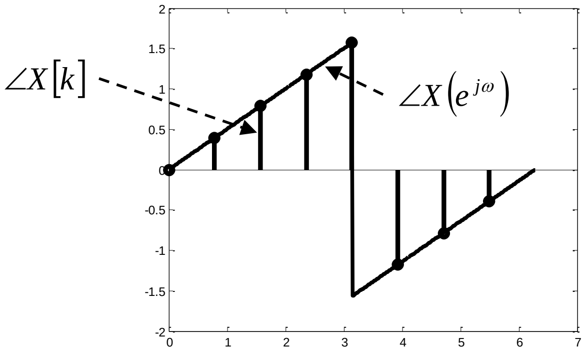


$\measuredangle X[k] = -\measuredangle X\left[\left((-k)\right)_8\right]$

$\measuredangle X[1] = -\measuredangle X[7]$

$\measuredangle X[2] = -\measuredangle X[6]$

$\measuredangle X[3] = -\measuredangle X[5]$



$$\sum_{r=-\infty}^{\infty} \delta(n - rN) = \frac{1}{N} \sum_{p=0}^{N-1} e^{\pm jp\frac{2\pi}{N}n}$$

6) Convolution Property

DFS: Periodic Convolution

DFT: Circular Convolution

$$x[n] * y[n] \quad \leftrightarrow \quad X\left(e^{j\omega}\right)Y\left(e^{j\omega}\right)$$

$$\sum_{r=-\infty}^{\infty} \delta(n - rN) = \frac{1}{N} \sum_{p=0}^{N-1} e^{\pm jp\frac{2\pi}{N}n}$$

DFS: Periodic Convolution

Let

$$\tilde{x}_1[n] \leftrightarrow \tilde{X}_1[k] \quad and \quad \tilde{x}_2[n] \leftrightarrow \tilde{X}_2[k]$$

(same fund. period)

and

$$\tilde{X}_3[k] \triangleq \tilde{X}_1[k]\tilde{X}_2[k]$$

$$\tilde{x}_3[n] = ???$$

$$\sum_{r=-\infty}^{\infty} \delta(n - rN) = \frac{1}{N} \sum_{p=0}^{N-1} e^{\pm jp\frac{2\pi}{N}n}$$

$$\begin{aligned}
\tilde{x}_3[n] &= \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}_1[k] \tilde{X}_2[k] W_N^{-kn} \\
&= \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}_1[k] \left(\sum_{p=0}^{N-1} \tilde{x}_2[p] W_N^{kp} \right) W_N^{-kn} \\
&= \frac{1}{N} \sum_{p=0}^{N-1} \tilde{x}_2[p] \underbrace{\left(\sum_{k=0}^{N-1} \tilde{X}_1[k] W_N^{k(p-n)} \right)}_{N\tilde{x}_1[n-p]}
\end{aligned}$$

Therefore

$$\tilde{x}_3[n] = \sum_{p=0}^{N-1} \tilde{x}_1[n-p] \tilde{x}_2[p]$$

periodic convolution of $\tilde{x}_1[n]$ and $\tilde{x}_2[n]$.

$$\sum_{r=-\infty}^{\infty} \delta(n - rN) = \frac{1}{N} \sum_{p=0}^{N-1} e^{\pm jp\frac{2\pi}{N}n}$$

Ex:

```
x=1:5;
y=[1 0 0 -2 0];
X=fft(x);
Y=fft(y);
Z=X.*Y;
z=ifft(Z)            (z = [-5 -6 -7 2 1])
stem(z)
```

1	2	3	4	5		
1	0	-2	0	0	→	-5
0	1	0	-2	0	→	-6
0	0	1	0	-2	→	-7
-2	0	0	1	0	→	2
0	-2	0	0	1	→	1

$$\sum_{r=-\infty}^{\infty} \delta(n-rN) = \frac{1}{N}\sum_{p=0}^{N-1} e^{\pm jp\frac{2\pi}{N}n}$$

DFT: Circular Convolution

$$x_1[n] \xleftrightarrow{N-point\ DFT} X_1[k] \quad and \quad x_2[n] \xleftrightarrow{N-point\ DFT} X_2[k]$$

$$X_3[k] = X_1[k]X_2[k]$$

$$x_3[n] = ?$$

----------*-----------

$$x_3[n] = \begin{cases} \tilde{x}_3[n] & n = 0,1,\dots,N-1 \\ 0 & o.w. \end{cases}$$

Using the result from DFS

$$x_3[n] = \begin{cases} \sum_{r=0}^{N-1} x_1\left[\left((n-r)\right)_N\right] x_2[r] & n = 0,1,\dots,N-1 \\ 0 & o.w. \end{cases}$$

This is called "$\boldsymbol{N}$**-point circular convolution**" of $x_1[n]$ and $x_2[n]$

$$x_3[n] = x_1[n] \circledast_4 x_2[n]$$

$$\sum_{r=-\infty}^{\infty} \delta(n - rN) = \frac{1}{N} \sum_{p=0}^{N-1} e^{\pm jp\frac{2\pi}{N}n}$$

Linear Convolution and Circular Convolution

We know the following: Let

$$x_1[n] \leftrightarrow X_1(e^{j\omega}) \quad \text{and} \quad x_2[n] \leftrightarrow X_2(e^{j\omega}),$$

also let

$$X_{LC}(e^{j\omega}) = X_1(e^{j\omega}) X_2(e^{j\omega}).$$

Then

$$x_{LC}[n] = IDTFT\{X_{LC}(e^{j\omega})\}$$

$$= x_1[n] * x_2[n]$$

$$\sum_{r=-\infty}^{\infty} \delta(n - rN) = \frac{1}{N} \sum_{p=0}^{N-1} e^{\pm jp\frac{2\pi}{N}n}$$

Now, let's ask the following question

$$x_3[n] = IDFT\left\{\underbrace{X_1[k]X_2[k]}_{X_3[k]}\right\} = ???$$

where

$$X_3[k] = \left(X_1(e^{j\omega})X_2(e^{j\omega})\right)\Big|_{\omega=k\frac{2\pi}{N}} \qquad k = 0,1,\dots,N-1.$$

$$= X_{LC}(e^{j\omega})\Big|_{\omega=k\frac{2\pi}{N}} \qquad k = 0,1,\dots,N-1.$$

$$\sum_{r=-\infty}^{\infty} \delta(n - rN) = \frac{1}{N} \sum_{p=0}^{N-1} e^{\pm jp\frac{2\pi}{N}n}$$

We know that IDFT yields

$$x_3[n] = \begin{cases} \displaystyle\sum_{r=-\infty}^{\infty} x_{LC}[n - rN] & n = 0,1,\dots,N-1 \\ 0 & o.w. \end{cases}$$

$$\sum_{r=-\infty}^{\infty} \delta(n-rN) = \frac{1}{N}\sum_{p=0}^{N-1} e^{\pm jp\frac{2\pi}{N}n}$$

Therefore N-point circular convolution of $x_1[n]$ and $x_2[n]$ can also be computed via linear convolution:

Compute

$$x_{LC}[n] = x_1[n] * x_2[n]$$

Then, you can find $x_1[n] \circledast_N x_2[n]$ as

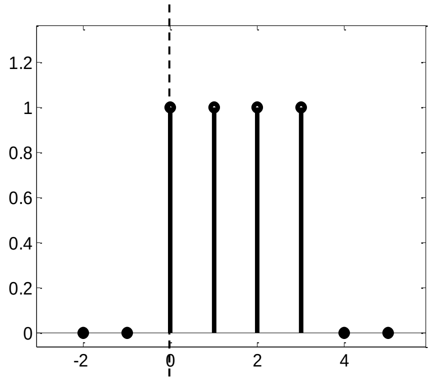
$$x_1[n] \circledast_N x_2[n] = \begin{cases} \sum_{r=-\infty}^{\infty} x_{LC}[n-rN] & n = 0,1,\ldots,N-1 \\ 0 & o.w. \end{cases}$$

or as

$$x_1[n] \circledast_N x_2[n] = IDFT\{X_1[k]X_2[k]\}$$

$$\sum_{r=-\infty}^{\infty} \delta(n - rN) = \frac{1}{N} \sum_{p=0}^{N-1} e^{\pm jp\frac{2\pi}{N}n}$$

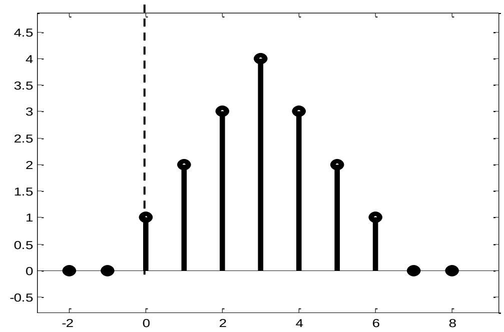
Ex: Linear convolution



$$x[n]: \ [\cdots 0 \ 0 \ \underset{\substack{\uparrow \\ n=0}}{1} \ 1 \ 1 \ 1 \ 0 \ 0 \cdots]$$

The linear convolution of $x[n]$ with itself is

$$y[n] = x[n] * x[n] \qquad y[n]: \ [\cdots 0 \ 0 \ \underset{\substack{\uparrow \\ n=0}}{1} \ 2 \ 3 \ 4 \ 3 \ 2 \ 1 \ 0 \ 0 \cdots]$$



Note that he length of $y[n]$ is 7 ($= 4 + 4 - 1$)

$$\sum_{r=-\infty}^{\infty} \delta(n-rN) = \frac{1}{N}\sum_{p=0}^{N-1} e^{\pm jp\frac{2\pi}{N}n}$$

Ex: Continued: 4-point circular convolution of $x[n]$ with itself

Let

$$W[k] = X_4[k]X_4[k]$$

and $w[n]$ be the IDFT of $W[k]$, then

$$w[n] = x[n] \circledast_4 x[n]$$

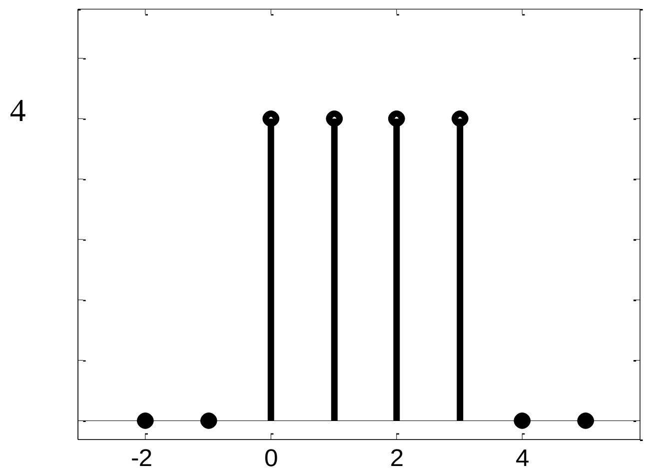
$$= \sum_{r=0}^{3} x\left[((n-r))_4\right] x[r]$$

To compute one needs

$$x\left[((-r))_4\right], x\left[((1-r))_4\right], x\left[((2-r))_4\right], x\left[((3-r))_4\right]$$

Then

$$w[n] = \sum_{r=0}^{3} 1 = 4, \qquad n = 0,1,2,3$$



This is not equal to the result of linear convolution!

$$\sum_{r=-\infty}^{\infty} \delta(n - rN) = \frac{1}{N} \sum_{p=0}^{N-1} e^{\pm jp\frac{2\pi}{N}n}$$

Ex: Continued

6-point circular convolution of $x[n]$ with itself

Let

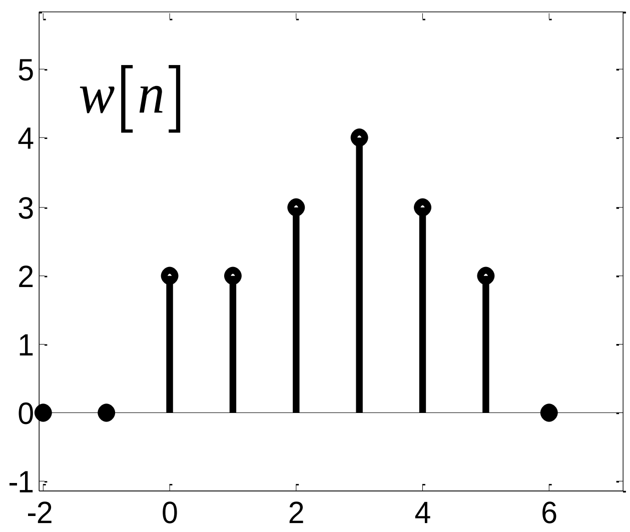
$$W[k] = X_6[k]X_6[k]$$

and $w[n]$ be the IDFT of $W[k]$, then

$$w[n] = x[n] \circledast_6 x[n]$$

$$= \sum_{r=0}^{5} x\left[\left((n-r)\right)_6\right] x[r]$$

$$w[n] = \left[\dots 0 \quad 0 \quad \underbrace{2}_{n=0} \quad 2 \quad 3 \quad 4 \quad 3 \quad 2 \quad 0 \quad 0 \dots\right]$$

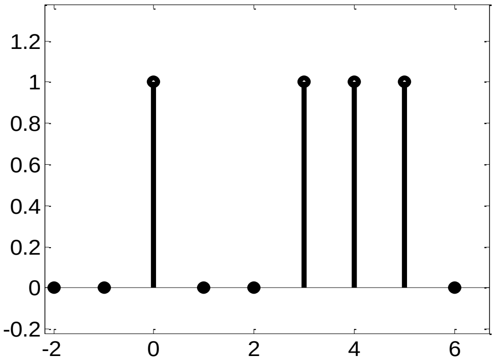


This is also not the same as the result of linear convolution!

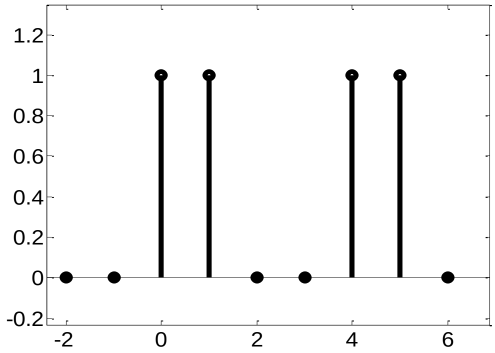
$$\sum_{r=-\infty}^{\infty} \delta(n-rN) = \frac{1}{N}\sum_{p=0}^{N-1} e^{\pm jp\frac{2\pi}{N}n}$$

However, partially correct!

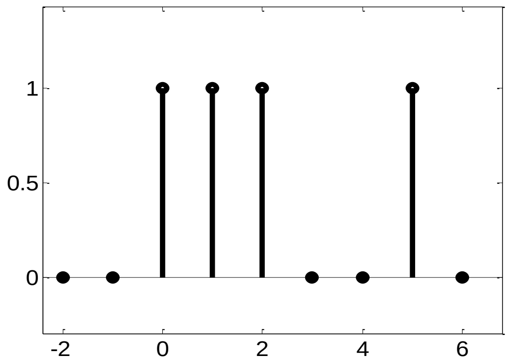
$x\left[\left((-r)\right)_6\right]$



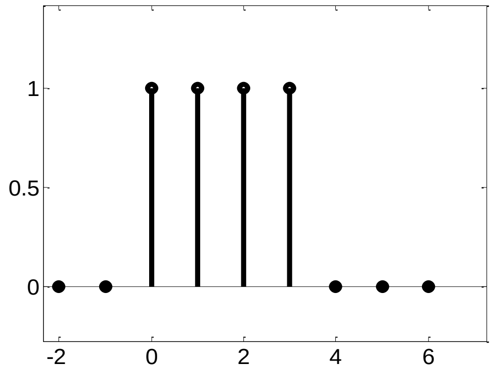
$x\left[\left((1-r)\right)_6\right]$



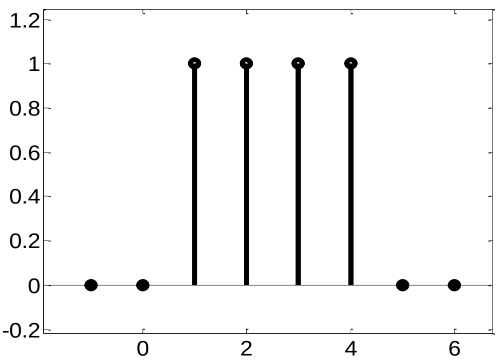
$x\left[\left((2-r)\right)_6\right]$



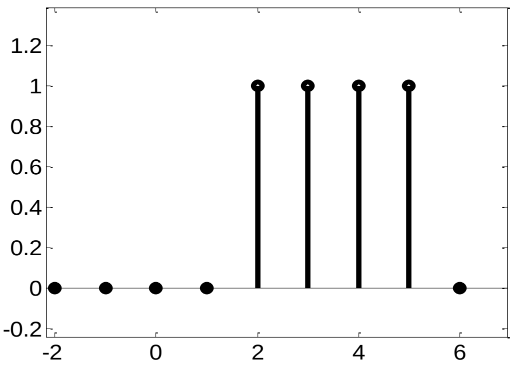
$x\left[\left((3-r)\right)_6\right]$



$x\left[\left((4-r)\right)_6\right]$



$x\left[\left((5-r)\right)_6\right]$



$$\sum_{r=-\infty}^{\infty} \delta(n-rN) = \frac{1}{N}\sum_{p=0}^{N-1} e^{\pm jp\frac{2\pi}{N}n}$$

Ex: Continued

7-point circular convolution of *x*[*n*] with itself

$$W[k] = X_7[k]X_7[k]$$

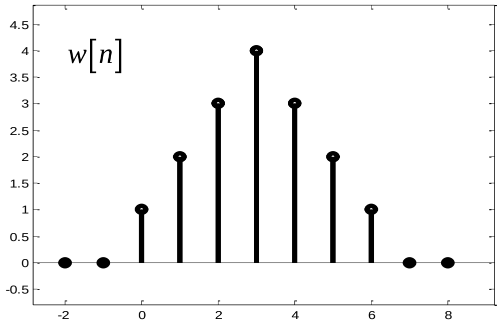
$$w[n] = x[n] \circledast_7 x[n]$$

$$= \sum_{r=0}^{6} x\left[\left((n-r)\right)_7\right]x[r]$$

$$w[n] = \left[\dots 0 \quad 0 \quad \underbrace{2}_{n=0} \quad 2 \quad 3 \quad 4 \quad 3 \quad 2 \quad 0 \quad 0 \dots\right]$$

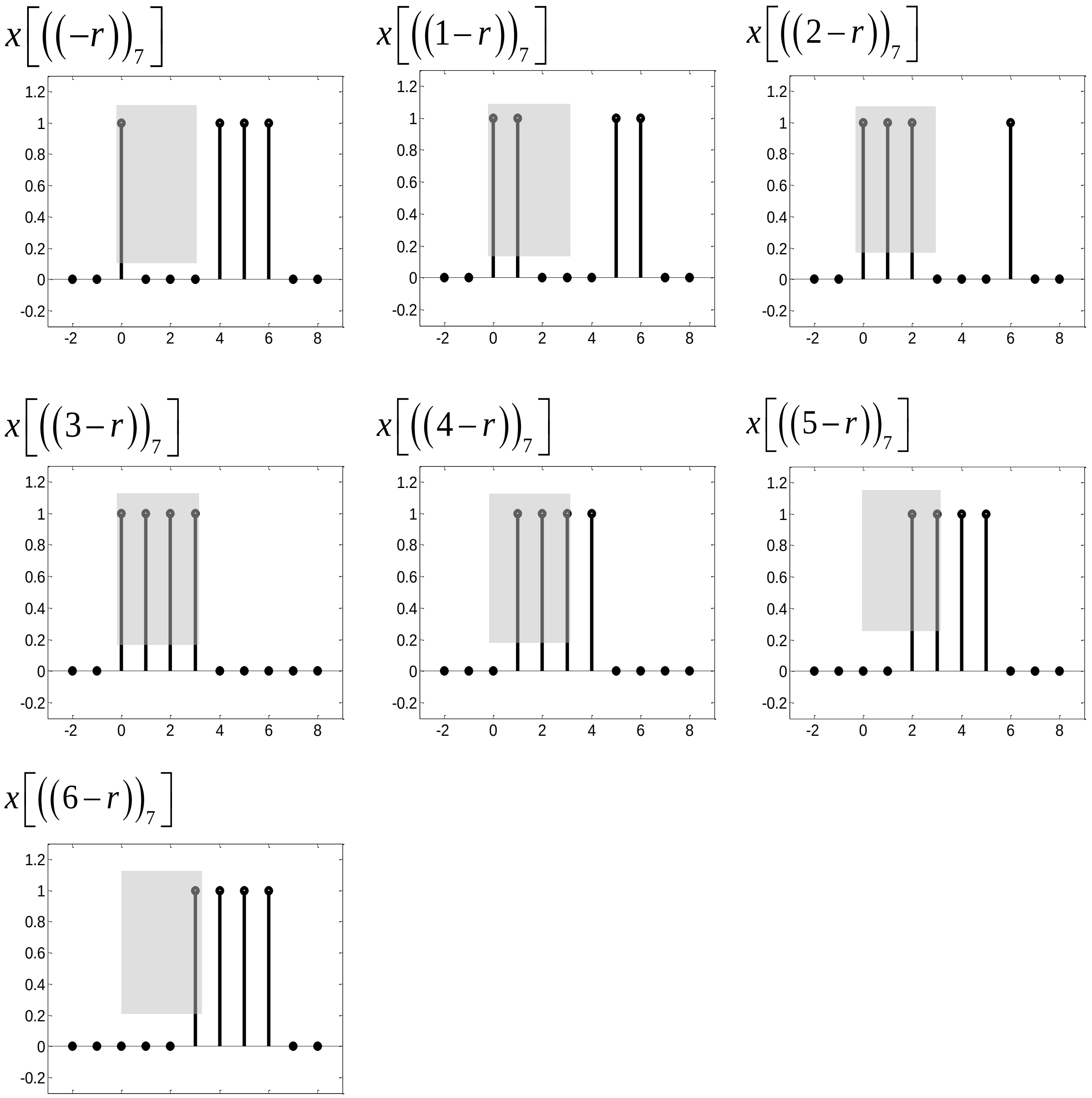
$w[n]$ is the IDFT of $X_7[k]X_7[k]$, $X_7[k]$: 7-point DFT

$$w[n] = \left[\dots 0 \quad 0 \quad \underbrace{1}_{n=0} \quad 2 \quad 3 \quad 4 \quad 3 \quad 2 \quad 1 \quad 0 \quad 0 \dots\right]$$



In this case the result is the same as that of linear convolution.

$$\sum_{r=-\infty}^{\infty} \delta(n - rN) = \frac{1}{N} \sum_{p=0}^{N-1} e^{\pm jp\frac{2\pi}{N}n}$$



Nonzero values of $x[r]$ are over the shaded region.

$$\sum_{r=-\infty}^{\infty} \delta(n-rN) = \frac{1}{N}\sum_{p=0}^{N-1} e^{\pm jp\frac{2\pi}{N}n}$$

Ex:

$$x[n] = \left[\dots \quad 0 \quad 0 \quad \underbrace{1}_{n=0} \quad 2 \quad 0 \quad 0 \quad \dots\right]$$

$$y[n] = \left[\dots \quad 0 \quad 0 \quad \underbrace{-2}_{n=0} \quad 1 \quad 1 \quad 0 \quad 0 \quad \dots\right]$$

a) Let $X[k]$ and $Y[k]$ be 5-point DFTs of $x[n]$ and $y[n]$, respectively.

Find the sequence $f[n] = IDFT\{X[k]\,Y[k]\}$.

b) Let $X[k]$ and $Y[k]$ be 3-point DFTs of $x[n]$ and $y[n]$, respectively. Find the sequence $w[n] = IDFT\{X[k]\,Y[k]\}$

c) Let

$$X[k] = X\left(e^{j\omega}\right)\Big|_{\omega=k\frac{2\pi}{2}} \qquad k = 0,1$$

$$Y[k] = Y\left(e^{j\omega}\right)\Big|_{\omega=k\frac{2\pi}{2}} \qquad k = 0,1$$

Find the sequence $p[n]$ obtained by applying 2-point inverse DFT operation to $X[k]\,Y[k]$.

$$\sum_{r=-\infty}^{\infty} \delta(n - rN) = \frac{1}{N}\sum_{p=0}^{N-1} e^{\pm jp\frac{2\pi}{N}n}$$

a)

$$f[n] = x[n] \circledast_5 y[n]$$

$$= \sum_{r=0}^{4} x[r]y\left[((n-r))_5\right] \qquad n = 0,1,2,3,4$$

$$= x[0]y\left[((n))_5\right] + x[1]y\left[((n-1))_5\right] \qquad n = 0,1,2,3,4$$

$$\begin{array}{rrrrrr} & -2 & 1 & 1 & 0 & 0 \\ + & 0 & -4 & 2 & 2 & 0 \\ \hline & -2 & -3 & 3 & 2 & 0 \end{array}$$

Therefore

$$f[n] = \left[\ldots \quad 0 \quad 0 \quad \underbrace{-2}_{n=0} \quad -3 \quad 3 \quad 2 \quad 0 \quad 0 \quad \ldots\right]$$

Or, since the linear convolution of $x[n]$ and $y[n]$ yields (3+2-1) 4-point sequence and the DFTs are 5-point, 5-point circular convolution and linear convolution yields the same result.

$$f[n] = x[n] \circledast_5 y[n]$$
$$= x[n] * y[n]$$
$$= z[n]$$

$$\sum_{r=-\infty}^{\infty} \delta(n-rN) = \frac{1}{N}\sum_{p=0}^{N-1} e^{\pm jp\frac{2\pi}{N}n}$$

b)

$$w[n] = x[n] \circledast_3 y[n]$$

$$= \sum_{r=0}^{2} x[r]y\left[\big((n-r)\big)_3\right] \qquad n = 0,1,2$$

$$= x[0]y\left[\big((n)\big)_3\right] + x[1]y\left[\big((n-1)\big)_3\right] \qquad n = 0,1,2$$

$$\begin{array}{cccc} & -2 & 1 & 1 \\ + & 2 & -4 & 2 \\ \hline & 0 & -3 & 3 \end{array}$$

Therefore

$$w[n] = \left[\ldots \quad 0 \quad 0 \quad \underbrace{-3}_{n=0} \quad 3 \quad 0 \quad 0 \quad \ldots\right]$$

$$\sum_{r=-\infty}^{\infty} \delta(n - rN) = \frac{1}{N} \sum_{p=0}^{N-1} e^{\pm jp\frac{2\pi}{N}n}$$

or

$$w[n] = x[n] \circledast_3 y[n]$$

$$= \begin{cases} \displaystyle\sum_{r=-\infty}^{\infty} z[n - r3] & n = 0,1,2 \\ 0 & o.w. \end{cases}$$

				$n=0$	$n=1$	$n=2$				
				-2	-3	3	2			
							-2	-3	3	2
$+$	-2	-3	3	2						
				0	-3	3				

$$\sum_{r=-\infty}^{\infty} \delta(n - rN) = \frac{1}{N} \sum_{p=0}^{N-1} e^{\pm jp\frac{2\pi}{N}n}$$

c)

$$p[n] = \begin{cases} \dfrac{1}{2} \displaystyle\sum_{k=0}^{1} X[k]\, Y[k] e^{jk\frac{2\pi}{2}n} & n = 0,1 \\ \\ 0 & o.w. \end{cases}$$

$$= \begin{cases} \displaystyle\sum_{r=0}^{1} z[n - r2] & n = 0,1 \\ \\ 0 & o.w. \end{cases}$$

where

$$z[n] = x[n] * y[n]$$

(linear conv.)

Therefore

$$\begin{array}{ccc|cc|cccc}
 & & & n=0 & n=1 & & & & \\
 & & & -2 & -3 & 3 & 2 & & \\
 & & & & & -2 & -3 & 3 & 2 \\
+ & -2 & -3 & 3 & 2 & & & & \\
\hline
 & & & 1 & -1 & & & & \\
\end{array}$$

$$p[n] = \left[\ldots \quad 0 \quad 0 \quad \underbrace{1}_{n=0} \quad -3 \quad 0 \quad 0 \quad \ldots \right]$$