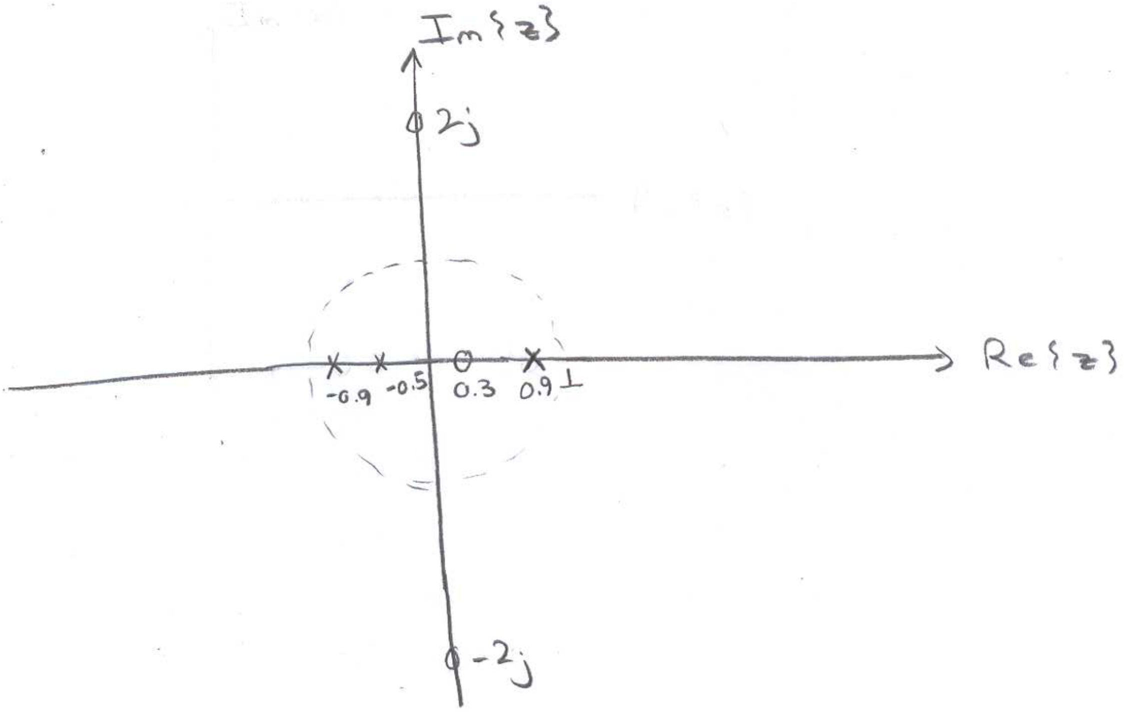
(1) $$H(z) = \frac{(z-0.3)(z+2j)(z-2j)}{(z+0.9)(z-0.9)(z+0.5)}$$

a)



1st Possible ROC : $|z| < 0.5$

⇒ ROC doesn't contain unit circle. System is unstable. Sequence is left sided since ROC is inner of innermost pole. System is anticausal.

2nd Possible ROC : $0.5 < |z| < 0.9$

⇒ ROC doesn't contain unit circle. System is unstable. Sequence is two sided since ROC is a ring.

3rd Possible ROC $|z| > 0.9$

⇒ ROC contains unit circle. System is stable. Also causal. Because ROC is outer of outermost pole.

b)

$$H_1(z) = \frac{(1-0.3z^{-1})\, z^{-4}}{(1+0.9z^{-1})(1-0.9z^{-1})(1+0.5z^{-1})(1+0.5jz^{-1})(1-0.5jz^{-1})}$$

$\Rightarrow$ All poles and zeros of $H_1(z)$ are inside the unit circle So $H_1(z)$ is minimum phase filter.

$$H_{lin}(z) = \underbrace{(1+2jz^{-1})(1-2jz^{-1})(1+0.5jz^{-1})(1-0.5jz^{-1})}_{\text{If } re^{j\theta} \text{ is zero, } \frac{1}{r}e^{-j\theta} \text{ is also zero}}\, z^{4}$$

$$= (1+4z^{-2})(1+0.25z^{-2})\, z^{4}$$

$$= \underbrace{z^4+4.25z^2+1}_{M=4 \rightarrow \text{even}} \rightarrow \text{Type I FIR Linear Phase Filter}$$

(

a) $\angle H(e^{jw}) = ARG[H(e^{jw})] + 2\pi \underbrace{r(w)}_{\text{positive or negative integer}}$

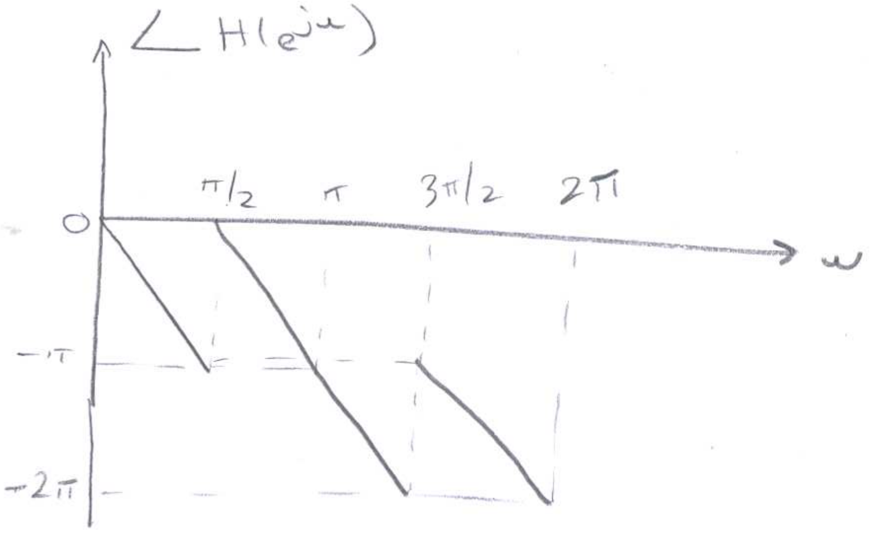
$\angle H(e^{jw})$ can exceed the range $-\pi$ to π.
The principal value of the phase has jumps of 2π, owing to the integer multiples of 2π that must be subtracted in certain regions to bring the phase curve within the range of the principal value. This is the reason of the jump at $w = \pi$.

Notice that;

$$H(e^{jw}) = \underbrace{A(e^{jw})}_{\text{real function}} e^{-j\frac{w}{2}}$$

At $w = \frac{\pi}{2}$ and $w = \frac{3\pi}{2}$, the sign of $A(e^{jw})$ changes. This means adding π to the phase.

b)



$$grd[H(e^{jw})] = \frac{-d}{dw}[\angle H(e^{jw})]$$

If we ignore discontinuties that result from the addition of constant phase (π).

$$\mathrm{grd}[H(e^{j\omega})] = \frac{-d}{d\omega}\left(-2w\right) = 2$$

As we said in part a) $H(e^{j\omega})$ can be expressed as,

$$H(e^{j\omega}) = \underbrace{A(e^{j\omega})}_{\text{real function.}} e^{-j2w}$$

$\Rightarrow$ This filter is a generalized linear phase.

$h[n]$ is nonzero $0 \leq n \leq M = 4$. $h[n]$ is Type I or Type III

<u>Type I FIR Linear-Phase System</u>

$h[n] = h[M-n]$ $\quad 0 \leq n \leq M$ $\quad M$ is even.

$$H(e^{j\omega}) = e^{-j\omega M/2}\left(\sum_{k=0}^{M/2} a[k]\cos(\omega k)\right)$$

$a[0] = h[M/2]$

$a[k] = 2h[M/2 - k]$ $\quad k = 1, 2, \ldots M/2$

<u>Type III FIR Linear-Phase System</u>

$h[n] = -h[M-n]$ $\quad 0 \leq n \leq M$ $\quad M$ is even.

$$H(e^{j\omega}) = j e^{-j\omega M/2}\left[\sum_{k=1}^{M/2} c[k]\sin(\omega k)\right]$$

$c[k] = 2h[M/2 - k]$ $\quad k = 1, 2, \ldots M/2$

We see that $H(e^{j\omega})$ is in the form of Type I FIR Linear Phase filter.

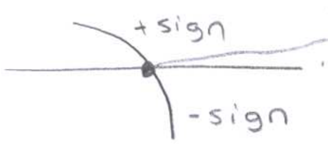
$$h[n] = [\,a \quad b \quad c \quad b \quad a\,] \qquad 0 \leq n \leq 4$$

c) If $z = \frac{1}{2}$ is a zero, then $z = 2$ is also zero.
The sign of $A(e^{j\omega})$ changes at $\omega = \frac{\pi}{2}$ and $\omega = \frac{3\pi}{2}$

Note that $A(e^{j\omega})$ is a continuous function since

$$A(e^{j\omega}) = \sum_{k=0}^{2} a[k]\cos(\omega k)$$

$a[0] = h[2]$
$a[1] = 2h[1]$
$a[2] = 2h[0]$



$\longrightarrow$ So $A(e^{j\omega})$ must be 0 at $\omega = \frac{\pi}{2}$ and $\omega = \frac{3\pi}{2}$

$\Rightarrow H(z) = 0$ where $z = e^{j\frac{\pi}{2}}, e^{j\frac{3\pi}{2}}$. The other zeros are $z = j, -j$.

$$H(z) = K\left(1 - \tfrac{1}{2}z^{-1}\right)\left(1 - 2z^{-1}\right)\left(1 - jz^{-1}\right)\left(1 + jz^{-1}\right)$$

$$= K\left(1 - \tfrac{5}{2}z^{-1} + z^{-2}\right)\left(1 + z^{-2}\right)$$

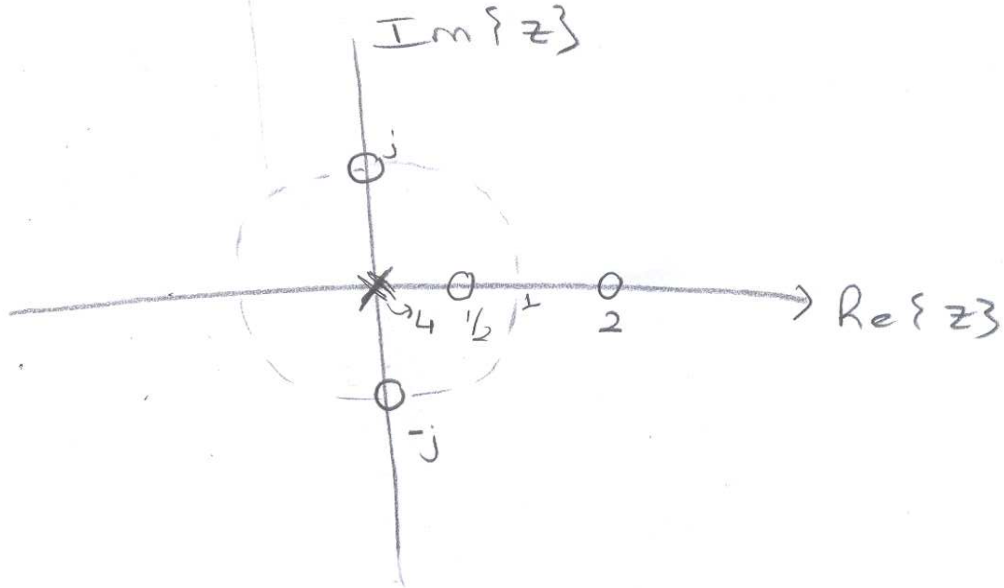
$$= K\left(1 - \tfrac{5}{2}z^{-1} + 2z^{-2} - \tfrac{5}{2}z^{-3} + z^{-4}\right)$$

$$H(1) = 1 = K\left(1 - \tfrac{5}{2} + 2 - \tfrac{5}{2} + 1\right) \Longrightarrow \boxed{K = -1}$$

$$H(z) = -1 + \frac{5}{2} z^{-1} - 2z^{-2} + \frac{5}{2} z^{-3} - z^{-4}$$

$$= \frac{-(z-\frac{1}{2})(z-2)(z-j)(z+j)}{z^4}$$

4 poles at $z=0$



(4) If $h[n] \neq 0$ for $0 \leq n \leq M$

Type I

$$H(e^{jw}) = e^{-jwM/2} \left(\sum_{k=0}^{M/2} a[k] \cos(wk) \right) , \quad a[0] = h[M/2]$$

$a[k] = 2h[M/2-k]$

$k = 1, 2, .. M/2$

M is even.

Type II

$$H(e^{jw}) = e^{-jwM/2} \left(\sum_{k=1}^{(M+1)/2} b[k] \cos\left[w(k-\tfrac{1}{2}) \right] \right)$$

$b[k] = 2h[\frac{M+1}{2} - k]$

$k = 1, 2, .. \frac{M+1}{2}$

M is odd

a) i) $N_1 = M_1 - 1 \quad , \quad N_2 = M_2 - 1$

$$H_1(e^{j\omega}) = e^{-j\omega\frac{N_1}{2}} \underbrace{\left(\sum_{k=0}^{N_1/2} a[k] \cos(\omega k) \right)}_{A_1(e^{j\omega})}$$

$a[0] = h_1[N_1/2]$

$a[k] = 2h_1[N_1/2 - k]$

$k = 1, 2, \ldots N_1/2$

$$H_2(e^{j\omega}) = e^{-j\omega\frac{N_2}{2}} \underbrace{\left(\sum_{k=1}^{(N_2+1)/2} b[k] \cos(\omega(k - \tfrac{1}{2})) \right)}_{A_2(e^{j\omega})}$$

$b[k] = 2h_2[\frac{N_2+1}{2} - k]$

$k = 1, 2, \ldots \frac{N_2+1}{2}$

$G(e^{j\omega}) = H_1(e^{j\omega}) H_2(e^{j\omega})$

overall system frequency response

$$= \underbrace{e^{-j\omega\left(\frac{N_1+N_2}{2}\right)}}_{e^{j(\omega\alpha+\beta)}} \underbrace{A_1(e^{j\omega}) A_2(e^{j\omega})}_{\text{real}} \Rightarrow \text{generalized Linear phase}$$

Determine its type, consider $G(z)$

$H_1(z) = h_1[0] + h_1[1] z^{-1} + \ldots \quad h_1[N_1] z^{-N_1}$

$H_2(z) = h_2[0] + h_2[1] z^{-1} + \quad h_2[N_2] z^{-N_2}$

$G(z) = H_1(z) H_2(z) = g[0] + g[1] z^{-1} + \ldots \quad g[N_1+N_2] z^{-(N_1+N_2)}$

$g[0] = h_1[0] h_2[0]$

$g[1] = h_1[0] h_2[1] + h_1[1] h_2[0]$

$g[2] = h_1[0] h_2[2] + h_1[1] h_2[1] + h_1[2] h_2[0]$

$$g[m] = \sum_{k=0}^{m} h_1[k] h_2[m-k]$$

$g[N_1+N_2] = h_1[N_1] h_2[N_2]$

$$g[N_1+N_2-m] = \sum_{k=0}^{m} h_1[N_1-k] h_2[N_2-m+k]$$

$h_1[N_1-k] = h_1[k]$

$h_2[N_2-m+k] = h_2[m-k]$

$\rightarrow g[m] = g[N_1+N_2-m]$

$N_1 \rightarrow$ even

$N_2 \rightarrow$ odd $\Rightarrow N_1+N_2$ odd $\Rightarrow$ Type II $\leftarrow$

b) A

There are 6 poles at infinity.
ROC contains unit circle (stable system)
ROC is inner of the poles. (noncausal)

$H(z) = a_0 + a_1 z + a_2 z^2 + \dots$

At $z = \infty$ $H(z) \nearrow \infty$.

i) Since all zeros and poles are not inside the unit circle, it is not minimum phase system.

ii) 6 zeros, noncausal

$$\Longrightarrow H(z) = \underbrace{a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + a_5 z^5 + a_6 z^6}_{\text{FIR}}$$

iii) There are zeros both at low frequencies and high frequencies. Neither lowpass nor highpass.
Also since for any zero at $z_0 = re^{j\theta}$, there doesn't exist a pole at $p_0 = \frac{1}{r}e^{j\theta}$, the system is not allpass

iv) Since zeros occur in conjugate pairs, $a_0, a_1, a_2, a_3, a_4, a_5$ and a_6 are real. Real impulse response.

v) noncausal

vi) For any zero $z_0 = re^{j\theta}$, $z_0 = \frac{1}{r}e^{-j\theta}$ is also zero.

$\Rightarrow$ linear phase, no nonlinear phase. This can be seen from;

$$H_k(z) = (z - r_k e^{j\theta_k})(z - \frac{1}{r_k} e^{-j\theta_k}) = z^2 - \underbrace{\left(r_k e^{j\theta_k} + \frac{1}{r_k} e^{-j\theta_k}\right) z + 1}_{\text{linear phase system}}$$

$$H(z) = \prod_k H_k(z)$$

As we see in part a) cascading linear phase systems result in linear phase systems

vii) Linear phase.

$M = 6$ (even) Type I or III

Since $H(z)$ has not a zero at $z = 1$

It is not Type III. $\Rightarrow$ Type I

B

i) not minimum phase

ii) Since ROC doesn't contain whole region except 0 or ∞

IIR system. Because it has a pole other than 0 or ∞.

iii) Allpass.

iv) Since poles and zeros do not occur in conjugate pairs, it has complex impulse response

v) Since ROC contains unit circle (stable), ROC is outer of outermost pole. $\Rightarrow$ causal.

vi)
$$H(z) = K\frac{z - \frac{1}{r}e^{j\theta}}{z - re^{j\theta}} = K\frac{(1 - \frac{1}{r}e^{j\theta}z^{-1})}{(1 - re^{j\theta}z^{-1})}$$

$$= -K\frac{1}{r}e^{j\theta}\frac{(z^{-1} - re^{-j\theta})}{(1 - re^{j\theta}z^{-1})}$$

$$\measuredangle H(z) = \angle -K\frac{1}{r}e^{j\theta} - \omega - 2\arctan\left[\frac{r\sin(\omega-\theta)}{1 - r\cos(\omega-\theta)}\right]$$

$\Rightarrow$ nonlinear phase.

C

i) minimum phase

ii) Since ROC contains whole region except 0 FIR system. There are 3 zeros.

$$\underbrace{H(z) = a_0 + a_1 z^{-1} + a_2 z^{-2}}_{\text{when } z=0 \quad H(z) \nearrow \infty} + a_3 z^{-3} \quad \Rightarrow \text{FIR}$$

iii) There are zeros at high frequencies. It attenuates high frequencies. Lowpass filter $\leftarrow$

iv) real impulse response

v) ROC is outer of outermost pole. causal $\leftarrow$

vi) For zeros $z_0 = re^{j\theta}$, there is no zero at $z_0 = \frac{1}{r}e^{-j\theta}$
$\Rightarrow$ non-linear phase.