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Section: 2

EE430 - HW2

1) a)

$$y[n]-\frac{1}{2}y[n-1]=x[n]-x[n-1]+x[n-2]$$

$$h[n]-\frac{1}{2}h[n-1]=\delta[n]-\delta[n-1]+\delta[n-2] \Rightarrow h[n]=0 \;, n<0 \;\; (\text{system is causal})$$

$$h[0]=1$$
$$h[1]=-1/2$$
$$h[2]=3/4$$
$$h[n]=\frac{1}{2}h[n-1] \;, \; n>2$$

$$\boxed{\Rightarrow h[n]=\left(\frac{1}{2}\right)^{n}u[n]-\left(\frac{1}{2}\right)^{n-1}u[n-1]+\left(\frac{1}{2}\right)^{n-2}u[n-2]}$$

b) $$H(e^{j\omega})=\sum_{n=-\infty}^{\infty}h[n]e^{-j\omega n}$$ where $$\begin{cases} \text{if } x[n]=\alpha^n u[n] \Rightarrow X(e^{j\omega})=\dfrac{1}{1-\alpha e^{-j\omega}} \\ \text{and } y[n]=x[n-n_0] \Rightarrow Y(e^{j\omega})=X(e^{j\omega}).e^{-j\omega n_0} \end{cases}$$

$$\Rightarrow H(e^{j\omega})=\frac{1}{1-\frac{1}{2}e^{-j\omega}}-\frac{e^{-j\omega}}{1-\frac{1}{2}e^{-j\omega}}+\frac{e^{-2j\omega}}{1-\frac{1}{2}e^{-j\omega}}=\frac{e^{j\omega}-1+e^{-j\omega}}{e^{j\omega}-\frac{1}{2}}$$

$$\Rightarrow \boxed{H(e^{j\omega})=\frac{2\cos(\omega)-1}{e^{j\omega}-\frac{1}{2}}}$$

c) Magnitude and phase responses are plotted using MATLAB and can be find at the end.

d) $$x[n]=\cos\left(\frac{\pi}{3}n\right)+\sin\left(\frac{\pi}{2}n+\frac{\pi}{4}\right)=\frac{1}{2}\left(e^{j\frac{\pi}{3}n}+e^{-j\frac{\pi}{3}n}\right)+\frac{1}{2j}e^{j\frac{\pi}{4}}\left(e^{j\frac{\pi}{2}n}-e^{-j\frac{\pi}{2}n}\right)$$

$\Rightarrow$ since the system is LTI

$$y[n]=\frac{1}{2}\left(\underbrace{\cancel{e^{j\frac{\pi}{3}n}\,H(e^{j\omega})\Big|_{\omega=\frac{\pi}{3}}}}_{0}+\underbrace{\cancel{e^{-j\frac{\pi}{3}n}\,H(e^{j\omega})\Big|_{\omega=-\frac{\pi}{3}}}}_{0}\right)+\frac{1}{2j}e^{j\frac{\pi}{4}}\left(e^{j\frac{\pi}{2}n}H(e^{j\omega})\Big|_{\omega=\frac{\pi}{2}}-e^{-j\frac{\pi}{2}n}H(e^{j\omega})\Big|_{\omega=-\frac{\pi}{2}}\right)$$

$$y[n]=\frac{1}{2j}e^{j\frac{\pi}{4}}\left(e^{j\frac{\pi}{2}n}\left(\frac{-1}{j-1/2}\right)-e^{-j\frac{\pi}{2}n}\left(\frac{-1}{-j-\frac{1}{2}}\right)\right)=\boxed{e^{j\frac{\pi}{4}}\,\mathrm{Im}\left(e^{j\frac{\pi}{2}n}\left(\frac{-1}{j-\frac{1}{2}}\right)\right)=y[n]}$$

e) $H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]\, e^{-j\omega n}$

$\Rightarrow H(e^{-j\omega}) = \sum_{n=-\infty}^{\infty} h[n]\, e^{+j\omega n} \Rightarrow H^*(e^{-j\omega}) = \sum_{n=-\infty}^{\infty} h^*[n]\, e^{-j\omega n}$ where $h[n] = h^*[n]$ since $h[n]$ is real.

$\Rightarrow H(e^{j\omega}) = H^*(e^{-j\omega}) = H^*(e^{-j\omega}\underbrace{e^{j2\pi}}_{1}) = H^*(e^{-j(2\pi-\omega)})$ is proved.

This does not hold for arbitrary $h[n]$. $h[n]$ must be real to hold.

2) Let $y[n] = a^n u[n] \Rightarrow$ if $|a|<1 \Rightarrow Y(e^{j\omega}) = \dfrac{1}{1-\alpha e^{-j\omega}}$

$\Rightarrow$ let $x_1[n] = y[n-2] = a^{n-2} u[n-2] = X_1(e^{j\omega}) = e^{-2j\omega} \cdot \dfrac{1}{1-\alpha e^{-j\omega}} = \dfrac{1}{e^{2j\omega} - \alpha e^{j\omega}}$

$\Rightarrow x[n] = n \cdot x_1[n] \Rightarrow X(e^{j\omega}) = j \cdot \dfrac{d X_1(e^{j\omega})}{d\omega} = j\dfrac{d}{d\omega}\left(\dfrac{1}{e^{2j\omega} - \alpha e^{j\omega}}\right)$

$\Rightarrow X(e^{j\omega}) = \dfrac{-j}{(e^{2j\omega} - \alpha e^{j\omega})^2} \cdot (2j e^{2j\omega} - \alpha j e^{j\omega}) = \boxed{\dfrac{2e^{2j\omega} - \alpha e^{j\omega}}{(e^{2j\omega} - \alpha e^{j\omega})^2} = X(e^{j\omega})}$

3) $x_1[n] = \sin\left(\frac{\pi}{7}n\right) + \sin\left(\frac{\pi}{3}n\right)$ $\quad x_2[n] = u[n] \cdot x_1[n]$

a) for $h[n] = [\frac{1}{2}\ \frac{1}{2}]$ for $n = [0\ 1] \Rightarrow$

$y_1[n] = \frac{1}{2}\left(x_1[n] + x_1[n-1]\right)$

$y_2[n] = \frac{1}{2}\left(x_1[n]u[n] + x_1[n-1] \cdot u[n-1]\right)$

$\Rightarrow y_1[n] = y_2[n]$ for $n \geq 1$

b) for $h[n] = [\frac{1}{2}\ \frac{1}{2}]$ for $n = [2\ 3] \Rightarrow$

$y_1[n] = \frac{1}{2}\left(x_1[n-2] + x_1[n-3]\right)$

$y_2[n] = \frac{1}{2}\left(x_1[n-2]u[n-2] + x_1[n-3]u[n-3]\right)$

$\Rightarrow y_1[n] = y_2[n]$ for $n \geq 3$

d) for $h[n] = [\frac{1}{6}\ \frac{1}{6}\ \frac{1}{6}\ \frac{1}{6}\ \frac{1}{6}\ \frac{1}{6}]$ for $n = [0\ 1\ 2\ 3\ 4\ 5]$

$\Rightarrow y_1[n] = \frac{1}{6}\sum_{k=0}^{5} x_1[n-k]$

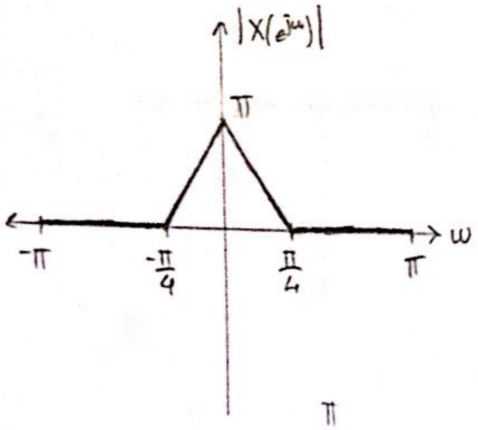
$y_2[n] = \frac{1}{6}\sum_{k=0}^{5} x_1[n-k]u[n-k]$

$\Rightarrow y_1[n] = y_2[n]$ for $n \geq 5$

4) a) since $x[n]$ is a real sequence ; $X(e^{j\omega}) = X^*(e^{-j\omega})$

$\Rightarrow$ $\mathrm{Re}\{X(e^{j\omega})\}$ is symmetric and $\mathrm{Im}\{X(e^{j\omega})\}$ is antisymmetric

$\Rightarrow$ $|X(e^{j\omega})|$ is symmetric and $\measuredangle X(e^{j\omega})$ is antisymmetric.



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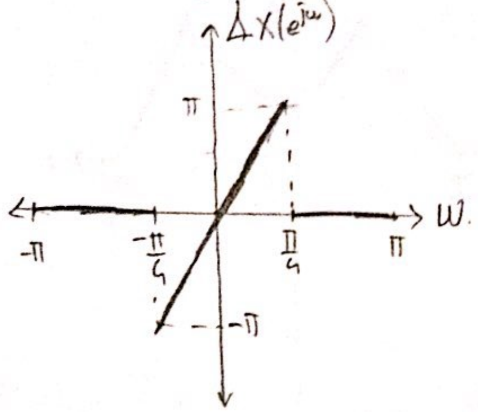


b) $$X_c(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X\left(e^{j(\omega-\Phi)}\right)\pi\left[\sum_{r=-\infty}^{\infty}\delta\left(\Phi-\frac{\pi}{5}+2\pi r\right)+\sum_{r=-\infty}^{\infty}\delta\left(\Phi+\frac{\pi}{5}+2\pi r\right)\right]d\Phi$$

$$= \frac{1}{2\pi}\cdot\pi.\int_{-\pi}^{\pi} X\left(e^{j(\omega-\Phi)}\right).\left\{\delta\left(\Phi-\frac{\pi}{5}\right)+\delta\left(\Phi+\frac{\pi}{5}\right)\right\}d\Phi$$

$$= \frac{1}{2}\left(X\left(e^{j(\omega-\frac{\pi}{5})}\right)+X\left(e^{j(\omega+\frac{\pi}{5})}\right)\right)$$

$$X_s(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X\left(e^{j(\omega-\Phi)}\right).j\pi.\left[\sum_{r=-\infty}^{\infty}\delta\left(\Phi+\frac{\pi}{5}+2\pi r\right)-\sum_{r=-\infty}^{\infty}\delta\left(\Phi-\frac{\pi}{5}+2\pi r\right)\right]d\Phi$$

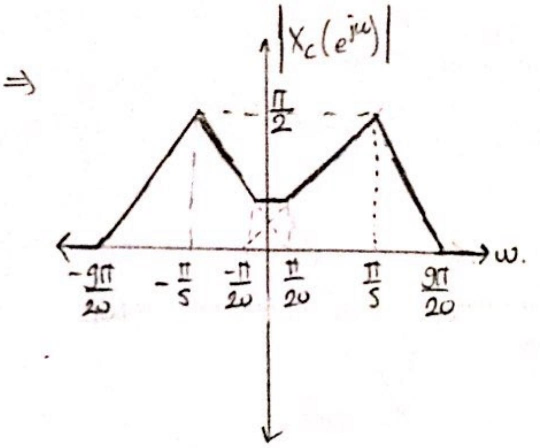
$$= \frac{1}{2\pi}.j\pi.\int_{-\pi}^{\pi} X\left(e^{j(\omega-\Phi)}\right).\left\{\delta\left(\Phi+\frac{\pi}{5}\right)-\delta\left(\Phi-\frac{\pi}{5}\right)\right\}d\Phi$$

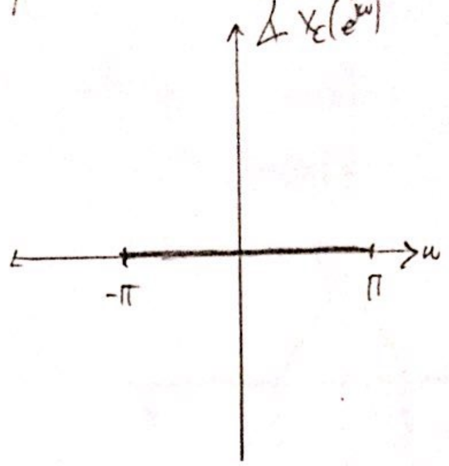
$$= \frac{j}{2}\left(X\left(e^{j(\omega+\frac{\pi}{5})}\right)-X\left(e^{j(\omega-\frac{\pi}{5})}\right)\right)$$

c) If $X(e^{j\omega})$ is real ; $|X(e^{j\omega})| = X(e^{j\omega})$ and $\measuredangle X(e^{j\omega}) = 0$

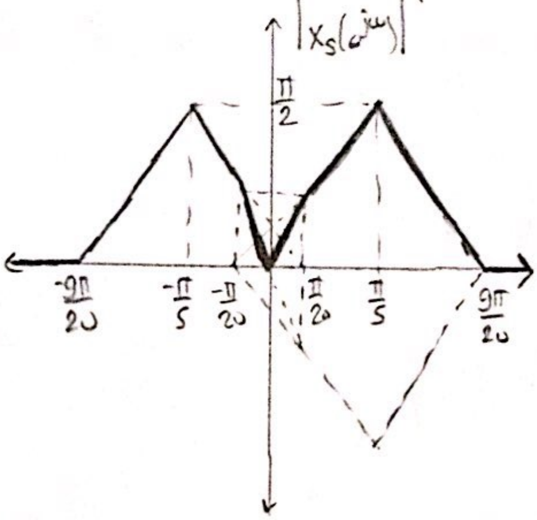
$$\Rightarrow X_c(e^{j\omega}) = \frac{1}{2}\left(X\left(e^{j(\omega-\frac{\pi}{5})}\right) + X\left(e^{j(\omega+\frac{\pi}{5})}\right)\right)$$

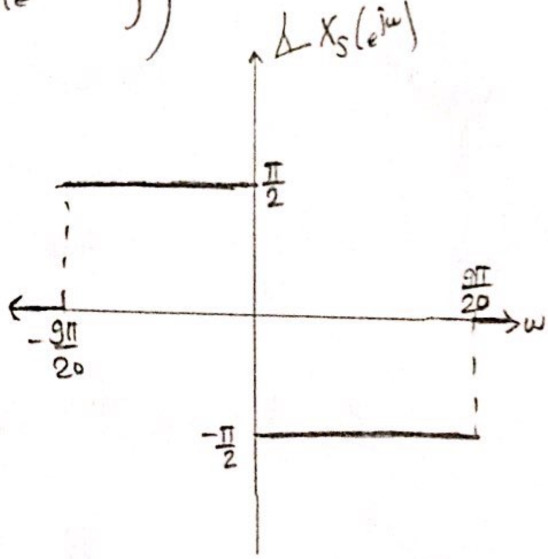
$\Rightarrow$





$$\Rightarrow X_s(e^{j\omega}) = \frac{j}{2}\left(X\left(e^{j(\omega+\frac{\pi}{5})}\right) - X\left(e^{j(\omega-\frac{\pi}{5})}\right)\right)$$





5) Let $X(e^{j\omega}) = \dfrac{\sin\left(\frac{5}{2}\omega\right)}{\sin\left(\frac{\omega}{2}\right)}$ be DTFT of the sequence $x[n]$

Note that this expression is equal to "0"

$$\Rightarrow X(e^{j\omega}) = \frac{e^{j\frac{5}{2}\omega} - e^{-j\frac{5}{2}\omega}}{e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}}} = \frac{e^{+2j\omega} - e^{-3j\omega}}{1 - e^{-j\omega}} = \frac{e^{+2j\omega}}{1-e^{-j\omega}} - \frac{e^{-3j\omega}}{1-e^{-j\omega}} + e^{+2j\omega}\pi\sum_{k=-\infty}^{\infty}\delta(\omega-2\pi k) - e^{-3j\omega}\pi\sum_{k=-\infty}^{\infty}\delta(\omega-2\pi k)$$

$$= \underbrace{\frac{e^{2j\omega}}{1-e^{-j\omega}} + e^{2j\omega}\pi\sum_{k=-\infty}^{\infty}\delta(\omega-2\pi k)}_{\text{DTFT of } u[n+2]} - \underbrace{\frac{e^{-3j\omega}}{1-e^{-j\omega}} - e^{-3j\omega}\pi\sum_{k=-\infty}^{\infty}\delta(\omega-2\pi k)}_{\text{DTFT of } u[n-3]}$$

$$\Rightarrow x[n] = u[n+2] - u[n-3] \text{ and } \sum_{n=-\infty}^{\infty}(x[n])^2 = 5 = \frac{1}{2\pi}\int_{-\pi}^{\pi}|X(e^{j\omega})|^2 d\omega = \frac{2}{2\pi}\int_{-\pi}^{0}|X(e^{j\omega})|^2 d\omega$$

b) *

* since $x[n]$ is real, $X(e^{jw})$ is conjugate symmetric. Therefore;

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|X(e^{jw})|^2\,dw = \frac{2}{2\pi}\int_{-\pi}^{0}|X(e^{jw})|^2\,dw$$

$$\Rightarrow\quad 5 = \frac{1}{\pi}\int_{-\pi}^{0}|X(e^{jw})|^2\,dw \Rightarrow \boxed{\int_{-\pi}^{0}\left|\frac{\sin(\frac{5}{2}w)}{\sin(\frac{w}{2})}\right|^2 dw = 5\pi}$$

6) $x[n] = e^{jw_0 n} \Rightarrow X(e^{jw}) = 2\pi\sum_{r=-\infty}^{\infty}\delta(w-w_0-2\pi r)$

$$\Rightarrow Y(e^{jw}) = H(e^{jw}).X(e^{jw})$$

$$\Rightarrow\quad y[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi}H(e^{jw}).X(e^{jw}).e^{jwn}\,dw = \frac{1}{\cancel{2\pi}}\int_{-\pi}^{\pi}H(e^{jw})e^{jwn}\sum_{r=-\infty}^{\infty}\cancel{2\pi}.\delta(w-w_0+2\pi r)$$

$\Downarrow$ only nonzero if $w = w_0$.

$$\boxed{\Rightarrow y[n] = H(e^{jw_0})\,e^{jw_0 n}}$$

7) a) $h[n] = \left(\frac{1}{2}\right)^n u[n] + 2^n u[n]$ and $x[n] = 3^n$

$$\Rightarrow y[n] = \sum_{k=-\infty}^{\infty}h[k]x[n-k] = \sum_{k=0}^{\infty}\left(\left(\frac{1}{2}\right)^k+2^k\right).3^{n-k} = 3^n\left[\overbrace{\sum_{k=0}^{\infty}\left(\frac{1}{6}\right)^k}^{\frac{6}{5}} + \overbrace{\sum_{k=0}^{\infty}\left(\frac{2}{3}\right)^k}^{3}\right]$$

$$\boxed{= 3^n.\frac{21}{5} = 3^n.C}\ \checkmark$$

b) $h[n] = \left(\frac{1}{2}\right)^n u[n] + 2^n u[-n]$ and $x[n] = 3^n$

$$\Rightarrow y[n] = \sum_{k=-\infty}^{\infty}h[k]x[n-k] = \underbrace{\sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^k.3^{n-k}}_{3^n.\frac{6}{5}} + \underbrace{\sum_{k=-\infty}^{0}2^k.3^{n-k}}_{3^n\sum_{k=0}^{\infty}\left(\frac{3}{2}\right)^k}$$

$3^n\sum_{k=0}^{\infty}\left(\frac{3}{2}\right)^k \Rightarrow$ infinite sum. $\left(\frac{3}{2}>1\right)$

$\Rightarrow y[n]$ cannot be expressed as $3^n.C$ where C is a complex constant.

8) a) $H(z) = \dfrac{2z^{-1}+1}{\left(1-\frac{1}{4}z^{-1}\right)\left(1+z^{-1}\right)\left(1+\frac{1}{4}z^{-1}\right)} = \dfrac{A}{1-\frac{1}{4}z^{-1}} + \dfrac{B}{1+z^{-1}} + \dfrac{C}{1+\frac{1}{4}z^{-1}}$

$\Rightarrow A\left(1+\frac{5}{4}z^{-1}+\frac{1}{4}z^{-2}\right) + B\left(1-\frac{1}{16}z^{-2}\right) + C\left(1+\frac{3}{4}z^{-1}-\frac{1}{4}z^{-2}\right)$

$= 2z^{-1}+1$

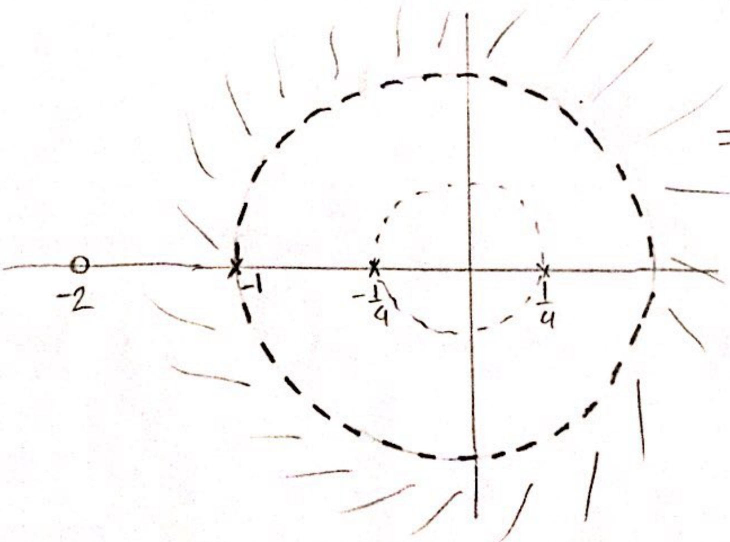
$$\Rightarrow \left.\begin{array}{l} A+B+C = 1 \\ \frac{5}{4}A + \frac{3C}{4} = 2 \\ \frac{1}{4}A - \frac{1}{16}B - \frac{1}{4}C = 0 \end{array}\right\} \begin{array}{l} 5A-3C=1 \\ \Rightarrow \frac{1+3C}{4} + \frac{3C}{4} = 2 \Rightarrow C = \frac{7}{6} \\ A = \frac{9}{10} \\ B = \frac{16}{15} \end{array}$$

$\Rightarrow H(z) = \dfrac{\frac{9}{10}}{1-\frac{1}{4}z^{-1}} + \dfrac{\frac{16}{15}}{1+z^{-1}} + \dfrac{\frac{7}{6}}{1+\frac{1}{4}z^{-1}}$

$\Rightarrow h[n] = \underbrace{\underbrace{\frac{9}{10}\left(\frac{1}{4}\right)^n u[n]}_{|z|>\left|\frac{1}{4}\right|} + \underbrace{\frac{16}{15}(-1)^n u[n]}_{|z|>1} + \underbrace{\frac{7}{6}\left(-\frac{1}{4}\right)^n u[n]}_{|z|>\left|-\frac{1}{4}\right|}}$

$h[n]=0$ for $n<0$
$\Rightarrow$ system is causal.

we have chosen these ROC's since it is given that the system is STABLE.



$\Rightarrow$ ROC

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9) c) $x[n] = 3 + j5 + \sin\left(\frac{\pi}{4}n\right) = 3e^{j0n} + 5je^{j0n} + \frac{1}{2j}e^{j\frac{\pi}{4}n} - \frac{1}{2j}e^{-j\frac{\pi}{4}n}$

$$\Rightarrow y[n] = 3.e^{j0n}H(e^{j\omega})\Big|_{\omega=0} + 5j.H(e^{j\omega})\Big|_{\omega=0} + \frac{1}{2j}e^{j\frac{\pi}{4}n}H(e^{j\omega})\Big|_{\omega=\frac{\pi}{4}} - \frac{1}{2j}e^{-j\frac{\pi}{4}n}H(e^{j\omega})\Big|_{\omega=-\frac{\pi}{4}}$$

$\Rightarrow$ using the given magnitude and phase responses:

$$y[n] = 3 + 5j + \frac{1}{2j}e^{j\frac{\pi}{4}n}\frac{1}{2}.e^{-j\frac{\pi}{8}} - \frac{1}{2j}.e^{-j\frac{\pi}{4}n}.\frac{1}{2}.e^{+j\frac{\pi}{8}}$$

$$\boxed{\Rightarrow y[n] = 3 + 5j + \frac{1}{2}\sin\left(\frac{\pi}{4}n - \frac{\pi}{8}\right)}$$