

## COMPUTATION OF DFT

### FAST FOURIER TRANSFORM (FFT)

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn} \quad k = 0, 1, \dots, N-1$$

For each  $k$ ,

$N$  complex multiplications

$N - 1$  complex additions

For all

$N^2$  complex multiplications

$N(N - 1)$  complex additions

Each complex multiplication = 4 real multiplications + 2 real additions

Each complex addition = 2 real additions

Therefore, for each  $k$

$4N$  real multiplications

$4N - 2$  real additions  $(2(N - 1) + 2N)$

In total

$4N^2$  real multiplications

$N(4N - 2)$  real additions

Let's have a close look at an 8-point DFT.

$$X[0] = x[0] + x[1] + x[2] + x[3] + x[4] + x[5] + x[6] + x[7]$$

$$X[1] = x[0] + x[1] W_8 + x[2] W_8^2 + x[3] W_8^3 + x[4] W_8^4 + x[5] W_8^5 + x[6] W_8^6 + x[7] W_8^7$$

$$X[2] = x[0] + x[1] W_8^2 + x[2] W_8^4 + x[3] W_8^6 + x[4] \underbrace{W_8^8}_1 + x[5] \underbrace{W_8^{10}}_{W_8^2} + x[6] \underbrace{W_8^{12}}_{W_8^4} + x[7] \underbrace{W_8^{14}}_{W_8^6}$$

$$X[3] = x[0] + x[1] W_8^3 + x[2] W_8^6 + x[3] \underbrace{W_8^9}_{W_8^1} + x[4] \underbrace{W_8^{12}}_{W_8^4} + x[5] \underbrace{W_8^{15}}_{W_8^7} + x[6] \underbrace{W_8^{18}}_{W_8^2} + x[7] \underbrace{W_8^{21}}_{W_8^5}$$

$$X[4] = x[0] + x[1] \underbrace{W_8^4}_{-1} + x[2] \underbrace{W_8^8}_1 + x[3] \underbrace{W_8^{12}}_{W_8^4=-1} + x[4] \underbrace{W_8^{16}}_1 + x[5] \underbrace{W_8^{20}}_{W_8^4=-1} + x[6] \underbrace{W_8^{24}}_1 + x[7] \underbrace{W_8^{28}}_{W_8^4=-1}$$

$$X[5] = x[0] + x[1] W_8^5 + x[2] \underbrace{W_8^{10}}_{W_8^2} + x[3] \underbrace{W_8^{15}}_{W_8^7} + x[4] \underbrace{W_8^{20}}_{W_8^4=-1} + x[5] \underbrace{W_8^{25}}_{W_8^1} + x[6] \underbrace{W_8^{30}}_{W_8^6} + x[7] \underbrace{W_8^{35}}_{W_8^3}$$

$$X[6] = x[0] + x[1] W_8^6 + x[2] \underbrace{W_8^{12}}_{W_8^4} + x[3] \underbrace{W_8^{18}}_{W_8^2} + x[4] \underbrace{W_8^{24}}_1 + x[5] \underbrace{W_8^{30}}_{W_8^6} + x[6] \underbrace{W_8^{36}}_{W_8^4} + x[7] \underbrace{W_8^{42}}_{W_8^2}$$

$$X[7] = x[0] + x[1] W_8^7 + x[2] \underbrace{W_8^{14}}_{W_8^6} + x[3] \underbrace{W_8^{21}}_{W_8^5} + x[4] \underbrace{W_8^{28}}_{W_8^4=-1} + x[5] \underbrace{W_8^{35}}_{W_8^3} + x[6] \underbrace{W_8^{42}}_{W_8^2} + x[7] \underbrace{W_8^{49}}_{W_8^1}$$

Note that

$$W_8^4 = -W_8^0$$

$$W_8^5 = -W_8^1$$

$$W_8^6 = -W_8^2$$

$$W_8^7 = -W_8^3$$

i.e.

$$W_N^a = -W_N^{a+\frac{N}{2}}$$

$$X[1] = (x[0] - x[4]) + (x[1] - x[5]) W_8^1 + (x[2] - x[6]) W_8^2 + (x[3] - x[7]) W_8^3$$

$$X[2] = (x[0] - x[2] + x[4] - x[6]) + (x[1] - x[3] + x[5] - x[7]) W_8^2$$

$$X[3] = (x[0] - x[4]) + (x[3] - x[7]) W_8^1 + (-x[2] + x[6]) W_8^2 + (x[1] - x[5]) W_8^3$$

$$X[4] = x[0] - x[1] + x[2] - x[3] + x[4] - x[5] + x[6] - x[7]$$

$$X[5] = (x[0] - x[4]) + (-x[1] + x[5]) W_8^1 + (x[2] - x[6]) W_8^2 + (-x[3] + x[7]) W_8^3$$

$$X[6] = x[0] - x[2] + x[4] - x[6] + (-x[1] + x[3] - x[5] + x[7]) W_8^2$$

$$X[7] = (x[0] - x[4]) + (-x[3] + x[7]) W_8^1 + (-x[2] + x[6]) W_8^2 + (-x[1] + x[5]) W_8^3$$

Therefore, the computation of the following is sufficient to get all 8-point DFT values...

$$x[0] + x[4] \qquad x[0] - x[4]$$

$$x[1] + x[5] \qquad x[1] - x[5]$$

$$x[2] + x[6] \qquad x[2] - x[6]$$

$$x[3] + x[7] \qquad x[3] - x[7]$$

$$(x[1] - x[5])W_8^1 \qquad (-x[1] + x[5])W_8^3$$

$$(x[3] - x[7])W_8^3 \qquad (-x[3] + x[7])W_8^1$$

$$(x[2] - x[6])W_8^2 \qquad (x[1] - x[3] + x[5] - x[7])W_8^2$$

We will study the two systematic methods

1) Decimation-in-time

2) Decimation-in-frequency

## Decimation-in-Time

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn} \quad k = 0, 1, \dots, N-1$$

Assume that  $N$  is a power of two,  $N = 2^m$ .



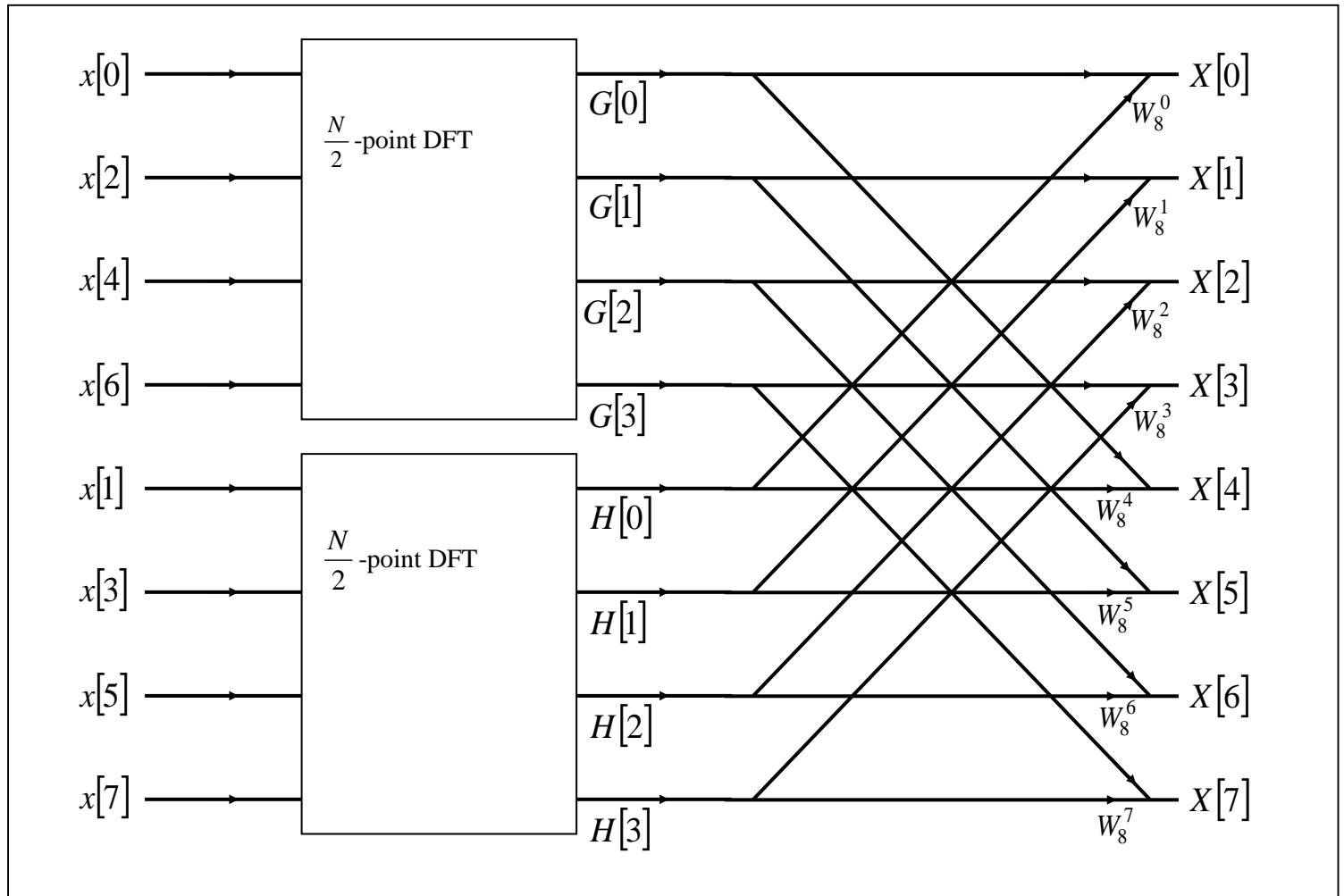
$$\begin{aligned}
X[k] &= \sum_{n:\text{even}}^{N-1} x[n]W_N^{kn} + \sum_{n:\text{odd}}^{N-1} x[n]W_N^{kn} \\
&= \sum_{r=0}^{\frac{N}{2}-1} x[2r]W_N^{2kr} + \sum_{r=0}^{\frac{N}{2}-1} x[2r+1]W_N^{k(2r+1)} \\
&= \underbrace{\sum_{r=0}^{\frac{N}{2}-1} x[2r]W_N^{\frac{kr}{2}}}_{G[k]} + W_N^k \underbrace{\sum_{r=0}^{\frac{N}{2}-1} x[2r+1]W_N^{\frac{kr}{2}}}_{H[k]}
\end{aligned}$$

$G[k]: \frac{N}{2} - \text{point DFT of even indexed samples}$

$H[k]: \frac{N}{2} - \text{point DFT of odd indexed samples}$

Therefore

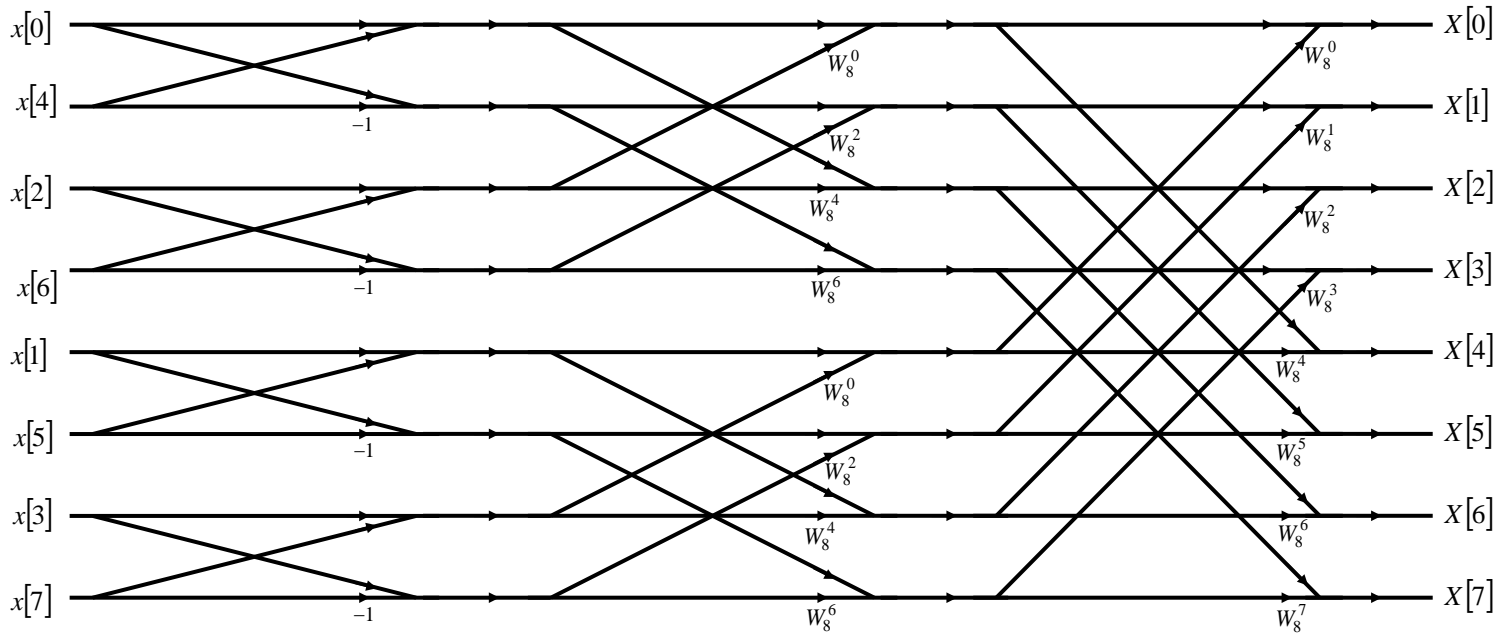
$$X[k] = G[k] + W_N^k H[k] \quad k = 0, \dots, N-1$$



$$2 \left( \frac{N}{2} \right)^2 + N = \frac{N^2}{2} + N \quad \text{complex multiplications required}$$

$\frac{N}{2}$  –point DFTs can be decomposed into  $\frac{N}{4}$  –point DFTs.

Then the following computation diagram arises.



$$4 \left( \frac{N}{4} \right)^2 + N + N = \frac{N^2}{4} + 2N$$

complex multiplications required

In general, the number of sections will be

$$\log_2 N = v$$

Therefore

$$\underbrace{\frac{N}{2} \left( \frac{N}{2} \right)^2}_{2N} + N + N + \cdots + N = (v + 1)N$$

complex multiplications required

Considering the multiplications by -1 and 1, this value is approximately referred to as

$$N \log_2 N$$

This is also the approximate number of additions.

Ex:

$$N = 2^{10} = 1024$$

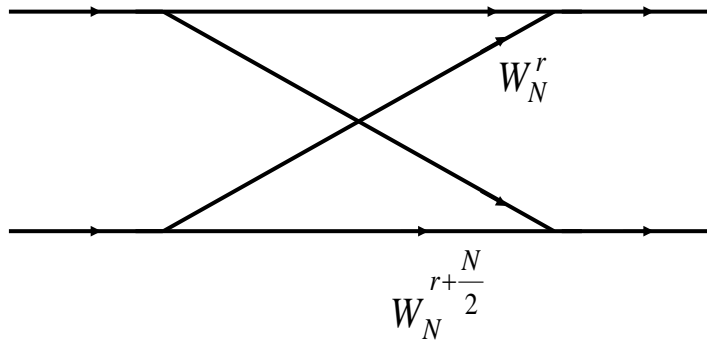
$$N^2 = 1\,048\,576 \approx 10^6$$

$$N \log_2 N = 10\,240 \approx 10^4$$

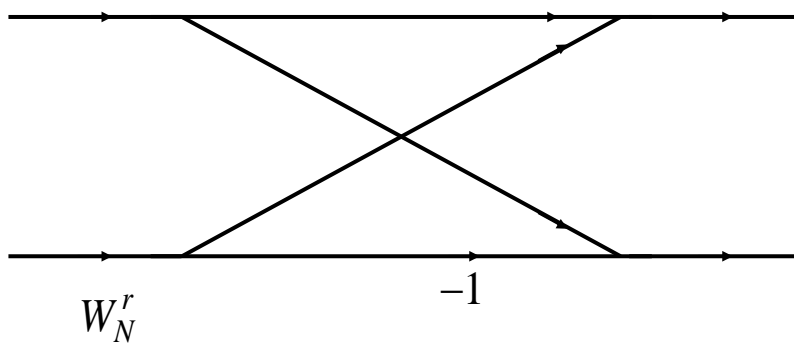
A reduction by a factor of 100 approximately.

## “BUTTERFLY”s

Indeed, above diagrams contain many “butterfly” structures,



Such butterflies can be simplified as





## DECIMATION-IN-FREQUENCY

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn} \quad k = 0, 1, \dots, N-1$$

Assume that  $N$  is a power of two,  $N = 2^m$ .

Consider “even” and “odd” indexed elements of the DFT:

$$X[k] \begin{cases} \rightarrow X[2r] \\ \rightarrow X[2r + 1] \end{cases} \quad r = 0, 1, \dots, \frac{N}{2} - 1$$

Even indexed elements:

$$\begin{aligned}
 X[2r] &= \sum_{n=0}^{N-1} x[n] W_N^{2rn} \quad r = 0, 1, \dots, \frac{N}{2} - 1 \\
 &= \sum_{n=0}^{\frac{N}{2}-1} x[n] W_N^{2rn} + \sum_{n=\frac{N}{2}}^{N-1} x[n] W_N^{2rn} \\
 &= \sum_{n=0}^{\frac{N}{2}-1} x[n] W_N^{2rn} + \sum_{n=0}^{\frac{N}{2}-1} x\left[n + \frac{N}{2}\right] W_N^{2r\left(n + \frac{N}{2}\right)} \\
 &= \sum_{n=0}^{\frac{N}{2}-1} \left( x[n] + x\left[n + \frac{N}{2}\right] \right) W_{\frac{N}{2}}^{rn}
 \end{aligned}$$

This is a  $\frac{N}{2}$ -point DFT of

$$\left( x[n] - x\left[n + \frac{N}{2}\right] \right) \quad n = 0, 1, \dots, \frac{N}{2} - 1.$$

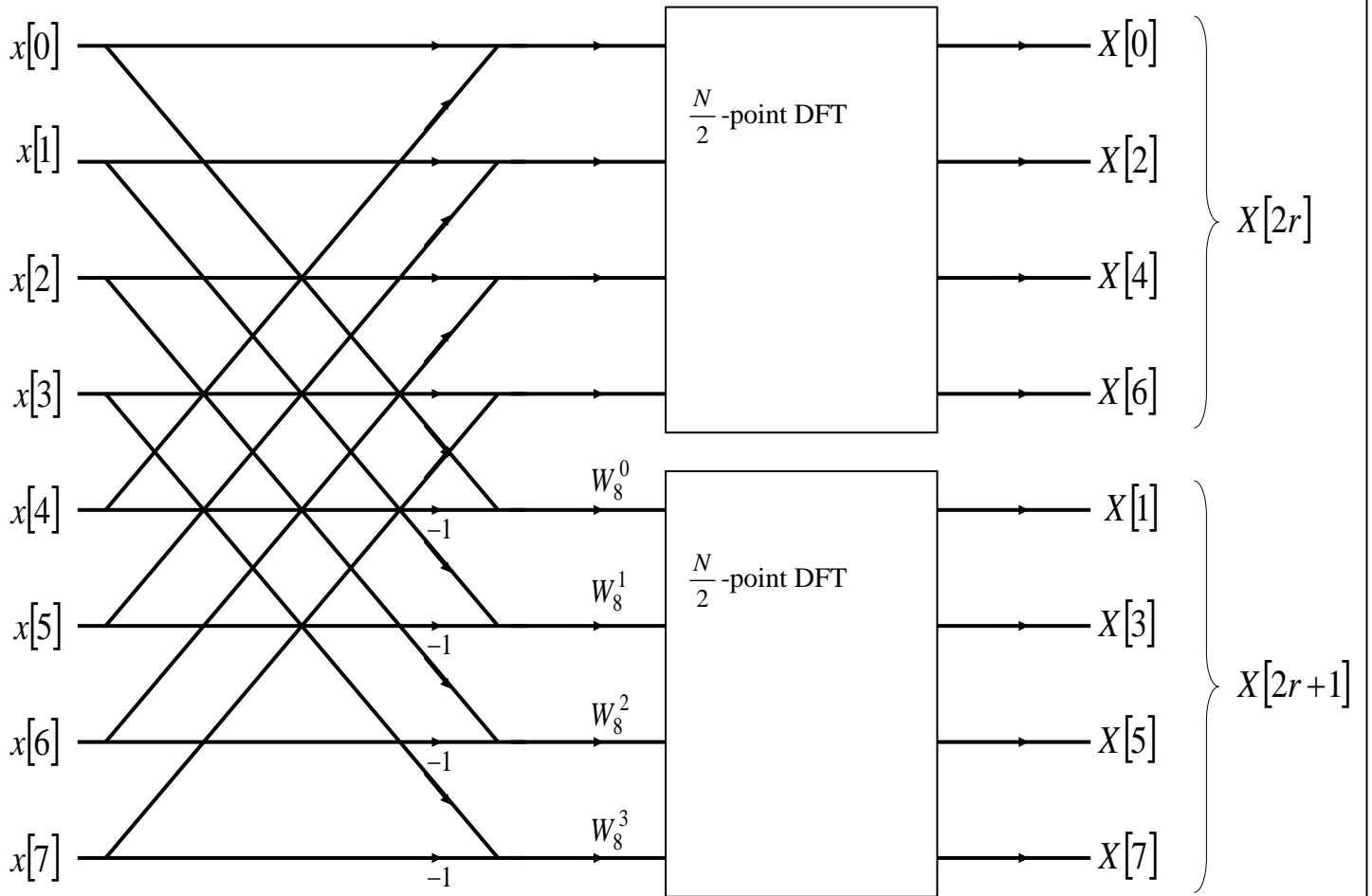
Odd indexed elements: Similarly,

$$X[2r + 1] = \sum_{n=0}^{\frac{N}{2}-1} \left( x[n] - x \left[ n + \frac{N}{2} \right] \right) W_N^n W_{\frac{N}{2}}^{rn}$$

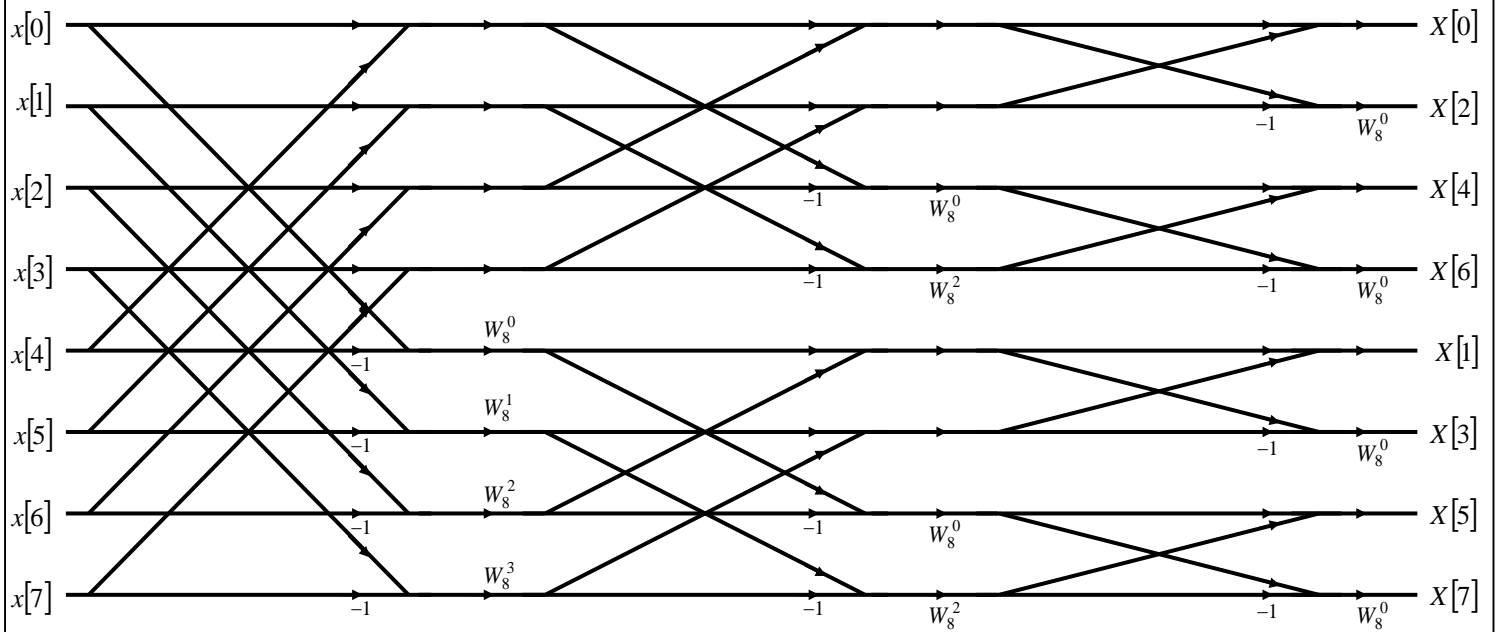
This is a  $\frac{N}{2}$ -point DFT of

$$\left( x[n] - x \left[ n + \frac{N}{2} \right] \right) W_N^n \quad n = 0, 1, \dots, \frac{N}{2} - 1.$$

For  $N=8$



Decomposing  $\frac{N}{2}$ -point DFT blocks down to 2-point DFTs we get the following  
flow diagram



## THE GOERTZEL ALGORITHM

The algorithm reduces the storage requirement

DFT expression:

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn} \quad k = 0, 1, \dots, N-1$$

can be written in a recursive form.

For example, let  $N=8$

$$X[k] = x[0] + x[1]W_N^k + x[2]W_N^{2k} + x[3]W_N^{3k} + x[4]W_N^{4k} + x[5]W_N^{5k} + x[6]W_N^{6k} + x[7]W_N^{7k}$$

$$= x[0] + W_N^k(x[1] + x[2]W_N^k + x[3]W_N^{2k} + x[4]W_N^{3k} + x[5]W_N^{4k} + x[6]W_N^{5k} + x[7]W_N^{6k})$$

$\vdots$

$$= x[0] + W_N^k(x[1] + W_N^k(x[2] + W_N^k(x[3] + W_N^k(x[4] + W_N^k(x[5] + W_N^k(x[6] + x[7]W_N^k))))))$$

OR

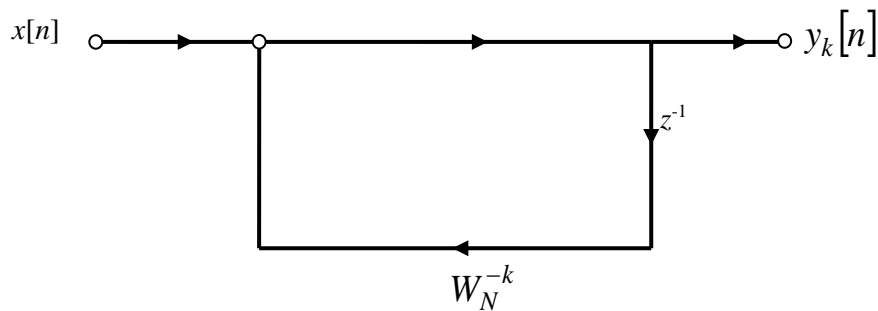


Since  $W_N^{-kN} = 1$

$$\begin{aligned} X[k] &= W_N^{-kN} \sum_{n=0}^{N-1} x[n] W_N^{kn} \\ &= \sum_{n=0}^{N-1} x[n] W_N^{-k(N-n)} \end{aligned}$$

For  $N=8$

$$\begin{aligned} W_8^{-8k} X[k] &= \\ (((((((x[0] W_8^{-7k} + x[1]) W_8^{-k} + x[2]) W_8^{-k} + x[3]) W_8^{-k} + x[4]) W_8^{-k} + x[5]) W_8^{-k} + x[6]) W_8^{-k} + x[7]) W_8^{-k} \end{aligned}$$



$$y_k[n] = W_N^{-k} y_k[n-1] + x[n]$$

Take  $X[k] = y_k[N]$

Note that  $X[k]$ 's are computed sequentially.

Computational demand:

For each  $k$ ,  $y_k[1], y_k[2], \dots, y_k[N - 1]$  must be computed to find  $y_k[N] = X[k]$ .

This requires, (assuming a complex input)  $4N$  multiplications and  $4N$  additions.

This amount is almost the same as that of direct computation.

However, the storage requirement of  $W_N^{kn}$  values vanishes.

## DOWNSAMPLING AND DECIMATION-IN-TIME

For the  $N$ -point DFT of  $x[n]$  we obtained

$$X[k] = G[k] + W_N^k H[k]$$

where  $G[k]$  and  $H[k]$  are the  $\frac{N}{2}$ -point DFTs, respectively, of even and odd indexed elements of the sequence  $x[n]$ , i.e.,

$$g[n] = x[2n]$$

$$h[n] = x_{-1}[2n] \quad x_{-1}[n] = x[n + 1]$$

$$G(e^{j\omega}) = \frac{1}{2} \left( X(e^{j\frac{\omega}{2}}) + X(e^{j\frac{1}{2}(\omega-2\pi)}) \right)$$

$$H(e^{j\omega}) = \frac{1}{2} \left( e^{j\frac{\omega}{2}} X(e^{j\frac{\omega}{2}}) + e^{j\frac{1}{2}(\omega-2\pi)} X(e^{j\frac{1}{2}(\omega-2\pi)}) \right)$$

Therefore

$$X(e^{j\omega}) = G(e^{j2\omega}) + e^{-j\omega} H(e^{j2\omega})$$

$$X[k] = X(e^{j\omega}) \Big|_{\omega=k\frac{2\pi}{N}}$$

$$\begin{aligned} G(e^{j2\omega}) &= \sum_{n=0}^{\frac{N}{2}-1} x[2n]e^{-j2\omega n} \\ G(e^{j2\omega}) \Big|_{\omega=k\frac{2\pi}{N}} &= \sum_{n=0}^{\frac{N}{2}-1} x[2n]e^{-j2k\frac{2\pi}{N}n} \\ &= \sum_{n=0}^{\frac{N}{2}-1} x[2n]W_{\frac{N}{2}}^{kn} \end{aligned}$$

Likewise

$$H(e^{j2\omega}) \Big|_{\omega=k\frac{2\pi}{N}} = \sum_{n=0}^{\frac{N}{2}-1} x[2n+1]W_{\frac{N}{2}}^{kn}$$

RADIX 3 vs, RADIX 2

In place computations

Chirp transform 9.6