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Section : 2

EE430 - HW2

1) a)

$$y[n] - \frac{1}{2}y[n-1] = x[n] - x[n-1] + x[n-2]$$

$$h[n] - \frac{1}{2}h[n-1] = \delta[n] - \delta[n-1] + \delta[n-2] \Rightarrow h[n] = 0 \ , n<0 \ \text{(system is causal)}$$

$h[0] = 1$

$h[1] = -1/2$

$h[2] = 3/4$

$h[n] = \frac{1}{2}h[n-1] \ , n>2$

$$\Rightarrow \boxed{h[n] = \left(\frac{1}{2}\right)^n u[n] - \left(\frac{1}{2}\right)^{n-1} u[n-1] + \left(\frac{1}{2}\right)^{n-2} u[n-2]}$$

b) $H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n}$ where $\begin{cases} \text{if } x[n] = \alpha^n u[n] \Rightarrow X(e^{j\omega}) = \dfrac{1}{1-\alpha e^{-j\omega}} \\ \text{and } y[n] = x[n-n_0] \Rightarrow Y(e^{j\omega}) = X(e^{j\omega}).e^{-j\omega n_0} \end{cases}$

$$\Rightarrow H(e^{j\omega}) = \frac{1}{1-\frac{1}{2}e^{-j\omega}} - \frac{e^{-j\omega}}{1-\frac{1}{2}e^{-j\omega}} + \frac{e^{-2j\omega}}{1-\frac{1}{2}e^{-j\omega}} = \frac{e^{j\omega} - 1 + e^{-j\omega}}{e^{j\omega} - \frac{1}{2}}$$

$$\Rightarrow \boxed{H(e^{j\omega}) = \frac{2\cos(\omega) - 1}{e^{j\omega} - \frac{1}{2}}}$$

c) Magnitude and phase responses are plotted using MATLAB and can be find at the end.

d) $x[n] = \cos\left(\frac{\pi}{3}n\right) + \sin\left(\frac{\pi}{2}n - \frac{\pi}{4}\right) = \frac{1}{2}\left(e^{j\frac{\pi}{3}n} + e^{-j\frac{\pi}{3}n}\right) + \frac{1}{2j}\left(e^{j\frac{\pi}{4}}.e^{j\frac{\pi}{2}n} - e^{-j\frac{\pi}{4}}.e^{-j\frac{\pi}{2}n}\right)$

$\Rightarrow$ Since the system is LTI: $y[n] = \frac{1}{2}e^{j\frac{\pi}{3}n}H(e^{j\frac{\pi}{3}}) + \frac{1}{2}e^{-j\frac{\pi}{3}n}H(e^{-j\frac{\pi}{3}}) + \frac{1}{2j}e^{j\frac{\pi}{4}}e^{j\frac{\pi}{2}n}H(e^{j\frac{\pi}{2}}) - \frac{1}{2j}e^{-j\frac{\pi}{4}}e^{-j\frac{\pi}{2}n}H(e^{-j\frac{\pi}{2}})$

0

$$\Rightarrow y[n] = \frac{1}{2j}\left(e^{j\left(\frac{\pi}{2}n+\frac{\pi}{4}\right)}\frac{2}{1-2j} - e^{-j\left(\frac{\pi}{2}n+\frac{\pi}{4}\right)}\frac{2}{1+2j}\right) = 2\,\mathrm{Im}\left(\frac{e^{j\left(\frac{\pi}{2}n+\frac{\pi}{4}\right)}}{1-2j}\right)$$

$$\Rightarrow y[n] = \frac{2}{5}\left(\sin\left(\frac{\pi}{2}n+\frac{\pi}{4}\right) + 2\cos\left(\frac{\pi}{2}n+\frac{\pi}{4}\right)\right)$$

e) $H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]\, e^{-j\omega n}$

$\Rightarrow H(e^{-j\omega}) = \sum_{n=-\infty}^{\infty} h[n]\, e^{+j\omega n} \Rightarrow H^*(e^{-j\omega}) = \sum_{n=-\infty}^{\infty} h^*[n]\, e^{-j\omega n}$ where $h[n] = h^*[n]$ since $h[n]$ is real

$\Rightarrow H(e^{j\omega}) = H^*(e^{-j\omega}) = H^*(e^{-j\omega} \underbrace{e^{j2\pi}}_{1}) = H^*(e^{-j(2\pi-\omega)})$ is proved.

This does not hold for arbitrary $h[n]$. $h[n]$ must be real to hold.

2) Let $y[n] = \alpha^n u[n] \Rightarrow$ if $|\alpha| < 1 \Rightarrow Y(e^{j\omega}) = \dfrac{1}{1-\alpha . e^{-j\omega}}$

$\Rightarrow$ let $x_1[n] = y[n-2] = \alpha^{n-2} u[n-2] = X_1(e^{j\omega}) = e^{-2j\omega} \cdot \dfrac{1}{1-\alpha e^{-j\omega}} = \dfrac{1}{e^{2j\omega} - \alpha e^{j\omega}}$

$\Rightarrow x[n] = n \cdot x_1[n] \Rightarrow X(e^{j\omega}) = j \cdot \dfrac{d\, X_1(e^{j\omega})}{d\omega} = j \dfrac{d}{d\omega}\left(\dfrac{1}{e^{2j\omega} - \alpha e^{j\omega}}\right)$

$\Rightarrow X(e^{j\omega}) = \dfrac{-j}{(e^{2j\omega} - \alpha e^{j\omega})^2} \cdot (2j e^{2j\omega} - \alpha j e^{j\omega}) = \boxed{\dfrac{2e^{2j\omega} - \alpha e^{j\omega}}{(e^{2j\omega} - \alpha e^{j\omega})^2} = X(e^{j\omega})}$

3) $x_1[n] = \sin\left(\frac{\pi}{7} n\right) + \sin\left(\frac{\pi}{3} n\right)$ $\quad x_2[n] = u[n] . x_1[n]$

a) for $h[n] = [\frac{1}{2}\ \frac{1}{2}]$ for $n = [0\ 1] \Rightarrow$ $y_1[n] = \frac{1}{2}\left(x_1[n] + x_1[n-1]\right)$

$y_2[n] = \frac{1}{2}\left(x_1[n] u[n] + x_1[n-1] . u[n-1]\right)$

$\Rightarrow y_1[n] = y_2[n]$ for $n \geq 1$

b) for $h[n] = [\frac{1}{2}\ \frac{1}{2}]$ for $n = [2\ 3] \Rightarrow$ $y_1[n] = \frac{1}{2}\left(x_1[n-2] + x_1[n-3]\right)$

$y_2[n] = \frac{1}{2}\left(x_1[n-2] u[n-2] + x_1[n-3] u[n-3]\right)$

$\Rightarrow y_1[n] = y_2[n]$ for $n \geq 3$

d) for $h[n] = [\frac{1}{6}\ \frac{1}{6}\ \frac{1}{6}\ \frac{1}{6}\ \frac{1}{6}\ \frac{1}{6}]$ for $n = [0\ 1\ 2\ 3\ 4\ 5]$

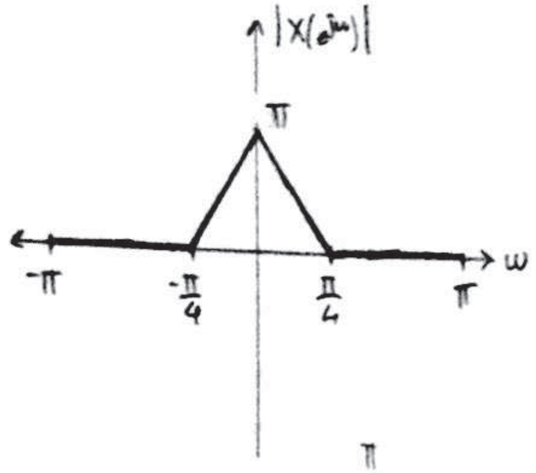
$\Rightarrow y_1[n] = \frac{1}{6} \sum_{k=0}^{5} x_1[n-k]$ $\quad \Rightarrow y_1[n] = y_2[n]$ for $n \geq 5$

$y_2[n] = \frac{1}{6} \sum_{k=0}^{5} x_1[n-k] u[n-k]$

4) a) since $x[n]$ is a real sequence ; $X(e^{j\omega}) = X^*(e^{-j\omega})$

$\Rightarrow$ $Re\{X(e^{j\omega})\}$ is symmetric and $Im\{X(e^{j\omega})\}$ is antisymmetric

$\Rightarrow$ $|X(e^{j\omega})|$ is symmetric and $\measuredangle X(e^{j\omega})$ is antisymmetric.



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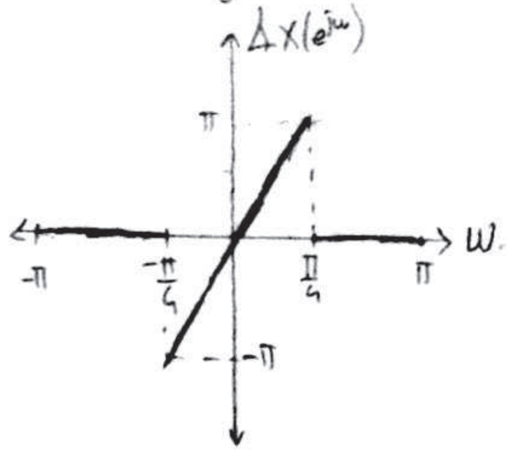


b) $$X_c(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j(\omega-\Phi)})\,\pi\left[\sum_{r=-\infty}^{\infty}\delta\left(\Phi-\frac{\pi}{5}+2\pi r\right)+\sum_{r=-\infty}^{\infty}\delta\left(\Phi+\frac{\pi}{5}+2\pi r\right)\right]d\Phi$$

$$= \frac{1}{2\pi}\cdot\pi\int_{-\pi}^{\pi} X(e^{j(\omega-\Phi)})\cdot\left[\delta\left(\Phi-\frac{\pi}{5}\right)+\delta\left(\Phi+\frac{\pi}{5}\right)\right]d\Phi$$

$$= \frac{1}{2}\left(X\left(e^{j(\omega-\frac{\pi}{5})}\right)+X\left(e^{j(\omega+\frac{\pi}{5})}\right)\right)$$

$$X_s(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j(\omega-\Phi)})\,j\pi\left[\sum_{r=-\infty}^{\infty}\delta\left(\Phi+\frac{\pi}{5}+2\pi r\right)-\sum_{r=-\infty}^{\infty}\delta\left(\Phi-\frac{\pi}{5}+2\pi r\right)\right]d\Phi$$

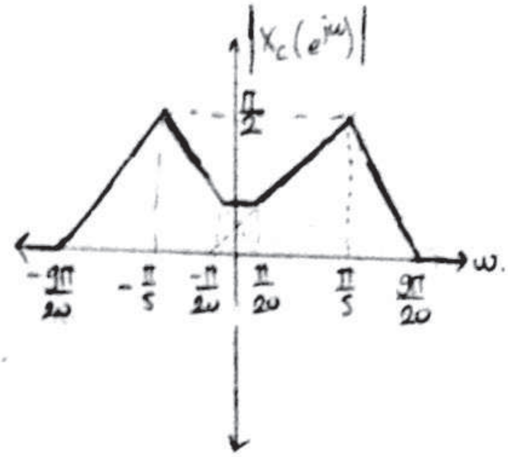
$$= \frac{1}{2\pi}\,j\pi\int_{-\pi}^{\pi} X(e^{j(\omega-\Phi)})\cdot\left[\delta\left(\Phi+\frac{\pi}{5}\right)-\delta\left(\Phi-\frac{\pi}{5}\right)\right]d\Phi$$

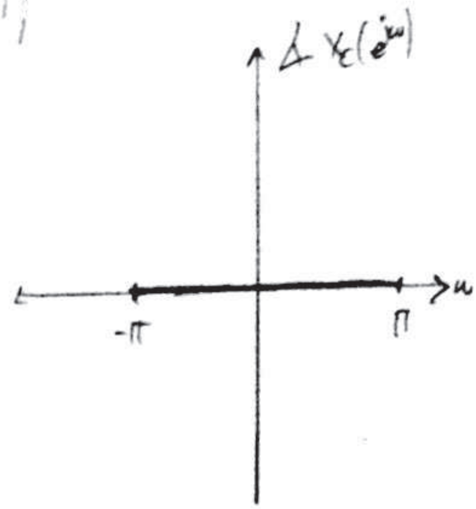
$$= \frac{j}{2}\left(X\left(e^{j(\omega+\frac{\pi}{5})}\right)-X\left(e^{j(\omega-\frac{\pi}{5})}\right)\right)$$

c) If $X(e^{j\omega})$ is real ; $|X(e^{j\omega})| = X(e^{j\omega})$ and $\measuredangle X(e^{j\omega}) = 0$

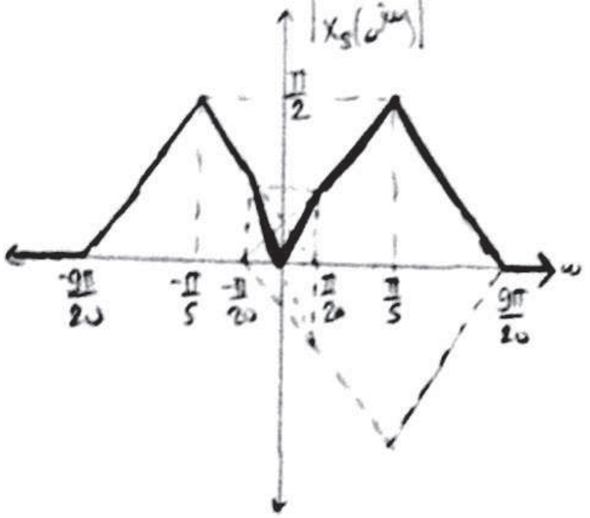
$$\Rightarrow X_c(e^{j\omega}) = \frac{1}{2}\left(X\left(e^{j(\omega-\frac{\pi}{5})}\right) + X\left(e^{j(\omega+\frac{\pi}{5})}\right) \right)$$

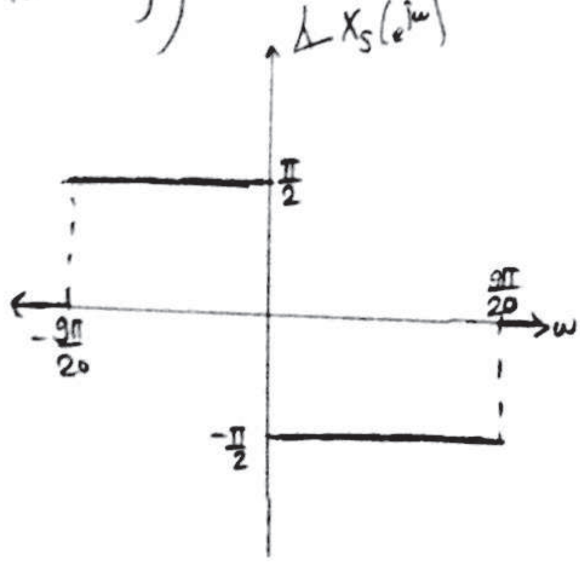
⇒





$$\Rightarrow X_s(e^{j\omega}) = \frac{j}{2}\left(X\left(e^{j(\omega+\frac{\pi}{5})}\right) - X\left(e^{j(\omega-\frac{\pi}{5})}\right) \right)$$





5) Let $X(e^{j\omega}) = \dfrac{\sin(\frac{5}{2}\omega)}{\sin(\frac{\omega}{2})}$ be DTFT of the sequence $x[n]$

Note that this expression is equal to "0"

$$\Rightarrow X(e^{j\omega}) = \frac{e^{j\frac{5}{2}\omega} - e^{-j\frac{5}{2}\omega}}{e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}}} = \frac{e^{+2j\omega} - e^{-3j\omega}}{1 - e^{-j\omega}} = \frac{e^{+2j\omega}}{1-e^{-j\omega}} - \frac{e^{-3j\omega}}{1-e^{-j\omega}} + e^{+2j\omega}\pi\sum_{k=-\infty}^{\infty}\delta(\omega-2\pi k) - e^{-3j\omega}\pi\sum_{k=-\infty}^{\infty}\delta(\omega-2\pi k)$$

$$= \underbrace{\frac{e^{2j\omega}}{1-e^{-j\omega}} + e^{2j\omega}\pi\sum_{k=-\infty}^{\infty}\delta(\omega-2\pi k)}_{\text{DTFT of } u[n+2]} - \underbrace{\frac{e^{-3j\omega}}{1-e^{-j\omega}} - e^{-3j\omega}\pi\sum_{k=-\infty}^{\infty}\delta(\omega-2\pi k)}_{\text{DTFT of } u[n-3]}$$

$$\Rightarrow x[n] = u[n+2] - u[n-3] \text{ and } \sum_{n=-\infty}^{\infty}(x[n])^2 = 5 = \frac{1}{2\pi}\int_{-\pi}^{\pi}|X(e^{j\omega})|^2 d\omega = \frac{2}{2\pi}\int_{-\pi}^{0}|X(e^{j\omega})|^2 d\omega$$

since $x[n]$ is real, $X(e^{j\omega})$ is conjugate symmetric. Therefore;

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|X(e^{j\omega})|^2 d\omega = \frac{2}{2\pi}\int_{-\pi}^{0}|X(e^{j\omega})|^2 d\omega$$

$$\Rightarrow 5 = \frac{1}{\pi}\int_{-\pi}^{0}|X(e^{j\omega})|^2 d\omega \Rightarrow \boxed{\int_{-\pi}^{0}\left|\frac{\sin(\frac{5}{2}\omega)}{\sin(\frac{\omega}{2})}\right|^2 d\omega = 5\pi}$$

6) $x[n] = e^{j\omega_0 n} \Rightarrow X(e^{j\omega}) = 2\pi\sum_{r=-\infty}^{\infty}\delta(\omega-\omega_0+2\pi r)$

$$\Rightarrow Y(e^{j\omega}) = H(e^{j\omega}).X(e^{j\omega})$$

$$\Rightarrow y[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi}H(e^{j\omega})X(e^{j\omega})e^{j\omega n}d\omega = \frac{1}{\cancel{2\pi}}\int_{-\pi}^{\pi}H(e^{j\omega})e^{j\omega n}\underbrace{\sum_{r=-\infty}^{\infty}\cancel{2\pi}.\delta(\omega-\omega_0+2\pi r)}_{\text{only nonzero if } \omega=\omega_0.}$$

$$\boxed{\Rightarrow y[n] = H(e^{j\omega_0})e^{j\omega_0 n}}$$

7) a) $h[n] = \left(\frac{1}{2}\right)^n u[n] + 2^n u[n]$ and $x[n] = 3^n$

$$\Rightarrow y[n] = \sum_{k=-\infty}^{\infty}h[k]x[n-k] = \sum_{k=0}^{\infty}\left(\left(\frac{1}{2}\right)^k+2^k\right)3^{n-k} = 3^n\left[\overbrace{\sum_{k=0}^{\infty}\left(\frac{1}{6}\right)^k}^{\frac{6}{5}} + \overbrace{\sum_{k=0}^{\infty}\left(\frac{2}{3}\right)^k}^{3}\right]$$

$$\boxed{= 3^n.\frac{21}{5} = 3^n.C} \checkmark$$

b) $h[n] = \left(\frac{1}{2}\right)^n u[n] + 2^n u[-n]$ and $x[n] = 3^n$

$$\Rightarrow y[n] = \sum_{k=-\infty}^{\infty}h[k]x[n-k] = \underbrace{\sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^k 3^{n-k}}_{3^n.\frac{6}{5}} + \underbrace{\sum_{k=-\infty}^{0}2^k 3^{n-k}}_{3^n\sum_{k=0}^{\infty}\left(\frac{3}{2}\right)^k}$$

$3^n\sum_{k=0}^{\infty}\left(\frac{3}{2}\right)^k \Rightarrow$ infinite sum $\left(\frac{3}{2}>1\right)$

$\Rightarrow y[n]$ cannot be expressed as $3^n.C$ where C is a complex constant

8) a) $$H(z) = \frac{2z^{-1}+1}{\left(1-\frac{1}{4}z^{-1}\right)\left(1+z^{-1}\right)\left(1+\frac{1}{4}z^{-1}\right)} = \frac{A}{1-\frac{1}{4}z^{-1}} + \frac{B}{1+z^{-1}} + \frac{C}{1+\frac{1}{4}z^{-1}}$$

$$\Rightarrow A\left(1+\frac{5}{4}z^{-1}+\frac{1}{4}z^{-2}\right) + B\left(1-\frac{1}{16}z^{-2}\right) + C\left(1+\frac{3}{4}z^{-1}-\frac{1}{4}z^{-2}\right)$$

$$= 2z^{-1}+1$$

$$\Rightarrow \left.\begin{aligned} A+B+C &= 1 \\ \frac{5}{4}A+\frac{3C}{4} &= 2 \\ \frac{1}{4}A-\frac{1}{16}B-\frac{1}{4}C &= 0 \end{aligned}\right\} \quad 5A-3C=1 \quad \Rightarrow \frac{1+3C}{4}+\frac{3C}{4} = 2 \Rightarrow \quad C=\frac{7}{6}$$

$$A = \frac{9}{10}$$

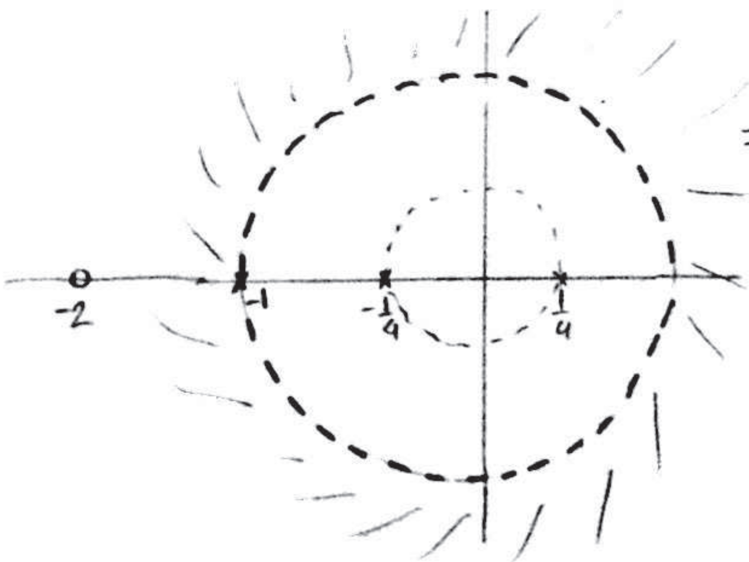
$$B = \frac{16}{15}$$

$$\Rightarrow H(z) = \frac{\frac{9}{10}}{1-\frac{1}{4}z^{-1}} + \frac{\frac{16}{15}}{1+z^{-1}} + \frac{\frac{7}{6}}{1+\frac{1}{4}z^{-1}}$$

$$\Rightarrow h[n] = \underbrace{\frac{9}{10}\left(\frac{1}{4}\right)^n u[n]}_{|z|>\left|\frac{1}{4}\right|} + \underbrace{\frac{16}{15}(-1)^n u[n]}_{|z|>1} + \underbrace{\frac{7}{6}\left(-\frac{1}{4}\right)^n u[n]}_{|z|>\left|-\frac{1}{4}\right|}$$

$h[n]=0$ for $n<0$
$\Rightarrow$ system is causal.

we have chosen these ROC's since it is given that the system is STABLE



$\Rightarrow$ ROC

9) c) $x[n] = 3 + j5 + \sin\left(\frac{\pi}{4}n\right) = 3e^{j0n} + 5je^{j0n} + \frac{1}{2j}e^{j\frac{\pi}{4}n} - \frac{1}{2j}e^{-j\frac{\pi}{4}n}$

$$\Rightarrow y[n] = 3.e^{j0n}H(e^{j\omega})\Big|_{\omega=0} + 5j.H(e^{j\omega})\Big|_{\omega=0} + \frac{1}{2j}e^{j\frac{\pi}{4}n}.H(e^{j\omega})\Big|_{\omega=\frac{\pi}{4}} - \frac{1}{2j}e^{-j\frac{\pi}{4}n}H(e^{j\omega})\Big|_{\omega=-\frac{\pi}{4}}$$

$\Rightarrow$ using the given magnitude and phase responses:

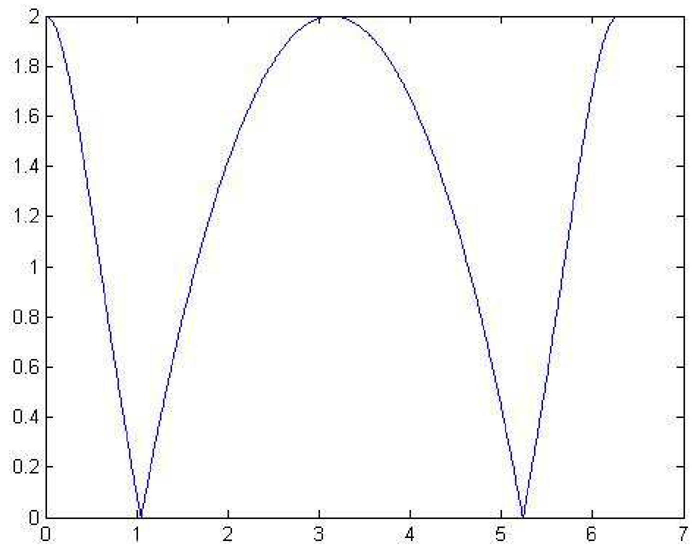
$$y[n] = 3 + 5j + \frac{1}{2j}e^{j\frac{\pi}{4}n}\cdot\frac{1}{2}.e^{-j\frac{\pi}{8}} - \frac{1}{2j}\cdot e^{-j\frac{\pi}{4}n}\cdot\frac{1}{2}.e^{+j\frac{\pi}{8}}$$

$$\boxed{\Rightarrow y[n] = 3 + 5j + \frac{1}{2}\sin\left(\frac{\pi}{4}n - \frac{\pi}{8}\right)}$$

3)c)

```
h=zeros(1,1000);
h(1)=1;
h(2)=-0.5;
h(3)=0.75;
for i=4:1:1000
    h(i)=h(i-1)/2;
end
[H,w] = freqz(h,1,1000,'whole');
plot(w,abs(H));
figure;
plot(w,angle(H));
```

Magnitude Response:



Phase Response: