GETTING THE RESULT OF LINEAR CONVOLUTION USING DFT

Let $x[n]$ and $y[n]$ be N_1-point and N_2-point sequences and $w[n]$ be their linear convolution.

Let $X[k]$ and $Y[k]$ be N-point DFTs of $x[n]$ and $y[n]$.

The IDFT of $W[k] = X[k]Y[k]$ is the same as the linear convolution of $x[n]$ and $y[n]$ if $N \geq N_1 + N_2 - 1$.

Ex: Let $x[n]$ and $y[n]$ be 5-point and 9-point sequences, respectively. At least how many point DFT has to be used to get the result of their linear convolution?

Answer: 5+9-1=13 point DFT.

However, if, for example, 10-point DFT used then

- The result has 10 samples in total
- The first 13-10 = 3 samples out of these 10 samples are not equal to the those of the linear convolution
- The remaining 10-3 samples are equal to those of the linear convolution.
- Therefore only 7 samples of the result of linear convolution (13 samples) will be obtained.

MATLAB Example

```
x = [1 2 3];       y = [-1 2 1 2];             z = conv(x,y);           z =   -1     0     2    10     7     6

X = fft(x,6)
                   X = 6    0.5-4.3i     -1.5 + 0.9i      2     -1.5-0.9i     0.5+4.3i

Y = fft(y,6)
                   Y = 4     -2.5 - 2.6i     -0.5 - 0.9i     -4     -0.5 +0. 9i     -2.5 + 2.6i

Z=X.*Y
                 Z =24     -12.5 +9.5i     1.5 + 0. 9i     -8     1.5 -0. 9i     -12.5 - 9.5i

z_=ifft(Z)       z =    -1     0     2    10     7     6


X = fft(x,5)
                 X =6       -0.8 -3.7i         0.3+1.7i          0.3-1.7i           -0.8+3. 7i

Y = fft(y,5)
                 Y =4       -2.8 - 1.3i        -1.7 - 2.1i        -1.7 + 2.1i        -2.8 + 1.3i

Z = X.*Y
                 Z =24      -2.5 + 11.4i      3.0 -  3.5i        3.0 + 3.5i         -2.5 - 11.4i

z_= ifft(Z)      z_ =    5     0     2    10     7
```

7) Sampling the DTFT

$$\underset{\substack{\uparrow \\ \text{length}=N_X}}{x[n]} \xleftrightarrow{DTFT} X\left(e^{j\omega}\right)$$

$$\hat{X}[k] = X\left(e^{j\omega}\right)\Big|_{\omega = k\frac{2\pi}{N}} \qquad k = 0,1,\ldots, N-1$$

Where N is arbitrary, i.e. not necessarily $N \geq N_x$!

$\hat{x}[n] = ?$

(N-point IDFT of $\hat{X}[k]$)

$$\hat{x}[n] = \begin{cases} \frac{1}{N}\sum_{k=0}^{N-1} \hat{X}[k] W_N^{-kn} & n = 0,1,\dots,N-1 \\ 0 & o.w. \end{cases}$$

$$\sum_{k=0}^{N-1} \hat{X}[k] W_N^{-kn} = \sum_{k=0}^{N-1} \left(\sum_{r=-\infty}^{\infty} x[r] W_N^{kr} \right) W_N^{-kn}$$

$$= \sum_{r=-\infty}^{\infty} x[r] \underbrace{\left(\sum_{k=0}^{N-1} W_N^{k(r-n)} \right)}_{N\sum_{m=-\infty}^{\infty} \delta[n-r-mN]}$$

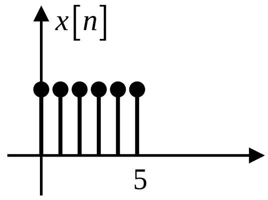
$$= N \left(x[n] * \sum_{m=-\infty}^{\infty} \delta[n-mN] \right)$$

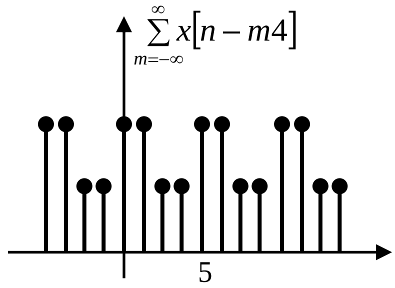
$$= N \sum_{m=-\infty}^{\infty} x[n-mN]$$

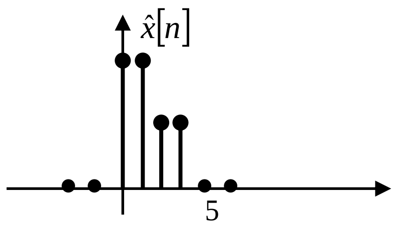
Therefore

$$\hat{x}[n] = \begin{cases} \sum_{m=-\infty}^{\infty} x[n-mN] & n = 0,1,\dots,N-1 \\ 0 & o.w. \end{cases}$$

Ex: $N_x = 6$, $N = 4$







$$X\left(e^{j\omega}\right) = \frac{1-e^{-j\omega 6}}{1-e^{-j\omega}} = \frac{e^{-j\omega 3}}{e^{-j\frac{\omega}{2}}} \frac{e^{j\omega 3} - e^{-j\omega 3}}{e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}}} = e^{-j\frac{5\omega}{2}} \frac{\sin 3\omega}{\sin \dfrac{\omega}{2}}$$

For $N = 4$

$$X[0] = 6 \quad X[1] = 1 + j \quad X[2] = 0 \qquad X[3] = 1 - j$$

and

$$x_4[n]: \ \cdots 0\ 0\ \underset{n=0}{2}\ 2\ 1\ 1\ 0\ 0 \cdots$$

Ex: Demonstration of the proof.

$$N_X = 8, \;\; N = 5$$

$$X\left(e^{j\omega}\right) = \sum_{n=0}^{7} x[n] e^{-j\omega n}$$
$$= \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

$$\hat{X}[k] = X\left(e^{j\omega}\right)\Big|_{\omega = k\frac{2\pi}{5}}$$
$$= \sum_{n=-\infty}^{\infty} x[n] W_5^{kn}$$
$$= x[0] + x[1]W_5^{k} + x[2]W_5^{k2} + x[3]W_5^{k3} + x[4]W_5^{k4} + x[5]\underbrace{W_5^{k5}}_{W_5^{k0}} + x[6]\underbrace{W_5^{k6}}_{W_5^{k1}} + x[7]\underbrace{W_5^{k7}}_{W_5^{k2}}$$
$$= \left(x[0] + x[5]\right)$$
$$+ \left(x[1] + x[6]\right)W_5^{k}$$
$$+ \left(x[2] + x[7]\right)W_5^{k2}$$
$$+ x[3]W_5^{k3}$$
$$+ x[4]W_5^{k4}$$

This is the 5-point DFT of

$$\hat{x}[0] = x[0] + x[5]$$
$$\hat{x}[1] = x[1] + x[6]$$
$$\hat{x}[2] = x[2] + x[7]$$
$$\hat{x}[3] = x[3]$$
$$\hat{x}[4] = x[4]$$

8) Multiplication in Time Domain

DFS

Let $\tilde{x}_1[n]$ and $\tilde{x}_2[n]$ be periodic with common period *N*.

$$\tilde{x}_3[n]=\tilde{x}_1[n]\tilde{x}_2[n] \quad \xleftrightarrow{\text{DFS}} \quad \tilde{X}_3[k]=\frac{1}{N}\sum_{l=0}^{N-1}\tilde{X}_1[l]\tilde{X}_2[k-l]$$

DFT

Let $x_1[n]$ and $x_2[n]$ be two finite length sequences and $X_1[k]$ and $X_2[k]$ their *N*-point DFTs, $N\geq \max(N_1,N_2)$.

$$x_3[n]=x_1[n]x_2[n] \quad \xleftrightarrow{N\text{-point DFT}} \quad X_3[k]=\frac{1}{N}\sum_{l=0}^{N-1}X_1[l]X_2\left[((k-l))_N\right] \qquad k=0,1,\ldots,N-1$$

IMPLEMENTING LTI SYSTEMS USING DFT

The output of an LTI system can be computed using DFT:

Multiply the DFTs of input and impulse response and compute the IDFT

There are efficient DFT computation techniques.

We consider FIR systems with “long” inputs.

BLOCK PROCESSING

Input is decomposed into short segments and the output is computed by properly "combining" the responses to the short segments.

Otherwise, it is not practical to consider DFTs of very long sequences. Furthermore the output of the system will be delayed excessively since all the input has to be collected first..

- $h[n]$: length P.
- The input is decomposed into blocks of length L.

Two Methods

Overlap-Add

DFT length: $L+P-1$ point

Overlap-Save

DFT length: L point

OVERLAP-ADD METHOD

Let's start with a linear convolution example

$h[n]$: $\ldots, 0, 0, \underset{\substack{\uparrow \\ n=0}}{1}, 2, 3, 0, 0, \ldots$ 3-point sequence (P = 3)

$x[n]$: $\ldots, 0, 0, \underset{\substack{\uparrow \\ n=0}}{1}, -1, 2, -2, -1, 1, 0, 0, \ldots$

Linear convolution: 6+3-1 = 8 points

$$
\begin{array}{lrrrrrrrr}
 & 1 & 2 & 3 & & & & & \\
 & & -1 & -2 & -3 & & & & \\
 & & & 2 & 4 & 6 & & & \\
 & & & & -2 & -4 & -6 & & \\
 & & & & & -1 & -2 & -3 & \\
+ & & & & & & 1 & 2 & 3 \\
\hline
 & \underset{\substack{\uparrow \\ n=0}}{1} & 1 & 3 & -1 & 1 & -7 & -1 & 3
\end{array}
$$

$y[n]$: $\ldots, 0, 0, \underset{\substack{\uparrow \\ n=0}}{1}, 1, 3, -1, 1, -7, -1, 3, 0, 0\ldots$

By Overlap-add:

$$h[n]:\ \ldots,0,\ 0,\ \underset{\substack{\uparrow\\ n=0}}{1},\ 2,\ 3,\ 0,\ 0,\ldots$$

3-point sequence (P = 3)

$$x[n]:\ \ldots,0,\ 0,\ \underset{\substack{\uparrow\\ n=0}}{1},\ -1,\ 2,\ -2,\ -1,\ 1,\ 0,\ 0\ ,\ldots$$

(In general, infinite length, starts at n = 0)

Decompose the input, $x[n]$, into L-point segments.

Let's choose L= 3; could be something else...

$$x[n]:\ \ldots,0,\ 0,\ \underbrace{1,\ -1,\ 2,}_{x_1}\ \underbrace{-2,\ -1,\ 1,}_{x_2}\ 0,\ 0\ ,\ldots$$

Compute the **<u>5-point</u>** (L+P-1) DFTs of x_1 and x_2 → X_1 and X_2

Compute the **<u>5-point</u>** DFT of h → H

Compute

y_1 = IDFT{ $H\,X_1$ } $\qquad y_1:\ \ldots,\ 0,\ 0,\ \underset{\uparrow}{1},\ 1,\ 3,\ 1,\ 6,\ 0,\ 0\ ,\ldots$

and

y_2 = IDFT{ $H\,X_2$ } $\qquad y_2:\ \ldots,\ 0,\ \underset{\uparrow}{0},\ 0,\ 0,\ -2,\ -5,\ -7,\ -1,\ 3,\ 0,\ 0\ ,\ldots$

Compute $y[n]$ by "**overlap-add**", i.e,

$$\begin{array}{lrrrrrrrr} & 1 & 1 & 3 & 1 & 6 & & & \\ + & & & & -2 & -5 & -7 & -1 & 3 \\ \hline & \underset{\substack{\uparrow\\ n=0}}{1} & 1 & 3 & -1 & 1 & -7 & -1 & 3 \end{array}$$

Overlap Add Formally

Let the impulse response of the system be of length *P* samples.

Decompose the input $x[n]$ into <u>nonoverlaping, consecutive</u> blocks of length L.

r^{th} block is $x_r[n] = \begin{cases} x[n+rL] & 0 \le n < L \\ 0 & o.w. \end{cases}$ so that $x[n] = \sum_{r=0}^{\infty} x_r[n-rL]$

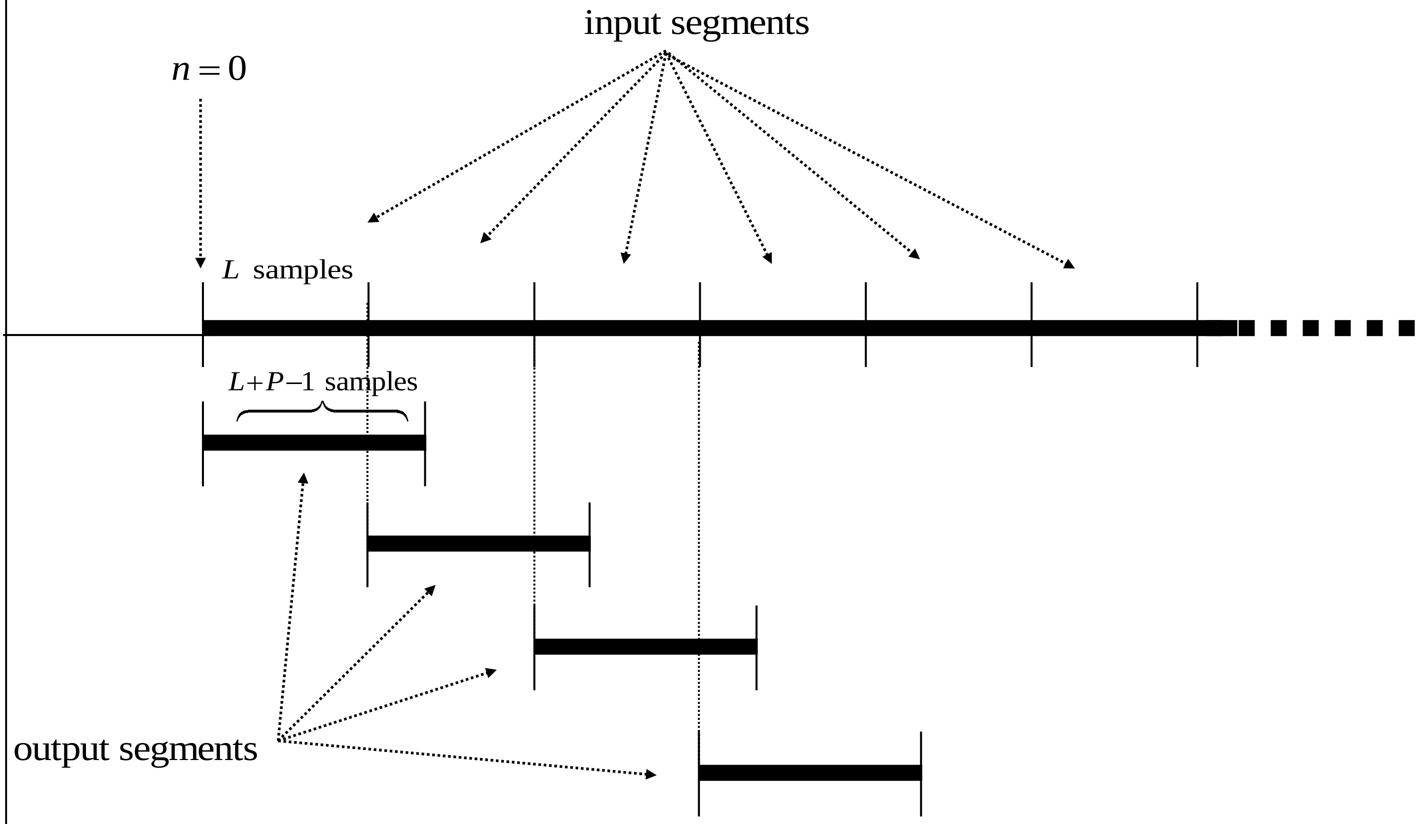
Compute the (*L*+*P*-1)-point DFT of $h[n]$, $H[k]$

For the r^{th} block, the output is found by (*L*+*P*-1)-point IDFT of $Y_r[k] = H[k]X_r[k]$, $y_r[n]$

$$\Rightarrow \quad y_0[n],\ y_1[n],\ y_2[n],\ \ldots$$

The overall output is formed as $y[n] = \sum_{r=0}^{\infty} y_r[n-rL] = y_0[n] + y_1[n-L] + y_2[n-2L] + \ldots$

Overlap Add Pictorially



OVERLAP-SAVE METHOD

Ex:

$$h[n]:\ \ldots, 0,\ 0,\ \underset{\underset{n=0}{\uparrow}}{1},\ 2,\ 3,\ 0,\ 0, \ldots$$

3-point sequence (P = 3)

$$x[n]:\ \ldots, 0,\ 0,\ \underset{\underset{n=0}{\uparrow}}{1},\ -1,\ 2,\ -2,\ -1,\ 1,\ 0,\ 0\ , \ldots$$

(In general, infinite length, starts at n = 0)

Decompose the input, $x[n]$, into L-point segments.

Choose L= 4, could be something else...

In overlap-save method, we use L-point DFTs

Therefore, the first (L+P-1)-L = P-1 samples for each output segment will be incorrect. In this example, the first 2 samples...

Because of the two incorrect samples at each output block, the first segment is choosen to start at n = - (P-1) = -2

$$x[n]:\ \ldots, 0,\ 0,\ \underbrace{0,\ 0,\ 1,\ -1}_{x_1},\ 2,\ -2,\ -1,\ 1,\ 0,\ 0\ , \ldots$$

then, the second segment is chosen as

$$x[n]:\ \ldots, 0,\ 0,\ \underbrace{0,\ 0,\ \overbrace{1,\ -1}_{}}_{x_1}\overbrace{,\ 2,\ -2}^{x_2},\ -1,\ 1,\ 0,\ 0\ , \ldots$$

which “**overlaps**” wih the previous segment by two samples

Compute the <u>**4-point**</u> DFTs of $x_1, x_2, x_3, x_4, x_5, \ldots$ $\rightarrow$ $X_1, X_2, X_3, X_4, X_5, \ldots$

Compute the <u>**4-point**</u> DFT of h $\rightarrow$ H

Compute

$$y_1 = IDFT\{H\,X_1\} \qquad y_1:\ \ldots, 0, 0, 1, -3, 1, 1, 0, 0, \ldots$$

$$y_2 = IDFT\{H\,X_2\} \qquad y_2:\ \ldots, 0, 0, 3, -5, 3, -1, 0, 0, \ldots$$

$$y_3 = IDFT\{H\,X_3\} \qquad y_3:\ \ldots, 0, 0, 1, 5, 1, -7, 0, 0, \ldots$$

$$y_4 = IDFT\{H\,X_4\} \qquad y_4:\ \ldots, 0, 0, -1, -1, -1, 3, 0, 0, \ldots$$

$$y_5 = IDFT\{H\,X_5\} \qquad y_5:\ \ldots, 0, 0, 0, 0, 0, 0, 0, 0, \ldots$$

Compute $y[n]$ by "**saving**" the correct samples,

saved

$$\begin{array}{ccccccccccc}
 & \cancel{1} & \cancel{-3} & \multicolumn{2}{c}{\boxed{1 \quad 1}} & & & & & & \\
 & & & \cancel{3} & \cancel{-5} & \multicolumn{2}{c}{\boxed{3 \quad -1}} & & & & \\
 & & & & & \cancel{1} & \cancel{5} & \multicolumn{2}{c}{\boxed{1 \quad -7}} & & \\
+ & & & & & & & \cancel{-1} & \cancel{-1} & \multicolumn{2}{c}{\boxed{-1 \quad 3}} \\
\hline
 & & & 1 & 1 & 3 & -1 & 1 & -7 & -1 & 3 \\
 & & & \underset{n=0}{\uparrow} & & & & & & &
\end{array}$$

$$y[n]:\ \ldots, 0, 0, \underset{\substack{\uparrow \\ n=0}}{1}, 1, 3, -1, 1, -7, -1, 3, 0, 0 \ldots$$

OVERLAP SAVE FORMALLY

Let $N < L + P - 1 \qquad (N \geq L \geq P)$

The first $(L + P - 1 - N)$ samples of N-point IDFT are incorrect.

In particular, if $N = L,\ P - 1$ samples are incorrect.

So, choose the first input segment, $x_0[n]$, as

$$\underbrace{x[-(P-1)], x[-(P-2)], \ldots, x[-1], x[0], x[1], \ldots, x[L-P]}_{L \text{ samples}}$$

i.e, $\quad x_0[n] = x[n-(P-1)] \qquad 0 \leq n \leq L-1$

Compute its *L*-point DFT $\rightarrow X_0[k]$

Compute the *L*-point DFT of *h* $\rightarrow H[k]$

Compute $\hat{y}_0[n] =$ *L*-point IDFT of $X_0[k]\,H[k]$.

Discard the first (*P*-1) samples to get the first ouput segment

$$y_0[n] = \begin{cases} \hat{y}_0[n] & P-1 \leq n \leq L-1 \\ 0 & o.w. \end{cases}$$

then the first (L-(P-1)) samples of the output are

$$y[n] = y_0[n+(P-1)] \qquad n = 0,1,\ldots,L-(P-1)-1$$

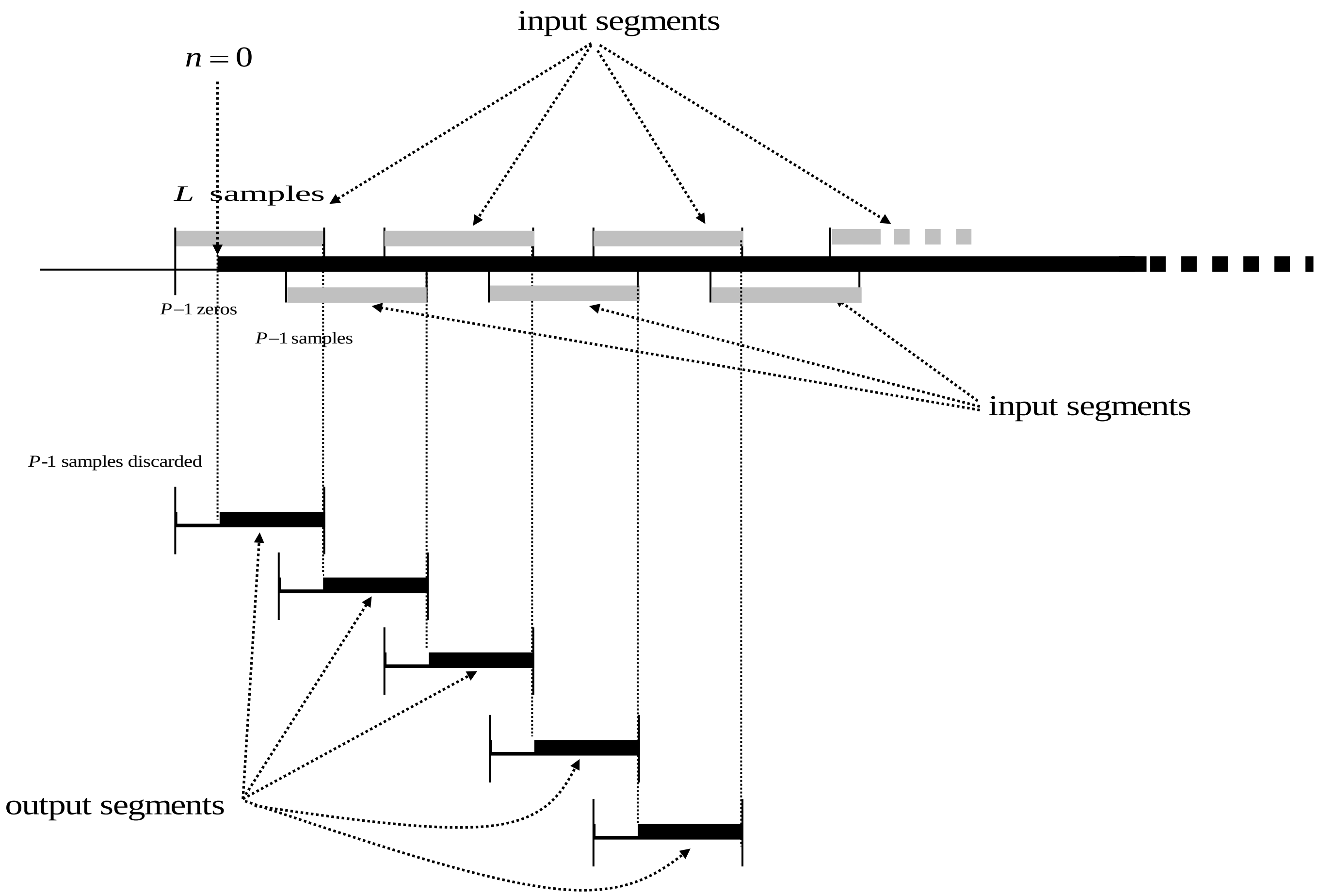
Then take the second input segment as

$$\underbrace{x[L-P-(P-1)+1], \ldots, x[2L-2P+1]}_{L \text{ samples}}$$

i.e. $\quad x[L-2P+2], \ldots, x[2L-2P+1]$

and continue the process...

Overlap Save Pictorially



LINEAR CONVOLUTION AND CIRCULAR CONVOLUTION

We know the following

$$x[n] \leftrightarrow X(e^{j\omega})$$

$$y[n] \leftrightarrow Y(e^{j\omega})$$

$$Z(e^{j\omega}) = X(e^{j\omega})Y(e^{j\omega})$$

$$z[n] = IDTFT\{Z(e^{j\omega})\}$$

$$= x[n] * y[n]$$

Now, the question is

$$\hat{z}[n] = ?$$

$$\hat{z}[n] = IDFT\left\{\underbrace{X[k]Y[k]}_{\hat{Z}[k]}\right\}$$

where

$$\hat{Z}[k] = \left(X\left(e^{j\omega}\right)Y\left(e^{j\omega}\right)\right)\Big|_{\omega=k\frac{2\pi}{N}}$$

$$k = 0,1,\dots,N-1$$

We know that IDFT yields

$$\hat{z}[n] = \begin{cases} \sum_{r=-\infty}^{\infty} z[n-rN] & n = 0,1,\dots,N-1 \\ 0 & o.w. \end{cases}$$

Therefore N-point circular convolution of $x[n]$ and $y[n]$ can also be computed via linear convolution:

Compute

$$z[n] = x[n] * y[n]$$

Then, you can find $x[n] \circledast_N y[n]$ as

$$x[n] \circledast_N y[n] = \begin{cases} \sum_{r=-\infty}^{\infty} z[n - rN] & n = 0,1,\dots,N-1 \\ 0 & o.w. \end{cases}$$

or as

$$x[n] \circledast_N y[n] = IDFT\{X[k]Y[k]\}$$

Ex:

$$x[n] = \left[\ldots \quad 0 \quad 0 \quad \underbrace{1}_{n=0} \quad 2 \quad 0 \quad 0 \quad \ldots\right]$$

$$y[n] = \left[\ldots \quad 0 \quad 0 \quad \underbrace{-2}_{n=0} \quad 1 \quad 1 \quad 0 \quad 0 \quad \ldots\right]$$

a) Let $X[k]$ and $Y[k]$ be 5-point DFTs of $x[n]$ and $y[n]$, respectively.

Find the sequence $f[n] = IDFT\{X[k]\, Y[k]\}$.

b) Let $X[k]$ and $Y[k]$ be 3-point DFTs of $x[n]$ and $y[n]$, respectively. Find the sequence $w[n] = IDFT\{X[k]\, Y[k]\}$

c) Let

$$X[k] = X(e^{j\omega})\Big|_{\omega=k\frac{2\pi}{2}} \qquad k = 0,1$$

$$Y[k] = X(e^{j\omega})\Big|_{\omega=k\frac{2\pi}{2}} \qquad k = 0,1$$

Find the sequence $p[n]$ obtained by applying 2-point inverse DFT operation to $X[k]\, Y[k]$.

a)

$$f[n] = x[n] \circledast_5 y[n]$$

$$= \sum_{r=0}^{4} x[r] y\left[\left((n-r)\right)_5\right] \qquad n = 0,1,2,3,4$$

$$= x[0] y\left[\left((n)\right)_5\right] + x[1] y\left[\left((n-1)\right)_5\right] \qquad n = 0,1,2,3,4$$

$$\begin{array}{rrrrrr} & -2 & 1 & 1 & 0 & 0 \\ + & 0 & -4 & 2 & 2 & 0 \\ \hline & -2 & -3 & 3 & 2 & 0 \end{array}$$

Therefore

$$f[n] = \left[\dots \;\; 0 \;\; 0 \;\; \underbrace{-2}_{n=0} \;\; -3 \;\; 3 \;\; 2 \;\; 0 \;\; 0 \;\; \dots\right]$$

Or, since the linear convolution of $x[n]$ and $y[n]$ yields (3+2-1) 4-point sequence and the DFTs are 5-point, 5-point circular convolution and linear convolution yields the same result.

$$f[n] = x[n] \circledast_5 y[n]$$

$$= x[n] * y[n]$$

$$= z[n]$$

b)

$$
\begin{aligned}
w[n] &= x[n] \circledast_3 y[n] \\
&= \sum_{r=0}^{2} x[r]\, y\left[((n-r))_3\right] \qquad\qquad n = 0,1,2 \\
&= x[0]\, y\left[((n))_3\right] + x[1]\, y\left[((n-1))_3\right] \qquad n = 0,1,2
\end{aligned}
$$

$$
\begin{array}{cccc}
 & -2 & 1 & 1 \\
+ & 2 & -4 & 2 \\
\hline
 & 0 & -3 & 3
\end{array}
$$

Therefore

$$
w[n] = \left[\ldots \quad 0 \quad 0 \quad \underbrace{-3}_{n=0} \quad 3 \quad 0 \quad 0 \quad \ldots \right]
$$

or

$$
w[n] = x[n] \circledast_3 y[n]
$$

$$
= \begin{cases} \displaystyle\sum_{r=-\infty}^{\infty} z[n - r3] & n = 0,1,2 \\ 0 & o.w. \end{cases}
$$

$$
\begin{array}{cccc:ccc:cccc}
 & & & & n=0 & n=1 & n=2 & & & & \\
 & & & & -2 & -3 & 3 & 2 & & & \\
 & & & & & & & -2 & -3 & 3 & 2 \\
+ & -2 & -3 & 3 & 2 & & & & & & \\
\hline
 & & & & 0 & -3 & 3 & & & &
\end{array}
$$

c)

$$p[n] = \begin{cases} \dfrac{1}{2}\displaystyle\sum_{k=0}^{1} X[k]\, Y[k] e^{jk\frac{2\pi}{2}n} & n = 0,1 \\ \\ 0 & o.w. \end{cases}$$

$$= \begin{cases} \displaystyle\sum_{r=0}^{1} z[n - r2] & n = 0,1 \\ \\ 0 & o.w. \end{cases}$$

where

$$z[n] = x[n] * y[n]$$

(linear conv.)

Therefore

$$\begin{array}{ccc|cc|cccc} & & & n=0 & n=1 & & & & \\ & & & -2 & -3 & 3 & 2 & & \\ & & & & & -2 & -3 & 3 & 2 \\ + & -2 & -3 & 3 & 2 & & & & \\ \hline & & & 1 & -3 & & & & \end{array}$$

$$p[n] = \left[\ldots \quad 0 \quad 0 \quad \underbrace{1}_{n=0} \quad -3 \quad 0 \quad 0 \quad \ldots\right]$$