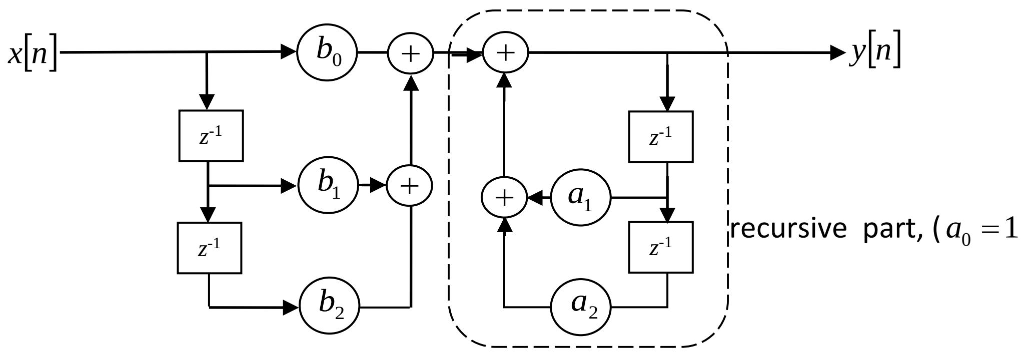
Linear Constant Coefficient Difference Equations

LTI systems may be represented by LCCDEs.

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

$$y[n] + a_1 y[n-1] + a_2 y[n-2] = b_0 x[n] + b_1 x[n-1] + b_2 x[n-2]$$



Ex: Accumulator

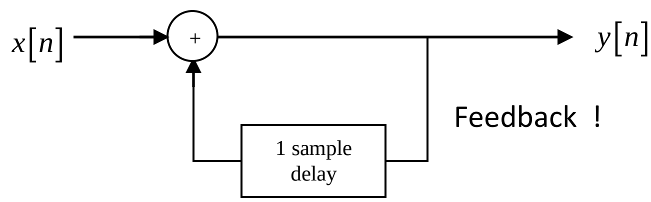
$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

It can be represented as

$$y[n] = y[n-1] + x[n]$$

or

$$y[n] - y[n-1] = x[n]$$



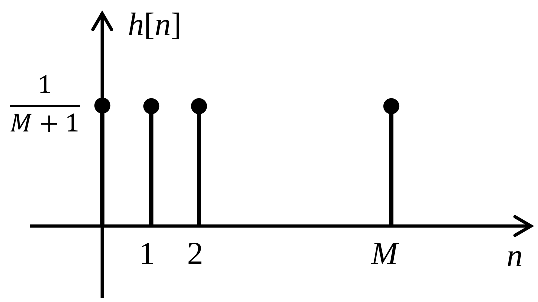
Impulse response of accumulator is $h[n] = u[n]$

Ex: Moving Average (MA) system

$$y[n] = \frac{1}{M+1}\sum_{k=0}^{M} x[n-k]$$

Impulse response

$$h[n] = \frac{1}{M+1}(u[n] - u[n-(M+1)\ ])$$

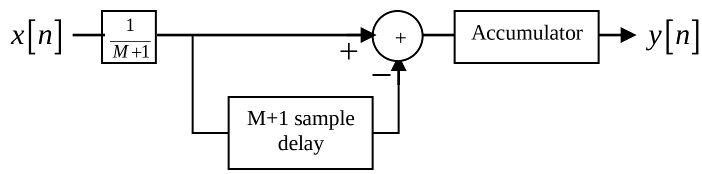


Impulse response of MA system can *also* be written as

$$h[n] = \frac{1}{M+1}\left(\delta[n] - \delta\left[n-(M+1)\right]\right) * u[n]$$

This expression reminds us that MA system can be considered as the cascade of an *accumulator* and another system with impulse response

$$h[n] = \frac{1}{M+1}\left(\delta[n] - \delta[n-(M+1)]\right)$$



Therefore MA system can also be described by

$$\rightarrow \; y[n] - y[n-1] = \frac{1}{M+1}\left(x[n] - x[n-(M+1)]\right)$$

"Less arithmetic operations in the implementation but recursion may cause numerical problems in finite precision."

Given a set of boundary conditions, a difference equation can be solved *recursively* in forward and backward directions.

For example, let $y[-1], y[-2], .., y[-N]$ be specified

Forward recursive solution:

$$y[n] = -\sum_{k=1}^{N} \frac{a_k}{a_0} y[n-k] + \sum_{k=0}^{M} \frac{b_k}{a_0} x[n-k] \qquad\qquad n = 0,1, \ldots$$

Backward recursive solution:

$$y[n-N] = -\sum_{k=0}^{N-1} \frac{a_k}{a_N} y[n-k] + \sum_{k=0}^{M} \frac{b_k}{a_N} x[n-k] \qquad\qquad n = 0, -1, -2, \ldots$$

Ex:

$$y[n] = -\frac{1}{2}y[n-1] + x[n],$$

$$y[-1] = c$$

$$x[n] = K\delta[n]$$

Forward:

$$y[0] = -\frac{1}{2}c + K$$

$$y[1] = \frac{1}{4}c - \frac{1}{2}K$$

$$y[n] = \underbrace{\left(-\frac{1}{2}\right)^{n+1} c}_{\text{hom. soln.}} + \underbrace{\left(-\frac{1}{2}\right)^{n} K}_{\text{particular soln.}}$$

Backward:

$$y[n-1] = -2y[n] + 2x[n]$$

$$y[-1] = c$$

$$y[-2] = -2c$$

$$\vdots$$

$$y[n] = \left(-\frac{1}{2}\right)^{n+1} c \qquad n \le -1$$

The general solution becomes $y[n] = \left(-\frac{1}{2}\right)^{n+1} c + \left(-\frac{1}{2}\right)^{n} K\, u[n]$

The Solution of LCCDEs

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

general solution = particular solution + homogeneous solution

$$y[n] = y_p[n] + y_h[n]$$

Particular solution: Given a particular input $x_p[n]$, particular solution $y_p[n]$ is any solution that satisfies the equation for this input.

Homogeneous solution: $y_h[n]$, satisfies $\sum_{k=0}^{N} a_k y_h[n-k] = 0$.

Homogeneous Solution

In general $y_h[n]$ is a *weighted* sum of z^n type signals where *z* is a(complex) constant.

$$\sum_{k=0}^{N} a_k z^{n-k} = 0$$

$$\sum_{k=0}^{N} a_k z^{-k} = 0$$

This equation has *N* roots, z_k , $k = 1, \ldots, N$.

So,

$$y_h[n] = \sum_{k=1}^{N} A_k z_k^n = 0$$

where A_k s can be determined according to the initial (auxiliary, boundary) conditions.

Initial Rest Assumption and LTI Systems

When a LCCDE is considered together with "initial rest" (zero initial conditions) assumption, the input-output ($x[n]$-$y[n]$) relationship becomes a *linear* and *time-invariant* one.

For nonzero initial conditions

1) Even if the input is zero, the output is nonzero → input-output relationship is nonlinear

2) If the input is shifted by n_0, the output is not shifted by the same amount → system is time-varying.

   (For example, in the above example the solution for a shifted input is

$$y[n]=\left(-\frac{1}{2}\right)^{n+1}c+\left(-\frac{1}{2}\right)^{n-n_0}K\ u[n-n_0]\ )$$

Therefore "A sytem described by a LCCDE is a LTI one if it is initially at rest, i.e. initial conditions are zero."

Note:

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

If N = 0,

$$y[n] = \sum_{k=0}^{M} \frac{b_k}{a_0} x[n-k]$$

no initial conditions are needed to solve.

Impulse response is $$h[n] = \sum_{k=0}^{M} \left(\frac{b_k}{a_0}\right) \delta[n-k].$$

It is a FIR sytem.

Causality

Given a system described by a LCCDE together with initial rest assumption.

One cannot determine whether the system is causal or not. Causality must be specified separately.

Therefore we have to state something like “ *a causal/noncausal system described by the following LCCDE…*”

Finding The Impulse Response From LCCDE

Problem Statement: Given the LCCDE describing a *causal LTI* system, find its impulse response.

Ex: $$y[n]-ay[n-1]=x[n]$$

We take $x[n]=\delta[n]$

$$h[n]-ah[n-1]=\delta[n] \qquad h[-1]=0$$

i) By recursion

$$h[0]=ah[-1]+\delta[0]$$
$$=1$$

$$h[1]=ah[0]+\delta[1]$$
$$=a$$

$$\vdots$$

$$h[n]=ah[n-1]+\delta[n]$$
$$=a^n$$

$$\Rightarrow h[n]=a^n u[n]$$

ii) By finding the homogeneous solution

$$x[n]=\delta[n] \rightarrow y[n]-ay[n-1]=0 \qquad n>0$$
$$y[n]=Kz^n \qquad n>0$$
$$Kz^n-aKz^{n-1}=0$$
$$Kz^n\left(1-az^{-1}\right)=0$$
$$\left(1-az^{-1}\right)=0$$
$$z=a$$
$$\Rightarrow y[n]=Ka^n \qquad n>0$$

Since $y[0]=1 \quad \Rightarrow \quad Ka^0=K=1$

$$\Rightarrow h[n]=a^n \qquad n\geq 0$$

Ex: $$y[n]-ay[n-1]=x[n-1]$$
$$\Rightarrow h[n]=a^{n-1}u[n-1]$$

Ex: $$y[n]-ay[n-1]=x[n]+x[n-1]$$
$$h[n]=a^{n}u[n]+a^{n-1}u[n-1]$$
$$=\delta[n]+(1+a)a^{n-1}u[n-1]$$

Ex: Homogeneous solution, repeated roots

$$y[n]-\frac{1}{2}y[n-1]-\frac{1}{4}y[n-2]-\frac{1}{4}y[n-3]=x[n]$$

$$y_h[n]-\frac{1}{2}y_h[n-1]-\frac{1}{4}y_h[n-2]-\frac{1}{4}y_h[n-3]=0$$

$$Kz^n\left(1-\frac{1}{2}z^{-1}-\frac{1}{4}z^{-2}-\frac{1}{4}z^{-3}\right)=0$$

$$z_1=\frac{1}{2},\ z_2=z_3=\frac{1}{4}$$

$$y_h[n]=K_1\frac{1}{2}^n u[n]+K_2\frac{1}{4}^n u[n]+K_3 n\frac{1}{4}^n u[n]$$

Exercise: Find the impulse response of the causal LTI system described by

$$y[n-1]-2y[n-2]=x[n-2]$$

Is it a stable system?

Exercise:

a) Write the difference equation that describes the LTI system with impulse response $h[n]=\left(\frac{2}{3}\right)^n u[n]$.

b) Repeat part-a for $h[n]=\left(\frac{2}{3}\right)^{n-1} u[n-1]$

c) Repeat part-a for $h[n]=-\left(\frac{2}{3}\right)^n u[-n-1]$