LTI SYSTEMS

CONVOLUTION AND IMPULSE RESPONSE

The Lengths of Input and Output Sequences

Two interpretations of its computation

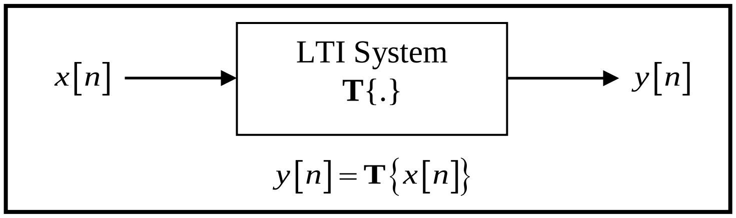
LINEAR BUT TIME-VARYING SYTEMS

FINITE/INFINITE IMPULSE RESPONSE SYSTEMS (FIR, IIR)

PROPERTIES OF LTI SYSTEMS

- Convolution is commutative
- Convolution is associative
- Cascading LTI Systems
- Parallel LTI Systems
- BIBO Stability
- Causal LTI Systems

IMPULSE RESPONSE AND CONVOLUTION



Linearity $\mathbf{T}\{a\,x_1[n]+b\,x_2[n]\} = a\mathbf{T}\{x_1[n]\}+b\mathbf{T}\{x_2[n]\}$

Time-invariance $\mathbf{T}\{x[n]\}=y[n] \quad \Rightarrow \quad \mathbf{T}\{x[n-n_0]\}=y[n-n_0]\,.$

An input signal can be written as

$$x[n] = \cdots + x[-1]\delta[n+1] + x[0]\delta[n] + x[1]\delta[n-1] + \cdots$$

$$= \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

Using the **linearity** and **time-invariance** of the system the output of the LTI system can be written as

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

where $h[n]$ is the “impulse response” of the LTI system.

This operation is called *convolution* of $x[n]$, and $h[n]$.

Derivation

$$y[n] = T\{x[n]\}$$

$$= T\{\ldots + x[-1]\delta[n+1] + x[0]\delta[n] + x[1]\delta[n-1] + \cdots\}$$

$$= \cdots + x[-1]T\{\delta[n+1]\} + x[0]T\{\delta[n]\} + x[1]T\{\delta[n-1]\} + \cdots$$

$$= \cdots + x[-1]h[n+1] + x[0]h[n] + x[1]h[n-1] + \cdots$$

or compactly,

$$y[n] = T\{x[n]\}$$

$$= T\left\{\sum_{k=-\infty}^{\infty} x[k]\delta[n-k]\right\}$$

$$= \sum_{k=-\infty}^{\infty} x[k]T\{\delta[n-k]\}$$

$$= \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Convolution of $x[n]$and $h[n]$

Convolution is shown by a $'*'$,

$$y[n] = x[n] * h[n]$$

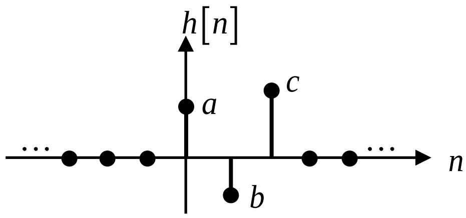
It is easy to show that

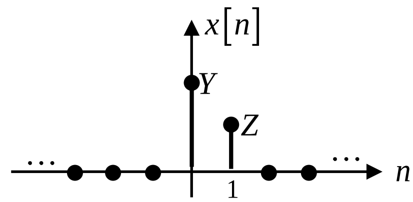
$$\sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

i.e.,

$$x[n] * h[n] = h[n] * x[n]$$

Ex:





The output can be considered as the superposition of responses to individual samples of the input.

$$y[n] = \underbrace{Yh[n]}_{the\ response\ to\ x[0]} + \underbrace{Zh[n-1]}_{the\ response\ to\ x[1]}$$

or equivalently

$$y[n] = ax[n] + bx[n-1] + cx[n-2]$$

Ex: The output can be considered as the superposition of responses to individual samples of the input.

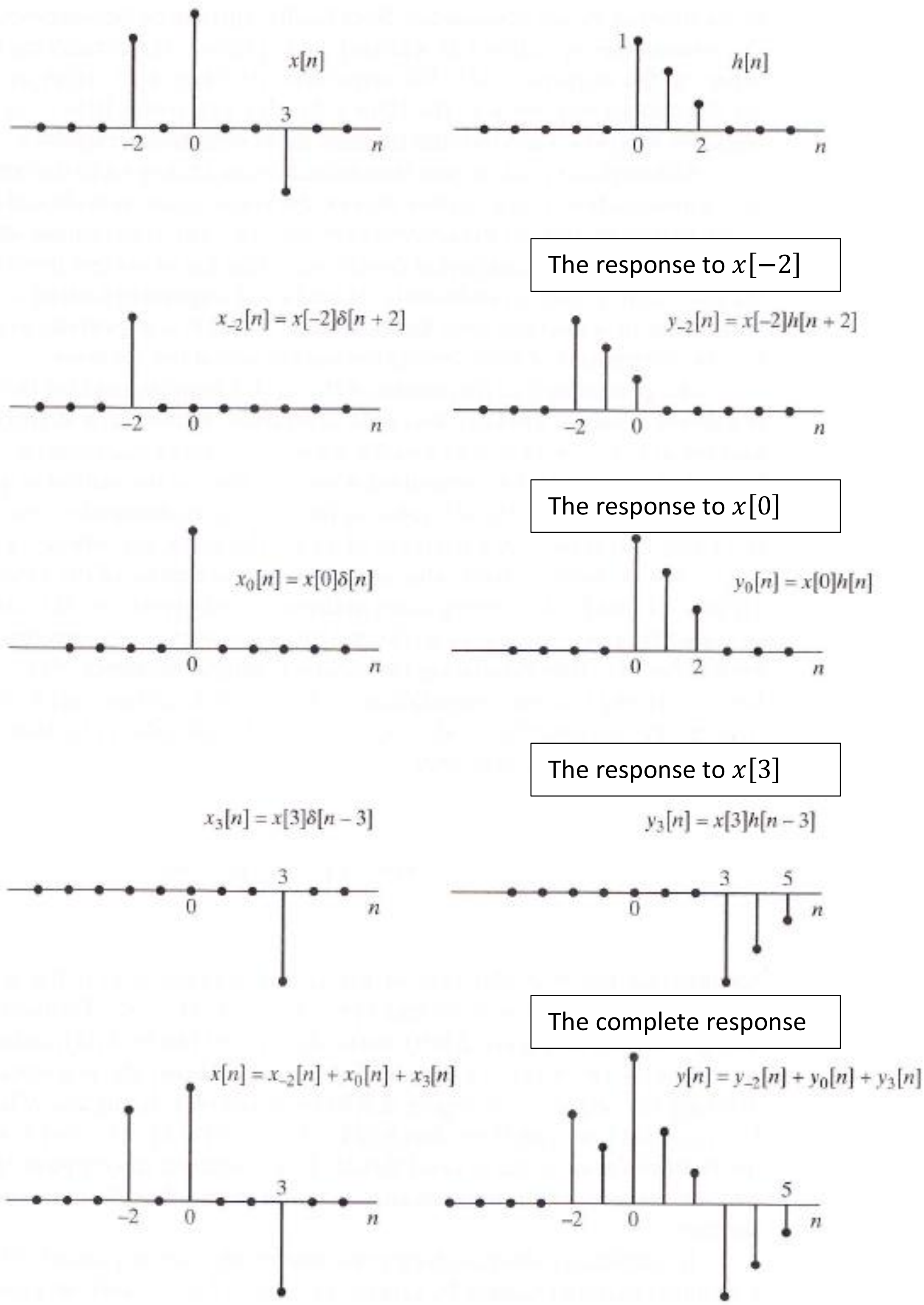


Figure 2.8 Representation of the output of an LTI system as the superposition of responses to individual samples of the input.

The second interpretation of convolution allows one to compute one output sample at a time.

For example, to get $y[n]$ (say $y[3]$),

1) multiply the two sequences $x[k]$ and $h[n-k]$ ($h[3-k]$), taking *k* as the independent variable.

2) Add the sample values of this product

Ex: Applying the second method to compute the whole output samples.

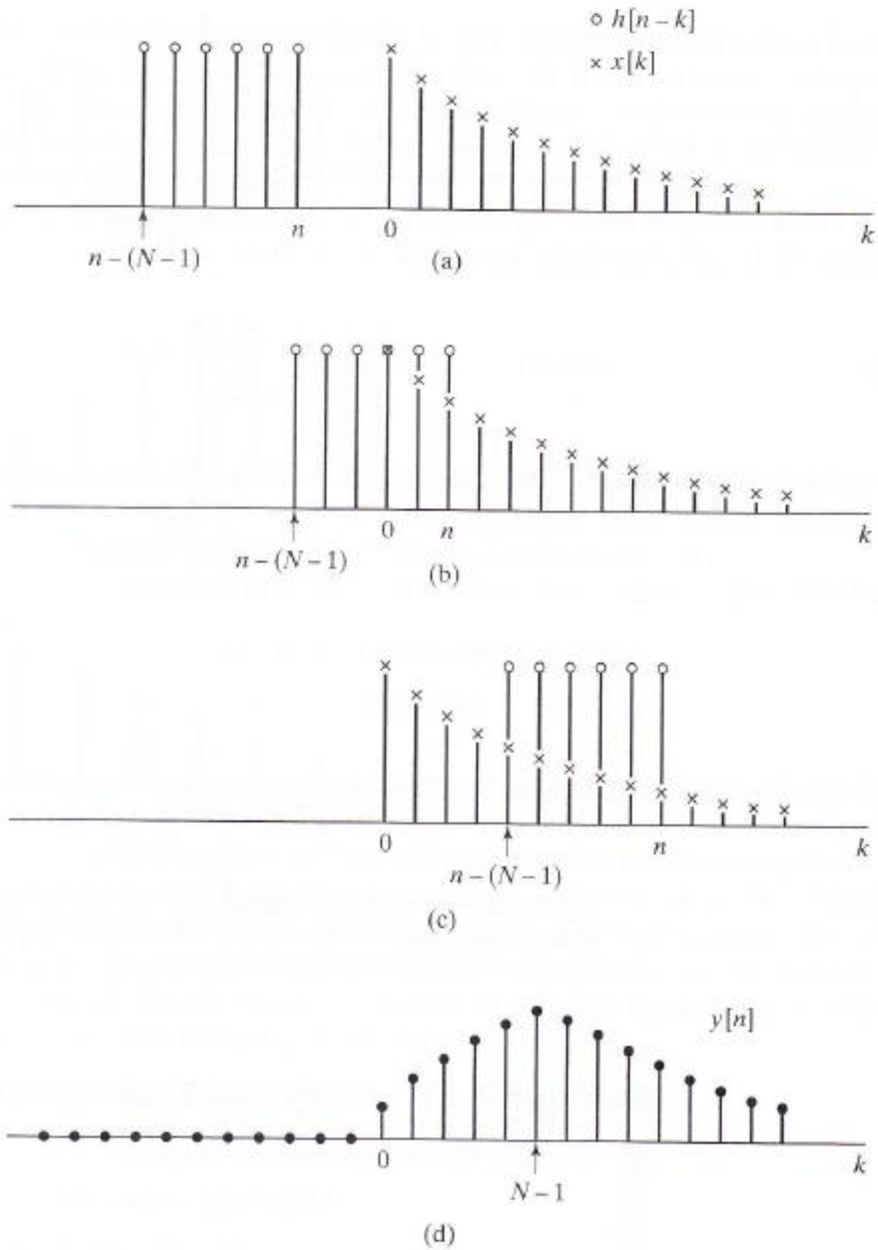


Figure 2.10 Sequence involved in computing a discrete convolution. (a)–(c) The sequences $x[k]$ and $h[n-k]$ as a function of k for different values of n. (Only nonzero samples are shown.) (d) Corresponding output sequence as a function of n.

THE LENGTHS OF INPUT AND OUTPUT SEQUENCES

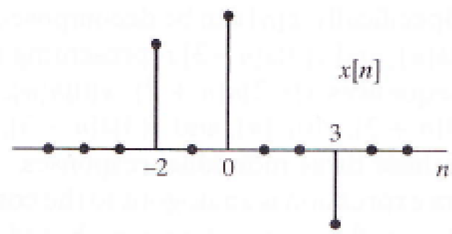
Suppose that the input signal and the impulse response have finite durations:

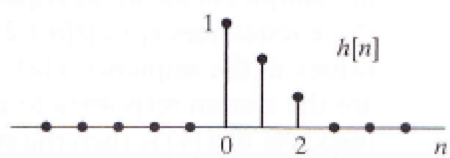
Input $x[n]$	: length N
Imp. Resp. $h[n]$	: length M

Then the length of the output signal is $N + M - 1$.

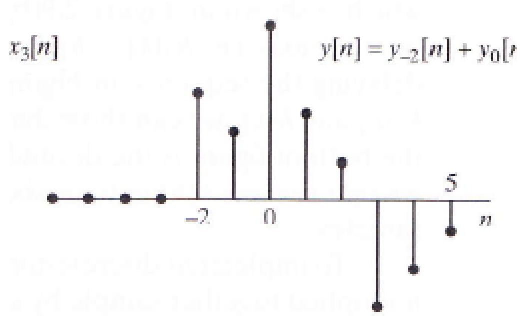
Output $y[n]$	: length $N + M - 1$

<u>Ex</u>:





Input $x[n]$	: length is 6 samples
Imp. Resp. $h[n]$	: length is 3 samples



Output $y[n]$: length is 8 samples

CONVOLUTION IN MATLAB

```
>> help conv
 conv Convolution and polynomial multiplication.

    C = conv(A, B) convolves vectors A and B.
     The resulting vector is length
     MAX([LENGTH(A)+LENGTH(B)-1,LENGTH(A),LENGTH(B)]).
     If A and B are vectors of polynomial coefficients, convolving them is
     equivalent to multiplying the two polynomials.

    C = conv(A, B, SHAPE) returns a subsection of the convolution with
     size

     specified by SHAPE:
      'full'  - (default) returns the full convolution,
      'same'  - returns the central part of the convolution
                that is the same size as A.
      'valid' - returns only those parts of the convolution
                that are computed without the zero-padded edges.
                LENGTH(C)is MAX(LENGTH(A)-MAX(0,LENGTH(B)-1),0).


    Class support for inputs A,B:
       float: double, single

    See also deconv, conv2, convn, filter and,
    in the signal Processing Toolbox, xcorr, convmtx.

    Overloaded methods:
       cvx/conv
       gf/conv
       gpuArray/conv

    Reference page in Help browser
       doc conv
```

Exercise: Compare “`conv`” and “`filter`” commands

Exercise:Study “`deconv`”

Play and experiment with code1_LN3.m

```
clear all
close all

h = [2 -3 1];
x = [1 0 0 0 -2];


% h = [2 -3 1 -1 4];
% x = rand(1,5);              % uniformly distributed numbers from [0,1]
% x = 2*(rand(1,5)-0.5);      % uniformly distributed numbers from [-1,1]%
% x = rand(1,randi(7,1))
% x = rand(1,randi(7,1))  % uniformly distributed numbers from [0,1], signal length is
also random
y = conv(x,h)

nh = 0:length(h)-1;
nx = 0:length(x)-1;
ny = 0:length(y)-1;

mini = min([h x y]);
maxi = max([h x y]);

figure
subplot(3,1,1); stem(nh,h,'linewidth',2);
v = axis;
v = [v(1)-1 length(y) mini maxi];
axis(v)
subplot(3,1,2); stem(nx,x,'r','linewidth',2)
v = axis;
v = [v(1)-1 length(y) mini maxi];
axis(v)
subplot(3,1,3); stem(ny,y,'k','linewidth',2)
v = axis;
v = [v(1)-1 length(y) mini maxi];
axis(v)
```

and also with code2_LN3.m

```
clear all
close all

h = [2 -3 1 ]
x = [1 0 0 0 -2]
y = conv(h,x)
yy = filter(h,1,x)
yyy = filter(x,1,h)
```

LINEAR BUT TIME-VARYING SYTEMS

If the system is linear but time-varying, the derivation on the first page yields

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h_k[n]$$

where

$h_k[n]$ (time-varying impulse response)

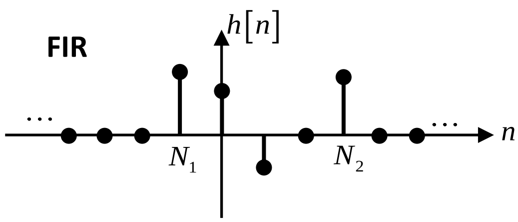
is the response of the system to an impulse at time k , i.e, $\delta[n-k]$.

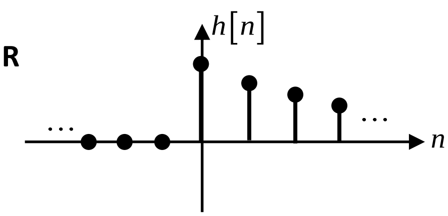
LTV systems are used, for example, to model mobile communication channels.

In mobile communication channel impulse response changes due to the motion of the transmitter and receiver, and also due to the change or moving objects in the environment.

FINITE/INFINITE IMPULSE RESPONSE SYSTEMS

If $h[n]$ has a finite number of samples ($h[n]=0 \quad n<N_1 \quad n>N_2, \quad N_1<N_2$) then the system is said to be a *Finite Impulse Response* (FIR) system, otherwise an *Infinite Impulse Response* (IIR) system.





PROPERTIES OF LTI SYSTEMS

CONVOLUTION IS COMMUTATIVE

$$x[n] * h[n] = h[n] * x[n]$$

$$x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Let $m = n - k$

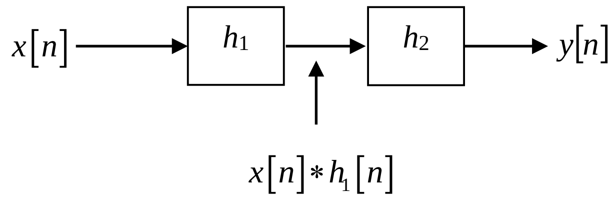
$$= \sum_{m=-\infty}^{\infty} x[n-m]h[m]$$

$$= h[n] * x[n]$$

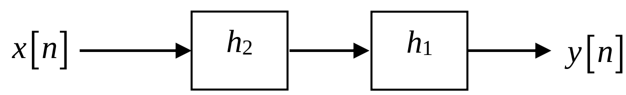
CONVOLUTION IS ASSOCIATIVE

$$\left(a[n]*b[n]\right)*c[n]=a[n]*\left(b[n]*c[n]\right)$$

Prove!



$$\begin{aligned} y[n] &= (x[n]*h_1[n])*h_2[n] \\ &= x[n]*(h_1[n]*h_2[n]) \\ &= x[n]*(h_2[n]*h_1[n]) \\ &= (x[n]*h_2[n])*h_1[n] \end{aligned}$$



→ If the systems are LTI, the order of cascade can be changed!

Ex: Consider the following two systems

$$y_1[n] = x[n] + x[n-1] + x[n-2]$$

and

$$y_2[n] = \sum_{k=0}^{\infty} a^k x[n-k] = x[n] + ax[n-1] + a^2 x[n-2] + \ldots$$

They are LTI (check!), so their order is arbitrary in their cascade connection.

Show that their impulse responses are

$$h_1[n] = u[n] - u[n-3] = \delta[n] + \delta[n-1] + \delta[n-2]$$

$$h_2[n] = a^n u[n]$$

What is the impulse response of their cascade?

Ex: Let two systems be described by

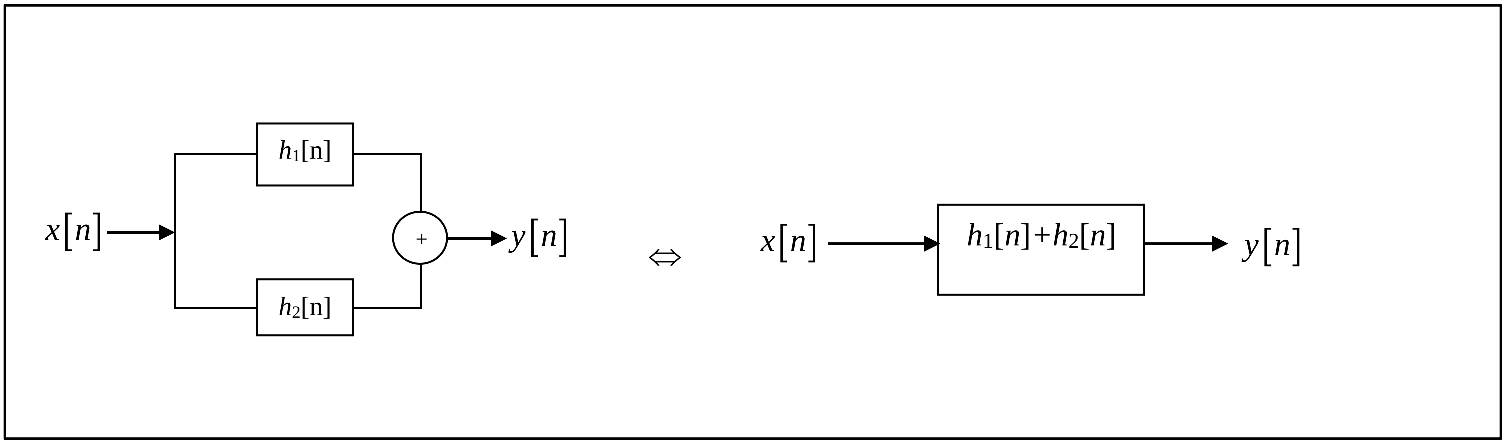
$$y_1[n] = x[n-2] \qquad \text{(impulse response is } h_1[n] = \delta[n-2]\text{)}$$

and

$$y_2[n] = x^2[n]. \qquad \text{(! impulse response is } h_2[n] = \delta^2[n] = \delta[n]\text{)}$$

One of them is nonlinear so their order cannot be changed in their cascade.

PARALLEL LTI SYSTEMS



BIBO STABILITY

For an LTI system

BIBO stability $\Leftrightarrow \sum_{n=-\infty}^{\infty} |h[n]| \le B_h$ (impulse response is absolutely summable)

<u>Proof:</u>

<u>Sufficiency</u>:

$$|y[n]| = \left| \sum_{k=-\infty}^{\infty} h[k] x[n-k] \right| \le \sum_{k=-\infty}^{\infty} |h[k]| \underbrace{x[n-k]}_{\le B_x} \le B_x \sum_{k=-\infty}^{\infty} |h[k]|$$

so if $\sum_{k=-\infty}^{\infty} |h[k]| \le B_h$ then $|y[n]| \le B_h B_x = B_y$

<u>Necessity</u>: (by contradiction)

Assume that the impulse response is not absolutely summable, i.e. $\sum_{n=-\infty}^{\infty} |h[n]| \to \infty$

Also let $x[n] = \begin{cases} \dfrac{h[-n]}{|h[-n]|} & h[-n] \ne 0 \\ 0 & h[-n] = 0 \end{cases}$ so that it is bounded.

Now consider $y[0] = \sum_{k=-\infty}^{\infty} h[k] x[-k] = \sum_{k=-\infty}^{\infty} \dfrac{h[k]}{|h[k]|} h[k] = \sum |h[k]| \to \infty$

FIR (LTI) SYSTEMS ARE ALWAYS STABLE

Why?

CAUSAL LTI SYSTEMS

The impulse response of a causal LTI system satisfies

$$h[n] = 0 \qquad n < 0$$