LTI Systems

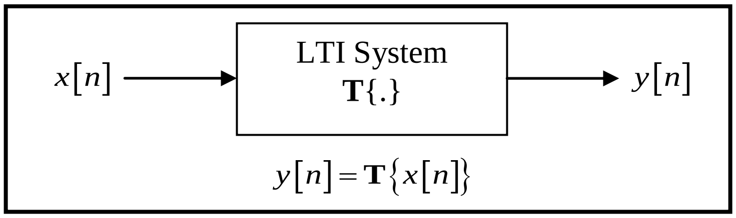
CONVOLUTION

Linearity and time-invariance of a system, and

representation of its input signal in terms of impulses ($x[n]=\sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$)

can be combined to express the output of that LTI system as the *convolution* of the input, $x[n]$, and the system's impulse response, $h[n]$, i.e.,

$$y[n]=\sum_{k=-\infty}^{\infty} x[k]h[n-k]=\sum_{k=-\infty}^{\infty} h[k]x[n-k]$$



Linearity $\quad \mathbf{T}\{a x_1[n]+b x_2[n]\} = a\mathbf{T}\{x_1[n]\}+b\mathbf{T}\{x_2[n]\}$

Time-invariance $\quad \mathbf{T}\{x[n]\}=y[n] \quad \Rightarrow \quad \mathbf{T}\{x[n-n_0]\} = y[n-n_0]$.

Derivation

$$
\begin{aligned}
y[\mathrm{n}] &= \mathrm{T}\left\{\sum_{k=-\infty}^{\infty} x[k]\delta[n-k]\right\} \\
&= \sum_{k=-\infty}^{\infty} x[k]T\{\delta[n-k]\} \\
&= \sum_{k=-\infty}^{\infty} x[k]h[n-k] \qquad \text{Convolution of } x[n] \text{and } h[n]
\end{aligned}
$$

It is written as

$$y[n] = x[n] * h[n]$$

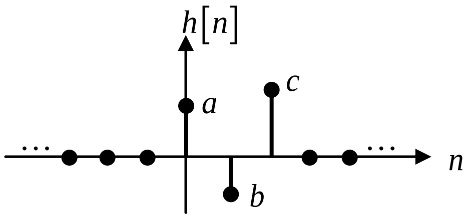
It is easy to show that

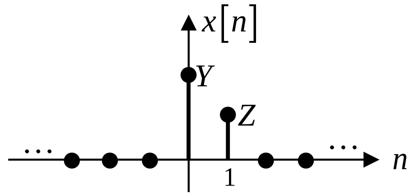
$$\sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

i.e.,

$$x[n] * h[n] = h[n] * x[n]$$

Ex:





The output can be considered as the superposition of responses to individual samples of the input.

$$y[n] = \underbrace{Yh[n]}_{the\ response\ to\ x[0]} + \underbrace{Zh[n-1]}_{the\ response\ to\ x[1]}$$

or equivalently

$$y[n] = ax[n] + bx[n-1] + cx[n-2]$$

Ex: The output can be considered as the superposition of responses to individual samples of the input.

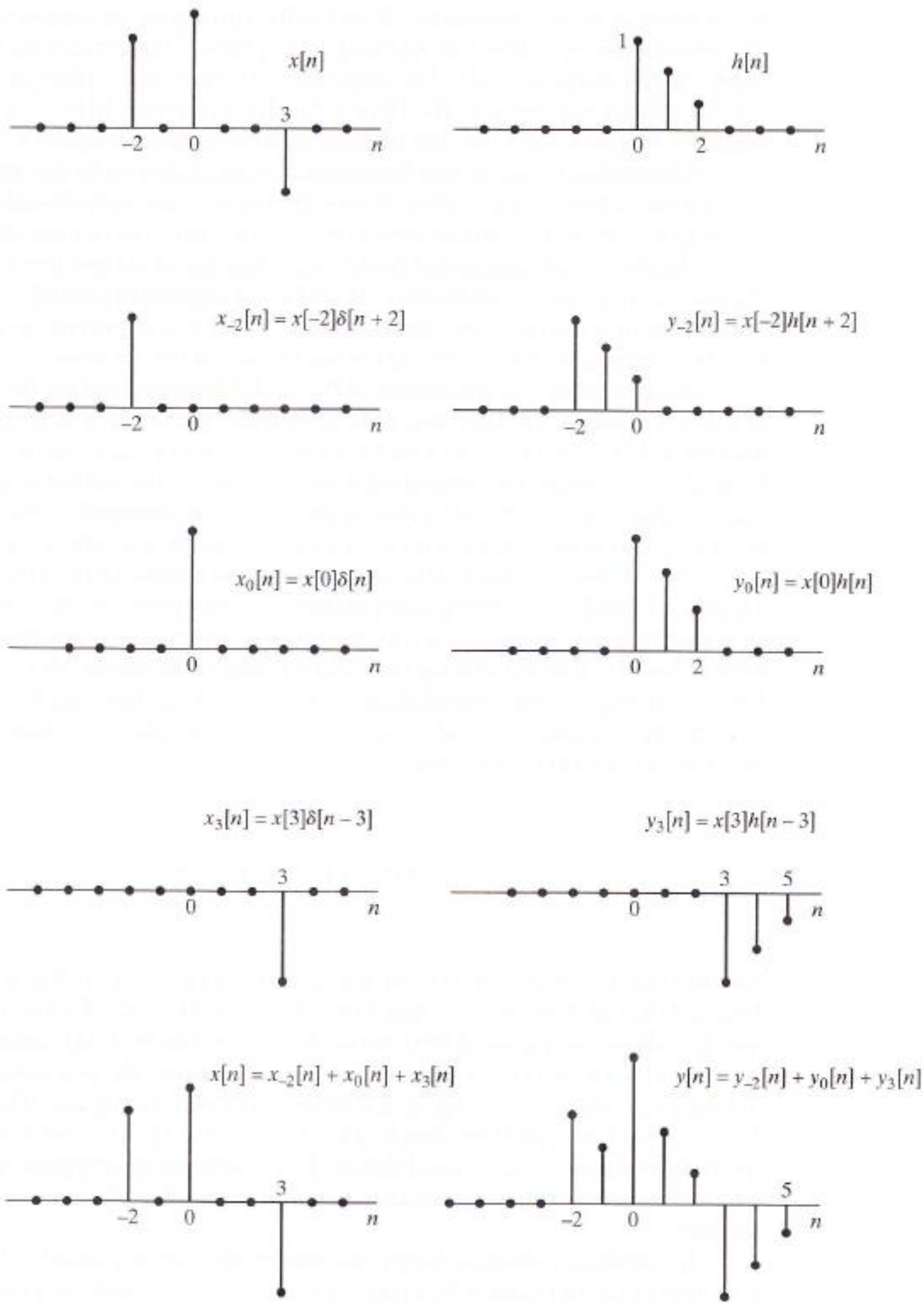


Figure 2.8 Representation of the output of an LTI system as the superposition of responses to individual samples of the input.

The other interpretation of convolution allows one to compute one output sample at a time.

For example, to get $y[n]$ (say $y[3]$),

1) multiply the two sequences $x[k]$ and $h[n-k]$ ($h[3-k]$), taking k as the independent variable.
2) Add the sample values of this product

Ex:Applying the second method to compute the whole output samples.

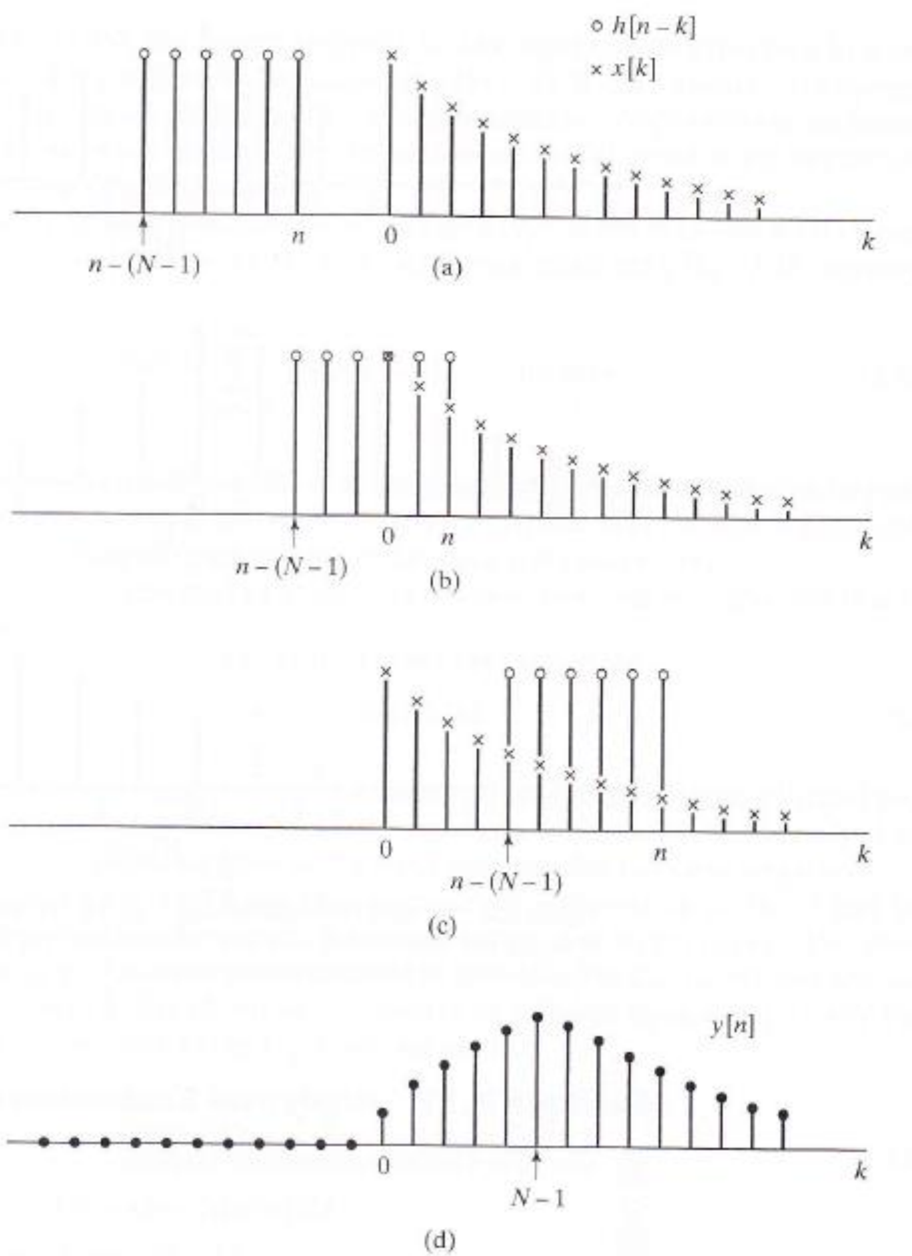


Figure 2.10 Sequence involved in computing a discrete convolution. (a)–(c) The sequences $x[k]$ and $h[n-k]$ as a function of k for different values of n. (Only nonzero samples are shown.) (d) Corresponding output sequence as a function of n.

The Lengths of Input and Output Sequences.

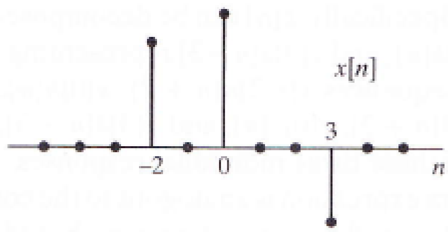
Suppose that the input signal and the impulse response have finite durations:

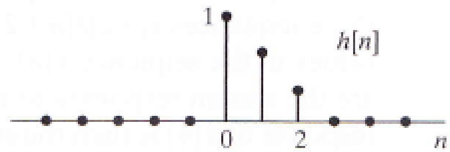
Input $x[n]$	: length N
Imp. Resp. $h[n]$	: length M

Then the length of the output signal is $N+M$-1.

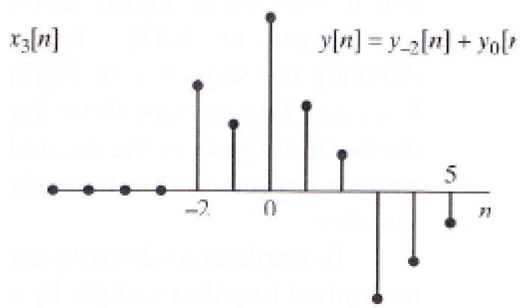
Output $y[n]$	: length $N+M$-1

<u>Ex</u>:





Input $x[n]$	: length is 6 samples
Imp. Resp. $h[n]$	: length is 3 samples



Output $y[n]$	: length is 8 samples

LINEAR BUT TIME-VARYING SYTEMS

Note that if the system is linear but time-varying, the derivation on the first page yields

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h_k[n]$$

where

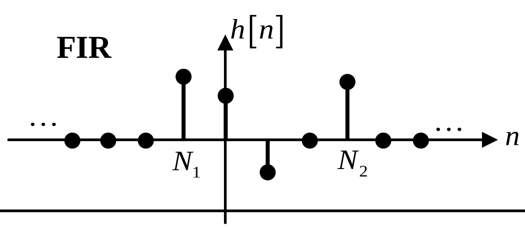
$$h_k[n] \text{ (time-varying impulse response)}$$

is the response of the system to an impulse at time k , i.e, $\delta[n-k]$.

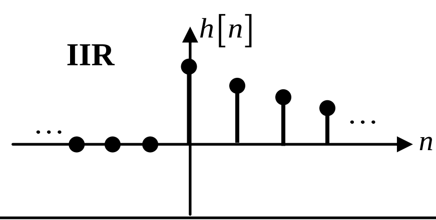
FINITE/INFINITE IMPULSE RESPONSE SYSTEMS

If $h[n]$ has a finite number of samples ($h[n]=0 \quad n<N_1 \quad n>N_2, \quad N_1<N_2$) then the system is said to be a *Finite Impulse Response* (FIR) system, otherwise an *Infinite Impulse Response* (IIR) one.

FIR



IIR



Properties of LTI Systems

Convolution is **commutative**

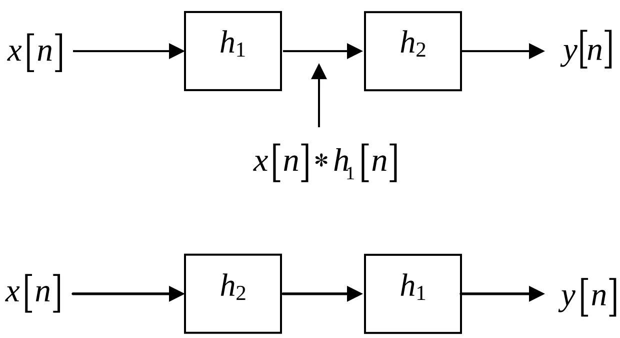
$$x[n]*h[n]=h[n]*x[n]$$

$$\sum_{k=-\infty}^{\infty} x[k]h[\underset{\substack{\uparrow \\ n-k=m}}{n-k}]=\sum_{m=-\infty}^{\infty} h[m]x[n-m]=h[n]*x[n]$$

Convolution is **associative**

$$\left(a[n]*b[n]\right)*c[n]=a[n]*\left(b[n]*c[n]\right)$$ Show! Exercise

Cascading LTI Systems



$$\begin{aligned} y[n] &= \left(x[n]*h_1[n]\right)*h_2[n] \\ &= x[n]*\left(h_1[n]*h_2[n]\right) \\ &= x[n]*\left(h_2[n]*h_1[n]\right) \\ &= \left(x[n]*h_2[n]\right)*h_1[n] \end{aligned}$$

→ If the systems are LTI, the order of cascade can be changed!

Ex: Let two systems be described by

$$y_1[n]=x[n]+x[n-1]+x[n-2]$$

and

$$\begin{aligned} y_2[n] &= \sum_{k=0}^{\infty} a^k x[n-k] \\ &= x[n]+ax[n-1]+a^2x[n-2]+\ldots \end{aligned}$$

They are LTI (check!), so their order is arbitrary in their cascade connection. (excluding possible practical concerns)

Show that their impulse responses are

$$\begin{aligned} h_1[n] &= u[n]-u[n-3] \\ &= \delta[n]+\delta[n-1]+\delta[n-2] \end{aligned}$$

$$h_2[n]=a^n u[n]$$

However,

Ex: Let two systems be described by

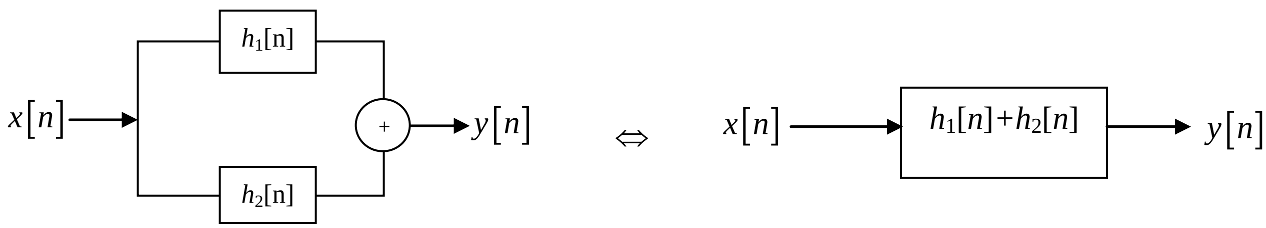
$$y_1[n] = x[n-2] \qquad \text{(impulse response is } h_1[n] = \delta[n-2]\text{)}$$

and

$$y_2[n] = x^2[n]. \qquad \text{(! impulse response is } h_2[n] = \delta^2[n] = \delta[n]\text{)}$$

One of them is nonlinear so, cannot be interchanged in their cascade.

Parallel LTI Systems



BIBO Stability

For an LTI system

$$\text{BIBO stability} \Leftrightarrow \sum_{n=-\infty}^{\infty} |h[n]| \le B_h \qquad \text{(impulse response is absolutely summable)}$$

<u>Proof:</u>
<u>Sufficiency</u>:

$$|y[n]| = \left| \sum_{k=-\infty}^{\infty} h[k]x[n-k] \right| \le \sum_{k=-\infty}^{\infty} |h[k]| \underbrace{|x[n-k]|}_{\le B_x} \le B_x \sum_{k=-\infty}^{\infty} |h[k]|$$

$$\text{so if } \sum_{k=-\infty}^{\infty} |h[k]| \le B_h \quad \text{then } |y[n]| \le B_h B_x = B_y$$

<u>Necessity</u>: (by contradiction)

Assume that the impulse response is not absolutely summable, i.e. $\sum_{n=-\infty}^{\infty} |h[n]| \to \infty$

Also let $x[n] = \begin{cases} \dfrac{h[-n]}{|h[-n]|} & h[-n] \ne 0 \\ 0 & h[-n] = 0 \end{cases}$ so that it is bounded.

Now consider $$y[0] = \sum_{k=-\infty}^{\infty} h[k]x[-k] = \sum_{k=-\infty}^{\infty} \frac{h[k]}{|h[k]|} h[k] = \sum |h[k]| \to \infty$$

Therefore FIR (LTI) systems are always stable!