LINEAR CONSTANT COEFFICIENT DIFFERENCE EQUATIONS

- LCCDEs CAN BE USED TO REPRESENT LTI SYSTEMS
- FORWARD/BACKWARD SOLVABILITY
- THE SOLUTION OF LCCDEs
  - Particular Solution, Homogeneous Solution: Complete Solution
- INITIAL REST ASSUMPTION AND LTI SYSTEMS
- LCCDE → IMPULSE RESPONSE
- RECURSIVE/NONRECURSIVE FORMS

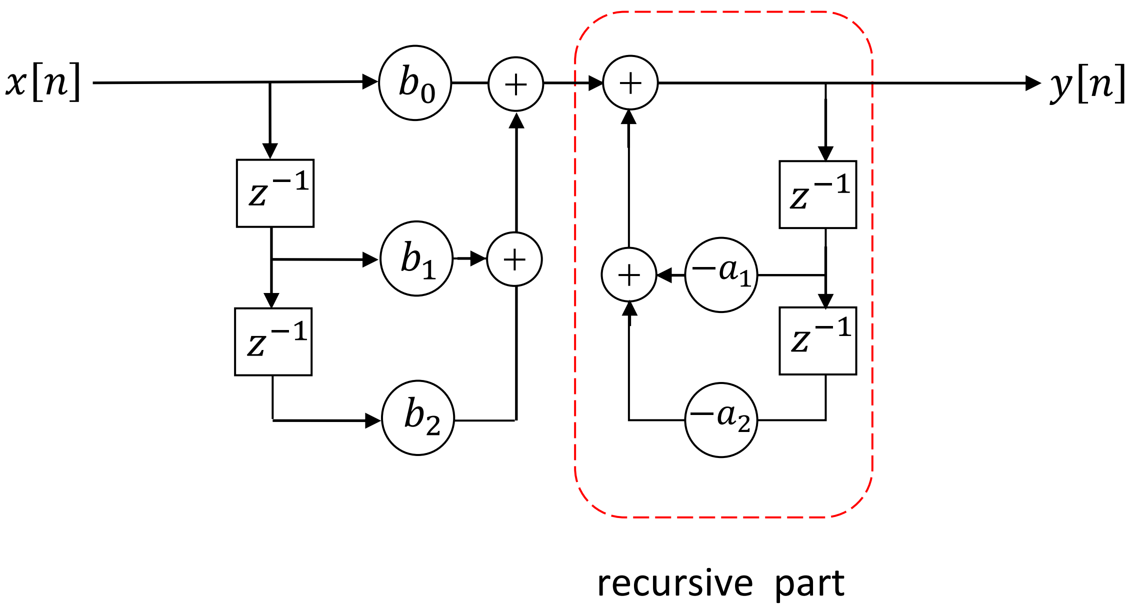
LCCDES CAN BE USED TO REPRESENT LTI SYSTEMS

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

$$N = M = 2$$

$$a_0 = 1$$

$$y[n] + a_1 y[n-1] + a_2 y[n-2] = b_0 x[n] + b_1 x[n-1] + b_2 x[n-2]$$



FORWARD/BACKWARD SOLVABILITY

Given N boundary conditions, a difference equation can be solved *recursively* in forward and backward "directions".

For example, let $y[-1], y[-2], \dots, y[-N]$ be specified.

Forward recursive solution:

$$y[n] = -\sum_{k=1}^{N} \frac{a_k}{a_0} y[n-k] + \sum_{k=0}^{M} \frac{b_k}{a_0} x[n-k] \qquad n = 0,1,\ldots$$

Backward recursive solution:

$$y[n-N] = -\sum_{k=0}^{N-1} \frac{a_k}{a_N} y[n-k] + \sum_{k=0}^{M} \frac{b_k}{a_N} x[n-k] \qquad\qquad n = -1, -2, \dots$$

Ex:

$$y[n] = -\frac{1}{2}y[n-1] + x[n]$$

$$y[-1] = c$$

$$x[n] = K\delta[n]$$

Forward:

$$y[0] = -\frac{1}{2}c + K$$

$$y[1] = \frac{1}{4}c - \frac{1}{2}K$$

$$\vdots$$

$$y[n] = \underbrace{\left(-\frac{1}{2}\right)^{n+1} c}_{\substack{homogeneous \\ solution}} + \underbrace{\left(-\frac{1}{2}\right)^{n} K}_{\substack{particular \\ solution}} \qquad n \geq 0$$

Backward:

$$y[n-1] = -2y[n] + 2x[n]$$

$$y[-2] = -2c$$

$$y[-3] = 4c$$

$$\vdots$$

$$y[n] = \left(-\frac{1}{2}\right)^{n+1} c \qquad n \leq -1$$

THE SOLUTION OF LCCDES

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

general solution = particular solution + homogeneous solution

$$y[n] = y_p[n] + y_h[n]$$

Particular solution:

Given a particular input $x_p[n]$, particular solution $y_p[n]$ is any solution that satisfies the equation for this input.

Homogeneous solution:

$y_h[n]$, satisfies

$$\sum_{k=0}^{N} a_k y[n-k] = 0$$

Homogeneous Solution

In general, $y_h[n]$ is a *weighted* sum of z^n type signals ($z \in \mathbb{C}$).

Therefore it has to satisfy the homogeneous equation:

$$\sum_{k=0}^{N} a_k z^{n-k} = 0$$

$$\sum_{k=0}^{N} a_k z^{-k} = 0$$

This equation has N roots, z_k, $k = 1, \dots, N$.

So,

$$y_h[n] = \sum_{k=1}^{N} A_k z_k^n = 0$$

where A_ks can be determined according to the initial (auxiliary, boundary) conditions.

Ex:

$$y[n]-\frac{1}{6}y[n-1]-\frac{1}{6}y[n-2]=x[n]$$

$$z^n-\frac{1}{6}z^{n-1}-\frac{1}{6}z^{n-3}=0$$

$$1-\frac{1}{6}z^{-1}-\frac{1}{6}z^{-3}=0$$

$$z^2-\frac{1}{6}z-\frac{1}{6}=0$$

$$z=\frac{1}{2},-\frac{1}{3}$$

$$y_h[n]=A_1\left(\frac{1}{2}\right)^n+A_2\left(-\frac{1}{3}\right)^n$$

Let the initial conditions be

$$y[0]=2,\qquad y[1]=-1$$

$$A_1+A_2=2$$

$$\frac{1}{2}A_1-\frac{1}{3}A_2=-1$$

$$\frac{1}{2}(2-A_2)-\frac{1}{3}A_2=-1$$

$$1-\frac{1}{2}A_2-\frac{1}{3}A_2=-1$$

$$-\frac{5}{6}A_2=-2$$

$$A_2=\frac{12}{5}$$

$$A_1=2-\frac{12}{5}=-\frac{2}{5}$$

INITIAL REST ASSUMPTION AND LTI SYSTEMS

When a LCCDE is considered together with “initial rest” (zero initial conditions) assumption, the input-output ($x[n] - y[n]$) relationship becomes a *linear* and *time-invariant* one.

For nonzero initial conditions

1) Even if the input is zero, the output is nonzero → input-output relationship is nonlinear

2) If the input is shifted by n_0, the output is not shifted by the same amount → system is time-varying.

   For example, in the above example the forward solution for a shifted input

$$\hat{x}[n] = K\delta[n - n_0]$$

   is

$$y[n] = \left(-\frac{1}{2}\right)^{n+1} c + \left(-\frac{1}{2}\right)^{n-n_0} Ku[n - n_0] \qquad n \geq 0$$

Therefore

“A sytem described by a LCCDE is a LTI one if it is initially at rest, i.e. initial conditions are zero.”

Note: When $N = 0$, a FIR is described.

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

If $N \; = \; 0$,

$$\Rightarrow \quad y[n] = \sum_{k=0}^{M} \frac{b_k}{a_0} x[n-k]$$

no initial conditions are needed to solve.

Impulse response is

$$h[n] = \sum_{k=0}^{M} \frac{b_k}{a_0} \delta[n-k].$$

This is a FIR sytem.

Causality

Given a system described by a LCCDE together with initial rest assumption.

One cannot determine whether the system is causal or not. Causality must be specified separately.

Therefore we have statements like “ *a causal/noncausal system described by the following LCCDE…*”

FINDING THE IMPULSE RESPONSE FROM LCCDE

Problem Statement: Given the LCCDE describing a *causal LTI* system, find its impulse response.

In finding the impulse response, initial conditions are taken as zero!

Ex: Consider a causal system described by

$$y[n] - ay[n-1] = x[n]$$

Let's find the impulse response of this system.

We take $\quad x[n] = \delta[n]$

$$\Rightarrow \qquad h[n] - ah[n-1] = \delta[n]$$

together with

$$h[-1] = 0.$$

i) By recursion

$$\begin{aligned} h[0] &= ah[-1] + \delta[0] \\ &= 1 \end{aligned}$$

$$\begin{aligned} h[1] &= ah[0] + \delta[1] \\ &= a \\ &\vdots \\ h[n] &= ah[n-1] + \delta[n] \\ &= a^n \end{aligned}$$

$$\Rightarrow h[n] = a^n u[n]$$

ii) By finding the homogeneous solution

$$x[n] = \delta[n] \quad \Rightarrow \quad y[n] - ay[n-1] = 0, \qquad n > 0$$

$$y[n] = Kz^n\,, \qquad n > 0$$

$$Kz^n - aKz^{n-1} = 0$$
$$1 - az^{-1} = 0$$
$$z = a$$
$$y[n] = Ka^n, \qquad n > 0$$

Since $h[0] = 1 \quad \Rightarrow \quad Ka^0 = K = 1$

$$\Rightarrow h[n] = a^n \qquad n > 0$$

Ex:

$$y[n] - ay[n-1] = x[n-1]$$

We can easily find the impulse response by using the result of the previous example and time-invariance of the system.

$$\Rightarrow \quad h[n] = a^{n-1}u[n-1]$$

Ex:

$$y[n] - ay[n-1] = x[n] + x[n-1]$$

Using the two results above and linearity of the system:

$$\Rightarrow \quad h[n] = a^n u[n] + a^{n-1} u[n-1]$$

$$= \delta[n] + (1+a) a^{n-1} u[n-1]$$

Ex: Causal system, homogeneous solution (repeated roots)

$$y[n]-\frac{1}{2}y[n-1]-\frac{1}{4}y[n-2]-\frac{1}{4}y[n-3]=x[n]$$

$$y_h[n]-\frac{1}{2}y_h[n-1]-\frac{1}{4}y_h[n-2]-\frac{1}{4}y_h[n-3]=0$$

$$Kz^n\left(1-\frac{1}{2}z^{-1}-\frac{1}{4}z^{-2}-\frac{1}{4}z^{-3}\right)=0$$

$$z_1=\frac{1}{2},\qquad z_2=z_3=\frac{1}{4}$$

$$y_h[n]=K_1\left(\frac{1}{2}\right)^n u[n]+K_2\left(\frac{1}{4}\right)^n u[n]+K_3\,n\left(\frac{1}{4}\right)^n u[n]$$

Ex: Impulse response, previous example.

$$h[n] = K_1 \frac{1}{2}^n u[n] + K_2 \frac{1}{4}^n u[n] + K_3 n \frac{1}{4}^n u[n]$$

Find $h[0], h[1], h[2]$, then find K_1, K_2, K_3.

From

$$h[n] - \frac{1}{2} h[n-1] - \frac{1}{4} h[n-2] = \delta[n]$$

and

$$h[-1] = h[-2] = 0$$

$$\Rightarrow \quad h[0] = 1 \qquad h[1] = \frac{1}{2} \qquad h[2] = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

$$\Rightarrow$$

$$\begin{aligned} h[0] &= K_1 + K_2 + K_3 = 1 \\ h[1] &= \frac{1}{2} K_1 + \frac{1}{4} K_2 + \frac{1}{4} K_3 = \frac{1}{2} \\ h[2] &= \frac{1}{4} K_1 + \frac{1}{16} K_2 + \frac{1}{8} K_3 = \frac{1}{2} \end{aligned}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{16} & \frac{1}{8} \end{bmatrix} \begin{bmatrix} K_1 \\ K_2 \\ K_3 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

$$\begin{bmatrix} K_1 \\ K_2 \\ K_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{16} & \frac{1}{8} \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \\ 4 \end{bmatrix}$$

Exercise: Find the impulse response of the causal LTI system described by

$$y[n-1] - 2y[n-2] = x[n-2]$$

Is it a stable system?

Ex: Impulse response of the noncausal first order system.

initial condition: $h[1] = 0$

$$\underbrace{h[1]}_{=0} - ah[0] = \delta[1]$$

$$\Rightarrow \quad h[0] = 0$$

$$h[0] - ah[-1] = \delta[0]$$

$$\Rightarrow \quad h[-1] = -\frac{1}{a}$$

$$Ka^{-1} = -\frac{1}{a}$$

$$\Rightarrow \quad K = -1$$

$$h[n] = -a^n u[-n-1]$$

Exercise:

a) Write the difference equation that describes the LTI system with impulse response $h[n] = \left(\frac{2}{3}\right)^n u[n]$.

b) Repeat part-a for $h[n] = \left(\frac{2}{3}\right)^{n-1} u[n-1]$

c) Repeat part-a for $h[n] = -\left(\frac{2}{3}\right)^n u[-n-1]$

lccde.m

```
clear all
close all

% N = 2;
% a = [1 -1.5 -1];
N = 1;
a = [1 -0.5];

% M = 1;
% b = [1 -2];
M = 0;
b = [1];

y = zeros(1,max(N,M)); % initial conditions
x = zeros(1,1000);
x(max(N,M)+1)=1;  % to find impulse response
% x = 2*(rand(1,1000)-0.5);

for n = max(N,M)+1:30
    D = (-y(n-N:n-1)*transpose(fliplr(a(2:end))) + x(n-
M:n)*transpose(fliplr(b))) / a(1);
    y = [y D];
end
stem(y)



1./roots(fliplr(a))
1./roots(fliplr(b))
```

Ex: Accumulator

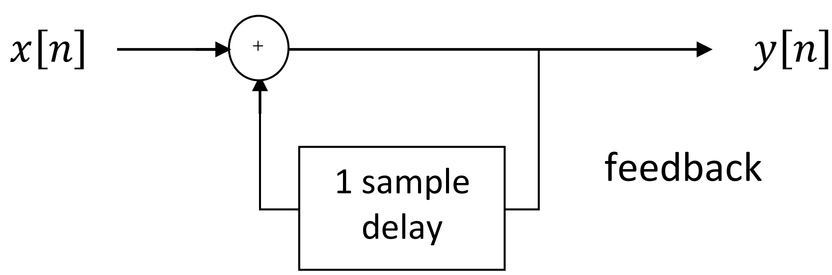
$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

It can be represented as

$$y[n] = y[n-1] + x[n]$$

or

$$y[n] - y[n-1] = x[n]$$



Impulse response of accumulator is

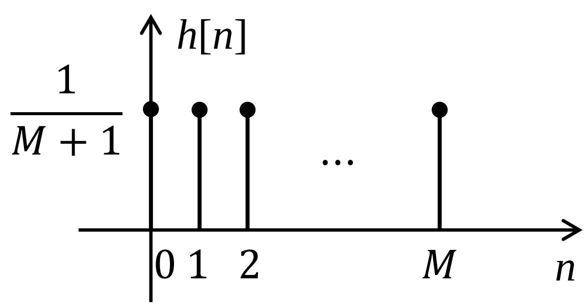
$$h[n] = u[n]$$

Ex: Moving Average (MA) system

$$y[n] = \frac{1}{M+1}\sum_{k=0}^{M} x[n-k]$$

Impulse response

$$h[n] = \frac{1}{M+1}\left(u[n] - u[n-(M+1)\,]\right)$$

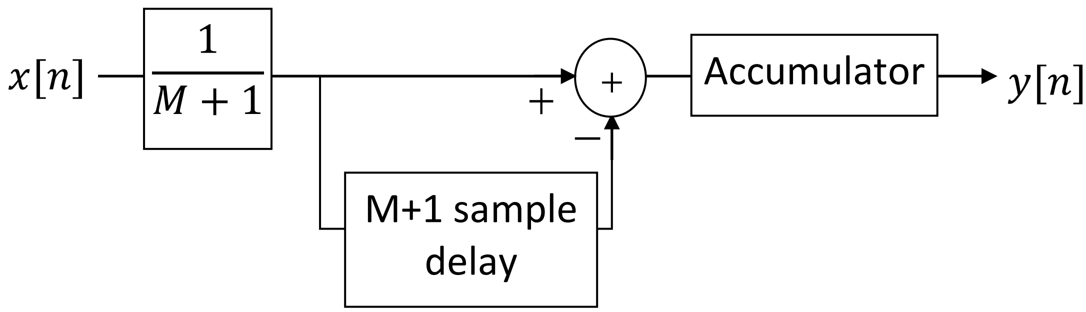


Impulse response of MA system can *also* be written as

$$h[n] = \frac{1}{M+1}(\delta[n] - \delta[n-(M+1)]) * u[n]$$

This expression reminds us that MA system can be considered as the cascade of an *accumulator* and another system with impulse response

$$h[n] = \frac{1}{M+1}(\delta[n] - \delta[n-(M+1)])$$



Therefore MA system can also be described by

$$y[n] - y[n-1] = \frac{1}{M+1}(x[n] - x[n-(M+1)])$$

"Less arithmetic operations in the implementation but recursion may cause numerical problems in finite precision."

```matlab
clear all
close all
% generate input
M1 = -30;
M2 = -21;
M = M2-M1+1;  % length of MA
w = 2*pi/7;
N = 300; % length of input=N+2M+1
n = 0:N;
x = [zeros(1,abs(2*M1)) sin(w*n)];
plot(x)
hold on
b = ones(1,M)/M; % impulse response of MA
y = conv(b,x);
yy = [y zeros(1,3*M)]
yy = circshift(yy,[0,M1]);
plot(yy,'r')
legend('input','output')
xlabel('n, sample no')
ylabel('amplitude, unitless')
title('response of a MA system to a sinusoid')
```