Lecture Notes

EE430 Digital Signal Processing

Department of Electrical and Electronics Engineering

Middle East Technical University (METU)

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Chapter 1

Discrete-time Signals and Systems

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This chapter covers the fundamental concepts of discrete-time (DT) signals and systems. In particular, we cover Sections 2.1 to 2.9 from our textbook.

Reading assignment for this chapter :

- Sections 2.1 to 2.9 from our textbook.

1.1 Discrete-time (DT) signals

A DT signal is simply a sequence of numbers indexed by integer n.

Our notation to show a DT signal is :

A DT signal can be obtained from

- an inherently discrete event (e.g. number of students attending lecture n)
- sampling of a continuous-time (CT) signal :

1.1.1 Basic sequences and sequence operations

Unit sample sequence

Any sequence $x[n]$ can be written in terms of delayed and scaled $\delta[n]$.

For an arbitrary $x[n]$, we have :

Unit step sequence

Relations between unit step and sample sequences:

Exponential sequences

General form :

If A and α are real, $x[n]$ is real.

- $A > 0$, $0 < \alpha < 1$: $x[n]$ decreases in time
- $A > 0$, $-1 < \alpha < 0$: $x[n]$ increases in time with alternating sign
- $|\alpha| > 1$: $x[n]$ grows in magnitude as n increases

If A and α are complex :

Complex exponentials

$$x[n] =$$

Properties :

1. Complex exponentials $Ae^{j(\omega_0+2\pi r)n}$ with frequencies $(\omega_0+2\pi r)$, $r \in \mathbb{Z}$ (e.g. ω_0, $\omega_0+2\pi$, $\omega_0+4\pi$, ...) are equivalent to each other:

2. Based on above property, when discussing complex exponentials $Ae^{j\omega_0 n}$ (or sinusoids $\cos(\omega_0 n+\phi)$), we only need to consider and interval of length 2π for frequency ω_0:

3. Complex exponentials $Ae^{j\omega_0 n}$ (or sinusoids $\cos(\omega_0 n + \phi)$) are <u>periodic only if</u> $\frac{2\pi}{\omega_0}$ is a ratio of integers, i.e.

   Remember periodicity requirement for any sequence $x[n]$:

   Ex: Are following sequences periodic ? If so, find the periods.

   $x_1[n] = \cos(n)$ $\quad$ $x_2[n] = \cos(\frac{2\pi}{8}n)$ $\quad$ $x_3[n] = \cos(\frac{3\pi}{8}n + \phi)$

4. (Prop.1 + Prop. 3) There are only N distinguishable frequencies for which the complex exponentials $Ae^{j\omega_0 n}$ (or sinusoids $\cos(\omega_0 n + \phi)$) are periodic with N:

5. For complex exponentials $Ae^{j\omega_0 n}$ (or sinusoids $\cos(\omega_0 n + \phi)$),
   - low frequencies are in the vicinity of $\omega_0 =$
   - high frequencies are in the vicinity of $\omega_0 =$

   Rate of oscillation of complex exp. (or sinusoid) determines whether frequency is high or low:

<u>Note :</u> For CT complex exponential $x(t) = Ae^{j\phi_0 t}$, <u>none</u> of the above 5 properties hold :

1.
2.
3.
4.
5.

Transformation of independent variable n

Time shift:

Time reversal:

Note : First time shift, then time reversal $\neq$ first time reversal, then time shift

1.2 Discrete-time (DT) systems

Notation:

1.2.1 Memoryless systems:

Output $y[n]$ does not depend on past or future values of input $x[n]$.
Ex:

1.2.2 Linear systems:

The systems satisfies the following relation for any $a, b, x_1[n], x_2[n]$:

In a linear system, if input

Ex: Are these systems linear ?

1.2.3 Time-invariant systems:

Any time shift at the input causes a time shift at the output by the same amount.

Ex: Are these systems time-invariant ? (Accumulator)

Ex: Are these systems time-invariant ? (Compressor)

1.2.4 Causality:

Current output sample $y[n]$ depends only on current and past input samples $x[n], x[n-1], x[n-2], ...$
Ex:

1.2.5 Stability:

A system is stable if and only if (iff) every bounded input (i.e.) produces a bounded output (i.e.).
Ex:

1.3 Linear time-invariant (LTI) systems

LTI systems have both linearity and time-invariance properties.
LTI systems are a very important class of systems.
The output of a LTI system to an arbitrary input can be calculated by the famous convolution sum:

Hence, for any input $x[n]$ to an LTI system, output

An LTI system is completely characterized by its impulse response $h[n] = T\{\delta[n]\}$.

1.3.1 Computation of convolution sum:

Ex: $x[n] = \delta[n+2] + 2\delta[n] - \delta[n-3]$ is input to LTI system with impulse response $h[n] = 3\delta[n] + 2\delta[n-1] + \delta[n-2]$. Find output $y[n]$ using two methods.

Echo method : Add outputs to each weighted and delayed delta function in the input. (Useful when input has few samples.)

Sliding average method : Apply definition of convolution sum.

Ex: Impulse response of LTI system is $h[n] = u[n] - u[n-N]$ and input $x[n] = a^n u[n], \quad 0 < a < 1$. Find output $y[n]$.

1.4 Properties of convolution and LTI systems

1.4.1 Properties of convolution

- **Distribution** over addition:

- **Commutative** property:

- **Associative** property:

Figure 1.1:
(Figure 2.11 in textbook) (a) Parallel combination of LTI systems (b) an equivalent system.

Figure 1.2:
(Figure 2.12 in textbook) (a) Cascade combination of LTI systems (b) equivalent cascade system (c) single equivalent system.

1.4.2 Properties of LTI systems

- **Impulse response property:** An LTI system is completely characterized/specified/determined by its impulse response $h[n]$.
  $\Longrightarrow$

- **Memory property:** LTI system is memoriless $\Longleftrightarrow$

- **Causality property:** LTI system is causal $\Longleftrightarrow$

- **Stability property:** LTI system is stable $\Longleftrightarrow$
  Proof given in two steps.
  Step-1 : Sufficiency, i.e. if $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$, then LTI system is stable.

  Step-2 : Necessity, i.e. for LTI system to be stable, we must have $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$.

- **Invertibility property:** LTI system ($h[n]$) is invertible $\Longleftrightarrow$ There is another LTI system ($g[n]$) such that $h[n] * g[n] = \delta[n]$.

1.4.3 FIR and IIR systems

FIR : **F**inite (-duration) **I**mpulse **R**esponse ($h[n]$ has finite number of nonzero samples)
Ex:

IIR : **I**nfinite (-duration) **I**mpulse **R**esponse ($h[n]$ has infinite number of nonzero samples)
Ex:

1.5 Linear constant-coefficient difference equations (LCCDE)

An important subclass of LTI systems exist, where the input $x[n]$ and output $y[n]$ satisfy an LCCDE.

Ex: Accumulator :

Ex: LTI system with impulse response $h[n] = \frac{1}{n^2}u[n-1]$. System is LTI, but cannot be represented with and LCCDE.

Initial rest conditions (IRC)

An LCCDE alone does not uniquely specify a system. (i.e. there may be multiple systems satisfying the same LCCDE)

- Auxiliary conditions are required together with LCCDE to uniquely specify the system.
- Some auxiliary conditions may result in non-LTI system.
- The so-called **"initial rest" auxiliary conditions** lead to a unique LTI and causal system.
  Initial rest conditions (IRC) :

In this course, we are mostly interested in finding impulse response $h[n]$ of LTI systems (and not necessarily the output $y[n]$ for an arbitrary $x[n]$, because we can then use convolution..).
$\Longrightarrow$

We can find the desired $h[n]$ in two steps :

1. Find **homogeneous solution** from

2. Find **initial conditions** for the homogeneous solution using

Ex: System satisfies : $y[n] - 3y[n-1] - 4y[n-2] = x[n] + 2x[n]$. Find $h[n]$ under IRC.

Recursive calculation from LCCDE

The **recursive nature of LCCDE** are very powerful/useful and can also be used to calculate the output $y[n]$ recursively, under many auxiliary conditions.
Ex: (Same LCCDE) $y[n] - 3y[n-1] - 4y[n-2] = x[n] + 2x[n]$

LCCDE of FIR and IIR systems

<u>If $N = 0$</u>, in the LCCDE equation, we have

- no recursion and thus no initial/auxiliary conditions are required to compute output

- actually, the LCCDE is in the form of a convolution where

If $N > 1$, in the LCCDE equation, we have

- recursion is required to compute output
- if IRC used, system is LTI and causal and IIR due to recursion

Transform domain approaches

Transform domain approaches are best/useful for LCCDE describing LTI systems.
Ex: LCCDE : $y[n] - 3y[n-1] - 4y[n-2] = x[n] + 2x[n]$

1.6 Frequency domain representation of DT signals and systems

Consider an LTI system with impulse response $h[n]$ and input $x[n]$. The output $y[n]$ is

If $x[n] = e^{j\omega n}$ for $-\infty < n < \infty$ (i.e. complex exponential with frequency ω)

$\Longrightarrow$
- $e^{j\omega n}$ is the **eigenfunction** for all LTI systems.
- The corresponding **eigenvalue** is $H(e^{j\omega})$, also called the **frequency response** of the system.

Ex: What is the frequency response of and ideal delay system ?

It will be shown that a broad class of signals can be represented by a sum of complex exponentials

Hence, for an LTI system, the output can be easily calculated

Note that the frequency response is periodic with 2π:

Therefore, frequency response can be defined only over a range of 2π :

Note that the signals $e^{j\omega n}$ and $e^{j(\omega+2\pi)n}$ are equal and hence the system cannot differentiate between these eigenfunctions.

Ex: Input to an LTI systems is $x[n] = A\cos(\omega_0 n + \phi)$. Find output in terms of $H(e^{j\omega})$.

An important class of LTI systems, called **frequency selective filters**, have frequency response $H(e^{j\omega})$ that is unity (i.e. 1) over a range of frequencies and 0 for the remaining frequencies.

Figure 1.3:
(Figure 2.17 in textbook) Ideal lowpass filter showing (a) periodicity of frequency response and (b) one period of frequncy response.

Figure 1.4:
(Figure 2.18 in textbook) Ideal frequency selective filters (a) Highpass filter (b) Bandstop filter (c) Bandpass filter.

Ex: Moving average system: $y[n] = \frac{1}{M_1+M_2+1}\sum_{k=-M_1}^{M2} x[n-k]$. LTI ? If so, find $h[n]$ and $H(e^{j\omega})$.

Suddenly applied complex exponential inputs

This subsection (Sec. 2.6.2 in textbook) is a reading assignment. It discusses LTI system when inputs are of the form $x[n] = e^{j\omega_0 n}u[n]$ instead of $x[n] = e^{j\omega_0 n}$.

1.7 Representation of sequences by Fourier transforms

1.8 Symmetry properties of DT Fourier transforms

1.9 DT Fourier transform theorems

1.10 Computing Linear Convolution using DFT

1.10.1 Linear Convolution of Two Finite Length Sequences

Figure 17

1.10.2 Circular Convolution as Linear Convolution with Aliasing

*****Remember Sect.8.4 Sampling the DTFT

1. For L-pt $x[n]$:
2. Sample DTFT $X(e^{jw})$ at $w_K = k\frac{2\pi}{N}$,$k \in Z$:
3. $\widetilde{X}[k]$ can be used as **DFS** coeffs:

******Consider similar discussion with DFT instead DFS:

1. For L-pt $x[n]$:
2. Sample DTFT $X(e^{jw})$ in $[0, 2\pi)$:
3. $X[k]$ can be viewed as **DFT** coeffs of an N-pt seq. $x_p[n]$:

*** Now consider similar discussion for $x_3[n] = x_1[n] * x_2[n]$, where $x_1[n]$ is $L - point$, $x_2[n]$ is $P - point$ and $x_3[n]$ is $(L + P - 1) - point$ seqs.

1. For $x_3[n] = x_1[n] * x_2[n]$:
2. Sample $X_3(e^{jw})$ in $[0, 2\pi)$:

3. $X_3[k]$ can be viewed as **DFT** coeffs of an N-pt seq. $x_{3p}[n]$:

4. But since $X_3[k] = X_1[k].X_2[k]$,from circular convolution theorem, we have: