EE 430

Section 2
HW 2
1 - 9

1) a) **First Method**

Since the system is causal, $h[n]=0$ for $n<0$.
Find $h[n]$ in the forward direction ($n \nearrow$)

$$h[n] - \frac{1}{2} h[n-1] = \delta[n] - \delta[n-1] + \delta[n-2]$$

Seperate $h[n] = h_1[n] + h_2[n] + h_3[n]$ where

$$h_1[n] - \frac{1}{2} h_1[n-1] = \delta[n]$$

$$h_2[n] - \frac{1}{2} h_2[n-1] = -\delta[n-1]$$

$$h_3[n] - \frac{1}{2} h_3[n-1] = \delta[n-2]$$

$h_1[n]$

$$h_1[0] = \frac{1}{2} h_1[-1] + \delta[0] = 1$$

$$h_1[1] = \frac{1}{2} h_1[0] + \delta[1] = \frac{1}{2}$$

$\vdots$

$$h_1[n] = \left(\frac{1}{2}\right)^n u[n]$$

$h_2[n]$

$$h_2[0] = \frac{1}{2} h_2[-1] - \delta[-1] = 0$$

$$h_2[1] = \frac{1}{2} h_2[0] - \delta[0] = -1$$

$$h_2[2] = \frac{1}{2} h_2[1] - \delta[1] = -\frac{1}{2}$$

$\vdots$

$$h_2[n] = -\left(\frac{1}{2}\right)^{n-1} u[n-1]$$

$h_3[n]$

$h_3[0] = \frac{1}{2} h_3[-1] + \delta[-2] = 0$

$h_3[1] = \frac{1}{2} h_3[0] + \delta[-1] = 0$

$h_3[2] = \frac{1}{2} h_3[1] + \delta[0] = 1$

$h_3[3] = \frac{1}{2} h_3[2] + \delta[1] = \frac{1}{2}$

$\vdots$

$h_3[n] = \left(\frac{1}{2}\right)^{n-2} u[n-2]$

$$h[n] = \left(\frac{1}{2}\right)^{n} u[n] - \left(\frac{1}{2}\right)^{n-1} u[n-1] + \left(\frac{1}{2}\right)^{n-2} u[n-2]$$

Second Method

$$Y(z) - \frac{1}{2} z^{-1} Y(z) = X(z)\left[1 - z^{-1} + z^{-2}\right]$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 - z^{-1} + z^{-2}}{1 - \frac{1}{2} z^{-1}} \qquad |z| > \frac{1}{2} \text{ (causal)}$$

$$h[n] = \left(\frac{1}{2}\right)^{n} u[n] - \left(\frac{1}{2}\right)^{n-1} u[n-1] + \left(\frac{1}{2}\right)^{n-2} u[n-2]$$

b) $H(z)$ exists at $|z| = 1$

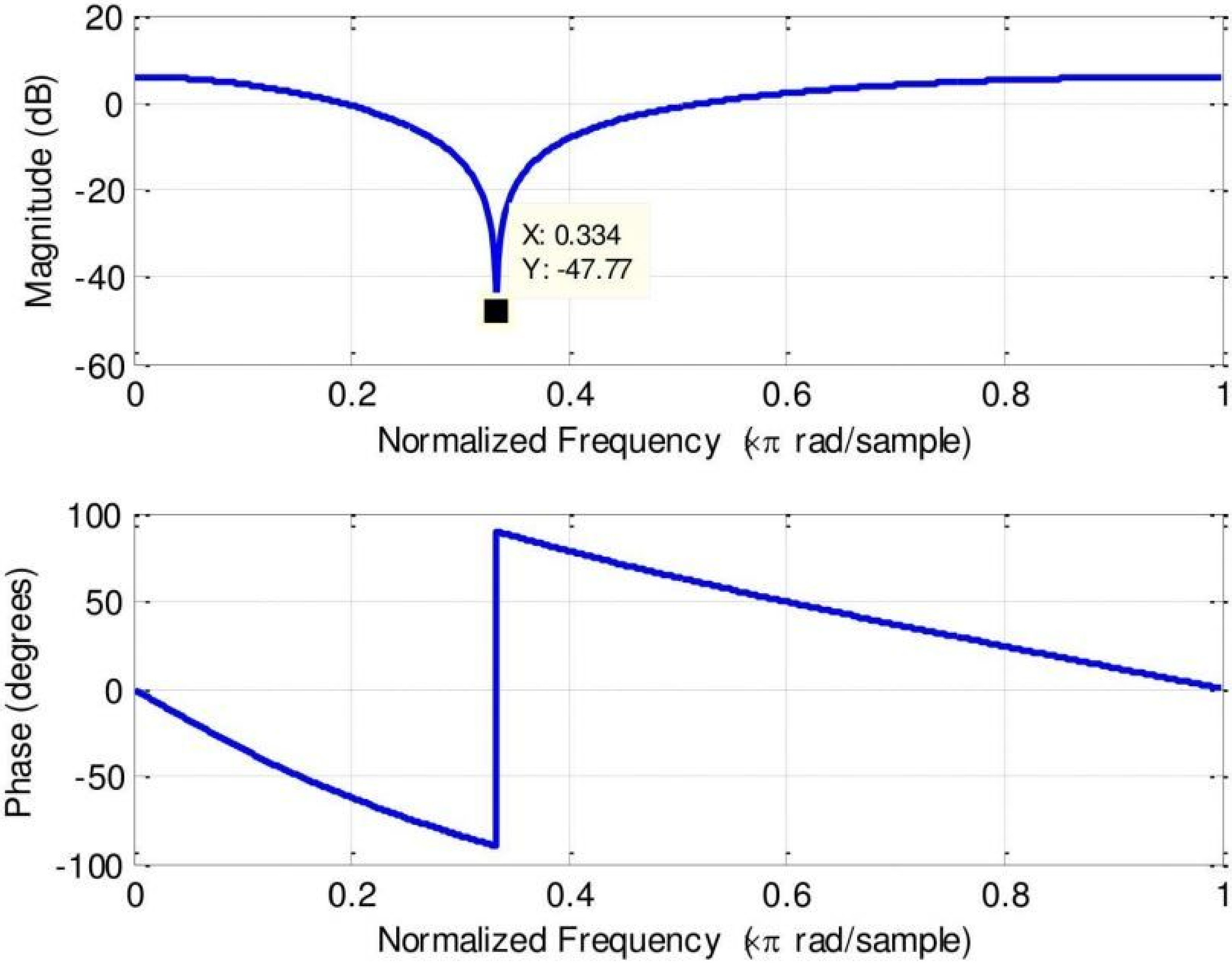
$$H(e^{jw}) = H(z)\Big|_{z=e^{jw}} = \frac{1 - e^{-jw} + e^{-j2w}}{1 - \frac{1}{2} e^{-jw}}$$

c) Use MATLAB command freqz (b,a) where b & a defines the frequency response of the system

$$H(e^{jw}) = \frac{b(1) + b(2)e^{-jw} + b(3)e^{-j2w}}{a(1) + a(2)e^{-jw}}$$

```
b= [1  -1  1];
a= [1 -0.5];
```

1.c)



20
0
-20
-40
-60
Magnitude (dB)
X: 0.334
Y: -47.77
0
0.2
0.4
0.6
0.8
1
Normalized Frequency (×π rad/sample)
100
50
0
-50
-100
Phase (degrees)
0
0.2
0.4
0.6
0.8
1
Normalized Frequency (×π rad/sample)

d) The frequency response of the system has a null at $w = \frac{\pi}{3}$ rad/sample, hence only the sine signal will appear at the output:

$$y[n] = \underbrace{\left|H(e^{j\frac{\pi}{3}})\right|}_{0} \cos\left(\frac{\pi}{3}n + \measuredangle H(e^{j\frac{\pi}{3}})\right)$$

$$+ \left|H(e^{j\frac{\pi}{2}})\right| \sin\left(\frac{\pi}{2}n + \frac{\pi}{4} + \measuredangle H(e^{j\frac{\pi}{2}})\right)$$

$$H(e^{j\frac{\pi}{3}}) = \frac{1 - \cos\left(\frac{\pi}{3}\right) + \cancel{j\sin\left(\frac{\pi}{3}\right)} + \cos\left(\frac{2\pi}{3}\right) - \cancel{j\sin\left(\frac{2\pi}{3}\right)}}{1 - \frac{1}{2}\cos\left(\frac{\pi}{3}\right) + \frac{1}{2}j\sin\left(\frac{\pi}{3}\right)}$$

$$= \frac{1 - \frac{1}{2} - \frac{1}{2}}{1 - \frac{1}{4} + j\frac{\sqrt{3}}{4}} = 0$$

$$H(e^{j\frac{\pi}{2}}) = \frac{1 + j - 1}{1 + \frac{1}{2}j} = \frac{j}{1 + \frac{1}{2}j} \bigg/ {1 - \frac{1}{2}j} = \frac{\frac{1}{2} + j}{\frac{5}{4}}$$

$$\left|H(e^{j\frac{\pi}{2}})\right| = \frac{\sqrt{\frac{1}{4} + 1}}{\frac{5}{4}} = \frac{2}{\sqrt{5}} \cong 0.8944$$

$$\measuredangle H(e^{j\frac{\pi}{2}}) \cong 0.3524\pi$$

$$\Rightarrow y[n] \cong 0.8944 \sin\left(\frac{\pi}{2}n + 0.6024\pi\right)$$

e) $$H^*(e^{j(2\pi - w)}) = \left(\frac{1 - e^{-j(2\pi - w)} + e^{-j2(2\pi - w)}}{1 - \frac{1}{2}e^{-j(2\pi - w)}}\right)^*$$

$$= \left(\frac{1 - e^{jw} + e^{j2w}}{1 - \frac{1}{2}e^{jw}}\right)^* = \frac{1 - e^{-jw} + e^{-j2w}}{1 - \frac{1}{2}e^{-jw}} = H(e^{jw}) \checkmark$$

Generally,

$$H(e^{jw}) = \sum_{n=-\infty}^{\infty} h[n] e^{-jwn}$$

$$H^*(e^{j(2\pi-w)}) = \left[\sum_{n=-\infty}^{\infty} h[n] \underbrace{e^{-j(2\pi-w)n}}_{=e^{jwn}} \right]^* = \sum_{n=-\infty}^{\infty} h^*[n] e^{-jwn}$$

The equality $H(e^{jw}) = H^*(e^{j(2\pi-w)})$ does not hold for an arbitrary $h[n]$. It is valid if $h[n]$ is real as in ar example, i.e., $h[n] = h^*[n]$.

(2) $\mathcal{F}\{a^n u[n]\} = \dfrac{1}{1-ae^{-j\omega}}, \quad |a|<1$

$\Downarrow$

$$\mathcal{F}\{a^{n-2}u[n-2]\} = \frac{e^{-j2\omega}}{1-ae^{-j\omega}}$$

Knowing that $nx[n] \longleftrightarrow j\dfrac{dX(e^{j\omega})}{d\omega}$

$$\mathcal{F}\{na^{n-2}u[n-2]\} = j\,\frac{-j2e^{-j2\omega}(1-ae^{-j\omega}) - aje^{-j\omega}e^{-j2\omega}}{(1-ae^{-j\omega})^2}$$

$$= \frac{2e^{-j2\omega} - 2ae^{-j3\omega} + ae^{-j3\omega}}{(1-ae^{-j\omega})^2}$$

$$= \frac{2e^{-j2\omega} - ae^{-j3\omega}}{(1-ae^{-j\omega})^2}$$

(3) a) i) From the eigenfunction property of complex exponentials,

$$y_1[n] = \left|H(e^{j\frac{\pi}{7}})\right| \sin\left(\frac{\pi}{7}n + \measuredangle H(e^{j\frac{\pi}{7}})\right)$$
$$+ \left|H(e^{j\frac{\pi}{3}})\right| \sin\left(\frac{\pi}{3}n + \measuredangle H(e^{j\frac{\pi}{3}})\right)$$

$$H(e^{jw}) = \frac{1}{2} + \frac{1}{2}e^{-jw}$$

$$H(e^{j\frac{\pi}{7}}) = \frac{1}{2} + \frac{1}{2}e^{-j\frac{\pi}{7}} \cong 0.9749\, e^{-j\,0.0714\pi}$$

$$H(e^{j\frac{\pi}{3}}) = \frac{1}{2} + \frac{1}{2}e^{-j\frac{\pi}{3}} \cong 0.8660\, e^{-j\,0.1667\pi}$$

$$y_1[n] \cong 0.9749 \sin\left(\frac{\pi}{7}n - 0.0714\pi\right)$$
$$+ 0.8660 \sin\left(\frac{\pi}{3}n - 0.1167\pi\right)$$

ii) Consider $x[n] = e^{jw_0 n}u[n]$,

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k] = \sum_{k=0}^{n} h[k]\, e^{jw_0(n-k)}$$

$\hookrightarrow$ h[n] is causal

$$y[n] = \begin{cases} 0 & \text{for } n < 0 \\ \left(\sum_{k=0}^{n} h[k] e^{-jw_0 k}\right) e^{jw_0 n} & n \geq 0 \end{cases}$$

For $n \geq 0$

$$y[n] = \underbrace{\left(\sum_{k=0}^{\infty} h[k]\, e^{-jw_0 k}\right)}_{H(e^{jw_0})} e^{jw_0 n} - \left(\sum_{k=n+1}^{\infty} h[k]\, e^{-jw_0 k}\right) e^{jw_0 n}$$

$$= H(e^{jw_0})\, e^{jw_0 n} - \left(\sum_{k=n+1}^{\infty} h[k]\, e^{-jw_0 k}\right) e^{jw_0 n}$$

$$x_2[n] = x_1[n]u[n] = \left(\frac{1}{2j}e^{j\frac{\pi}{7}n} - \frac{1}{2j}e^{-j\frac{\pi}{7}n} + \frac{1}{2j}e^{j\frac{\pi}{3}n} - \frac{1}{2j}e^{-j\frac{\pi}{3}n}\right)u[n]$$

$$\left.\begin{aligned} y_2[n] = {} & \frac{1}{2j}H(e^{j\frac{\pi}{7}})e^{j\frac{\pi}{7}n} - \frac{1}{2j}H(e^{-j\frac{\pi}{7}})e^{-j\frac{\pi}{7}n} \\ & + \frac{1}{2j}H(e^{j\frac{\pi}{3}})e^{j\frac{\pi}{3}n} - \frac{1}{2j}H(e^{-j\frac{\pi}{3}})e^{-j\frac{\pi}{3}n} \end{aligned}\right\} \text{same as in the first part}$$

$$-\frac{1}{2j}\left(\sum_{k=n+1}^{\infty} h[k]e^{-j\frac{\pi}{7}k}\right)e^{j\frac{\pi}{7}n} + \frac{1}{2j}\left(\sum_{k=n+1}^{\infty} h[k]e^{j\frac{\pi}{7}k}\right)e^{-j\frac{\pi}{7}n}$$

$$-\frac{1}{2j}\left(\sum_{k=n+1}^{\infty} h[k]e^{-j\frac{\pi}{3}k}\right)e^{j\frac{\pi}{3}n} + \frac{1}{2j}\left(\sum_{k=n+1}^{\infty} h[k]e^{j\frac{\pi}{3}k}\right)e^{-j\frac{\pi}{3}n}$$

If $n+1 > \underset{\substack{\downarrow \\ h[k]=0 \text{ for } k>1}}{1} \Rightarrow y_2[n] = y_1[n] \cong 0.9749\sin\left(\frac{\pi}{7}n - 0.0714\pi\right) + 0.8660\sin\left(\frac{\pi}{3}n - 0.1167\pi\right)$

If $n=0$

$$y_2[0] = y_1[0] - \frac{1}{2j}\left(h[1]e^{-j\frac{\pi}{7}}\right)e^{j\frac{\pi}{7}\cdot 0} + \frac{1}{2j}\left(h[1]e^{j\frac{\pi}{7}}\right)e^{-j\frac{\pi}{7}\cdot 0}$$

$$-\frac{1}{2j}\left(h[1]e^{-j\frac{\pi}{3}}\right)e^{j\frac{\pi}{3}0} + \frac{1}{2j}\left(h[1]e^{j\frac{\pi}{3}}\right)e^{-j\frac{\pi}{3}0}$$

$$= y_1[0] + \frac{1}{2}\sin\left(\frac{\pi}{7}\right) + \frac{1}{2}\sin\left(\frac{\pi}{3}\right) = 0.9749\sin(-0.0714\pi) + 0.8660\sin(-0.1167\pi) + \frac{1}{2}\sin\left(\frac{\pi}{7}\right) + \frac{1}{2}\sin\left(\frac{\pi}{3}\right)$$

$$y_2[n] = \begin{cases} 0 & \text{for } n<0 \\ 0.9749\sin(-0.0714\pi) + 0.8660\sin(-0.1167\pi) + \frac{1}{2}\sin\left(\frac{\pi}{7}\right) + \frac{1}{2}\sin\left(\frac{\pi}{3}\right) & \text{for } n=0 \\ y_1[n] = 0.9749\sin\left(\frac{\pi}{7}n - 0.0714\pi\right) + 0.8660\sin\left(\frac{\pi}{3}n - 0.1167\pi\right) & \text{for } n\geq 1 \end{cases}$$

In general for $x[n] = e^{j w_0 n} u[n]$

$$y[n] = \sum_{k=-\infty}^{\infty} h[k] e^{j w_0 (n-k)} - \sum_{k=n+1}^{\infty} h[k] e^{j w_0 (n-k)}$$

$$= H(e^{j w_0}) e^{j w_0 n} - \underbrace{\left(\sum_{k=n+1}^{\infty} h[k] e^{-j w_0 k} \right)}_{\text{This part is zero if } n+1 > m \text{ where } h[k]=0 \text{ for } k>m} e^{j w_0 n}$$

Hence $y_2[n] = y_1[n]$ if $n+1 > m$

Here $m = 1$, so $y_2[n] = y_1[n]$ if $n > 0$

b) i) $y_1[n] = \left| H(e^{j\frac{\pi}{7}}) \right| \sin\left(\frac{\pi}{7} n + \measuredangle H(e^{j\frac{\pi}{7}}) \right)$

$$+ \left| H(e^{j\frac{\pi}{3}}) \right| \sin\left(\frac{\pi}{3} n + \measuredangle H(e^{j\frac{\pi}{3}}) \right)$$

$$H(e^{jw}) = \frac{1}{2} e^{-j2w} + \frac{1}{2} e^{-j3w} = \frac{1}{2} e^{-j2.5w} \left(e^{j0.5w} + e^{-j0.5w} \right)$$

$$= \frac{1}{2} e^{-j2.5w} \left(2\cos(0.5w) \right)$$

$$= e^{-j2.5w} \cos(0.5w)$$

$$H(e^{j\frac{\pi}{7}}) = e^{-j\frac{5\pi}{14}} \cos\left(\frac{\pi}{14}\right) \cong 0.9749\, e^{-j\frac{5\pi}{14}}$$

$$H(e^{j\frac{\pi}{3}}) = e^{-j\frac{5\pi}{6}} \cos\left(\frac{\pi}{6}\right) \cong 0.8660\, e^{-j\frac{5\pi}{6}}$$

$$y_1[n] \cong 0.9749 \sin\left(\frac{\pi}{7} n - \frac{5\pi}{14} \right) + 0.8660 \sin\left(\frac{\pi}{3} n - \frac{5\pi}{6} \right)$$

ii) $y_2[n] = 0$ for $n<0$ since $h[n]$ is causal.

For $n \geq 0$

$$y_2[n] = y_1[n] - \frac{1}{2j}\left(\sum_{k=n+1}^{\infty} h[k] e^{-j\frac{\pi}{7}k}\right) e^{j\frac{\pi}{7}n}$$

$$+ \frac{1}{2j}\left(\sum_{k=n+1}^{\infty} h[k] e^{j\frac{\pi}{7}k}\right) e^{-j\frac{\pi}{7}n} - \frac{1}{2j}\left(\sum_{k=n+1}^{\infty} h[k] e^{-j\frac{\pi}{3}k}\right) e^{j\frac{\pi}{3}n}$$

$$+ \frac{1}{2j}\left(\sum_{k=n+1}^{\infty} h[k] e^{j\frac{\pi}{3}k}\right) e^{-j\frac{\pi}{3}n}$$

If $\underset{\substack{\downarrow \\ h[k]=0 \text{ for } k>3}}{n+1>3} \Longrightarrow y_2[n] = y_1[n]$

If $n=0$

$$y_2[0] = y_1[0] - \frac{1}{2j}\left(h[2] e^{j\frac{-\pi}{7}\cdot 2} + h[3] e^{j\frac{-\pi}{7}\cdot 3}\right) e^{+j\frac{\pi}{7}\cdot 0}$$

$$+ \frac{1}{2j}\left(h[2] e^{+j\frac{\pi}{7}\cdot 2} + h[3] e^{+j\frac{\pi}{7}\cdot 3}\right) e^{-j\frac{\pi}{7}\cdot 0}$$

$$- \frac{1}{2j}\left(h[2] e^{-j\frac{2\pi}{3}} + h[3] e^{-j\pi}\right) e^{j\frac{\pi}{3}\cdot 0}$$

$$+ \frac{1}{2j}\left(h[2] e^{j\frac{2\pi}{3}} + h[3] e^{+j\pi}\right) e^{-j\frac{\pi}{3}\cdot 0}$$

$$y_2[0] = 0.9749 \sin\left(\frac{-5\pi}{14}\right) + 0.8660 \sin\left(\frac{-5\pi}{6}\right)$$

$$+ \frac{1}{2}\sin\left(\frac{2\pi}{7}\right) + \frac{1}{2}\sin\left(\frac{3\pi}{7}\right) + \frac{1}{2}\sin\left(\frac{2\pi}{3}\right) + \frac{1}{2}\sin(\pi)$$

If $n=1$

$$y_2[1] = y_1[1] - \frac{1}{2j}\left(h[2] e^{-j\frac{\pi}{7}\cdot 2} + h[3] e^{-j\frac{\pi}{7}\cdot 3}\right) e^{+j\frac{\pi}{7}\cdot 1}$$

$$+ \frac{1}{2j}\left(h[2] e^{j\frac{2\pi}{7}} + h[3] e^{j\frac{3\pi}{7}}\right) e^{-j\frac{\pi}{7}} - \frac{1}{2j}\left(h[2] e^{-j\frac{2\pi}{3}} + h[3] e^{-j\pi}\right) e^{j\frac{\pi}{3}\cdot 1}$$

$$+ \frac{1}{2j}\left(h[2] e^{+j\frac{2\pi}{3}} + h[3] e^{j\pi}\right) e^{-j\frac{\pi}{3}}$$

$$y_2[1] = 0.9749 \sin\left(\frac{\pi}{7} - \frac{5\pi}{14}\right) + 0.8660 \sin\left(\frac{\pi}{3} - \frac{5\pi}{6}\right)$$

$$+ \frac{1}{2}\sin\left(\frac{\pi}{7}\right) + \frac{1}{2}\sin\left(\frac{2\pi}{7}\right) + \frac{1}{2}\sin\left(\frac{\pi}{3}\right) + \frac{1}{2}\sin\left(\frac{2\pi}{3}\right)$$

$n = 2$

$$y_2[2] = y_1[2] - \frac{1}{2j}\left(h[3]\, e^{-j\frac{\pi}{7}\cdot 3}\right) e^{j\frac{\pi}{7}\cdot 2}$$

$$+ \frac{1}{2j}\left(h[3]\, e^{j\frac{3\pi}{7}}\right) e^{-j\frac{2\pi}{7}} - \frac{1}{2j}\left(h[3]\, e^{-j\pi}\right) e^{j\frac{2\pi}{3}}$$

$$+ \frac{1}{2j}\left(h[3]\, e^{j\pi}\right) e^{-j\frac{2\pi}{3}}$$

$$= 0.9749 \sin\left(\frac{2\pi}{7} - \frac{5\pi}{14}\right) + 0.8660 \sin\left(\frac{2\pi}{3} - \frac{5\pi}{6}\right)$$

$$+ \frac{1}{2}\sin\left(\frac{\pi}{7}\right) + \frac{1}{2}\sin\left(\frac{\pi}{3}\right)$$

$y_2[n] = y_1[n]$ for $n \geq 3$ $\left(n+1 > m \text{ where } h[k] = 0 \text{ for } k > m\right)$

Here $m = 3$

$$y_2[n] = \begin{cases} 0 & \text{for } n < 0 \\ 0.9749 \sin\left(\frac{-5\pi}{14}\right) + 0.8660 \sin\left(\frac{-5\pi}{6}\right) + \frac{1}{2}\sin\left(\frac{2\pi}{7}\right) \\ \quad + \frac{1}{2}\sin\left(\frac{3\pi}{7}\right) + \frac{1}{2}\sin\left(\frac{2\pi}{3}\right) + \frac{1}{2}\sin(\pi) & , n = 0 \\ 0.9749 \sin\left(\frac{\pi}{7} - \frac{5\pi}{14}\right) + 0.8660 \sin\left(\frac{\pi}{3} - \frac{5\pi}{6}\right) \\ \quad + \frac{1}{2}\sin\left(\frac{\pi}{7}\right) + \frac{1}{2}\sin\left(\frac{2\pi}{7}\right) + \frac{1}{2}\sin\left(\frac{\pi}{3}\right) + \frac{1}{2}\sin\left(\frac{2\pi}{3}\right) & , n = 1 \\ 0.9749 \sin\left(\frac{2\pi}{7} - \frac{5\pi}{14}\right) + 0.8660 \sin\left(\frac{2\pi}{3} - \frac{5\pi}{6}\right) \\ \quad + \frac{1}{2}\sin\left(\frac{\pi}{7}\right) + \frac{1}{2}\sin\left(\frac{\pi}{3}\right) & , n = 2 \\ 0.9749 \sin\left(\frac{\pi}{7}n - \frac{5\pi}{14}\right) + 0.8660 \sin\left(\frac{\pi}{3}n - \frac{5\pi}{6}\right) & , n \geq 3 \end{cases}$$

c) i) $y_1[n] = \left|H(e^{j\frac{\pi}{7}})\right| \sin\left(\frac{\pi}{7}n + \measuredangle H(e^{j\frac{\pi}{7}})\right)$

$+ \left|H(e^{j\frac{\pi}{3}})\right| \sin\left(\frac{\pi}{3}n + \measuredangle H(e^{j\frac{\pi}{3}})\right)$

$$H(e^{jw}) = \frac{1}{6} + \frac{1}{6}e^{-jw} + \frac{1}{6}e^{-j2w} + \frac{1}{6}e^{-j3w} + \frac{1}{6}e^{-j4w} + \frac{1}{6}e^{-j5w}$$

$$= e^{-j2.5w}\left(\frac{1}{3}\cos(2.5w) + \frac{1}{3}\cos(1.5w) + \frac{1}{3}\cos(0.5w)\right)$$

$$H(e^{j\frac{\pi}{7}}) = e^{-j\frac{5\pi}{14}}\left(\frac{1}{3}\cos\left(\frac{5\pi}{14}\right) + \frac{1}{3}\cos\left(\frac{3\pi}{14}\right) + \frac{1}{3}\cos\left(\frac{\pi}{14}\right)\right)$$

$$= 0.7302\, e^{-j\frac{5\pi}{14}}$$

$$H(e^{j\frac{\pi}{3}}) = e^{-j\frac{5\pi}{6}}\left(\frac{1}{3}\cos\left(\frac{5\pi}{6}\right) + \frac{1}{3}\cos\left(\frac{\pi}{2}\right) + \frac{1}{3}\cos\left(\frac{\pi}{6}\right)\right)$$

$$= 0$$

$$y_1[n] = 0.7302 \sin\left(\frac{\pi}{7}n - \frac{5\pi}{14}\right)$$

ii) $y_2[n] = 0$ for $n<0$ since $h[n]$ is causal.

$y_2[n] = y_1[n]$ for $n+1 > 5$ (since $h[k]=0$ for $k>5$)

$$y_2[0] = y_1[0] - \frac{1}{2j}\left(\sum_{k=1}^{5} \frac{1}{6} e^{-j\frac{\pi}{7}k}\right) + \frac{1}{2j}\left(\sum_{k=1}^{5} \frac{1}{6} e^{j\frac{\pi}{7}k}\right)$$

$$- \frac{1}{2j}\left(\sum_{k=1}^{5} \frac{1}{6} e^{-j\frac{\pi}{3}k}\right) + \frac{1}{2j}\left(\sum_{k=1}^{5} \frac{1}{6} e^{j\frac{\pi}{3}k}\right)$$

$$= 0.7302 \sin\left(\frac{-5\pi}{14}\right) + \frac{1}{6}\sum_{k=1}^{5}\left(\sin\left(\frac{\pi}{7}k\right) + \sin\left(\frac{\pi}{3}k\right)\right)$$

$$y_2[1] = 0.7302 \sin\left(\frac{\pi}{7} - \frac{5\pi}{14}\right) + \frac{1}{6}\sum_{k=1}^{4}\left(\sin\left(\frac{\pi}{7}k\right) + \sin\left(\frac{\pi}{3}k\right)\right)$$

$\vdots$

$$y_2[n] = \begin{cases} 0 & , n < 0 \\ 0.7302 \sin\left(\frac{\pi}{7}n - \frac{5\pi}{14}\right) + \frac{1}{6} \sum_{k=1}^{5-n} \left(\sin\left(\frac{\pi}{7}k\right) + \sin\left(\frac{\pi}{3}k\right)\right) & 0 \leq n \leq 4 \\ 0.7302 \sin\left(\frac{\pi}{7}n - \frac{5\pi}{14}\right) & n \geq 5 \end{cases}$$

(4) a) $X(e^{jw}) = \sum_{n=-\infty}^{\infty} x[n] e^{-jwn}$

$$X^*(e^{jw}) = \sum_{n=-\infty}^{\infty} \underbrace{x^*[n]}_{=x[n] \text{ (It is real)}} e^{jwn}$$

$$X^*(e^{-jw}) = \sum_{n=-\infty}^{\infty} x[n] e^{-jwn} = X(e^{jw})$$

$$X(e^{-jw}) = X^*(e^{jw}) \Longrightarrow |X(e^{-jw})| = |X(e^{jw})|$$

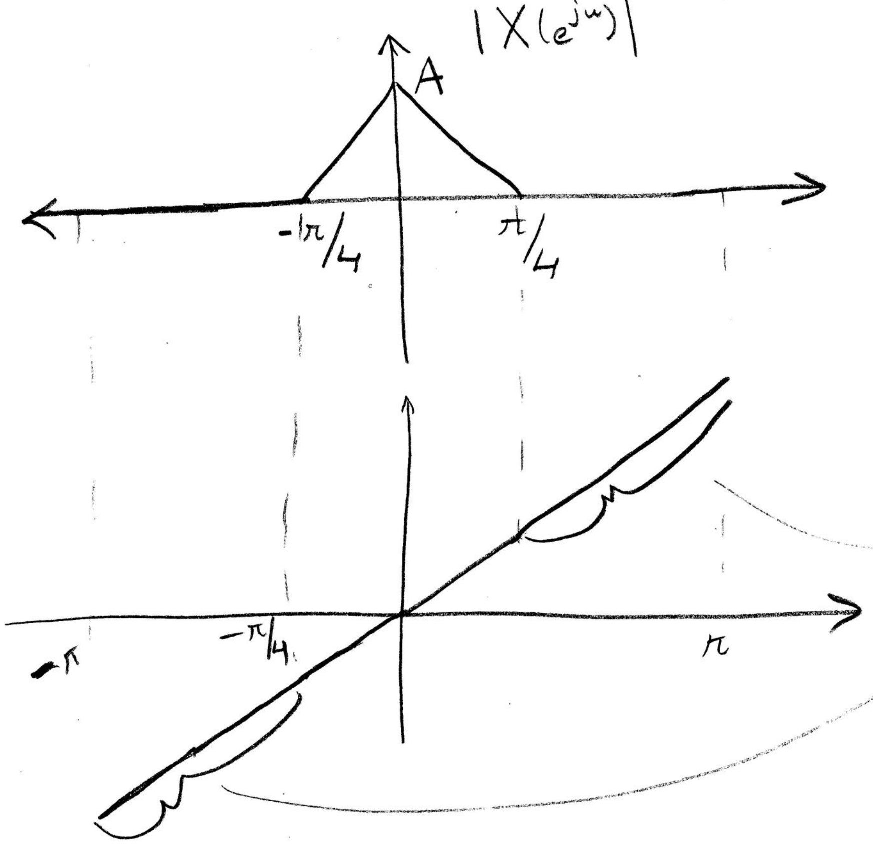
magnitude is even function

$$\angle X(e^{-jw}) = -\angle X(e^{jw})$$

phase is odd function

$\Downarrow$

$a = -b \Rightarrow 2b = \frac{\pi}{2}, \ b = \frac{\pi}{4}, \ a = -\frac{\pi}{4}$



The phase response here is not important since $|X(e^{jw})| = 0$ at $\frac{\pi}{4} < |w| \leq \pi$.

It is an only arbitrary plotting (odd function of course)

b) From modulation theory,

$$\mathcal{F}\{x[n]y[n]\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(w-\theta)})\,d\theta$$

$$\mathcal{F}\{\cos(\tfrac{\pi}{5}n)\} = \sum_{k=-\infty}^{\infty} \pi\delta(w-\tfrac{\pi}{5}+2\pi k) + \pi\delta(w+\tfrac{\pi}{5}+2\pi k)$$

$$\mathcal{F}\{\sin(\tfrac{\pi}{5}n)\} = \sum_{k=-\infty}^{\infty} \frac{\pi}{j}\delta(w-\tfrac{\pi}{5}+2\pi k) - \frac{\pi}{j}\delta(w+\tfrac{\pi}{5}+2\pi k)$$

$$X_c(e^{jw}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta}) \sum_{k=-\infty}^{\infty} \pi\delta(w-\theta-\tfrac{\pi}{5}+2\pi k)+\pi\delta(w-\theta+\tfrac{\pi}{5}+2\pi k)\,d\theta$$

$$= \frac{1}{2}\sum_{k=-\infty}^{\infty}\int_{-\pi}^{\pi} X(e^{j\theta})\left(\delta(w-\theta-\tfrac{\pi}{5}+2\pi k)+\delta(w-\theta+\tfrac{\pi}{5}+2\pi k)\right)d\theta$$

$$= \frac{1}{2}\sum_{k=-\infty}^{\infty} X(e^{j(w-\frac{\pi}{5}+2\pi k)}) + X(e^{j(w+\frac{\pi}{5}+2\pi k)})$$

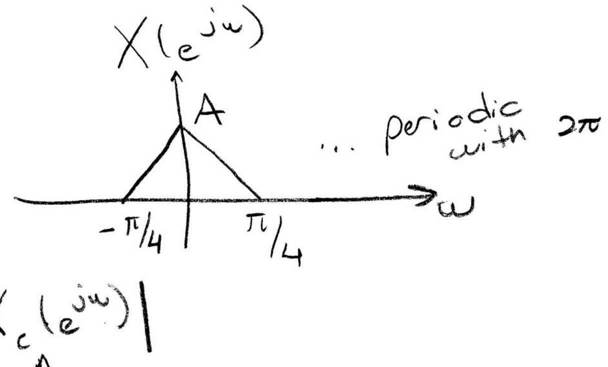
$$= \frac{1}{2}X(e^{j(w-\frac{\pi}{5})}) + \frac{1}{2}X(e^{j(w+\frac{\pi}{5})})$$

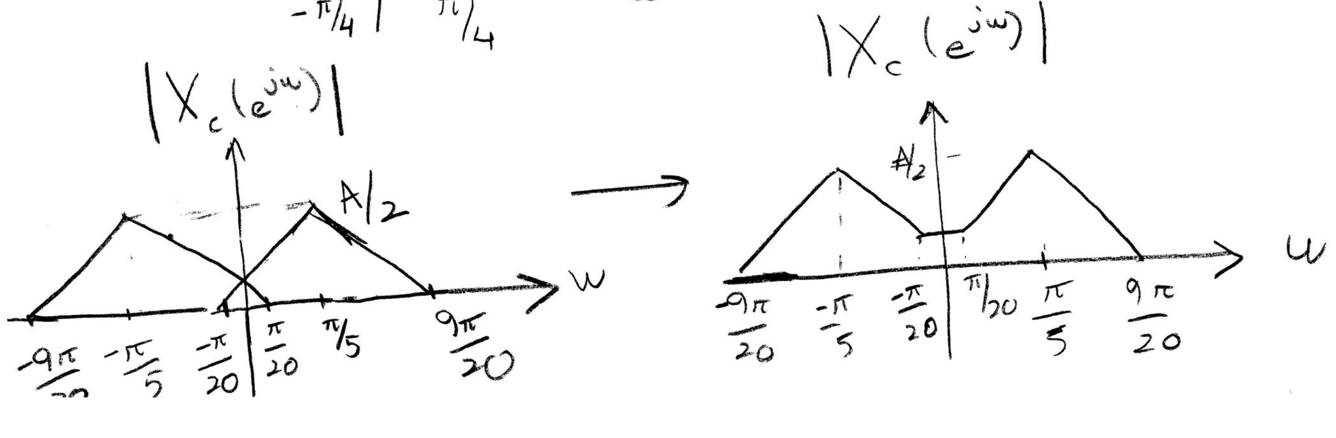
Similarly,

$$X_s(e^{jw}) = \frac{1}{2j}X(e^{j(w-\frac{\pi}{5})}) - \frac{1}{2j}X(e^{j(w+\frac{\pi}{5})})$$

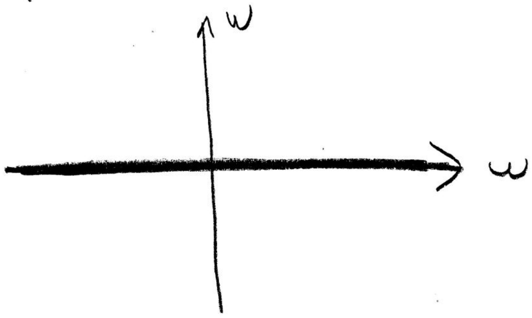
c) Assume that $X(e^{jw}) \geq 0$ So $|X(e^{jw})| = X(e^{jw})$

$\angle X(e^{jw}) = 0$

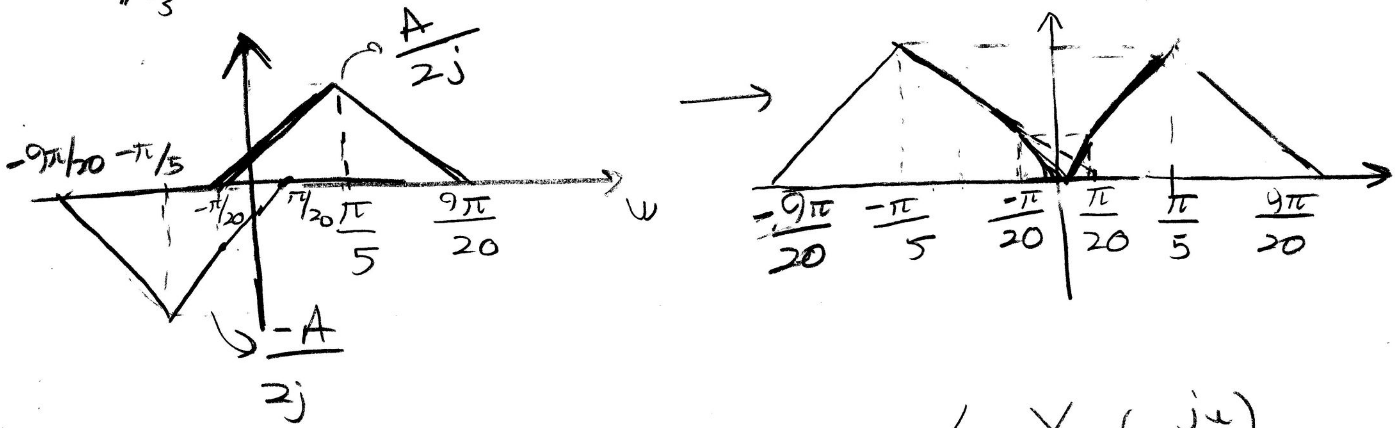




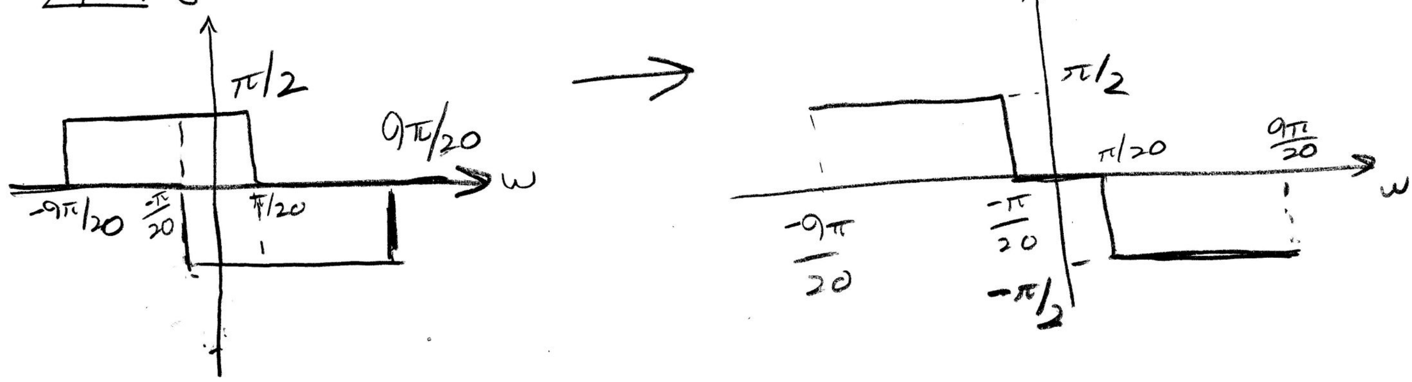
$\angle X_c(e^{j\omega}) = 0$



$X_s(e^{j\omega})$ $\quad\quad$ $|X_s(e^{j\omega})|$



$\angle X_s(e^{j\omega})$ $\quad\quad$ $\angle X_s(e^{j\omega})$



★ Please note that all graphs are periodic with 2π. For simplicity, only one period is shown.

(5) $$\int_{-\pi}^{0} \left| \frac{\sin\left(\frac{5w}{2}\right)}{\sin\left(\frac{w}{2}\right)} \right|^2 dw$$

$\left| \frac{\sin\left(\frac{5w}{2}\right)}{\sin\left(\frac{w}{2}\right)} \right|^2$ is an even function of w. Therefore,

$$\int_{-\pi}^{0} \left| \frac{\sin\left(\frac{5w}{2}\right)}{\sin\left(\frac{w}{2}\right)} \right|^2 dw = \frac{1}{2} \int_{-\pi}^{\pi} \left| \frac{\sin\left(\frac{5w}{2}\right)}{\sin\left(\frac{w}{2}\right)} \right|^2 dw.$$

Let's say $x[n] = \text{IDTFT}\left\{ \frac{\sin(5w/2)}{\sin(w/2)} \right\}$

Then From Parseval's relation;

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \frac{\sin(5w/2)}{\sin(w/2)} \right|^2 dw = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\sin(5w/2)}{\sin(w/2)} e^{jwn} dw = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{j\frac{5w}{2}} - e^{-j\frac{5w}{2}}}{e^{jw/2} - e^{-jw/2}} e^{jwn} dw$$

From finite sum formula

$$\frac{e^{j\frac{5w}{2}} - e^{-j\frac{5w}{2}}}{e^{jw/2} - e^{-jw/2}} = \frac{e^{j2w} - e^{-j3w}}{1 - e^{-jw}} = \sum_{m=-2}^{2} e^{-jwm}$$

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{m=-2}^{2} e^{-jwm} e^{jwn} dw$$

$$x[n] = \frac{1}{2\pi} \sum_{m=-2}^{2} \underbrace{\int_{-\pi}^{\pi} e^{jw(n-m)} dw}_{2\pi\delta[n-m]} = \delta[n+2] + \delta[n+1] + \delta[n] + \delta[n-1] + \delta[n-2]$$

$$\int_{-\pi}^{0} \left| \frac{\sin(5w/2)}{\sin(w/2)} \right|^2 dw = \pi \sum_{n=-\infty}^{\infty} |x[n]|^2 = \boxed{5\pi}$$

6 $x[n] = e^{j\omega_0 n} \longrightarrow \boxed{\text{LTI}} \longrightarrow y[n] = H(e^{j\omega_0})e^{j\omega_0 n}$

$Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$

$$y[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} Y(e^{j\omega})\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})H(e^{j\omega})\, e^{j\omega n}\, d\omega$$

$X(e^{j\omega}) = \delta(\omega - \omega_0)$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \delta(\omega-\omega_0)H(e^{j\omega})\, e^{j\omega n}\, d\omega = H(e^{j\omega_0})e^{j\omega_0 n}$$ ✓

7) a) $h[n] = \left(\frac{1}{2}\right)^n u[n] + 2^n u[n]$

$$y[n] = x[n] * h[n]$$
$$= \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$$

$$= \sum_{k=0}^{\infty} \left[\left(\frac{1}{2}\right)^k + 2^k\right] 3^{n-k} = 3^n \left[\underbrace{\sum_{k=0}^{\infty} \left(\frac{1}{6}\right)^k + \left(\frac{2}{3}\right)^k}_{\text{convergent}}\right] = C3^n$$

$$= C = \frac{1}{1-\frac{1}{6}} + \frac{1}{1-\frac{2}{3}}$$

* Note that $|z| = 3$ is inside the ROC of $H(z)$ $(|z| > 2)$

Hence, $y[n]$ can be expressed as

$y = C3^n$

b) $h[n] = \left(\frac{1}{2}\right)^n u[n] + 2^n u[-n]$

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k] = \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k 3^{n-k} + \sum_{k=-\infty}^{0} (2)^k 3^{n-k}$$

$$= 3^n \left[\underbrace{\sum_{k=0}^{\infty} \left(\frac{1}{6}\right)^k + \left(\frac{3}{2}\right)^k}_{\text{not convergent}}\right]$$

since $\left|\frac{3}{2}\right| \geq 1$

* Note that $|z| = 3$ is not inside the ROC of $H(z)$ $\left(\frac{1}{2} < |z| < 2\right)$

* We cannot express $y[n]$ as $y[n] = C3^n$

c) $h[n] = 5^n u[-n]$

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]\,x[n-k] = \sum_{k=-\infty}^{0} 5^k\, 3^{n-k} = 3^n \Bigg[\underbrace{\sum_{k=0}^{\infty} \left(\frac{3}{5}\right)^k}_{\text{convergent}} \Bigg]$$

$$= C = \frac{1}{1-\frac{3}{5}}$$

$$y[n] = C\,3^n \checkmark$$

* Note that $|z| = 3$ is inside the ROC of $H(z)$.
$(|z| < 5)$

d) $y[n] = \sum_{k=-\infty}^{\infty} h[k]\,x[n-k]$

$$= \sum_{k=0}^{\infty} 3^k\, 3^{n-k} = 3^n \Bigg[\underbrace{\sum_{k=0}^{\infty} 1}_{\text{not convergent}} \Bigg]$$

*We cannot express $y[n]$ as $y[n] = C\,3^n$ ✗

* $|z| = 3$ is not inside the ROC $(|z| > 3)$

8) a) $$H(z) = \frac{2z^{-1}+1}{\left(1-\frac{1}{4}z^{-1}\right)\left(1+\frac{2}{3}z^{-1}\right)\left(1+\frac{1}{4}z^{-1}\right)}$$

$$= \frac{z^2(z+2)}{\left(z-\frac{1}{4}\right)\left(z+\frac{2}{3}\right)\left(z+\frac{1}{4}\right)}$$

zeros: $z_1 = z_2 = 0$, $z_3 = -2$

poles $p_1 = \frac{1}{4}$, $p_2 = -\frac{2}{3}$, $p_3 = -\frac{1}{4}$

ROC: $|z| > \frac{2}{3}$

The LTI system is stable. So the impulse response is absolutely summable. ROC must include unit circle.

The sequence is right sided. In addition as $z \nearrow \infty$ $H(z)$ does not blow up. There is no pole at $z=\infty$. This means that $h[n]=0$ for $n<0$. Consider $H(z)$,

$$H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n} = \ldots\ldots \underbrace{h[-3]z^3 + h[-2]z^2 + h[-1]z}\ldots$$

If $h[n] \neq 0$ for $n<0$ then as $z \nearrow \infty$ $H(z) \nearrow \infty$. So $H(z)$ does not exist for $z=\infty$. In this case if $h[n]=0$ for some $n<n_0$ where n_0 is a negative integer then the sequence may be still right sided but not causal.

Here, the system is causal. For example, consider the following system,

$$h_1[n] = \left(\frac{1}{2}\right)^{n+3} u[n+3]$$ $\Rightarrow$ It is a right sided sequence since $h_1[n]=0$ for $n<-3$ and as $n \nearrow \infty$ it is still nonzero. However the system is not causal.

Consider $H_1(z)$

$$H_1(z) = \frac{z^3}{1-\frac{1}{2}z^{-1}} = \frac{z^4}{z-\frac{1}{2}}$$

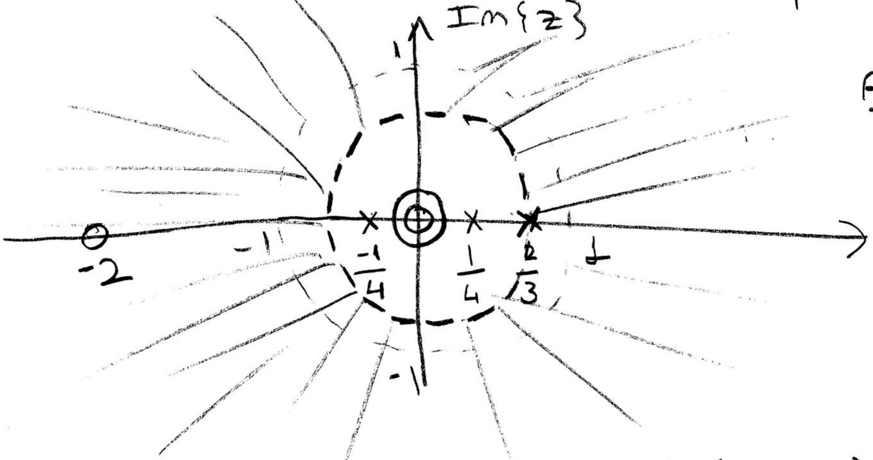
zeros: $z_1 = z_2 = z_3 = z_4 = 0$

poles: $p_1 = \frac{1}{2}$, $p_2 = p_3 = p_4 = \infty$

ROC: $\frac{1}{2} < |z| < \infty$

We understand from ROC that $h_1[n]$ is a right sided sequence but not causal !!!

Let's now turn back to the question.



ROC: $|z| > \frac{2}{3}$

b) $$H(z) = \frac{z-4}{(1-3z^{-1})(1-5z^{-1})} = \frac{z^2(z-4)}{(z-3)(z-5)}$$

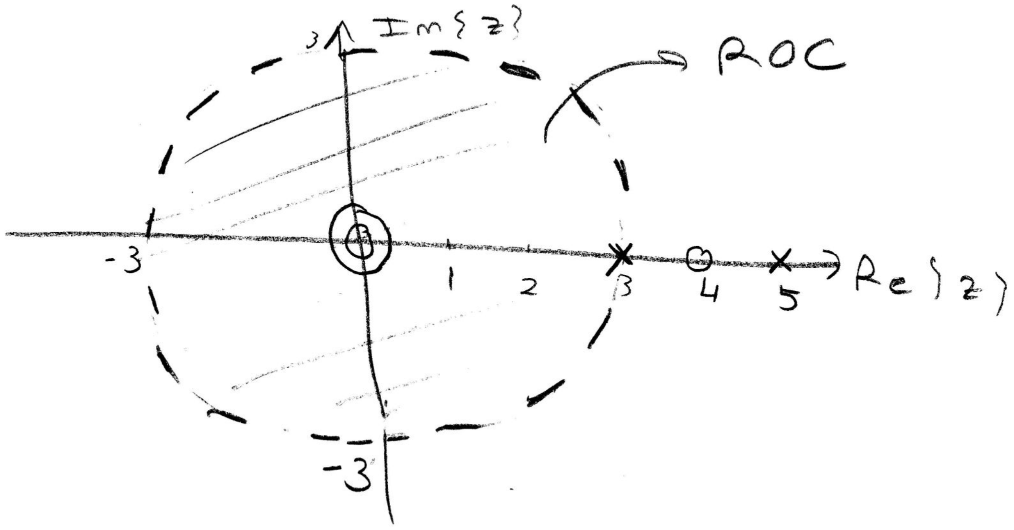
zeros: $z_1 = z_2 = 0$, $z_2 = 4$

poles: $p_1 = 3$, $p_2 = 5$, $p_3 = \infty$

ROC: $|z| < 3$

(for stability)

not causal



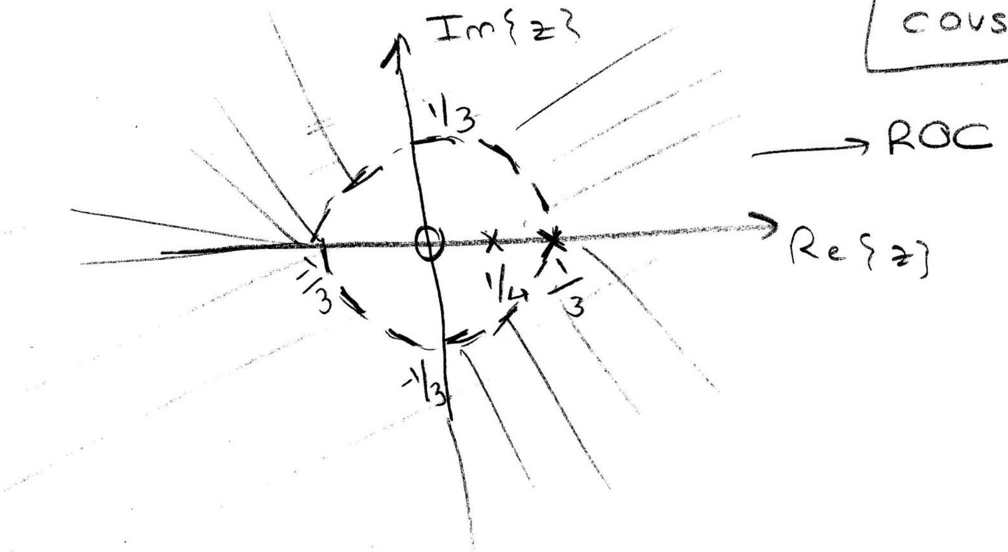
c) $H(z) = \dfrac{z^{-1}}{\left(1-\frac{1}{4}z^{-1}\right)\left(1-\frac{1}{3}z^{-1}\right)} = \dfrac{z}{\left(z-\frac{1}{4}\right)\left(z-\frac{1}{3}\right)}$

zeros: $z_1 = 0$, $z_2 = \infty$

poles: $p_1 = \frac{1}{4}$, $p_2 = \frac{1}{3}$

ROC: $|z| > \frac{1}{3}$ for stability

$\boxed{\text{causal}}$



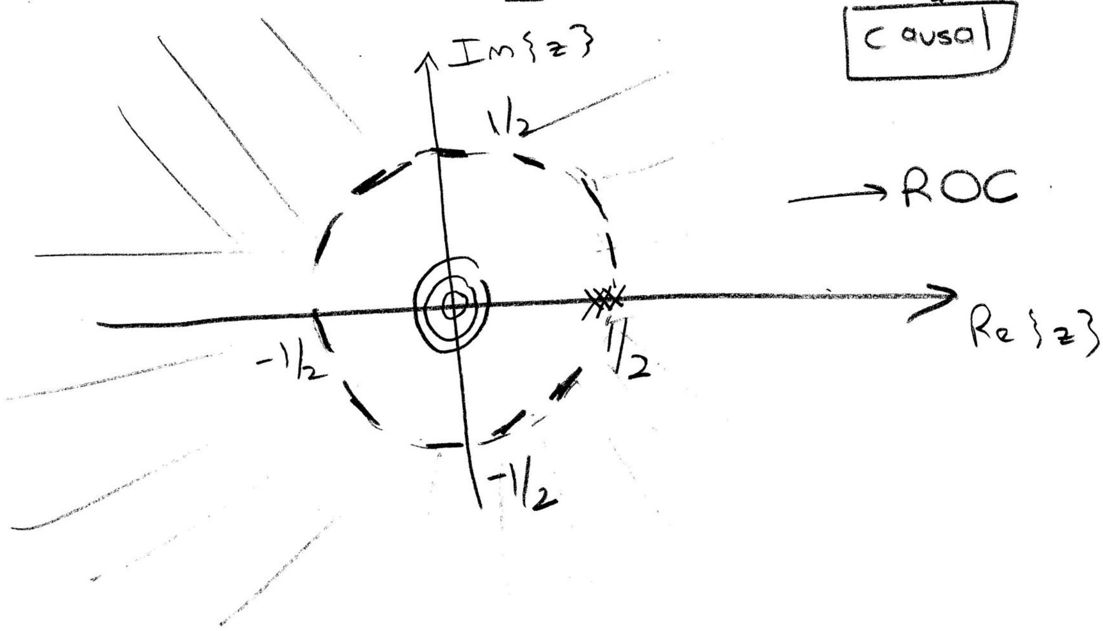
d) $H(z) = \dfrac{1}{\left(1-\frac{1}{2}z^{-1}\right)^3} = \dfrac{z^3}{\left(z-\frac{1}{2}\right)^3}$

zeros: $z_1 = z_2 = z_3 = 0$

poles: $p_1 = p_2 = p_3 = \frac{1}{2}$

ROC: $|z| > \frac{1}{2}$ for stability

$\boxed{\text{causal}}$



(9) Use the eigenfunction property of complex exponentials:

$$e^{j\omega_0 n} \longrightarrow \boxed{\text{LTI}} \longrightarrow H(e^{j\omega_0})\, e^{j\omega_0 n}$$

From linearity;

$$\sum_{k=0}^{M} a_k e^{j\omega_k n} \longrightarrow \boxed{\text{LTI}} \longrightarrow \sum_{k=0}^{M} a_k H(e^{j\omega_k})\, e^{j\omega_k n}$$

a) $x[n] = \cos(1.6\pi n) = \frac{1}{2} e^{j1.6\pi n} + \frac{1}{2} e^{-j1.6\pi n}$

$$y[n] = \frac{1}{2} H(e^{j1.6\pi})\, e^{j1.6\pi n} + \frac{1}{2} H(e^{-j1.6\pi n})\, e^{-j1.6\pi n}$$

$$H(e^{j1.6\pi}) = H(e^{-j0.4\pi}) = \frac{1}{5} e^{j0.2\pi}$$

$$H(e^{-j1.6\pi}) = H(e^{j0.4\pi}) = \frac{1}{5} e^{-j0.2\pi}$$

$$y[n] = \frac{1}{10} e^{j(1.6\pi n + 0.2\pi)} + \frac{1}{10} e^{-j(1.6\pi n + 0.2\pi)}$$

$$= \boxed{\frac{1}{5}\cos(1.6\pi n + 0.2\pi)}$$

$\longrightarrow$ We obtained a real output for real input. Because the system has a real impulse response ($H(e^{-j\omega}) = H^*(e^{j\omega})$)

b) $x[n] = \sin(30.4\pi n) = \frac{1}{2j} e^{j0.4\pi n} - \frac{1}{2j} e^{-j0.4\pi n}$

$$y[n] = \frac{1}{2j} H(e^{j0.4\pi})\, e^{j0.4\pi n} - \frac{1}{2j} H(e^{-j0.4\pi})\, e^{-j0.4\pi n}$$

$$= \frac{1}{10j} e^{j(0.4\pi n - 0.2\pi)} - \frac{1}{10j} e^{-j(0.4\pi n - 0.2\pi)}$$

$$y[n] = \frac{1}{5}\sin(0.4\pi n - 0.2\pi)$$

c) $x[n] = (3+j5)\, e^{j0n} + \frac{1}{2j} e^{j0.25\pi n} - \frac{1}{2j} e^{-j0.25\pi n}$

$$y[n] = (3+j5)\, H(e^{j0}) + \underbrace{\left|H(e^{j0.25\pi})\right| \sin\left(0.25\pi n + \angle H(e^{j0.25\pi})\right)}_{\text{since } h[n] \text{ is real, look at part a and b}}$$

$$\boxed{y[n] = (3+j5) + \frac{1}{2} \sin\left(0.25\pi n - \frac{\pi}{8}\right)}$$