Homework 2

Due Date: November,13 23:55

1) Impulse responses of some LTI systems are given below. Let $x[n] = 3^n$ be the input signal of these systems. Determine those systems for which their output signals can be expressed as $y[n] = C\ 3^n$ where C is a complex constant. Explain formally.

a) $h[n] = \left(\frac{1}{2}\right)^n u[n] + 2^n u[n]$

b) $h[n] = \left(\frac{1}{2}\right)^n u[n] + 2^n u[-n]$

c) $h[n] = 5^n u[-n]$

d) $h[n] = 3^n u[n]$

2) Find the impulse responses of the stable LTI systems having the following system functions. Which of them are causal? Plot the pole-zero diagrams and show their ROCs.

a) $H(z) = \frac{2z^{-1}+1}{\left(1-\frac{1}{4}z^{-1}\right)\left(1+\frac{2}{3}z^{-1}\right)\left(1+\frac{1}{4}z^{-1}\right)}$

b) $H(z) = \frac{z-4}{(1-3z^{-1})(1-5z^{-1})}$

c) $H(z) = \frac{z^{-1}}{\left(1-\frac{1}{4}z^{-1}\right)\left(1-\frac{1}{3}z^{-1}\right)}$

d) $H(z) = \frac{1}{\left(1-\frac{1}{2}z^{-1}\right)^3}$

3) The impulse response of a LTI system is

$$h[n] = \delta[n] - \sqrt{2}\delta[n-1] + \delta[n-2].$$

a) Find the system function $H(z)$. Plot the pole-zero diagram, indicate ALL poles and zeros, show the ROC.

b) Does this system have a frequency response? Why? If yes, plot its magnitude and phase.

c) Find the output of this system to the following input signals

$$x_1[n] = \sin\left(\frac{\pi}{4}n + \frac{\pi}{4}\right)$$

$$x_2[n] = \sin\left(\frac{\pi}{4}n + \frac{\pi}{4}\right)u[n]$$

$$x_3[n] = \sin\left(\frac{\pi}{4}n + \frac{\pi}{4}\right) + \sin\left(\frac{3\pi}{4}n\right)$$

d) Comment on the relationship between the frequency response and zero locations of $H(z)$.

4) The system function of a LTI system is

$$H(z) = \frac{1 - \sqrt{2}z^{-1} + z^{-2}}{1 - 2z^{-1}}.$$

When the input is $\sin\left(\frac{\pi}{2}n\right)$, the output of this system is $\sqrt{\frac{2}{5}}\sin\left(\frac{\pi}{2}n + \tan^{-1}\frac{1}{2}\right)$.

a) Find the impulse response of this system.
b) Is the system causal?
c) Find the difference equation for this system.

6) The z-transform, $X(z)$, of a right –sided sequence $x[n]$ exists for $z = 4e^{j\omega},\ \ 0 \le \omega < 2\pi$. Show that $X(z)$ exists for $= 4.1e^{j\omega},\ \ 0 \le \omega < 2\pi$, but not necessarily for $z = 3.9e^{j\omega},\ \ 0 \le \omega < 2\pi$.

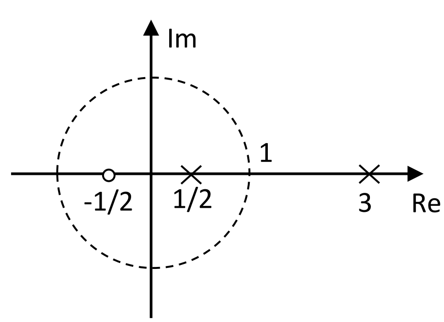
7) What are the ROCs of the z-transforms of the following sequences?

a) $x[n] = \delta[n+3] + \delta[n-3]$
b) $x[n] = \delta[n+3]$
c) $x[n] = \delta[n-3]$

8) Let $x[n] = \delta[n+1] + \left(\frac{1}{2}\right)^n u[n]$. Find the z-transforms of the following sequences. What are the ROCs? State all poles and zeros.

a) $x[n]$
b) $x[n-5]$
c) $nx[n]$
d) $\cos\left(\frac{\pi}{2}n\right)x[n]$

9) The pole-zero plot of the system function, $H(z)$, of a <u>stable</u> LTI system is shown. It is known that $H(1) = 1$.



a. Show the ROC. Determine the impulse response $h[n]$.
b. Let $h_1[n] = h[-n+2]$. Sketch the pole-zero plot for $H_1(z)$ show its ROC.

10) The output of a stable LTI system is $y[n] = \delta[n+1] + 2\delta[n] + \delta[n-1]$ when its input is $x[n] = -2\delta[n+2] - 4\delta[n+1] + 4\delta[n-1] + 2\delta[n-2]$. Find its impulse response $h[n]$.

11) Problem 3.30 of textbook.

12) Problem 3.52 of textbook.

13) Problem 3.58 of textbook.

14) Let $x[n] = \delta[n] + 3\delta[n-1] + \delta[n-2]$.

a. Plot $x[n]$ and its periodic extension, $\tilde{x}[n]$, for $N = 3$ and $N = 5$.

$$\tilde{x}_N[n] = \sum_{k=-\infty}^{\infty} x[n-kN] = x\left[((n))_N\right]$$

b. Find the Discrete Fourier Series (DFS) coefficients, $\tilde{X}_3[k]$, of $\tilde{x}_3[n]$. Write the DFS representation of $\tilde{x}_3[n]$.

c. Find the Discrete Fourier Series (DFS) coefficients, $\tilde{X}_5[k]$, of $\tilde{x}_5[n]$. Write the DFS representation of $\tilde{x}_5[n]$.

d. Find the DTFT, $X(e^{j\omega})$, of $x[n]$. Plot its magnitude and phase.

e. Verify that $\tilde{X}_3[k] = X(e^{j\omega})\big|_{\omega=k\frac{2\pi}{3}}$ and $\tilde{X}_5[k] = X(e^{j\omega})\big|_{\omega=k\frac{2\pi}{5}}$, i.e., uniformly spaced samples of DTFT of $x[n]$. Show these samples on the magnitude and phase plot of $X(e^{j\omega})$.

f. Compute the sample values of $\tilde{x}_3[n]$ and $\tilde{x}_5[n]$ using their DFS representations and their DFS coefficients you found in parts (b) and (c), respectively.

15)

a. Find the 3-point and 5-point DFTs ($X_3[k]$ and $X_5[k]$) of $x[n]$ given in Question-14.

b. What is the relationship between $X_3[k]$ and $\tilde{X}_3[k]$, and $X_5[k]$ and $\tilde{X}_5[k]$?

c. How would you find $x[n]$ using its 3-point and 5-point DFTs?

d. Find $X_3[k]$ and $X_5[k]$ using MATLAB.

16) Let $y[n] = \delta[n-2] + 3\delta[n-3] + \delta[n-4]$ and $z[n] = 3\delta[n] + \delta[n-1] + \delta[n-4]$

a. Plot $y[n]$ and $z[n]$.

b. Relate $y[n]$ and $z[n]$ to $x[n]$ of Question-14

c. Find the 5-point DFTs, $Y_5[k]$ and $Z_5[k]$, of $y[n]$ and $z[n]$. Do they have 3-point DFTs? Why?

17) Let $\tilde{W}[k] = X(e^{j\omega})\big|_{\omega=k\frac{2\pi}{2}}$, i.e., two (uniformly spaced) samples from each period of $X(e^{j\omega})$, DTFT of $x[n]$ in Question-14.

a. Find the periodic sequence $\tilde{w}[n]$ whose DFS coefficients are $\tilde{W}[k]$.

b. Find the relationship between $\tilde{w}[n]$ and $x[n]$.

18) (Generalization of the result in Question-17) Let $x[n]$ be an arbitrary sequence with a DTFT $X(e^{j\omega})$and

$$\widetilde{W}[k] = X(e^{j\omega})\big|_{\omega=k\frac{2\pi}{M}}$$

i.e., M (uniformly spaced) samples from each period of $X(e^{j\omega})$. Also let $\widetilde{w}[n]$ be the periodic sequence whose DFS coefficients are $\widetilde{W}[k]$.

a) Show that

$$\widetilde{w}[n] = \sum_{k=-\infty}^{\infty} x[n-kM]$$

b) Assuming that $x[n]$ has finite length N. Comment on the cases $M \geq N$ and $M < N$.

c) Verification using MATLAB. You may use the following code to verify for different values of M and N.

```
clear all
close all
N = 10;
n = 0:(N-1);
x = 1:N;
M = 3;
% M = 5;
% M = 7;
% M = 10;
% M = 15;
W_M = exp(-j*2*pi/M);
F = W_M .^ n ;
for k = 0:(M-1)
   DFT_matrix(k+1,:) = F.^k;
end
Z = DFT_matrix * x';
z = ifft(Z)
```

19) Let $x[n] = 3\delta[n] - 2\delta[n-1] + \delta[n-2] + \delta[n-3] - 2\delta[n-4] - \delta[n-5]$.

You do not need to compute any DFTs in parts (a)-(c)!

a) Let $W_3[k] = X(e^{j\omega})|_{\omega=k\frac{2\pi}{3}}$, $k = 0,1,2$. Find the sequence $w_3[n]$ whose 3-point DFT is $W_3[k]$.

b) Let $W_5[k] = X(e^{j\omega})|_{\omega=k\frac{2\pi}{5}}$, $k = 0,1,2,3,4$. Find the sequence $w_5[n]$ whose 5-point DFT is $W_5[k]$.

c) Let $W_8[k] = X(e^{j\omega})|_{\omega=k\frac{2\pi}{8}}$, $k = 0,1,2,3,4,5,6,7$. Find the sequence $w_8[n]$ whose 8-point DFT is $W_8[k]$.

Let $h[n] = 2\delta[n] - 1\delta[n-1]$ be the impulse response of a LTI system.

You do not need to compute any DFTs in parts (d)-(f)!

d) Let $H_3[k]$ be the 3-point DFT of $h[n]$. Find the sequence $y_3[n]$ whose 3-point DFT is $W_3[k]H_3[k]$.
e) Let $H_5[k]$ be the 5-point DFT of $h[n]$. Find the sequence $y_5[n]$ whose 5-point DFT is $W_5[k]H_5[k]$.
f) Let $H_8[k]$ be the 8-point DFT of $h[n]$. Find the sequence $y_8[n]$ whose 8-point DFT is $W_8[k]H_8[k]$.
g) Describe the relationships between the sequence $y[n] = x[n] * h[n]$ and the sequences $y_3[n], y_5[n], y_8[n]$.

20) Let $x[n]$ be a sequence of length N; N is even. Let $X[k]$ be its N-point DFT.

a) Show that $X[k]$ can be written as

$$X[k] = E\left[((k))_{\frac{N}{2}}\right] + e^{-jk\frac{2\pi}{N}} O\left[((k))_{\frac{N}{2}}\right] \qquad k = 0,1,\dots,N-1$$

where $E[k]$ and $O[k]$ are the $\frac{N}{2}$-point DFTs of $e[n] = x[2n]$ and $o[n] = x[2n+1]$, respectively.

b) Assume that $x[n]$ is real.

i. Count the number of real multiplications and real additions in the direct computation of $X[k]$.

ii. Count the number of real multiplications and real additions in the computation of $[k]$ according to the right hand side of the above expression.

iii. Compare the numbers of arithmetic operations in these two cases.

21) Let $h[n] = 2\delta[n] - \delta[n-1] + \delta[n-2]$ be the impulse response of a LTI system and

$$x[n] = [1 \ \ 2 \ \ 3 \ \ 4 \ \ -1 \ \ -2 \ \ -3 \ \ -4 \ \ 1 \ \ 2 \ \ 3 \ \ 4] \qquad 0 \le n < 11$$

be an input to this system. The output $y[n]$ will be found by using the overlap-add method. Take the length, L, of the input segments as $L = 4$.

a) How many point DFTs will be used in this computation?
b) How many input segments are there? Write all of them.
c) Find the response of the system to the individual input segments (use MATLAB; find DFTs, multiply them and then take inverse DFT).
d) Obtain the whole output sequence by using the responses to input segments.

22) Overlap-save method will be used for the setting in Question-16.

Take the length, L, of the input segments as $L = 6$. Use 7-point DFTs. Answer parts (b)-(d).

23) $x[n]$ is a finite length sequence defined to be zero for $n < 0$ and $n > N - 1$. DTFT of $x[n]$ is $X(e^{j\omega})$.

a) $Y_1[k]$ is obtained by taking L , ($\frac{N}{2} < L < N$), uniform samples of $X(e^{j\omega})$ in $0 \le \omega < 2$. $y_1[n]$ is the result of L-point IDFT of $Y_1[k]$. Find the indices of samples of $y_1[n]$ such that $y_1[n] = x[n]$.

b) Let $x[n] = [1\ \ 2\ -1\ -2\ \ 3]$, $0 \le n \le 4$ and $Y_2[k] = e^{-j\frac{4\pi}{5}} X[k]$ where $X[k]$ is the 5-point DFT of $x[n]$. Find the 5-point IDFT of $Y_2[k]$.

c) Two finite length sequences $x_1[n] = [1 -1\ -2\ \ 3\ \ 2\ \ 1]$, $0 \le n \le 5$, and $x_2[n] = [2\ \ 1\ \ 3\ \ 4\ -1\ \ 2]$, $0 \le n \le 5$ are given. Find the 6-point circular convolution of these sequences.

d) Let $x[n] = [3\ -1\ \ 4\ \ 1\ \ 2\ -1\ -3]$, $0 \le n \le 6$. Find the periodic even part of this sequence, $x_{pe}[n]$. Also find the 7-point DFT of $x_{pe}[n]$.

24) Let $x[n] = [0\ \ 1\ \ 3\ -2\ \ \ 2\ \ \ 4]$, $0 \le n \le 5$, and $X[k]$ be its 10-point DFT.

Answer the following questions by using the DFT properties.

a) Find the sequence whose 10-point DFT is $X^2[k]$.

b) Find the sequence whose 10-point DFT is $e^{j\pi k}X[k]$.

c) Find the sequence whose 5-point DFT is $Z[k] = X[2k], k = 0,1, \dots ,4$.

25) Let $v[n]$ be a real sequence of length $2N$ with a $2N$-point DFT $V[k]$. $g_1[n]$ and $g_2[n]$ are obtained from the even and odd indexed sequences of $v[n]$ as,

$$g_1[n] = v[2n], \ \ g_2[n] = v[2n+1], 0 \le n \le N-1$$

with N-point DFT's $G_1[k]$ and $G_2[k]$ respectively. Define a new complex sequence

$$x[n] = g_1[n] + j\, g_2[n], 0 \le n \le N-1\,.$$

a) Find $X^*\left[\left((-k)\right)_N\right]$, in terms of $G_1[k]$ and $G_2[k]$.

b) Determine $G_1[k]$ and $G_2[k]$, in terms of $X[k]$.

c) Determine the $2N$-point DFT $V[k]$ in terms of N-point DFTs $G_1[k]$ and $G_1[k]$.

26) $x[n]$ is a finite length sequence, nonzero only for $0 \le n \le 5$ and $h[n]$ is a sequence nonzero only for $0 \le n \le 4$. Let $Y_6[k] = X_6[k]H_6[k]$ where $X_6[k]$ and $H_6[k]$ are the 6-point DFTs of $x[n]$ and $h[n]$, respectively. 6-point IDFT of $Y_6[k]$, $y_6[n]$, is

$$y_6[n] = [-4\ -6\ -5\ \ \ 1\ \ \ 14\ \ \ 0\,]\quad 0 \le n \le 7.$$

Similarly, $y_8[n]$ is the 8-point IDFT of $Y_8[k] = X_8[k]H_8[k]$ where $X_8[k]$ and $H_8[k]$ are the 8-point DFTs of $x[n]$ and $h[n]$, respectively. $y_8[n]$ is

$$y_8[n] = [1\ \ \ 7\ \ \ 0\ -3\ \ -1\ \ \ 14\ \ \ 0\ -7\ -11]\quad\ 0 \le n \le 7$$

Find the sample values of the sequence, $y[n] = x[n] * h[n]$, which is the linear convolution of $x[n]$ and $h[n]$.

27) The Fourier series coefficients of a periodic sequence $\tilde{x}[n]$ are

$$\tilde{X}[k] = [6\quad 4\quad -2\quad 4]\ \ \text{for } k = 0,1,2,3$$

a) Find $\tilde{x}[n]$.

b) Find the sequence $x[n]$ whose whose DFT is $X[k] = \begin{cases} \tilde{X}[k] & k = 0,1,2,3 \\ 0 & o.w. \end{cases}$

c) Find the sequence $y[n]$ whose DFT is

$$Y[k] = \begin{cases} \tilde{X}[k-2] & k = 0,1,2,3 \\ 0 & o.w. \end{cases}$$

d) Find the sequence $w[n]$ whose DFT is

$$W[k] = \begin{cases} X\left[\left((k+2)\right)_4\right] & k = 0,1,2,3 \\ 0 & o.w. \end{cases}$$

28) Let $x[0] = a, x[1] = b, x[2] = c, x[3] = d$ and $x[n] = 0$ for $n < 0$ and $n > 3$. Let

$$y[n] = X(e^{j\omega})\big|_{\omega = n\frac{2\pi}{3}} \qquad n = 0,1,2$$

Find the 3-point DFT of $y[n]$ in terms of $\{a, b, c, d\}$

29) Let $A_N[k]$ denote N-point DFT of a sequence $a[n]$.

Let $p[n] \triangleq IDFT\{X_4[k]H_4[k]\}$ and $s[n] \triangleq IDFT\{X_5[k]H_5[k]\}$

$$p[n] = \begin{cases} [1 \quad 3 \quad 0 \quad -2] & n = 0,1,2,3 \\ 0 & o.w. \end{cases}$$

$$s[n] = \begin{cases} [6 \quad 0 \quad -1 \quad -2 \quad -1] & n = 0,1,2,3,4 \\ 0 & o.w. \end{cases}$$

Find the output of the system with impulse response $h[n]$ when $x[n]$ is its input.

30) $x[n]$ and $h[n]$ are 4-point sequences. Their linear convolution is $y[n]$, i.e. $y[n] = x[n] * h[n]$.

$y[n]$ is given as $y[n] = [-2 \quad 1 \quad 5 \quad 1 \quad -2 \quad -2 \quad 1]$ for $n = 0,1, \dots 6$.

Let $z[n] = x\left[((n-2))_4\right]$.

Find the 4-point circular convolution of $z[n]$ and $h[n]$.

31) Let $x[n]$ be a N-point real sequence ($x[n] = 0,\ \ n \notin \{0,1,\dots,N-1\}$).

a) What is the periodic conjugate symmetric part, $x_{pe}[n]$, (periodic even part in this case) of $x[n]$? Write its definition.

   Let $x[n] = \delta[n] - \delta[n-1] + 2\delta[n-2]$ for the following parts.

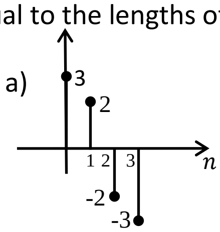
b) Find and plot $x_{pe}[n]$.
c) Express the DFT, $\hat{X}[k]$, of $x_{pe}[n]$ in terms of $X[k]$, the DFT of $x[n]$. Also,
   i. Find $\hat{X}[k]$ of your $x[n]$ in part (a).
   ii. Plot $\hat{X}[k]$.
d) Plot the periodic conjugate anti-symmetric part, $X_{po}[k]$, of $X[k]$.
e) Find the sequence, $\bar{\bar{x}}[n]$ whose DFT is $X_{po}[k]$.

32) Let $x[n] = 3\delta[n] + 2\delta[n-1] + \delta[n-2] + 2\delta[n-3]$

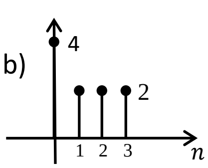
a) Plot $x[n]$.
b) Find and plot the DFT, $X[k]$, (magnitude and phase) of $x[n]$.
c) Find the sequence, $y[n]$, whose DFT is $x[k] = 3\delta[k] + 2\delta[k-1] + \delta[k-2] + 2\delta[k-3]$.
d) You are given a software that computes N-point DFTs of complex sequences. You are not allowed to modify it. How can you use it to compute inverse-DFTs. Clearly describe any pre- and/or post-processing needed.

33) Which of the following sequences' DFTs can be expressed as $A[k]e^{j\alpha k}$ where $A[k]$ is real or imaginary, and α is a constant. For those that can be expressed as such find $A[k]$ and α. (DFT sizes are equal to the lengths of the sequences.)

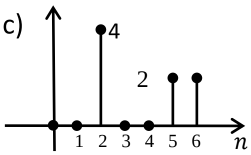
a)



b)



c)



34)

a) Determine the polynomial result of (1+3z^{-1}-4z^{-2})(-1+2z^{-1}-3z^{-2}+z^{-3}+7z^{-4}) using "`conv`" command in MATLAB.

b) Let us consider

$$X(z) = \frac{1-3z^{-1}+4z^{-2}}{1-z^{-1}+z^{-2}-z^{-3}}$$

Using "`residuez`" command, determine the inverse z transform of $X(z)$.

c) Let us consider

$$X(z) = \frac{1-0.2z^{-1}-1.2z^{-2}}{1-0.9z^{-1}+0.81z^{-2}}.$$

Using "`zplane`" command, plot the pole-zero plot of $X(z)$. Plot also the magnitude and phase characteristics of it using "`freqz`" command. Comment on the relationship between the frequency response and zero & pole locations of $X(z)$.

35)

a) Write a MATLAB function named "`mydft`" which computes the N-point DFT of a given sequence. Find the 9-point DFT of x[n]=[1 2 3 4 5 6 7] using this function. Plot the result.

b) Find the 9-point DFT of x[n] in part a) using "`fft`" function. Verify your function comparing the results of part a) and part b).

c) Consider $x[n] = 2\cos\left(\frac{\pi}{13}n - \frac{\pi}{8}\right) + 5\sin\left(\frac{\pi}{5}n + \frac{\pi}{2}\right).$

Obtain 130-point DFT of $x[n]$ for $n = 0, \ldots, 129$. Plot the magnitude and phase of DFT and explain your observations. At which frequency bins the peaks of the magnitude of DFT occur? What are the magnitudes at these frequency bins? What are the phases of DFT at these frequency bins? What are the magnitudes at the other frequency bins? Comment on your observations.