SAMPLING

UNIFORM SAMPLING

C/D, D/C (A/D, D/A)

A MATHEMATICAL MODEL OF SAMPLING

IMPULSE SAMPLING

ALIASING

EXPRESSING $X(e^{j\omega})$ IN TERMS OF $X_c(\Omega)$

NYQUIST-SHANNON SAMPLING THEOREM

RECONSTRUCTION OF A CT SIGNAL FROM A DT SIGNAL

DISCRETE-TIME PROCESSING OF CONTINUOUS-TIME SIGNALS

IMPULSE RESPONSES OF EQUIVALENT CT AND DT SYSTEMS

CHANGING THE SAMPLING RATE IN DISCRETE-TIME

RATE REDUCTION BY AN INTEGER FACTOR

RATE INCREASE BY AN INTEGER FACTOR

CHANGING THE SAMPLING RATE BY A NONINTEGER (RATIONAL) FACTOR

DIGITAL PROCESSING OF ANALOG SIGNALS

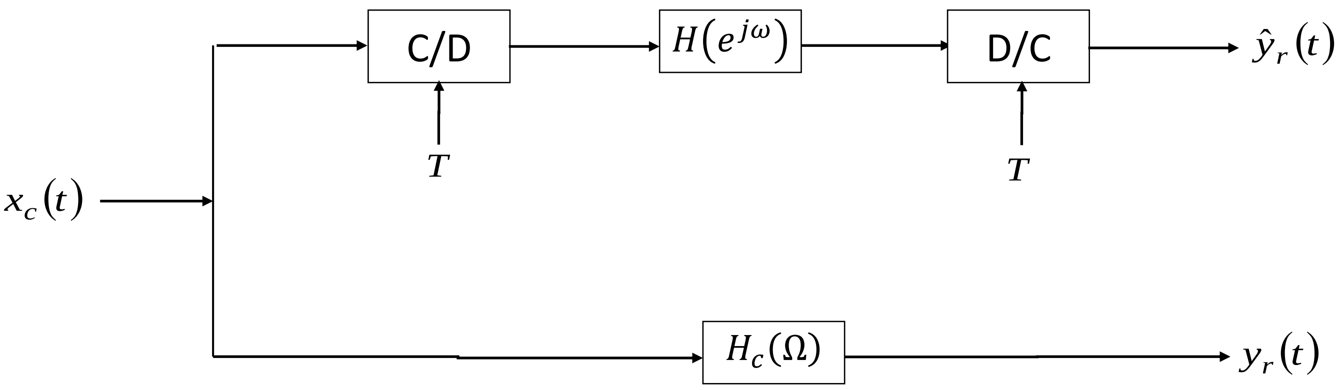
ANTI-ALIASING FILTER

ANALOG TO DIGITAL CONVERSION

QUANTIZATION

DIGITAL TO ANALOG CONVERSION

IMPULSE RESPONSES OF EQUIVALENT CT AND DT SYSTEMS



Does

$$y_r(t) = \hat{y}_r(t)$$

hold if

$$h[n] = h_c(nT) \quad ?$$

Let's see, if

$$h[n] = h_c(nT)$$

$$\Rightarrow \qquad H(e^{j\omega}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} H_c\left(\frac{\omega}{T} - k\,\frac{2\pi}{T}\right) \ .$$

Then, since

$$H_{eff}(\Omega) = \begin{cases} H(e^{j\Omega \mathrm{T}}) & |\Omega| < \dfrac{\pi}{T} \\ 0 & o.w. \end{cases}$$

If $H_c(\Omega)$ is bandlimited to $\frac{\pi}{T}$,

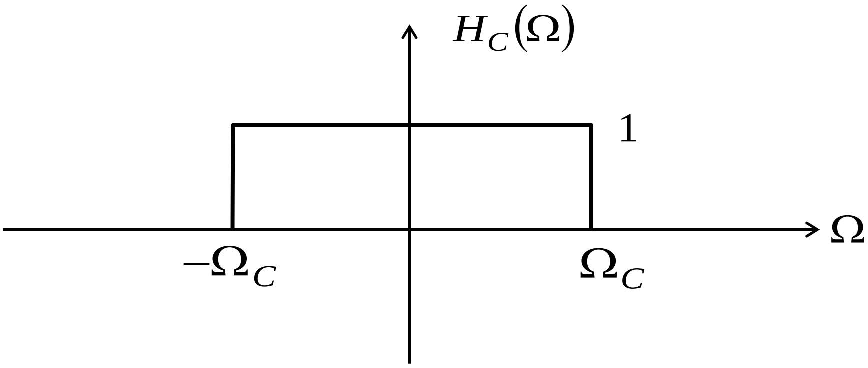
$$H_{eff}(\Omega) = \frac{1}{T} H_c(\Omega)$$

Therefore, upper and lower paths are equivalent, i.e. $y_r(t) = \hat{y}_r(t)$, if

$$h[n] = Th_c(nT)$$

Note that, it is assumed that $x_c(t)$ is bandlimited to $\frac{\pi}{T}$.

Ex:



$$h_c(t) = \frac{sin(\Omega_c t)}{\pi t}$$

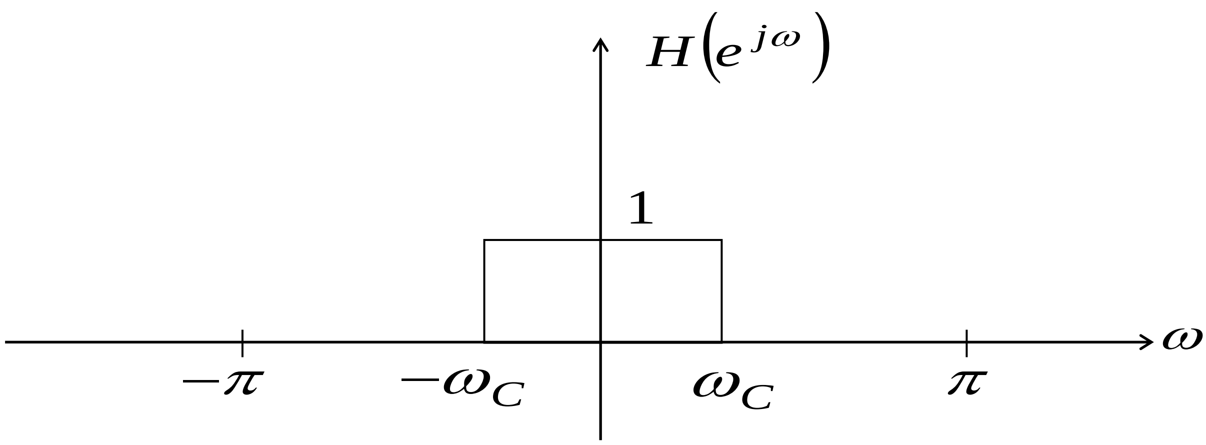
If

$$h[n] = T h_c(nT)$$

$$= T\,\frac{sin(\Omega_c T n)}{\pi T n}$$

$$= \frac{sin(\Omega_c T n)}{\pi n}$$

$$= \frac{sin(\omega_c n)}{\pi n}$$



Ex: Rational system functions

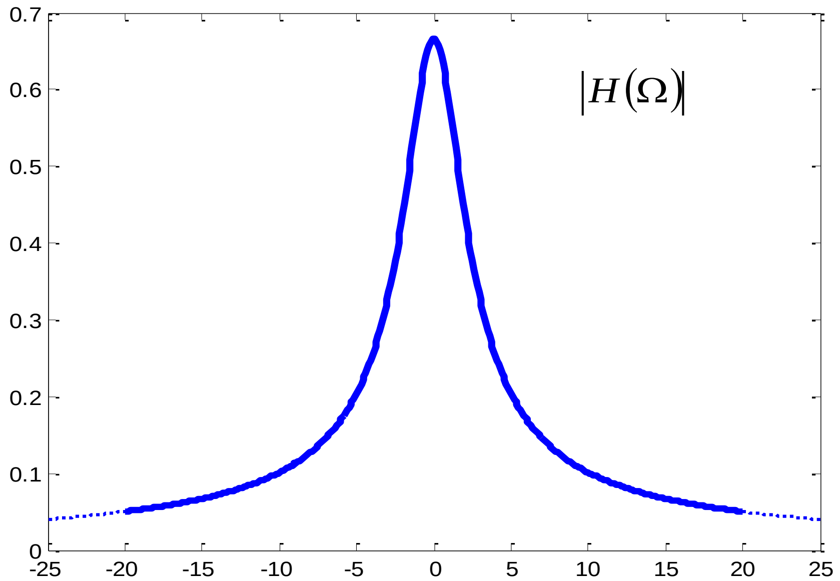
Let

$$H(s) = \frac{s+4}{s^2+5s+6}$$
$$= \frac{2}{s+2} - \frac{1}{s+3}$$

for which

$$h(t) = 2e^{-2t}u(t) - e^{-3t}u(t).$$

$$H(\Omega) = \frac{j\Omega+4}{-\Omega^2 + j5\Omega + 6}$$



$H(\Omega)$ is not bandlimited!

Ex: The frequency response of a LTI discrete-time system is $H(e^{j\omega}) = e^{-j\alpha\omega}$.

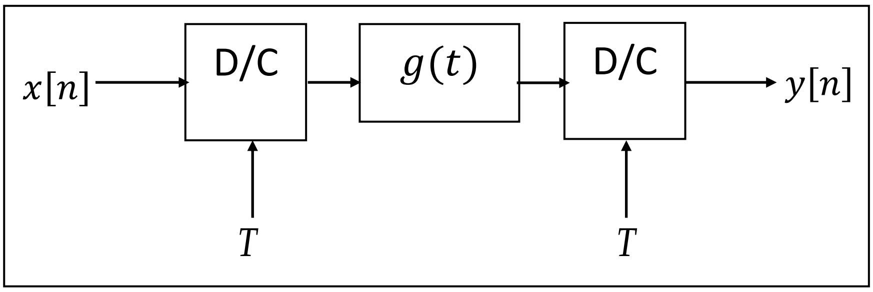
a) Plot the magnitude, phase and phase delay $\left(-\frac{\angle H(e^{j\omega})}{\omega}\right)$.

b) Find the impulse response of this system by carrying out the inverse DTFT.

c) Plot the impulse response for $\alpha = 3$.

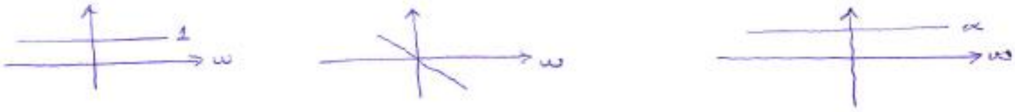
d) Plot the impulse repsonse for $\alpha = 3.5$.

e) What is the impulse response $g(t)$ so that $\frac{Y(e^{j\omega})}{X(e^{j\omega})} = e^{-j3.5\omega}$?



Comment on the results of part-d and part-e. (What is the function/purpose of the discrete time system whose frequency response is $e^{-j3.5\omega}$?)

a) $|H(e^{j\omega})| = 1$ $\quad\quad$ $\measuredangle H(e^{j\omega}) = -\alpha\omega$ $\quad\quad$ $\tau_{ph}(\omega) = -\dfrac{\measuredangle H}{\omega} = \alpha$

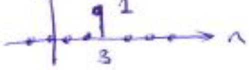


b) $$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-j\alpha\omega}\, e^{j\omega n}\, d\omega$$

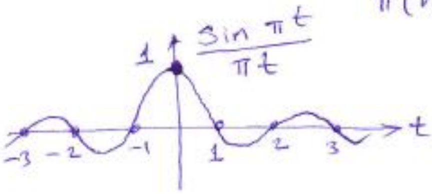
$$= \frac{1}{2\pi}\,\frac{1}{\pi(n-\alpha)}\; 2j\,\sin(\pi(n-\alpha))$$

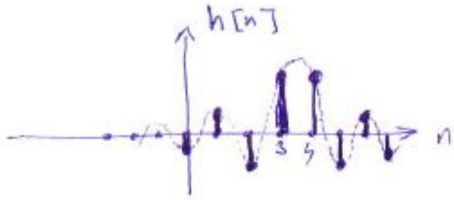
$$= \frac{\sin(\pi(n-\alpha))}{\pi(n-\alpha)} = h[n]$$

c) $\alpha = 3$ $\quad\quad$ $h[n] = \delta[n-3]$



d) $\alpha = 3.5$ $\quad\quad$ $h[n] = \dfrac{\sin(\pi(n-3.5))}{\pi(n-3.5)}$





e) $$X(e^{j\omega}) \rightarrow \underbrace{X(e^{j\Omega T})\, G(\Omega)}_{|\Omega| < \frac{\pi}{T}} \rightarrow X(e^{j\omega})\, G\left(\frac{\omega}{T}\right) = Y(e^{j\omega})$$

$$\Rightarrow G\left(\frac{\omega}{T}\right) = e^{-j3.5\omega} \quad \Rightarrow \quad G(\Omega) = e^{-j3.5T\Omega} \quad \Rightarrow \quad g(t) = \delta(t - 3.5T)$$

Comment: $e^{-j3.5\omega}$ (system) yields an output which can be obtained by resampling the continuous-time counterpart of its input signal after delaying by $3.5T$ seconds.

Problem Statement:

Given $x[n]$. Consider

$$x_c(t) = \sum_{n=-\infty}^{\infty} x[n]h_r(t - nT)$$

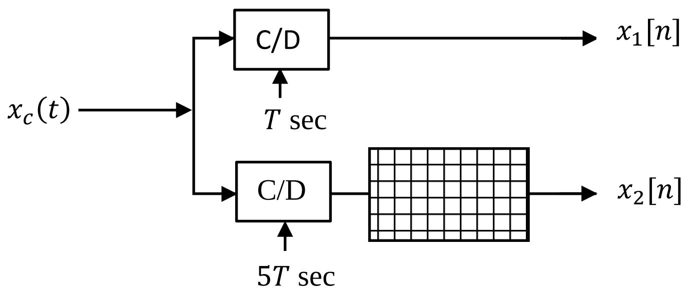
obtained from $x[n]$ by bandlimited interpolation.

We wish to construct another discrete-time sequence $y[n]$ such that

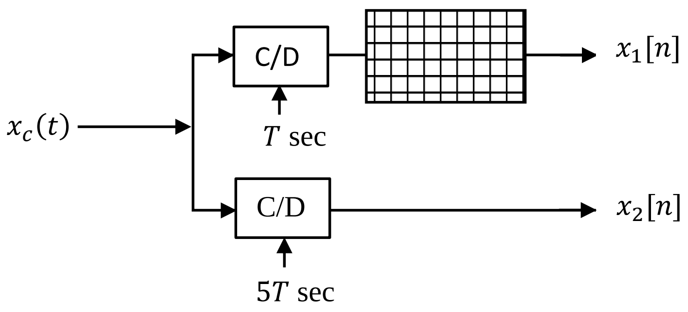
$$y[n] = x_c[nT']$$

Following examples may help to understand the rate change problem.

Find a discrete-time system to place into the grid box so that $x_2[n] = x_1[n]$.



OR



Note that $x_c(t)$ has to be bandlimited to …………

- We aim to do this in discrete-time, i.e., without generating $x_c(t)$ and resampling it with T', i.e., we want to obtain $y[n]$ from $x[n]$ by discrete-time processing.

- We will consider rational $\frac{T'}{T}$ rate changes.

- To do so, first, we will study rate increase

$$\frac{T'}{T} < 1$$

and rate decrease

$$\frac{T'}{T} > 1$$

by integer factors, then by rational factors.

RATE INCREASE BY AN INTEGER FACTOR (INTERPOLATION)

$$T' = \frac{T}{L} \qquad L\text{: positive integer, } L > 1$$

Define

$$x_i[n] = x_c(nT')$$

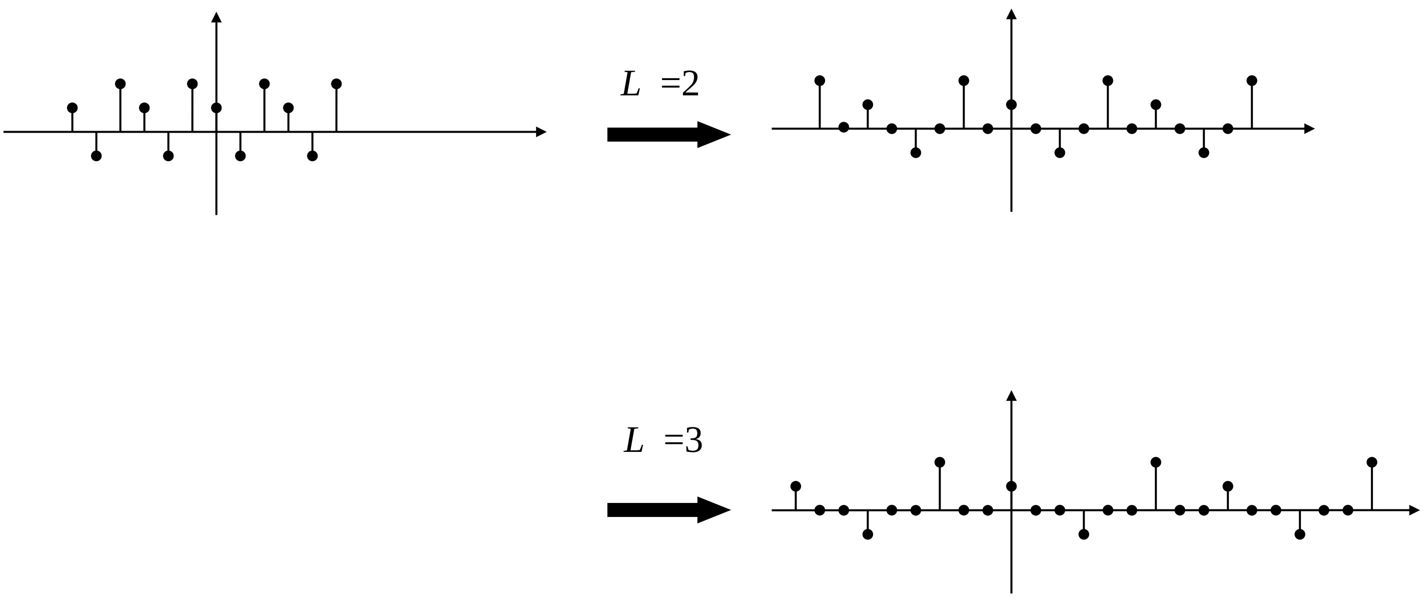
First, consider the "expander"

$$x[n] \longrightarrow \boxed{\uparrow L} \longrightarrow x_e[n]$$

L-fold

"upsampler"/ "expander"

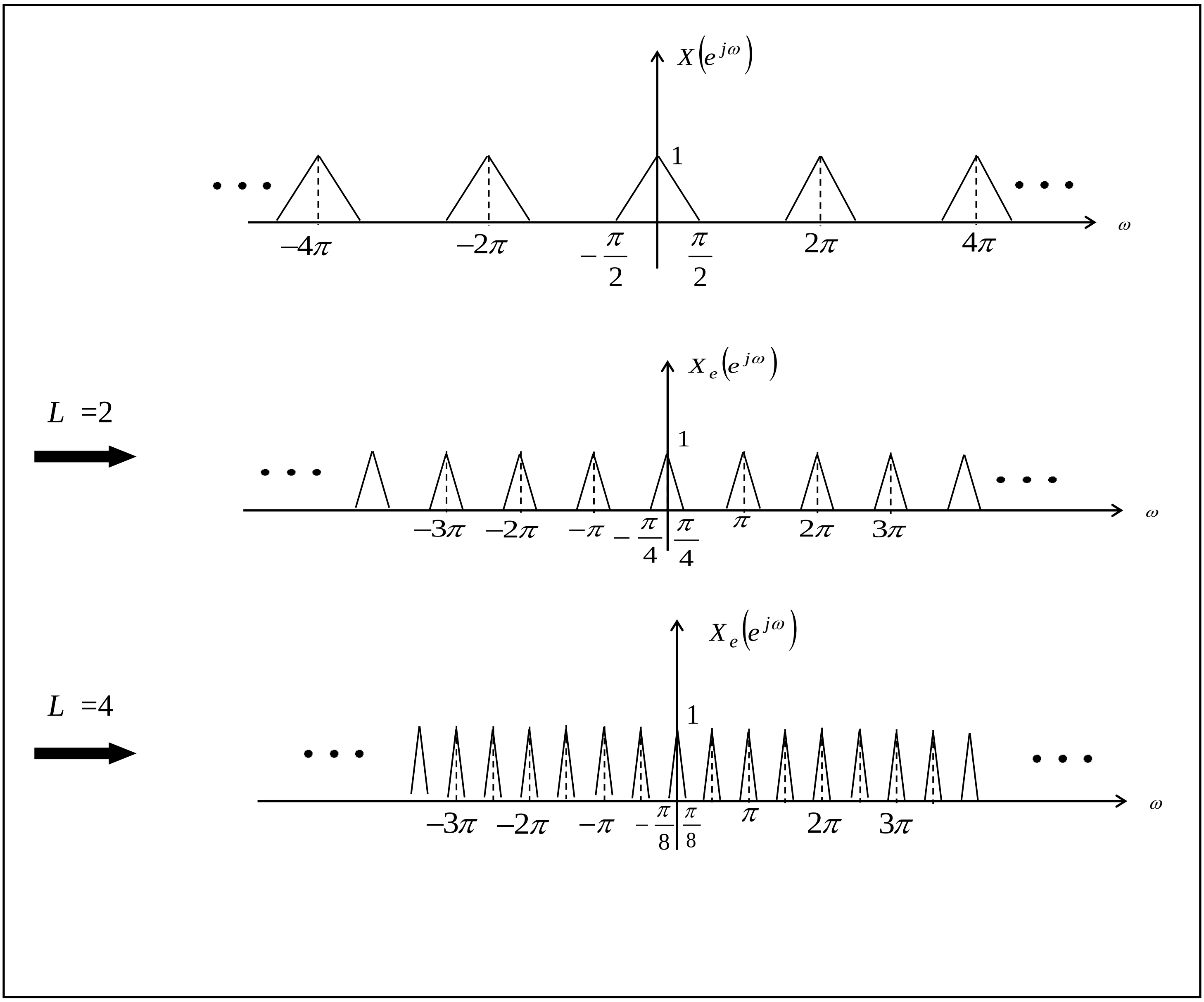
$$x_e[n] = \begin{cases} x\left[\frac{n}{L}\right] & n = 0, \pm L, \pm 2L, \ldots \\ 0 & otherwise \end{cases}$$



$$x_e[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-kL]$$

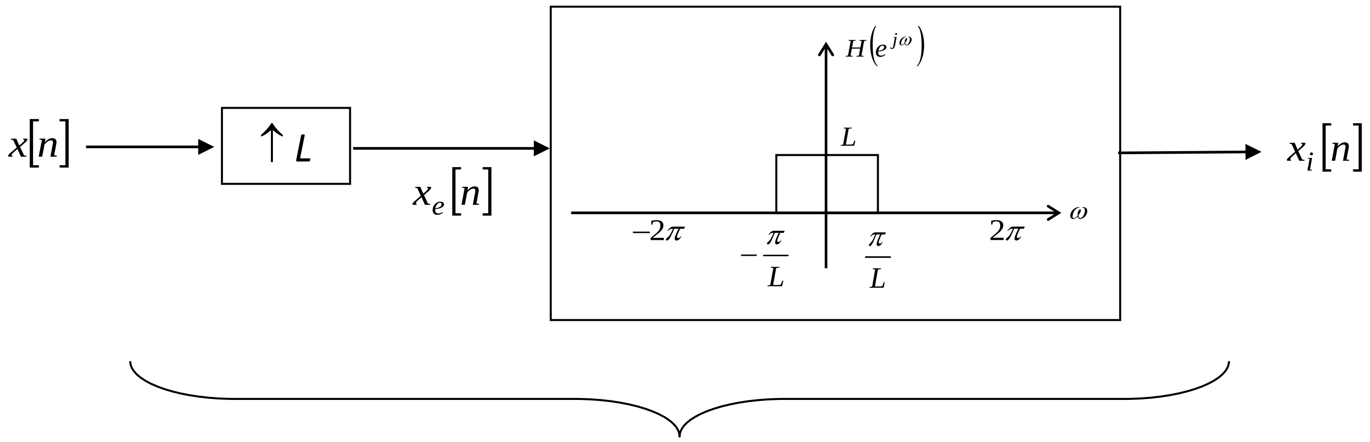
$$X_e\left(e^{j\omega}\right) = \sum_{k=-\infty}^{\infty} x[k]\, e^{-j\omega kL}$$

$$= X\left(e^{j\omega L}\right)$$

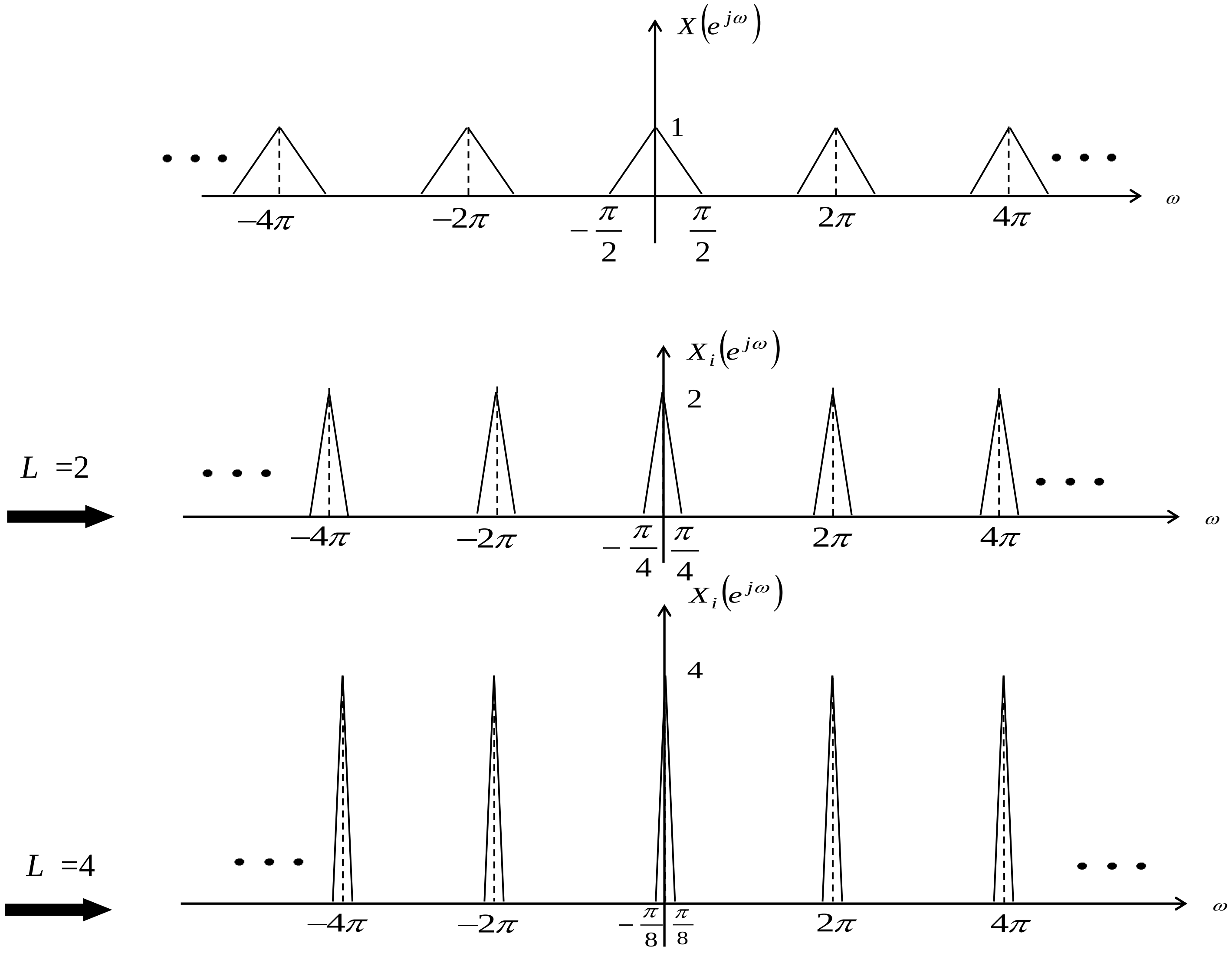


$X(e^{j\omega})$
1
-4π
-2π
$-\frac{\pi}{2}$
$\frac{\pi}{2}$
2π
4π
ω
$L = 2$
$X_e(e^{j\omega})$
1
-3π
-2π
$-\pi$
$-\frac{\pi}{4}$
$\frac{\pi}{4}$
π
2π
3π
ω
$L = 4$
$X_e(e^{j\omega})$
1
-3π
-2π
$-\pi$
$-\frac{\pi}{8}$
$\frac{\pi}{8}$
π
2π
3π
ω

Now, let's remove undesired components by lowpass filtering and provide a gain of L:



INTERPOLATION



So that

$$X_i(e^{j\omega}) = \frac{L}{T} \sum_{k=-\infty}^{\infty} X_c\left(\frac{L}{T}(\omega - k2\pi)\right)$$

as desired.

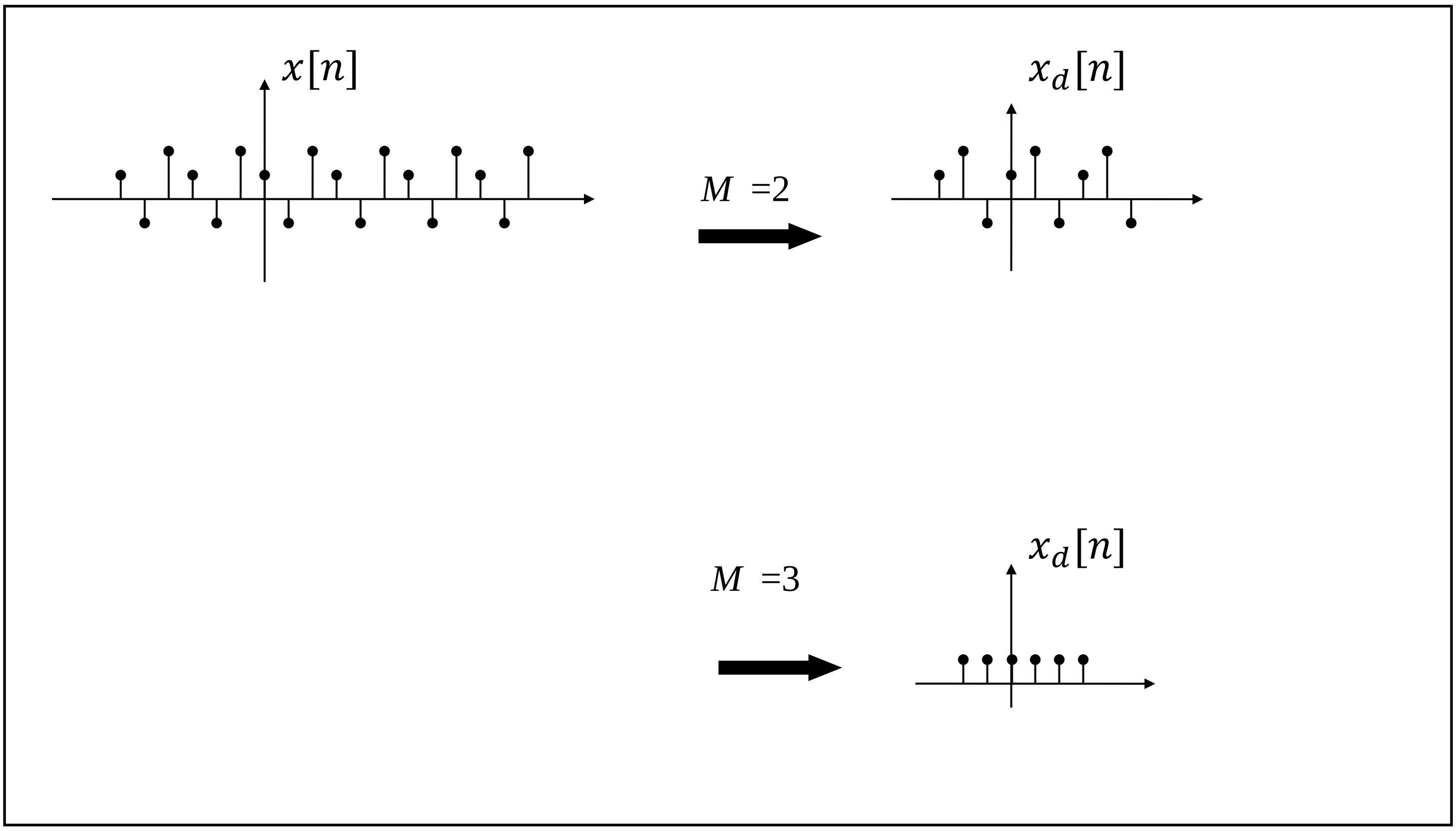
RATE REDUCTION BY AN INTEGER FACTOR (DECIMATION)

$$T' = MT$$

$$x_d[n] = x[Mn]$$

$$x[n] \longrightarrow \boxed{\downarrow M} \longrightarrow x_d[n]$$

M-fold
“downsampler”/ “compressor”



$x[n]$
M =2
$x_d[n]$
M =3
$x_d[n]$

Now we will relate $X\left(e^{j\omega}\right)$ and $X_d\left(e^{j\omega}\right)$.

Let

$$x_c(t) \triangleq \sum_{n=-\infty}^{\infty} x[n]h_r(t-nT)$$

We know that

$$X(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(\frac{1}{T}(\omega - k2\pi)\right)$$

since $\quad x[n] = x_c(nT),$

and

$$X_d(e^{j\omega}) = \frac{1}{MT}\sum_{k=-\infty}^{\infty} X_c\left(\frac{1}{MT}(\omega - k2\pi)\right) \qquad (*)$$

$$= \frac{1}{MT}\left(\cdots + X_c\left(\frac{1}{MT}(\omega+2\pi)\right) + X_c\left(\frac{\omega}{MT}\right) + X_c\left(\frac{1}{MT}(\omega-2\pi)\right) + \cdots\right)$$

Since $x_d[n] = x_c(nMT)$.

Here we have periodic extension of $X_c\left(\frac{\omega}{MT}\right)$ with period 2π.

Note that,

$$\frac{1}{M} X\left(e^{j\frac{\omega}{M}}\right) = \frac{1}{MT} \sum_{k=-\infty}^{\infty} X_c\left(\frac{1}{MT}(\omega - k2\pi M)\right) \qquad (**)$$

$$= \frac{1}{MT}\left(\cdots + X_c\left(\frac{1}{MT}(\omega + 2\pi M)\right) + X_c\left(\frac{\omega}{MT}\right)\right.$$

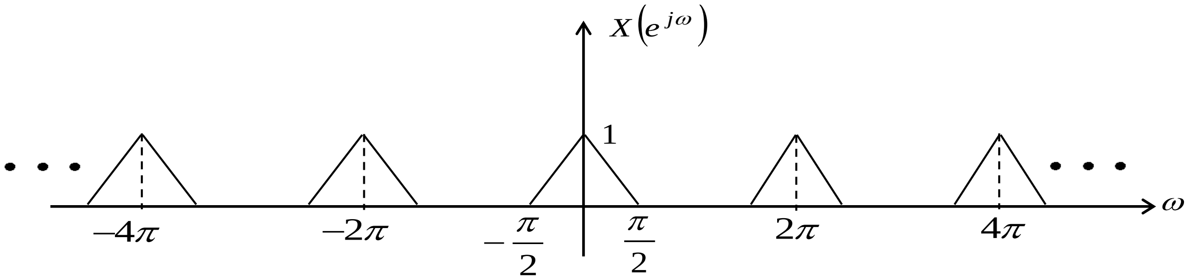
$$\left. + X_c\left(\frac{1}{MT}(\omega - 2\pi M)\right) + \cdots\right)$$

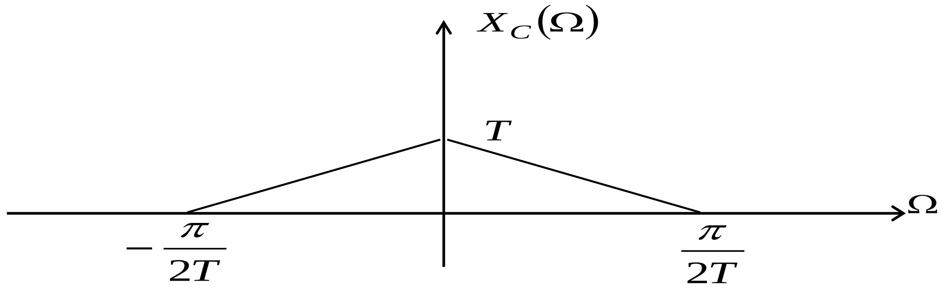
Here we have periodic extension of $X_c\left(\frac{\omega}{MT}\right)$ with period $2\pi M$.

Comparing (*) and (**)

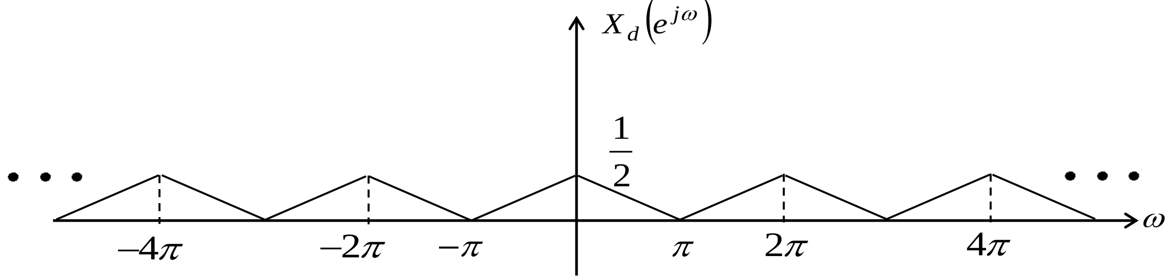
$$X_d\left(e^{j\omega}\right) = \frac{1}{M} \sum_{k=0}^{M-1} X\left(e^{j\left(\frac{\omega}{M} - k\frac{2\pi}{M}\right)}\right)$$

Ex:

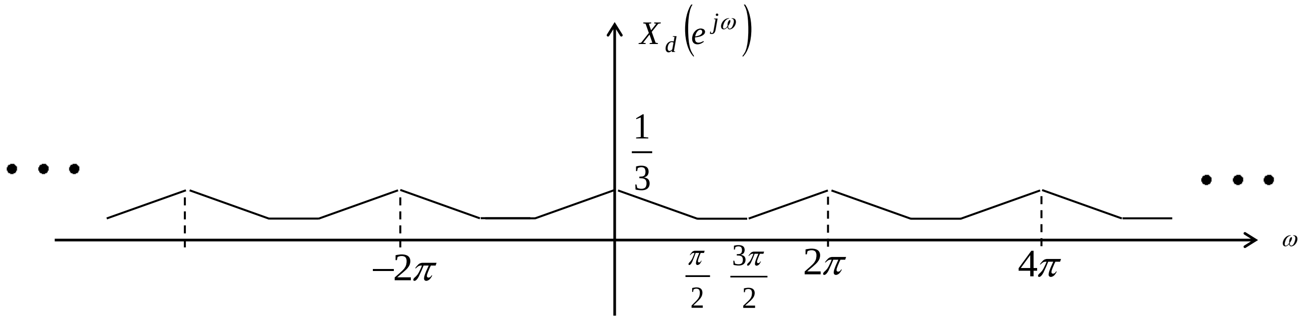




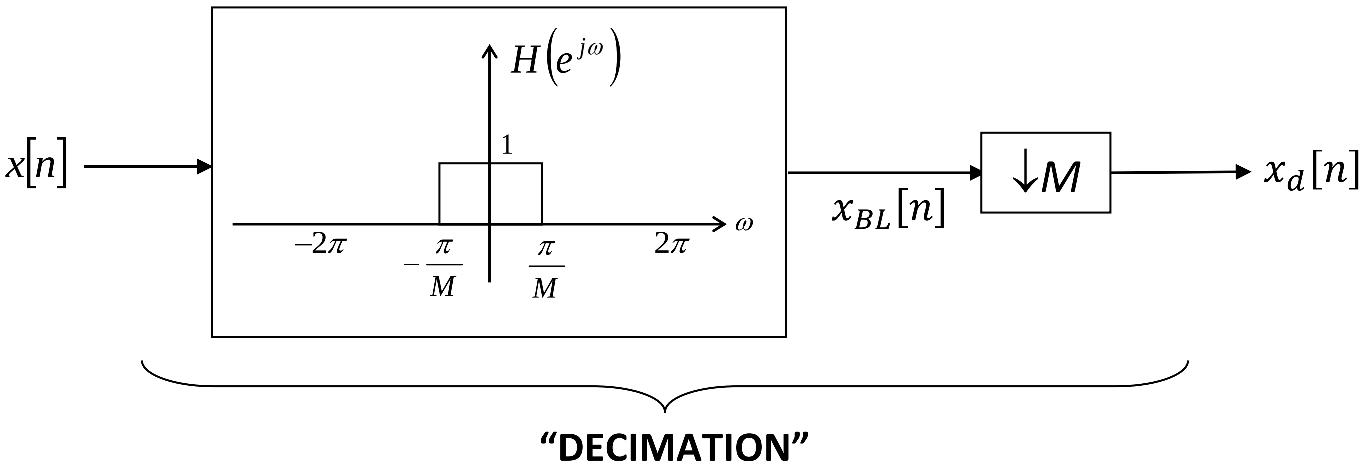
Let $M = 2$



If $M = 3$



Therefore, to avoid aliasing, before M-fold sampling rate reduction, an ideal lowpass filter having a cutoff frequency of $\frac{\pi}{M}$ has to be used!



Note that if $x[n]$ is not bandlimited to $\frac{\pi}{M}$, anti-aliasing filter causes distortion. However this is preferred against aliasing.

ANOTHER WAY TO RELATE $X\left(e^{j\omega}\right)$ AND $X_d\left(e^{j\omega}\right)$

Consider

$$x_d[n] \longrightarrow \boxed{\uparrow M} \longrightarrow \hat{x}[n]$$

Let

$$\hat{x}[n] \triangleq x[n] \left(\sum_{k=-\infty}^{\infty} \delta[n-kM] \right)$$

$$= x[n] \left(\frac{1}{M} \sum_{p=0}^{M-1} e^{jp\frac{2\pi}{M}n} \right)$$

$$\hat{X}\left(e^{j\omega}\right) = \frac{1}{M} \sum_{p=0}^{M-1} X\left(e^{j\left(\omega - p\frac{2\pi}{M}\right)}\right)$$

$$\hat{X}\left(e^{j\omega}\right) = X_d\left(e^{jM\omega}\right)$$

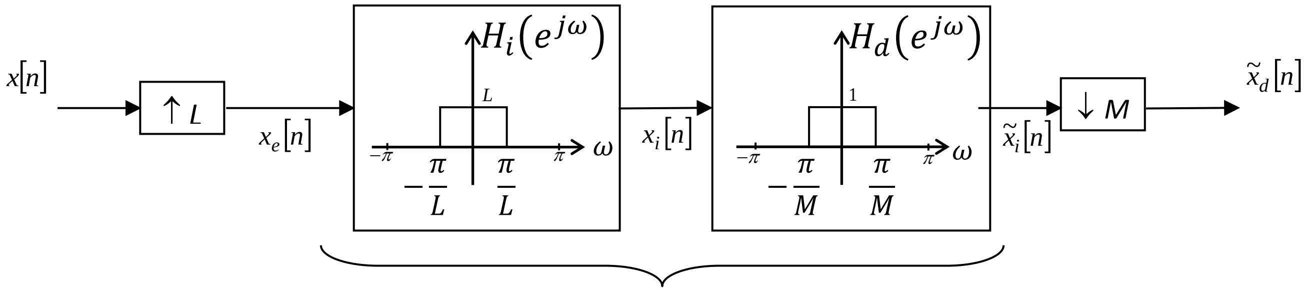
$$X_d\left(e^{j\omega}\right) = \hat{X}\left(e^{j\frac{\omega}{M}}\right)$$

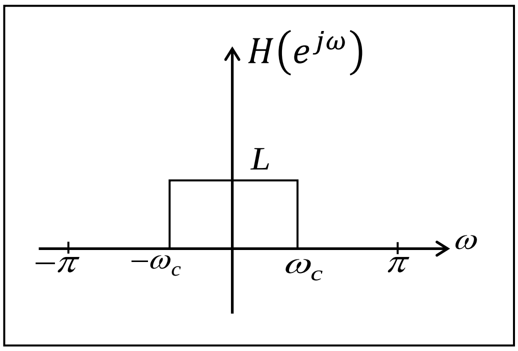
$$= \frac{1}{M} \sum_{p=0}^{M-1} X\left(e^{j\left(\frac{\omega}{M} - p\frac{2\pi}{M}\right)}\right)$$

CHANGING THE SAMPLING RATE BY A NONINTEGER (RATIONAL) FACTOR

First upsample (interpolate) by a factor of L, then downsample (decimate) by a factor of M.

Priority of upsampling is important!





$$\omega_c = \min\left\{\frac{\pi}{L}, \frac{\pi}{M}\right\}$$

If $\frac{L}{M} > 1$, sampling rate is increased.
If $\frac{L}{M} < 1$, sampling rate is decreased.

In either case, upsampling must be performed first!

Otherwise, $x[n]$ has to be bandlimited to $\frac{\pi}{M}$ although

no bandlimit is required for $\frac{L}{M} > 1$

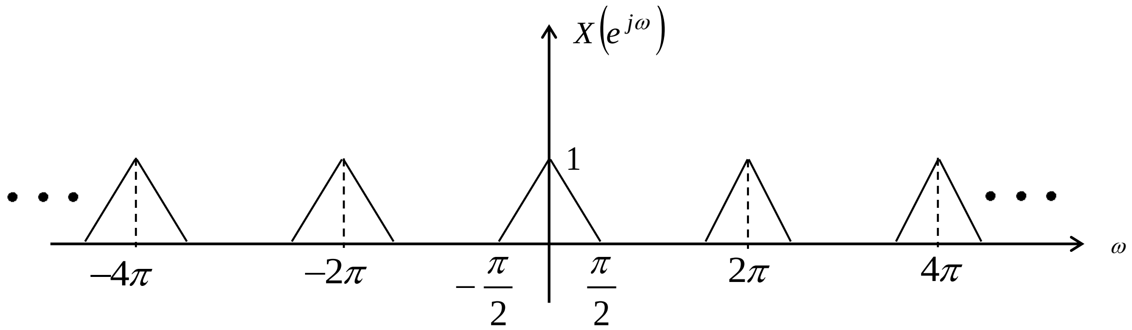
or

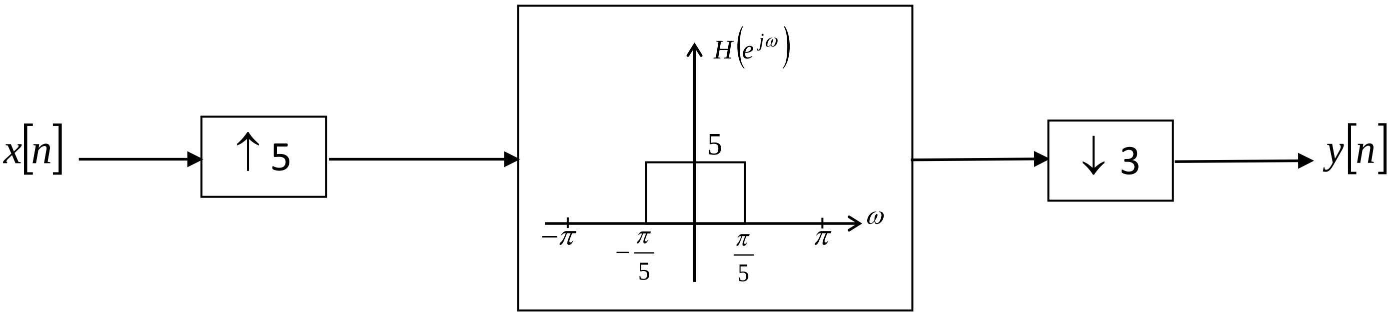
a bandlimit of $\frac{\pi L}{M}$ is sufficient for $\frac{L}{M} < 1$.

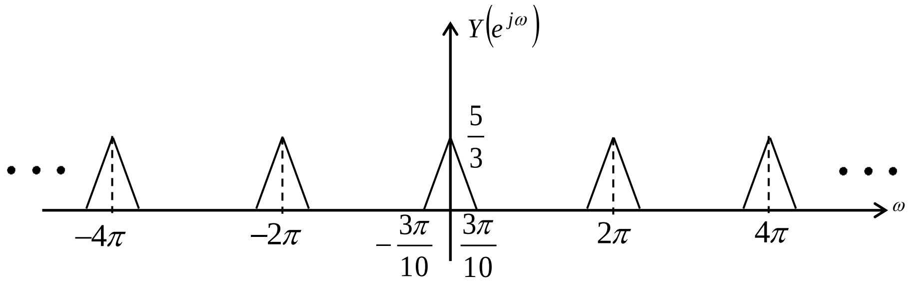
Ex: Let $L = 5$ and $M = 3$

Since sampling rate is increased, no bandlimit is required.

Assume that the input has the following spectrum,

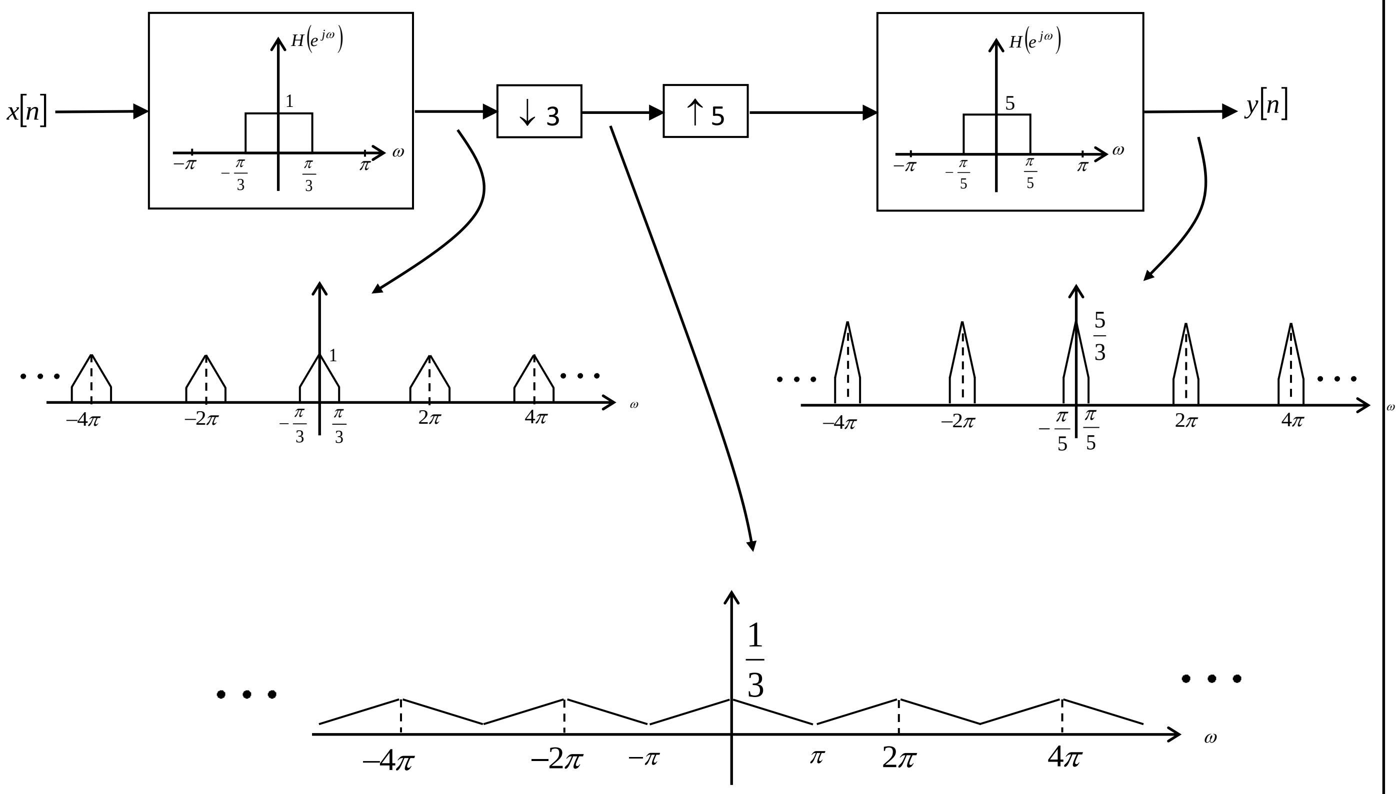






Rate change is achieved without signal distortion.

On the other hand, if downsampling is performed first



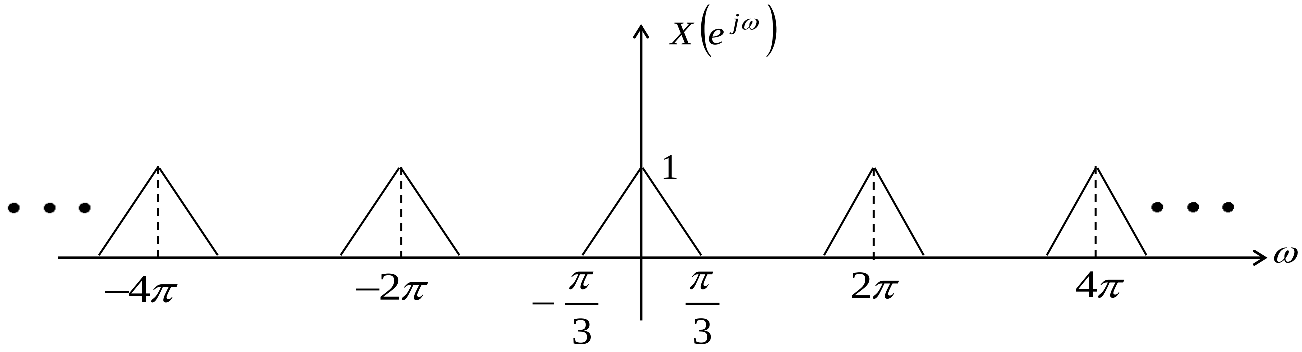
Rate change is achieved however signal is distorted!

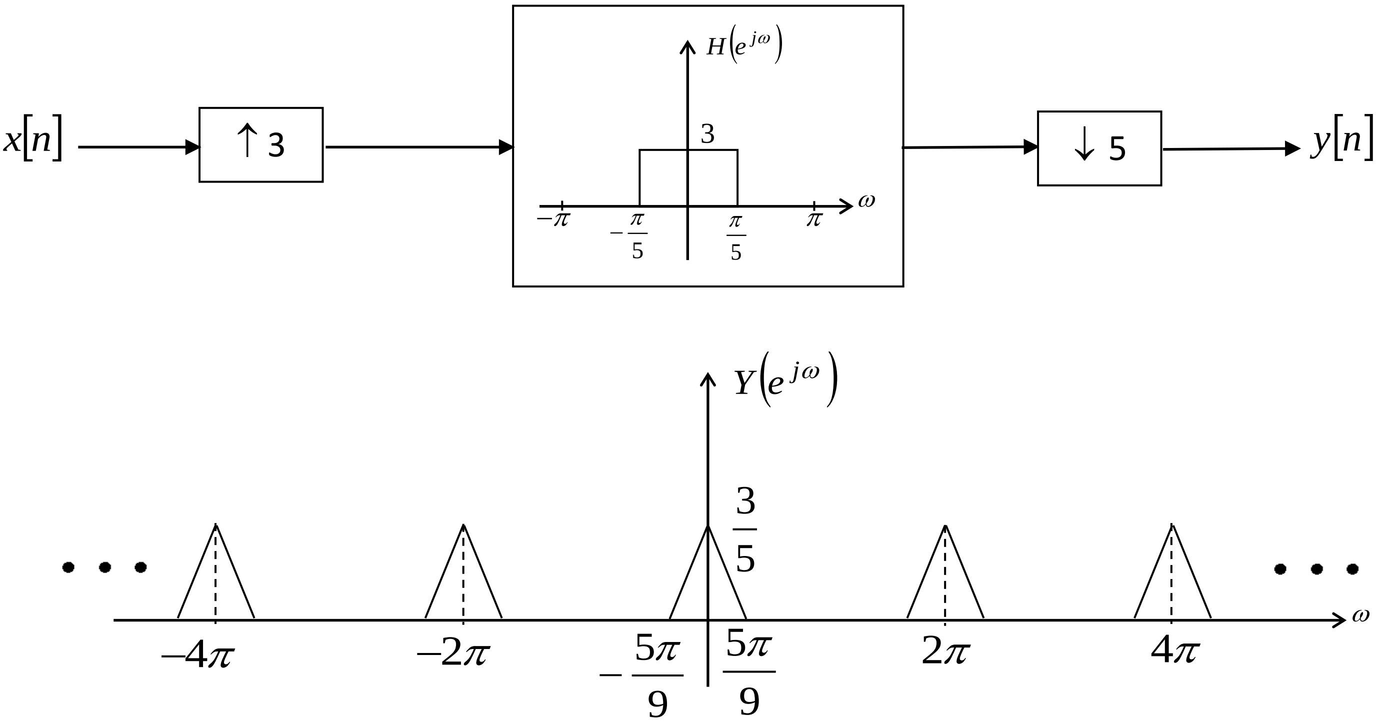
END OF THE EXAMPLE

Ex: Let $L = 3$ and $M = 5$

This time, SR is decreased so $x[n]$ has to be bandlimited to $\frac{3\pi}{5}$, otherwise aliasing distortion occurs.

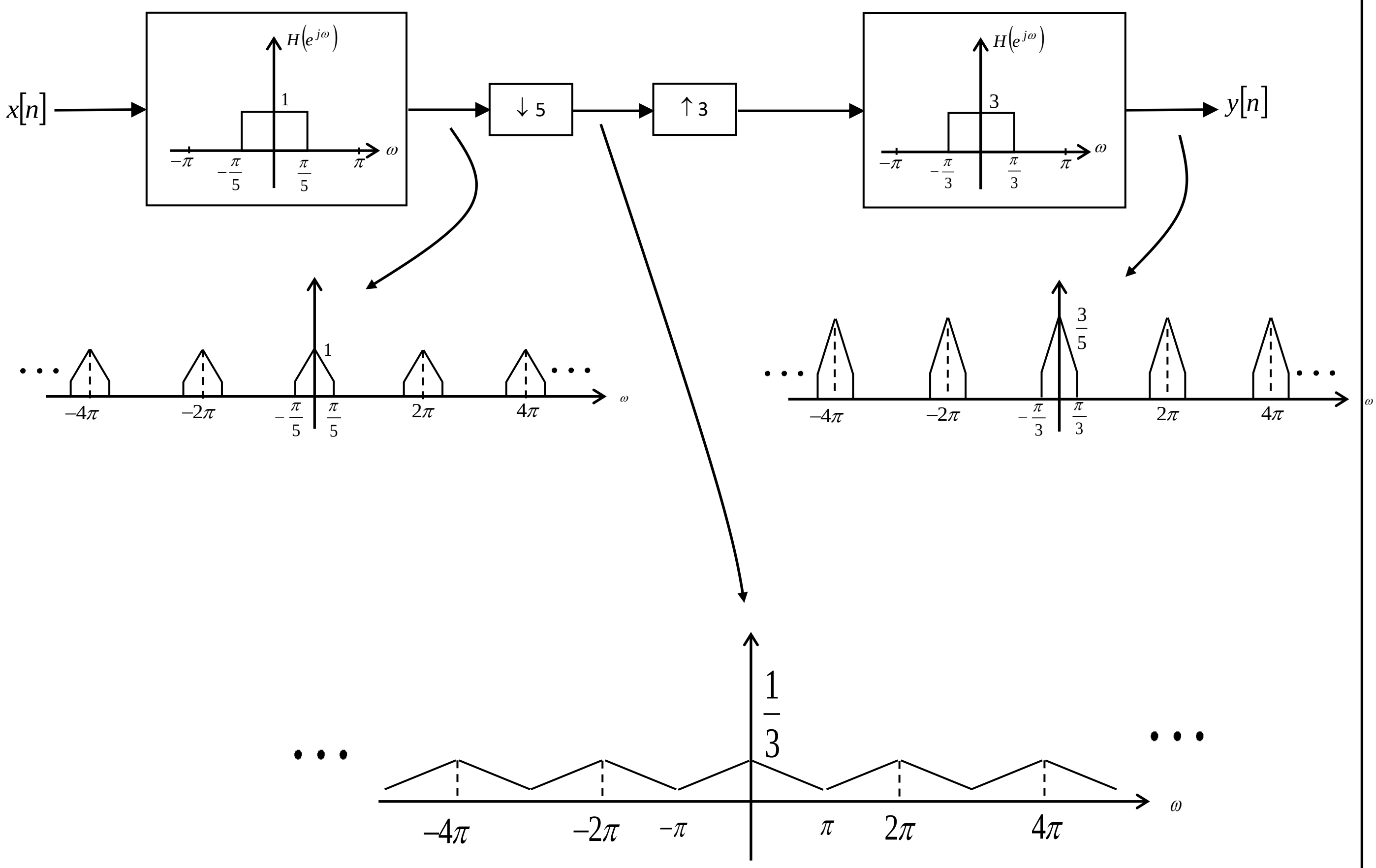
Assume that the input has the following spectrum,





Rate change is achieved without signal distortion.

On the other hand, if downsampling is performed first



Rate change is achieved, however, signal is distorted.

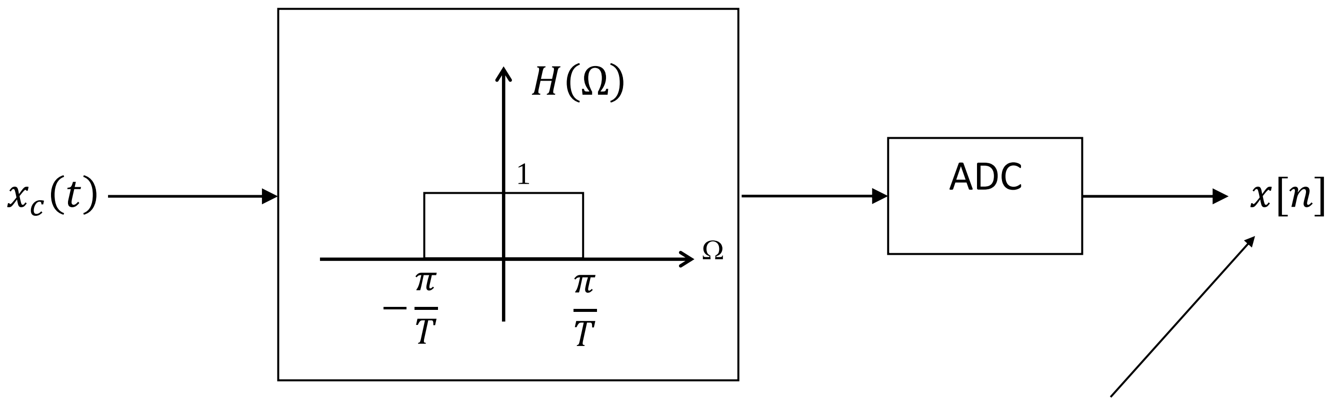
END OF THE EXAMPLE

ANTI-ALIASING FILTER

Anti-aliasing filter is a lowpass filter with a cutoff frequency of

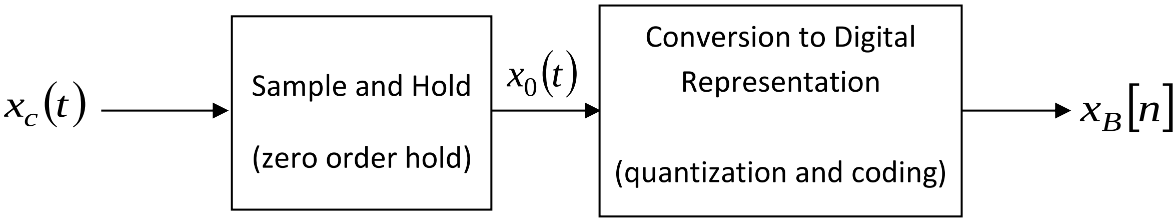
$$\frac{\pi}{T} = \frac{\Omega_s}{2}$$

ANTI-ALIASING FILTER



quantized values
(8, 12, 16, … bits)

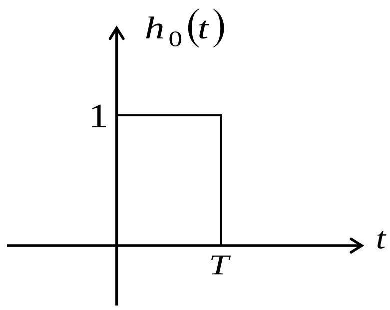
Ideal filter characteristic cannot be achieved in practice.
The distortion of the nonideal filter can be be taken into account in DT system design.

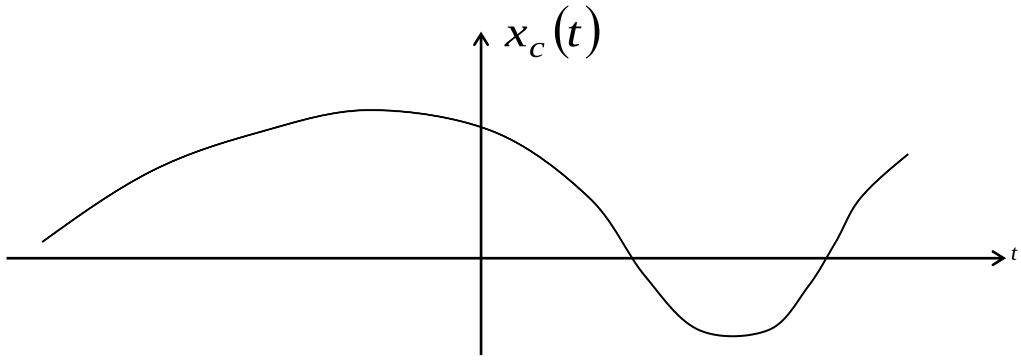


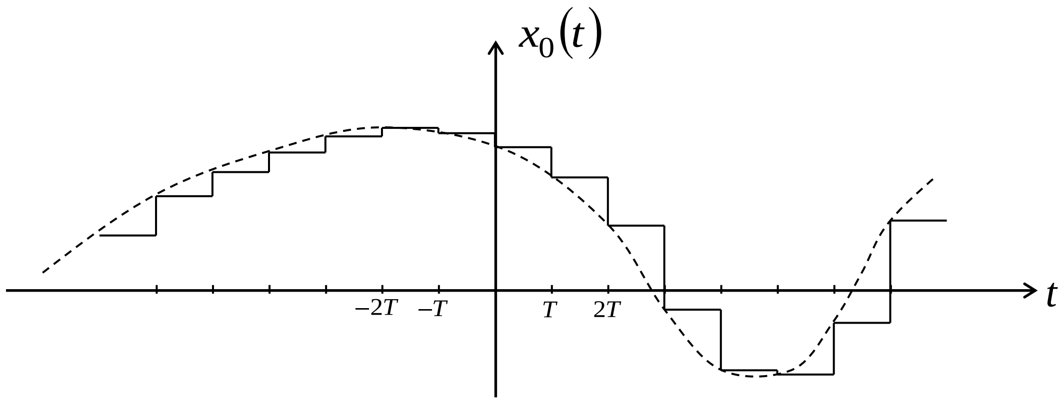
Zero order hold produces

$$x_0(t) = \sum_{n=-\infty}^{\infty} x_c(nT) h_0(t - nT)$$

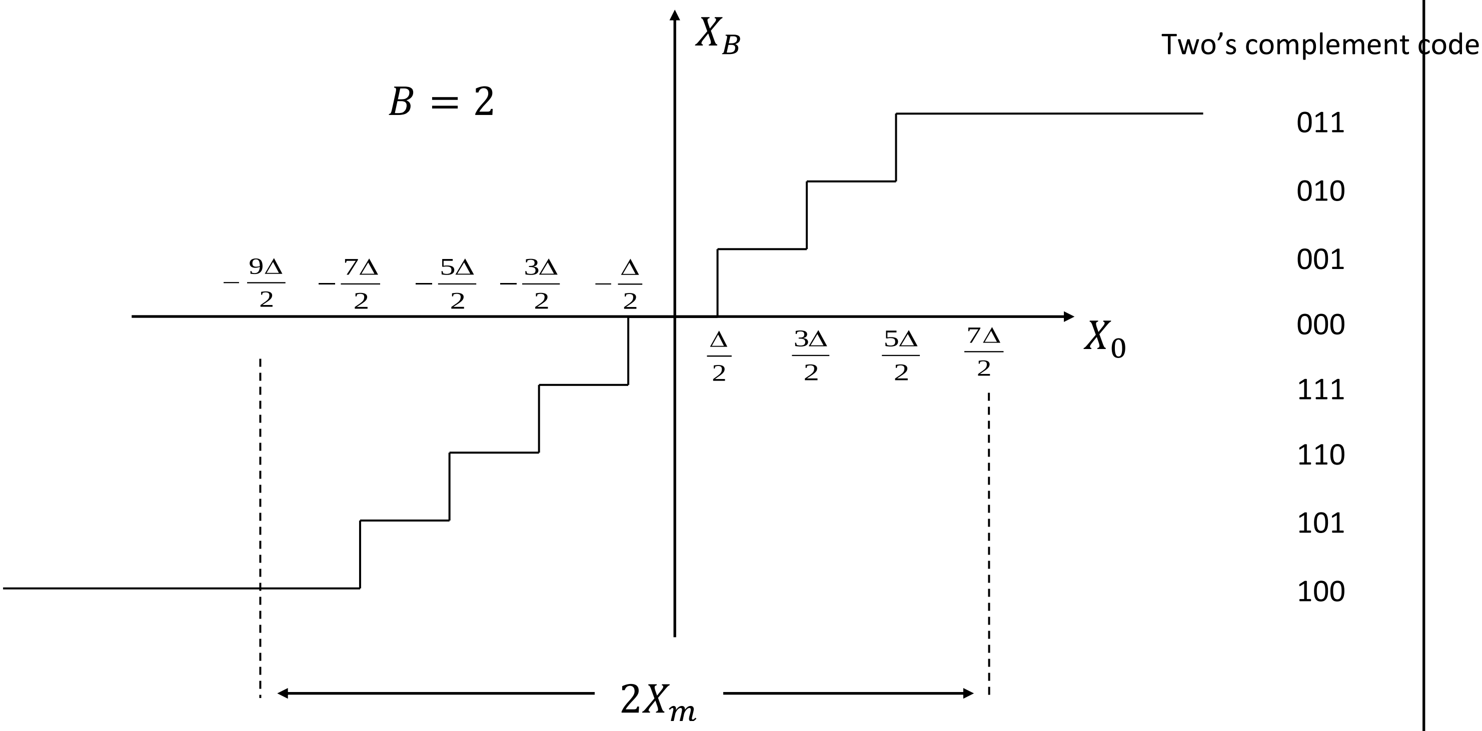
where




$x_c(t)$
t


$x_0(t)$
$-2T$
$-T$
T
$2T$
t

$B + 1$ bit uniform quantization.



Δ : quantization level

$2\ X_m$: dynamic range

$$\Delta = \frac{2X_m}{2^{B+1}}$$

$$= \frac{X_m}{2^B}$$

$$
X_B = \begin{cases}
011 & \frac{5\Delta}{2} \le X_0 \\
010 & \frac{3\Delta}{2} \le X_0 < \frac{5\Delta}{2} \\
001 & \frac{\Delta}{2} \le X_0 < \frac{3\Delta}{2} \\
000 & -\frac{\Delta}{2} \le X_0 < \frac{\Delta}{2} \\
111 & -\frac{3\Delta}{2} \le X_0 < -\frac{\Delta}{2} \\
110 & -\frac{5\Delta}{2} \le X_0 < -\frac{3\Delta}{2} \\
101 & -\frac{7\Delta}{2} \le X_0 < -\frac{5\Delta}{2} \\
100 & X_0 < -\frac{7\Delta}{2}
\end{cases}
$$

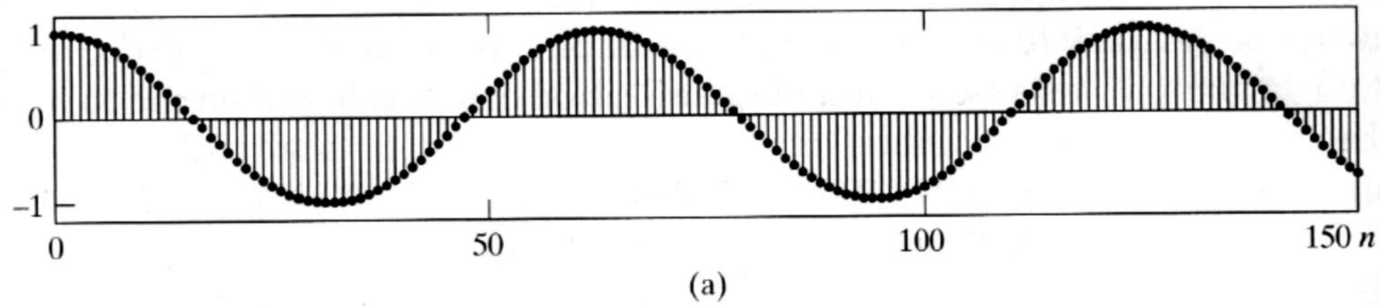


Figure 4.57 Example of quantization noise. (a) Unquantized samples of the signal $x[n] = 0.99\cos(n/10)$.

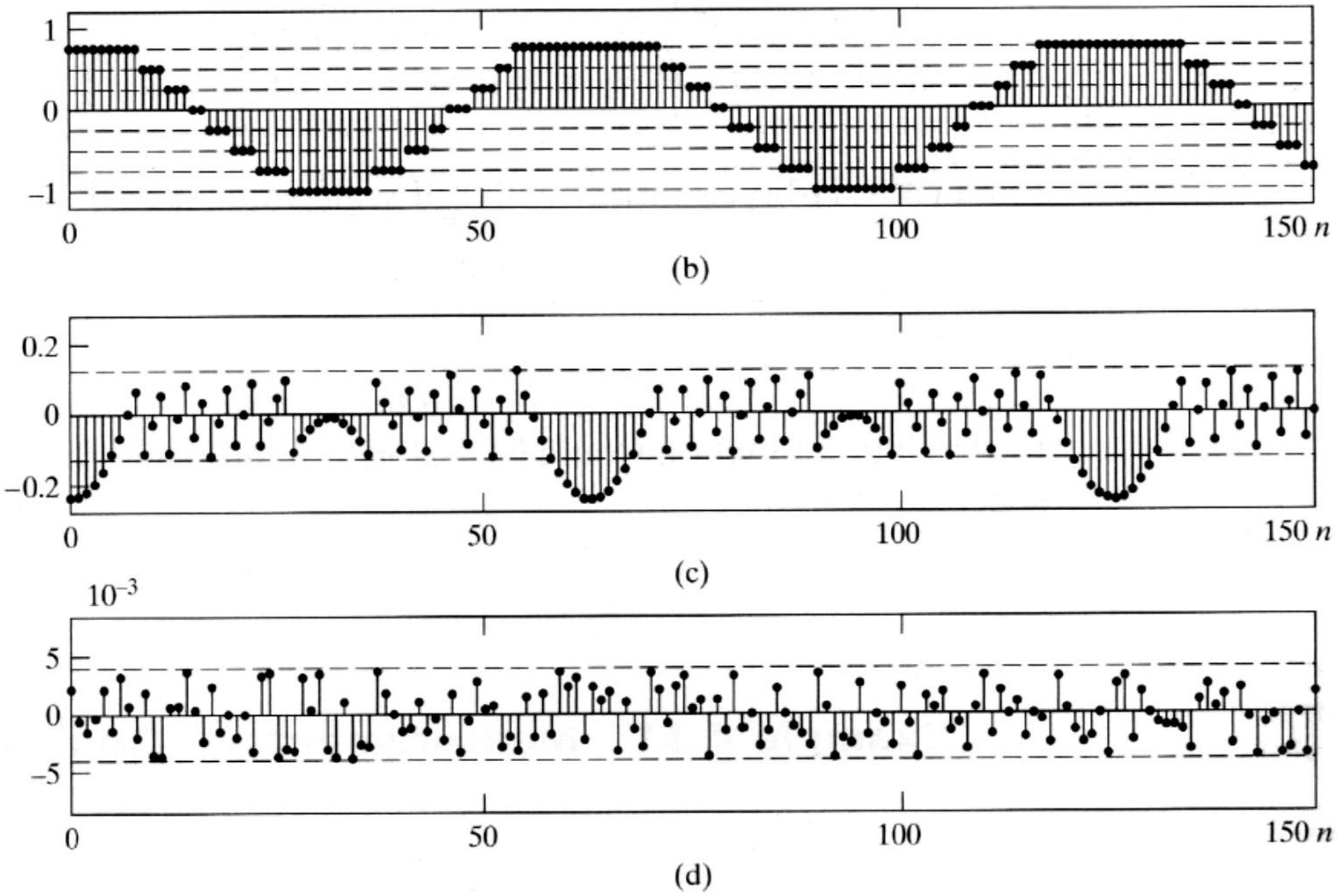
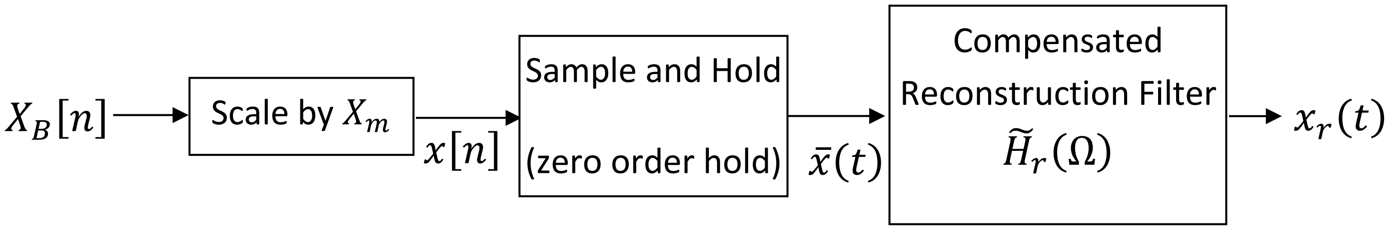
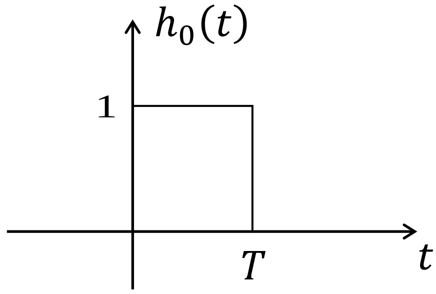
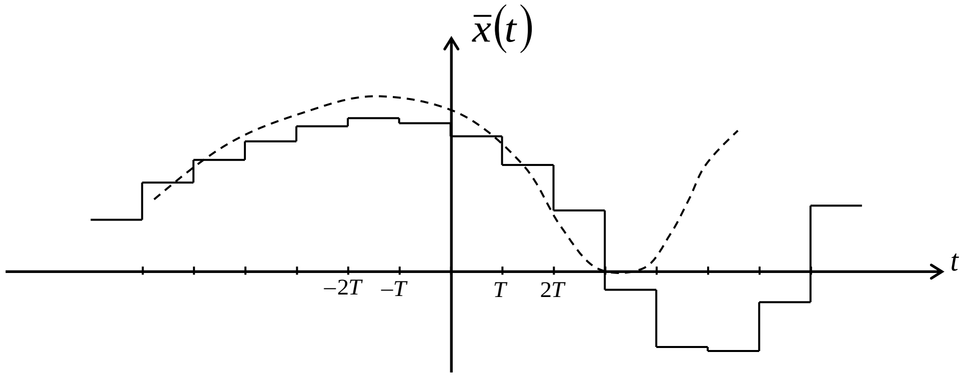


Figure 4.57 (*continued*) (b) Quantized samples of the cosine waveform in part (a) with a 3-bit quantizer. (c) Quantization error sequence for 3-bit quantization of the signal in (a). (d) Quantization error sequence for 8-bit quantization of the signal in (a).

DIGITAL TO ANALOG CONVERSION







$$\bar{x}(t) = \sum_{n=-\infty}^{\infty} x[n]h_0(t-nT)$$

$$\bar{X}(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega nT}H_0(\Omega)$$

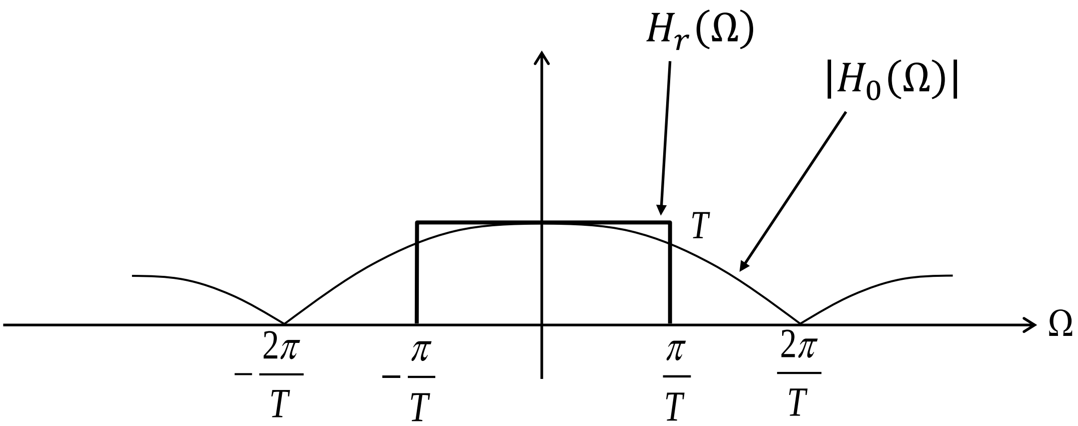
$$= X\left(e^{j\Omega T}\right)H_0(\Omega)$$

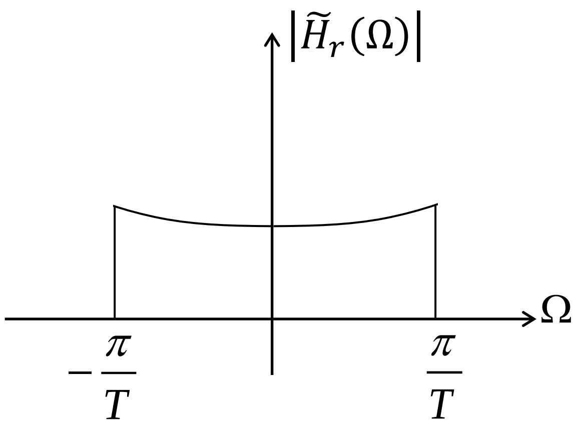
$$H_0(\Omega) = \frac{\sin\left(T\dfrac{\Omega}{2}\right)}{\dfrac{\Omega}{2}}e^{-j\frac{\Omega T}{2}}$$

Therefore (compensated) reconstruction filter can be specified as

$$\widetilde{H}_r(\Omega) = \frac{H_r(\Omega)}{H_0(\Omega)}$$

$$= \begin{cases} \dfrac{T\,\frac{\Omega}{2}}{\sin\left(\frac{\Omega T}{2}\right)} e^{j\frac{\Omega T}{2}} & |\Omega| < \dfrac{\pi}{T} \\ 0 & otherwise \end{cases}$$





Ex: $x_c(t) = \cos(\Omega_0 t)$

$$X_c(\Omega) = \pi\,\delta(\Omega - \Omega_0) + \pi\,\delta(\Omega + \Omega_0)$$

If $x_c(t)$ is sampled with sampling period *T*.

$$\Rightarrow x[n] = \cos(\Omega_0 T n)$$
$$= \cos(\omega_0 n)$$

$$X(e^{j\omega}) = \pi\delta(\omega - \Omega_0 T) + \pi\delta(\omega + \Omega_0 T) \qquad \text{in } (-\pi, \pi] \qquad (1)$$

and periodic with 2π.

Or we can use,

$$X(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(\frac{\omega}{T} - k\frac{2\pi}{T}\right)$$
$$= \frac{1}{T}\sum_{k=-\infty}^{\infty} \pi\,\delta\left(\frac{\omega}{T} - k\frac{2\pi}{T} - \Omega_0\right) + \frac{1}{T}\sum_{k=-\infty}^{\infty} \pi\,\delta\left(\frac{\omega}{T} - k\frac{2\pi}{T} + \Omega_0\right)$$

$$\delta\left(\frac{\omega}{T} - k\frac{2\pi}{T} - \Omega_0\right) = \delta\left(\frac{1}{T}(\omega - k2\pi - \Omega_0 T)\right) \underset{\uparrow}{=} T\delta(\omega - k2\pi - \Omega_0 T)$$

$$\text{since} \quad \delta(a\,\omega) = \frac{1}{|a|}\delta(\omega)$$

Therefore

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} \pi\delta(\omega - k2\pi - \Omega_0 T) + \sum_{k=-\infty}^{\infty} \pi\,\delta(\omega - k2\pi + \Omega_0 T) \qquad (2)$$

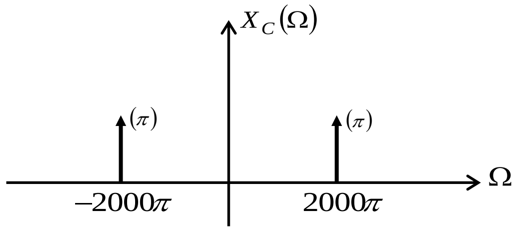
Obviously, (1) and (2) are the same.

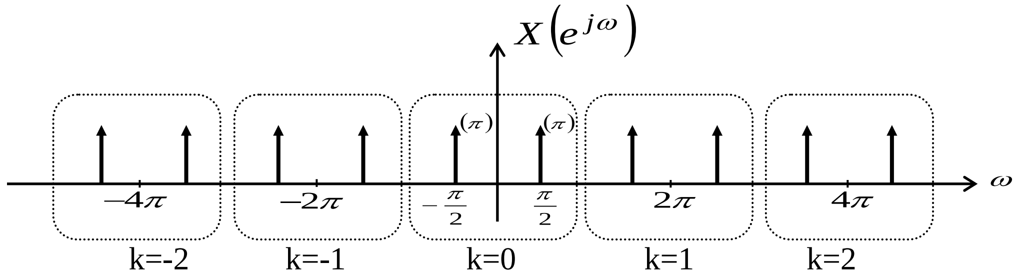
a) Let $\quad \Omega_0 = 2000\pi \quad (1\ \text{kHz})$

$\qquad T = 0.25\ \text{ms} \quad (4\ \text{kHz})$

$$\Rightarrow \Omega_0 T = \frac{\pi}{2}$$

$$X\left(e^{j\omega}\right) = \sum_{k=-\infty}^{\infty} \pi\delta\left(\omega - k2\pi - \frac{\pi}{2}\right) + \sum_{k=-\infty}^{\infty} \pi\,\delta\left(\omega - k2\pi + \frac{\pi}{2}\right)$$



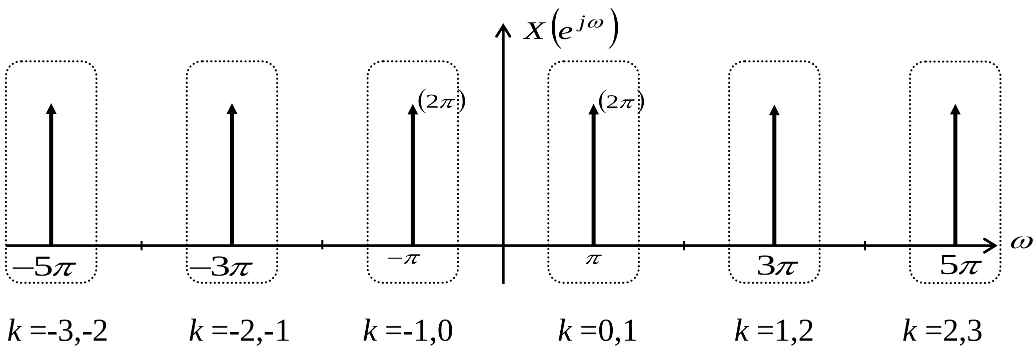


b) Let $\quad \Omega_0 = 2000\pi \quad (1\ \text{kHz})$

$\qquad T = 0.5\ \text{ms} \quad (2\ \text{kHz})$

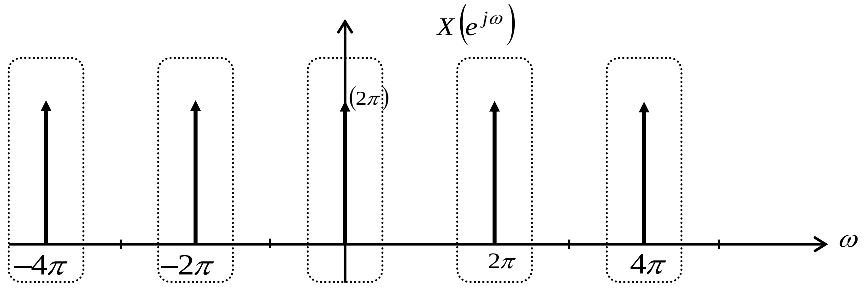
$$\Rightarrow \Omega_0 T = \pi$$

$$X\left(e^{j\omega}\right) = \sum_{k=-\infty}^{\infty} \pi\delta(\omega - k2\pi - \pi) + \sum_{k=-\infty}^{\infty} \pi\,\delta(\omega - k2\pi + \pi) \quad \left(= \sum_{k=-\infty}^{\infty} 2\pi\,\delta(\omega - k2\pi - \pi) \right)$$



c) Let $\begin{array}{ll}\Omega_0 = 2000\pi & (1\ \text{kHz}) \\ T = 1\ \text{ms} & (1\ \text{kHz})\end{array}$ $\Rightarrow \Omega_0 T = 2\pi$

$$X\left(e^{j\omega}\right) = \sum_{k=-\infty}^{\infty} \pi\delta(\omega - k2\pi - 2\pi) + \sum_{k=-\infty}^{\infty} \pi\,\delta(\omega - k2\pi + 2\pi) \quad \left(= \sum_{k=-\infty}^{\infty} 2\pi\,\delta(\omega - k2\pi) \right)$$

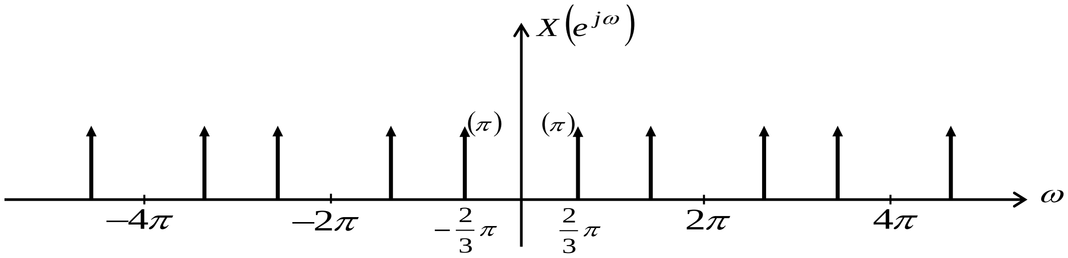


k =-3,-1 k =-2,0 k =-1,1 k =0,2 k =1,3

Indeed, in this case $x[n]=1$. (Reconstruction at any T yields a cont-time DC!)

d) Let $\begin{array}{ll}\Omega_0 = 2000\pi & (1\,\text{kHz}) \\ T = \dfrac{1}{300}\ \text{s} & (300\text{Hz})\end{array}$ $\Rightarrow \Omega_0 T = \dfrac{20}{3}\pi = \left(6 + \dfrac{2}{3}\right)\pi$

$$X\left(e^{j\omega}\right) = \sum_{k=-\infty}^{\infty} \pi\delta\left(\omega - k2\pi - \frac{20}{3}\pi\right) + \sum_{k=-\infty}^{\infty} \pi\,\delta\left(\omega - k2\pi + \frac{20}{3}\pi\right)$$

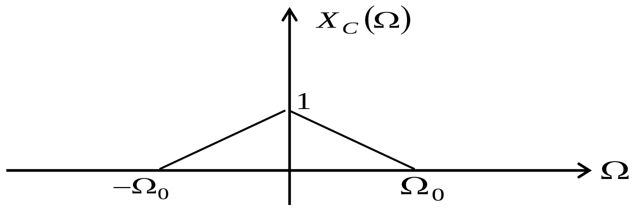


k=1 k=-5 k=2 k=-4 k=3 k=-3 k=4 k=-2 k=5 k=-1

END OF THE EXAMPLE

What do you get if you reconstruct at 300 Hz.?

Ex: Let



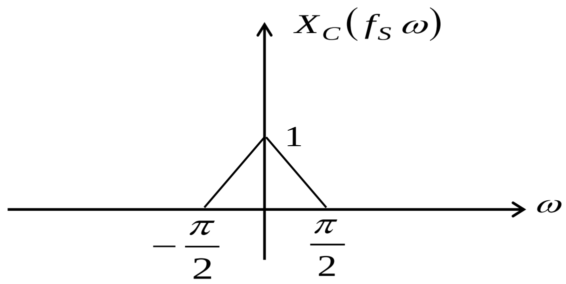
$$X\left(e^{j\omega}\right) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(\frac{\omega}{T} - k\frac{2\pi}{T}\right)$$

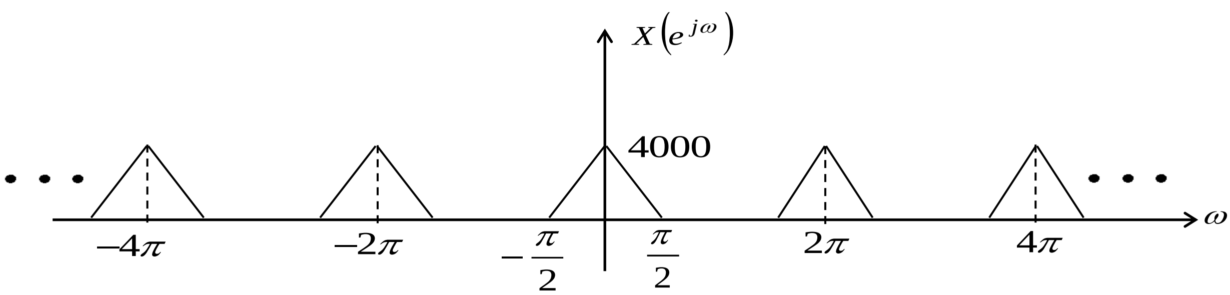
$$= \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(\frac{1}{T}(\omega - k2\pi)\right)$$

$$= \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(f_s(\omega - k2\pi)\right)$$

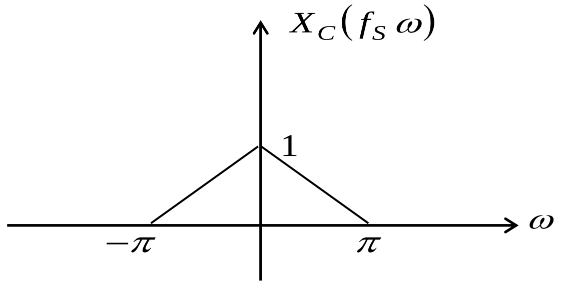
a) Let $\Omega_0 = 2000\pi$ (1 kHz) $\quad \Rightarrow \Omega_0 T = \dfrac{\pi}{2}$

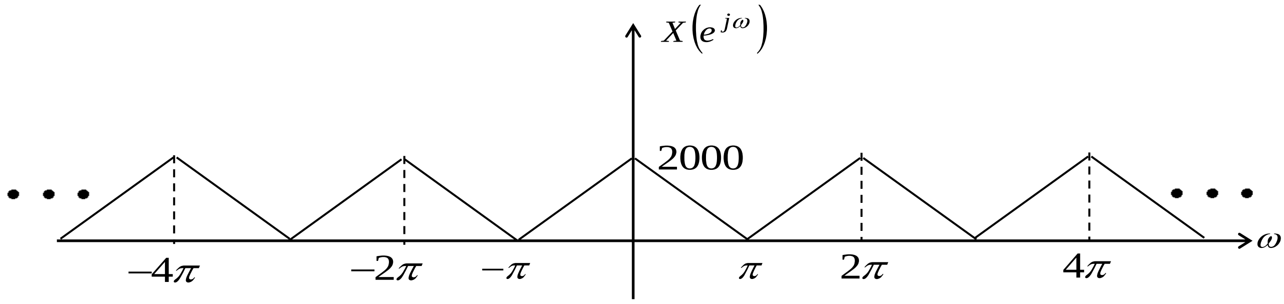
$T = 0.25$ ms (4 kHz)



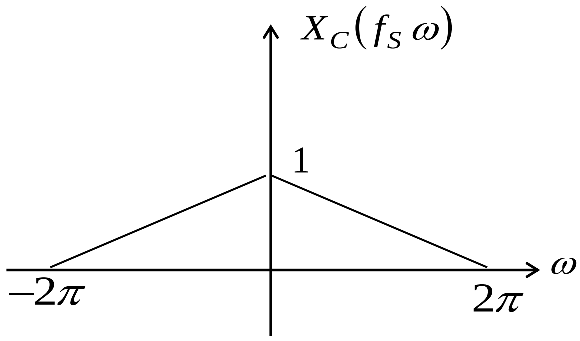


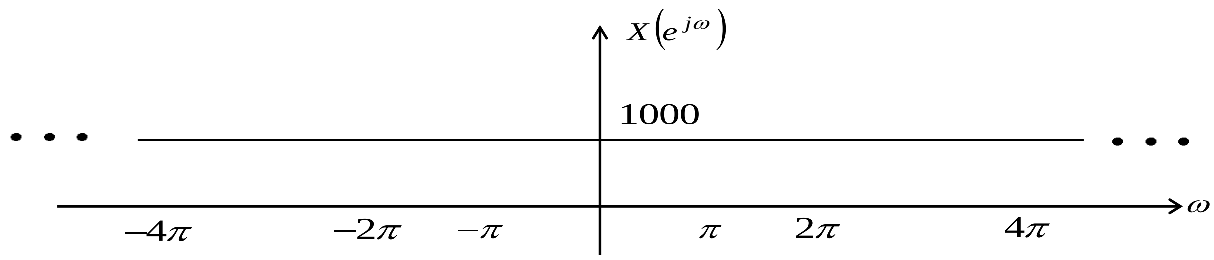
b) Let $\Omega_0 = 2000\pi \quad (1\ \text{kHz})$
$T = 0.5\ \text{ms} \quad (2\ \text{kHz})$ $\Rightarrow \Omega_0 T = \pi$





c) Let $\Omega_0 = 2000\pi \quad (1\ \text{kHz})$
$T = 1\ \text{ms} \quad (1\ \text{kHz})$ $\Rightarrow \Omega_0 T = 2\pi$





d) Let $\Omega_0 = 2000\pi$ (1 kHz)

$T = \frac{1}{300}$ s (300Hz) $\quad \Rightarrow \Omega_0 T = \frac{20}{3}\pi = \left(6 + \frac{2}{3}\right)\pi$