Homework 3 Solutions

Question 3

For a given input $x[n]$, we have

$$Y_1(e^{j\omega}) = S_1(e^{j\omega L})$$

where

$$S_1(e^{j\omega}) \triangleq H(e^{j\omega})X(e^{j\omega})$$

and

$$Y_2(e^{j\omega}) = G(e^{j\omega})S_2(e^{j\omega})$$

where

$$S_2(e^{j\omega}) \triangleq X(e^{j\omega L}).$$

We need

$$H(e^{j\omega L})X(e^{j\omega L}) = S_1(e^{j\omega L}) = Y_1(e^{j\omega}) = Y_2(e^{j\omega}) = G(e^{j\omega})S_2(e^{j\omega}) = G(e^{j\omega})X(e^{j\omega L}).$$

Since $x[n]$ is arbitrary,

$$H(e^{j\omega L}) = G(e^{j\omega}).$$

Question 8

Part a: Nyquist-rate=30KHz. In this case, $\Omega_s = 2\pi \cdot 15 \cdot 10^3 \text{ rad/s}$.

Part b: $\Omega_s = 2\pi \cdot 20 \cdot 10^3 \text{ rad/s}$.

Part c: $\Omega_s = 2\pi \cdot 29 \cdot 10^3 \text{ rad/s}$. Since this has the widest transition band, it is the easiest to implement.

Question 14

Part a:

$$\begin{aligned} H(z) &= \frac{\alpha(z-z_1)(z-z_1^*)(z-z_3)(z-z_3^*)}{(z-p_1)(z-p_1^*)(z-p_3)(z-p_3^*)} \\ &= \frac{\alpha(z-z_1)(z-z_1^*)(z-z_3)(z-z_3^*)(1-z_1^*z)(1-z_1z)(1-z_3^*z)(1-z_3z)}{(z-p_1)(z-p_1^*)(z-p_3)(z-p_3^*)(1-z_1^*z)(1-z_1z)(1-z_3^*z)(1-z_3z)} \\ &= H_{\text{min}}(z)H_{\text{ap}}(z) \end{aligned}$$

where

$$H_{\text{min}}(z) = \frac{\alpha(1-z_1z)(1-z_1^*z)(1-z_3z)(1-z_3^*z)}{(z-p_1)(z-p_1^*)(z-p_3)(z-p_3^*)}$$

$$H_{\mathrm{ap}}(z) \triangleq \frac{(z-z_1)(z-z_1^*)(z-z_3)(z-z_3^*)}{(1-z_1^*z)(1-z_1z)(1-z_3^*z)(1-z_3z)}.$$

$H_{\min}(z)$ has zeros at $\frac{e^{\pm j\pi/4}}{1.2}, \frac{e^{\pm j3\pi/4}}{1.2}$ and poles at $0.8e^{\pm j\pi/6}, 0.8e^{\pm j5\pi/6}$.

$H_{\mathrm{ap}}(z)$ has zeros at $1.2e^{\pm j\pi/4}, 1.2e^{\pm j3\pi/4}$ and poles at $\frac{e^{\pm j\pi/4}}{1.2}, \frac{e^{\pm j3\pi/4}}{1.2}$.

Part b:

For lowpass, we need poles that are nearby $z=1$ and/or zeros that are nearby $z=-1$.

For highpass, we need poles that are nearby $z=-1$ and/or zeros that are nearby $z=1$.

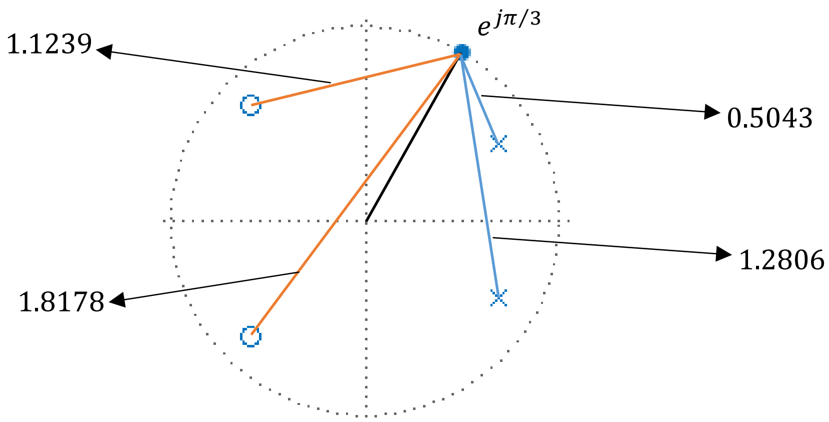
Therefore,

$$H_1(z) \triangleq \frac{(1-z_3z)(1-z_3^*z)}{(z-p_1)(z-p_1^*)}$$

$$H_2(z) \triangleq \frac{\alpha(1-z_1z)(1-z_1^*z)}{(z-p_3)(z-p_3^*)}.$$

Part c:

$$H_1(z) = \frac{1.44\left(z-\frac{1}{z_3}\right)\left(z-\frac{1}{z_3^*}\right)}{(z-p_1)(z-p_1^*)}$$



$$\left|H_1(e^{j\pi/3})\right| \approx \frac{1.44 \cdot 1.1239 \cdot 1.8178}{0.5043 \cdot 1.2806} \approx 4.56\,.$$

Question 15

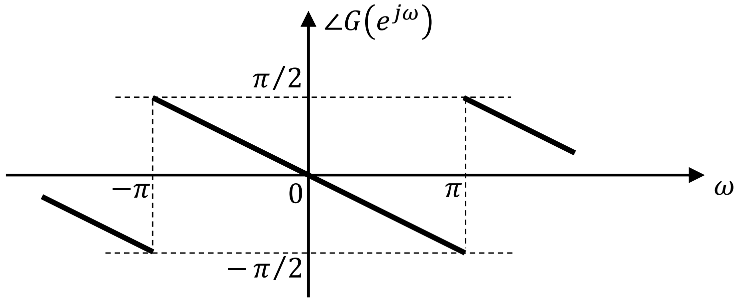
Part a:

$$G(z) = 1 + z^{-1} = \frac{z+1}{z}.$$

$G(z)$ has a pole at $z = 0$ and a zero at $z = -1$.

$$G(e^{j\omega}) = 1 + e^{-j\omega} = 2e^{-j\omega/2}\cos(\omega/2)$$

$$\angle G(e^{j\omega}) = -\omega/2\,, \qquad -\pi < \omega < \pi\,.$$



Part b:

i) The discontinuities at $\omega = \pi/2$ and $\omega = 3\pi/2$ are due to zeros on the unit circle, specifically at $z = \pm j$. The discontinuity at $\omega = \pi$ is due to plotting only the principal value of the argument.

ii) If ω is not at a discontinuity,

$$\tau(\omega) = -\frac{\mathrm{d}}{\mathrm{d}\omega}\angle H(e^{j\omega}) = 2\,.$$

Taking care of the discontinuities, the group delay is equal to 2 everywhere.
$H(e^{j\omega})$ is of the form

$$H(e^{j\omega}) = A(e^{j\omega})e^{-j2\omega}$$

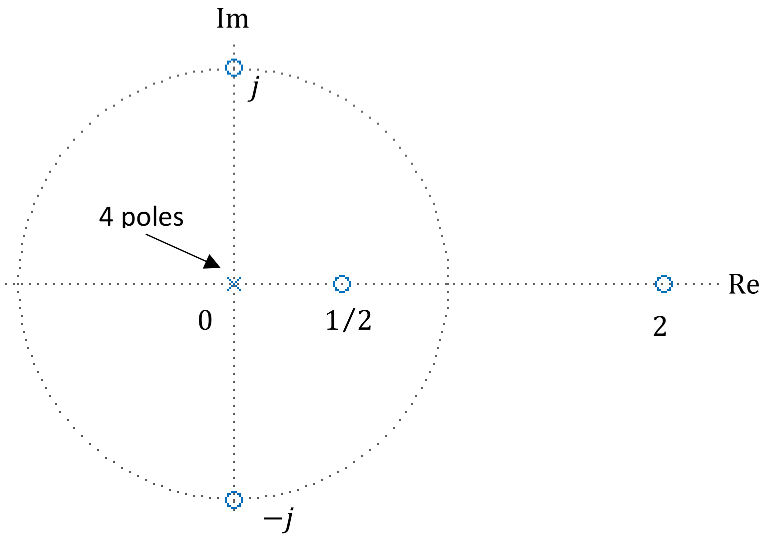
where $A(e^{j\omega})$ is real. It is of Type 1.

Part c:

Since $h[n]$ is causal, $h[n] = 0$ for $n < 0$. Therefore, $H(z)$ has four zeros and

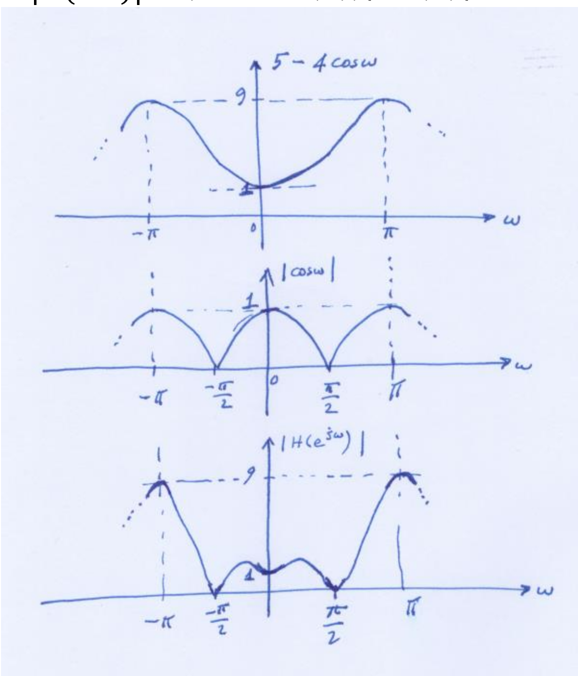
$$H(z) = \frac{\alpha(z-2)(z-1/2)(z-j)(z+j)}{z^4}$$

The constraint $1 = H(1)$ implies that $\alpha = -1$.



Part d:

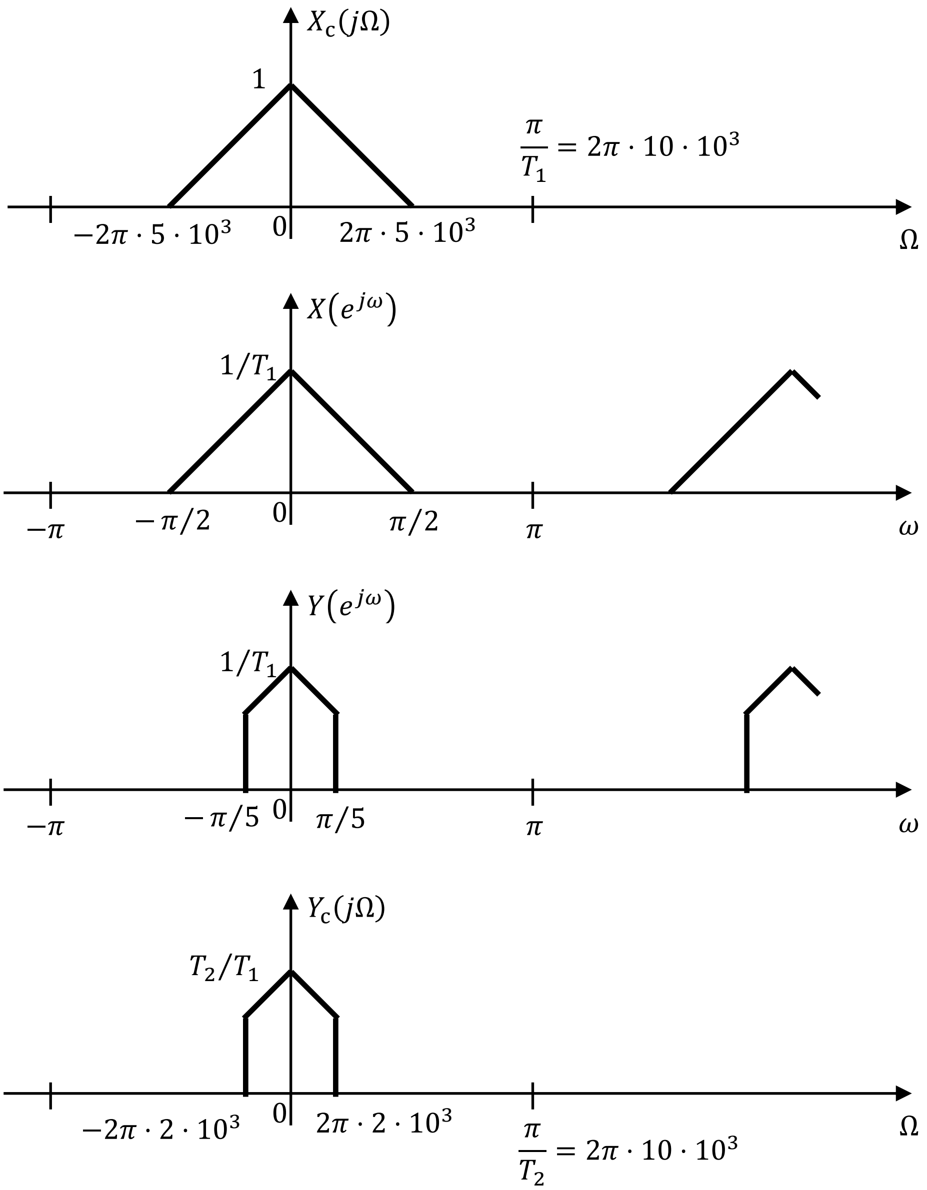
$$H(z) = -1 + (5/2)z^{-1} - 2z^{-2} + (5/2)z^{-3} - z^{-4}.$$
$$H(e^{j\omega}) = (-2\cos(2\omega) + 5\cos(\omega) - 2)e^{-j2\omega}$$
$$= (5 - 4\cos(\omega))\cos(\omega)\, e^{-j2\omega}.$$
$$|H(e^{j\omega})| = (5 - 4\cos(\omega))|\cos(\omega)|.$$



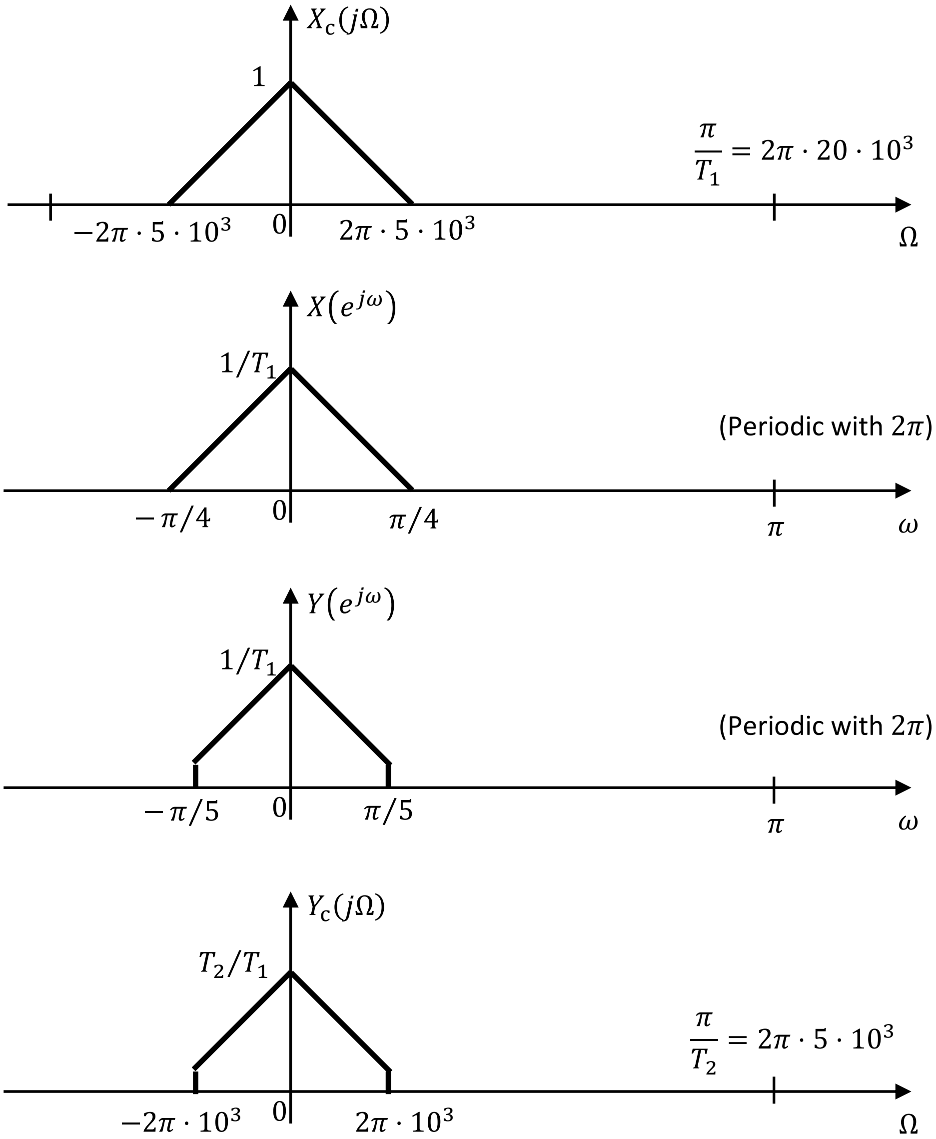
Question 17

Part a:

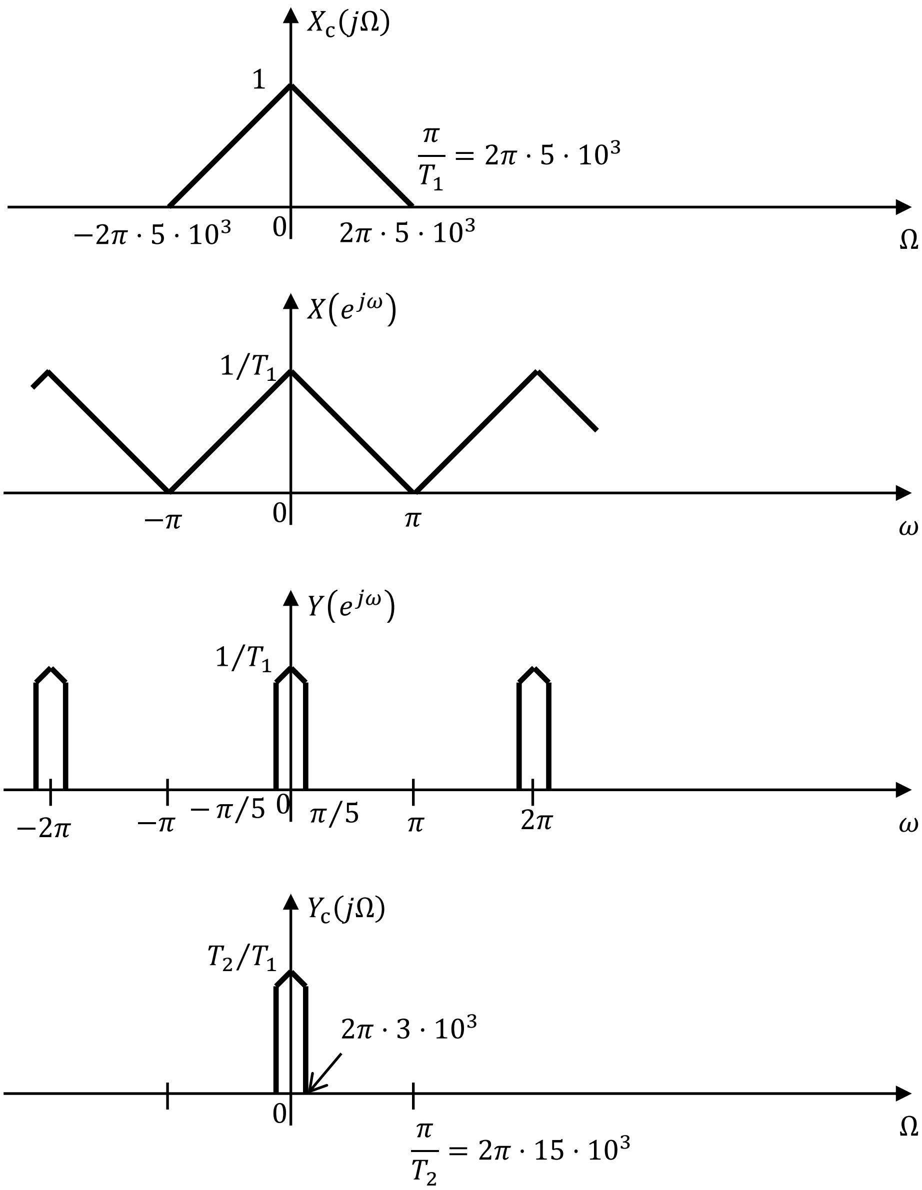
(i)



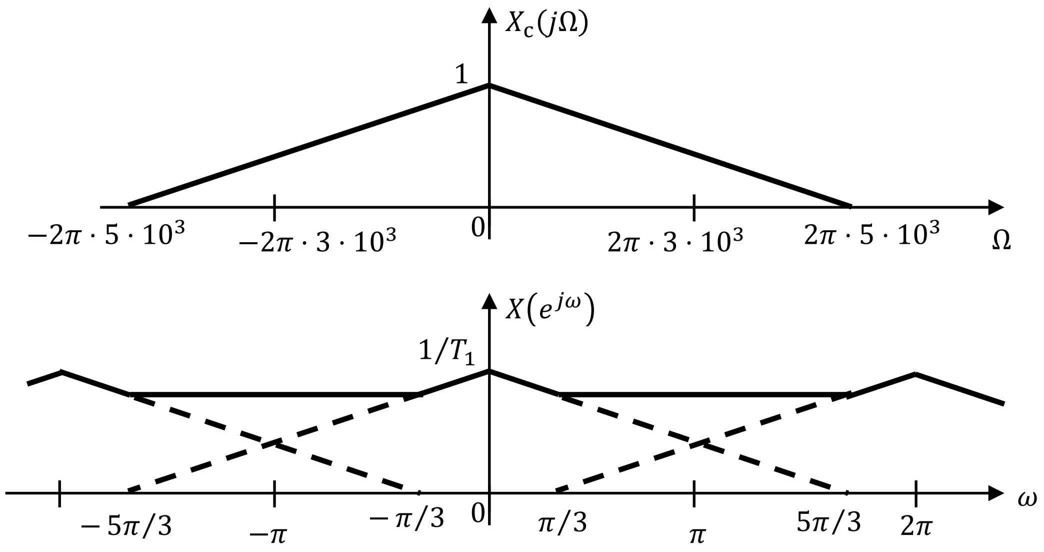
(ii)


$X_c(j\Omega)$
1
$\frac{\pi}{T_1} = 2\pi \cdot 20 \cdot 10^3$
$-2\pi \cdot 5 \cdot 10^3$
0
$2\pi \cdot 5 \cdot 10^3$
Ω
$X(e^{j\omega})$
$1/T_1$
(Periodic with 2π)
$-\pi/4$
0
$\pi/4$
π
ω
$Y(e^{j\omega})$
$1/T_1$
(Periodic with 2π)
$-\pi/5$
0
$\pi/5$
π
ω
$Y_c(j\Omega)$
T_2/T_1
$\frac{\pi}{T_2} = 2\pi \cdot 5 \cdot 10^3$
$-2\pi \cdot 10^3$
0
$2\pi \cdot 10^3$
Ω

(iii)


$X_c(j\Omega)$
1
$\frac{\pi}{T_1} = 2\pi \cdot 5 \cdot 10^3$
$-2\pi \cdot 5 \cdot 10^3$
0
$2\pi \cdot 5 \cdot 10^3$
Ω
$X(e^{j\omega})$
$1/T_1$
$-\pi$
0
π
ω
$Y(e^{j\omega})$
$1/T_1$
-2π
$-\pi$
$-\pi/5$
0
$\pi/5$
π
2π
ω
$Y_c(j\Omega)$
T_2/T_1
$2\pi \cdot 3 \cdot 10^3$
0
Ω
$\frac{\pi}{T_2} = 2\pi \cdot 15 \cdot 10^3$

Part b:



The maximum choice of ω_c is $\pi/3$ and for this choice,

$$H_c(j\Omega) = \begin{cases} T_2/T_1, & |\Omega| < 2\pi \cdot 10^3, \\ 0, & |\Omega| > 2\pi \cdot 10^3. \end{cases}$$

Question 21

Part a: B, C, D, E

Part b: A, F

Part c: A, B, C, E, F

Part d: E

Part e: A, F

Part f: C

Part g: E

Part h: F

Part i: F

Part j: E