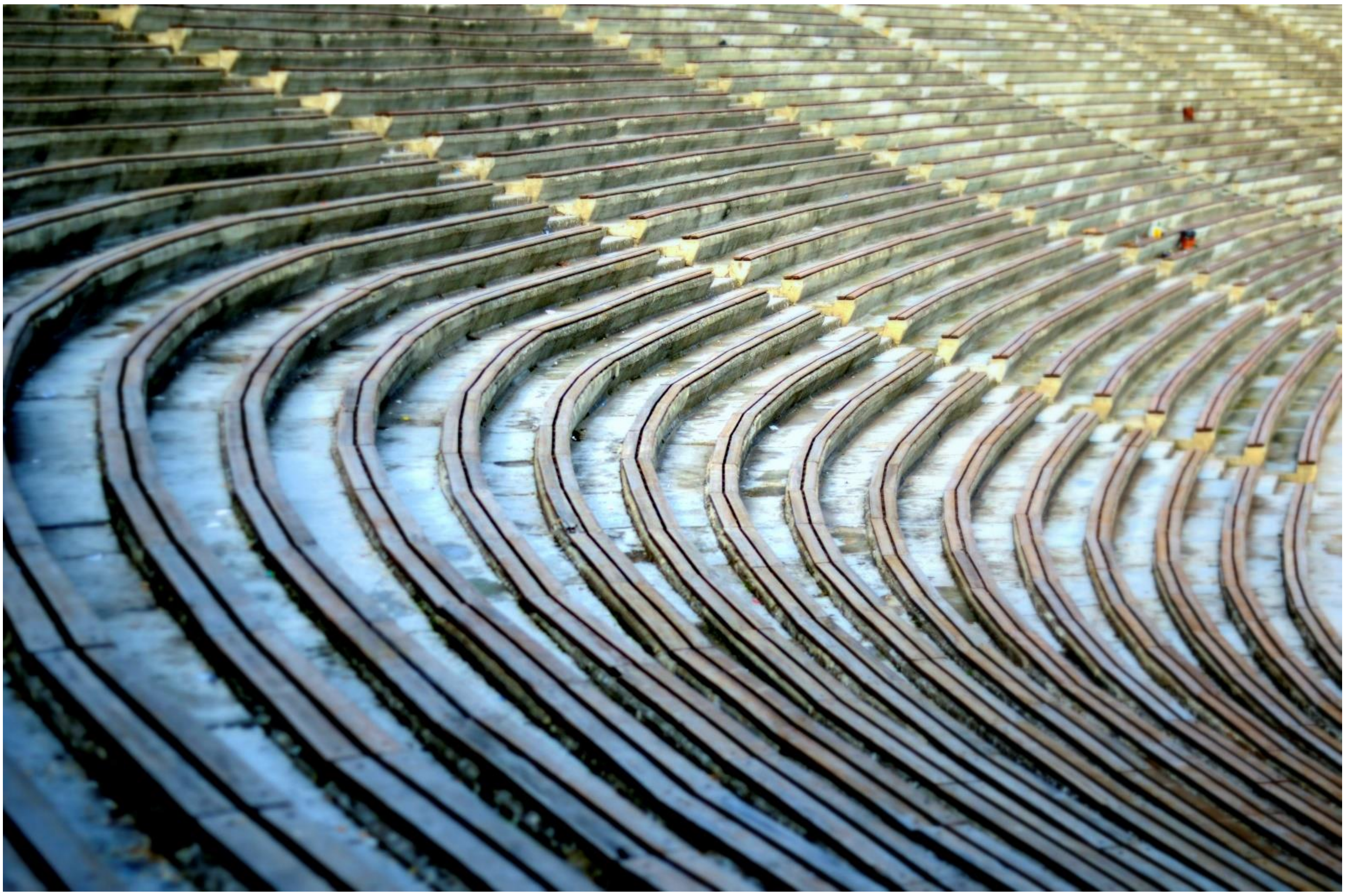


SAMPLING

UNIFORM SAMPLING OF SINUSOIDAL SIGNALS

UNIFORM SAMPLING

C/D, D/C (A/D, D/A)

A MATHEMATICAL MODEL OF SAMPLING

IMPULSE SAMPLING

ALIASING

EXPRESSING $X(e^{j\omega})$ IN TERMS OF $X_c(\Omega)$

NYQUIST-SHANNON SAMPLING THEOREM

RECONSTRUCTION OF A C-Time SIGNAL FROM A D-Time SIGNAL

D-Time PROCESSING OF C-Time SIGNALS

IMPULSE RESPONSES OF EQUIVALENT C-Time AND D-Time SYSTEMS

CHANGING THE SAMPLING RATE IN D-TIME

RATE REDUCTION BY AN INTEGER FACTOR

RATE INCREASE BY AN INTEGER FACTOR

CHANGING THE SAMPLING RATE BY A NONINTEGER (RATIONAL) FACTOR

DIGITAL PROCESSING OF ANALOG SIGNALS

ANALOG TO DIGITAL CONVERSION

QUANTIZATION

DIGITAL TO ANALOG CONVERSION

UNIFORM SAMPLING OF A SINUSOIDAL SIGNAL,

$$x_c(t) = \cos(2\pi f_0 t)$$

$$\begin{aligned} x[n] &= x_c(nT) \\ &= x_c\left(\frac{n}{f_s}\right) \\ &= \cos(2\pi \frac{f_0}{f_s} n) \\ &= \cos(\omega_0 n) \end{aligned}$$

ALIASING

If

$$\frac{f_0}{f_s} \geq 1$$

Let

$$f_0 = \hat{f}_0 + kf_s \qquad \hat{f}_0 < f_s \,, \;\; k \in Z^+$$

$$x[n] = \cos\left(2\pi \frac{f_0}{f_s} n\right)$$

$$= \cos\left(2\pi \frac{\hat{f}_0}{f_s} n\right)$$

$$= \cos(\widehat{\omega}_0 n)$$

where

$$\omega_0 = 2\pi f_0, \qquad \widehat{\omega}_0 = 2\pi \hat{f}_0 \,,$$

$$\omega_0 = \widehat{\omega}_0 + k2\pi \,.$$

Furthermore, if

$$\frac{1}{2} < \frac{\hat{f}_0}{f_s} < 1$$

i.e.,

$$\pi < \widehat{\omega}_0 < 2\pi$$

$$\cos(\widehat{\omega}_0 n) = \cos(2\pi n - \widehat{\omega}_0 n)$$

$$= \cos\big((2\pi - \widehat{\omega}_0)n\big)$$

Equivalently,

$$\sin(\widehat{\omega}_0 n) = -\sin\big((2\pi - \widehat{\omega}_0)n\big)$$

Ex:

$$f_s = 2000 \text{ Hz}$$

$f_0 = 3400$ Hz $\quad \omega_0 = 3.4\pi \quad x[n] = \cos(3.4\pi n) = \cos(1.4\pi n) = \cos(0.6\pi n)$

$f = 1400$ Hz $\quad \omega_0 = 1.4\pi \quad x[n] = \cos(1.4\pi n) = \cos(0.6\pi n)$

$f = 600$ Hz $\quad \omega_0 = 0.6\pi \quad x[n] = \cos(0.6\pi n)$

$f = 3000$ Hz $\quad \omega_0 = 3\pi \quad x[n] = \cos(3\pi n)$

$f = 1000$ Hz $\quad \omega_0 = \pi \quad x[n] = \cos(\pi n)$

$f = 2800$ Hz $\quad \omega_0 = 2.8\pi \quad x[n] = \cos(2.8\pi n) = \cos(0.8\pi n)$

$f = 800$ Hz $\quad \omega_0 = 0.8\pi \quad x[n] = \cos(0.8\pi n)$

$f = 2000$ Hz $\quad \omega_0 = 2\pi \quad x[n] = \cos(2\pi n) = \cos(0) = 1$

Sine counterparts;

$f_0 = 3400$ Hz $\quad \omega_0 = 3.4\pi \quad x[n] = \sin(3.4\pi n) = \sin(1.4\pi n) = -\sin(0.6\pi n)$

$f = 1400$ Hz $\quad \omega_0 = 1.4\pi \quad x[n] = \sin(1.4\pi n) = -\sin(0.6\pi n)$

$f = 600$ Hz $\quad \omega_0 = 0.6\pi \quad x[n] = \sin(0.6\pi n)$

TERMINOLOGY

"analog-to-digital" versus "C-Time- to-D-Time"

- Continuous-time signals are commonly processed by using discrete-time systems.
- Mobile devices, TV receivers and displays, radar, sonar, music recording/editing, image processing…

"**analog to digital**" conversion: A/D

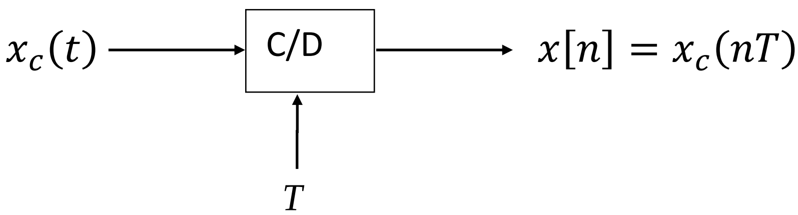
"**continuous-time to discrete-time**" conversion: C/D

A/D = {C/D} and {quantization of sample values}

UNIFORM SAMPLING

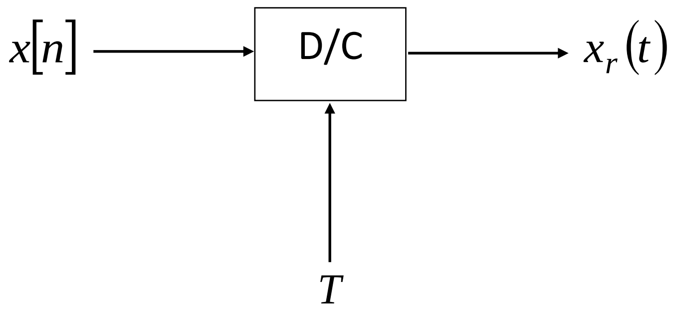
Samples are spaced uniformly in time.

Continuous-time to discrete-time conversion



T: sampling period

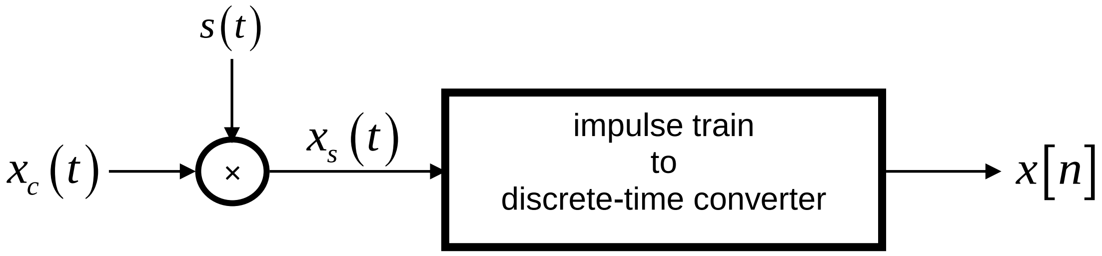
Discrete -time to continuous-time conversion



In practice, we have the task of “digital to analog” (D/A) conversion.

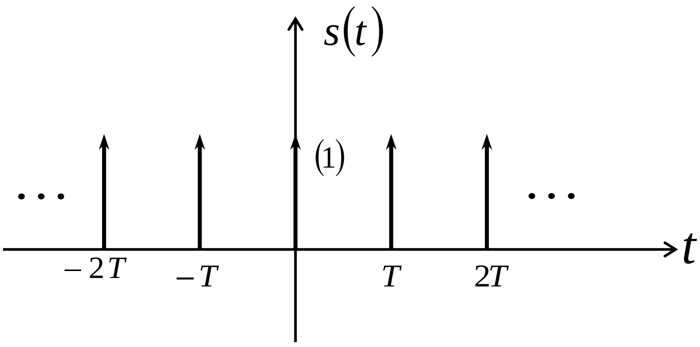
A central question is about determining, if exist, the mathematical conditions required to recover a continuous-time signal from its uniformly spaced samples.

A MATHEMATICAL MODEL OF SAMPLING



$$s(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$$

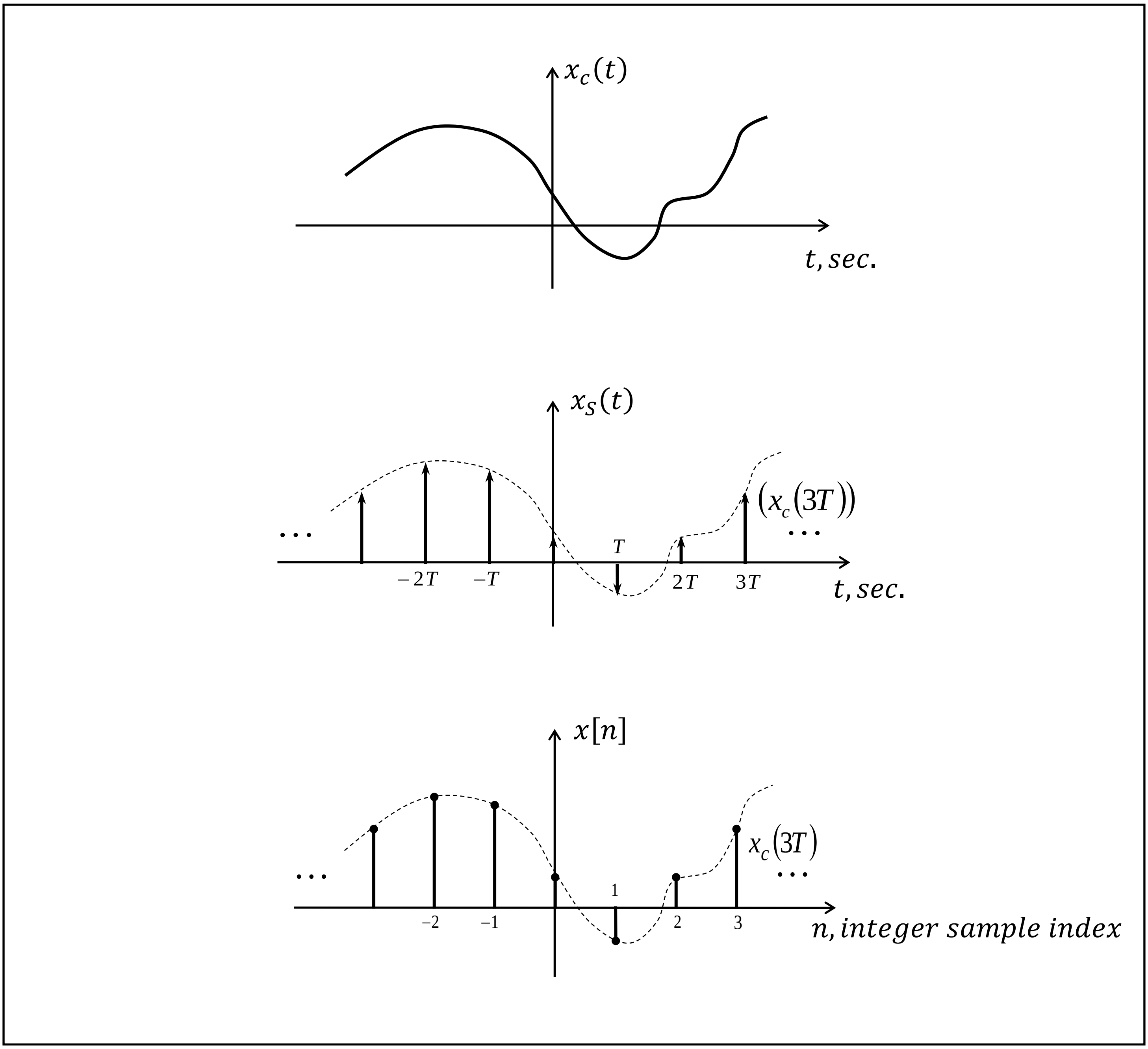
impulse train



T : sampling period

$f_s = \frac{1}{T}$: sampling frequency (Hz)

$\Omega_s = \frac{2\pi}{T}$: sampling frequency (rad/sec)



$x_c(t)$
$t, sec.$
$x_S(t)$
$\cdots$
T
$-2T$
$-T$
$2T$
$3T$
$(x_c(3T))$
$\cdots$
$t, sec.$
$x[n]$
$\cdots$
1
−2
−1
2
3
$x_c(3T)$
$\cdots$
$n, integer\ sample\ index$

$$x_s(t) = \sum_{n=-\infty}^{\infty} x_c(nT)\delta(t - nT)$$

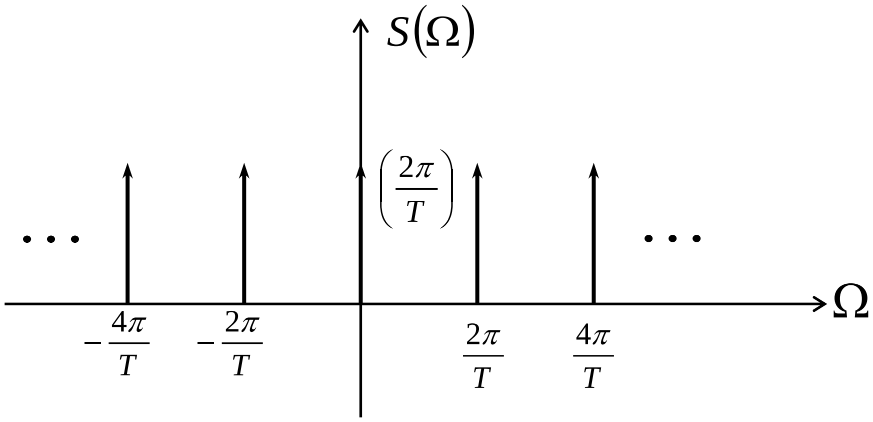
$$x[n] = x_c(nT)$$

IMPULSE TRAIN IN FREQUENCY DOMAIN

Using (continuous-time) Fourier series representation of $s(t)$ with coefficients $a_k = \frac{1}{T}$ and

$$e^{jk\frac{2\pi}{T}t} \quad \xleftrightarrow{CTFT} \quad 2\pi\delta\left(\Omega - k\frac{2\pi}{T}\right)$$

$$S(\Omega) = \frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta\left(\Omega - k\frac{2\pi}{T}\right)$$



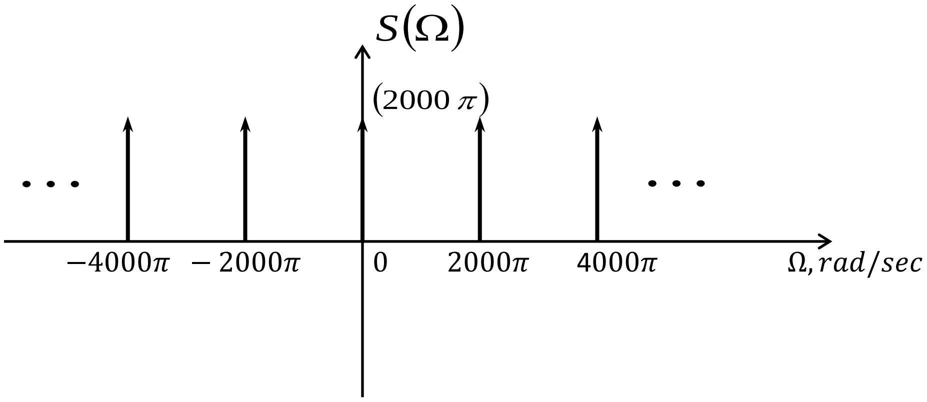
$$s(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{2\pi}{T}t} \qquad a_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} s(t)e^{-jk\frac{2\pi}{T}t}dt$$

Ex:

$T = 1$ ms,

$f_s = 1000$ Hz,

$\Omega_s = \frac{2\pi}{T} = 2000\pi$ rad/sec

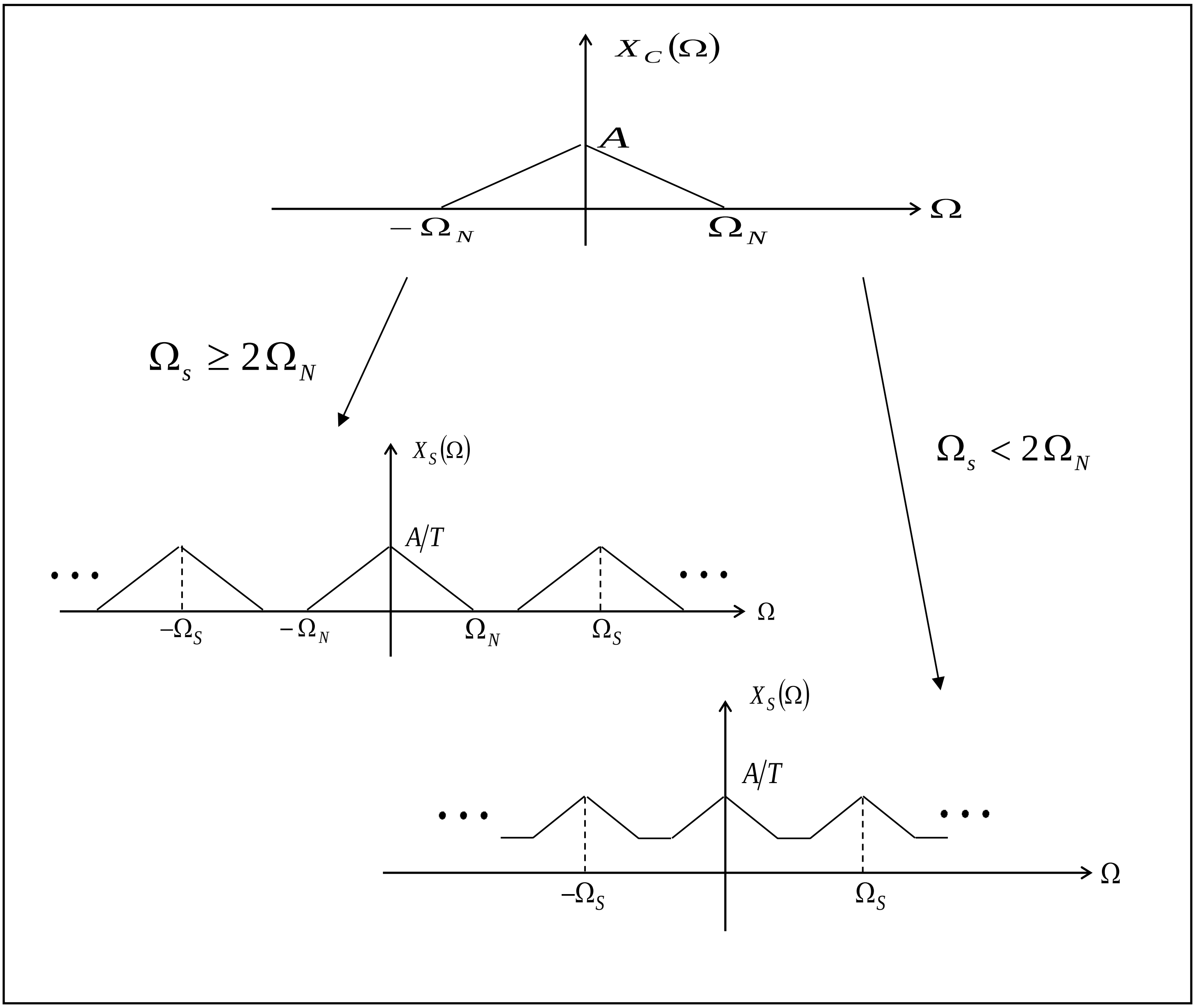


After multiplication with Impulse Train

$$X_s(\Omega) = \frac{1}{2\pi} X_c(\Omega) * S(\Omega)$$

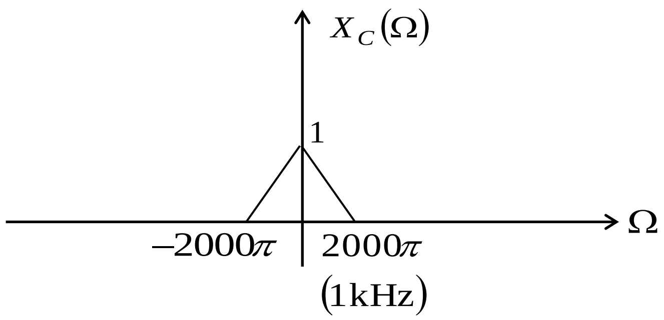
Since $S(\Omega)$ is an impulse train

$$X_s(\Omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c\left(\Omega - k\frac{2\pi}{T}\right)$$

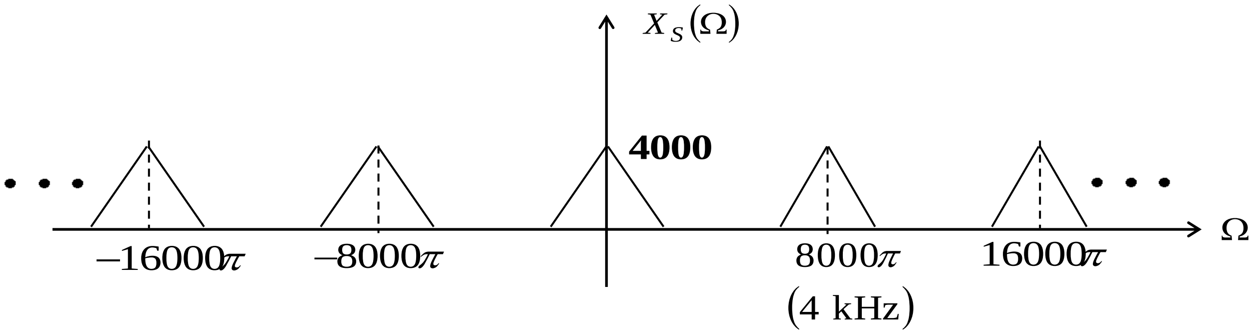


$X_C(\Omega)$
A
Ω
$-\Omega_N$
Ω_N
$\Omega_s \geq 2\Omega_N$
$\Omega_s < 2\Omega_N$
$X_S(\Omega)$
A/T
$\cdots$
$\cdots$
Ω
$-\Omega_S$
$-\Omega_N$
Ω_N
Ω_S
$X_S(\Omega)$
A/T
$\cdots$
$\cdots$
Ω
$-\Omega_S$
Ω_S

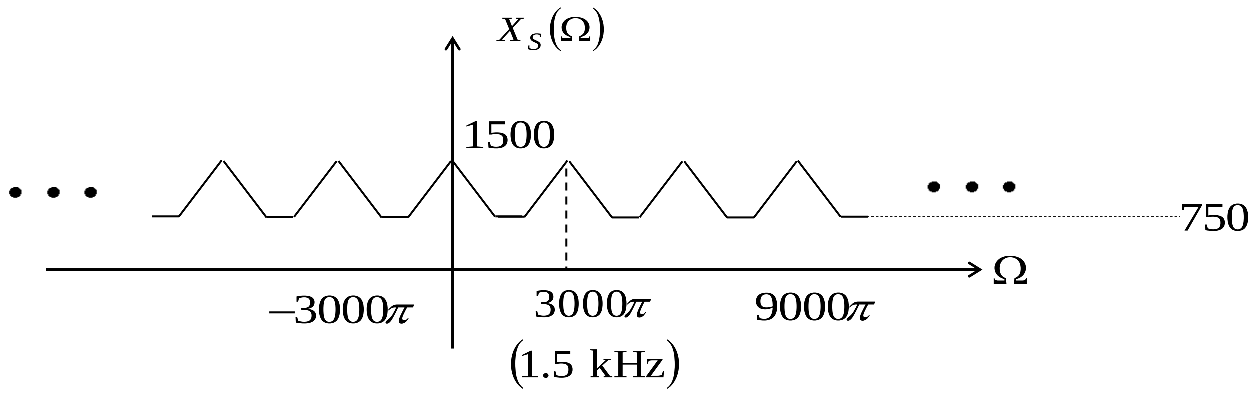
Ex: Let $x_c(t)$ be "*bandlimited*" to 1 kHz or equivalently to 2000π rad/sec., i.e. $\Omega_N = 2000\pi$



Let $T = 0.25$ ms ($f_s = 4$ kHz) $\Omega_s = 8000\pi$



However, if $T = 2/3$ ms $f_s = 1.5$ kHz) $\Omega_s = 3000\pi$



Definition

The overlap (distortion) of the spectrum when $\Omega_s < 2\Omega_N$ is called "**aliasing**".

EXPRESSING $X\left(e^{j\omega}\right)$ IN TERMS OF $X_c(\Omega)$

$$x_s(t) = \sum_{n=-\infty}^{\infty} x_c(nT)\delta(t - nT)$$

$$= \sum_{n=-\infty}^{\infty} x[n]\delta(t - nT)$$

Time-shift property of CTFT,

$$X_s(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega T n}$$

Compare this expression to

$$X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

Conclude that

$$X\left(e^{j\omega}\right) = X_s(\Omega)\big|_{\Omega=\frac{\omega}{T}}$$

Hence

$$X\left(e^{j\omega}\right) = \frac{1}{T}\sum_{n=-\infty}^{\infty} X_c\left(\frac{\omega}{T} - k\frac{2\pi}{T}\right)$$

A “linear warping” of the frequency scale and its periodic extension.

$$X\left(e^{j\omega}\right) = \frac{1}{T}\Bigg(\ldots + X_c\left(\frac{1}{T}(\omega + 4\pi)\right)$$

$$+X_c\left(\frac{1}{T}(\omega + 2\pi)\right)$$

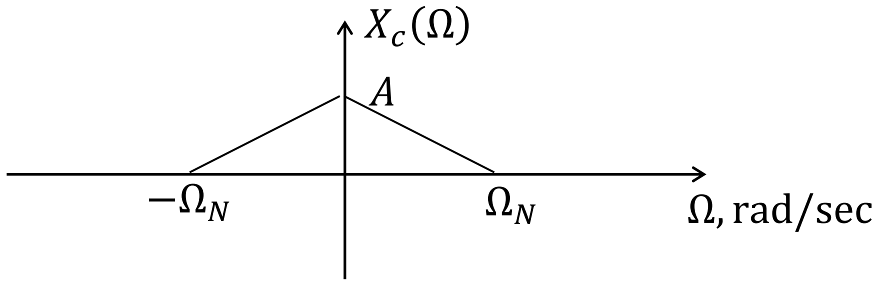
$$+X_c\left(\frac{\omega}{T}\right)$$

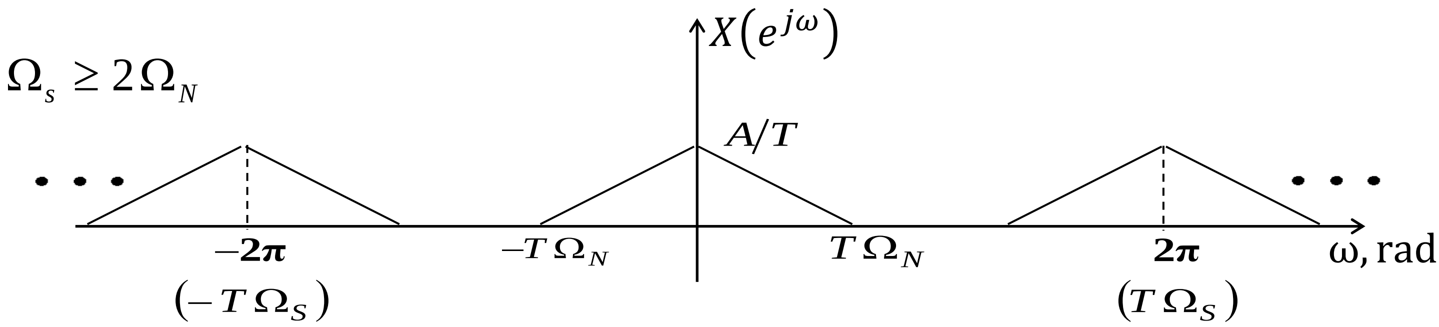
$$+X_c\left(\frac{1}{T}(\omega - 2\pi)\right)$$

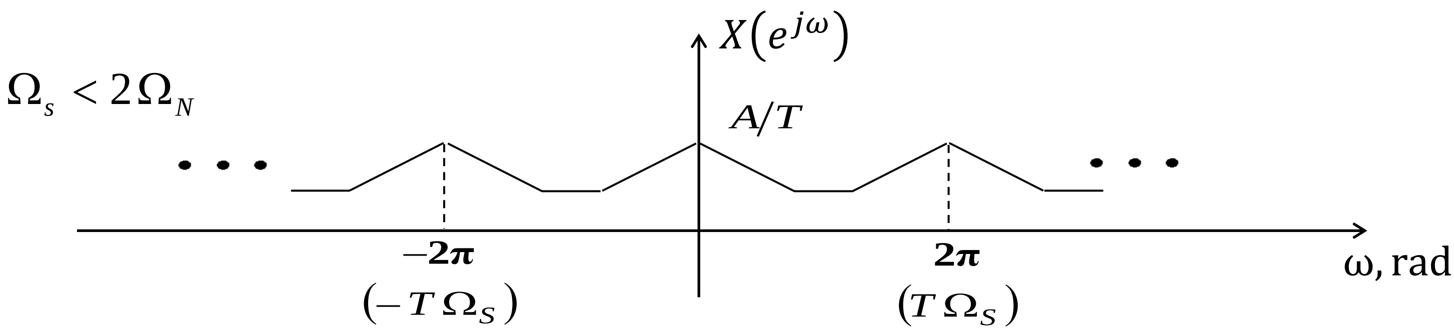
$$+X_c\left(\frac{1}{T}(\omega - 4\pi)\right)$$

$$+\cdots\Bigg)$$

Therefore

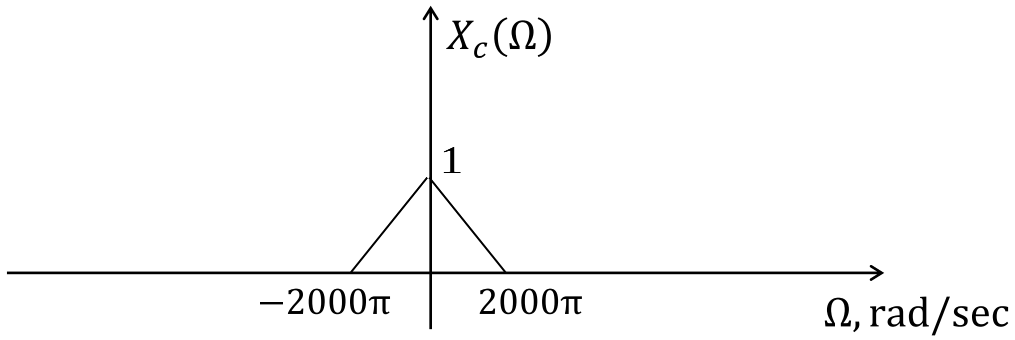

$X_c(\Omega)$
A
$-\Omega_N$
Ω_N
Ω, rad/sec

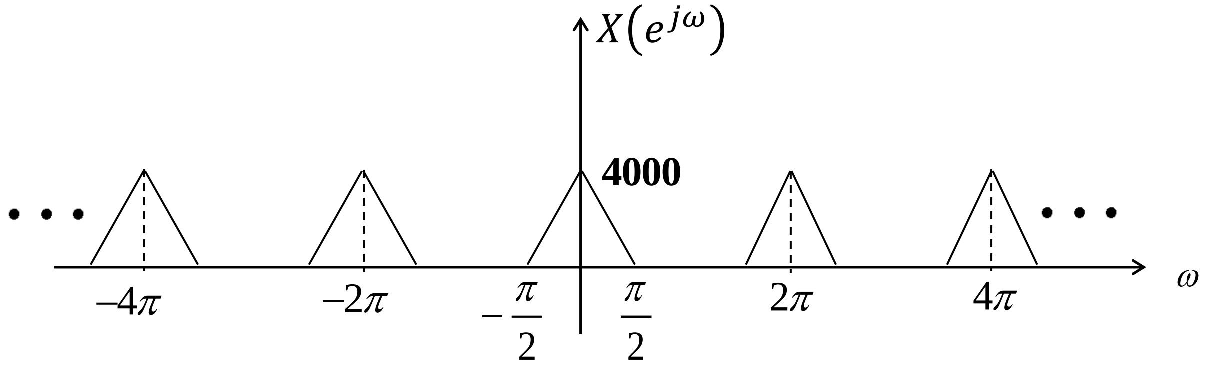

$\Omega_s \geq 2\Omega_N$
$X(e^{j\omega})$
A/T
-2π
$(-T\Omega_S)$
$-T\Omega_N$
$T\Omega_N$
2π
$(T\Omega_S)$
ω, rad


$\Omega_s < 2\Omega_N$
$X(e^{j\omega})$
A/T
-2π
$(-T\Omega_S)$
2π
$(T\Omega_S)$
ω, rad

Ex: Let $x_c(t)$ be "*bandlimited*" to 1 kHz or equivalently to 2000π rad/sec., i.e., $\Omega_N = 2000\pi$

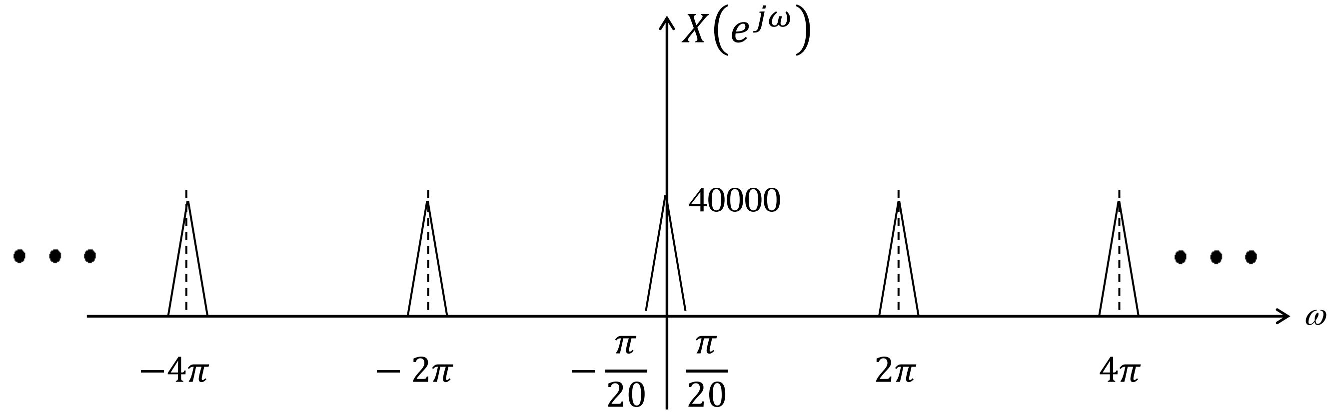
Let $T = 0.25$ ms ($f_s = 4$ kHz) $\Omega_s = 8000\pi$





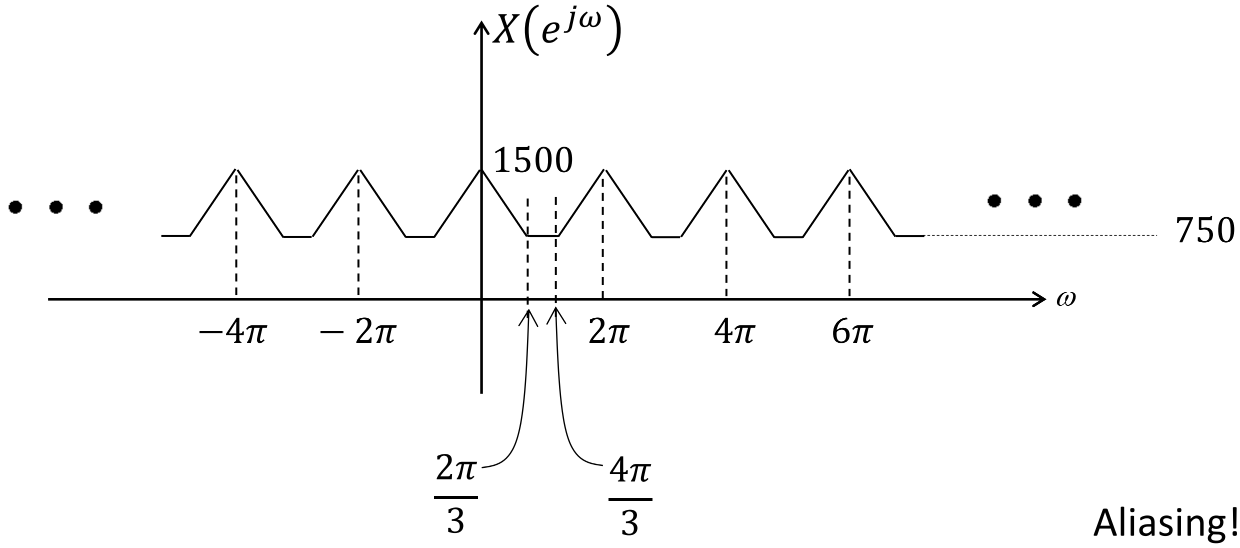
If sampling frequency is increased ten times:

$T = 0.025$ ms ($f_s = 40$ kHz) $\qquad \Omega_s = 80000\pi$



If sampling frequency is decreased,

$T = \frac{2}{3}$ ms ($f_s = 1.5$ kHz) $\Omega_s = 3000\pi$



Aliasing!

Notes

- Under no aliasing, discrete-time and continuous-time frequency scales are related by

$$\omega = \Omega\, T$$

$$\Omega = f_s\, \omega$$

$$f_s = \frac{1}{T}$$

- Sampling frequency, Ω_s, is "always" mapped to 2π in discrete-time frequency scale.

- Discrete-time frequency "equivalent", ω_a , of any continuous-time frequency, Ω_a $\left(\Omega_a \leq \frac{\Omega_s}{2}\right)$, can be found using the ratio $\frac{\Omega_a}{\Omega_s}$:

  "In the last example, the band edge frequency, 1 kHz, is one fourth of the sampling frequency, 4 kHz. Therefore, the band edge of $X\left(e^{j\omega}\right)$ is at $\frac{\pi}{2}$, one fourth of 2π."

NYQUIST-SHANNON SAMPLING THEOREM

Let $x_c(t)$ be bandlimited to Ω_N, i.e.,

$$X_c(\Omega) = 0 \quad |\Omega| \geq \Omega_N$$

$x_c(t)$ can be determined uniquely from its samples

$$x[n] = x_c(nT)$$

if

$$\Omega_s \geq 2\Omega_N$$

Pay attention to the equality!

Ω_N: Nyquist frequency

$2\Omega_N$: Nyquist rate

https://www.wikiwand.com/en/Nyquist%E2%80%93Shannon_sampling_theorem

Historical background

The sampling theorem was implied by the work of Harry Nyquist in 1928,[9] in which he showed that up to 2B independent pulse samples could be sent through a system of bandwidth B; but he did not explicitly consider the problem of sampling and reconstruction of continuous signals. About the same time, Karl Küpfmüller showed a similar result,[10] and discussed the sinc-function impulse response of a band-limiting filter, via its integral, the step response *Integralsinus*; this bandlimiting and reconstruction filter that is so central to the sampling theorem is sometimes referred to as a *Küpfmüller filter* (but seldom so in English).

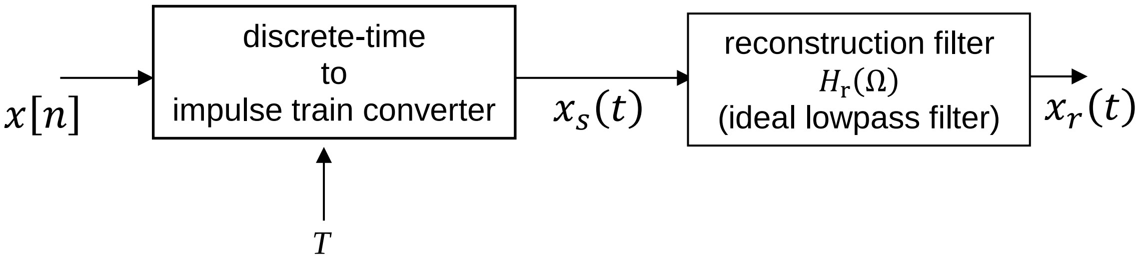
The sampling theorem, essentially a dual of Nyquist's result, was proved by Claude E. Shannon.[2] V. A. Kotelnikov published similar results in 1933,[11] as did the mathematician E. T. Whittaker in 1915,[12] J. M. Whittaker in 1935,[13] and Gabor in 1946 ("Theory of communication"). In 1999, the Eduard Rhein Foundation awarded Kotelnikov their Basic Research Award "for the first theoretically exact formulation of the sampling theorem."

In 1948 and 1949, Claude E. Shannon published the two revolutionary papers in which he founded the information theory. [14][15][2] In Shannon 1948 the sampling theorem is formulated as "Theorem 13": Let f(t) contain no frequencies over W. Then

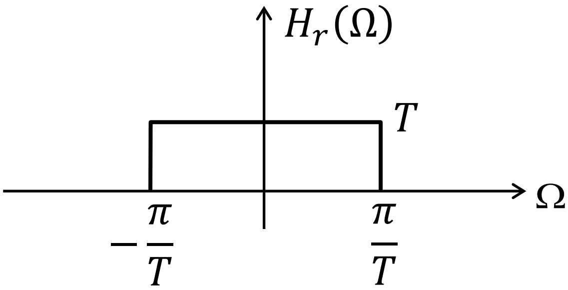
$$f(t) = \sum_{n=-\infty}^{\infty} X_n \frac{\sin \pi(2Wt - n)}{\pi(2Wt - n)},$$

where $X_n = f(n/2W)$. It was not until these papers were published that the theorem known as "Shannon's sampling theorem" became common property among communication engineers, although Shannon himself writes that this is a fact which is common knowledge in the communication art.[note 2] A few lines further on, however, he adds: ... "but in spite of its evident importance [it] seems not to have appeared explicitly in the literature of communication theory".

RECONSTRUCTION OF A CT SIGNAL FROM A DT SIGNAL

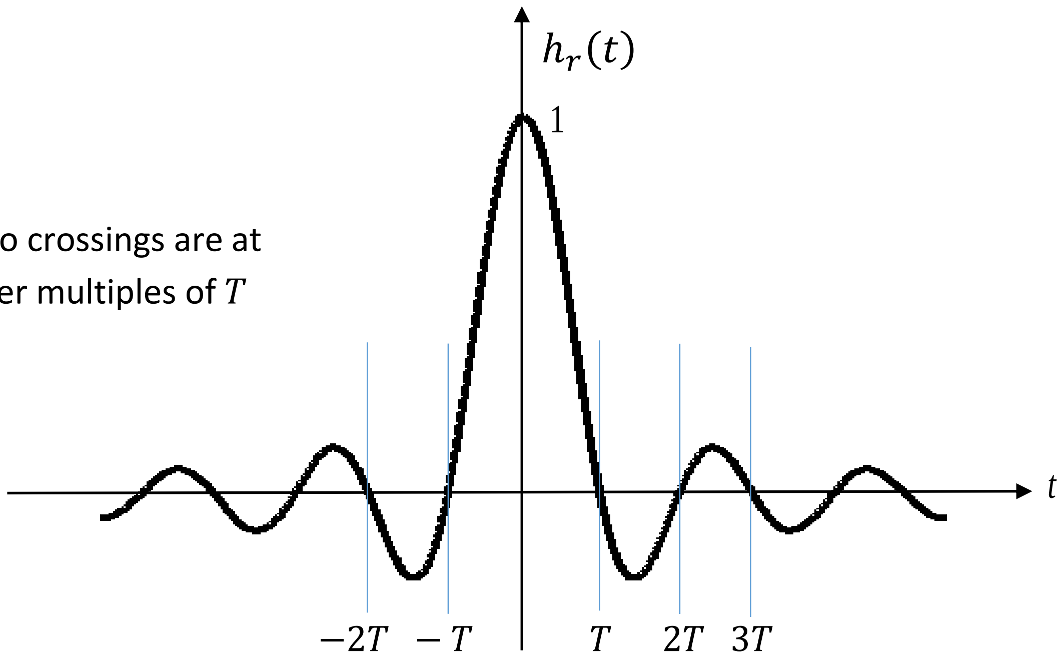


$$x_s(t) = \sum_{n=-\infty}^{\infty} x_c(nT)\delta(t - nT)$$



$$h_r(t) = \frac{\sin\left(\frac{\pi}{T}t\right)}{\frac{\pi}{T}t}$$

All zero crossings are at integer multiples of T



$$x_r(t) = x_s(t) * h_r(t)$$

$$= \left(\sum_{n=-\infty}^{\infty} x[n]\delta(t - nT) \right) * h_r(t)$$

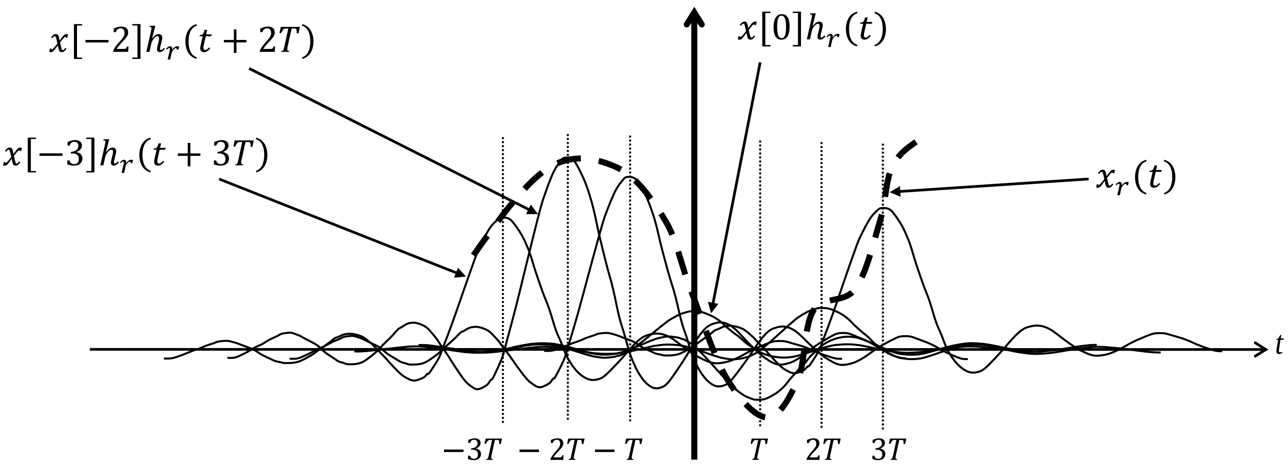
$$= \sum_{n=-\infty}^{\infty} x[n]h_r(t - nT)$$

$$X_r(\Omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-jnT\Omega} H_r(\Omega)$$

$$= \left(\sum_{n=-\infty}^{\infty} x[n]\, e^{-jnT\Omega} \right) H_r(\Omega)$$

$$= X\left(e^{j\Omega \mathrm{T}}\right) H_r(\Omega)$$

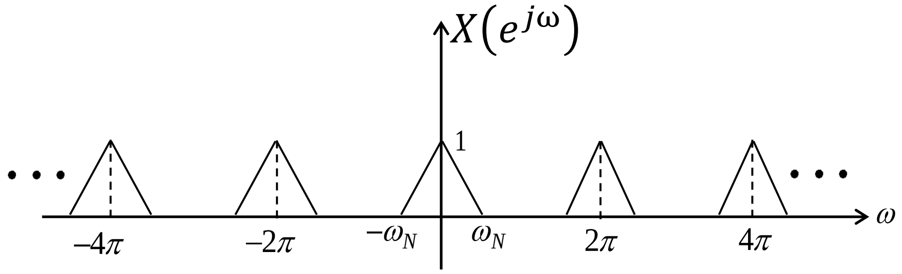
$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n]h_r(t - nT)$$

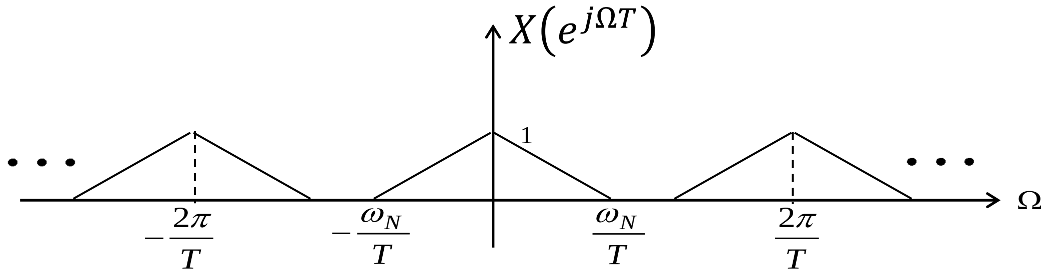


```
% bandlimited interpolation
% using 2N+1 terms in the summation

clear all
close all

N = 25  ;
f = 1   ;               % frequency of continuous time sinusoid to be sampled
Omega = 2 * pi * f ;
f_s   = 7          ;    % sampling frequency
T_s   = 1/f_s      ;
k     = -N:N       ;
x     = sin(Omega*k*T_s)   ; % samples of the continuous time sinusoid

delta = 4;
t = -delta:0.01:delta  ; % time interval in which we plot our results

% computing the sinc signals in the summation
for n = -N:N
   h_r(n+N+1,:) = x(n+N+1) * sin(f_s*pi*(t-n*T_s))./(f_s*pi*(t-n*T_s));
   h_r(n+N+1,find(isnan(h_r(n+N+1,:)))) = x(n+N+1) ;
end

 x_r = sum(h_r);   % reconstructed cont-time signal

plot(t,sin(Omega*t),'k','LineWidth',3)
hold
pause
plot(t,h_r(N+1-3,:),'g','LineWidth',3)
pause
plot(t,h_r(N+1-2,:),'r','LineWidth',3)
pause
plot(t,h_r(N+1-1,:),'c','LineWidth',3)
pause
plot(t,h_r(N+1,:),'m','LineWidth',3)
pause
plot(t,h_r(N+1+1,:),'y','LineWidth',3)
pause
plot(t,h_r(N+1+2,:),'b','LineWidth',3)
pause
plot(t,x_r,'r--','LineWidth',1.5)
```

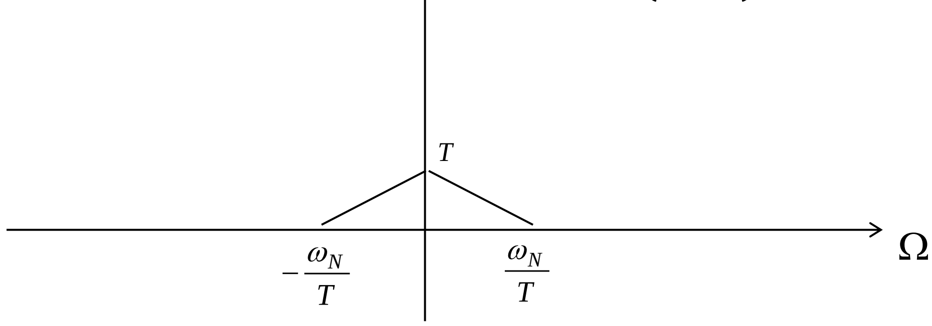
Ex:

$$X_r(\Omega) = X\left(e^{j\Omega T}\right) H_r(\Omega)$$

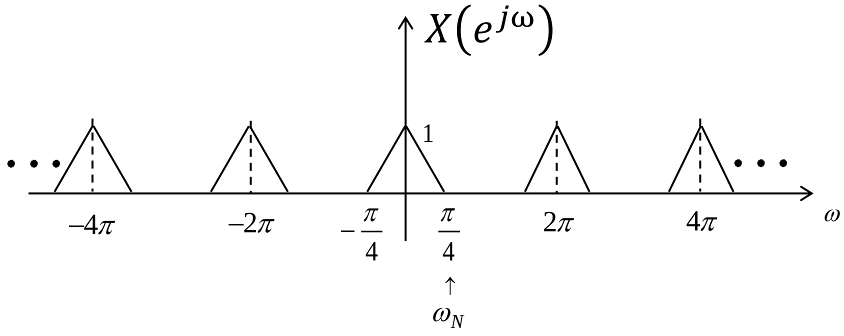




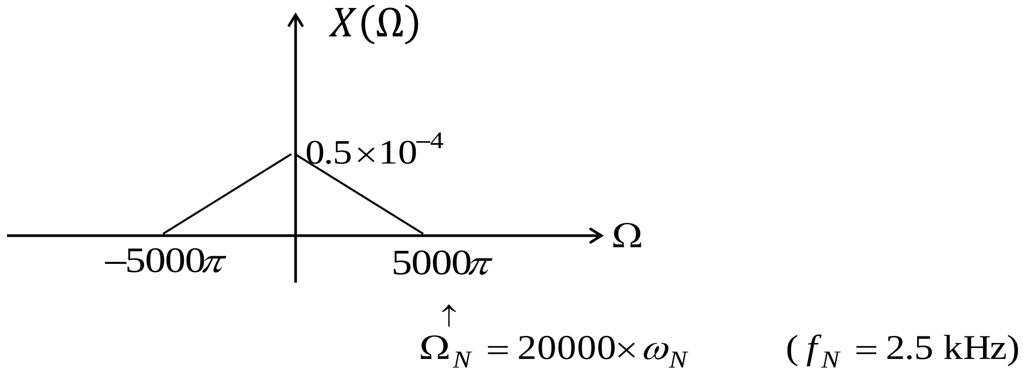
$$X_r(\Omega) = X\left(e^{j\Omega T}\right) H_r(\Omega)$$



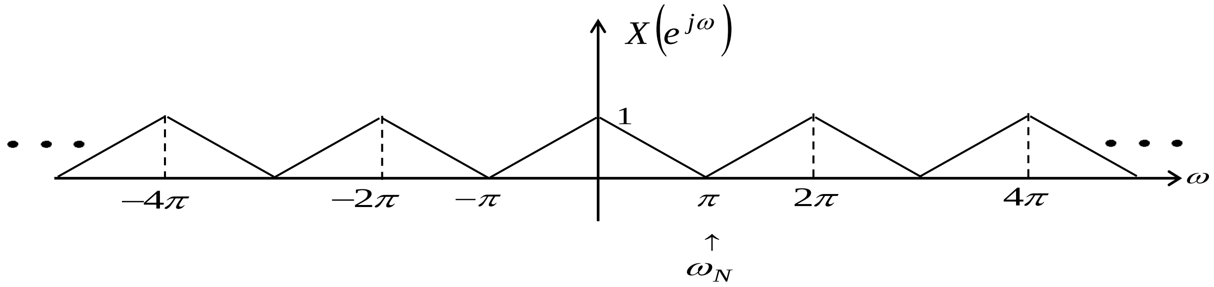
Ex: Given



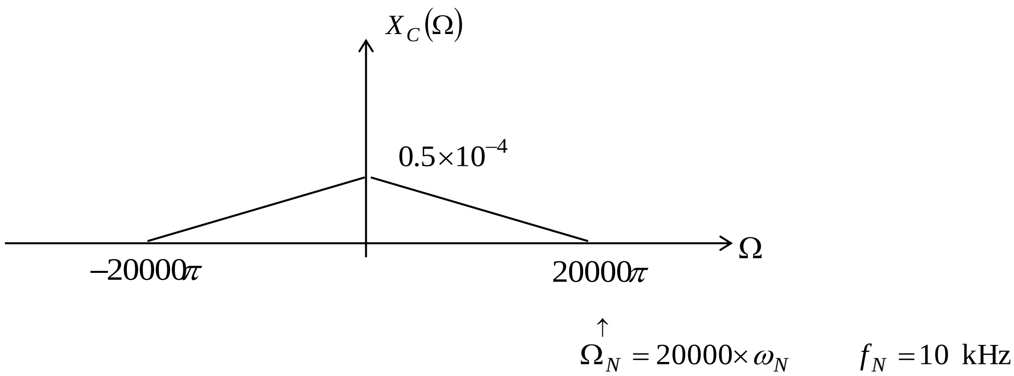
If $f_s = 20$ kHz, i.e., $\Omega_s = 40\pi$ krad/s, $T = \frac{1}{20}$ ms



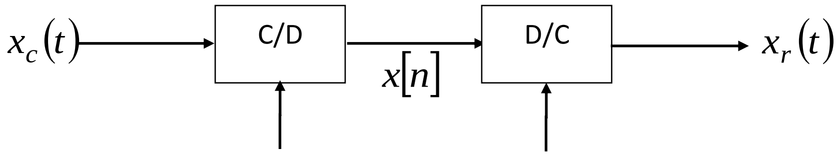
Ex: Given



If $f_s = 20$ kHz, i.e., $\Omega_s = 40\pi$ krad/s, $T = \frac{1}{20}$ ms

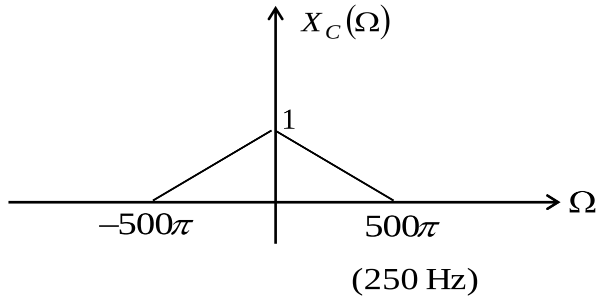


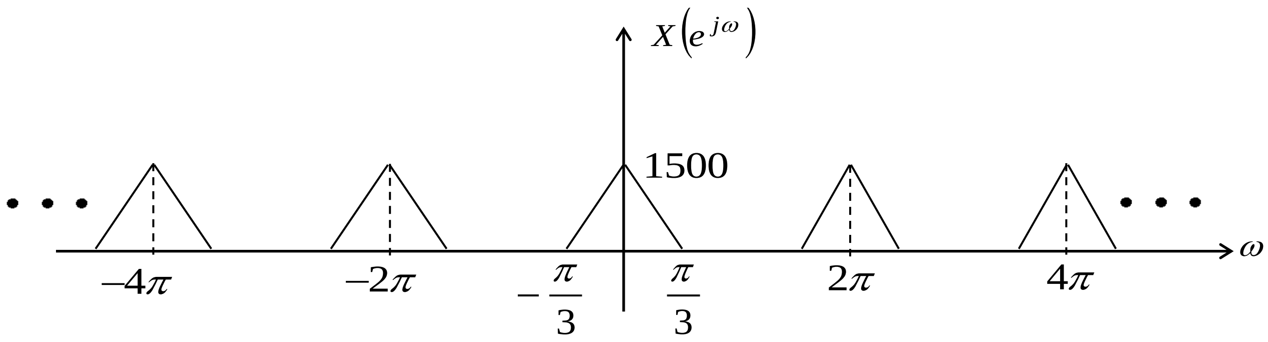
Ex:

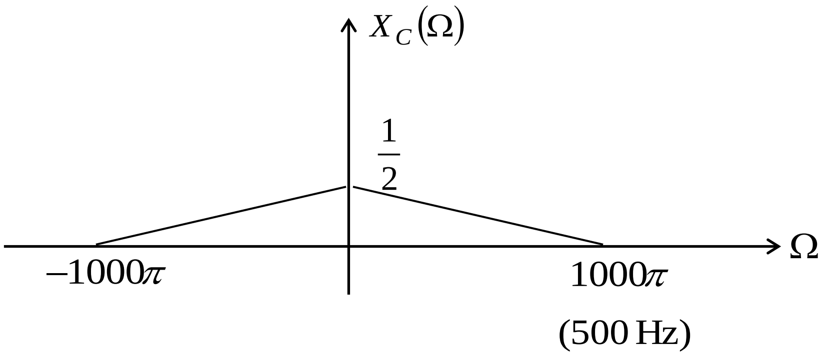


$$T_1 = \frac{1}{1500} \qquad T_2 = \frac{1}{3000}$$

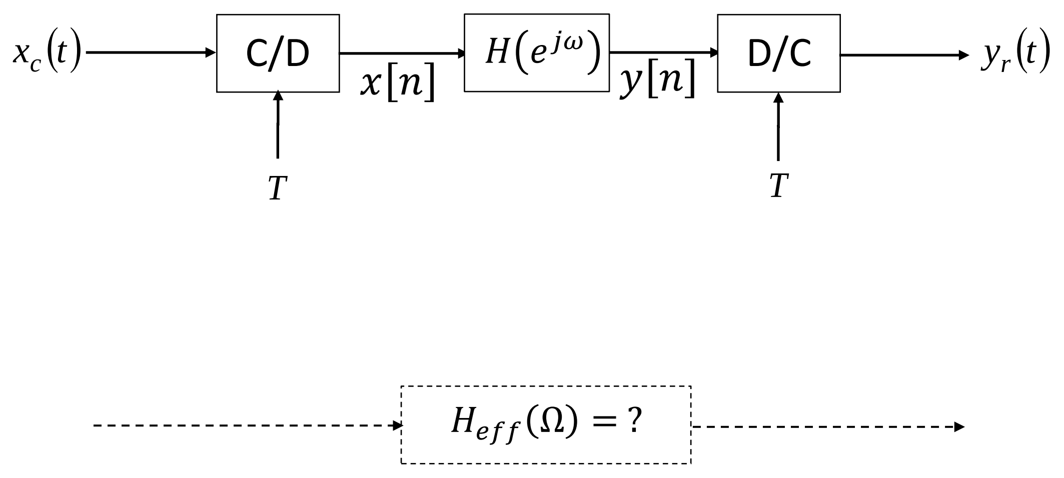
$$f_1 = 1500 \text{ Hz} \qquad f_2 = 3000 \text{ Hz}$$







DISCRETE-TIME PROCESSING OF CONTINUOUS-TIME SIGNALS



$$Y_r(\Omega) = H_r(\Omega) Y\left(e^{j\Omega T}\right)$$

$$= \begin{cases} TY\left(e^{j\Omega T}\right) & |\Omega| < \dfrac{\pi}{T} \\ 0 & o.w. \end{cases}$$

$$Y\left(e^{j\Omega T}\right) = H\left(e^{j\Omega T}\right) X\left(e^{j\Omega T}\right)$$

$$= H\left(e^{j\Omega T}\right) \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c\left(\Omega - k\frac{2\pi}{T}\right)$$

$$Y_r(\Omega) = \begin{cases} H\left(e^{j\Omega T}\right) \displaystyle\sum_{k=-\infty}^{\infty} X_c\left(\Omega - k\frac{2\pi}{T}\right) & |\Omega| < \dfrac{\pi}{T} \\ 0 & o.w. \end{cases}$$

If $x_c(t)$ is badlimited to $\frac{\pi}{T}$

$$Y_r(\Omega) = \begin{cases} H\left(e^{j\Omega\mathrm{T}}\right) X_c(\Omega) & |\Omega| < \dfrac{\pi}{T} \\ 0 & o.w. \end{cases}$$

Hence

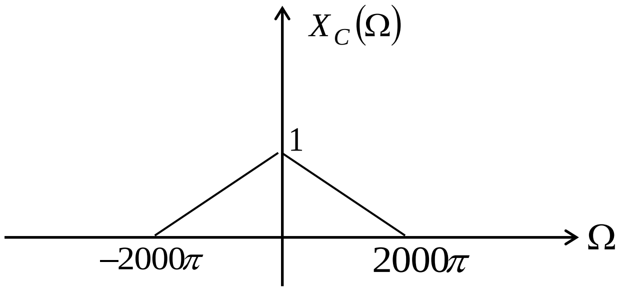
$$Y_r(\Omega) = H_{eff}(\Omega) X_c(\Omega)$$

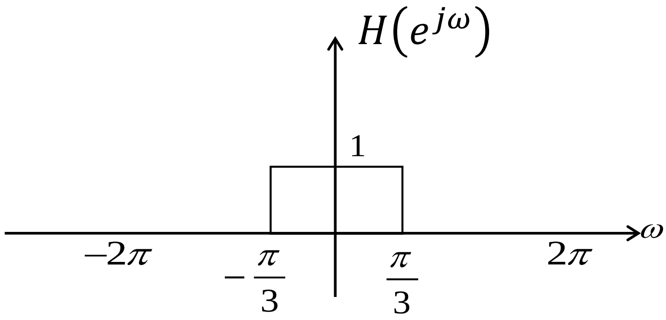
where

$$H_{eff}(\Omega) = \begin{cases} H\left(e^{j\Omega\mathrm{T}}\right) & |\Omega| < \dfrac{\pi}{T} \\ 0 & o.w. \end{cases}$$

$\rightarrow$ Note that $H_{eff}(\Omega)$ is valid if $x_c(t)$ is badlimited to $\frac{\pi}{T}$ $\leftarrow$

Ex: Let $T = 0.25$ ms (4 kHz) and

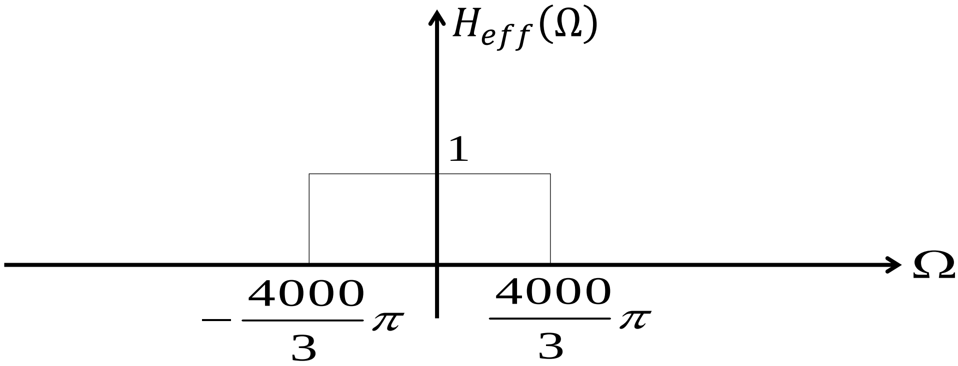




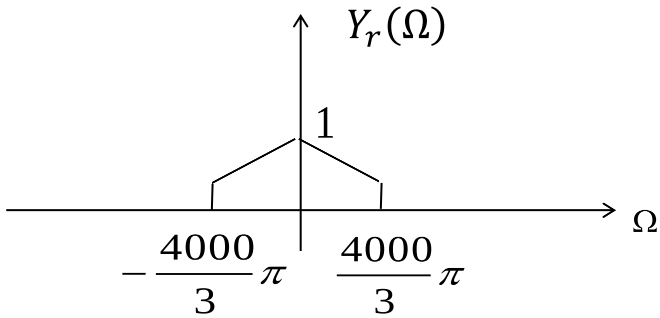
Therefore

$$H_{eff}(\Omega) = \begin{cases} H\left(e^{j\Omega T}\right) & |\Omega| < 4000\pi \\ 0 & o.w. \end{cases}$$

$$= \begin{cases} 1 & |\Omega| < \frac{4000}{3}\pi \\ 0 & o.w. \end{cases}$$



and



Ex: What should the cut-off frequency of the discrete-time ideal lowpass filter be so that the continuous-time signal is lowpass filtered with a cut-off frequency of 3 kHz when sampling frequency 8 kHz?

$$\omega_C = \frac{3}{8} 2\pi = \frac{3}{4} \pi$$