EE 430 Sec 1&3 Homework 3 Solutions

(Contact Erdal Epçaçan, epcacan@metu.edu.tr, D-122, for any questions)

1. Remember the definition of N point DFT for a sequence

$$\mathrm{X[k]} = \sum_{\mathrm{n=0}}^{\mathrm{N-1}} x[n] W_N^{kn} \ , W_N = e^{-j\left(\frac{2\pi}{N}\right)}$$

$$x[n] = \frac{1}{N} \sum_{\mathrm{k=0}}^{\mathrm{N-1}} X[k] W_N^{-kn}$$

a. Note that

$$x[n] = \frac{e^{\frac{j\pi n}{2}} - e^{\frac{-j\pi n}{2}}}{2j} = \frac{e^{\frac{j\pi n}{2}} - e^{\frac{3\pi n}{2}}}{2j}$$

So $[1] = 2/j = -2j$, $X[3] = -2/j = 2j$ and $X[0] = 0 = X[2]$

$$H[k] = \sum_{\mathrm{n=0}}^{\mathrm{N-1}} 2^n e^{-\frac{j\pi kn}{2}}$$

$$= \sum_{\mathrm{n=0}}^{\mathrm{N-1}} \left(2\, e^{-\frac{j\pi k}{2}}\right)^n = \frac{1 - \left(2\, e^{-\frac{j\pi k}{2}}\right)^4}{1 - \left(2\, e^{-\frac{j\pi k}{2}}\right)} = \frac{-15}{1 - \left(2\, e^{-\frac{j\pi k}{2}}\right)} = \begin{cases} 15, & k = 0 \\ -3 + 6j, & k = 1 \\ -5\,, & k = 2 \\ -3 - 6j\,, & k = 3 \end{cases}$$

b. $x[n] ④ h[n] = \sum_{m=0}^{3} x[m] h\left[((n-m))_4\right] = \ y[n] = [6 - 3 \ \ -6 \ \ 3], n = 0,1,2,3$

c. From part a and b

$$y[n] = \frac{1}{4} \sum_{\mathrm{k=0}}^{3} X[k] H[k] \ W_N^{-kn} \,, n = 0,1,2,3$$

$$y[n] = [6 - 3 \ \ -6 \ \ 3], n = 0,1,2,3$$

d. We know that linear convolution of x[n] and h[n] will have a length of 4+4-1=7 so. By multiplying 7 point DFT of x[n] and h[n] and taking inverse DFT we can find linear convolution result

2.

a. Using shifting property of DFT $y[n] \ = \ x\left[((n+3))_8\right]$

b. Noting that $W_N^{nk} = W_N^{(n+N)k} \ G[k] = X[2k] \ with \ N = 4$ means g[n] = x[n]+x[n+4]

$$g[n] = [5 \ \ 3 - 1.5 - 1.5] \quad n = 0,1,2,3$$

c. $(-1)^k = e^{-\frac{2\pi}{8}4k}$ also $W_8^{2n} x[n] \rightarrow X\left[((k+2))_8\right]$ so

$$W_8^{2(n-4)} x[n-4] \rightarrow (-1)^k X\left[((k+2))_8\right]$$

So $g_2[n] = x[n] - e^{-\frac{j\pi n}{2}} x[n] = [\,3, 1 - 2j\,, -1.5\,, -2.5 + 1j, -3\,, 2 - 1j\,], n = 0, \ldots, 7$

d. Since the length of linear convolution will be 8+3-1= 10 and the length of circular convolution is 8 the last two samples in linear convolution will be added to first two terms and the remaining ones will be the same.

3.

$$\sum_{n=0}^{N-1} x[n]y[n]^* = \sum_{n=0}^{N-1} x[n]\left(\frac{1}{N}\sum_{k=0}^{N-1} Y[k]\, W_N^{-nk}\right)^* = \sum_{n=0}^{N-1} x[n]\left(\frac{1}{N}\sum_{k=0}^{N-1} Y\,[k]\, W_N^{-nk}\right)^*$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} Y\,[k]^* \sum_{n=0}^{N-1} W_N^{k} x[n] = \frac{1}{N}\sum_{k=0}^{N-1} Y\,[k]^* X[k]$$

4.

a.

$$x_1[n] = [\,4\ \ 1\ \ -3.5\ \ -2.5\ \ 1\,]$$
$$x_2[n] = [2\ \ 2\ \ 1\ \ 0\ \ 0\,]$$
$$y_1[n] = [\,12\ \ 7\ \ -17.5\ \ -13\ \ 7.5\ \ 6\ \ -2\ ]$$
$$y_2[n] = [\,6\ \ 8\ \ 1\ \ 3\ \ -2\ \ 0\ \ 0]$$

Overlapping 3-1 = 2 points of $y_1[n]$ and $y_2[n]$ and adding them

$$y[n] = [12\ \ 7\ \ -17.5\ \ -13\ \ 7.5\ \ 12\ \ 6\ \ 1\ \ 3\ \ -2\ \ 0\ \ 0]$$

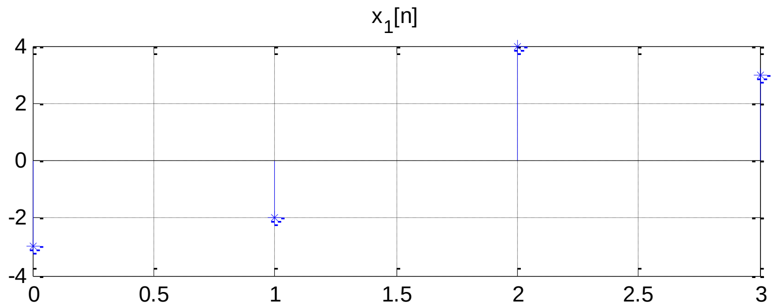
b.

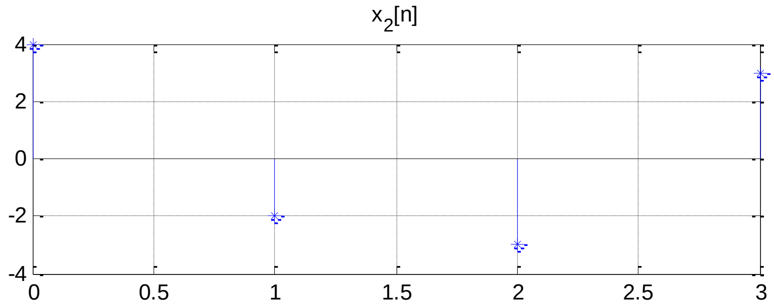
$$x_1[n] = [\,0\ \ 0\ \ 4\ \ 1\ \ -3.5\ \ -2.5\ \ 1\,]$$
$$x_2[n] = [-2.5\ \ 1\ \ 2\ \ 2\ \ 1\ \ 0\ \ 0\,]$$
$$y_1[n] = [\,0\ \ 0\ \ 12\ \ 7\ \ -17.5\ \ -13\ \ 7.5\ \ ]$$
$$y_2[n] = [-7.5\ \ 0.5\ \ 12\ \ 6\ \ 1\ \ 3\ \ -2\ \ 0\ \ 0]$$

Discarding first 3-1=2 points of $y_1[n]$ and $y_2[n]$ and adding them

$$y[n] = [12\ \ 7\ \ -17.5\ \ -13\ \ 7.5\ \ 12\ \ 6\ \ 1\ \ 3\ \ -2\ \ 0\ \ 0]$$

5.



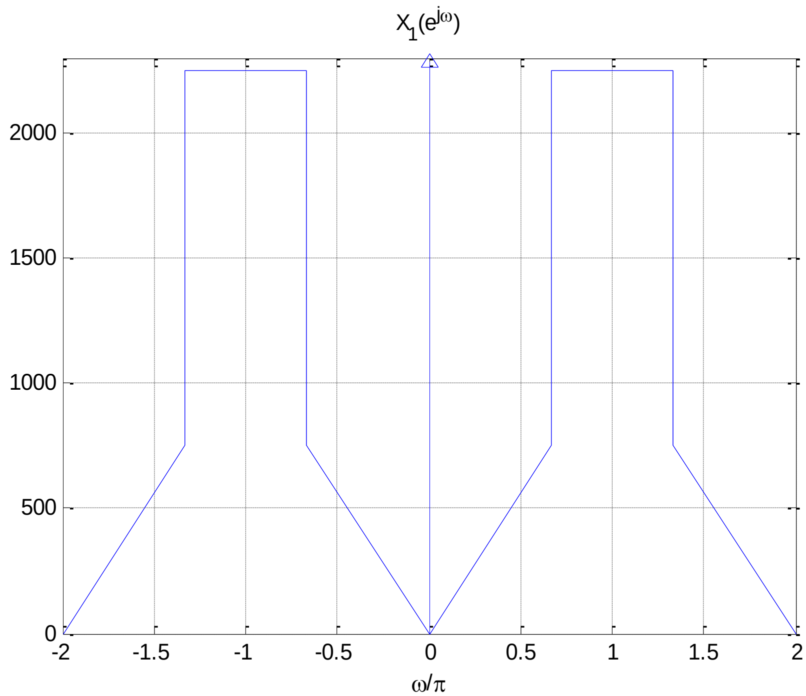


6.

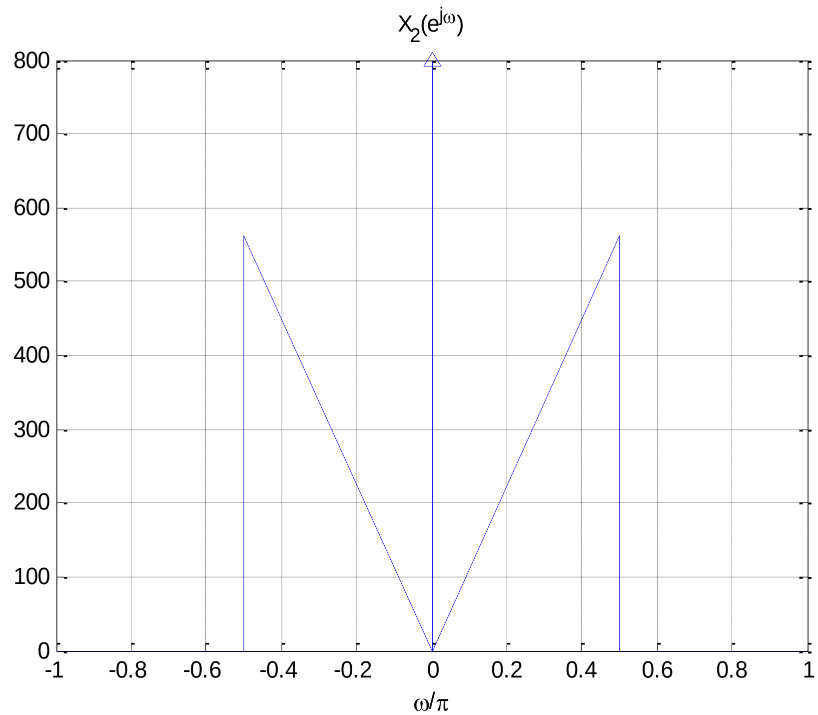
a. Nyquist frequency is the highest frequency that can be represented in a discrete system with a specified sampling frequency, in other words it is half of sampling frequency. For $X_1(j\Omega), \Omega_s \geq 4000\pi \; so \; f_s \geq 2000$ for $X_2(j\Omega), \Omega_s > 4000\pi$ due to impulse.

b.

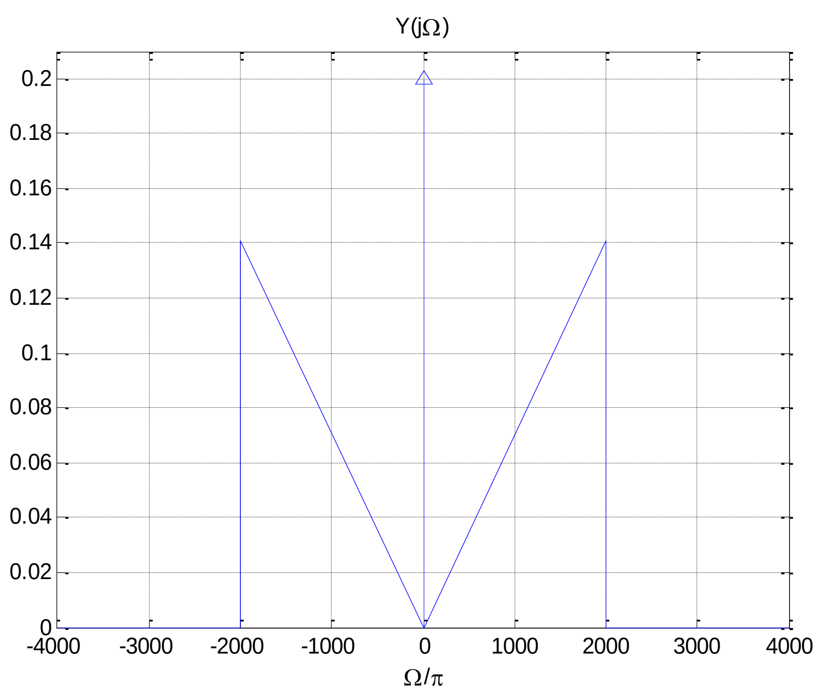
$$X_1(e^{j\omega}) = \frac{1}{T_1}\sum X\left(\frac{J\omega}{T_1} - \frac{2\pi k}{T_1}\right)$$



$$X_2(e^{j\omega}) = X_1(e^{j\omega})H(e^{j\omega})$$

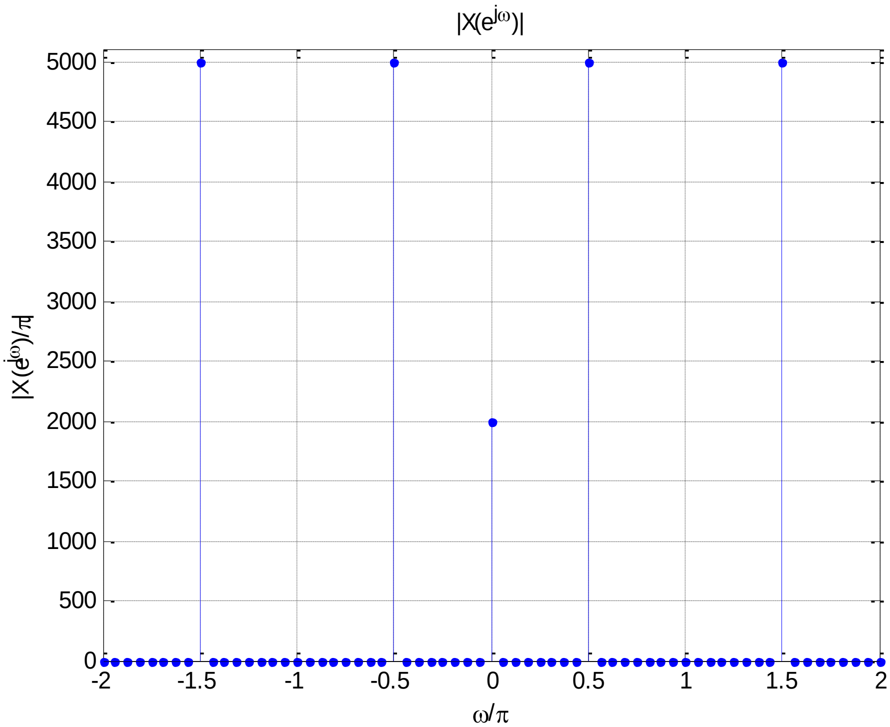


$$Y(j\Omega) = T_2 X_2(e^{j\Omega T_2})$$



7.

a. Nyquist rate should be greater than the biggest frequency component in the sampled signal so Nyquist rate $> 880\pi$.



b.

$$X\left(e^{j\omega}\right) = \frac{1}{T}\sum X_c\left(\frac{J\omega}{T} - \frac{2\pi k}{T}\right)$$

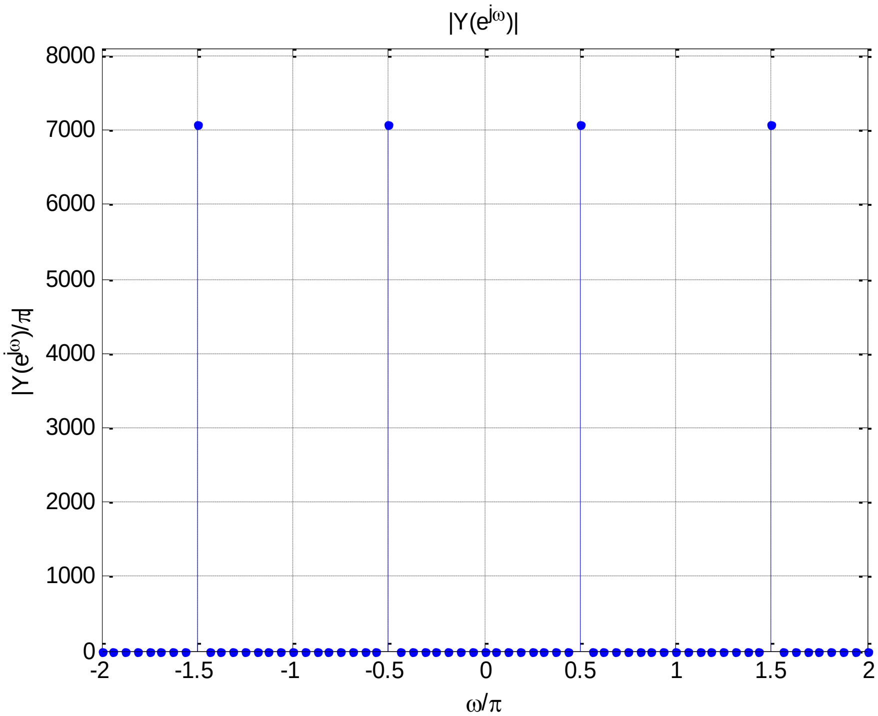
$$X_c(j\Omega) = \left(2\pi\delta(\Omega) + 3\pi[\delta(\Omega + 500\pi) + \delta(\Omega - 500\pi)] + \frac{4\pi}{j}[\delta(\Omega + 1500\pi) - \delta(\Omega - 1500\pi)]\right)$$

Note that aliasing occurs for the third term and due to aliasing the impulses are seen at $\frac{\pi}{2}$ as for the second term.

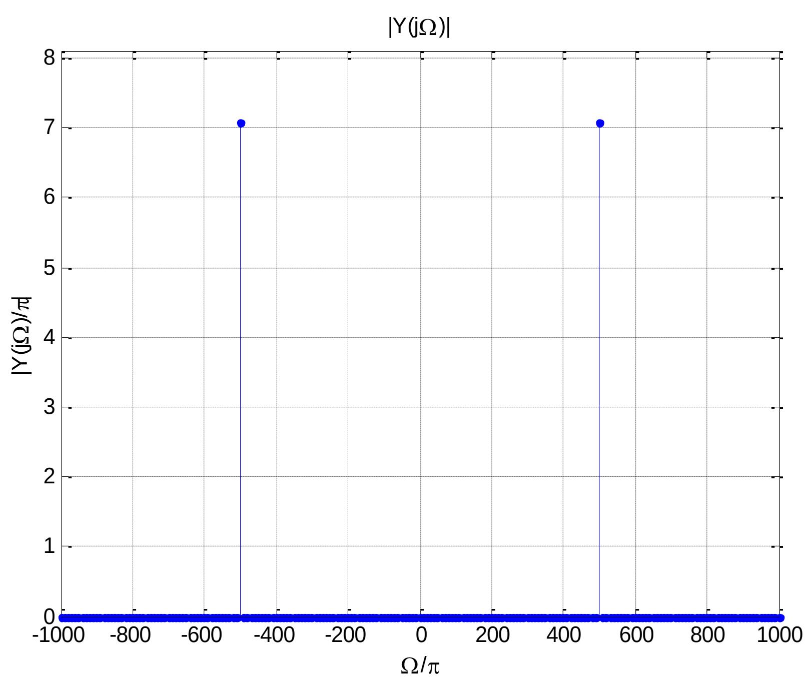
$$Y\left(e^{j\omega}\right) = X\left(e^{j\omega}\right)H\left(e^{j\omega}\right)$$

$$H\left(e^{j0}\right) = 0$$

$$H\left(e^{\frac{j\pi}{2}}\right) = 1 - j, \left|H\left(e^{\frac{j\pi}{2}}\right)\right| = \sqrt{2}$$



$$Y_r(j\Omega) = TY\left(e^{j\Omega T_2}\right)$$



c. We can obtain such a case when $H\left(e^{j\omega}\right) = 0$ at the frequencies of last two term.

$$H\left(e^{j\omega}\right) = 0 \; when \; \omega = \frac{2\pi}{5}k \rightarrow \omega_1 = \frac{2\pi}{5}, \omega_2 = \frac{4\pi}{5}$$

$$\Omega_1 = T\omega_1 = 400\pi, \qquad \Omega_2 = T\omega_2 = 800\pi$$

8.

```
N = 32;
n = 0:N-1;
k = 0:N-1;
W = exp(1j*2*pi/N);

x = (n ==0);
% x = ones(1,N);
% x = sin(pi*n/5+pi/8);
% x = sin(pi*(n-N/2)/8)./pi.*(n-N/2);

X = zeros(size(x));

tic
for m =1:32
    X(m)  = sum(x.*W.^(n*(m-1)));
end
toc

tic
fft(x);
toc
subplot(211)
stem(abs(X))
title('|X|')
subplot(212)
stem(angle(X))
title('Phase(X)')
```