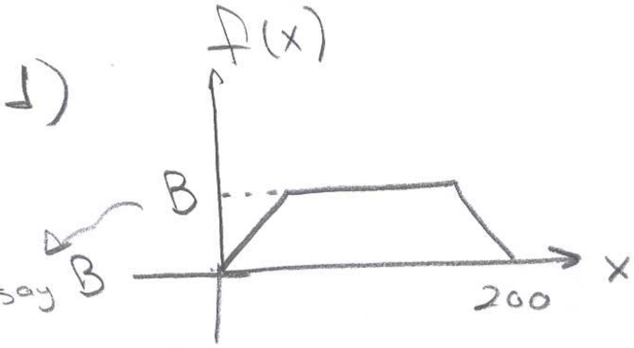
EE 430

Digital Signal Processing

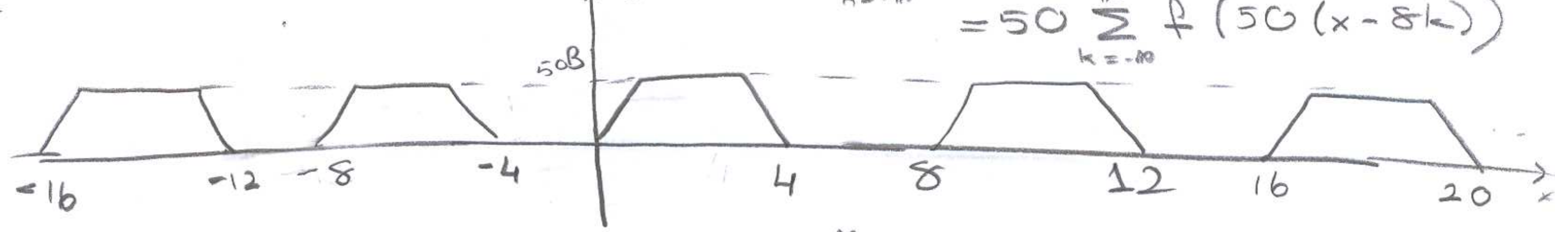
Section 2

HW 4

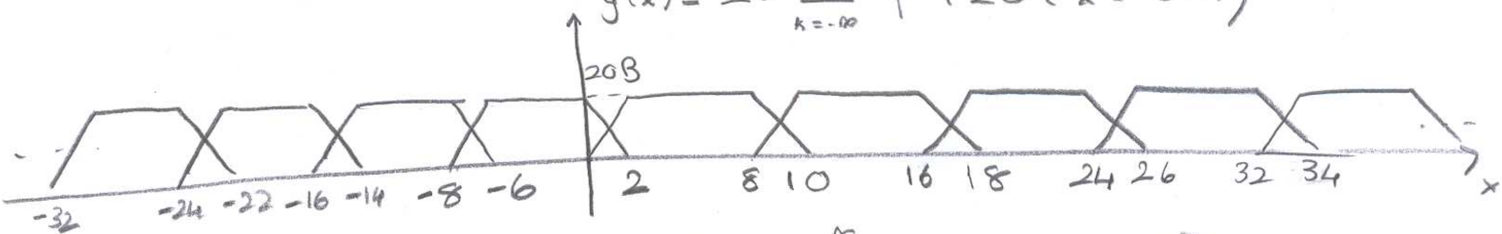
1)



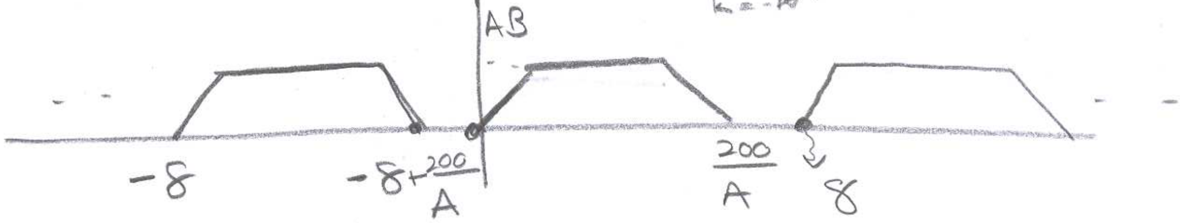
$$g(x) = 50 \sum_{k=-\infty}^{\infty} f(50x - 400k) = 50 \sum_{k=-\infty}^{\infty} f(50(x-8k))$$



$$g(x) = 20 \sum_{k=-\infty}^{\infty} f(20(x-8k))$$



$$g(x) = A \sum_{k=-\infty}^{\infty} f(A(x-8k))$$

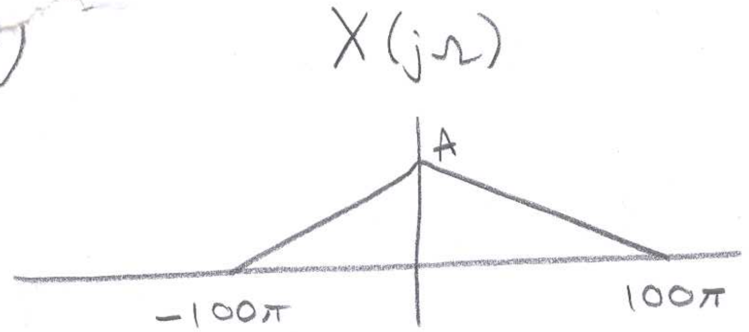


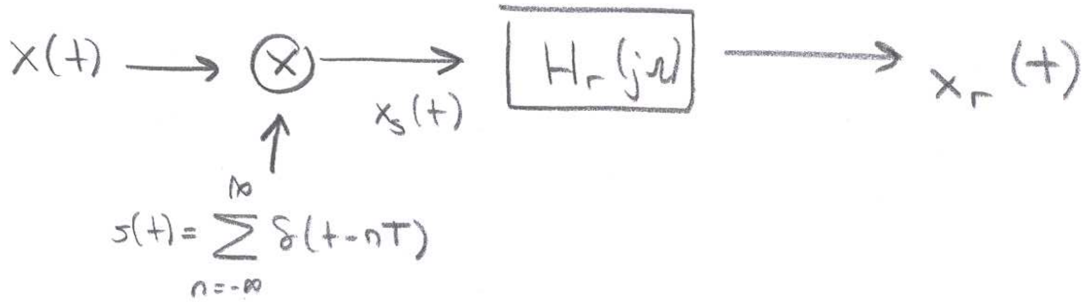
$$-8 + \frac{200}{A} \leq 0$$

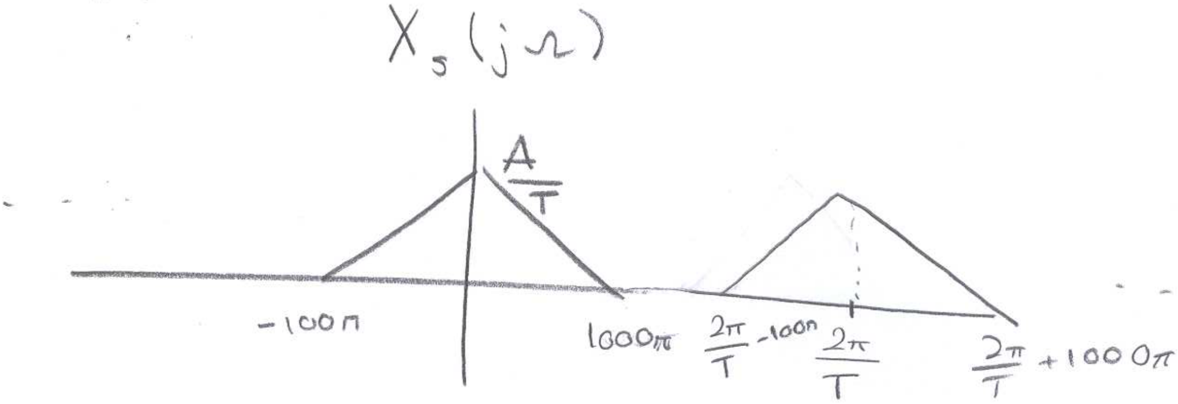
$$\frac{200}{A} \leq 8$$

equivalent $\rightarrow$ $\boxed{A \geq 25}$

2)







To be able to recover $x(t)$ from $x[n]$ there should be no aliasing in $X_s(j\Omega)$.

$$\frac{2\pi}{T} - 100\pi > 1000\pi$$

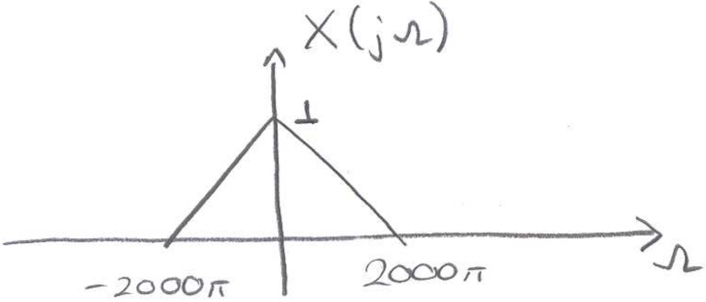
$$\frac{1}{T} > 550$$

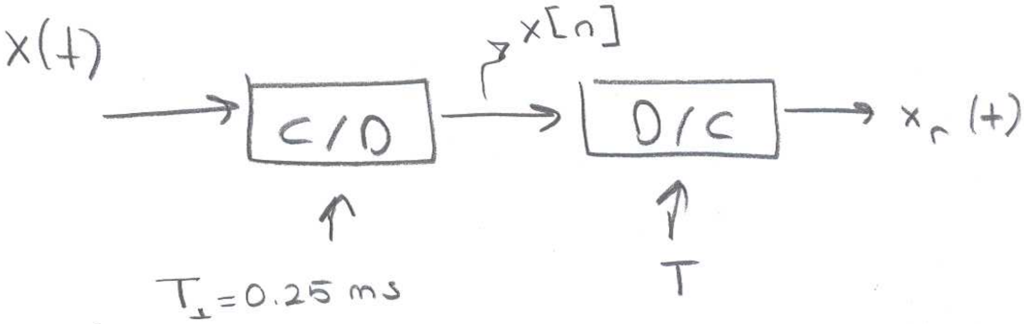
$\frac{1}{T} \rightarrow$ sampling frequency in Hz.

Minimum sampling frequency in Hz is 550 Hz.

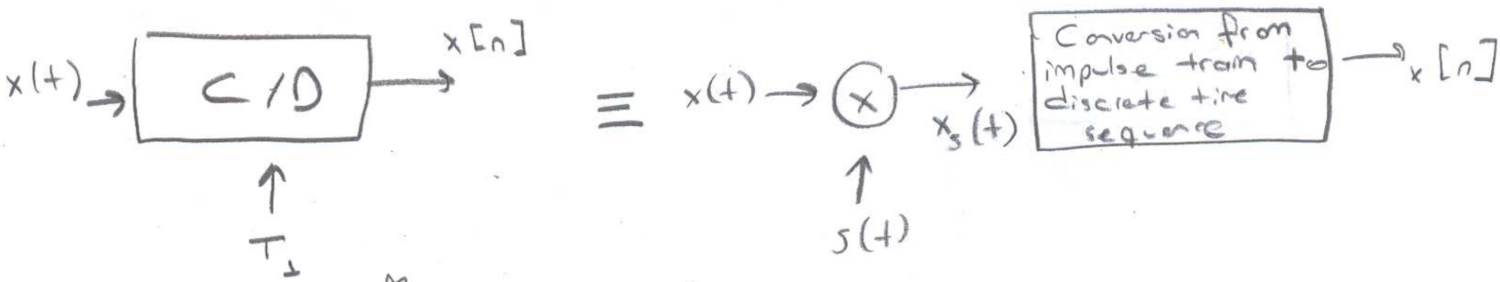
3

3)





$$x[n] = x(nT_1)$$



$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_1)$$

$$x_s(t) = x(t)\, s(t) = \sum_{n=-\infty}^{\infty} x(t)\, \delta(t - nT_1)$$

$$= \sum_{n=-\infty}^{\infty} x(nT_1)\, \delta(t - nT_1)$$

$$S(j\Omega) = \frac{2\pi}{T_1} \sum_{k=-\infty}^{\infty} \delta\left(\Omega - k\frac{2\pi}{T_1}\right)$$

$$X_s(j\Omega) = \frac{1}{2\pi} X(j\Omega) * S(j\Omega)$$

$$= \frac{1}{T_1} \sum_{k=-\infty}^{\infty} X\left(j\left(\Omega - k\frac{2\pi}{T}\right)\right)$$

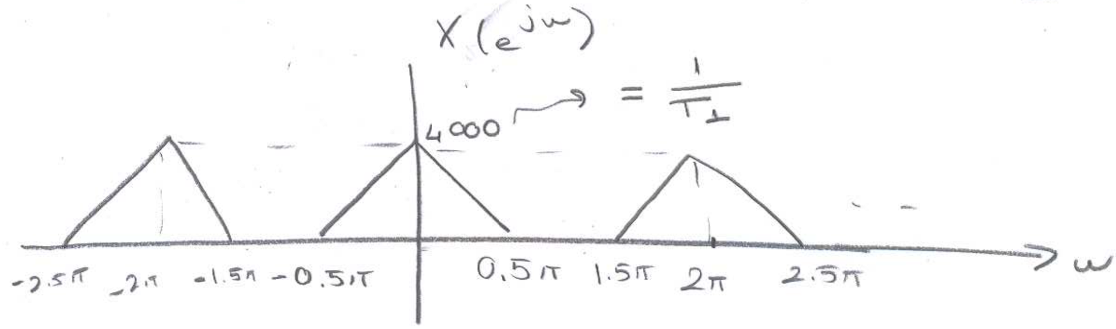
$$X_s(j\Omega) = \sum_{n=-\infty}^{\infty} \underbrace{x(nT_1)}_{x[n]} e^{-j\Omega n T_1}$$

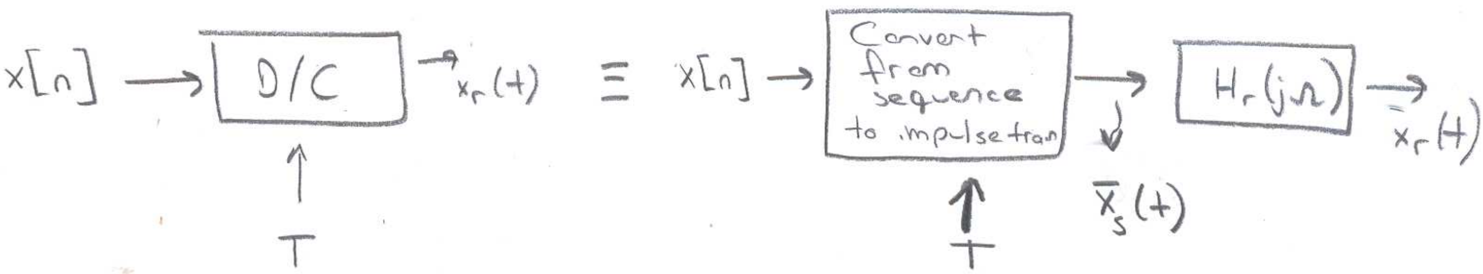
$$= \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n T_1} = X(e^{j\omega})\Big|_{\omega = \Omega T_1}$$

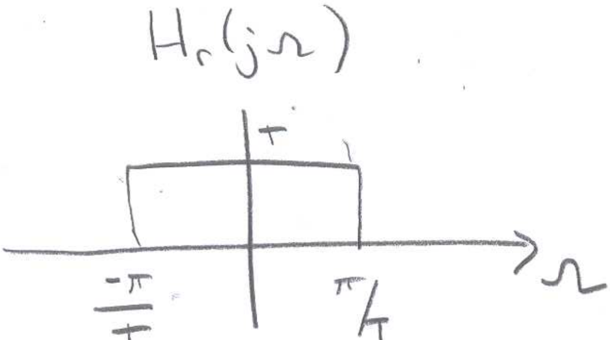
$$= X(e^{j\Omega T_1})$$

$$X(e^{j\Omega T_1}) = \frac{1}{T_1} \sum_{k=-\infty}^{\infty} X\left(j\left(\Omega - k\frac{2\pi}{T_1}\right)\right)$$

$$X(e^{j\omega}) = \frac{1}{T_1} \sum_{k=-\infty}^{\infty} X\left[j\left(\frac{\omega}{T_1} - k\frac{2\pi}{T_1}\right)\right]$$



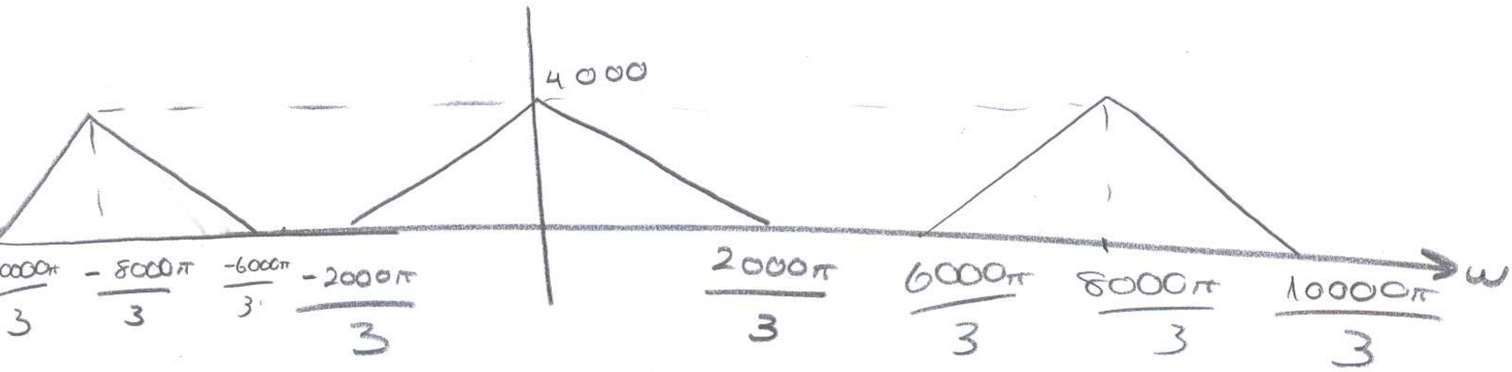




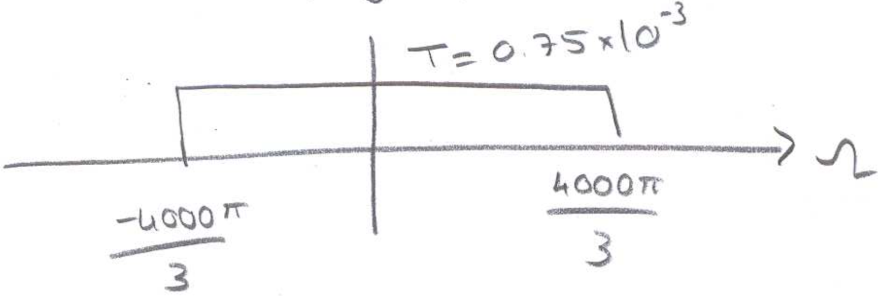
$$\bar{x}_s(t) = \sum_{n=-\infty}^{\infty} x[n]\,\delta(t-nT)$$

$$\bar{X}_s(j\Omega) = \sum_{n=-\infty}^{\infty} x[n]\,e^{-j\Omega nT} = X(e^{j\omega})\Big|_{\omega=\Omega T}$$

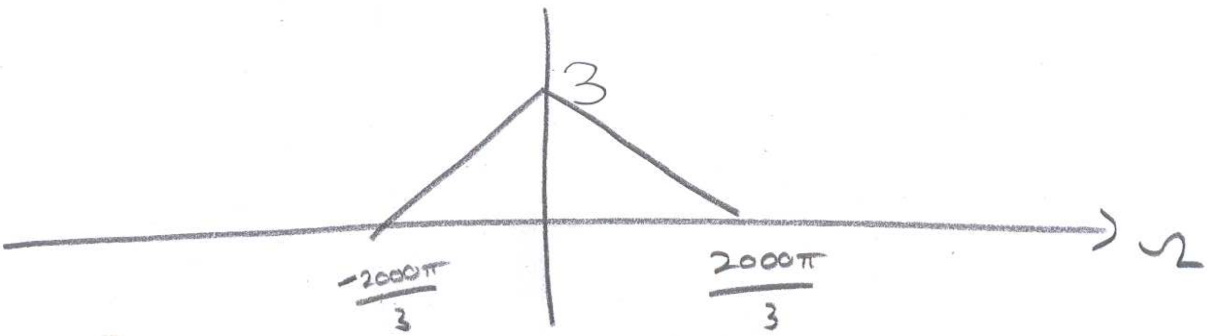
a) $T = 0.75\,ms$ $\quad \bar{X}_s(j\Omega)$



$H_r(j\Omega)$

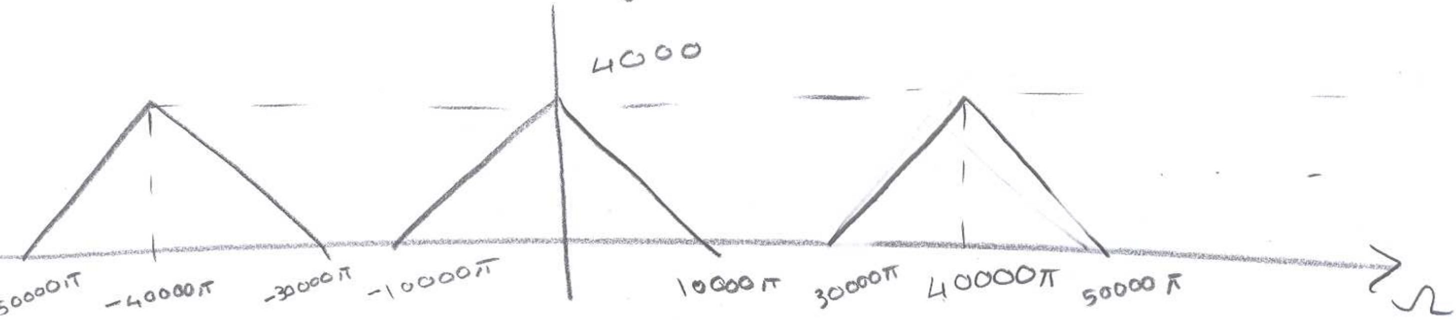


$X_r(j\Omega)$

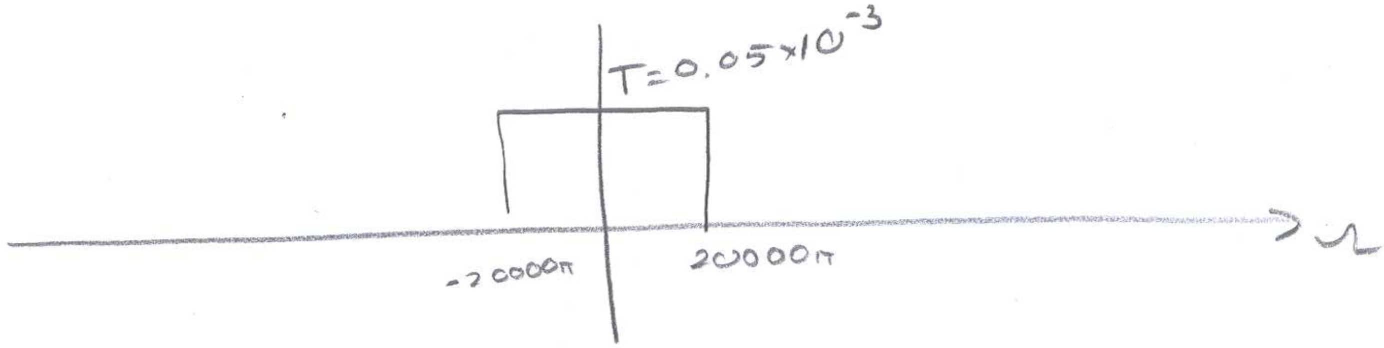


b) $T = 0,05$ ms

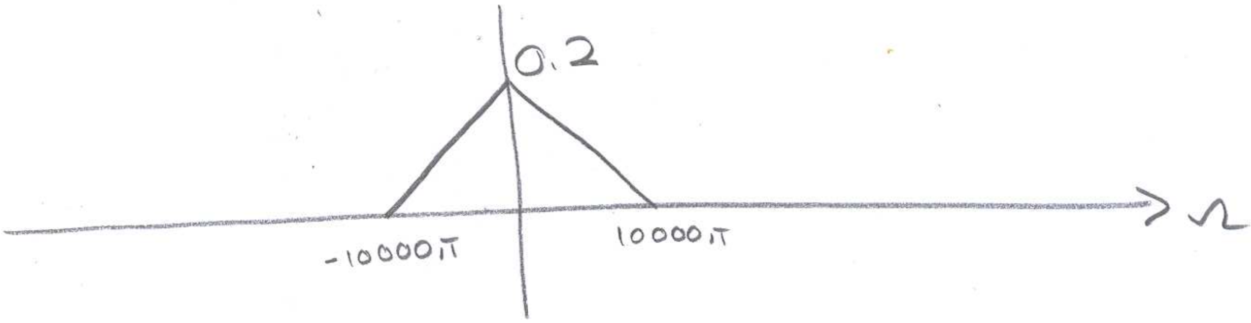
$\overline{X}_s(j\Omega)$



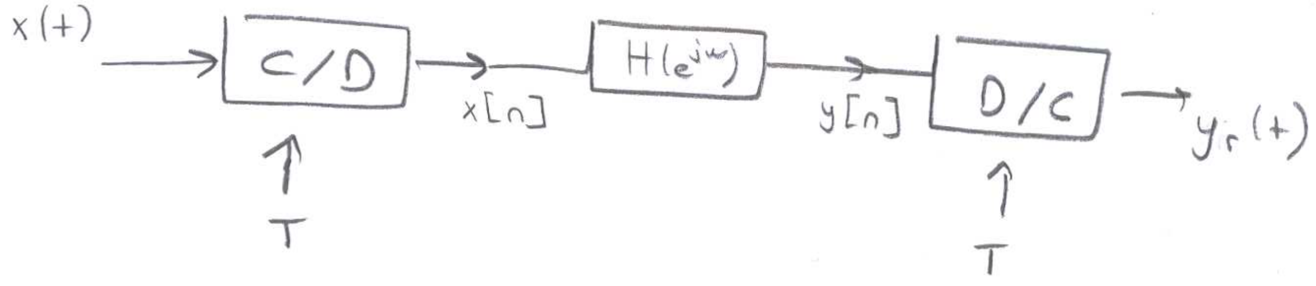
$H_r(j\Omega)$



$X_r(j\Omega)$



5)



$$x[n] = x(nT)$$

$$X(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X\left[j\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)\right]$$

$$Y_r(j\Omega) = \underbrace{H_r(j\Omega)}_{\substack{\text{reconstruction} \\ \text{filter} \\ \text{in } D/C}} Y(e^{j\Omega T}) = \begin{cases} T\,Y(e^{j\Omega T}) & |\Omega| < \frac{\pi}{T} \\ 0 & \text{o.w} \end{cases}$$

$$Y(e^{j\omega}) = H(e^{j\omega})\,X(e^{j\omega})$$

$$Y_r(j\Omega) = H_r(j\Omega)\,H(e^{j\Omega T})\,X(e^{j\Omega T})$$

$$Y_r(j\Omega) = H_r(j\Omega)\,H(e^{j\Omega T})\,\frac{1}{T}\sum_{k=-\infty}^{\infty} X\left(j\left(\Omega - \frac{2\pi k}{T}\right)\right)$$

If $X(j\Omega) = 0$ for $|\Omega| \geq \frac{\pi}{T}$ then

$H_r(j\Omega)$ cancels the factor $\frac{1}{T}$ and selects only the term for $k=0$;

$$Y_r(j\Omega) = \begin{cases} H(e^{j\Omega T})\,X(j\Omega) & |\Omega| < \pi/T \\ 0 & \text{o.w} \end{cases}$$

* If $X(j\Omega)$ is bandlimited and the sampling rate is above the Nyquist rate the output is;

$$Y_r(j\Omega) = H_{eff}(j\Omega)\, X(j\Omega)$$

$$H_{eff}(j\Omega) = \begin{cases} H(e^{j\Omega T}) & |\Omega| < \frac{\pi}{T} \\ 0 & \text{o.w} \end{cases}$$

a) $10000\pi T \leq \pi$

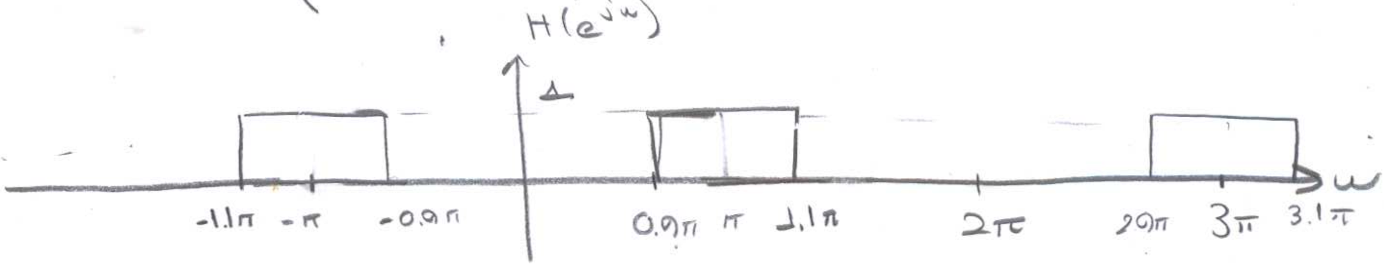
$$T \leq \frac{1}{10000}$$

Because the filter is a highpass filter. H(e^(jw)) cannot be zero at w=pi.

Take $T = \frac{1}{10000} = 0.1 \text{ ms}$

$$H_{eff}(j\Omega) = \begin{cases} 1 & 9000\pi \leq |\Omega| \leq 10000\pi \\ 0 & \text{o.w} \end{cases}$$

$$H(e^{jw}) = \begin{cases} 1 & 0.9\pi \leq |w| \leq \pi \\ 0 & \text{o.w} \end{cases}$$



* $X(j\Omega)$ should be bandlimited to 10000π so that the system acts as the specified bandpass filter.

$$X(j\Omega) = 0 \quad \text{for } |\Omega| > 10000\pi$$

b) $1000\pi \, T \leqslant \pi$

$$T \leqslant \frac{1}{1000}$$

Take $T = \frac{1}{1000} = 1 \text{ ms}$

$$H_{eff}(j\Omega) = \begin{cases} 1 & 900\pi \leqslant |\Omega| \leqslant 1000\pi \\ 0 & \text{o.w} \end{cases}$$

$$H(e^{jw}) = \begin{cases} 1 & 0.9\pi \leqslant |w| \leqslant \pi \\ 0 & \text{o.w} \end{cases}$$

* $X(j\Omega)$ should be bandlimited to 1000π.

$$X(j\Omega) = 0 \quad \text{for } |\Omega| > 1000\pi$$

7)

a) $$x[n] \longrightarrow \boxed{\uparrow 2} \xrightarrow{z_1[n]} \boxed{\downarrow 5} \longrightarrow y_1[n]$$

$$z_1[n] = \begin{cases} x[n/2] & n = 0, \pm 2, \pm 4, \\ 0 & \text{o.w} \end{cases}$$

$$z_1[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-2k]$$

$$Z_1(e^{jw}) = \sum_{n=-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} x[k]\,\delta[n-2k] \right) e^{-jwn}$$

$$= \sum_{k=-\infty}^{\infty} x[k]\, e^{-jw2k} = X(e^{j2w})$$

$$y_1[n] = z_1[5n]$$

$$Z_1(e^{jw}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} Z_{1_c}\left[j\left(\frac{w}{T} - \frac{2\pi k}{T}\right)\right]$$

$$Y_1(e^{jw}) = \frac{1}{5T} \sum_{r=-\infty}^{\infty} Z_{1_c}\left[j\left(\frac{w}{5T} - \frac{2\pi r}{5T}\right)\right]$$

$r = i + k5 \qquad i = 0, 1, \ldots 4 \qquad k: -\infty \to \infty$

$$Y_1(e^{jw}) = \frac{1}{5} \sum_{i=0}^{4} \left\{ \frac{1}{T} \sum_{k=-\infty}^{\infty} Z_{1_c}\left[j\left(\frac{w}{5T} - \frac{2\pi k}{T} - \frac{2\pi i}{5T}\right)\right] \right\}$$

$$Z_1(e^{j(w-2\pi i)/5}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} Z_{1_c}\left[j\left(\frac{w-2\pi i}{5T} - \frac{2\pi k}{T}\right)\right]$$

$$Y_1(e^{jw}) = \frac{1}{5} \sum_{i=0}^{4} Z_1\left(e^{j\left(\frac{w-2\pi i}{5}\right)}\right)$$

$$Y_1(e^{jw}) = \frac{1}{5} \sum_{i=0}^{4} X\left(e^{j\left(\frac{2}{5}(w-2\pi i)\right)}\right)$$

b) Similarly;

$$Z_2(e^{jw}) = \frac{1}{5} \sum_{i=0}^{4} X\left(e^{j\left(\frac{w-2\pi i}{5}\right)}\right)$$

$$Y_2(e^{jw}) = Z_2(e^{jw2}) = \frac{1}{5} \sum_{i=0}^{4} X\left(e^{j\left(\frac{2w-2\pi i}{5}\right)}\right)$$

c)

$$Y_1(e^{j\omega}) = \frac{1}{5}\Big[\underline{\underline{X(e^{j\frac{2\omega}{5}})}} + X(e^{j(\frac{2\omega}{5}-\frac{4\pi}{5})}) + \underset{*}{X(e^{j(\frac{2\omega}{5}-\frac{8\pi}{5})})}$$

$$+ \underset{**}{X(e^{j(\frac{2\omega}{5}-\frac{12\pi}{5})})} + \underset{***}{X(e^{j(\frac{2\omega}{5}-\frac{16\pi}{5})})}\Big]$$

$$Y_2(e^{j\omega}) = \frac{1}{5}\Big[\underline{\underline{X(e^{j\frac{2\omega}{5}})}} + \underset{**}{X(e^{j(\frac{2\omega}{5}-\frac{2\pi}{5})})} + X(e^{j(\frac{2\omega}{5}-\frac{4\pi}{5})})$$

$$+ \underset{***}{X(e^{j(\frac{2\omega}{5}-\frac{6\pi}{5})})} + \underset{*}{X(e^{j(\frac{2\omega}{5}-\frac{8\pi}{5})})}$$

$$X(e^{j(\frac{2\omega}{5}-\frac{12\pi}{5})}) = X(e^{j(\frac{2\omega}{5}-\frac{2\pi}{5})})$$

$$X(e^{j(\frac{2\omega}{5}-\frac{16\pi}{5})}) = X(e^{j(\frac{2\omega}{5}-\frac{6\pi}{5})})$$

$$Y_1(e^{j\omega}) = Y_2(e^{j\omega}) \Longrightarrow y_1[n] = y_2[n]$$

d) In this case,

$$Y_1(e^{j\omega}) = \frac{1}{4}\sum_{i=0}^{3} X(e^{j(\frac{2}{4}(\omega - 2\pi i))})$$

$$Y_2(e^{j\omega}) = \frac{1}{4}\sum_{i=0}^{3} X(e^{j(\frac{2}{4}\omega - \frac{2\pi i}{4})})$$

$$Y_1(e^{j\omega}) = \frac{1}{4}\Big[X(e^{j\frac{\omega}{2}}) + X(e^{j(\frac{\omega}{2}-\pi)}) + X(e^{j(\frac{\omega}{2}-2\pi)}) + X(e^{j(\frac{\omega}{2}-3\pi)})$$

$$Y_2(e^{j\omega}) = \frac{1}{4}\Big[X(e^{j\frac{\omega}{2}}) + X(e^{j(\frac{\omega}{2}-\frac{\pi}{2})}) + X(e^{j(\frac{\omega}{2}-\pi)}) + X(e^{j(\frac{\omega}{2}-\frac{3\pi}{2})})$$

$$Y_1(e^{j\omega}) \neq Y_2(e^{j\omega}) \Longrightarrow y_1[n] \neq y_2[n]$$

8)

$$x[n] \longrightarrow \boxed{H(e^{j\omega})} \xrightarrow{\ z_1[n]\ } \boxed{\downarrow M} \longrightarrow y_1[n]$$

$$x[n] \longrightarrow \boxed{\downarrow M} \xrightarrow{\ z_2[n]\ } \boxed{G(e^{j\omega})} \longrightarrow y_2[n]$$

$$Z_1(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$$

$$Y_1(e^{j\omega}) = \frac{1}{M}\sum_{i=0}^{M-1} Z_1\left(e^{j\left(\frac{\omega-2\pi i}{M}\right)}\right)$$

$$= \frac{1}{M}\sum_{i=0}^{M-1} X\left(e^{j\left(\frac{\omega-2\pi i}{M}\right)}\right) H\left(e^{j\left(\frac{\omega-2\pi i}{M}\right)}\right)$$

$$Z_2(e^{j\omega}) = \frac{1}{M}\sum_{i=0}^{M-1} X\left(e^{j\left(\frac{\omega-2\pi i}{M}\right)}\right)$$

$$Y_2(e^{j\omega}) = G(e^{j\omega})Z_2(e^{j\omega}) = \frac{1}{M}\sum_{i=0}^{M-1} X\left(e^{j\left(\frac{\omega-2\pi i}{M}\right)}\right) G(e^{j\omega})$$

If $x[n]$ is bandlimited to $\frac{\pi}{M}$,

$$\underbrace{X\left(e^{j\left(\frac{\omega-2\pi i}{M}\right)}\right)}_{} \qquad i = 0, 1, \ldots M-1$$

These are not overlapping.

$$H_{eff}(e^{j\omega}) = \begin{cases} H(e^{j\omega}) & |\omega| \leq \frac{\pi}{M} \\ 0 & \text{o.w} \end{cases}$$

$$X\left(e^{j\left(\frac{\omega-2\pi i}{M}\right)}\right) H_{eff}\left(e^{j\left(\frac{\omega-2\pi k}{M}\right)}\right) = 0 \text{ for } i \neq k.$$

(no overlapping region)

15

$$Y_1(e^{j\omega}) = \left[\sum_{k=0}^{M-1} H_{eff}\left(e^{j\left(\frac{\omega-2\pi k}{M}\right)}\right)\right] \frac{1}{M}\sum_{i=0}^{M-1} X\left(e^{j\left(\frac{\omega-2\pi i}{M}\right)}\right)$$

$$Y_2(e^{j\omega}) = G(e^{j\omega}) \frac{1}{M}\sum_{i=0}^{M-1} X\left(e^{j\left(\frac{\omega-2\pi i}{M}\right)}\right)$$

If $G(e^{j\omega}) = \sum_{k=0}^{M-1} H_{eff}\left(e^{j\left(\frac{\omega-2\pi k}{M}\right)}\right)$

Then $Y_1(e^{j\omega}) = Y_2(e^{j\omega}) \Longrightarrow y_1[n] = y_2[n]$

9)

$$x[n] \longrightarrow \boxed{H(e^{j\omega})} \xrightarrow[z_1[n]]{} \boxed{\uparrow L} \longrightarrow y_1[n]$$

$$x[n] \longrightarrow \boxed{\uparrow L} \xrightarrow[z_2[n]]{} \boxed{G(e^{j\omega})} \longrightarrow y_2[n]$$

$$Z_1(e^{j\omega}) = X(e^{j\omega}) H(e^{j\omega})$$

$$Y_1(e^{j\omega}) = Z_1(e^{j\omega L}) = X(e^{j\omega L}) H(e^{j\omega L})$$

$$Z_2(e^{j\omega}) = X(e^{j\omega L})$$

$$Y_2(e^{j\omega}) = Z_2(e^{j\omega}) G(e^{j\omega}) = X(e^{j\omega L}) G(e^{j\omega})$$

If $G(e^{j\omega}) = H(e^{j\omega L})$

$$Y_1(e^{j\omega}) = Y_2(e^{j\omega}) \Longrightarrow y_1[n] = y_2[n]$$