Aykut DEMİREL
1813948
Section: 2

# EE-430 – HW5

0)

2) Yes, it can have if it is a linear phase system.

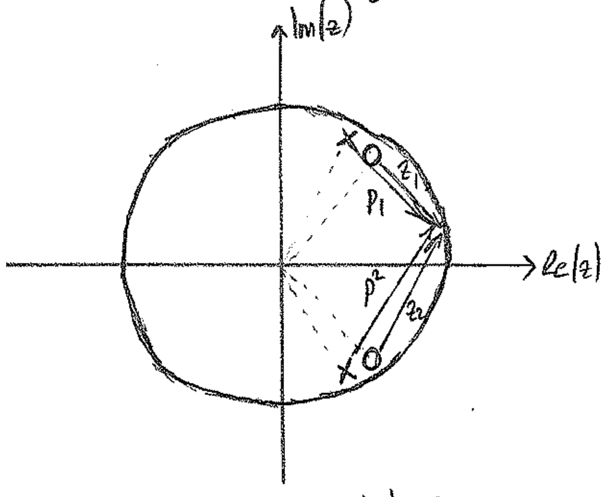
$$H(e^{j\omega}) = e^{-j\alpha\omega} A(e^{j\omega}) \quad \text{where} \quad A(e^{j\omega}) > 0 \text{ and real.}$$

$$\Rightarrow \operatorname{Arg}(H(e^{j\omega})) = -\alpha\omega \Rightarrow \left.\begin{aligned} \text{phase-delay} &= -\frac{\operatorname{Arg}(H(e^{j\omega}))}{\omega} = \alpha \\ \text{group-delay} &= -\frac{d\operatorname{Arg}(H(e^{j\omega}))}{d\omega} = \alpha \end{aligned}\right\} \text{They are identical}$$

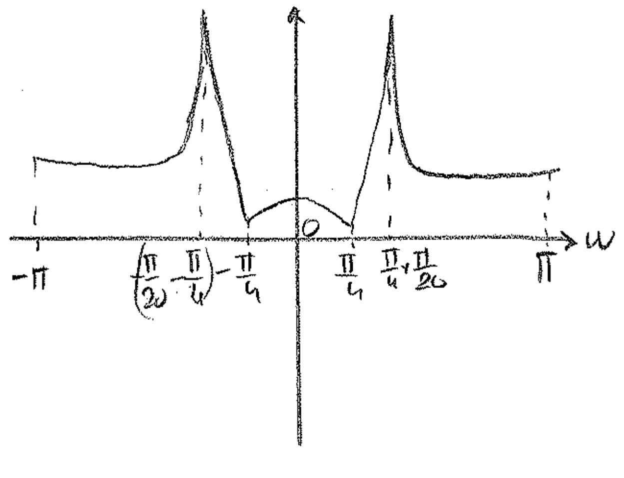
5)

$$H_1(z) = \frac{\left(1 - 0.95e^{j\frac{\pi}{4}} z^{-1}\right)\left(1 - 0.95 e^{-j\frac{\pi}{4}} z^{-1}\right)}{\left(1 - 0.95e^{j\left(\frac{\pi}{4}+\frac{\pi}{20}\right)} z^{-1}\right)\left(1 - 0.95e^{-j\left(\frac{\pi}{4}+\frac{\pi}{20}\right)} z^{-1}\right)}$$

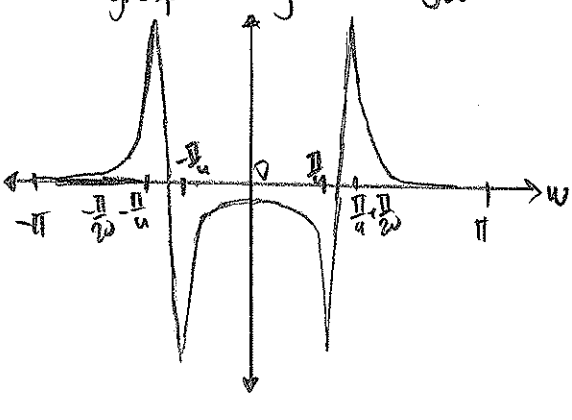
⇒ Pole-zero diagram



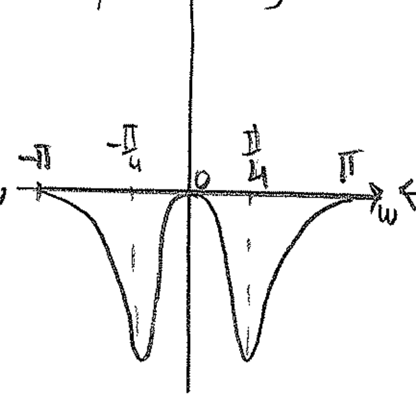

magnitude response:



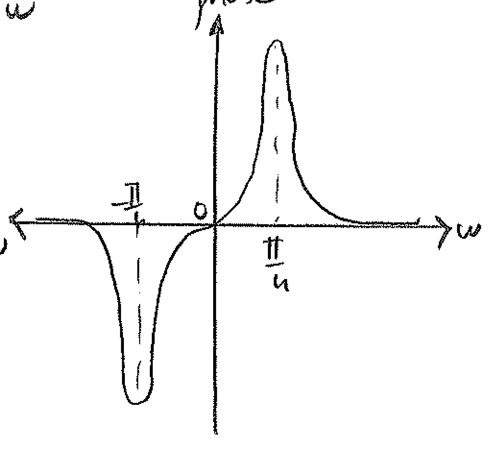

group delay = $-\frac{d\,\text{phase}}{d\omega}$




phase-delay = $-\frac{\text{phase}}{\omega}$

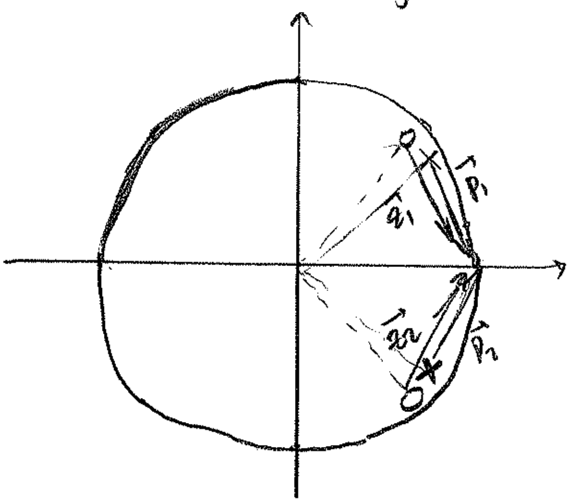



phase



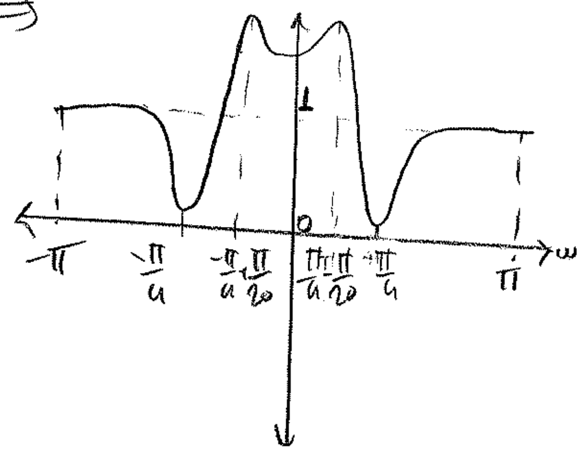
$H_e(z) \Rightarrow \quad z_1, z_2 = 0.95 e^{\pm j\frac{\pi}{4}} \qquad p_1, p_2 = 0.95 e^{\pm j\left(\frac{\pi}{4} - \frac{\pi}{20}\right)}$

pole-zero diagram

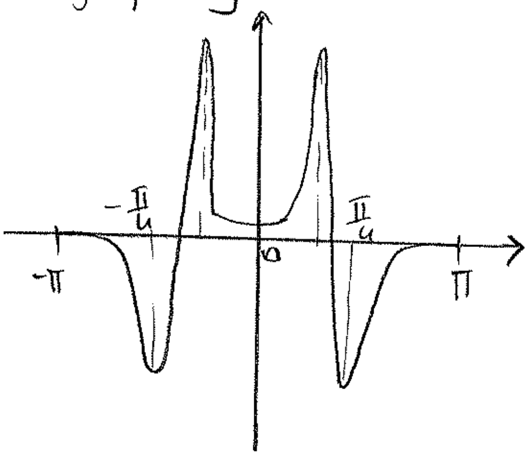



Magnitude response

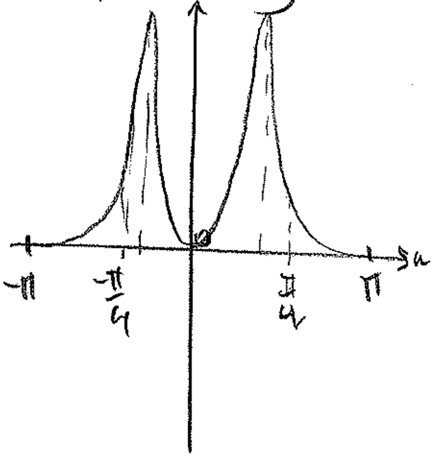
$\Rightarrow$



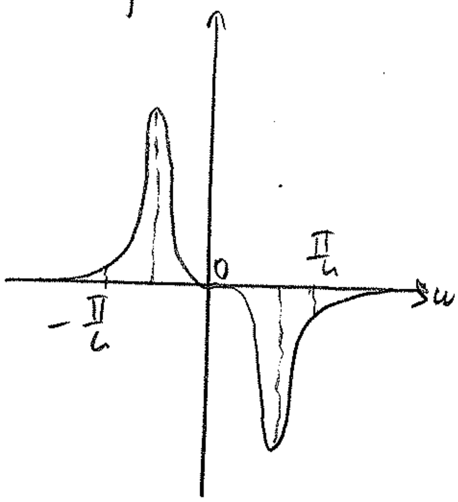

group-delay




phase delay




phase



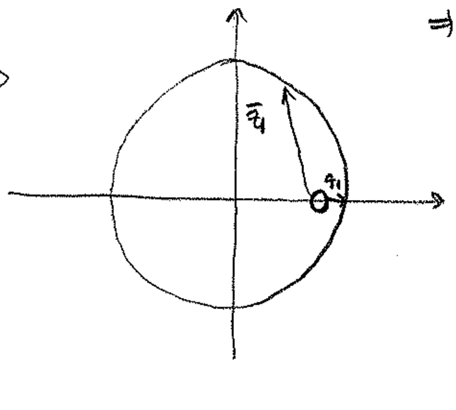

$$\text{phase} = \left(\mathrm{Arg}(\vec{z_1}) + \mathrm{Arg}(\vec{z_2})\right) - \left(\mathrm{Arg}(\vec{p_1}) + \mathrm{Arg}(\vec{p_2})\right) \pm \pi \text{ maybe}$$

So phase tends to decrease when we are getting closer to a pole since angle of pole vector increases more rapidly than other pole and zero vectors. For zeros; vice versa.

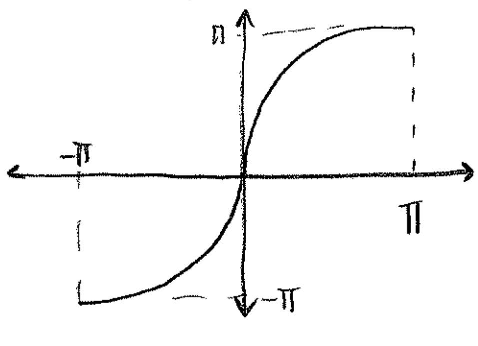
7) For $H(z) = 1 - 0.9z^{-1}$

$\Rightarrow$ as $\omega$ increases from 0 to $\pi$, angle of $\vec{z_1}$ increases; always.

$\Rightarrow$

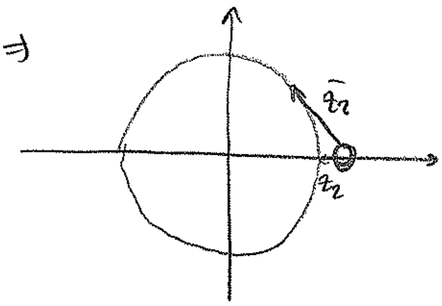



phase

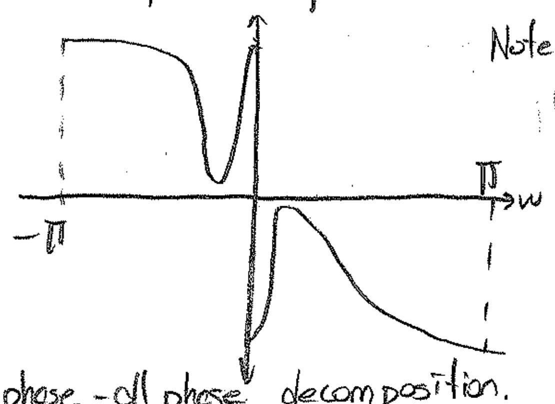


For $H(z) = 1 - \frac{10}{9} z^{-1}$ $\Rightarrow$ as $\omega$ increases from $0$ to $\pi$, angle of $\bar{z}_2^{-1}$ first increases and than decreases.

$\Rightarrow$




$\Rightarrow$ phase response




Note that $-\pi = \pi$

11) All system functions have a minimum phase - all phase decomposition.

$$H(z) = \frac{(1-2z^{-1})(z+2)(z-\frac{1}{2})}{(1-\frac{1}{2}z^{-1}+\frac{1}{2}z^{-2})}$$

$(1-2z^{-1})(z+2)$ → zeros at 2 and -2 outside the unit circle

$(z-\frac{1}{2})$ → zero at $(\frac{1}{2})$ inside the unit circle

$(1-\frac{1}{2}z^{-1}+\frac{1}{2}z^{-2})$ } poles inside the unit circle.

$$\Rightarrow H_{ap}(z) = \left\{ \frac{z^{-1}-\frac{1}{2}}{1-\frac{1}{2}z^{-1}} \cdot \frac{z^{-1}+\frac{1}{2}}{1+\frac{1}{2}z^{-1}} \right\}$$

$\Rightarrow$ has all-pass characteristics and has zeros at "2 and -2".

$$\Rightarrow H_{min}(z) = \frac{H(z)}{H_{ap}(z)} = \frac{\overset{-2}{\cancel{(1-2z^{-1})}}\overset{2z}{\cancel{(z+2)}}(z-\frac{1}{2})}{(1-\frac{1}{2}z^{-1}+\frac{1}{2}z^{-2})} \cdot \frac{(1-\frac{1}{2}z^{-1})(1+\frac{1}{2}z^{-1})}{\cancel{(z^{-1}-\frac{1}{2})}\cancel{(z^{-1}+\frac{1}{2})}}$$

$$= \frac{-4z(1-\frac{1}{2}z^{-1})(1+\frac{1}{2}z^{-1})}{(1-\frac{1}{2}z^{-1}+\frac{1}{2}z^{-2})}$$

13) Phase lag function : $-Arg(H(e^{j\omega}))$

Phase-delay function: $-\frac{Arg(H(e^{j\omega}))}{\omega}$

15) a) $H(z) = \dfrac{(1-z_1 z^{-1})(1-z_1^* z^{-1})}{(1-p_1 z^{-1})(1-p_1^* z^{-1})}$ where $z_1 = 1.2e^{j\frac{3\pi}{4}}$ $p_1 = 0.5e^{j\frac{\pi}{2}} = j0.5$

Seperate $H(z) = H_1(z).H_2(z)$ ;

$H_1(z) = (1-z_1 z^{-1})(1-z_1^* z^{-1})$ and $H_2(z) = \dfrac{1}{(1-p_1 z^{-1})(1-p_1^* z^{-1})}$ $p_1 = 0.5e^{j\frac{\pi}{2}} = r.e^{j\theta}$

$= 1 - 2.4\cos\frac{3\pi}{4} z^{-1} + 1.44 z^{-2}$ $= \dfrac{1}{1-2r\cos\theta z^{-1} + r^2 z^{-2}}$

$\Rightarrow h_1[n] = \delta[n] + 1.2\sqrt{2}\,\delta[n-1] + 1.44\,\delta[n-2]$

$h_2[n] = \dfrac{1}{\sin\theta}(0.5)^n \sin\left(\frac{\pi}{2}(n+1)\right)u[n] = (0.5)^n \sin\left(\frac{\pi}{2}(n+1)\right)u[n]$

$\Rightarrow$ impulse response $=$ $h[n] = h_1[n] * h_2[n] = h_2[n] + 1.2\sqrt{2}\,h_2[n-1] + 1.44\,h_2[n-2]$

b) $H_{ap}(z) = \left(\dfrac{z^{-1} - \frac{1}{1.2}e^{-j\frac{3\pi}{4}}}{1-\frac{1}{1.2}e^{j\frac{3\pi}{4}}z^{-1}}\right)\left(\dfrac{z^{-1} - \frac{1}{1.2}e^{+j\frac{3\pi}{4}}}{1-\frac{1}{1.2}e^{-j\frac{3\pi}{4}}z^{-1}}\right) = \dfrac{\left(z^{-1} - \frac{1}{z_1}\right)}{\left(1-\frac{1}{z_1^*}z^{-1}\right)} \cdot \dfrac{\left(z^{-1} - \frac{1}{z_1^*}\right)}{\left(1-\frac{1}{z_1}z^{-1}\right)}$

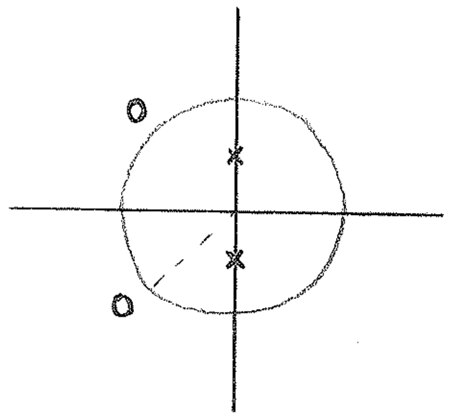
$\Rightarrow H_{min}(z) = \dfrac{H(z)}{H_{ap}(z)} = \dfrac{\overset{-z_1}{\cancel{(1-z_1 z^{-1})}}\,\overset{-z_1^*}{\cancel{(1-z_1^* z^{-1})}}}{(1-p_1 z^{-1})(1-p_1^* z^{-1})} \cdot \dfrac{\left(1-\frac{1}{z_1^*}z^{-1}\right)\left(1-\frac{1}{z_1}z^{-1}\right)}{\cancel{\left(z^{-1}-\frac{1}{z_1}\right)}\,\cancel{\left(z^{-1}-\frac{1}{z_1^*}\right)}}$

$H_{min}(z) = |z_1|^2 \cdot \dfrac{\left(1-\frac{1}{z_1}z^{-1}\right)\left(1-\frac{1}{z_1^*}z^{-1}\right)}{(1-p_1 z^{-1})(1-p_1^* z^{-1})} = \dfrac{1.44\left(1 + \frac{1}{1.44}z^{-2} - \left(\frac{1}{z_1}+\frac{1}{z_1^*}\right)z^{-1}\right)}{(1-p_1 z^{-1})(1-p_1^* z^{-1})}$

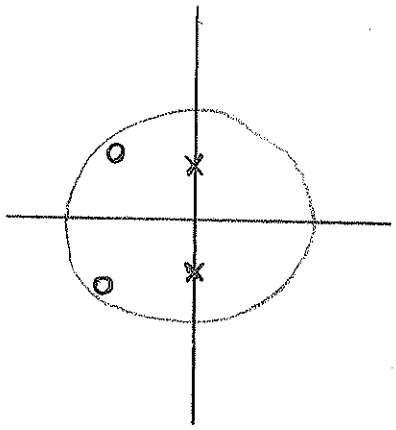
$= 1.44 + 1.2\sqrt{2}\,z^{-1} + z^{-2}$

$\Rightarrow h_{min}[n] = 1.44\,h_2[n] + 1.2\sqrt{2}\,h_2[n-1] + h_2[n-2]$ where $h_2[n]$ is as found in part a.

c) $\underline{H(z)}$ $\underline{H_{min}(z)}$

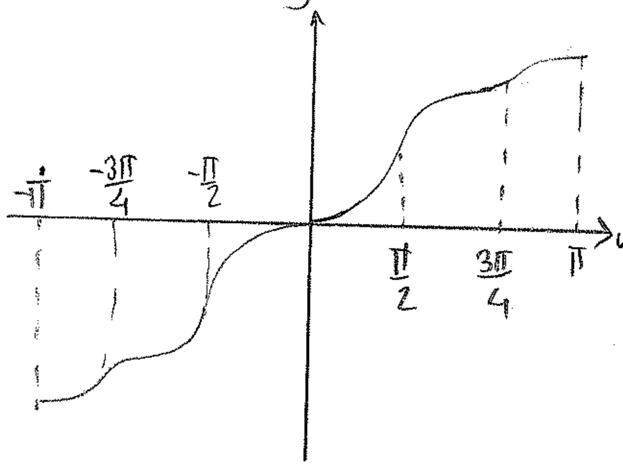


pole-zero diagram

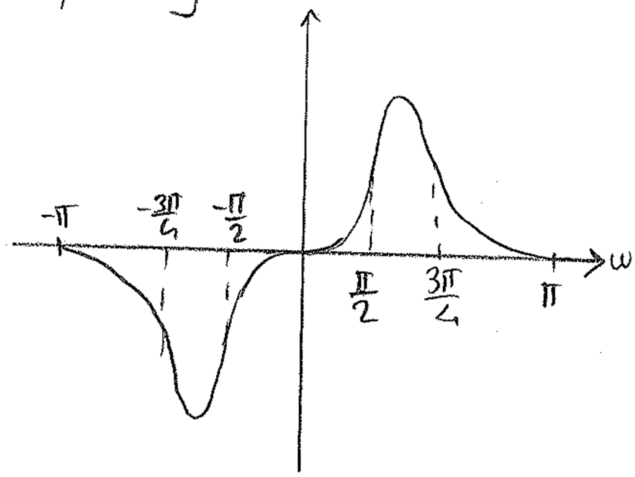


pole-zero diagram.

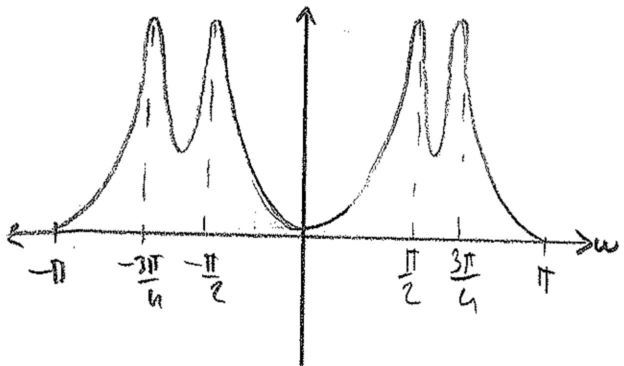
phase-lag function $= -\angle H(e^{j\omega})$



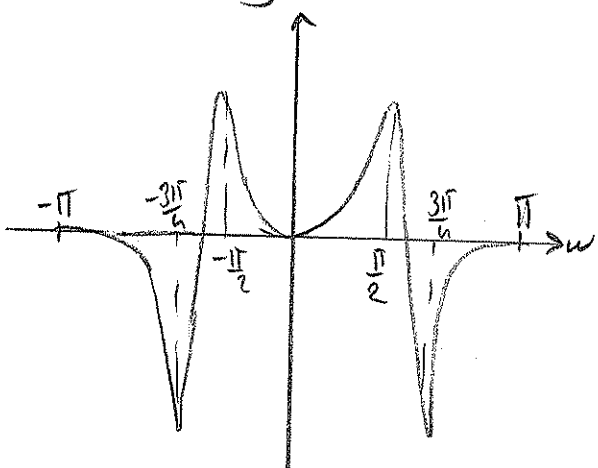

phase lag $= -\angle H_{min}(e^{j\omega})$




d) group delay of $H(z)$




group delay of $H_{min}(z)$




e) $H(z) = H_{min}(z)\, H_{ap}(z) \Rightarrow -\angle H(z) = -\angle H_{min}(z) \underbrace{- \angle H_{ap}(z)}_{>0 \text{ for } 0<\omega<\pi}$

$\Rightarrow -\angle H_{min}(z) < -\angle H(z)$ for $0<\omega<\pi$

phase lag of $H_{min}(z)$ is smaller than that of $H(z)$

7) $H(z) = H_{min}(z) \cdot H_{ap}(z)$

$$\Rightarrow -\frac{d\angle H(z)}{dw} = -\frac{d\angle H_{min}(z)}{dw} - \underbrace{\frac{d\angle H_{ap}(z)}{dw}}_{\geq 0} \Rightarrow \text{grpdelay}(H_{min}(z)) \leq \text{groupdelay}(H(z))$$

9) $\sum_{n=0}^{M} |h[n]|^2 = \sum_{n=0}^{M} | h_2[n] + 1{,}2\sqrt{2}\, h_2[n-1] + 1{,}44\, h_2[n-2]|^2$

and $\sum_{n=0}^{M} |h_{min}[n]|^2 = \sum_{n=0}^{M} | 1{,}44\, h_2[n] + 1{,}2\sqrt{2}\, h_2[n-1] + h_2[n-2]|^2$

given that $h_2[n] = (0.5)^n \sin\left(\frac{\pi}{2}(n+1)\right) u[n] \Rightarrow |h_2[n]|$ is decreasing

Since we have $h_2[n]$ term in energy of $h_{min}[n]$ multiplied with 1,44

$$\sum_{n=0}^{M} |h_{min}[n]|^2 > \sum_{n=0}^{M} |h[n]|^2 \quad \#$$

23) No, $s = f(z)$ transformation must satisfy certain specifications.

$\Rightarrow$ it must result in a stable and causal $H(z)$ from a stable and causal $H_c(s)$ to ensure stability.

$\Rightarrow$ it must map unit circle onthe imaginary axis, left-half-s plane to inside the unit circle and right-half-s plane to outside the unit circle.

$\Rightarrow$ Mapping must be one-to-one between "s" and "z".

25) Yes. Since bilinear transformation does not cause aliasing and all-pass continuous filter can be passed to discrete time using it.

29) a) According to figure given in the question;

$$0.8 \leq |H(e^{jw})| \leq 1 \qquad 0 \leq \omega \leq \frac{3\pi}{10}$$

$$0 \leq |H(e^{jw})| \leq 0.2 \qquad \frac{6\pi}{10} \leq \omega \leq \pi$$

b) Specifications are given in DT $\Rightarrow$ take $T=1$ for simplicity.

$\Rightarrow$ Specs in continuous time;

i) $0.8 \leq |H(\Omega)| \leq 1$ for $0 \leq \Omega \leq \frac{3\pi}{10}$

$0 \leq |H(\Omega)| \leq 0.2$ for $\frac{6\pi}{10} \leq \Omega \leq \pi$

$\Rightarrow$ apply boundary conditions as:

$$H(s)H(-s) = |H(\Omega)|^2 = \frac{1}{1+\left(\frac{j\Omega}{j\Omega_c}\right)^{2N}}$$ to find $N$ and $\Omega_c$

$\Rightarrow$ since Butterworth filters are monotonic everywhere in magnitude, we should consider only 2 conditions:

$$\left.\begin{aligned} 1+\left(\frac{0.3\pi}{\Omega_c}\right)^{2N} &= \frac{1}{(0.8)^2} = 1.5625 \\ 1+\left(\frac{0.6\pi}{\Omega_c}\right)^{2N} &= \frac{1}{(0.2)^2} = 25 \end{aligned}\right\} \quad \left(\frac{1}{2}\right)^{2N} = \frac{0.5625}{24} \Rightarrow 2^{2N} = 42.667$$

$$\Rightarrow N = \frac{1}{2}\log_2 42.667 = 2.71$$

$N$ must be integer $\Rightarrow N=3$

$$\Rightarrow \left(\frac{0.3\pi}{\Omega_c}\right)^6 = 0.5625 \Rightarrow \Omega_c = \frac{0.3\pi}{\sqrt[6]{0.5625}} \cong 1{,}04.$$

$\Rightarrow$ poles are at $s = 1{,}04\, e^{j\frac{\pi}{3}k}$ where $k=\{0,1,2,3,4,5\}$.

$\Rightarrow$ Take left-half plane poles;

$$H(s) = \frac{A}{\left(s-1.04e^{j\frac{2\pi}{3}}\right)\left(s-1.04e^{j\pi}\right)\left(s-1.04e^{j\frac{4\pi}{3}}\right)} \Rightarrow H(s)\Big|_{s=j\omega\,|_{\omega=0}}=1 \Rightarrow A=1{,}125.$$

ii) We transform continuous time filter to discrete time one by impulse invariance as follows;

* Express $H(s)$ in partial fractions as $H(s) = \sum_{k=1}^{3} \frac{A_k}{s-s_k}$ } $s_k$'s are poles found.

* Write $H(z)$ as $H(z) = \frac{A_k \cdot T}{1-e^{s_k T}z^{-1}}$ { $T=1$.

c) $0.8 \leq |H(e^{j\omega})| \leq 1 \qquad 0 \leq \omega \leq 0.3\pi$

$0 \leq |H(e^{j\omega})| \leq 0.2 \qquad 0.6\pi \leq \omega \leq \pi$

use Bilinear Transformation as;

$\Omega = \frac{2}{T} \tan\left(\frac{\omega}{2}\right)$ and take $T=1$ for simplicity.

$2\tan\left(\frac{0}{2}\right) = 0$

$2\tan\left(\frac{0.3\pi}{2}\right) = 1{,}02$

$2\tan\left(\frac{0.6\pi}{2}\right) = 2{,}753$

$2\tan\left(\frac{\pi}{2}\right) = \infty$

Continuous time specs are:

$0.8 \leq |H(\Omega)| \leq 1 \quad , \quad 0 \leq \Omega \leq 1{,}02$

$0 \leq |H(\Omega)| < 0.2 \quad , \quad \Omega \geq 2{,}753$