EE430 Digital Signal Processing
HW 2

1. 1st Question

3) The impulse response of a LTI system is

$$h[n] = \delta[n] - \sqrt{2}\delta[n-1] + \delta[n-2].$$

a) Find the system function $H(z)$. Plot the pole-zero diagram, indicate ALL poles and zeros, show the ROC.

b) Does this system have a frequency response? Why? If yes, plot its magnitude and phase.

c) Find the output of this system to the following input signals

$$x_1[n] = \sin\left(\frac{\pi}{4}n + \frac{\pi}{4}\right)$$

$$x_2[n] = \sin\left(\frac{\pi}{4}n + \frac{\pi}{4}\right)u[n]$$

$$x_3[n] = \sin\left(\frac{\pi}{4}n + \frac{\pi}{4}\right) + \sin\left(\frac{3\pi}{4}n\right)$$

d) Comment on the relationship between the frequency response and zero locations of $H(z)$.

Figure 1: Q1

EE 430

Fall 2014

HW 2 (Section 2)

Solutions for 10-29

10)

$$h[n] = \delta[n] - \sqrt{2}\,\delta[n-1] + \delta[n-2]$$

a)

$$H(z) = \sum_{n=-\infty}^{\infty} h[n]\, z^{-n}$$

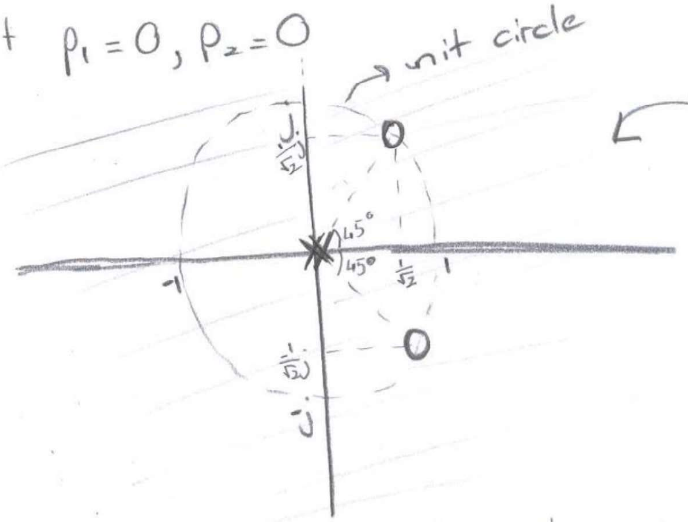
$$= \sum_{n=0}^{2} h[n]\, z^{-n} = 1 - \sqrt{2}\, z^{-1} + z^{-2}$$

$$= \left(1 - \left(\tfrac{1}{\sqrt{2}} + \tfrac{1}{\sqrt{2}}j\right) z^{-1}\right)\left(1 - \left(\tfrac{1}{\sqrt{2}} - \tfrac{1}{\sqrt{2}}j\right) z^{-1}\right)$$

$$= \frac{\left(z - \left(\tfrac{1}{\sqrt{2}} + \tfrac{1}{\sqrt{2}}j\right)\right)\left(z - \left(\tfrac{1}{\sqrt{2}} - \tfrac{1}{\sqrt{2}}j\right)\right)}{z^2}$$

zeros at $z_1 = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}j$, $z_2 = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}j$

poles at $p_1 = 0$, $p_2 = 0$



ROC: $|z| > 0$

(since the only poles at $p_1 = p_2 = 0$)

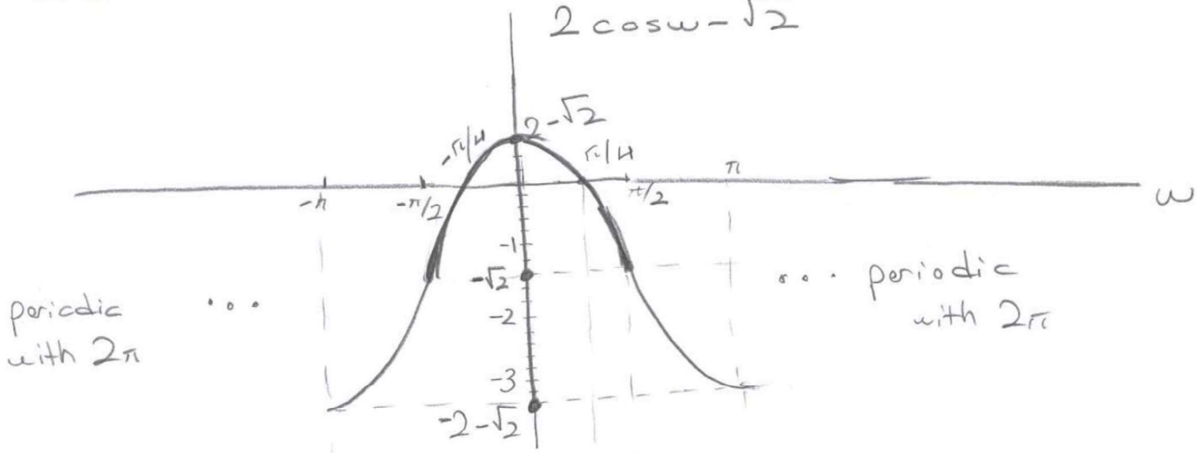
b) Since ROC contains the unit circle, this system has a frequency response.

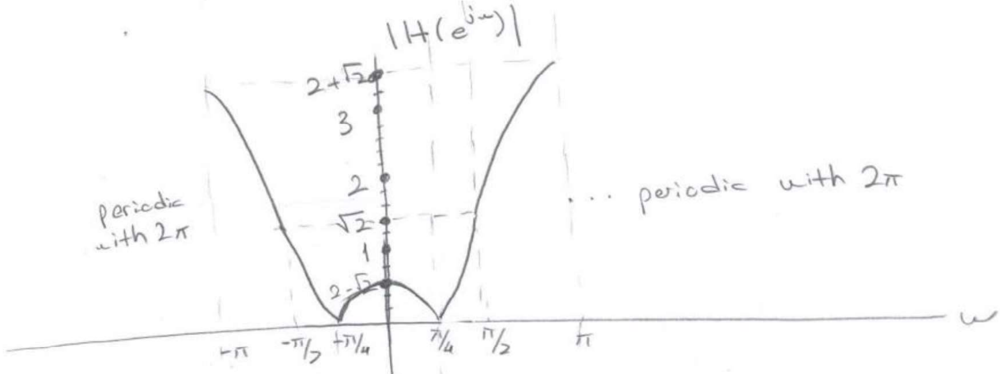
$$H(e^{jw}) = H(z)\Big|_{z=e^{jw}} = 1 - \sqrt{2}e^{-jw} + 1e^{-j2w}$$

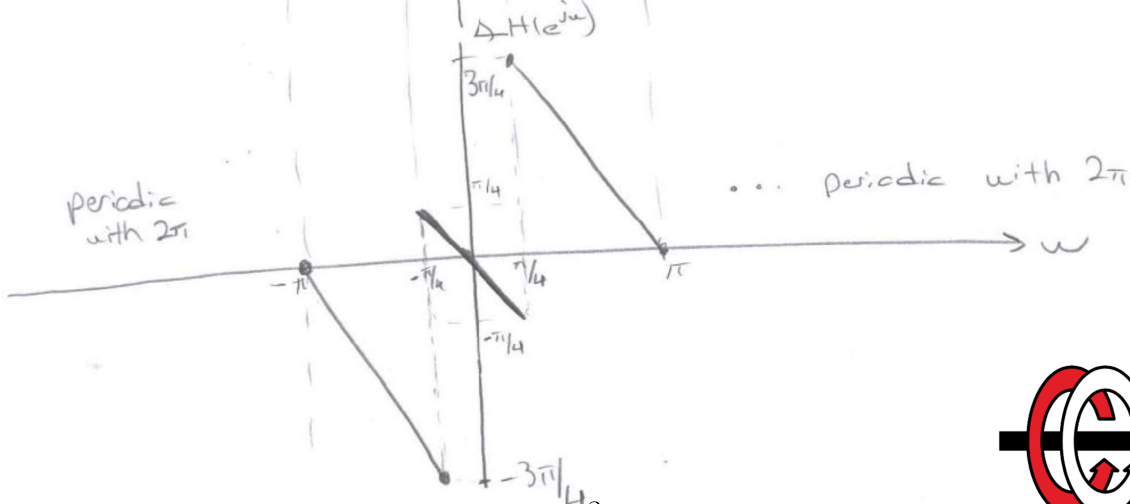
$$= e^{-jw}\left(e^{jw} - \sqrt{2} + e^{-jw}\right) = e^{-jw}\left(2\cos w - \sqrt{2}\right)$$

$$|H(e^{jw})| = |2\cos w - \sqrt{2}|$$

$$\measuredangle H(e^{jw}) = -w + \measuredangle(2\cos w - \sqrt{2})$$







c) Using the eigenfunction property;

$$x[n] = \sum_k a_k e^{j\omega_k n} \longrightarrow \boxed{h[n]} \longrightarrow y[n] = \sum_k a_k H(e^{j\omega_k}) e^{j\omega_k n}$$

i) $$x_1[n] = \sin\left(\frac{\pi}{4}n + \frac{\pi}{4}\right) = \frac{e^{j\frac{\pi}{4}n+\frac{\pi}{4}} - e^{-j\frac{\pi}{4}n-\frac{\pi}{4}}}{2j}$$

$$y_1[n] = \frac{e^{j\frac{\pi}{4}}}{2j} \underbrace{H(e^{j\frac{\pi}{4}})}_{0} e^{j\frac{\pi}{4}n} - \frac{e^{-j\frac{\pi}{4}}}{2j} \underbrace{H(e^{-j\frac{\pi}{4}})}_{0} e^{-j\frac{\pi}{4}n}$$

$$y_1[n] = 0$$

ii) $$x[n] = e^{j\omega n} u[n]$$

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k] = \sum_{k=-\infty}^{n} h[k]\, e^{j\omega(n-k)}$$

Since $h[n]$ is causal;

$$y[n] = \begin{cases} 0 & n < 0 \\ \left(\sum_{k=0}^{n} h[k] e^{-j\omega k}\right) e^{j\omega n} & n \geq 0 \end{cases}$$

For $n \geq 0$

$$y[n] = \left(\sum_{k=0}^{\infty} h[k] e^{-j\omega k}\right) e^{j\omega n} - \left(\sum_{k=n+1}^{\infty} h[k] e^{-j\omega k}\right) e^{j\omega n}$$

$$= H(e^{j\omega}) e^{j\omega n} - \left(\sum_{k=n+1}^{\infty} h[k] e^{-j\omega k}\right) e^{j\omega n}$$

$$x_2[n] = \sin\left(\frac{\pi}{4}n + \frac{\pi}{4}\right)u[n] = \left(\frac{e^{j\pi/4}}{2j}e^{j\frac{\pi}{4}n} - \frac{e^{-j\pi/4}}{2j}e^{-j\frac{\pi}{4}n}\right)u[n]$$

$$y_2[n] = \frac{e^{j\pi/4}}{2j}e^{j\frac{\pi}{4}n}\underbrace{H(e^{j\pi/4})}_{0} - \frac{e^{-j\pi/4}}{2j}e^{-j\pi/4 n}\underbrace{H(e^{-j\pi/4})}_{0}$$

$$-\left(\sum_{k=n+1}^{\infty} h[k]e^{-j\frac{\pi}{4}k}\right)\frac{e^{j\pi/4}}{2j}e^{j\pi/4 n} + \left(\sum_{k=n+1}^{\infty} h[k]e^{+j\frac{\pi}{4}k}\right)\frac{e^{-j\pi/4}}{2j}e^{-j\pi/4 n}$$

$n \geq 0$

$y_2[n] = 0 \qquad n < 0$

$y_2[n] = 0 \qquad n < 0$

$$y_2[0] = -\left(\sum_{k=1}^{2} h[k]e^{-j\pi/4 k}\right)\frac{e^{j\pi/4}}{2j}e^{j\pi/4 \cdot 0} + \left(\sum_{k=1}^{2} h[k]e^{+j\frac{\pi}{4}k}\right)\frac{e^{-j\pi/4}}{2j}e^{-j\pi/4 \cdot 0}$$

$$= -\left(-\sqrt{2}e^{-j\pi/4} + 1e^{-j\pi/2}\right)\frac{e^{j\pi/4}}{2j} + \left(-\sqrt{2}e^{j\pi/4} + 1e^{j\pi/2}\right)\frac{e^{-j\pi/4}}{2j}$$

$$= \cancel{-j\frac{\sqrt{2}}{2}} + j\frac{e^{-j\pi/4}}{2} \cancel{+j\frac{\sqrt{2}}{2}} - j\frac{e^{j\pi/4}}{2} = \frac{j}{2}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}j\right) - \frac{j}{2}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}j\right)$$

$$= \frac{1}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$y_2[1] = -\left(\sum_{k=2}^{2} h[k]\, e^{-j\pi/4 k}\right) \frac{e^{j\pi/4}}{2j} e^{j\pi/4} + \left(\sum_{k=2}^{2} h[k] e^{j\pi/4 k}\right) \frac{e^{-j\pi/4}}{2j} e^{-j\pi/4}$$

$$= -1\, e^{-j\pi/2} \frac{e^{j\pi/2}}{2j} + e^{j\pi/2} \frac{e^{-j\pi/2}}{2j} = 0$$

$y_2[n] = 0 \quad n \geq 2$

$$y_2[n] = \frac{1}{\sqrt{2}}\, \delta[n]$$

iii) $x_3[n] = \underbrace{\sin\left(\frac{\pi}{4} n + \frac{\pi}{4}\right)}_{x_4[n]} + \underbrace{\sin\left(\frac{3\pi}{4} n\right)}_{x_5[n]}$

$y_3[n] = y_4[n] + y_5[n]$

$y_4[n]$: $x_4[n] * h[n] = 0$ (from i))

$y_5[n]$: $x_5[n] * h[n]$

$$y_3[n] = \frac{e^{j\,3\pi/4 n}}{2j} H(e^{j\,3\pi/4}) - \frac{e^{-j\,3\pi/4 n}}{2j} H(e^{-j\,3\pi/4})$$

$$H(e^{j\frac{3\pi}{4}}) = e^{-j\frac{3\pi}{4}}\left(2\underbrace{\cos\frac{3\pi}{4}}_{\frac{-1}{\sqrt{2}}} - \sqrt{2}\right) = 2\sqrt{2}\,e^{j\pi/4}$$

$$H(e^{-j3\pi/4}) = e^{j3\pi/4}\left(2\cos\left(\frac{-3\pi}{4}\right) - \sqrt{2}\right) = 2\sqrt{2}\,e^{-j\pi/4}$$

$$y_3[n] = 2\sqrt{2}\,\frac{e^{j\left(\frac{3\pi}{4}n + \frac{\pi}{4}\right)}}{2j} - 2\sqrt{2}\,\frac{e^{j\left(\frac{-3\pi}{4}n - \frac{\pi}{4}\right)}}{2j}$$

$$\boxed{y_3[n] = 2\sqrt{2}\sin\left(\frac{3\pi}{4}n + \frac{\pi}{4}\right)}$$

d) When the zeros of $H(z)$ are on the unit circle, at the frequencies where these zeros are located $H(e^{j\omega})$ (frequency response) is 0. For example when $H(z)$ has a zero where $z_0 = e^{j\omega_0}$; $H(e^{j\omega_0})$ is 0.

2. 2nd Question

4) The system function of a LTI system is

$$H(z) = \frac{1 - \sqrt{2}z^{-1} + z^{-2}}{1 - 2z^{-1}}.$$

When the input is $\sin\left(\frac{\pi}{2}n\right)$, the output of this system is $\sqrt{\frac{2}{5}}\sin\left(\frac{\pi}{2}n + \tan^{-1}\frac{1}{2}\right)$.

a) Find the impulse response of this system.
b) Is the system causal?
c) Find the difference equation for this system.

Figure 2: Q1

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a) $h[n] = -2^{n}u[-n-1] + \sqrt{2}.2^{n-1}u[-n] - 2^{n-2}u[-n+1]$

b) Since $h[n] \neq 0$ for $n<0$, the system is not causal.

c) $y[n] - 2y[n-1] = x[n] - \sqrt{2}x[n-1] + x[n-2]$

Figure 3: Q2s

3. 3rd Question

6) The z-transform, $X(z)$, of a right –sided sequence $x[n]$ exists for $z = 4e^{j\omega}$, $0 \le \omega < 2\pi$. Show that $X(z)$ exists for $= 4.1e^{j\omega}$, $0 \le \omega < 2\pi$, but not necessarily for $z = 3.9e^{j\omega}$, $0 \le \omega < 2\pi$.

Figure 4: Q3

$\boxed{15}$ Assume that $x[n] = 0$ for some $n < n_0$. 2

Then

$$X(z) = \sum_{n=n_0}^{\infty} x[n] z^{-n} \quad \text{and } X(4e^{j\omega}) \text{ exists.}$$

This means $\mathcal{F}\{x[n]4^{-n}\}$ exists $\Longrightarrow$

$$\sum_{n=n_0}^{\infty} |x[n]|\, 4^{-n} < \infty \equiv x[n]4^{-n} \text{ is absolutely summable}$$

Now consider the sequence $x[n]4.1^{-n}$.

$$\sum_{n=n_0}^{\infty} |x[n]|\, 4.1^{-n} = \sum_{n=n_0}^{\infty} |x[n]|\, 4^{-n} \left(\frac{4.1}{4}\right)^{-n}$$

$$= \underbrace{\sum_{\substack{n=n_0 \\ \text{if } n_0<0}}^{0} |x[n]|\, 4^{-n} \left(\frac{4.1}{4}\right)^{-n}}_{=A \;<\infty \text{ (finite sum)}} + \sum_{n=1}^{\infty} |x[n]|\, 4^{-n} \underbrace{\left(\frac{4.1}{4}\right)^{-n}}_{<1 \text{ for } n\geq 1}$$

$$\leq A + \sum_{n=1}^{\infty} |x[n]|\, 4^{-n} < \infty$$

So $X(z)$ exists for $z = 4.1 e^{j\omega}$ $\quad 0 \leq \omega < 2\pi$

Now consider $z = 3.9 e^{j\omega}$. The sequence $x[n]\, 3.9^{-n}$ is not necessarily absolutely summable, since

$$\sum_{n=1}^{\infty} |x[n]|\, 4^{-n} \underbrace{\left(\frac{3.9}{4}\right)^{-n}}_{>1 \text{ for } n\geq 1} \text{ is not necessarily finite.}$$

Figure 5: Q3s

4. 4th Question

8) Let $x[n] = \delta[n+1] + \left(\frac{1}{2}\right)^n u[n]$. Find the z-transforms of the following sequences. What are the ROCs? State all poles and zeros.

a) $x[n]$
b) $x[n-5]$
c) $nx[n]$
d) $\cos\left(\frac{\pi}{2}n\right)x[n]$

Figure 6: Q4

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a) $X(z) = \dfrac{z\left(z+\frac{1}{2}\right)}{z-\frac{1}{2}}$ ROC: $\frac{1}{2} < |z| < \infty$

zeros: $z_1 = 0$, $z_2 = \frac{-1}{2}$

poles $p_1 = \frac{1}{2}$, $p_2 = \infty$

b) $y[n] = x[n-5]$

$Y(z) = X(z)\,z^{-5} = \dfrac{\left(z+\frac{1}{2}\right)}{z^4\left(z-\frac{1}{2}\right)}$ ROC: $|z| > \frac{1}{2}$

(Because $y[n]$ is a right sided sequence)

zeros: $z_1 = \frac{-1}{2}$, $z_2 = z_3 = z_4 = z_5 = \infty$ (4 zeros at ∞)

poles: $p_1 = \frac{1}{2}$, $p_2 = p_3 = p_4 = p_5 = 0$ (4 poles at 0)

c) $y[n] = n\,x[n]$

$Y(z) = -z\,\dfrac{dX(z)}{dz} = \dfrac{-z\left(z-\left(\frac{1}{2}+\frac{1}{\sqrt{2}}\right)\right)\left(z-\left(\frac{1}{2}-\frac{1}{\sqrt{2}}\right)\right)}{\left(z-\frac{1}{2}\right)^2}$

ROC: $\frac{1}{2} < |z| < \infty$

($n\,x[n]$ is a right sided sequence)

zeros: $z_1 = 0$, $z_2 = \frac{1}{2}+\frac{1}{\sqrt{2}}$, $z_3 = \frac{1}{2}-\frac{1}{\sqrt{2}}$

poles: $p_1 = p_2 = \frac{1}{2}$, $p_3 = \infty$

d) $y[n] = \cos\left(\frac{\pi}{2}n\right)x[n]$ $\longrightarrow$ right sided sequence

$Y(z) = \dfrac{z^2}{\left(z-\frac{1}{2}j\right)\left(z+\frac{1}{2}j\right)}$

ROC: $|z| > \frac{1}{2}$

zeros: $z_1 = z_2 = 0$

poles $p_1 = \frac{1}{2}j$, $p_2 = \frac{-1}{\ldots}$

Figure 7: Q4s

5. 5th Question

9) The pole-zero plot of the system function, $H(z)$, of a stable LTI system is shown. It is known that $H(1) = 1$.

a. Show the ROC. Determine the impulse response $h[n]$.
b. Let $h_1[n] = h[-n+2]$. Sketch the pole-zero plot for $H_1(z)$ show its ROC.

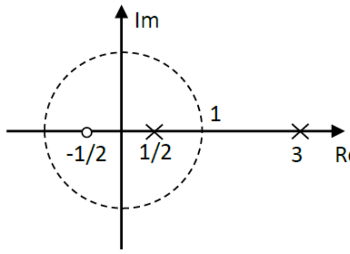


Figure 8: Q5

22)

a) ROC: $\frac{1}{2} < |z| < 3$

$$h[n] = \frac{4}{15}\left(\frac{1}{2}\right)^{n-1} u[n-1] + \frac{14}{15} 3^{n-1} u[-n]$$

b) $$H_1(z) = \frac{z^{-2}\left[\frac{-2}{9} - \frac{4}{9} z^{-1}\right]}{(1-2z^{-1})(1-\frac{1}{3}z^{-1})} = \frac{\frac{-2}{9}(z+2)}{(z-2)(z-\frac{1}{3})z}$$

zeros: $z_1 = -2$, $z_2 = z_3 = \infty$

poles: $p_1 = 2$, $p_2 = \frac{1}{3}$, $p_3 = 0$

ROC: $\frac{1}{3} < |z| < 2$

Figure 9: Q5s

11) Problem 3.30 of textbook.

12) Problem 3.52 of textbook.

13) Problem 3.58 of textbook.

Figure 10: Q11-12-13

6. 6th Question

30. A causal LTI system has system function

$$H(z) = \frac{1 - z^{-1}}{1 - 0.25z^{-2}} = \frac{1 - z^{-1}}{(1 - 0.5z^{-1})(1 + 0.5z^{-1})}.$$

(a) Determine the output of the system when the input is $x[n] = u[n]$.
(b) Determine the input $x[n]$ so that the corresponding output of the above system is $y[n] = \delta[n] - \delta[n-1]$.
(c) Determine the output $y[n]$ when the input is $x[n] = \cos(0.5\pi n)$ for $-\infty < n < \infty$. You may leave your answer in any convenient form.

Figure 11: Q6

7. 7th Question

52. Let $x[n]$ be a causal stable sequence with z-transform $X(z)$. The *complex cepstrum* $\hat{x}[n]$ is defined as the inverse transform of the logarithm of $X(z)$; i.e.,

$$\hat{X}(z) = \log X(z) \stackrel{\mathcal{Z}}{\longleftrightarrow} \hat{x}[n],$$

where the ROC of $\hat{X}(z)$ includes the unit circle. (Strictly speaking, taking the logarithm of a complex number requires some careful considerations. Furthermore, the logarithm of a valid z-transform may not be a valid z-transform. For now, we assume that this operation is valid.)

Determine the complex cepstrum for the sequence

$$x[n] = \delta[n] + a\delta[n - N], \qquad \text{where } |a| < 1.$$

Figure 12: Q7

3.49.

$$\begin{aligned} x[n] &= \delta[n] + a\delta(n-N) \quad |a| < 1 \\ X(z) &= 1 + az^{-N} \\ \hat{X}(z) &= \log X(z) = \log(1 + az^{-N}) = az^{-N} - \frac{a^2 z^{-2N}}{2} + \frac{a^3 z^{-3N}}{3} - \ldots \end{aligned}$$

Therefore,

$$\hat{x}[n] = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} a^k \delta[n - kN]$$

Figure 13: Q7s

8. 8th Question

58. The aperiodic autocorrelation function for a real-valued stable sequence $x[n]$ is defined as

$$c_{xx}[n] = \sum_{k=-\infty}^{\infty} x[k]x[n+k].$$

(a) Show that the z-transform of $c_{xx}[n]$ is

$$C_{xx}(z) = X(z)X(z^{-1}).$$

Determine the ROC for $C_{xx}(z)$.

(b) Suppose that $x[n] = a^n u[n]$. Sketch the pole–zero plot for $C_{xx}(z)$, including the ROC. Also, find $c_{xx}[n]$ by evaluating the inverse z-transform of $C_{xx}(z)$.

(c) Specify another sequence, $x_1[n]$, that is not equal to $x[n]$ in part (b), but that has the same autocorrelation function, $c_{xx}[n]$, as $x[n]$ in part (b).

(d) Specify a third sequence, $x_2[n]$, that is not equal to $x[n]$ or $x_1[n]$, but that has the same autocorrelation function as $x[n]$ in part (b).

Figure 14: Q8

3.55. **(a)**

$$c_{xx}[n] = \sum_{k=-\infty}^{\infty} x[k]x[n+k] = \sum_{k=-\infty}^{\infty} x[-k]x[n-k] = x[-n] * x[n]$$

$$C_{xx}(z) = X(z^{-1})X(z) = X(z)X(z^{-1})$$

$X(z)$ has ROC: $r_R < |z| < r_L$ and therefore $X(z^{-1})$ has ROC: $r_L^{-1} < |z| < r_R^{-1}$. Therefore $C_{xx}(z)$ has ROC: $\max[r_L^{-1}, r_R] < |z| < \min[r_R^{-1}, r_L]$

(b) $x[n] = a^n u[n]$ is stable if $|a| < 1$. In this case

$$X(z) = \frac{1}{1-az^{-1}} \qquad |a| < |z| \quad \text{and} \quad X(z^{-1}) = \frac{1}{1-az} \qquad |z| < |a^{-1}|$$

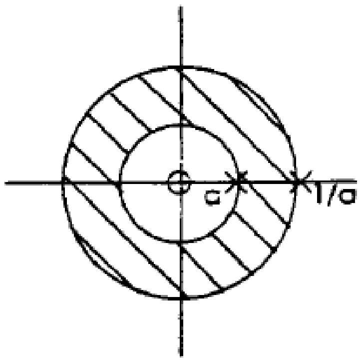
Therefore

$$\begin{aligned} C_{xx}(z) &= \frac{1}{1-az^{-1}}\frac{1}{1-az} = \frac{-az^{-1}}{(1-az^{-1})(1-a^{-1}z^{-1})} \\ &= \frac{\frac{1}{1-a^2}}{1-az^{-1}} - \frac{\text{[illegible]}}{1-a^{-1}z^{-1}} \qquad |a| < |z| < |a^{-1}| \end{aligned}$$

This implies that

$$c_{xx}[n] = \frac{1}{1-a^2}\left[a^n u[n] + a^{-n}u[-n-1]\right]$$

Thus, in summary, the poles are at a and a^{-1}; the zeros are at 0 and ∞; and the ROC of $C_{xx}(z)$ is $|a| < |z| < |a^{-1}|$.



(c) Clearly, $x_1[n] = x[-n]$ will have the same autocorrelation function. For example,

$$X_1(z) = \frac{1}{1-az} \qquad |z| < |a^{-1}| \Longrightarrow C_{x_1x_1}(z) = \frac{1}{1-az}\frac{1}{1-az^{-1}} = C_{xx}(z)$$

(d) Also, any delayed version of $x[n]$ will have the same autocorrelation function; e.g., $x_2[n] = x[n-m]$ implies

$$X_2(z) = \frac{z^{-m}}{1-az^{-1}} \qquad |a| < |z| \Longrightarrow C_{x_2x_2}(z) = \frac{z^{-m}}{1-az^{-1}}\frac{z^{m}}{1-az} = C_{xx}(z)$$

Figure 15: Q8s

9. 9th Question

34)

a) Determine the polynomial result of $(1+3z^{-1}-4z^{-2})(-1+2z^{-1}-3z^{-2}+z^{-3}+7z^{-4})$ using "conv" command in MATLAB.

b) Let us consider

$$X(z) = \frac{1-3z^{-1}+4z^{-2}}{1-z^{-1}+z^{-2}-z^{-3}}$$

Using "residuez" command, determine the inverse z transform of $X(z)$.

c) Let us consider

$$X(z) = \frac{1-0.2z^{-1}-1.2z^{-2}}{1-0.9z^{-1}+0.81z^{-2}}.$$

Using "zplane" command, plot the pole-zero plot of $X(z)$. Plot also the magnitude and phase characteristics of it using "freqz" command. Comment on the relationship between the frequency response and zero & pole locations of $X(z)$.

Figure 16: Q9