

SAMPLING

UNIFORM SAMPLING

C/D, D/C (A/D, D/A)

A MATHEMATICAL MODEL OF SAMPLING

IMPULSE SAMPLING

ALIASING

EXPRESSING $X(e^{j\omega})$ IN TERMS OF $X_c(\Omega)$

NYQUIST-SHANNON SAMPLING THEOREM

RECONSTRUCTION OF A CT SIGNAL FROM A DT SIGNAL

DISCRETE-TIME PROCESSING OF CONTINUOUS-TIME SIGNALS

IMPULSE RESPONSES OF EQUIVALENT CT AND DT SYSTEMS

CHANGING THE SAMPLING RATE IN DISCRETE-TIME

RATE REDUCTION BY AN INTEGER FACTOR

RATE INCREASE BY AN INTEGER FACTOR

CHANGING THE SAMPLING RATE BY A NONINTEGER (RATIONAL) FACTOR

DIGITAL PROCESSING OF ANALOG SIGNALS

ANALOG TO DIGITAL CONVERSION

QUANTIZATION

DIGITAL TO ANALOG CONVERSION

Ex:

Consider sampling of a sinusoidal signal,

$$x_c(t) = \cos(2\pi f_0 t)$$

$$x[n] = x_c(nT) = x_c\left(\frac{n}{f_s}\right) = \cos\left(2\pi \frac{f_0}{f_s} n\right)$$

$$x[n] = \cos\left(2\pi \frac{f_0}{f_s} n\right) = \cos(\omega_0 n)$$

If

$$\frac{f_0}{f_s} \geq 1$$

Let

$$\hat{f}_0 = f_0 \text{ modulo } f_s$$

$$x[n] = \cos\left(2\pi \frac{f_0}{f_s} n\right) = \cos\left(2\pi \frac{\hat{f}_0}{f_s} n\right) = \cos(\omega_0 n)$$

Furthermore, if

$$\frac{\hat{f}_0}{f_s} > \frac{1}{2}$$

$$x[n] = \cos\left(2\pi \frac{f_0}{f_s} n\right) = \cos\left(2\pi \frac{\hat{f}_0}{f_s} n\right) = \cos(\omega_0 n)$$

Ex:

$$f_s = 2000 \text{ Hz}$$

$$f_0 = 3400 \text{ Hz} \quad \omega_0 = 3.4\pi \quad x[n] = \cos(3.4\pi n) = \cos(1.4\pi n) = \cos(0.6\pi n)$$

$$f = 3000 \text{ Hz} \quad \omega_0 = 3\pi \quad x[n] = \cos(3\pi n) = \cos(\pi n)$$

$$f = 2800 \text{ Hz} \quad \omega_0 = 2.8\pi \quad x[n] = \cos(2.8\pi n) = \cos(0.8\pi n)$$

$$f = 2000 \text{ Hz} \quad \omega_0 = 2\pi \quad x[n] = \cos(2\pi n) = \cos(0) = 1$$

$$f = 1400 \text{ Hz} \quad \omega_0 = 1.4\pi \quad x[n] = \cos(1.4\pi n) = \cos(0.6\pi n)$$

$$f = 1000 \text{ Hz} \quad \omega_0 = \pi \quad x[n] = \cos(\pi n)$$

$$f = 800 \text{ Hz} \quad \omega_0 = 0.8\pi \quad x[n] = \cos(0.8\pi n)$$

$$f = 600 \text{ Hz} \quad \omega_0 = 0.6\pi \quad x[n] = \cos(0.6\pi n)$$

Same color --> same DT frequency

Continuous-time signals are commonly processed by using digital systems.

Mobile devices, TV receivers and displays, radar, sonar,...

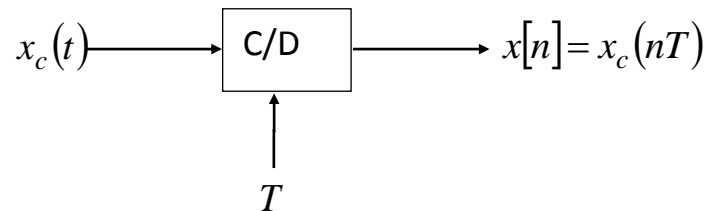
“analog to digital” conversion: A/D

“continuous-time to discrete-time” conversion: C/D

A/D = C/D and quantization of sample values

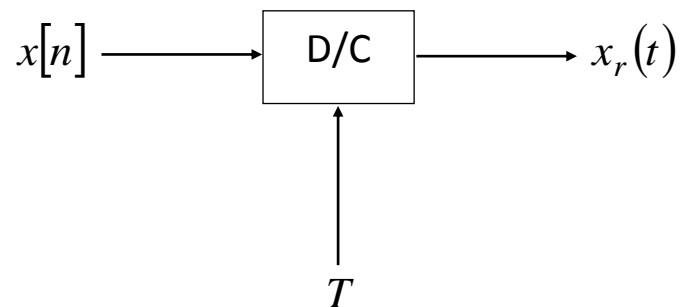
UNIFORM SAMPLING

Samples are spaced uniformly in time.



T : sampling period

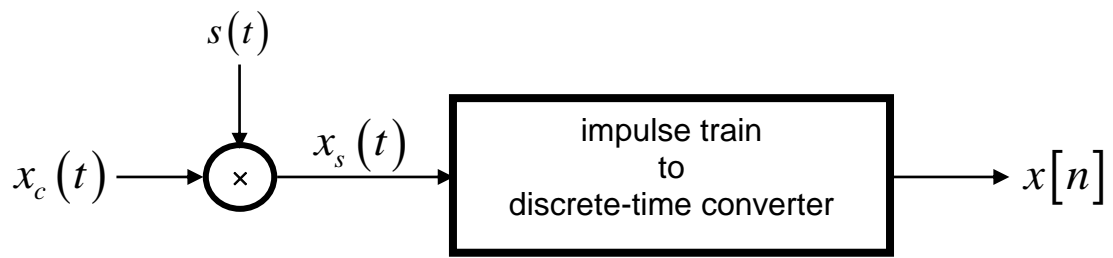
“discrete-time to continuous-time conversion”, D/C.



In practice, we have the term “digital to analog” (D/A) conversion.

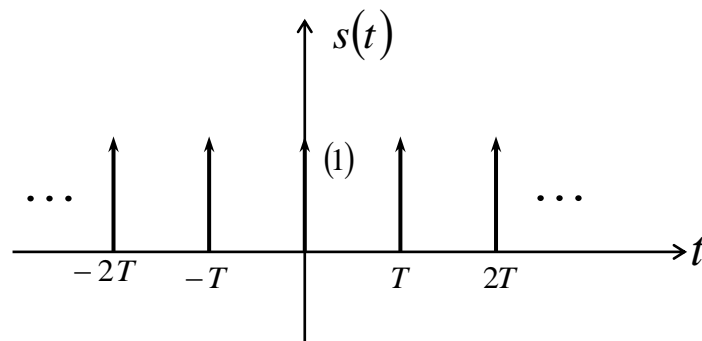
A central question is about determining, if exist, the conditions required to recover a continuous-time signal from its uniformly acquired samples.

A MATHEMATICAL MODEL OF SAMPLING



$$s(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$$

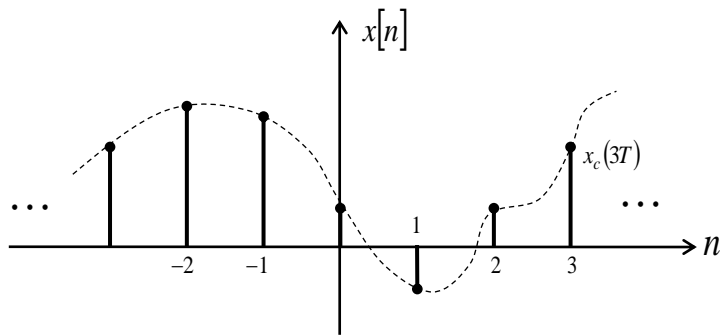
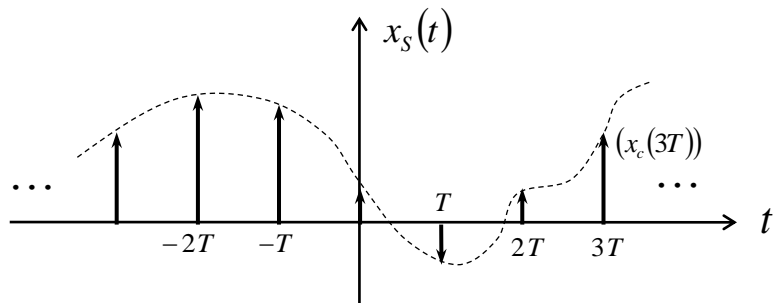
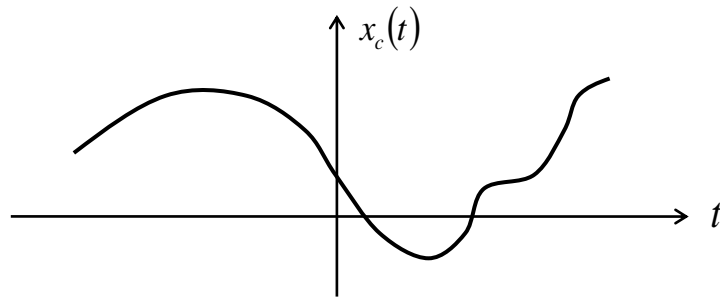
impulse train



T : sampling period

$f_s = \frac{1}{T}$: sampling frequency (Hz)

$\Omega_s = \frac{2\pi}{T}$: sampling frequency (rad/sec)



$$x_s(t) = \sum_{n=-\infty}^{\infty} x_c(nT)\delta(t - nT)$$

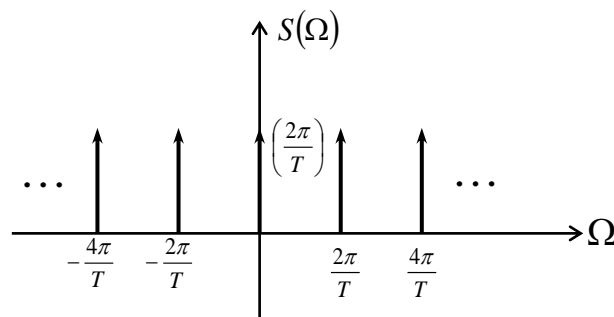
$$x[n] = x_c(nT)$$

IMPULSE TRAIN IN FREQUENCY DOMAIN

Using Fourier series representation of $s(t)$ with coefficients $a_k = \frac{1}{T}$ and

$$e^{-jk\frac{2\pi}{T}t} \xleftrightarrow{CTFT} 2\pi\delta\left(\Omega - k\frac{2\pi}{T}\right)$$

$$S(\Omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta\left(\Omega - k\frac{2\pi}{T}\right)$$



$$a_k = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} s(t) e^{-jk\frac{2\pi}{T}t} dt$$

Note that, an alternative expression is

$$S(\Omega) = \sum_{k=-\infty}^{\infty} e^{-jkT\Omega}$$

since

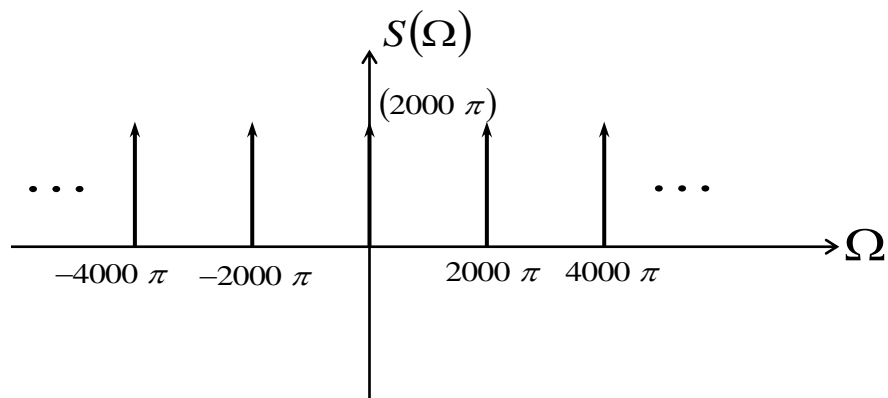
$$\delta(t - kT) \xleftrightarrow{CTFT} e^{-jkT\Omega}$$

Ex:

$$T = 1 \text{ ms},$$

$$f = 1000 \text{ Hz},$$

$$\Omega = \frac{2\pi}{T} = 2000\pi \text{ rad/sec}$$

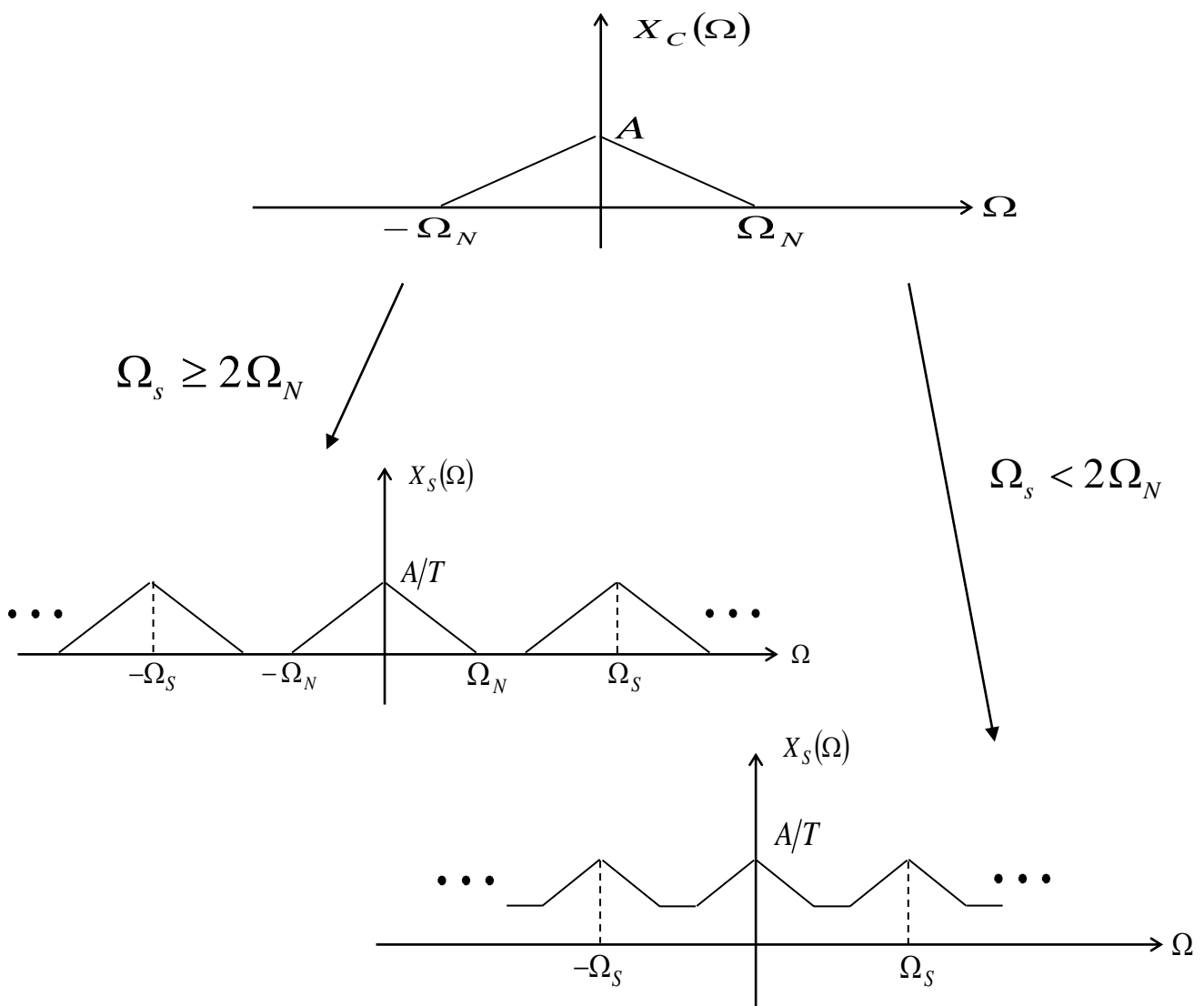


After multiplication with Impulse Train

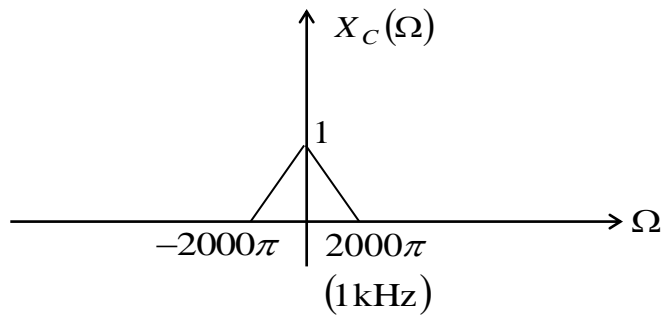
$$X_s(\Omega) = \frac{1}{2\pi} X_c(\Omega) * S(\Omega)$$

Since $S(\Omega)$ is an impulse train

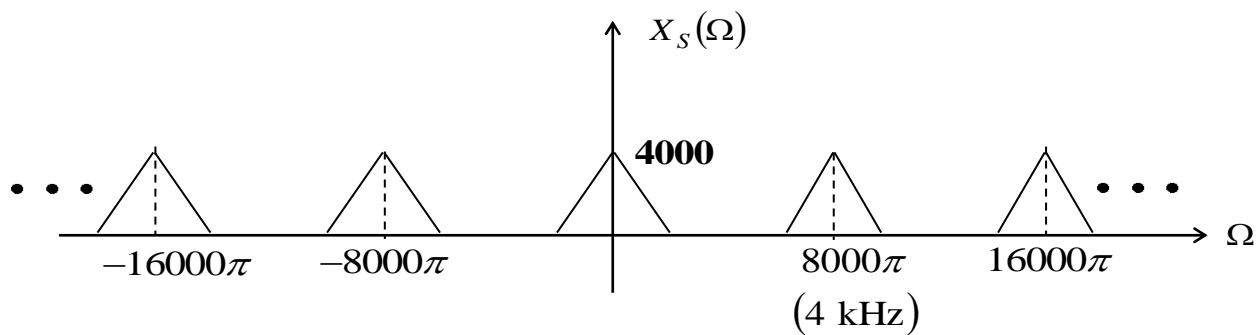
$$X_s(\Omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c\left(\Omega - k \frac{2\pi}{T}\right)$$



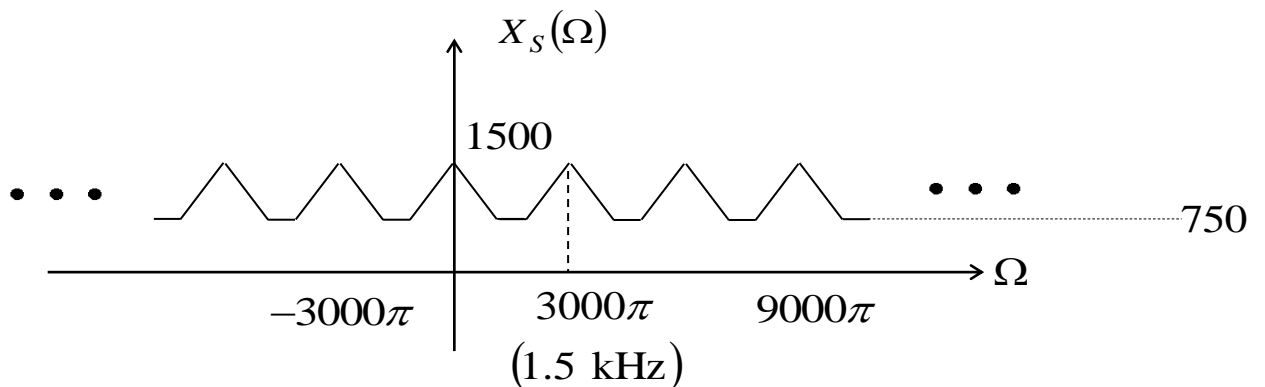
Ex: Let $x_c(t)$ be “bandlimited” to 1 kHz or equivalently to 2000π rad/sec., i.e.
 $\Omega_N = 2000\pi$



Let $T = 0.25$ ms ($f_s = 4$ kHz) $\Omega_s = 8000\pi$



However, if $T = 2/3$ ms ($f_s = 1.5$ kHz) $\Omega_s = 3000\pi$



Definition

The overlap (distortion) of the spectrum when $\Omega_S < 2 \Omega_N$ is called “**aliasing**”.

EXPRESSING $X(e^{j\omega})$ IN TERMS OF $X_c(\Omega)$

$$\begin{aligned}x_s(t) &= \sum_{n=-\infty}^{\infty} x_c(nT)\delta(t - nT) \\&= \sum_{n=-\infty}^{\infty} x[n]\delta(t - nT)\end{aligned}$$

Time-shift property of CTFT,

$$X_s(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega Tn}$$

Compare this expression to

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

Conclude that

$$X(e^{j\omega}) = X_s(\Omega)|_{\Omega=\frac{\omega}{T}}$$

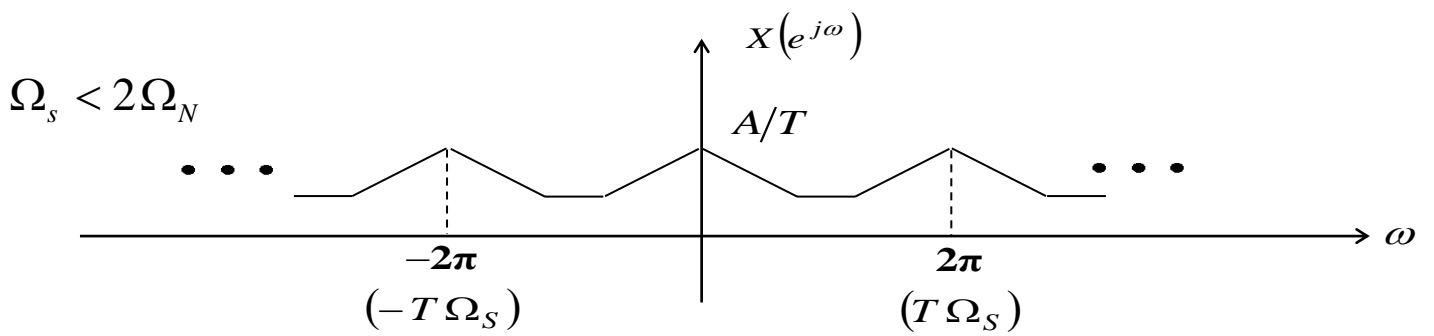
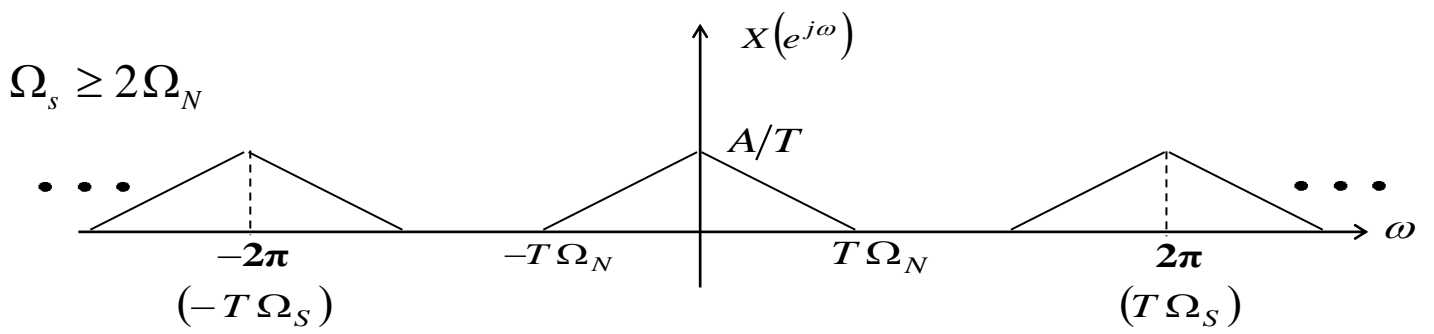
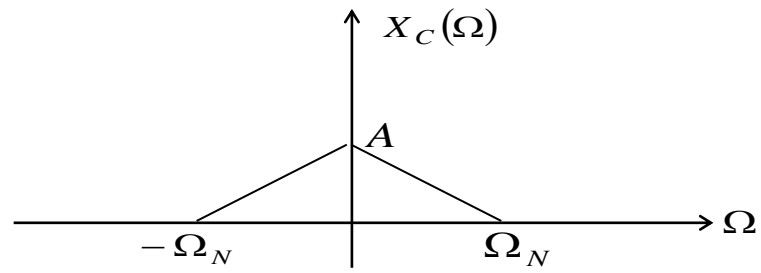
Hence

$$X(e^{j\omega}) = \frac{1}{T} \sum_{n=-\infty}^{\infty} X_c\left(\frac{\omega}{T} - k\frac{2\pi}{T}\right)$$

A “linear warping” of the frequency scale and its periodic extension.

$$\begin{aligned}
X(e^{j\omega}) = & \dots + \frac{1}{T} X_c \left(\frac{1}{T} (\omega + 4\pi) \right) \\
& + \frac{1}{T} X_c \left(\frac{1}{T} (\omega + 2\pi) \right) \\
& + \frac{1}{T} X_c \left(\frac{\omega}{T} \right) \\
& + \frac{1}{T} X_c \left(\frac{1}{T} (\omega - 2\pi) \right) \\
& + \frac{1}{T} X_c \left(\frac{1}{T} (\omega - 4\pi) \right) \\
& + \dots
\end{aligned}$$

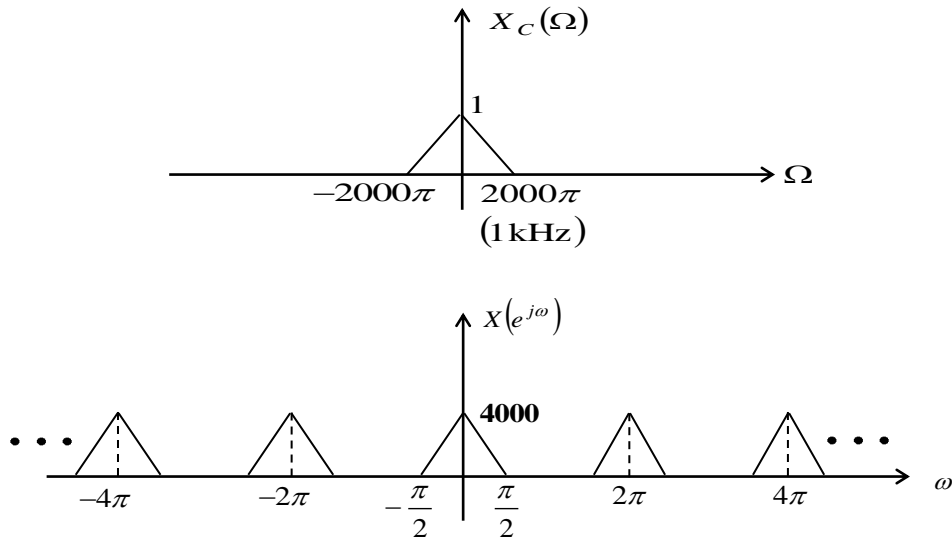
Therefore



Ex: Let $x_c(t)$ be “bandlimited” to 1 kHz or equivalently to 2000π rad/sec.,
i.e., $\Omega_N = 2000\pi$

Let $T = 0.25$ ms ($f_s = 4$ kHz)

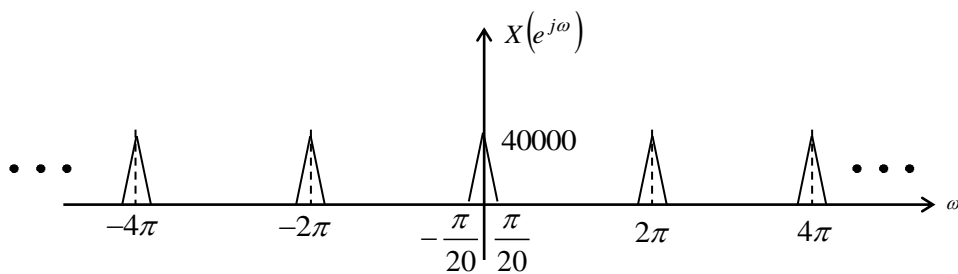
$$\Omega_s = 8000\pi$$



If sampling frequency is increased ten times:

$T = 0.025$ ms ($f_s = 40$ kHz)

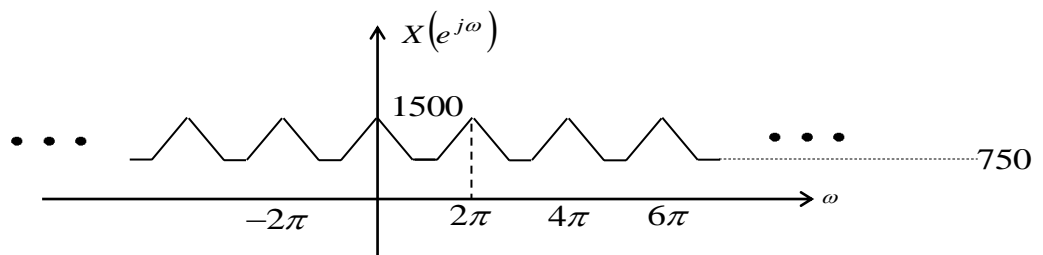
$$\Omega_s = 80000\pi$$



If sampling frequency is decreased,

$T = \frac{2}{3}$ ms ($f_s = 1.5$ kHz)

$$\Omega_s = 3000\pi$$



Aliasing!

Notes

- The discrete-time and continuous-time frequency scales are related by $\omega = \Omega T$
- Sampling frequency, Ω_s , is “always” mapped to 2π in discrete-time frequency scale.
- Discrete-time frequency “equivalent”, ω_a , of any continuous-time frequency, Ω_a , can be found using the ratio $\frac{\Omega_a}{\Omega_s}$:

“In the last example, the band edge frequency, 1 kHz, is one fourth of the sampling frequency, 4 kHz. Therefore, the band edge of $X(e^{j\omega})$ is at $\frac{\pi}{2}$, one fourth of 2π .”

NYQUIST-SHANNON SAMPLING THEOREM

Let $x_c(t)$ be bandlimited to Ω_N , i.e.,

$$X_c(\Omega) = 0 \quad |\Omega| \geq \Omega_N$$

$x_c(t)$ can be determined uniquely from its samples

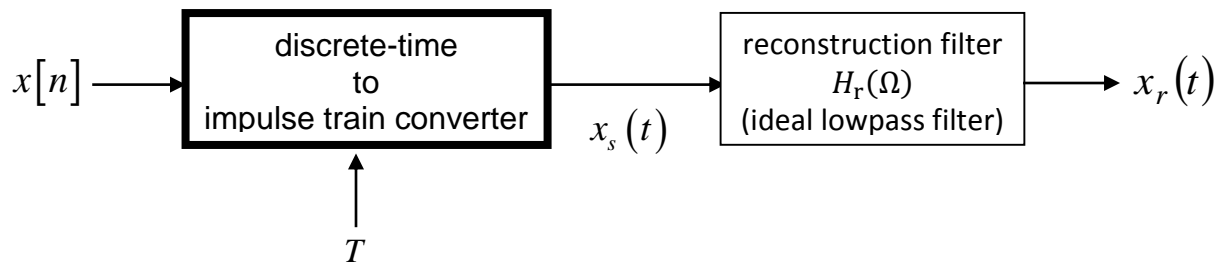
$$x[n] = x_c(nT)$$

if

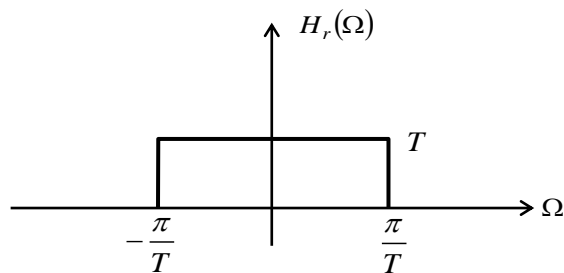
$$\Omega_s \geq 2\Omega_N$$

Pay attention to the equality!

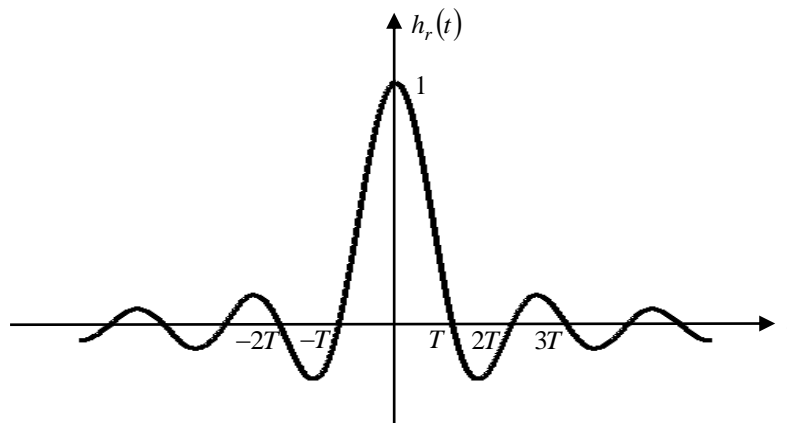
RECONSTRUCTION OF A CT SIGNAL FROM A DT SIGNAL



$$x_s(t) = \sum_{n=-\infty}^{\infty} x_c(nT)\delta(t - nT)$$



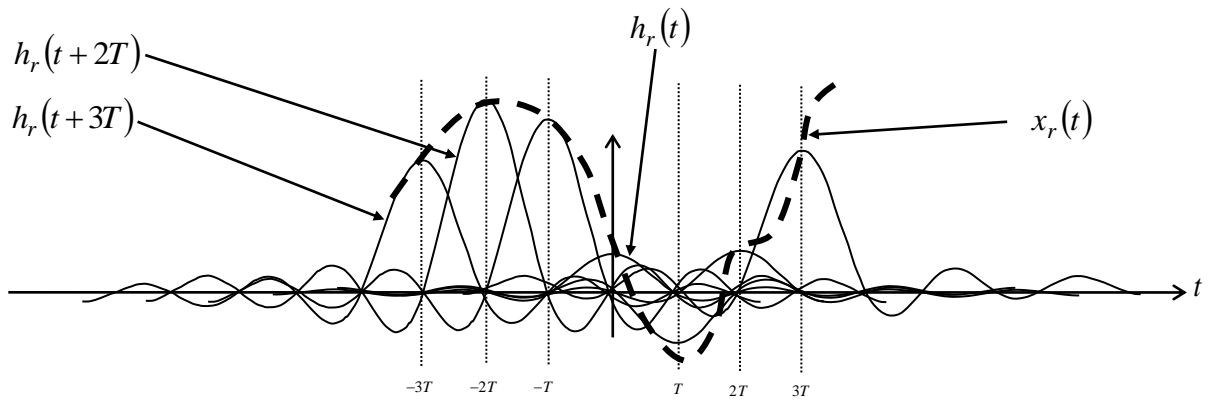
$$h_r(t) = \frac{\sin\left(\frac{\pi}{T}t\right)}{\frac{\pi}{T}t}$$



$$\begin{aligned}
 x_r(t) &= x_s(t) * h_r(t) \\
 &= \left(\sum_{n=-\infty}^{\infty} x[n] \delta(t - nT) \right) * h_r(t) \\
 &= \sum_{n=-\infty}^{\infty} x[n] h_r(t - nT)
 \end{aligned}$$

$$\begin{aligned}
 X_r(\Omega) &= \sum_{n=-\infty}^{\infty} x[n] e^{-jnT\Omega} H_r(\Omega) \\
 &= \left(\sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega T n} \right) H_r(\Omega) \\
 &= X(e^{j\Omega T}) H_r(\Omega)
 \end{aligned}$$

$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n] h_r(t - nT)$$



```

% bandlimited interpolation
% using 2N+1 terms in the summation

clear all
close all

N = 25 ;
f = 1 ;      % frequency of continuous time sinusoid to be sampled
Omega = 2 * pi * f ;
f_s = 7 ;    % sampling frequency
T_s = 1/f_s ;
k = -N:N ;
x = sin(Omega*k*T_s) ; % samples of the continuous time sinusoid

delta = 4;
t = -delta:0.01:delta ; % time interval in which we plot our results

% computing the sinc signals in the summation
for n = -N:N
    h_r(n+N+1,:) = x(n+N+1) * sin(f_s*pi*(t-n*T_s))./(f_s*pi*(t-n*T_s));
    h_r(n+N+1,find(isnan(h_r(n+N+1,:)))) = x(n+N+1) ;
end

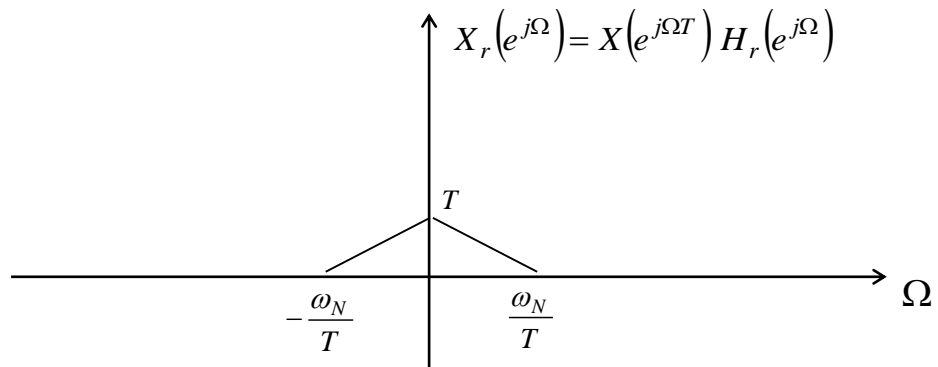
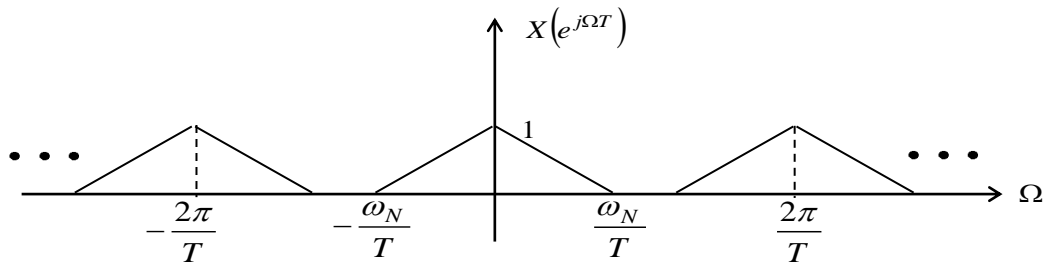
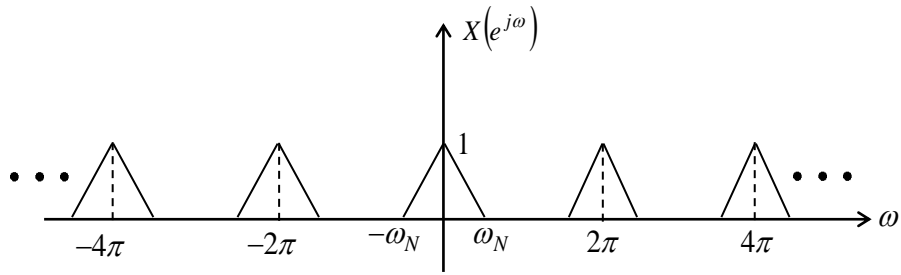
x_r = sum(h_r); % reconstructed cont-time signal

plot(t,sin(Omega*t),'k','LineWidth',3)
hold
pause
plot(t,h_r(N+1-3,:),'g','LineWidth',3)
pause
plot(t,h_r(N+1-2,:),'r','LineWidth',3)
pause
plot(t,h_r(N+1-1,:),'c','LineWidth',3)
pause
plot(t,h_r(N+1,:),'m','LineWidth',3)
pause
plot(t,h_r(N+1+1,:),'y','LineWidth',3)
pause
plot(t,h_r(N+1+2,:),'b','LineWidth',3)
pause
plot(t,x_r,'r--','LineWidth',1.5)

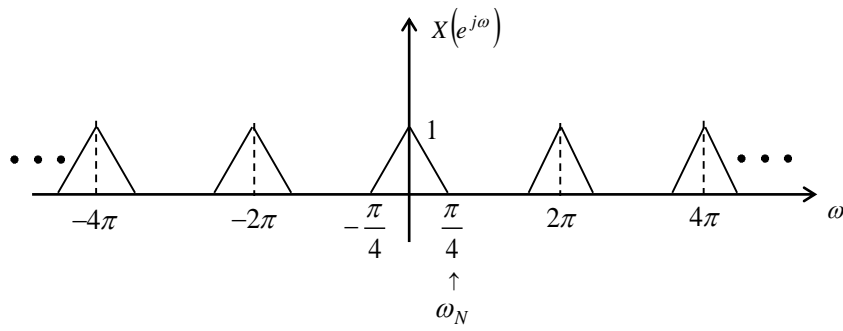
```

Ex:

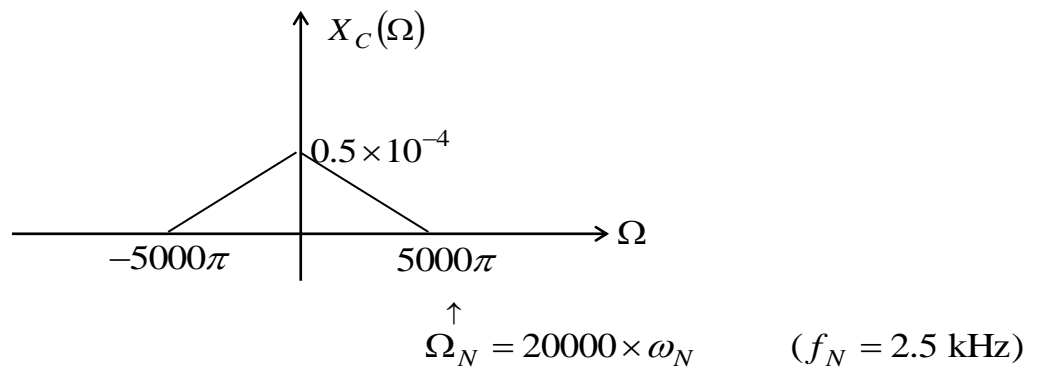
$$X_r(\Omega) = X(e^{j\Omega T}) H_r(\Omega)$$



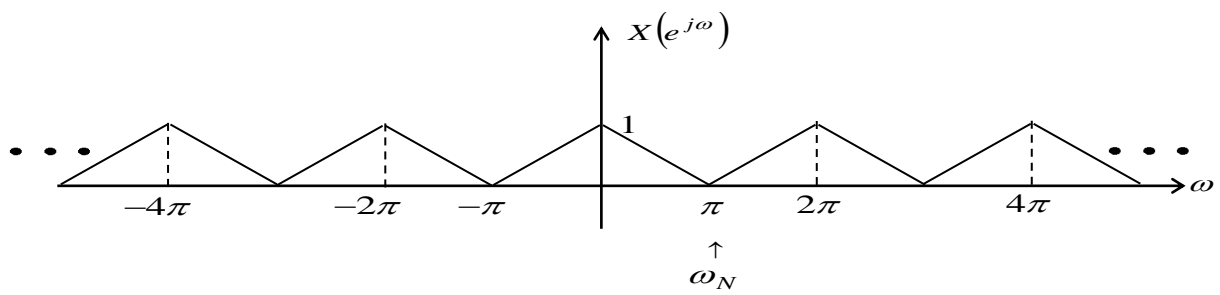
Ex: Given



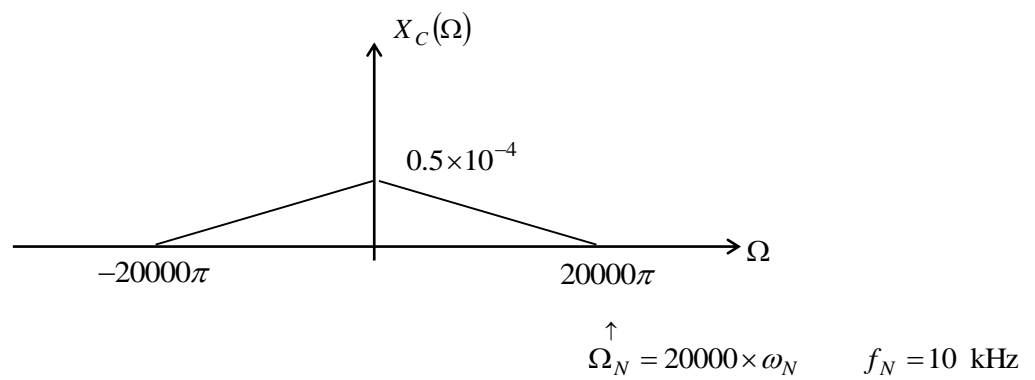
If $f_s = 20$ kHz, i.e., $\Omega_s = 40\pi$ krad/s, $T = \frac{1}{20}$ ms



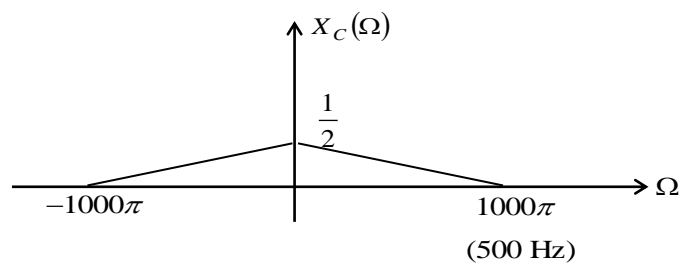
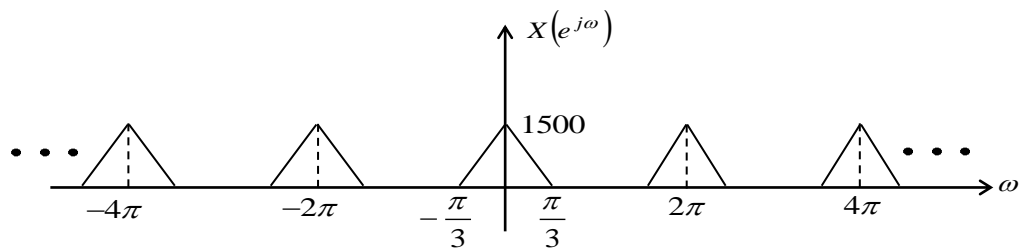
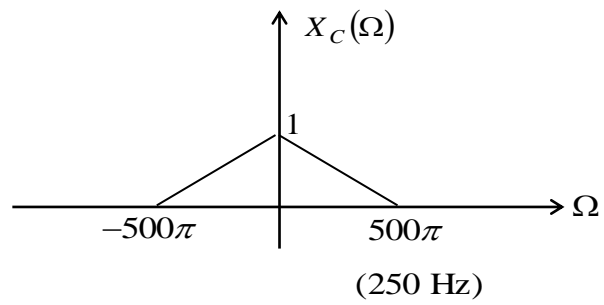
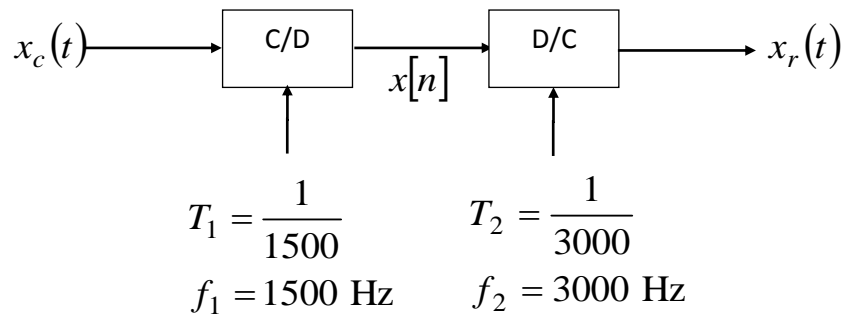
Ex: Given



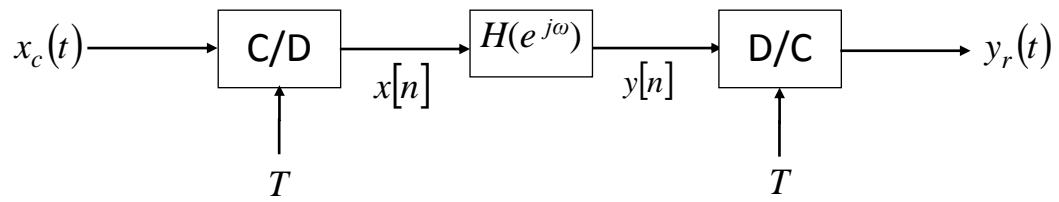
If $f_s = 20$ kHz, i.e., $\Omega_s = 40\pi$ krad/s, $T = \frac{1}{20}$ ms



Ex:



DISCRETE-TIME PROCESSING OF CONTINUOUS-TIME SIGNALS



$$Y_r(\Omega) = H_r(\Omega)Y(e^{j\Omega T})$$

$$= \begin{cases} TY(e^{j\Omega T}) & |\Omega| < \frac{\pi}{T} \\ 0 & o.w. \end{cases}$$

$$Y(e^{j\Omega T}) = H(e^{j\Omega T})X(e^{j\Omega T})$$

$$= H(e^{j\Omega T}) \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c\left(\Omega - k \frac{2\pi}{T}\right)$$

$$Y_r(\Omega) = \begin{cases} H(e^{j\Omega T}) \sum_{k=-\infty}^{\infty} X_c\left(\Omega - k \frac{2\pi}{T}\right) & |\Omega| < \frac{\pi}{T} \\ 0 & o.w. \end{cases}$$

Assuming that $x_c(t)$ is bandlimited to $\frac{\pi}{T}$

$$Y_r(\Omega) = \begin{cases} H(e^{j\Omega T}) X_c(\Omega) & |\Omega| < \frac{\pi}{T} \\ 0 & \text{o.w.} \end{cases}$$

Hence

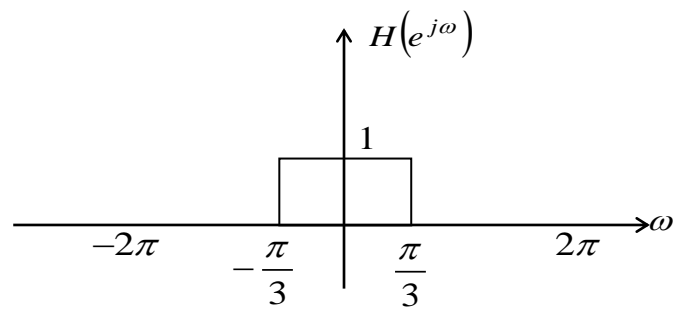
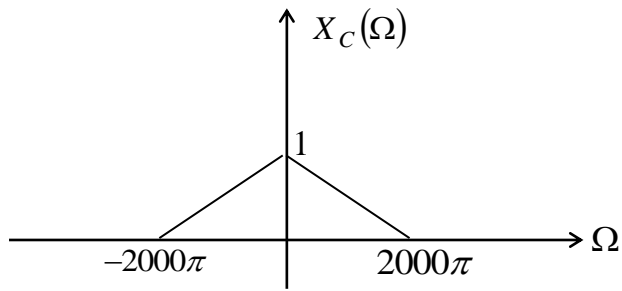
$$Y_r(\Omega) = H_{eff}(\Omega) X_c(\Omega)$$

where

$$H_{eff}(\Omega) = \begin{cases} H(e^{j\Omega T}) & |\Omega| < \frac{\pi}{T} \\ 0 & \text{o.w.} \end{cases}$$

→ Note that $H_{eff}(\Omega)$ is valid if $x_c(t)$ is bandlimited to $\frac{\pi}{T}$ ←

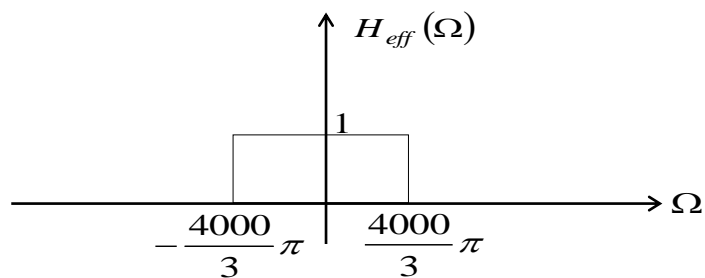
Ex: Let $T = 0.25$ ms (4 kHz) and



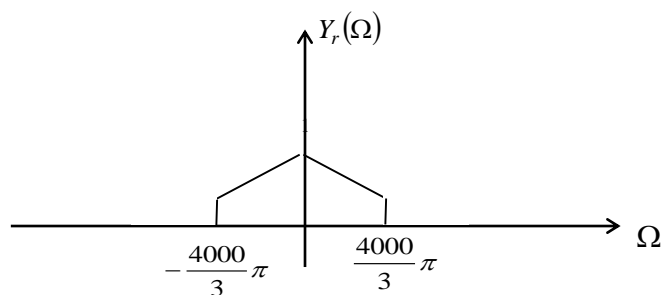
Therefore

$$H_{eff}(\Omega) = \begin{cases} H(e^{j\Omega T}) & |\Omega| < 4000\pi \\ 0 & o.w. \end{cases}$$

$$= \begin{cases} 1 & |\Omega| < \frac{4000}{3}\pi \\ 0 & o.w. \end{cases}$$



and



Ex: What should the cut-off frequency of the discrete-time ideal lowpass filter be so that the continuous-time signal is lowpass filtered with a cut-off frequency of 3 kHz when the sampling frequency 8 kHz?

$$\omega_C = \frac{3}{8} 2\pi = \frac{3}{4} \pi$$