

## GETTING THE RESULT OF LINEAR CONVOLUTION USING DFT

Let  $x[n]$  and  $y[n]$  be  $N_1$ -point and  $N_2$ -point sequences and  $w[n]$  be their linear convolution.

Let  $X[k]$  and  $Y[k]$  be  $N$ -point DFTs of  $x[n]$  and  $y[n]$ .

**The IDFT of  $W[k] = X[k]Y[k]$  is the same as the linear convolution of  $x[n]$  and  $y[n]$  if  $N \geq N_1 + N_2 - 1$ .**

**Ex:** Let  $x[n]$  and  $y[n]$  be 5-point and 9-point sequences, respectively. At least how many point DFT has to be used to get the result of their linear convolution?

Answer:  $5+9-1=13$  point DFT.

However, if, for example, 10-point DFT used then

- The result has 10 samples in total
- The first  $13-10 = 3$  samples out of these 10 samples are not equal to the those of the linear convolution
- The remaining  $10-3$  samples are equal to those of the linear convolution.
- Therefore only 7 samples of the result of linear convolution (13 samples) will be obtained.

## MATLAB Example

$x = [1 \ 2 \ 3];$      $y = [-1 \ 2 \ 1 \ 2];$      $z = \text{conv}(x,y);$      $z = -1 \ 0 \ 2 \ 10 \ 7 \ 6$

$X = \text{fft}(x,6)$

$X = 6 \quad 0.5-4.3i \quad -1.5+0.9i \quad 2 \quad -1.5-0.9i \quad 0.5+4.3i$

$Y = \text{fft}(y,6)$

$Y = 4 \quad -2.5-2.6i \quad -0.5-0.9i \quad -4 \quad -0.5+0.9i \quad -2.5+2.6i$

$Z = X.*Y$

$Z = 24 \quad -12.5+9.5i \quad 1.5+0.9i \quad -8 \quad 1.5-0.9i \quad -12.5-9.5i$

$z_ = \text{ifft}(Z) \quad z = -1 \quad 0 \quad 2 \quad 10 \quad 7 \quad 6$

$X = \text{fft}(x,5)$

$X = 6 \quad -0.8-3.7i \quad 0.3+1.7i \quad 0.3-1.7i \quad -0.8+3.7i$

$Y = \text{fft}(y,5)$

$Y = 4 \quad -2.8-1.3i \quad -1.7-2.1i \quad -1.7+2.1i \quad -2.8+1.3i$

$Z = X.*Y$

$Z = 24 \quad -2.5+11.4i \quad 3.0-3.5i \quad 3.0+3.5i \quad -2.5-11.4i$

$z_ = \text{ifft}(Z) \quad z_ = 5 \quad 0 \quad 2 \quad 10 \quad 7$

$X = \text{fft}(x,4)$

$X = 6.0000 + 0.0000i \quad -2.0000 - 2.0000i \quad 2.0000 + 0.0000i \quad -2.0000 + 2.0000i$

$Y = \text{fft}(y,4)$

$Y = 4 \quad -2 \quad -4 \quad -2$

$Z = X.*Y$

$Z = 24.0000 + 0.0000i \quad 4.0000 + 4.0000i \quad -8.0000 + 0.0000i \quad 4.0000 - 4.0000i$

$z_ = \text{ifft}(Z) \quad z_ = 6 \quad 6 \quad 2 \quad 10$

## 7) Sampling the DTFT

$$\underbrace{x[n]}_{\text{length}=N_x} \xleftrightarrow{DTFT} X(e^{j\omega})$$

Let

$$\hat{X}[k] \triangleq X(e^{j\omega}) \Big|_{\omega=k\frac{2\pi}{N}} \quad k = 0, 1, \dots, N-1$$

where  $N$  is arbitrary,

i.e. not necessarily  $\geq N_x$  !

$$\hat{x}[n] = ???$$

$\hat{x}[n]$  is  $N$ -point IDFT of  $\hat{X}[k]$

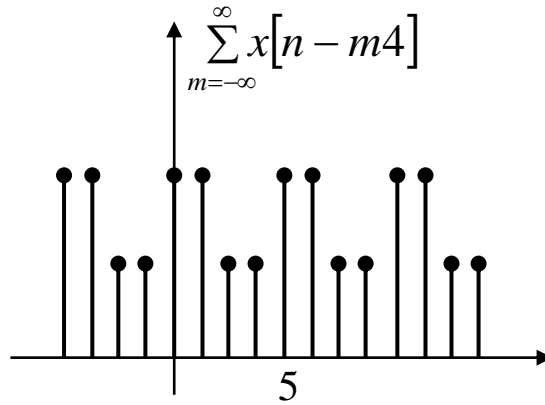
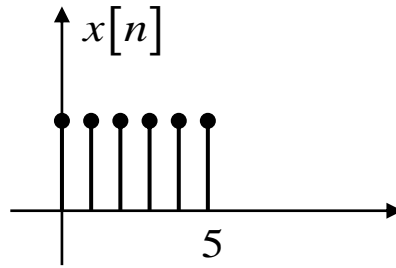
$$\hat{x}[n] = \begin{cases} \frac{1}{N} \sum_{k=0}^{N-1} \hat{X}[k] W_N^{-kn} & n = 0, 1, \dots, N-1 \\ 0 & \text{o.w.} \end{cases}$$

$$\begin{aligned} \sum_{k=0}^{N-1} \hat{X}[k] W_N^{-kn} &= \sum_{k=0}^{N-1} \left( \sum_{r=-\infty}^{\infty} x[r] W_N^{kr} \right) W_N^{-kn} \\ &= \sum_{r=-\infty}^{\infty} x[r] \underbrace{\left( \sum_{k=0}^{N-1} W_N^{k(r-n)} \right)}_{N \sum_{m=-\infty}^{\infty} \delta[n-r-mN]} \\ &= N \left( x[n] * \sum_{m=-\infty}^{\infty} \delta[n-mN] \right) \\ &= N \sum_{m=-\infty}^{\infty} x[n-mN] \end{aligned}$$

Therefore

$$\hat{x}[n] = \begin{cases} \sum_{m=-\infty}^{\infty} x[n - mN] & n = 0, 1, \dots, N - 1 \\ 0 & o.w. \end{cases}$$

**Ex:**  $N_x = 6, N = 4$



**Ex:** Demonstration of the proof.  $N_X = 8, N = 5$

$$X(e^{j\omega}) = \sum_{n=0}^7 x[n]e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

$$\hat{X}[k] = X(e^{j\omega})|_{\omega=k\frac{2\pi}{5}}$$

$$= \sum_{n=-\infty}^{\infty} x[n]e^{-jk\frac{2\pi}{5}n}$$

$$= x[0] + x[1]e^{-jk\frac{2\pi}{5}} + x[2]e^{-jk\frac{2\pi}{5}2} + x[3]e^{-jk\frac{2\pi}{5}3} + x[4]e^{-jk\frac{2\pi}{5}4}$$

$$+ x[5]\underbrace{e^{-jk\frac{2\pi}{5}5}}_{=1} + x[6]\underbrace{e^{-jk\frac{2\pi}{5}6}}_{=e^{-jk\frac{2\pi}{5}}} + x[7]\underbrace{e^{-jk\frac{2\pi}{5}7}}_{=e^{-jk\frac{2\pi}{5}2}}$$

$$= (x[0] + x[5]) + (x[1] + x[6])e^{-jk\frac{2\pi}{5}} + (x[2] + x[7])e^{-jk\frac{2\pi}{5}2} \\ + x[3]e^{-jk\frac{2\pi}{5}3} + x[4]e^{-jk\frac{2\pi}{5}4}$$

This is the 5-point DFT of

$$\hat{x}[0] = x[0] + x[5]$$

$$\hat{x}[1] = x[1] + x[6]$$

$$\hat{x}[2] = x[2] + x[7]$$

$$\hat{x}[3] = x[3]$$

$$\hat{x}[4] = x[4]$$



## 8) Multiplication in Time Domain

### DFS

Let  $\tilde{x}_1[n]$  and  $\tilde{x}_2[n]$  be periodic with a common period  $N$ .

$$\tilde{x}_3[n] = \tilde{x}_1[n]\tilde{x}_2[n] \xleftrightarrow{DFS} \tilde{X}_3[k] = \frac{1}{N} \sum_{l=0}^{N-1} \tilde{X}_1[l]\tilde{X}_2[k-l]$$

Proof: Exercise

### DFT

Let  $x_1[n]$  and  $x_2[n]$  be two finite length sequences of length  $N_1$  and  $N_2$ , and  $X_1[k]$  and  $X_2[k]$  be their  $N$ -point DFTs,  $N \geq \max(N_1, N_2)$ .

$$x_3[n] = x_1[n]x_2[n] \xleftrightarrow{N\text{-point DFT}} X_3[k] = \frac{1}{N} \sum_{l=0}^{N-1} X_1[l]X_2[(k-l)_N]$$
$$k = 0, 1, \dots, N-1$$

## IMPLEMENTING LTI SYSTEMS USING DFT

The output of an LTI system can be computed using DFT:

Multiply the DFTs of input and impulse response and compute the IDFT

There are efficient DFT computation techniques.

We consider FIR systems with “long” inputs.

## BLOCK PROCESSING

Input is decomposed into short segments and the output is computed by properly “combining” the responses to the short segments.

Otherwise, it is not practical to consider DFTs of very long sequences.  
Furthermore the output of the system will be delayed excessively since all the input has to be collected first.

Let

The length of the impulse response of the system to be implemented  $h[n] = P$ .

The input is decomposed into blocks of length  $L$ .

## Two Methods

### Overlap-Add

DFT length:  $L + P - 1$  point

### Overlap-Save

DFT length:  $L$  point

# OVERLAP-ADD METHOD

Let's start with a linear convolution example

$$h[n]: \dots, 0, 0, \underset{\substack{\uparrow \\ n=0}}{1}, 2, 3, 0, 0, \dots \quad \text{3-point sequence } (P=3)$$

$$x[n]: \dots, 0, 0, \underset{\substack{\uparrow \\ n=0}}{1}, -1, 2, -2, -1, 1, 0, 0, \dots$$

Linear convolution:  $6+3-1 = 8$  points

$$\begin{array}{cccccccc}
 & & 1 & 2 & 3 & & & \\
 & & & -1 & -2 & -3 & & \\
 & & & & 2 & 4 & 6 & \\
 & & & & & -2 & -4 & -6 \\
 & & & & & & -1 & -2 & -3 \\
 & & & & & & & 1 & 2 & 3 \\
 + & & & & & & & & & \\
 \hline
 & 1 & 1 & 3 & -1 & 1 & -7 & -1 & 3 & 
 \end{array}$$

$\uparrow$   
 $n=0$

$$y[n]: \dots, 0, 0, \underset{\substack{\uparrow \\ n=0}}{1}, 1, 3, -1, 1, -7, -1, 3, 0, 0, \dots$$

By Overlap-add:

$$h[n]: \dots, 0, 0, \underset{\substack{\uparrow \\ n=0}}{1}, 2, 3, 0, 0, \dots$$

3-point sequence ( $P = 3$ )

$$x[n]: \dots, 0, 0, \underset{\substack{\uparrow \\ n=0}}{1}, -1, 2, -2, -1, 1, 0, 0, \dots$$

(In general, infinite length, starts at  $n = 0$ )

Decompose the input,  $x[n]$ , into  $L$ -point segments.

Let's choose  $L = 3$ ; could be something else...

$$x[n]: \dots, 0, 0, \underbrace{1, -1, 2}_{x_1}, \underbrace{-2, -1, 1}_{x_2}, 0, 0, \dots$$

Compute the **5-point**  $(L + P - 1)$  DFTs of  $x_1$  and  $x_2 \rightarrow X_1$  and  $X_2$

Compute the **5-point** DFT of  $h \rightarrow H$

Compute

$$y_1 = IDFT\{HX_1\} \quad y_1: \dots, 0, 0, \underset{\uparrow}{1}, 1, 3, 1, 6, 0, 0, \dots$$

and

$$y_2 = IDFT\{H X_2\} \quad y_2: \dots, 0, \underset{\uparrow}{0}, 0, 0, -2, -5, -7, -1, 3, 0, 0, \dots$$

Compute  $y[n]$  by “**overlap-add**”, i.e,

$$\begin{array}{cccccccc}
 & 1 & 1 & 3 & 1 & 6 & & \\
 + & & & & -2 & -5 & -7 & -1 & 3 \\
 \hline
 & 1 & 1 & 3 & -1 & 1 & -7 & -1 & 3 \\
 & \uparrow & & & & & & & \\
 & n=0 & & & & & & & 
 \end{array}$$



## Overlap Add Formally

Let the impulse response of the system be of length  $P$  samples.

Decompose the input  $x[n]$  into nonoverlapping, consecutive blocks of length  $L$ .

$$r^{th} \text{ block is } x_r[n] = \begin{cases} x[n + rL] & 0 \leq n < L \\ 0 & \text{otherwise} \end{cases}$$

so that  $x[n] = \sum_{r=0}^{\infty} x_r[n - rL]$ .

Compute the  $(L + P - 1)$ -point DFT of  $h[n]$ ,  $H[k]$ .

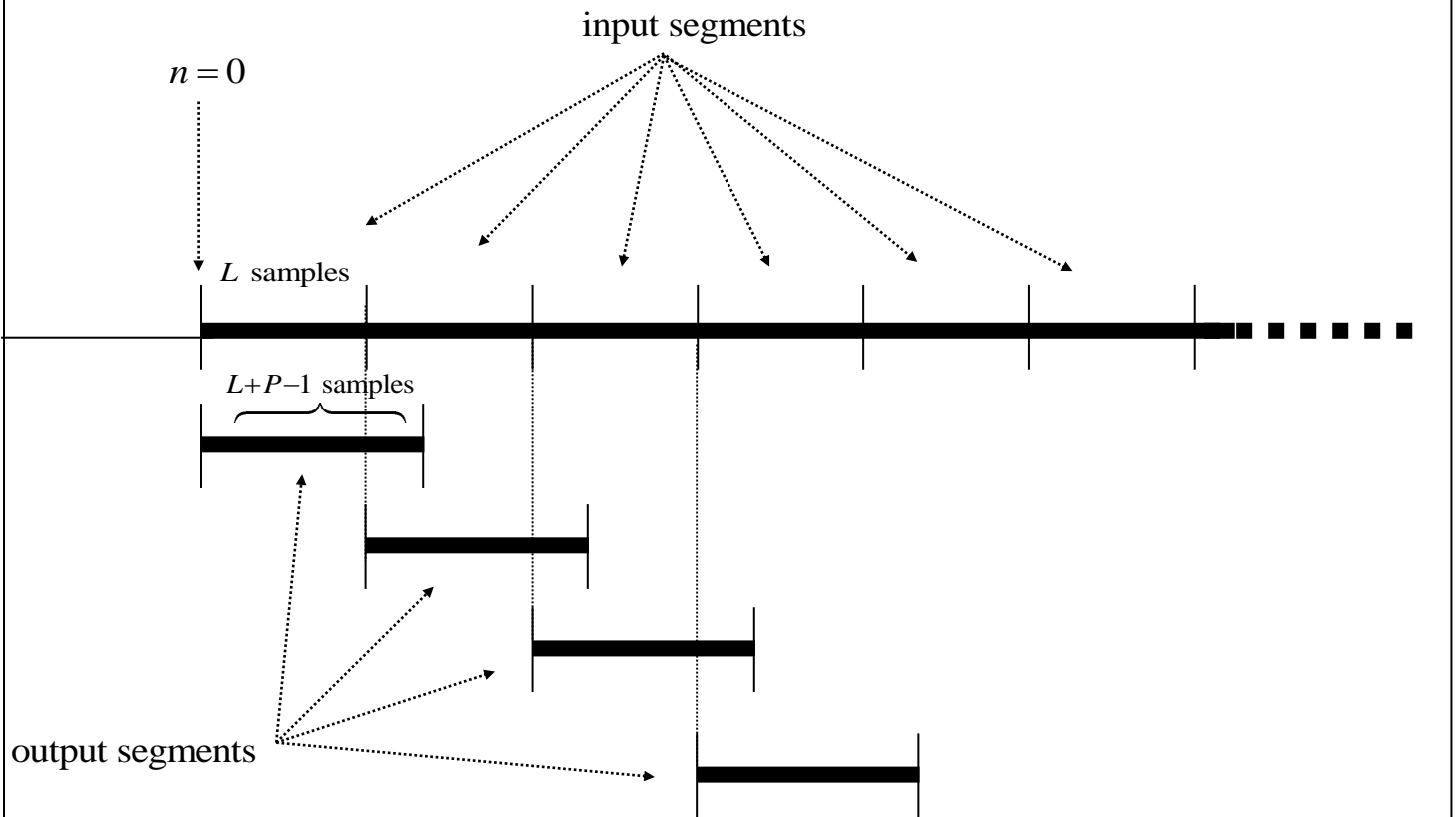
For the  $r^{th}$  block, the output,  $y_r[n]$ , is found by  $(L + P - 1)$ -point IDFT of  $Y_r[k] = H[k]X_r[k]$ .

$$\Rightarrow y_0[n], y_1[n], y_2[n], \dots$$

The overall output is formed as

$$\begin{aligned} y[n] &= \sum_{r=0}^{\infty} y_r[n - rL] \\ &= y_0[n] + y_1[n - L] + y_2[n - 2L] + \dots \end{aligned}$$

## Overlap Add Pictorially



# OVERLAP-SAVE METHOD

Ex:

$$h[n]: \dots, 0, 0, \underset{\substack{\uparrow \\ n=0}}{1}, 2, 3, 0, 0, \dots$$

3-point sequence ( $P = 3$ )

$$x[n]: \dots, 0, 0, \underset{\substack{\uparrow \\ n=0}}{1}, -1, 2, -2, -1, 1, 0, 0, \dots$$

(In general, infinite length, starts at  $n = 0$ )

Decompose the input,  $x[n]$ , into  $L$ -point segments ( $L \geq P$ ).

Choose  $L = 4$ , could be something else...

In overlap-save method, we use  $L$ -point DFTs.

Therefore, the first  $(L + P - 1) - L = P - 1$  samples for each output segment will be incorrect. In this example, the first 2 samples...

Because of the two incorrect samples at each output block, the first segment is chosen to start at  $n = -(P - 1) = -2$

$$x[n]: \dots, 0, 0, \underbrace{0, 0, 1, -1, 2, -2, -1, 1, 0, 0}_{x_1}, \dots$$

then, the second segment is chosen as

$$x[n]: \dots, 0, 0, 0, 0, \underbrace{1, -1, 2, -2, -1, 1, 0, 0}_{x_1}, \dots$$

which “**overlaps**” with the previous segment by two samples

Compute the **4-point** DFTs of  $x_1, x_2, x_3, x_4, x_5, \dots$

$\rightarrow X_1, X_2, X_3, X_4, X_5, \dots$

Compute the **4-point** DFT of  $h$

$\rightarrow H$

Compute

$$y_1 = \text{IDFT}\{H X_1\}$$

$$y_1: \dots, 0, 0, 1, -3, 1, 1, 0, 0, \dots$$

$$y_2 = \text{IDFT}\{H X_2\}$$

$$y_2: \dots, 0, 0, 3, -5, 3, -1, 0, 0, \dots$$

$$y_3 = \text{IDFT}\{H X_3\}$$

$$y_3: \dots, 0, 0, 1, 5, 1, -7, 0, 0, \dots$$

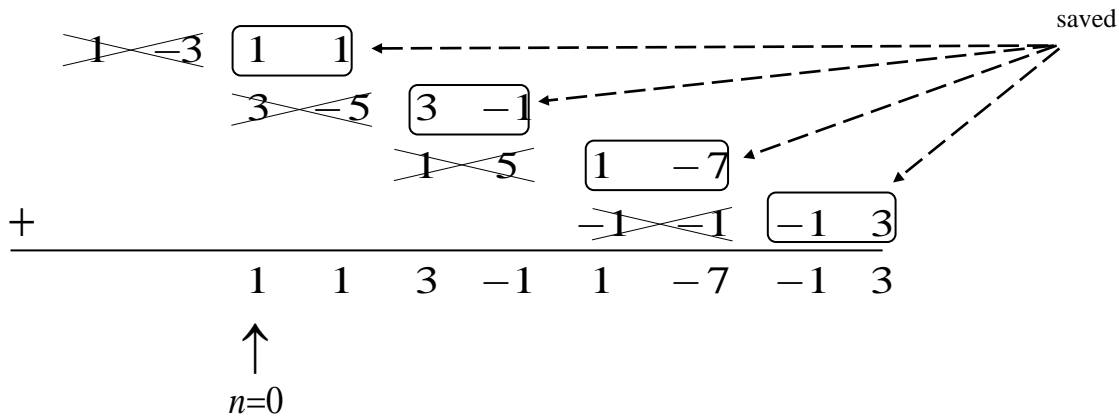
$$y_4 = \text{IDFT}\{H X_4\}$$

$$y_4: \dots, 0, 0, -1, -1, -1, 3, 0, 0, \dots$$

$$y_5 = \text{IDFT}\{H X_5\}$$

$$y_5: \dots, 0, 0, 0, 0, 0, 0, 0, 0, \dots$$

Compute  $y[n]$  by “**saving**” the correct samples,



$$y[n]: \dots, 0, 0, \underset{\substack{\uparrow \\ n=0}}{1}, 1, 3, -1, 1, -7, -1, 3, 0, 0, \dots$$

## OVERLAP SAVE FORMALLY

Let  $N < L + P - 1$  ( $N \geq L \geq P$ )

The first  $(L + P - 1 - N)$  samples of  $N$ -point IDFT are incorrect.

In particular, if  $N = L$ ,  $P - 1$  samples are incorrect.

So, choose the first input segment,  $x_0[n]$ , as

$$\underbrace{x[-(P-1)], x[-(P-2)], \dots, x[-1], x[0], x[1], \dots, x[L-P]}_{L \text{ samples}}$$

i.e.,  $x_0[n] = x[n - (P-1)] \quad 0 \leq n \leq L-1$

Compute its  $L$ -point DFT  $\rightarrow X_0[k]$

Compute the  $L$ -point DFT of  $h \rightarrow H[k]$

Compute  $\hat{y}_0[n] = L\text{-point IDFT of } X_0[k] H[k].$

Discard the first  $(P-1)$  samples to get the first output segment

$$y_0[n] = \begin{cases} \hat{y}_0[n] & P-1 \leq n \leq L-1 \\ 0 & \text{o.w.} \end{cases}$$

then the first  $(L-(P-1))$  samples of the output are

$$y[n] = y_0[n + (P-1)] \quad n = 0, 1, \dots, L - (P-1) - 1$$

Then take the second input segment as

$$\underbrace{x[L-P-(P-1)+1], \dots, x[2L-2P+1]}_{L \text{ samples}}$$

i.e.  $x[L-2P+2], \dots, x[2L-2P+1]$

and continue the process...

## Overlap Save Pictorially

