Z-TRANSFORM

REGION OF CONVERGENCE (ROC)

$X(z)$ CAN BE VIEWED AS THE DTFT OF THE SEQUENCE $x[n]r^{-n}$

$X\left(e^{j\omega}\right)$ FROM $X(z)$

DIFFERENT SEQUENCES MAY HAVE THE SAME $H(z)$ WITH DIFFERENT ROC

POLES AND ZEROS

ROC OF FINITE LENGTH SEQUENCES

PROPERTIES OF ROC

PROPERTIES OF Z-TRANSFORM

SYSTEM FUNCTION

POLES AND ZEROS OF CAUSAL AND STABLE SYSTEMS

LCCDEs AND SYSTEM FUNCTIONS

INVERSE Z-TRANSFORM

  INVERSE TRANSFORM BY PARTIAL FRACTION EXPANSION

  INVERSION BY LONG DIVISION

TRANSFORM PAIRS

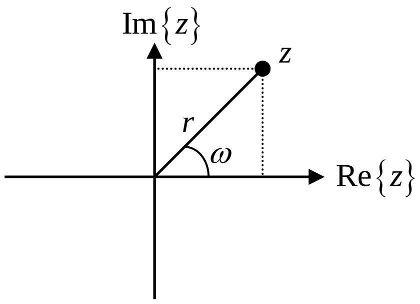
MATLAB LINEAR SYSTEM TRANSFORMATIONS

Z-TRANSFORM

z-transform of a sequence $x[n]$ is defined as

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} \qquad z \in C$$

$z = re^{j\omega}$



$X(z)$ is a complex valued function.

It takes complex values on the complex domain.

$$\ldots + x[-2]z^2 + x[-1]z^1 + x[0] + x[1]z^{-1} + x[2]z^{-2} + \cdots$$

A series containing powers of z and z^{-1}

REGION OF CONVERGENCE (ROC)

The set of complex numbers for which

$$\ldots + x[-2]z^{2} + x[-1]z^{1} + x[0] + x[1]z^{-1} + x[2]z^{-2} + \cdots$$

is finite.

May or may not exist!

Ex: $x[n] = \delta[n] + 2\delta[n-1]$

$$X(z) = 1 + 2z^{-1}$$

Let's evaluate $X(z)$ at a few values of z.

$$X(1) = 1 + 2 = 3$$

$$X(1+j) = X\left(\sqrt{2}e^{j\frac{\pi}{4}}\right) = 1 + 2\frac{1}{\sqrt{2}}e^{-j\frac{\pi}{4}} = 1 + \sqrt{2}e^{-j\frac{\pi}{4}} = 1 + (1-j)$$

$$= 2 - j \qquad \left(\sqrt{5}e^{j\tan^{-1}\left(-\frac{1}{2}\right)}\right)$$

$$X(-4+j3) = X\left(5e^{j\tan^{-1}\frac{3}{-4}}\right) = 1 + \frac{2}{5}\left(\frac{-4}{5} - j\frac{3}{5}\right)$$

$$= \frac{17}{25} - j\frac{6}{25}$$

$$X(-2) = 0$$

$X(0) \to \infty$!!!

Ex: $x[n] = 2\delta[n+1] + \delta[n]$

$$X(z) = 2z + 1$$

Let's evaluate $X(z)$ at a few values of z.

$X(1) = 1 + 2 = 3$

$X(1+j) = 1 + 2(1+j) = 3 + j2$

$X(-4+j3) = 1 + 2(-4+3j) = -7 + j6$

$X\left(-\frac{1}{2}\right) = 0$

$X(\infty) \rightarrow \infty$!!!

Ex: $x[n] = 2\delta[n+1] + \delta[n] - \delta[n-1]$

$$X(z) = 2z + 1 - z^{-1}$$

Let's evaluate $X(z)$ at a few values of z.

$$X(1) = 2 + 1 - 1$$

$$= 2$$

$$X(1+j) = 2(1+j) + 1 - \frac{1}{2}(1-j)$$

$$= \frac{5}{2}(1+j)$$

$$X(-4+j3) = 2(-4+3j) + 1 - \frac{1}{-4+j3}$$

$$= -\frac{-171}{25} + j\frac{153}{25}$$

$$X\left(\frac{1}{2}\right) = 0$$

$$X(-1) = 0$$

$$X(0) \to \infty \qquad !!!$$

$$X(\infty) \to \infty \qquad !!!$$

Ex: $x[n] = \left(\frac{1}{2}\right)^n u[n]$

$$X(z) = 1 + \frac{1}{2}z^{-1} + \frac{1}{4}z^{-2} + \frac{1}{8}z^{-3} + \cdots$$

Let's evaluate $X(z)$ at a few values of z.

$$X(1) = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 2$$

$$X(1+j) = 1 + \frac{1}{2}(1+j)^{-1} + \frac{1}{4}(1+j)^{-2} + \frac{1}{8}(1+j)^{-3} + \cdots = \frac{2}{5}(3-j)$$

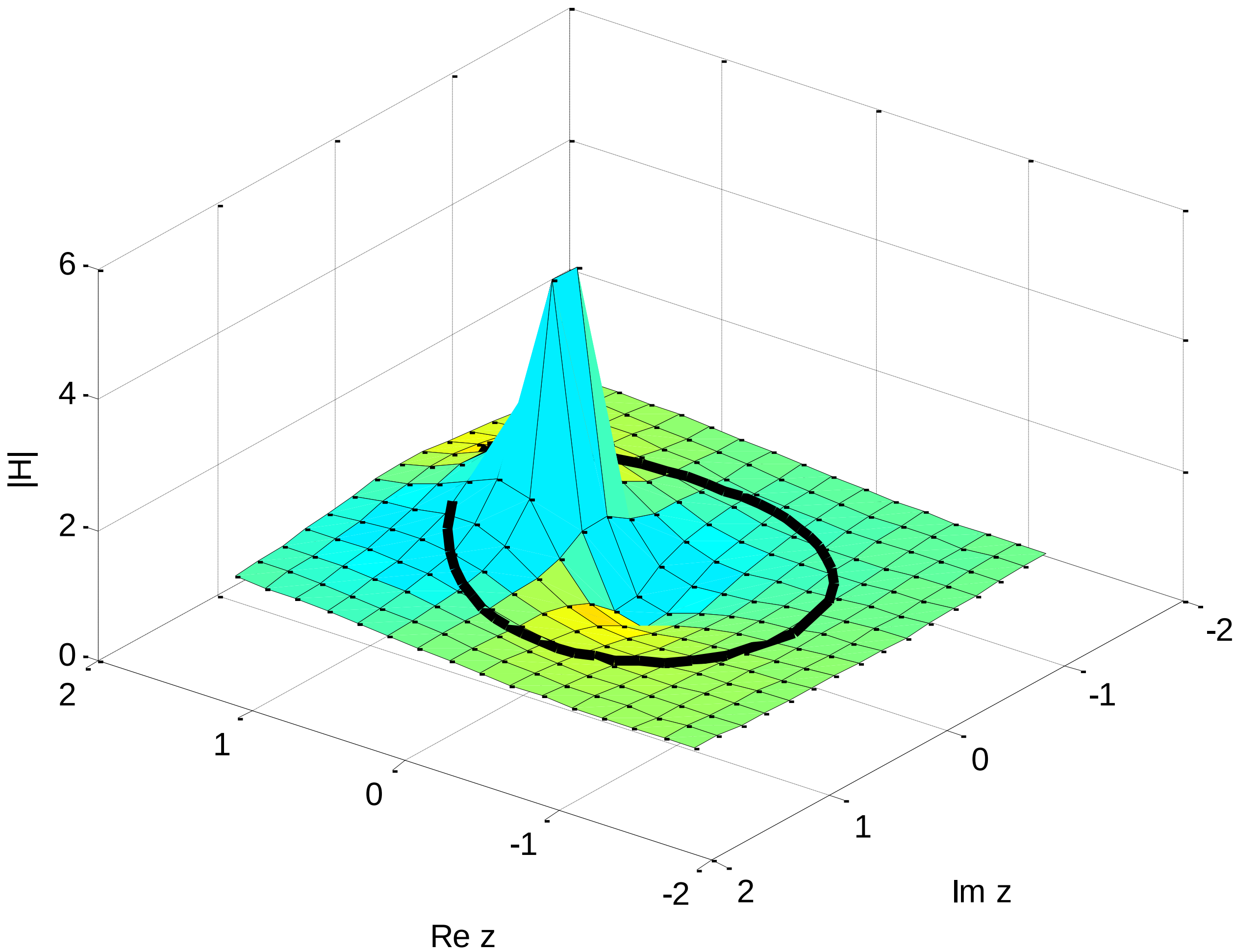
$$X(-4+j3) = \frac{2}{117}(54-j3)$$

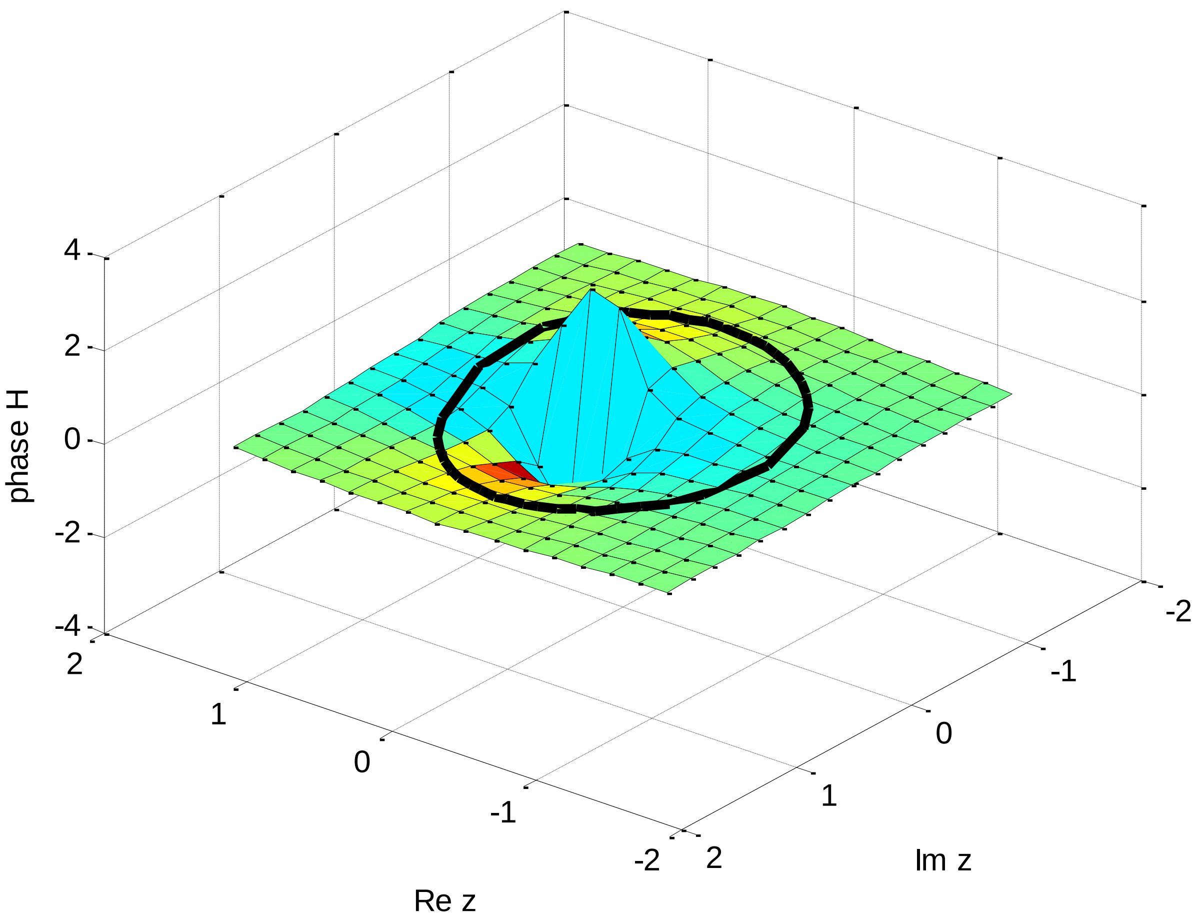
$$X\left(\frac{1}{2}\right) \to \infty$$

Indeed $\quad X(z) \to \infty \qquad |z| < \frac{1}{2}$

LN5_code1_z_trans_plot.m

```
clear all
close all
a = 0.5;
prec =0.2;
sinir = 1.5;
[rr,ii] = meshgrid(-sinir:prec:sinir, -sinir:prec:sinir);
z = rr+i*ii;    %z-plane grid
z1 = 1./z;
% inks = find(abs(z)<=abs(a));
H = 1./(1-a*z1);
% H(inks) = 0;

precH = 0.1;
x = -1:precH:1;
y = sqrt(1-x.^2);
zz = x+i*y;
zz = 1./zz;
H_UC_1 = 1./(1-a*zz);

ey = -1*sqrt(1-x.^2);
zz = x+i*ey;
zz = 1./zz;
H_UC_2 = 1./(1-a*zz);

figure
surfl(rr,ii,angle(H))
xlabel('Re z')
ylabel('Im z')
zlabel('phase H')
hold on;
plot3(x,y,angle(H_UC_1)+0.05,'k','linewidth',3);
plot3(x,ey,angle(H_UC_2)+0.05,'k','linewidth',3);

inks = find(abs(H)>= 10);
H(inks) = 10;
figure
surfl(rr,ii,abs(H))
hold on;
plot3(x,y,abs(H_UC_1)+0.05,'k','linewidth',3);
plot3(x,ey,abs(H_UC_2)+0.05,'k','linewidth',3);
xlabel('Re z')
ylabel('Im z')
zlabel('|H|')
```


6
4
2
0
|H|
2
1
0
-1
-2
Re z
-2
-1
0
1
2
Im z


4
2
0
-2
-4
phase H
2
1
0
-1
-2
Re z
-2
-1
0
1
2
Im z

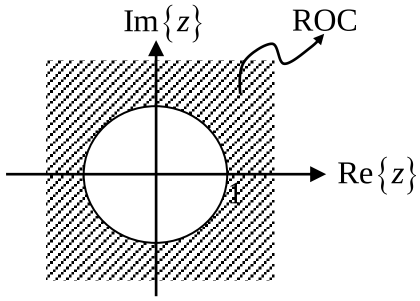
$X(z)$ CAN BE VIEWED AS THE DTFT OF THE SEQUENCE $x[n]r^{-n}$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\left(re^{j\omega}\right)^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \left(x[n]r^{-n}\right) e^{-j\omega n}$$

Obviously the convergence of the above series depends also on “r”.

<u>Ex</u>: The DTFT of $u[n]$ does not exist in the formal sense. However, the DTFT of $r^{-n}u[n] = \left(r^{-1}\right)^n u[n]$ exists if $r > 1$ i.e. $|z| > 1$.

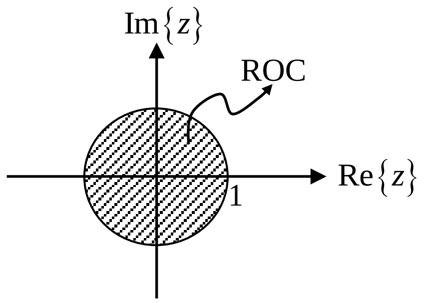


ROC: Region Of Convergence. The set of *z* values for which the series converges

$$U(z) = \sum_{n=0}^{\infty} z^{-n} = \sum_{n=0}^{\infty} \left(z^{-1}\right)^n = \frac{1}{1 - z^{-1}} \qquad |z| > 1$$

Ex: The DTFT of $x[n]=-u[-n-1]$ does not exist in the formal sense. However, the DTFT of $-r^{-n}u[-n-1]=-\left(r^{-1}\right)^n u[-n-1]$ exists if $r<1$ i.e. $|z|<1$.

$$
\begin{aligned}
X(z) &= -\sum_{n=-\infty}^{-1} z^{-n} \\
&= -\sum_{n=1}^{\infty} z^{n} \\
&= 1-\sum_{n=0}^{\infty} z^{n} \\
&= 1-\frac{1}{1-z} \qquad |z|<1 \\
&= -\frac{z}{1-z} \qquad |z|<1 \\
&= \frac{1}{1-z^{-1}} \qquad |z|<1
\end{aligned}
$$



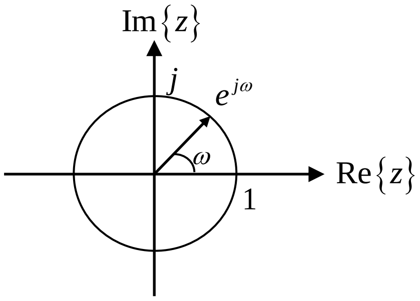
$X(z)$ expressions in the above examples are the same but their ROCs are different. Therefore, without knowing ROC, $X(z)$ expression is not sufficient to identify the sequence!

$X\left(e^{j\omega}\right)$ FROM $X(z)$

Note that whenever $X(z)$ exists for $|z| = 1$

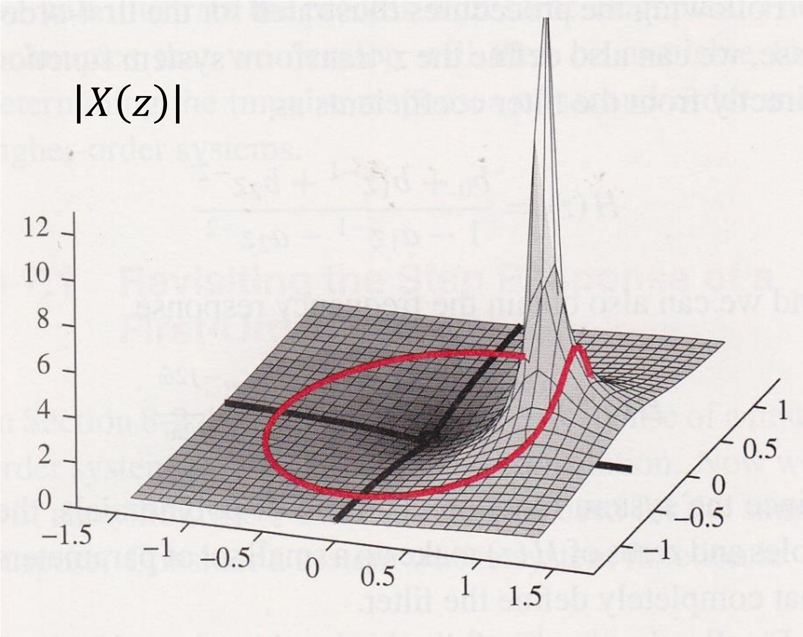
If it is evaluated for $|z| = 1 = e^{-j\omega}$ we get the DTFT of $x[n]$!

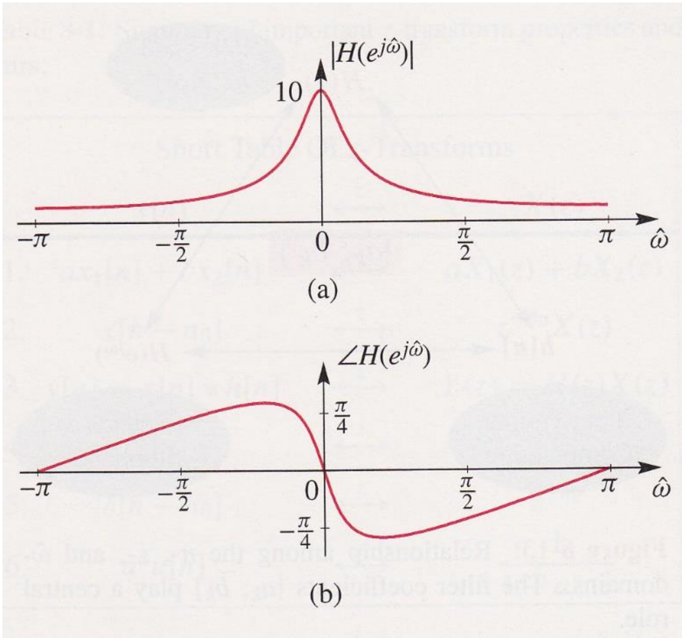
$$X(z)\Big|_{z=e^{j\omega}} = X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

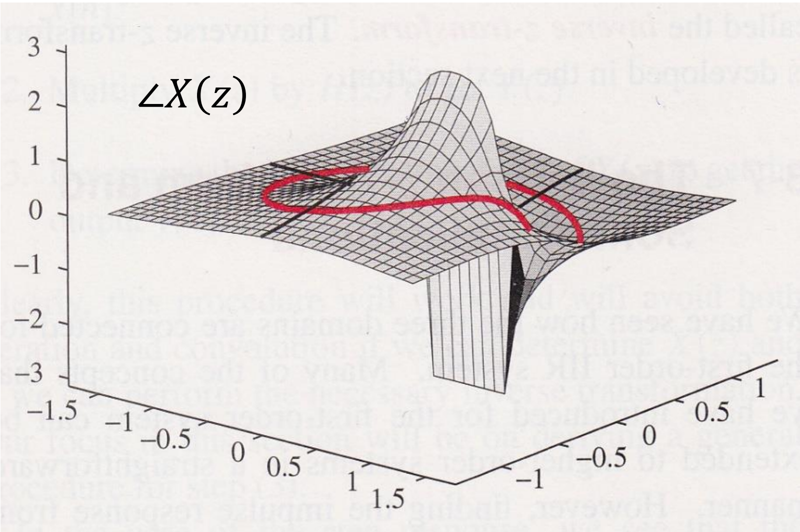


Ex:

$$x[n] = 0.8^n u[n] \qquad\qquad X(z) = \frac{1}{1 - 0.8z^{-1}} \qquad |z| > 0.8$$







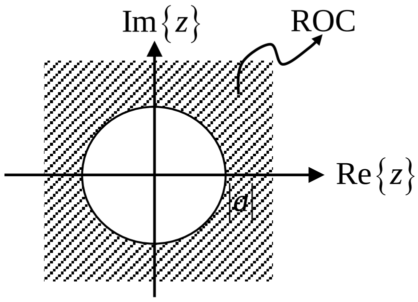
DIFFERENT SEQUENCES MAY HAVE THE SAME $H(z)$ WITH DIFFERENT ROC

$$x[n] = a^n u[n] \qquad \leftrightarrow \qquad \frac{1}{1 - az^{-1}} \quad |z| > 1$$

$$x[n] = -a^n u[-n-1] \qquad \leftrightarrow \qquad \frac{1}{1 - az^{-1}} \quad |z| < 1$$

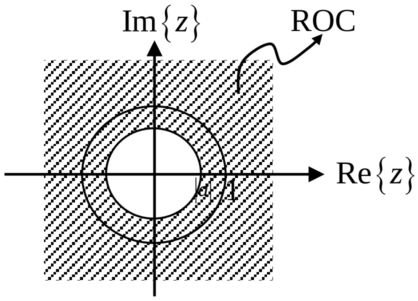
Ex: z-transform of $x[n] = a^n u[n]$

$$X(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} \left(az^{-1}\right)^n = \frac{1}{1-az^{-1}} \qquad \left|az^{-1}\right| < 1 \quad \text{i.e.} \quad |z| > |a|$$



Note that if $|a|<1$ then ROC includes the unit circle. Then, DTFT of $x[n]=a^n u[n]$ exists (as we know).

DTFT: $$X(z)\Big|_{z=e^{j\omega}} = X\left(e^{j\omega}\right) = \frac{1}{1-ae^{-j\omega}} \qquad \text{when} \qquad |a|<1$$



Ex: z-transform of $x[n] = -a^n u[-n-1]$

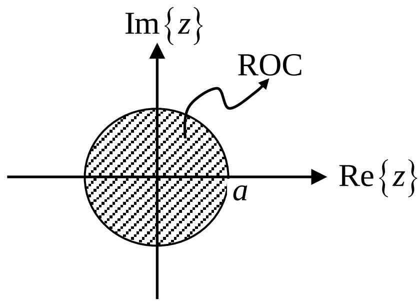
$$X(z) = -\sum_{n=-\infty}^{-1} a^n z^{-n}$$

$$= -\sum_{n=1}^{\infty} \left(a^{-1} z\right)^n$$

$$= 1 - \sum_{n=0}^{\infty} \left(a^{-1} z\right)^n$$

$$= 1 - \frac{1}{1 - a^{-1} z} \qquad \left|a^{-1} z\right| < 1$$

$$= \frac{1}{1 - a z^{-1}} \qquad |z| < |a|$$



Note that closed form $X(z)$ expressions in the above examples are the same but their ROCs are different. Therefore without knowing ROC, $X(z)$ expression is not sufficient to identify the sequence!
Also, if $|a| > 1$ ROC includes the unit circle. Then, DTFT of $x[n] = -a^n u[-n-1]$ exists.

POLES AND ZEROS

Definition:

Let

$$X(z) = \frac{\mathrm{P(z)}}{\mathrm{Q}(z)}$$

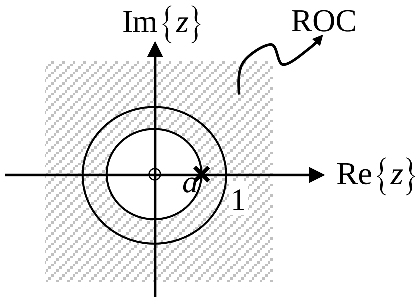
i.e. ratio of polynomials in z,

z_0 : a **zero** of $X(z)$ if $P(z_0) = 0$

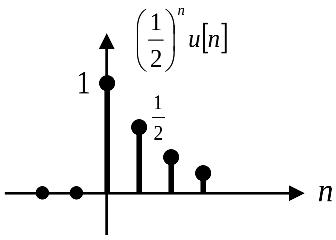
z_0 : a **pole** of $X(z)$ if $Q(z_0) = 0$

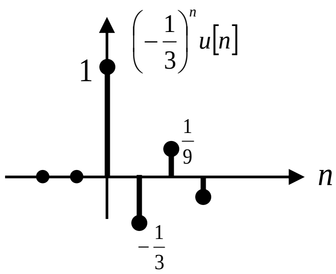
Ex: $X(z) = \dfrac{1}{1 - a z^{-1}} = \dfrac{z}{z - a}$ one zero at 0, one pole at $z = a$.

If $x[n] = a^n u[n]$ $\quad X(z) = \dfrac{1}{1 - az^{-1}} \quad |z| > |a|$



Ex A: $x[n]=\left(\frac{1}{2}\right)^n u[n]+\left(-\frac{1}{3}\right)^n u[n]$ (right-sided)





z-transform is a linear operation therefore

$$X(z)=\mathrm{Z}\left\{\left(\frac{1}{2}\right)^n u[n]\right\}+\mathrm{Z}\left\{\left(-\frac{1}{3}\right)^n u[n]\right\}$$

$$\mathrm{Z}\left\{\left(\frac{1}{2}\right)^n u[n]\right\}=\frac{1}{1-\frac{1}{2}z^{-1}} \qquad |z|>\frac{1}{2}:ROC_1$$

$$\mathrm{Z}\left\{\left(-\frac{1}{3}\right)^n u[n]\right\}=\frac{1}{1+\frac{1}{3}z^{-1}} \qquad |z|>\frac{1}{3}:ROC_2$$

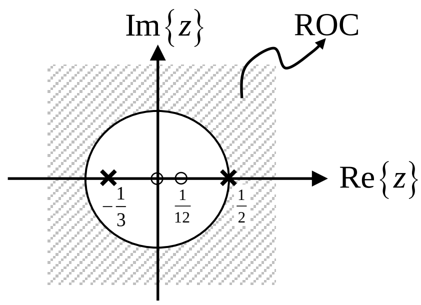
ROC of $X(z)$ is the intersection of the ROCs of the components.

$$X(z)=\frac{1}{1-\frac{1}{2}z^{-1}}+\frac{1}{1+\frac{1}{3}z^{-1}}=\frac{2\left(1-\frac{1}{12}z^{-1}\right)}{\left(1-\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{3}z^{-1}\right)}=\frac{2\left(1-\frac{1}{12}z^{-1}\right)}{1-\frac{1}{6}z^{-1}-\frac{1}{6}z^{-2}} \qquad ROC=ROC_1\cap ROC_2=|z|>\frac{1}{2}$$

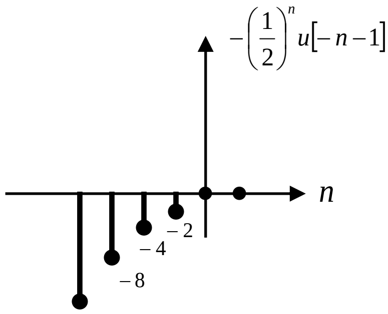
$$=\frac{2z\left(z-\frac{1}{12}\right)}{\left(z-\frac{1}{2}\right)\left(z+\frac{1}{3}\right)}=\frac{2z\left(z-\frac{1}{12}\right)}{z^2-\frac{1}{6}z-\frac{1}{6}}$$

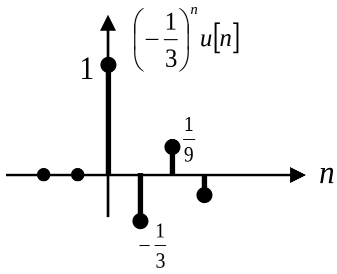
two zeroes at z = 0 and $z=\frac{1}{12}$

two poles at $z=\frac{1}{2}$ and $z=-\frac{1}{3}$.



Ex B: $x[n] = -\left(\frac{1}{2}\right)^n u[-n-1] + \left(-\frac{1}{3}\right)^n u[n]$ (two-sided)





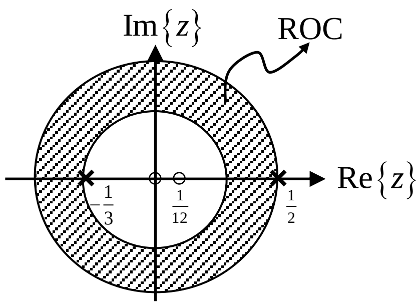
$$X(z) = \mathrm{Z}\left\{-\left(\frac{1}{2}\right)^n u[-n-1]\right\} + \mathrm{Z}\left\{\left(-\frac{1}{3}\right)^n u[n]\right\}$$

$\mathrm{Z}\left\{-\left(\frac{1}{2}\right)^n u[-n-1]\right\} = \frac{1}{1-\frac{1}{2}z^{-1}} \qquad \lvert z\rvert < \frac{1}{2} : ROC_1$	$\mathrm{Z}\left\{\left(-\frac{1}{3}\right)^n u[n]\right\} = \frac{1}{1+\frac{1}{3}z^{-1}} \qquad \lvert z\rvert > \frac{1}{3} : ROC_2$

ROC of *X*(*z*) is the intersection of the ROCs of the components.

$$X(z) = \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{1}{1+\frac{1}{3}z^{-1}} = \frac{2\left(1-\frac{1}{12}z^{-1}\right)}{\left(1-\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{3}z^{-1}\right)} \qquad ROC = ROC_1 \cap ROC_2 = \frac{1}{3} < \lvert z\rvert < \frac{1}{2}$$

$$= \frac{2z\left(z-\frac{1}{12}\right)}{\left(z-\frac{1}{2}\right)\left(z+\frac{1}{3}\right)}$$

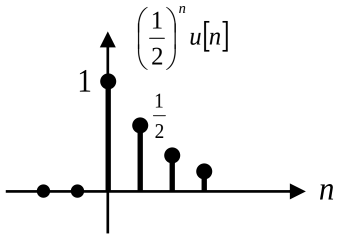


two zeroes at z = 0 and $z = \frac{1}{12}$

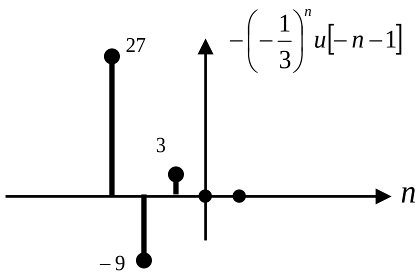
two poles at $z = \frac{1}{2}$ and $z = -\frac{1}{3}$.

Ex C: $x[n]=\left(\frac{1}{2}\right)^n u[n]-\left(-\frac{1}{3}\right)^n u[-n-1]$

$$X(z)=\mathrm{Z}\left\{-\left(\frac{1}{2}\right)^n u[-n-1]\right\}+\mathrm{Z}\left\{\left(-\frac{1}{3}\right)^n u[n]\right\}$$



$$\boxed{\mathrm{Z}\left\{\left(\frac{1}{2}\right)^n u[n]\right\}=\frac{1}{1-\frac{1}{2}z^{-1}} \qquad |z|>\frac{1}{2}:ROC_1}$$



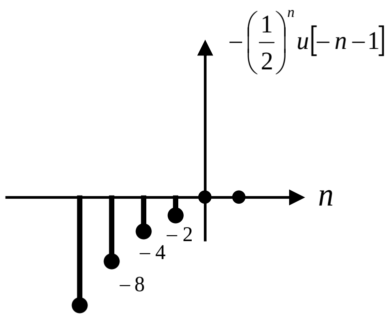
$$\boxed{\mathrm{Z}\left\{-\left(-\frac{1}{3}\right)^n u[-n-1]\right\}=\frac{1}{1+\frac{1}{3}z^{-1}} \qquad |z|<\frac{1}{3}:ROC_2}$$

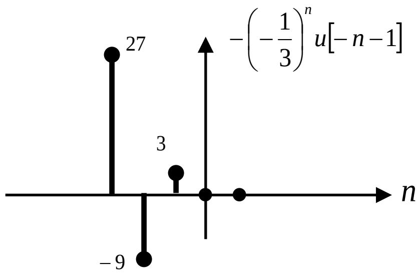
ROC of $X(z)$ is the intersection of the ROCs of the components.

$ROC=ROC_1\cap ROC_2=\varnothing$

Hence z-transform of $x[n]$ does not exist!

Ex D: $x[n] = -\left(\frac{1}{2}\right)^n u[-n-1] - \left(-\frac{1}{3}\right)^n u[-n-1]$ (left-sided)





$$X(z) = \mathrm{Z}\left\{-\left(\frac{1}{2}\right)^n u[-n-1]\right\} + \mathrm{Z}\left\{-\left(-\frac{1}{3}\right)^n u[-n-1]\right\}$$

$$\mathrm{Z}\left\{-\left(\frac{1}{2}\right)^n u[-n-1]\right\} = \frac{1}{1-\frac{1}{2}z^{-1}} \qquad |z| < \frac{1}{2} : ROC_1$$

$$\mathrm{Z}\left\{-\left(-\frac{1}{3}\right)^n u[-n-1]\right\} = \frac{1}{1+\frac{1}{3}z^{-1}} \qquad |z| < \frac{1}{3} : ROC_2$$

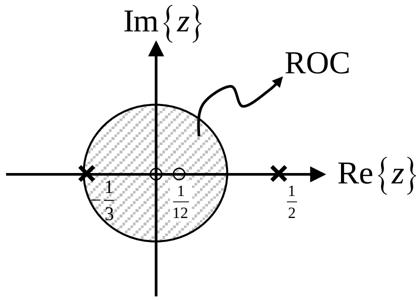
ROC of $X(z)$ is the intersection of the ROCs of the components.

$$X(z) = \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{1}{1+\frac{1}{3}z^{-1}} = \frac{2\left(1-\frac{1}{12}z^{-1}\right)}{\left(1-\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{3}z^{-1}\right)} \qquad ROC = ROC_1 \cap ROC_2 = |z| < \frac{1}{3}$$

$$= \frac{2z\left(z-\frac{1}{12}\right)}{\left(z-\frac{1}{2}\right)\left(z+\frac{1}{3}\right)}$$

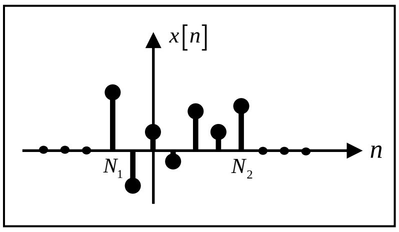
two zeroes at $z = 0$ and $z = \frac{1}{12}$

two poles at $z = \frac{1}{2}$ and $z = -\frac{1}{3}$.



ROC OF FINITE LENGTH SEQUENCES

Finite length sequence: $x[n]=0 \qquad n<N_1,\ n>N_2$



$$X(z)=\sum_{n=N_1}^{N_2} x[n]z^{-n}$$

$$= x[N_1]z^{-N_1}+x[N_1+1]z^{-N_1-1}+\ldots+x[N_2-1]z^{-N_2+1}+x[N_2]z^{-N_2}$$

If $N_1=N_2=0$, i.e. $x[n]=\delta[n]$ $X(z)=\sum_{n=-\infty}^{\infty}\delta[n]z^{-n}=1$

ROC is the entire *z*-plane.

Otherwise,

A) If $N_2>N_1>0$ only negative powers of *z* → poles at 0

Ex: $x[n]=\delta[n-1] \qquad X(z)=z^{-1}$

ROC is the entire *z*-plane except *z* =0

B) If $N_1<N_2<0$ only positive powers of *z* → poles at ∞

Ex: $x[n]=\delta[n+1] \quad X(z)=z$

ROC is the entire *z*-plane except *z* → ∞

C) If $N_1<0,\quad N_2>0$

negative and positive powers of *z* → poles at 0 and ∞

Ex: $x[n]=\delta[n+1]+\delta[n]+\delta[n-1] \qquad X(z)=z+1+z^{-1}$

ROC is the entire *z*-plane except *z* = 0 and *z* → ∞

Ex: $x[n]=\delta[n]+2\delta[n-1]-3\delta[n-5]$

$$X(z)=\sum_{n=-\infty}^{\infty}x[n]z^{-n}=1+2z^{-1}-3z^{-5}=\frac{z^5+2z^4-3}{z^5}$$

ROC is the entire *z*-plane except $z = 0$.

5th order pole at $z = 0$

5 zeros at $z_{1,2} = -1.6220 \pm j\,0.1877$,

$z_{3,4} = 0.1220 \pm j1.0538$,

$z_5 = 1$

Ex: $x[n]=\begin{cases} a^n & 0\le n\le N-1 \\ 0 & \text{ow}\end{cases}$

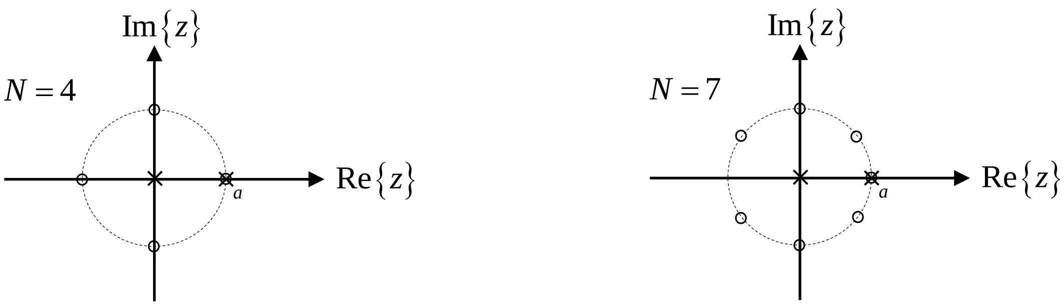
$$X(z)=\sum_{n=0}^{N-1}a^n z^{-n}=1+az^{-1}+a^2z^{-2}+\ldots a^{N-1}z^{-N+1}\qquad =\frac{1-\left(az^{-1}\right)^N}{1-az^{-1}}\qquad =\frac{z^N-a^N}{z^{N-1}\left(z-a\right)}$$

There are *N* zeros:

$$z^N=a^N=a^Ne^{jk2\pi}\qquad\Rightarrow\qquad z=a\,e^{jk\frac{2\pi}{N}}\qquad k=0,1,\ldots,N-1$$

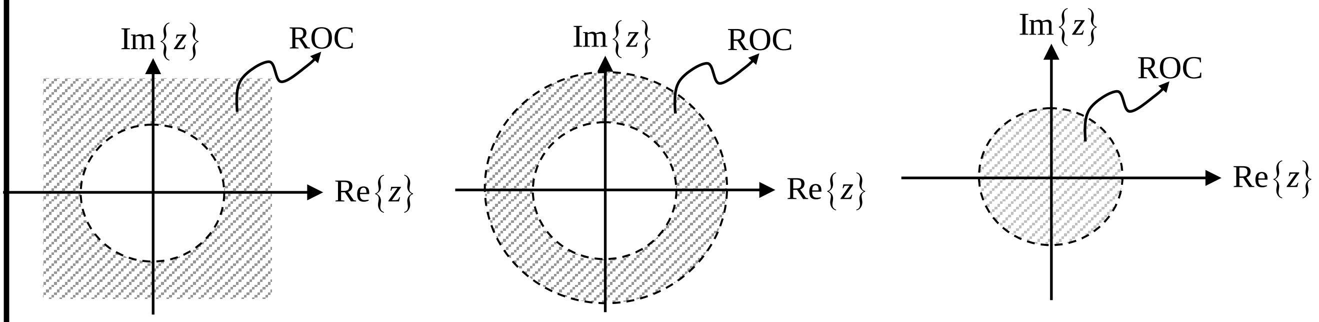
There is a (N-1)st order pole at *z* = 0

There is a pole at *z* = *a* however it is cancelled by a zero at *z* = *a*.



PROPERTIES OF ROC

1) ROC is a ring or a disk centered around origin.



ROC of a right- ROC of a two-sided seq. ROC of a left-sided seq.

2) ROC does not contain poles. All poles are outside ROC. <u>Poles exist at the boundaries</u>.

3) For finte length seq. ROC is the entire z-plane except <u>possibly</u> $z = 0$ and/or $z = \infty$.

4) DTFT of the seq. exist if ROC contains unit circle.

PROPERTIES OF Z-TRANSFORM

1) Linearity

$$X(z)=\sum_{n=-\infty}^{\infty}x[n]z^{-n} \;\Rightarrow\; Z\{ax[n]+by[n]\}=\sum_{n=-\infty}^{\infty}\left(ax[n]+by[n]\right)z^{-n}$$

$$=a\sum_{n=-\infty}^{\infty}x[n]z^{-n}+b\sum_{n=-\infty}^{\infty}y[n]z^{-n}=a\,X(z)+bY(z) \qquad ROC=ROC_1\cap ROC_2$$

<u>Ex</u>:

$$x[n]=u[n]+\delta[n+1] \quad\Rightarrow\quad X(z)=\frac{1}{1-z^{-1}}+z=\frac{z}{1-z^{-1}} \qquad ROC:|z|>1 \text{ except } \infty$$

2) Time Shifting

$$x[n] \leftrightarrow X(z) \qquad x[n-n_0] \leftrightarrow z^{-n_0} X(z)$$

ROC remains the same except possible addition or deletion of $z \to \infty$ or $z = 0$

Ex: $\delta[n] \leftrightarrow 1 \Rightarrow \delta[n-1] \leftrightarrow z^{-1}$ $\quad ROC:$ entire z-plane except $z=0$

Ex: $\delta[n] \leftrightarrow 1 \Rightarrow \delta[n+1] \leftrightarrow z$ $\quad ROC:$ entire z-plane except $z \to \infty$

Ex:

$$a^n u[n] \leftrightarrow \underbrace{\frac{1}{1-az^{-1}}}_{\substack{\text{zero at } z=0 \\ \text{pole at } z=a}} \Rightarrow a^{n-3}u[n-3] \leftrightarrow \underbrace{\frac{z^{-3}}{1-az^{-1}} = \frac{1}{z^2(z-a)}}_{\substack{\text{three zeros at } z\to\infty \\ \text{poles at } z=0 \text{ and } z=a}} \qquad ROC: |z| > a$$

$$\delta[n-3] + a\delta[n-4] + a^2\delta[n-5] + \ldots \leftrightarrow z^{-3} + az^{-4} + a^2z^5 + \ldots$$

Ex:

$$a^n u[n] \leftrightarrow \frac{1}{1-az^{-1}} \Rightarrow a^{n+2}u[n+2] \leftrightarrow \underbrace{\frac{z^2}{1-az^{-1}} = \frac{z^3}{z-a}}_{\substack{\text{three zeros at } z=0 \\ \text{pole at } z=a \text{ and } z\to\infty}} \qquad ROC: |z| > a \text{ except } z \to \infty$$

$$\delta[n+2] + a\delta[n+1] + a^2\delta[n] + a^3\delta[n-1] + \ldots \leftrightarrow z^2 + az + a^2 + a^3z^{-1} + \ldots$$

3) Multiplication by an exponential sequence

$$x[n] \leftrightarrow X(z) \quad \text{ROC:} \quad a < |z| < b$$

$$z_0^n x[n] \leftrightarrow X\left(\frac{z}{z_0}\right) \quad \text{ROC: } |z_0|a < |z| < |z_0|b$$

$$Z\left\{z_0^n\, x[n]\right\} = \sum_{n=-\infty}^{\infty} z_0^n\, x[n] z^{-n} = \sum_{n=-\infty}^{\infty} x[n]\left(\frac{z}{z_0}\right)^{-n} = X\left(\frac{z}{z_0}\right)$$

To see why ROC is modified so:

$$Z\left\{z_0^n\, x[n]\right\} = \sum_{n=-\infty}^{\infty} x[n]\left(\frac{r}{|z_0|}\right)^{-n} e^{-j\omega n} e^{j\omega_0 n} \qquad z_0 = re^{j\omega_0} \qquad z_0 = |z_0| e^{j\omega_0}$$

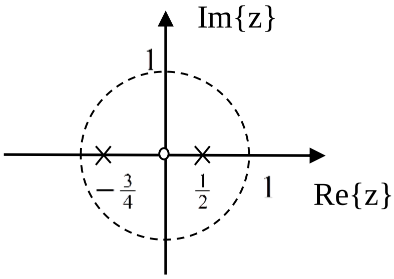
Ex :

$$x[n] = \left(\frac{1}{3}\right)^n u[n] \qquad \text{ROC: } \frac{1}{3} < |z|$$

$$x[n] = 2^n \left(\frac{1}{3}\right)^n u[n] \qquad \text{ROC: } \frac{2}{3} < |z|$$

Exercise:

Let $x[n]$ be a stable sequence with Z-transform $X(z)$. $X(z)$ has the following pole-zero plot ,



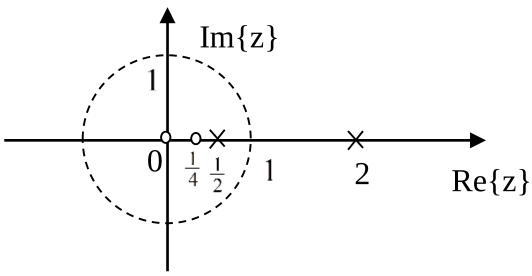
a) Find and plot poles and zeros of $W(z)$ where $w[n]=\cos\left(\frac{\pi}{2}n\right)\ x[n]$.

b) Let $y[n]$ be a stable sequence at the output of a LTI system (impulse response $h[n]$) when the input is $x[n]$ (above). $Y(z)$ has the following pole-zero plot.

i)What is the (region of conv.)ROC of $Y(z)$?

ii) Find the ROC of $H(z)$

iii) Find $h[n]$ if $H(z)|_{z=1}=-\frac{7}{16}$



4) Differentiation of $X(z)$

$$x[n] \leftrightarrow X(z) \qquad n\,x[n] \leftrightarrow -z\,\frac{dX(z)}{dz}$$

$$ROC_{nx[n]} = ROC_{x[n]}$$

$$-z\,\frac{dX(z)}{dz} = \sum_{n=-\infty}^{\infty} z\,n\,x[n]\,z^{-n-1} = \sum_{n=-\infty}^{\infty} n\,x[n]\,z^{-n} = Z\{n\,x[n]\}$$

Ex: Find the sequence whose z-transform is $X(z) = \log\left(1 + az^{-1}\right) \quad |z| > a$

$$X(z) = \log\left(1 + az^{-1}\right) \quad |z| > a \qquad \Rightarrow -z\frac{dX(z)}{dz} = \frac{az^{-1}}{1 + az^{-1}} \leftrightarrow nx[n]$$

Remember

$$\frac{1}{1 + az^{-1}} \leftrightarrow (-a)^n u[n] \quad \underset{\substack{\text{using the}\\\text{time-shift}\\\text{and}\\\text{linearity}\\\text{properties}}}{\Rightarrow} \quad \frac{az^{-1}}{1 + az^{-1}} \leftrightarrow a(-a)^{n-1} u[n-1] = nx[n] \quad \Rightarrow \ x[n] = (-1)^{n-1}\frac{a^n}{n}u[n-1]$$

$$\Rightarrow \qquad X(z) = \log\left(1 + az^{-1}\right) \quad |z| > a \quad \leftrightarrow \quad x[n] = (-1)^{n-1}\frac{a^n}{n}u[n-1]$$

Ex: Find the sequence $x[n] \leftrightarrow X(z) = \dfrac{1}{(1-az^{-1})^3} \qquad |z| > a$

$$a^n u[n] \leftrightarrow \frac{1}{(1-az^{-1})} \qquad |z| > a$$

$$\Rightarrow \quad na^n u[n] \leftrightarrow \frac{az^{-1}}{(1-az^{-1})^2} \qquad |z| > a$$

$$\Rightarrow na^{n-1} u[n] \leftrightarrow \frac{z^{-1}}{(1-az^{-1})^2} \qquad |z| > a$$

$$\Rightarrow (n+1)a^n u[n+1] = (n+1)a^n u[n] \leftrightarrow \frac{1}{(1-az^{-1})^2} \qquad |z| > a$$

(take the derivative, multiply by $-z$)

$$\Rightarrow n(n+1)a^n u[n+1] \leftrightarrow \frac{2az^{-1}}{(1-az^{-1})^3} \qquad |z| > a$$

$$\Rightarrow \frac{1}{2}(n+1)(n+2)a^n u[n+2] = \frac{1}{2}(n+1)(n+2)a^n u[n] \leftrightarrow \frac{1}{(1-az^{-1})^3} \qquad |z| > a$$

In general, $\Rightarrow \dfrac{(n+K-1)!}{n!\,(K-1)!} a^n u[n] \leftrightarrow \dfrac{1}{(1-az^{-1})^k} \qquad |z| > a$

5) Conjugation of *x*[*n*]

$x[n] \leftrightarrow X(z) \Leftrightarrow x^*[n] \leftrightarrow X^*(z^*)$ ROC does not change

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

$$X^*(z) = \sum_{n=-\infty}^{\infty} x^*[n](z^{-n})^*$$

$$X^*(z^*) = \sum_{n=-\infty}^{\infty} x^*[n]z^{-n}$$

Remember the corresponding “symmetry properties” of DTFT.

5) Time reversal of $x[n]$

$$x[n] \leftrightarrow X(z) \Leftrightarrow x[-n] \leftrightarrow X\left(\frac{1}{z}\right)$$

$$ROC_{x[-n]} = \frac{1}{ROC_{z[n]}}: \qquad a < |z| < b \Rightarrow \frac{1}{b} < |z| < \frac{1}{a}$$

<u>Ex</u>:

$$a^n u[n] \leftrightarrow \frac{1}{1-az^{-1}} \qquad |z| > a \qquad \Rightarrow \qquad a^{-n} u[-n] \leftrightarrow \frac{1}{1-az} \qquad |z| < \frac{1}{a}$$

6) Convolution

$$x[n] \leftrightarrow X(z) \qquad ROC_x$$
$$y[n] \leftrightarrow Y(z) \qquad ROC_y$$

$$x[n] * y[n] \leftrightarrow X(z)Y(z) \qquad ROC_x \cap ROC_y$$

SYSTEM FUNCTION

Let $h[n]$ be the impulse response of a LTI system. Then, the z-transform of $h[n]$, $H(z)$, is called the **system function**.

Remember that

$$e^{j\omega n} \longrightarrow \boxed{h[n]} \longrightarrow e^{j\omega n}\ H\left(e^{j\omega}\right)$$

$$\boxed{\text{If } H\left(e^{j\omega}\right) \text{ exists}}$$

Similarly if $x[n] = z^n$ then

$$\begin{aligned} y[n] &= z^n * h[n] \\ &= \sum_{k=-\infty}^{\infty} h[k] z^{n-k} \\ &= z^n \sum_{k=-\infty}^{\infty} h[k] z^{-k} \\ &= z^n H(z) \end{aligned}$$

$$\boxed{\text{If } \sum_{k=-\infty}^{\infty} h[k] z^{-k} \text{ converges}}$$

Therefore we can **generalize the concept of eigenfunction** for LTI systems.

Exponential functions, not only complex exponential functions, are eigenfunctions of LTI systems.

Stable systems:

$h[n]$ is absolutely summable.
ROC includes unit circle and DTFT of $h[n]$, $H\left(e^{j\omega}\right)$, exists.
$H\left(e^{j\omega}\right)$ is the frequency response function of the system.

Causal systems:

$h[n], \quad n < 0 \quad \Rightarrow$ There are no *z* terms with positive powers in $H(z)$.
$\Rightarrow$ ROC includes z → ∞.

Also, $h[n]$ is right –sided $\Rightarrow$ ROC is outside of the circular boundary determined by the pole with the largest magnitude.

Causal and Stable Systems:

$\Rightarrow$ All poles are inside the unit circle.

LCCDEs AND SYSTEM FUNCTIONS

$$y[n] = x[n] * h[n] \leftrightarrow Y(z) = X(z)H(z) \qquad \Rightarrow H(z) = \frac{Y(z)}{X(z)}$$

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k] \leftrightarrow \sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z)$$

$$\Rightarrow Y(z) \sum_{k=0}^{N} a_k z^{-k} Y(z) = X(z) \sum_{k=0}^{M} b_k z^{-k}$$

$$\Rightarrow H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

$$H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \ldots + b_M z^{-M}}{a_0 + a_1 z^{-1} + a_2 z^{-2} + \ldots + a_N z^{-N}}$$

ROC of the system function obtained from a LCCDE has to be determined according to the information about the causality of the system. Remember that such information must be provided independently for a complete characterization of the system.

Ex: $y[n]-\frac{1}{6}y[n-1]-\frac{1}{6}y[n-2]=x[n]+x[n-1]-\frac{1}{3}x[n-2]$

It is known that the system is causal, find the system function and its ROC.

$$Y(z)\left(1-\frac{1}{6}z^{-1}-\frac{1}{6}z^{-2}\right)=X(z)\left(1+z^{-1}-\frac{1}{3}z^{-2}\right)$$

$$H(z)=\frac{Y(z)}{X(z)}=\frac{1+z^{-1}-\frac{1}{3}z^{-2}}{1-\frac{1}{6}z^{-1}-\frac{1}{6}z^{-2}}$$

$$H(z)=K+\frac{a+bz^{-1}}{1-\frac{1}{6}z^{-1}-\frac{1}{6}z^{-2}}=2+\frac{1}{1-\frac{1}{2}z^{-1}}-\frac{2}{1+\frac{1}{3}z^{-1}}$$

Since it is given that the system is causal, impulse response satisfies $h[n]=0, \quad n<0$ and therefore it is right sided. There are two poles at $\frac{1}{2}$ and $-\frac{1}{3}$.

So ROC is $|z|>\frac{1}{2}$.

Furthermore impulse response is

$$h[n]=2\delta[n]+\left(\frac{1}{2}\right)^n u[n]-2\left(-\frac{1}{3}\right)^n u[n]$$

INVERSE Z-TRANSFORM

A sequence can be obtained from its z-transform as

$$x[n]=\frac{1}{2\pi j}\oint_C X(z)z^{n-1}dz$$

This is a contour integral over the complex plane.

The contour is in the ROC of $X(z)$ and it is traced in the counter clockwise direction.

Remember that *X*(*z*) is the DTFT of $x[n]r^{-n}$. In particular, the contour integral above can be obtained by writing the inverse DTFT of $X(re^{j\omega})$.

The contour integral can be evaluated using the "Cauchy's residue theorem":

$$x[n]=\sum\left[\text{residues of } G(z)z^{n-1} \text{ at the poles inside } C\right]$$

INVERSE TRANSFORM BY PARTIAL FRACTION EXPANSION

Let

$$X(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

Assuming that $M < N$, and all poles are simple,

$$X(z) = \frac{b_0}{a_0} \frac{\prod_{k=1}^{M}\left(1 - c_k z^{-1}\right)}{\prod_{k=1}^{N}\left(1 - d_k z^{-1}\right)}$$

Neglecting the constant factor, *X*(*z*) can be written as

$$X(z) = \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}} \ ,$$

where

$$A_k = \left(1 - d_k z^{-1}\right) X(z)\Big|_{z=d_k}$$

If $M \geq N$ and all poles are simple,

$$X(z) = q_0 z^{M-N} + q_1 z^{M-N-1} + \ldots + q_{M-N} + \frac{\sum_{k=0}^{N-1} \tilde{b}_k z^{-k}}{\sum_{k=0}^{N} \tilde{a}_k z^{-k}},$$

then, partial fraction expansion is applied to the last term.

If there is a pole, d_i, of ordes $s > 1$

$$X(z) = \sum_{k=1}^{N-s} \frac{A_k}{1 - d_k z^{-1}} + \sum_{k=1}^{s} \frac{C_k}{(1 - d_i z^{-1})^m}$$

where

$$C_k = \frac{1}{(s-m)!\,(-d_i)^{s-m}} \left\{ \frac{d^{s-m}}{dy^{s-m}} \left((1 - d_i y)^s X(y^{-1}) \right) \right\} \Big|_{y = d_i^{-1}}$$

Ex: Find $x[n]$ if $X(z)=\dfrac{4+11z^{-1}-11z^{-2}}{2-3z^{-1}-3z^{-2}+2z^{-3}}$.

$$X(z)=2\,\frac{1+\frac{11}{4}z^{-1}-\frac{11}{4}z^{-2}}{1-\frac{3}{2}z^{-1}-\frac{3}{2}z^{-2}+z^{-3}}$$

$$=2\left(\frac{1/2}{1-\frac{1}{2}z^{-1}}-\frac{1}{1+z^{-1}}+\frac{3/2}{1-2z^{-1}}\right)$$

$$=\frac{1}{1-\frac{1}{2}z^{-1}}-\frac{2}{1+z^{-1}}+\frac{3}{1-2z^{-1}}$$

$x[n]$ depends on ROC.

There are three poles at $z=\frac{1}{2}$, $z=-1$, $z=2$.

There are four possibilities:

a) If $ROC:\ |z|>2$ Then $x[n]$ is right-sided.

$$x[n]=\left(\frac{1}{2}\right)^n u[n]-2(-1)^n u[n]+3\left(2^n\right)u[n]$$

b) If $ROC:\ |z|<\frac{1}{2}$ Then $x[n]$ is left-sided.

$$x[n]=-\left(\frac{1}{2}\right)^n u[-n-1]+2(-1)^n u[-n-1]-3\left(2^n\right)u[-n-1]$$

c) If $ROC:\ 1<|z|<2$ Then $x[n]$ is two-sided.

$$x[n]=\left(\frac{1}{2}\right)^n u[n]-2(-1)^n u[n]-3\left(2^n\right)u[-n-1]$$

d) If $ROC:\ \frac{1}{2}<|z|<1$ Then $x[n]$ is two-sided.

$$x[n]=\left(\frac{1}{2}\right)^n u[n]+2(-1)^n u[-n-1]-3\left(2^n\right)u[-n-1]$$

<u>Ex</u>: What is $x[n]$ if $X(z) = \dfrac{3z^{-3}}{\left(1 - \frac{1}{4}z^{-1}\right)^2}$ and $x[n]$ is left sided ?

We know that $$-na^n u[-n-1] \leftrightarrow \frac{az^{-1}}{\left(1-az^{-1}\right)^2} \qquad |z| < |a|$$

$$X(z) = 12z^{-2}\frac{\frac{1}{4}z^{-1}}{\left(1-\frac{1}{4}z^{-1}\right)^2} \qquad |z| < \frac{1}{4} \Rightarrow x[n] = -12(n-2)\left(\frac{1}{4}\right)^{n-2} u[-(n-2)-1]$$

<u>Ex</u>: What is $x[n]$ if $X(z) = \dfrac{z^7 - 2}{\left(1 - z^{-7}\right)} \qquad |z| > 1$?

$$X(z) = \frac{z^7 - 2}{\left(1-z^{-7}\right)} = z^7 - \frac{1}{\left(1-z^{-7}\right)} = z^7 - \sum_{m=0}^{\infty} z^{-7m} \leftrightarrow \delta[n+7] - \sum_{m=0}^{\infty}\delta[n-7m]$$

Ex: What is $x[n]$ if $X(z)=\dfrac{\left(1+z^{-1}\right)^2}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-z^{-1}\right)} \qquad |z|>1$?

$$X(z)=B_0+\frac{A_1}{\left(1-\frac{1}{2}z^{-1}\right)}+\frac{A_2}{\left(1-z^{-1}\right)}=2+\frac{-9}{\left(1-\frac{1}{2}z^{-1}\right)}+\frac{8}{\left(1-z^{-1}\right)}$$

$$\Rightarrow x[n]=2\delta[n]-9\left(\frac{1}{2}\right)^n u[n]+8u[n]$$

Ex:

$$X(z)=\frac{z^{-1}+2}{z^{-2}-2z^{-1}-3} \qquad \frac{1}{3}<|z|<1$$

$$=\frac{z^{-1}+2}{\left(z^{-1}+1\right)\left(z^{-1}-3\right)}$$

$$=\frac{-\frac{1}{3}z^{-1}-\frac{2}{3}}{\left(1+z^{-1}\right)\left(1-\frac{1}{3}z^{-1}\right)}$$

$$=\frac{A}{\left(1+z^{-1}\right)}+\frac{B}{\left(1-\frac{1}{3}z^{-1}\right)} \qquad \frac{1}{3}<|z|<1 \qquad \left(A=\left(1+z^{-1}\right)X(z)\Big|_{z=-1}=-\frac{1}{4} \qquad B=\left(1-\frac{1}{3}z^{-1}\right)X(z)\Big|_{z=\frac{1}{3}}=-\frac{5}{12}\right)$$

$$\Leftrightarrow x[n]=-A\,(-1)^n\,u[-n-1]\;+\;B\left(\frac{1}{3}\right)^n u[n]$$

Ex:

$$X(z)=\frac{z^{-1}+2}{z^{-3}-5z^{-2}+3z^{-1}+9} \qquad \frac{1}{3}<|z|<1$$

$$=\frac{z^{-1}+2}{\left(z^{-1}+1\right)\left(z^{-1}-3\right)^2}$$

$$=\frac{\frac{1}{9}z^{-1}+\frac{2}{9}}{\left(1+z^{-1}\right)\left(1-\frac{1}{3}z^{-1}\right)^2}$$

$$=\frac{A}{\left(1+z^{-1}\right)}+\frac{B}{\left(1-\frac{1}{3}z^{-1}\right)}+\frac{C}{\left(1-\frac{1}{3}z^{-1}\right)^2} \qquad \frac{1}{3}<|z|<1$$

$$\left(A=\left(1+z^{-1}\right)\ X(z)\Big|_{z=-1}=\frac{1}{16} \qquad C=\left(1-\frac{1}{3}z^{-1}\right)^2\ X(z)\Big|_{z=\frac{1}{3}}=\frac{5}{36} \qquad B=-3\ \frac{d}{d\left(z^{-1}\right)}\left(1-\frac{1}{3}z^{-1}\right)^2\ X(z)\Big|_{z=\frac{1}{3}}=\frac{1}{48}\right)$$

$$\Leftrightarrow \qquad x[n]=-A\ (-1)^n\ u[-n-1]\ +\ B\left(\frac{1}{3}\right)^n u[n]\ +\ \underbrace{C\frac{1}{1/3}\ (n+1)\left(\frac{1}{3}\right)^{n+1} u[n+1]}_{C\ (n+1)\left(\frac{1}{3}\right)^n u[n+1]}$$

Ex:

$$X(z)=\frac{z^{-4}-2z^{-3}-3z^{-2}+\ z^{-1}+2}{z^{-2}-2z^{-1}-3} \qquad \frac{1}{3}<|z|<1$$

$$=\ z^{-2}\ +\ \frac{z^{-1}+2}{\left(z^{-1}+1\right)\left(z^{-1}-3\right)}$$

$$=\ z^{-2}\ +\frac{-\frac{1}{3}z^{-1}-\frac{2}{3}}{\left(1+z^{-1}\right)\left(1-\frac{1}{3}z^{-1}\right)}$$

$$=\ z^{-2}\ +\frac{A}{\left(1+z^{-1}\right)}+\frac{B}{\left(1-\frac{1}{3}z^{-1}\right)} \qquad \frac{1}{3}<|z|<1 \qquad \left(A=\left(1+z^{-1}\right)\ X(z)\Big|_{z=-1}=-\frac{1}{4} \qquad B=\left(1-\frac{1}{3}z^{-1}\right)\ X(z)\Big|_{z=\frac{1}{3}}=-\frac{5}{12}\right)$$

$$\Leftrightarrow\ x[n]=\delta[n-2]\ -\ A\ (-1)^n\ u[-n-1]\ +\ B\left(\frac{1}{3}\right)^n u[n]$$

INVERSION BY LONG DIVISION

Ex: $X(z) = \dfrac{1}{1 - az^{-1}}$ $\quad |z| > a$

$$
\begin{array}{l|l}
1 & 1 - az^{-1} \\
\hline
1 - az^{-1} & \underbrace{1 + az^{-1} + a^2 z^{-2} + a^3 z^{-3} \ldots}_{x[n]=a^n u[n]} \\
az^{-1} & \\
az^{-1} - a^2 z^{-2} & \\
a^2 z^{-2} &
\end{array}
$$

Ex: $X(z) = \dfrac{1}{1 - az^{-1}}$ $\quad |z| < a$

$$
\begin{array}{l|l}
1 & -az^{-1} + 1 \\
\hline
1 - az^{-1} & \underbrace{-a^{-1}z - a^{-2}z^2 - a^{-3}z^3 \ldots}_{x[n]=-a^n u[-n-1]} \\
a^{-1}z & \\
a^{-1}z - a^{-2}z^2 & \\
a^{-2}z^2 &
\end{array}
$$

Z-TRANSFORM PAIRS

$\delta[n]$	$\leftrightarrow$	1	*entire z−plane*
$u[n]$	$\leftrightarrow$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert>1$
$-u[-n-1]$	$\leftrightarrow$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert<1$
$\delta[n-m]$	$\leftrightarrow$	z^{-m}	*entire z−plane except origin (if* $m>0$*) or except* ∞ *(if* $m<0$*)*
$a^n u[n]$	$\leftrightarrow$	$\dfrac{1}{1-a\,z^{-1}}$	$\lvert z\rvert>\lvert a\rvert$
$-a^n u[-n-1]$	$\leftrightarrow$	$\dfrac{1}{1-a\,z^{-1}}$	$\lvert z\rvert<\lvert a\rvert$
$n\ \ a^n u[n]$	$\leftrightarrow$	$\dfrac{a\,z^{-1}}{\left(1-a\,z^{-1}\right)^2}$	$\lvert z\rvert>\lvert a\rvert$ $\quad \left((n+1)\ \ a^n\ \ u[n+1] \leftrightarrow \dfrac{1}{\left(1-a\,z^{-1}\right)^2}\quad \lvert z\rvert>\lvert a\rvert\right)$
$-n\ \ a^n u[-n-1]$	$\leftrightarrow$	$\dfrac{a\,z^{-1}}{\left(1-a\,z^{-1}\right)^2}$	$\lvert z\rvert<\lvert a\rvert$
$\cos(\omega_0 n)\ \ u[n]$	$\leftrightarrow$	$\dfrac{1-\cos(\omega_0)\ z^{-1}}{1-2\cos(\omega_0)\ z^{-1}+z^{-2}}$	$\lvert z\rvert>1$ $\quad \left(\begin{array}{l}\cos\ (\omega_0 n)\ \ u[n]=\frac{1}{2}\left(e^{j\omega_0 n}u[n]+e^{-j\omega_0 n}u[n]\right)\\ use\quad a^n u[n]\quad with\quad a=e^{j\omega_0}\end{array}\right)$
$\sin(\omega_0 n)\ \ u[n]$	$\leftrightarrow$	$\dfrac{\sin(\omega_0)\ z^{-1}}{1-2\cos(\omega_0)\ z^{-1}+z^{-2}}$	$\lvert z\rvert>1$
$r^n\cos(\omega_0 n)\ \ u[n]$	$\leftrightarrow$	$\dfrac{1-r\cos(\omega_0)\ z^{-1}}{1-2r\cos(\omega_0)\ z^{-1}+r^2z^{-2}}$	$\lvert z\rvert>1$ $\quad \left(\begin{array}{l}r^n\cos\ (\omega_0 n)\ \ u[n]=\frac{1}{2}r^n\left(e^{j\omega_0 n}u[n]+e^{-j\omega_0 n}u[n]\right)\\ use\quad a^n u[n]\quad with\quad a=re^{j\omega_0}\end{array}\right)$
$r^n\sin(\omega_0 n)\ \ u[n]$	$\leftrightarrow$	$\dfrac{r\sin(\omega_0)z^{-1}}{1-2r\cos(\omega_0)\ z^{-1}+r^2z^{-2}}$	$\lvert z\rvert>1$

MATLAB LINEAR SYSTEM TRANSFORMATIONS

latc2tf - Lattice or lattice ladder to transfer function conversion
polyscale - Scale roots of polynomial
polystab - Polynomial stabilization
<u>residuez - Z-transform partial fraction expansion</u>
sos2ss - Second-order sections to state-space conversion
sos2tf - Second-order sections to transfer function conversion
sos2zp - Second-order sections to zero-pole conversion
ss2sos - State-space to second-order sections conversion
ss2tf - State-space to transfer function conversion
ss2zp - State-space to zero-pole conversion
tf2latc - Transfer function to lattice or lattice ladder conversion
<u>tf2sos - Transfer Function to second-order sections conversion</u>
tf2ss - Transfer function to state-space conversion
<u>tf2zpk - Discrete-time transfer function to zero-pole conversion</u>
<u>zp2sos - Zero-pole to second-order sections conversion</u>
zp2ss - Zero-pole to state-space conversion
<u>zp2tf - Zero-pole to transfer function conversion</u>

RESIDUEZ Z-transform partial-fraction expansion.
[R,P,K] = RESIDUEZ(B,A) finds the residues, poles and direct terms of the partial-fraction expansion of B(z)/A(z),

```
 B(z)      r(1)             r(n)
 ---- = ------------ +...  ------------ + k(1) + k(2)z^(-1) ...
 A(z)   1-p(1)z^(-1)      1-p(n)z^(-1)
```

B and A are the numerator and denominator polynomial coefficients, respectively, in ascending powers of z^(-1). R and P are column vectors containing the residues and poles, respectively. K contains the direct terms in a row vector. The number of poles is

n = length(A)-1 = length(R) = length(P)

The direct term coefficient vector is empty if length(B) < length(A); otherwise,

length(K) = length(B)-length(A)+1

If P(j) = ... = P(j+m-1) is a pole of multiplicity m, then the expansion includes terms of the form

```
      R(j)            R(j+1)                  R(j+m-1)
 -------------- + ------------------   + ... + ------------------
 1 - P(j)z^(-1)   (1 - P(j)z^(-1))^2          (1 - P(j)z^(-1))^m
```

[B,A] = RESIDUEZ(R,P,K) converts the partial-fraction expansion back to B/A form.

Warning: Numerically, the partial fraction expansion of a ratio of polynomials represents an ill-posed problem. If the denominator polynomial, A(s), is near a polynomial with multiple roots, then small changes in the data, including roundoff errors, can make arbitrarily large changes in the resulting poles and residues. Problem formulations making use of state-space or zero-pole representations are preferable.

TF2SOS Transfer Function to Second Order Section conversion.

[SOS,G] = TF2SOS(B,A) finds a matrix SOS in second-order sections form and a gain G which represent the same system H(z) as the one with numerator B and denominator A. The poles and zeros of H(z) must be in complex conjugate pairs.

SOS is an L by 6 matrix with the following structure:

```
SOS = [ b01 b11 b21  1 a11 a21
        b02 b12 b22  1 a12 a22
        ...
        b0L b1L b2L  1 a1L a2L ]
```

Each row of the SOS matrix describes a 2nd order transfer function:

```
          b0k +  b1k z^-1 +  b2k  z^-2
Hk(z) =  ----------------------------
           1 +  a1k z^-1 +  a2k  z^-2
```

where k is the row index.

G is a scalar which accounts for the overall gain of the system. If G is not specified, the gain is embedded in the first section.

The second order structure thus describes the system H(z) as:

H(z) = G*H1(z)*H2(z)*...*HL(z)

NOTE: Embedding the gain in the first section when scaling a direct-form II structure is not recommended and may result in erratic scaling. To avoid embedding the gain, use tf2sos with two outputs.

TF2SOS(B,A,DIR_FLAG) specifies the ordering of the 2nd order sections. If DIR_FLAG is equal to 'UP', the first row will contain the poles closest to the origin, and the last row will contain the poles closest to the unit circle. If DIR_FLAG is equal to 'DOWN', the sections are ordered in the opposite direction. The zeros are always paired with the poles closest to them. DIR_FLAG defaults to 'UP'.

TF2SOS(B,A,DIR_FLAG,SCALE) specifies the desired scaling of the gain and the numerator coefficients of all 2nd order sections. SCALE can be either 'NONE', Inf or 2 which correspond to no scaling, infinity norm scaling and 2-norm scaling respectively. SCALE defaults to 'NONE'. The filter must be stable in order to scale in the 2-norm or inf-norm sense. Using infinity-norm scaling in conjunction with 'UP' ordering will minimize the probability of overflow in the realization. On the other hand, using 2-norm scaling in conjunction with 'DOWN' ordering will minimize the peak roundoff noise.

```
TF2ZPK  Discrete-time transfer function to zero-pole conversion.
   [Z,P,K] = TF2ZPK(NUM,DEN)  finds the zeros, poles, and gain:

                (z-Z(1))(z-Z(2))...(z-Z(n))
      H(z) =  K ---------------------------
                (z-P(1))(z-P(2))...(z-P(n))

   from a single-input, single-output transfer function in polynomial form:

              NUM(z)
      H(z) = --------
              DEN(z)

   EXAMPLE:
     [b,a] = butter(3,.4);
     [z,p,k] = tf2zpk(b,a)

   See also tf2zp, zplane.

   Reference page in Help browser
      doc tf2zpk
```

ZP2SOS(Z,P,K,DIR_FLAG,SCALE) specifies the desired scaling of the gain and the numerator coefficients of all 2nd order sections. SCALE can be either 'NONE', Inf or 2 which correspond to no scaling, infinity norm scaling and 2-norm scaling respectively. SCALE defaults to 'NONE'. The filter must be stable in order to scale in the 2-norm or inf-norm sense. Using infinity-norm scaling in conjunction with 'UP' ordering will minimize the probability of overflow in the realization. On the other hand, using 2-norm scaling in conjunction with 'DOWN' ordering will minimize the peak roundoff noise.

ZP2SOS(Z,P,K,DIR_FLAG,SCALE,KRZFLAG) specifies whether or not to keep real zeros that are the negative of each other together rather than ordering according to their proximity to poles. If KRZFLAG is true, this is done and the result is a numerator with a middle coefficient equal to zero. The default is false.

NOTE: Infinity-norm and 2-norm scaling are appropriate only for direct form II structures.

See also tf2sos, sos2zp, sos2tf, sos2ss, ss2sos, cplxpair.

Reference page in Help browser
doc zp2sos

ZP2TF Zero-pole to transfer function conversion.

[NUM,DEN] = ZP2TF(Z,P,K) forms the transfer function:

```
        NUM(s)
H(s) = --------
        DEN(s)
```

given a set of zero locations in vector Z, a set of pole locations in vector P, and a gain in scalar K. Vectors NUM and DEN are returned with numerator and denominator coefficients in descending powers of s.

See also tf2zp.