Middle East Technical University
Dept. of EEE

HW 2

EE 430 Digital Signal Processing – 2014 Fall

Due: Oct. 31, 2013

Submit for the first 9 problems. (29 in total)

1) The following difference equation for a causal LTI system is given,

$$y[n]-\frac{1}{2}y[n-1]=x[n]-x[n-1]+x[n-2]$$

a) Find the impulse response, $h[n]$.

b) Find the frequency response, $H(e^{j\omega})$.

c) Plot the magnitude response, $|H(e^{j\omega})|$ and phase response, $\measuredangle H(e^{j\omega})$, in MATLAB using “freqz” command.

d) Let $x[n]=\cos\left(\frac{\pi}{3}n\right)+\sin\left(\frac{\pi}{2}n+\frac{\pi}{4}\right)$ be the system input. Find the output $y[n]$.

e) Prove that $H(e^{j\omega})=H^*(e^{j(2\pi-\omega)})$. Does this equality hold for an arbitrary $h[n]$? Explain.

2) Find the DTFT of the sequence $x[n]=na^{n-2}u[n-2]$.

3) Let $x_1[n]=\sin\left(\frac{\pi}{7}n\right)+\sin\left(\frac{\pi}{3}n\right)$ and $x_2[n]=x_1[n]u[n]$. Find the responses, $y_1[n]$ and $y_2[n]$ to $x_1[n]$ and $x_2[n]$, respectively, of the following systems.

a) A moving average system with impulse response $h[n]=[\frac{1}{2}\ \ \frac{1}{2}]$ for $n=[0\ \ 1]$.
b) A moving average system with impulse response $h[n]=[\frac{1}{2}\ \ \frac{1}{2}]$ for $n=[2\ \ 3]$.
c) A moving average system with impulse response $h[n]=[\frac{1}{6}\ \ \frac{1}{6}\ \ \frac{1}{6}\ \ \frac{1}{6}\ \ \frac{1}{6}\ \ \frac{1}{6}]$ for $n=[0\ \ 1\ \ 2\ \ 3\ \ 4\ \ 5]$.

Compare $y_1[n]$ to $y_2[n]$ in each case; can you identify an interval on which $y_1[n]=y_2[n]$. How is that interval related to the particular impulse response?

4) In the $[-\pi,\pi)$ interval, the DTFT, $X(e^{j\omega})$, of a real sequence $x[n]$ is nonzero only in $[a,b], a<0<b, b-a=\frac{\pi}{2}$; otherwise it is arbitrary.

a) Plot the magnitude and phase of a candidate $X(e^{j\omega})$.
b) Express the DTFTs, $X_C(e^{j\omega})$ and $X_S(e^{j\omega})$, respectively, of $\cos\left(\frac{\pi}{5}n\right)x[n]$ and $\sin\left(\frac{\pi}{5}n\right)x[n]$ in terms of $X(e^{j\omega})$.
c) Assume that $X(e^{j\omega})$ is real valued and still complies with the specifications above. Plot the magnitudes and phases of $X_C(e^{j\omega})$ and $X_S(e^{j\omega})$ using your $X(e^{j\omega})$ in part-a.

5) Find $\int_{-\pi}^{0}\left|\frac{\sin\left(\frac{5}{2}\omega\right)}{\sin\left(\frac{\omega}{2}\right)}\right|^2 d\omega$.

6) Show that the response of a LTI system to a complex exponential $x[n]=e^{j\omega_0 n}$ is $H(e^{j\omega_0})\,e^{j\omega_0 n}$ by taking the inverse DTFT of $Y(e^{j\omega})=H(e^{j\omega})X(e^{j\omega})$ where $X(e^{j\omega})$ is the DTFT of $x[n]=e^{j\omega_0 n}$.

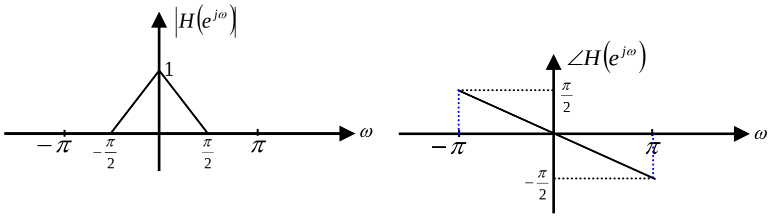
7) Impulse responses of some LTI systems are given below. Let $x[n] = 3^n$ be the input signal of these systems. Determine those systems for which their output signals can be expressed as $y[n] = C\ 3^n$ where C is a complex constant. Explain formally. **Submit parts-a and -b only.**

a) $h[n] = \left(\frac{1}{2}\right)^n u[n] + 2^n u[n]$
b) $h[n] = \left(\frac{1}{2}\right)^n u[n] + 2^n u[-n]$
c) $h[n] = 5^n u[-n]$
d) $h[n] = 3^n u[n]$

8) Find the impulse responses of the stable LTI systems having the following system functions. Which of them are causal? Plot the pole-zero diagrams and show the ROCs. **Submit part-a only.**

a) $H(z) = \frac{2z^{-1}+1}{\left(1-\frac{1}{4}z^{-1}\right)(1+z^{-1})\left(1+\frac{1}{4}z^{-1}\right)}$
b) $H(z) = \frac{z-4}{(1-3z^{-1})(1-5z^{-1})}$
c) $H(z) = \frac{z^{-1}}{\left(1-\frac{1}{4}z^{-1}\right)\left(1-\frac{1}{3}z^{-1}\right)}$
d) $H(z) = \frac{1}{\left(1-\frac{1}{2}z^{-1}\right)^3}$

9) The magnitude and the phase of the frequency response of a LTI system are given. Find the output signal of this system for each of the following input signals. **Submit part-c only.**



a. $x[n] = \cos(1.6\pi n)$
b. $x[n] = \sin(30.4\pi n)$
c. $x[n] = 3 + j5 + \sin(0.25\pi n)$

10) The impulse response of a LTI system is

$$h[n] = \delta[n] - \sqrt{2}\delta[n-1] + \delta[n-2].$$

a) Find the system function $H(z)$. Plot the pole-zero diagram, indicate ALL poles and zeros, show the ROC.
b) Does this system have a frequency response? Why? If yes, plot its magnitude and phase.
c) Find the output of this system to the following input signals

$$x_1[n] = \sin\left(\frac{\pi}{4}n + \frac{\pi}{4}\right)$$

$$x_2[n] = \sin\left(\frac{\pi}{4}n + \frac{\pi}{4}\right)u[n]$$

$$x_3[n] = \sin\left(\frac{\pi}{4}n + \frac{\pi}{4}\right) + \sin\left(\frac{3\pi}{4}n\right)$$

d) Comment on the relationship between the frequency response and zero locations of $H(z)$.

11) The system function of a LTI system is

$$H(z) = \frac{1 - \sqrt{2}z^{-1} + z^{-2}}{1 - 2z^{-1}}.$$

When the input is $\sin\left(\frac{\pi}{2}n\right)$, the output of this system is $\sqrt{\frac{2}{5}}\sin\left(\frac{\pi}{2}n + \tan^{-1}\frac{1}{2}\right)$.

a) Find the impulse response of this system.
b) Is the system causal?
c) Find the difference equation for this system.

12) Prove the modulation/windowing property of DTFT.

13) Let

$$x[n] = \cos\left(\frac{\pi}{4}n\right)$$

$$y[n] = w[n]x[n]$$

a) Find the DTFT, $Y(e^{j\omega})$, of $y[n]$ in terms of $W(e^{j\omega})$ using the modulation/windowing property of DTFT.
b) Let

$$w[n] = \begin{cases} 1 & n \in [-M, M] \\ 0 & otherwise \end{cases}$$

Find $W(e^{j\omega})$ and plot (MATLAB) its magnitude in $\omega \in [-\pi, \pi]$ for $M = 3, 10, 50$.

c)
   i. Plot (MATLAB) $|Y(e^{j\omega})|$ in $\omega \in [-\pi, \pi]$ for $M = 3, 10, 50$.
   ii. Note the frequencies at which the peak of $|Y(e^{j\omega})|$ is observed for $M = 3, 10, 50$. Are they the same? Comment on your observation and explain why it is so.

d) Let

$$x[n] = \cos\left(\frac{\pi}{4}n\right) + \cos\left(\frac{\pi}{30}n\right)$$

   i. Plot (MATLAB) $|Y(e^{j\omega})|$ in $\omega \in [-\pi, \pi]$ for $M = 3, 10, 50$.
   ii. Describe the differences you observe in the three plots. Explain the reasons of these differences.

e) Based on your experience in this item, comment on the results of items (e) and (f) of Homework 2.

14) The z-transform, $X(z)$, of a sequence $x[n]$ exists for $z = 4e^{j\pi}$. Show that $X(z)$ exists for $z = 4e^{j\frac{2\pi}{7}}$ and in general for $z = 4e^{j\omega},\ \ 0 \le \omega < 2\pi$.

15) The z-transform, $X(z)$, of a right –sided sequence $x[n]$ exists for $z = 4e^{j\omega},\ \ 0 \le \omega < 2\pi$. Show that $X(z)$ exists for $= 4.1e^{j\omega},\ \ 0 \le \omega < 2\pi$, but not necessarily for $z = 3.9e^{j\omega},\ \ 0 \le \omega < 2\pi$.

16) Find (if exists) the outputs of the systems in question-7 to the inputs $x_1[n] = \cos\left(\frac{\pi}{2}n\right)$ and $x_2[n] = \cos\left(\frac{\pi}{2}n\right)u[n]$.

17) Find the impulse responses of the stable LTI systems having the following system functions. Which of them are causal? Plot the pole-zero diagrams and show the ROCs.

e) $H(z) = \frac{2z^{-1}+1}{\left(1-\frac{1}{4}z^{-1}\right)(1+z^{-1})\left(1+\frac{1}{4}z^{-1}\right)}$
f) $H(z) = \frac{z-4}{(1-3z^{-1})(1-5z^{-1})}$
g) $H(z) = \frac{z^{-1}}{\left(1-\frac{1}{4}z^{-1}\right)\left(1-\frac{1}{3}z^{-1}\right)}$
h) $H(z) = \frac{1}{\left(1-\frac{1}{2}z^{-1}\right)^3}$

18) What the ROCs of the z-transforms of the following sequences?
a) $x[n] = \delta[n+3] + \delta[n-3]$
b) $x[n] = \delta[n+3]$
c) $x[n] = \delta[n-3]$

19) Let $x[n] = \delta[n+1] + \left(\frac{1}{2}\right)^n u[n]$. Find the z-transforms of the following sequences. What are the ROCs? State all poles and zeros.

a) $x[n]$
b) $x[n-5]$
c) $nx[n]$
d) $\cos\left(\frac{\pi}{2}n\right)x[n]$

20) Let the frequency response of a LTI system be

$$H(e^{j\omega}) = \begin{cases} 1 & -\frac{\pi}{10} < \omega < \frac{\pi}{10} \\ 0 & otherwise \end{cases}$$

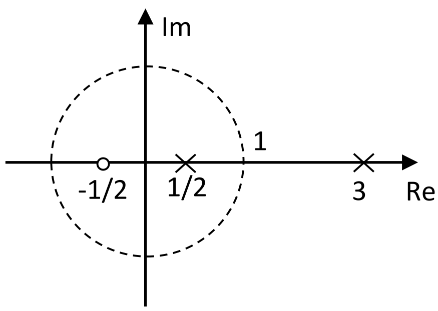
a) Find the impulse response, $h[n]$, of this system. Is it a causal one?
b) Let $h_1[n] = h[n]\cos\left(\frac{\pi}{4}n\right)$. Find and plot $H_1(e^{j\omega})$.

21) Impulse responses of some LTI systems are given. Find the difference equation that represents the input-output relationship of each system.

a) $h[n] = \left(\frac{1}{2}\right)^n u[n]$
b) $h[n] = -\left(\frac{1}{2}\right)^n u[-n-1]$
c) $h[n] = \left(\frac{1}{2}\right)^{n-1} u[n-1]$
d) $h[n] = \left(\frac{1}{2}\right)^n u[n-4]$
e) $h[n] = \left(\frac{1}{2}\right)^n u[n] + 2^n u[n]$

22) The pole-zero plot of the system function, $H(z)$, of a <u>stable</u> LTI system is shown. It is known that $H(1) = 1$.

a. Show the ROC. Determine the impulse response $h[n]$.
b. Let $h_1[n] = h[-n+2]$. Sketch the pole-zero plot for $H_1(z)$ show its ROC.



23) The output of a stable LTI system is $y[n] = \delta[n+1] + 2\delta[n] + \delta[n-1]$ when its input is $x[n] = -2\delta[n+2] - 4\delta[n+1] + 4\delta[n-1] + 2\delta[n-2]$. Find its impulse response $h[n]$.

24) Evaluate

$$\sum_{n=-\infty}^{\infty} \left| \frac{\sin\left(\frac{\pi n}{4}\right)}{\pi n} \right|^2$$

25) MATLAB – Let

$$x[n] = [1\ 2\ 3\ 4\ 5\ 6\ 7\ 6\ 5\ 4\ 3\ 2\ 1] \quad \text{for } n = 0,1,\dots,12$$

and

$$h[n] = [\tfrac{1}{5}\ \tfrac{1}{5}\ \tfrac{1}{5}\ \tfrac{1}{5}\ \tfrac{1}{5}] \qquad \text{for } n = 0,1,\dots,4.$$

i.e, $h[n]$ is a moving average filter of length 5.

a) Generate x and h sequences (vectors) and convolve them; $\mathrm{y} = \mathrm{conv(h, x)}$. Plot $x[n]$ and $y[n]$ in the same panel. Comment on the effect of the filter $h[n]$ on $x[n]$.

b) Compute and plot the magnitudes of DTFTs of $x[n]$ and $y[n]$ as follows

```
X_mag =  abs(fft(x,1024));
Y_mag =  abs(fft(y,1024));
```

To plot in $[0,2\pi]$

```
w = 0:1023;
w = 2 * pi * w / 1024;
plot(w, X_mag);
hold
plot(w, Y_mag,'r');
```

To plot in $[-\pi, \pi]$

```
w = -512:511;
w =  pi * w / 512;
plot(w, fftshift(X_mag));
hold
plot(w, fftshift(Y_mag),'r');
```

Comment on the differences between X_mag and Y_mag.

26) MATLAB – In this item, the moving average filter of item-3 will be applied to the following (chirp) signal

$$x[n] = \cos\big((\omega_0 + k_\omega n)n\big), \quad n = 0,1,\dots 999$$

where

$$\omega_0 = 0.33\pi$$
$$\omega_f = 1.1\omega_0$$
$$k_\omega = \frac{(\omega_f - \omega_0)}{999} \; .$$

a) Generate x and h sequences (vectors) and convolve them; $y = \mathrm{conv}(\mathrm{h}, \mathrm{x})$..
b) Plot x and y sequences.
c) Plot the magnitude response of the moving average filter of length 5 in $[0, \pi]$.

You may use the code below:

```
clear all
close all

filt_length = 5;
h =  ones(1,filt_length) / filt_length;

N     = 1000         ;
w_0   = 0.33  * pi   ;
w_f   = w_0 * 1.1    ;
n     = 0:(N-1)      ;
k_w   = (w_f - w_0) / (N-1)   ;
x = cos((w_0 + k_w * n) .* n) ;

y = conv(h,x);

plot(x)
figure
plot(y)

[H,W] = freqz(h,1,1000);
figure
plot(W/pi,abs(H))
```

d) Now, set $\omega_0 = 0.4\pi$ and repeat parts a,b.
e) You can make your own trials with different filter lengths or other parameter values.
f) Comment on your results.

27) MATLAB - Using the relevant parts of the code in item-4, compare the magnitudes of the frequency responses of the filters that have the following impulse responses:

$$h_1[n] = [\tfrac{1}{5}\ \tfrac{1}{5}\ \tfrac{1}{5}\ \tfrac{1}{5}\ \tfrac{1}{5}] \quad \text{for } n = 0,1,\dots,4$$

$$h_2[n] = [\tfrac{1}{5}\ \tfrac{-1}{5}\ \tfrac{1}{5}\ \tfrac{-1}{5}\ \tfrac{1}{5}] \quad \text{for } n = 0,1,\dots,4$$

28) MATLAB – Generate

$$x[n] = \sin\left(\frac{2\pi}{9}n\right) + \cos\left(\frac{10\pi}{17}n\right), \qquad n = 0,1,\dots,152$$

a) What are the fundamental periods of the sinusoidal components.
b) Design a Butterwoth filter using the 'butter' command. Set the filter order to 8 and the cutoff frequency to $\frac{\pi}{4}$. The command is '[b,a]=butter(8 , 0.25)'. (Note that 0.25 corresponds to $\frac{\pi}{4}$. Explain why?)
This is an IIR filter. The coefficients of the numerator and denominator polynomials of its rational transfer function are in vectors b and a, respectively.

c) Filter $x[n]$ by the this Butterworth filter using the 'filter' command. (This time we do not use the 'conv' command. Explain why?)
d) Plot x and y (the output of the Butterworth filter) using both 'plot' and 'stem' commands.
e) Plot x and y on the same panel.
f) Plot the magnitude response of the Butterworth filter.
g) Comment on the effect of this filter on the input signal.
You may use the code below:

```
clear all
close all

n = 0:152;
x = sin(n*2*pi/9) + cos(n*10*pi/17);

[b,a] = butter(8,0.25);

y = filter(b,a,x);

plot(x);
figure
stem(x)

figure
plot(y)
figure
plot(y)

figure
plot(x)
hold
plot(y,'r')

[H,W] = freqz(b,a,1000);
figure
plot(W,abs(H))
figure
plot(W,20*log10(abs(H)))
```

29) MATLAB - Spectrogram is a convenient mathematical tool to observe the "time-varying" frequency content of signals. Spectrogram is closely related to Fourier transform.

a) Generate

$$x[n] = \begin{cases} \sin\left(\frac{2\pi}{9}n\right), & n = 0,1,\ldots,199 \\ \cos\left(\frac{10\pi}{17}n\right), & n = 200,\ldots,399 \\ \sin\left(\frac{2\pi}{9}n\right) + \cos\left(\frac{10\pi}{17}n\right), & n = 400,\ldots,699 \end{cases}$$

b) Plot and observe the spectrogram of $x[n]$ using "`spectrogram(x)`" command.
c) State your interpretation of the picture displayed.
d) Describe how the "spectrogram" command might have generated the observed pictures.
e) Now try "`spectrogram(x,50,0)`" and "`spectrogram(x,50,25)`. What are the roles of these parameters? How do they affect the spectrogram picture?
f) Now try "`spectrogram(x,10,0)`". State your observation on how the resulting picture differs from the previous ones.
You may use the code below:

```
clear all
close all

n1 = 0:199;
x1 = sin(n1*2*pi/9);
n2 = 200:399;
x2 = cos(n2*10*pi/17);
n3 = 400:799;
x3 = sin(n3*2*pi/9) + cos(n3*10*pi/17);

x = [x1 x2 x3];

plot(x)
figure
spectrogram(x)
figure
spectrogram(x,50,0)
figure
spectrogram(x,50,25)
figure
spectrogram(x,10,0)
```