Discrete-Time Signals

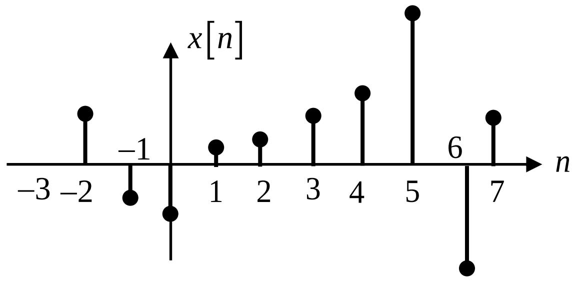
- Mathematical Representation of DT Signals
- Unit Sample Sequence
- Unit Step Sequence
- Exponential Sequences
- Sinusoidal Sequences
- Two Fundamental Properties of DT Sinusoidal Sequences

Discrete-Time Signals

A discrete-time signal is a sequence of numbers.

Its n^{th} element is $x[n], \; n \in Z$.

$x[n]$ may be real or complex.



Ex: Let T = 0.001 sec = 1 msec.

We do NOT write $\ldots, x[-0.001], x[0], x[0.001], x[0.002]\ldots!$

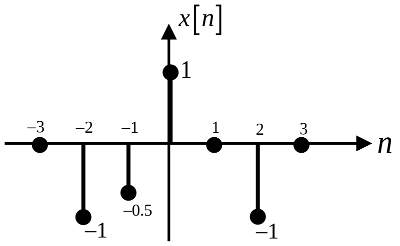
We write $\ldots, x[-1], x[0], x[1], x[2]\ldots$

$x[n]$ may have been obtained by sampling a continuous-time signal, i.e.,

$$x[n] = x_C(t)|_{t=nT} \, , \quad n \in Z$$

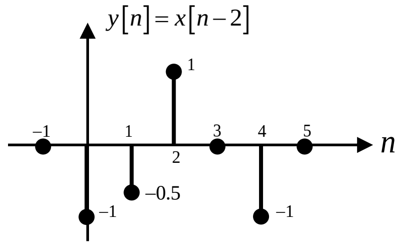
Time Shift of a Signal

Ex: Delay



$y[n] = x[n - n_0]$

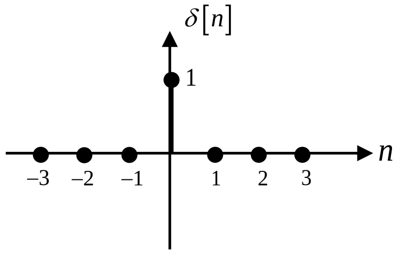
n_0 is always an integer!



We do NOT write sth. like $x[n - 2.15]$

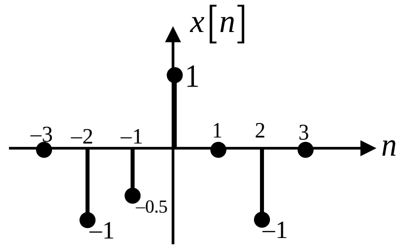
(continuous shift/interpolation…)

UNIT SAMPLE SEQUENCE



$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

Ex: Let $x[n]$ be



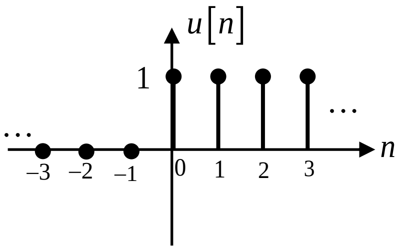
Can be written as:

$$x[n] = -\delta[n+2] - 0.5\delta[n+1] + \delta[n] - \delta[n-2]$$

In general, any seq. can be written as $\quad x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]$

This is the fundamental expression in the derivation of the fact that the output of a LTI system is the convolution of the input and the system's impulse response.

UNIT STEP SEQUENCE



$$u[n]=\begin{cases}1 & n\geq 0\\ 0 & n<0\end{cases}$$

<u>Ex</u>: $u[n]=\sum_{k=0}^{\infty}\delta[n-k]$ (convolution of $u[n]$ and $\delta[n]$)

or

<u>Ex</u>: $u[n]=\sum_{k=-\infty}^{n}\delta[k]$ (like integration in cont. time)

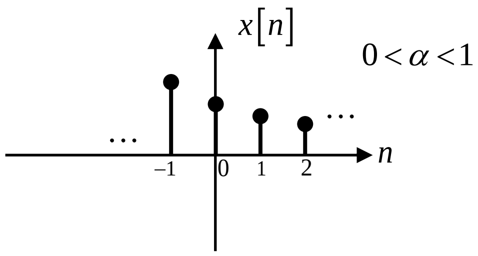
on the other hand

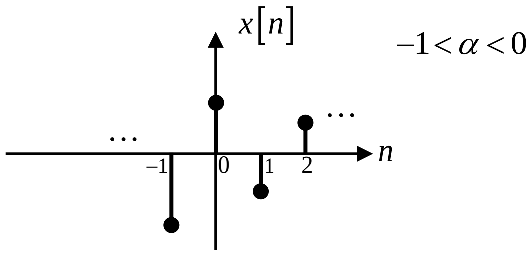
<u>Ex</u>: $\delta[n]=u[n]-u[n-1]$ (like differentation in cont. time)

EXPONENTIAL SEQUENCES (real valued)

They appear in the solution and analysis of LTI systems.

$$x[n] = A\alpha^n$$

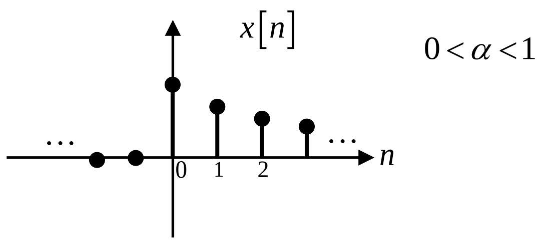




if $|\alpha|>1$ then $|x[n]|$ grows as $n \to \infty$

TRUNCATED EXPONENTIAL SEQUENCE:

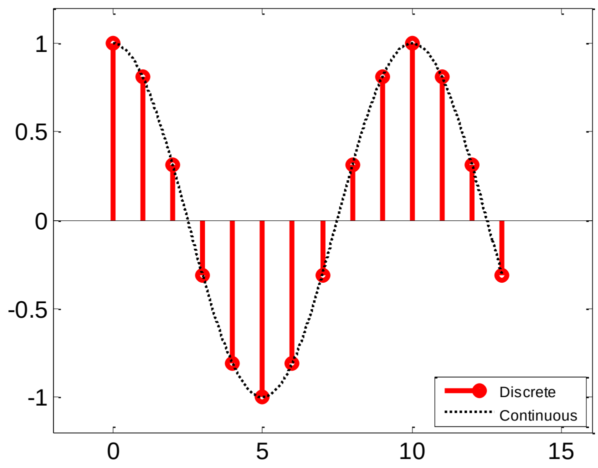
$$x[n] = \begin{cases} A\alpha^n & n \geq 0 \\ 0 & n < 0 \end{cases} \quad = \quad A\alpha^n \, u[n]$$



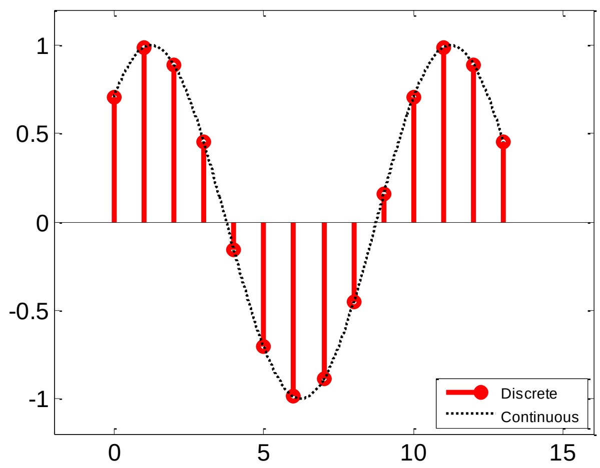
SINUSOIDAL SEQUENCES

$$x[n] = A\cos(\omega_0 n + \phi)$$

$\omega_0 = \frac{2\pi}{10} \quad \phi = 0$



$\omega_0 = \frac{2\pi}{10} \quad \phi = \frac{\pi}{4}$



Note that, $x_1[n]$ and $x_2[n]$ cannot be related by a simple time shift.

$$\cancel{x_2[n] = x_1\left[n - \frac{5}{4}\right]} \qquad \cancel{x_2[n] \neq x_1\left[n - \frac{5}{4}\right]}$$

code_1.m

```
clear all
close all

N = 10;
w0 = 2*pi / N ;
alfa = 1;
phase = 0;  %pi/4;

M = N + 3;
n =  [0:0.01:M];
y = alfa.^n.*cos(w0*n-phase);

nn = [0:M];
x = alfa.^nn.*cos(w0*nn-phase);

stem(nn,x,'r','LineWidth',3)
hold
plot(n,y,'k:','LineWidth',2)
v = axis;
dV = v(4)-v(3);
v = [v(1)-2 v(2)+2 v(3)-0.1*dV v(4)+0.1*dV];
axis(v)

set(gca,'fontsize',14)
hleg = legend('Discrete','Continuous','location','southeast');
set(hleg,'fontsize', 9)
```

EXPONENTIAL SEQUENCES (complex valued)

$$x[n] = A\,\alpha^n \qquad A, \alpha \in C$$

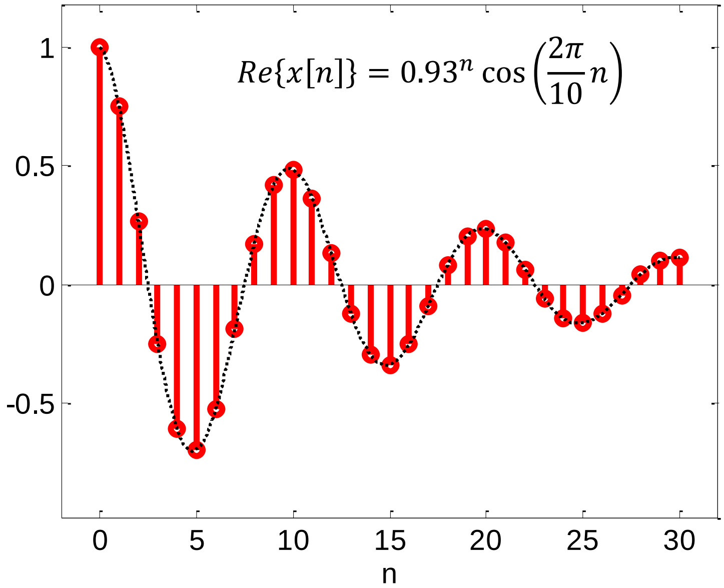
$$x[n] = \underbrace{0.93^n}_{|\alpha|} \cos\left(\frac{2\pi}{10} n\right)$$

$$A = |A| e^{j\phi} \qquad \alpha = |\alpha| e^{j\omega_0}$$

$$\Rightarrow x[n] = |A||\alpha|^n \cos(\omega_0 n + \phi) + j|A||\alpha|^n \sin(\omega_0 n + \phi)$$

For example,

$$A = 1 \qquad \alpha = 0.93 \qquad \phi = 0 \qquad \omega_0 = \frac{2\pi}{10}$$



COMPLEX EXPONENTIAL SEQUENCES:

Let

$$|\alpha| = 1$$

in

$$x[n] = A\,\alpha^n \qquad A, \alpha \in C$$

Then,

$$|A|e^{j(\omega_0 n+\phi)}$$

is called a complex exponential sequence.

$$\Rightarrow \quad Ae^{j\omega_0 n} = |A|\cos(\omega_0 n+\phi) + j|A|\sin(\omega_0 n+\phi)$$

A sinusoidal sequence can be expressed in terms of a complex exponential sequence.

$$M\cos(\omega_0 n+\phi) = Re\{Ae^{j\omega_0 n}\} = \frac{1}{2}\left(Ae^{j\omega_0 n} + A^* e^{-j\omega_0 n}\right); \quad A = Me^{j\phi}, \qquad M \in R$$

$$M\sin(\omega_0 n+\phi) = Im\{Ae^{j\omega_0 n}\} = \frac{1}{2j}\left(Ae^{j\omega_0 n} - A^* e^{-j\omega_0 n}\right); \quad A = Me^{j\phi}, \qquad M \in R$$

ω_0 : frequency (radians/sample or, shortly, radians, NOT rad/sec!)

ϕ : phase shift (radians)

TWO FUNDAMENTAL PROPERTIES OF COMPLEX EXPONENTIAL (SINUSOIDAL) DISCRETE-TIME SEQUENCES

FIRST: For any frequency value ω_0, $\omega_0 + k2\pi$ $(k\!: integer)$ is an equivalent frequency value, i.e.,

if $x[n] = Ae^{j\omega_0 n}$ and $y[n] = Ae^{j(\omega_0 + k2\pi)n}$

then $x[n] = y[n] \quad \forall n \in Z$

$$\cos(\omega_0 n) = \cos(\omega_0 n + k2\pi n)$$

$$\sin(\omega_0 n) = \sin(\omega_0 n + k2\pi n)$$

In other words, the elements of the set $\{\omega | \omega = \omega_0 + k2\pi, \omega_0 \in R, k \in Z\}$ are equivalent if they are considered as the frequencies of discrete-time complex exponentials/sinusoids.

$$\ldots = \cos\left(-\frac{9\pi}{5}n\right) = \cos\left(\frac{\pi}{5}n\right) = \cos\left(\frac{11\pi}{5}n\right) = \cos\left(\frac{21\pi}{5}n\right) = \cdots$$

$$\ldots = e^{-j\frac{9\pi}{5}n} = e^{j\frac{\pi}{5}n} = e^{j\frac{11\pi}{5}n} = e^{j\frac{21\pi}{5}n} = \cdots$$

Therefore an interval of 2π covers all distinct frequencies.

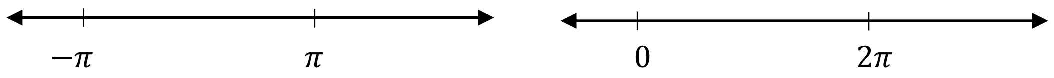
(effectively an interval of π! Why?)

Convention for the Reference Frequency Interval

Since frequencies are equivalent when multiples of 2π is added/subtracted, convention is to use either of the following as the basic interval

$$-\pi \leq \omega < \pi$$

$$0 \leq \omega < 2\pi$$



Ex:

i) $\cos(416.31\pi n) = \cos(208.155(2\pi n)) = \cos(0.155(2\pi n)) = \cos(0.31\pi n)$

ii) $\sin(416.31\pi n) = \sin(208.155(2\pi n)) = \sin(0.155(2\pi n)) = \sin(0.31\pi n)$

Ex:

i) $\cos(417.31\pi n) = \cos(208.655(2\pi n)) = \cos(0.655(2\pi n)) = \cos(1.31\pi n) = \cos(0.69\pi n)$

ii) $\sin(417.31\pi n) = \sin(208.655(2\pi n)) = \sin(0.655(2\pi n)) = \sin(1.31\pi n) = -\sin(0.69\pi n)$ (minus sign !!!)

<u>Practically</u>, it is sufficient consider

$$\cos(f\pi n), \sin(f\pi n) \quad \text{for} \quad 0 \le f \le 1$$

since

$$\cos(f\pi n) = \cos\big((2-f)\pi n\big)$$

and

$$\sin(f\pi n) = -\sin\big((2-f)\pi n\big)$$

Ex:

A 100 MHz signal $x_C(t) = \cos(2 \times 10^8 \pi t)$ is sampled at a rate of 250 MHz

(i.e. sampling period is $T_S = \frac{1}{250\ 000\ 000} = 4$ pico sec.)

$$x[n] = \cos(0.8\pi n)$$

Find another continuous-time (CT) sinusoid that would yield the same discrete-time sinusoid (i.e., $x[n]$) at this sampling frequency.

How many other CT sinusoids would yield the same DT sequence?

Answer:

$$\cos(\omega_0 n) = \cos((\omega_0 + k2\pi)\, n)$$

$$= \cos\left(\frac{(\omega_0 + k2\pi)}{T_S} T_S n\right)$$

$$\to \cos\big((2\pi f_0 f_S + k2\pi f_S)t\big)$$

where

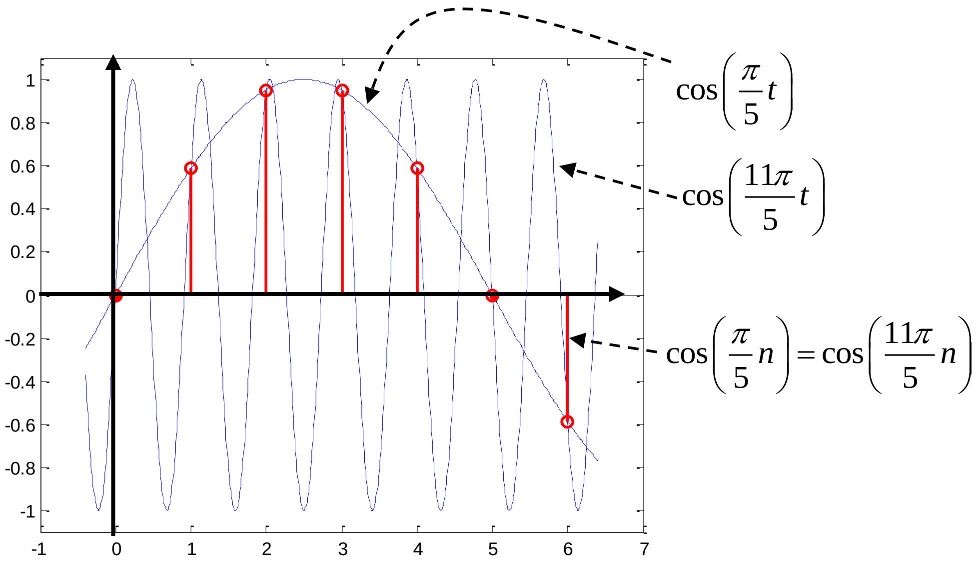
$$f_0 = 0.4 \qquad\qquad f_S = 250\ MHz$$

Therefore all CT sinusoids at frequencies 100 MHz, 350 MHz, 600 MHz, 850 MHz, … yield the same DT sinusoid for this sampling frequency.

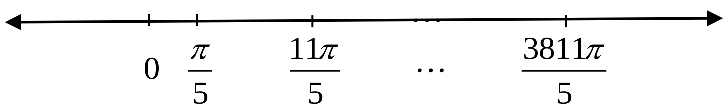
Their frequencies can be expressed as

$$\frac{(k+0.4)}{T_S} = (k+0.4)250 \text{ MHz}$$

Ex:




```
clear all
close all
n = [-0.4:0.01:6.4];
nn = [0:6];
x = sin(n*pi/5);
y = sin(n*11*pi/5);
z = sin(nn*11*pi/5);
plot(n,x);
hold
plot(n,y)
stem(nn,z,'r','LineWidth',2
)
v=axis;
v=[v(1) v(2) -1.1 1.1];
axis(v)
```



SECOND:

A DT sinusodal ($\cos(\omega_0 n + \phi)$) or complex exponential signal $e^{j(\omega_0 n+\phi)}$ is not necessarily periodic!

To be periodic,

ω_0 must be a *rational* multiple of π,

i.e.,

$$\omega_0 = \frac{p}{q}\pi, \qquad p, q \in Z$$

Proof:

$$A\cos(\omega_0 n + \phi) \stackrel{?}{=} A\cos(\omega_0 (n+N) + \phi)$$

$$A\cos(\omega_0 (n+N) + \phi) = A\cos(\omega_0 n + \omega_0 N + \phi)$$

For periodicity $\omega_0 N = k2\pi \quad \Rightarrow \quad \omega_0 = \frac{k}{N}2\pi \quad \text{or} \quad \frac{\omega_0}{2\pi} = \frac{k}{N} \qquad k \in Z$

has to be satisfied.

Ex: $\cos(5n) \qquad \omega_0 = 5 \qquad \frac{\omega_0}{2\pi} = \frac{5}{2\pi}$ is not rational so it is not periodic.

FUNDAMENTAL PERIOD, *N*, IS NOT NECESSARILY EQUAL TO $\frac{2\pi}{\omega_0}$

Since, for periodic sinusoids,

$$\omega_0 N = k2\pi$$

i.e,

$$N = \frac{k2\pi}{\omega_0},$$

fundamental period, N, is not necessarily equal to $\frac{2\pi}{\omega_0}$.

Finding the Fundamental Period of a Sinusoid

Find the smallest k, k_{min} , so that $k_{min}\frac{2\pi}{\omega_0}$ is an integer.

Then, the fundamental period is

$$N = k_{min}\frac{2\pi}{\omega_0} \quad .$$

Ex: $\cos\left(\frac{\pi}{5}n\right)$ $\qquad \omega_0 = \frac{\pi}{5}$ $\qquad \frac{\omega_0}{2\pi} = \frac{1}{10}$ $\qquad N = k\frac{2\pi}{\omega_0} = k\frac{2\pi}{\frac{\pi}{5}} = 10 \quad (k=1)$

Ex: $\cos\left(\frac{5\pi}{17}n\right)$ $\qquad \omega_0 = \frac{5\pi}{17}$ $\qquad \frac{\omega_0}{2\pi} = \frac{5}{34}$ $\qquad N = k\frac{34}{5} = 34 \quad (k=5)$

Ex: $\cos\left(\frac{6\pi}{5}n\right)$ $\qquad \omega_0 = \frac{6\pi}{5}$ $\qquad \frac{\omega_0}{2\pi} = \frac{3}{5}$ $\qquad N = k\frac{5}{3} = 5 \quad (k=3)$

Ex: Let $x_1[n]=\cos(\omega_1 n)$ and $x_2[n]=\cos(\omega_2 n)$. Find two "frequencies" ω_1 and ω_2 such that $\omega_1 \neq \omega_2 + k2\pi$ for any integer *k*, and $x_1[n]$ and $x_2[n]$ are both periodic with fundamental period *N*=13.

$$N = 13 = k\frac{2\pi}{\omega}, \ k\text{: integer}$$

$$\Rightarrow \omega = k\frac{2\pi}{13}$$

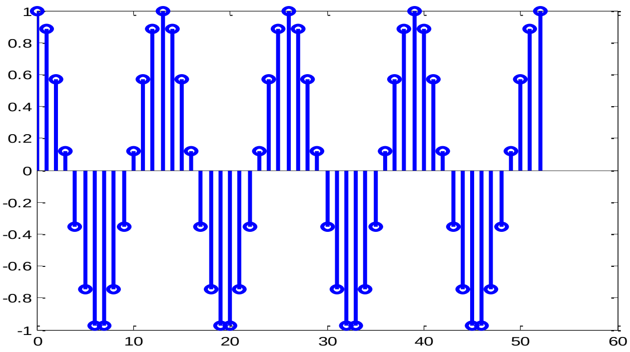
Choose, for example, *k* =1 and *k* =2 $\Rightarrow \omega_1 = \frac{2\pi}{13}, \ \omega_2 = \frac{4\pi}{13}$

Therefore,
DT sinusoids may have different "frequencies"
although
their fundamental periods are the same!

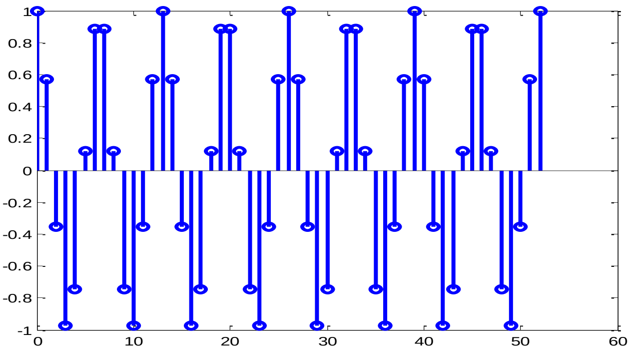
What do the discrete-time sinusoids look like?

Some frequencies between 0 and π

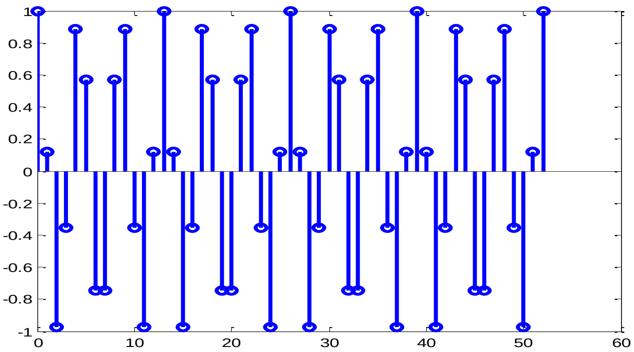
$\cos\left(\frac{2\pi}{13}n\right)$ *N*=13



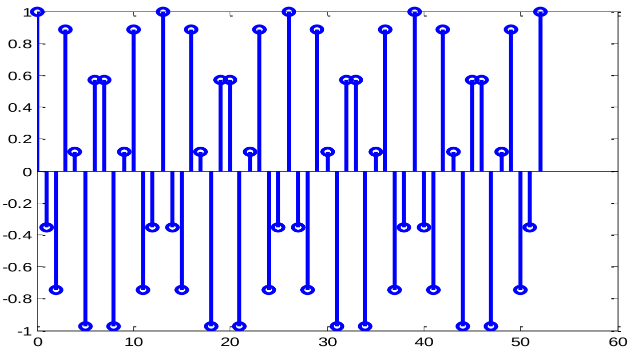
$\cos\left(\frac{4\pi}{13}n\right)$ *N*=13



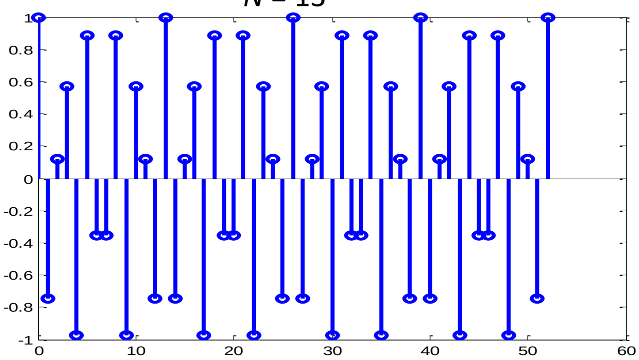
$\cos\left(\frac{6\pi}{13}n\right)$ *N*= 13



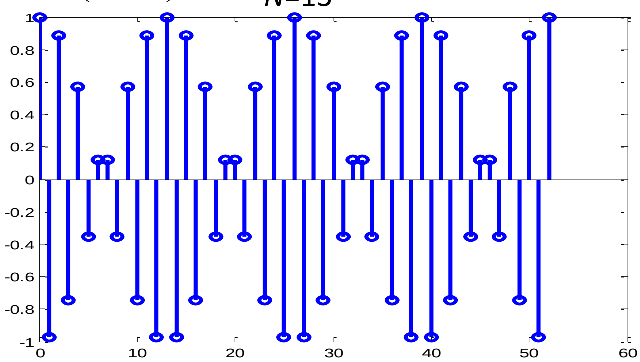
$\cos\left(\frac{8\pi}{13}n\right)$ *N*=13



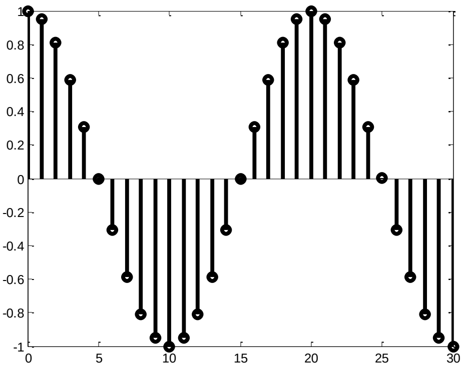
$\cos\left(\frac{10\pi}{13}n\right)$ *N* = 13



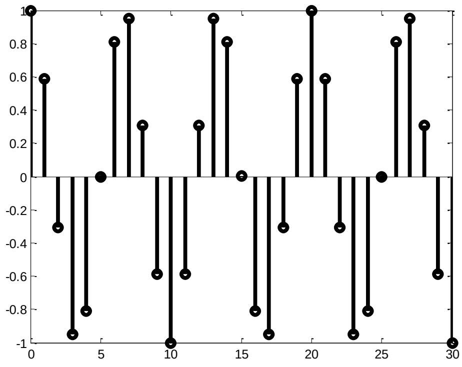
$\cos\left(\frac{12\pi}{13}n\right)$ *N*=13



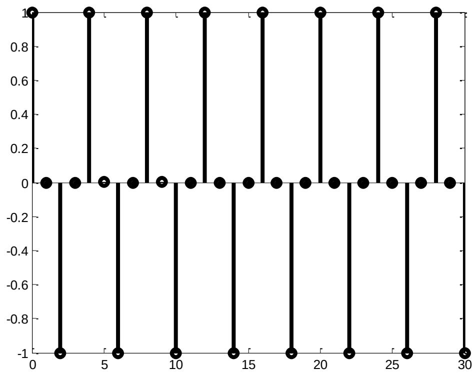
$\cos(0.1\pi n)$ N=20



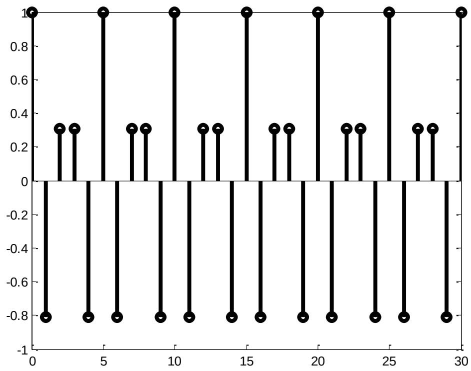
$\cos(0.3\pi n)$ N=20



$\cos(0.5\pi n)$ N=4

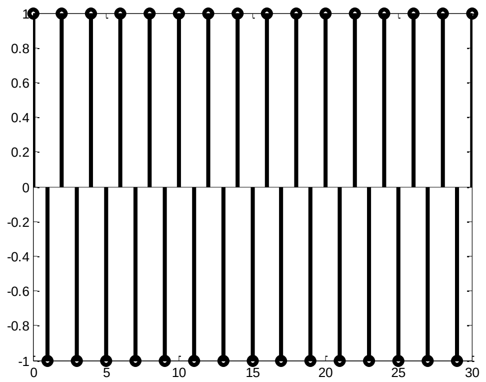


$\cos(0.8\pi n)$ N=5



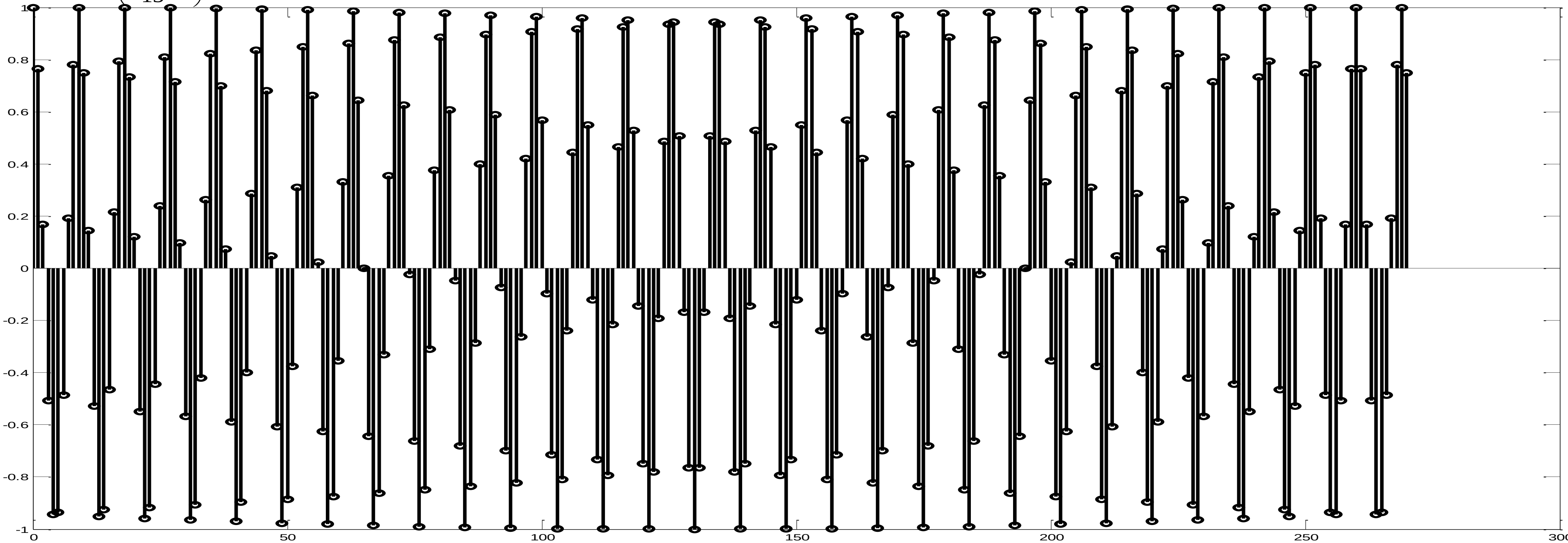
THE HIGHEST FREQUENCY SIGNAL IN DISCRETE-TIME

$\cos(\pi n)$ N= 2

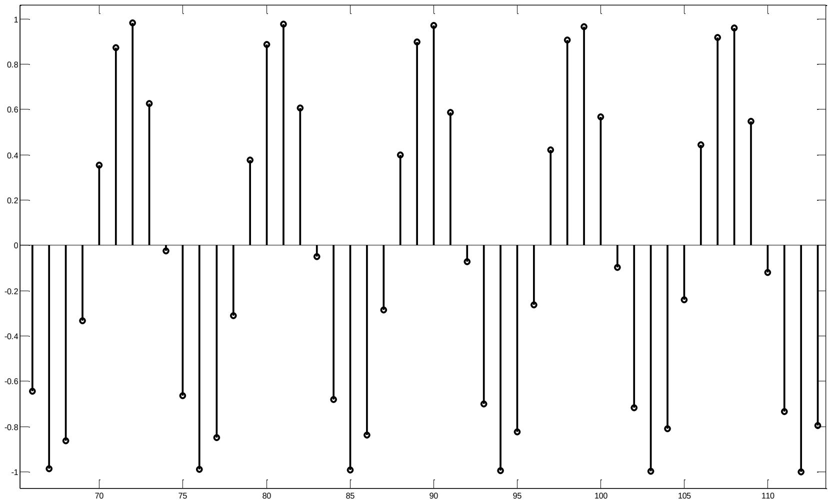


$\cos\left(\dfrac{2.9\pi}{13}n\right)$

Fundamental period is N=260

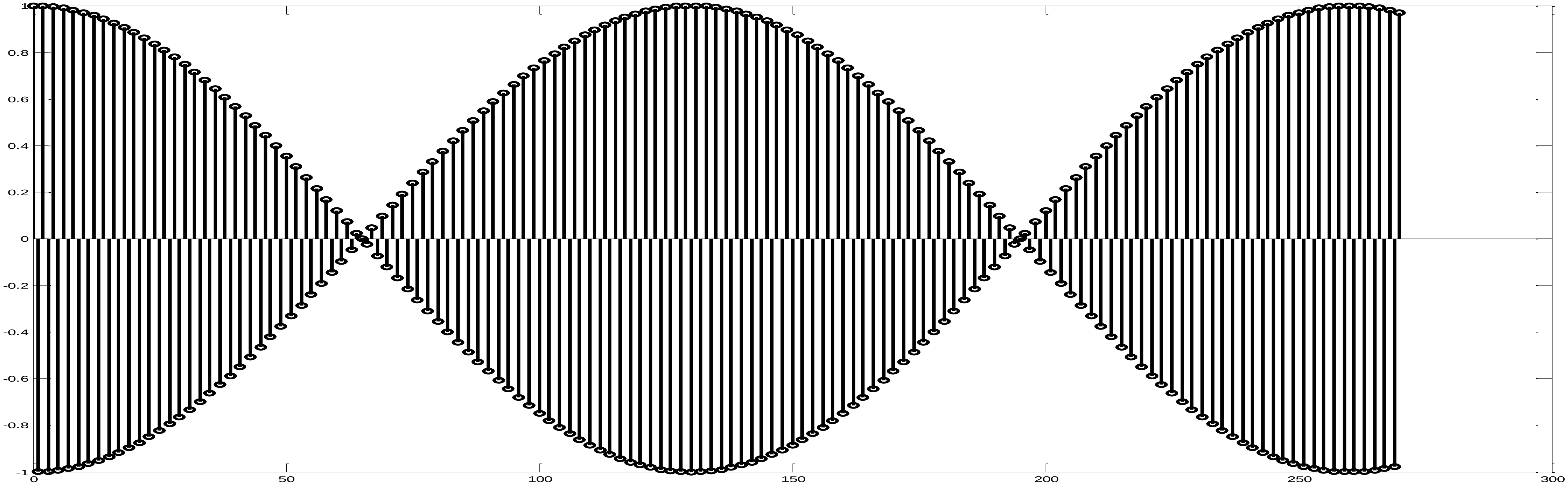


detail



$$\cos\left(\frac{12.9\pi}{13}n\right)$$

Fundamental period is N=260



detail