1) The impulse response, $h[n]$, of an LTI system is zero outside the interval $0 \le n \le 3$. The 4-point DFT of $h[n]$ is given as

$$H[k] = 2\delta[k] + \delta[k-1] + \delta[k-3].$$

The input, $x[n]$, to the LTI system is given by

$$x[n] = \delta[n] + \delta[n-1] + \delta[n-2] + \delta[n-3].$$

a) Find and plot $h[n]$ for all n.

b) If we denote the N-point DFT of $x[n]$ and $h[n]$ by $X_N[k]$ and $H_N[k]$, respectively, find and plot the N-point IDFT $m_N[n]$ of $M_N[k] = X_N[k]H_N[k]$ for $N = 7$ and $N = 4$.

c) Describe clearly and briefly a method to find the output $y[n]$ of the LTI system where you can use only 5-point DFT and IDFT operations (other length DFT and IDFT operations are not allowed). You are also not allowed to directly use any convolution operation.

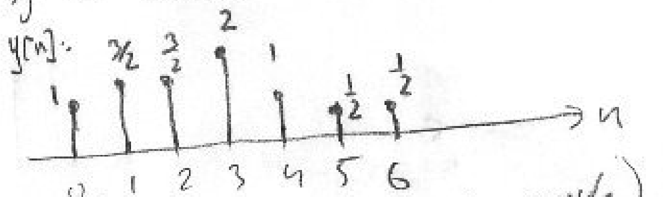
+6

a) $\frac{1}{4}\sum_{k=0}^{3} H[k] e^{jk\frac{2\pi}{4}n} = \frac{1}{4}\left(2 + e^{j\frac{2\pi}{4}n} + e^{j3\frac{2\pi}{4}n}\right) = \frac{1}{2} + \frac{1}{2}\cos\left(\frac{\pi}{2}n\right)$ $e^{-j\frac{2\pi}{4}}$

$\Rightarrow h[n] = \begin{cases} \frac{1}{2} + \frac{1}{2}\cos\left(\frac{\pi}{2}n\right), & 0 \le n \le 3 \\ 0, & \text{o.w.} \end{cases}$

+8 b) $m_N[n] = \mathrm{IDFT}_N\{X_N[k]H_N[k]\} = x[n] \circledN h[n]$ by the circular convolution theorem.

First find linear convolution $y[n] = x[n] * h[n]$



$\underline{N=7}$: $m_7[n] = x[n] \circled{7} h[n] = \begin{cases} \sum_{r=-\infty}^{\infty} y[n-7r], & 0 \le n \le 6 \\ 0, & \text{o.w.} \end{cases} = y[n]$ (equal to linear convol.)

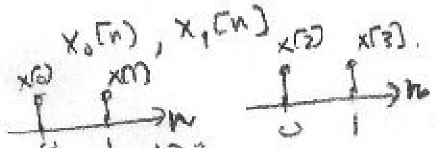
$\underline{N=4}$: $m_4[n] = x[n] \circled{4} h[n] = \begin{cases} \sum_{r=-\infty}^{\infty} y[n-4r], & 0 \le n \le 3 \\ 0, & \text{o.w.} \end{cases} = \begin{cases} y[n] + y[n+4], & 0 \le n \le 3 \\ 0, & \text{o.w.} \end{cases}$

+8 c) We can use overlap-add or overlap save methods:

Overlap-add: ① Choose blocksize B such that $N \ge B + 4 - 1 \Rightarrow \boxed{B \le 2} \Rightarrow$ let $\boxed{B=2}$

N: DFT size 5; B: $x[n]$ size; 4: $h[n]$ size

② Divide $x[n]$ into non-overlapping blocks of size $B=2$: $x_0[n]$, $x_1[n]$

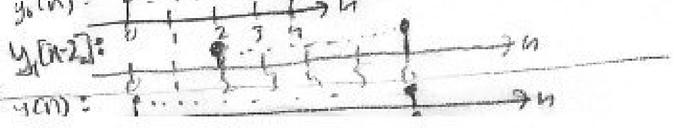


③ Calculate 5-pt DFT of $x_0[n], x_1[n], h[n]$: $X_0[k], X_1[k], H[k]$.

④ Calculate $y_0[n] = \mathrm{IDFT}_5\{X_0[k]H[k]\}$, $y_1[n] = \mathrm{IDFT}_5\{X_1[k]H[k]\}$ to find result $y[n]$:

⑤ Add overlapping regions of $y_0[n], y_1[n]$

$y[n] = y_0[n] + y_1[n-2]$



2) Consider a signal $x_c(t)$ that is bandlimited to 5 kHz with its spectrum, $X_c(j\Omega)$, plotted below. We want to filter $x_c(t)$ by using a discrete-time system and obtain a continuous-time output $y_c(t)$. The system is shown as the upper branch of the figure below. The desired effective frequency response $H_{eff}(j\Omega) = \frac{Y_c(j\Omega)}{X_c(j\Omega)}$ is constrained to be nonzero for $|\Omega| < \pi 10^4$.

a) Consider the upper branch. Let $H_0(e^{j\omega}) = \frac{\sin(\omega)}{\omega}$, $|\omega| < \pi$ and $T_1 = 10^{-4}$ sec.

+6 i) Obtain $X(e^{j\omega})$ and $Z_0(e^{j\omega})$ in terms of $X_c(j\Omega)$. Roughly plot $X(e^{j\omega})$, $H_0(e^{j\omega})$ and $Z_0(e^{j\omega})$, and label the axes clearly.

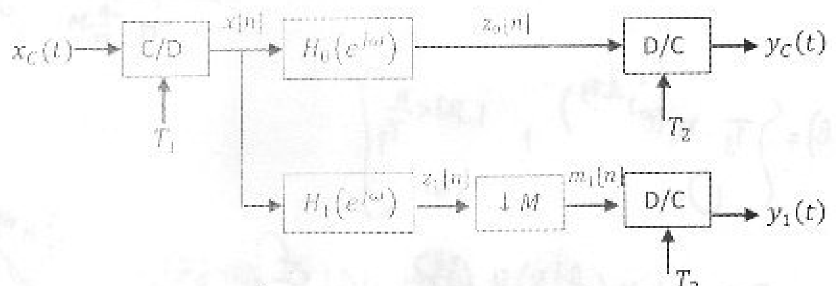
+6 ii) Write $Y_c(j\Omega)$ in terms of $Z_0(e^{j\omega})$. Determine T_2 and $H_{eff}(j\Omega)$. Plot $H_{eff}(j\Omega)$.

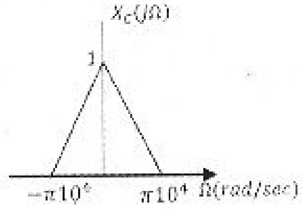
b) Consider the lower branch now. Let $T_1 = 10^{-5}$ seconds and $T_3 = 10^{-4}$ sec. In order to obtain $y_1(t) = y_c(\alpha t)$ for some positive real number α, find

+3 i) the frequency response $H_1(e^{j\omega})$,

+4 ii) conditions on M and α

+3 iii) and a relation between M and α.

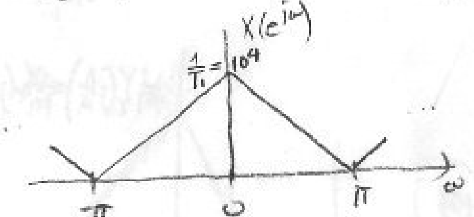


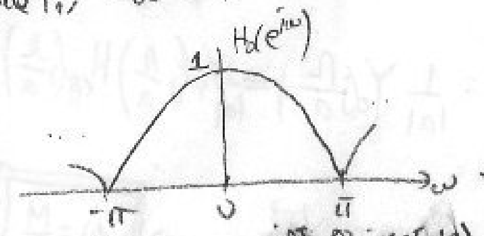


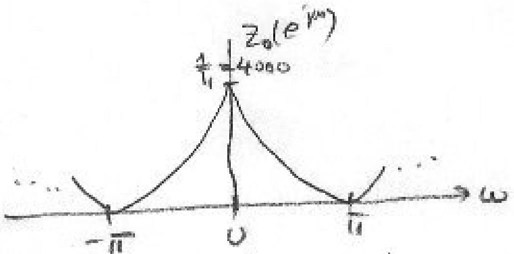
a) $\boxed{i}$ $X(e^{j\omega}) = \frac{1}{T_1}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{T_1} - k\frac{2\pi}{T_1}\right)\right)$ Since no aliasing occurs (i.e. $\frac{2\pi}{T_1} \geq 2\cdot(\pi 10^4)$), $X(e^{j\omega}) = \frac{1}{T_1} X_c\left(j\frac{\omega}{T_1}\right), |\omega| < \pi$

$Z_0(e^{j\omega}) = X(e^{j\omega})\cdot H_0(e^{j\omega}) = \frac{1}{T_1} X_c\left(j\frac{\omega}{T_1}\right)\cdot\frac{\sin(\omega)}{\omega}, \quad |\omega| < \pi.$

+6



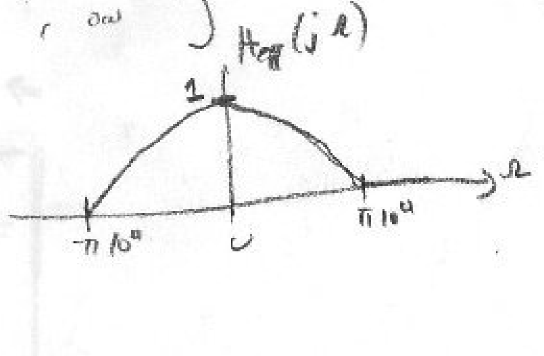




$\boxed{ii}$ $Y_c(j\Omega) = \begin{cases} T_2\cdot Z_0(e^{j\Omega T_2}), & |\Omega| < \frac{\pi}{T_2} \\ 0, & o.w. \end{cases} = \begin{cases} T_2\frac{1}{T_1} H_0(e^{j\Omega T_2}) \sum_{k=-\infty}^{\infty} X_c\left(j\left(\frac{\Omega T_2 - k2\pi}{T_1}\right)\right), & |\Omega| < \frac{\pi}{T_2} \\ 0, & o.w. \end{cases} = \underbrace{X_c(j\Omega)\, H_{eff}(j\Omega)}_{\text{bandlimited to } \pi\cdot 10^4 \text{ rad/s}}$

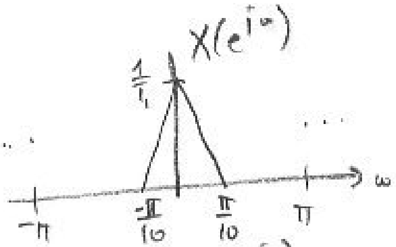
let $\boxed{T_2 = T_1 = 10^{-4} \text{ sec.}}$ $\Rightarrow$ $Y_c(j\Omega) = \begin{cases} H_0(e^{j\Omega T_2})\cdot X_c(\Omega), & |\Omega| < \pi\cdot 10^4 \\ 0, & o.w. \end{cases}$

+6 $\Rightarrow$ $\boxed{H_{eff}(j\Omega) = \begin{cases} H_0(e^{j\Omega T_2}) = \frac{\sin(\Omega\cdot 10^{-4})}{\Omega\cdot 10^{-4}}, & |\Omega| < \pi\cdot 10^4 \\ 0, & o.w. \end{cases}}$

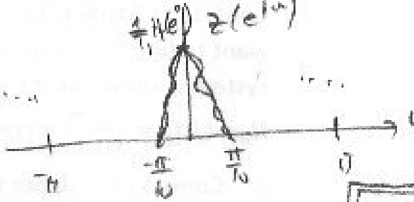


b) $T_1 = 10^{-5}$ sec.

$$X(e^{j\omega}) = \frac{1}{T_1} X_c\left(j\frac{\omega}{T_1}\right), \quad |\omega| < \pi$$



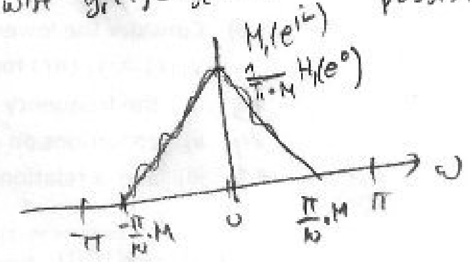
$$Z_1(e^{j\omega}) = X(e^{j\omega}) H_1(e^{j\omega}) = \frac{1}{T_1} X_c\left(j\frac{\omega}{T_1}\right) H_1(e^{j\omega}), \quad |\omega| < \pi \quad \left(|\omega| < \frac{\pi}{10}\right)$$



+2

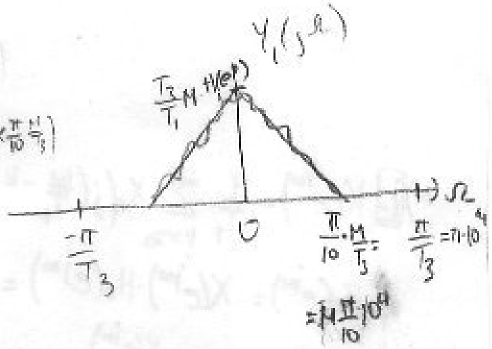
$$M_1(e^{j\omega}) = \frac{1}{M} Z_1\left(e^{j\frac{\omega}{M}}\right) = \frac{1}{MT_1} X_c\left(j\frac{\omega}{MT_1}\right) H_1\left(e^{j\frac{\omega}{M}}\right), \quad |\omega| < \pi \quad \left(|\omega| < \frac{\pi}{10} \cdot M\right)$$

if no aliasing occurs, i.e. $\boxed{M \leq 10}$. Otherwise $y_1(t) = y_c(at)$ would not be possible.



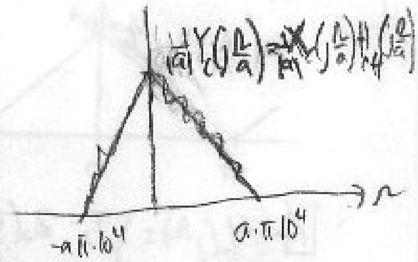
$$Y_1(j\Omega) = \begin{cases} T_3 \cdot M_1(e^{j\Omega T_3}), & |\Omega| < \frac{\pi}{T_3} \\ 0, & \text{o.w.} \end{cases}$$

$$= \begin{cases} T_3 \frac{1}{M} \frac{1}{T_1} X_c\left(\frac{\Omega T_3}{MT_1}\right) H_1\left(e^{j\frac{\Omega T_3}{M}}\right), & |\Omega| < \frac{\pi}{T_3} \quad \left(|\Omega| < \frac{\pi}{10}\frac{M}{T_3}\right) \\ 0, & \text{o.w.} \end{cases}$$



$$= \frac{1}{|a|} Y_c\left(j\frac{\Omega}{a}\right) = \frac{1}{|a|} X_c\left(j\frac{\Omega}{a}\right) H_{cH}\left(j\frac{\Omega}{a}\right)$$

$$\Rightarrow \frac{T_3}{MT_1} = \frac{1}{a} \Rightarrow \boxed{10 = \frac{M}{a}} \quad +3$$



$$\Rightarrow H_{cH}\left(j\frac{\Omega}{a}\right) = \begin{cases} H_1\left(e^{j\frac{\Omega T_3}{M}}\right), & |\Omega| < \frac{\pi}{10}\frac{M}{T_3} = \pi 10^3 M \\ 0, & \text{o.w.} \end{cases}$$

$$\Rightarrow a\pi \cdot 10^4 < \frac{\pi}{T_3} = \pi \cdot 10^4 \Rightarrow \boxed{a < 1} \quad +2$$

$$\Rightarrow H_{cH}(j\Omega) = \begin{cases} H_1\left(e^{j\frac{\Omega a T_3}{M}}\right), & |\Omega| < \frac{\pi \cdot 10^3 M}{a} = \pi \cdot 10^4 \\ 0, & \text{o.w.} \end{cases}$$

$$\boxed{\Rightarrow H_1(e^{j\omega}) = \begin{cases} H_{cH}\left(j\frac{\omega M}{aT_3}\right), & |\omega| < \frac{\pi}{10} \\ \text{arbitrary}, & \frac{\pi}{10} < |\omega| < \pi \end{cases} = \begin{cases} \frac{\sin(\omega 10^5 \cdot 10^{-4})}{\omega 10^5 \cdot 10^{-4}} = \frac{\sin(\omega \cdot 10)}{\omega \cdot 10}, & |\omega| < \frac{\pi}{10} \\ \text{arbitrary}, & \frac{\pi}{10} \leq |\omega| < \pi \end{cases}}$$

10^5

+3