DFT: DISCRETE FOURIER TRANSFORM

DFS: DISCRETE FOURIER SERIES

DFT: DISCRETE FOURIER TRANSFORM

IDFT: INVERSE DFT

THE SIGNIFICANCE OF DFT AND DFS: SAMPLING THE DTFT

PROPERTIES OF DFS AND DFT

1) Linearity
2) Time Shift Property (DFS) – Circular Time Shift Property (DFT)
3) Multiplication by a complex exponential
4) Duality
5) Symmetry Properties Real Sequences
6) Convolution Property
    - Circular Convolution
    - Getting the Result of Linear Convolution Using DFT
7) Sampling the DTFT
8) Multiplication in Time Domain

IMPLEMENTING LTI SYSTEMS USING DFT

- Overlap-Add
- Overlap-Save

LINEAR CONVOLUTION AND CIRCULAR CONVOLUTION

DFT: DISCRETE FOURIER TRANSFORM

DTFT, $X(e^{j\omega})$, is a function of a continuous variable, ω.

However, if $x[n]$ is of finite length *N*, it can be considered as one period of its periodic extension

$$\tilde{x}[n] = \sum_{k=-\infty}^{\infty} x[n - kN].$$

Then, Fourier series representation of $\tilde{x}[n]$ can be used to represent $x[n]$ as well.

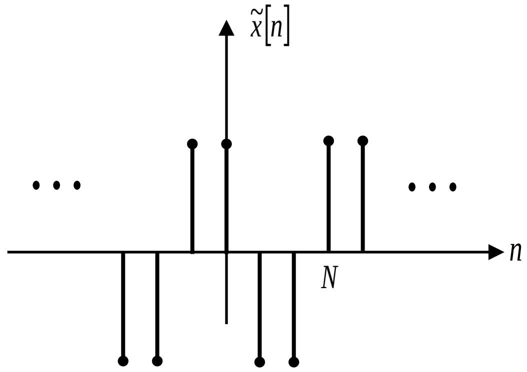
Note that the DFS representation of a periodic sequence of period N has N coefficients.

Therefore, instead of an infinite set of numbers as required by DTFT, a finite length sequence of length a finite length sequence, $x(n)$, of length N can be represented by N complex values (Fourier series coefficients).

Now, let's review Fourier series representation of periodic sequences.

DFS: DISCRETE FOURIER SERIES

Let $\tilde{x}[n]$ be a periodic sequence with fundamental period *N*;



Its DFS representation is

$$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}[k]e^{jk\frac{2\pi}{N}n}$$

$$= \frac{1}{N}\left(\tilde{X}[0] + \tilde{X}[1]e^{j\frac{2\pi}{N}n} + \tilde{X}[2]e^{j2\frac{2\pi}{N}n} + \ldots + \tilde{X}[N-1]e^{j(N-1)\frac{2\pi}{N}n}\right)$$

This is a representation in terms of sinusoidal sequences at the fundamental frequency and its multiples (harmonic components).

$$\left\{1,\ e^{j\frac{2\pi}{N}n},\ e^{j\frac{2\pi}{N}2n}, \ldots, e^{j\frac{2\pi}{N}(N-1)n}\right\} = \left\{1,\ \underbrace{e^{j\omega_0 n}}_{\text{fundamental}},\ \underbrace{e^{j2\omega_0 n}, \ldots, e^{j(N-1)\omega_0 n}}_{\text{harmonics}}\right\} \qquad \omega_0 = \frac{2\pi}{N}$$

Note that the number of frequency components depends on signal period, *N*.

Note also that, in this set $e^{jk\frac{2\pi}{N}n}$ and $e^{j(N-k)\frac{2\pi}{N}n}$ are complex conjugates of each other, i.e.

$$e^{jk\frac{2\pi}{N}n} = e^{-j(N-k)\frac{2\pi}{N}n}$$

The DFS coefficients $\tilde{X}[k], k=0,1,\ldots,N-1$, are obtained as

$$\tilde{X}[k]=\sum_{n=0}^{N-1}\tilde{x}[n]e^{-jk\frac{2\pi}{N}n}$$

$$=\tilde{x}[0]+\tilde{x}[1]e^{-jk\frac{2\pi}{N}}+\tilde{x}[2]e^{-jk\frac{2\pi}{N}2}+\ldots+\tilde{x}[N-1]e^{-jk\frac{2\pi}{N}(N-1)}$$

To obtain this expression multiply

$$\tilde{x}[n]=\frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}[k]e^{jk\frac{2\pi}{N}n}$$

by

$$e^{-jm\frac{2\pi}{N}n}$$

and sum over n =0,1,2,...,N-1.

$$\sum_{n=0}^{N-1}\tilde{x}[n]e^{-jm\frac{2\pi}{N}n}$$

$$=\frac{1}{N}\sum_{n=0}^{N-1}\sum_{k=0}^{N-1}\tilde{X}[k]e^{jk\frac{2\pi}{N}n}\,e^{-jm\frac{2\pi}{N}n}$$

$$=\frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}[k]\underbrace{\sum_{n=0}^{N-1}e^{j(k-m)\frac{2\pi}{N}n}}_{=\begin{cases}N & if\quad k-m=0 \text{ or an integer multiple of } 2\pi\\ 0 & if\quad k-m\neq 0\end{cases}}$$

$$\Rightarrow\qquad \sum_{n=0}^{N-1}\tilde{x}[n]e^{-j\frac{2\pi}{N}mn}=\tilde{X}[m]$$

Note that $\tilde{X}[k]$ is periodic with *N* since

$$\tilde{X}[k+rN] = \sum_{n=0}^{N-1} \tilde{x}[n] e^{-j\frac{2\pi}{N}kn} \underbrace{e^{-j\frac{2\pi}{N}rNn}}_{1} = \tilde{X}[k]$$

So it is sufficient to keep *N* values.

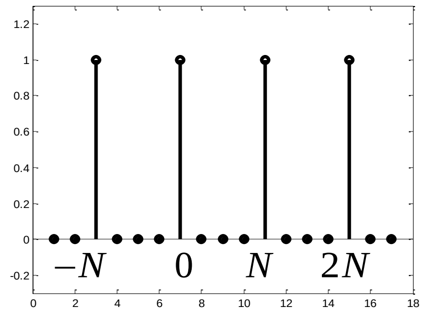
<u>Notation</u>: For convenience define $W_N \triangleq e^{-j\frac{2\pi}{N}}$

$$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1} \tilde{X}[k] W_N^{-kn} \qquad \tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n] W_N^{kn}$$

$W_N^{-kn} = e^{j\frac{2\pi}{N}kn}$ is the k^{th} harmonic.

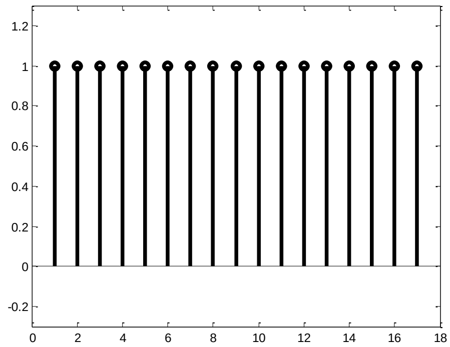
Ex:

$$\tilde{x}[n] = \sum_{r=-\infty}^{\infty} \delta[n - rN]$$



DFS coefficients of $\tilde{x}[n]$

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \delta[n] W_N^{kn} = 1$$ (independent of N)



DFS representation of $\tilde{x}[n]$

$$\tilde{x}[n] = \frac{1}{N} \sum_{k=0}^{N-1} W_N^{-kn} = \frac{1}{N} \sum_{k=0}^{N-1} e^{j\frac{2\pi}{N}kn} = \frac{1}{N} N \sum_{k=0}^{N-1} \delta[n - kN] = \sum_{k=0}^{N-1} \delta[n - kN]$$

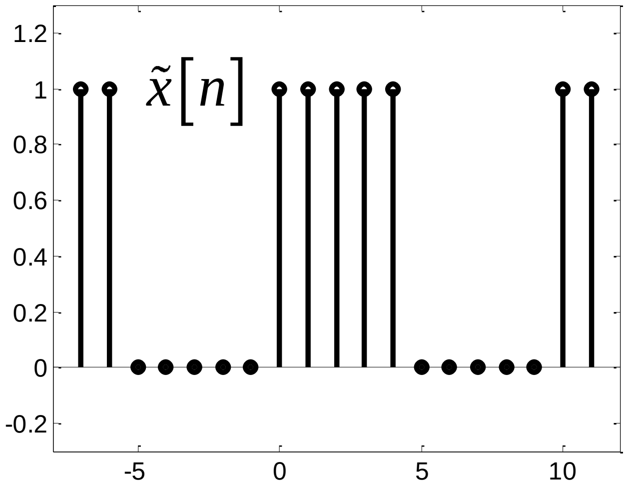
For $N = 2$

$$\tilde{x}[n] = \frac{1}{2} \sum_{k=0}^{1} W_N^{-kn} = \frac{1}{2}\left(1 + e^{j\pi n}\right) = \begin{cases} 1 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

For $N = 3$

$$\tilde{x}[n] = \frac{1}{3} \sum_{k=0}^{2} W_N^{-kn} = \frac{1}{3}\left(1 + e^{j\frac{2\pi}{3}n} + e^{j\frac{4\pi}{3}n}\right) = \begin{cases} 1 & \text{if } n \text{ is a multiple of 3} \\ 0 & \text{ow} \end{cases}$$

Ex:



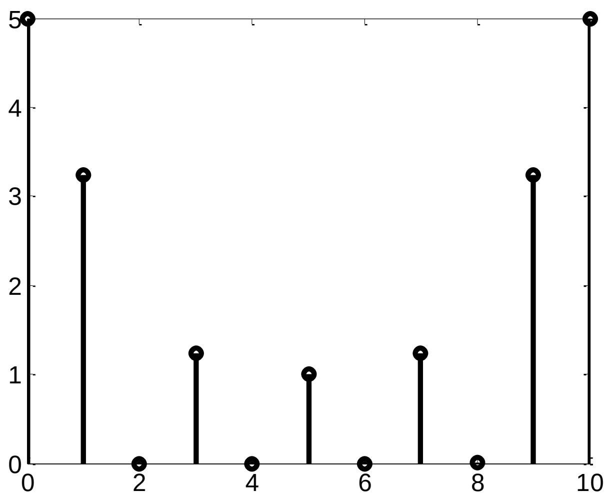
$N = 10$

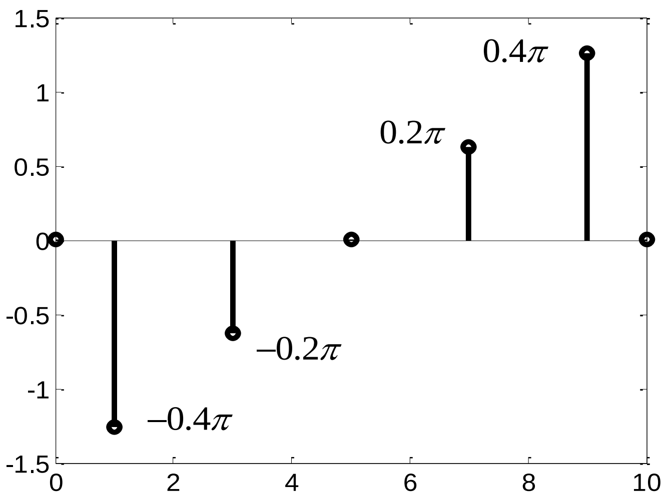
DFS coefficients of $\tilde{x}[n]$

$$\tilde{X}[k] = \sum_{n=0}^{4} W_{10}^{kn} = \frac{1-W_{10}^{5k}}{1-W_{10}^{k}} = \frac{1-e^{-j\pi k}}{1-e^{-j\frac{\pi}{5}k}}$$

$$= \frac{e^{-j\frac{\pi}{2}k}}{e^{-j\frac{\pi}{10}k}} \frac{e^{j\frac{\pi}{2}k} - e^{-j\frac{\pi}{2}k}}{e^{j\frac{\pi}{10}k} - e^{-j\frac{\pi}{10}k}} = e^{-j\frac{4\pi}{10}k} \frac{\sin\frac{\pi}{2}k}{\sin\frac{\pi}{10}k}$$

$\tilde{X}[k]$ is periodic with $N = 10$.



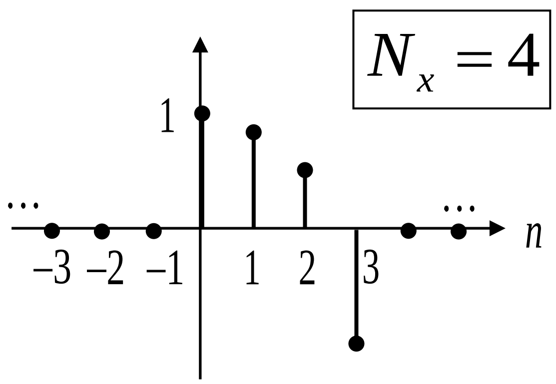


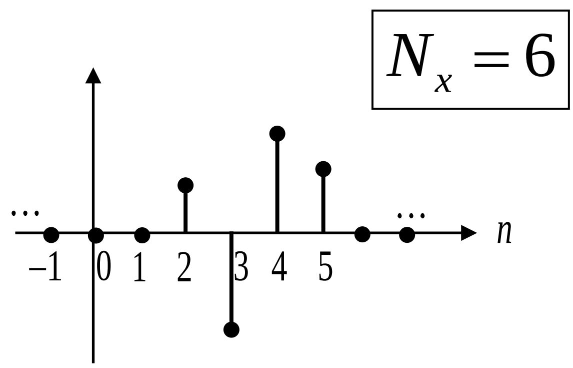
DFT: DISCRETE FOURIER TRANSFORM

Let $x[n]$ be a finite length (length = N_x) sequence such that

$$x[n]=0 \quad \text{for} \quad n<0 \quad \text{and} \quad n>N_x-1$$

Ex:

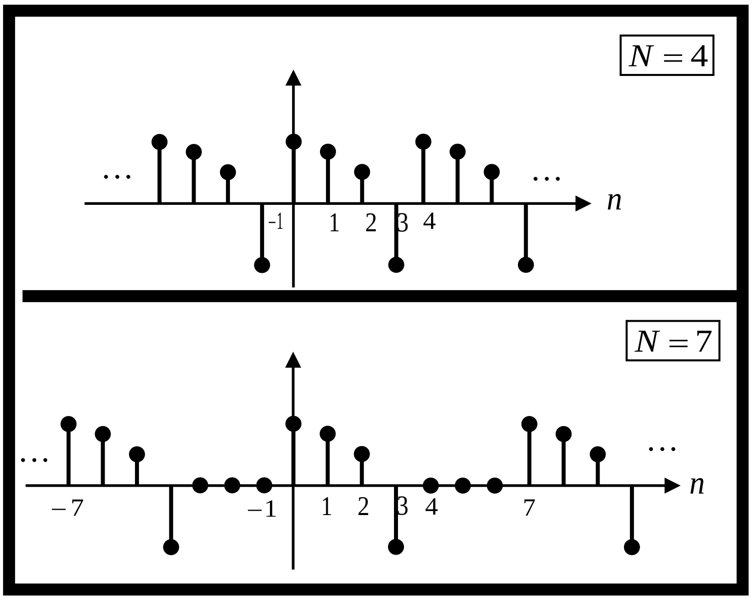


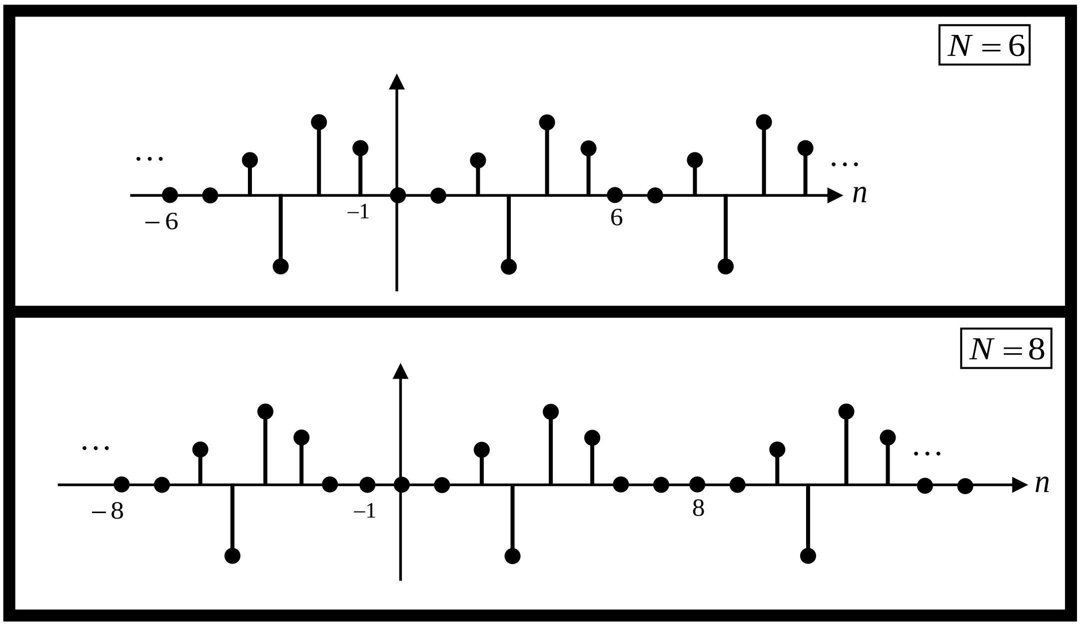


Let

$$\tilde{x}[n] = \sum_{p=-\infty}^{\infty} x[n - pN], \quad N \geq N_x,$$

be its periodic extension with perion N:





Let

$$\tilde{X}[k],\ k \in Z$$

be the DFS coefficients of

$$\tilde{x}[n].$$

Then, ***N*-point DFT** of $x[n]$ is defined as

$$X[k] = \begin{cases} \sum_{n=0}^{N-1} x[n] e^{j\frac{2\pi}{N}kn} & k = 0,1,\dots,N-1 \\ 0 & ow. \end{cases}$$

or

$$X[k] = \begin{cases} \tilde{X}[k] & k = 0,1,\dots,N-1 \\ 0 & o.w. \end{cases}$$

IDFT: INVERSE DFT

$$x[n] = \begin{cases} \frac{1}{N}\sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi}{N}kn} & n = 0,1,\ldots,N-1 \\ 0 & n \neq 0,1,\ldots,N-1 \end{cases}$$

Notation: Modulo

Define

$$((n))_N \stackrel{\Delta}{=} (n) \bmod N$$

then

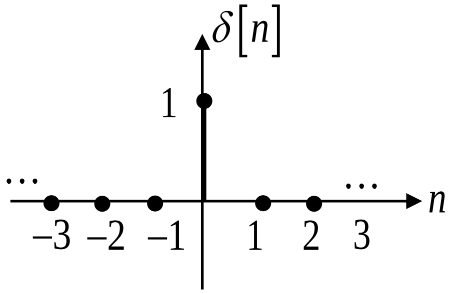
$$\tilde{X}[k] = X\left[((k))_N\right]$$

and

$$\tilde{x}[n] = x\left[((n))_N\right]$$

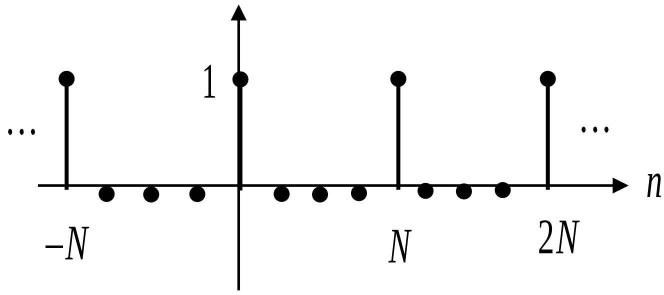
```
>> x=1:5            x =    1    2    3    4    5
>>    for i=1:5
            y(i) = x(mod(-(i-1),5)+1);          (since vector indices start from 1 in MATLAB)
      end
>> y  y =    1    5    4    3    2
```

Ex: $x[n]=\delta[n]$

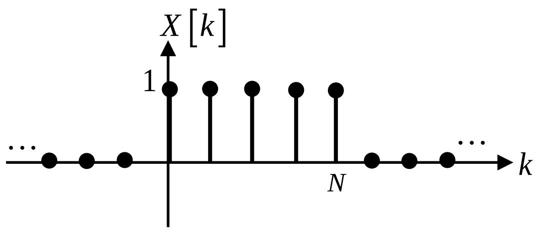


$N_x = 1$

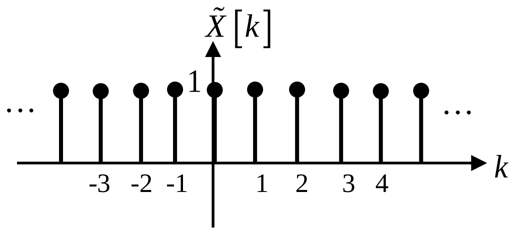
$$\tilde{x}[n]=\sum_{p=-\infty}^{\infty}\delta[n-pN]$$



$$X[k]=\begin{cases}\tilde{X}[k] & k=0,1,2,\ldots,N-1\\ 0 & \text{ow.}\end{cases}$$



$$\tilde{X}[k]=\sum_{n=0}^{N-1}\delta[n]e^{-j\frac{2\pi}{N}kn}=1 \qquad k\in\mathbb{Z}$$



Using 1-point DFT, $x[n]=\delta[n]$ can be obtained by IDFT as,

$$x[n]=\begin{cases}\frac{1}{1}\left(X[0]\,W_1^{-0n}\right)=\frac{1}{1}(1\times 1)=1 & n=0\\ 0 & o.w.\end{cases}$$

Using 2-point DFT, $x[n]=\delta[n]$ can be obtained by IDFT as,

$$x[n]=\begin{cases}\frac{1}{2}\left(X[0]\,W_2^{-0n}+X[1]\,W_2^{-1n}\right) & n=0,1\\ 0 & o.w.\end{cases}$$

$$x[n]=\begin{cases}\frac{1}{2}\left(X[0]\times 1+X[1]\,e^{j\pi n}\right) & n=0,1\\ 0 & o.w.\end{cases}$$

$$x[0]=\frac{1}{2}(1+1)=1 \qquad x[1]=\frac{1}{2}(1-1)=0$$

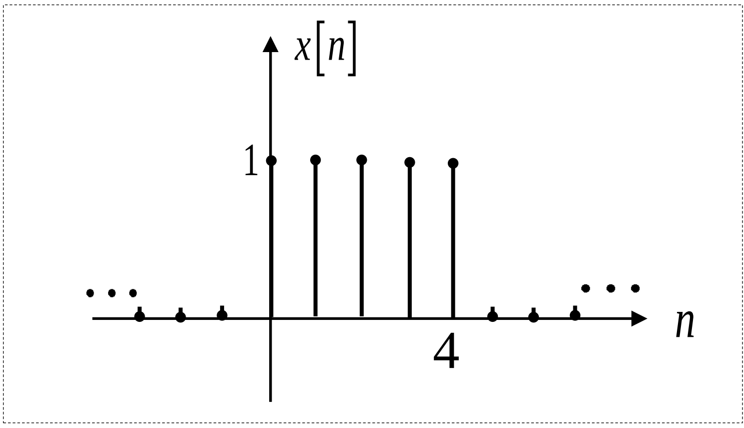
Using 3-point DFT, $x[n]=\delta[n]$ can be obtained by IDFT as,

$$x[n]=\begin{cases}\frac{1}{3}\left(X[0]\,W_3^{-0n}+X[1]\,W_3^{-1n}+X[2]\,W_3^{-2n}\right) & n=0,1,2\\ 0 & o.w.\end{cases}$$

$$x[n]=\begin{cases}\frac{1}{3}\left(X[0]\times 1+X[1]\,e^{j\frac{2\pi}{3}n}+X[2]\,e^{j2\frac{2\pi}{3}n}\right) & n=0,1,2\\ 0 & o.w.\end{cases}$$

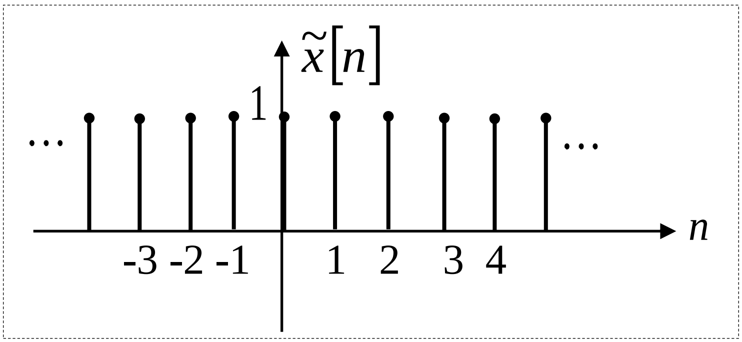
$$x[0]=\frac{1}{3}(1+1+1)=1 \qquad x[1]=\frac{1}{3}\left(1+e^{j\frac{2\pi}{3}}+e^{j2\frac{2\pi}{3}}\right)=0 \qquad x[2]=\frac{1}{3}\left(1+e^{j\frac{2\pi}{3}2}+e^{j2\frac{2\pi}{3}2}\right)=0$$

Ex: $x[n]:\ \ldots 0\ 0\ \underset{\substack{\uparrow \\ n=0}}{1}\ 1\ 1\ 1\ 1\ 0\ 0 \ldots$



Length of $x[n]$ is 5.

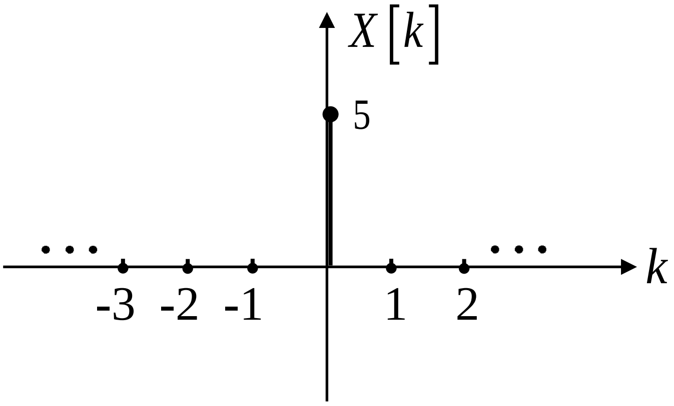
$$\tilde{x}[n]=\sum_{p=-\infty}^{\infty} x[n-5p]$$



Let's consider 5-point DFT ($N=5$), so

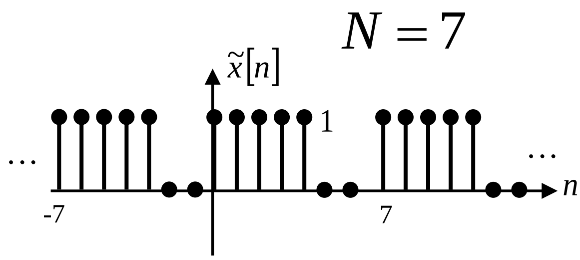
$$X[k]=\begin{cases}\displaystyle\sum_{n=0}^{4} x[n]W_5^{kn} & k=0,1,\ldots,4 \\ 0 & \text{o.w.}\end{cases}$$

$$\sum_{n=0}^{4} x[n]W_5^{kn}=\frac{1-W_5^{k5}}{1-W_5^{k}}=\frac{1-e^{-j2\pi k}}{1-e^{-j\frac{2\pi}{5}k}}=\begin{cases}5 & k=0 \\ 0 & k=1,2,3,4\end{cases}=5\delta[k]$$



Ex: Continued

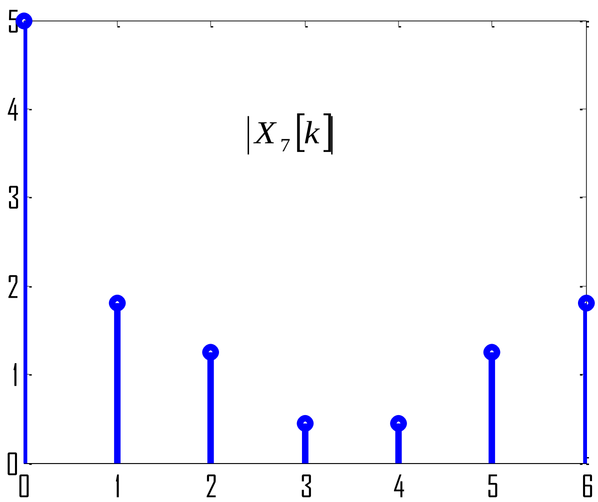
$$\tilde{x}[n] = \sum_{p=-\infty}^{\infty} x[n-7p]$$

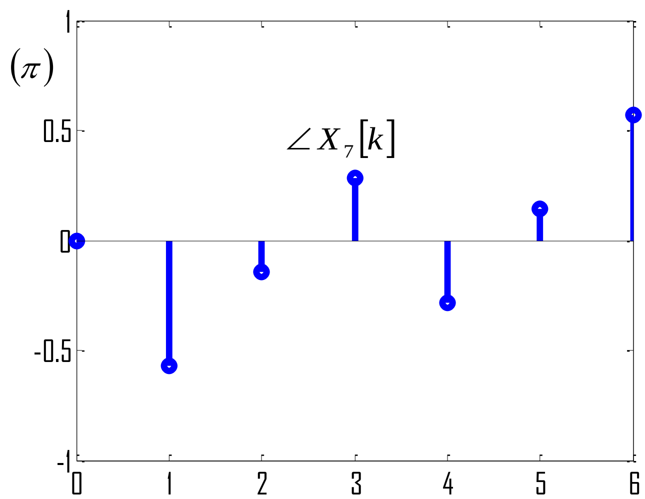


Now consider 7-point DFT (N =7), so

$$X[k] = \begin{cases} \sum_{n=0}^{4} x[n] W_7^{kn} & k = 0,1,\ldots,6 \\ 0 & \text{ow.} \end{cases}$$

$$\sum_{n=0}^{4} x[n] W_7^{kn} = \frac{1-W_7^{k5}}{1-W_7^{k}} = \frac{1-e^{-j\frac{10\pi}{7}k}}{1-e^{-j\frac{2\pi}{7}k}} = \frac{e^{-j\frac{5\pi}{7}k}(e^{j\frac{5\pi}{7}k} - e^{-j\frac{5\pi}{7}k})}{e^{-j\frac{\pi}{7}k}(e^{j\frac{\pi}{7}k} - e^{-j\frac{\pi}{7}k})} = e^{-j\frac{4\pi}{7}k} \frac{\sin(\frac{5\pi}{7}k)}{\sin(\frac{\pi}{7}k)}$$

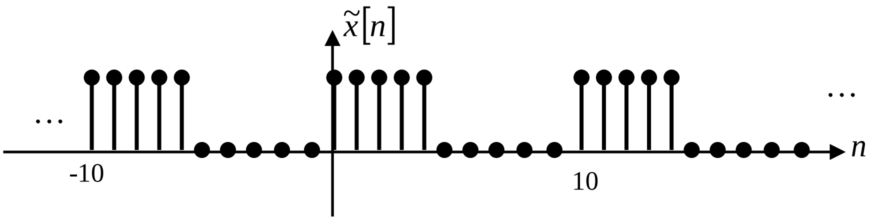




Now consider 10-point DFT ($N = 10$)

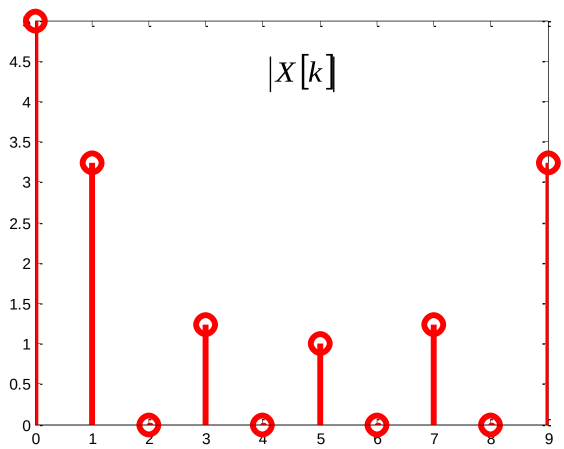
$$\tilde{x}[n] = \sum_{p=-\infty}^{\infty} x[n-10p] \qquad N = 10$$

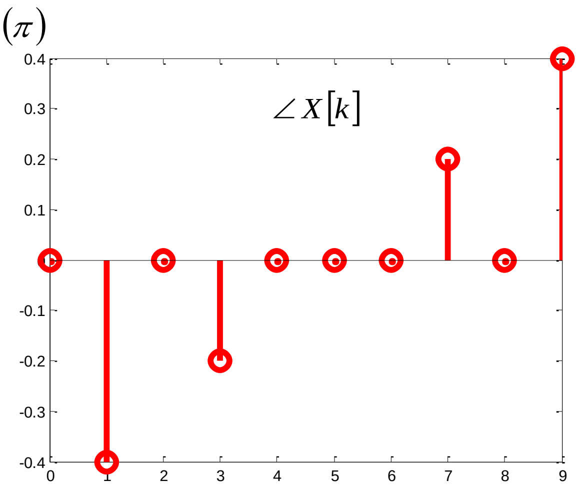
So



$$X[k] = \begin{cases} \sum_{n=0}^{4} x[n] W_{10}^{kn} & k = 0,1,\ldots,9 \\ 0 & \text{ow.} \end{cases}$$

$$\sum_{n=0}^{4} x[n] W_{10}^{kn} = \frac{1-W_{10}^{k5}}{1-W_{10}^{k}} = \frac{1-e^{-j\pi k}}{1-e^{-j\frac{\pi k}{5}}} = \frac{e^{-j\frac{\pi k}{2}}\left(e^{j\frac{\pi k}{2}} - e^{-j\frac{\pi k}{2}}\right)}{e^{-j\frac{\pi k}{10}}\left(e^{j\frac{\pi k}{10}} - e^{-j\frac{\pi k}{10}}\right)} = e^{-j\frac{4\pi}{10}k} \frac{\sin(\frac{\pi k}{2})}{\sin(\frac{\pi}{10}k)}$$





Note that all of the above DFTs can be used to get $x[n]$ back!

THE SIGNIFICANCE OF DFT AND DFS: SAMPLING THE DTFT

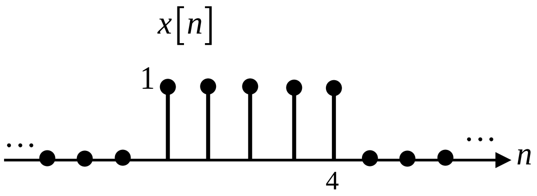
$\tilde{X}[k]$ are N uniformly spaced samples of $X(e^{j\omega})$.

Therefore, also are $X[k]$.

$$X[k] = X(e^{j\omega})\Big|_{\omega=\frac{2\pi}{N}k}$$

$$= \sum_{n=0}^{N-1} x[n]e^{-j\left(\frac{2\pi}{N}k\right)n} \qquad\qquad k = 0,1,\dots,N-1$$

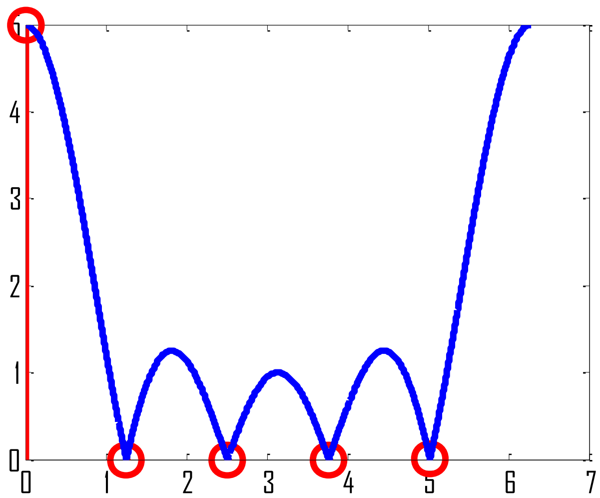
Ex: Continued



$$X\left(e^{j\omega}\right)=\sum_{n=0}^{4}e^{-j\omega n}=e^{-j2\omega}\frac{\sin\left(\frac{5\omega}{2}\right)}{\sin\left(\frac{\omega}{2}\right)}$$

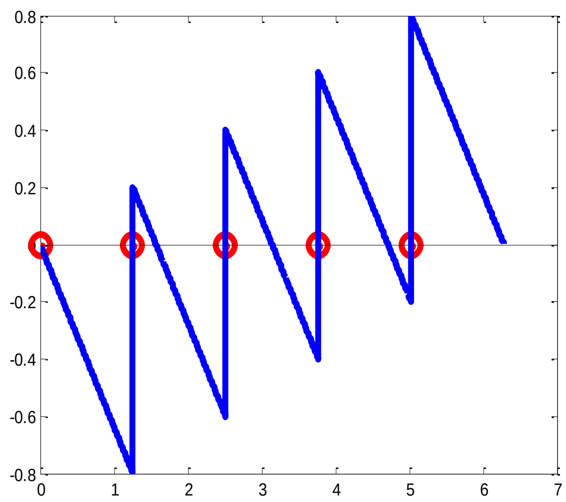
5-point DFT

$|X_5[k]|$

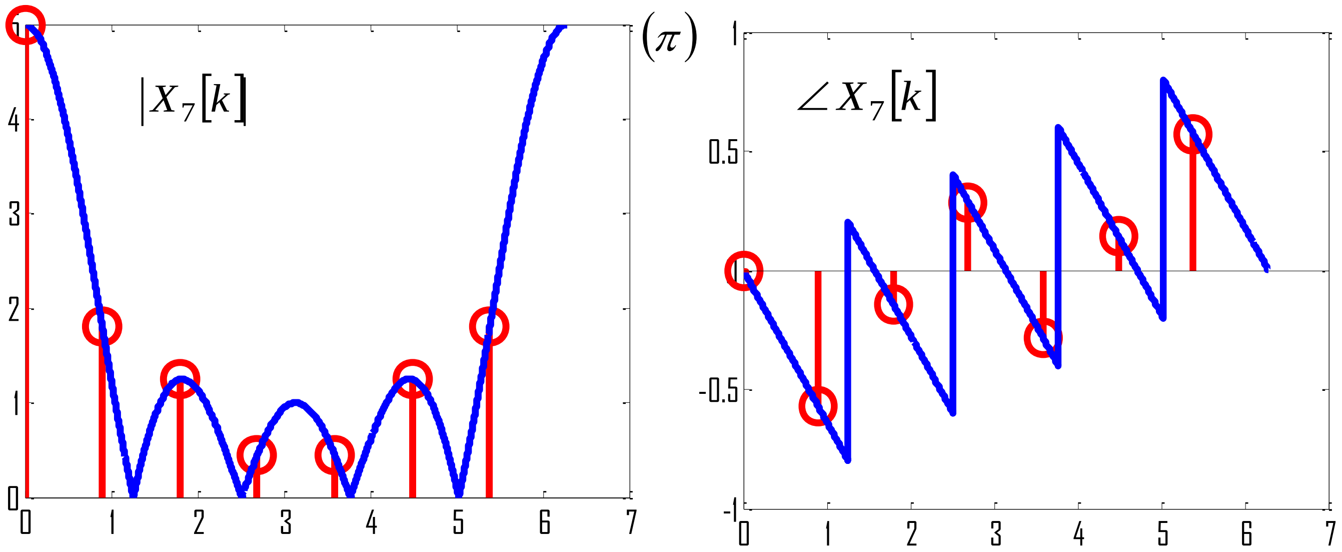


$\angle X_5[k]$

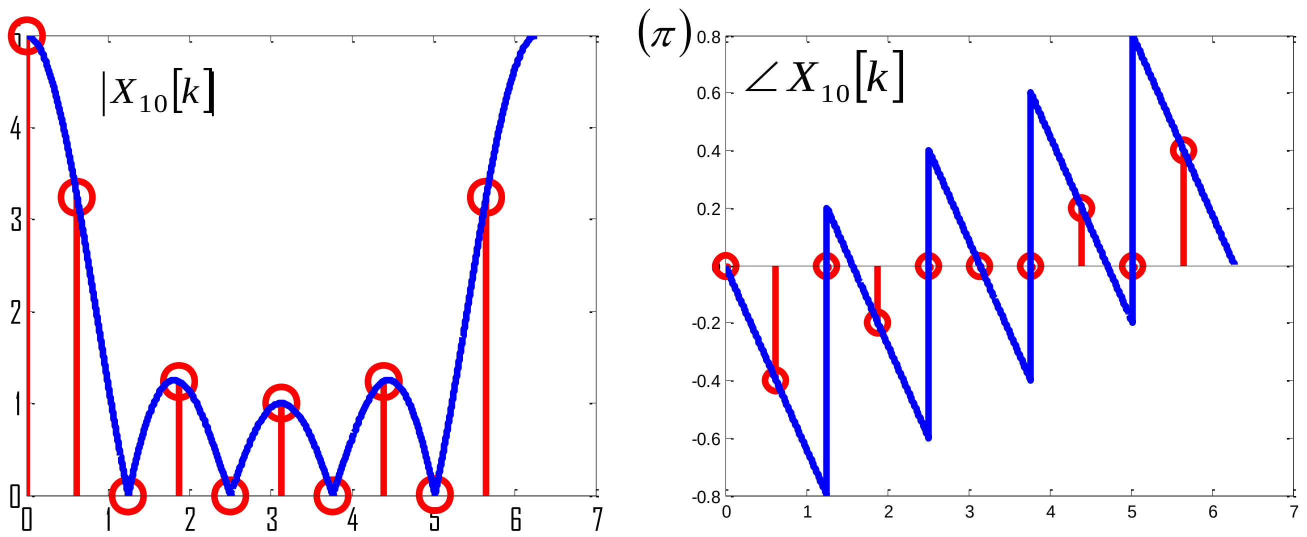
(π)



7-point DFT



10-point DFT



HORIZONTAL SCALE !

PROPERTIES OF DFS AND DFT

Effectively, properties of DFT and DFS are the same.

We just need to fit the notation!

1) Linearity

DFS

Let $\tilde{x}_1[n]$ and $\tilde{x}_2[n]$ be periodic sequences with the same period *N*.

$$\tilde{x}_1[n] \xleftrightarrow{\text{DFS}} \tilde{X}_1[k]$$

$$\tilde{x}_2[n] \xleftrightarrow{\text{DFS}} \tilde{X}_2[k]$$

Then

$$\tilde{x}_3[n] = a\tilde{x}_1[n] + b\tilde{x}_2[n] \xleftrightarrow{\text{DFS}} \tilde{X}_3[k] = a\tilde{X}_1[k] + b\tilde{X}_2[k]$$

DFT

Let $x_1[n]$ and $x_2[n]$ be finite-length sequences

$$x_1[n] \xleftrightarrow{N\text{-point DFT}} X_1[k]$$

$$x_2[n] \xleftrightarrow{N\text{-point DFT}} X_2[k]$$

Then

$$x_3[n] = ax_1[n] + bx_2[n] \xleftrightarrow{N\text{-point DFT}} X_3[k] = aX_1[k] + bX_2[k]$$

Ex: Let

$$x_1[n]=\delta[n]+\delta[n-1] \xleftrightarrow{\text{4-point DFT}} X_1[k]=1+e^{-j\frac{2\pi}{4}k}$$

(note that $x_1[n]$ is a 2-point sequence)

and

$$x_2[n]=\delta[n]+\delta[n-1]+\delta[n-2]+\delta[n-3] \xleftrightarrow{\text{4-point DFT}} X_1[k]=1+e^{-j\frac{2\pi}{4}k}+e^{-j\frac{2\pi}{4}k2}+e^{-j\frac{2\pi}{4}k3}$$
$$=1+e^{-j\frac{\pi}{2}k}+e^{-j\pi k}+e^{-j\frac{3\pi}{2}k}$$

$$x_3[n]=x_1[n]+x_2[n]=2\delta[n]+2\delta[n-1]+\delta[n-2]+\delta[n-3]$$
$$\xleftrightarrow{\text{4-point DFT}} X_3[k]=2+2e^{-j\frac{\pi}{2}k}+e^{-j\pi k}+e^{-j\frac{3\pi}{2}k}=X_1[k]+X_2[k]$$

2) Time Shift Property (DFS) – Circular Time Shift Property (DFT)

Shift Property (DFS)

$$\tilde{x}[n] \xleftrightarrow{\text{DFS}} \tilde{X}[k] \quad \Rightarrow \quad \tilde{x}[n-\Delta] \xleftrightarrow{\text{DFS}} e^{-j\frac{2\pi}{N}k\Delta}\tilde{X}[k] = W_N^{k\Delta}\tilde{X}[k]$$

Proof:

$$\tilde{y}[n] = \tilde{x}[n-\Delta]$$

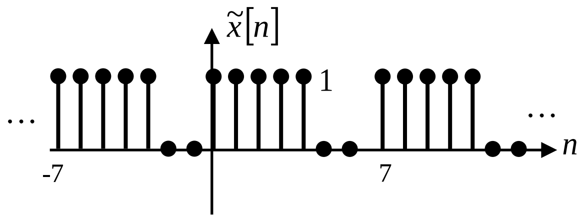
$$\tilde{Y}[k] = \sum_{n=0}^{N-1} \tilde{x}[n-\Delta]\, e^{-jk\frac{2\pi}{N}n} \quad \text{let } m = n-\Delta$$

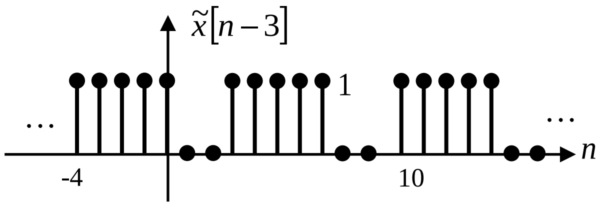
$$\tilde{Y}[k] = e^{-j\frac{2\pi}{N}\Delta} \sum_{m=-\Delta}^{N-1-\Delta} \tilde{x}[m]\, e^{-j\frac{2\pi}{N}m}$$

Since $\tilde{x}[m]$ and $e^{-j\frac{2\pi}{N}m}$ are periodic with N and the summation is over N consecutive values

$$\tilde{Y}[k] = e^{-j\frac{2\pi}{N}\Delta} \sum_{m=0}^{N-1} \tilde{x}[m]\, e^{-j\frac{2\pi}{N}m} = e^{-j\frac{2\pi}{N}\Delta}\, \tilde{X}[k]$$

Ex:





$$\tilde{x}[n-3] \xleftrightarrow{DFS} W_7^{3k}\,\tilde{X}[k]$$

$$\tilde{X}[k] = e^{-j\frac{4\pi}{7}k}\,\frac{\sin\left(\frac{5\pi}{7}k\right)}{\sin\left(\frac{\pi}{7}k\right)}$$

$$= \underbrace{W_7^{3k}\; e^{-j\frac{4\pi}{7}k}}_{e^{-j\frac{2\pi}{7}5k}}\,\frac{\sin\left(\frac{5\pi}{7}k\right)}{\sin\left(\frac{\pi}{7}k\right)}$$

Circular Shift Property (DFT)

$$x[n] \xleftrightarrow{\text{DFT}} X[k]$$

$$a[n] = ? \xleftrightarrow{\text{DFT}} W_N^{k\Delta} X[k]$$

Since

$$x[n] \triangleq \begin{cases} \tilde{x}[n] & n = 0,1,\ldots,N-1 \\ 0 & n \neq 0,1,\ldots,N-1 \end{cases}$$

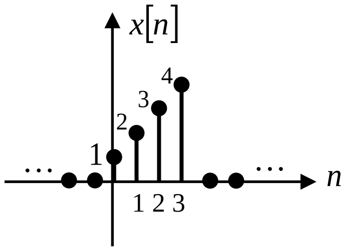
$$\Rightarrow \quad a[n] \triangleq \begin{cases} \tilde{x}[n-\Delta] & n = 0,1,\ldots,N-1 \\ 0 & n \neq 0,1,\ldots,N-1 \end{cases}$$

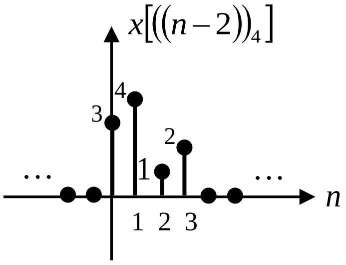
Or can be written as

$$a[n] = \begin{cases} x[((n-\Delta))_N] & n = 0,1,\ldots,N-1 \\ 0 & o.w. \end{cases}$$

Ex: If “signal length = DFT length”.

$$x[n] \xleftrightarrow{\text{4-point DFT}} X[k] \qquad x\left[((n-2))_4\right] \xleftrightarrow{\text{4-point DFT}} W_4^{k2} X[k]$$





Distorted

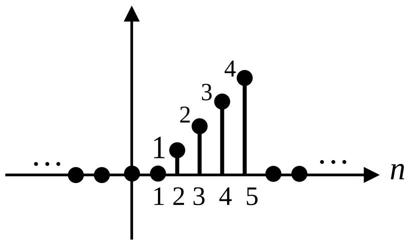
Ex: continued, “signal length < DFT length”.

If 6-point DFT is used $x[n] \xleftrightarrow{\text{6–pointDFT}} X[k]$

$$x[(n-2)_6] \xleftrightarrow{\text{6–pointDFT}} W_6^{2k} X[k]$$

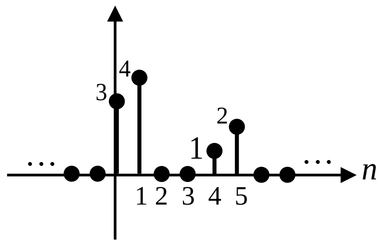
$$x\left[\left(\left(n+2\right)\right)_6\right] \xleftrightarrow{\text{6-point DFT}} W_6^{-k2} X[k]$$

$$x[((n-2))_6]$$



Undistorted

$$x[((n-4))_6] = x[((n+2))_6]$$



Distorted

3) Multiplication by a complex exponential

DFS

For a periodic sequence with period *N*

$$e^{j\frac{\Delta 2\pi}{N}n}\tilde{x}[n] = W_N^{-\Delta n}\tilde{x}[n] \qquad \leftrightarrow \qquad \tilde{X}[k-\Delta]$$

DFT

$$e^{j\Delta\frac{2\pi}{N}n}x[n] = W_N^{-\Delta n}x[n] \xleftrightarrow{N-\text{point DFT}} X\left[\left(\left(k-\Delta\right)\right)_N\right] \quad k = 0, 1, \ldots, N-1$$

Note that the length, N_x, of $x[n]$ has to satisfy $N \geq N_x$.

Ex: Let $x[n]$ be of length 7.

Find N and Δ so that

$$e^{j\frac{2\pi}{3}n}x[n]=W_N^{-\Delta n}x[n]\xleftrightarrow{N-\text{pointDFT}}X\left[((k-\Delta))_N\right] \quad k=0,1,\ldots,N-1$$

For $N = 9$ and $\Delta = 3$

$$e^{j3\frac{2\pi}{9}n}x[n]=W_9^{-3n}x[n]\xleftrightarrow{9-\text{pointDFT}}X\left[((k-3))_9\right] \quad k=0,1,\ldots,N-1$$

where $X[k]$ is the 9-point DFT of $x[n]$.

Other solutions: $N = 12 \quad \Delta = 4,$

$N = 15 \quad \Delta = 5, \quad \ldots$

In general $e^{j\frac{2\pi}{3}} = e^{j\Delta\frac{2\pi}{N}\underbrace{\left(\frac{N}{3\Delta}\right)}_{1}}$

Therefore

$$N = multiple\ of\ 3, \quad N > 7, \quad \Delta = \frac{N}{3}$$

4) Duality

DFS

$$\tilde{x}[n] \xleftrightarrow{DFS} \tilde{X}[k] \quad \Leftrightarrow \quad \tilde{X}[n] \xleftrightarrow{DFS} N\,\tilde{x}[-k]$$

Proof:

$$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1} \tilde{X}[k] W_N^{-kn} \quad \leftrightarrow \quad \boxed{\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n] W_N^{kn}}$$

$$\Downarrow$$

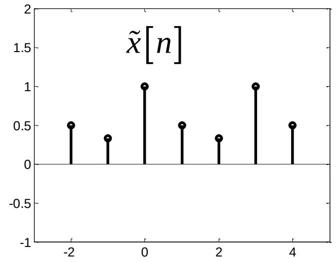
compare

$$\boxed{\underbrace{N\,\tilde{x}[-n]}_{\text{DFS coefficients of the sequence } \tilde{X}[k]} = \sum_{k=0}^{N-1} \tilde{X}[k] W_N^{kn}} \quad \Rightarrow \quad \tilde{X}[n] \xleftrightarrow{DFS} N\,\tilde{x}[-k]$$

DFT

$$x[n] \xleftrightarrow{DFT} X[k] \quad \Leftrightarrow \quad X[n] \xleftrightarrow{DFT} N\,x[((-k))_N]$$

Ex: Let $\tilde{x}[n] = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \end{bmatrix}$ periodic with 3
$\uparrow$ (under 1)



$$\tilde{X}[k] = 1 + \frac{1}{2} e^{-j\frac{2\pi}{3}k} + \frac{1}{3} e^{-j\frac{2\pi}{3}2k}$$

$$= \begin{bmatrix} \underset{\uparrow}{\frac{11}{6}} & \frac{7 - j\sqrt{3}}{12} & \frac{7 + j\sqrt{3}}{12} \end{bmatrix} \quad \text{periodic with } 3$$

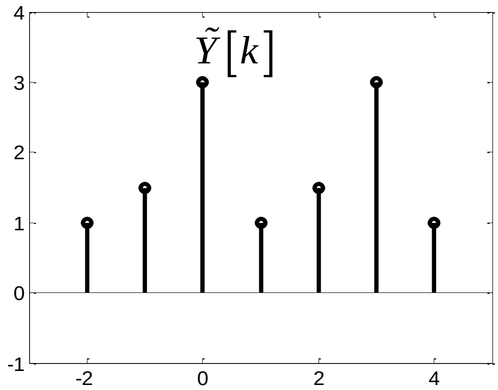
Then the DFS coefficients of the periodic sequence

$$\tilde{y}[n] = \tilde{X}[n] = \begin{bmatrix} \frac{11}{6} & \frac{7 - j\sqrt{3}}{12} & \frac{7 + j\sqrt{3}}{12} \end{bmatrix} \quad \text{periodic with } 3$$

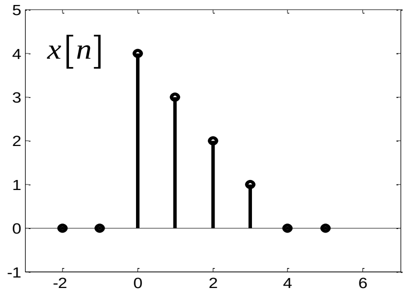
are

$$Y[k] = 3\tilde{x}[-k]$$

$$= 3\begin{bmatrix} \underset{\uparrow}{1} & \frac{1}{3} & \frac{1}{2} \end{bmatrix}$$



Ex: Let $x[n]$ be

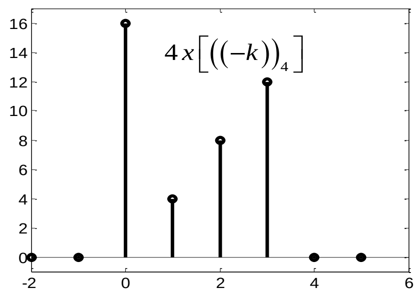


Using 4-point DFT,

$$4x\left[((-k))_4\right]$$

is the DFT of

$$X[n] = 4 + 3e^{-j\frac{2\pi}{4}n} + 2e^{-j\frac{2\pi}{4}2n} + e^{-j\frac{2\pi}{4}3n}$$



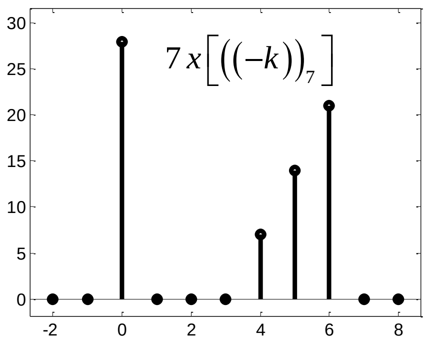
However, if, for example, 7-point DFT is used

Then

$$7\,x\left[((-k))_7\right]$$

is the DFT of

$$X[n] = 4 + 3e^{-j\frac{2\pi}{7}n} + 2e^{-j\frac{2\pi}{7}2n} + e^{-j\frac{2\pi}{7}3n}$$



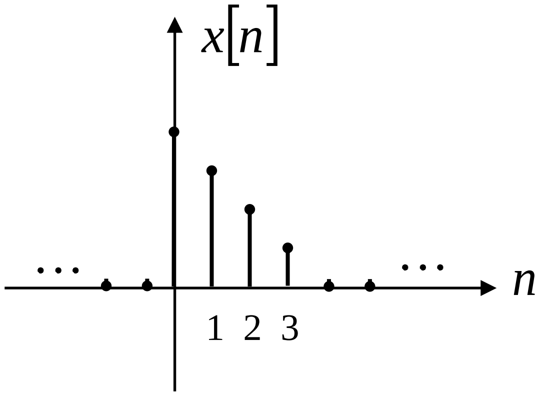
5) Symmetry Properties

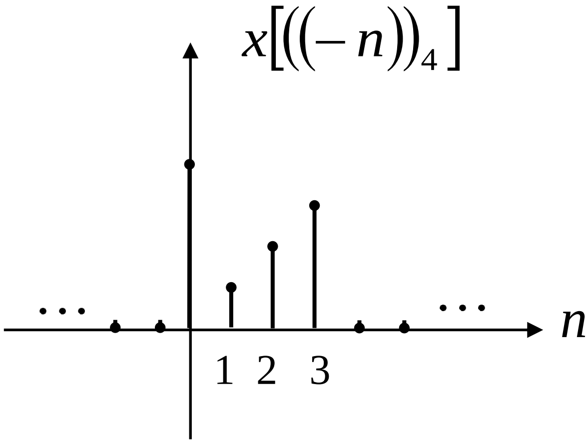
Symmetry Properties of DFT are strictly related to those of DTFT

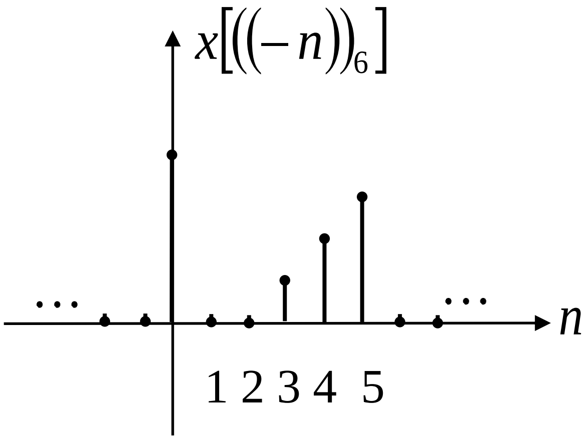
$$\underset{DFT}{X[k]} = \underbrace{X\left(e^{j\omega}\right)}_{DTFT}\Big|_{\omega=\frac{2\pi}{N}k}$$

DFS	DFT n =0,1,…,N-1 k =0,1,…,N-1
$\tilde{x}[n] \leftrightarrow \tilde{X}[k]$	$x[n] \leftrightarrow X[k]$
$\tilde{x}^*[n] \leftrightarrow \tilde{X}^*[-k]$	$x^*[n] \leftrightarrow X^*[((-k))_N]$
$\tilde{x}[-n] \leftrightarrow \tilde{X}[-k]$	$x[((-n))_N] \leftrightarrow X[((-k))_N]$
$\mathrm{Re}\{\tilde{x}[n]\} \leftrightarrow \tilde{X}_{\mathrm{e}}[k] = \frac{1}{2}\left(\tilde{X}[k] + \tilde{X}^*[-k]\right)$ conjugate symmetric part	$\mathrm{Re}\{x[n]\} \leftrightarrow X_{\mathrm{pe}}[k] = \frac{1}{2}\left(X[k] + X^*[((-k))_N]\right)$ periodic conjugate symmetric part
$j\,\mathrm{Im}\{\tilde{x}[n]\} \leftrightarrow \tilde{X}_{\mathrm{o}}[k] = \frac{1}{2}\left(\tilde{X}[k] - \tilde{X}^*[-k]\right)$ conjugate antisymmetric part	$j\,\mathrm{Im}\{x[n]\} \leftrightarrow X_{\mathrm{po}}[k] = \frac{1}{2}\left(X[k] - X^*[((-k))_N]\right)$ periodic conjugate antisymmetric part
$\tilde{x}_{\mathrm{e}}[n] = \frac{1}{2}\left(\tilde{x}[n] + \tilde{x}^*[-n]\right) \leftrightarrow \mathrm{Re}\{\tilde{X}[k]\}$ conjugate symmetric part	$x_{\mathrm{pe}}[n] = \frac{1}{2}\left(x[n] + x^*[((-n))_N]\right) \leftrightarrow \mathrm{Re}\{X[k]\}$ periodic conjugate symmetric part
$\tilde{x}_{\mathrm{o}}[n] = \frac{1}{2}\left(\tilde{x}[n] - \tilde{x}^*[-n]\right) \leftrightarrow j\,\mathrm{Im}\{\tilde{X}[k]\}$ conjugate antisymmetric part	$x_{\mathrm{po}}[n] = \frac{1}{2}\left(x[n] - x^*[((-n))_N]\right) \leftrightarrow j\,\mathrm{Im}\{X[k]\}$ periodic conjugate antisymmetric part

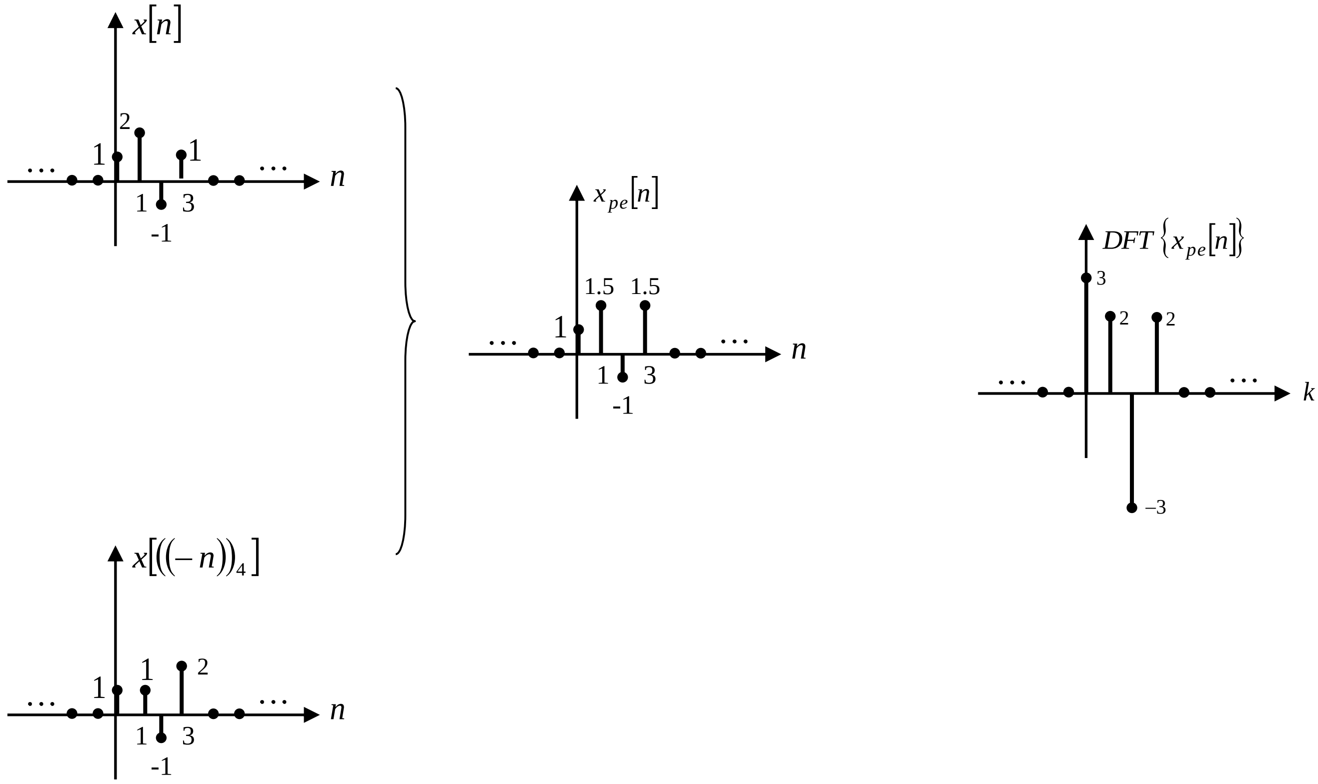
Ex: What is $x\left[((-n))_N\right]$?







Ex: The DFT of a periodic even sequence is real valued.



Real Sequences

$$x[n] = x^*[n] \quad \Rightarrow \quad X\left(e^{j\omega}\right) = X^*\left(e^{-j\omega}\right)$$

We know that DTFT is also conjugate symmetric wrt π.

Then, since DFT is obtained by uniformly sampling DTFT,

$X[k]$ is conjugate symmetric over k = 1,2,...,N-1 .

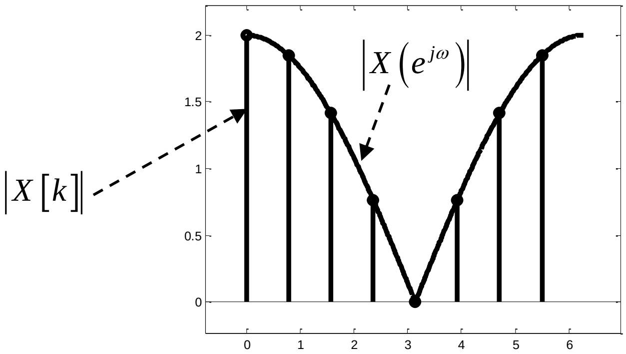
DFS	DFT
$\tilde{X}[k] = \tilde{X}^*[-k]$	$X[k] = X^*\left[((-k))_N\right]$ or $X[k] = X[N-k] \quad k = 1, 2, \ldots, N-1$
$\mathrm{Re}\{\tilde{X}[k]\} = \mathrm{Re}\{\tilde{X}[-k]\}$	$\mathrm{Re}\{X[k]\} = \mathrm{Re}\{X[N-k]\}$ $k = 1, 2, \ldots, N-1$
$\mathrm{Im}\{\tilde{X}[k]\} = -\mathrm{Im}\{\tilde{X}[-k]\}$	$\mathrm{Im}\{X[k]\} = -\mathrm{Im}\{X[N-k]\}$ $k = 1, 2, \ldots, N-1$
$\lvert\tilde{X}[k]\rvert = \lvert\tilde{X}[-k]\rvert$	$\lvert X[k]\rvert = \lvert X[N-k]\rvert$ $k = 1, 2, \ldots, N-1$
$\tilde{X}[k] = -\angle\tilde{X}[-k]$	$\angle X[k] = -\angle X[N-k]$ $k = 1, 2, \ldots, N-1$

Ex: $x[n] = \delta[n] + \delta[n-1]$ → $X\left(e^{j\omega}\right) = 1 + e^{j\omega} = e^{-j\frac{\omega}{2}} \cos\left(\frac{\omega}{2}\right)$

For 8-point DFT

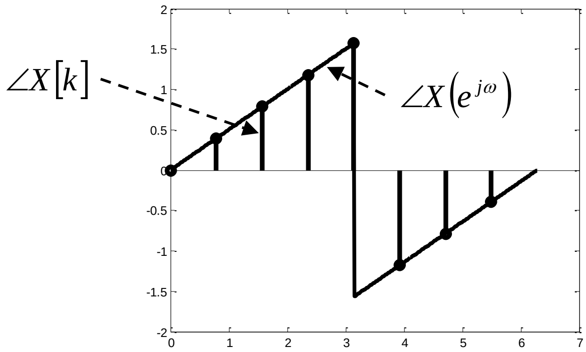
$$X[k] = 1 + e^{j\frac{2\pi}{8}k}$$

$$\left|X[k]\right| = \left|X\left[((-k))_8\right]\right|$$



$$\left|X[1]\right| = \left|X[7]\right|$$
$$\left|X[2]\right| = \left|X[6]\right|$$
$$\left|X[3]\right| = \left|X[5]\right|$$

$$\angle X[k] = -\angle X\left[((-k))_8\right]$$



$$\angle X[1] = -\angle X[7]$$
$$\angle X[2] = -\angle X[6]$$
$$\angle X[3] = -\angle X[5]$$

6) Convolution Property

DFS: Periodic Convolution

DFT: Circular Convolution

$$x[n] * y[n] \leftrightarrow X\left(e^{j\omega}\right)Y\left(e^{j\omega}\right)$$

$$\tilde{x}_1[n] \leftrightarrow \tilde{X}_1[k] \quad \text{and} \quad \tilde{x}_2[n] \leftrightarrow \tilde{X}_2[k]$$

(same fund. period)

$$\tilde{X}_3[k] = \tilde{X}_1[k]\tilde{X}_2[k]$$

$$\tilde{x}_3[n] = ?$$

$$\begin{aligned}
\tilde{x}_3[n] &= \frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}_1[k]\tilde{X}_2[k]W_N^{-kn} \\
&= \frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}_1[k]\left(\sum_{k=0}^{N-1}\tilde{x}_2[r]W_N^{kr}\right)W_N^{-kn} \\
&= \frac{1}{N}\sum_{r=0}^{N-1}\tilde{x}_2[r]\underbrace{\left(\sum_{k=0}^{N-1}\tilde{X}_1[k]W_N^{k(r-n)}\right)}_{N\tilde{x}_1[n-r]}
\end{aligned}$$

herefore

$$\tilde{x}_3[n] = \sum_{r=0}^{N-1}\tilde{x}_1[n-r]\,\tilde{x}_2[r]$$

periodic convolution of $\tilde{x}_1[n]$ and $\tilde{x}_2[n]$.

Ex:

```
x=1:5;
y=[1 0 0 -2 0];
X=fft(x);
Y=fft(y);
Z=X.*Y;
z=ifft(Z)          (z = [-5 -6 -7 2 1])
stem(z)
```

$$x_1[n] \xleftrightarrow{N\text{-point DFT}} X_1[k] \quad \text{and} \quad x_2[n] \xleftrightarrow{N\text{-point DFT}} X_2[k],$$

$$X_3[k] = X_1[k] X_2[k]$$

$$x_3[n] = ?$$

$$x_3[n] = \begin{cases} \tilde{x}_3[n] & n = 0,1,\ldots,N-1 \\ 0 & o.w. \end{cases}$$

Using the result from DFS

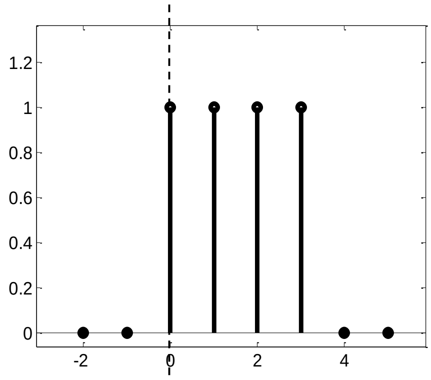
$$x_3[n] = \begin{cases} \sum_{r=0}^{N-1} x_1[((n-r))_N]\, x_2[r] & n = 0,1,\ldots,N-1 \\ 0 & o.w. \end{cases}$$

This is called "***N*-point circular convolution**" of $x_1[n]$ and $x_2[n]$

$$x_3[n] = x_1[n] \circledast_4 x_2[n]$$

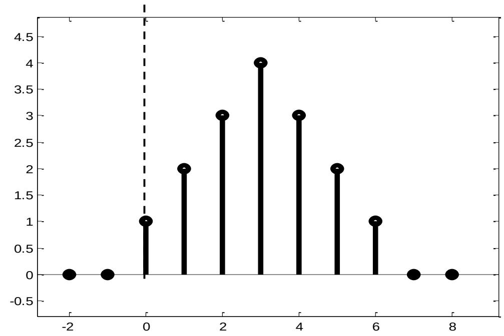
Ex: Linear convolution

$$x[n]:\ [\cdots 0\ 0\ \underset{\substack{\uparrow\\ n=0}}{1}\ 1\ 1\ 1\ 0\ 0 \cdots]$$



The linear convolution of *x*[*n*] with itself is

$$y[n] = x[n] * x[n] \qquad\qquad y[n]:\ [\cdots 0\ 0\ \underset{\substack{\uparrow\\ n=0}}{1}\ 2\ 3\ 4\ 3\ 2\ 1\ 0\ 0 \cdots]$$



Note that he length of $y[n]$ is 7 (= 4+4-1)

Ex: Continued: 4-point circular convolution of $x[n]$ with itself

Let

$$W[k] = X_4[k]X_4[k]$$

and $w[n]$ be the IDFT of $W[k]$, then

$$w[n] = x[n] \circledast_4 x[n]$$

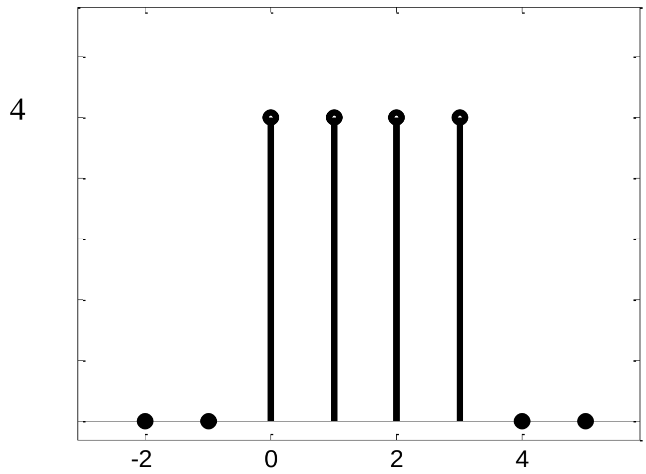
$$= \sum_{r=0}^{3} x\left[((n-r))_4\right] x[r]$$

To compute one needs

$$x\left[((-r))_4\right], x\left[((1-r))_4\right], x\left[((2-r))_4\right], x\left[((3-r))_4\right]$$

Then

$$w[n] = \sum_{r=0}^{3} 1 = 4\,, \qquad n = 0,1,2,3$$



This is not equal to the result of linear convolution!

Ex: Continued

6-point circular convolution of $x[n]$ with itself

Let

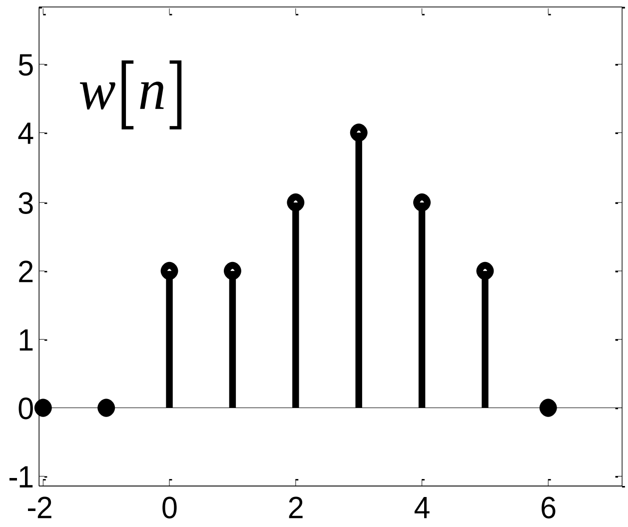
$$W[k] = X_6[k]X_6[k]$$

and $w[n]$ be the IDFT of $W[k]$, then

$$w[n] = x[n] \circledast_6 x[n]$$

$$= \sum_{r=0}^{5} x\left[\left((n-r)\right)_6\right] x[r]$$

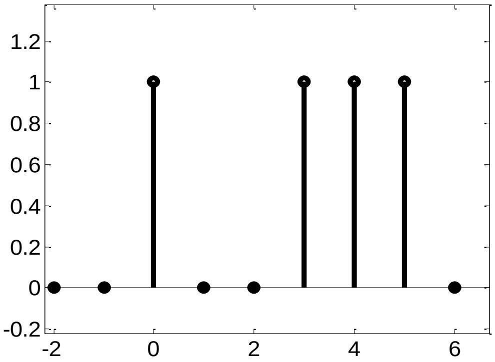
$$w[n] = \left[\dots 0 \quad 0 \quad \underbrace{2}_{n=0} \quad 2 \quad 3 \quad 4 \quad 3 \quad 2 \quad 0 \quad 0 \dots\right]$$



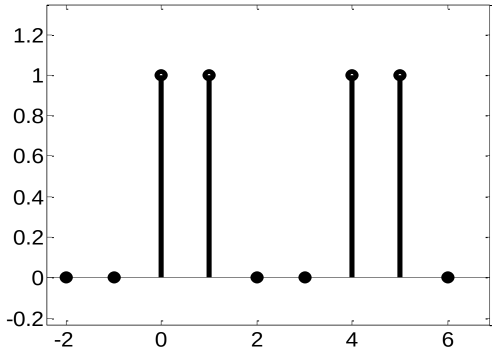
This is also not the same as the result of linear convolution!

However, partially correct!

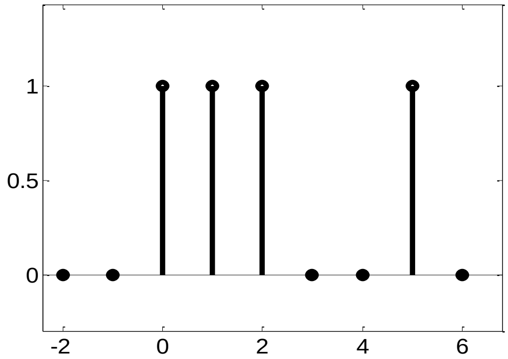
$$x\left[\left(\left(-r\right)\right)_6\right]$$



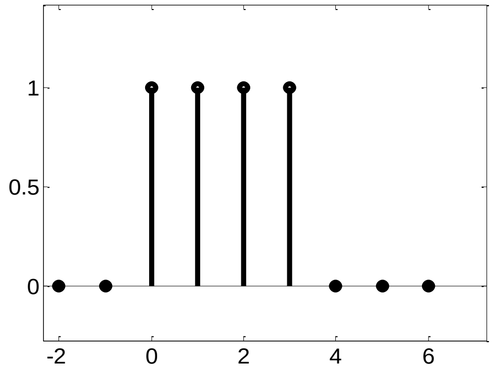
$$x\left[\left(\left(1-r\right)\right)_6\right]$$



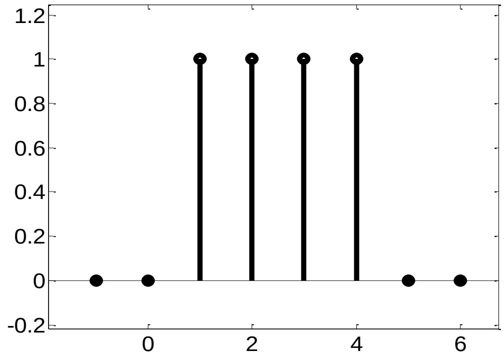
$$x\left[\left(\left(2-r\right)\right)_6\right]$$



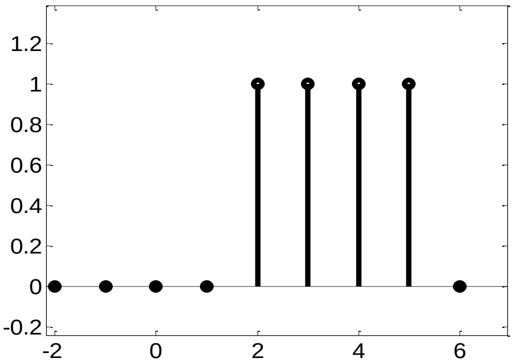
$$x\left[\left(\left(3-r\right)\right)_6\right]$$



$$x\left[\left(\left(4-r\right)\right)_6\right]$$



$$x\left[\left(\left(5-r\right)\right)_6\right]$$



Ex: Continued

7-point circular convolution of *x*[*n*] with itself

$$W[k] = X_7[k]X_7[k]$$

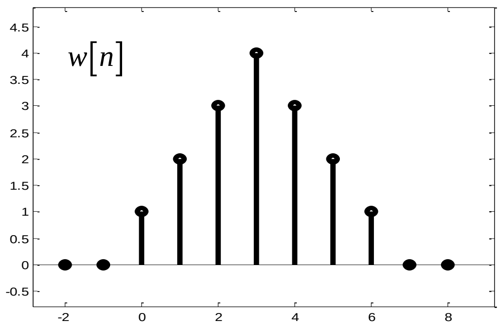
$$w[n] = x[n] \circledast_7 x[n]$$

$$= \sum_{r=0}^{6} x\left[((n-r))_7\right] x[r]$$

$$w[n] = \left[\dots 0 \quad 0 \quad \underbrace{2}_{n=0} \quad 2 \quad 3 \quad 4 \quad 3 \quad 2 \quad 0 \quad 0 \dots\right]$$

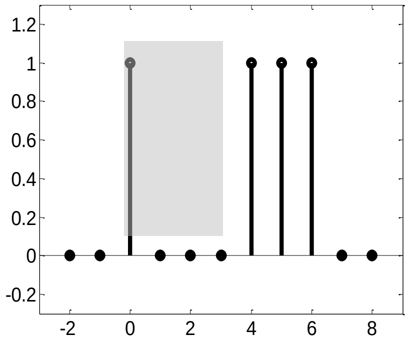
$w[n]$ is the IDFT of $X_7[k]X_7[k]$, $X_7[k]$: 7-point DFT

$$w[n] = \left[\dots 0 \quad 0 \quad \underbrace{1}_{n=0} \quad 2 \quad 3 \quad 4 \quad 3 \quad 2 \quad 1 \quad 0 \quad 0 \dots\right]$$

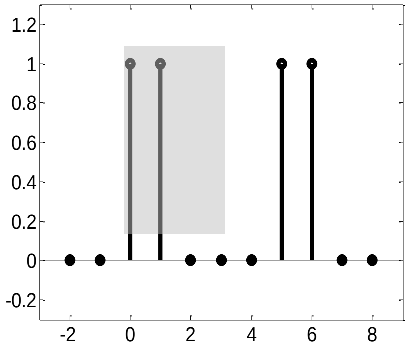


In this case the result is the same as that of linear convolution.

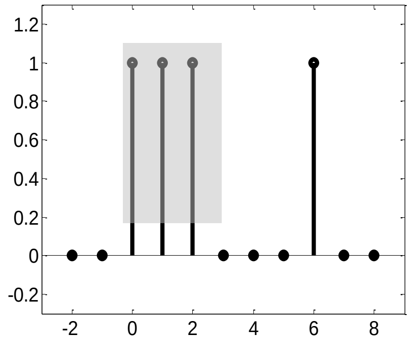
$x\left[\left(\left(-r\right)\right)_7\right]$



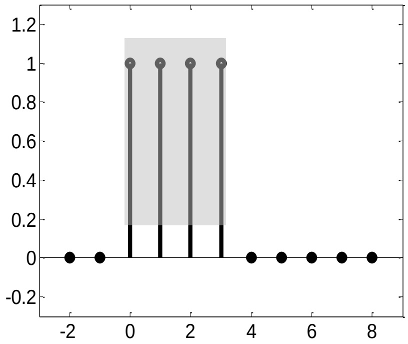
$x\left[\left(\left(1-r\right)\right)_7\right]$



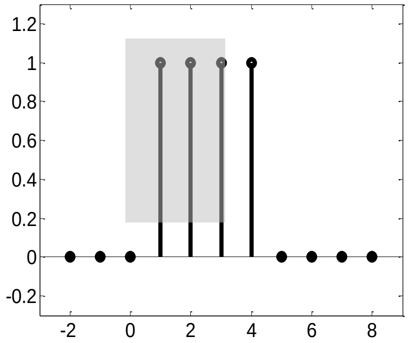
$x\left[\left(\left(2-r\right)\right)_7\right]$



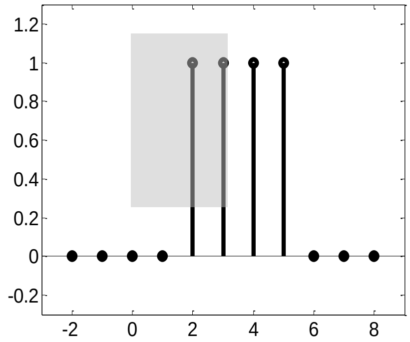
$x\left[\left(\left(3-r\right)\right)_7\right]$



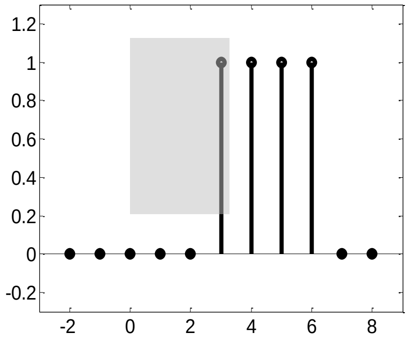
$x\left[\left(\left(4-r\right)\right)_7\right]$



$x\left[\left(\left(5-r\right)\right)_7\right]$



$x\left[\left(\left(6-r\right)\right)_7\right]$



Nonzero values of $x[r]$ are over the shaded region.