The Significance of Linear Phase

```
clear all
close all
b = [1.2 0 1.2];
a = [1 0 0.81];
[H,w]=freqz(b,a,1024);
plot(w/pi,abs(H))

% [b,a] = cheby2(9,30,0.25);
% [H,w] = freqz(b,a,1024);
% plot(w/pi,20*log10(abs(H)))
title('magnitude')
figure
plot(w/pi,unwrap(angle(H))/pi)
grid
title('phase')
noOFperiods = 20;
fS = 10000;
f1 = 0.5*fS / 10;
f2 = 0.5*fS / 5;
n = 0:round(noOFperiods * fS / f1);
x = cos(n * 2*pi*f1 / fS) + 0.3 * cos(n * 2*pi*f2 / fS);
y = filter(b,a,x);
figure
plot(x)
hold on
plot(y,'r')
title('input-output')
legend('input','output');
```

GENERALIZED LINEAR PHASE SYSTEMS

Linear Phase Systems

$$\measuredangle H\left(e^{j\omega}\right) = -\alpha\omega$$

Ex: α: integer

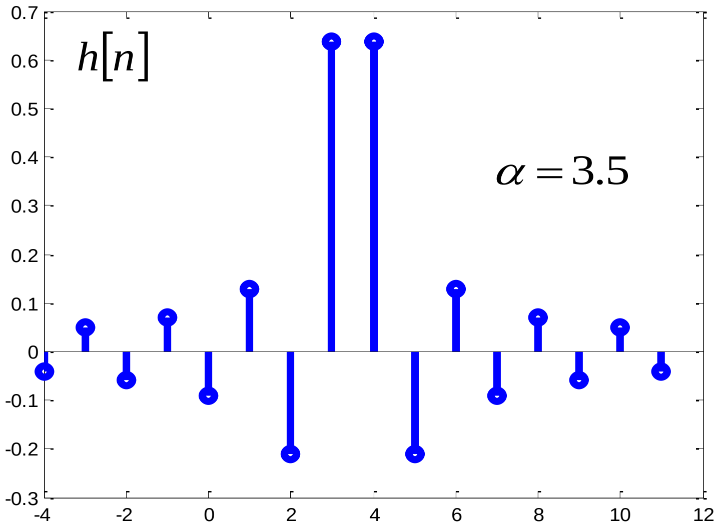
$$h[n] = \delta[n-\alpha]$$

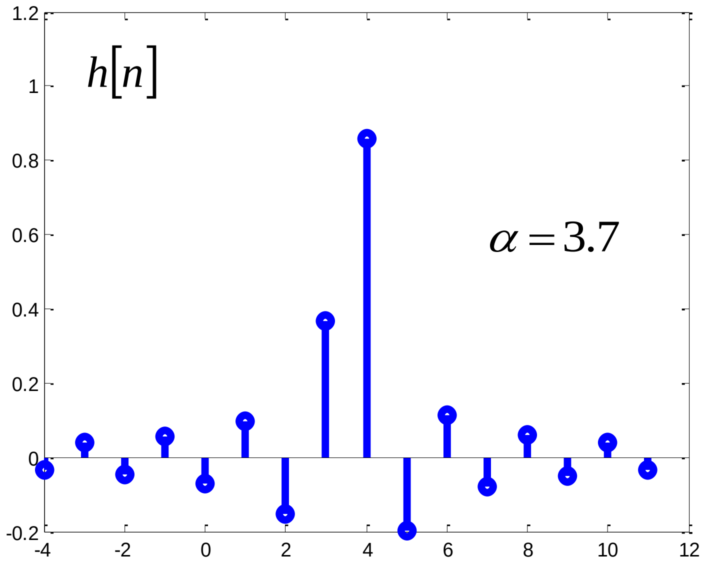
$$H\left(e^{j\omega}\right) = e^{-j\alpha\omega}$$

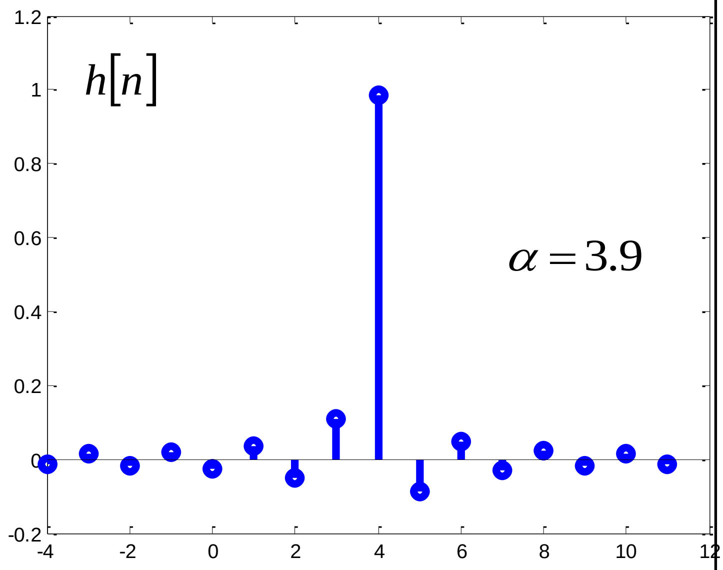
Ex: α: noninteger

$$H\left(e^{j\omega}\right) = e^{-j\alpha\omega}$$

$$h[n] = \frac{\sin\left(\pi(n-\alpha)\right)}{\pi(n-\alpha)}$$







There are systems having “piecewise linear” phase responses and constant group delay.

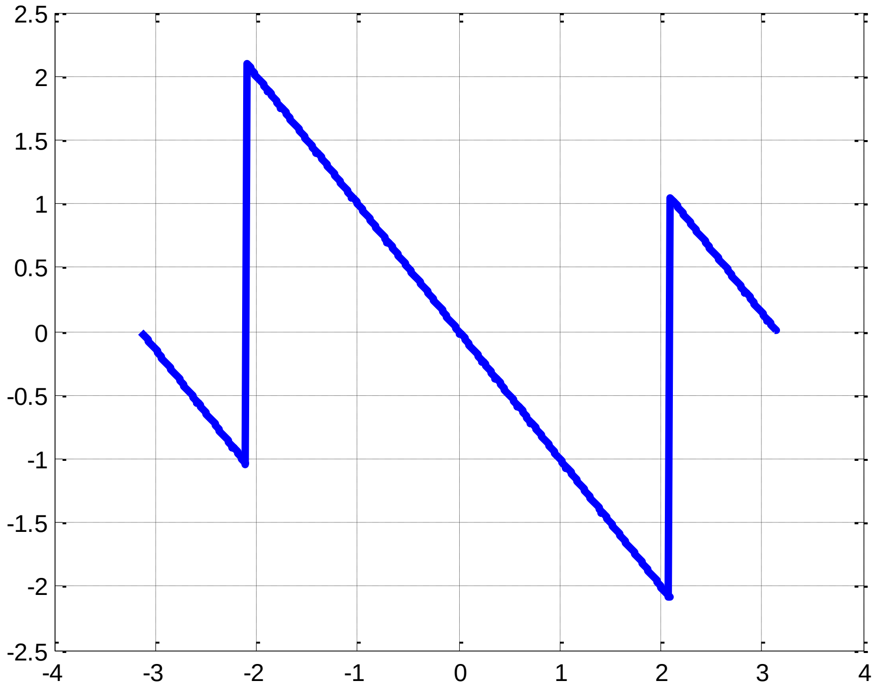
```
clear all
close all
N=40;
n = -N:N;
alpha=3.9;
b = (sin(pi*(n-alpha)))./(pi*(n-alpha));
a = 1;
[H,w]=freqz(b,a,1024);
plot(w/pi,abs(H))

title('magnitude')
figure
plot(w/pi,unwrap(angle(H))/pi)
grid
title('phase')
noOFperiods = 100;
fS = 10000;
f1 = 0.5*fS / 10;
f2 = 0.5*fS / 5;
n = 0:round(noOFperiods * fS / f1);
x = cos(n * 2*pi*f1 / fS) + 0.3 * cos(n * 2*pi*f2 / fS);
y = filter(b,a,x);
figure
plot(x)
hold on
plot(y,'r')
title('input-output')
legend('input','output');
```

$$h[n] = [1 \quad 1 \quad 1]$$

$$H(e^{j\omega}) = e^{-j\omega}(1 + 2\cos\omega)$$

$$ARG\left(H(e^{j\omega})\right)$$

$(\times\, \pi)\ rad$



$\omega, (\times\, \pi)\ rad$

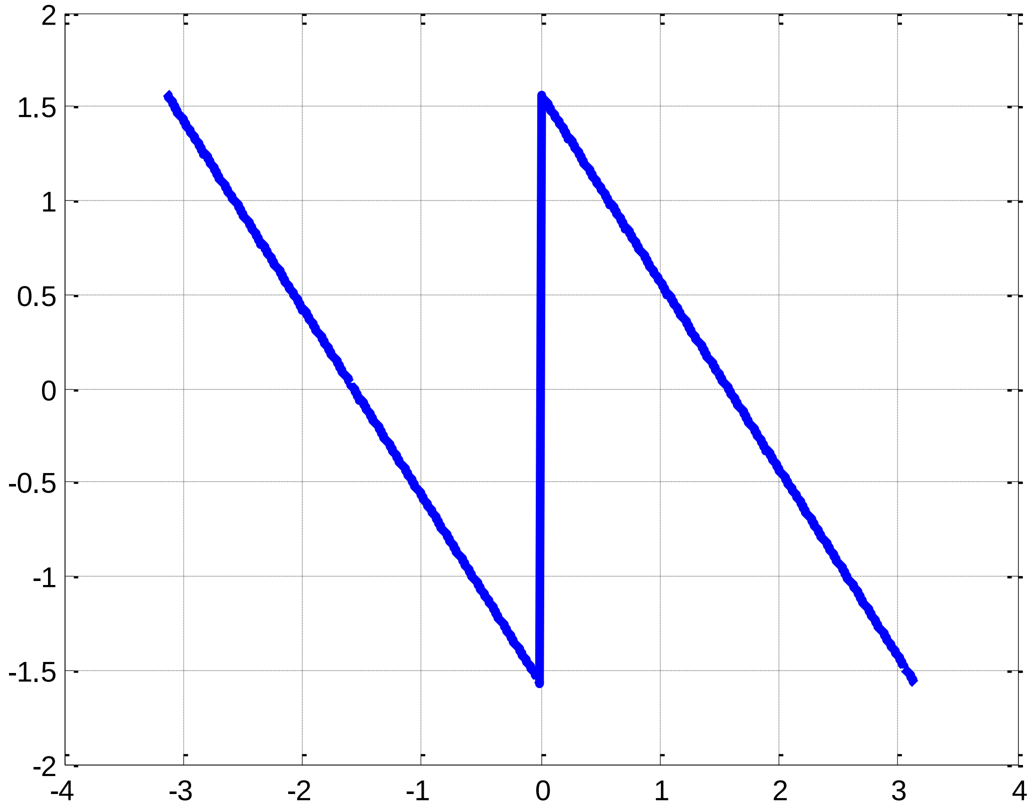
$$h[n] = [1 \quad 0 \quad -1]$$

$$H(e^{j\omega}) = je^{-j\omega}(2\sin\omega)$$

$$= e^{-j\left(\omega-\frac{\pi}{2}\right)}(2\sin\omega)$$

$$ARG\left(H(e^{j\omega})\right)$$

$(\times\,\pi)\;rad$

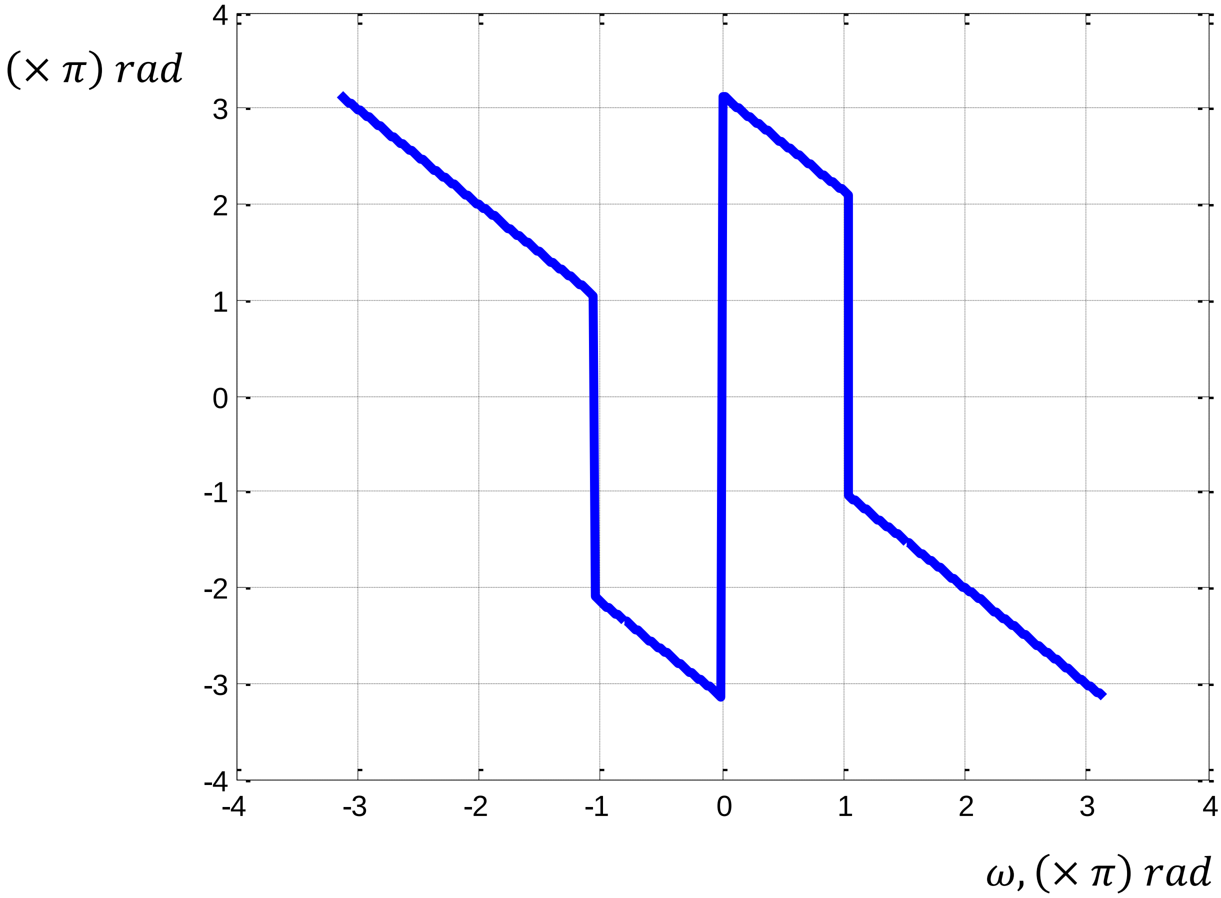


$\omega, (\times\,\pi)\;rad$

$$h[n] = [-1 \quad 1 \quad -1]$$

$$H(e^{j\omega}) = e^{-j\omega}(1 - 2\cos\omega)$$

$$ARG\left(H(e^{j\omega})\right)$$



GENERALIZED LINEAR PHASE SYSTEMS

$$H\left(e^{j\omega}\right) = A(\omega)\, e^{-j(\alpha\omega-\beta)}$$

LINEAR PHASE IF $A(\omega) > 0$ $\beta = 0$

GENERALIZED LINEAR PHASE IF $A(\omega) \in R$ (bipolar)

Group Delay of a GLP System

$$\tau_{gr}(\omega) = \alpha \qquad\qquad CONSTANT$$

The Impulse Response of a GLP System Satisfies

$$\sum_n h[n]\sin(\omega(n-\alpha)+\beta)=0$$

since

$$H(e^{j\omega})=A(\omega)\cos(\beta-\omega\alpha)+jA(\omega)\sin(\beta-\omega\alpha)$$

$$H(e^{j\omega})=\sum_n h[n]\cos(\omega n)$$

$$-j\sum_n h[n]\sin(\omega n) \qquad (Fourier\ transform)$$

(Equate real and imaginary parts, form the ratio both sides, ...)

```
clear all
close all
n = -10:10;
omeg = pi/3.03;
alph = 0.1;
bet = pi/2;
s = sin(omeg*(n-alph)+bet);
stem(n,s)
```

They have (even or odd) “symmetric” (!) impulse responses.

Even Symmetric (Type I and Type II)

$$h[n] = \begin{cases} h[M-n] & 0 \leq n \leq M \\ 0 & o.w. \end{cases}$$

Ex:

Type I	odd length	$[1 \quad 2 \quad 1]$	$[3 \quad -2 \quad 2 \quad -2 \quad 3]$
Type II	even length	$[3 \quad 3]$	$[-1 \quad 2 \quad 2 \quad -1]$

Odd Symmetric (Type III and Type IV)

$$h[n] = \begin{cases} -h[M-n] & 0 \leq n \leq M \\ 0 & o.w. \end{cases}$$

Type III	odd length	$[1 \quad 0 \quad -1]$	$[3 \quad -2 \quad 0 \quad -2 \quad 3]$
Type IV	even length	$[3 \quad -3]$	$[-1 \quad 2 \quad -2 \quad 1]$

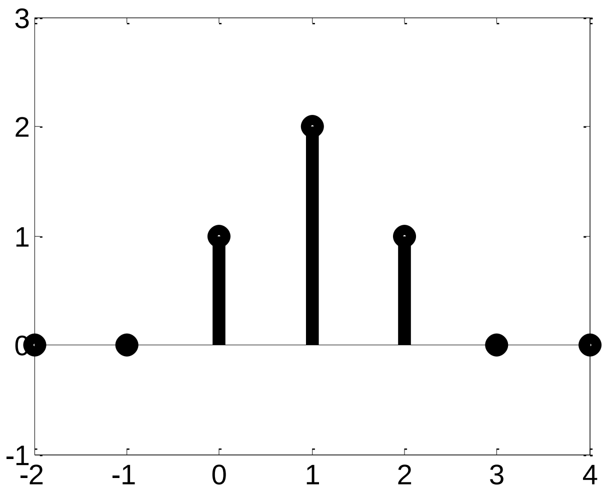
Examples

Type I

$[1 \quad 2 \quad 1]$

$$H(e^{j\omega}) = 1 + 2e^{-j\omega} + e^{-j2\omega}$$
$$= e^{-j\omega}(2 + 2\cos\omega)$$

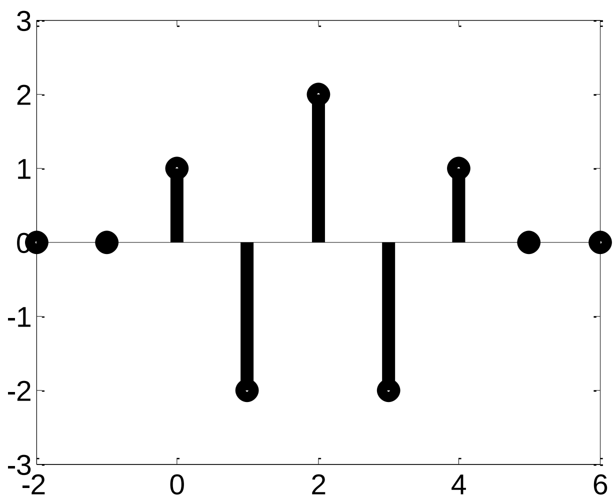
$\alpha = 1 \qquad \beta = 0$



$[1 \quad -2 \quad 2 \quad -2 \quad 1]$

$$H(e^{j\omega}) = e^{-j2\omega}(2 - 4\cos\omega + 2\cos 2\omega)$$

$\alpha = 2 \qquad \beta = 0$

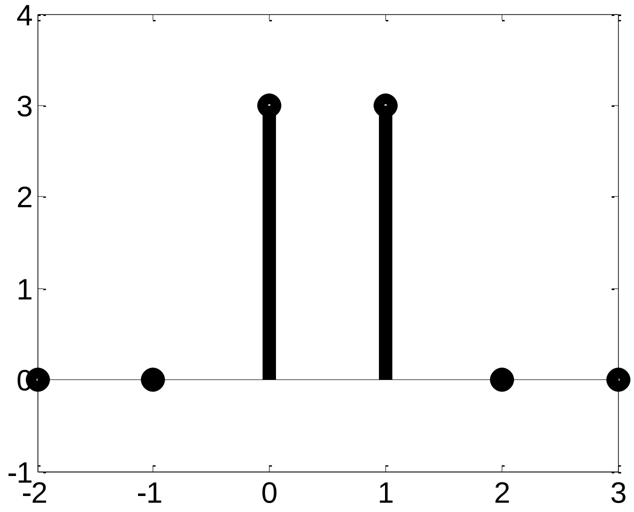


Type II

[3 3]

$$H(e^{j\omega}) = e^{-j\frac{\omega}{2}}\left(6\cos\frac{\omega}{2}\right)$$

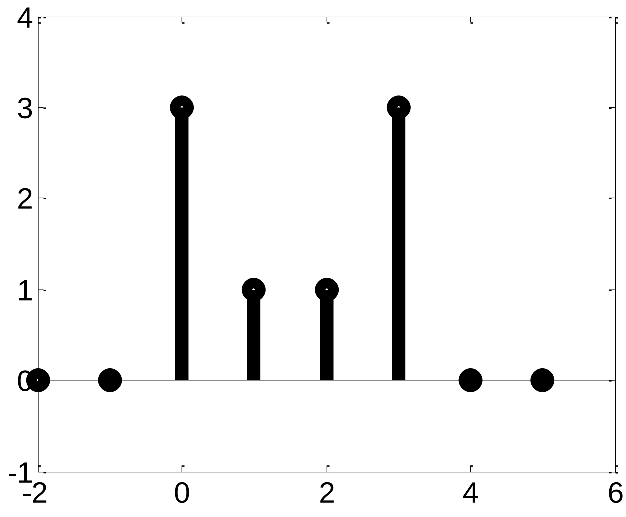
$\alpha = \frac{1}{2} \qquad \beta = 0$



[3 1 1 3]

$$H(e^{j\omega}) = e^{-j\frac{3\omega}{2}}\left(2\cos\frac{\omega}{2} + 6\cos\frac{3\omega}{2}\right)$$

$\alpha = -\frac{3}{2} \qquad \beta = 0$

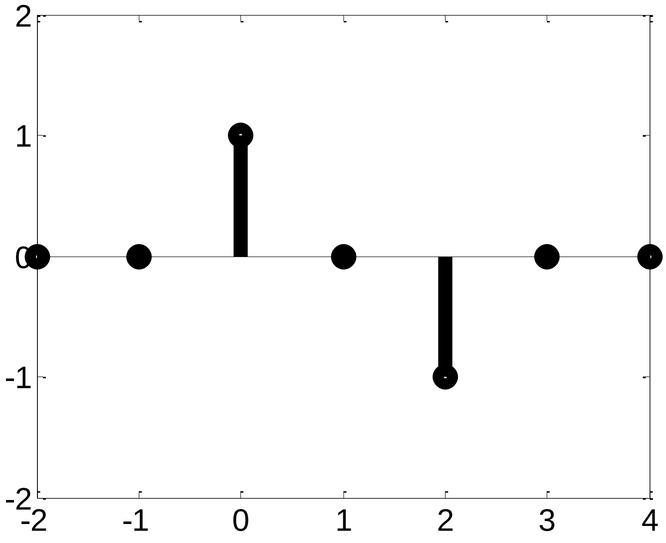


Type III

$[1 \quad 0 \quad -1]$

$$H(e^{j\omega}) = je^{-j\omega}(2\sin\omega)$$
$$= e^{-j\left(\omega-\frac{\pi}{2}\right)}(2\sin\omega)$$

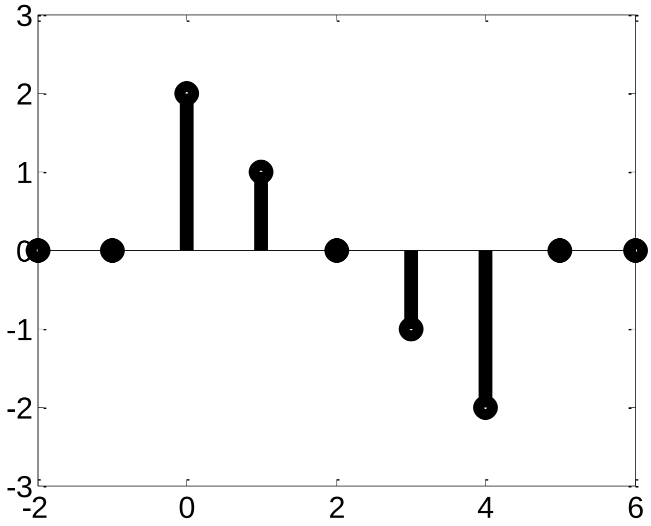
$\alpha = 1 \qquad \beta = \dfrac{\pi}{2}$



$[2 \quad 1 \quad 0 \quad -1 \quad -2]$

$$H(e^{j\omega}) = e^{-j\left(2\omega-\frac{\pi}{2}\right)}(2\sin\omega + 4\sin 2\omega)$$

$\alpha = 2 \qquad \beta = \dfrac{\pi}{2}$

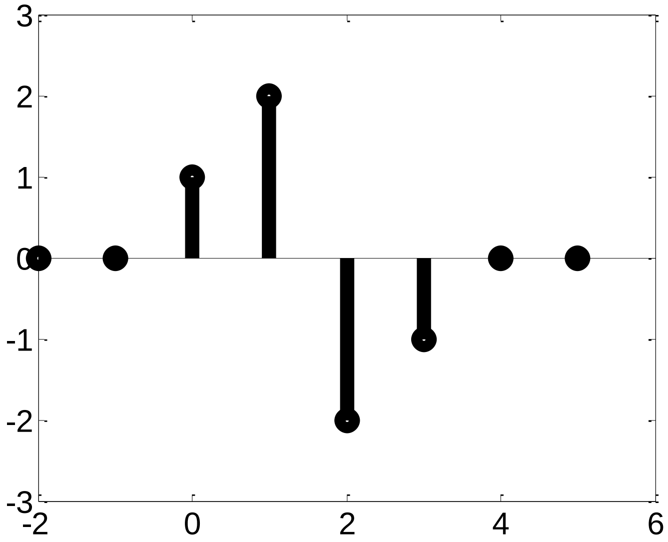


Type IV

$[1 \quad 2 \quad -2 \quad -1]$

$$H(e^{j\omega}) = e^{-j\left(\frac{3\omega}{2}-\frac{\pi}{2}\right)}\left(4\sin\frac{\omega}{2} + 2\sin\frac{3\omega}{2}\right)$$

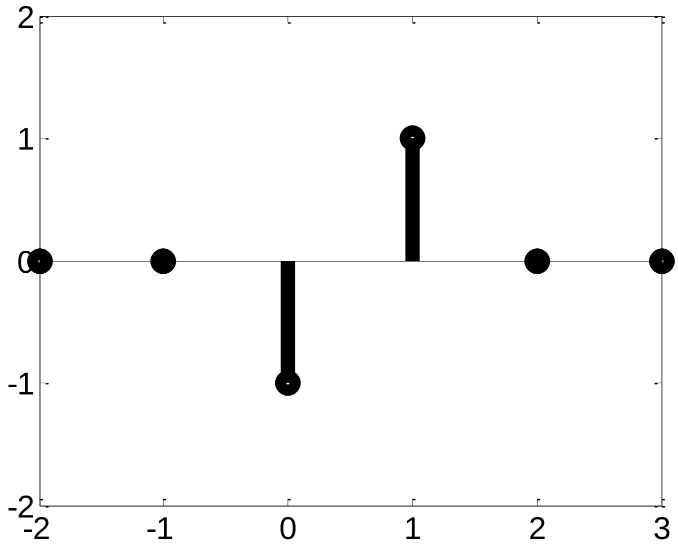
$$\alpha = \frac{3}{2} \qquad \beta = \frac{\pi}{2}$$



$[-1 \quad 1]$

$$H(e^{j\omega}) = e^{-j\left(\frac{\omega}{2}+\frac{\pi}{2}\right)}\left(2\sin\frac{\omega}{2}\right)$$

$$\alpha = \frac{1}{2} \qquad \beta = -\frac{\pi}{2}$$



```
%[H,w]=freqz(h,1,1024);

clear all
close all

N = 1024;
h = [1 1 1];

H = fft(h,N);
H = fftshift(H);
w = 2*pi*[-N/2:N/2-1]/N;

subplot(2,1,1);plot(w,abs(H));
title('magnitude of H(e^{j\omega})');
xlabel('\omega , (\times \pi) rad  ')
ylabel('|H|')
subplot(2,1,2);plot(w,angle(H)/pi);
title('phase of H(e^{j\omega})   (principal value)');
xlabel('\omega , (\times \pi) rad  ')
ylabel('(\times \pi) rad  ')
```

ZERO LOCATIONS

Even symmetric filters (Type I and Type II)

$$
\begin{aligned}
H(z) &= \sum_{n=0}^{M} h[n] z^{-n} \\
&= \sum_{n=0}^{M} h[M-n] z^{-n} \\
&= z^{-M} \sum_{n=0}^{M} h[n] z^{n} \\
&= z^{-M} H(z^{-1})
\end{aligned}
$$

Therefore, for even symmetric filters (Type I and Type II)

$$H(z) = z^{-M} H(z^{-1})$$

Similarly, for odd symmetric filters (Type III and Type IV)

$$H(z) = -z^{-M}H(z^{-1})$$

Both equalities,

$$H(z) = z^{-M}H(z^{-1}) \,,$$

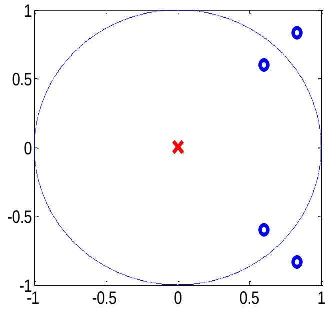
$$H(z) = -z^{-M}H(z^{-1}) \,,$$

indicate that, zeros of all types are in pairs:

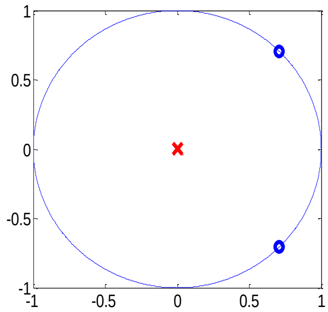
Therefore, a zero at $z = z_0$ is always accompanied by a zero at $z = \frac{1}{z_0}$

Therefore, zeros of a FIR, causal, real GLP filter will be a combination of the forms SIMILAR to those shown in the following figures.

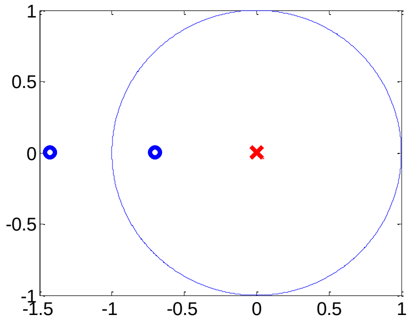
quadruple zeros



zeros in pairs



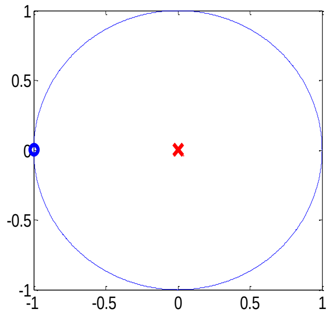
or



on the unit circle

real

single zeros



at 1 and/or -1

Above equalities also yield the following:

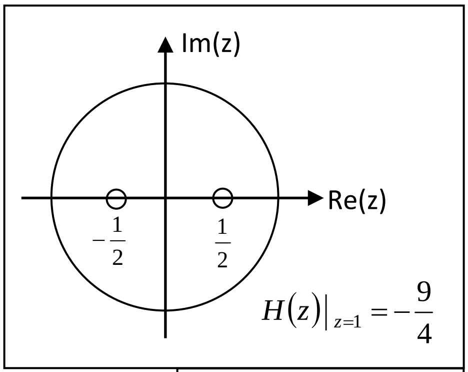
Type	M	symmetry	length	zero at $z=1$ ($\omega=0$)	zero at $z=-1$ ($\omega=\pi$)
I	even	even	odd	not necessarily[1]	not necessarily[2]
II	odd	even	even	not necessarily[3]	YES
III	even	odd	odd	YES	YES
IV	odd	odd	even	YES	not necessarily[4]

Restrictions on types of GLP filters

	Lowpass	Highpass
Type-I		
Type-II		No
Type-III	No	No
Type-IV	No	

[1] $(1-z^{-1})^2$
[2] $(1+z^{-1})^2$
[3] $(1-z^{-1})^2(1+z^{-1})$
[4] $(1+z^{-1})^2(1-z^{-1})$

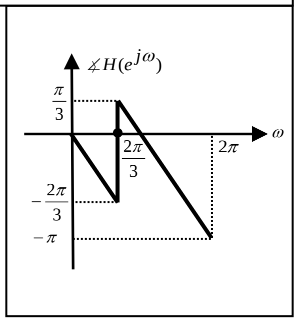
Ex: Consider a causal, generalized linear phase system. The length of the impulse response is 5. Some of the zeros of the transfer function, $H(z)$, of this system are shown in the *upper* panel.



a) Find and plot the impulse response of this system.

b) Find the frequency response and plot its magnitude and phase.

c) Find and plot the impulse response of the minimum phase system that have the same magnitude response.

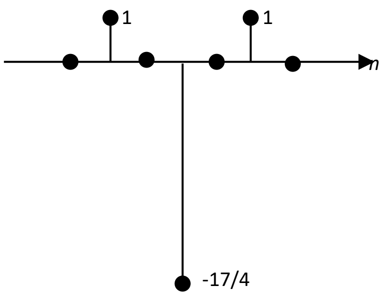


d) (**This part is independent of the previous parts.**) The phase of the frequecy response of an *even symmetric* generalized linear phase system is shown in the *lower* panel.

i) What is the length of the impulse response? Why?

ii) Let $h[0]=1$. Write the frequency response in terms of $h[0]$ and the other elements of the impulse response. Plot the magnitude of the frequency response.

iii) Find the whole impulse response.



a) The other zeros are at z = 2 and z = -2.

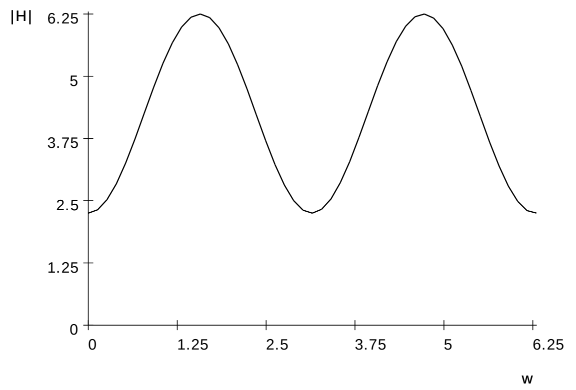
$$\Rightarrow H(z) = A \left(1-\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{2}z^{-1}\right)\left(1-2z^{-1}\right)\left(1+2z^{-1}\right)$$

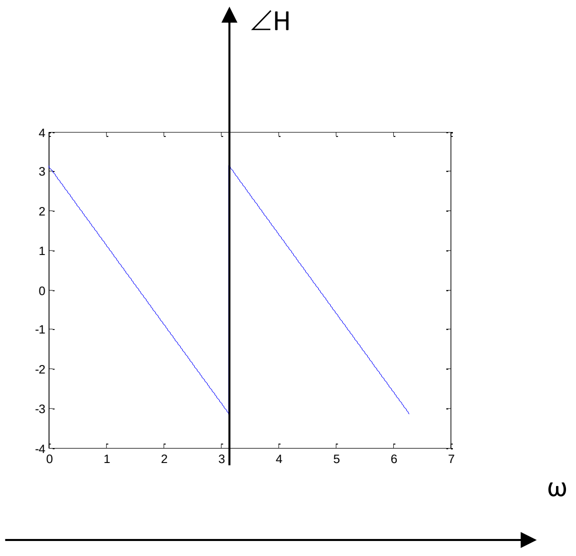
$$H(z) = A \left(1-\frac{17}{4}z^{-2}+z^{-4}\right), \qquad H(1) = A \quad -\frac{9}{4} \quad \Rightarrow A=1.$$

$$\Rightarrow h[n] = \delta[n] - \frac{17}{4}\delta[n-2] + \delta[n-4]$$

b) $H\left(e^{j\omega}\right) = 1-\frac{17}{4}e^{-j2\omega}+e^{-j4\omega} = e^{-j2\omega}\left(-\frac{17}{4}+e^{j2\omega}+e^{-j2\omega}\right) = e^{-j2\omega}\left(-\frac{17}{4}+2\cos\left(2\omega\right)\right)$

$$H\left(e^{j\omega}\right)=1-\frac{17}{4}e^{-j2\omega}+e^{-j4\omega}=e^{-j2\omega}\left(-\frac{17}{4}+e^{j2\omega}+e^{-j2\omega}\right)=e^{-j2\omega}\left(-\frac{17}{4}+2\cos\left(2\omega\right)\right)$$





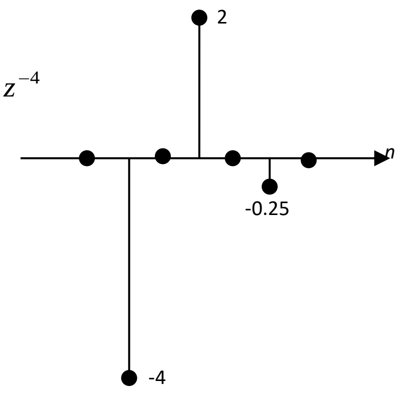
c) $H(z)=\left(1-\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{2}z^{-1}\right)\left(1-2z^{-1}\right)\left(1+2z^{-1}\right)=\left(1-\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{2}z^{-1}\right)(-2)\left(z^{-1}-\frac{1}{2}\right)(2)\left(z^{-1}+\frac{1}{2}\right)$

$$=\left(1-\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{2}z^{-1}\right)(-2)\left(z^{-1}-\frac{1}{2}\right)(2)\left(z^{-1}+\frac{1}{2}\right)\frac{\left(1-\frac{1}{2}z^{-1}\right)}{\left(1-\frac{1}{2}z^{-1}\right)}\frac{\left(1+\frac{1}{2}z^{-1}\right)}{\left(1+\frac{1}{2}z^{-1}\right)}$$

$$=\underbrace{(-4)\left(1-\frac{1}{2}z^{-1}\right)^2\left(1+\frac{1}{2}z^{-1}\right)^2}_{H_{\min}(z)}\underbrace{\left(\frac{\left(z^{-1}-\frac{1}{2}\right)}{\left(1-\frac{1}{2}z^{-1}\right)}\frac{\left(z^{-1}+\frac{1}{2}\right)}{\left(1+\frac{1}{2}z^{-1}\right)}\right)}_{H_{ap}(z)}$$

$$H_{\min}(z)=(-4)\left(1-\frac{1}{2}z^{-1}\right)^2\left(1+\frac{1}{2}z^{-1}\right)^2=(-4)\left(1-0.5z^{-1}+\frac{1}{16}z^{-4}\right)=-4+2z^{-1}-\frac{1}{4}z^{-4}$$

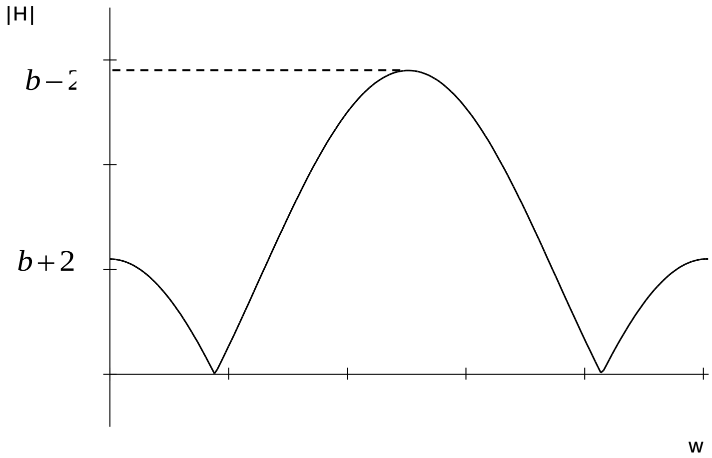
$$h_{\min}[n]=-4\delta[n]+2\delta[n-2]-0.25\delta[n-4]$$



d) **i**)The slope of the phase is -1 (-1x group delay) so the filter length is 3.

$$h[n]=a\delta[n]+b\delta[n-1]+a\delta[n-2]$$

$$H\left(e^{j\omega}\right)=a+be^{-j\omega}+ae^{-j2\omega}=e^{-j\omega}\left(b+2a\cos(\omega)\right)$$



ii) $h[0]=1 \Rightarrow$ a=1

$$H\left(e^{j\omega}\right)=e^{-j\omega}\left(b+2\cos(\omega)\right)$$

$$\measuredangle H\left(e^{j0}\right)=0 \Rightarrow b>-2$$

iii) The first zero crossing must be at 3

$$b+2\cos\left(\frac{2\pi}{3}\right)=0 \Rightarrow b=1$$