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(1)

EE 430 HW #2

1) a) First find it for $y[n] - \frac{1}{2}y[n-1] = x[n]$

$\hat{y}_s[n] - \frac{1}{2}\hat{y}_s[n-1] = 0$

$\rightarrow Az^n(1-\frac{1}{2}z^{-1}) = 0 \qquad z = \frac{1}{2}$

$\Rightarrow \hat{y}_s[n] = c_1\left(\frac{1}{2}\right)^n u[n]$

$\rightarrow h_s[n] - \frac{1}{2}h_s[n-1] = \delta[n] \quad ; \quad h_s[0] = 1, \; h_s[1] = 1/2$

$h_s[0] = c_1 = 1 \rightarrow c_1 = 1$

$h_s[n] = \left(\frac{1}{2}\right)^n u[n]$

and as LTI $\Rightarrow$ $h[n] = h_s[n] - h[n-1] + h[n+1]$

$= \left(\frac{1}{2}\right)^n \left(\delta[n] - \frac{1}{2}\delta[n-1] + 3u[n-2]\right)$

b) $H(e^{jw}) = \sum_{k=-\infty}^{\infty} h[k]e^{-jwk}$

$= 1 - \frac{1}{2}e^{-jw} + 3\sum_{k=2}^{\infty}\left(\frac{1}{2}\right)^k \cdot e^{-jwk}$

$= 1 - \frac{1}{2}e^{-jw} + 3\left[\sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^k e^{-jwk} - 1 - \frac{1}{2}e^{-jw}\right]$

$= 1 - \frac{1}{2}e^{-jw} + 3\frac{1}{1-\frac{1}{2}e^{-jw}} - 3 - \frac{3}{2}e^{-jw}$

$= -2 - 2e^{-jw} + \frac{6}{2-e^{-jw}}$

c) we need z transform

$Y(z) - \frac{1}{2} Y(z) z^{-1} = X(z) - X(z) z^{-1} + X(z) z^{-2}$

$H(z) = \frac{Y(z)}{X(z)} = \frac{1 - z^{-1} + z^{-2}}{1 - \frac{1}{2} z^{-1}}$

A and B vectors are such that

$A = [1 \ -0.5] \qquad B = [1 \ -1 \ 1]$

freqz (B, A, 'whole')

Plots are given in the page 4.

h[n] is real, as expected, magnitude plot is even and phase plot is odd

d) $y[n] = x[n] * h[n]$

Convolution in time domain is not so easy here

$x[n] = \cos(\frac{\pi}{3} n) + \sin(\frac{\pi}{2} n + \frac{\pi}{4})$

$= \cos(\frac{\pi}{3} n) + \sin(\frac{\pi}{2} n) \cdot \frac{1}{\sqrt{2}} + \cos(\frac{\pi}{2} n) \cdot \frac{1}{\sqrt{2}}$ (this is easier)

$X(e^{j\omega}) = \pi \{ \delta(\omega - \frac{\pi}{3}) + \delta(\omega + \frac{\pi}{3}) \} + \frac{\pi}{j\sqrt{2}} [\delta(\omega - \frac{\pi}{2}) - \delta(\omega + \frac{\pi}{2})]$

$+ \frac{\pi}{\sqrt{2}} [\delta(\omega - \frac{\pi}{2}) + \delta(\omega + \frac{\pi}{2})]$

From the convolution property $Y(e^{j\omega}) = X(e^{j\omega}) H(e^{j\omega})$

$$Y(e^{j\omega}) = \left(-2 - 2e^{-j\omega} + \frac{6}{2 - e^{-j\omega}}\right) \left[\frac{\pi(1-j)}{\sqrt{2}} \delta(\omega - \frac{\pi}{2}) + \pi\delta(\omega - \frac{\pi}{3}) + \pi\delta(\omega + \frac{\pi}{3}) + \left(\frac{(1+j)\pi}{\sqrt{2}}\right) \delta(\omega + \frac{\pi}{2}) \right]$$

$$= \left(-2 + j2 + \frac{6}{2+j}\right) \frac{\pi(1-j)}{\sqrt{2}} \delta(\omega - \frac{\pi}{2}) + \left(-2 - j2 + \frac{6}{2-j}\right) \frac{\pi(1+j)}{\sqrt{2}} \delta(\omega + \frac{\pi}{2})$$

$$+ \underbrace{\left(-2 - 2e^{-j\pi/3} + \frac{6}{2 - e^{-j\pi/3}}\right)}_{0} \pi\delta(\omega - \frac{\pi}{3}) + \underbrace{\left(-2 - 2e^{j\pi/3} + \frac{6}{2 - e^{j\pi/3}}\right)}_{0} \pi\delta(\omega + \frac{\pi}{3})$$

$$= \left(\frac{3\sqrt{2}}{5} + j\frac{\sqrt{2}}{5}\right) \pi\delta(\omega - \frac{\pi}{2}) + \left(\frac{3\sqrt{2}}{5} - j\frac{\sqrt{2}}{5}\right) \pi\delta(\omega + \frac{\pi}{2})$$

Now inverse transform

$$y[n] = \left(\frac{3\sqrt{2}}{5} + j\frac{\sqrt{2}}{5}\right)\pi \cdot \frac{1}{2\pi} e^{j\frac{\pi}{2}n} + \left(\frac{3\sqrt{2}}{5} - j\frac{\sqrt{2}}{5}\right)\pi \cdot \frac{1}{2\pi} e^{-j\frac{\pi}{2}n}$$

$$y[n] = \left(\frac{3\sqrt{2}}{10} + j\frac{\sqrt{2}}{10}\right)(j)^n + \left(\frac{3\sqrt{2}}{10} - j\frac{\sqrt{2}}{10}\right)(-1)^n \cdot (j)^n$$

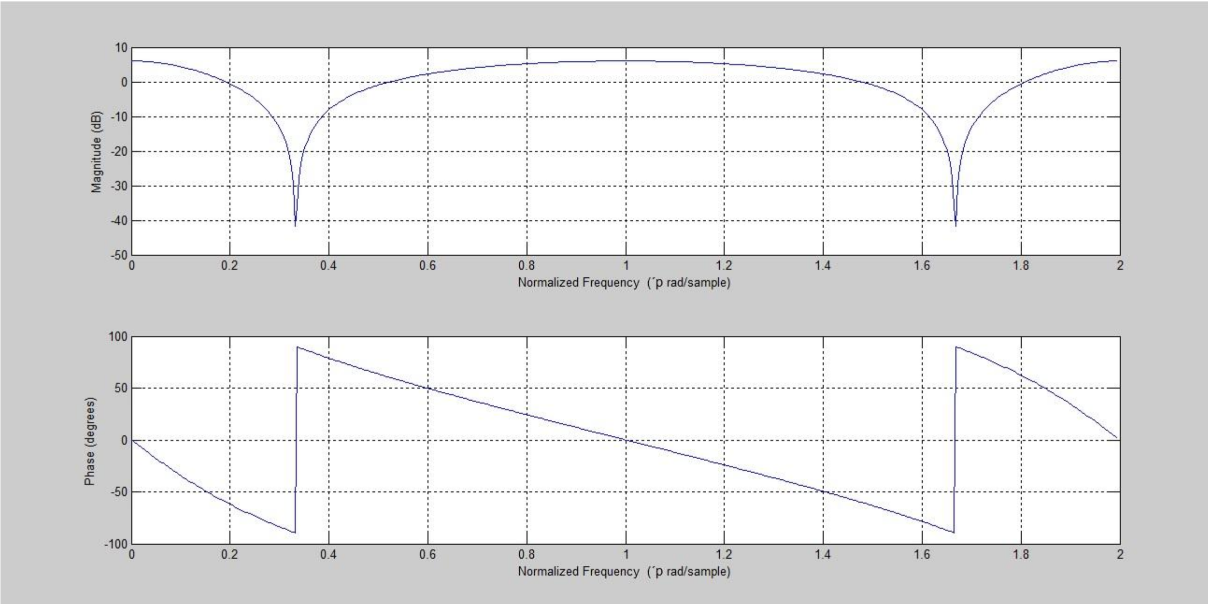
$$\Rightarrow y[n] = \begin{cases} \frac{3\sqrt{2}}{5} & 0 \equiv \mathrm{mod}(n) \\ -\frac{\sqrt{2}}{5} & 1 \equiv \mathrm{mod}(n) \\ -\frac{3\sqrt{2}}{5} & 2 \equiv \mathrm{mod}(n) \\ \frac{\sqrt{2}}{5} & 3 \equiv \mathrm{mod}(n) \end{cases}$$

e) $H(e^{j\omega}) = -2 - 2e^{-j\omega} + \frac{6}{2 - e^{-j\omega}}$

$$H^*(e^{j(2\pi-\omega)}) = \left(-2 - 2\underbrace{e^{-j2\pi}}_{1} e^{j\omega} + \frac{6}{2 - \underbrace{e^{-j2\pi}}_{1} \cdot e^{j\omega}}\right)^*$$

$$= \left(-2 - 2e^{j\omega} + \frac{6}{2 - e^{j\omega}}\right)^* = -2 - 2e^{-j\omega} + \frac{6}{2 - e^{-j\omega}}$$

At the last step we can see that, the equality $H(e^{j\omega}) = H^*(e^{j(2\pi-\omega)})$ because $H(e^{j\omega})$ is even symmetric. If it wasn't, this wouldn't satisfy. That means, for this equation to hold $h[n]$ must be real,



Magnitude (dB)
Normalized Frequency (´p rad/sample)
Phase (degrees)
Normalized Frequency (´p rad/sample)

(7)

(5)

2) $x[n] = n a^{n-2} u[n-2]$

$$a^n u[n] \longleftrightarrow \frac{1}{1-ae^{-jw}}$$

$$a^{n-2} u[n-2] \longleftrightarrow \frac{e^{-jw2}}{1-ae^{-jw}}$$

we know differentiation in the frequency domain bring multiplication of n

$$-jna^{n-2}u[n-2] \longleftrightarrow \frac{d}{dw}\left(\frac{e^{-jw2}}{1-ae^{-jw}}\right)$$

$$na^{n-2}u[n-2] \longleftrightarrow j\,\frac{-2w\,e^{-jw2}(1-ae^{-jw}) - jae^{-jw}\,e^{-jw2}}{(1-ae^{-jw})^2}$$

$$\boxed{X(e^{jw}) = \frac{2e^{-j2w} - ae^{-j3w}}{(1-ae^{-jw})^2}}$$

⑥

3) a) $h[n] = (\delta[n] + \delta[n-1])$

$\Rightarrow y_1[n] = x_1[n] + x_1[n-1]$

$$y_1[n] = \sin\left(\frac{\pi}{7}(n-1)\right) + \sin\left(\frac{\pi}{3}(n-1)\right) + \sin\left(\frac{\pi}{7}n\right) + \sin\left(\frac{\pi}{3}n\right)$$

similarly $y_2[n] = x_2[n] + x_2[n-1]$

$$y_2[n] = \left(\sin\left(\frac{\pi}{7}n\right) + \sin\left(\frac{\pi}{3}n\right)\right)u[n] + \left(\sin\left(\frac{\pi}{7}(n-1)\right) + \sin\left(\frac{\pi}{3}(n-1)\right)\right) \cdot u[n-1]$$

!! $\rightarrow y_2[n] = y_1[n]$ for $n \geq 1$

b) $h[n] = \frac{1}{2}(\delta[n-2] + \delta[n-3])$

$\Rightarrow y_1[n] = (x_1[n-2] + x_1[n-3]) \cdot \frac{1}{2}$

$$= \left[\sin\left(\frac{\pi}{7}(n-2)\right) + \sin\left(\frac{\pi}{3}(n-2)\right)\right] \times \frac{1}{2} + \frac{1}{2} \times \left[\sin\left(\frac{\pi}{7}(n-3)\right) + \underbrace{\sin\left(\frac{\pi}{3}(n-3)\right)}_{-\sin\left(\frac{\pi n}{3}\right)}\right]$$

similarly $y_2[n] = x_2[n-2] + x_2[n-3]$

and as above $y_2[n] = y_1[n]$ for $n \geq 3$

c) $y_1[n] = \frac{1}{6} \sum_{i=0}^{5} x_1[n-i]$

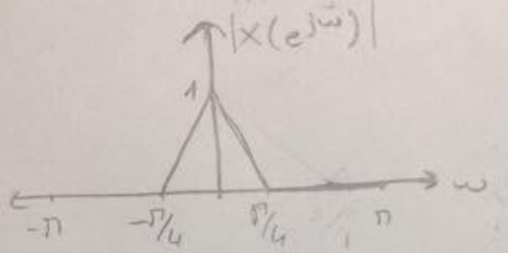
$y_2[n] = \frac{1}{6} \sum_{i=0}^{5} x_1[n-i]u[n-i]$

I am not going to write it openly, but if I did, we would see $y_1[n] = y_2[n]$ for $n \geq 5$

We can generally say that, suddenly applied inputs have the same output from the last nonzero n-value of the impulse response.

(7)

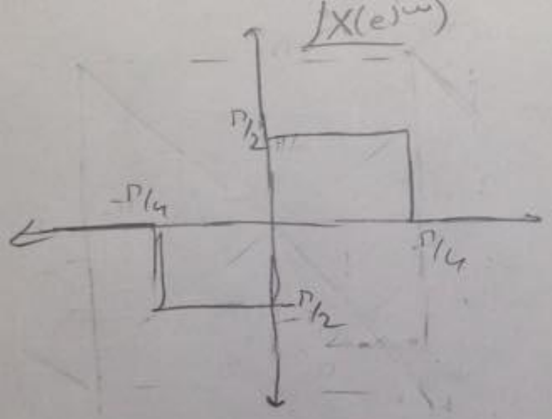
4) a)



$a = -\frac{\pi}{4}$

$b = \frac{3\pi}{4}$

must be even as $x[n]$ is real. However, phase diagram must be odd.



b) • $\cos(\frac{\pi}{5}n) \leftrightarrow \left(\delta(\omega-\frac{\pi}{5}) + \delta(\omega+\frac{\pi}{5})\right) \times \pi$

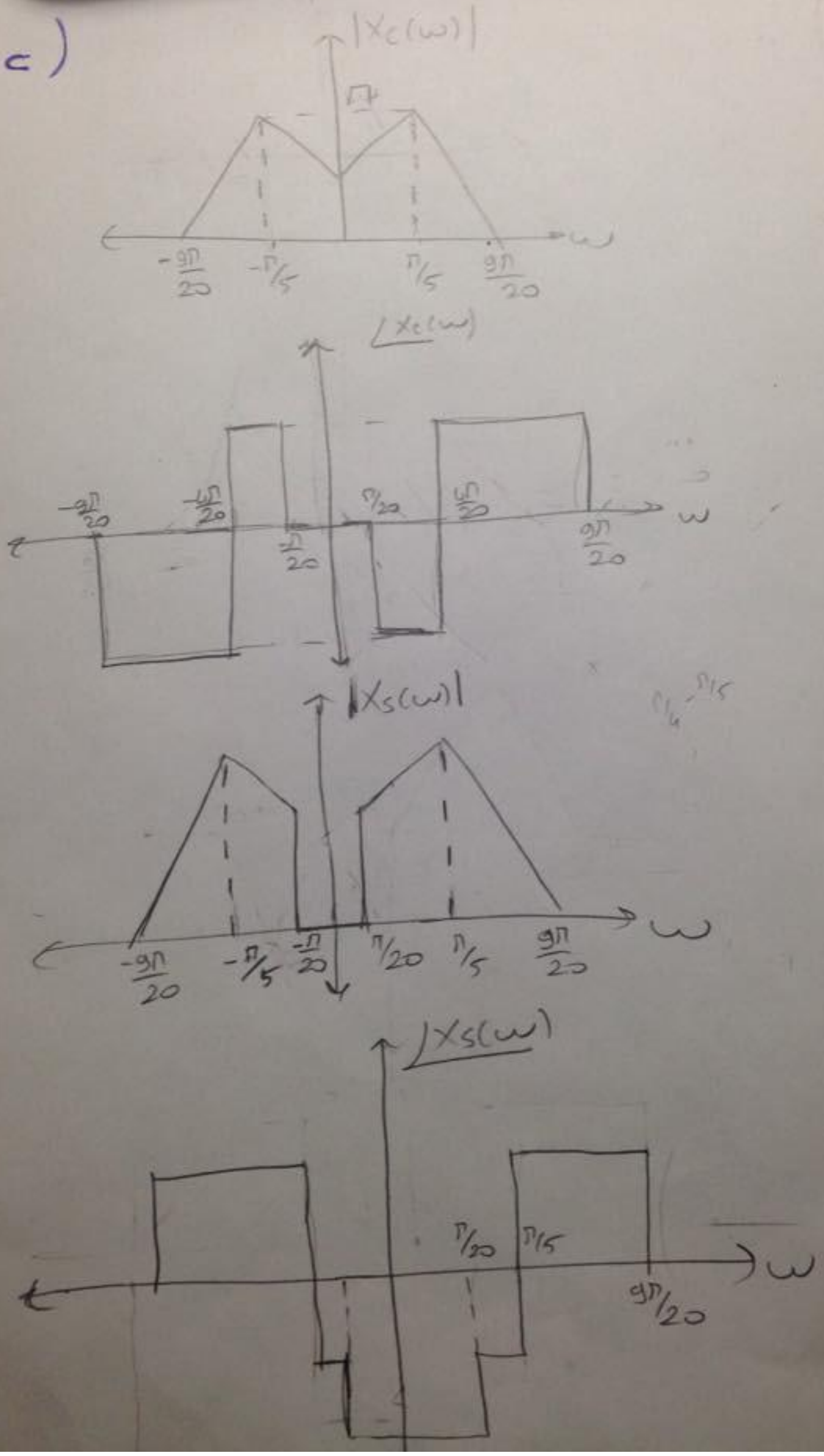
$\Rightarrow \cos(\frac{\pi}{5}n)\, x[n] \leftrightarrow \mathcal{F}\{\cos(\frac{\pi}{5}n)\} * X(e^{j\omega})$

$X_c(\omega) = \left[X(\omega-\frac{\pi}{5}) + X(\omega+\frac{\pi}{5})\right] \times \pi$

• $\sin(\frac{\pi}{5}n) \leftrightarrow \frac{\pi}{j}\left(\delta(\omega-\frac{\pi}{5}) - \delta(\omega+\frac{\pi}{5})\right)$

$X_s(\omega) = \frac{\pi}{j}\left[X(\omega-\frac{\pi}{5}) - X(\omega+\frac{\pi}{5})\right]$

c)



|Xc(ω)|
ω
-9π/20
-π/5
π/5
9π/20
∠Xc(ω)
ω
-9π/20
-4π/20
-π/20
π/20
4π/20
9π/20
|Xs(ω)|
ω
-9π/20
-π/5
-π/20
π/20
π/5
9π/20
∠Xs(ω)
ω
π/20
π/5
9π/20

9

5) as it is real

$$\int_{-\pi}^{0}\left|\frac{\sin\left(\frac{5}{2}w\right)}{\sin\left(\frac{w}{2}\right)}\right|^2 dw = \int_{-\pi}^{0}\frac{\sin^2\left(\frac{5w}{2}\right)}{\sin^2\left(\frac{w}{2}\right)}dw$$

$$= \int\frac{\left(\sin\left(\frac{w}{2}\right)\cos(2w)+\cos\left(\frac{w}{2}\right)\sin(2w)\right)^2}{\sin^2\left(\frac{w}{2}\right)}dw$$

$$= \int\left(\cos(2w)+\frac{\cos\frac{w}{2}\sin 2w}{\sin\frac{w}{2}}\right)^2 dw$$

$$= \int\left(\cos(2w)+4\cos^2\left(\frac{w}{2}\right)\cos(w)\right)^2 dw$$

$$= \int\left(\cos(2w)+2(\cos(w)+1)\cos(w)\right)^2 dw$$

$$= \int\left(\cos(2w)+2\cos^2(w)+2\cos(w)\right)^2 dw = \int\left(2\cos(2w)+1+2\cos(w)\right)^2 dw$$

$$= \int_{-\pi}^{0}\left(4\cos^2(2w)+\underbrace{4\cos(2w)}_{0}+\underbrace{8\cos(2w)\cos(w)}_{0}+\underbrace{4\cos(w)}_{0}+4\cos^2(w)+1\right)dw$$

$$= 4\int_{-\pi}^{0}\cos^2(2w)dw+4\int_{-\pi}^{0}\cos^2(w)dw+\int_{-\pi}^{0}dw = 5\pi$$

(10)

6/ we know

$X(e^{jw}) = 2\pi\delta(w-w_0)$ when $x[n] = e^{jw_0 n}$

and $y[n] = h[n] * x[n]$

↳ convolution property

$Y(e^{jw}) = H(e^{jw})\underbrace{X(e^{jw})}$

↳ nonzero only for $w=w_0$

$\Rightarrow Y(e^{jw}) = H(e^{jw_0})\,X(e^{jw})$

$= H(e^{jw_0}) \times 2\pi\delta(w-w_0)$

now take the inverse DTFT

$$y[n] = \frac{1}{2\pi}\int_0^{2\pi} Y(e^{jw})e^{jwn}\,dw$$

$$= \frac{1}{2\pi}\int_0^{2\pi} H(e^{jw_0}) \times 2\pi\,\delta(w-w_0)e^{jwn}\,dw$$

$$= H(e^{jw_0})\int_{w_0^-}^{w_0^+} \delta(w-w_0)\,e^{jw_0 n}\,dw$$

$$= H(e^{jw_0})\,e^{jw_0 n} = y[n]$$

7) a) $y[n] = \sum_{k=-\infty}^{\infty} x[n-k]h[k]$

$$= \sum_{k=0}^{\infty} 3^{n-k}\left[\left(\tfrac{1}{2}\right)^k + 2^k\right] = 3^n \underbrace{\sum_{k=0}^{\infty}\left[\left(\tfrac{1}{6}\right)^k + \left(\tfrac{2}{3}\right)^k\right]}_{\text{obviously a constant}}$$

$$C = \frac{21}{5} \qquad C = \frac{1}{1-\frac{1}{6}} + \frac{1}{1-\frac{2}{3}} \quad \text{constant}$$

b) $y[n] = \sum_{k=-\infty}^{\infty} x[n-k]h[k]$

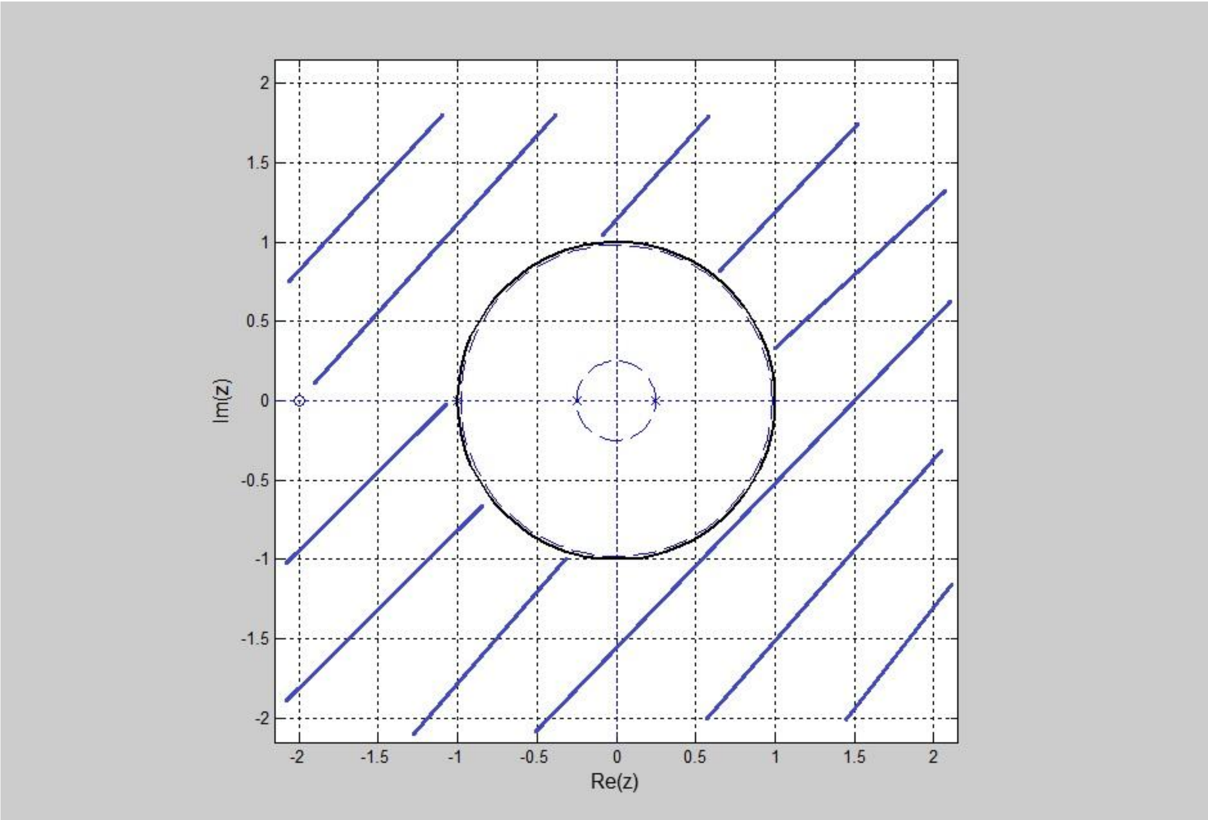
$$= 3^n \sum_{k=0}^{\infty}\left(\tfrac{2}{3}\right)^k + 3^n \sum_{k=-\infty}^{0} 3^{-k} \cdot 2^k = 3^{n+1} + 3^n \underbrace{\sum_{k=0}^{\infty}\left(\tfrac{3}{2}\right)^k}_{\text{diverges to} \to \infty}$$

cannot be expressed as $C \times 3^n$

8) a) zero $\to -2$

pole $\to 1/4, -1/4, -1$

The system is not stable as we have pole at unit circle, ROC doesn't contain it. This may be seen at page 12



Re(z)
Im(z)

9) c) $x[n] = 3+j5 + \sin\left(\frac{\pi n}{4}\right)$ (13)

$X(e^{j\omega}) = 2\pi(3+j5)\,\delta(\omega) - j\pi\left[\delta\left(\omega-\frac{\pi}{4}\right) - \delta\left(\omega+\frac{\pi}{4}\right)\right]$

$Y(e^{j\omega}) = X(e^{j\omega})\,H(e^{j\omega})$ and $\angle H(e^{j\omega}) = \frac{-\omega}{2}$

$\hookrightarrow = 2\pi(3+j5)\,\delta(\omega) - j\pi\left[\frac{1}{2}\delta\left(\omega-\frac{\pi}{4}\right).e^{-\frac{j\pi}{8}} - \frac{1}{2}\delta\left(\omega+\frac{\pi}{4}\right).e^{j\frac{\pi}{8}}\right.$

$\rightarrow y[n] = 3+j5 - \frac{j}{4}e^{j\left(\frac{\pi}{4}n-\frac{\pi}{8}\right)} + \frac{j}{4}e^{j\left(\frac{-\pi}{4}n+\frac{\pi}{8}\right)}$

$= 3+j5 + \frac{j}{4}\left[-\cos\left(\frac{\pi n}{4}-\frac{\pi}{8}\right) - j\sin\left(\frac{\pi n}{4}-\frac{\pi}{8}\right) + \cos\left(\frac{\pi}{4}n-\frac{\pi}{8}\right) - j\sin\left(\frac{\pi}{4}n-\right.\right.$

$= 3+j5 + \frac{1}{2}\sin\left(\frac{\pi}{4}n-\frac{\pi}{8}\right)$