

EE430 Digital Signal Processing

HW 2

1. The impulse response of the system is given as $h[n] = \delta[n] - \sqrt{2}\delta[n-1] + \delta[n-2]$

(a) The system function $H(z)$ can be found as follows;

$$\begin{aligned}\mathcal{Z}\{\delta[n]\} &= 1 \quad \forall z \\ \mathcal{Z}\{x[n-n_0]\} &= z^{-n_0}X(z)\end{aligned}$$

Thus,

$$H(z) = 1 - \sqrt{2}z^{-1} + z^{-2}$$

$H(z)$ can be put in more classical transfer function form to find poles and zeros.

$$H(z) = \frac{z^2 - \sqrt{2}z + 1}{z^2} = \frac{(z - (\frac{1}{2} + \frac{j}{2}))(z - (\frac{1}{2} - \frac{j}{2}))}{z^2}$$

Therefore, poles and zeros of the system function can be easily found to be as,

$$z_{1,2} = \frac{1}{2} \mp \frac{j}{2}, \quad p_{1,2} = 0$$

Since the impulse response of the system is finite-length, the region of convergence of system function is **All Z Plane** except $z = 0$ due to poles. Pole-zero plot can be seen at *Figure 2*.

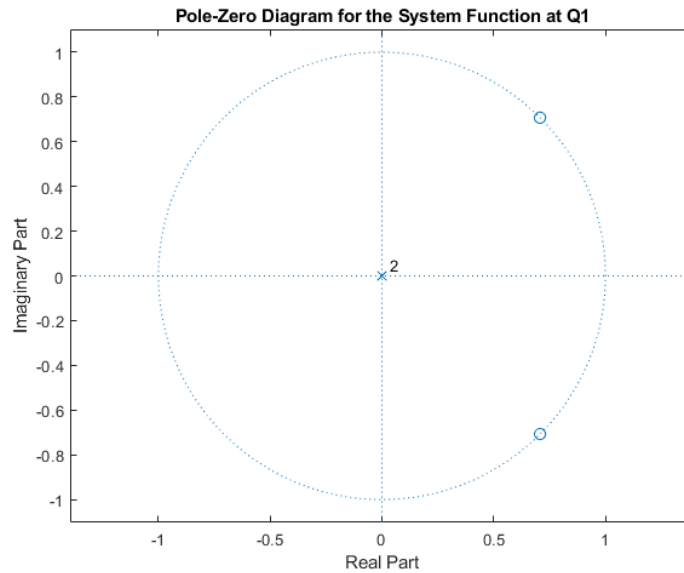


Figure 1: Pole-Zero Diagram for the System Function



- (b) Since the ROC contains the unit circle, the system has a frequency response and it can be found by simply replacing z with e^{jw} .

$$H(e^{jw}) = H(z)|_{z=e^{jw}} = 1 - \sqrt{2}e^{-jw} + e^{-j2w}$$

$$H(e^{jw}) = e^{-jw}[e^{jw} + e^{-jw} - \sqrt{2}]$$

$$H(e^{jw}) = e^{-jw} [\cos(w) - \sqrt{2}]$$

Magnitude and phase response of the frequency response can be seen at *Figure 2a* and *Figure 2b* respectively.

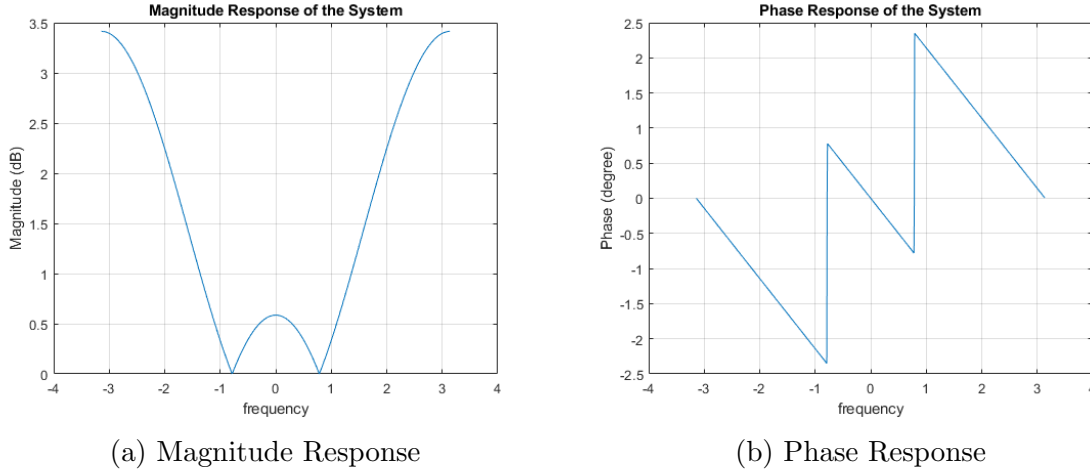


Figure 2: Magnitude and Phase Response of the System

- (c) The output of the system for the different inputs can be found using Z-transform of given inputs.
- i. $x_1[n] = \sin(\frac{\pi}{4}n + \frac{\pi}{4}) = x_1[n](u[-n-1] + u[n])$, since the ROC of the z transform of this input would be empty set, let us use another property of DTFT since we already have frequency response.

$$y[n] = \sum_{k=-\infty}^{\infty} a_k H(e^{jw}) e^{jwn}$$

$$x_1[n] = \sin(\frac{\pi}{4}n) \cos(\frac{\pi}{4}) + \sin(\frac{\pi}{4}) \cos(\frac{\pi}{4}n) = \frac{1}{\sqrt{2}} \left(\sin(\frac{\pi}{4}n) + \cos(\frac{\pi}{4}n) \right)$$

$$x_1[n] = \frac{1}{\sqrt{2}} \left(\frac{1}{2j} (e^{j\frac{\pi}{4}n} - e^{-j\frac{\pi}{4}n}) + \frac{1}{2} (e^{j\frac{\pi}{4}n} + e^{-j\frac{\pi}{4}n}) \right)$$

$$x_1[n] = \frac{1}{2\sqrt{2}} (e^{j\frac{\pi}{4}n}(1-j) + e^{-j\frac{\pi}{4}n}(1+j))$$



Thus,

$$y_1[n] = \frac{1}{2\sqrt{2}} \left(e^{j\frac{\pi}{4}n}(1-j)H(e^{j\frac{\pi}{4}}) + e^{-j\frac{\pi}{4}n}(1+j)H(e^{-j\frac{\pi}{4}}) \right)$$

Since $H(e^{j\frac{\pi}{4}}) = H(e^{-j\frac{\pi}{4}}) = 0$

$$\boxed{y_1[n] = 0}$$

ii. $x_2[n] = \sin(\frac{\pi}{4}n + \frac{\pi}{4})u[n]$ can be simplified further as follows;

$$x_2[n] = \frac{1}{\sqrt{2}} \left(\sin(\frac{\pi}{4}n) + \cos(\frac{\pi}{4}n) \right)$$

using

$$\mathcal{Z}\{u[n]\} = \frac{1}{1-z^{-1}}, \text{ ROC : } |z| > |a|$$

$$\mathcal{Z}\{z_0^n x[n]\} = X(z/z_0), \text{ ROC : } R_x|z_0|$$

$$\mathcal{Z}\{\cos(w_0)n\}u[n] = \frac{1 - \cos(w_0)z^{-1}}{1 - 2\cos(w_0)z^{-1} + z^{-2}}, |z| > 1$$

$$\mathcal{Z}\{\sin(w_0)n\}u[n] = \frac{\sin(w_0)z^{-1}}{1 - 2\cos(w_0)z^{-1} + z^{-2}}, |z| > 1$$

the z-transform $X_2(z)$ can be found as follow;

$$X_2(z) = \frac{1}{\sqrt{2}} \left(\frac{1 - \cos(\frac{\pi}{4})z^{-1}}{1 - 2\cos(\frac{\pi}{4})z^{-1} + z^{-2}} + \frac{\sin(\frac{\pi}{4})z^{-1}}{1 - 2\cos(\frac{\pi}{4})z^{-1} + z^{-2}} \right)$$

$$X_2(z) = \frac{1}{\sqrt{2}} \left(\frac{1 - \frac{1}{\sqrt{2}}}{1 - \sqrt{2}z^{-1} + z^{-2}} + \frac{\frac{1}{\sqrt{2}}z^{-1}}{1 - \sqrt{2}z^{-1} + z^{-2}} \right)$$

$$X_2(z) = \frac{1}{\sqrt{2}} \left(\frac{1}{1 - \sqrt{2}z^{-1} + z^{-2}} \right)$$

$$Y_2(z) = X_2(z)H(z) = \frac{1}{\sqrt{2}}$$

$$\boxed{y[n] = \frac{1}{\sqrt{2}}\delta[n]}$$

iii. $x_3[n] = \sin(\frac{\pi}{4}n + \frac{\pi}{4}) + \sin(\frac{3\pi}{4}n)$ Using linearity property of Z-Transform, we can safely say that the output for the first term is zero, let us find the output for the second term using the eigenfunction property again,

$$\sin(\frac{3\pi}{4}n) = \frac{1}{2j} \left(e^{j\frac{3\pi}{4}n} - e^{-j\frac{3\pi}{4}n} \right)$$



$$y_3[n] = \frac{1}{2j} \left(e^{j\frac{3\pi}{4}n} H(e^{j\frac{3\pi}{4}}) - e^{-j\frac{3\pi}{4}n} H(e^{-j\frac{3\pi}{4}}) \right)$$

$$H(e^{j\frac{3\pi}{4}}) = e^{-j\frac{3\pi}{4}} \left[\cos\left(\frac{3\pi}{4}\right) - \sqrt{2} \right] = 2\sqrt{2}e^{j\frac{\pi}{4}}$$

$$H(e^{-j\frac{3\pi}{4}}) = e^{j\frac{3\pi}{4}} \left[\cos\left(\frac{3\pi}{4}\right) - \sqrt{2} \right] = 2\sqrt{2}e^{-j\frac{\pi}{4}}$$

$$y_3[n] = \frac{2\sqrt{2}}{2j} \left(e^{j(\frac{3\pi}{4}n + \frac{\pi}{4})} - e^{-j(\frac{3\pi}{4}n + \frac{\pi}{4})} \right)$$

$$y_3[n] = 2\sqrt{2} \sin\left(\frac{3\pi}{4}n + \frac{\pi}{4}\right)$$

(d) Since on the unit circle, the Z-Transform is equivalent to the DTFT, the zeros of the Z-Transform at unit circle is also the zeros of the DTFT.

2. The system function is given as $H(z) = \frac{1 - \sqrt{2}z^{-1} + z^{-2}}{1 - 2z^{-1}}$

(a) a

(b) b

(c)

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 - \sqrt{2}z^{-1} + z^{-2}}{1 - 2z^{-1}}$$

Thus,

$$y[n] - 2y[n-1] = x[n] - \sqrt{2}x[n-1] + x[n-2]$$

3. Since $x[n]$ is a right-sided sequence, the ROC of the $H(z)$ would be outward. It is implied that the ROC of the $H(z)$ includes $|z| = 4$ circle. This means that the ROC can be something close to a ROC at *Figure 3*. The circle can be smaller but can not be larger than $|z| = 4$ since it is given that the Z Transform exist for that circle.

Thus, in any circumstances, $X(z)$ exists for $z = 4.1e^{jw}$ since the ROC includes $|z| = 4.1$ circle in any case. However, it is possible that $X(z)$ may not exist for $z = 3.9e^{jw}$ since the ROC may not include $|z| = 3.9$ circle as in *Figure 3*.



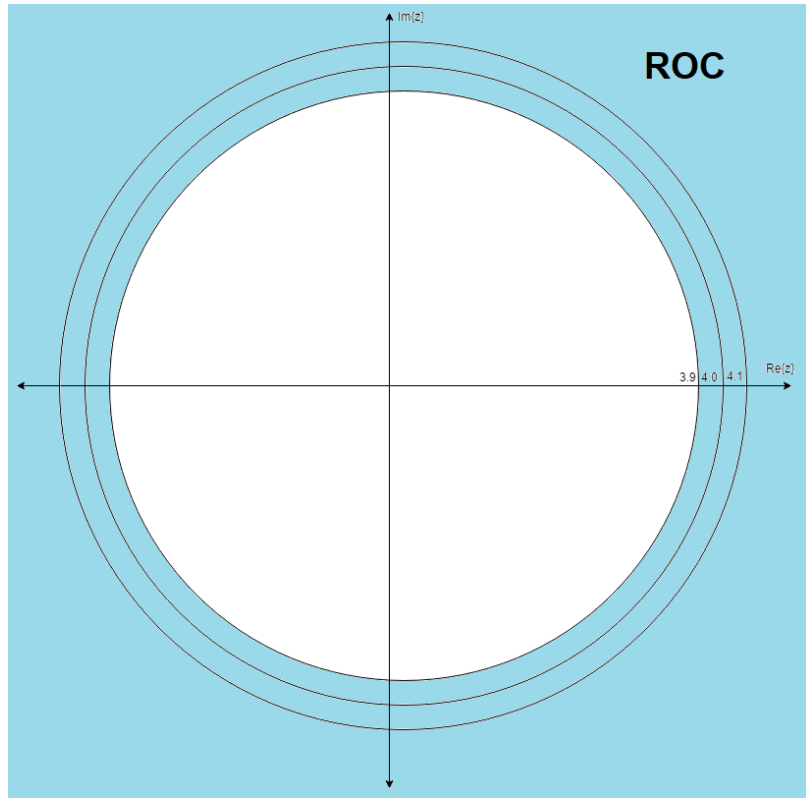


Figure 3: Possible ROC for the Given System

4. Given that $x[n] = \delta[n + 1] + (\frac{1}{2})^n u[n]$

(a)

$$X(z) = z - \frac{1}{1 - \frac{1}{2}z^{-1}} = \frac{z(z + \frac{1}{2})}{z - \frac{1}{2}}, \quad |z| > \frac{1}{2}$$

$$p_{a-1} = \frac{1}{2}, \quad p_{a-2} = \inf, \quad z_{a-1} = 0, \quad z_{a-2} = \frac{-1}{2}$$

(b)

$$\mathcal{Z}\{x[n - 5]\} = z^{-5}X(z) = \frac{z + \frac{1}{2}}{z^4(z - \frac{1}{2})}, \quad |z| > \frac{1}{2}$$

$$p_{b-1} = \frac{1}{2}, \quad p_{b-2,3,4,5} = 0, \quad z_{b-1} = \frac{-1}{2}, \quad z_{b-2,3,4,5} = \inf$$

(c)

$$\mathcal{Z}\{nx[n]\} = -z \frac{d}{dz} X(z) = -z \frac{d}{dz} \left(\frac{z + \frac{1}{2}}{z^4(z - \frac{1}{2})} \right), \quad |z| > \frac{1}{2}$$



$$\mathcal{Z}\{nx[n]\} = -z \left(-\frac{1+z^2}{z^5(z-0.5^2)} \right)$$

$$\boxed{\mathcal{Z}\{nx[n]\} = \frac{1+z^2}{z^4(z-0.5^2)}}$$

$$\boxed{p_{c-1,2} = \frac{1}{2}}, \quad \boxed{p_{c-3,4,5,6} = 0}, \quad \boxed{z_{c-1,2} = \mp j}, \quad \boxed{p_{c-3,4,5,6} = \inf}$$

(d)

$$\cos\left(\frac{\pi}{2}n\right)x[n] = \frac{1}{2} \left(e^{j\frac{\pi}{2}n}x[n] + e^{-j\frac{\pi}{2}n}x[n] \right)$$

$$\mathcal{Z}\left\{\cos\left(\frac{\pi}{2}n\right)x[n]\right\} = \frac{1}{2} \left(X\left(\frac{z}{e^{j\frac{\pi}{2}}}\right) + X\left(\frac{z}{e^{-j\frac{\pi}{2}}}\right) \right)$$

$$\boxed{\mathcal{Z}\left\{\cos\left(\frac{\pi}{2}n\right)x[n]\right\} = \frac{1}{2} \left(\frac{\left(\frac{z}{e^{j\frac{\pi}{2}}}\right)\left(\frac{z}{e^{j\frac{\pi}{2}}} + \frac{1}{2}\right)}{\frac{z}{e^{j\frac{\pi}{2}}} - \frac{1}{2}} + \frac{\left(\frac{z}{e^{-j\frac{\pi}{2}}}\right)\left(\frac{z}{e^{-j\frac{\pi}{2}}} + \frac{1}{2}\right)}{\frac{z}{e^{-j\frac{\pi}{2}}} - \frac{1}{2}} \right)}}$$

5. $H(z)$ will be in the form of

$$\frac{\alpha(z + \frac{1}{2})}{(z - 3)(z - \frac{1}{2})}$$

$$H(1) = \frac{\alpha(3/2)}{(-2)(1/2)} = 1$$

$$\boxed{\alpha = \frac{-2}{3}}$$

$$\boxed{H(z) = -\frac{2}{3} \frac{(z + \frac{1}{2})}{(z - 3)(z - \frac{1}{2})}}$$

(a) System to be stable, the ROC must include unit circle. The ROC can not also include any pole and should be one piece. Thus the ROC can only be

$$\boxed{ROC : \frac{1}{2} < |z| < 3}$$

The pole-zero diagram for this system and its ROC can be seen at *Figure 4*.



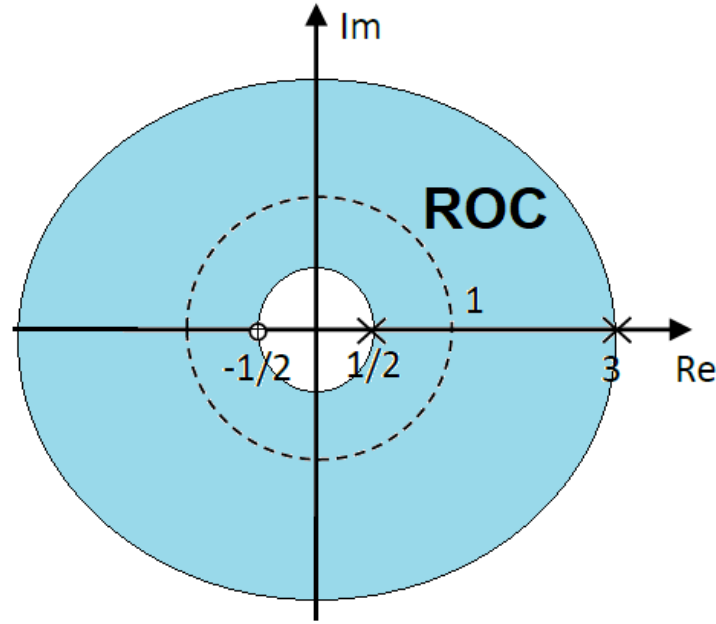


Figure 4: ROC for the Given System

To find $h[n]$, partial fraction method can be used;

$$H(z) = \left[\frac{a}{z-3} + \frac{b}{z-\frac{1}{2}} \right]$$

where a and b can be found easily as

$$a = -\frac{14}{15}, \quad b = \frac{4}{15}$$

Thus, considering the ROC, $h[n]$ becomes;

$$h[n] = \frac{4}{5} \left(\frac{1}{2} \right)^n u[n] - \frac{14}{15} (3)^n u[-n-1]$$

(b) Given that,

$$h_2[n] = h[-n+2]$$

Z-Transform of $h_2[n]$ can be calculated as follows,

$$H_2(z) = z^2 X(1/z) = H(z) = -\frac{2}{3} \frac{z^2(z^{-1} + \frac{1}{2})}{(z^{-1} - 3)(z^{-1} - \frac{1}{2})}$$



$$H_2(z) = -\frac{2}{9} \frac{z^3(z+2)}{(z-\frac{1}{3})(z-2)}, \quad \frac{1}{3} < |z| < 2$$

Pole-zero diagram for this transform and its ROC can be seen at *Figure 5*.

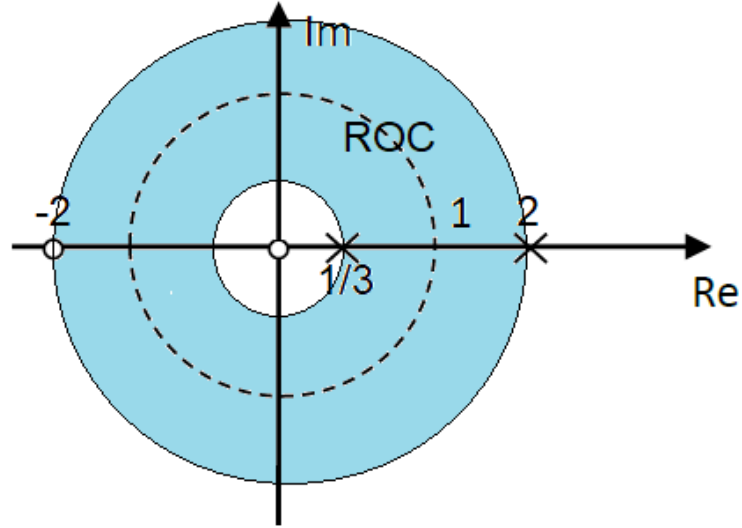


Figure 5: ROC for the Updated System

6. Given that

$$H(z) = \frac{1 - z^{-1}}{1 - 0.25z^{-2}} = \frac{1 - z^{-1}}{(1 - 0.5z^{-1})(1 + 0.5z^{-1})}$$

(a)

$$X(z) = \frac{1}{1 - z^{-1}}$$

$$Y_1(z) = X_1(z)H(z) = \frac{1}{(1 - 0.5z^{-1})(1 + 0.5z^{-1})} = \frac{a}{1 - 0.5z^{-1}} + \frac{b}{1 + 0.5z^{-1}}$$

$$a = b = \frac{1}{2}$$

$$Y_1(z) = \frac{1}{2} \left(\frac{1}{1 - 0.5z^{-1}} \right) + \frac{1}{2} \left(\frac{1}{1 + 0.5z^{-1}} \right)$$

$$y_1[n] = \frac{1}{2} [(-0.5)^n + (0.5)^n] u[n]$$



(b)

$$Y_2(z) = 1 - z^{-1} = X_2(z)H(z)$$

$$X_2(z) = 1 - 0.25z^{-2}$$

$$\boxed{x_2[n] = \delta[n] - 0.25\delta[n-2]}$$

(c) Since the system is casual

$$h[n] = 0 \text{ for } n < 0$$

Thus, the input can behave as $\cos(0.5\pi n)u[n] = x_3[n]$.

$$X_3(z) = \frac{1 - \cos(0.5\pi)z^{-1}}{1 - 2\cos(0.5\pi)z^{-1} + z^{-2}}, \quad |z| > 1$$

$$X_3(z) = \frac{1}{1 + z^{-2}}$$

$$Y_3(z) = X_3(z)H(z) = \frac{1 - z^{-1}}{(1 + z^{-2})(1 - 0.25z^{-2})}$$

$$\boxed{y_3[n] = \mathcal{Z}^{-1}\{Y_3(z)\}}$$

7. Given that

$$\hat{X}(z) = \log X(z)$$

and

$$x[n] = \delta[n] + a\delta[n-N]$$

 $\hat{x}[n]$ can be found as follows,

$$X(z) = 1 + az^{-N}$$

$$\hat{X}(z) = \mathcal{J}\{1 + az^{-N}\}$$

Using Taylor Series Expansion, more specifically Mercator Series expansion,

$$\log(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} \dots$$

assume, $x = az^{-N}$ for our case,

$$\hat{X}(z) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{az^{-Nn}}{n}$$

Thus,

$$\boxed{\hat{x}[n] = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \delta[n - Nn]}$$



8. Given that

$$c_{xx}[n] = \sum_{k=-\infty}^{\infty} x[k]x[k+n]$$

(a)

$$C_{xx}(z) = \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x[k]x[k+n]z^{-n}$$

$$C_{xx}(z) = \sum_{k=-\infty}^{\infty} x[k] \sum_{n=-\infty}^{\infty} x[k+n]z^{-n}$$

Let $m = k + n$

$$C_{xx}(z) = \sum_{k=-\infty}^{\infty} x[k]z^k \sum_{m=-\infty}^{\infty} x[m]z^{-m}$$

$$C_{xx}(z) = \sum_{k=-\infty}^{\infty} x[k]z^k X(z)$$

Let $\hat{k} = -k$

$$C_{xx}(z) = \sum_{\hat{k}=-\infty}^{\infty} x[-\hat{k}]z^{-\hat{k}}X(z)$$

$$C_{xx}(z) = \mathcal{Z}\{x[-k]\}X(z)$$

$$\boxed{C_{xx}(z) = X\left(\frac{1}{z}\right)X(z)}$$

(b)

$$X(z) = \frac{1}{1 - az^{-1}}$$

$$X(z^{-1}) = \frac{1}{1 - az}$$

$$C_{xx} = X(z)X(z^{-1}) = \frac{1}{(1 - az^{-1})(1 - az)} = \frac{-az^{-1}}{(1 - az^{-1})(1 - a^{-1}z^{-1})}$$

$$C_{xx} = \frac{a_1}{1 - az^{-1}} + \frac{a_2}{1 - a^{-1}z^{-1}}$$

$$\boxed{a_1 = -\frac{1}{1 - a^{-2}}}, \quad \boxed{a_2 = -\frac{a^2}{1 - a^2}}$$

Since it is stable, the ROC includes unit circle. Thus, ROC would be in a ring form. Assuming $a < 1$ and $1/a > 1$, the ROC becomes $\frac{1}{a} < |z| < a$. Considering the ROC, c_{xx} becomes,



$$c_{xx}[n] = -\frac{1}{1-a^{-2}}a^n u[n] - \frac{a^2}{1-a^2}a^{-n}u[-n-1]$$

The pole-zero diagram and ROC for the given system can be seen at *Figure 6*.

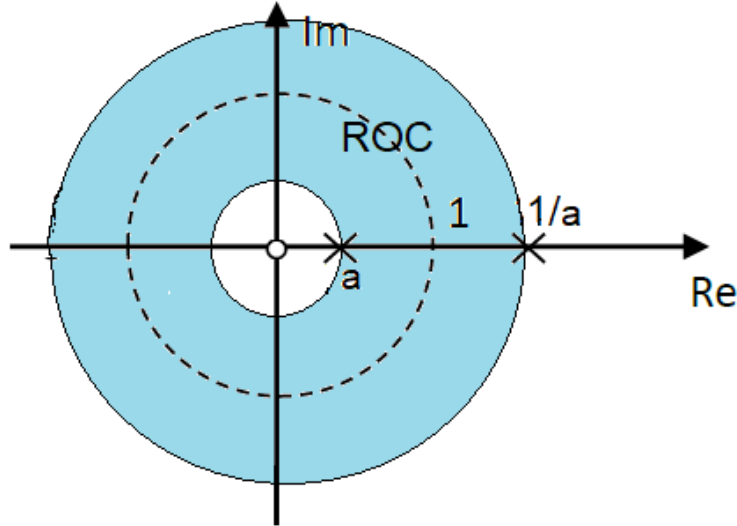


Figure 6: ROC for Given System

- (c) It is clear that $c_{xx}(z) = c_{xx}(z^{-1})$. Thus, $x[-n]$ would have same autocorrelation function with $x[n]$.
- (d) Also notice that $C_{xx} = X(z)X(z^{-1}) = \frac{1}{(1-az^{-1})(1-az)}$, if we could have only extra polynomial at the dominator like z^{-l} , we could potentially get rid of it as we multiply with z^l term of $X(z^{-1})$. This can be satisfied with any shifted version of $x[n]$, i.e., $x[n-l]$

9. 9th Question

- (a) Z-Transfer found using conv command can be seen at *Figure 7*. The source code for this part can be found in **Appendix B**.

```
x_z =
-z^-1 - z^-2 + 7 z^-3 - 16 z^-4 + 22 z^-5 + 17 z^-6 - 28 z^-7
```

Figure 7: Z-Transfer using conv command



- (b) Inverse Z-Transform of the given system function using `residuez` can be seen at *Figure 8*. The source code for this part can be found in **Appendix B**.

```
The inverse z-transform of given function is:
x[1]= 1.000000 (1.000000)^(n)
x[2]= 0.000000 (0.000000)^(n)
x[3]= 0.000000 (0.000000)^(n)
```

Figure 8: Inverse Z-Transform of the Given System Function

- (c) Pole-Zero diagram for the system function given can be seen at *Figure 9*. The source code for this part can be found in **Appendix B**.

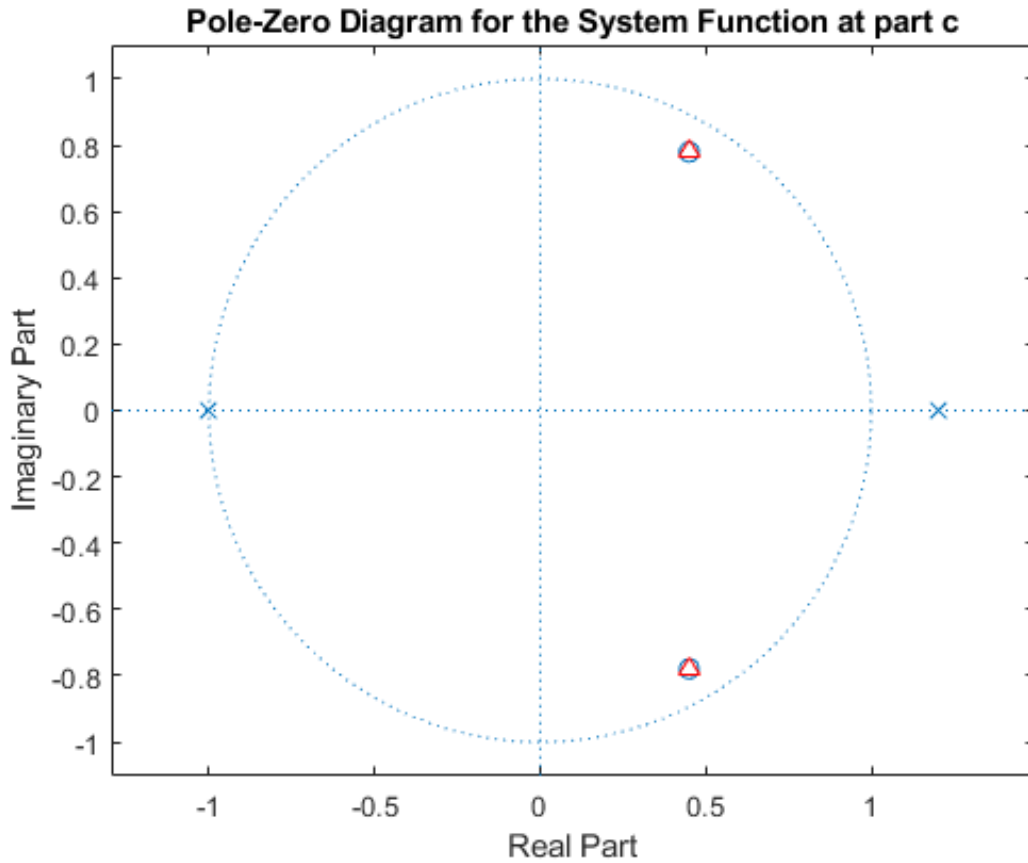


Figure 9: Pole-Zero Diagram for the System Function Given at part c



Appendices

A Source Code for Question 1

```

1 %% Q1a
2 num =[1 -sqrt(2) 1]
3 den=[1 0 0]
4 x = tf (num,den)
5 figure(1)
6 zplane(num,den)
7 title("Pole-Zero Diagram for the System Function at Q1")
8
9
10 %% Q1b
11 L=1000;
12 dw=2*pi/L;
13 w = -pi:dw:pi-dw;
14
15 HH=freqz (num,den,w);
16
17 figure(2)
18 mag=abs(HH)
19 plot(w,mag)
20 title('Magnitude Response of the System')
21 grid on
22 xlabel("frequency")
23 ylabel("Magnitude (dB)")
24
25 figure(3)
26 phase=angle(HH)
27 plot(w,phase)
28 title('Phase Response of the System')
29 grid on
30 xlabel("frequency")
31 ylabel("Phase (degree)")

```



B Source Code for Question 9

```

1 %% Q9a
2 x_1=[ 1 3 -4]
3 x_2=[-1 2 -3 1 7]
4 x_k=conv(x_1,x_2)
5
6 x_z=filt([0 x_k] , [1])
7
8 %% Q9b
9
10 a= [1 -3 4];
11 b= [1 -1 1 -1];
12 [r,p,k]=residuez(a , b);
13 [rsize1 , rsize2]=size(r);
14
15 i=1;
16 fprintf('The inverse z-transform of given function is:\n')
17 while i<(rsize1+1)
18     fprintf('x[%i]= %f (%f)^(n)\n',i , r(i),p(i));
19     i=i+1;
20 end
21
22 %% Q9c
23
24 a_c= [1 -0.2 -1.2 ]
25 b_c= [1 -0.9 0.81]
26 [r_c , p_c , k_c]=residuez(a_c , b_c)
27
28
29 zplane(b_c , a_c)
30 hold on
31 plot(p_c , '^r')
32 title(" Pole-Zero Diagram for the System Function at part c")
33 hold off

```

