EE 430

Fall 2014

HW 2 (Section 2)

Solutions for 10-29

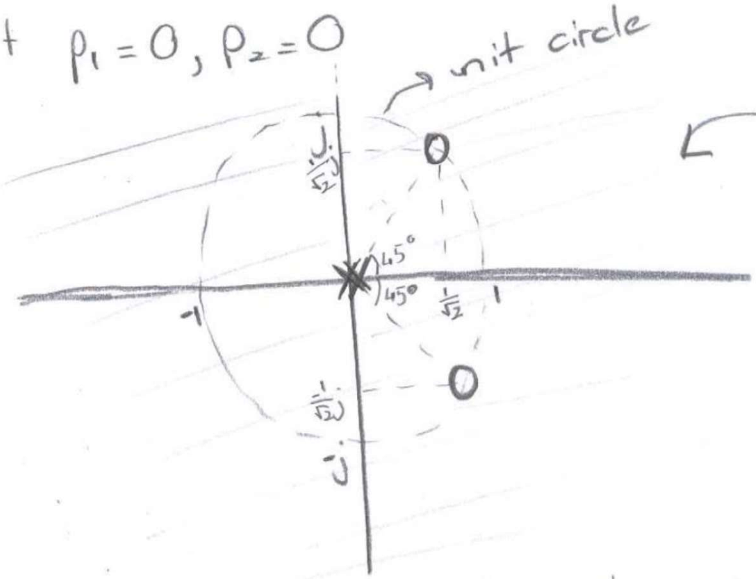
10)

$$h[n] = \delta[n] - \sqrt{2}\,\delta[n-1] + \delta[n-2]$$

a)
$$H(z) = \sum_{n=-\infty}^{\infty} h[n]\, z^{-n}$$
$$= \sum_{n=0}^{2} h[n]\, z^{-n} = 1 - \sqrt{2}\, z^{-1} + z^{-2}$$
$$= \left(1 - \left(\tfrac{1}{\sqrt{2}} + \tfrac{1}{\sqrt{2}}j\right)z^{-1}\right)\left(1 - \left(\tfrac{1}{\sqrt{2}} - \tfrac{1}{\sqrt{2}}j\right)z^{-1}\right)$$
$$= \frac{\left(z - \left(\tfrac{1}{\sqrt{2}} + \tfrac{1}{\sqrt{2}}j\right)\right)\left(z - \left(\tfrac{1}{\sqrt{2}} - \tfrac{1}{\sqrt{2}}j\right)\right)}{z^2}$$

zeros at $z_1 = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}j$, $z_2 = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}j$

poles at $p_1 = 0$, $p_2 = 0$



ROC: $|z| > 0$ (since the only poles at $p_1 = p_2 = 0$)

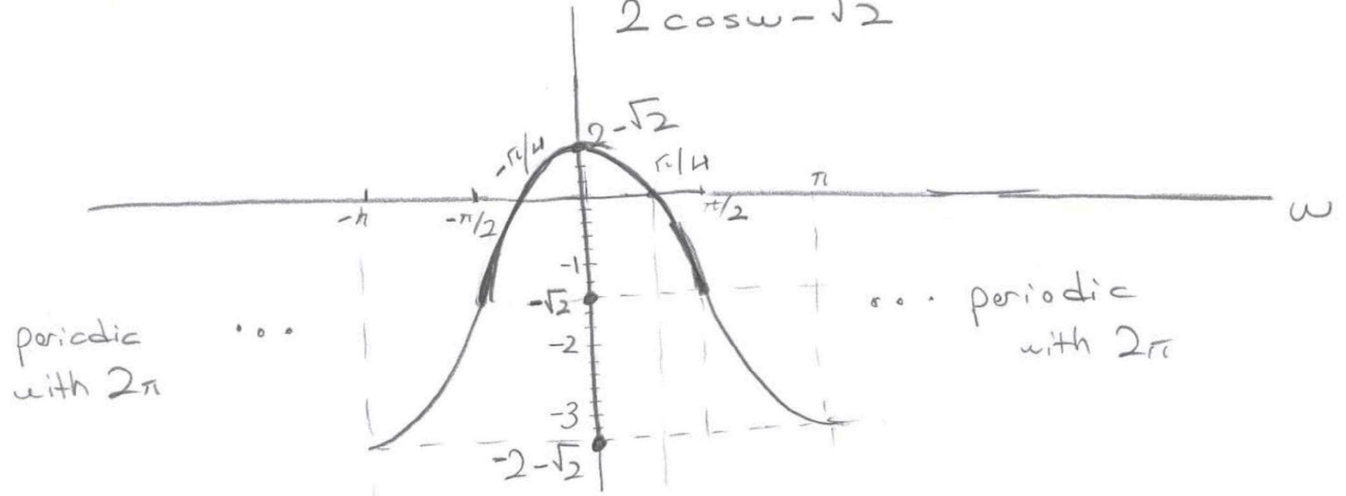
b) Since ROC contains the unit circle, this system has a frequency response.

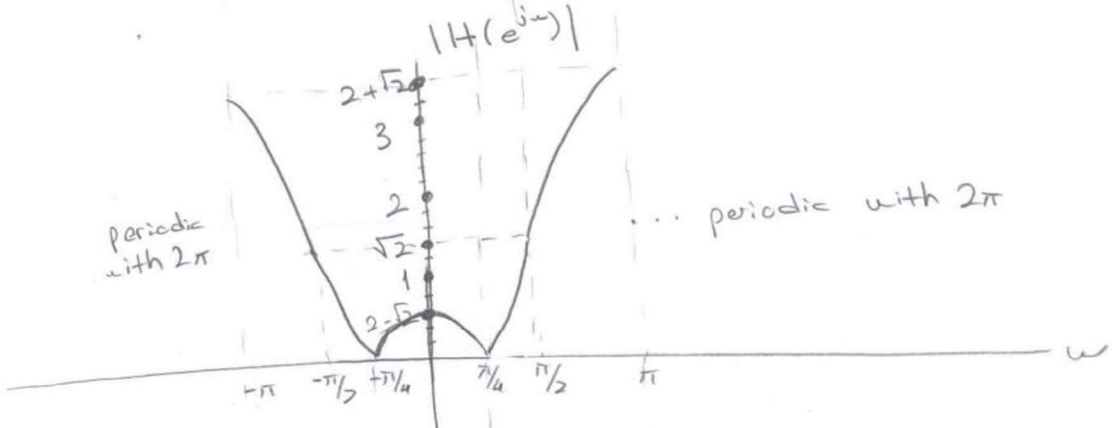
$$H(e^{j\omega}) = H(z)\Big|_{z=e^{j\omega}} = 1 - \sqrt{2}e^{-j\omega} + 1e^{-j2\omega}$$

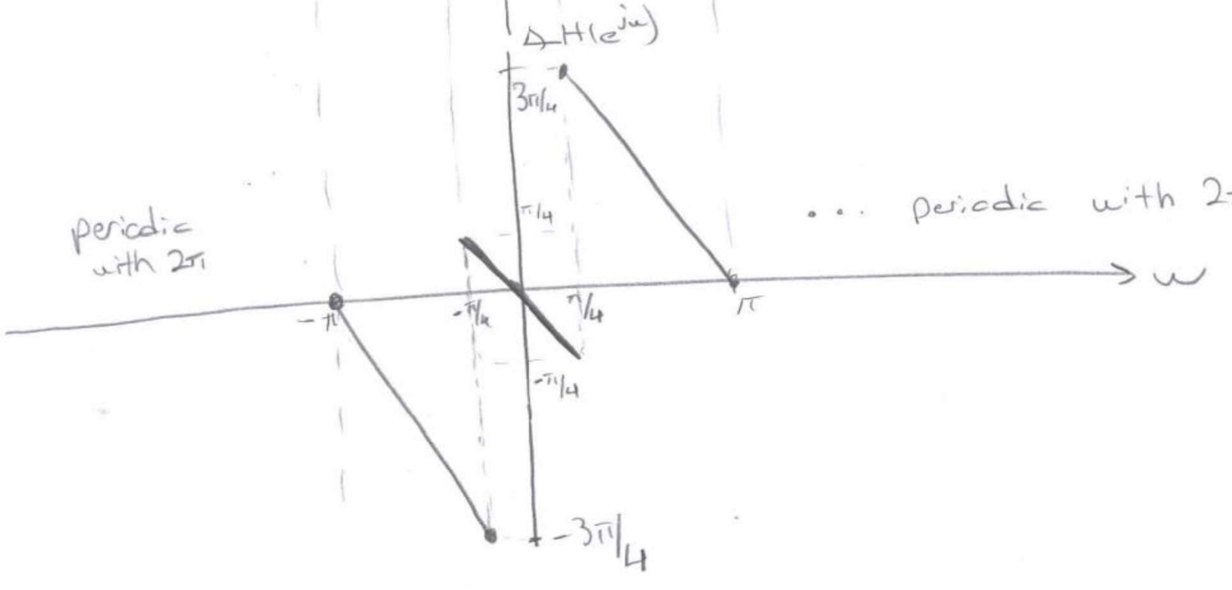
$$= e^{-j\omega}(e^{j\omega} - \sqrt{2} + e^{-j\omega}) = e^{-j\omega}(2\cos\omega - \sqrt{2})$$

$$|H(e^{j\omega})| = |2\cos\omega - \sqrt{2}|$$

$$\measuredangle H(e^{j\omega}) = -\omega + \measuredangle(2\cos\omega - \sqrt{2})$$







c) Using the eigenfunction property;

$$x[n] = \sum_k a_k e^{j\omega_k n} \longrightarrow \boxed{h[n]} \longrightarrow y[n] = \sum_k a_k H(e^{j\omega_k}) e^{j\omega_k n}$$

i) $x_1[n] = \sin\left(\frac{\pi}{4} n + \frac{\pi}{4}\right) = \dfrac{e^{j\frac{\pi}{4} n + \frac{\pi}{4}} - e^{-j\frac{\pi}{4} n - \frac{\pi}{4}}}{2j}$

$$y_1[n] = \frac{e^{j\frac{\pi}{4}}}{2j} \underbrace{H(e^{j\frac{\pi}{4}})}_{0} e^{j\frac{\pi}{4} n} - \frac{e^{-j\frac{\pi}{4}}}{2j} \underbrace{H(e^{-j\frac{\pi}{4}})}_{0} e^{-j\frac{\pi}{4} n}$$

$y_1[n] = 0$

ii) $x[n] = e^{j\omega n} u[n]$

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k] = \sum_{k=-\infty}^{n} h[k]\, e^{j\omega(n-k)}$$

Since $h[n]$ is causal;

$$y[n] = \begin{cases} 0 & n < 0 \\ \left(\sum_{k=0}^{n} h[k] e^{-j\omega k} \right) e^{j\omega n} & n \geq 0 \end{cases}$$

For $n \geq 0$

$$y[n] = \left(\sum_{k=0}^{\infty} h[k] e^{-j\omega k} \right) e^{j\omega n} - \left(\sum_{k=n+1}^{\infty} h[k] e^{-j\omega k} \right) e^{j\omega n}$$

$$= H(e^{j\omega}) e^{j\omega n} - \left(\sum_{k=n+1}^{\infty} h[k] e^{-j\omega k} \right) e^{j\omega n}$$

$$x_2[n] = \sin\left(\frac{\pi}{4}n + \frac{\pi}{4}\right)u[n] = \left(\frac{e^{j\pi/4}}{2j}e^{j\frac{\pi}{4}n} - \frac{e^{-j\pi/4}}{2j}e^{-j\frac{\pi}{4}n}\right)u[n]$$

$$y_2[n] = \frac{e^{j\pi/4}}{2j}e^{j\frac{\pi}{4}n}\underbrace{H(e^{j\pi/4})}_{0} - \frac{e^{-j\pi/4}}{2j}e^{-j\pi/4 n}\underbrace{H(e^{-j\pi/4})}_{0}$$

$$-\left(\sum_{k=n+1}^{\infty} h[k]e^{-j\frac{\pi}{4}k}\right)\frac{e^{j\pi/4}}{2j}e^{j\pi/4 n} + \left(\sum_{k=n+1}^{\infty} h[k]e^{+j\frac{\pi}{4}k}\right)\frac{e^{-j\pi/4}}{2j}e^{-j\pi/4 n}$$

$n \geq 0$

$y_2[n] = 0 \quad n < 0$

$y_2[n] = 0 \quad n < 0$

$$y_2[0] = -\left(\sum_{k=1}^{2} h[k]e^{-j\pi/4 k}\right)\frac{e^{j\pi/4}}{2j}e^{j\pi/4\cdot 0} + \left(\sum_{k=1}^{2} h[k]e^{+j\frac{\pi}{4}k}\right)\frac{e^{-j\pi/4}}{2j}e^{-j\pi/4\, 0}$$

$$= -\left(-\sqrt{2}e^{-j\pi/4} + 1e^{-j\pi/2}\right)\frac{e^{j\pi/4}}{2j} + \left(-\sqrt{2}e^{j\pi/4} + 1e^{j\pi/2}\right)\frac{e^{-j\pi/4}}{2j}$$

$$= \cancel{-j\frac{\sqrt{2}}{2}} + j\frac{e^{-j\pi/4}}{2} \cancel{+j\frac{\sqrt{2}}{2}} - j\frac{e^{j\pi/4}}{2} = \frac{j}{2}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}j\right) - \frac{j}{2}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}j\right)$$

$$= \frac{1}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$y_2[1] = -\left(\sum_{k=2}^{2} h[k]e^{-j\pi/4 k}\right)\frac{e^{j\pi/4}}{2j}e^{j\pi/4} + \left(\sum_{k=2}^{2} h[k]e^{j\pi/4 k}\right)\frac{e^{-j\pi/4}}{2j}e^{-j\pi/4}$$

$$= -1e^{-j\pi/2}\frac{e^{j\pi/2}}{2j} + e^{j\pi/2}\frac{e^{-j\pi/2}}{2j} = 0$$

$$y_2[n] = 0 \quad n \geq 2$$

$$y_2[n] = \frac{1}{\sqrt{2}}\delta[n]$$

iii) $x_3[n] = \underbrace{\sin\left(\frac{\pi}{4}n + \frac{\pi}{4}\right)}_{x_4[n]} + \underbrace{\sin\left(\frac{3\pi}{4}n\right)}_{x_5[n]}$

$$y_3[n] = y_4[n] + y_5[n]$$

$y_4[n]$: $x_4[n] * h[n] = 0$ (from i))

$y_5[n]$: $x_5[n] * h[n]$

$$y_3[n] = \frac{e^{j3\pi/4 n}}{2j}H(e^{j3\pi/4}) - \frac{e^{-j3\pi/4 n}}{2j}H(e^{-j3\pi/4})$$

$$H(e^{j\frac{3\pi}{4}}) = e^{-j\frac{3\pi}{4}}\Big(2\underbrace{\cos\frac{3\pi}{4}}_{\frac{-1}{\sqrt{2}}} - \sqrt{2}\Big) = 2\sqrt{2}\,e^{j\pi/4}$$

$$H(e^{-j3\pi/4}) = e^{j3\pi/4}\Big(2\cos\Big(\frac{-3\pi}{4}\Big) - \sqrt{2}\Big) = 2\sqrt{2}\,e^{-j\pi/4}$$

$$y_3[n] = 2\sqrt{2}\,\frac{e^{j\left(\frac{3\pi}{4}n + \frac{\pi}{4}\right)}}{2j} - 2\sqrt{2}\,\frac{e^{j\left(\frac{-3\pi}{4}n - \frac{\pi}{4}\right)}}{2j}$$

$$\boxed{y_3[n] = 2\sqrt{2}\sin\Big(\frac{3\pi}{4}n + \frac{\pi}{4}\Big)}$$

d) When the zeros of $H(z)$ are on the unit circle, at the frequencies where these zeros are located $H(e^{j\omega})$ (frequency response) is 0. For example when $H(z)$ has a zero where $z_0 = e^{j\omega_0}$; $H(e^{j\omega_0})$ is 0.