A brief introduction to gauges

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Introduction.

In this paper we explore Maxwell's equations in terms of gauge theory.

Reference to gauge freedom in electromagnetism can be found as early as Maxwell. In deriving his equations Maxwell noted that there was some freedom of choice in the magnetic vector potential. Choosing to keep things simple, he used what became known as the Coulomb gauge, $\nabla \cdot \mathbf{A} = 0$. Later on, Lorenz used his eponymous gauge, $\nabla \cdot \mathbf{A} = -\mu_0 \varepsilon_0 \frac{\partial V}{\partial t}$, to produce a general solution to Maxwell's equations (the derivation of which we will reproduce below) (Jackson, 2001).[1] Amidst discussions among the community of which of these potentials was "correct", Fock (1926; see translation in Fock, 2010) wrote down equations for the transformation from one gauge to another, as well as a similar equation which transformed the wave equation, thus resolving the question. All this was completed before the name of gauge was ever uttered.

Beginning in 1928 with his book *Gruppentheorie und Quantenmechanik*, Hermann Weyl finally coined the term whilst he, according to Jackson, "enshrined as fundamental the principle of gauge invariance". On top of repeating Fock's result, Weyl first suggested the interpretation that these equations governed the relation between electromagnetism and matter.

Definition and examples.

The first question one might ask about gauge theory is "what is a gauge?" As it turns out, the answer is not particularly illuminating - a gauge is a function chosen to modify some physical quantity whilst keeping other quantities the same. A more intuitive answer can be found by first adding adjectives.

Maxwell's and Schrödinger's equations.

Mathematical formalism.

The intrinsic difficulty in understanding differential geometry is that its notation is so difficult to parse. Before we begin to discuss gauges in terms of their modern formulation, there are several important pieces of terminology - and, indeed, mathematics - which we must work to understand. To that end, we devote this section to the differential geometric framework of gauge theory, and of theoretical physics in general.[2]

We begin, as we must, with the concept of a *manifold*. Simply put, the manifold in a problem is the space on which we are working. This could be anything from the Earth to a small copper wire to all of spacetime. More important than the shape or size of the manifold is that we can *describe* it in terms of

[1] This was eventually attributed to H. A. Lorentz, although Lorenz (and Riemann as well) had proved this long before his time (Jackson, 2001).

[2] I have derived much of my understanding of this subject from the book by Baez and Muniain (1994).

something we know more about - n-dimensional Euclidean space being the preference. Mathematically, we say that an object is a manifold if it is locally equivalent to $\mathbb{R}^n$ for some n.[3]

We use the surface of the Earth as an example. If we were to look at the Earth from space, we would see a sphere (roughly). But when we look at it from *on the surface*, we see what looks like a flat plane. In fact, we can define a coordinate system centered at our current location. This coordinate can extend as far as the horizon, or further, without any problems - we could even extend our coordinate system fully halfway around the earth. In this way, we have defined a *chart* based on our current location. If we choose enough charts, and space them out so that every point on the earth has a reasonably sized region around it that fits inside at least one chart, we have what is called an *atlas*. In this way, the surface of the Earth is locally equivalent to Euclidean space - or locally Euclidean - and is a manifold.

[3]here $\mathbb{R}^n$ denotes the n-dimensional vector space described by n-tuples of real numbers. Common examples include the Cartesian plane and 3D space.