Problem 1

The volume of an N-dimensional ball.

Let us first consider the given equation:

$$\int_{\mathbb{R}} \exp(-x^2/2)dx = \sqrt{2\pi} \tag{1}$$

As our problem space involves N dimensions, we can consider this equation in a N-dimensional context:

$$\int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} \exp(-x_1^2/2) * ... * \exp(-x_n^2/2)dx_1...dx_n = \Pi_{i=1}^{n} \int_{-\infty}^{\infty} \exp(-x_i^2/2)dx_i \tag{2}$$

$$= (2\pi)^{n/2} \tag{3}$$

We can also consider this integral in spherical coordinates by dividing $\mathbb{R}^n$ into shells of dimension $\mathbb{R}^{n-1}$:

$$\Pi_{i=1}^{n} \int_{-\infty}^{\infty} \exp(-x_i^2/2)dx_i = \int_0^{\infty} \int_{Shell^{n-1}} \exp(-r^2/2)dAdr \tag{4}$$

$$= \int_0^{\infty} \exp(-r^2/2) \int_{Shell^{n-1}} dAdr \tag{5}$$

Let the area (using this term for generality as it has dimension $n-1$) of our shell be $A_{n-1}(r) = A_{n-1}(1)r^{n-1}$. Then, we have:

$$\int_0^{\infty} \exp(-r^2/2) \int_{Shell^{n-1}} dAdr = A_{n-1}(1) \int_0^{\infty} r^{n-1} \exp(-r^2/2)dr \tag{6}$$

Finally, noting the similarity to the Γ function we plug in $t = \frac{r^2}{2}$ through change of variables to achieve a similar form:

$$A_{n-1}(1) \int_0^{\infty} r^{n-1} \exp(-r^2/2)dr = A_{n-1}(1)2^{(n-2)/2} \int_0^{\infty} t^{(n-2)/2}e^{-t}dt \tag{7}$$

$$= A_{n-1}(1)2^{(n-2)/2}\Gamma(n/2) \tag{8}$$

Comparing with our earlier result through integration by products, we can solve for $A_{n-1}(1)$, $A_{n-1}(1)$

(by proportionality), and then $V_n(1) = w_n$:

$$\begin{aligned}
(2\pi)^{n/2} &= A_{n-1}(1)2^{(n-2)/2}\Gamma(n/2) && (9)\\
A_{n-1}(1) &= \frac{(2\pi)^{n/2}}{2^{(n-2)/2}\Gamma(n/2)} && (10)\\
&= \frac{2\pi^{n/2}}{\Gamma(n/2)} && (11)\\
A_{n-1}(r) &= \frac{2\pi^{n/2}}{\Gamma(n/2)}r^{n-1} && (12)\\
V_n(1) = w_n &= \int_0^1 A_{n-1}(r)dr && (13)\\
&= \int_0^1 \frac{2\pi^{n/2}}{\Gamma(n/2)}r^{n-1}dr && (14)\\
&= \frac{2\pi^{n/2}}{n\Gamma(n/2)} && (15)\\
&= \frac{\pi^{n/2}}{\Gamma(n/2+1)} && (16)\\
& && (17)
\end{aligned}$$

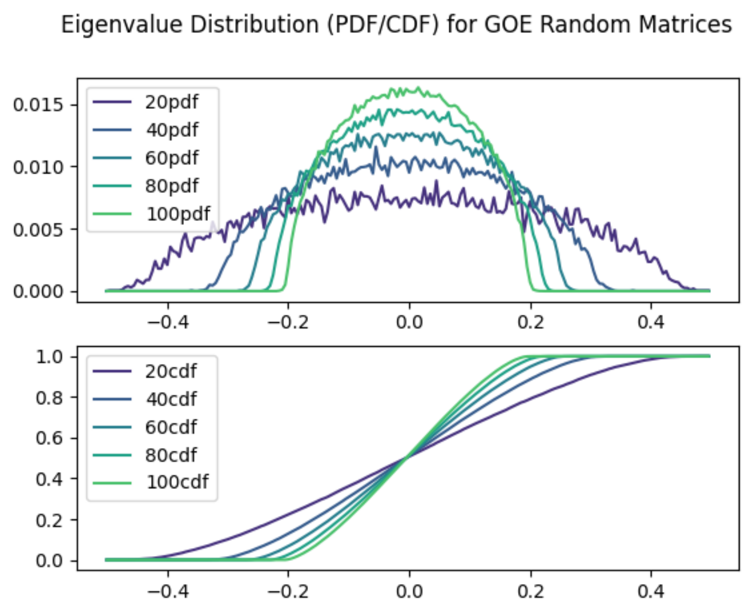
And we have our answer.

Although this solution required reading multiple related resources online, I wrote the final draft without consulting those resources.

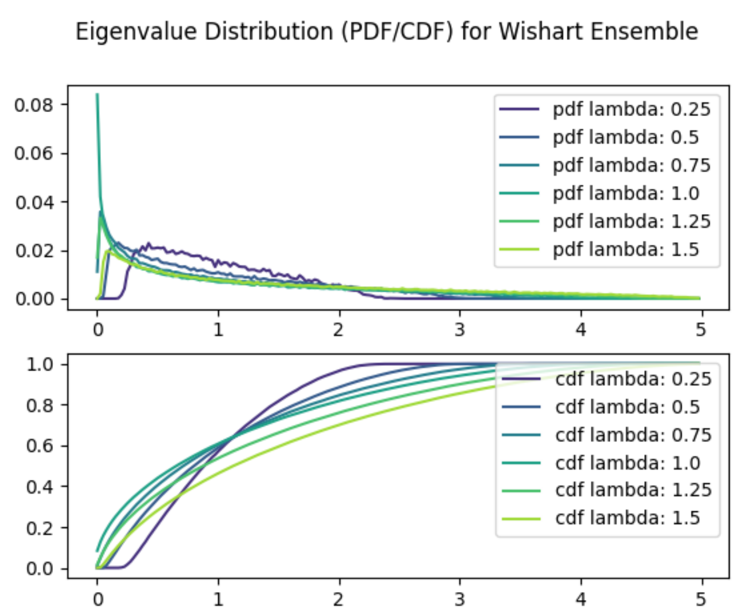
Problem 2

Limit laws in random matrix theory.

a. Here we start to observe the Wigner semicircle law.



b. I'm not sure if the scaling is exactly correct here, but we can clearly see the shape of the Marchenko-Pastur distribution.



Problem 3

Matrix completion.

a. This can be done quite easily with a line such as

```
r = np.power(np.random.standard_normal(n), 2)
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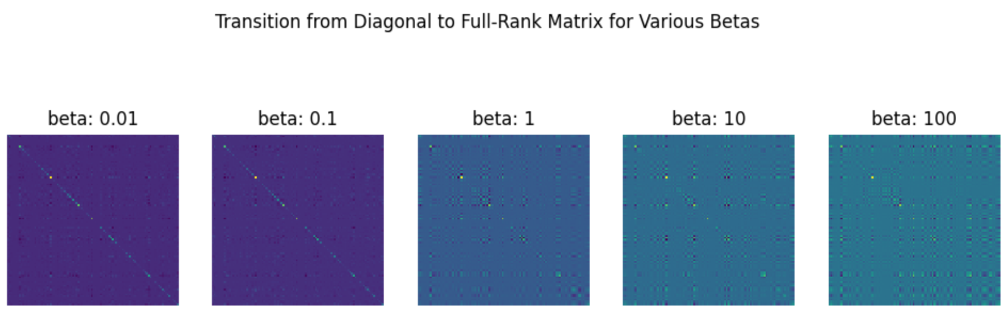
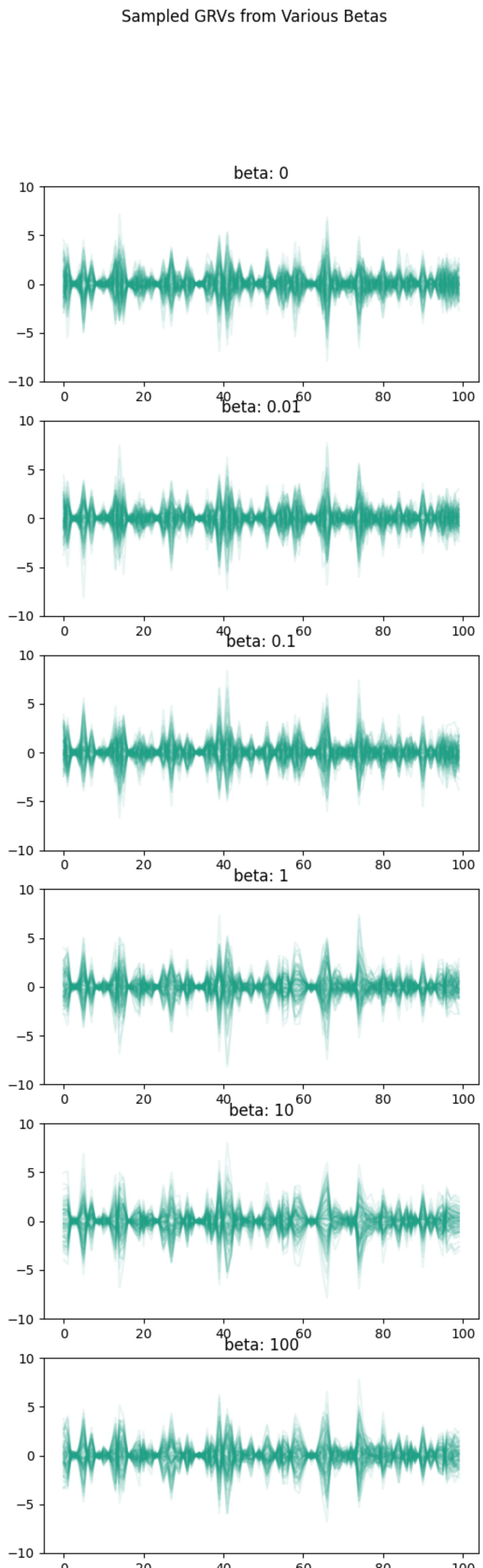
b. Here darker colors represent smaller values; the matrix with $\beta = 0.01$ has non-diagonal values nearing 0. There is an evident transition to a higher-rank matrix as beta increases.



c. For various betas, I've sampled random GRVs with P_β as the covariance matrix. It seems that higher beta values have individual points that vary less between themselves and other points, leading to less centered distributions (lighter color near $y = 0$), as a point that is a bit off will lead other points to also be off. Especially in the areas with higher variance (scaling) such as $x = 42, 65$ this effect becomes more evident.


Sampled GRVs from Various Betas
beta: 0
beta: 0.01
beta: 0.1
beta: 1
beta: 10
beta: 100