Topics in Optimal Control

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Chapter 1

Dynamic Programming

1.1 Discrete time systems

1.1.1 A Fundamental Lemma

Consider the minimization problem

$$\inf_{(x,y)\in D} f(x,y)$$

where $D \subset X \times X$, and X is a metric space. Denote $D_x := \{y \in X : (x,y) \in D\}$, $D^y := \{x \in X : (x,y) \in D\}$.

Lemma 1.1 *The following equality holds*

$$\inf_{(x,y)\in D} f(x,y) = \inf_x \inf_{y \in D_x} f(x,y) = \inf_y \inf_{x\in D^y} f(x,y).$$

In addition, the infimums can be replaced by minimum when either of the three infimum is achieved for some $(x_*, y_*) \in D$.

Proof 1 *Let $I_1 = \inf_{(x,y)\in D} f(x,y)$, $I_2 = \inf_x \inf_{y\in D_x} f(x,y)$. It suffices to prove $I_1 = I_2$. By definition, there exists a sequence $\{(x_k, y_k)\}_{k=1}^{\infty} \subset D$, such that $\lim_{k\to\infty} f(x_k, y_k) \searrow I_1$ [1]. Then*

$$I_2 \le \inf_{y\in D_{x_k}} f(x_k, y) \le f(x_k, y_k), \ \forall k \ge 1.$$

Letting $k \to \infty$, we get $I_2 \le I_1$. For the inverse direction, note that for any $\varepsilon > 0$, one can find a pair $(x'', y'') \in D$ such that

$$\inf_{(x',y')\in D} f(x', y') \ge f(x'', y'') - \varepsilon \tag{1.1}$$

and $f(x'', y'') \ge I_1$. On the other hand, there exists an integer $K > 0$, such that for all $k \ge K$, $f(x_k, y_k) \le I_1 + \varepsilon \le f(x'', y'') + \varepsilon$. It follows from (1.1) that for any $(x,y) \in D$,

$$f(x,y) \ge \inf_{(x',y')\in D} f(x', y') \ge f(x_k, y_k) - 2\varepsilon, \ \forall k \ge K.$$

Hence

$$I_2 = \inf_x \inf_{y\in D_x} f(x,y) \ge \lim_{k\to\infty} f(x_k, y_k) - 2\varepsilon = I_1 - 2\varepsilon.$$

Since ε is arbitrary, we get $I_2 \ge I_1$, which implies that $I_1 = I_2$.

[1] $\lim_{k\to\infty} a_k \searrow a$ means that a_k is decreasing and converges to a.

This lemma is extremely powerful as we will see in subsequent sections. We underscore that the interchangeability of two inf are essential. On the other hand, it is usually illegal to interchange inf and sup as in general

$$\inf_x \sup_y f(x,y) \neq \sup_y \inf_x f(x,y).$$

which makes differential games different from classical optimal control as we will see later.

1.1.2 Multistage Minimization

For most of the time, we will consider "min" in lieu of "inf", due to the consideration that the difference between the two are not essential for our discussions.

Consider the function

$$J(x,y,z) = f(x,y) + g(y,z)$$

and the minimization problem

$$J^*(x) = \min_{(y,z)\in D} f(x,y) + g(y,z).$$

Invoking Lemma 1.1, J can be rewritten as

$$\begin{aligned} J^*(x) &= \min_y \min_{z\in D_y} f(x,y) + g(y,z) \\ &= \min_y (f(x,y) + \min_{z\in D_y} g(y,z)) \end{aligned}$$

The minimization has been divided into two steps and that is why we call it a twostage minimization problem. In the first step, y is fixed, and we minimize $g(y,z)$ over D_y to get a function $\varphi(y) = \min_{z\in D_y} g(y,z)$. The second step is to minimize the function $f(x,y) + \varphi(y)$.

For multistage minimization, we consider

$$J(x_0) = \sum_{k=1}^{N} g_k(x_{k-1}, x_k) \tag{1.2}$$

$$J^*(x_0) = \min_{(x_1,\cdots x_N)} J(x_0, x_1, \cdots, x_N)$$

This is a multistage minimization problem. Using Lemma 1.1, we know

$$J_i(x) = \left.\sum_{k=i}^{N} g_k(x_{k-1}, x_k)\right|_{x_{i-1}=x}$$

$$J_i^*(x) = \min_{(x_i,\cdots,x_N)} J_i(x), \quad 1 \le i \le N$$

rewrite J as

$$\begin{aligned} J_1^*(x_0) &= \min_{x_1} \min_{(x_2,\cdots,x_N)} \sum_{k=1}^{N} g_k(x_{k-1}, x_k) \\ &= \min_{x_1} \left[g_1(x_0, x_1) + \min_{(x_2,\cdots,x_N)} \sum_{k=2}^{N} g_k(x_{k-1}, x_k) \right] \\ &= \min_{x_1} [g_1(x_0, x_1) + J_2^*(x_1)] \end{aligned}$$

Similarly,

$$\begin{aligned} J_m^*(x) &= \min_{x_m} [g_m(x, x_m) + J_{m+1}^*(x_m)], \quad 1 \le m \le N-1 \\ J_N^*(x) &= \min_{x_N} g_N(x, x_N) \end{aligned}$$

The above algorithm is nothing but the celebrated *Bellman's principle*, or *Bellman's principle of optimality* or *dynamic programming* which is formally stated as follows:

> An optimal policy has the property that no matter what the previous decision (i.e., controls) have been, the remaining decisions must constitute an optimal policy with regard to the state resulting from those previous decisions.

1.1.3 Dynamic Programming and Optimal Control

In the previous subsection, we have considered a minimization problem where the cost functions are in a multistage structure such that the minimization can be solved recursively. Now we move from pure optimization problems to optimal control of discrete time systems. We will see that the principle of optimility derived in the previous subsection can be successfully applied to optimal control.

Consider the nonlinear discrete time system

$$x_{k+1} = f_k(x_k, u_k), \tag{1.3}$$

where $x_k \in X$ (the system state at time instant k), $u_k \in U$ (the input at time instant k), and the cost function

$$\begin{aligned} J_1(x_1) &= \varphi(x_{N+1}) + \sum_{k=1}^{N} L_k(x_k, u_k), \\ J^* &= \min_{(u_1,\cdots,u_N)} J_1(x_1) \end{aligned} \tag{1.4}$$

with $1 \le N \in \mathbb{Z}$, and the initial state x_1 is assumed to fixed.

Define

$$\begin{aligned} J_i(x) &= \sum_{k=i}^{N} L_k(x_k, u_k)\Big|_{x_i = x} + \varphi(x_{N+1}), \\ J_i^*(x) &= \min_{(u_i,\cdots,u_N)} J_i(x) \end{aligned}$$

then

$$\begin{aligned} J_1^*(x_1) &= \min_{u_1}\left[L_1(x_1,u_1) + \min_{(u_2,\cdots,u_N)} \sum_{k=2}^{N} L_k(x_k,u_k) + \varphi(x_{N+1})\right] \\ &= \min_{u_1}(L_1(x_1,u_1) + J_2^*(x_2)) \\ &= \min_{u_1}[L_1(x_1,u_1) + J_2^*(f_1(x_1,u_1))]. \end{aligned}$$

Doing this recursively, we get

$$\begin{aligned} J_i^*(x) &= \min_{u_i}[L_i(x,u_i) + J_{i+1}^*(f_i(x,u_i))], \quad 1 \le i \le N-1 \\ J_N^*(x) &= \min_{u_N}[L_N(x,u_N) + \varphi(f_N(x,u_N))] \end{aligned} \tag{1.5}$$

This is the so called *Bellman's equation*. The optimal policy at the i-th stage is $u_i^* = \arg\min_{u_i} J_i(x_i)$.

1.1.4 LQG controller and Kalman filter

Example 1 (LQG Control (full-state info)) *We consider the system*

$$x_{k+1} = A_k x_k + B_k u_k + D_k w_k$$

where $w_k \sim N(0, D\sigma_k)$ *are independent Gaussian noise and* $x_0 \sim N(0, \Sigma_0)$ *which is independent of* $\{w_k\}_0^{N-1}$. *Define the finite horizen cost functional*

$$J = E\left[x_N^T Q_N x_N + \sum_{k=0}^{N-1}(x_k^T Q_k x_k + u_k^T R_k u_k)\right].$$

The objective is to find a Markov control process $u = (u_0, \cdots, u_{N-1})$ *such that*

$$J \to \min.$$

Define

$$J_i(x) = \min_{(u_i,\cdots,u_{N-1})} E\left[x_N^T Q_N x_N + \sum_{k=i}^{N-1}(x_k^T Q_k x_k + u_k^T R_k u_k)\middle| x_i = x\right], \quad 0 \le i \le N-1$$

Then

$$J = E^{x_0}[J_0(x_0)]$$

and

$$\begin{aligned}
J_i(x) &= \min_{u_i} \min_{(u_{i+1},\cdots,u_{N-1})} E\left[x_i^T Q_i x_i + u_i^T R_i u_i + x_N^T Q_N x_N + \sum_{k=i+1}^{N-1}(x_k^T Q_k x_k + u_k^T R_k u_k)\middle| x_i = x\right] \\
&= \min_{u_i} E\left[x_i^T Q_i x_i + u_i^T R_i u_i + \min_{(u_{i+1},\cdots,u_{N-1})} x_N^T Q_N x_N + \sum_{k=i+1}^{N-1}(x_k^T Q_k x_k + u_k^T R_k u_k)\middle| x_i = x\right] \\
&= \min_{u_i}\left\{\begin{array}{c} x^T Q_i x + u_i^T(x) R_i u_i(x) \\ +E\left[\min_{(u_{i+1},\cdots,u_{N-1})} x_N^T Q_N x_N + \sum_{k=i+1}^{N-1}(x_k^T Q_k x_k + u_k^T R_k u_k)\middle| x_i = x\right]\end{array}\right\} \\
&= \min_{u_i}\left\{x^T Q_i x + u_i^T(x) R_i u_i(x) + E^{w_i}\left[J_{i+1}(A_i x + B_i u_i(x) + D_k w_i)\right]\right\}, \quad 0 \le i \le N-1
\end{aligned}$$

$$J_N(x) = x^T Q_N x$$

It can be easily seen that $J_i(x)$ *is quadratic in* x, *e.g.,*

$$\begin{aligned}
&\frac{\partial}{2\partial u_{N-1}}\left\{\begin{array}{c} x^T Q_{N-1} x + u_{N-1}^T(x) R_{N-1} u_{N-1}(x) \\ +E^{w_{N-1}}\left[(A_{N-1}x + B_{N-1}u_{N-1}(x) + w_{N-1})^T Q_N (A_{N-1}x + B_{N-1}u_{N-1}(x) + w_{N-1})\right]\end{array}\right\} \\
&= u_{N-1}^T R_{N-1} + E^{w_{N-1}}[(A_{N-1}x + B_{N-1}u_{N-1}(x) + w_{N-1})^T Q_N B_{N-1}] \\
&= u_{N-1}^T R_{N-1} + x^T A_{N-1}^T Q_N B_{N-1} + u_{N-1}^T B_{N-1}^T Q_N B_{N-1} \\
&= u_{N-1}^T (R_{N-1} + B_{N-1}^T Q_N B_{N-1}) + x^T A_{N-1}^T Q_N B_{N-1} = 0 \\
&\Downarrow
\end{aligned}$$

$$u_{N-1} = -(R_{N-1} + B_{N-1}^T Q_N B_{N-1})^{-1} B_{N-1}^T Q_N A_{N-1} x$$

and one can readily check that $J_{N-1}(x) = x^T S_{N-1} x + p_{N-1}$ *in which* S_{N-1} *and* p_{N-1} *are constant. Thus by induction, we may assume*

$$J_i(x) = x^T S_i x + p_i$$

by Bellman's equation, we have

$$
\begin{aligned}
x^T S_i x + p_i = \min_{u_i} \{ & x^T Q_i x + u_i^T R_i u_i \\
& + E^{w_i} \left[(A_i x + B_i u_i(x) + w_i)^T S_{i+1} (A_i x + B_i u_i(x) + w_i) + p_{i+1} \right] \}
\end{aligned}
$$

Differentiate w.r.t. to u_i of the above to get

$$
\begin{aligned}
2u_i^T R_i + E^{w_i} \left\{ 2(A_i x + B_i u_i(x) + w_i)^T S_{i+1} B_i \right\} &= 2u_i^T R_i + 2x^T A_i^T S_{i+1} B_i + 2u_i^T B_i^T S_{i+1} B_i \\
&\Downarrow \\
u_i^* &= -(R_i + B_i^T S_{i+1} B_i)^{-1} B_i^T S_{i+1} A_i x
\end{aligned}
$$

Substituting back, we obtain

$$
\begin{aligned}
S_i &= Q_i + A_i^T S_{i+1} B_i (R_i + B_i^T S_{i+1} B_i)^{-1} R_i (R_i + B_i^T S_{i+1} B_i)^{-1} B_i^T S_{i+1} A_i \\
&\quad + [A_i - B_i (R_i + B_i^T S_{i+1} B_i)^{-1} B_i^T S_{i+1} A_i]^T S_{i+1} [A_i - B_i (R_i + B_i^T S_{i+1} B_i)^{-1} B_i^T S_{i+1} A_i] \\
&= Q_i + A_i^T [S_{i+1} - S_{i+1} B_i (R_i + B_i^T S_{i+1} B_i)^{-1} B_i^T S_{i+1}] A_i \\
S_N &= Q_N
\end{aligned}
$$

Notice that the (full-information) feedback control law does not depend on the covariance of the noise.

Example 2 (Kalman filter) *Consider*

$$
\begin{aligned}
x_{k+1} &= A_k x_k + D_k w_k \\
y_k &= C_k x_k + H_k v_k.
\end{aligned}
$$

Define the conditional distribution

$$
x_i | y_0, \cdots, y_{i-1} := x_i | Y_{i-1}
$$

whose mean and covariance are denoted $\hat{x}_{i|i-1}$ and $\Sigma_{i|i-1}$ respectively. In the similar fashion, define $x_i|Y_i$ with mean $\hat{x}_{i|i}$ and covariance $\Sigma_{i|i}$. Notice that

$$
x_i | Y_i = (x_i | Y_{i-1}) \mid (y_i | Y_{i-1}) \tag{1.6}
$$

in which

$$
y_i | Y_{i-1} = C_i x_i | Y_{i-1} + H_k v_k
$$

Thus we can calculate $x_i|Y_i$ through the joint distribution

$$
\begin{aligned}
\begin{bmatrix} x_i | Y_{i-1} \\ y_i | Y_{i-1} \end{bmatrix} &= \begin{bmatrix} I \\ C_i \end{bmatrix} x_i | Y_{i-1} + \begin{bmatrix} 0 \\ H_i \end{bmatrix} v_i \\
&\sim N \left(\begin{bmatrix} I \\ C \end{bmatrix} \hat{x}_{i|i-1}, \begin{bmatrix} \Sigma_{i|i-1} & \Sigma_{i|i-1} C_i^T \\ C_i \Sigma_{i|i-1} & C_i \Sigma_{i|i-1} C_i^T + H_i V H_i^T \end{bmatrix} \right)
\end{aligned}
$$

to obtain

$$
\begin{aligned}
\hat{x}_{i|i} &= \hat{x}_{i|i-1} + \Sigma_{i|i-1} C_i^T (C_i \Sigma_{i|i-1} C_i^T + H_i V H_i^T)^{-1} (y_i - C_i \hat{x}_{i|i-1}), \\
\Sigma_{i|i} &= \Sigma_{i|i-1} - \Sigma_{i|i-1} C_i^T (C_i \Sigma_{i|i-1} C_i^T + H_i V H_i^T)^{-1} C_i \Sigma_{i|i-1} \\
&\to E \left\{ [(x_i | Y_i) - \hat{x}_{i|i}][(x_i | Y_i) - \hat{x}_{i|i}]^T \right\} \\
&\to E \left\{ [x_i - \hat{x}_{i|i}][x_i - \hat{x}_{i|i}]^T \mid Y_i \right\}
\end{aligned} \tag{1.7}
$$

Furthermore,

$$\begin{aligned}
x_{i+1}|Y_i &= A_i x_i + D_i w_i|Y_i = A_i x_i|Y_i + D_i w_i \\
\hat{x}_{i+1|i} &= A_i \hat{x}_{i|i} = A_i\left[\hat{x}_{i|i-1} + \Sigma_{i|i-1}C_i^T(C_i\Sigma_{i|i-1}C_i^T + H_iVH_i^T)^{-1}(y_i - C_i\hat{x}_{i|i-1})\right] \\
&= A_i\hat{x}_{i|i-1} + A_i\Sigma_{i|i-1}C_i^T(C_i\Sigma_{i|i-1}C_i^T + H_iVH_i^T)^{-1}(y_i - C_i\hat{x}_{i|i-1}) \\
\Sigma_{i+1|i} &= A_i\Sigma_{i|i}A_i^T + D_iWD_i^T \\
&= A_i\left[\Sigma_{i|i-1} - \Sigma_{i|i-1}C_i^T(C_i\Sigma_{i|i-1}C_i^T + H_iVH_i^T)^{-1}C_i\Sigma_{i|i-1}\right]A_i^T + D_iWD_i^T \\
&\to E\left\{[(x_{i+1}|Y_i - \hat{x}_{i+1|i}][(x_{i+1}|Y_i - \hat{x}_{i+1|i}]^T\right\} \\
&\to E\left\{[(x_{i+1} - \hat{x}_{i+1|i}][(x_{i+1} - \hat{x}_{i+1|i}]^T \mid Y_i\right\}
\end{aligned}$$

Example 3 (LQG (partial-state info)) *We consider partial state information LQG control:*

$$\begin{aligned}
x_{k+1} &= A_kx_k + B_ku_k + D_kw_k \\
y_k &= C_kx_k + H_kv_k
\end{aligned}$$

where $w_k \sim N(0, W)$, $v_k \sim N(0, V)$, $x_0 \sim N(0, X)$ *and they are independent. The cost is as before, namely,*

$$J = E\left[x_N^TQ_Nx_N + \sum_{k=0}^{N-1}(x_k^TQ_kx_k + u_k^TR_ku_k)\right].$$

But the Markov control process $u = (u_0, \cdots, u_{N-1})$ *should only use the information of* y*, i.e., if we denote*

$$\mathcal{Y}_k = \sigma(y_0, \cdots, y_k)$$

then $u_k \in \mathcal{Y}_k$*.*

Let $Y_k = [y_0, \cdots, y_k] \in \mathbb{R}^{d\times(k+1)}$*, and*

$$J_i(Y_i) = \min_{(u_i,\cdots,u_{N-1})} E\left[x_N^TQ_Nx_N + \sum_{k=i}^{N-1}(x_k^TQ_kx_k + u_k^TR_ku_k)\middle| Y_k\right], \quad 0 \le i \le N-1.$$

Then

$$\begin{aligned}
J_i(Y_i) &= \min_{u_i} E\left[x_i^TQ_ix_i + u_i^TR_iu_i + \min_{(u_{i+1},\cdots,u_{N-1})} x_N^TQ_Nx_N + \sum_{k=i}^{N-1}(x_k^TQ_kx_k + u_k^TR_ku_k)\middle| Y_i\right] \\
&= \min_{u_i} E\left[x_i^TQ_ix_i + u_i^TR_iu_i + J_{i+1}(Y_{i+1}) \mid Y_i\right]
\end{aligned}$$

We remark that $J_{i+1}(Y_{i+1})$ *within the bracket should be understood as a random viariable in* $\sigma(y_{i+1})$*.*

As in Kalman filter, let $\hat{x}_{i|i} = E[x_i|Y_i]$ *or in short* $\hat{x}_i$*. Then*

$$J_i(Y_i) = \hat{x}_i^TQ_i\hat{x}_i + tr(Q_i\Sigma_{i|i}) + \min_{u_i} u_i^TR_iu_i + E[J_{i+1}(Y_{i+1}) \mid Y_i]$$

since $E[x_i^TQ_ix_i] = \hat{x}_i^TQ_i\hat{x}_i + tr(Q_i\Sigma_{i|i})$*. Notice that* $J_N(Y_N) = E[x_N^TQ_Nx_N|Y_N] = \hat{x}_N^TQ_N\hat{x}_N + tr(Q_N\Sigma_{N|N})$*, we may assume*

$$J_i(Y_i) = \hat{x}_i^TP_i\hat{x}_i + q_i$$

Substitude this into the Bellman equation to get

$$\hat{x}_i^TP_i\hat{x}_i + q_i = \hat{x}_i^TQ_i\hat{x}_i + tr(Q_i\Sigma_{i|i}) + \min_{u_i}\left\{u_i^TR_iu_i + E[\hat{x}_{i+1}^TP_{i+1}\hat{x}_{i+1} + q_{i+1} \mid Y_i]\right\}$$

We should be careful about the notation $E[\hat{x}_{i+1}^T P_{i+1} \hat{x}_{i+1} + q_{i+1} \mid Y_i]$, *as* $\hat{x}_{i+1}$ *is a r.v. instead of a deterministic number! Thus*

$$\begin{aligned} f(Y_i) &= E[\hat{x}_{i+1}^T P_{i+1} \hat{x}_{i+1} \mid Y_i] \\ &= \int \hat{x}_{i+1}^T (Y_i, y_{i+1}) P_{i+1} \hat{x}_{i+1}(Y_i, y_{i+1}) \mu(dy_{i+1}) \end{aligned}$$

Recall that

$$\hat{x}_{i+1|i+1} = A_i \hat{x}_{i|i} + B_i u_i + L_{i+1} e_{i+1}$$

which in more precisely language reads $\hat{x}_{i+1|i+1}(Y_i, y_i) = A_i \hat{x}_{i|i}(Y_i) + B_i u_i(Y_i) + L_{i+1} e_{i+1}(Y_i, y_{i+1})$, *and*

$$\begin{aligned} L_{i+1} &= \Sigma_{i+1|i} C_{i+1}^T (C_{i+1} \Sigma_{i+1|i} C_{i+1}^T + H_{i+1} V H_{i+1}^T)^{-1} \\ e_{i+1} &= y_{i+1} - C_{i+1}(A_i \hat{x}_i + B_i u_i) \\ &= C_{i+1} x_{i+1} + H_{i+1} v_{i+1} - C_{i+1}(A_i \hat{x}_i + B_i u_i) \\ &= C_{i+1}(A_i x_i + B_i u_i + D_i w_i) + H_{i+1} v_{i+1} - C_{i+1}(A_i \hat{x}_i + B_i u_i) \\ &= C_{i+1}[A_i (x_i - \hat{x}_i) + D_i w_i] + H_{i+1} v_{i+1} \end{aligned}$$

Now since e_{i+1} *and* Y_i *are independent, we deduce that* $E[e_{i+1}|Y_i] = E[e_{i+1}]$ *(*$\hat{x}_i$ *should be seen as a r.v. of* Y_i*). Recall that*

$$E[(x_i - \hat{x}_i)(x_i - \hat{x}_i)^T | Y_i] = \Sigma_{i|i}$$

it follows that

$$e_{i+1} \sim N(0, C_{i+1} \Sigma_{i+1|i} C_{i+1}^T + H_{i+1} V H_{i+1}^T)$$

Thus

$$\begin{aligned} &E[\hat{x}_{i+1}^T P_{i+1} \hat{x}_{i+1} \mid Y_i] \\ &= E[(A_i \hat{x}_{i|i} + B_i u_i + L_{i+1} e_{i+1})^T P_{i+1} (A_i \hat{x}_{i|i} + B_i u_i + L_{i+1} e_{i+1}) \mid Y_i] \\ &= \hat{x}_{i|i}^T A_i^T P_{i+1} A_i \hat{x}_{i|i} + u_i^T (2 B_i^T P_{i+1} A_i \hat{x}_{i|i} + B_i^T P_{i+1} B_i u_i) + \\ &+ tr(L_{i+1}^T P_{i+1} L_{i+1} Var(e_{i+1})) \end{aligned}$$

$$\begin{aligned} &tr(L_{i+1}^T P_{i+1} L_{i+1} Var(e_{i+1})) \\ &= tr\left(L_{i+1}^T P_{i+1} \Sigma_{i+1|i} C_{i+1}^T\right) \\ &= tr(P_{i+1} \Sigma_{i+1|i} C_{i+1}^T (C_{i+1} \Sigma_{i+1|i} C_{i+1}^T + H_{i+1} V H_{i+1}^T)^{-1} C_{i+1} \Sigma_{i+1|i}) \\ &= tr(P_{i+1}(\Sigma_{i+1|i} - \Sigma_{i+1|i+1})) \end{aligned}$$

Combining these terms, we get

$$\begin{aligned} J_i(Y_i) &= \hat{x}_i^T P_i \hat{x}_i + q_i \\ &= \hat{x}_i^T (Q_i + A_i^T P_{i+1} A_i) \hat{x}_i + tr(Q_i \Sigma_{i|i}) + q_{t+1} \\ &+ tr(P_{i+1}(\Sigma_{i+1|i} - \Sigma_{i+1|i+1})) \\ &+ \min_{u_i} \left\{ u_i^T R_i u_i + 2 u_i^T B_i^T P_{i+1} A_i \hat{x}_{i|i} + u_i^T (R_i + B_i^T P_{i+1} B_i u_i) \right\} \end{aligned}$$

thus the optimal policy is

$$\begin{aligned} u_i^* &= -(R_i + B_i^T P_{i+1} B_i)^{-1} B_i^T P_{i+1} A_i \hat{x}_i \\ &= -(R_i + B_i^T P_{i+1} B_i)^{-1} B_i^T P_{i+1} A_i E[x_i | Y_i] \end{aligned}$$

and P_t satisfies

$$P_i = Q_i + A_i^T P_{i+1} A_i - A_i^T P_{i+1} B_i (R_i + B_i^T P_{i+1} B_i)^{-1} B_i^T P_{i+1} A_i$$
$$q_i = q_{i+1} + tr[Q_i \Sigma_{i|i} + P_{i+1}(\Sigma_{i+1|i} - \Sigma_{i+1|i+1})]$$

And the optimal cost is

$$J = EJ_0(Y_0) = q_0 + E[\hat{x}_0^T P_0 \hat{x}_0]$$

1.1.5 Infinite Horizen Dynamic Programming

So far, we have only considered "finite horizen" problem, let's see what happens if we work with infinite horizen minimization problems. Consider

$$J_1(x_1) = \sum_{k=1}^{\infty} L_k(x_k, u_k)$$
$$J_1^*(x_1) = \min_{(u_1,\cdots)} J_1(x_1)$$

and let

$$J_i(x) = \left. \sum_{k=i}^{\infty} L_k(x_k, u_k) \right|_{x_i = x}$$
$$J_i^*(x) = \min_{(u_i,\cdots)} J_i(x), \quad i \geq 1$$

then it follows from Bellman's principle that

$$J_i^*(x) = \min_{u_i} [L_i(x, u_i) + J_{i+1}^*(f_i(x, u_i))]. \tag{1.8}$$

Unfortunately, there is no terminal condition for (1.8) so that the recursive algorithm is not implementable, except an important special case: time-invariant system with stationary cost function. More precisely, we consider the following system

$$x_{k+1} = f(x_k, u_k) \tag{1.9}$$

with cost function

$$J_1(x_1) = \sum_{k=1}^{\infty} L(x_k, u_k)$$

Keeping the notations as before, we have

$$\begin{aligned} J_i^*(x) &= \min_{(u_i,\cdots)} \left. \sum_{k=i}^{\infty} L(x_k, u_k) \right|_{x_i = x} \\ &= \min_{(u_1,\cdots)} \left. \sum_{k=1}^{\infty} L(x_k, u_k) \right|_{x_1 = x} \\ &= J_1^*(x), \ \forall i \geq 1. \end{aligned}$$

Write $J_i^*(x) := J^*(x)$, the Bellman equation (1.8) becomes

$$J^*(x) = \min_{u} [L(x, u) + J^*(f(x, u))]. \tag{1.10}$$

Let $BC(\mathbb{R})$ denote the set of bounded real valued continuous functions on $\mathbb{R}$ equipped with the norm $||f|| := \sup_{x\in\mathbb{R}} |f(x)|$. It is well-known that $BC(\mathbb{R})$ is a Banach space under this norm. Define the operator $T : BC(\mathbb{R}) \to BC(\mathbb{R})$ according to (1.10) as

$$TJ(x) = \min_u [L(x,u) + J(f(x,u))].$$

We show that T is a nonexpansive mapping. In fact, for any $\tilde{J} \in BC(\mathbb{R})$, there holds

$$\begin{aligned} TJ(x) &= \min_u [L(x,u) + \tilde{J}(f(x,u)) + (J(f(x,u)) - \tilde{J}(f(x,u)))] \\ &\le \min_u [L(x,u) + \tilde{J}(f(x,u))] + ||J - \tilde{J}|| \\ &= T\tilde{J}(x) + ||J - \tilde{J}||, \end{aligned}$$

or

$$||TJ - T\tilde{J}|| \le ||J - \tilde{J}||.$$

Since the mapping T is only non-expansive, it is not clear wether a fixed point exists or not without further assumption on the system and cost function.

Example 4 (Infinite Horizen LQR) *Consider the linear time-invariant discrete time system*

$$x_{k+1} = Ax_k + Bu_k$$

with quadratic cost

$$J_1(x_1) = \sum_{k=1}^{\infty} x_k^T Q x_k + u_k^T R u_k$$

where $R > 0$. Choose an initial function as $J_0(x) = x^T P x$ with $P > 0$. Then

$$\begin{aligned} J_1(x) = TJ_0(x) &= \min_u [x^T Q x + u^T R u + (Ax + Bu)^T P (Ax + Bu)] \\ &= x^T (Q + A^T P A) x + \min_u [u^T (R + B^T P B) u + 2u^T B^T P A x] \\ &= x^T \Pi_1 x + \min_u (u + K_1 x)^T (R + B^T P B)(u + K_1 x) \\ &= x^T \Pi_1 x \end{aligned}$$

where

$$\begin{aligned} \Pi_1 &= Q + A^T P A - A^T P B (R + B^T P B)^{-1} B^T P A \\ K_1 &= (R + B^T P B)^{-1} B^T P A \end{aligned}$$

and the optimal control is

$$u_1 = \arg\min_u TJ_0(x) = -K_1 x_1.$$

Similarly,

$$\begin{aligned} J_k(x) = TJ_{k-1}(x) &= x^T \Pi_k x + \min_u (u + K_k x)^T (R + B^T \Pi_{k-1} B)(u + K_k x) \\ &= x^T \Pi_k x \end{aligned}$$

where

$$\begin{aligned} \Pi_k &= Q + A^T \Pi_{k-1} A - A^T \Pi_{k-1} B (R + B^T \Pi_{k-1} B)^{-1} B^T \Pi_{k-1} A \\ K_k &= (R + B^T \Pi_{k-1} B)^{-1} B^T \Pi_{k-1} A \end{aligned} \tag{1.11}$$

and

$$u_k = -K_k x_k, \quad \forall k \geq 1.$$

The equation (1.11) is an iterated Riccati equation, which admits a fixed point when (A, B) *is controllable. And*

$$J^*(x) = x^T(\lim_{k\to\infty} \Pi_k)x$$

From this example we see from above that for LQR for discrete time system, the key additional assumption is controllability, which makes the non-expansive mapping defined by the Bellman equation a contractive one.

One may also consider a modified cost function with a *discount factor* $\alpha \in (0, 1)$ so that the Bellman equation always admits a solution when the minimization is feasible:

$$J(x) = \sum_{k=0}^{\infty} \alpha^k L(x_k, u_k)\Big|_{x_0=x}.$$

In this case,

$$\begin{aligned}
J^*_{i+1}(x) &= \min_{(u_{i+1},\cdots)} \sum_{k=i+1}^{\infty} \alpha^k L(x_k, u_k)\Big|_{x_{i+1}=x} \\
&= \min_{(u_{i+1},\cdots)} \sum_{k=i}^{\infty} \alpha^{k+1} L(x_{k+1}, u_{k+1})\Big|_{x_{i+1}=x} \\
&= \alpha \min_{(u_{i+1},\cdots)} \sum_{k=i}^{\infty} \alpha^{k} L(x_{k+1}, u_{k+1})\Big|_{x_{i+1}=x} \\
&= \alpha J^*_i(x).
\end{aligned}$$

Hence the Bellman's equation (1.8) becomes

$$J^*(x) = \min_u [L(x, u) + \alpha J^*(f(x, u))]$$

It is easily verified as above that the mapping $\bar{T}J(x) = \min_u[L(x, u) + \alpha J(f(x, u))]$ is contractive, i.e.,

$$||\bar{T}J_1 - \bar{T}J_2|| \leq \alpha||J_1 - J_2||, \quad \forall J_1, J_2 \in BC(\mathbb{R})$$

Hence there exists a unique $J^* \in BC(\mathbb{R})$ such that

$$\bar{T}J^* = J^*$$

Thus for any initial function $J_0 \in BC(\mathbb{R})$, lettin $J_n = T^n J_0$, we have

$$||J_n - J^*|| \leq \alpha^n ||J_0 - J^*||$$

Thus J_n converges to the optimal cost exponentially as $n \to \infty$.

1.2 Continuous time systems

1.2.1 Bellman principle of optimality

The following theorem can be viewed as the foundation of optimal control of continuous time systems using dynamic programming.

Theorem 1.1 *Let $J(u(\cdot))$ be a cost function, with $u(\cdot) \in U_{\text{admis}}[t_0, t_1]$. Assume that*

1. *$J(u(\cdot))$ is separable for any time $t \in [t_0, t_1]$ in the sense that there exist functions $J_1 : U \times \mathbb{R} \to \mathbb{R}$, $J_2 : U \to \mathbb{R}$ such that*
$$J(u(\cdot)) = J_1(u_1(\cdot), J_2(u_2(\cdot)))$$
where $u_1 = u1_{[t_0,t)}$ and $u_2 = u1_{[t,t_1]}$ for all $t \in [t_0, t_1]$, i.e., the truncations of u on the interval $[t_0, t)$ and $[t, t_1]$ respectively;

2. *J_1 is nondecreasing with respect to the second argument.*

Then Bellman's principle of optimality holds for the cost function $J(u(\cdot))$:

$$J^* = \min_{u(\cdot)\in U_{\text{admis}}[t_0,t_1]} J(u(\cdot)) = \min_{u_1(\cdot)\in U_{\text{admis}}[t_0,t)} J_1\left(u_1(\cdot), \min_{u_2(\cdot)\in U_{\text{admis}}[t,t_1]} J_2(u_2(\cdot))\right), \quad \forall t \in [t_0, t_1]$$

Proof 2 *For any $u_1(\cdot), u_2(\cdot)$, we have*

$$J^* \le J_1(u_1(\cdot), J_2(u_2(\cdot)))$$

hence

$$J^* \le J_1\left(u_1(\cdot), \min_{u_2(\cdot)\in U_{\text{admis}}[t,t_1]} J_2(u_2(\cdot))\right)$$

and

$$J^* \le \min_{u_1(\cdot)\in U_{\text{admis}}[t_0,t)} J_1\left(u_1(\cdot), \min_{u_2(\cdot)\in U_{\text{admis}}[t,t_1]} J_2(u_2(\cdot))\right).$$

On the other hand,

$$\begin{aligned}
&\min_{u_1(\cdot)\in U_{\text{admis}}[t_0,t)} J_1\left(u_1(\cdot), \min_{u_2(\cdot)\in U_{\text{admis}}[t,t_1]} J_2(u_2(\cdot))\right) \\
\le &\min_{u_1(\cdot)\in U_{\text{admis}}[t_0,t)} J_1\left(u_1(\cdot), J_2(u_2(\cdot))\right) \quad \textit{(monotonicity)} \\
\le &\min_{u_2(\cdot)\in U_{\text{admis}}[t,t_1]} \min_{u_1(\cdot)\in U_{\text{admis}}[t_0,t)} J_1\left(u_1(\cdot), J_2(u_2(\cdot))\right) \\
= &\min_{u_2(\cdot)\in U_{\text{admis}}[t,t_1]} \min_{u_1(\cdot)\in U_{\text{admis}}[t_0,t)} J(u(\cdot)) \\
= &\min_{u(\cdot)\in U_{\text{admis}}[t_0,t_1]} J(u(\cdot)) = J^*.
\end{aligned}$$

Consider the cost function in Bolza form

$$J(u(\cdot)) = \varphi(x(T)) + \int_0^T L(x(s), u(s))ds.$$

Let

$$J_1(u_1, y) = y + \int_0^t L(x(s), u(s))ds$$
$$J_2(u_2) = \varphi(x(T)) + \int_t^T L(x(s), u(s))ds$$

Then obviously $J(u(\cdot)) = J_1(u_1, J_2(u_2))$ and J_1 is nondecreasing with respect to the second argument. Then we immediately obtain the following result.

Let $x(t)$ be the solution to this Cauchy problem

$$\dot{x} = f(x, u, t),$$
$$x(s) = y$$

and

$$J(s,y;u(\cdot)) = \varphi(x(T)) + \int_s^T L(x(t),u(t),t)dt.$$

Define the so called value function

$$\begin{aligned} V(s,y) &:= \min_{u(\cdot)\in U_{\text{admis}}[s,T]} J(s,y;u(\cdot)), \\ V(T,y) &= \varphi(y). \end{aligned} \tag{1.12}$$

Then taking J_1 and J_2 as above and invoking Theorem 1.1, we get

$$V(s,y) = \min_{u(\cdot)\in U_{\text{admis}}[s,\hat{s}]} \left\{ \int_s^{\hat{s}} L(x(t,s,y,u(\cdot)),u(t),t)dt + V(\hat{s},x(\hat{s},s,y,u(\cdot)) \right\}, \quad \forall \hat{s}\in[s,T]. \tag{1.13}$$

Now for any give $u \in U$, and $\hat{s} > s$, we have

$$\frac{V(s,y)-V(\hat{s},x(\hat{s},s,y,u(\cdot))}{\hat{s}-s} - \frac{1}{\hat{s}-s}\int_s^{\hat{s}} L(x(t,s,y,u(\cdot)),u(t),t)dt \le 0$$

which implies

$$-\frac{\partial V}{\partial s}(s,y) - \frac{\partial V}{\partial x}(s,y)f(y,u,s) - L(y,u,s) \le 0$$

resulting in

$$-\frac{\partial V}{\partial s}(s,y) + \sup_{u\in U} H\left(-\frac{\partial V(s,y)}{\partial x}, y, u, s\right) \le 0. \tag{1.14}$$

where

$$H(p,x,u,t) = pf(x,u,t) - L(x,u,t) \tag{1.15}$$

On the other hand, for any $\varepsilon > 0$, there exists a control u_ε such that

$$V(s,y) + \varepsilon(\hat{s}-s) \ge \int_s^{\hat{s}} L(x(t,s,y,u_\varepsilon(\cdot)),u_\varepsilon(t),t)dt + V(\hat{s},x(\hat{s},s,y,u_\varepsilon(\cdot))$$

or

$$\begin{aligned} -\varepsilon &\le \frac{V(s,y)-V(\hat{s},x(\hat{s},s,y,u_\varepsilon(\cdot))}{\hat{s}-s} - \frac{1}{\hat{s}-s}\int_s^{\hat{s}} L(x(t,s,y,u_\varepsilon(\cdot)),u_\varepsilon(t),t)dt \\ &= \frac{1}{\hat{s}-s}\int_s^{\hat{s}} -\frac{\partial V}{\partial s}(t,x(t,s,y,u_\varepsilon(\cdot)) - \frac{\partial V}{\partial x}(t,x(t,s,y,u_\varepsilon(\cdot))f(x(t,s,y,u_\varepsilon(\cdot)),u_\varepsilon(t),t) \\ &- L(x(t,s,y,u_\varepsilon(\cdot)),u_\varepsilon(t),t)\mathrm{d}t \\ &= \frac{1}{\hat{s}-s}\int_s^{\hat{s}} -\frac{\partial V}{\partial s}(t,x(t,s,y,u_\varepsilon(\cdot)) + H\left(-\frac{\partial V}{\partial x}(t,x(t,s,y,u_\varepsilon(\cdot)),x(t,s,y,u_\varepsilon(\cdot),u_\varepsilon(t),t\right)\mathrm{d}t \end{aligned}$$

Let $\hat{s} \to s$, we get

$$\begin{aligned} -\varepsilon &\le -\frac{\partial V}{\partial s}(s,y) + H\left(-\frac{\partial V}{\partial x}(s,y),y,u_\varepsilon(s),s\right) \\ &\le -\frac{\partial V}{\partial s}(s,y) + \sup_{u\in U} H\left(-\frac{\partial V}{\partial x}(s,y),y,u,s\right) \end{aligned} \tag{1.16}$$

Since ε is arbitrary, (1.16) and (1.14) imply

$$-\frac{\partial V}{\partial s}(s,y) + \sup_{u\in U} H\left(-\frac{\partial V}{\partial x}(s,y),y,u,s\right) = 0, \quad \forall s\in[0,T], \ \forall y$$

This is a PDE with boundary condition

$$V(T, y) = \varphi(y).$$

Thus the optimal control problem reduces to solving a PDE.

Theorem 1.2 *Suppose that $V(s, y)$ defined as (1.12) is continuously differentiable. Then $V(t, y)$ is a solution to the following Hamilton-Jacobi-Bellman (HJB) PDE:*

$$\begin{aligned} -V_t(t, y) + \sup_{u \in U} H\left(-\nabla V(t, y), y, u, t\right) = 0, \\ V(T, y) = \varphi(y) \end{aligned} \tag{1.17}$$

where H is defined in (1.17) and we have adopted the notations $\frac{\partial V}{\partial t} = V_t$ and $\frac{\partial V}{\partial x} = \nabla V$.

To obtain the optimal control law based on the solution of the HJB equation (1.17). We can follow the following steps:

1. Let
$$u_*(t) = u_*(t, x, \nabla V(t, x)) = \arg \sup_{u \in U} H(-\nabla V(t, x), x, u, t)$$
with fixed x, t and $\nabla V(t, x)$.

2. Find a continuously differentiable solution $V(t, x)$ to
$$\begin{aligned} -V_t(t, y) + &\sup_{u \in U} H\left(-\nabla V(t, y), y, u_*(\cdot), t\right) = 0, \\ V(T, y) = &\varphi(y), \quad (t, y) \in [0, T) \times \mathbb{R}^n \end{aligned}$$

3. Solve for the solution $x_*(t) =: x_*(t; s, x)$ to the Cauchy problem of the following ODE:
$$\begin{aligned} \dot{x}_* &= f(x_*(t), u_*(t, x_*(t), \nabla V(t, x_*(t))), t) \\ x_*(s) &= x \end{aligned}$$

   Then
$$u_*(t, x_*(t; 0, x), \nabla V(t, x_*(t; 0, x)))$$
is an optimal control and $x_*(t; 0, x)$ is the corresponding optimal process.

1.2.2 Solutions of HJB: method of characteristics

The HJB equation can be studied under a more general form of PDE:

$$F(x, u, \nabla u) = 0, \quad x \in \Omega \subset \mathbb{R}^n. \tag{1.18}$$

with boundary condition

$$u(x) = \tilde{u}(x), \quad x \in \partial\Omega,$$

and u is a real-valued function. The function F is assumed to be a continuous mapping from $\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$ to $\mathbb{R}$.

A classical method tackling this kind of PDE is the so called *method of characteristics*, the idea of which is as follows. Given a point $y \in \partial\Omega$ and a curve $x : [0, 1] \to \bar{\Omega}$, with $x(0) = y$. Let us examine the values of $u(x)$ along this curve, see Figure 1.2.2.

Introduce the notation

$$(p_1, \cdots, p_n) = (u_{x_1}, \cdots, u_{x_n}).$$

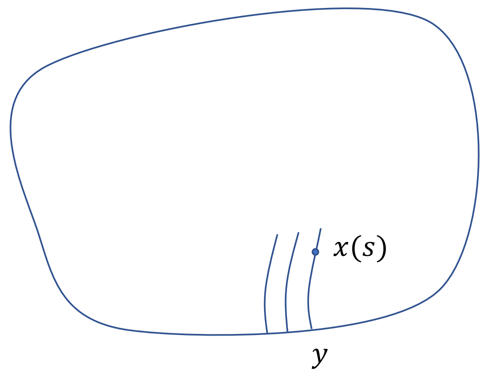

Figure 1.1: Method of characteristics.

For convenience, denote

$$u(s) =: u(x(s))$$
$$p(s) =: p(x(s)) = \nabla u(x(s)).$$

Then

$$\dot{u} = \sum_{i=1}^{n} u_{x_i}\dot{x}_i = \sum_{i=1}^{n} p_i \dot{x}_i$$
$$\dot{p}_i = \sum_{j=1}^{n} u_{x_i x_j}\dot{x}_i$$

Differentiating (1.18) w.r.t. x_i we get

$$\frac{\partial F}{\partial x_i} + \frac{\partial F}{\partial u}u_{x_i} + \sum_{i=1}^{n}\frac{\partial F}{\partial p_i}u_{x_i x_j} = 0.$$

If the curve $x(s)$ is chosen such that $\dot{x}_i = \partial F/\partial p_i$, then one can easily obtain the following

$$\dot{u} = \sum_{i=1}^{n} p_i \frac{\partial F}{\partial p_i},$$
$$\dot{p}_i = -\frac{\partial F}{\partial x_i} - \frac{\partial F}{\partial u}p_i, \quad i = 1, \cdots, n$$

or in more compact form

$$\dot{x} = \frac{\partial F}{\partial p}$$
$$\dot{u} = p\frac{\partial F}{\partial p}$$
$$\dot{p} = -\frac{\partial F}{\partial x} - \frac{\partial F}{\partial u}p$$

The above equation is a system of ordinary differential equations with boundary condition

$$x(0) = y$$
$$u(0) = \bar{u}(y)$$
$$p(0) = \nabla u(y)$$

for $y \in \partial\Omega$. Thus by varying the initial condition y, we can obtain *local solutions* near $\partial\Omega$ of the PDE (1.18).

1.2.3 Viscosity solution of HJB equation

Our next aim is to show that the value function defined in (1.12) is a *viscosity solution* of the HJB equation (1.17).

Define the set of super-differentials of a function $g : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ at x as

$$D^+g(x) =: \left\{ p \in (\mathbb{R}^n)^* : \limsup_{y \to x} \frac{g(y) - g(x) - p(y-x)}{|y-x|} \leq 0 \right\}$$

and the set of sub-differentials at x as

$$D^-g(x) =: \left\{ p \in (\mathbb{R}^n)^* : \liminf_{y \to x} \frac{g(y) - g(x) - p(y-x)}{|y-x|} \geq 0 \right\}$$

As shown in the following figures.

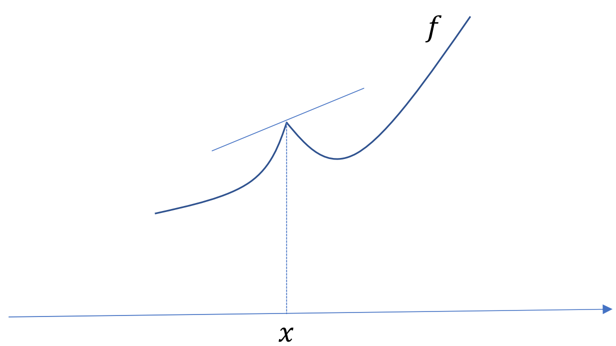

Figure 1.2: Super-differential.

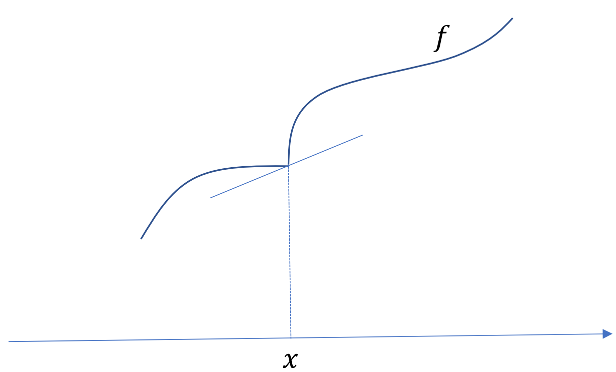

Figure 1.3: Sub-differential.

The crucial characterization of super- and sub-differentials for us is the following.

Lemma 1.2 *Let $g \in C(\Omega)$. Then*

1. *$p \in D^+g(x)$ iff there exists a function $\varphi \in C^1(\Omega)$ such that $\nabla\varphi(x) = p$ and $g - \varphi$ has a local maximum at x;*

2. *$p \in D^-g(x)$ iff there exists a function $\varphi \in C^1(\Omega)$ such that $\nabla\varphi(x) = p$ and $g - \varphi$ has a local minimum at x.*

The proof of this lemma is easy and hence omitted. Notice that the mere regularity assumption on g is only continuity! Now we are ready to give the definition of the celebrated *viscosity solution*. One should keep in mind that there is no differentiability assumption on the solution.

Definition 1.1 *Consider the first order PDE (1.18), in which F is continuous. A function $g \in C(\Omega)$ is a viscosity sub-solution of the PDE if*

$$F(x, g(x), p) \leq 0, \quad \forall x \in \Omega, \quad p \in D^+g(x).$$

It is a viscosity super-solution if

$$F(x, g(x), p) \geq 0, \quad \forall x \in \Omega, \quad p \in D^-g(x).$$

It is a viscosity solution if it is both a viscosity supersolution and a viscosity subsolution.

Due to Lemma 1.2, g is a viscosity sub-solution if, for each $\varphi \in C^1(\Omega)$ such that $u - \varphi$ has a local maximum at x, there holds

$$F(x, g(x), \nabla\varphi(x)) \leq 0$$

and it is a viscosity super-solution if, for each $\varphi \in C^1(\Omega)$ such that $u - \varphi$ has a local minimum at x, there holds

$$F(x, g(x), \nabla\varphi(x)) \geq 0.$$

Theorem 1.3 *Consider the system $\dot{x} = f(x, u)$ with $x \in \mathbb{R}^n$ and $u \in U \subset \mathbb{R}^m$, and cost function $J(u(\cdot)) = \varphi(x(T)) + \int_0^T L(x(s), u(s))ds$. Let $V(s, y)$ be the value function defined as (1.12). Suppose that f and L are continuous. Then V is a viscosity solution of the HJB equation*

$$-V_t + H(x, -\nabla V) = 0, \quad (t, x) \in (0, T) \times \mathbb{R}^n$$

with terminal condition

$$V(T, x) = g(x), \quad x \in \mathbb{R}^n$$

and Hamiltonian

$$H(x, p) = \sup_{u \in U}\{pf(x, u) - L(x, u)\}.$$

Proof 3 *(We follow [1].) Let $\varphi \in C^1((0, T) \times \mathbb{R}^n)$. We need to show*

1) If $V - \varphi$ attains a local maximum at $(t_0, x_0) \in (0, T) \times \mathbb{R}^n$, then

$$-\varphi_t(t_0, x_0) + \sup_{u \in U}\{-\nabla\varphi(t_0, x_0)f(x_0, u) - L(x_0, u)\} \leq 0$$

or

$$\varphi_t(t_0, x_0) + \inf_{u \in U}\{\nabla\varphi(t_0, x_0)f(x_0, u) + L(x_0, u)\} \geq 0 \tag{1.19}$$

2) If $V - \varphi$ attains a local minimum at $(t_0, x_0) \in (0, T) \times \mathbb{R}^n$, then

$$-\varphi_t(t_0, x_0) + \sup_{u \in U}\{-\nabla\varphi(t_0, x_0)f(x_0, u) - L(x_0, u)\} \geq 0$$

or

$$\varphi_t(t_0, x_0) + \inf_{u \in U}\{\nabla\varphi(t_0, x_0)f(x_0, u) + L(x_0, u)\} \leq 0. \tag{1.20}$$

To prove 1), assume that $V(t_0, x_0) = \varphi(t_0, x_0)$ and $V(t, x) \le \varphi(t, x)$ for all t, x. If (1.19) is not true, then there exist $\omega \in U$, $\theta > 0$ such that

$$\varphi_t(t_0, x_0) + \inf_{u \in U} \{\nabla\varphi(t_0, x_0) f(x_0, u) + L(x_0, u)\} < -\theta.$$

By continuity, this inequality implies

$$\varphi_t(t, x) + \{\nabla\varphi(t, x) f(x, \omega) + L(x, \omega)\} < -\theta - L(x, \omega) \tag{1.21}$$

when

$$|t - t_0| < \delta, \quad |x - x_0| < \delta,$$

for some $\delta > 0$. Call $x(t) := x(t; t_0, x_0, \omega)$ the solution to

$$\dot{x} = f(x(t), \omega), \quad x(t_0) = x_0.$$

We then have

$$\begin{aligned} V(t_0 + \delta, x(t_0 + \delta)) - V(t_0, x_0) &\le \varphi(t_0 + \delta, x(t_0 + \delta)) - \varphi(t_0, x_0) \\ &= \int_{t_0}^{t_0+\delta} \frac{d}{dt} \varphi(t, x(t)) dt \\ &= \int_{t_0}^{t_0+\delta} \{\varphi_t(t, x(t)) + \nabla\varphi(t, x(t)) f(x(t), \omega)\}\, dt \\ &\le - \int_{t_0}^{t_0+\delta} L(x(t), \omega) dt - \delta\theta. \quad \textit{(due to (1.21))}. \end{aligned}$$

On the other hand, by the definition of value function,

$$V(t_0 + \delta, x(t_0 + \delta)) - V(t_0, x_0) \ge \int_{t_0}^{t_0+\delta} L(x(t), \omega) dt$$

which induces a contradiction. Thus $V(t, y)$ is indeed a viscosity sub-solution. Part 2) can be proved similarly.

1.3 Game Theory and Minimax Control

1.3.1 Interchangeability of "inf" and "sup"

Consider the function $J : X \times X \to \mathbb{R}$ and two quatity

$$\bar{J} = \inf_x \sup_y J(x, y)$$
$$\underline{J} = \sup_y \inf_x J(x, y)$$

In general these two quatities are not equal, but we have the obvious inequality $\bar{J} \ge \underline{J}$ since $\bar{J} \ge \inf_x J(x, y)$ for any y.

We say that (x_*, y_*) is a *saddle point* of J if

$$J(x_*, y) \le J(x_*, y_*) \le J(x, y_*), \quad \forall x, y.$$

The existence of a saddle point implies that $\bar{J} = \underline{J}$ since

$$\underline{J} \ge \inf_x J(x, y_*) \ge \sup_y J(x_*, y) \ge \inf_x \sup_y J(x, y) = \bar{J}.$$

But the converse is in general not true as shown in the following simple example.

Example 5 *Consider the function* $J : \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$, $J(x,y) = e^{-y^2}\max\{0,x\}$. *Then* $\bar{J} = \underline{J} = 0$ *since*

$$\inf_x \sup_y J(x,y) = \sup_y \inf_x J(x,y) = 0$$

but it is obvious that there exists no saddle point.

However, we have the following lemma, whose proof is left as an excercise (use Brouwer's fixed point theorem).

Lemma 1.3 *Suppose that* $X \subseteq \mathbb{R}^m$ *and* $Y \subseteq \mathbb{R}^n$ *are two compact, convex sets. If* $x \mapsto J(x,y_0)$ *and* $y \mapsto -J(x_0,y)$ *are convex functions for any* $(x_0,y_0) \in X \times Y$, *then* $\bar{J} = \underline{J}$ *and* J *has a saddle point.*

This lemma will be crucial for us to develop dynamic programming algorithms.

1.3.2 Linear Quadratic Dynamic Game

In this subsection, we study the minimax control of (time-dependent) linear discrete time system of the form

$$x_{k+1} = A_k x_k + B_k u_k + D_k w_k \tag{1.22}$$

with cost function

$$J_\gamma(x_1) = x_{N+1}^T Q_{N+1} x_{N+1} + \sum_{k=1}^{N}(x_k^T Q_k x_k + u_k^T R_k u_k - \gamma^2 w_k^T w_k). \tag{1.23}$$

The minimization problem is to find (u_*, w_*) such that

$$J_\gamma^*(x_1) = \min_u \max_w J_\gamma(x_1)$$

In order to apply Lemma 1.3, we need to check the strict convexity of the mapping $w \mapsto -J_\gamma(x_1)$ or the strict "concavity" of the mapping $w \mapsto J_\gamma(x_1)$. It is well-known that the function $\varphi(x) = x^T P x + a^T x + b$ is strictly convex if and only if $P + P^T > 0$, therefore, all we need to check is the positive definiteness of the matrix $-\frac{\partial^2 J}{\partial w^2}$.

When $N = 1$,

$$J_\gamma(x_1) = -\gamma^2 w_1^T w_1 + w_1^T D_1^T Q_2 D_1 w_1 + *$$

where $*$ stands for the first-order term of w. Hence $-\frac{\partial^2 J}{\partial w^2} = \gamma^2 I - D_1^T Q_2 D_1$ and $w \mapsto J_\gamma(x_1)$ is strictly concave if and only if $\gamma^2 I - D_1^T Q_2 D_1 > 0$.

When $N = 2$,

$$J_\gamma(x_1) = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}^T M_2 \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} + *$$

where

$$M_2 = \begin{bmatrix} \gamma^2 I - D_1^T(Q_2 + A_2^T Q_3 A_2)D_1 & -D_1^T A_2^T Q_3 D_2 \\ -D_2^T Q_3 A_2 D_1 & \gamma^2 I - D_2^T Q_3 D_2 \end{bmatrix}.$$

Using Schur's complement, it follows that $M_2 > 0$ if and only if

$$\gamma^2 I - D_2^T Q_3 D_2 > 0$$
$$\gamma^2 I - D_1^T[Q_2 + A_2^T Q_3 A_2 + A_2^T Q_3 D_2(\gamma^2 I - D_2^T Q_3 D_2)^{-1} D_2^T Q_3 A_2] D_1 > 0$$

which can also be written as

$$\gamma^2 I - D_k^T S_{k+1} D_k > 0, \quad k = 1, 2$$

with

$$
\begin{aligned}
S_2 &= Q_2 + A_2^T S_3 A_2 + A_2^T S_3 D_2 (\gamma^2 I - D_2^T S_3 D_2)^{-1} D_2^T S_3 A_2 \\
S_3 &= Q_3
\end{aligned}
$$

It turns out that more general is true:

Lemma 1.4 *The cost function (1.23) is strictly concave in w if and only if*

$$\gamma^2 I - D_k^T S_{k+1} D_k > 0, \quad k \in \{1, \cdots, N\}$$

where S_{k+1}, $k \in \{1, \cdots, N\}$ is generated by the Riccati equation

$$
\begin{aligned}
S_k &= Q_k + A_k^T S_{k+1} A_k + A_k^T S_{k+1} D_k (\gamma^2 I - D_k^T S_{k+1} D_k)^{-1} D_k^T S_{k+1} A_k \\
S_{N+1} &= Q_{N+1}
\end{aligned}
$$

Assume the conditions in Lemma 1.4 hold, then $w_i \mapsto x_{N+1}^T Q_{N+1} x_{N+1} + \sum_{k=i}^N (x_k^T Q_k x_k + u_k^T R_k u_k - \gamma^2 w_k^T w_k)$ is also strictly concave for $i = \{1, \cdots, N\}$. Therefore, the minimization problem can be solved using dynamic programming as follows (the strict convexity of $u \mapsto J_\gamma(u, w)$ is obvious and):

$$\min_u \max_w J_\gamma(x) = \min_{u_1} \max_{w_1} \left[-\gamma^2 w_1^T w_1 + u_1^T R_1 u_1 + \min_{(u_2, \cdots, u_N)} \max_{(w_2, \cdots, w_N)} J_2(A_1 x + B_1 u_1 + D_1 w_1) \right]$$

where

$$J_i(x) = x_{N+1}^T Q_{N+1} x_{N+1} + \sum_{k=i}^N (x_k^T Q_k x_k + u_k^T R_k u_k - \gamma^2 w_k^T w_k) \Bigg|_{x_i = x}$$

Denote $J_i^*(x) = \min_{(u_i, \cdots, u_N)} \max_{(w_i, \cdots, w_N)} J_i(x)$, then $J_\gamma^*(x) = J_1^*(x)$ where

$$
\begin{aligned}
J_i^*(x) &= \min_{u_i} \max_{w_i} (-\gamma^2 w_i^T w_i + u_i^T R_i u_i + J_{i+1}^*(A_i x + B_i u_i + D_i w_i)), \quad i = 1, \cdots, N-1 \\
J_N^*(x) &= \min_{u_N} \max_{w_N} [-\gamma^2 w_N^T w_N + u_N^T R_N u_N \\
&\quad + (A_N x + B_N u_N + D_N w_N)^T Q_{N+1} (A_N x + B_N u_N + D_N w_N)],
\end{aligned}
$$

To find the optimal control and the worst disturbance, one needs to find the saddle point of the quadratic function $(w_i, u_i) \mapsto -\gamma^2 w_i^T w_i + u_i^T R_i u_i + J_{i+1}^*(A_i x + B_i u_i + D_i w_i)$. However, a quadratic function of this form may have multiple saddle points.

Chapter 2

Maximum Principle

2.1 The method of tent and the proof of the MP

2.1.1 The separability of tents

Definition 2.1 (Separability) *Let $K_0, \cdots, K_s$ be some closed, convex cones with a common apex x in $\mathbb{R}^n$. They are said to be separable if there exists a hyperplane Γ through x that separates one of the cones from the intersection of the others.*

Theorem 2.1 (The criterion of separability) *Let $K_0, \cdots, K_s$ be some closed, convex cones with a common apex x in $\mathbb{R}^n$. Then they are separable if and only if there exist dual vectors $a_i,\ i = 0, 1, \cdots, s$ fulfilling*

$$\langle a_i, y - x\rangle \le 0$$

for all $y \in K_i$ at least one of which is not zero and such that

$$a_0 + \cdots + a_s = 0.$$

Theorem 2.2 (Necessary condition of separability) *Let $\Omega_0, \cdots, \Omega_s$ be sets in $\mathbb{R}^n$ satisfying*

$$\Omega_0 \cap \cdots \cap \Omega_s = \{x\},$$

and $K_0, \cdots, K_s$ be tents of these sets at x. Assume that at least one of the tents is not a plane. Then $K_0, \cdots, K_s$ is separable.

2.1.2 The Lagrange multiplier

To see how the method of tent works, we study the minimization problem under constraints. That is, we consider the following

$$\begin{aligned} &\min\ g_0(x) \\ &\text{subject to } g_i(x) = 0, \quad i = 1, \cdots, m \end{aligned} \tag{LM}$$

in which $\{g_i\}_{i=0}^m \in C^1(\mathbb{R}^n; \mathbb{R})$. Assume $\operatorname{rank}(Dg(x)) = m$ on the set $S = \{x : g_i(x) = 0,\ i = 1, \cdots, m\}$, with the notation

$$g(x) = \begin{bmatrix} g_1(x) \\ \vdots \\ g_m(x) \end{bmatrix}.$$

Suppose that the problem is feasible and there exists an admissible x_* which minimizes $g_0(x)$. To apply the tent method, define

$$\Omega_i = \{x \in \mathbb{R}^n : g_i(x) = 0\}, \quad i = 1, \cdots, m$$

and

$$\Omega_0 = \{x : g_0(x) < g_0(x_*)\} \cup \{x_*\},$$

then it is necessary that

$$\Omega_0 \cap \cdots \cap \Omega_m = \{x_*\}.$$

On the other hand,

$$\begin{aligned} K_0 &= \{x : \langle x - x_*, \operatorname{grad} g_0(x_*)\rangle \leq 0\} \\ K_i &= \{x : \langle x - x_*, \operatorname{grad} g_i(x_*)\rangle = 0\}, \quad i = 1, \cdots, m \end{aligned}$$

are tents of Ω_i. Since K_0 is a half space, not a flat, Theorem 2.2 applies. Namely, there exist $m+1$ covectors, a_i, not all zero, satisfying

$$\sum_{i=0}^{m} a_i = 0.$$

and $a_0 = \lambda_0 \operatorname{grad} g_0(x_*)$ for some real constants $\lambda_0 \geq 0$. Similarly, $a_i = \lambda_i \operatorname{grad} g_i(x_*)$ for some reals λ_i. Therefore, the above condition becomes

$$\sum_{i=0}^{m} \lambda_i \operatorname{grad} g_i(x_*) = 0$$

Because $Dg(x_*)$ has constant rank, $\{\operatorname{grad} g_i(x_*)\}_{i=1}^m$ is linearly independent and $\lambda_0 \neq 0$. To summarize, we have shown the following:

Theorem 2.3 *If x_* solves the minimization problem (LM), then there exist m constants, not all zero, such that*

$$\operatorname{grad} g_0(x_*) = \sum_{i=1}^{m} \lambda_i \operatorname{grad} g_i(x_*).$$

2.1.3 The optimal control problem

We start by introducing the optimal control problem under fixed terminal time.

Problem 1 *Consider the control system*

$$\dot{x} = f(x, u), \tag{2.1}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in U \subset \mathbb{R}^m$ for all t, a cost function

$$J(u(\cdot)) = \varphi(x(t_1)) + \int_{t_0}^{t_1} L(x(s), u(s))ds,$$

with φ, L smooth and t_0, t_1 fixed. The optimal control problem consists in finding a process $u_*(t)$, $x_*(t)$, $t_0 \leq t \leq t_1$, with a measurable controller $u_*(t)$ such that $x_*(t_0) \in M_0$, $x_*(t_1) \in M_1$, and $J(u_*(\cdot))$ attains a minimum. We say that the problem is in

- *Mayer form if $L = 0$,*
- *Lagrange form if $\varphi = 0$,*

- *Bolza form if neither L nor φ is zero.*

Let

$$x_{n+1}(t) = \int_{t_0}^{t} L(x(s), u(s))ds$$

then Bolza form reduces to Mayer form. Due to this reason, it suffices to study the optimal control problem under the Mayer form.

Introduce the notations:

$$\begin{aligned}
&x_1 := x_*(t_1) \\
&\Omega_0 = \{x_1\} \cup \{x : \varphi(x) < \varphi(x_1)\} \\
&\Omega_1 : \text{ reachability set from } x_0 \\
&\Omega_2 = M_1 : \text{ the terminal manifold}
\end{aligned}$$

Let $u_*(t)$, $x_*(t)$, $t_0 \leq t \leq t_1$ be an optimal process. Then it is easily seen that

$$\Omega_0 \cap \Omega_1 \cap \Omega_2 = \{x_*(t_1)\}. \tag{2.2}$$

We derive the MP based on this condition. Let

$$\begin{aligned}
&K_0 = \{x : \langle x - x_1, \operatorname{grad} \varphi(x_1)\rangle \leq 0\} \\
&K_2 = T_{x_1}\Omega_2
\end{aligned}$$

then K_0 and K_2 are tents of Ω_0 and Ω_2 respectively. The necessary conditions are derived once a tent of Ω_1 is worked out.

2.1.4 Needle variation

Suppose at the moment that $u_* : [t_0, t_1] \to U$ is continuous. Fix $\tau \in (t_0, t_1]$. Consider a needle variation of u_* for small $\varepsilon > 0$:

$$u_\varepsilon(t) = \begin{cases} w, & t \in (\tau - \varepsilon, \tau] \\ u_*(t), & \text{otherwise} \end{cases}$$

where $w \in U$ is some constant, see Figure 2.1.4. Denote $t \mapsto x_\varepsilon(t)$ the solution to $\dot{x} = f(x, u_\varepsilon)$.

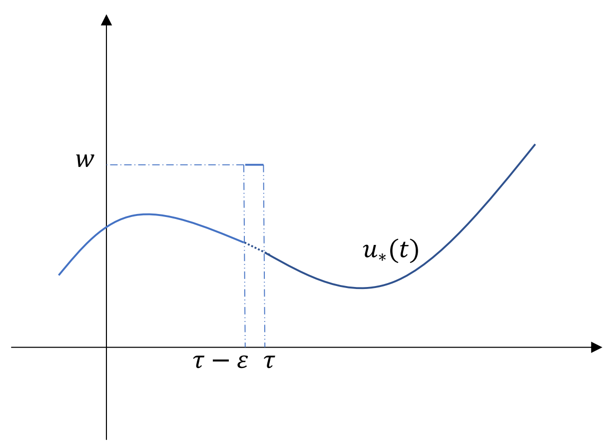

Figure 2.1: Needle variation.

Then when $t \in [0, \tau - \varepsilon]$, $x_\varepsilon(t) = x_*(t)$. Since

$$x_\varepsilon(\tau) = x_*(\tau) + \int_{\tau-\varepsilon}^{\tau} f(x_\varepsilon(t), w) - f(x_*(t), u_*(t))dt.$$

at time $t = \tau$, we have

$$\begin{aligned} v(\tau) &:= \lim_{\varepsilon \to 0+} \frac{x_\varepsilon(\tau) - x_*(\tau)}{\varepsilon} \\ &= \lim_{\varepsilon \to 0+} \frac{1}{\varepsilon} \left[\int_{\tau-\varepsilon}^{\tau} f(x_\varepsilon(t), w)dt - \int_{\tau-\varepsilon}^{\tau} f(x_*(t), u_*(t))dt \right] \\ &= f(x_*(\tau), w) - f(x_*(\tau), u_*(\tau)). \end{aligned} \tag{2.3}$$

Let $v : [\tau, t_1] \to \mathbb{R}^n$ (rigorously, $v(t) \in T_{x(t)}\mathbb{R}^n$) be defined as [1]

$$v(t) = \left. \frac{\partial x_\varepsilon(t)}{\partial \varepsilon} \right|_{\varepsilon = 0+}, \quad t \in [\tau, t_1]$$

Then $v(t_1)$ is a tangential vector to Ω_1 which we call a *deviation vector*. Now let $v_1(t_1), \cdots, v_r(t_1)$ be some different deviation vectors obtained in this way corresponding to some distinct time instants $\tau_1 < \cdots < \tau_r$ and constant inputs $w_1, \cdots, w_r$. Consider the combined needle variation

$$u_{\varepsilon,k}(t) = \begin{cases} w_i, & t \in (\tau_i - k_i\varepsilon, \tau_i] \text{ for some } i \in \{1, \cdots, r\} \\ u_*(t), & \text{otherwise} \end{cases}$$

where k_i are non-negative constants satisfying $\sum_{i=1}^r k_i = 1$. One can show that

$$\sum_{i=1}^r k_i v_i(t_1) = \left. \frac{\partial x(t_1, u_{\varepsilon,k})}{\partial \varepsilon} \right|_{\varepsilon = 0+}$$

which implies that $\sum_{i=1}^r k_i v_i(t_1)$ is also a tangential vector to Ω_1. Define K_1 as the convex cone generated by the deviation vectors.

2.1.5 The adjoint equation and the MP

Condition (2.2) implies that K_0, K_1, K_2 are separable. Invoking Theorem 2.1 and Theorem 2.2, we deduce that there exist three dual vectors $a_i \in K_i^*$, such that

$$a_0 + a_1 + a_2 = 0. \tag{2.4}$$

Recall the obvious lemma:

Lemma 2.1 *Consider two linear ODE*

$$\begin{aligned} \dot{x} &= A(t)x \\ \dot{p} &= -pA(t) \end{aligned}$$

where $x \in \mathbb{R}^n$, $p \in (\mathbb{R}^n)^*$ (x and p are understood as column and row vectors respectively). Then $p(t)x(t) = p(t')x(t')$ for any $t, t' \in \mathbb{R}$.

[1] Equivalently, $v(t)$ is the solution to the Cauchy problem

$$\begin{aligned} \dot{v} &= \frac{\partial f}{\partial x}(x_*(t), u_*(t))v, \quad t \in [\tau, t_1] \\ v(\tau) &= f(\tau, x_*(\tau), w) - f(\tau, x_*(\tau), u_*(\tau)). \end{aligned}$$

To apply this lemma, construct the following system

$$\dot{p} = -p\frac{\partial f}{\partial x}(x_*(t), u_*(t)) \tag{2.5}$$

with terminal state $p(t_1) = a_1$. By definition, $a_1 v(t_1) \geq 0$ for any deviation vector $v(t_1)$, where the latter is the solution to the variational system

$$\begin{aligned} \dot{v} &= \frac{\partial f}{\partial x}(x_*(t), u_*(t))v, \\ v(\tau) &= f(x_*(\tau), w) - f(x_*(\tau), u_*(\tau)) \end{aligned} \tag{2.6}$$

hence invoking the above lemma, we have

$$H(p(t_1), x_*(t_1), u_*(t_1)) = \text{constant}, \tag{2.7}$$

and

$$p(\tau)[f(x_*(\tau), w) - f(x_*(\tau), u_*(\tau)] = p(t_1)v(t_1) = a_1 v(t_1) \geq 0$$

or

$$p(\tau)f(x_*(\tau), w) \geq p(\tau)f(x_*(\tau), u_*(\tau)) \tag{2.8}$$

If we denote

$$H(p, x, u) := pf(x, u)$$

then (2.8) is equivalent to saying

$$H(p(t), x_*(t), u_*(t)) = \min_{u \in U} H(p(t), x_*(t), u), \tag{2.9}$$

for all t since τ is arbitrary in (2.8) and the assumption that u is continuous.

Since $x_0 \in M_0$ is not fixed, we can take $v(t_0) = \xi \in T_{x_*(t_0)}M_0$ such that $a_1 v(t_1) \geq 0$. Similarly, take $v(t_0) = -\xi$, from which we deduce $a_1 v(t_1) = 0 = p(t_0)\xi$ or

$$p(t_0) \perp M_0. \tag{2.10}$$

For a_0, since K_0 is a half space, $a_0 = \lambda \operatorname{grad} \varphi(x_1)$ with someconstant $\lambda \leq 0$. For a_2, since K_2 is a plane, then $a_2 \in K_2^*$ if and only if $a_2 \perp K_2$. It follows from (2.4) that

$$\lambda \operatorname{grad} \varphi(x_1) + p(t_1) \perp \Omega_2. \tag{2.11}$$

If $\lambda = 0$, then $0 \neq p(t_1) \perp \Omega_2$.

For u not continuous, only the condition (2.9) needs to be changed by noticing that the limits in (2.3) exist for almost all $t \in [t_0, t_1]$. Summarizing, we have proved the following.

Theorem 2.4 *Suppose that the Mayer form optimal control Problem 1 admits an admissable optimal law u_* with corresponding trajectory $x_*(t)$. Then there is a solution to the adjoint equation (2.5), such that the triple $(p(t), x_*(t), u_*(t))$ satisfies the maximum condition (2.9) for almost all t and the transversality condition (2.7), (2.10) and there exists a constant $\lambda \geq 0$, such that (2.11) is satisfied ($p(t_1) \neq 0$ as long as $\lambda = 0$).*

Remark 2.1 *The reader should now be able to derive the necessary conditions of the optimal control in the Bolza form.*

We have so far considered the optimal control problem under the condition that t_1 is fixed. It can be easily extened to the case of free terminal time: it is obvious that all the necessary conditions of Theorem 2.4 still need to be hold. The mere difference is that now one can also make the variation of the terminal time. Therefore, the reacheability tent can be taken as

$$K_1' = \{v(t_1) + \mu f(x_*(t_1), u_*(t_1)) : v(t_1) \in K_1,\ \mu \in \mathbb{R}\}$$

since at $x_*(t_1)$ one can move along the curve $(t_1 - \varepsilon, t_1 + \varepsilon) \ni t \mapsto f(x_*(t), u_*(t))$, or in a more rigorous way,

$$v(t_1) + \mu f(x_*(t_1), u_*(t_1)) = \left.\frac{\partial x_\varepsilon(t_1 + \varepsilon\mu)}{\partial \varepsilon}\right|_{\varepsilon=0+}.$$

It follows that one can obtain a finer condition than (2.7):

$$H(p(t), x_*(t), u_*(t)) = 0,\ \forall t \in [t_0, t_1].$$

Indeed, since $a_1(\mu f(x_*(t_1), u_*(t_1))) \leq 0$ for any $\mu \in \mathbb{R}$, it is necessary that $a_1 f(x_*(t_1), u_*(t_1)) = 0$.

2.2 Maximum principle on manifolds

In this section, we study maximum principle on manifolds. For this, we need some preparation of symplectic geometry.

2.2.1 Hamiltonian vector fields and symplectic geometry

We include a short introduction to symplectic geometry. The main reference of this subsection is [3].

Definition 2.2 *Let M be a manifold and $\mathcal{F}(M)$ the set of smooth real-valued functions on M (an $\mathbb{R}$-algebra under piecewise product and sum). A Poisson bracket is a binary operation*

$$\{,\} : \mathcal{F}(M) \times \mathcal{F}(M) \to \mathcal{F}(M)$$

which satisfies the following properties for all $f, g, h \in \mathcal{F}(M)$:

1. *bilinearity: $\{f, g\}$ is $\mathbb{R}$-bilinear in f and g;*
2. *anticommutativity: $\{f, g\} = -\{g, f\}$;*
3. *Jacobi's identity: $\{\{f, g\}, h\} + \{\{h, f\}, g\} + \{\{g, h\}, f\} = 0$;*
4. *Leibnitz' rule $\{fg, h\} = f\{g, h\} + g\{f, h\}$.*

The manifold M is said to be a Poisson manifold if it is equipped with a Poisson bracket.

Definition 2.3 *Let $(M_1, \{,\}_1)$ and $(M_2, \{,\})$ be two Poisson manifolds. A mapping $\varphi : M_1 \to M_2$ is called Poisson if for all $f, h \in \mathcal{F}(M_2)$, we have*

$$\{f, h\}_2 \circ \varphi = \{f \circ \varphi, h \circ \varphi\}_1.$$

(φ is a morphism in the language of category theory).

Definition 2.4 *Let M be a manifold and Ω is a 2-form ($(0, 2)$-tensor). The pair (M, Ω) is called a symplectic manifold if Ω satisfies*

1. *$d\Omega = 0$ (i.e., Ω is closed) and*

2. *Ω is nondegenerate in the sense that $\Omega(v, w) = 0$ for all w implies that v is a zero tangent vector.*

Definition 2.5 *Let (M, Ω) be a symplectic manifold and let $f \in \mathcal{F}(M)$. Let X_f be the unique vector field on M satisfying*

$$\Omega_z(X_f(z), v) = df(z)(v), \quad \textit{for all } v \in T_z M.$$

We call X_f the Hamiltonian vector field of f. Hamilton's equations are the differential equations on M given by

$$\dot{z} = X_f(z).$$

Remark 2.2 *The existence and uniqueness of X_f is guaranteed by the non-degeneracy of Ω.*

If (M, Ω) is a symplectic manifold. Then one can define a Poisson bracket as

$$\{f, g\} = \Omega(X_f, X_g)$$

which we call the Poisson bracket associated with the symplectic manifold (M, Ω). Therefore, every symplectic manifold is also Poisson. The converse is not true. However, Hamiltonian vector fields can still be defined on Poisson manifold.

Definition 2.6 *Let $(M, \{,\})$ be a Poisson manifold and let $f \in \mathcal{F}(M)$. Define X_f be the unique vector field on M satisfying*

$$dk(X_f) = \{k, f\} \quad \textit{for all } k \in \mathcal{F}(M)$$

we call X_f the Hamiltonian vector field of f.

This definition coincides with Definition 2.5 when the Poisson manifold is symplectic. Now given $H \in \mathcal{F}(M)$, the Hamilton's equation is $\dot{z} = X_H(z)$. By the chain rule, for any $f \in \mathcal{F}(M)$, we have $\frac{df(z(t))}{dt} = df(X_H(z)) = \{f, H\}$. Thus the Hamilton's equation can be written in the following three (equivalent) ways:

1. $\dot{z} = X_H(z)$;
2. $\dot{f} = df(X_H(z))$ for all $f \in \mathcal{F}(M)$;
3. $\dot{f} = \{f, H\}$ for all $f \in \mathcal{F}(M)$.

The following is a basic fact about Hamiltonian system.

Proposition 2.1 *Let $\phi_t : M \to M$ be the flow of the Hamilton's equation $\dot{z} = X_H(z)$. Then*

1. *ϕ_t is a Poisson map;*
2. *$H \circ \phi_t = H$ (conservation of energy).*

We now come to one of the most important constructions of symplectic manifold, namely, the cotangent bundle.

Consider an n dimensional manifold Q and its cotangent bundle T^*Q. Let (q_i) be the coordinates on Q and (q^i, p_j) the induced coordinate on T^*Q. More precisely, for any $\omega \in T^*Q$, $p_j(\omega) = \omega\left(\frac{\partial}{\partial q^j}\right)$. Next, define a 2-form ω on T^*Q by

$$\omega = \sum_{i=1}^{n} dq^i \wedge dp_i \tag{2.12}$$

One can check that ω is well-defined (coordinate-free). As an alternative, we consider the 1-form

$$\Theta = \sum_{i=1}^{n} p_i dq^i$$

and $\omega = -d\Theta$. Thus, it suffices to show that Θ is coordinate-free. (The notation $p_i dq^i$ is a little ambiguous since it may also be understood as a dual vector in T^*Q instead of in T^*T^*Q! We adopt this notation anyway since it is standard. The function p_i in front of dq^i should remind the reader that it is a dual vector in T^*T^*Q.) To show that Θ is well-defined, let $(\tilde{q}^i, \tilde{p}_j)$ be another coordinate, where $p_i = \tilde{p}_j \frac{\partial \tilde{q}_j}{\partial q^i}$. Since $dq^i = \sum_{j=1}^n \frac{\partial q^i}{\partial \tilde{q}^j} d\tilde{q}^j$, we have

$$\Theta = \sum_{i=1}^{n} p_i dq^i = \sum_{i,j,r=1}^{n} \tilde{p}_j \frac{\partial \tilde{q}^j}{\partial q^i} \frac{\partial q^i}{\partial \tilde{q}^r} d\tilde{q}^r = \sum_{j,r=1}^{n} \delta_j^r \tilde{p}_j d\tilde{q}^r = \sum_{i=1}^{n} \tilde{p}_i d\tilde{q}^i$$

The 1-form Θ is the *tautological form* or *Liouville 1-form* and the 2-form $\omega = -d\Theta$ is the *canonical symplectic form*. To summarize:

Proposition 2.2 *Let Q be a smooth manifold. Then $(T^*Q,\ \omega)$ is a symplectic manifold, where ω is defined as (2.12).*

Let's calculate in coordinates. Let $H \in \mathcal{F}(T^*Q)$, and $X_H = X^i \frac{\partial}{\partial q^i} + X^{n+i} \frac{\partial}{\partial p_i}$ (we use Einstein summation natation: the repeated index is summed). By definition, for any $v = v^i \frac{\partial}{\partial q^i} + v^{n+i} \frac{\partial}{\partial p_i}$, there holds

$$dq^i \wedge dp_i \,(X_H, v) = dH(v),$$

or

$$X^i v^{n+i} - X^{n+i} v^i = \frac{\partial H}{\partial q^i} v^i + \frac{\partial H}{\partial p^i} v^{n+i},$$

from which it follows that

$$X^i = \frac{\partial H}{\partial p^i}, \quad X^{n+i} = -\frac{\partial H}{\partial q^i}$$

Hence

$$X_H(q,p) = \sum_{i=1}^{n} \left(\frac{\partial H}{\partial p^i} \frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i} \frac{\partial}{\partial p_i} \right).$$

And the Hamilton's equation reads

$$\dot{q}^i = \frac{\partial H}{\partial p^i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q^i}.$$

Further more, for $H_1, H_2 \in \mathcal{F}(T^*Q)$, the Poisson bracket reads

$$\{H_1, H_2\} = dH_1(X_{H_2}) = \sum_{i=1}^{n} \left(\frac{\partial H_1}{\partial q^i} \frac{\partial H_2}{\partial p^i} - \frac{\partial H_1}{\partial p^i} \frac{\partial H_2}{\partial q_i} \right).$$

With these preparations, we are ready to state the maximum principle on manifolds.

2.2.2 MP in the language of differential geometry

Consider the optimal control problem in Mayer form, i.e., Problem 1 with $L = 0$. Then the maximum principle can be stated as follows (we omit the transversal condition as they are the same as before).

Theorem 2.5 *Equip T^*M with the canonical symplectic structure. Let $u_* : [t_0, t_1] \to U$ be the optimal control and $x_* : [t_0, t_1] \to M$ the optimal process of the Mayer problem. Define a Hamiltonian $H : T^*M \times U \to \mathbb{R}$ by $H(x, \lambda, u) = \langle \lambda, f(x, u) \rangle$. Then there exists a curve $\Lambda : [t_0, t_1] \to T^*M$, with $\Lambda(t) = (x_*(t), \lambda(t))$ for $t \in [t_0, t_1]$ such that Λ is the solution to the Hamilton's equation $\dot{\Lambda} = X_H(\Lambda)$. Moreover, along Λ the Hamiltonian H satisfies the maximum principle*

$$H(x_*(t), \lambda(t), u_*(t)) = \max_{u \in U} H(x_*(t), \lambda(t), u).$$

2.2.3 Non-holonomic systems and sub-Riemannian geometry

2.2.3.1 The isoperimetric problem

2.2.3.2 Non-holonomic systems

2.2.3.3 Sub-Riemannian geometry

2.3 Maximum Principle of Discrete Time Systems

2.3.1 Fixed control region

Historically, the dynamic principle was first developped for continuous time systems. This however, doesn't mean that MP for discrete time system is harder. We will see now for fixed control region, the problem is in fact easy.

Consider the discrete time system

$$x_{k+1} = f_k(x_k, u_k), \tag{2.13}$$

with input sequence

$$(u_1, \cdots, u_N)$$

and resulting process

$$(x_1, \cdots, x_N, x_{N+1}).$$

Assume

$$u_t \in U_t \subset \mathbb{R}^m, t = 1, \cdots, N \tag{2.14}$$

in which U_t can be state dependent. But in this subsection, we assume that $U_t \equiv U$ is fixed. The state is under constraint

$$x_t \in M_t \subset \mathbb{R}^n, \quad t = 1, \cdots, N+1. \tag{2.15}$$

Problem 2 *The optimal control problem of the discrete time system (2.13) with cost function*

$$J(u) = \varphi(x_{N+1}) + \sum_{k=1}^{N} L_k(x_k, u_k)$$

consists in finding a policy

$$u^* = (u_1^*, u_2^*, \cdots, u_N^*)$$

with $u_i^* \in U$ and N fixed, such that $x_t^* \in M_t$ for $t = 1, \cdots, N+1$, and $J(u^*)$ attains a minimum. We say that the problem is in

- *Mayer form if $L = 0$,*
- *Lagrange form if $\varphi = 0$,*
- *Bolza form if neither L nor φ is zero.*

Like in the continuous time case, we consider only the Mayer form as the other two forms are equivalent to it.

For discrete time system, the sets Ω_0, Ω_2 and the tents K_0, K_2 are the same as before. The only difference is the reacheability region Ω_1 and its tent K_1. To calculate K_1, define a variation similar to the needle variation of continuous signal at the instant $i \in \{1, \cdots, N\}$:

$$u_k^{(i,\varepsilon)} = \begin{cases} u_k^* + \varepsilon u, & k = i, \\ u_k^*, & \text{otherwise} \end{cases}$$

where $u \in U$. Let $x_{k+1}^{(i,\varepsilon)}$ be the solution to

$$\begin{aligned} x_{k+1}^{(i,\varepsilon)} &= f_k(x_k^{(i,\varepsilon)}, u_k^{(i,\varepsilon)}), \quad k = \{i, \cdots, N\}. \\ x_i^{(i,\varepsilon)} &= x_i^*. \end{aligned}$$

Then

$$\left. \frac{\partial x_k^{(i,\varepsilon)}}{\partial \varepsilon} \right|_{\varepsilon=0}, \quad k \in \{i+1, \cdots, N\}$$

is the solution to the discrete time variational system

$$\begin{aligned} v_{k+1} &= \frac{\partial f_k}{\partial x}(x_k^*, u_k^*) v_k, \quad k \in \{i, \cdots, N\} \\ v_{i+1} &= \frac{\partial f_i}{\partial u_i}(x_k^*, u_k^*) u \end{aligned}$$

Similar to the continuous time case, we call v_{N+1} a *deviation vector* under the variation. Then the convex cone $K_1^{''}$ generated by the deviation vectors is a tent of the reacheability region. By Theorem 2.2, there exist three covectors $a_0 \in K_0^*$, $a_1 \in (K_1'')^*$, $a_2 \in K_2^*$, not all zero, such that $a_0 + a_1 + a_2 = 0$ and $a_0 = \lambda \operatorname{grad} \varphi(x_{N+1})$ with $\lambda \geq 0$. To characterize K_1'', introduce the *adjoint system*

$$\begin{aligned} p_k &= p_{k+1} \frac{\partial f_k}{\partial x_k}(x_k^*, u_k^*), \quad k \in \{i+1, \cdots, N\} \\ p_{N+1} &= a_1 \end{aligned}$$

The following lemma is obvious, which is the discrete time version of Lemma 2.1:

Lemma 2.2 *Consider the system*

$$\begin{aligned} x_{k+1} &= A_k x_k \\ p_k &= p_{k+1} A_k \end{aligned}$$

where $x_k \in \mathbb{R}^n$, $p_k \in (\mathbb{R}^n)^*$. Then $p_k x_k = p_{k'} x_{k'}$ for all integers k, k'.

Then

$$0 \leq a_1 v_{N+1} = p_{N+1} v_{N+1} = p_{i+1} v_{i+1} = p_{i+1} \frac{\partial f_i}{\partial u_i}(x_i^*, u_i^*) u, \quad \forall u \in U.$$

which implies

$$p_{k+1} \frac{\partial f_k}{\partial u_k}(x_k^*, u_k^*) = 0, \ \forall k \in \{1, \cdots, N\},$$

since U is open. Since x_1 is not fixed, we can take $\pm v_1 \in T_{x_1^*} M_0$. Then $0 \leq a_1 v_{N+1} = p_1 v_1 \leq 0$, or $p_1 v_1 = 0$, which is equivalent to

$$p_1 \perp M_0$$

The transversal condition is

$$\lambda \operatorname{grad} \varphi(x_{N+1}) + p_{N+1} \perp M_1$$

with $\lambda \leq 0$. If the terminal state is free, i.e., $M_1 = \mathbb{R}^n$, then $p_{N+1} = -\lambda \operatorname{grad} \varphi(x_{N+1})$.

2.3.2 Variable control region

In this subsection we consider the Mayer problem with $J(u) = J(x_{N+1})$.

Let

$$\Phi_t(x(t)) = \bigcup_{u \in U_t} \{f_t(x(t), u)\} \subset \mathbb{R}^m,\ t = 1, \cdots, N.$$

Assume that the sets $\Phi_t(x)$ are compact, convex and continuously dependent on $x \in \mathbb{R}^n$ for every $t = 1, \cdots, N$. We say a trajectory

$$(x(1), \cdots, x(N))$$

admits a local section if for every $t = 1, \cdots, N$, there is a smooth function $\sigma_t : U \subset \mathbb{R}^m \to \mathbb{R}^m$, where U is a neighbourhood of $x(t)$, such that

$$\sigma_t(x) \in \Phi_t(x),\ \forall x \in U \ \text{ and } \sigma_t(x(t)) = x(t+1) = f_t(x(t), u(t))$$

Introduce the following notations for each $\theta \in \{1, \cdots, N\}$ in which $n = (N+1)m$:

$$\begin{aligned}
&z = (x_1, \cdots, x_{N+1}) \in \mathbb{R}^n, \quad \text{with } x_\theta \in \mathbb{R}^m, \quad \theta \in \{1, \cdots, N+1\} \\
&\Xi_\theta = \{z \in \mathbb{R}^n : x_{\theta+1} \in \Phi_\theta(x_\theta)\}, \quad \theta \in \{1, \cdots, N\} \\
&\Omega^*_\theta = \{z \in \mathbb{R}^n : x_\theta \in M_\theta\}, \quad \theta \in \{1, \cdots, N+1\} \\
&P_\theta : \ \text{a tent of } M_\theta \text{ at } x_\theta, \quad \theta \in \{1, \cdots, N+1\} \\
&P^*_\theta = \{z \in \mathbb{R}^n : x_\theta \in P_\theta\} \text{ (then } P^*_\theta \text{ is a tent of } \Omega^*_\theta), \quad \theta \in \{1, \cdots, N+1\}
\end{aligned}$$

Assume that

$$\bar{z} = (\bar{x}_1, \cdots, \bar{x}_{N+1})$$

is the optimal process.

With these notations, the problem of finding an optimal trajectory for the system (2.13) reduces to the problem of minimizing $J(x(N+1))$ on the set

$$\Sigma = \left(\bigcap_{k=1}^{N} \Xi_k \right) \cap \left(\bigcap_{k=1}^{N+1} \Omega^*_k \right).$$

Since the tents of Ω^*_θ are known as P^*_θ for $\theta = 1, \cdots, N+1$, it remains to calculate the tents of Ξ_θ. Define

$$\begin{aligned}
Q_\theta = &\left\{ \bar{z} + \delta z : \bar{x}_{\theta+1} + \delta x_{\theta+1} - \frac{\partial \sigma_\theta(\bar{x}_\theta)}{\partial x} \delta x_\theta \in L_\theta \right\}, \\
&\theta = 1, \cdots, N
\end{aligned}$$

where

$$L_\theta : \ \text{supporting cone of } \Phi_\theta(\bar{x}_\theta) \text{ at } \bar{x}_{\theta+1}, \quad \theta \in \{1, \cdots, N\}$$

We claim that Q_θ is a tent of Ξ_θ. Assume this fact, we would deduce the following.

There is a number $\psi_0 \leq 0$ and dual vectors

$$\begin{aligned}
a_t &\in D(P^*_t) \subset \mathbb{R}^{m(N+1)}, \quad t = 1, \cdots, N+1 \\
b_t &\in D(Q_t) \subset \mathbb{R}^{mN}, \quad t = 1, \cdots, N
\end{aligned}$$

such that

$$\psi_0 \operatorname{grad}_z J(\bar{x}_{N+1}) + \sum_{t=1}^{N+1} a_t + \sum_{t=1}^{N} b_t = 0 \tag{2.16}$$

If we write

$$\begin{aligned} a_t &= (a_t^1, \cdots, a_t^{N+1}) \\ b_t &= (b_t^1, \cdots, b_t^N) \end{aligned}$$

then $a_t^i \neq 0$ only when $i = t$, and $b_t^i \neq 0$ only when $i = t,\ t+1$. By assumption $\langle b_t, \delta z \rangle \geq 0$ for any $\delta z \in Q_t$. Take a δz satisfying $\delta x_{t+1} = \frac{\partial \sigma_t(\bar{x}_t)}{\partial x} \delta x_t$ and $\delta x_i = 0$ for $i \neq t$, then

$$b_t^t \delta x_t + b_t^{t+1} \frac{\partial \sigma_t(\bar{x}_t)}{\partial x} \delta x_t \geq 0, \quad \delta x_t \in \mathbb{R}^m$$

Let $\varphi_t = b_t^t$ and $\psi_t = b_t^{t+1}$, the above implies

$$\varphi_t + \psi_t \frac{\partial \sigma_t(\bar{x}_t)}{\partial x} = 0.$$

Hence the condition (2.16) can be written as

$$\begin{aligned} \psi_{t-1} &= -\lambda_t + \psi_t \frac{\partial \sigma_t(\bar{x}_t)}{\partial x}, \quad t = 1, \cdots, N \\ \psi_0 &= 0 \\ \psi_N &= -\lambda_{N+1} - \psi_0 \operatorname{grad}_x J(\bar{x}_{N+1}) \end{aligned}$$

since

$$\begin{aligned} &\operatorname{grad}_z J(\bar{x}_{N+1}) = (0, \cdots, 0, \operatorname{grad}_x J(\bar{x}_{N+1})) \\ &\sum_{t=1}^{N+1} a_t = (\lambda_1, \cdots, \lambda_{N+1}) \\ &\sum_{t=1}^{N} b_t = (\varphi_1, \psi_1 + \varphi_2, \cdots, \psi_{N-1} + \varphi_N, \psi_N) \end{aligned}$$

where we have denoted $\lambda_t = a_t^t$.

Further, choose δz is such a way that $x_{t+1} = \bar{x}_{t+1} + \delta x_{t+1} \in L_t$ and $\delta x_i = 0$ for $i \neq t+1$. Therefore

$$0 \leq \psi_t \delta x_{t+1}$$

In other words, the function $\eta_t(v) = \psi_t v$ achieves minimum at the point $\bar{x}_{t+1}$. Since $\Phi_t(\bar{x}_t)$ is contained in L_t, it follows that

$$\psi_t \bar{x}_{t+1} = \min_{x \in \Phi_t(\bar{x}_t)} \psi_t x = \min_{u \in U_t} \psi_t f_t(\bar{x}_t, u), \quad t = 1, \cdots, N$$

Thus we are left to show that Q_θ is a tent of Ξ_θ.

Choose $z \in Q_\theta$ arbitrary ($x_{\theta+1}$ is not necessarily in $\Phi_\theta(x_\theta)$). Define

$$\varphi_\theta(z) \text{ the projection of } x_{\theta+1} \text{ to } \Phi_\theta(x_\theta),$$

and

$$\Psi_\theta(z) = (x_0, \cdots, x_\theta, \varphi_\theta(z), x_{\theta+2}, \cdots, x_N) \in \mathbb{R}^n$$

Then since $\varphi_\theta(z) \in \Phi_\theta(x_\theta)$, $\Psi_\theta(z) \in \Xi_\theta$ for any $z \in \mathbb{R}^n$. It remains to show

$$\Psi_\theta(z) = z + o(z - \bar{z})$$

or

$$\varphi_\theta(z) = x_{\theta+1} + o(z - \bar{z}).$$

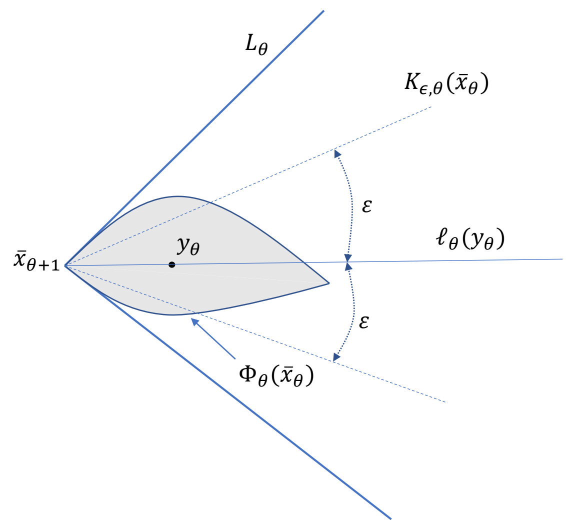

Figure 2.2: Illustration

Consider the point

$$s_\theta(x_\theta) = \sigma(x_\theta) + \delta x_{\theta+1} - \frac{\partial \sigma_\theta(\bar{x}_\theta)}{\partial x}\delta x_\theta.$$

By definition of Q_θ, $s_\theta(x_\theta) \in L_\theta$, the supporting cone of $\Phi_\theta(\bar{x}_\theta)$ at $\bar{x}_{\theta+1}$. Since z is close to $\bar{z}$, we have

$$\begin{aligned} s_\theta(\delta x_\theta) &= \sigma(\bar{x}_\theta + \delta x_\theta) + \delta x_{\theta+1} - \frac{\partial \sigma_\theta(\bar{x}_\theta)}{\partial x}\delta x_\theta \\ &= \sigma(\bar{x}_\theta) + \delta x_{\theta+1} + o(\delta x_\theta) \\ &= \bar{x}_{\theta+1} + \delta x_{\theta+1} + o(\delta x_\theta) \\ &= x_{\theta+1} + o(\delta x_\theta). \end{aligned}$$

Suppose now that $s_\theta \in \Phi_\theta(x_\theta)$, then the conclusion would follow as $|\varphi_\theta(z) - x_{\theta+1}| \le |s_\theta - x_{\theta+1}|$ since $\varphi_\theta(z)$ is the projection of $x_{\theta+1}$ to $\Phi_\theta(x_\theta)$. Unfortunately, s_θ may not be in $\Phi_\theta(x_\theta)$, therefore, it should be replaced by some other point s'_θ. For this, we notice that $s_\theta(\bar{x}_\theta)$ is in L_θ. We draw a ray emanating from $\bar{x}_{\theta+1}$ passing through this point $(s_\theta(\bar{x}_\theta))$ (see Figure 2.3.2), and then the rays emanating from $\bar{x}_{\theta+1}$ with angle $\varepsilon > 0$ (sufficiently small) form a cone, which we denote as $K_{\varepsilon,\theta}(\bar{x}_\theta)$ and that

$$\mathrm{Int}K_{\varepsilon,\theta}(\bar{x}_\theta) \cap \Phi_\theta(\bar{x}_\theta) \neq \varnothing.$$

In the similar way, one can define a cone at $\sigma_\theta(x_\theta)$ with the direction of $s_\theta(x_\theta)$ as axis and angle radius ε, see Figure 2.3.2. By continuity and compactness, one can then show that

$$\mathrm{Int}K_{\varepsilon,\theta}(x_\theta) \cap \Phi_\theta(x_\theta) \neq \varnothing.$$

Now we project $s_\theta(x_\theta)$ to an interior point of $\Phi_\theta(x_\theta)$, say s'_θ, and $|s_\theta(x_\theta) - s'_\theta| \le \left|\delta x_{\theta+1} - \frac{\partial \sigma_\theta(\bar{x}_\theta)}{\partial x}\delta x_\theta\right| \sin(\varepsilon)$. The conclusion follows by noticing that

$$|\varphi_\theta(z) - x_{\theta+1}| \le |s'_\theta - x_{\theta+1}| \le |s'_\theta - s_\theta(x_\theta)| + |s'_\theta(x_\theta) - x_{\theta+1}| = O(\varepsilon) + o(\delta x_\theta)$$

and $\varepsilon > 0$ is arbitrary.

To summarize, we have proven the following theorem.

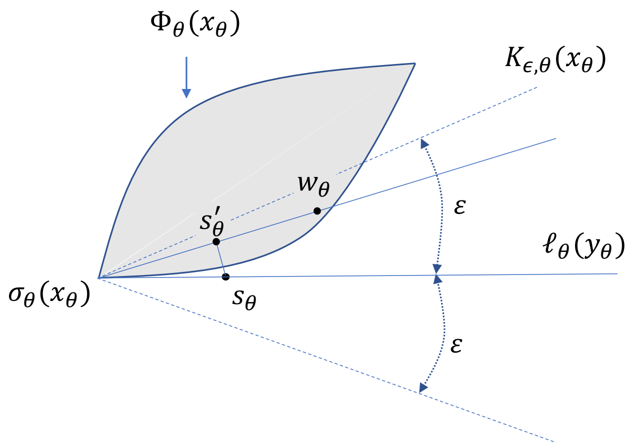

Figure 2.3: Illustration

Theorem 2.6 *Consider the control system (2.13), with state constraint (2.15) and input constraint (2.14). Assume that $\Phi_t(x)$ are compact, convex and continuously dependent on $x \in \mathbb{R}^m$ for all $t = 1, \cdots, N$. Assume P_t is a tent of M_t for each $t = 1, \cdots, N$. Let $(x_1, \cdots, x_{N+1})$ be an optimal process under control sequence $(u_1, \cdots, u_N)$ which minimizes the cost $J(u) = J(x_{N+1})$. Then there is a number $\lambda_0 \geq 0$ and vectors $\psi_t \in (\mathbb{R}^m)^*$, $\lambda_t \in D(P_t)$ such that*

$$\begin{aligned}
&\psi_{t-1} = -\lambda_t + \psi_t \frac{\partial f_t(x_t, x_t)}{\partial x}, \quad t = 1, \cdots, N \\
&\psi_0 = 0 \\
&\psi_N = -\lambda_{N+1} + \lambda_0 \operatorname{grad}_x J(x_{N+1})
\end{aligned}$$

and

$$H(t, x_t, u_t) = \min_{u \in U_t} H(t, x_t, u).$$

where $H(t, x, u) = \psi_t f_t(x, u)$.

Remark 2.3 *It is immediately to see that when the initial state x_1 is fixed, $x_2, \cdots, x_{N+1}$ are not constraint and $U_t = U$ is an open set, then the above condition reduces to*

$$\begin{aligned}
&\psi_{t-1} = \psi_t \frac{\partial f_t(x_t, u_t)}{\partial x}, \quad t = 2, \cdots, N \\
&\psi_N = \lambda_0 \operatorname{grad}_x J(x_{N+1}) \\
&\psi_t \frac{\partial f_t(x_t, u_t)}{\partial u_t} = 0, \quad t = 1, \cdots, N
\end{aligned}$$

2.3.3 Discussions

2.3.3.1 Bolza form

Now we return to the general form of optimal control, i.e, the Bolza form

$$J(u) = \varphi(x_{N+1}) + \sum_{k=1}^{N} L_k(x_k, u_k).$$

To transform it into the Mayer form, let $y_{k+1} = y_k + \tilde{L}_k(x_k, u_k)$ where

$$\tilde{L}_k(x_k, u_k) = \begin{cases} L_k(x_k, u_k), & k = 1, \cdots, N-1 \\ L_N(x_N, u_N) + \varphi(f_N(x_N, u_N)) & k = N \end{cases}$$

Therefore we obtain an augmented system in $\mathbb{R}^{m+1}$:

$$\begin{aligned} x_{k+1} &= f_k(x_k, u_k) \\ y_{k+1} &= y_k + \tilde{L}_k(x_k, u_k) \end{aligned}$$

with $y_1 = 0$. Let

$$z_k = \begin{bmatrix} x_k \\ y_k \end{bmatrix} \in \mathbb{R}^{m+1}$$

The cost function becomes

$$J(u) = \tilde{\varphi}(z_{N+1}) = y_{N+1}$$

Suppose that x_1 is fixed, invoking Theorem 2.6, there is a number $\lambda_0 \geq 0$, vectors $\psi_t = (\alpha_t, \beta_t) \in (\mathbb{R}^m)^* \times (\mathbb{R}^1)^*$, $\lambda_t \in D(P_t)$ such that

$$\begin{aligned} &(\alpha_{t-1}, \beta_{t-1}) = (\alpha_t, \beta_t) \begin{bmatrix} \frac{\partial f_t(x_t, u_t)}{\partial x} & 0 \\ \frac{\partial \tilde{L}_t(x_t, u_t)}{\partial x} & 1 \end{bmatrix}, \quad t = 2, \cdots, N \\ &(\alpha_N, \beta_N) = \lambda_0(0, 1) \\ &(\alpha_t, \beta_t) \begin{bmatrix} \frac{\partial f_t}{\partial u} \\ \frac{\partial L_t}{\partial u} \end{bmatrix} = 0. \end{aligned}$$

from which we see

$$\begin{aligned} \alpha_{N-1} &= \lambda_0 \left(\operatorname{grad} \varphi(x_{N+1}) \frac{\partial f_N(x_N, u_N)}{\partial x} + \frac{\partial L_N(x_N, u_N)}{\partial x} \right) \\ &= \lambda_0 \operatorname{grad} \varphi(x_{N+1}) \frac{\partial f_N(x_N, u_N)}{\partial x} + \lambda_0 \frac{\partial L_N(x_N, u_N)}{\partial x} \end{aligned}$$

Thus redefining $\alpha_N =: \lambda_0 \operatorname{grad} \varphi(x_{N+1})$ we obtain

$$\begin{aligned} &\alpha_{t-1} = \alpha_t \frac{\partial f_t(x_t, u_t)}{\partial x} + \lambda_0 \frac{\partial L_t(x_t, u_t)}{\partial x} = \frac{\partial H_t}{\partial x}, \quad t = 2, \cdots, N \\ &\alpha_N = \lambda_0 \operatorname{grad} \varphi(x_{N+1}), \\ &\alpha_t \frac{\partial f_t}{\partial u} + \lambda_0 \frac{\partial L_t}{\partial u} = \frac{\partial H_t}{\partial u} = 0, \quad t = 1, \cdots, N \end{aligned} \tag{2.17}$$

The Hamiltonian function is $H_k(x, y), = \alpha_k f_k(x, u) + \lambda_0(y + L_k(x, u))$. But since y is independent of u, one can also define $H_k(x, y, u) = \alpha_k f_k(x, u) + \lambda L_k(x, u)$ and the maximum condition becomes

$$H_k(x_k, y_k, u_k) = \min_{u \in U} H_k(x_k, y_k, u).$$

2.3.3.2 Connections to DP

To illustrate the connections to dynamic programming, we show that dynamic programming algorithm (1.5) and discrete time maximum principle (2.17) give the same optimal control law. We consider only the Mayer case as it is equivalent to Bolza form.

(DP $\Rightarrow$ MP): Assume that

$$u_t(x) = \arg\min_u [J^*_{t+1}(f_t(x, u))]$$

then we have

$$J_t^*(x) = J_{t+1}^*(f_t(x, u_t(x)))$$

from which it follows

$$\begin{aligned}
\frac{\partial J_t^*}{\partial x} &= \frac{\partial J_{t+1}^*}{\partial x}\left(\frac{\partial f_t}{\partial x} + \frac{\partial f_t}{\partial u_t}\frac{\partial u_t}{\partial x}\right) \\
&= \frac{\partial J_{t+1}^*}{\partial x}\frac{\partial f_t}{\partial x} + \frac{\partial J_{t+1}^*}{\partial x}\frac{\partial f_t}{\partial u_t}\frac{\partial u_t}{\partial x} \\
&= \frac{\partial J_{t+1}^*}{\partial x}\frac{\partial f_t}{\partial x}
\end{aligned}$$

since $\frac{\partial J_{t+1}^*}{\partial u} = 0$. Letting $\lambda_0 = 1$, $\alpha_t = \frac{\partial J_{t+1}^*}{\partial x}$, we deduce

$$\begin{aligned}
\alpha_{t-1} &= \alpha_t \frac{\partial f_t}{\partial x} \\
0 &= \alpha_t \frac{\partial f_t}{\partial u_t}
\end{aligned}$$

Let $H_t(x, u) = \frac{\partial J_{t+1}^*(x_{t+1})}{\partial x} f_t(x, u)$. Since $J_{t+1}^*(f_t(x_t, u_t)) \leq J_{t+1}^*(f_t(x_t, u))$ or $J_{t+1}^*(v_t) \leq J_{t+1}^*(v), \ \forall v \in V_t = \cup_{u \in U_t}\{f_t(x_t, u)\}$ from which it follows

$$\frac{\partial J_{t+1}^*(x_{t+1})}{\partial x}(v_t - v) \leq 0, \quad \forall v \in V_t$$

or $\frac{\partial J_{t+1}^*(x_{t+1})}{\partial x} v_t \leq \frac{\partial J_{t+1}^*(x_{t+1})}{\partial x} v$ (we have used the fact that V_t is convex). Hence $H_t(x_t, u_t) = \min_{u \in U_t} H_t(x_t, u)$.

(MP $\Rightarrow$ DP) It is sufficient to notice that $\frac{\partial J_{t+1}^*(x_{t+1})}{\partial x}(v_t - v) \leq 0, \quad \forall v \in V_t$ implies $J_{t+1}^*(v_t) \leq J_{t+1}^*(v), \quad \forall v \in V_t$.

Remark 2.4 *Notice that in the discrete time maximum principle, we need the assumption of convexity, which is not the case for dynamic programming! Consider for example (a common case), when the input set U_t is only a finite set, then V_t won't be convex and the discrete time maximum principle does not say anything!*

Chapter 3

Optimal Filtering and LQG control

3.1 A short review of stochastic calculus

3.1.1 Motivations

Consider the system $\dot{x} = f(t, x(t))$ with a noise $v : \mathbb{R}_+ \to \mathbb{R}^n$

$$\dot{x} = f(t, x) + v, \quad x \in \mathbb{R}^n, \ t \geq 0$$

We sample the system under sample time Δt, and let $x_k = x(k\Delta t)$ for $k \in \mathbb{N}$.

Then

$$\begin{aligned} x_{k+1} &= x_k + \int_{k\Delta t}^{(k+1)\Delta t} f(s, x(s)) + v(s) ds \\ &= x_k + \int_{k\Delta t}^{(k+1)\Delta t} f(s, x(s)) ds + \int_{k\Delta t}^{(k+1)\Delta t} v(s) ds \end{aligned}$$

For discrete time system, it is custumary to model a system with noise as

$$x_{k+1} = f(k\Delta t, x_k) + n_k \tag{3.1}$$

where n_k is a "white noise" in the sense that $n_k \sim N(0, \sigma^2)$ and that $n_1, \cdots, n_k, \cdots$ are independent. If the above is a sample system of the continuous time system, then the variance of the Gaussian variable n_k should be made to depend on the sample time since if

$$\int_{k\Delta t}^{(k+1)\Delta t} v(s) ds \sim N(0, \sigma^2)$$

then

$$\begin{aligned} \int_{k\Delta t}^{(k+2)\Delta t} v(s) ds &= \int_{k\Delta t}^{(k+1)\Delta t} v(s) ds + \int_{(k+1)\Delta t}^{(k+2)\Delta t} v(s) ds \\ &= n_k + n_{k+1} \sim N(0, 2\sigma^2) \end{aligned}$$

which can be viewed as n_j under sample time $2\Delta t$ for some j. Thus the variance of n_k should be proportional to the square root of the sample time. Hence we may assume $n_k \sim N(0, c\Delta t)$ for some $c > 0$. Now that

$$N(0, c\Delta t) \sim \int_{k\Delta t}^{(k+1)\Delta t} v(s) ds$$

it is reasonable to come up with a function $w : \mathbb{R}_+ \to \mathbb{R}^n$ with $\frac{dw(s)}{ds} = v(s)$ and that

$$\int_{k\Delta t}^{(k+1)\Delta t} v(s)ds = \int_{k\Delta t}^{(k+1)\Delta t} \frac{dw(s)}{ds} ds = w((k+1)\Delta t) - w(k\Delta t)$$
$$:= w_{k+1} - w_k \sim N(0, c\Delta t).$$

Thus a reasonable noise model could be written as

$$\dot{x} = f(t, x(t)) + \frac{dw(t)}{dt}$$

where w should have the following property: $w(t_m) - w(t_{m-1}),\ w(t_{m-1}) - w(t_{m-2}),\ \cdots$ are independent Gaussian variables and that $w(t) - w(s) \sim N(0, c(t-s))$. By doing this, we are in fact constructing a stochastic process: namely, a Brownian motion. It is called a standard Brownian motion when $c = 1$. The above equation is usually written in the following form

$$dx(t) = f(t, x(t))dt + dw(t). \tag{3.2}$$

Suppose now that in (3.1), the variance of n_k is time dependent, namely $n_k \sim N(0, \sigma^2(k\Delta t)\Delta t)$ for some real function σ. Hence

$$\int_{k\Delta t}^{(k+1)\Delta t} v(t, x(t))dt \sim N(0, \sigma(k\Delta t, x(k\Delta t)))\Delta t)$$

which implies

$$\int_{k\Delta t}^{(k+1)\Delta t} v(t, x(t))dt = \sigma(k\Delta t, x(k\Delta t))[w((k+1)\Delta t) - w(k\Delta t)], \tag{3.3}$$

where w is the standard Brownian motion. Therefore it is suggestive to write

$$\int_{k\Delta t}^{(k+1)\Delta t} v(t, x(t))dt =: \int_{k\Delta t}^{(k+1)\Delta t} \sigma(t, x(t))dw(t)$$

when Δt is small. We underscore that the integral on the right hand side is not a Stieltjes integral as the Brownian motion does not have finite variation. Instead, the integral should be exactly understood as the right hand side of (3.3). The above discussions motivate to write down the following equation as an extension of (3.2) with a diffusion coefficient σ:

$$dx(t) = f(t, x(t))dt + \sigma(t, x(t))dw(t). \tag{3.4}$$

We call the equation (3.4) a stochastic differential equation (SDE). The solution to this SDE is written as

$$x(t) = x(s) + \int_s^t f(r, x(r))dr + \int_s^t \sigma(r, x(r))dw(r)$$

and the integral of the last term on the right hand side is understood as (3.3) when $|t - s|$ is small. Now since

$$\int_s^t \sigma(r, x(r))dw(r) = \sum_k \sigma(k\Delta t, x(k\Delta t))(w((k+1)\Delta t) - w(k\Delta t)) \tag{3.5}$$

the integral on the left hand side for arbitrary $s < t$ should be defined as the limit (in certain sense) of the right hand side when $\Delta t \to 0$.

Remark 3.1 *We call the integral defined above Itô integral of σ. It is important to keep in mind that the Itô integral should always be evaluated at the left end points of the partitioned intervals, as in (3.3). One would obtain a totally different integral if one evaluate at the right end points, which is a clear difference between Rieman-Stietjes integral.*

The rigorous constructions of Brownian motion and Itô integral are quite technical and out of the scope of this note. We refer the reader to the excellent text [2]. In the next part, we review some important notions from stochastic calculus, especially the Itô formula and the notion of Markov process.

3.1.2 Martingale

Throughout this subsection, we consider a probability space $(\Omega, \mathcal{F}, P)$ with Ω the sample space, $\mathcal{F}$ the sigmal algebra and P the probability measure.

Definition 3.1 *A filtration on $(\Omega, \mathcal{F}, P)$ is a collection $(\mathcal{F}_t)_{0\le t\le\infty}$ indexed by $[0, +\infty]$ of sub-σ-algebras of $\mathcal{F}$, such that for every $0 \le s \le t$*

$$\mathcal{F}_0 \subset \mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}_\infty \subset \mathcal{F}$$

Definition 3.2 *A stochastic process $(X_t)_{t\ge0}$ with values in a measurable space $(E, \mathcal{E})$ ($\mathcal{E}$ is the σ-algebra on E) is called adapted (to $(\mathcal{F}_t)_{0\le t\le\infty}$) if for every $t \ge 0$, X_t is $\mathcal{F}_t$-measurable. This process is progressive if, for every $t \ge 0$, the mapping*

$$(\omega, s) \mapsto X_s(\omega)$$

defined on $\Omega \times [0, t]$ is measurable w.r.t. the σ-algebra $\mathcal{F}_t \otimes \mathcal{B}([0, t])$. ($\mathcal{B}([0, t])$ is the Borel algebra on $[0, t]$)

Another important notion is stopping time.

Definition 3.3 *A r.v. $T : \Omega \to [0, \infty]$ is a stopping time of the filtration $(\mathcal{F})_t$ if $\{T \le t\} \in \mathcal{F}_t$ for every $t \ge 0$. The σ-algebra of the past before T is then defined by*

$$\mathcal{F}_T = \{A \in \mathcal{F}_\infty : \forall t \ge 0,\ A \cap \{T \le t\} \in \mathcal{F}_t\}.$$

As usual, for a r.v. X, we say that $X \in L^p$ if $E|X|^p < \infty$. Given a process $(X_t)_{t\ge0}$ adapted to $(\mathcal{F}_t)_{t\ge0}$, we adopt the notation $E_s[X_t]$ to mean $E[X_t|\mathcal{F}_s]$. Next we introduce one of the most important notions in stochastic calculus: martingale.

Definition 3.4 *An adapted real-valued process $(X_t)_{t\ge0}$ such that $X_t \in L^1$ for every $t \ge 0$ is called*

1. *a martingale if, for every $0 \le s < t$, $E_s[X_t] = X_s$; (implies $EX_t = EX_s$])*
2. *a supermartingale if, for every $0 \le s < t$, $E_s[X_t] \le X_s$; (implies $EX_t \le EX_s$)*
3. *a submartingale if, for every $0 \le s < t$, $E_s[X_t] \ge X_s$ (implies $EX_t \ge EX_s$)*

Definition 3.5 *A real-valued process $B = (B_t)_{t\ge0}$ is a Brownian motion started from 0 if*

1. *$B_0 = 0$ almost surely (a.s.);*
2. *for every $0 \le s < t$, the r.v. $B_t - B_s$ is independent of $\sigma(B_r,\ r \le s)$ and distributed according to $N(0, t - s)$;*
3. *all sample paths $(t \mapsto B_t(\omega))$ of B are continuous;*

if additionally B is adapted to $(\mathcal{F}_t)_{t\ge0}$, we say that B is an $(\mathcal{F}_t)$-Browninan motion. Similarly, a process $B = (B_t)_{t\ge0}$ with values in $\mathbb{R}^d$ is a d-dimensional (F_t)-Brownian motion if its components are independent Brownian motion and B is adapted to $(\mathcal{F}_t)$ and has independent increments with respect to $(\mathcal{F}_t)$.

Obviously, a Brownian motion is a martingale. But one can construct many more martingales using Brownian motion, among which the most important one is the stochastic integral that we will construct later. For the moment, one can easily verify that both $B_t^2 - t$ and $e^{\theta B_t - \frac{\theta^2}{2}t}$ for any $\theta \in \mathbb{R}$ are martingales.

Given a stochastic process (X_t), there is an obvious way of constructing a filtration such that the process is adapted: $\mathcal{F}_t = \sigma(X_s;\ s \le t)$. Hence, when not specified, one may always assume that a process is adapted to the filtration constructed such. Due to this reason, in the rest of this note, a process is always assume to be adpated.

Proposition 3.1 *Consider a real process* $(X_t)_{t\geq 0}$ *and a convex function* $f:\mathbb{R}\to\mathbb{R}_+$ *such that* $E[f(X_t)]<\infty$ *for every* $t\geq 0$.

1. *If* (X_t) *is a martingale, then* $(f(X_t))$ *is a submartingale;*
2. *If* (X_t) *is a submartingale, and if* f *is nondecreasing, then* $f(X_t)$ *is a submartingale.*

3.1.3 Stochastic integration

As we know from integration theory, to define abstract integration, one starts with some kind of simple functions. Then since the integration is a linear operator, there is a unique extension of this operator to the closure (under certain topology) of simple functions. The stochastic integration is also defined in this way. But what kind of "simple functions" should we start with? More generally, the stochastic integration should be defined for what kind of functions?

To get some intuiation, we go back to the formula

$$x(t)=x(s)+\int_s^t f(r,x(r))dr+\int_s^t \sigma(r,x(r))dw(r).$$

The last term on the right hand side suggests that the stochastic integration should preserve certain properties of stochastic process. For example, take $\sigma(r,x)=x$, $f=0$, and $x(0)=0$, then $x(t)=\int_0^t x(r)dw(r)$. Then if $x(t)$ is a martingale, $\int_s^t x(r)dw(r)$ should also be a martingale.

The goal of this subsection is to define stochastic integration for a rather general class of functions – semimartingales.

Consider an "elementary process"

$$H_s(\omega)=\sum_{i=0}^{p-1} H_i(\omega)1_{(t_i,t_{i+1}]}(s) \tag{3.6}$$

where $0=t_0<t_1<\cdots<t_p$ and for each $i\in\{0,\cdots,p-1\}$, H_i is bounded and $\mathcal{F}_{t_i}$-measurable. Obviously, H is a progressive process (Definition 3.2). Then invoking the formula (3.5), the integration of H w.r.t. a process $M=(M_t)_{t\geq 0}$ should be defined as

$$\left(\int HdM\right)_t =: \sum_{i=0}^{p-1} H_i(M_{t_{i+1}\wedge t}-M_{t_i\wedge t}). \tag{3.7}$$

Easy calculations show that $\int HdM$ so defined is a martingale (since $H_i(M_{t_{i+1}\wedge t}-M_{t_i\wedge t})$ is for each i).

It remains to extend the "elementary processes" to some closed set under certain norm. Some preparations are needed.

Definition 3.6 *An adapted continuous process* $A=(A_t)_{t\geq 0}$ *is called a finite variation process if all its sample paths are finite variation functions*[1] *on* $\mathbb{R}_+$*. If in addition the sample paths are nondecreasing functions, the process* A *is called an increasing process.*

Given a process $M=(M_t)_{t\geq 0}$ and a stopping time T, define the stopped process at T as

$$M_t^T=M_{t\wedge T}$$

more precisely, letting $X=:M^T$, then $X_t(\omega)=M_{t\wedge T(\omega)}(\omega)$.

[1] We say that a right continuous function $a:[0,T]\to\mathbb{R}$ with $a(0)=0$ has finite variation if there exists a signed measure μ on $[0,T]$ such that $a(t)=\mu([0,t])$ for every $t\in[0,T]$.

Definition 3.7 *A continuous adpated process $M = (M_t)_{t\geq 0}$ with $M_0 = 0$ a.s. is called a continuous local martingale if there exists a nondecreasing sequence $(T_n)_{n\geq 0}$ of stopping times such that $T_n \uparrow \infty$ and for every n, the stopped process (M^{T_n}) is a uniformly integrable martingale. When $M_0 \neq 0$, M is called a continuous local martingale if $M - M_0$ is such. In both cases, we say that the sequence of stopping times (T_n) reduces M.*

Definition 3.8 *A process $X = (X_t)_{t\geq 0}$ is a continuous semimartingale if it can be written in the form*

$$X_t = M_t + A_t$$

where M is a continuous local martingale and A a finite variation process.

The next lemma indicates that the decomposition above is unique up to indistinguishability.

Lemma 3.1 *Let M be a CLM. Assume that M is also a FVP with $M_0 = 0$. Then $M_t = 0$ for every $t \geq 0$ a.s.*

Now we go back to define a norm for the "elementary processes", a crucial task toward to definition of stochastic integration.

Theorem 3.1 *Let $M = (M_t)_{\geq 0}$ be a continuous local martingale. There exists an increasing process denoted by $(\langle M, M\rangle_t)_{t\geq 0}$, which is unique up to indistinguishability, such that $M_t^2 - \langle M, M\rangle_t$ is a continuous local martingale. Furthermore, for every fixed $t > 0$, if $0 = t_0^n < t_1^n < \cdots < t_{p_n}^n = t$ is an increasing sequence of subdivisions of $[0, t]$ with mesh tending to 0, we have*

$$\langle M, M\rangle_t = \lim_{n\to\infty} \sum_{i=1}^{p_n} (M_{t_i^n} - M_{t_{i-1}^n})^2 \tag{3.8}$$

in probability. The process $\langle M, M\rangle$ is called the quadratic variation of M.

We can make the following observations.

- It can be easily checked that for a standard Brownian motion B, we have $\langle B, B\rangle_t = t$.
- The quadratic variation of a process does not depend on the initial value M_0 by (3.8). In fact, if $M_t = M_0 + N_t$, then $\langle M, M\rangle = \langle N, N\rangle$.
- In the formula (3.8), if M is a finite variation process, then

$$\begin{aligned}\sum_{i=1}^{p_n} (M_{t_i^n} - M_{t_{i-1}^n})^2 &\leq \left(\sup_{1\leq i\leq p_n} |M_{t_i^n} - M_{t_{i-1}^n}|\right) \sum_{i=1}^{p_n} |M_{t_i^n} - M_{t_{i-1}^n}| \\ &\leq \left(\sup_{1\leq i\leq p_n} |M_{t_i^n} - M_{t_{i-1}^n}|\right) \left(\int_0^t |dM_s|\right) \to 0\end{aligned}$$

  in probability as $n \to \infty$. Hence we can define quadratic variation for finite variation process. But can we define it for semimartingales?

  That is, if $X = M + A$, with M a local continuous martingale and A a finite variation process. Then to define $\langle X, X\rangle = \langle M + A, M + A\rangle$, we shall define $\langle M, A\rangle$ (the impose linearlity on the bracket is "natural") i.e., the "bracket" between a local martingale and a finite variation process. But this can be simply defined as

$$\langle M, A\rangle = \lim_{n\to\infty} \sum_{i=1}^{p_n} (M_{t_i^n} - M_{t_{i-1}^n})(A_{t_i^n} - A_{t_{i-1}^n}).$$

But
$$\left|\sum_{i=1}^{p_n}(M_{t_i^n}-M_{t_{i-1}^n})(A_{t_i^n}-A_{t_{i-1}^n})\right| \le \left(\int_0^t |dA_s|\right)\sup_{1\le i\le p_n}|M_{t_i^n}-M_{t_{i-1}^n}|\to 0$$
in probability as $n\to\infty$.

To go one step further, this motivates us to define the bracket between two local continuous martingale as
$$\langle M,N\rangle = \lim_{n\to\infty}\sum_{i=1}^{p_n}(M_{t_i^n}-M_{t_{i-1}^n})(N_{t_i^n}-N_{t_{i-1}^n})$$
with mesh tending to 0. The above discussions show that the finite variation parts of M and N do not contribute to the bracket, i.e., if $M=X+A$, $N=X'+A'$, with X,X' CLM and A,A' FVP. Then $\langle M,N\rangle=\langle X,X'\rangle$.

Theorem 3.2 *Given two CLMs M,N. Then*

1. *$\langle M,N\rangle$ is the unique (up to indistinguishability) FVP such that $M_tN_t-\langle M,N\rangle_t$ is a CLM.*
2. *The mapping $(M,N)\mapsto\langle M,N\rangle$ is bilinear and symmetric.*
3. *For every stopping time T, $\langle M^T,N^T\rangle_t=\langle M^T,N\rangle_t=\langle M,N\rangle_{t\wedge T}$.*
4. *If M and N are two continuous martingales bounded in L^2, $M_tN_t-\langle M,N\rangle_t$ is a uniformly integrable martingale. Consequently, $\langle M,N\rangle_\infty$ is well defined as the almost sure limit of $\langle M,N\rangle_t$ as $t\to\infty$ is integrable, and satisfies*
$$E[M_\infty N_\infty]=E[M_0N_0]+E[\langle M,N\rangle_\infty].$$

Consider the space of all CLM bounded in L^2 with 0 as initial distribution, which we denote as $\mathbb{H}$. Define an inner product on $\mathbb{H}$ as
$$(M,N)_{\mathbb{H}}=E[M_\infty N_\infty]=E[\langle M,N\rangle_\infty]$$
then one can show that $\mathbb{H}$ is a Hilbert space under this inner product. Now fix a CLM M, define an inner product on the space of progressive processs as
$$(H,K)_{L^2(M)}=E\left[\int_0^\infty H_sK_s d\langle M,M\rangle_s\right] \tag{3.9}$$
where
$$L^2(M)=\left\{H \text{ is progressive and } (H,H)_{L^2(M)}<\infty\right\}.$$
As usual, $L^2(M)$ is a Hilbert space. Note that in (3.9), because $t\mapsto\langle M,M\rangle_t$ is a continuous increasing function, the integral inside the expectation is Stieltjes integral and hence well-defined. Thus, we have constructed two Hilbert spaces, namely, $L^2(M)$ and $\mathbb{H}$. Recall that the RHS of (3.7) is a martingale. Further more, it is bounded in $\mathbb{H}$, more precisely
$$\begin{aligned}\left(\int_0^\cdot HdM,\int_0^\cdot HdM\right)_{\mathbb{H}} &= \left(\sum_{i=0}^{p-1}H_i(M_{t_{i+1}\wedge\cdot}-M_{t_i\wedge\cdot}),\sum_{i=0}^{p-1}H_i(M_{t_{i+1}\wedge\cdot}-M_{t_i\wedge\cdot})\right)_{\mathbb{H}}\\ &= E\left[\left\langle\sum_{i=0}^{p-1}H_i(M_{t_{i+1}}-M_{t_i}),\sum_{i=0}^{p-1}H_i(M_{t_{i+1}}-M_{t_i})\right\rangle\right]\\ &= E\left[\sum_{i=0}^{p-1}H_i^2(\langle M,M\rangle_{t_{i+1}}-\langle M,M\rangle_{t_i})\right]\\ &= \int_0^\infty H_s^2 d\langle M,M\rangle_s\\ &= (H,H)_{L^2(M)}\end{aligned}$$

Thus the linear mapping

$$H \mapsto \int_0^{\cdot} HdM$$

is an isometry (hence continuous) from the set of elementary processes $\subset L^2(M)$ into $\mathbb{H}$. Then one can extend the integral to $L^2(M)$ in a unique way if elementary processes are dense in $L^2(M)$, which is indeed the case. Thus for any $H \in L^2(M)$, the integral $\int HdM$ is defined as the limit of $\int H_n dM$ where H is the limit of elementary processes in $L^2(M)$. (Note that since $\mathbb{H}$ is Hilbert (complete), the limit is still a martingale!) For convenience, $\int_0^{\cdot} HdM$ is also written as $H \cdot M$.

The following are some properties of the stochastic integral:

- Let $H \in L^2(M)$, $M, N \in \mathbb{H}$. Then

$$\langle H \cdot M, N \rangle = H \cdot \langle M, N \rangle$$

  and $H \cdot M$ is the unique element in $\mathbb{H}$ such that the above holds for all $N \in \mathbb{H}$. From this formula, we can deduce that

$$\begin{aligned} \langle H \cdot M, H \cdot M \rangle &= H \cdot \langle M, H \cdot M \rangle \\ &= H \cdot \left(\int_0^{\cdot} H_s d \langle M, M \rangle_s \right) \\ &= H^2 \cdot \langle M, M \rangle \end{aligned} \tag{3.10}$$

  where the equality (3.10) is justified first for elementary processes and then one extend it to $L^2(M)$. Written out explicitly ,the above relation reads

$$\left\langle \int_0^{\cdot} HdM, \int_0^{\cdot} HdM \right\rangle_t = \int_0^t H_s^2 d \langle M, M \rangle_s .$$

  More generally, for $K \in L^2(N)$, we have

$$\langle H \cdot M, K \cdot N \rangle = HK \cdot \langle M, N \rangle .$$

- Let $M, N \in \mathbb{H}$, and $H \in L^2(M)$, $K \in L^2(N)$. Then since $H \cdot M$ and $K \cdot N$ are martingales in $\mathbb{H}$, we have for every $t \in [0, \infty]$,

$$\begin{aligned} E\left[\int_0^t H_s dM_s\right] &= 0, \\ E_s\left[\int_0^t H_r dM_r\right] &= \int_0^s H_r dM_r, \quad \forall 0 \le s \le t \\ E_s\left[\int_s^t H_r dM_r\right] &= 0 \end{aligned}$$

- More over

$$E\left[(H \cdot M)_t (K \cdot N)_t\right] = E[((HK) \cdot \langle M, N \rangle)_t]$$

  or

$$E\left[\left(\int_0^t H_s dM_s\right)\left(\int_0^t K_s dN_s\right)\right] = E\left[\int_0^t H_s K_s d \langle M, N \rangle_s\right].$$

  In particular

$$E\left[\left(\int_0^t H_s dM_s\right)^2\right] = E\left[\int_0^t H_s^2 d \langle M, M \rangle_s\right]$$

In the above, we have defined stochastic integral for martingales bounded in L^2, i.e. $\mathbb{H}$. Now we generalize the stochastic integral to CLMs.

Given a CLM M, define

$$L^2_{\text{loc}}(M) = \left\{ H : \int_0^t H_s^2 d\langle M, M\rangle_s < \infty, \quad \forall t \geq 0 \right\} \quad \text{a.s.}$$
$$L^2(M) = \left\{ H : \int_0^\infty H_s^2 d\langle M, M\rangle_s < \infty \right\}$$

(Since $\langle M, M\rangle$ is FVP, both spaces are well defined). We point out that $L^2(M)$ is still a Hilbert space.

Theorem 3.3 *Let M be a CLM. For every $H \in L^2_{loc}(M)$, there exists a unique CLM with initial value 0, which is denoted by $H \cdot M$, such that, for every CLM N,*

$$\langle H \cdot M, N\rangle = H \cdot \langle M, N\rangle .$$

If $H \in L^2_{loc}(M)$ and K is a progressive process, we have $K \in L^2_{loc}(H \cdot M)$ if and only if $HK \in L^2_{loc}(M)$ and then

$$H \cdot (K \cdot M) = HK \cdot M.$$

We write

$$(H \cdot M)_t = \int_0^t H_s dM_s$$

and call it the stochastic integral of H w.r.t. M.

Now that a semimartingale X can be decomposted as the sum of a CLM and a FVP, namely, $X = M + V$. Then for any locally bounded progressive process H, one can define

$$H \cdot X := H \cdot M + \int H_s dV_s$$

where $\int H_s dV_s$ is the usual Stieltjes integral.

As before, this integral has the following properties:

1. Let X be a continuous semimartingale, and K, H two locally bounded progressive processes. Then $KH \cdot X = K \cdot (H \cdot X)$.
2. Let H be a locally bounded progressive process. If X is a CLM or FVP, the same holds for $H \cdot X$.

3.1.4 Itô's formula

Itô's formula will be our most useful tool in this text. Even if one does not know the rigorous construction of stochastic integral, Itô's formula will be sufficient for the study of stochastic optimal control.

Theorem 3.4 *Let $X^1, \cdots, X^p$ be p continuous semimartingales, and let F be a twice continuously differentiable real function on $\mathbb{R}^p$. Then for every $t \geq 0$,*

$$\begin{aligned} F(X_t^1, \cdots, X_t^p) &= F(X_0^1, \cdots, X_0^p) \\ &\quad + \sum_{i=1}^p \int_0^t \frac{\partial F}{\partial x^i}(X_s^1, \cdots, X_s^p) dX_s^i \\ &\quad + \frac{1}{2} \sum_{i,j=1}^p \int_0^t \frac{\partial^2 F}{\partial x^i \partial x^j}(X_s^1, \cdots, X_s^p) d\left\langle X^i, X^j \right\rangle_s . \end{aligned}$$

We mention a few consequences of Itô's formula.

1. $F(X_t^1, \cdots, X_t^p)$ is a semimartingale. This is what we had expected in the beginning of the last subsubsection!

2. Let $F(x,y) = xy$. Then we see that

$$\begin{aligned} X_t Y_t &= X_0 Y_0 + \int_0^t X_s dY_s + \int_0^t Y_s dX_s + \int_0^t d\langle X, Y\rangle_s \\ &= X_0 Y_0 + \int_0^t X_s dY_s + \int_0^t Y_s dX_s + \langle X, Y\rangle_t \end{aligned}$$

   This formula can be viewed as the *formula of integration by parts*. In particular, if $Y = X$,

$$X_t^2 = X_0^2 + 2\int_0^t X_s dX_s + \langle X, X\rangle_t .$$

   We know that when X is a CLM, then $\langle X, X\rangle$ is the unique FVP such that $X^2 - \langle X, X\rangle$ is a CLM. The above formula tells us that

$$\langle X, X\rangle_t = X_t^2 - X_0^2 - 2\int_0^t X_s dX_s.$$

3. Let $X_t^1 = t$, $X_t^2 = B_t$ (standard Brownian motion), and $F \in C^2(\mathbb{R}_+ \times \mathbb{R})$. Then

$$F(t, B_t) = F(0, B_0) + \int_0^t \frac{\partial F}{\partial x}(s, B_s) dB_s + \int_0^t \left(\frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial x^2} \right)(s, B_s) ds.$$

3.1.5 Theory of Markov process

Let $(E, \mathcal{E})$ be a measurable space. A *Markovian transition kernel* from E into E is a mapping $Q : E \times \mathcal{E} \to [0,1]$ satisfying the following properties:

1. For every $x \in E$, the mapping $\mathcal{E} \ni A \mapsto Q(x, A)$ is a probability measure on $(E, \mathcal{E})$.

2. For every $A \in \mathcal{E}$, the mapping $E \ni x \mapsto Q(x, A)$ is $\mathcal{E}$-measurable.

Given a transition kernel Q, if $f : E \to \mathbb{R}$ is bounded measurable, we define the function $Qf : E \to \mathbb{R}$ by

$$Qf(x) = \int_E Q(x, dy) f(y) \tag{3.11}$$

which is still bounded measurable.

Definition 3.9 *A collection $(Q_{s,t})_{0 \le s \le t}$ of transition kernels on E is called a transition semigroup if the following properties hold.*

1. *For every $x \in E$ and $t \in \mathbb{R}$, $Q_{t,t}(x, dy) = \delta_x(dy)$.*

2. *For all $0 \le s \le r \le t$ and $A \in \mathcal{E}$,*

$$Q_{s,t}(x, A) = \int_E Q_{s,r}(x, dy) Q_{r,t}(y, A) \tag{3.12}$$

   (Chapman-Kolmogorov identity).

3. *For every $A \in \mathcal{E}$, the function $(s,t,x) \mapsto Q_{s,t}(x,A)$ is measurable w.r.t. the σ-algebra $\mathcal{B}(\mathbb{R}_+) \times \mathcal{B}(\mathbb{R}_+) \times \mathcal{E}$.*

When $Q_{s,t} = Q_{s+r,t+r}$ for all $r \in \mathbb{R}$, we say that the transition semigroup is time independent and we simply write $Q_{t-s} := Q_{s,t}$. Now given $f \in B(E)$, $0 \le s \le r \le t$, by Chapman-Kolmogorov identity, we have

$$\begin{aligned}
Q_{s,r}Q_{r,t}f(x) &= \int_E Q_{s,r}(x,dy)Q_{r,t}f(y) \\
&= \int_E Q_{s,r}(x,dy)\int_E Q_{r,t}(y,dw)f(w) \\
&= \int_E f(w)\int_E Q_{s,r}(x,dy)Q_{r,t}(y,dw) \\
&= \int_E f(w)Q_{s,t}(x,dw) \\
&= Q_{s,t}f(x).
\end{aligned}$$

Hence we get the identity

$$Q_{s,r}Q_{r,t} = Q_{s,t}, \quad \forall 0 \le s \le r \le t$$

which is equivalent to the Chapman-Kolmogorov identity when $Q_{s,t}$ is understood as operators from $B(E)$ to $B(E)$. Since $A \mapsto Q_{s,t}(x,A)$ is a probability measure, it is easily seen from (3.11) that $Q_{s,t} : B(E) \to B(E)$ is non-expansive (i.e., $||Q_{s,t}|| \le 1$) when $B(E)$ is equipped with norm $||f|| = \sup\{|f(x)| : x \in E\}$.

Now we are ready to define Markov process.

Definition 3.10 *A process $(X_t)_{t\ge 0}$ with values in E is called a Markov process with transition semigroup $(Q_{s,t})_{0\le s\le t}$ if*

$$E_s[f(X_t)] = Q_{s,t}f(X_s), \quad \forall s,t \ge 0 \tag{3.13}$$

for each $f \in B(E)$.

When the transition semigroup is time independent, the Markov property (3.13) becomes

$$E_s[f(X_{s+t})] = Q_t f(X_s), \quad \forall s,t \ge 0.$$

Now take $f = 1_A$, with $A \in \mathcal{E}$. Then (3.13) implies

$$P(X_{t+s} \in A|\mathcal{F}_s) = Q_{s,s+t}1_A(X_s) = Q_{s,s+t}(X_s, A)$$

from which we deduce that

$$P(X_t \in A|X_{t_1}, \cdots, X_{t_m}) = P(X_t \in A|X_{t_m})$$

whenever $t_1 \le \cdots \le t_m \le t$. In other words, the conditional distribution of X_{s+t} knowing the past $(X_r,\ 0 \le r \le s)$ before time s depends only on the present state X_s. In particular, when $X_s = x$, we get

$$Q_{s,t}(x,A) = P(X_t \in A|X_s = x)$$

Let $C_0(E)$ be the set of continuous real functions on E that vanish at infinity. It is common knowledge that $C_0(E)$ is a Banach space for the norm $||f|| = \sup\{|f(x)| : x \in E\}$.

Definition 3.11 *Let $(Q_{s,t})$ be a transition semigroup on E. We say that it is a Feller semigroup if*

1. *$\forall f \in C_0(E)$, $Q_{s,t}f \in C_0(E)$ for all $0 \le s \le t$.*

2. $\forall f \in C_0(E),\ ||Q_{s,s+h}f - f|| \to 0$ *as* $h \to 0$.

Define the operators $A(t)$ by

$$A(t)f = \lim_{h\to 0+} \frac{Q_{t,t+h}f - f}{h}$$

where the limit is taken in $C_0(E)$ and the domain of $A(t)$ is such that the above limit exists, i.e.,

$$D(A(t)) = \left\{ f \in C_0(E) : \frac{Q_{t,t+h}f - f}{h} \text{ converges in } C_0(E) \text{ when } h \to 0+ \right\}.$$

3.1.6 Stochastic differential equation

Let d and m be positive integers, and let σ and b be locally bounded measurable functions defined on $\mathbb{R}_+ \times \mathbb{R}^d$ and taking values in $\mathbb{R}^{d\times m}$ and in $\mathbb{R}^d$ respectively. We write $\sigma = (\sigma_{ij})_{1\le i\le d, 1\le j\le m}$ and $b = (b_i)_{1\le i\le d}$.

A *solution of the stochastic differential equation*

$$\begin{aligned} &dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t \\ &X_0 \text{ is } \mathcal{F}_0\text{-measurable} \end{aligned} \tag{3.14}$$

consists of

1. a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,\infty]}, P)$ (where the filtration is always assumed to be complete);
2. an m-dimensional $(\mathcal{F}_t)$-Brownian motion $B = (B^1, \cdots, B^m)$ started from 0;

an $(\mathcal{F}_t)$-adapted process $X = (X^1, \cdots, X^d)$ with values in $\mathbb{R}^d$, with continuous sample paths, such that

$$X_t = X_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dB_s.$$

The solution is called a *strong solution* if $(\mathcal{F}_t)_{t\in[0,\infty]}$ is specified *a priori*. Otherwise it is called a *weak solution*, i.e., the filtration is part of the solution. In this note, we are mainly interested in strong solution. If for any two strong solutions X, Y we have

$$P(X(t) = Y(t),\ 0 \le t < \infty) = 1,$$

we say that the solution is unique.

Theorem 3.5 *If there exists a constant* $K > 0$ *such that for every* $t \ge 0$*,* $x, y \in \mathbb{R}^d$*,*

$$|b(t,x) - b(t,y)| + |\sigma(t,x) - \sigma(t,y)| \le K|x - y|$$

then (3.14) has a unique strong solution.

We show that the solution of SDE is a Markov process (Definition 3.10).

To that end, define

$$Q_{s,t}(x, A) := P(X(t; s, x) \in A)$$

we show that

$$Q_{s,t}f(X_s) = E_s[f(X_t)] \tag{3.15}$$

(note that this would imply $Q_{s,t}f(x) = E[f(X(t; s, x))]$).

In fact,

$$\begin{aligned}
Q_{s,t}1_A(X_s) &= Q_{s,t}(X_s, A) \\
&= P(X(t; s, x) \in A)|_{x=X_s} \\
&\quad (X_s \text{ is a r.v. so must be put outside } P(\cdot)!) \\
&= E_s 1_A(X(t; s, X_s)) \\
&\quad (X_s \text{ is } \mathcal{F}_s\text{-measurable}) \\
&= E_s 1_A(X_t). \\
&\quad (\text{uniqueness of the solution enforces that } X(t; s, X_s) = X_t \text{ for } t \geq s)
\end{aligned}$$

A standard argument using monotone class lemma will finalize the proof of the formula (3.15). It remains to show that $Q_{s,t}$ is a transition group, i.e.,

$$Q_{s,r}Q_{r,t} = Q_{s,t}. \tag{3.16}$$

But

$$\begin{aligned}
Q_{s,t}f(x) &= E[f(X(t; s, x)) \\
&\quad (\text{see the remark after (3.15)}) \\
&= E[E_r[f(X(t; s, x))]] \\
&= E[Q_{r,t}f(X_r)] \\
&\quad (\text{again, by (3.15)}) \\
&= \int Q_{r,t}f(y)P(X_r \in dy) \\
&= \int Q_{r,t}f(y)Q_{s,r}(x, dy) \\
&\quad (\text{since } P(X_r \in A) = P(X(r; s, x) \in A) = Q_{s,r}(x, A)) \\
&= \int Q_{s,r}(x, dy)Q_{r,t}f(y)
\end{aligned}$$

which indeed verifies (3.16). For time dependent function, evidently we should define $Q_{s,t}f(s, t) := E[f(t, X(t; s, x))]$.

In the literature, it is common to denote

$$P(s, x; t, A) := Q_{s,t}(x, A) = P(X_t \in A | X_s = x)$$

and the property (3.16) can now be expressed as

$$P(s, x; t, A) = \int P(s, x; r, dy)P(r, y; t, A).$$

Our next task is to find the generator of the Markov process (transition group) $(Q_{s,t})_{0 \leq s \leq t}$. Since

$$X_t^x = x + \int_0^t b(r, X_r^x)dr + \int_0^t \sigma(r, X_r^x)dB_r.$$

we find the quadratic variation (when X is of one-dimension)

$$\begin{aligned}
\langle X^x, X^x \rangle_t &= \left\langle \int_0^\cdot \sigma(r, X_r^x)dB_r, \int_0^\cdot \sigma(r, X_r^x)dB_r \right\rangle_t \\
&= \int_0^t \sigma(r, X_r^x)^2 dr.
\end{aligned}$$

More generally, we have $d\langle X^x, X^x\rangle_t = \sigma(t, X_t^x)\sigma^T(t, X_t^x)dt =: (a_{ij})dt$.

Now given a function $\varphi \in C^{1,2}$ (C^1 in w.r.t. to the first variable and C^2 w.r.t the second), by Itô's formula

$$\begin{aligned}\varphi(t, X_t) &= \varphi(s, X_s) + \int_s^t \frac{\partial\varphi}{\partial t}(r, X_r)dr + \int_s^t \frac{\partial\varphi}{\partial x}(r, X_r)dX_r + \frac{1}{2}\sum_{i,j}\int_s^t \frac{\partial^2\varphi}{\partial x^2}(r, X_r)a_{ij}(r, X_r)dr \\ &= X_s + \int_s^t \frac{\partial\varphi}{\partial t}(r, X_r)dr + \int_s^t \frac{\partial\varphi}{\partial x}(r, X_r)b(r, X_r)dr + \int_s^t \frac{\partial\varphi}{\partial x}(r, X_r)\sigma(r, X_r)dB_r \\ &+ \frac{1}{2}\int_s^t \operatorname{tr}\left(\frac{\partial^2\varphi}{\partial x^2}(r, X_r)\sigma(r, X_r)\sigma^T(r, X_r)\right)dr\end{aligned}$$

Then

$$E_s[\varphi(t, X_t) - \varphi(s, X_s)] = E_s\left[\int_s^t \frac{\partial\varphi}{\partial t}(r, X_r) + A(r)\left(\varphi(r, X_r)\right)dr\right]$$

in which

$$\begin{aligned}A(r)(\varphi(r, x)) &:= \frac{1}{2}\operatorname{tr}\left(\sigma\sigma^T\Delta\varphi\right) + (\nabla\varphi)b(t, x) \\ &= \frac{1}{2}\sum a_{ij}(t, x)\frac{\partial^2\varphi}{\partial x_i\partial x_j}(t, x) + \sum b_i(t, x)\frac{\partial\varphi}{\partial x_i}.\end{aligned} \tag{3.17}$$

Therefore, by fixing $X_s = x$, we obtain

$$\begin{aligned}\frac{Q_{s,s+h}\varphi(s, x) - \varphi(s, x)}{h} &= \frac{E[\varphi(s+h, X(s+h; s, x))] - \varphi(s, x)}{h} \\ &= \frac{1}{h}E\left[\int_s^{s+h} \frac{\partial\varphi}{\partial t}(r, X_r) + A(r)\left(\varphi(r, X_r)\right)dr\right] \\ &\to \varphi_s(s, x) + A(s)\left(\varphi(s, x)\right) \quad \text{as } h \to 0+\end{aligned}$$

Hence the generator of $Q_{s,t}$ is $\varphi_s + A(s)\varphi$ where A is defined as (3.17). When considering only time independent functions φ, then $A(s)$ alone is the generator since $\varphi_s = 0$ for all $s \geq 0$.

3.1.7 Girsanov theorem

$\mathcal{M}_{\text{loc}}^c$: continuous local martingale

$\mathcal{M}_{\text{loc}}^{2,c}$: continuous local martingale s.t. $\sup_{0\le s\le t} E|X_s|^2 < \infty, \forall t \in \mathbb{R}_+$

For $X \in \mathcal{M}_{\text{loc}}^{2,c}$ with $X_0 = 0$, define

$$\mathcal{E}(X)_t =: \exp\left(X_t - \frac{1}{2}\langle X, X\rangle_t\right) \tag{3.18}$$

where $\langle X, X\rangle$ is the quadratic variation.

Lemma 3.2 $\mathcal{E}(X) \in \mathcal{M}_{loc}^c$.

Proof 4 *By Ito formula,*

$$\begin{aligned}\mathcal{E}(X)_t &= 1 + \int_0^t \mathcal{E}(X)_s(dX_s - \frac{1}{2}d\langle X, X\rangle_s) \\ &+ \frac{1}{2}\int_0^t \mathcal{E}(X)_s d\left\langle X_s - \frac{1}{2}\langle X, X\rangle_s, X_s - \frac{1}{2}\langle X, X\rangle_s\right\rangle\end{aligned}$$

but

$$\left\langle X_s - \frac{1}{2}\langle X, X\rangle_s, X_s - \frac{1}{2}\langle X, X\rangle_s \right\rangle = \langle X, X\rangle_s$$

therefore

$$\mathcal{E}(X)_t = 1 + \int_0^t \mathcal{E}(X)_s dX_s$$

which is a local continuous martingale.

Theorem 3.6 *Let $X \in \mathcal{M}^c_{loc}$ with $X_0 = 0$. Consider the following properties:*

1. *$E[\exp \frac{1}{2}\langle X, X\rangle_\infty] < \infty$ (Novikov's condition);*
2. *X is a uniformly integrable martingale, and $E[\exp \frac{1}{2} L_\infty] < \infty$ (Kazamaki's condition);*
3. *$\mathcal{E}(X)$ is a uniformly integrable martingale.*

Remark 3.2 *When we consider local martingales on finte interval, say on $[0, T]$, the conditions in the above theorem changes accordingly, e.g., the Novikov condition becomes $E[\exp \frac{1}{2}\langle X, X\rangle_T] < \infty$.*

Theorem 3.7 (Girsanov) *Let $(X_t)_{t\in[0,T]}$ be a continous local martingale, and assume that $\mathcal{E}(X)_t$ is a martingale on $[0, T]$. Define a process*

$$D_t =: \mathcal{E}(X)_t = E[\mathcal{E}(X)_T|\mathcal{F}_t]$$

then (D_t) is a uniformly integrable martingale. Further, define a probability measure Q by

$$\frac{dQ}{dP} = D_T$$

Then for any martingale Y on $[0, T]$, the process $\tilde{Y}_t = Y_t - \langle X, Y\rangle_t$ is a martingale under Q on $[0, T]$.

Example 6 *Let $Y = W$ be a Brownian motion, and*

$$X_t = \int_0^t \beta_s dW_s$$

then

$$\tilde{W}_t = W_t - \int_0^t \beta_s ds$$

is a martingale under $dQ = z_T dP$ where

$$z_T = \mathcal{E}\left(\int_0^{\cdot} \beta_s dW_s\right)_T = \exp\left(\int_0^T \beta_s dW_s - \frac{1}{2}\int_0^T |\beta_s|^2 ds\right).$$

Clearly, $\mathcal{E}(X)$ is a martingale if $E \exp \frac{1}{2}\langle X, X\rangle_T = E \exp \frac{1}{2}\int_0^T |\beta_s|^2 ds < \infty$. In fact, we can say more: $(\tilde{W}_t)_{[0,T]}$ is a Brownian motion. In particular, $\tilde{W}_t$ is independent of $\mathcal{F}_0$.

Let $\mathcal{G} \subset \mathcal{F}$, and $P << Q$ such that $dP = M dQ$, then

$$E^P[X|\mathcal{G}] = \frac{E^Q[X\frac{dP}{dQ}|\mathcal{G}]}{E^Q[\frac{dP}{dQ}|\mathcal{G}]} \tag{3.19}$$

This is called the abstract *Bayes formula*.

3.2 Stochastic optimal control

3.2.1 Stochastic principle of optimality

The formulation of stochastic optimal control problem is somewhat the same as the deterministic case. Given a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ on which an m-dimensional standard Brownian motion B is defined. Consider the following controlled SDE:

$$\begin{aligned} dx(t) &= b(t, x(t), u(t))dt + \sigma(t, x(t), u(t))dB_t \\ x(0) &= x_0 \in \mathbb{R}^n \end{aligned} \tag{3.20}$$

where $b : [0, T] \times \mathbb{R}^n \times U \to \mathbb{R}^n$, $\sigma : [0, T] \times \mathbb{R}^n \times U \to \mathbb{R}^{n\times m}$, with U being a given separable metric space and $u : [0, T] \times \Omega \to U$ is called the control. Define the *feasible control* set as

$$\mathcal{U}[0, T] = \{u : [0, T] \times \Omega \to U \mid u(\cdot) \text{ is } (\mathcal{F}_t)\text{-adapted}\}.$$

The cost functional for stochastic optimal control is defined as

$$J(u(\cdot)) = E\left\{h(x(T)) + \int_0^T L(t, x(t), u(t))dt\right\} \tag{3.21}$$

and call

$$\mathcal{U}_{\mathrm{ad}}[0, T] = \{u \in \mathcal{U}[0, T] : \text{ the solution of (3.20) is unique and } J(u(\cdot)) < \infty\}$$

the s-admissible control set. It is also natural to consider state feedback control and call

$$\mathcal{U}_{\mathrm{f}}[0, T] = \{u \in \mathcal{U}_{\mathrm{ad}}[0, T] : \ u(t) = \phi(t, X_t) \text{ for some continuous function } \phi\}$$

the f-admissible control set.

As in the deterministic case, we derive the principle of optimality, i.e., the stochastic version of (1.13). The stochastic optimal control problem is find $\bar{u}(\cdot) \in \mathcal{U}_{\mathrm{f}}[0, T]$ (if exists) such that

$$J(\bar{u}(\cdot)) = \inf_{u(\cdot)\in\mathcal{U}_{\mathrm{f}}[0,T]} J(u(\cdot))$$

Assumption 1 *(A1) U is a Polish space (separable Banach space).*

(A2) The maps b, σ, h, L are uniformly continuous, and there exists a constant $K > 0$, such that for $\varphi(t, x, u) = b(t, x, u), \sigma(t, x, u), h(x), L(t, x, u)$,

$$\begin{gathered} |\varphi(t, x, u) - \varphi(t, y, u)| \leq K|x - y|, \quad \forall t \in [0, T],\ x, y \in \mathbb{R}^n,\ u \in U \\ |\varphi(t, 0, u)| \leq K, \quad \forall (t, u) \in [0, T] \times U \end{gathered}$$

Let

$$J(s, y; u(\cdot)) = E\left\{h(x(T)) + \int_s^T L(t, x(t), u(t))dt\right\}$$

and define the value function as

$$\begin{aligned} V(s, y) &= \inf_{u(\cdot)\in\mathcal{U}_{\mathrm{f}}[s,T]} J(s, y; u(\cdot)), \quad \forall (s, y) \in [0, T) \times \mathbb{R}^n, \\ V(T, y) &= h(y), \quad \forall y \in \mathbb{R}^n. \end{aligned}$$

We have the following proposition.

Proposition 3.2 *Let (A1)-(A2) hold. Then for any $(s,y) \in [0,T) \times \mathbb{R}^n$ and $s \leq \hat{s} \leq T$*

$$V(s,y) = \inf_{u(\cdot)\in\mathcal{U}_f[s,T]} E\left\{\int_s^{\hat{s}} L(t, X(t;s,y,u(\cdot)), u(t))dt + V(\hat{s}, X(t;s,y,u(\cdot)))\right\}. \tag{3.22}$$

Formula (3.22) enjoys the same structure as (1.13), but since the cost function (3.21) does not admit the splitting in Theorem (1.1), it is not a immediate consequence of that theorem.

Proof 5 *Let $\mathcal{F}^s_{\hat{s}} = \sigma\{B_r : s \leq r \leq \hat{s}\}$. Denote right hand side of (3.22) by $\bar{V}(s,y)$. For any $\varepsilon \geq 0$, there exists $u(\cdot) \in \mathcal{U}_f[s,T]$ such that*

$$\begin{aligned}
V(s,y) + \varepsilon > J(s,y;u(\cdot)) & \\
&= E\left\{\int_s^T L(t, X(t;s,y,u(\cdot)), u(t))dt + h(X(T;s,y,u(\cdot))\right\} \\
&= E\left\{\int_s^{\hat{s}} L(t, X(t;s,y,u(\cdot)), u(t))dt\right\} \\
&\quad + EE_{\mathcal{F}^s_{\hat{s}}}\left[\int_{\hat{s}}^T L(t, X(t;s,y,u(\cdot)), u(t))dt + h(X(T;s,y,u(\cdot))\right] \\
&= E\left\{\int_s^{\hat{s}} L(t, X(t;s,y,u(\cdot)), u(t))dt\right\} \\
&\quad + EE_{\mathcal{F}^s_{\hat{s}}}\left[\int_{\hat{s}}^T L(t, X(t;\hat{s},X_{\hat{s}},u(\cdot)), u(t))dt + h(X(T;\hat{s},X_{\hat{s}},u(\cdot))\right] \\
&\quad \textit{(uniqueness of solution)} \\
&= E\left\{\int_s^{\hat{s}} L(t, X(t;s,y,u(\cdot)), u(t))dt + J(\hat{s}, X(\hat{s};s,y,u(\cdot));u(\cdot))\right\} \\
&\geq E\left\{\int_s^{\hat{s}} L(t, X(t;s,y,u(\cdot)), u(t))dt + V(\hat{s}, X(\hat{s};s,y,u(\cdot)))\right\} \\
&\geq \bar{V}(s,y).
\end{aligned}$$

To prove the converse, we need a technical result regarding the regularity of J and V: Given a constant $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that whenever $|x-y| < \delta$,

$$|J(\hat{s},y;u(\cdot)) - J(\hat{s},x;u(\cdot))| + |V(\hat{s},y) - V(\hat{s},x)| \leq \varepsilon, \quad \forall u(\cdot) \in \mathcal{U}_f[\hat{s},T].$$

Next, choose a partition of $\mathbb{R}^n$ with $\mathbb{R}^n = \cup_j D_j$, $D_i \cap D_j = \varnothing$ if $i \neq j$ and diam$(D_j) < \delta$. Then there exist $(u_j)_{j\geq 1} \in \mathcal{U}_f[\hat{s},T]$ such that

$$J(\hat{s},x_j;u_j(\cdot)) \leq V(\hat{s},x_j) + \varepsilon, \quad \forall x_j \in D_j.$$

Hence for any $x \in D_j$, we have

$$J(\hat{s},x,u_j(\cdot)) \leq J(\hat{s},x_j,u_j(\cdot)) + \varepsilon \leq V(\hat{s},x_j) + 2\varepsilon \leq V(\hat{s},x) + 3\varepsilon.$$

Now for any $u(\cdot) \in \mathcal{U}_f[s,T]$, define

$$\tilde{u}(t) = \begin{cases} u(t), & t \in [s,\hat{s}) \\ u_j(t), & t \in [\hat{s},T] \text{ and } x(t) \in D_j \end{cases}$$

Then

$$
\begin{aligned}
V(s,y) &\le J(s,y;\tilde{u}(\cdot)) \\
&= E\left\{\int_s^T L(t,X(t;s,y,\tilde{u}(\cdot)),u(t))dt + h(X(T;s,y,\tilde{u}(\cdot))\right\} \\
&= E\left\{\int_s^{\hat{s}} L(t,X(t;s,y,u(\cdot)),u(t))dtx\right\} \\
&\quad + EE_{\mathcal{F}^s_{\hat{s}}}\left[\int_{\hat{s}}^T L(t,X(t;s,y,\tilde{u}(\cdot)),u(t))dt + h(X(T;s,y,\tilde{u}(\cdot))\right] \\
&= E\left\{\int_s^{\hat{s}} L(t,X(t;s,y,u(\cdot)),u(t))dt + J(\hat{s},X(\hat{s};s,y,u(\cdot));\tilde{u}(\cdot))\right\} \\
&\le E\left\{\int_s^{\hat{s}} L(t,X(t;s,y,u(\cdot)),u(t))dt + V(\hat{s},X(\hat{s};s,y,u(\cdot))) + 3\varepsilon\right\}.
\end{aligned}
$$

Again, as in the deterministic case, based on the above proposition, one can easily prove the following theorem.

Theorem 3.8 *Suppose that (A1)-(A2) hold and the value function $V \in C^{1,2}([0,T]\times\mathbb{R}^n)$. Then V is a solution of the following PDE (stochastic HJB equation):*

$$
\begin{aligned}
&-V_t + \sup_{u\in U} G(t,x,u,-V_x,-V_{xx}) = 0 \\
&V(x,T) = h(x), \quad x \in \mathbb{R}^n
\end{aligned}
\tag{3.23}
$$

where

$$G(t,x,u,p,P) = \frac{1}{2}\operatorname{tr}\left(P\sigma(t,x,u)\sigma(t,x,u)^T\right) + \langle p, b(t,x,u)\rangle - L(t,x,u).$$

Invoking the infinitesimal generator $A(\cdot)$ defined as (3.17), the stochastic HJB equation can also be written as

$$
\begin{aligned}
0 &= V_t + \inf_{u\in U}[A^u(t)V + L(t,x,u)], \\
V(x,T) &= h(x), \quad x \in \mathbb{R}^n.
\end{aligned}
\tag{3.24}
$$

where

$$A^u(t) := \frac{1}{2}\sum a_{ij}(t,x,u)\frac{\partial^2}{\partial x_i \partial x_j}(t,x) + \sum b_i(t,x,u)\frac{\partial}{\partial x_i}.$$

Notice that when $\sigma = 0$, (3.23) reduces exactly to the deterministic HJB (c.f. (1.17)). Thus the stochastic principle of optimality is a generalization of the deterministic one.

3.2.2 Full state LQG control

Consider now the linear controlled stochastic system

$$dx(t) = [A(t)x(t) + B(t)u(t)]dt + \sigma(t)dB_t \tag{3.25}$$

on the interval $[0,T]$ with $A(\cdot) \in L^\infty([0,T];\mathbb{R}^{n\times n})$, $B(\cdot) \in L^\infty([0,T];\mathbb{R}^{n\times m})$, $u(\cdot) \in \mathcal{U}_{\mathrm{f}}[0,T]$ and $\sigma \in L^\infty([0,T];\mathbb{R}^{n\times d})$, $B.$ is a d-dimensional Brownian motion. The cost function of interest for this system is

$$J(s,x,u) = E\left\{x(T)^T D x(T) + \int_s^T [x(t)^T M(t) x(t) + u(t)^T R(t) u(t)]dt\right\},$$

in which $x(s) = x$, $M(t) \geq aI_{n\times n}$, $R(t) \geq bI_{m\times m}$ and $D > cI_{n\times n}$ for some constants $a, b, c \in \mathbb{R}_{>0}$.

In order to solve the stochastic HJB (3.23) or (3.24), it is natural to propose the following candidate

$$V(t,x) = x^T K(t)x + q(t)$$

for some functions $K : [0,T] \to \mathbb{R}^{n\times n}$ (symmetric), and $q : [0,T] \to \mathbb{R}$.

Now

$$\begin{aligned}
&A^u(t)V(t,x) + L(t,x,u)\\
=&A^u(t)[x^T K(t)x + q(t)] + x^T M(t)x + u^T R(t)u\\
=&2x^T K(t)[A(t)x + B(t)u] + \mathrm{tr}(\sigma(t)\sigma(t)^T K(t)) + x^T M(t)x + u^T R(t)u
\end{aligned}$$

which is a quadratic function of u. By the fact that $R(t) \geq bI_{m\times m}$ we know $\inf[A^u(t)V(t,x) + L(t,x,u)]$ is achieved at

$$\{u : \frac{\partial}{\partial u}[A^u(t)V(t,x) + L(t,x,u)] = 0\}$$

or

$$2R(t)u_* + 2B(t)^T K(t)x = 0$$

which results in a static feedback control law

$$u_*(t,x) = -R(t)^{-1}B(t)^T K(t)x.$$

Substituting u_* into the stochastic HJB, we get

$$0 = x^T[\dot{K} + KA + A^T K - KBR^{-1}B^T K + M]x + \dot{q}(t) + \mathrm{tr}(\sigma\sigma^T K).$$

Hence a sufficient condition for the optimal law is

$$\begin{aligned}
\dot{K}(t) &= -K(t)A(t) - A(t)^T K(t) + K(t)B(t)R^{-1}(t)B(t)^T K(t) - M(t)\\
K(T) &= D\\
\dot{q}(t) &= \mathrm{tr}(\sigma(t)\sigma(t)^T K(t))\\
q(T) &= 0
\end{aligned}$$

and that the resulting solution $K(t)$ being symmetric positive definite.

3.2.3 Revisit of viscosity solution of HJB

Let us consider two systems

$$\begin{aligned}
\mathrm{S}_1 &: dx(t) = f(t,x(t),u(t))dt\\
\mathrm{S}_2 &: dx(t) = f(t,x(t),u(t))dt + \sqrt{2\varepsilon}dB_t
\end{aligned}$$

i.e., S_2 is obtained by adding a stochastic term $\sqrt{2\varepsilon}dB_t$ on S_1.

Consider the cost function for the two systems

$$\begin{aligned}
J_1(s,y,u(\cdot)) &= \int_s^T L(t,x(t),u(t))dt + \varphi(x(T)), \quad x(t) \text{ solves } \mathrm{S}_1\\
J_2(s,y,u(\cdot)) &= E\left[\int_s^T L(t,x(t),u(t))dt + \varphi(x(T))\right], \quad x(t) \text{ solves } \mathrm{S}_2
\end{aligned}$$

respectively.

The HJB for the two systems are

$$0 = V_t + \inf_u \left(\frac{\partial V(t,x)}{\partial x} f(t,x,u) + L(t,x,u) \right) \tag{3.26}$$

$$0 = W_t + \inf_u \left(\frac{\partial W(t,x)}{\partial x} f(t,x,u) + L(t,x,u) \right) + \varepsilon \frac{\partial^2 W(x,t)}{\partial x^2} \tag{3.27}$$

We observe that the stochastic HJB can be obtained from the deterministic HJB by adding the term $\varepsilon \Delta W$. It is reasonable to expect that when $\varepsilon \to 0$, W^ε (the solution to (3.27) with a given ε) converges to V in certain sense (in fact, uniformly) since the term $\varepsilon \Delta W^\varepsilon$ vanishes as $\varepsilon \to 0$. From parabolic PDE theory, (3.27) admits smooth solutions (while (3.26) doesn't! Thus the term $\varepsilon \Delta W$ regularizes the HJB (3.26)). Since the convergence of W^ε is uniform, V should be continuous. One can show that this V is indeed the viscosity solution that we have introduced in Section 1.2.3. On the other hand, the construction of the viscosity solution in Section 1.2.3 has nothing to do with the discussion here. It is indeed a more intrinsic way of construction.

3.3 Kallianpur-Striebel formula

Give a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$ and

$$\begin{aligned} &\text{System: } dX_t = b(t,X_t)dt + \sigma(t,X_t)dW_t \\ &\text{Observable: } dY_t = h(t,X_t)dt + dB_t \end{aligned} \tag{3.28}$$

Assume $(B_t)_{t\in[0,T]}$ and $(W_t)_{t\in[0,T]}$ are independent d and p dimensional Brownian motions adapted to $(\mathcal{F}_t)$, $X_0 \in \mathcal{F}_0$ and $Y_0 = 0$ a.s.

The mappings

$$\begin{aligned} b &: [0,\infty) \times \mathbb{R}^d \to \mathbb{R}^d \\ \sigma &: [0,\infty) \times \mathbb{R}^d \to \mathbb{R}^{m\times d} \\ h &: [0,\infty) \times \mathbb{R}^d \to \mathbb{R}^p \end{aligned}$$

are assumed to be measurable. Without further assumption, we assume that the equation for (X_t, Y_t) has a unique (strong) solution.

Denote

$$\mathcal{F}_t^Y = \sigma\{Y_s : 0 \le s \le t\}$$

The goal of the filtering problem is to compute the optimal estimates $\pi_t(f) := E[f(X_t)|\mathcal{F}_t^Y]$ when $f(X_t) \in L^1$.

The idea is to construct a probability measure Q, such that X and Y are independent under Q. Then by the Bayes formula (3.19), we would have

$$\begin{aligned} \pi_t(f)(\omega) &= \frac{E^Q[f(X_t)\frac{dP}{dQ}|\mathcal{F}_t^Y]}{E^Q[\frac{dP}{dQ}|\mathcal{F}_t^Y]} \\ &= \frac{\tilde{E}^Q[f(X_t(\tilde{\omega}))\frac{dP}{dQ}(X(\tilde{\omega}), Y(\omega))]}{\tilde{E}^Q[\frac{dP}{dQ}(X(\tilde{\omega}), Y(\omega))]} \end{aligned}$$

where $X(\tilde{\omega}), Y(\omega) \in C[0,T]$. Our main tool to construct Q is the Girsanov theorem (see Theorem

3.7 in the Appendix). Define

$$\begin{aligned}\Lambda_t &= \mathcal{E}\left(-\int_0^t h(s, X_s)dB_s\right)_t \\ &= \exp\left(-\int_0^t h(s, X_s)dB_s - \frac{1}{2}\int_0^t |h(s, X_s)|^2 ds\right) \\ &= \exp\left(-\int_0^t h(s, X_s)dY_s + \frac{1}{2}\int_0^t |h(s, X_s)|^2 ds\right)\end{aligned}$$

see (3.18). Then since $Y_t = W_t - \left(-\int_0^t h(s, X_s)dB_s\right)$, it follows from Girsanov theorem (see Example 6) that (Y_t) is a Brownian motion under Q defined by $dQ = \Lambda_T dP$ whenever

$$E\left[\exp\left(\frac{1}{2}\int_0^T |h(s, X_s)|^2 ds\right)\right] < \infty$$

Next, we show that X and Y are indeed independent under Q. We have to prove

$$E^Q[\Phi(X)\Psi(Y)] = E^Q[\Phi(X)]E^Q[\Psi(Y)]$$

for any bounded measurable functions Φ and Ψ on $C[0, T]$. The following relations are trivial:

$$\begin{aligned}E^Q[\Phi(X)\Psi(Y)] &= E^P[\Lambda_T(X, Y)\Phi(X)\Psi(Y)] \\ &= E^P[E^P[\Lambda_T(X, Y)\Phi(X)\Psi(Y)|X]] \\ &= E^P[\Phi(X)E^P[\Lambda_T(X, Y)\Psi(Y)|X]]\end{aligned}$$

To continure, observe that

$$\begin{aligned}E^P[\Lambda_T(X, Y)\Psi(Y)|X](\omega) &= \tilde{E}^P[\Lambda_T(X(\omega), Y^{X(\omega)}(\tilde{\omega}))\Psi(Y^{X(\omega)}(\tilde{\omega}))] \\ &= \tilde{E}^{\tilde{Q}}[\Psi(Y^{X(\omega)}(\tilde{\omega}))] \\ &= \int_{C[0,T]} \Psi(y)\mu^{\bar{W}}(y)\end{aligned}$$

where

$$Y_t^{X(\omega)}(\tilde{\omega}) = \int_0^t h(s, X_s(\omega))ds + B_t(\tilde{\omega}),$$

$\mu^{\bar{W}}$ is the measure on $C[0, T]$ induced by a Brownian motion $\bar{W}$ and that $Y_t^{X(\omega)}(\tilde{\omega})$ is a Brownian motion under $d\tilde{Q} = \Lambda_T(X(\omega), Y^{X(\omega)}(\tilde{\omega}))dP$ by Girsanov theorem. Also, we see that $E^P[\Lambda_T(X, Y)\Psi(Y)|X](\omega)$ does not dependent on ω and hence is deterministic! Thus we obtain

$$E^Q[\Phi(X)\Psi(Y)] = E^P[\Phi(X)]\int_{C[0,T]} \Psi(y)\mu^W(y).$$

Choose $\Psi \equiv 1$, we get $E^Q[\Phi(X)] = E^P[\Phi(X)]$. Choose $\Phi \equiv 1$, we get $E^Q[\Psi(Y)] = \int_{C[0,T]} \Psi(y)\mu^W(y)$, which shows that Y is independent of X.

Therefore

$$
\begin{aligned}
\pi_t(f)(\omega) &= \frac{E^Q\left[f(X_t)\Lambda_T^{-1}|\mathcal{F}_t^Y\right]}{E^Q[\Lambda_T^{-1}|\mathcal{F}_t^Y]} \\
&= \frac{E^Q\left[f(X_t)\Lambda_t^{-1}|\mathcal{F}_t^Y\right]}{E^Q[\Lambda_t^{-1}|\mathcal{F}_t^Y]}(\omega) \\
&= \frac{\tilde{E}^Q\left[f(X_t(\tilde{\omega})\Lambda_t^{-1}(X(\tilde{\omega}),Y(\omega))\right]}{\tilde{E}^Q[\Lambda_t^{-1}(X(\tilde{\omega}),Y(\omega))]} \\
&= \frac{\int_{C[0,T]} f(\iota_t(x))\Lambda_t^{-1}(x,Y(\omega))\mu^X(dx)}{\int_{C[0,T]} \Lambda_t^{-1}(x,Y(\omega))\mu^X(dx)}
\end{aligned}
$$

due to the independence of X and Y. $\iota_t(x) = x_t$. The second equality follows from the following fact:

$$
\begin{aligned}
E^Q[Z\Lambda_t^{-1}] &= E^P[Z\Lambda_T\Lambda_t^{-1}] \\
&= E^P[Z\Lambda_t^{-1}E^P[\Lambda_T|\mathcal{F}_t]] \\
&= E^P[Z] \\
&= E^Q[Z\Lambda_T^{-1}] \\
&= E^Q[ZE^Q[\Lambda_T^{-1}|\mathcal{F}_t]]
\end{aligned}
$$

hence $E^Q[\Lambda_T^{-1}|\mathcal{F}_t] = \Lambda_t^{-1}$, i.e., Λ_t is an $\mathcal{F}_t$ martingale under Q.

Kallianpur-Striebel formula

$$
E[f(X_t)|\mathcal{F}_t^Y](\omega) = \frac{\tilde{E}^Q\left[f(X_t(\tilde{\omega})\Lambda_t^{-1}(X(\tilde{\omega}),Y(\omega))\right]}{\tilde{E}^Q[\Lambda_t^{-1}(X(\tilde{\omega}),Y(\omega))]}
$$

where

$$
\Lambda_t^{-1} = \exp\left(\int_0^t h(s,X_s)dY_s - \frac{1}{2}\int_0^t |h(s,X_s)|^2 ds\right)
$$

As a biproduct, we also see

$$
\begin{aligned}
\Lambda_t^{-1} &= \exp\left(\int_0^t h(s,X_s)dY_s - \frac{1}{2}\int_0^t |h(s,X_s)|^2 ds\right) \\
&= \mathcal{E}\left(\int_0^\cdot h(s,X_s)dY_s\right)_t
\end{aligned}
$$

Hence Λ_t^{-1} is an $\mathcal{F}_t$ martingale under P on $[0,T]$ i.e, $E^P[\Lambda_T^{-1}|\mathcal{F}_t] = \Lambda_t^{-1}$.

3.4 Zakai and FKK equation

Keep the notations as in the previous section and introduce a new one:

$$
\sigma_t(f) = E^Q\left[f(X_t)\Lambda_t^{-1}|\mathcal{F}_t^Y\right]
$$

then $\pi_t(f) = \frac{\sigma_t(f)}{\sigma_t(1)}$. We derive an equation for $\sigma_t(f)$. For convenience, put $z_t = \Lambda_t^{-1}$, then $dz_t = z_t h_t^T dY_t$ where we write for convenience $h_s = h_s(s, X_s)$.

then by Ito's formula

$$
\begin{aligned}
df(X_t)z_t &= z_t \nabla f(X_t)^T dX_t + f(X_t)dz_t + \\
&\quad + \frac{1}{2}\sum_{i,j=1}^{d} z_t \partial_{ij} f(X_t) d\left\langle X^i, X^j \right\rangle_t + \frac{1}{2}\sum_{i=1}^{d} \partial_i f(X_t) d\left\langle X_t^i, z_t \right\rangle \\
&= z_t \nabla f(X_t)^T [b(t,X_t)dt + \sigma(t,X_t)dW_t] + f(X_t) z_t h_t^T dY_t + \frac{1}{2}\sum_{i,j,k=1}^{d} \sigma^{ik}\sigma^{jk}\partial_{ij} f(X_t)dt \\
&= z_t \left[\nabla f(X_t)^T b(t,X_t) + \frac{1}{2}\sum_{i,j,k=1}^{d} \sigma^{ik}\sigma^{jk}\partial_{ij} f(X_t) \right] dt \\
&\quad + z_t \nabla f(X_t)^T \sigma(t,X_t)dW_t + f(X_t) z_t h_t^T dY_t \\
&= z_t L f(X_t) + z_t \nabla f(X_t)^T \sigma(t,X_t)dW_t + f(X_t) z_t h_t^T dY_t
\end{aligned}
$$

or

$$
\begin{aligned}
f(X_t)z_t &= f(X_0) + \int_0^t \Lambda_s^{-1} L f(X_s) ds \\
&\quad + \int_0^t \Lambda_s^{-1} \nabla f(X_s)^T \sigma(s,X_s) dW_s + \int_0^t f(X_s) z_s h_s^T(s,X_s) dY_s.
\end{aligned} \tag{3.29}
$$

where we have used:

$$
\begin{aligned}
Lf &= \frac{1}{2}\sum_{i,j,k=1}^{d} \sigma^{ik}\sigma^{jk}\partial_{ij}^2 f + \sum_{i=1}^{d} b_i \partial_i f \\
\left\langle X^i, X^j \right\rangle_t &= \left\langle \int b^i dt + \int \sum_k \sigma^{ik} dW_t^k, \int b^j dt + \int \sum_k \sigma^{jk} dW_t^k \right\rangle \\
&= \left\langle \int \sum_k \sigma^{ik} dW_t^k, \int \sum_k \sigma^{jk} dW_t^k \right\rangle = \sum_k \int \sigma^{ik}\sigma^{jk} dt \\
\left\langle X_t^i, z_t \right\rangle &= 0
\end{aligned}
$$

Take the conditional expectation on (3.29), we obtain

$$
\begin{aligned}
&E^Q[f(X_t)\Lambda_t^{-1} | \mathcal{F}_t^Y] \\
=&E^Q[f(X_0)] + \int_0^t E^Q[\Lambda_s^{-1} L f(X_s) | \mathcal{F}_s^Y] ds + \int_0^t E^Q[\Lambda_s^{-1} f(X_s) h^T(s,X_s) | \mathcal{F}_s^Y] dY_s
\end{aligned}
$$

or

$$
\sigma_t(f) = \sigma_0(f) + \int_0^t \sigma_s(Lf) ds + \int_0^t \sigma_s(h_s f)^T dY_s \tag{3.30}
$$

where $(h_s f)(x) = f(x)h(s,x)$, $\sigma_0(f) = E^P[f(X_0)]$. In differential form, it also reads

$$
d\sigma_t(f) = \sigma_t(Lf)dt + \sigma_t(h_t f)dY_t \tag{3.31}
$$

which is an SDE. Equation (3.30) is called the Zakai equation.

Zakai equation

$$\sigma_t(f) = \sigma_0(f) + \int_0^t \sigma_s(Lf)ds + \int_0^t \sigma_s(h_s f)dY_s$$

Differential form

$$d\sigma_t(f) = \sigma_t(Lf)dt + \sigma_t(h_t f)dY_t$$

where

$$Lf = \frac{1}{2}\sum_{i,j,k=1}^{d} \sigma^{ik}\sigma^{jk}\partial_{ij}^2 f + \sum_{i=1}^{d} b_i\partial_i f$$
$$(h_s f)(x) = h^T(s,x)f(x)$$

With the Zakai equation, we can now derive a equation for $\pi_t(f)$ using Ito's formula:

$$\begin{aligned} d\pi_t(f) &= d\left(\frac{\sigma_t(f)}{\sigma_t(1)}\right) \\ &= \frac{d\sigma_t(f)}{\sigma_t(1)} - \frac{\sigma_t(f)d\sigma_t(1)}{\sigma_t(1)^2} + \frac{\sigma_t(f)|\sigma_t(h)|^2}{\sigma_t(1)^3}dt - \frac{\sigma_t(h)^T\sigma_t(h_t f)}{\sigma_t^2(1)}dt \\ &= \pi_t(Lf) + [\pi_t(h_t f) - \pi_t(f)\pi_t(h)]^T[dY_t - \pi_t(h)dt] \end{aligned}$$

or

$$\pi_t(f) = \pi_0(f) + \int_0^t \pi_s(L_s f)ds + \int_0^t [\pi_s(h_s f) - \pi_s(f)\pi_s(h)]^T d\bar{B}_s \tag{3.32}$$

where

$$\bar{B}_t = Y_t - \int_0^t \pi_s(h)ds \tag{3.33}$$

or

$$d\bar{B}_t = dY_t - \pi_s(h)dt$$

and we have used:

$$\begin{aligned} &L1 = 0 \\ &h_t 1(x) = h(t,x)^T \Rightarrow \sigma_t(h_t 1) = \sigma_t(h^T) \\ &d\sigma_t(1) = \sigma_t(h^T)dY_t \\ &d\left(\frac{x_t}{y_t}\right) = \frac{dx_t}{y_t} - \frac{x_t dy_t}{y_t^2} - \frac{d\langle x,y\rangle_t}{y_t^2} + \frac{x_t d\langle y,y\rangle_t}{y_t^3} \\ &d\langle\sigma_t(f),\sigma_t(1)\rangle_t = |\sigma_t(h_t f)|^2 d\langle Y,Y\rangle_t = |\sigma_t(h_t f)|^2 dt \end{aligned}$$

The process $\bar{B}_t$ is so important that it has a name: the *innovation process* of the filter.
The formula (3.32) is called te *FKK equation*.

FKK equation

$$\pi_t(f) = \pi_0(f) + \int_0^t \pi_s(L_s f)ds + \int_0^t [\pi_s(h_s f) - \pi_s(f)\pi_s(h)]^T d\bar{B}_s$$

where

$$\bar{B}_t = Y_t - \int_0^t \pi_s(h)ds$$

Proposition 3.3 *The innovation process* $(\bar{B}_t)_{t\in[0,T]}$ *is a Brownian motion adpated to* $(\mathcal{F}_t^Y)_{t\in[0,T]}$.

Proof 6 *Clearly* $\bar{B}_0 = 0$ *a.s, and*

$$\bar{B}_t = \int_0^t [h_s - \pi_s(h)]ds + B_t$$

where we have written $h_s = h(s, X_s)$ *for convenience. Recall that* $\pi_t(h) = E[h_t|\mathcal{F}_t^Y]$. *It suffices to show*

$$E\left[e^{i\alpha^T(\bar{B}_t-\bar{B}_s)}|\mathcal{F}_t^Y\right] = e^{-|\alpha|^2(t-s)/2}.$$

For this, we apply Ito's formula to $\eta_t = \exp(i\alpha^T\bar{B}_t)$:

$$\begin{aligned} e^{i\alpha^T\bar{B}_t} &= e^{i\alpha^T\bar{B}_s} + i\int_s^t e^{i\alpha^T\bar{B}_u}\alpha^T dB_u \\ &\quad + i\int_s^t e^{i\alpha^T\bar{B}_u}\alpha^T(h_u - \pi_u(h))du - \frac{1}{2}|\alpha|^2\int_s^t e^{i\alpha^T\bar{B}_u}du \end{aligned}$$

An immediate observation is that $E\left[\int_s^t e^{i\alpha^T\bar{B}_u}\alpha^T dB_u|\mathcal{F}_s^Y\right] = 0$ *since* $\int_s^t e^{i\alpha^T\bar{B}_u}\alpha^T dB_u$ *is an* $\mathcal{F}_t \supset \mathcal{F}_t^Y$ *martingale. Further, for* $u \geq s$,

$$\begin{aligned} E\left[e^{i\alpha^T\bar{B}_u}\pi_u(h)|\mathcal{F}_s^Y\right] &= E\left[e^{i\alpha^T\bar{B}_u}E[h_u|\mathcal{F}_u^Y]|\mathcal{F}_s^Y\right] \\ &= E\left[e^{i\alpha^T\bar{B}_u}h_u|\mathcal{F}_s^Y\right] \end{aligned}$$

thus $E\left[\int_s^t e^{i\alpha^T\bar{B}_u}\alpha^T(h_u - \pi_u(h))du|\mathcal{F}_s^Y\right] = \alpha^T\int_s^t E\left[e^{i\alpha^T\bar{B}_u}h_u - \pi_u(h)|\mathcal{F}_s^Y\right]du = 0$. *Combining these two, we arrive at*

$$E\left[e^{i\alpha^T\bar{B}_t}h_t|\mathcal{F}_s^Y\right] = e^{i\alpha^T\bar{B}_s} - \frac{1}{2}|\alpha|^2\int_s^t E\left[e^{i\alpha^T\bar{B}_u}h_u|\mathcal{F}_s^Y\right]du$$

and the proof is completed.

Suppose that there is a density $p_t(x)$ such that

$$p_t(x) = \frac{dP(X_t \preceq x|\mathcal{F}_t^Y)}{dx}$$

then

$$\pi_t(f) = E[f(X_t)|\mathcal{F}_t^Y] = \int_{\mathbb{R}^d} f(x)p_t(x)\mathrm{d}x$$

Substitute this into (3.32) and suppose that $f \in C_c^2(\mathbb{R}^d)$, then

$$\begin{aligned} \int_{\mathbb{R}^d} f(x)p_t(x)dx &= \int_{\mathbb{R}^d} f(x)p_0(x)dx + \int_0^t\int_{\mathbb{R}^d}(L_sf)(x)p_s(x)dxds \\ &\quad + \int_0^t\left[\int_{\mathbb{R}^d}(h_sf)(x)p_s(x)dx - \left(\int_{\mathbb{R}^d} f(x)p_s(x)dx\right)\pi_s(h)^T\right]d\bar{B}_s \\ &= \int_{\mathbb{R}^d} f(x)p_0(x)dx + \int_{\mathbb{R}^d} f(x)\left(\int_0^t L_s^*p_s(x)ds\right)dx \\ &\quad + \int_{\mathbb{R}^d} f(x)\left(\int_0^t h_s^Tp_s(x)d\bar{B}_s\right)dx - \int_{\mathbb{R}^d} f(x)\left(\int_0^t p_s(x)\pi_s(h)^Td\bar{B}_s\right)dx \\ &= \int_{\mathbb{R}^d} f(x)\left[p_0(x) + \int_0^t L_s^*p_s(x)ds + \int_0^t p_s(x)[h_s - \pi_s(h)]^Td\bar{B}_s\right]dx \end{aligned}$$

Thus

$$dp_t(x) = L_t^* p_t(x)dt + p_t(x)[h(t,x) - \pi_t(h)]^T d\bar{B}_t. \tag{3.34}$$

Since we have assumed that $f \in C_c^2(\mathbb{R}^d)$, the above equation should be understood in the week sense.

If however, the density for the unnormalized quantity $\sigma_t(f)$ is quested, i.e., search for $q_t(x)$ such that

$$\sigma_t(f) = \int_{\mathbb{R}^d} f(x)q_t(x)dx, \quad \forall f \in C_c^2(\mathbb{R}^d)$$

Note that if this is the case, then

$$\int_{\mathbb{R}^d} f(x)p_t(x)\mathrm{d}x = \int_{\mathbb{R}^d} f(x)\frac{q_t(x)}{\int_{\mathbb{R}^d} q_t(x)dx}dx$$

hence

$$p_t(x) = \frac{q_t(x)}{\int_{\mathbb{R}^d} q_t(x)dx}$$

which implies

$$\pi_t(f) = \int_{\mathbb{R}^d} f(x)p_t(x)\mathrm{d}x = \frac{\int_{\mathbb{R}^d} f(x)q_t(x)\mathrm{d}x}{\int_{\mathbb{R}^d} q_t(x)dx}$$

Thus, if the equation for $q_t(x)$ is simpler than (3.34), we can calculate $q_t(x)$ first and then use the last formula to calculate $p_t(x)$. Using (3.31), we easily find

$$dq_t(x) = L_t^* q_t(x)dt + h(t,x)^T dY_t \tag{3.35}$$

of which the initial distribution $q_0(x)$ is determined by the distribution of X_0. This equation is called the *Zakai-PDE*.

Equations for condtional density

Define

$$\pi_t(f) = \int_{\mathbb{R}^d} f(x)p_t(x)\mathrm{d}x$$
$$\sigma_t(f) = \int_{\mathbb{R}^d} f(x)q_t(x)\mathrm{d}x$$

then

$$\begin{aligned} \text{normalized: } & dp_t(x) = L_t^* p_t(x)dt + p_t(x)[h(t,x) - \pi_t(h)]^T d\bar{B}_t \\ \text{unnormalized: } & dq_t(x) = L_t^* q_t(x)dt + h(t,x)^T dY_t \end{aligned}$$

3.5 Kalman-Bucy filter

3.5.1 Zero input

In this section, we consider filtering problem of the linear model (zero input):

$$\begin{aligned} dX_t &= [A(t)X_t + +D(t)u_t]dt + C(t)dW_t \\ dY_t &= H(t)X_tdt + dB_t \end{aligned} \tag{3.36}$$

where $A(t), C(t), H(t)$ are deterministic real matrices of dimensions $n \times n$, $n \times m$, $l \times n$ respectively. W_t and B_t are Brownian motions adpated to filtration $\mathcal{F}_t$ (w.l.o.g, one can take $\mathcal{F}_t = \mathcal{F}_t^W \vee \mathcal{F}_t^B$). $Y_0 = 0$ a.e. and X_0, Y_0 are independent. u_t is the control input adpated to $\mathcal{F}_t^Y$.

It is customary in the linear case to assume $X_0 \sim N(\hat{X}_0, \hat{P}_0)$, i.e., the initial distribution of X_t is Gaussian and that the equation for (X_t, Y_t) has a unique strong solution adapted to $\mathcal{F}_t$.

In this subsection, we restrict ourselves to the zero input case, i.e., $u_t \equiv 0$ for all $t \geq 0$.

Define $\hat{X}_t := E[X_t|\mathcal{F}_t^Y]$ (notice that this is consistent with the notation $\hat{X}_0$ introduced earlier). Since the system (3.36) forms a Gaussian system, we are also interested in the covariance matrix of the conditional mean: $\hat{P}_t := E[(X_t - \hat{X}_t)(X_t - \hat{X}_t)^T|\mathcal{F}_t^Y]$. Due to Proposition 3.4 below, $\hat{P}_t$ is a deterministic function, i.e., $\hat{P}_t := E[(X_t - \hat{X}_t)(X_t - \hat{X}_t)^T]$.

To apply FKK equation, let $f(x) = x^i$, then

$$Lf(x) = \sum_k A_{ik} x_k, \quad h_t f(x) = x^i H(t) x, \quad h(t,x) = H(t)x$$

hence

$$\hat{X}_t^i = E[X_t^i|\mathcal{F}_t^Y] = E[X_0^i] + \int_0^t \sum_k A_{ik} \hat{X}_s^k ds + \int_0^t \left[E[X_s^i H(s) X_s|\mathcal{F}_s^Y] - \hat{X}_s^i H(s) \hat{X}_s \right]^T d\bar{B}_s.$$

Align all $\hat{X}_t^i$ as a column vector, we get

$$\begin{aligned}
\hat{X}_t &= E[X_0] + \int_0^t A\hat{X}_s ds + \int_0^t \left[E[H(s) X_s X_s^T|\mathcal{F}_s^Y] - H(s)\hat{X}_s \hat{X}_s \right]^T d\bar{B}_s \\
&= E[X_0] + \int_0^t A\hat{X}_s ds + \int_0^t \left[E[X_s X_s^T|\mathcal{F}_s^Y] - \hat{X}_s \hat{X}_s \right] H(s)^T d\bar{B}_s \\
&= E[X_0] + \int_0^t A\hat{X}_s ds + \int_0^t E[(X_s - \hat{X}_s)(X_s - \hat{X}_s)^T|\mathcal{F}_s^Y] H(s)^T d\bar{B}_s \\
&= E[X_0] + \int_0^t A\hat{X}_s ds + \int_0^t \hat{P}_s H(s)^T d\bar{B}_s
\end{aligned}$$

or equivalently

$$d\hat{X}_t = A(t)\hat{X}_t dt + \hat{P}_t H(t)^T d\bar{B}_t \tag{3.37}$$

$$d\bar{B}_t = dY_t - H(t)\hat{X}_t dt \tag{3.38}$$

with $\hat{X}_0 = E[X_0]$.

To derive the equation for $\hat{P}_t$, first notice that

$$\begin{aligned}
\hat{P}_t &= E[(X_t - \hat{X}_t)(X_t - \hat{X}_t)^T] \\
&= EX_t X_t^T - E[\hat{X}_t \hat{X}_t^T],
\end{aligned}$$

and then we apply Ito's formula to $\hat{P}_t$:

$$\begin{aligned}
dX_t^i X_t^j &= X_t^i dX_t^j + X_t^j dX_t^i + d\left\langle X^i, X^i \right\rangle_t \\
&= X_t^i dX_t^j + X_t^j dX_t^i + C_i C_j^T dt \\
&= X_t^i (A^j X_t dt + C^j dW_t) + X_t^j (A^i X_t dt + C^i dW_t) + C_i C_j^T dt \\
d\hat{X}_t^i \hat{X}_t^j &= \hat{X}_t^i d\hat{X}_t^j + \hat{X}_t^j d\hat{X}_t^i + d\left\langle \hat{X}^i, \hat{X}^j \right\rangle_t \\
&= \hat{X}_t^i d\hat{X}_t^j + \hat{X}_t^j d\hat{X}_t^i + \hat{P}_t^i H(t)^T H(t) (\hat{P}_t^j)^T dt \\
&= \hat{X}_t^i (A^j \hat{X}_t dt + \hat{P}_t^j H^T d\bar{B}_t) + \hat{X}_t^j (A^i \hat{X}_t + \hat{P}_t^i H^T d\bar{B}_t) + \hat{P}_t^i H(t)^T H(t) (\hat{P}_t^j)^T dt
\end{aligned}$$

hence

$$dE[X_t^i X_t^j] = [E(A^j X_t^i X_t + A^i X_t^j X_t) + C_i C_j^T]dt$$
$$dE[\hat{X}_t^i \hat{X}_t^j] = [E(A^j \hat{X}_t^i \hat{X}_t + A^i \hat{X}_t^j \hat{X}_t) + \hat{P}_t^i H(t)^T H(t)(\hat{P}_t^j)^T]dt$$

and

$$\frac{d\hat{P}_t^{ij}}{dt} = A^j E[X_t^i X_t - \hat{X}_t^i \hat{X}_t] + A^i E[X_t^j X_t - \hat{X}_t^j \hat{X}_t] + C_i C_j^T - \hat{P}_t^i H(t)^T H(t)(\hat{P}_t^j)^T$$

or equivalently

$$\frac{d\hat{P}_t}{dt} = A(t)\hat{P}_t + \hat{P}_t(t)A(t)^T + C(t)C(t)^T - \hat{P}_t H(t)^T H(t)\hat{P}_t \tag{3.39}$$

with $\hat{P}_0 = P_0$, i.e., the covariance matrix of X_0.

Kalman-Bucy filter

System:

$$dX_t = A(t)X_t dt + C(t)dW_t$$
$$dY_t = H(t)X_t dt + dB_t$$

Filter:

$$d\hat{X}_t = A(t)\hat{X}_t dt + \hat{P}_t H(t)^T d\bar{B}_t$$
$$d\bar{B}_t = dY_t - H(t)\hat{X}_t dt$$

where

$$\frac{d\hat{P}_t}{dt} = A(t)\hat{P}_t + \hat{P}_t(t)A(t)^T + C(t)C(t)^T - \hat{P}_t H(t)^T H(t)\hat{P}_t$$

Proposition 3.4 *The process $X_t - \hat{X}_t$ is independent of $\mathcal{F}_t^Y$, i.e., $E[f(X_s - \hat{X}_s)|\mathcal{F}_s^Y] = E[f(X_s - \hat{X}_s)]$ a.s. for any bounded measurable f.*

Proof 7 *Step 1: we show that the conditional distribution $X_t|\mathcal{F}_t^Y$ is Gaussian. Fix t and let $Y_n^k = X_{kt/2^n}$, $n \geq 1$. Define*

$$\mathcal{E}_n = \sigma\left\{Y_n^k : k = 1, \cdots, 2^n\right\}$$

Then $\mathcal{E}_n$ is a filtration ($\mathcal{E}_\infty = \mathcal{F}_t^Y$). Since (X, Y) is a Gaussian process on $[0, t]$, the joint distribution of $\{(X_{kt/2^n}, Y_{kt/2^n})\}_{k=1}^{2^n}$ is a Gaussian vector and thus the conditional expectation $X_t|\mathcal{E}_n$ is Gaussian with mean $E[X_t|\mathcal{E}_n] =: \hat{X}_t^n$ and covariance $\hat{P}_t^n$. Let $\pi_t^n(A) := P\{X_t \in A|\mathcal{E}_n\}$, then

$$\begin{aligned}\phi_n(\lambda) = E[\exp(i\lambda^T X_t)|\mathcal{E}_n] &= \int_{\mathbb{R}^d} \exp(i\lambda^T x)\pi_t^n(dx)\\ &= \exp\left(\lambda^T \hat{X}_t^n - \frac{1}{2}\lambda^T \hat{P}_t^n \lambda\right).\end{aligned}$$

Since $\{\phi_n(\lambda)\}_{n=1}^\infty$ and $\{\hat{X}_t^n\}_{n=1}^\infty$ are both uniformly integrable martingales adapted to $\mathcal{E}_n$, $\phi_\infty(\lambda)$ and $\hat{X}_t^\infty$ exist and are in L^1. Thus $\hat{P}_t^\infty$ also exists. To sum up, in a.s. sense,

$$\begin{aligned}E[\exp(i\lambda^T X_t)|\mathcal{F}_t^Y] &= E[\exp(i\lambda^T X_t)|\mathcal{E}_\infty]\\ &= \lim_{n\to\infty} \phi_n(\lambda)\\ &= \exp\left(\lambda^T \hat{X}_t - \frac{1}{2}\lambda^T \hat{P}_t \lambda\right)\end{aligned}$$

here we have omitted the superscript "∞". Thus $X_t|\mathcal{F}_t^Y \sim N(\hat{X}_t, \hat{P}_t)$.

Step 2: $X_t - \hat{X}_t$ is independent of $\mathcal{F}_t^Y$. It is known from elementary probability theory that when (X,Y) are jointly Gaussian, then $X - E[X|Y]$ is independent of Y. From Step 1, we know that $X_t - \hat{X}_t^n$ is independent of $\mathcal{E}_n$ for all n. For any bounded measurable function f and $A \in \mathcal{F}_t^Y$, let $A_n = E[1_A|\mathcal{E}_n]$, we have

$$E[f(X_t - \hat{X}_t^n)1_{A_n}] = E[f(X_t - \hat{X}_t^n)]P(A_n)$$

but

$$\lim_{n\to\infty} E[f(X_t - \hat{X}_t^n)1_{A_n}] = E[f(X_t - \hat{X}_t)1_A]$$
$$\lim_{n\to\infty} E[f(X_t - \hat{X}_t^n)]P(A_n) = E[f(X_t - \hat{X}_t)]P(A)$$

thus $E[f(X_t - \hat{X}_t)1_A] = E[f(X_t - \hat{X}_t)]P(A)$. The conclusion now follows.

To find the conditional density, we use the Zakai-PDE, which reads

$$dq_t(x) = q_t(x)x^T H(t)^T dY_t + \mathrm{tr}\left[\frac{1}{2}C(t)C(t)^T \mathrm{Hess}(q_t(x)) - \nabla(q_t(x)A(t)x)\right] dt$$

which has a solution of the form

$$q_t(x) = \mathrm{const} \times \exp\left(-\frac{1}{2}(x - \hat{X}_t)^T \hat{P}_t^{-1}(x - \hat{X}_t)\right).$$

3.5.2 Non-zero input

The non-zero input case is also important, which is not evident right now but will be clear in the next section.

We use a superscript "u" to indicate the signal under control input u. A first observation is that

$$X_t^u = \int_0^t A(s)X_s \mathrm{d}s + \int_0^t D(s)u_s \mathrm{d}s + \int_0^t C(s)dW_s$$
$$\hat{X}_t^u = E\left[\int_0^t A(s)X_s \mathrm{d}s|\mathcal{F}_t^Y\right] + \int_0^t D(s)u_s \mathrm{d}s + E\left[\int_0^t C(s)dW_s|\mathcal{F}_t^Y\right]$$

and then

$$X_t^u - \hat{X}_t^u = X_t^0 - \hat{X}_t^0.$$

There are two implifications from the above formula: first, the covariance matrix $\hat{P}_t^u$ does not depend on u, i.e., $\hat{P}_t^u = \hat{P}_t$; second, the differential of the above formula results in

$$\begin{aligned} d\hat{X}_t^u &= d\hat{X}_t^0 + dX_t^u - dX_t^0 \\ &= d\hat{X}_t^0 + D(t)u_t dt. \end{aligned}$$

Hence, the only thing we need to do to obtain the Kalman-Bucy filter with control is to add a term $D(t)u_t\mathrm{d}t$ in (3.37) and keep the innovation process and covariance matrix unchanged.

Kalman-Bucy filter with input

System:

$$\begin{aligned} dX_t &= [A(t)X_t + D(t)u_t]dt + C(t)dW_t \\ dY_t &= H(t)X_t dt + dB_t \end{aligned}$$

Filter:

$$\begin{aligned} d\hat{X}_t &= [A(t)\hat{X}_t + D(t)u_t]dt + \hat{P}_t H(t)^T d\bar{B}_t \\ d\bar{B}_t &= dY_t - H(t)\hat{X}_t dt \end{aligned}$$

where

$$\frac{d\hat{P}_t}{dt} = A(t)\hat{P}_t + \hat{P}_t(t)A(t)^T + C(t)C(t)^T - \hat{P}_t H(t)^T H(t)\hat{P}_t$$

3.6 Numerical method

3.6.1 Particle filter

3.6.2 Monte Carlo method

3.7 Partial State LQG and Separation Principle

This section is devoted to linear quadratic Gaussian control of the linear system

$$\begin{aligned} &\text{system: } dX_t = (A(t)X_t + B(t)u_t)dt + C(t)dW_t \\ &\text{observable: } dY_t = H(t)X_t dt + dB_t \end{aligned} \tag{3.40}$$

under the optimal cost

$$J[u] = E\left[\int_0^T (X_t^u)^T Q(t) X_t^u + u_t^T R(t) u_t \mathrm{d}t + X_T^u Q_f X_T^u\right] \tag{3.41}$$

where $Q(t)$, $R(t)$ and Q_f are all semi-positive definite. The term X_t^u represents the solution of the system under control u_t, which is required to depend only on the information $\{Y_s\}_{s\in[0,t]}$. In other words, u_t is $\mathcal{F}_t^Y$ measurable.

The first and the third terms in the cost functional are somewhat annoying since they are not observable. However, we can perform an easy manipulation to transform $J[u]$ into a more tractable form. This is achieved by applying the tower property of conditional expection:

$$J[u] = E\left[\int_0^T E[(X_t^u)^T Q(t) X_t^u|\mathcal{F}_t^Y] + E[u_t^T R(t) u_t] \mathrm{d}t + E[X_T^u Q_f X_T^u|\mathcal{F}_t^Y]\right]$$

Now the terms in the cost function are all observable! Instead of viewing this as a “magic”, we would rather say that this somewhat natural. If we admit this fact, then the famous “separation principle" will come as a natural consequence. To see this, let $\hat{X}_t^u = E[X_t^u|\mathcal{F}_t^Y]$, then

$$\begin{aligned}
&E[(X_t^u)^T Q(t) X_t^u|\mathcal{F}_t^Y] \\
&= E[(X_t^u - \hat{X}_t^u + \hat{X}_t^u)^T Q(t)(X_t^u - \hat{X}_t^u + \hat{X}_t^u)|\mathcal{F}_t^Y] \\
&= E[(X_t^u - \hat{X}_t^u)^T Q(t)(X_t^u - \hat{X}_t^u)|\mathcal{F}_t^Y] \\
&\quad + 2E[(X_t^u - \hat{X}_t^u)^T Q(t)\hat{X}_t^u|\mathcal{F}_t^Y] + E[(\hat{X}_t^u)^T Q(t)\hat{X}_t^u|\mathcal{F}_t^Y] \\
&= \mathrm{tr}(E[Q(t)(X_t^u - \hat{X}_t^u)(X_t^u - \hat{X}_t^u)^T|\mathcal{F}_t^Y]) + E[(\hat{X}_t^u)^T Q(t)\hat{X}_t^u] \\
&= \mathrm{tr}(E[Q(t)\hat{P}_t^u]) + E[(\hat{X}_t^u)^T Q(t)\hat{X}_t^u]
\end{aligned}$$

As mentioned in the previous subsection, the term $\mathrm{tr}(E[Q(t)\hat{P}_t^u])$ is deterministic and does not depend on u. Hence it does not affect the optimal value of $J[u]$. To make this precise, rewrite $J[u]$ as

$$J[u] = E\left[\int_0^T (\hat{X}_t^u)^T Q(t)\hat{X}_t^u + u_t^T R(t) u_t \mathrm{d}t + \hat{X}_T^u Q_f \hat{X}_T^u\right] + \mathrm{tr}(E[Q(t)\hat{P}_t] + E[Q_f \hat{P}_T])$$

and we can claim

$$\arg\min_u J[u] = \arg\min_u \bar{J}[u]$$

where $\bar{J}(u)$ is

$$\bar{J}(u) = E\left[\int_0^T (\hat{X}_t^u)^T Q(t)\hat{X}_t^u + u_t^T R(t) u_t \mathrm{d}t + \hat{X}_T^u Q_f \hat{X}_T^u\right].$$

Now the optimal control problem has been transformed into a "full-state observable" one. Thus invoking the results for full-state LQG, we can immediately state the following theorem.

Theorem 3.9 *Let $\hat{P}_t$ be the solution of the Riccati equation*

$$\begin{aligned}
\frac{d\hat{P}_t}{dt} &= A(t)\hat{P}_t + \hat{P}_t A^T(t) - \hat{P}_t H(t)^T H(t)\hat{P}_t + C(t)C(t)^T \\
\hat{P}_0 &= \mathrm{Cov}(X_0)
\end{aligned}$$

and K_t be the solution of the time-reversed Riccati equation

$$\begin{aligned}
\frac{dK_t}{dt} &= -A(t)^T K_t - K_t A(t) + K_t D(t) R^{-1}(t) D(t)^T K_t - R(t) \\
K_T &= Q_f
\end{aligned}$$

Then the partial state LQG has a solution

$$u_t^* = -R^{-1}(t) D(t)^T K_t \hat{X}_t$$

where $\hat{X}_t$ satisfies

$$\begin{aligned}
d\hat{X}_t &= (A(t) - D(t)R^{-1}(t)D(t)^T K_t)\hat{X}_t dt + \hat{P}_t H(t)^T d\bar{B}_t \\
d\bar{B}_t &= dY_t - H(t)\hat{X}_t dt.
\end{aligned}$$

This theorem has the spirit of "separation" since as we know K_t is the optimal gain for full-state LQG and $\hat{X}_t$ is the output of the optimal filter. Thus the theorem suggests that we can divide the design of the partial-state LQG into two parts. The first part is filtering, i.e., to obtain $\hat{X}_t$ and the filter gain $\hat{P}_t$, the second part amounts to the design of a full-state LQG based on the filter state $\hat{X}_t$. These two parts can be designed separately.

Chapter 4

Optimal Steering

4.1 An Introduction to Optimal Transport

4.1.1 Wasserstein distance

4.1.2 Monge-Kantorovich duality

4.2 Optimal Steering and the Schrödinger Bridge problem

Chapter 5

Appendix

5.0.1 ODE

The solution $\phi(t;0,x)$ with initial condition $x(0)=x$ of the ODE

$$\dot{x} = f(t,x)$$

satisfies the semigroup property

$$\phi(t;s,\phi(s,0,x)) = \phi(t;0,x).$$

Proof 8 *Let*

$$\varphi(t,s) = \phi(t;s,\phi(s;0,x)), \quad t \geq s$$

We have to show

$$\varphi(t,s) = \varphi(t,0).$$

It suffices to show that

$$\frac{\partial\varphi(t,s)}{\partial s} = 0, \quad \forall s \leq t.$$

We calculate

$$\varphi(t,s) = \phi(s;0,x) + \int_s^t f(r,\varphi(r,s))dr$$

Then

$$\begin{aligned}
\frac{\partial\varphi(t,s)}{\partial s} &= f(s,\varphi(s,s)) - f(s,\varphi(s,s)) + \int_s^t \frac{\partial f}{\partial x}(r,\varphi(r,s))\frac{\partial\varphi(r,s)}{\partial s}dr \\
&= \int_s^t \frac{\partial f}{\partial x}(r,\varphi(r,s))\frac{\partial\varphi(r,s)}{\partial s}dr \\
\frac{d}{dt}\frac{\partial\varphi(t,s)}{\partial s} &= \frac{\partial f}{\partial x}(t,\varphi(t,s))\frac{\partial\varphi(t,s)}{\partial s}, \quad \left.\frac{\partial\varphi(t,s)}{\partial s}\right|_{t=s} = 0
\end{aligned}$$

Hence $\frac{\partial\varphi(t,s)}{\partial s} = 0$ *for all* $s \leq t$.

5.0.2 Gaussian vectors

We gather some freqently used properties of Gaussian variables.

- Let x and y be independent Gaussian variables

  $$x \sim N(\mu_x, \Sigma_x), \quad y \sim N(\mu_y, \Sigma_y)$$

  then

  $$\begin{bmatrix} x \\ y \end{bmatrix} \sim N\left(\begin{bmatrix} \mu_x \\ \mu_y \end{bmatrix}, \begin{bmatrix} \Sigma_x & \\ & \Sigma_y \end{bmatrix} \right)$$

- If $x \sim N(\mu, \Sigma)$, let $y = Ax$, then

  $$y \sim N(A\mu, A\Sigma A^T)$$

- Suppose x and y are jointly Gaussian

  $$\begin{bmatrix} x \\ y \end{bmatrix} \sim N\left(\begin{bmatrix} \mu_x \\ \mu_y \end{bmatrix}, \begin{bmatrix} \Sigma_x & \Sigma_{xy} \\ \Sigma_{xy}^T & \Sigma_y \end{bmatrix} \right)$$

  then

  $$x|y \sim N\left(\mu + \Sigma_{xy}\Sigma_y^{-1}(y - \mu_y),\ \Sigma_x - \Sigma_{xy}\Sigma_y^{-1}\Sigma_{xy}^T\right)$$

  Hence

  $$E[x|y] = \mu + \Sigma_{xy}\Sigma_y^{-1}(y - \mu_y)$$

  Furthermore, $x|y$ and $x - E[x|y]$ are independent.

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