Algorithm simplification for Heisenberg group case

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1 Introduction and Background

Following [**?**, Algorithm 1] we implement a Python program to implement said algorithm in the setting of a Heisenberg substitution. We explain the variation from Algorithm 1 in our python program implementation.
Let us recall shortly the setting from [**?**, Section 3.3]. The action in the Heisenberg group $G := H_3(\mathbb{R})$, is given by

$$(x,y,z)\cdot(a,b,c) := \big(x+a, y+b, z+c+\frac{1}{2}(xb-ay)\big) \quad \text{for any} \quad (x,y,z),(a,b,c)\in H_3(\mathbb{R}).$$

The metric d_H is left invariant metric on $H_3(\mathbb{R})$ satisfying $d_H\big((x,y,z),(a,b,c)\big) = \|(x,y,z)^{-1}\cdot(a,b,c)\|_{CK}$, with $\|\cdot\|_{CK}$ being the Cygan-Koranyi norm on $H_3(\mathbb{R})$ given by

$$\|(x,y,z)\|_{CK} := \sqrt[4]{(x^2+y^2)^2+z^2}.$$

An underlying dilation family on $H_3(\mathbb{R})$ is given by $(D_\lambda)_{\lambda>0}$ where

$$D_\lambda : H_3(\mathbb{R}) \to H_3(\mathbb{R}), \quad D_\lambda(x,y,z) := (\lambda x, \lambda y, \lambda^2 z),$$

The lattice in $H_3(\mathbb{R})$ we consider is $\Gamma := H_3(2\mathbb{Z})$, which is invariant under D_λ for any $\lambda\in\mathbb{N}$. $V := [-1,1)^3$ defines a fundamental domain for Γ in G. Altogether $\mathcal{D}_H = \big(G, d_H, (D_\lambda)_{\lambda>0}, \Gamma, V\big)$ is a dilation datum.
We are interested in checking whether inclusions of the form

$$xF_1 \subseteq D^N(\gamma)V(N,F_2) \tag{1.1}$$

hold, for $x,\gamma\in\Gamma$, finite nonempty sets $F_1,F_2\Subset\Gamma$ and $N\in\mathbb{N}$.

2 Simplifications for the Heisenberg setting

We make some basic observations regarding the Heisenberg substitution case. First, it is easily verifiable that $V(1) = [-\lambda_0,\lambda_0)^2\times[-\lambda_0^2,\lambda_0^2)$. We now turn to establish some estimates regarding $V(n)$.

Lemma 2.1. *Let $n \in \mathbb{N}$ and $\lambda_0 \geq 3$. Then th following estimates hold. Let $\pi_X : H_3(\mathbb{R}) \to \mathbb{R}$, $\pi_Y : H_3(\mathbb{R}) \to \mathbb{R}$ and $\pi_Z : H_3(\mathbb{R}) \to \mathbb{R}$ be projections of the first, second and third coordinates accordingly.*

- *$\pi_X\big(V(n)\big) = \pi_Y\big(V(n)\big)$ and*

$$\pi_X\big(V(n)\big) = \begin{cases} \big[-\lambda_0^n, \lambda_0^n\big), & \text{if } \lambda_0 \notin 2\mathbb{N} \\ \big[-\lambda_0^n - \sum_{k=1}^{n-1} \lambda_0^k, \lambda_0^n - \sum_{k=1}^{n-1} \lambda_0^k\big), & \text{if } \lambda_0 \in 2\mathbb{N} \end{cases}$$

- *For all $(x, y) \in \pi_X\big(V(n)\big) \times \pi_Y\big(V(n)\big)$, the set $\{z \in \mathbb{R} : (x, y, z) \in V(n)\}$ is a connected set of length $2\lambda_0^{2n}$.*

One can verify the previous lemma inductively, which is left to the interested reader.

2.1 Shifting sets with respect to the Heisenberg action

Looking at Algorithm 1, we must verify inclusion (1.1) for $|D^N[V] \cap \Gamma|$. We need to check for each such x, at most $|B(e, R_{F_2} + C_+) \cap \Gamma|$ values of γ_x for (1.1). For each such γ_x, we must also compute $D^N(\gamma_x)V(N, F_2)$. This last computation demands naively $|V(N)|$ many steps. This implies that a naive application of algorithm has a run time of the order λ_0^{8N}.

Luckily, our sets are of rectangular form. i.e., there are $a_1, a_2, c_1, c_2 \in 2\mathbb{Z}$ such that F is of the form

$$F = \Big([a_1, a_2)^2 \times [c_1, c_2)\Big) \cap \Gamma.$$

It can be also represented as a union of "discrete" intervals,

$$F = \bigsqcup_{m_1, m_2 \in 2\mathbb{Z}, a_1 \leq m_i < a_2} \{(m_1, m_2)\} \times \big([c_1, c_2) \cap 2\mathbb{Z}\big).$$

In general, sets of the form

$$F = \bigcup_{(m_1, m_2) \in W} \{(m_1, m_2)\} \times \Big([c_1^{(m_1, m_2)}, c_2^{(m_1, m_2)}) \cap 2\mathbb{Z}\Big) \tag{2.1}$$

where $W \subseteq \big(2\mathbb{Z}\big)^2$, will be called a union of discrete intervals, or UDI in short.

Lemma 2.2. *Let F be a UDI, $\gamma \in \Gamma$ and $N \in \mathbb{N}$. Then, both γF and $V(N, F) \cap \Gamma$ are UDIs.*

Proof. Note first that the map $\eta \mapsto \gamma \cdot \gamma$ is a bijection for any γ. Note also that

$$V(N+1, F) = D\big[V(N, F) \cap \Gamma\big] \cdot V(1).$$

It holds for distinct $\eta_1, \eta_2 \in \Gamma$ that $D(\eta_1)V(1) \cap D(\eta_2)V(1) = \emptyset$. It then follows that is enough to verify the claim for sets of the form $F = \{(m_1, m_2)\} \times \big([c_1, c_2) \cap 2\mathbb{Z}\big)$. For such sets, computing γF gives that $\gamma F = \{(\gamma_1 + m_1, \gamma_2 + m_2)\} \times \big(\gamma_3 + \Delta(\gamma_1, \gamma_2; m_1, m_2) + [c_1, c_2) \cap 2\mathbb{Z}\big)$, where $\Delta(\gamma_1, \gamma_2; m_1, m_2) := \frac{1}{2}(\gamma_1 m_2 - \gamma_2 m_1)$. By an inductive argument, it is enough to show that $V(1, F) \cap \Gamma$ is a UDI. Note that

$$D[F] = \big\{(\lambda_0 m_1, \lambda_0 m_2)\big\} \times \Big([\lambda_0^2 c_1, \lambda_0^2 c_2 - 2\lambda_0^2] \cap 2\lambda_0^2\mathbb{Z}\Big),$$

which in turn implies for any $w_1, w_2 \in [-\lambda_0, \lambda_0) \cap 2\mathbb{Z}$ we have

$$D[F]\cdot\Big(\{(w_1,w_2)\}\times[-\lambda_0^2,\lambda_0^2)\Big) = \big\{(\lambda_0 m_1+w_1, \lambda_0 m_2+w_2)\big\}\times\Big(\lambda_0\Delta(m_1,m_2;w_1,w_2)+\big[\lambda_0^2(c_1-1),\lambda_0^2(c_2-1)\big)\Big).$$

We deduce that

$$V(1,F) = \bigcup_{w_1,w_2\in[-\lambda_0,\lambda_0)\cap 2\mathbb{Z}} \big\{(\lambda_0 m_1+w_1, \lambda_0 m_2+w_2)\big\}\times\Big(\lambda_0\Delta(m_1,m_2;w_1,w_2)+\big[\lambda_0^2(c_1-1),\lambda_0^2(c_2-1)\big)\Big),$$

which is a UDI. □

We therefore prefer to operate on sets more efficiently by saving them as UDIs.

2.2 Computing the search radius R_F

In algorithm 1 line 11, we wish to compute a search radius to find possible γ_x for each x, for a susupected testing domain $F \Subset \Gamma$. Note first, that as long as we find a $\gamma_x = (\gamma_1, \gamma_2, \gamma_3)$ satisfying inclusion (1.1) for each x, it does not matter what R_F was by [**?**, Lemma 6.3]. Therefore, if the algorithm returns True for a smaller than theoretically needed search radius, it still implies that F is a testing domain.
However, given that the dilation with respect to the XY coordinates beahves as in the case of Euclidean block substitutions, we may compute efficiently what $\gamma_1, \gamma_2 \in 2\mathbb{Z}$ may be. This gives us a set $A_{XY}(x;N) \in \big(2\mathbb{Z}\big)^2$ such that $(\gamma_1,\gamma_2) \in A_{XY}(x;N)$ and $|A_{XY}(x;N)| \leq 4$. It then remains to find whether there exists values of $\gamma_3 \in 2\mathbb{Z}$ satisfying inclusion (1.1).
Let us denote $I(c_1, c_2) := [c_1, c_2] \cap 2\mathbb{Z}$. Then, we obtain that

$$xF = \bigsqcup_{a_1\leq m_i\leq a_2} \{(m_1+x_1, m_2+x_2)\} \times \Big(x_3 + \Delta\big(x_1,x_2;m_1,m_2\big) + I(c_1,c_2)\cap 2\mathbb{Z}\Big).$$

An inductive argument on N, shows that for any N we have that

$$V(N,F) = \bigsqcup_{(m_1,m_2)\in W_N(a_1,a_2)} \{(m_1,m_2)\} \times \Big(\delta_{m_1,m_2} + I_n(\lambda_0)\cap 2\mathbb{Z}\Big),$$

where $\delta_{m_1,m_2} \in 2\mathbb{Z}$ It follows that

$$D^N(\gamma)V(N,F) = \bigcup_{(m_1,m_2)\in W_N(a_1,a_2)} \{(\lambda_0^N\gamma_1+m_1, \lambda_0^N\gamma_2+m_2)\}\times\Big(\lambda_0^{2N}\gamma_3+\lambda_0^N\cdot\Delta\big((\gamma_1,\gamma_2),(m_1,m_2)\big)+\delta_{m_1,m_2}+I_n(\lambda_0)\cap 2\mathbb{Z}\Big).$$

Hence, there exists $\gamma_x \in \Gamma$ satisfying inclusion (1.1) if and only if for some $(\gamma_1,\gamma_2) \in A_{XY}(x;N)$ there exists $\gamma_3 \in 2\mathbb{Z}$ such that

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