28 Mar 2022 L1 Intro to Waves AMATH 353

<<< Intro to Waves

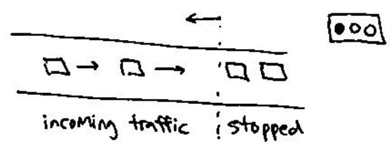
→ Stadium waves

- medium (fans) moves up & down
- wave moves left & right

} → transverse wave (traveling pulse)

→ Traffic density waves

- car moves forward
- red light signal moves backward



→ Water waves

> frequency - # of peaks / troughs that pass a fixed point per second

> period - reciprical of frequency

- $T = f^{-1}$

> wavelength - distance between two successive waves

- diff waves are formed due to diff disturbing forces, restoring forces on diff. length & time scales

> progressive (traveling) waves - moves horizontally

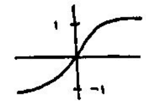
> standing waves - no horizontal movement, oscillates vertically.

<<< Intro to Waves

→ Water Waves (Mathematical Description)

- properties of water waves
  - > period T
  - > frequency $f = \frac{1}{T}$
  - > angular frequency $\omega = 2\pi f$
  - > wavelength λ
  - > wave number $k = \frac{2\pi}{\lambda}$
  - > phase speed $c = \frac{\lambda}{T}$ (celerity of propagation)
  - > depth of water

- hyperbolic functions
  - $\sinh x = \frac{e^x - e^{-x}}{2}$
  - $\cosh x = \frac{e^x + e^{-x}}{2}$
  - $\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$



  - small x approx : $\tanh x \approx x$ if $|x| < 0.05$
  - large x approx : $\tanh x \approx 1$ if $x > \pi$
  - neg. large x approx : $\tanh x \approx -1$ if $x < -\pi$

- speed of water waves is depth-dependent
  - $c = \sqrt{\left(\frac{g\lambda}{2\pi} + \frac{T 2\pi}{\lambda\rho}\right) \tanh\left(\frac{2\pi}{\lambda} d\right)}$ gravity & surface tension
  - $c \approx \sqrt{\frac{g\lambda}{2\pi} \tanh\left(\frac{2\pi}{\lambda} d\right)}$ neglegible surface tension : $\frac{T 2\pi}{\lambda\rho} \to 0$ as $\lambda \uparrow$

→ Deep Water waves ($d > \frac{\lambda}{2}$)

$$\left.\begin{aligned} c &= \sqrt{\frac{g\lambda}{2\pi}} \\ c &= \frac{\lambda}{T} \end{aligned}\right\} \quad \frac{g\lambda}{2\pi} = \frac{\lambda^2}{T} \quad \Rightarrow \quad \boxed{\lambda = \frac{gT^2}{2\pi}} \quad \boxed{c = \frac{gT}{2\pi}}$$

> dispersion - phase velocity depend on frequency (period)

→ Shallow water waves ($d \ll \frac{\lambda}{2}$)

- $\boxed{c = \sqrt{gd}}$ → no dispersion
- $\boxed{\lambda = \sqrt{gd}\, T}$

- tsunami formation and danger
  - tsunami initially moves fast in deep water (large T)
  - deep → shallow
  - fast → slow (loss of T dependence)
  - E flux is constant
    - $c \downarrow$, E density $\uparrow$, mech. E dissapates thru height $\uparrow$

→ Seismic Waves

> P wave - longitudal, body wave, spherical dissipation $\propto \frac{1}{r^2}$

> S wave - transverse, body wave, spherical dissipation $\propto \frac{1}{r^2}$, ½ speed of P wave

> Love wave - surface wave, dissipation $\propto \frac{1}{r}$

> Rayleigh wave - surface wave, dissipation $\propto \frac{1}{r}$

<<< Intro to Waves

→ Waves of Biological Invasion

- medium moves with waves
- bird migration, disease spread

→ Other Examples of Waves

- light, sound, brain waves

<<< Definition of Waves

1. Disturbance ⇒ signal
   - measurable diagnostic signal
   - e.g. height, density
2. Speed of propagation

> Wave – any recognizable signal that is transferred from one part of a medium to another with recognizable velocity of propagation.

<<< Mathematical Representation of Waves

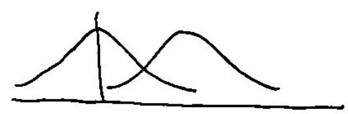
$\boxed{\text{Ex1}}$ $u(x,t) = f(x - ct)$

- moves to the right at speed c
- at $t=0$, $u(x,0) = f(x)$
- track $f(x=0)$ at later t,

$$x - ct = 0$$
$$x = ct$$
$$\frac{dx}{dt} = c$$

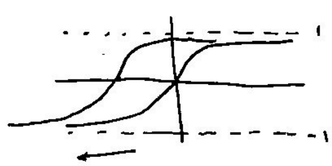
$\boxed{\text{Ex2}}$ $u(x,t) = e^{-(x-t)^2}$

- moves to the right at speed 1
- $u(x,0) = e^{-x^2}$



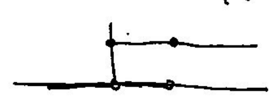
$\boxed{\text{Ex3}}$ $u(x,t) = \tanh(x+t)$

- moves to the left at speed 1
- $u(x,0) = \tanh(x)$



$\boxed{\text{Ex4}}$ $u(x,t) = H(x-2t)$

- moves to the right at speed 2
- $u(x,0) = H(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$



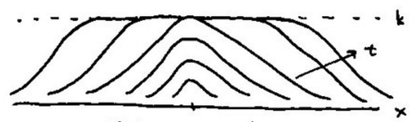
→ Visualization

- Snapshots
- Animations
- Slice Plot
- Surface Plot
- Density Plot (x-t)

<<< Examples of PDE

- advection eqn - $u_t + cu_x = 0$ linear 1st order homogeneous PDE
- diffusion eqn - $u_t = Du_{xx}$ linear 2nd order homogeneous PDE
- convection eqn - $u_t + cu_x = Du_{xx}$ linear 2nd order homogeneous PDE
- Burger's eqn - $u_t + uu_x = Du_{xx}$ nonlinear 2nd order PDE
- reaction-diffusion model - $u_t = ru + Du_{xx}$ linear 2nd order homogeneous PDE

reaction diffusion model



Fisher eqn (logistic growth)

- Fisher eqn - $u_t = ru\left(1-\frac{u}{k}\right) + u_{xx}$ nonlinear 2nd order PDE
- Nagumo eqn - $u_t = u(u-a)(1-u) + u_{xx}$ nonlinear 2nd order PDE
- wave eqn - $u_{tt} = c^2 u_{xx}$ linear 2nd order homogeneous PDE

- PDEs are derived from first principles
  - mass balance eqn
  - conservation laws
  - other physical laws (e.g. Newton's 2nd law)

<<< Classification of PDE

- order - order of the highest partial derivative
  - $u_t + cu_x = 0$ - 1st order
  - $u_t = Du_{xx}$ - 2nd order
  - $u_t + uu_x + u_{xxx} = 0$ - 3rd order
- number of variables
  - $u_t = Du_{xx}$ - 2 var
  - $u_t = D(u_{xx} + u_{yy})$ - 3 var
- linearity - dependent variable u are not multiplied together or in function arguments
  - 1st order linear PDE : $au_t + bu_x + cu = h$
  - 2nd order linear PDE : $au_{tt} + bu_{tx} + cu_{xx} + du_t + eu_x + fu + h$

→ Linear PDE Classification

- homogeneity - homogeneous if $h = 0$
- Coefficient - constant or variable (does not depend on u either way!)
- basic type - classifies 2nd order PDE
  - hyperbolic - $b^2 - 4ac > 0$ disturbing-preserving waves
  - parabolic - $b^2 - 4ac = 0$ smoothing, spreading flows
  - elliptic - $b^2 - 4ac < 0$ equilibrial, energy minimizing surfaces

6 Apr 2022 L5 Basic Types, Physical Interpretation AMATH 353

<<< Example of Basic Types

$\boxed{Ex1}$ $u_t = u_{xx}$ $u_t - u_{xx} = 0$

$b^2 - 4ac = 0 - 4(1)(0) = 0 \Rightarrow$ parabolic

$\boxed{Ex2}$ $u_{tt} = u_{xx}$ $u_{tt} - u_{xx} = 0$

$b^2 - 4ac = 0 - 4(1)(-1) = 4 > 0 \Rightarrow$ hyperbolic

$\boxed{Ex3}$ $u_{tx} = 0$

$b^2 - 4ac = 1^2 - 4(0)(0) = 1 > 0 \Rightarrow$ hyperbolic

$\boxed{Ex4}$ $u_{xx} + u_{yy} = 0$

$b^2 - 4ac = 0 - 4(1)(1) = -4 < 0 \Rightarrow$ elliptic

$\boxed{Ex5}$ $y u_{xx} + u_{yy} = 0$

$b^2 - 4ac = 0 - 4(y)(1) = -4y$

$-4y = 0$

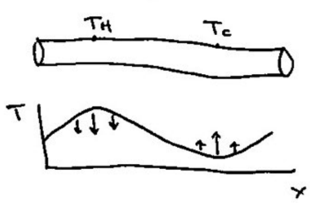
$$y = 0 \Rightarrow \begin{cases} y > 0 & \text{hyperbolic} \\ y = 0 & \text{parabolic} \\ y < 0 & \text{elliptic} \end{cases}$$

<<< Physical Interpretation of Derivatives

- u_t - rate of change (ROC) of u (e.g. velocity)
- u_{tt} - ROC of ROC of u (e.g. acceleration)
- u_x - slope
- u_{xx} - concavity

$\boxed{Ex6}$ Interpret $u_t = D u_{xx}$ of Temperature of a 1D rod

- Rate of change of Temp is proportional to concavity of Temp curve.



- at max, concave down, T decrease
- at min, concave up, T increase

<<< Traveling Waves

> traveling wave solution - $u(x,t) = f(x-ct)$

- depending on PDE, may be restriction on f and c.

[EX] Formulate advection eqn using traveling wave soln.

- Let $z \equiv x-ct$, so $u(x,t) = f(x-ct) = f(z)$

- $\frac{\partial u}{\partial t} = \frac{\partial f}{\partial z}\frac{\partial z}{\partial t} = -c\frac{df}{dz} = -c f'(z)$

- $\frac{\partial u}{\partial x} = \frac{\partial f}{\partial z}\frac{\partial z}{\partial x} = \frac{df}{dz} = f'(z)$

$$-cf'(z) + cf'(z) = 0$$

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$$

- Function-independent

[EX] Verify traveling wave is soln to wave eqn $u_{tt} = a^2 u_{xx}$

- $\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial t}[-cf'(z)] = -c \cdot \frac{\partial}{\partial z}f'(z)\frac{\partial z}{\partial t} = +c^2 f''(z)$

- $\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x} f'(z) = \frac{\partial}{\partial z} f'(z)\frac{\partial z}{\partial x} = f''(z)$

- Substitute, $+c^2 f''(z) = a^2 f''(z)$

$$(c^2-a^2) f''(z) = 0$$

$c^2 = a^2$ or $f''(z) = 0$

$c = \pm a$ $\qquad$ $f(z) = A + Bz$

$\boxed{u(x,t) = f(x \pm at)}$ or $\boxed{u(x,t) = A + B(x-ct)}$

- restriction on $c = \pm a$
- no restriction on f

- restriction on f
- no restriction on c

- general soln : $\boxed{u(x,t) = c_1 f_1(x-at) + c_2 f_2(x+at) + c_3[A+B(x-ct)]}$

  - does not have to be traveling wave after linear combination.
  - could have diff speed and/or direction.

<<< Types of Travelling Wave

→ Wave front

· diagnostic signal approaches diff limit in front of and behind the wave

> $\lim_{x\to\infty} u(x,t) = k_1$

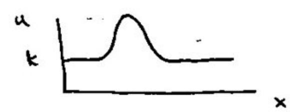
> $\lim_{x\to -\infty} u(x,t) = k_2$

$\Big\}\ k_1 \neq k_2$

→ Pulse

· diagnostic signal approaches same limit in front of and behind the wave

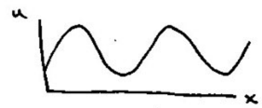
> $\lim_{x\to\infty} u(x,t) = \lim_{x\to -\infty} u(x,t) = k$



→ Wave trains

> $u(x,t) = f(kx + \omega t)$

· oscillates indefinitely



$\boxed{Ex}$ $u(x,t) = A\cos(kx - \omega t)$

$= A\cos\left[k\left(x - \frac{\omega}{k}t\right)\right]$, where $c = \frac{\omega}{k}$

> amplitude A - largest value of wave train

> wave number $k = \frac{2\pi}{\lambda}$

> angular frequency $\omega = 2\pi f$

<<< Example of Wave Train and Dispersion

$\boxed{Ex}$ Advection eqn $u_t + a u_x = 0$ is nondispersive. Find its dispersion relation.

· $u(x,t) = A\cos(kx - \omega t)$

· $u_t = -A\sin(kx-\omega t)(-\omega)$

· $u_x = -A\sin(kx-\omega t)(k)$

$A\omega\sin(kx-\omega t) - aAk\sin(kx-\omega t) = 0$

$A(\omega - ak)\sin(kx-\omega t) = 0$

$\omega - ak = 0$

$\omega = ak$

· speed c does not depend on k

· $c = \frac{\omega}{k} = \frac{ak}{k} = a$

$\boxed{Ex}$ Find the above dispersion relation using complex wave train.

· $u(x,t) = \cos(kx-\omega t) + i\sin(kx-\omega t) = e^{i(kx-\omega t)}$

· $u_x = ik\,e^{i(kx-\omega t)}$

· $u_t = -i\omega\,e^{i(kx-\omega t)}$

$-i\omega e^{i(kx-\omega t)} + a i k e^{i(kx-\omega t)} = 0$

$i(-\omega + ak)\,e^{i(kx-\omega t)} = 0$

$\omega = ak$

· $c = \frac{\omega}{k} = a$

<< Example of Wave Train and Dispersion (cont.)

$\boxed{EX}$ Find dispersion relation of Klein-Gordon eqn $u_{tt} = a^2 u_{xx} - bu$

- $u(x,t) = A\cos(kx - \omega t)$
- $u_t = -A\sin(kx-\omega t)(-\omega)$
- $u_{tt} = -A\omega^2\cos(kx-\omega t)$
- $u_x = -Ak\sin(kx-\omega t)$
- $u_{xx} = -Ak^2\cos(kx-\omega t)$

$$-A\omega^2\cos(kx-\omega t) = -a^2Ak^2\cos(kx-\omega t) - bA\cos(kx-\omega t)$$
$$A(\omega^2 - a^2k^2 - b)\cos(kx-\omega t) = 0$$
$$\omega^2 - a^2k^2 - b = 0$$
$$\omega = \pm\sqrt{a^2k^2+b}$$

- phase speed $c = \frac{\omega}{k} = \pm\sqrt{\frac{a^2k^2+b}{k^2}} = \pm\sqrt{a^2+\frac{b}{k^2}} = \pm\sqrt{a^2+\frac{a^2b}{\omega^2-b}}$
    - PDE is dispersive $\because$ $c(k)$ or $c(\omega)$
    - normal dispersion - $\omega\uparrow, c\downarrow$

$\boxed{EX}$ Find dispersion relation of beam eqn $u_{tt} + a^2u_{xxxx} = 0$

- $u_{xxx} = +Ak^3\sin(kx-\omega t)$
- $u_{xxxx} = Ak^4\cos(kx-\omega t)$
- Substitute, $-A\omega^2\cos(kx-\omega t) + a^2Ak^4\cos(kx-\omega t) = 0$

$$A(-\omega^2 + a^2k^4)\cos(kx-\omega t) = 0$$
$$\omega^2 = a^2k^4$$
$$\omega = \pm ak^2$$

- $c = \frac{\omega}{k} = \pm ak = \pm\sqrt{a\omega}$
    - PDE is dispersive
    - anomolous dispersion - $\omega\uparrow, c\uparrow$

→ Dispersion

- dispersion relation can be complex
- dispersive only for phase speed that's real and nonconstant
- In dispersion, each component move in its own phase speed without preserving shape.
    - certain wave packets preserve shape and move in group velocity.
    - beats - wave packets

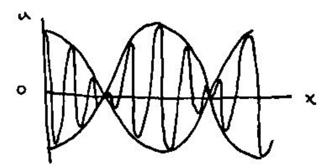
<<< Dispersion & Group Velocity

$\boxed{\text{EX}}$ Derive expression of group velocity of dispersive PDE traveling wave soln.

- Assume $u(x,t) = u_1(x,t) + u_2(x,t)$
  - $u_1(x,t) = a\cos(k_1 x - \omega_1 t)$ $k_1 \equiv k + \Delta k$ $\omega_1 \equiv \omega + \Delta\omega$
  - $u_2(x,t) = a\cos(k_2 x - \omega_2 t)$ $k_2 \equiv k - \Delta k$ $\omega_2 \equiv \omega - \Delta\omega$
- By trig identity $\cos A + \cos B = 2\cos[\frac{1}{2}(A-B)]\cos[\frac{1}{2}(A+B)]$
  - $u(x,t) = 2a\cos[\frac{1}{2}(k_1x - \omega_1 t - k_2 x + \omega_2 t)]\cos[\frac{1}{2}(k_1 x - \omega_1 t + k_2 x - \omega_2 t)]$

$$= 2a\cos[\tfrac{1}{2}(2\Delta k x - 2\Delta\omega t)]\cos[\tfrac{1}{2}(2kx - 2\omega t)]$$

$$= \underbrace{2a\cos(\Delta k x - \Delta\omega t)}_{\text{Amplitude (envelope)}}\underbrace{\cos(kx - \omega t)}_{\text{traveling wave}}$$



- phase speed $c = \frac{\omega}{k}$
- group speed $c_g = \frac{\Delta\omega}{\Delta k} \Rightarrow \boxed{c_g = \frac{d\omega}{dk}}$

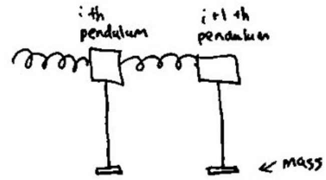
$\boxed{\text{EX}}$ Find group speed of dispersion relation $\omega = ak^2$ of beam eqn $u_{tt} + a^2 u_{xxxx} = 0$

- $c = \frac{\omega}{k} = \frac{ak^2}{k} = ak$
- $c_g = \frac{d\omega}{dk} = \frac{d}{dk}(ak^2) = 2ak$

13 Apr 2022 L8 Sine - Gordon Equation AMATH 353

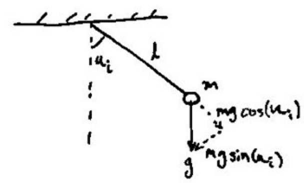
<-- Derivation of the Sine-Gordon Eqn

- crystal defect, coupled torsion pendula
- Let $u_i(t)$ be angle of rotation of i th pendulum



- $\sum \tau = I \frac{d^2u}{dt^2} = m l^2 \frac{d^2u}{dt^2}$ (Newton's second law)

  - torque of gravity $= -mgl\sin(u_i(t))$
  - torque of $i+1$ th pendulum $= K \dfrac{u_{i+1}(t) - u_i(t)}{\Delta x}$
  - torque of $i-1$ th pendulum $= K \dfrac{u_{i-1}(t) - u_i(t)}{\Delta x}$



$$m l^2 \frac{d^2u}{dt^2} = -mgl\sin(u_i(t)) + K\frac{u_{i+1} - 2u_i + u_{i-1}}{(\Delta x)^2}$$

- Let $\lim_{x\to 0} \frac{m}{\Delta x} = M$ be mass density
- Note the second order central difference term

$$\begin{aligned} & u(x+\Delta x) = u(x) + u'(x)\Delta x + \tfrac{1}{2}u''(x)(\Delta x)^2 + \cdots \\ + \; & u(x-\Delta x) = u(x) - u'(x)\Delta x + \tfrac{1}{2}u''(x)(\Delta x)^2 + \cdots \\ \hline & u(x+\Delta x) + u(x-\Delta x) = 2u(x) + u''(x)(\Delta x)^2 \end{aligned}$$

$$\Rightarrow \quad u''(x) = \frac{u(x+\Delta x) - 2u(x) + u(x-\Delta x)}{(\Delta x)^2}$$

$$Ml^2 u_{tt} = -Mgl\sin(u) + Ku_{xx}$$

- Let $A \equiv Ml^2$
- Let $T \equiv mgl$

$$Au_{tt} - Ku_{xx} + T\sin(u) = 0$$

$$\boxed{u_{tt} - u_{xx} + \sin(u) = 0}$$ change of variables

sine - Gordon equation

... Traveling Wave Solution of Sine-Gordon eqn

- $u(x,t) = f(z) = f(x-ct)$
  - $u_t = -c f'(z)$
  - $u_{tt} = c^2 f''(z)$
  - $u_x = f'(z)$
  - $u_{xx} = f''(z)$
- $u_{tt} - u_{xx} + \sin(u) = 0$

$$c^2 f'' - f'' + \sin(f) = 0$$

$$(c^2-1) f'' + \sin(f) = 0$$

$$\int (c^2-1) f'' f' + \sin(f) f' = \int 0$$

$$\tfrac{1}{2}(c^2-1)(f')^2 - \cos(f) = a = -1$$

b.c. $\begin{cases} \lim_{z\to\infty} f(z) = 0 \\ \lim_{z\to\infty} f'(z) = 0 \end{cases}$ for unpreturbed at ∞

$$(f')^2 = \frac{2}{1-c^2}(1-\cos(f))$$

$$(f')^2 = \frac{4}{1-c^2}\sin^2\left(\frac{f}{2}\right)$$

$2\sin^2\left(\frac{x}{2}\right) = 1-\cos(x)$ half angle formula

$$f' = -\frac{2}{\sqrt{1-c^2}}\sin\left(\frac{f}{2}\right)$$

choose negative $\because$ expect decrease to zero

$$g' = -\frac{1}{\sqrt{1-c^2}}\sin(g)$$

Let $g \equiv \frac{f}{2}$, $g' = \frac{1}{2}f'$

$$\int \frac{1}{\sin(g)}\,dg = \int -\frac{1}{\sqrt{1-c^2}}\,dz$$

$\int \frac{1}{\cos x}\,dx = \int \sec x\,dx = \ln(\sec x + \tan x) = \ln\left(\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)\right)$

$\int \frac{1}{\sin x}\,dx = \int \csc x\,dx = \ln(\csc x - \cot x) = \ln\left(\tan\left(\frac{x}{2}\right)\right)$

$$\ln\left(\tan\left(\frac{g}{2}\right)\right) = -\frac{1}{\sqrt{1-c^2}}z$$

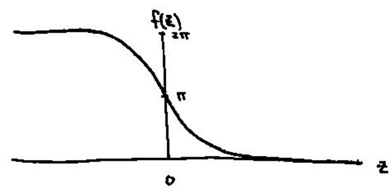
$$g = 2\tan^{-1}\left(\exp\left(-\frac{z}{\sqrt{1-c^2}}\right)\right)$$

$$f = 4\tan^{-1}\left(\exp\left(-\frac{z}{\sqrt{1-c^2}}\right)\right)$$

$$u(x,t) = 4\tan^{-1}\left(\exp\left(-\frac{x-ct}{\sqrt{1-c^2}}\right)\right)$$

$\begin{cases} \lim_{z\to\infty} f(z) = 0 \\ \lim_{z\to-\infty} f(z) = 2\pi \end{cases}$

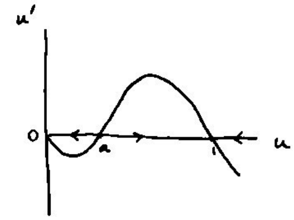
- physical interpretation
  - Far ahead the front, pendula are unperturbed
  - Far behind the front, pendula are at angle of 2π
  - Pendula in the back has rotated counterclockwize exactly one revolution
    - antikink traveling wave
  - Steepness of the front determined by speed c
    - $\uparrow c$, $\uparrow$ steepness



15 Apr 2022 L9 The Nagumo Equation AMATH 353

<<< Qualitative Understanding of Nagumo Eqn

- reaction-diffusion model for modeling spread of invading organisms.
    - $u(x,t)$ is population density



- $\boxed{u_t = \underbrace{u(u-a)(1-u)}_{\text{reaction}} + \underbrace{u_{xx}}_{\text{diffusion}}}$
    - ODE - $\frac{du}{dt} = u(u-a)(1-u)$
        - threshold at $u=a$ called strong Allee effect

<<< Traveling Wave Soln of Nagumo eqn

- $u(x,t) = f(z) = f(x-ct)$
    - $u_t = -cf'(z)$
    - $u_x = f'(z)$
    - $u_{xx} = f''(z)$

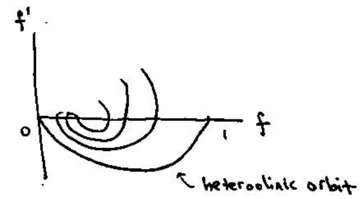
$$u_t = u(u-a)(1-u) + u_{xx}$$
$$-cf' = f(f-a)(1-f) + f''$$
$$f'' + cf' + f(f-a)(1-f) = 0$$

b.c. $\begin{cases} \lim_{z\to\infty} f(z) = 0 , & \lim_{z\to\infty} f'(z) = 0 \\ \lim_{z\to-\infty} f(z) = 1 , & \lim_{z\to-\infty} f'(z) = 0 \end{cases}$

→ Parabolic heteroclinic orbit

> heteroclinic orbit - connection between eqm pts in phase plot

- connects saddle eqm pts $(0,0)$, $(1,0)$



- Verify parabolic guess $f' = bf(f-1)$
    - $f'' = bff' + bf'(f-1)$
      $= bf'(2f-1)$
      $= b^2 f(f-1)(2f-1)$
    - Substitute in PDE,

$$b^2 f(f-1)(2f-1) + cbf(f-1) + f(f-a)(1-f) = 0$$
$$f(f-1)\left[b^2(2f-1) + cb - (f-a)\right] = 0$$
$$(-b^2 + cb + a) + (2b^2 - 1)f = 0$$

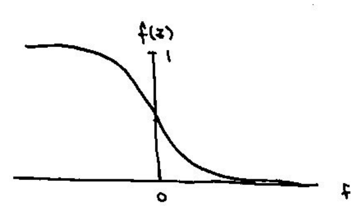
$-b^2 + cb + a = 0$ or $2b^2 - 1 = 0$

$c = \sqrt{2}\left(\frac{1}{2} - a\right)$ $\qquad b = \frac{1}{\sqrt{2}}$

$\begin{cases} c < 0, \text{ move fwd}, & a < \frac{1}{2} \\ c > 0, \text{ move backward}, & a > \frac{1}{2} \end{cases}$

→ Traveling Wave Soln

- The parabolic guess ODE $\quad f' = bf(f-1)$



$$\int \frac{df}{f(f-1)} = \int \frac{1}{\sqrt{2}}\, dz$$
$$\int \left(\frac{1}{f-1} - \frac{1}{f}\right) df = \int \frac{1}{\sqrt{2}}\, dz$$
$$\ln\left|\frac{f-1}{f}\right| = \frac{1}{\sqrt{2}} z + \cancel{c}^{\,0}$$
$$\frac{1-f}{f} = \exp\left(\frac{1}{\sqrt{2}} z\right)$$

choose flip sign for continuous value at 0.

$$f = \frac{1}{1 + e^{z/\sqrt{2}}}$$

<<< Observation of Traveling Pulse

- Russell observed solitary wave (soliton)
  - shape of hyperbolic secant
  - large enough water can produce 2 or more independent solitary waves
  - solitary waves cross each other without shape change
  - speed given by $c^2 = g(d+a)$, shallow water wave (d, depth; a, amplitude)
- Korteweg-de Vries eqn (kdV) describes long wavelength, small amplitude waves in shallow water
  - $\boxed{u_t + uu_x + u_{xxx} = 0}$
  - $u(x,t)$ is height of water

<<< Traveling Wave Solution of kdV eqn

- $u(x,t) = f(z) = f(x-ct)$
  - $u_t = -c f'(z)$
  - $u_x = f'(z)$
  - $u_{xx} = f''(z)$
  - $u_{xxx} = f'''(z)$

$$u_t + uu_x + u_{xxx} = 0$$

$$\int -cf' + ff' + f''' = \int 0$$

$$-cf + \tfrac{1}{2}f^2 + f'' = a = 0$$

$$\int -cff' + \tfrac{1}{2}f^2 f' + f''f' = \int 0$$

$$-\tfrac{1}{2}cf^2 + \tfrac{1}{6}f^3 + \tfrac{1}{2}(f')^2 = b = 0$$

b.c.
$$\begin{cases} \lim_{z\to\pm\infty} f(z) = 0 \\ \lim_{z\to\pm\infty} f'(z) = 0 \\ \lim_{z\to\pm\infty} f''(z) = 0 \end{cases}$$

$$3(f')^2 = (3c-f)f^2 \quad \Rightarrow \quad f(z) \in [0, 3c]$$

$$f' = \pm\sqrt{\frac{3c-f}{3}f^2}$$

$$\frac{df}{dz} = \pm\sqrt{\frac{3c-f}{3}}\, f$$

$$\int \frac{\sqrt{3}}{\sqrt{3c-f}\,f}\, df = \pm\int dz$$

$$\begin{cases} g = \sqrt{3c-f} \\ f = 3c - g^2 \\ df = -2g\,dg \end{cases}$$

$$2\sqrt{3}\int \frac{1}{3c-g^2}\, dg = -z$$

$$\int \left(\frac{1}{\sqrt{3c}+g} + \frac{1}{\sqrt{3c}-g}\right) dg = -\sqrt{c}\, z$$

$$\ln\left|\frac{\sqrt{3c}+g}{\sqrt{3c}-g}\right| = -\sqrt{c}\, z$$

$$g(z) = \sqrt{3c}\,\frac{e^{-\sqrt{c}z}-1}{e^{-\sqrt{c}z}+1}\cdot\frac{e^{\sqrt{c}z/2}}{e^{\sqrt{c}z/2}}$$

partial fraction

$$\frac{1}{3c-g^2} = \frac{A}{\sqrt{3c}+g} + \frac{B}{\sqrt{3c}-g}$$

$$A(\sqrt{3c}-g) + B(\sqrt{3c}+g) = 1$$

$$(A+B)\sqrt{3c} + (B-A)\, g = 1$$

$$\begin{cases} (A+B)\sqrt{3c} = 1 \\ B - A = 0 \end{cases} \Rightarrow 2A = \frac{1}{\sqrt{3c}}$$

$$A = \frac{1}{2\sqrt{3c}} \qquad B = \frac{1}{2\sqrt{3c}}$$

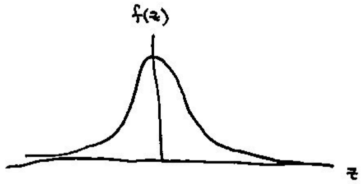
$$g(z) = -\sqrt{3c}\,\tanh\left(\frac{\sqrt{c}\,z}{2}\right)$$

$$f(z) = 3c - g^2 = 3c - 3c\tanh^2\left(\frac{\sqrt{c}\,z}{2}\right)$$

$$\boxed{f(z) = \underbrace{3c}_{\text{Amplitude}}\,\mathrm{sech}^2\left(\frac{\sqrt{c}\,z}{2}\right)}$$

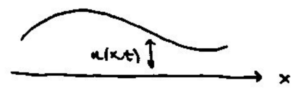
$$\mathrm{sech}^2 x = 1 - \tanh^2 x$$

- Amplitude depends on speed
  - $A = 3c$



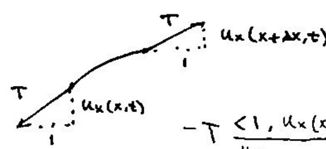
<<< Derivation of the Wave Equation

- Context: tightly stretched vibrating string
  - $u(x,t)$ is vertical displacement
  - u_t is vertical velocity
  - u_{tt} is vertical acceleration
  - u_x is slope of string



- Simplifying assumptions
  - uniform string - constant density ρ
  - planar vibration - string remains in vertical plane
  - uniform tension - tension is same magnitude (but diff direction) for any piece of string
  - no other force - tension only. no gravity, friction, restoring force, etc.
  - small vibration - ignore nonlinear effects, $u(x,t)$, $u_x(x,t)$ are small ($u(x,t) \ll 1$, $u_x(x,t) \ll 1$)

- Newton's second law on small segment
  - $F = ma$
    - F = vertical tension at two ends
      - left end = $-T u_x(x,t)$
      - right end = $T u_x(x+\Delta x, t)$
    - $m = \rho \Delta x$
    - $a = u_{tt}$



$$-T \frac{\langle 1, u_x(x,t)\rangle}{\|\langle 1, u_x(x,t)\rangle\|}$$

y-comp
$$= -T \frac{u_x(x,t)}{\sqrt{1^2 + \cancel{u_x^2}^{\approx 0}}}$$

$$= -T u_x(x,t)$$

$$m = \rho \int_x^{x+\Delta x} \sqrt{1 + \cancel{u_x^2(s,t)}^{\approx 0}}\, ds$$

$$= \rho \Delta x$$

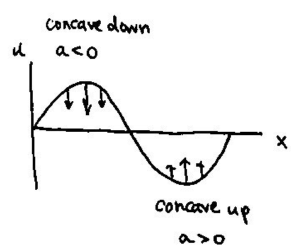
  - $\rho \Delta x\, u_{tt} = -T u_x(x,t) + T u_x(x+\Delta x, t)$

$$\rho u_{tt} = \frac{-T u_x(x,t) + T u_x(x+\Delta x, t)}{\Delta x}$$

$$\rho u_{tt} = T u_{xx}$$

$$\boxed{u_{tt} = c^2 u_{xx}} \qquad \text{let } c \geq \sqrt{\frac{T}{\rho}}$$

- Physical significance
  - Acceleration proportional to concavity
    - concave up accelerates upward
    - concave down accelerates downward



22 Apr 2022 L12 d'Alembert's Solution AMATH 353

<-- Derivation of the d'Alembert's solution

→ Change of variables

- $\xi \equiv x - ct$
- $\eta \equiv x + ct$
- $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\cancelto{1}{\frac{\partial \xi}{\partial x}} + \frac{\partial u}{\partial \eta}\cancelto{1}{\frac{\partial \eta}{\partial x}} = u_\xi + u_\eta$
- $$\begin{aligned}\frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x}(u_\xi + u_\eta)\\ &= \frac{\partial}{\partial \xi}(u_\xi + u_\eta)\cancelto{1}{\frac{\partial \xi}{\partial x}} + \frac{\partial}{\partial \eta}(u_\xi + u_\eta)\cancelto{1}{\frac{\partial \eta}{\partial x}}\\ &= u_{\xi\xi} + u_{\eta\xi} + u_{\xi\eta} + u_{\eta\eta}\\ &= u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}\end{aligned}$$

  $\left\{\begin{array}{l}\cdot \text{ smooth for } u_{\eta\xi} = u_{\xi\eta}\\ \cdot \text{ twice continuously differentiable}\end{array}\right.$
- $$\begin{aligned}\frac{\partial u}{\partial t} &= \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial t}\\ &= -c u_\xi + c u_\eta\end{aligned}$$
- $$\begin{aligned}\frac{\partial^2 u}{\partial t^2} &= \frac{\partial}{\partial t}(-c u_\xi + c u_\eta)\\ &= \frac{\partial}{\partial \xi}(-c u_\xi + c u_\eta)\frac{\partial \xi}{\partial t} + \frac{\partial}{\partial \eta}(-c u_\xi + c u_\eta)\frac{\partial \eta}{\partial t}\\ &= (-c u_{\xi\xi} + c u_{\eta\xi})(-c) + (-c u_{\xi\eta} + c u_{\eta\eta})(c)\\ &= c^2 u_{\xi\xi} - 2c^2 u_{\xi\eta} + c^2 u_{\eta\eta}\end{aligned}$$

→ Substitution into the wave equation

$$\begin{aligned} u_{tt} &= c^2 u_{xx}\\ c^2 u_{\xi\xi} - 2c^2 u_{\xi\eta} + c^2 u_{\eta\eta} &= c^2(u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta})\\ 4u_{\xi\eta} &= 0\\ \int u_{\xi\eta} &= \int 0\\ \int u_\xi &= \int \phi(\xi)\\ u(\xi,\eta) &= \int \phi(\xi)\, d\xi + G(\eta)\\ u(\xi,\eta) &= F(\xi) + G(\eta)\\ &\boxed{u(x,t) = F(x-ct) + G(x+ct)}\end{aligned}$$

- One wave moves to right, one moves to left
  - both with speed c
- F and G may be any twice differentiable function.
  - need initial condition on ∞ domain
  - need boundary condition on finite or semi-∞ domain

<<< d'Alembert's Solution for Initial Value Problem for Wave Eqn

- Cauchy problem : wave eqn on an infinite domain
  - Solve the wave eqn $u_{tt} = c^2 u_{xx}$ for $t>0$ and $x \in (-\infty, \infty)$ with initial conditions
    - $u(x,0) = f(x)$
    - $u_t(x,0) = g(x)$

- At $t=0$, $u(x,0) = f(x) = F(x) + G(x)$

- Differentiate general soln.

$$\frac{\partial u}{\partial t} = \frac{dF}{d\xi}\frac{\partial \xi}{\partial t} + \frac{dG}{d\eta}\frac{\partial \eta}{\partial t}$$

$$= -c\frac{dF}{d\xi} + c\frac{dG}{d\eta}$$

$$= -c F'(x) + c G'(x) \qquad \begin{cases} F'(x) \equiv \frac{dF}{d\xi} \\ G'(x) \equiv \frac{dG}{d\eta} \end{cases}$$

- At $t=0$, $u_t(x,0) = g(x) = -cF'(x) + cG'(x)$

$$\frac{1}{c} g(x) = -F'(x) + G'(x)$$

system of eqns of F and G

- Solve the system $\begin{cases} F(x) + G(x) = f(x) \\ -F'(x) + G'(x) = \frac{1}{c} g(x) \end{cases} \Rightarrow F'(x) + G'(x) = f'(x)$

add & subtract, $\Rightarrow \begin{cases} F'(x) = \frac{1}{2} f'(x) - \frac{1}{2c} g(x) \\ G'(x) = \frac{1}{2} f'(x) + \frac{1}{2c} g(x) \end{cases}$

integrate, $\Rightarrow \begin{cases} F(x) = \frac{1}{2} f(x) - \frac{1}{2c}\int_0^x g(y)\,dy + k_1 \\ G(x) = \frac{1}{2} f(x) + \frac{1}{2c}\int_0^x g(y)\,dy + k_2 \end{cases}$ $\quad \begin{cases} F(x) + G(x) = f(x), \text{ so} \\ k_1 + k_2 = 0 \\ k \equiv k_1 = -k_2 \end{cases}$

- change variables $\Rightarrow \begin{cases} F(x-ct) = \frac{1}{2} f(x-ct) + \int_{x-ct}^{0} g(y)\,dy + k \\ G(x+ct) = \frac{1}{2} f(x+ct) + \int_0^{x+ct} g(y)\,dy - k \end{cases}$

add the eqns $\quad u(x,t) = F(x-ct) + G(x+ct)$

$$\boxed{u(x,t) = \frac{1}{2}\left[f(x-ct) + f(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)\,dy}$$

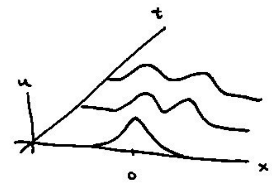
→ Restriction on f and g
- twice continuously differentiable (classical soln)
  - discontinuity and oscillations can be introduced.

<<< Examples of d'Alembert's solution

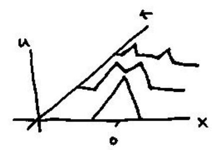
$\boxed{\text{Ex}}$ For $u_{tt} = c^2 u_{xx}$ with initial conditions $u(x,0) = e^{-x^2}$, $u_t(x,0) = 0$ for $t>0$, $x \in (-\infty, \infty)$, find the d'Alembert's solution.

- $f(x) = e^{-x^2}$
- $g(x) = 0$
- $u(x,t) = \frac{1}{2}\left[f(x-ct) + f(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)\,dy$

$u(x,t) = \frac{1}{2}\left[e^{-(x-ct)^2} + e^{-(x+ct)^2}\right]$



$\boxed{\text{Ex}}$ For $u_{tt} = c^2 u_{xx}$ with $u(x,0) = f(x) = \begin{cases} 1-|x| & |x| \le 1 \\ 0 & |x| > 1 \end{cases}$ and $u_t(x,0) = g(x) = 0$,
the d'Alembert's soln gives two triangular pulse, each 1/2 as high as initial.



$\boxed{\text{Ex}}$ Verify $u(x,t) = \cos(ct)\sin(x)$ is a d'Alembert's Solution to $u_{tt} = c^2 u_{xx}$.

- $u(x,0) = \sin(x)$
- $u_t(x,0) = \left[-c\sin(ct)\sin(x)\right]_{t=0} = 0$
- $u(x,t) = \frac{1}{2}\left[\sin(x-ct) + \sin(x+ct)\right]$
  - $\sin(x-ct) = \sin(x)\cos(ct) - \cos(x)\sin(ct)$
  - $\sin(x+ct) = \sin(x)\cos(ct) + \cos(x)\sin(ct)$ trig addition formula
- $u(x,t) = \cos(ct)\sin(x)$

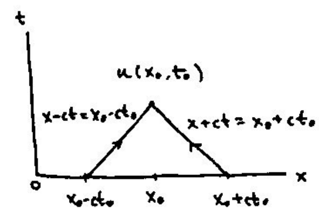
$\boxed{\text{Ex}}$ Find d'Alembert's soln of $u_{tt} = c^2 u_{xx}$ with init condition $u(x,0) = 0$, $u_t(x,0) = \sin x$ for $t>0$, $x \in (-\infty, \infty)$

- $f(x) = 0$
- $g(x) = \sin x$
- $u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \sin y\,dy$

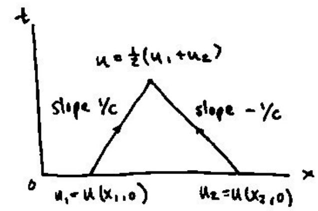
$= \frac{1}{2c}\left[\cos(x+ct) - \cos(x-ct)\right]$

∴ The Characteristic Triangle

→ Zero initial velocity



- $u_{tt} = c^2 u_{xx}$, $t>0$, $x \in (-\infty,\infty)$
- $u(x,0) = f(x)$ $\Big\}$ $u(x,t) = \frac{1}{2}[f(x-ct) + f(x+ct)]$
- $u_t(x,0) = 0$
- need two values at initial time to know value at current time
  - soln is two waves one to left, one to right
  - trace back the waves to $t=0$

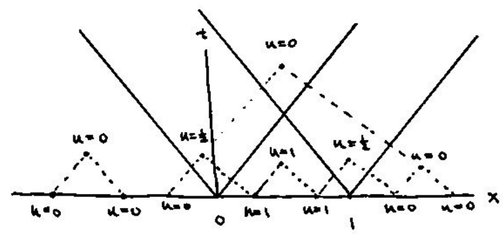


> Characteristics - lines connecting $u(x,t)$ to $u(x_1,0)$ and $u(x_2,0)$
  - $x-ct = x_0-ct_0$
  - $x+ct = x_0+ct_0$

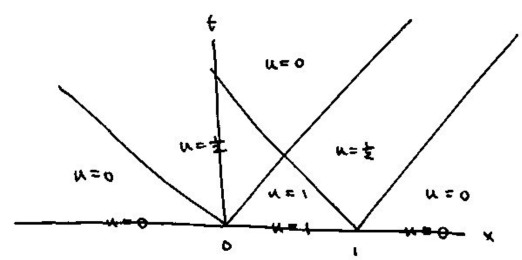
- Average of initial values $u(x,0)$ following the characteristic lines is the current value.

$\boxed{\text{EX}}$ $u_{tt} = c^2u_{xx}$, $t>0$, $x \in (-\infty,\infty)$

- $u(x,0) = f(x) = \begin{cases} 1 & x \in [0,1] \\ 0 & \text{otherwise} \end{cases}$
- $u_t(x,0) = g(x) = 0$

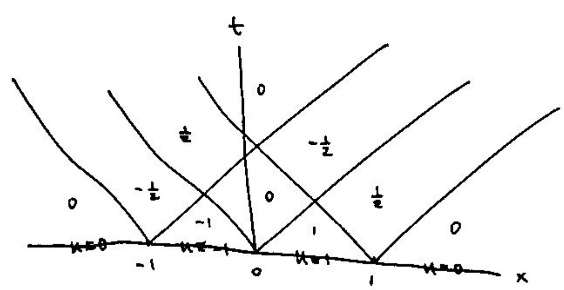


- At initial time $t=0$, identify boundaries that hass diff. $u(x,0)$ values
- Draw left and right characteristic lines to identify range of influence.
- Value of arbitrary pt $u(x,t)$ is average of $u(x,0)$ following the two characteristic lines.



$\boxed{\text{EX}}$ $u_{tt} = c^2u_{xx}$, $t>0$, $x \in (-\infty,\infty)$

- $u(x,0) = f(x) = \begin{cases} -1 & x \in [-1,0] \\ 1 & x \in [0,1] \\ 0 & \text{otherwise} \end{cases}$
- $u_t(x,0) = g(x) = 0$

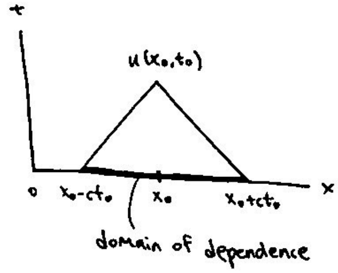


<<< The Characteristic Triangle

→ Zero initial function value

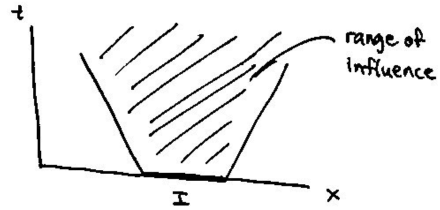
- $u_{tt} = c^2 u_{xx}$, $t > 0$, $x \in (-\infty, \infty)$
- $u(x,0) = 0$
- $u_t(x,0) = g(x)$

$$\left.\begin{array}{l} u(x,0)=0 \\ u_t(x,0)=g(x) \end{array}\right\} \quad u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)\, dy$$



- Value at current time $u(x,t)$ is the integral of initial velocity between $x-ct$ and $x+ct$ (domain of dependence)

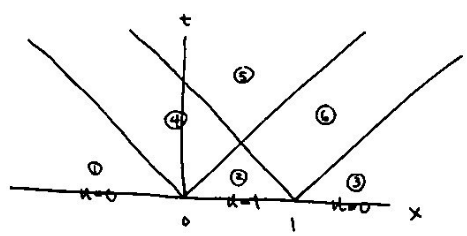
> domain of dependence - $[x_0-ct_0, x_0+ct_0]$
  - only initial values in domain of dependence affect solution at (x_0,t_0).

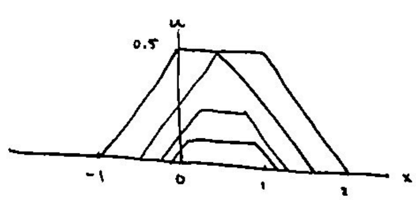


> range of influence - disturbance in interval I at $t=0$ influences all points whose domain of dependence includes I.
  - bounded by
    - $-1/c$ characteristic line at left bound
    - $1/c$ characteristic line at right bound

$\boxed{\text{EX}}$ $u_{tt} = c^2 u_{xx}$, $t > 0$, $x \in (-\infty, \infty)$

- $u(x,0) = f(x) = 0$
- $u_t(x,0) = g(x) = \begin{cases} 1 & x \in [0,1] \\ 0 & \text{otherwise} \end{cases}$
- Consider 6 diff regions :





① $u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} 0\, dy = 0$

② $u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} 1\, dy = t$

③ $u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} 0\, dy = 0$

④ $u(x,t) = \frac{1}{2c}\int_{0}^{x+ct} 1\, dy = \frac{x+ct}{2c}$

⑤ $u(x,t) = \frac{1}{2c}\int_{0}^{1} 1\, dy = \frac{1}{2c}$

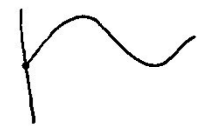
⑥ $u(x,t) = \frac{1}{2c}\int_{x-ct}^{1} 1\, dy = \frac{1-x+ct}{2c}$

} bounds changes to ignore part where $g(x)=0$, since its integral is zero anyways

<<< Semi-infinite string with one fixed end

→ Problem

- $u_{tt} = c^2 u_{xx}$, $t > 0$, $x \in [0, \infty)$
- initial conditions - $u(x,0) = f(x)$, $u_t(x,0) = g(x)$
- boundary condition - $u(0,t) = 0$
  - $u(0,0) = f(0) = 0$ ⇒ fixed end



$x=0$

→ $x \geq ct$

- d'Alembert's soln - $u(x,t) = F(x-ct) + G(x+ct)$
- substitute init cond, $\begin{cases} F(x) = \frac{1}{2} f(x) - \frac{1}{2c}\int_0^x g(s)\,ds + k_1 \\ G(x) = \frac{1}{2} f(x) + \frac{1}{2c}\int_0^x g(s)\,ds + k_2 \end{cases}$ (see derivation of d'Alembert's soln for init value problem)
  - restriction: $x > 0$
    - substitute $x-ct$ restricts $x-ct > 0$ ⇒ $x > ct$
- same soln as d'Alembert's soln for init value problem ($x > ct$)
  - $u(x,t) = \frac{1}{2}\left[f(x-ct) + f(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds$

→ $x < ct$

- substitute d'Alembert's soln into boundary cond.

$u(0,t) = 0 = F(-ct) + G(ct)$

$F(-ct) = -G(ct)$

$F(z) = -G(-z)$ $\quad z \equiv -ct$ right moving wave is negative of left moving wave

- For positive argument of G ($-z > 0$, $z < 0$), we can determine F for negative argument ($z < 0$)
  - $F(x-ct) = -G(-(x-ct))$

$= -G(ct-x)$

$= -\frac{1}{2} f(ct-x) - \frac{1}{2c}\int_0^{ct-x} g(s)\,ds - k_2$ reflected wave

- total soln for $x<ct$ is sum of left-traveling unreflected wave and right-traveling reflected wave
  - $u(x,t) = F(x-ct) + G(x+ct)$

$= -G(ct-x) + G(x+ct)$

$= \frac{1}{2}\left[-f(ct-x) + f(x+ct)\right] + \frac{1}{2c}\int_{ct-x}^{x+ct} g(s)\,ds$

→ Solution (full range)

$$\boxed{u(x,t) = \begin{cases} \frac{1}{2}\left[f(x-ct) + f(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds & x \geq ct \\ \frac{1}{2}\left[-f(ct-x) + f(x+ct)\right] + \frac{1}{2c}\int_{ct-x}^{x+ct} g(s)\,ds & x < ct \end{cases}}$$

<<< Example of fixed end semi-infinite string

$\boxed{\text{EX}}$ $u_{tt} = c^2 u_{xx}$, $t > 0$, $x \in [0, \infty)$

- Init cond - $u(x,0) = f(x) = xe^{-x^2}$

  $u_t(x,0) = g(x) = 0$

- bound. cond - $u(0,t) = 0$

- $x > ct$
  - $u(x,t) = \frac{1}{2}[f(x-ct) + f(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds$

    $= \frac{1}{2}\left[(x-ct)e^{-(x-ct)^2} + (x+ct)e^{-(x+ct)^2}\right]$

- $x < ct$
  - $u(x,t) = \frac{1}{2}[-f(ct-x) + f(x+ct)] + \frac{1}{2c}\int_{ct-x}^{x+ct} g(s)\,ds$

    $= \frac{1}{2}\left[-(ct-x)e^{-(ct-x)^2} + (x+ct)e^{-(x+ct)^2}\right]$

    $= \frac{1}{2}\left[(x-ct)e^{-(x-ct)^2} + (x+ct)e^{-(x+ct)^2}\right]$

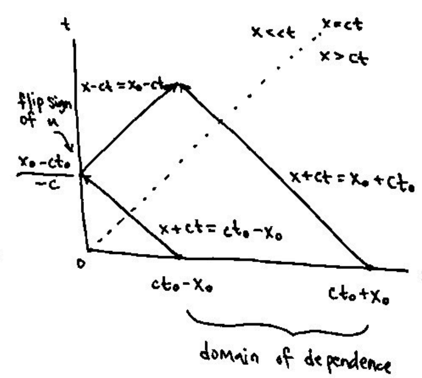
same functional form

but have diff sign of $x-ct$

- soln
  - $u(x,t) = \frac{1}{2}\left[(x-ct)e^{-(x-ct)^2} + (x+ct)e^{-(x+ct)^2}\right]$ in all $x \in [0, \infty)$

<<< Graphical Interpretation with Characteristics

- characteristic $x = ct$ from origin separates the two region
  - $x > ct$ - same as init value problem
  - $x < ct$ - reflection (sign change) at boundary $x = 0$
    - right characteristic - $x + ct = x_0 + ct_0$
    - left characteristic - $x - ct = x_0 - ct_0$
      - sign change at boundary, where $x = 0$, $t = \frac{x_0 - ct_0}{-c}$
        - $x + ct = ct_0 - x_0$
    - domain of dependence - $[ct_0 - x_0, ct_0 + x_0]$

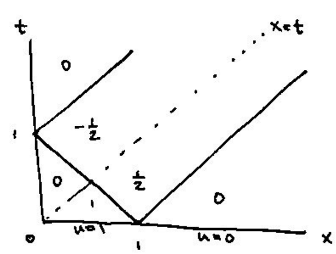


$\boxed{\text{EX}}$ $u_{tt} = u_{xx}$, $t > 0$, $x \in [0, \infty)$

- $u(x,0) = \begin{cases} 1 & x \in (0,1] \\ 0 & \text{otherwise} \end{cases}$

- $u_t(x,0) = 0$

- $u(0,t) = 0$



- Note that u changes sign after bouncing off $x = 0$

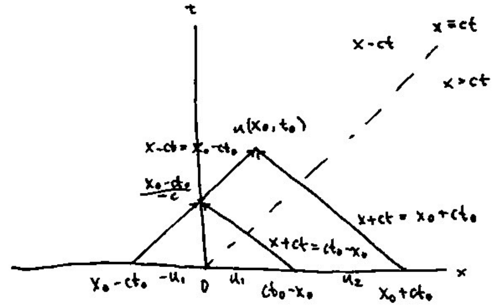
... Alternative Graphical Interpretation

- Since sign of u flips at $x=0$, we can assign values to negative x at $t=0$
  - $F(x-ct) = -G(-(x-ct)) = -G(ct-x)$

    $F(x) = -G(-x)$

    $\frac{1}{2}f(x) - \frac{1}{2c}\int_0^x g(s)\,ds = -\frac{1}{2}f(-x) - \frac{1}{2c}\int_0^{-x} g(s)\,ds$

    $\frac{1}{2}f(x) - \frac{1}{2c}\int_0^x g(s)\,ds = -\frac{1}{2}f(-x) - \frac{1}{2c}\int_0^{x} -g(-s)\,ds$



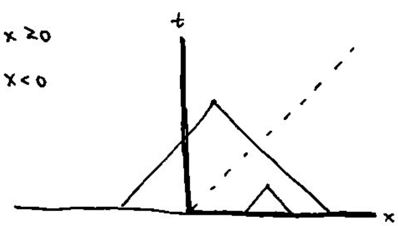
- Use odd extension of original functions
  - $f_o(x) = \begin{cases} f(x) & x \geq 0 \\ -f(-x) & x < 0 \end{cases}$
  - $g_o(x) = \begin{cases} g(x) & x \geq 0 \\ -g(-x) & x < 0 \end{cases}$

- two ways to think
  - fuctional - use odd extension f_o, g_o
  - graphical - use negative values of u in $x<0$ given u in $x>0$
    - only consider 1st quadrant
    - solve like init value prob.

$\boxed{EX}$ $u_{tt} = c^2 u_{xx}$

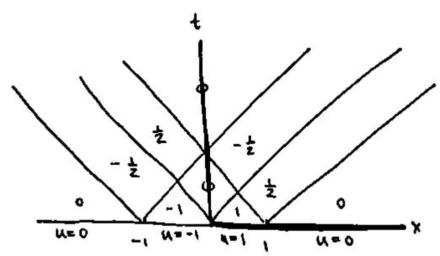
- IC $\begin{cases} u(x,0) = f(x) = xe^{-x^2} \\ u_t(x,0) = g(x) = 0 \end{cases}$ $\xrightarrow[\text{extension}]{\text{odd}}$ $f_o = \begin{cases} f(x) = xe^{-x^2} & x \geq 0 \\ -f(-x) = xe^{-x^2} & x < 0 \end{cases}$

  $g_o = 0$
- fixed BC : $u(0,t) = 0$
- $u(x,t) = \frac{1}{2}[f_o(x-ct) + f_o(x+ct)]$



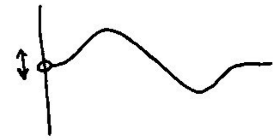
$\boxed{EX}$ $u_{tt} = c^2 u_{xx}$

- IC $\begin{cases} u(x,0) = \begin{cases} 1 & x \in (0,1) \\ 0 & \text{otherwise} \end{cases} \\ u_t(x,0) = 0 \end{cases}$
- fixed BC : $u(0,t) = 0$



<<< Semi-infinite string with a free end

→ Problem



x=0

- $u_{tt} = c^2 u_{xx}$, $t > 0$, $x \in (0, \infty)$
- initial conditions : $u(x,0) = f(x)$, $u_t(x,0) = g(x)$
- free boundary conditions : $u_x(0,t) = 0$
  - semiinfinite string attached to a ring moves vertically
  - end of string is horizontal at $x=0$ (no tensile force)

→ $x \geq ct$

- d'Alembert's soln ~ $u(x,t) = F(x-ct) + G(x+ct)$
  - $u(x,t) = \frac{1}{2}\left[f(x-ct) + f(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds$

→ $x < ct$

- substitute d'Alembert's soln to free B.C.
  - $u_x(0,t) = 0 = F'(-ct) + G'(ct)$

$$F'(-ct) = -G'(ct)$$

$$F'(z) = -G'(-z) \qquad z \equiv -ct$$

- Calculate $F'(z)$ from G
  - $G(z) = \frac{1}{2}f(z) + \frac{1}{2c}\int_0^z g(s)\,ds + k_2$
  - $F'(z) = -G'(-z)$

$$\int F'(z) = \int -\frac{1}{2}f'(-z) - \frac{1}{2c}g(-z)$$

$$F(z) = \frac{1}{2}f(-z) + \frac{1}{2c}\int_0^{-z} g(s)\,ds - k_2$$

$$F(x-ct) = \frac{1}{2}f(ct-x) + \frac{1}{2c}\int_0^{ct-x} g(s)\,ds - k_2$$

- total soln
  - $u(x,t) = F(x-ct) + G(x+ct)$

$$= \frac{1}{2}\left[f(ct-x) + f(x+ct)\right] + \frac{1}{2c}\left[\int_0^{ct-x} g(s)\,ds + \int_0^{x+ct} g(s)\,ds\right]$$

→ soln (full range)

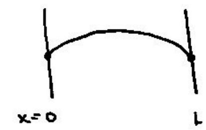
$$\boxed{u(x,t) = \begin{cases} \frac{1}{2}\left[f(x-ct) + f(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds & x \geq ct \\ \frac{1}{2}\left[f(ct-x) + f(x+ct)\right] + \frac{1}{2c}\left[\int_0^{ct-x} g(s)\,ds + \int_0^{x+ct} g(s)\,ds\right] & x < ct \end{cases}}$$

⇐ Finite string (1st try)

- solve finite string with d'Alembert's soln is challenging

→ Problem

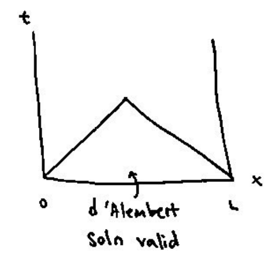
- $u_{tt} = c^2 u_{xx}$, $x \in (0,L)$, $t > 0$
- initial conditions : $u(x,0) = f(x)$, $u_t(x,0) = g(x)$
- fixed boundaries : $u(0,t) = 0$, $u(L,t) = 0$
  - $f(0) = 0$, $f(L) = 0$



→ d'Alembert soln

- d'Alembert soln - $u(x,t) = F(x-ct) + G(x+ct)$
  - $F(x) = \frac{1}{2} f(x) - \frac{1}{2c}\int_0^x g(s)\,ds + k_1$
  - $G(x) = \frac{1}{2} f(x) + \frac{1}{2c}\int_0^x g(s)\,ds + k_2$

  $\Big\}\ x \in (0,L)$
- restriction : $0 < x < L$
  - $0 < x - ct < L$
  - $0 < x + ct < L$

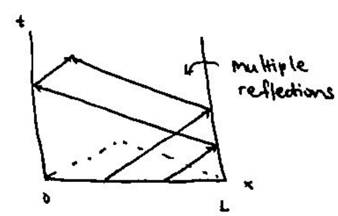


→ non-d'Alembert region

- substitute d'Alembert soln into B.C.
  - $u(0,t) = 0 = F(-ct) + G(ct)$
  - $u(L,t) = 0 = F(L-ct) + G(L+ct)$

  $\xrightarrow[\text{at boundary}]{\text{reflection}}$ $F(-ct) = -G(ct)$

  $G(L+ct) = -F(L-ct)$
- multiple reflection can occur
  - hard to keep track of
  - need alternative method of soln



<<< Standing Waves

- d'Alembert used standing waves to better solve finite string problem
- Motivate standing waves from d'Alembert soln
  - $u(x,t) = \sin(x-t) + \sin(x+t)$ sum of left and right moving waves

$$= [\sin(x)\cos(t) - \cos(x)\sin(t)] + [\sin(x)\cos(t) + \cos(x)\sin(t)]$$ trig identity

$$= 2\sin(x)\cos(t)$$

$$= \underbrace{2\cos(t)}_{\text{time-varying amplitude}}\ \underbrace{\sin(x)}_{\text{stationary sine}}$$

> standing waves - nonconstant function of the form

$$\boxed{u(x,t) = w(t)\,v(x)}$$

- basic shape $v(x)$ scaled vertically by $w(t)$

<<< Separation of variables

- not every standing wave is a soln of wave eqn
  - focus on PDE first without B.C.
- standing wave - $u(x,t) = w(t)v(x)$
  - $u_{tt} = v(x)\ddot{w}(t)$ (dot denote $\frac{d}{dt}$)
  - $u_{xx} = w(t)v''(x)$ (prime denote $\frac{d}{dx}$)
- wave eqn - $u_{tt} = c^2 u_{xx}$

$$v(x)\ddot{w}(t) = c^2 w(t)v''(x)$$

$$\frac{\ddot{w}(t)}{w(t)} = c^2\frac{v''(x)}{v(x)} = \lambda$$ ← for $f(t) = g(x)$ of diff variable to equal, they must be constant

$$\frac{\ddot{w}(t)}{w(t)} = \lambda \qquad \frac{v''(x)}{v(x)} = \lambda$$

$$\ddot{w}(t) = \lambda w(t) \qquad v''(x) = \frac{\lambda}{c^2}v(x)$$

- diff soln for $\lambda > 0$, $\lambda = 0$, $\lambda < 0$.

<<< Separation of Variables (cont.)

→ $\lambda = 0$

- $\ddot{w}(t) = \lambda w(t)$ • $v''(x) = \frac{\lambda}{c^2} v(x)$

$\ddot{w}(t) = 0$ $v''(x) = 0$

$w(t) = A + Bt$ $v(x) = C + Dx$

- standing wave - $u(x,t) = w(t)\,v(x)$

$$\boxed{u(x,t) = (A+Bt)(C+Dx)}$$

→ $\lambda > 0$

- $\lambda = r^2 > 0$
- $\ddot{w}(t) = \lambda w(t)$ • $v''(x) = \frac{\lambda}{c^2} v(x)$

$\ddot{w}(t) = r^2 w(t)$ $v''(x) = \frac{r^2}{c^2} v(x)$ ← linear, 2nd order const-coeff ODE

$w(t) = Ae^{rt} + Be^{-rt}$ $v(x) = Ce^{rx/c} + De^{-rx/c}$

- standing wave - $u(x,t) = w(t)\,v(x)$

$$\boxed{u(x,t) = (Ae^{rt} + Be^{-rt})(Ce^{rx/c} + De^{-rx/c})}$$

→ $\lambda < 0$

- $\lambda = -r^2 < 0$
- $\ddot{w}(t) = \lambda w(t)$ • $v''(x) = \frac{\lambda}{c^2} v(x)$

$\ddot{w}(t) = -r^2 w(t)$ $v''(x) = -\frac{r^2}{c^2} v(x)$

$\ddot{w}(t) + r^2 w(t) = 0$ $v''(x) + \left(\frac{r^2}{c^2}\right) v(x) = 0$ ← harmonic oscillator eqn

$w(t) = A\cos(rt) + B\sin(rt)$ $v(x) = C\cos\left(\frac{r}{c}x\right) + D\sin\left(\frac{r}{c}x\right)$

- standing wave - $u(x,t) = w(t)\,v(x)$

$$\boxed{u(x,t) = (A\cos(rt) + B\sin(rt))\left(C\cos\left(\frac{r}{c}x\right) + D\sin\left(\frac{r}{c}x\right)\right)}$$

<<< Standing Waves for Finite String

- impose fixed boundary conditions on standing waves

→ Problem

- $u_{tt} = c^2 u_{xx}$, $t > 0$, $x \in (0, L)$
- initial conditions : arbitrary
- fixed boundaries : $u(0,t) = 0$, $u(L,t) = 0$

→ Standing Wave Soln

- standing wave - $u(x,t) = w(t)\, v(x)$
  - Sub into B.C. - $u(0,t) = 0 = w(t)\, v(0)$ $\quad \underset{\text{trivial}}{w(t) = 0}$, $\underset{\text{B.C.}}{v(0) = 0}$

    $u(L,t) = 0 = w(t)\, v(L)$ $\quad w(t) = 0$, $v(L) = 0$

→ $\lambda = 0$

- $v(x) = C + Dx$

  $$\left.\begin{array}{ll} v(0) = 0 = C & \Rightarrow C = 0 \\ v(L) = C + DL = 0 & \Rightarrow D = 0 \end{array}\right\} v(x) = 0$$

- $u(x,t) = w(t)\, v(x) = 0$
  - trivial soln

→ $\lambda > 0$

- $v(x) = Ce^{rx/c} + De^{-rx/c}$
- $v(0) = 0 = C + D \Rightarrow D = -C$

  $v(L) = 0 = Ce^{rL/c} + De^{-rL/c}$

  $0 = Ce^{rL/c} - Ce^{-rL/c}$

  $0 = C(\underbrace{e^{rL/c} - e^{-rL/c}}_{\text{non-zero}})$

  $C = 0$, $D = 0$

- $v(x) = 0$
- $u(x,t) = w(t)v(x) = 0$
  - trivial soln

→ $\lambda < 0$

- $v(x) = C\cos\left(\frac{r}{c}x\right) + D\sin\left(\frac{r}{c}x\right)$
- $v(0) = 0 = C$
- $v(L) = 0 = D\sin\left(\frac{rL}{c}\right)$

  $D = 0$, $\sin\left(\frac{rL}{c}\right) = 0$

  $\frac{rL}{c} = n\pi$

  $\boxed{r = \frac{n\pi c}{L}}$

- $v(x) = D\sin\left(\frac{n\pi x}{L}\right)$
- $w(t) = A\cos\left(\frac{n\pi ct}{L}\right) + B\sin\left(\frac{n\pi ct}{L}\right)$

- $u(x,t) = w(t)v(x)$

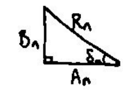
  $u(x,t) = \left[A\cos\left(\frac{n\pi ct}{L}\right) + B\sin\left(\frac{n\pi ct}{L}\right)\right] D\sin\left(\frac{n\pi x}{L}\right)$

  $$\boxed{u_n(x,t) = \underbrace{\left[A_n\cos\left(\frac{n\pi ct}{L}\right) + B_n\sin\left(\frac{n\pi ct}{L}\right)\right]}_{\text{amplitude}} \underbrace{\sin\left(\frac{n\pi x}{L}\right)}_{\text{shape}}}$$

  - nth mode of vibration
  - nth harmonic

<<< Standing Waves for Finite String

→ Modes of vibration



- $u_n(x,t) = \left[A_n \cos\left(\frac{n\pi ct}{L}\right) + B_n \sin\left(\frac{n\pi ct}{L}\right)\right] \sin\left(\frac{n\pi x}{L}\right)$

$$= \sqrt{A_n^2+B_n^2}\left[\frac{A_n}{\sqrt{A_n^2+B_n^2}} \cos\left(\frac{n\pi ct}{L}\right) + \frac{B_n}{\sqrt{A_n^2+B_n^2}} \sin\left(\frac{n\pi ct}{L}\right)\right] \sin\left(\frac{n\pi x}{L}\right)$$

$$= R_n \left[\cos(\delta_n)\cos\left(\frac{n\pi ct}{L}\right) + \sin(\delta_n)\sin\left(\frac{n\pi ct}{L}\right)\right] \sin\left(\frac{n\pi x}{L}\right)$$

$R_n \equiv \sqrt{A_n^2+B_n^2}$

$\cos(\delta_n) \equiv \frac{A_n}{R_n} = \frac{A_n}{\sqrt{A_n^2+B_n^2}}$

$\sin(\delta_n) \equiv \frac{B_n}{R_n} = \frac{B_n}{\sqrt{A_n^2+B_n^2}}$

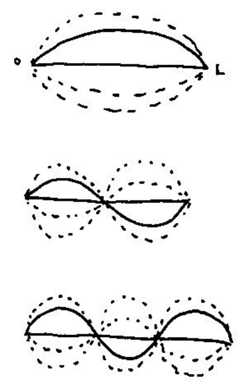
$$\boxed{u_n(x,t) = \underbrace{R_n \cos\left(\frac{n\pi ct}{L} - \delta_n\right)}_{\text{Amplitude}} \underbrace{\sin\left(\frac{n\pi x}{L}\right)}_{\text{shape}}}$$ nth harmonic

- angular frequency - $\boxed{\omega_n = \frac{n\pi c}{L}}$
- period - $\boxed{T_n = \frac{2\pi}{\omega_n} = \frac{2L}{nc}}$
- frequency - $\boxed{f_n = \frac{1}{T_n} = \frac{nc}{2L} = \frac{n}{2L}\sqrt{\frac{T}{\rho}}}$
  - > fundamental frequency - $n=1$, f_1
  - > overtones - $n>1$, f_n

n = 1

n = 2

n = 3



<<< Superposition

> principle of superposition - linear combination of solutions is also a solution if

1. linear PDE
2. homogeneous PDE
3. homogeneous boundary condition - boundary condition is 0.

- Verify wave eqn with fixed end satisfies condition
  - $u_{tt} = c^2 u_{xx}$ is linear and homogeneous PDE
  - $u(0,t) = u(L,t) = 0$ is homogeneous B.C.

- Verify superposition of two known soln u_1 and u_2 gives another soln $u = c_1u_1 + c_2u_2$

  - PDE:
  $$u_{tt} = (c_1u_1 + c_2u_2)_{tt}$$
  $$= c_1u_{1tt} + c_2u_{2tt}$$
  $$= c_1(c^2u_{1xx}) + c_2(c^2u_{2xx})$$
  $$= c^2(c_1u_1 + c_2u_2)_{xx}$$
  $$= c^2u_{xx} \quad ✓$$

  - B.C.1:
  $$u(0,t) = c_1u_1(0,t) + c_2u_2(0,t)$$
  $$= c_1(0) + c_2(0)$$
  $$= 0 \quad ✓$$

  - B.C.2:
  $$u(L,t) = c_1u_1(L,t) + c_2u_2(L,t)$$
  $$= c_1(0) + c_2(0)$$
  $$= 0 \quad ✓$$

> compound wave - superposition of N harmonics

- $u(x,t) = u_1(x,t) + u_2(x,t) + \cdots + u_N(x,t)$

$$= \sum_{n=1}^{N} u_n(x,t)$$

$$\boxed{u(x,t) = \sum_{n=1}^{N}\left[A_n\cos\left(\frac{n\pi ct}{L}\right) + B_n\sin\left(\frac{n\pi ct}{L}\right)\right]\sin\left(\frac{n\pi x}{L}\right)}$$

for standing wave on finite string $u_{tt} = c^2u_{xx}$ $\quad N \to \infty$

<<< Superposition Example

$\boxed{EX}$ · $u_{tt} = u_{xx}$, $c=1$, $x \in (0,1)$, $t>0$, $L=1$

· $u(0,t) = 0$, $u(1,t) = 0$

· $u(x,0) = f(x) = 0$

· $u_t(x,0) = g(x) = 2\sin(\pi x) - 3\sin(2\pi x)$

· general soln - $u(x,t) = \sum_{n=1}^{N} [A_n \cancel{\cos(n\pi t)}^{1} + B_n \cancel{\sin(n\pi t)}^{0}] \sin(n\pi x)$ $(L=1)$ $(c=1)$

$$u(x,0) = \sum_{n=1}^{N} A_n \sin(n\pi x) = f(x) = 0$$

$$A_n = 0$$

· time derivative - $u_t(x,t) = \sum_{n=1}^{N} [-A_n n\pi \cancel{\sin(n\pi t)}^{0} + B_n n\pi \cos\cancel{(n\pi t)}^{1}] \sin(n\pi x)$

$$u_t(x,0) = \sum_{n=1}^{N} B_n n\pi \sin(n\pi x) = g(x) = 2\sin(\pi x) - 3\sin(2\pi x)$$

$$B_1 \pi \sin(\pi x) + B_2 2\pi \sin(2\pi x) = 2\sin(\pi x) - 3\sin(2\pi x)$$

$$B_1 \pi = 2 \qquad B_2 2\pi = -3$$

$$B_1 = \frac{2}{\pi} \qquad B_2 = -\frac{3}{2\pi}$$

· soln - $u(x,t) = \sum_{n=1}^{N} [\cancel{A_n \cos}^{0}(n\pi t) + B_n \sin(n\pi t)] \sin(n\pi x)$

$$= \frac{2}{\pi}\sin(\pi t)\sin(\pi x) - \frac{3}{2\pi}\sin(2\pi t) \sin(2\pi x)$$

→ Infinite series

· generally, infinite series allows more complicated functions

· $u(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos\left(\frac{n\pi ct}{L}\right) + B_n \sin\left(\frac{n\pi ct}{L}\right)\right] \sin\left(\frac{n\pi x}{L}\right)$

· A_n, B_n decreases rapidly → series converges

$\boxed{EX}$ · $u_{tt} = u_{xx}$, $c=1$, $x \in (0,1)$, $t>0$, $L=1$

· $u(x,0) = f(x) = \sin(2\pi x)$

· $u_t(x,0) = g(x) = -3\sin(2\pi x)$

· general soln - $u(x,0) = \sum_{n=1}^{N} A_n \sin(n\pi x) = f(x) = \sin(2\pi x)$

$$A_2 = 1$$

· time derivative - $u_t(x,0) = \sum_{n=1}^{N} B_n n\pi \sin(n\pi x) = g(x) = -3\sin(2\pi x)$

$$B_2 2\pi = -3$$

$$B_2 = -\frac{3}{2\pi}$$

· soln - $u(x,t) = \sum_{n=1}^{N} [A_n \cos(n\pi t) + B_n \sin(n\pi t)] \sin(n\pi x)$

$$= \left[\cos(2\pi t) - \frac{3}{2\pi}\sin(2\pi t)\right] \sin(2\pi x)$$

<<< Fourier Series

- Solve the wave eqn with fixed BC with general initial conditions.

  - $u_{tt} = c^2 u_{xx}$, $x \in (0, L)$, $t > 0$
    - BC : $u(0,t) = 0$, $u(L,t) = 0$
    - IC $\begin{cases} u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases}$

  - general soln - $u(x,t) = \sum_{n=1}^{\infty} \left[A_n \cancel{\cos\left(\frac{n\pi ct}{L}\right)}^{1} + B_n \cancel{\sin\left(\frac{n\pi ct}{L}\right)}^{0} \right] \sin\left(\frac{n\pi x}{L}\right)$

$$u(x,0) = \boxed{\sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) = f(x)}$$ Fourier sine series

    - Fourier sine series requires $\begin{cases} f(0) = f(L) = 0 \\ f(x) \text{ continuous} \end{cases}$

  - time derivative ~ $u_t(x,t) = \sum_{n=1}^{\infty} \left[-\frac{n\pi c}{L} A_n \sin\left(\frac{n\pi ct}{L}\right) + \frac{n\pi c}{L} B_n \cos\left(\frac{n\pi ct}{L}\right) \right] \sin\left(\frac{n\pi x}{L}\right)$

$$u_t(x,t) = \sum_{n=1}^{\infty} \frac{n\pi c}{L} B_n \cos\left(\frac{n\pi ct}{L}\right) = g(x)$$

$$\sum_{n=1}^{\infty} b_n \cos\left(\frac{n\pi ct}{L}\right) = g(x)$$ Fourier sine series

$$b_n \equiv \frac{n\pi c}{L} B_n$$

→ Solving coefficients of Fourier Sine Series

  - orthogonality - $\int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx$

$$= \int_0^L \frac{1}{2} \left[\cos\left(\frac{(n-m)\pi x}{L}\right) - \cos\left(\frac{(n+m)\pi x}{L}\right) \right] dx$$

$$= \begin{cases} 0 & m \neq n \\ L/2 & m = n \end{cases}$$

  - Solve for A_n : $\sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) = f(x)$

$$\sum_{n=1}^{\infty} A_n \int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx = \int_0^L f(x) \sin\left(\frac{m\pi x}{L}\right) dx$$

$$A_n \frac{L}{2} = \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$ orthogonality , $m = n$

$$\boxed{A_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx}$$

  - Solve for B_n : $b_n = \frac{2}{L} \int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) dx$

$$B_n = \frac{L}{n\pi c} b_n = \frac{2}{n\pi c} \int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

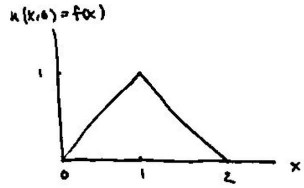
→ General Soln

$$\boxed{\begin{aligned} & u(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos\left(\frac{n\pi ct}{L}\right) + B_n \sin\left(\frac{n\pi ct}{L}\right) \right] \sin\left(\frac{n\pi x}{L}\right) \\ & A_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx \\ & B_n = \frac{2}{n\pi c} \int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) dx \end{aligned}}$$

<<< Fourier Series Example

$\boxed{\text{Ex}}$

- $u_{tt} = u_{xx}$, $c=1$, $x \in (0,2)$, $t>0$, $L=2$
  - BC – $u(0,t)=0$, $u(2,t)=0$
  - IC $\begin{cases} u(x,0) = f(x) = 1-|x-1| \\ u_t(x,0) = g(x) = 0 \end{cases}$



- $A_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$

$$= \int_0^2 \left[1-|x-1|\right] \sin\left(\frac{n\pi x}{2}\right) dx$$

$$= \int_0^1 x \sin\left(\frac{n\pi x}{2}\right) dx + \int_1^2 (2-x) \sin\left(\frac{n\pi x}{2}\right) dx$$

$$\vdots$$

$$= \frac{8}{(n\pi)^2} \sin\left(\frac{n\pi}{2}\right)$$

- $B_n = \frac{2}{n\pi c}\int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) dx = 0$
- $f(x) = 1-|x-1| = \sum_{n=1}^{\infty} \frac{8}{(n\pi)^2} \sin\left(\frac{n\pi}{2}\right) \sin\left(\frac{n\pi x}{2}\right)$
- $u(x,t) = \sum_{n=1}^{\infty} \frac{8}{(n\pi)^2} \sin\left(\frac{n\pi}{2}\right) \cos\left(\frac{n\pi t}{2}\right) \sin\left(\frac{n\pi x}{2}\right)$

<<< Other Boundary Conditions

- homogeneous boundary conditions of wave eqn (finite string)
  - fixed endpoints
    - $u(0,t)=0$, $u(L,t)=0$
    - BC of 1st kind, homogeneous Dirichlet BC
  - free endpoints
    - $u_x(0,t)=0$, $u_x(L,t)=0$
    - BC of 2nd kind, homogeneous Neumann BC
  - ellastic attachment
    - $-T u_x(0,t) = -k u(0,t)$, $T u_x(L,t) = -k u(L,t)$

      $u_x(0,t) = \alpha u(0,t)$ $\qquad$ $u_x(L,t) = -\alpha u(L,t)$, $\alpha \equiv \frac{k}{T}$
    - BC of 3rd kind, homogeneous Robin BC
- mixed boundary conditions
  - mixing fixed & free BC.

<<< Example of two free ends

- $u_{tt} = c^2 u_{xx}$, $x \in (0,L)$, $t > 0$.
  - IC: $u(x,0) = f(x)$, $u_t(x,0) = g(x)$
  - BC: $u_x(0,t) = u_x(L,t) = 0$ (free ends, homogeneous Newmann)
- Standing wave - $u(x,t) = w(t)v(x)$
  - sub into PDE: $u_{tt} = c^2 u_{xx}$

BC: $u_x(0,t) = 0 = w(t)v'(0)$, $u_x(L,t) = 0 = w(t)v'(L)$ $\Rightarrow$ $v'(0)=0$, $v'(L)=0$

$$\ddot{w}(t)v(x) = c^2 w(t) v''(x)$$

$$\frac{\ddot{w}(t)}{w(t)} = c^2\frac{v''(x)}{v(x)} = \lambda$$

for diff variable of x and t to equal, they must equal to a constant.

$$\ddot{w}(t) = \lambda w(t) \qquad v''(x) = \frac{\lambda}{c^2} v(x)$$

$\rightarrow \lambda = 0$

- $\ddot{w}(t) = 0$

  $w(t) = A + Bt$

- $v''(x) = 0$

  $v(x) = C + Dx$

  $v'(x) = D$

  $v'(0) = D = 0$

  $v(x) = C$

- $u(x,t) = w(t)v(x)$

  $= (A+Bt)C$

  $= A_0 + B_0 t$

$\rightarrow \lambda > 0$

- $\ddot{w}(t) = r^2 w(t)$

  $w(t) = Ae^{rt} + Be^{-rt}$

- $v''(x) = \frac{r^2}{c^2}v(x)$

  $v(x) = Ce^{rx/c} + De^{-rx/c}$

- $v'(x) = C\frac{r}{c}e^{rx/c} + D\left(-\frac{r}{c}\right)e^{-rx/c}$
  - $v'(0) = 0 = C\frac{r}{c} - D\frac{r}{c}$

    $0 = \frac{r}{c}(C - D)$

    $C = D$
  - $v'(L) = 0 = C\frac{r}{c}e^{rL/c} + D\left(-\frac{r}{c}\right)e^{-rL/c}$

    $0 = \frac{r}{c}C\underbrace{\left(e^{rL/c} - e^{-rL/c}\right)}_{>0}$

    $C = 0 \Rightarrow D = 0$

- $v(x) = 0$
- $u(x,t) = w(t)v(x) = 0$ $\Rightarrow$ trivial soln

... Example of two free ends (cont.)

→ $\lambda < 0$

- $\ddot{w}(t) = -r^2 w(t)$

  $w(t) = A\cos(rt) + B\sin(rt)$

- $v''(x) = -\frac{r^2}{c^2} v(x)$

  $v(x) = C\cos\left(\frac{r}{c}t\right) + D\sin\left(\frac{r}{c}t\right)$

- $v'(x) = -C\frac{r}{c}\sin\left(\frac{r}{c}t\right) + D\frac{r}{c}\cos\left(\frac{r}{c}t\right)$

  - $v'(0) = 0 = D\frac{r}{c} \Rightarrow D = 0$
  - $v'(L) = 0 = -C\frac{r}{c}\sin\left(\frac{rL}{c}\right) + \cancelto{0}{D\frac{r}{c}\cos\left(\frac{rL}{c}\right)}$

    $C = 0, \quad \sin\left(\frac{rL}{c}\right) = 0$

    $\frac{rL}{c} = n\pi$

    $r = \frac{n\pi c}{L}$

- $v_n(x) = C\cos\left(\frac{n\pi x}{L}\right)$
- $w_n(t) = A\cos\left(\frac{n\pi c t}{L}\right) + B\sin\left(\frac{n\pi c}{L}t\right)$
- $u_n(x,t) = w_n(t)v_n(t)$

  $= \left[A_n\cos\left(\frac{n\pi c t}{L}\right) + B_n\sin\left(\frac{n\pi c t}{L}\right)\right]\cos\left(\frac{n\pi x}{L}\right)$

→ General solution

- Superposition principle

  - $u_n(x,t) = A_0 + B_0 t + \sum_{n=1}^{\infty}\left[A_n\cos\left(\frac{n\pi c t}{L}\right) + B_n\sin\left(\frac{n\pi c t}{L}\right)\right]\cos\left(\frac{n\pi x}{L}\right)$

  $\boxed{u_n(x,t) = A_0 + B_0 t + \sum_{n=1}^{\infty}\left[A_n\cos(\omega_n t) + B_n\sin(\omega_n t)\right]\cos(k_n x)}$, $\omega_n \equiv \frac{n\pi c}{L}$, $k_n \equiv \frac{n\pi}{L}$

- Initial conditions

  - $u(x,0) = \boxed{f(x) = A_0 + \sum_{n=1}^{\infty} A_n\cos(k_n x)}$ Fourier cosine series
  - $u_t(x,0) = g(x) = B_0 + \sum_{n=1}^{\infty} B_n\omega_n\cos(k_n x)$

→ Solve for constants of Fourier cosine series

- $n=0$: $f(x) = A_0$

  $\int_0^L f(x)\,dx = \int_0^L A_0\,dx$

  $\boxed{A_0 = \frac{1}{L}\int_0^L f(x)\,dx}$

  $g(x) = B_0$

  $\int_0^L g(x)\,dx = \int_0^L B_0\,dx$

  $\boxed{B_0 = \frac{1}{L}\int_0^L g(x)\,dx}$

- $n \geq 1$: $f(x) = A_0 + \sum_{n=1}^{\infty} A_n\cos(k_n x)$

  $\int_0^L f(x)\cos(k_m x)\,dx = \int_0^L \left[A_0 + \sum_{n=1}^{\infty} A_n\cos(k_n x)\right]\cos(k_m x)\,dx$

  $\int_0^L f(x)\cos(k_m x)\,dx = A_n\frac{L}{2}$ orthogonality

  $\boxed{A_n = \frac{2}{L}\int_0^L f(x)\cos(k_n x)\,dx}$

  $B_n = \frac{2}{L\omega_n}\int_0^L g(x)\cos(k_n x)\,dx$

  $\boxed{B_n = \frac{2}{n\pi c}\int_0^L g(x)\cos(k_n x)\,dx}$

<< The Heat Equation

- $u_t = D u_{xx}$, $x \in (0, L)$ — diffusion eqn - describes temperature distribution - parabolic PDE
  - IC: $u(x,0) = f(x)$
  - BC: $u(0,t) = u(L,t) = 0$
- Standing wave soln - $u(x,t) = w(t)\,v(x)$
  - sub into PDE: $u_t = D u_{xx}$

$$\dot{w}(t)\,v(x) = D\,w(t)\,v''(x)$$

$$\frac{1}{D}\frac{\dot{w}(t)}{w(t)} = \frac{v''(x)}{v(x)} = \lambda = -r^2$$

- temp decrease with time $\dot{w}(t) < 0$
- temp always positive $w(t) > 0$
- choose $\lambda = -r^2 < 0$.
- $\lambda \geq 0$ gives trivial soln

→ $\lambda < 0$

- $\dot{w}(t) = -Dr^2 w(t)$

$$w(t) = C e^{-Dr^2 t}$$

- $v''(x) = -r^2 v(x)$

$$v(x) = A \sin(rx) + B\cos(rx)$$

BC: $v(0) = 0 = B\cos(rx) \Rightarrow B = 0$

$v(x) = A\sin(rx)$

$v(L) = 0 = A\sin(rL)$

$rL = n\pi$

$r = \frac{n\pi}{L}$

- $u(x,t) = w(t)\,v(x)$

$$= Ce^{-Dr^2 t}\left[A\sin(rx) + \cancel{B\cos(rx)}^{\,0}\right]$$

$$= A e^{-Dr^2 t}\sin(rx)$$

$$= A_n e^{-D(n\pi/L)^2 t}\sin\left(\frac{n\pi}{L}x\right)$$

→ General Soln

- Superposition principle

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} A_n e^{-D(n\pi/L)^2 t}\sin\left(\frac{n\pi x}{L}\right)}$$

← independent spacial modes with their own characteristic decay const.

- $\lim_{t\to\infty} u(x,t) \approx A_1 \sin\left(\frac{\pi x}{L}\right)$

→ Calculate constant A_n

- IC: $u(x,0) = f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)$ — Fourier sine series

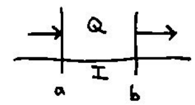
$$\int_0^L f(x)\sin\left(\frac{m\pi x}{L}\right)dx = \sum_{n=1}^{\infty} A_n \int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx$$

$$\int_0^L f(x)\sin\left(\frac{m\pi x}{L}\right)dx = A_n \frac{L}{2}$$ orthogonality

$$\boxed{A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{m\pi x}{L}\right)dx}$$

<<< Scalar Conservation Laws

$$\begin{pmatrix}\text{rate of change}\\ \text{of total quantity}\end{pmatrix} = \begin{pmatrix}\text{rate of production}\\ \text{in I}\end{pmatrix} + \begin{pmatrix}\text{rate of entry}\\ \text{at a}\end{pmatrix} - \begin{pmatrix}\text{rate of departure}\\ \text{at b}\end{pmatrix}$$



$$\frac{dQ}{dt} = \int_a^b f(x,t)\,dx + \phi(a,t) - \phi(b,t)$$

$$\boxed{\frac{d}{dt}\int_a^b u(x,t)\,dx = \int_a^b f(x,t)\,dx + \phi(a,t) - \phi(b,t)}$$

general balance eqn
(conservation law with sources)
integral form

- Interval - $I \in [a,b]$
- Density / concentration of quantity - $u(x,t)$
- Total quantity - $Q = \int_a^b u(x,t)\,dx$
- Rate of production of quantity per unit length - $f(x,t)$
- Left-to-right flux of quantity - $\phi(x,t)$

- works with discontinuity in $f(x,t)$
- works with not continuously differentiable $u(x,t)$ and $\phi(x,t)$

- If continously differentiable $u(x,t)$ and $\phi(x,t)$,
  - $\phi(a,t) - \phi(b,t) = -\int_a^b \frac{\partial \phi}{\partial x}\,dx$
  - $\frac{d}{dt}\int_a^b u(x,t)\,dx = \int_a^b \frac{\partial u}{\partial t}\,dx$
- If continuous $f(x,t)$, then
  - $\int_a^b \left[\frac{\partial u}{\partial t} - f(x,t) + \frac{\partial \phi}{\partial x}\right] dx = 0$

- continuous integrand
- arbitrary interval

$\Rightarrow$ integrand is 0.

$$\frac{\partial u}{\partial t} - f(x,t) + \frac{\partial \phi}{\partial x} = 0$$

$$\frac{\partial u}{\partial t} + \frac{\partial \phi}{\partial x} = f(x,t)$$

$$\boxed{u_t + \phi_x = f}$$

general balance eqn
differential form

<<< Source Terms

- source / reaction / generation
- no production : $f = 0$
- constant production rate : $f = r$
- exponential growth : $f = ru$
- logistic growth : $f = ru\left(1 - \frac{u}{K}\right)$
- logistic growth (with spatial dependence) : $f = r(x)\,u(x,t)\left[1 - \frac{u(x,t)}{K(x)}\right]$

<<< Constitutive Eqn for Flux

> constitutive eqn - flux ϕ dependence on u.

→ Advection and/or convection

- quantity moves horizontally (advection) or vertically (convection) with velocity $v(x,t)$
- $\phi = v(x,t)\, u(x,t)$
- general balance : $$\frac{\partial u}{\partial t} + \frac{\partial \phi}{\partial x} = f(x,t)$$

$$\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial x} + u\frac{\partial v}{\partial x} = f(x,t)$$

$\boxed{\text{Ex}}$ If no source term $f=0$

If constant velocity $v(x) = c \Rightarrow \frac{\partial v}{\partial x} = 0$

then balance is $\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$

→ Density-dependent advection

- advection speed depends on density $v(x,t) = v(u(x,t))$
- $\Phi = v(u(x,t))\, u(x,t)$

$\boxed{\text{Ex}}$ If $u(x,t)$ is density of cars, then the unit of flux is

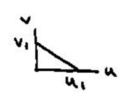
$$\phi \; [=] \; \frac{km}{hr} \times \frac{cars}{km} = \frac{cars}{hr}$$

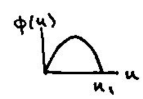
$\boxed{\text{Ex}}$ If $v = au$, then $\phi = v(u)u = au^2$

Balance eqn is $u_t + 2auu_x = 0$ $\Rightarrow$ nonlinear, but quasilinear

> quasilinear PDE - PDE that's linear with respect to highest order derivative

$\boxed{\text{Ex}}$ If $v(u) = v_1\left(1 - \frac{u}{u_1}\right)$, $u \in [0, u_1]$





then $\phi = v(u)u = v_1\left(1 - \frac{u}{u_1}\right)u$

Balance eqn is $u_t + v_1\left(1 - \frac{2u}{u_1}\right)u_x = 0$

→ Diffusion

- Fickian diffusion - flux proportional to negative gradient of density
- $\Phi = -D\frac{\partial u}{\partial x}$
- general balance : $$\frac{\partial u}{\partial t} - D\frac{\partial^2 u}{\partial x^2} = f(x,t)$$

$\boxed{\text{Ex}}$ Heat eqn - $\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2}$ $\qquad f = 0$

$\boxed{\text{Ex}}$ KISS model - $\frac{\partial u}{\partial t} = ru + D\frac{\partial^2 u}{\partial x^2}$ $\qquad f = ru$

$\boxed{\text{Ex}}$ Fisher model - $\frac{\partial u}{\partial t} = ru\left(1 - \frac{u}{K}\right) + D\frac{\partial^2 u}{\partial x^2}$ $\qquad f = ru\left(1 - \frac{u}{K}\right)$

$\boxed{\text{Ex}}$ Nagumo eqn (bistable) - $\frac{\partial u}{\partial t} = u(u-a)(1-u) + D\frac{\partial^2 u}{\partial x^2}$ $\qquad f = u(u-a)(1-u)$

⋯ Method of Characteristics

- reduce PDE into ODE by changing variables
  - let $x(s)$, $t(s)$, so $u(x,t) = u(x(s), t(s)) = u(s)$

$\boxed{\text{EX}}$ If $v = c$, then $\phi = cu$

PDE : $u_t + cu_x = 0$

IC : $u(x,0) = u_0(x)$

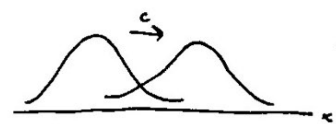
- chain rule : $\underbrace{\frac{\partial u}{\partial s}}_{0} = \frac{\partial u}{\partial t}\underbrace{\frac{\partial t}{\partial s}}_{1} + \frac{\partial u}{\partial x}\underbrace{\frac{\partial x}{\partial s}}_{c}$

$$0 = u_t + cu_x$$

$$\Rightarrow \begin{cases} \frac{\partial t}{\partial s} = 1 & t(s=0) = 0 \\ \frac{\partial x}{\partial s} = c & x(s=0) = x_0 \\ \frac{\partial u}{\partial s} = 0 & u(s=0) = u_0(x_0) \end{cases}$$

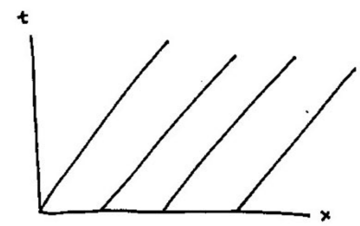
new variables s, x_0

- integrate ODE :
  - $t = s \rightarrow s = t$
  - $x = cs + x_0 \Rightarrow x_0 = x - ct$
  - $u = u_0(x_0) \Rightarrow u = u_0(x-ct)$



- right-moving traveling soln – initial distribution moves to the right at constant speed c.
- u is constant for fixed x_0
  - $x - ct = x_0$ is leveling curve of $u(x-ct)$ in the x-t plane

  > characteristic – line at which u has the same value



$\boxed{\text{Method}}$ linear homogeneous first-order PDE

$$a(x,t)u_t + b(x,t)u_x + c(x,t)u = 0$$

IC: $u(x,0) = u_0(x)$

Method of characteristic
$$\begin{cases} \frac{dt}{ds} = a(x,t) & t(0) = 0 \\ \frac{dx}{ds} = b(x,t) & x(0) = x_0 \\ \frac{du}{ds} + c(x,t)u = 0 & u(0) = u_0(x_0) \end{cases}$$

<<< Examples of Method of Characteristics

EX $u_t + u_x + 2u = 0$

$u(x,0) = u_0(x) = \sin(x)$

$t = x - x_0$

$$\begin{cases} \frac{dt}{ds} = 1 & t(0) = 0 & \Rightarrow & t = s & \Rightarrow & s = t \\ \frac{dx}{ds} = 1 & x(0) = x_0 & \Rightarrow & x = s + x_0 & \Rightarrow & x_0 = x - s = x - t \\ \frac{du}{ds} = -2u & u(0) = u_0(x_0) = \sin(x_0) & \Rightarrow & \int \frac{du}{u} = \int -2\,ds & & \end{cases}$$

$\ln(u) = -2s + C$

$u = Ae^{-2s} = \sin(x_0)e^{-2t} \Rightarrow u = \sin(x-t)e^{-2t}$

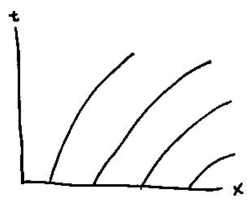
* u does not stay const along the characteristics ∵ decaying term of e^{-2t}

EX $u_t + xu_x + tu = 0$ $\quad u(x,0) = u_0(x)$

$$\begin{cases} \frac{dt}{ds} = 1 & t(0) = 0 & \Rightarrow & t = s & \Rightarrow & s = t \\ \frac{dx}{ds} = x & x(0) = x_0 & \Rightarrow & \int \frac{dx}{x} = \int ds & & \\ \frac{du}{ds} = -tu & u(0) = u_0(x_0) & & & & \end{cases}$$

$\ln(x) = s + C$

$x = Ae^{s} = x_0 e^{s} \Rightarrow x_0 = xe^{-s} = xe^{-t}$



$t = \ln\left(\frac{x}{x_0}\right)$

$\Rightarrow \int \frac{du}{u} = \int -t\,dt$

$\ln(u) = -\frac{1}{2}t^2 + C$

$u = Ae^{-\frac{1}{2}t^2} = u_0(x_0)e^{-\frac{1}{2}t^2} \Rightarrow u(x,t) = u_0(xe^{-t})\,e^{-\frac{1}{2}t^2}$

- characteristics is not straight line
- u decreases along characteristics due to decay term $e^{-\frac{1}{2}t^2}$

<<< Linear, Nonhomogeneous Problems

- $a(x,t)u_t + b(x,t)u_x + c(x,t)u = f(x,t)$
- $u(x,0) = u_0(x)$

EX $u_t + cu_x = e^{-3x}$

$u(x,0) = u_0(x)$

$$\begin{cases} \frac{dt}{ds} = 1 & t(0) = 0 & \Rightarrow & t = s & \Rightarrow & s = t \\ \frac{dx}{ds} = c & x(0) = x_0 & \Rightarrow & x = cs + x_0 & \Rightarrow & x_0 = x - cs = x - ct \\ \frac{du}{ds} = e^{-3x} & u(0) = u_0(x_0) & \Rightarrow & \frac{du}{ds} = e^{-3(cs+x_0)} & & \end{cases}$$

$\int du = \int e^{-3(cs+x_0)}\,ds$

$u = -\frac{1}{3c}e^{-3(cs+x_0)} + g(x_0)$

$u(0) = u_0(x_0) = -\frac{1}{3c}e^{-3x_0} + g(x_0)$

$g(x_0) = u_0(x_0) + \frac{1}{3c}e^{-3x_0}$

$u = -\frac{1}{3c}e^{-3(cs+x_0)} + u_0(x_0) + \frac{1}{3c}e^{-3x_0}$

$u(x,t) = \frac{1}{3c}e^{-3(x-ct)}\left(1 - e^{-3ct}\right) + u_0(x-ct)$

25 May 2022 L25 Nonlinear Problems AMATH 353

<<< Linear, Nonhomogeneous Problem

$\boxed{EX}$ $u_t + x u_t = t$

$u(x,0) = u_0(x)$

$$\begin{cases} \frac{dt}{ds} = 1 & t(0)=0 & \Rightarrow t = s & \Rightarrow s = t \\ \frac{dx}{ds} = x & x(0) = x_0 & \Rightarrow x = x_0 e^s = x_0 e^t & \Rightarrow x_0 = x e^{-t} \\ \frac{du}{ds} = t & u(0) = u_0(x_0) & \Rightarrow \frac{du}{ds} = s & \end{cases}$$

$$u = \frac{1}{2}s^2 + g(x_0)$$

$$u(x,0) = u_0(x_0) = g(x_0)$$

$$u = \frac{1}{2}s^2 + g(x_0)$$

$$u(x,t) = \frac{1}{2}t^2 + g(xe^{-t})$$

<<< Nonlinear Problems

- $u_t + c(u) u_x = 0$ → nonlinear, but quasilinear
- $u(x,0) = u_0(x)$

$$\begin{cases} \frac{dt}{ds} = 1 & t(0)=0 & \Rightarrow t = s & \Rightarrow s = t \\ \frac{dx}{ds} = c(u) & x(0) = x_0 & \Rightarrow \frac{dx}{dt} = c(u_0(x_0)) & \Rightarrow x = x_0 + c(u_0(x_0))t \\ \frac{du}{ds} = 0 & u(0) = u_0(x_0) & \Rightarrow u = u_0(x_0) & \end{cases}$$

← u is const along characteristics

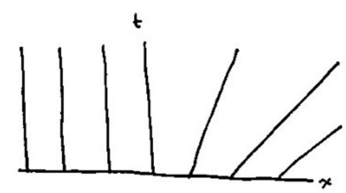
← straight line characteristics, varying slope $c(u_0(x_0))$ depending on x_0, $u(x_0)$

$$t = \frac{1}{c(u_0(x_0))}(x - x_0)$$

$\boxed{EX}$ $u_t + u u_x = 0$, $x \in (-\infty, \infty)$, $t > 0$

$$u(x,0) = \begin{cases} 0 & x \le 0 \\ e^{-1/x} & x > 0 \end{cases}$$

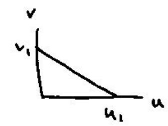
$$\begin{cases} \frac{dt}{ds} = 1 & t(0)=0 & \rightarrow t = s & \Rightarrow s = t \\ \frac{dx}{ds} = u & x(0) = x_0 & \rightarrow \frac{dx}{dt} = u_0(x_0) & \Rightarrow x = \begin{cases} x_0 & x_0 \le 0 \\ x_0 + e^{-1/x_0} t & x_0 > 0 \end{cases} \\ \frac{du}{ds} = 0 & u(0) = u_0(x_0) & \Rightarrow u = u_0(x_0) = \begin{cases} 0 & x_0 \le 0 \\ e^{-1/x_0} & x_0 > 0 \end{cases} & \end{cases}$$



$$t = e^{1/x_0}(x - x_0)$$

<<< The Traffic Problem

- car speed - $v(u) = v_1\left(1 - \frac{u}{u_1}\right)$, $u \in [0, u_1]$
- flux - $\phi(u) = v(u)u = v_1\left(1 - \frac{u}{u_1}\right)u$
  - $\phi_x = v_1\left(1 - \frac{2u}{u_1}\right)u_x$



- PDE - $u_t + \phi_x = f$ $(f=0)$

$$\underbrace{u_t + v_1\left(1 - \frac{2u}{u_1}\right)}_{c(u)} u_x = 0$$

- speed of density wave - $c(u) = v_1\left(1 - \frac{2u}{u_1}\right)$

→ Problem

- $u_t + c(u)u_x = 0$
- $u(x,0) = u_0(x)$

→ Method of Characteristics

$$\begin{cases} \frac{dt}{ds} = 1 & t(0)=0 & \Rightarrow t = s & \Rightarrow s = t \\ \frac{dx}{ds} = c(u) & x(0) = x_0 & \Rightarrow x = c(u_0(x_0))s + x_0 & \Rightarrow x = \left[v_1\left(1 - \frac{2u}{u_1}\right)\right]t + x_0 \\ \frac{du}{ds} = 0 & u(0) = u_0(x_0) & \Rightarrow u = u_0(x_0) & \quad x_0 = x - \left[v_1\left(1 - \frac{2u}{u_1}\right)\right]t \end{cases}$$

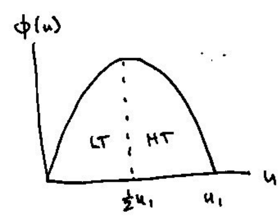
- u const along characteristics

$$\Rightarrow u = u_0\left(x - \left[v_1\left(1 - \frac{2u}{u_1}\right)\right]t\right)$$

→ speeds of interest

- car speed - $v(u) = v_1\left(1 - \frac{u}{u_1}\right)$
- density wave speed - $c(u) = v_1\left(1 - \frac{2u_0(x_0)}{u_1}\right)$

→ traffic regions



→ Light traffic (LT)

- $u_0(x_0) < \frac{1}{2}u_1 \Rightarrow c(u) = v_1\left(1 - \frac{2u_0(x_0)}{u_1}\right) > 0$
- cars forward, density wave forward

→ Heavy traffic (HT)

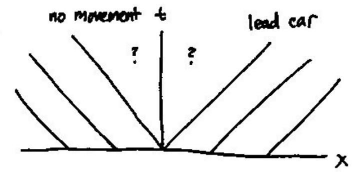
- $u_0(x_0) > \frac{1}{2}u_1 \Rightarrow c(u) = v_1\left(1 - \frac{2u_0(x_0)}{u_1}\right) < 0$
- cars forward, density wave backward
  - info of bumper to bumper signal moves back.

<<< Zone of Rarefaction

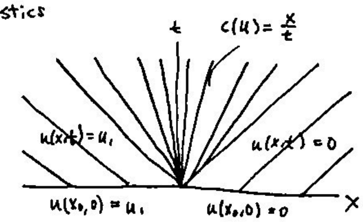
- traffic at red light

- $u(x,0) = u_0(x) = \begin{cases} u_1 & x < 0 \\ 0 & x > 0 \end{cases}$

- slope of characteristics $\dfrac{dt}{dx} = \dfrac{1}{c(u)} = \dfrac{1}{v_1\left(1 - \frac{2u_0(x_0)}{u_1}\right)} = \begin{cases} -\frac{1}{v_1} & x_0 < 0 \\ \frac{1}{v_1} & x_0 > 0 \end{cases}$



  - left - max density u_1, zero car velocity, negative slope of characteristics
  - right - zero density $u=0$, no car, positive slope of characteristics
  - middle - cone-shaped zone of rarefaction
    - density u slowly decrease



> zone of rarefaction

- $x = \alpha t \Rightarrow \alpha = \frac{x}{t}$
- $u(x,t) = u(\alpha t, t) = g(\alpha) = g\left(\frac{x}{t}\right)$ — u is const and indep. on t along the characteristics
- $u_t + c(u)u_x = 0$

$$-\frac{x}{t^2} g'(\alpha) + c(g(\alpha)) \frac{1}{t} g'(\alpha) = 0$$

$$\frac{1}{t} g'(\alpha) \left[c(g(\alpha)) - \frac{x}{t} \right] = 0$$

$$c(g(\alpha)) - \frac{x}{t} = 0$$

$$\boxed{c(u) = \frac{x}{t}}$$

- $c(u) = v_1\left(1 - \frac{2u}{u_1}\right) = \frac{x}{t}$

$$u = \frac{u_1}{2}\left(1 - \frac{x}{v_1 t}\right)$$

- $\dfrac{dx}{dt} = v(u) = v_1\left(1 - \dfrac{u}{u_1}\right)$

$$\frac{dx}{dt} = \frac{v_1}{2} + \frac{x}{2t}$$

- at $v=0$, $u=u_1$, $x\left(t = \frac{x_0}{v_1}\right) = -x_0$.

- $\dfrac{dx}{dt} - \dfrac{1}{2t} x = \dfrac{v_1}{2}$

- $\mu = e^{-\int \frac{1}{2t} dt} = \frac{1}{\sqrt{t}}$

$$\frac{dx}{dt}\frac{1}{\sqrt{t}} - \frac{x}{2t}\frac{1}{\sqrt{t}} = \frac{v_1}{2}\frac{1}{\sqrt{t}}$$

$$\frac{d}{dt}\left(\frac{1}{\sqrt{t}} x\right) = \frac{v_1}{2}\frac{1}{\sqrt{t}}$$

$$x(t) = v_1 t + k\sqrt{t}$$

$$x\left(t = \frac{x_0}{v_1}\right) = -x_0 = x_0 + k\sqrt{\frac{x_0}{v_1}}$$

$$k = -2\sqrt{v_1 x_0}$$

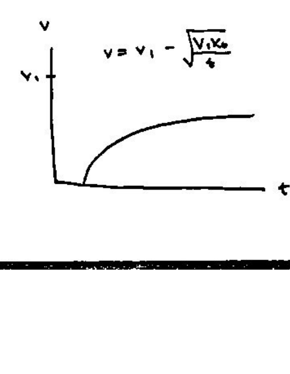
$$\boxed{x(t) = v_1 t - 2\sqrt{v_1 x_0 t}}$$

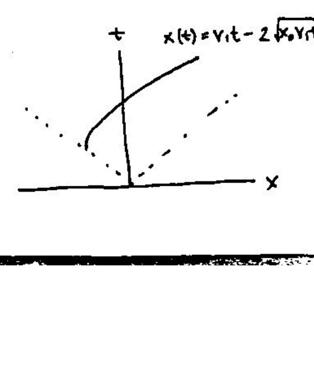
$$\boxed{v = \frac{dx}{dt} = v_1 - \sqrt{\frac{v_1 x_0}{t}}}$$

- time for car to pass the light is 4x the max speed:

$$x = 0 = v_1 t - 2\sqrt{v_1 x_0 t}$$

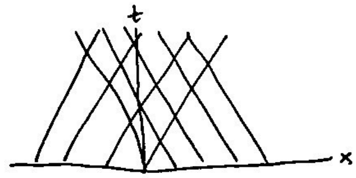
$$\boxed{t = 4\frac{x_0}{v_1}}$$



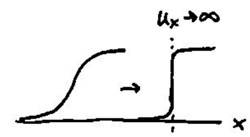


<<< Gradient Catastrophe

- Foggy day with traffic ahead
- $u(x,0) = u_0(x_0) = \begin{cases} \frac{u_1}{4} & x_0 < 0 \\ u_1 & x_0 > 0 \end{cases}$
- $\frac{dx}{dt} = c(u) = v_1\left(1 - \frac{2u_0(x_0)}{u_1}\right) = \begin{cases} \frac{v_1}{2} & x_0 < 0 \\ -v_1 & x_0 > 0 \end{cases}$
- $\frac{dt}{dx} = \left(\frac{dx}{dt}\right)^{-1} = \begin{cases} \frac{2}{v_1} & x_0 < 0 \\ -\frac{1}{v_1} & x_0 > 0 \end{cases}$



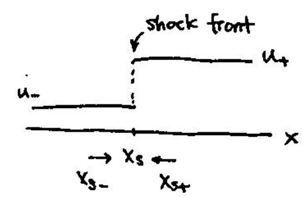
characteristics colliding



> gradient catastrophe - sudden change (discontinuity) in u that makes $u_x \to \infty$
  - approach 1 - shock wave soln
    - acknowledge discontinuity, let it be piecewise continuous
    - use integral form of conservation law
  - approach 2 - viscocity soln
    - smooth out by adding "look ahead" term that's proportional to gradient of density

<<< Shock Wave

> shock wave - discontinuity in soln $u(x,t)$



- integral form of conservation law $\quad \frac{d}{dt}\int_a^b u(x,t)\,dx = \phi(a,t) - \phi(b,t)$

$$\frac{d}{dt}\left[\int_a^{x_s^-(t)} u(x,t)\,dx + \int_{x_s^+(t)}^b u(x,t)\,dx\right] = \phi(a,t) - \phi(b,t)$$

$$\int_a^{x_s^-(t)} \frac{\partial u}{\partial t}\,dx + u(x_s^-,t)\frac{dx_s^-}{dt} + \int_{x_s^+(t)}^b \frac{\partial u}{\partial t}\,dx + u(x_s^+,t)\frac{dx_s^+}{dt} = \phi(a,t) - \phi(b,t)$$ Leibnitz's rule

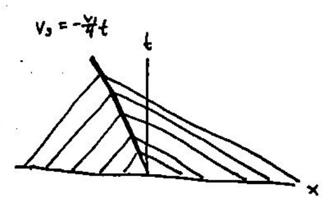
$\lim_{a \to x_s^-(t)} \lim_{b \to x_s^+(t)}$: $\quad u(x_s^-,t)\frac{dx_s}{dt} + u(x_s^+,t)\frac{dx_s}{dt} = \phi(x_s^-,t) - \phi(x_s^+,t)$

$$\boxed{\frac{dx_s}{dt} = \frac{[\phi]}{[u]} = \frac{\phi(x_s^-,t) - \phi(x_s^+,t)}{u(x_s^-,t) - u(x_s^+,t)}}$$ Rankine-Hugoniot jump condition

> shock wave - piecewise smooth soln with a jump along a curve $x_s(t)$ satisfying the Rankine-Hugoniot jump condition

> shock path - curve $x_s(t)$

Ex $\quad u(x,0) = u_0(x_0) = \begin{cases} \frac{u_1}{4} & x_0 < 0 \\ u_1 & x_0 > 0 \end{cases} \quad \begin{matrix} \Rightarrow u^- = \frac{u_1}{4} \\ \Rightarrow u^+ = u_1 \end{matrix}$



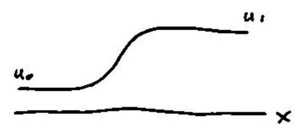
$\phi = v_1\left(1 - \frac{u}{u_1}\right)u$

$$\frac{dx_s}{dt} = \frac{v_1\left(1-\frac{u^+}{u_1}\right)u^+ - v_1\left(1-\frac{u^-}{u_1}\right)u^-}{u^+ - u^-} = \frac{v_1\cancel{\left(1-\frac{u_1}{u_1}\right)}^{0}u_1 - v_1\left(1-\frac{u_1}{4u_1}\right)\frac{u_1}{4}}{u_1 - \frac{u_1}{4}} = -\frac{v_1}{4}$$

$x_s = -\frac{v_1}{4}t$

<<< Viscocity Soln

- allow drivers to look ahead and respond to relative changes in u (gradient of u).
- $v = v_1\left(1 - \frac{u}{u_1}\right) - r\frac{1}{u}\frac{\partial u}{\partial x}$
- $\phi = v(u)u = v_1\left(1 - \frac{u}{u_1}\right)u - r\frac{\partial u}{\partial x}$
- $\phi_x = v_1\left(1 - \frac{2u}{u_1}\right)u_x - r u_{xx}$
- PDE - $u_t + v_1\left(1 - \frac{2u}{u_1}\right)u_x = r u_{xx}$ $\quad \lim_{x\to\infty} u(x,0) = u_1 \quad \lim_{x\to\infty} u_x(x,0) = 0$



- travelling wave soln - $u(x,t) = f(x-ct) = f(z)$ $\quad \lim_{x\to-\infty} u(x,0) = u_0 \quad \lim_{x\to\infty} u_x(x,0) = 0$
  - $u_t = -cf'$
  - $u_x = f'$
  - sub in PDE : $\int -cf' + v_1\left(1 - \frac{2f}{u_1}\right)f' = \int r f''$

$$-cf + v_1\left(1 - \frac{f}{u_1}\right)f = rf'' + k$$

$z \to \infty$: $\quad -cu_1 + v_1\cancel{\left(1 - \frac{u_1}{u_1}\right)}^{\to 0} u_1 = 0 + k$

$$-cu_1 = k$$

$z \to -\infty$:

$$-cu_0 + v_1\left(1 - \frac{u_0}{u_1}\right)u_0 = 0 + k$$

$$-cu_0 + v_1\left(1 - \frac{u_0}{u_1}\right)u_0 = k$$

$$\rightarrow \begin{cases} c = -v_1\frac{u_0}{u_1} \\ k = v_1 u_0 \end{cases}$$

$$v_1\frac{u_0}{u_1}f + v_1 f\left(1 - \frac{f}{u_1}\right) = rf' + v_1 u_0$$

$$u_0 f + f(u_1 - f) = \frac{ru_1}{v_1}f' + u_0 u_1$$

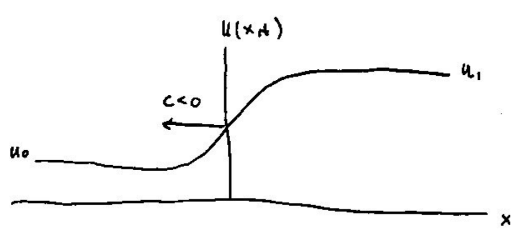
$$-\frac{ru_1}{v_1}f' = f^2 - (u_1 + u_0)f + u_0 u_1$$

$$-\frac{ru_1}{v_1}f' = (f - u_1)(f - u_0)$$

$$\int \frac{df}{(f-u_1)(f-u_0)} = \int -\frac{v_1}{ru_1}\,dz$$

$\vdots$ partial fraction

$$u(x,t) = f(z) = u_1 + \frac{u_0 - u_1}{1 + \exp\left[\frac{v_1(u_1 - u_0)}{ru_1}z\right]} \quad , \quad z \equiv x - ct = x + v_1\frac{u_0}{u_1}t$$



$r \to 0$, steeper wave front, recover shock wave

<<< Breaking Times and Earliest Shock Wave

- smooth initial conditions can still cause shock waves later

$\boxed{\text{EX}}$ antisocial speed maniac

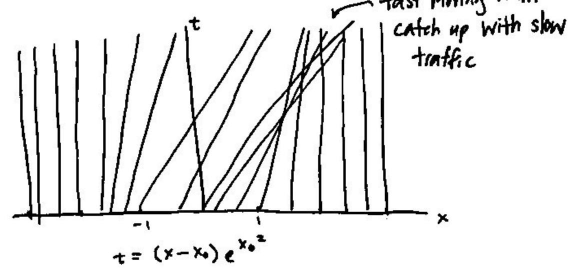
- $v(u) = au$
- $\phi = v(u)u = au^2$
- $\phi_x = 2auu_x$
- PDE - $u_t + \phi_x = 0$

$$u_t + 2auu_x = 0 \qquad (a = \tfrac{1}{2})$$

$$u_t + uu_x = 0$$

- IC : $u(x,0) = u_0(x) = e^{-x^2}$

$$\begin{cases} \frac{dt}{ds} = 1 & t(0) = 1 & \Rightarrow t = s \\ \frac{dx}{ds} = u & x(0) = x_0 & \Rightarrow x = u_0(x_0)t + x_0 = e^{-x_0^2}t + x_0 \\ \frac{du}{ds} = 0 & u(0) = u_0(x_0) & \Rightarrow u = u_0(x_0) = e^{-x_0^2} \end{cases}$$



> breaking time - earliest time $t_b > 0$ at which characteristics intersect corresponds to a gradient catastrophe ($u_x \to \infty$)

→ Break time calculation

- $u_t + c(u)u_x = 0$
- $u(x,0) = u_0(x)$

$$\begin{cases} \frac{dt}{ds} = 1 & t(0) = 0 & \Rightarrow t = s \\ \frac{dx}{ds} = c(u) & x(0) = x_0 & \Rightarrow x = x_0 + c(u_0(x_0))t \\ \frac{du}{ds} = 0 & u(0) = u_0(x_0) & \Rightarrow u = u_0(x_0) \end{cases}$$

- $u_x = u_0'(x_0)\frac{\partial x_0}{\partial x}$

  - $\frac{\partial}{\partial x} x = \frac{\partial}{\partial x}(x_0 + c(u_0(x_0))t)$

$$1 = \frac{\partial x_0}{\partial x} + t\frac{\partial x_0}{\partial x}\frac{\partial}{\partial x_0}c(u_0(x_0))$$

$$\frac{\partial x_0}{\partial x} = \frac{1}{1 + t\frac{d}{dx_0}c(u_0(x_0))}$$

$$u_x = \frac{u_0'(x_0)}{1 + t\frac{d}{dx_0}c(u_0(x_0))} \to \infty$$

$$1 + t\frac{d}{dx_0}c(u_0(x_0)) = 0$$

$$\boxed{t_b = -\frac{1}{\frac{d}{dx_0}c(u_0(x_0))}}$$

- $x_0(t_b)$ is x_0 that makes $\frac{d}{dx_0}c(u_0(x_0))$ most negative.

$\boxed{\text{EX}}$ antisocial speed maniac

- $c(u) = u$,
- $u_0(x) = e^{-x^2}$

$$\left.\begin{matrix} c(u) = u \\ u_0(x) = e^{-x^2} \end{matrix}\right\} c(u_0(x_0)) = e^{-x_0^2}$$

- $F(x_0) = \frac{d}{dx_0}e^{-x_0^2} = -2x_0e^{-x_0^2}$
- $F'(x_0) = (-2 + 4x_0^2)e^{-x_0^2} = 0$

$$x_0 = \pm\frac{1}{\sqrt{2}}$$

- choose $x_0 = \frac{1}{\sqrt{2}}$ that makes $F(x_0)$ most negative.

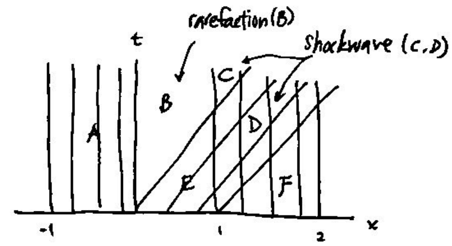
- $t_b = \frac{-1}{F(x_0)} = \frac{-1}{-2x_0e^{-x_0^2}} = \frac{-1}{\frac{-2}{\sqrt{2}}e^{-1/2}} = \sqrt{\frac{e}{2}} \approx 1.16$

<<< Combination of Rarefaction & Shock Waves

$\boxed{EX}$. $u_t + uu_x = 0$, $x \in (-\infty, \infty)$, $t > 0$

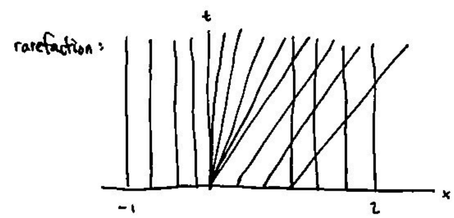
- $u(x,0) = u_0(x) = \begin{cases} 0 & x \le 0 \\ 1 & 0 < x < 1 \\ 0 & x \ge 1 \end{cases}$

$$\begin{cases} \frac{dt}{ds} = 1 & t(0) = 0 & \Rightarrow t = s \\ \frac{dx}{ds} = u & x(0) = x_0 & \Rightarrow x = u_0(x_0)t + x_0 = \begin{cases} x_0 + t & 0 < x_0 < 1 \\ x_0 & \text{otherwise} \end{cases} \\ \frac{du}{ds} = 0 & u(0) = u_0(x_0) & \Rightarrow u = u_0(x_0) \end{cases}$$



- zone of rarefaction : $c(u) = u(x,t) = \frac{x}{t}$

$x = ut$



- shock path 1 : $u_t + uu_x = 0$

$\phi_x = uu_x$

$\phi = \frac{1}{2}u^2$

$$\frac{dx_s}{dt} = \frac{[\phi]}{[u]} = \frac{\frac{1}{2}(u^+)^2 - \frac{1}{2}(u^-)^2}{u^+ - u^-} = \frac{1}{2}(u^+ + u^-) = \frac{1}{2}(0+1) = \frac{1}{2}$$

$x_s = \frac{1}{2}t + c_1$ shock starts at $x_s(0) = 1 \Rightarrow c_1 = 1$

$x_s = \frac{1}{2}t + 1$, $x \in [0,2]$

- shock path 2 :

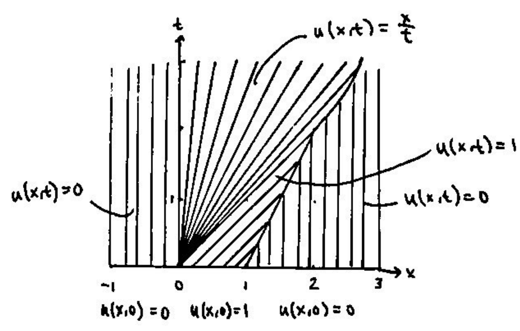
$$\frac{dx_s}{dt} = \frac{[\phi]}{[u]} = \frac{\frac{1}{2}(u^+)^2 - \frac{1}{2}(u^-)^2}{u^+ - u^-} = \frac{1}{2}(u^+ + u^-) = \frac{1}{2}(0 + \frac{x_s}{t}) = \frac{x_s}{2t}$$

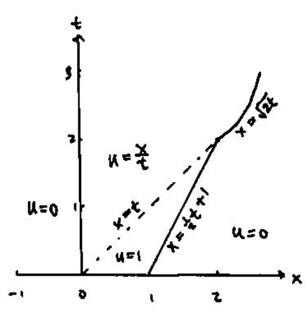
$$\int \frac{dx_s}{x_s} = \int \frac{dt}{2t}$$

$\ln(x_s) = \ln(\sqrt{t}) + c_1$

$x_s = c_2\sqrt{t}$ shock starts at $x_s(2) = 2 \Rightarrow c_2 = \sqrt{2}$

$x_s = \sqrt{2t}$ $t \ge 2$





For $t < 2$:

$$u(x,t) = \begin{cases} 0 & x < 0 \\ \frac{x}{t} & 0 < x < t \\ 1 & t < x < (\frac{1}{2}t + 1) \\ 0 & \frac{1}{2}t + 1 < x \end{cases}$$

For $t > 2$:

$$u(x,t) = \begin{cases} 0 & x < 0 \\ \frac{x}{t} & 0 < x < \sqrt{2t} \\ 0 & \sqrt{2t} < x \end{cases}$$