29 Sep 2021 L1 Intro to Transport Phenomena CHEME 330

... Course Outline & History

Raw Material, energy $\xrightarrow[\text{process}]{}$ products

↓

Many interconnected steps
represented by flowsheet (pictorially)
with units → unit operations

- transport phenomena looks inside each unit operation.
  - Momentum transport - CHEME 330
  - heat transport - CHEME 340
  - Mass (species) transport - CHEME 435
  - (electric charge transport)

History

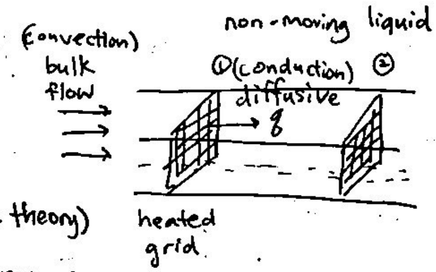
Flowsheet
↓
unit operation
↓
transport pheeomena
↓
Molecular level ?

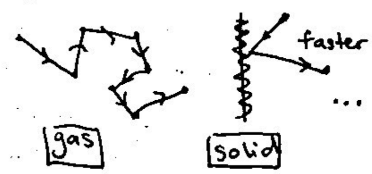
$\boxed{\text{I}}$ Transport phenomena in heat, mass, momentum

$\boxed{\text{I.A}}$ Modes of Transport

I.A.1. heat transport ~~diffusive mode~~

- diffusive mode (conduction)
  - $KE = \frac{3}{2}k_B T = \frac{3}{2}m\vec{v}^2$ (kinetic theory)
  - $KE_x = \frac{1}{2}k_B T = \frac{1}{2}mv_x^2$ (x-direction)
  - molecular motion - molecules translate kinetic energy to each other

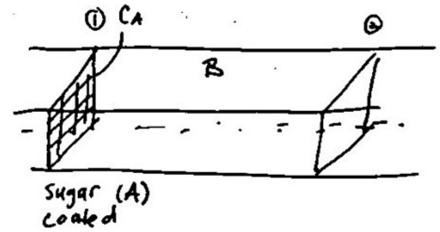


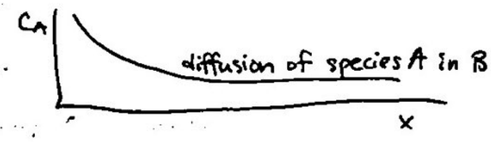


- convective mode
  - bulk Motion (of fluid/gas)
- radiation
  - radiation of EM radiation by objects > 0K

I.? Mass (Species) Transport

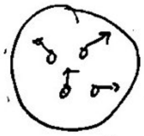
- diffusion of species A in B (Figure →)





IA3 continuum description of matter, "fluid particle"

- T or C_A at a point
- average property



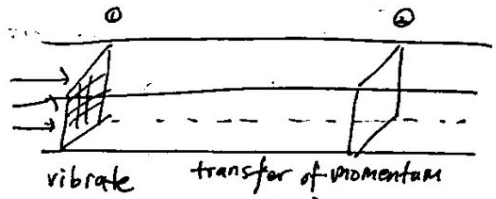
$n : 10^5 - 10^6$

$d_{part} \approx 10\,nm$

$d_{water} < d_{particle} < d$

IA4 Momentum Transport

- momentum $= m\vec{v}$
- momentum concentration $= \dfrac{m\vec{v}}{V} = \rho\vec{v}$



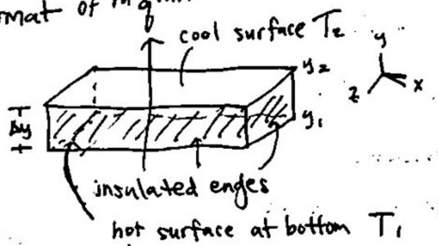
ρv_x

ρv_y

ρv_z

I.B The Phenomenological Rate Law of Diffusive Transport

I.B.1 Format of rate law



$\Delta y = y_2 - y_1$

$\Delta T = T_2 - T_1$

$$q \propto \frac{T_1 - T_2}{y_2 - y_1}$$

$$q = (\text{coeff}) \frac{T_1 - T_2}{y_2 - y_1}$$

$$= -(\text{coeff}) \frac{\Delta T}{\Delta y}$$

$$= -k \frac{\Delta T}{\Delta y}$$

$$= -k \frac{dT}{dy}$$

- heat: Q [=] energy : cal, J, erg, BTU
- heat flow: $\dot{Q}$ [=] $\frac{Q}{t}$ [=] $\frac{\text{energy}}{\text{time}}$: $\frac{\text{cal}}{s}$, $\frac{J}{s} = W$
- heat flux: $q = \frac{\dot{Q}}{A}$ [=] $\frac{\text{energy}}{\text{area} \cdot \text{time}}$: $\frac{\text{cal}}{\text{cm}^2 \cdot s}$, $\frac{W}{\text{cm}^2}$

  ↳ refer to specific area

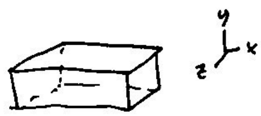
- $\boxed{q = -k \frac{dT}{dy}}$

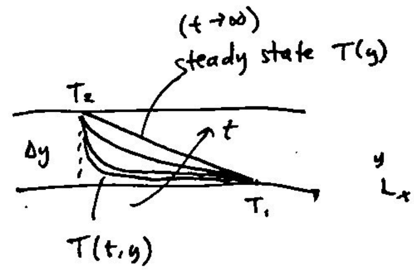
- Flux = (coeff) (driving force) #

↓

$\dfrac{\text{transported quantity}}{\text{area} \cdot \text{time}}$

$\boxed{I.8}$ Rate Law for heat conduction



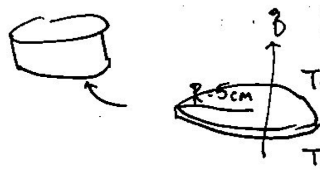


> Fourier's law $\boxed{q = -k \frac{dT}{dy}}$ 1-D rectilinear form

- k - thermoconductivity (a transport property)
- $k \, [=] \, \frac{[\text{flux}]}{[\text{driving force}]}$

$$\begin{cases} \text{cgs}: & [=] \dfrac{\frac{cal}{cm^2 s}}{\frac{K}{cm}} = \dfrac{cal}{cm \cdot s \cdot K} \\ \text{SI}: & [=] \dfrac{\frac{W}{m^2}}{\frac{K}{m}} = \dfrac{W}{m \cdot K} \end{cases}$$

$\boxed{\text{EX1}}$



q $k = 0.12 \frac{cal}{cm \cdot K \cdot s}$

$T_2 = 100°C$

$\Delta y = 0.5 cm$

$T_1 = 110°C$

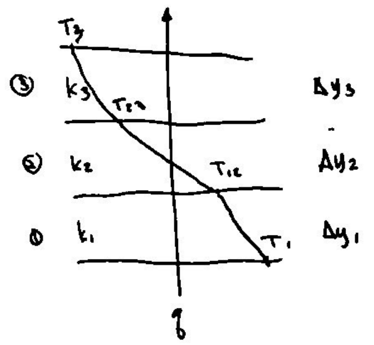
$\dot{Q} = ?$

$\lambda = 539 \frac{cal}{g}$ heat of vap. of H_2O

$$\dot{Q} = qA$$
$$= -k \frac{\Delta T}{\Delta y} A$$
$$= -\left(0.12 \frac{cal}{cm\, s\, K}\right)\left(\frac{100°C - 110°C}{0.5 cm}\right) \pi (5cm)^2$$
$$= 188.5 \; cal/s$$

$$\dot{m} = \frac{\dot{Q}}{\lambda} = \frac{188.5 \; cal/s}{5 \; (539 \; cal/g)} = 0.35 \; g/s$$

· Steady state conduction through layers in series



$$T_1 - T_{12} = q\frac{\Delta y_1}{k_1}$$

$$T_{12} - T_{23} = q\frac{\Delta y_2}{k_2}$$

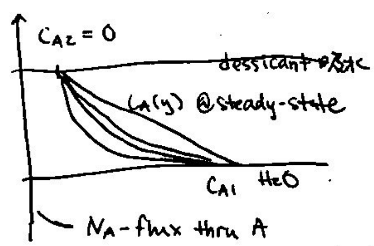
$$T_{23} - T_3 = q\frac{\Delta y_3}{k_3}$$

Add any combination to solve for any temperature difference

$$T_1 - T_3 = q\left[\frac{\Delta y_1}{k_1} + \frac{\Delta y_2}{k_2} + \frac{\Delta y_3}{k_3}\right]$$

$$q = \frac{T_1 - T_3}{[\cdots]}$$

$\boxed{\text{I.B.3}}$ Rate law for mass (species) transport



$T = 25°C$
$P = 1\,atm$

$$C_A^s = \frac{P_A^s}{RT} = \cdots$$

Diffusivity of A in B

$$N_A = -D_{AB}\frac{\Delta C_A}{\Delta y} = -D_{AB}\frac{dC_A}{dy}$$

$$D_{AB} [=] \frac{N_A}{\frac{dC_A}{dy}} = \frac{\frac{mol\,A}{cm^2 \cdot s}}{\frac{mol\,A}{cm^3 \cdot cm}} \quad [=]\ cm^2/s$$

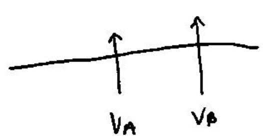
flux = −(coeff)(driving force)

$$N_A \approx -D_{AB}\frac{dC_A}{dy} = \text{constant} \quad (\text{Approx!})$$

• general case with species A & B

$$N_A = C_A V_A \ [=]\ \frac{mol\,A}{vol.} \cdot \frac{length}{time} = \frac{mol\,A}{area \cdot time} \quad (\text{flux unit})$$

$$N_B = C_B V_B$$



• v^* avg molar velocity: $v^* = \dfrac{C_A V_A + C_B V_B}{C_A + C_B} = \dfrac{C_A V_A + C_B V_B}{c} = \dfrac{C_A}{c}V_A + \dfrac{C_B}{c}V_B$

(let $c = C_A + C_B$)

$$= x_A V_A + x_B V_B = \frac{N_A + N_B}{c} = \frac{\Sigma N_i}{\Sigma c_i} \Rightarrow \boxed{v^* = \frac{N_A + N_B}{c}}$$

• approximation of N_A

$$N_A = C_A V_A = \underbrace{C_A v^*}_{\text{a surface moving at avg molar velocity}} + \underbrace{C_A(V_A - v^*)}_{\text{diffusion of A across the surface } (J_A^*)} = C_A\left(\frac{N_A + N_B}{c}\right) + J_A^* = x_A(N_A + N_B) + J_A^*$$

$$\boxed{J_A^* = -D_{AB}\frac{dC_A}{dy}}$$

Fick's law : 1D rectilinear form

diffusion of A across the surface moving @ v^*

I 3 Rate Law of Mass transport (cont.)

→ Case 1 : diffusion of A through a stagnant layer of B

- $N_B = 0$

$$N_A = x_A(N_A + \cancel{N_B}^{\,0}) + J_A^* \qquad (c_A = x_A c)$$

$$N_A = x_A N_A - D_{AB}\frac{dc_A}{dy}$$

$$N_A = x_A N_A - D_{AB}\, c\,\frac{dx_A}{dy}$$

$$N_A = \frac{c D_{AB}}{1-x_A}\frac{dx_A}{dy} = \text{const.}$$

$$\int_{y_1}^{y_2}\frac{N_A}{cD_{AB}}\,dy = \int_{x_{A1}}^{x_{A2}}\frac{1}{1-x_A}\,dx_A$$

$$\frac{N_A}{cD_{AB}}\overbrace{(y_2-y_1)}^{\Delta y} = \ln\left(\frac{1-x_{A2}}{1-x_{A1}}\right)$$

$$N_A = \frac{cD_{AB}}{\Delta y}\ln\left(\frac{1-x_{A2}}{1-x_{A1}}\right) \qquad \text{integral form for case 1}$$

→ Case of ours

$$\Delta y = L \qquad N_A = +\frac{cD_{AB}}{L}\ln\left(\frac{1-\cancel{x_{A2}}^{\,0}}{1-x_A^*}\right) = -\frac{cD_{AB}}{L}\ln(1-x_A^*) \qquad \text{integral form}$$

$x_{A2} = 0$

$x_{A1} = x_A^*$

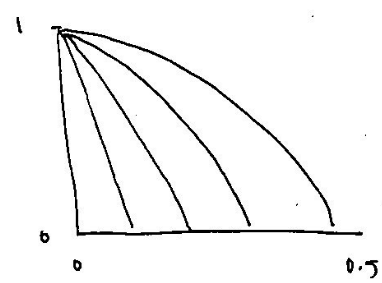
→ mol frac profile $x_A(y)$

$$\int_{0}^{y}\frac{N_A}{cD_{AB}}\,dy = \int_{x_A^*}^{x_A}\frac{1}{1-x_A}\,dx_A$$

$$\frac{N_A\, y}{cD_{AB}} = \ln\left(\frac{1-x_A}{1-x_A^*}\right)$$

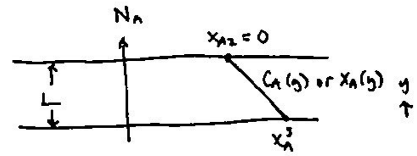
$$N_A = -\frac{\cancel{cD_{AB}}}{L}\ln(1-x_A^*)\frac{y}{\cancel{cD_{AB}}} = \ln(1-x_A) - \ln(1-x_A^*)$$

$$\boxed{x_A(y) = 1-(1-x_A^*)^{(1-y/L)}}$$



I.B.3 Rate Law of Mass Transport (cont.)

→ Case 1: Diffusion of A thru stagnant B (gas stripping)

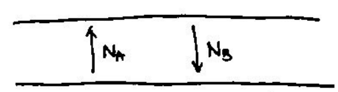


- $N_A = \underbrace{x_A [N_A + \cancel{N_B}^{0}]}_{\text{account for overall flow}} - \underbrace{c \mathcal{D}_{AB} \frac{dx_A}{dy}}_{\text{Fick's law}}$
- generalized: $\boxed{N_A = x_A [\Sigma N_i] - c \mathcal{D}_{AB} \frac{dx_A}{dy}}$

$$N_A = -\frac{c \mathcal{D}_{AB}}{1 - x_A} \frac{dx_A}{dy}$$

- integrated to: $N_A = -\frac{c \mathcal{D}_{AB}}{L} \ln(1 - x_A^s)$

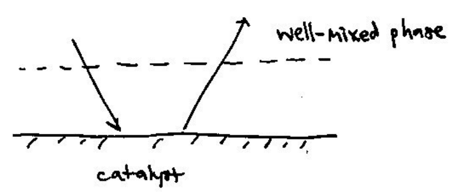
→ Case 2: Equimolar counter diffusion (distillation)



- $N_B = -N_A$ (equimolar)
- $N_A = x_A [\cancel{N_A + N_B}^{0}] - c \mathcal{D}_{AB} \frac{dx_A}{dy}$

$$\boxed{N_A = -c \mathcal{D}_{AB} \frac{dx_A}{dy}}$$

→ Case 3: Reaction at catalytic surface



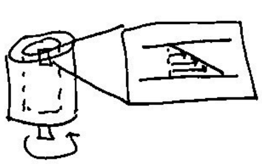
- $A \rightarrow 2B$ (example stoicheometry)
- $N_B = -2N_A$ (relation based on stoich)
- $N_A = x_A [N_A + \underbrace{\cancel{N_B}}_{-2N_A}] - c \mathcal{D}_{AB} \frac{dx_A}{dy}$

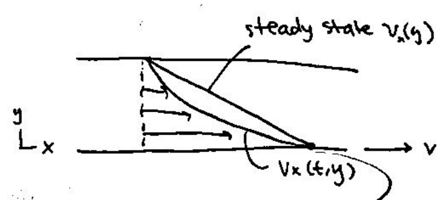
$$\boxed{N_A = -x_A N_A - c \mathcal{D}_{AB} \frac{dx_A}{dy}}$$ * for $A \rightarrow 2B$

I.B.4 Rate Law for Momentum transfer (Newton's Law of Viscocity)

→ 1D shear flow (couette flow)

- "cup & bob" couette viscometer



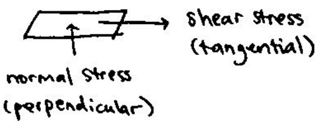


- stress $= \frac{F}{A}$
- shear stress $= \left(\frac{F}{A}\right)_x = \frac{\text{shear force}}{\text{area}}$
- stress must act on surfaces

> no-slip condition
- fluid at surface moves at same v as surface

- transport of x-momentum in y-direction

→ Stress on surface



→ Newton's law of viscosity

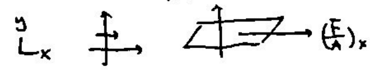
$$\left(\frac{F}{A}\right)_x = (\text{coeff})\,\frac{V}{\Delta y} = -\,\text{coeff}\,\frac{V_{x2}-V_{x1}}{y_2-y_1}$$

$$\boxed{\left(\frac{F}{A}\right)_x = -\underbrace{\mu}_{\text{viscocity}}\,\frac{dv_x}{dy}}$$ 1D shear flow

Newton's law of viscocity

→ Transport of x-momentum

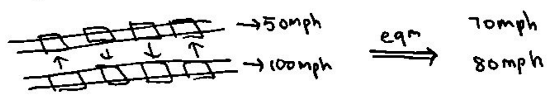
flux of x-momentum



- x-Mom = MV_x
- mom-flux $[=]$ $\frac{\text{momentum}}{\text{area}\cdot\text{time}}$ $[=]$ $\frac{\text{mass}\cdot\text{velocity}}{\text{area}\cdot\text{time}}$ $=$ $\frac{\overbrace{\text{mass}\cdot\text{length}}^{\text{force}}}{\text{area}\cdot\text{time}^2}$ $=$ $\frac{\text{force}}{\text{area}}$
- mom-flux $=$ $\frac{\text{force}}{\text{area}}$
- $\mu \;[=]\; \dfrac{(\frac{F}{A})_x}{\frac{dv_x}{dy}} \;[=]\; \begin{cases} \text{cgs}: \dfrac{\frac{g\ cm}{s^2\ cm^2}}{\frac{cm}{s\cdot cm}} = \boxed{\dfrac{g}{cm\cdot s} = \text{Poise}} \\ \text{SI}: \dfrac{\frac{kg\ m}{s^2 m^2}}{\frac{m}{s\cdot m}} = \dfrac{kg}{m\cdot s} = \boxed{Pa\cdot s} \end{cases}$ (centipoise cP used)
- $\boxed{1\ Pa\cdot s = 10\ \text{Poise} = 10^3\ cP}$
- $\mu\,(H_2O @ 25°C) = 1.0\ cP$

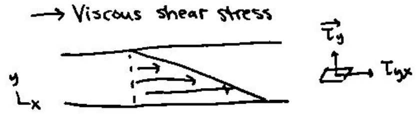
→ Mechanism of x-momentum transport

- molecules jump between diff y-planes, transfering momentum with them.
- by analogy, people jumping between trains.



[I.B.5] Designation of (viscous shear stress) = (momentum flux)

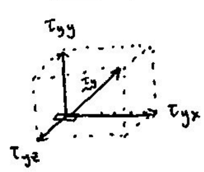
→ Viscous shear stress



$$\left(\frac{F}{A}\right)_x = \tau \rightarrow \begin{cases} \text{a viscous shear stress} \\ \text{a momentum flux} \end{cases}$$

$\underset{\sim}{\tau_y}$ = viscous stress on a y-plane

$$\boxed{\begin{array}{l}\tau_{yx} = \text{viscous shear stress exerted on a y-plane in the +x-direction} \\ \quad\text{by the fluid of lesser y on that of greater y}\end{array}}$$



$$\underset{\sim}{\tau_y} = \underset{\sim}{\delta_x}\tau_{yx} + \underset{\sim}{\delta_y}\tau_{yy} + \underset{\sim}{\delta_z}\tau_{yz}$$ $\underset{\sim}{\delta_i}$ are unit vectors

→ Momentum flux interpretation

- $\boxed{\tau_{yx} = \text{flux of x-momentum across a y-plane in +y-direction}}$

- $\boxed{\tau_{yx} = -\mu \frac{dv_x}{dy}}$ Newton's law of viscocity (1D rectilinear)

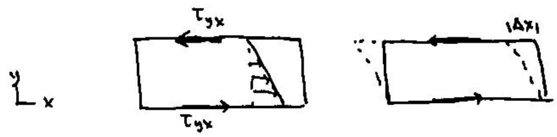
  "shear rate"
  shear strain rate $= \dot{\gamma}$ [=] s^{-1}

- $\dot{\gamma} = \frac{dv_x}{dy}$

> Newtonian fluid - fluid that obeys Newton's law of viscocity

I.B.6 Comparison of Fluid & Solid

> fluid - a material that can support no shear stress in a state of stable eqm (at rest)



fluid solid

- solid : $\boxed{\tau_{yx} = -G \frac{dx}{dy} = -G\gamma}$ Hooke's law

  > Hookean solids - solids that distort upon stress, and come back when stress disappears

- fluid : $\tau_{yx} = -\mu \frac{dv_x}{dy} = -\mu \frac{d\left(\frac{dx}{dt}\right)}{dy} = -\mu \frac{d}{dt}\left(\frac{dx}{dy}\right) = -\mu\dot{\gamma}$

  $$\boxed{\tau_{yx} = -\mu\dot{\gamma}}$$

- Response of solid to force is <u>extent</u> of distortion γ

- Response of fluid to force is <u>rate</u> of distortion $\dot{\gamma}$

I.B.7 Non-Newtonian fluid

- e.g. polymeric liquid, paste, slurries, emulsions, foams

  $\boxed{\tau_{yx} = -\eta \frac{dv_x}{dy}}$ 1D shear flow

  viscocity function

  - often, $\eta = \eta(\dot{\gamma})$

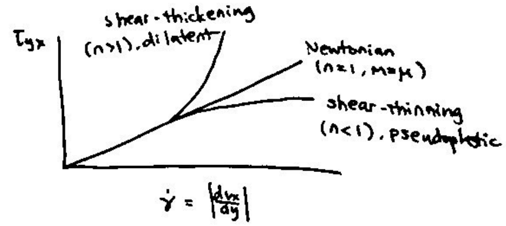
> Power-law fluid - fluid with $\boxed{\eta = m\left|\frac{dv_x}{dy}\right|^{n-1} = m\dot{\gamma}^{n-1}}$

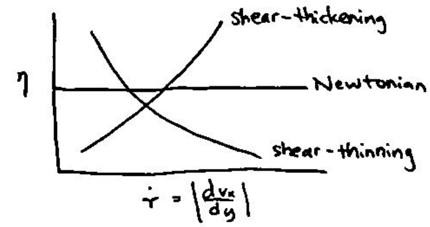
- shear stress : $\tau_{yx} = -\eta \frac{dv_x}{dy} = -m\left|\frac{dv_x}{dy}\right|^{n-1} \frac{dv_x}{dy} = -m\left(\frac{dv_x}{dy}\right)^n$

  $\boxed{\tau_{yx} = -m\left(\frac{dv_x}{dy}\right)^n = -m\dot{\gamma}^n}$ power-law fluid

→ Viscometric fluid

> viscometric fluid - fluid that $\eta \propto \dot{\gamma}$



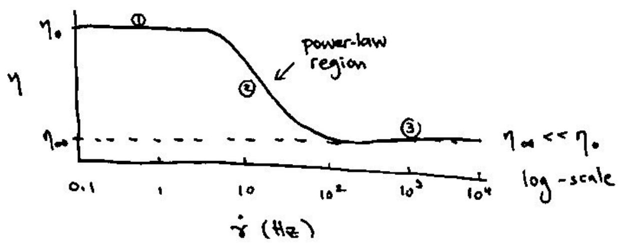


> shear-thickening (dilatent) - $n>1$

> shear-thinning (pseudoplastic) - $n<1$

> Newtonian - $n=1$, $m=\mu$

→ Slurries



> Carreau equation $\boxed{\dfrac{\eta-\eta_\infty}{\eta_0-\eta_\infty} = \left[1+[\lambda\dot{\gamma}]^2\right]^{\frac{n-1}{2}}}$ λ, n are empirical constants

① $\dot{\gamma} \to 0$

$$\frac{\eta-\eta_\infty}{\eta_0-\eta_\infty} \approx \frac{\eta}{\eta_0} = 1^{\frac{n-1}{2}} = 1 \quad \Rightarrow \quad \eta = \eta_0$$

② $\dot{\gamma}$ intermediate

· Note $(\lambda\dot{\gamma})^2 >> 1$

$$\frac{\eta-\eta_\infty}{\eta_0-\eta_\infty} \cong \frac{\eta}{\eta_0} = (\lambda\dot{\gamma})^{n-1} \quad \Rightarrow \quad \eta = \eta_0(\lambda\dot{\gamma})^{n-1} = m\dot{\gamma}^{n-1}, \text{ where } m \equiv \lambda^{n-1}$$

(power law)

③ $\dot{\gamma} \to \infty$

$$\frac{\eta-\eta_\infty}{\eta_0-\eta_\infty} = \frac{\eta-\eta_\infty}{\eta_0} = m\dot{\gamma}^{n-1} \approx 0 \quad \Rightarrow \quad \eta = \eta_\infty$$

negative exponent

→ Time effect

> thixotropic - η decrease with time

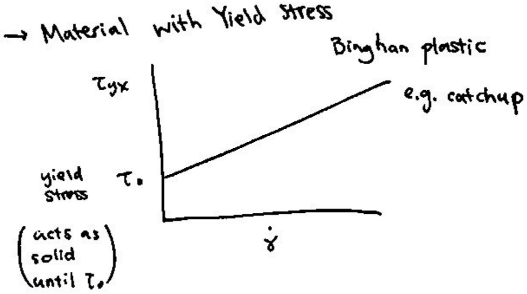
> anthixotropic - η increase with time

→ Viscoelastic material

· $\tau = \tau$ (strain, strain rate, time)

· constitutive eqn → rheology

→ Material with Yield Stress



I.B.8 Comparison of Rate Laws

- General Form: $\text{Flux} = -(\text{coeff})(\text{driving force})$
- Newton's law of viscosity: $\tau_{yx} = -\mu \frac{dv_x}{dy}$
- Fourier's law: $q = -k \frac{dT}{dy}$
- Fick's law: $J_A^* = -D_{AB} \frac{dc_A}{dy}$

- 1D rectilinear
- driving force $\propto$ concentration of transported quantity
- concentration = $\frac{\text{amount}}{\text{volume}}$

→ Concentration of x-momentum

- $c_{x\text{-mom}} = \frac{mv_x}{vol} = \rho v_x$
- $\frac{dc_{x\text{-mom}}}{dy} = \rho \frac{dv_x}{dy}$

$$\frac{dv_x}{dy} = \frac{1}{\rho} \frac{dc_{x\text{-mom}}}{dy}$$

- In Newton's law of viscosity:

$$\tau_{yx} = -\mu \frac{1}{\rho} \frac{dc_{x\text{-mom}}}{dy}$$

$$\boxed{\tau_{yx} = -\nu \frac{dc_{x\text{-mom}}}{dy}}$$

$\boxed{\nu \equiv \frac{\mu}{\rho}}$ kinematic viscosity

$\nu \; [=] \; \frac{\mu}{\rho} = cm^2/s$

→ Concentration of thermal energy

- $c_H = \rho \hat{H} \qquad (\hat{H} = \hat{H}_o)$

$$c_H = \rho \hat{H}_o + \rho \hat{C}_p (T - T_o)$$

- $\frac{dc_H}{dy} = \rho \hat{C}_p \frac{dT}{dy}$

$$\frac{dT}{dy} = \frac{1}{\rho \hat{C}_p} \frac{dc_H}{dy}$$

- In Fourier's law:

$$q_y = -k \frac{1}{\rho \hat{C}_p} \frac{dc_H}{dy}$$

$$\boxed{q_y = -\alpha \frac{dc_H}{dy}}$$

$\boxed{\alpha \equiv \frac{k}{\rho \hat{C}_p}}$ thermal diffusivity

$\alpha \; [=] \; cm^2/s$

→ concentration of species

- Fick's law is already in form of concentration gradient:

$\boxed{J_A^* = -D_{AB} \frac{dc_A}{dy}}$, $D_{AB} \; [=] \; cm^2/s$

→ Dimensionless group

> Prandtl number - $\boxed{Pr = \frac{\nu}{\alpha}} = \frac{\hat{C}_p \mu}{k} = \frac{\text{mom. diff}}{\text{therm. E diff}}$

> Schmidt number - $\boxed{Sc = \frac{\nu}{D_{AB}}} = \frac{\mu}{\rho D_{AB}} = \frac{\text{mom. diff}}{\text{species diff}}$

species	Pr	Sc
gas	~1	~1
liquid	~10-10^3	~10^2-10^5
liquid metal	~10^{-1}-1	"

I.C Transport Coefficients For Fluids : μ, k, D_{AB}

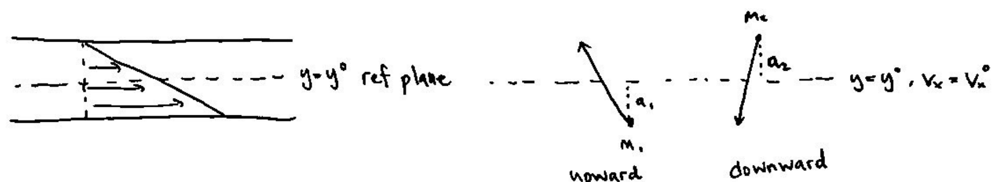
· Objective
- · predict & correlate μ, k, D_{AB}
- · dependence on T, P
- · relation to each other
- · dependence on composition

I.C.1 Ideal Gas (modest P)

→ Simple Kinetic Theory

- · molecules are structureless spheres of mass m and diameter d
- · no intermolecular forces
- · perfectly elastic collision
- · Maxwell-Boltzmann distribution of velocity
  - > average velocity $\boxed{\bar{u} = \sqrt{\dfrac{8 K T}{\pi m}}}$
  - > Boltzmann's constant $K = \dfrac{R}{N_A} = 1.38 \times 10^{-23}\ J/K = 1.38 \times 10^{-16}\ erg/K$
  - > Mean free path $\boxed{\lambda = \dfrac{1}{\sqrt{2}\pi d^2 n}}$
  - > number density $\boxed{n = \dfrac{N}{V}}$

→ Viscocity μ of ideal gas



· Velocity of particle : $(v_x)_1 = v_x^\circ - a_1 \dfrac{dv_x}{dy}$

· Momentum of particle
- upward : $m(v_x)_1 = mv_x^\circ - a_1 m \dfrac{dv_x}{dy}$
- downward : $m(v_x)_2 = mv_x^\circ + a_2 m \dfrac{dv_x}{dy}$

· Need
- · molecular flux in y-dir $z = z\uparrow = z\downarrow$; $\boxed{z = \frac{1}{4} n\bar{u}}$
- · avg distance of molecules from y° when they initiate their jump $\boxed{\bar{a} = \frac{2}{3}\lambda}$

$$\begin{bmatrix}\text{x-mom}\\ \text{flux}\end{bmatrix} = \tau_{yx} = z\left\{\left(\frac{\text{avg x-mom}}{\text{molecule}}\right)\uparrow - \left(\frac{\text{avg x-mom}}{\text{molecule}}\right)\downarrow\right\} = z\left\{m\left[v_x^\circ - \bar{a}\frac{dv_x}{dy}\right] - m\left[v_x^\circ + \bar{a}\frac{dv_x}{dy}\right]\right\}$$

$$= zm(-2\bar{a})\frac{dv_x}{dy} = -\mu\frac{dv_x}{dy} \quad \text{(Rate law)}$$

$$\Rightarrow \mu = 2mz\bar{a} = 2m\left(\tfrac{1}{4}n\bar{u}\right)\left(\tfrac{2}{3}\lambda\right) = \tfrac{1}{3}\rho\bar{u}\lambda = \tfrac{1}{3}\rho\sqrt{\frac{8KT}{\pi m}}\,\frac{1}{\sqrt{2}\pi d^2 n} = \tfrac{2}{3}\pi^{-\frac{3}{2}}\frac{\sqrt{mKT}}{d^2} \qquad (\rho \equiv mn)$$

$$\boxed{\mu = \frac{2}{3}\pi^{-\frac{3}{2}}\frac{\sqrt{mKT}}{d^2}}$$

→ Example of viscosity calculation

EX1 CO_2 @ 20°C

$$m = \frac{M}{N_A} = \frac{44.01}{6.023\times10^{23}} = 7.31\times10^{-23}\,g$$

$$d = 3.996\,\text{Å} = 3.996\times10^{-8}\,cm$$

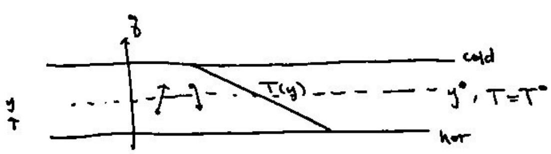
$$k = 1.38\times10^{-16}\,erg/K = \cdots\frac{g\,cm^2}{s^2\,K}$$

$$T = 20°C = 293.2K$$

$\mu = 1.29\times10^{-4}\,\frac{g}{cm\cdot s} = \cdots$ poise $= 0.0129$ cP

(experimental : $\mu = 0.0146$ cP)

→ Thermal conductivity k of ideal gas



- monoatomic ideal gas
- $\frac{\text{energy}}{\text{molecule}} = \frac{3}{2}kT$
- $$q = z\left\{\left(\frac{\text{avg energy}}{\text{molecule}}\right)\uparrow - \left(\frac{\text{avg energy}}{\text{molecule}}\right)\downarrow\right\}$$

$$= z\left[\frac{3}{2}k\bar{T}_{below} - \frac{3}{2}k\bar{T}_{above}\right]$$

$$= z\,\frac{3}{2}k\left(\bar{T}_{below} - \bar{T}_{above}\right)$$

$$= z\,\frac{3}{2}k\left[\left(T^\circ - \bar{a}\frac{dT}{dy}\right) - \left(T^\circ + \bar{a}\frac{dT}{dy}\right)\right]$$

$$= z\,\frac{3}{2}k\left(-2\bar{a}\frac{dT}{dy}\right)$$

$$= -3kz\bar{a}\frac{dT}{dy} = -k\frac{dT}{dy} \qquad \text{(Fourier's law)}$$

- $k = 3kz\bar{a}$

$$= 3\left(\frac{2}{3}m\hat{c}_v\right)z\bar{a} \qquad \left(c_v = \frac{3}{2}k,\ \hat{c}_v = \frac{3k}{2m} \Rightarrow k = \frac{2m\hat{c}_v}{3}\right)$$

$$= 2m\hat{c}_v\left(\frac{1}{4}n\bar{u}\right)\left(\frac{2}{3}\lambda\right) \qquad \left(z = \frac{1}{4}n\bar{u},\ \bar{a} = \frac{2}{3}\lambda\right)$$

$$= \frac{1}{3}\rho\hat{c}_v\bar{u}\lambda \qquad (\rho = mn)$$

$$\boxed{k = \frac{2\hat{c}_v}{3\pi^{3/2}}\frac{\sqrt{mkT}}{d^2}}$$

→ Diffusivity D_{AB} of ideal gas

- Without proof,

$$D_{AB} = \frac{1}{3}\bar{u}_{AB}\lambda_{AB}$$

$$\boxed{D_{AB} = \frac{2}{3\pi^{3/2}}\frac{\sqrt{(kT)^3/m_{AB}}}{d_{AB}^2\,P}} \quad , \quad \begin{cases} m_{AB} = \dfrac{2m_Am_B}{m_A+m_B} \\ d_{AB} = \dfrac{1}{2}(d_A+d_B)\end{cases}$$

→ Summary of transport coefficients of ideal gas : kinetic theory

$$\boxed{\mu = \frac{1}{3}\rho\bar{u}\lambda = \frac{2}{3\pi^{3/2}}\frac{\sqrt{m k T}}{d^2}} \Rightarrow \boxed{\mu \propto \sqrt{T}}$$

$$\boxed{k_{mono} = \frac{1}{3}\rho\hat{C}_v\bar{u}\lambda = \frac{2\hat{C}_v}{3\pi^{3/2}}\frac{\sqrt{m k T}}{d^2} = \hat{C}_v\mu} \Rightarrow \boxed{k_{mono} \propto \sqrt{T}}$$

$$\boxed{D_{AB} = \frac{1}{3}\bar{u}_{AB}\lambda_{AB} = \frac{2}{3\pi^{3/2}}\frac{\sqrt{(kT)^3/m_{AB}}}{d_{AB}^2 P}} \quad \begin{cases} m_{AB} = \frac{2m_A m_B}{m_A + m_B} \\ d_{AB} = \frac{1}{2}(d_A + d_B) \end{cases} \Rightarrow \boxed{D_{AB} \propto T^{\frac{3}{2}} P^{-1}}$$

$$Pr = \frac{\nu}{\alpha} = \frac{\hat{C}_v\mu}{k_{mono}} = \frac{\hat{C}_v\frac{1}{3}\rho\bar{u}\lambda}{\frac{1}{3}\rho\hat{C}_v\bar{u}\lambda} = 1 \quad \text{(monoatomic ideal gas)}$$

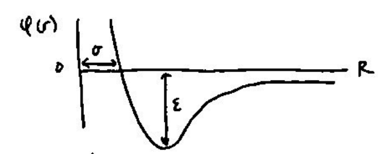
$$Sc = \frac{\nu}{D_{AB}} = \frac{\mu}{\rho D_{AB}} = \frac{\frac{1}{3}\rho\bar{u}\lambda}{\rho\frac{1}{3}\bar{u}\lambda} = 1 \quad \text{(general ideal gas)}$$

I.C.2 Refinement of Simple Kinetic Theory

→ Intermolecular Force

· potential energy of interaction $\varphi(r)$ – reversible work to bring a pair of molecule from ∞ separation to distance R from each other

> Lenard–Jones potential $\boxed{\varphi(r) = 4\varepsilon\left[\left(\frac{\sigma}{R}\right)^{12} - \left(\frac{\sigma}{R}\right)^{6}\right]}$



· attractive force $F_{attract} = +\frac{d\varphi(r)}{dr}$

$$\boxed{F_{attr} = \frac{24\varepsilon}{R}\left[\left(\frac{\sigma}{R}\right)^{6} - 2\left(\frac{\sigma}{R}\right)^{12}\right]}$$

→ Chapman – Enskog Equations

$$\mu = \frac{5}{16\pi}\frac{\sqrt{\pi m k T}}{\sigma^2\Omega_\mu}$$

$\Omega_\mu = \Omega_k$ = collision integrals, dimensionless, $\propto \frac{kT}{\varepsilon}$

$$k_{mono} = \frac{25}{32\pi}\frac{\sqrt{\pi m k T}}{\sigma^2\Omega_k}\hat{C}_v = \frac{5}{2}\hat{C}_v\mu$$

temperature dependence

$$\boxed{\mu\left(\frac{g}{cm\cdot s}\right) = 2.6693\times10^{-5}\frac{\sqrt{MT}}{\sigma^2\Omega_\mu}} \longrightarrow \boxed{\mu(T_2) = \mu(T_1)\sqrt{\frac{T_2}{T_1}}\frac{\Omega_{\mu 1}}{\Omega_{\mu 2}}}$$

$$\boxed{k_{mono}\left(\frac{cal}{cm\cdot s\cdot K}\right) = \frac{5}{2}\hat{C}_v\mu = 1.989\times10^{-4}\frac{\sqrt{T/M}}{\sigma^2\Omega_k}}$$

$$\boxed{D_{AB}(T_2) = D_{AB}(T_1)\left(\frac{T_2}{T_1}\right)^{\frac{3}{2}}\frac{\Omega_{D1}}{\Omega_{D2}}}$$

$$\boxed{k_{poly}\left(\frac{cal}{cm\cdot s\cdot K}\right) = \left[\hat{C}_p + \frac{5}{4}\frac{R}{M}\right]\mu}$$

$$\boxed{D_{AB}\left(\frac{cm^2}{s}\right) = 2.63\times10^{-3}\frac{\sqrt{T^3/M_{AB}}}{P\sigma_{AB}^2\Omega_D}} \quad \begin{cases} M_{AB} = \frac{2M_A M_B}{M_A + M_B} \\ \sigma_{AB} = \frac{1}{2}(\sigma_A + \sigma_B) \end{cases}$$

T [=] K
P [=] atm
σ_{AB} [=] Å

$\boxed{\text{I.C.3}}$ Transport properties of ideal gas mixtures

› Wilkie Eqn

$$\boxed{\mu_{mix} = \sum_{i=1}^{N} \frac{x_i \mu_i}{\sum_{j=1}^{N} x_j \Phi_{ij}}}$$

$$\boxed{k_{mix} = \sum_{i=1}^{N} \frac{x_i k_i}{\sum_{j=1}^{N} x_j \Phi_{ij}}}$$

$$\Phi_{ij} = \frac{1}{\sqrt{8}}\left[1+\frac{M_i}{M_j}\right]^{-\frac{1}{2}}\left[1+\left(\frac{\mu_i}{\mu_j}\right)^{\frac{1}{2}}\left(\frac{M_j}{M_i}\right)^{\frac{1}{4}}\right]^2$$

› Blanc's Eqn

$$\boxed{D_{i,mix} = \left[\sum_{j\neq i}^{N} \frac{x_j}{D_{ij}}\right]^{-1}}$$

D_{ij} determined by Chapman-Enskog Eqn

$\boxed{\text{I.C.4}}$ Transport Properties of Liquids

- Mainly T dependence
- P dependence at high P

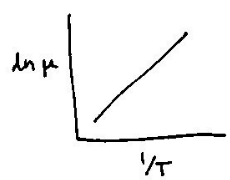
→ Viscosity μ

› Eyring model -

$$\boxed{\mu = \frac{N_A h}{\hat{V}} \exp\left[0.408 \frac{\Delta U_{vap}}{RT}\right]}$$

$$\boxed{\mu = A e^{B/T}}$$

$$\ln \mu = \ln A + \frac{B}{T}$$



→ Thermal Conductivity k

- Use the format of k for ideal gas.

$$k_{gas} = \frac{1}{3}\rho \hat{c}_v \bar{u} \lambda$$
$$= \frac{1}{3}\rho \hat{c}_v V_s a_{mol}$$
$$= \left(\frac{N_A m}{\tilde{V}}\right)\left(3\frac{K}{m}\right) v_s \left(\frac{\tilde{V}}{N_A}\right)^{\frac{1}{3}}$$
$$= 3\left(\frac{N_A}{\tilde{V}}\right)^{\frac{2}{3}} K v_s$$

$\bar{u}$ - characteristic velocity
λ - characteristic length
V_s - speed of sound
a_{mol} - molecular radius

$$\hat{c}_v = 3\left(\frac{R}{M}\right) = 3\frac{K}{m}$$
$$a = \left(\frac{\tilde{V}}{N_A}\right)^{\frac{1}{3}}$$
$$\rho = \frac{M}{\tilde{V}} = \frac{N_A m}{\tilde{V}}$$

› Bridgeman eqn

$$\boxed{k = 2.8\left(\frac{N_A}{\tilde{V}}\right)^{\frac{2}{3}} K V_s}$$

- weakly dependent on T (up or down)

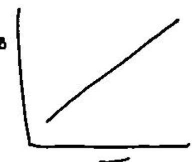
→ Diffusivity D_{AB}° for dilute A $(x_A \le 0.1)$

› Einstein equation $\boxed{D_{AB} \approx \frac{kT}{f}}$ f - hydrodynamic friction factor

· $f = \begin{cases} 6\pi \mu_B R_A & \text{no slip} \\ 4\pi \mu_B R_A & \text{free slip} \end{cases}$

› Stoke-Einstein eqn $\boxed{D_{AB} = \frac{kT}{4\pi \mu_B R_A}}$

· Temperature T dependence $\boxed{D_{AB}\mu_B \propto T}$



› Wilke-Chang correlation $\boxed{D_{AB} = 7.4\times10^{-8}\ \frac{(\gamma_B M_B)^{\frac{1}{2}}\,T}{\mu\ \tilde{V}_A^{0.6}}\ [=]\ cm^2/s}$

M_B - molecular weight of B
γ_B - association parameter
$\tilde{V}_A$ - molar vol of A at T_{bp} [=] cm^3/mol
T [=] K
$\mu \approx \mu_B$ [=] cP

→ Diffusivity D_{AB} in mixture media

› Vigne's eqn $\boxed{D_{AB} = (D_{AB}^{\circ})^{x_B}\,(D_{BA}^{\circ})^{x_A}}$

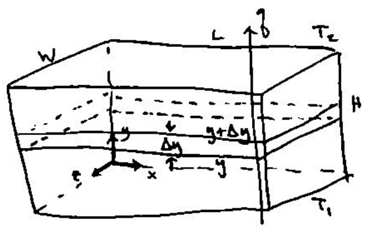
· For liquids, $D_{AB} \neq D_{BA}$

I.D The conservation principle : "Shell balance" for flux distribution & profile

- conservation of energy (thermal E)
- conservation of species (species A)
- conservation of momentum

I.D.1 Construction & Use of Shell Balances

› shell › control volume thin in the direction of transport



insulated edges

$\Delta V = LW\Delta y$

1D rectilinear case

flux distribution	⟶	profile
$q_y(y)$ = const		$T(y)$
$N_{Ay}(y)$ = const	⇨	$C_A(y)$ or $x_A(y)$
$\tau_{yx}(y)$ = const		$v_x(y)$

OIGA

O − I = G − A

out − in = generation − accumulation

$$\begin{bmatrix}\text{rate of heat}\\\text{flowing out}\end{bmatrix} - \begin{bmatrix}\text{rate of heat}\\\text{flowing in}\end{bmatrix} = \begin{bmatrix}\text{rate of heat}\\\text{generation}\end{bmatrix} - \begin{bmatrix}\text{rate of heat}\\\text{accumulation}\end{bmatrix}$$

$$\lim_{\Delta y \to 0} \frac{[qLW]_{y+\Delta y} - [qLW]_y}{LW\Delta y} = \frac{0 - 0}{LW\Delta y}$$

$\boxed{\frac{dq}{dy} = 0}$ differential equation of flux distribution

q = const.

→ Temperature profile

- boundary conditions (b.c.) $\begin{cases} y=0, & T=T_1 \\ y=H, & T=T_2 \end{cases}$

→ separate variable (method 1)

$$q = \text{const} = -k\frac{dT}{dy}$$

$$\int q\,dy = \int -k\,dT$$

$$qy = -kT + c_1$$

$$q(0) = -kT_1 + c_1 \quad \text{(b.c.1)}$$

$$c_1 = kT_1$$

$$qy = -kT + kT_1$$

$$\boxed{T(y) = T_1 - \left(\frac{q}{k}\right)y}$$ linear temp profile

→ integrate with indefinite bounds (method 2)

$$q = \text{const} = -k\frac{dT}{dy}$$

$$\int_0^y q\,dy = \int_{T_1}^{T} -k\,dT \quad \text{(b.c.1)}$$

$$qy = -k(T-T_1)$$

$$\boxed{T(y) = T_1 - \left(\frac{q}{k}\right)y}$$ linear temp profile

→ Get q by integrate whole layer.

$$q = \text{const.} = -k\frac{dT}{dy}$$

$$\int_0^H q\,dy = \int_{T_1}^{T_2} -k\,dT \quad \text{(b.c.1,2)}$$

$$qH = -k(T_2-T_1)$$

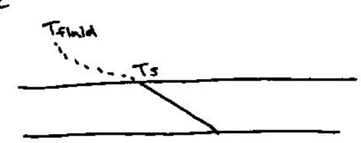
$$\boxed{q = \frac{k(T_1-T_2)}{H}}$$

I.D.2 Heat Transfer Boundary Conditions

- Type I: specify temperature at given surface
- Type II: specify flux q at a surface $\xLeftrightarrow{q=-k\frac{dT}{dy}}$ specify $\frac{dT}{dy}$ at surface (e.g. insulated $q=0$)
- Type III: Specify relationship between q and T at surface

  > Newton's law of cooling

  $$\boxed{q = h(T_s - T_{fluid})}$$



h - heat transfer coefficient, a lumped parameter that includes flow condition

$h[=]\ \frac{cal}{cm^2 \cdot s \cdot K}$

I.D.3 Mass Transfer Boundary Conditions.

- Type I: specify C_A or X_A at surface
- Type II: specify N_A at surface $\Longleftrightarrow$ $\frac{dc_A}{dy}$ or $\frac{dx_A}{dy}$ at surface
- Type III: specify relationship between N_A and C_A at surface

  > $\boxed{N_A = k_m(C_{As} - C_{A\,fluid})}$ k_m - Mass transfer coefficient

  $k_m[=]$ cm/s

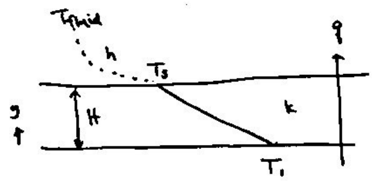
I.D.4 Examples of 1D Transport

EX1 Heat transfer through a solid bounded by a stirred fluid layer

$H = 4.0\ cm$ $k = 0.92\ \frac{cal}{cm \cdot s \cdot K}$ (aluminum)

$T_1 = 100°C$ $h = 0.05\ \frac{cal}{cm^2 \cdot s \cdot K}$

$T_{fluid} = 20°C$



Calculate: heat flux q and surface temperature T_s

$$q = h(T_s - T_{fluid}) \Rightarrow T_s - T_{fluid} = q\left(\frac{1}{h}\right)$$

$$q = -k\frac{dT}{dy} = k\,\frac{T_1 - T_s}{H} \Rightarrow T_1 - T_s = q\left(\frac{H}{k}\right) \quad +$$

$$T_1 - T_{fluid} = q\left(\frac{1}{h} + \frac{H}{k}\right)$$

$$q = \frac{T_1 - T_{fluid}}{\frac{1}{h} + \frac{H}{k}} = 3.29\ \frac{cal}{cm^2 \cdot s}$$

$$T_s = T_1 - \frac{q}{k}H = 85.7°C$$

EX2 Diffusion in capillary tube: measuring D_{AB}

Acetone (A) in air (B):

$\mathcal{M}_A = 58.08\ g/mol$

$\rho_A = 0.792\ g/cm^3$

$p_A^s = 0.243\ atm$

$x_A^s = \frac{p_A^s}{p} = \frac{0.243\ atm}{1\ atm} = 0.243$

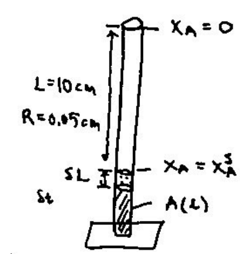
$c = \frac{n}{V} = \frac{p}{RT} = 4.16 \times 10^{-5}\ \frac{mol}{cm^3}$

Experiment

$\delta t = 17h = 61200s$

$\delta L = 0.6\ cm$

Setup



Shell Balance

$N_A = $ const.

Diffusion of A thru stagnant B

$$N_A = -\frac{cD_{AB}}{1 - x_A}\frac{dx_A}{dy} \xrightarrow{\text{integrate}} N_A = -\frac{cD_{AB}}{L}\ln(1 - x_A^s) \Rightarrow D_{AB} = -\frac{N_A L}{c\ln(1 - x_A^s)}$$

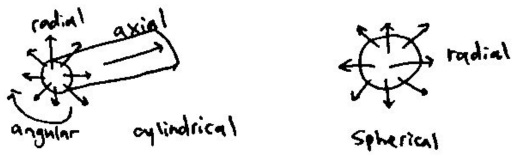
From experiment

$$N_A = \frac{\text{mol evap}}{(\pi R^2)\delta t} = \frac{(\text{vol evap})(\rho_A/\mathcal{M}_A)}{\pi R^2 \delta t} = \frac{(\pi R^2 \delta L)(\rho_A/\mathcal{M}_A)}{\pi R^2 \delta t} = \frac{\delta L\, \rho_A}{\delta t\, \mathcal{M}_A} = 1.337 \times 10^{-7}\ \frac{mol}{cm^2 \cdot s}$$

$$L = \bar{L} = L(0) + \frac{1}{2}\delta L = 10.3\ cm$$

$$D_{AB} = \frac{-N_A L}{c\ln(1 - x_A^s)} = 0.119\ cm^2/s$$

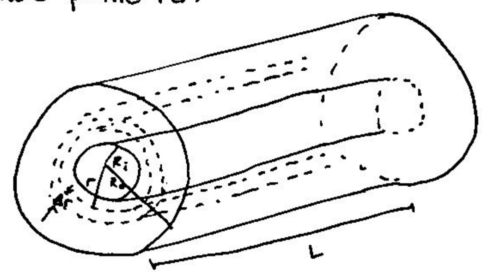
I.D.5 Radial Transport in Cylindrical & Spherical Coordinates



→ Radial Transport in Cylinder

· Determine flux distribution $q(r)$, radial temperature profile $T(r)$

· b.c. $\begin{cases} r = R_i\ , & T = T_i \\ r = R_o\ , & T = T_o \end{cases}$



OIGA

$$[q]_{r+\Delta r}\, 2\pi(r+\Delta r)L - [q]_r\, 2\pi r L = 0 - 0$$

$$\lim_{\Delta r \to 0} \frac{\{[rq]_{r+\Delta r} - [rq]_r\}\ 2\pi L}{2\pi r L \Delta r} = \frac{0}{2\pi r L \Delta r}$$

$\Delta V = 2\pi r L \Delta r$

· end insulated

· no generation

· steady state

$$\frac{1}{r}\frac{d(rq)}{dr} = 0$$

$$\boxed{\frac{d(rq)}{dr} = 0}$$

$$rq = c_1$$

$$q(r) = \frac{c_1}{r} = -k\frac{dT}{dr} \quad \leftarrow \text{Fourier's law}$$

$$\int_{R_i}^{R_o} \frac{c_1}{r}\, dr = \int_{T_i}^{T_o} -k\, dT \qquad \text{(b.c. 1, 2)}$$

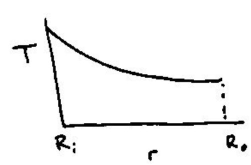
$$c_1 = \frac{-k(T_o - T_i)}{\ln\left(\frac{R_o}{R_i}\right)}$$

$$\boxed{q(r) = \frac{k(T_i - T_o)}{r\ln\left(\frac{R_o}{R_i}\right)}} = -k\frac{dT}{dr} \qquad \text{Fourier's law}$$

$$\frac{T_i - T_o}{\ln\left(\frac{R_o}{R_i}\right)} \int_{R_i}^{r} \frac{1}{r}\, dr = \int_{T_i}^{T} -dT$$

$$\frac{T_i - T_o}{\ln\left(\frac{R_o}{R_i}\right)} \ln\left(\frac{r}{R_i}\right) = T_i - T$$

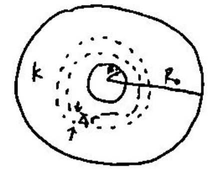
$$\boxed{T(r) = T_i - \frac{T_i - T_o}{\ln\left(\frac{R_o}{R_i}\right)} \ln\left(\frac{r}{R_i}\right)}$$



→ Radial transport in sphere

- Determine flux distribution $q(r)$, radial temperature profile $T(r)$

  - b.c. $\begin{cases} r = R_i, & T = T_i \\ r = R_o, & T = T_o \end{cases}$



$\Delta V = 4\pi r^2 \Delta r$

OIGA

$$[q]_{r+\Delta r}\, 4\pi (r+\Delta r)^2 - [q]_r\, 4\pi r^2 = 0 - 0$$

$$\lim_{\Delta r \to 0} \frac{[r^2 q]_{r+\Delta r}\, 4\pi - [r^2 q]_r\, 4\pi}{4\pi r^2 \Delta r} = \frac{0}{4\pi r^2 \Delta r}$$

$$\frac{1}{r^2}\frac{d(r^2 q)}{dr} = 0$$

$$\boxed{\frac{d(r^2 q)}{dr} = 0}$$

$$r^2 q = c_1$$

$$q = \frac{c_1}{r^2} = -k\frac{dT}{dr} \quad \text{Fourier's law}$$

$$\int_{R_i}^{R_o} \frac{c_1}{r^2}\, dr = \int_{T_i}^{T_o} -k\, dT \quad \text{(b.c. 1, 2)}$$

$$c_1 \left[-\frac{1}{r}\right]_{R_i}^{R_o} = -k(T_o - T_i)$$

$$c_1 = \frac{-k(T_o - T_i)}{\frac{1}{R_i} - \frac{1}{R_o}}$$

$$\boxed{q(r) = \frac{k(T_i - T_o)}{r^2\left(\frac{1}{R_i} - \frac{1}{R_o}\right)}} = -k\frac{dT}{dr} \quad \text{Fourier's law}$$

$$\frac{T_i - T_o}{\left(\frac{1}{R_i} - \frac{1}{R_o}\right)} \int_{R_i}^{r} \frac{1}{r^2}\, dr = \int_{T_i}^{T} -dT$$

$$\frac{T_i - T_o}{\left(\frac{1}{R_i} - \frac{1}{R_o}\right)} \left(\frac{1}{R_i} - \frac{1}{r}\right) = T_i - T$$

$$\boxed{T(r) = T_i + \frac{T_i - T_o}{\frac{1}{R_i} - \frac{1}{R_o}} \left[\frac{1}{r} - \frac{1}{R_i}\right]}$$

I.D.6 Functional Forms of Flux Distribution & Profiles

- Assumptions : 1D, no generation, steady state

coordinates	heat transport	species transport
rectangular	$q(y) = c_1$ $T(y) = c_1 y + c_2$	$N_A(y) = c_1$ $C_A(y) = c_1 y + c_2$ *
cylindrical	$q(r) = \frac{c_1}{r}$ $T(r) = c_1 \ln r + c_2$	$N_A(r) = \frac{c_1}{r}$ $C_A(r) = c_1 \ln r + c_2$ *
spherical	$q(r) = \frac{c_1}{r^2}$ $T(r) = \frac{c_1}{r} + c_2$	$N_A(r) = \frac{c_1}{r^2}$ $C_A(r) = \frac{c_1}{r} + c_2$ *

* Assume $N_A \approx J_A^*$ in dilute systems ($x_A \lesssim 0.1$), or equimolar counterdiffusion

I.D.7 Generation Terms

- Internal heat (thermal energy) generation
  - electric current
  - chemical reaction
  - nuclear reaction
  - absorbed radiation
  - viscous dissipation

> volumetric generation $S \; [=] \; \frac{\text{energy}}{\text{vol} \cdot \text{time}}$
  - uniform $S = \text{const}$
  - variable $S(x,y,z)$

) heat generation

> volumetric generation $R_A = \frac{\text{mol A}}{\text{vol time}}$
  - uniform
  - variable

) species generation

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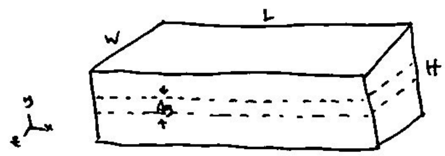
→ Generation in Rectilinear Coordinates

- Given uniform volumetric generation S, insulated edges
- Determine flux distribution $q(y)$ and temperature profile $T(y)$
- b.c. $\begin{cases} y=0, & T=T_1 \\ y=h, & T=T_2 \end{cases}$



$\Delta V = WL\Delta y$

OIGA

$$\lim_{\Delta y \to 0} \frac{[q]_{y+\Delta y} LW - [q]_y LW}{LW\Delta y} = \frac{SLW\Delta y - 0}{LW\Delta y}$$

$$\frac{dq}{dy} = S$$

$$q = Sy + c_1 = -k\frac{dT}{dy}$$ Fourier's law

$$\int_0^h \frac{Sy}{k} + \frac{c_1}{k}\, dy = \int_{T_1}^{T_2} -\, dT$$ (b.c. 1,2)

$$\left[\frac{Sy^2}{2k} + \frac{c_1}{k}y\right]_0^H = -(T_2 - T_1)$$

$$-\frac{S}{2k}H^2 - \frac{c_1}{k}H = T_2 - T_1$$

$$c_1 = \frac{k}{H}(T_2 - T_1) - \frac{SH}{2}$$

$$\boxed{q(y) = Sy + \frac{k}{H}(T_2 - T_1) - \frac{SH}{2}}$$

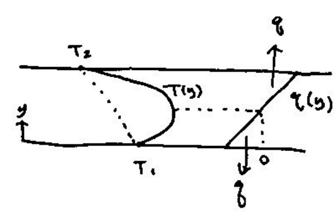
$$\int_0^y \frac{Sy}{k} + \frac{c_1}{k}\, dy = \int_{T_1}^{T} -\, dT$$ (b.c. 1)

$$-\left[\frac{Sy^2}{2k} + \frac{c_1}{k}y\right]_0^y = T - T_1$$

$$-\frac{Sy^2}{2k} - \frac{c_1}{k}y = T - T_1$$

$$T = T_1 - \frac{Sy^2}{2k} - \frac{c_1}{k}y$$

$$\boxed{T(y) = T_1 - \frac{S}{2k}y^2 + \left[\frac{SH}{2k} - \frac{T_2 - T_1}{H}\right]y}$$



Alternative approach to get $T(y)$

$$\frac{dq}{dy} = S$$

$$\frac{d}{dy}\left(-k\frac{dT}{dy}\right) = S$$

$$-k\frac{d^2T}{dy^2} = S$$

$$\Rightarrow T(y)$$

Species transport is analogous

$$\frac{dN_A}{dy} = R_A$$

$$N_A \cong -D_{AB}\frac{dc_A}{dy}$$

→ Review : The Shell Balance Method

1. Sketch system with coordinate system
2. Sketch "shell" , thin in direction of transport
3. Write shell volume ΔV
4. Write shell balance OIGA of transported quantity
5. Divide through by ΔV
6. Take limit as shell thickness → 0

↓

$\boxed{\text{Differential Equation of Flux Distribution}}$

7. Separate variable & integrate

↓

$\boxed{\text{Flux distribution, } C_1}$

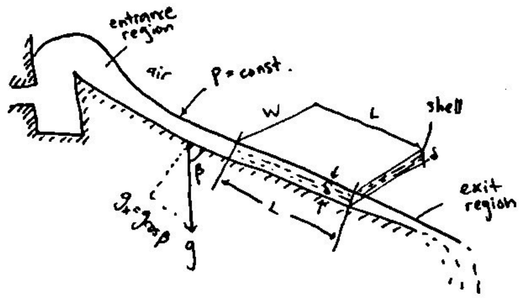
8. Substitute rate law
9. Separate variable & integrate

↓

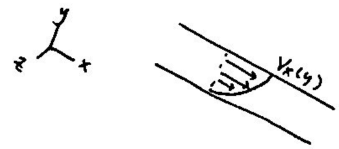
$\boxed{\text{Profile, } C_1, C_2}$

- C_1, C_2 evaluated by boundary conditions

$\boxed{\text{I.D.8}}$ Flow down an inclined plane ("falling film")



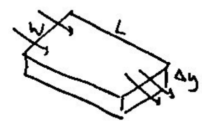
- steady-state
- no end effect (entrance & exit)
- no edge effect (no z dependence)



- b.c. $\begin{cases} y=0, & v_x = 0 \quad \text{(no slip condition)} \\ y=\delta, & \tau_{yx} = 0 \quad \text{(free slip condition)} \end{cases}$

► Obtain $\tau_{yx}(y)$, $v_x(y)$

- shell volume & balance



OIGA

$$\begin{bmatrix}\text{x-mom out}\\ \text{by diffusion}\end{bmatrix} + \begin{bmatrix}\text{x-mom out}\\ \text{by convection}\end{bmatrix} - \begin{bmatrix}\text{x-mom in}\\ \text{by diffusion}\end{bmatrix} - \begin{bmatrix}\text{x-mom in}\\ \text{by convection}\end{bmatrix}$$

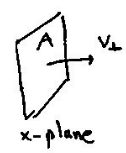
$$= \begin{bmatrix}\text{generation}\\ \text{of x-mom}\end{bmatrix} - \begin{bmatrix}\text{accumulation}\\ \text{of x-mom}\end{bmatrix}$$

→ x-momentum convection

$$\begin{bmatrix}\text{convective flow}\\ \text{of anything}\\ \text{across a surface}\end{bmatrix} = \underbrace{C_{any}}_{\text{concentration of anything}} \cdot \underbrace{Q}_{\text{volumetric flow rate}} \quad [=] \; \frac{\text{vol}}{\text{time}}$$

$$[=] \; \frac{\text{any}}{\text{vol}} \cdot \frac{\text{vol}}{\text{time}} = \frac{\text{any}}{\text{time}}$$

$$= C_{any} \, v_\perp \cdot A$$



For x-momentum, $C_{x\text{-mom}} = \rho v_x$, $v_\perp = v_x$, so

$$\begin{bmatrix}\text{convective flow}\\ \text{of x-mom}\end{bmatrix} = (\rho v_x) \cdot v_x A$$

$$\begin{bmatrix}\text{convective flux}\\ \text{of x-mom}\end{bmatrix} = \rho v_x v_x$$

across A (x-plane)

→ x-momentum generation

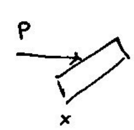
$$\text{mom generation} = \frac{d(mv)}{dt} = m\frac{dv}{dt} = ma = F$$

$$\text{x-mom generation} = \Sigma F_x \quad \text{(sum of all external x-forces acting on ctrl volume)}$$

body forces (gravity) | pressure forces (pressure on x-surf)

$$mg_x = (\rho L W \Delta y)(g \cos\beta)$$

$$PA = P_x W\Delta y - P_{x+L} W \Delta y$$
$$= (P_x - P_{x+L}) W \Delta y$$



OIGA (cont.)

$$[\tau_{yx}]_{y+\Delta y} LW + \cancel{[\rho v_x v_x]_x} W\Delta y - [\tau_{yx}]_y LW - \cancel{[\rho v_x v_x]_{x+L}} W\Delta y = \rho L W \Delta y\, g\cos(\beta) - 0 + \cancel{P_x W\Delta y} - \cancel{P_{x+L} W \Delta y}$$

(cancelled by $v_x(y)$ has no x-dependence, P = const.)

$$\lim_{\Delta y \to 0} \frac{\left([\tau_{yx}]_{y+\Delta y} - [\tau_{yx}]_y\right)\cancel{LW}}{\cancel{LW}\Delta y} = \frac{\rho \cancel{LW\Delta y}\, g\cos(\beta)}{\cancel{LW\Delta y}}$$

$$\boxed{\frac{d\tau_{yx}}{dy} = \rho g \cos(\beta)}$$ D.E. of flux distribution

$$d\tau_{yx} = \rho g\cos(\beta)\, dy$$

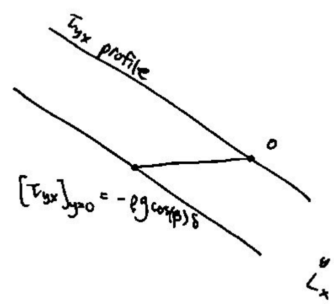
By b.c. 2 : $y = \delta$, $\tau_{yx} = 0$

$$\tau_{yx} = (\rho g \cos(\beta))y + c_1$$

$$0 = (\rho g\cos(\beta))\delta + c_1$$

$$c_1 = -\rho g \cos(\beta)\delta$$

$$\boxed{\tau_{yx} = -\rho g\cos(\beta)(\delta - y)}$$



Assume Newtonian fluid, use Newton's law of viscosity :

$$\tau_{yx} = -\mu\frac{dv_x}{dy} = -\rho g\cos(\beta)(\delta - y)$$

$$dv_x = \frac{\rho g\cos(\beta)}{\mu}(\delta - y)\, dy$$

$$v_x = \frac{\rho g \cos(\beta)}{\mu}\left[\delta y - \frac{y^2}{2}\right] + c_2$$

By b.c. 1 : $y = 0$, $v_x = 0$

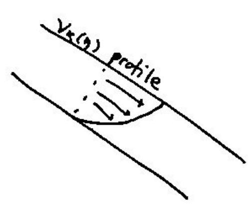
$$0 = 0 + c_2$$

$$c_2 = 0$$

$$v_x(y) = \frac{\rho g\cos(\beta)}{\mu}\left[\delta y - \tfrac{1}{2}y^2\right]$$

$\nu = \frac{\mu}{\rho}$

$$\boxed{v_x(y) = \frac{g\cos(\beta)}{2\nu}\left[2\delta y - y^2\right]}$$



→ Assumptions

- no entry length effect : $L \gg \delta$
- no edge effect : $W \gg \delta$
- incompressible Newtonian fluid : const μ, ρ
- no end effect : no ripple : $(Re)_{rippling} \lesssim 20$

> Renolds Number

$$\boxed{Re = \frac{L_{char} v_{char} \rho}{\mu}}$$

$L_{char} = 4\delta$

$v_{char} = \langle v_x \rangle$

$Re = \frac{4\delta\langle v_x\rangle \rho}{\mu}$ in this case

→ Evaluation of various descriptors of the flow

(a) skin friction τ° - stress of a moving fluid on a solid surface

$$\tau^\circ = -(\tau_{yx})_{y=0} = -\left[-(\rho g\cos\beta)(\delta-y)\right]_{y=0} = \rho g\cos(\beta)\delta$$

$$\boxed{\tau^\circ = \rho g\cos(\beta)\delta}$$

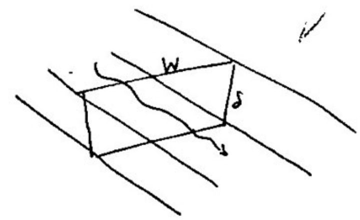
(b) free surface velocity v_x^{surf}

$$v_x^{surf} = v_x(y=\delta) = \frac{g\cos\beta}{2\nu}\left[2\delta y - y^2\right]_{y=\delta} = \frac{g(\cos\beta)}{2\nu}\delta^2$$

$$\boxed{v_x^{surf} = \frac{g(\cos\beta)}{2\nu}\delta^2}$$

(c) Volumetric flow rate Q

good drawing!!!



$$Q = \int v_\perp \, dA \qquad \begin{cases} v_\perp = v_x \\ dA = W dy \end{cases}$$

$$Q = \int v_x W \, dy$$

$$\frac{Q}{W} = \int_0^\delta v_x \, dy$$

$$\frac{Q}{W} = \frac{g\cos\beta}{2\nu}\int_0^\delta 2\delta y - y^2 \, dy$$

$$\frac{Q}{W} = \frac{g\cos\beta}{2\nu}\left[2\delta\frac{y^2}{2} - \frac{y^3}{3}\right]$$

$$\boxed{\frac{Q}{W} = \frac{(g\cos\beta)\delta^3}{3\nu}}$$

(d) Average velocity $\langle v_x \rangle$

$$\langle v_x \rangle = \frac{Q}{A} = \frac{Q}{W\delta} = \frac{g(\cos\beta)\delta^2}{3\nu}$$

$$\boxed{\langle v_x \rangle = \frac{g(\cos\beta)\delta^2}{3\nu}}$$

(e) mass flow rate $\dot{m}$

$$\dot{m} = \rho Q$$

(f) mass flow rate per unit width : Γ

$$\boxed{\Gamma = \frac{\rho Q}{W} = \frac{(\rho g\cos\beta)\delta^3}{3\nu}}$$

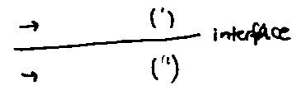
(e) Film thickness given Γ

$$\boxed{\delta = \sqrt[3]{\frac{3\nu\Gamma}{\rho g\cos\beta}}}$$

$\boxed{\text{I.D.9}}$ Boundary Condition for fluid flow

→ No slip condition

· velocity is continuous (same) at interface



$$\underset{\sim}{V}' = \underset{\sim}{V}''$$

· at solid surface,

$$V_{fluid} = V_{solid}$$

→ Continuity of stress

· stress is continuous (same) at interface, else interface will move

$$\underset{\sim}{\tau_y}' = \underset{\sim}{\tau_y}''$$

· the components, shear stress is also continuous.

$$\tau_{yx}' = \tau_{yx}''$$

$$-\mu'\left(\frac{dv_x}{dy}\right)' = -\mu''\left(\frac{dv_x}{dy}\right)''$$

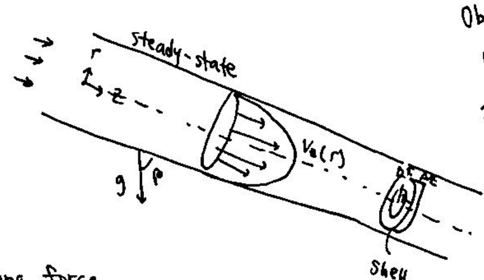
· If (") is air, $\mu'' \approx 0$, so

$$-\mu'\left(\frac{dv_x}{dy}\right)' = 0$$

is the free slip condition.

→ Symmetry condition

$\boxed{\text{I.D.10}}$ Flow in round tube : Hagen - Poiseiulle



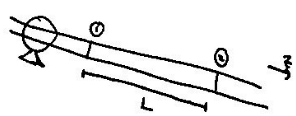
Obtain:

1. shear stress distribution $\tau_{rz}(r)$
2. velocity profile $v_z(r)$

→ Driving force

· Gravity

· Work done by pump creates pressure gradient



$$\frac{P_1 - P_2}{L} = -\frac{P_2 - P_1}{z_2 - z_1} = -\frac{\Delta P}{\Delta z} = -\frac{dP}{dz} = \text{const.} \quad \text{(pressure gradient)}$$

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→ Shell Balance

- thin in direction of transport
  - radial transport : Δr
  - convection in z-dir : Δz
- b.c. $\begin{cases} r = R, & v_z = 0 \quad (1) \\ r = 0, & \tau_{rz} = 0 \quad (2) \end{cases}$
- shell volume
  - $\Delta V = 2\pi r \Delta r \Delta z$

- OIGA shell balance

cancel by same flow

steady state

$$\begin{bmatrix}\text{rate z-mom}\\ \text{out by diffusion}\end{bmatrix} + \cancel{\begin{bmatrix}\text{rate z-mom}\\ \text{out by convection}\end{bmatrix}} - \begin{bmatrix}\text{rate z-mom}\\ \text{in by diffusion}\end{bmatrix} - \cancel{\begin{bmatrix}\text{rate z-mom}\\ \text{in by convection}\end{bmatrix}} = \begin{bmatrix}\text{z-mom}\\ \text{generation}\end{bmatrix} - \cancel{\begin{bmatrix}\text{z-mom}\\ \text{accumulation}\end{bmatrix}}^{\,0}$$

$$[r\tau_{rz}]_{r+\Delta r} 2\pi\Delta z - [r\tau_{rz}]_r 2\pi \Delta z = \Sigma F_z$$

$$\lim_{\Delta r, \Delta z \to 0} \frac{[r\tau_{rz}]_{r+\Delta r} 2\pi\Delta z - [r\tau_{rz}]_r 2\pi\Delta z}{2\pi r \Delta r \Delta z} = \frac{[P]_z 2\pi r \Delta r - [P]_{z+\Delta z} 2\pi r\Delta r + (\overset{\frac{m}{V}}{\rho}\, \overbrace{g\cos\beta}^{a})\, \overbrace{2\pi r \Delta r \Delta z}^{V}}{2\pi r \Delta r \Delta z}$$

$$\frac{1}{r}\frac{d(r\tau_{rz})}{dr} = \underbrace{-\frac{dP}{dz} + \rho g \cos\beta}_{} = \text{const.}$$

pressure-gravity driving force = const.
(written as [ ] hereinafter)

$$\frac{d(r\tau_{rz})}{dr} = [\ ]\, r$$

$$\int d(r\tau_{rz}) = \int [\ ]\, r\, dr$$

$$r\tau_{rz} = [\ ]\frac{r^2}{2} + c_1$$

$$\tau_{rz} = [\ ]\frac{r}{2} + \frac{c_1}{r} \quad \Rightarrow c_1 = 0 \text{ by b.c.2}$$

because $\tau_{rz} \not\to \infty$ at $r = 0$

$$\boxed{\tau_{rz} = \frac{1}{2}\left[-\frac{dP}{dz} + \rho g\cos\beta\right] r}$$

- skin friction $\tau^{\circ} = [\tau_{rz}]_{r=R} = \frac{1}{2}\left[-\frac{dP}{dz} + \rho g \cos\beta\right] R$

→ velocity profile

- Assume Newtonian fluid

- $\tau_{rz} = -\mu \frac{dv_z}{dr} = \frac{1}{2} [\ ] r$

$$\int dv_z = \int -\frac{1}{2\mu} [\ ] r\, dr$$

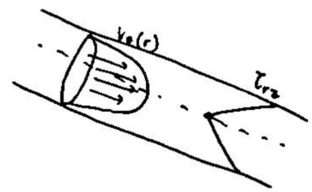
$$v_z = -\frac{1}{2\mu} [\ ] \frac{r^2}{2} + c_2$$

$$0 = -\frac{1}{2\mu} [\ ] \frac{R^2}{2} + c_2 \qquad \text{(b.c. 1)}$$

$$c_2 = \frac{1}{2\mu} [\ ] \frac{R^2}{2}$$

$$v_z = -\frac{1}{4\mu} [\ ] (R^2 - r^2)$$

$$\boxed{v_z = \frac{R^2}{4\mu} \left[-\frac{dP}{dz} + \rho g \cos\beta \right] \left[1 - \left(\frac{r}{R}\right)^2 \right]}$$



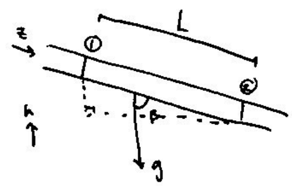
→ Driving Force (pressure-gravity)

- Rewrite the driving force in terms of modified pressure $\mathcal{P}$, where

$$\boxed{\mathcal{P} = P + \rho g h}$$

- $-\frac{dP}{dz} + \rho g \cos\beta$

$$\left\{ \begin{array}{l} \cdot\ -\frac{dP}{dz} = \frac{P_1 - P_2}{L} \\ \cdot\ \cos\beta = \frac{h_1 - h_2}{L} \end{array} \right.$$



$$= \frac{P_1 - P_2}{L} + \rho g \frac{h_1 - h_2}{L}$$

$$= \frac{(P_1 + \rho g h_1) - (P_2 + \rho g h_2)}{L}$$

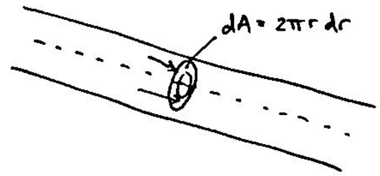
$$= \frac{\mathcal{P}_1 - \mathcal{P}_2}{L} \quad \Rightarrow \quad \boxed{-\frac{dP}{dz} + \rho g \cos\beta = \frac{\mathcal{P}_1 - \mathcal{P}_2}{L}}$$

$$= -\frac{d\mathcal{P}}{dz}$$

- Substitute in velocity profile,

$$\boxed{v_z = \frac{R^2}{4\mu} \left(\frac{\mathcal{P}_1 - \mathcal{P}_2}{L} \right) \left(1 - \left(\frac{r}{R}\right)^2 \right)}$$

→ Volumetric Flow Rate



$$Q = \int_A v_z \, dA \qquad \begin{cases} v_z = v_z \\ dA = 2\pi r\, dr \end{cases}$$

$$= \int_0^R v_z \, 2\pi r \, dr$$

$$= \int_0^R \frac{R^2}{4\mu} \frac{\mathcal{P}_1 - \mathcal{P}_2}{L} \left(1 - \left(\frac{r}{R}\right)^2\right) 2\pi r \, dr$$

$$= \frac{R^2}{4\mu} \frac{\mathcal{P}_1 - \mathcal{P}_2}{L} \, 2\pi \int_0^R \left(1 - \left(\frac{r}{R}\right)^2\right) r \, dr$$

$$= \frac{R^2}{4\mu} \frac{\mathcal{P}_1 - \mathcal{P}_2}{L} \, 2\pi \left[\frac{r^2}{2} - \frac{1}{R^2}\frac{r^4}{4} \right]_0^R$$

$$= \frac{R^2}{4\mu} \frac{\mathcal{P}_1 - \mathcal{P}_2}{L} \, 2\pi \, \frac{R^2}{4}$$

$$\boxed{Q = \frac{R^4 \pi}{8\mu} \frac{\mathcal{P}_1 - \mathcal{P}_2}{L}} \qquad \text{Hagen-Poiseuille Law}$$

→ Average velocity

$$\langle v_z \rangle = \frac{Q}{A} = \frac{Q}{\pi R^2} \quad \Rightarrow \quad \boxed{\langle v_z \rangle = \frac{R^2}{8\mu} \frac{\mathcal{P}_1 - \mathcal{P}_2}{L}}$$

→ Mass flow rate

$$\dot{m} = Q\rho \quad \Rightarrow \quad \boxed{\dot{m} = \frac{R^4 \pi \rho}{8\mu} \frac{\mathcal{P}_1 - \mathcal{P}_2}{L}}$$

→ Assumptions of Hagen-Poiseuille Flow

- Newtonian incompressible fluid (const. μ, ρ)
- steady state
- constant cross section
- no tube bents
- neglegible P-dependence with r
- laminar flow
  - eddies may form when velocity is high → turbulence
  - Reynolds's experiment
  - Reynold's number $\boxed{Re = \frac{L_{char} V_{char} \rho}{\mu}}$
    - for pipe flow $\boxed{Re\,(pipe) = \frac{D \langle v_z \rangle \rho}{\mu}}$
  - critical Re : $\boxed{Re_{critical} \cong 2100}$ turbulence for $Re \geq 2100$.

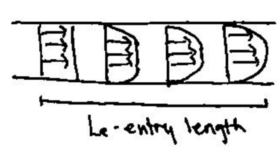
EX1 $\langle v \rangle = 1\,m/s$, $D = 1\,cm$

$\rho = 1\,g/cm^3$, $\mu = 0.01\,g/cm \cdot s$

for faucet

$Re = 10^4$ → turbulent

- fully developed flow
  - laminar flow : entry length $\boxed{L_e \cong 0.035\, D\, Re}$

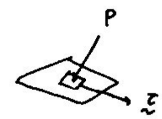


EX2 $D = 1\,cm$

$Re = 2000$

$L_e = 70\,cm$

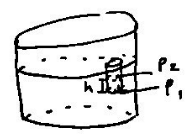
$\boxed{\text{I.D.11}}$ Fluid Pressure, Hydrostatics, Manometer



> shear stress - tangential force on surface
  - if fluid at rest, $\underset{\sim}{\tau} = 0$

> pressure - normal stress on surface per unit area $= P = \frac{F_\perp}{A}$
  - P is an isotropic scalar
  - P @ sea level : $1\,\text{atm} = 14.7\ \text{lb}_f/\text{in}^2$
  - P @ top of Mt. Rainier : 0.5 atm
  - P @ 1 mile below surface of ocean : 156 atm
  - P in tire : 2 atm

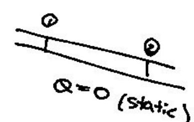
→ Equation of Hydrostatics



- $P_1 = P_2 + \frac{\rho (A h) g}{A} = P_2 + \rho g h$

$$\boxed{P_1 - P_2 = \rho g (h_2 - h_1)}$$ Eqn of hydrostatic

- Alternative derivation from Hagen-Poiseiulle Flow



- $Q = \frac{\pi R^4}{8\mu} \frac{\mathcal{P}_1 - \mathcal{P}_2}{L} = 0$ (at rest - hydrostatic)

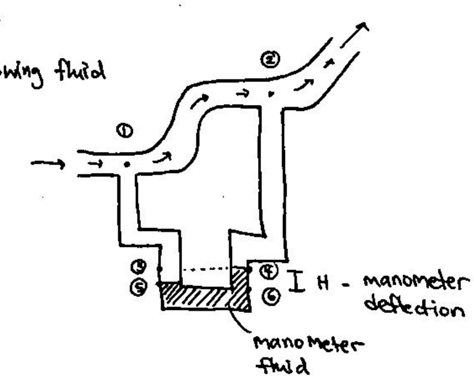
$\Rightarrow \mathcal{P}_1 = \mathcal{P}_2$

$P_1 + \rho g h_1 = P_2 + \rho g h_2$

$$\boxed{P_1 - P_2 = \rho g (h_2 - h_1)}$$ Eqn of hydrostatic

→ Manometer Eqn

- U-tube manometer can measure pressure difference of fluid flow.
  > manometer deflection H - diff in height of manometer
  > Manometer fluid - fluid with higher density than flowing fluid
    - $\rho_m > \rho$
  - fluid level higher at lower pressure (downstream)



- $P_5 = P_6$ (hydrostatic eqn)

$$\ominus \begin{cases} P_1 = P_5 - \rho g H - \rho g (h_1 - h_3) \\ P_2 = P_6 - \rho_m g H - \rho g (h_1 - h_4) - \rho g (h_2 - h_1) \end{cases}$$

$P_1 - P_2 = \cancel{P_5 - P_6}^{\,0} + (\rho_m - \rho) g H + \rho g (h_2 - h_1)$

$$\boxed{P_1 - P_2 = (\rho_m - \rho) g H + \rho g (h_2 - h_1)}$$ Manometer eqn

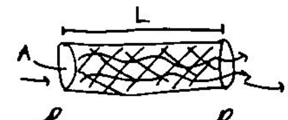
$$\boxed{\mathcal{P}_1 - \mathcal{P}_2 = (\rho_m - \rho) g H}$$

$\boxed{\text{I.D.12}}$ Laminar flow through porous media (Darcy eqn)

- Model as a bundle of straight circular cross section capillary tubes (H-P flow in tube)

  - cappillary tube: $\langle v_z \rangle = \dfrac{R^2}{8\mu} \dfrac{\mathcal{P}_1 - \mathcal{P}_2}{L}$
  - bed: $\langle v \rangle = \dfrac{R_{eff}^2}{8\mu} \dfrac{\mathcal{P}_1 - \mathcal{P}_2}{L}$

  } Assume laminar flow



- packing elements $D_p \ll D_{bed}$

> Darcy's law $\boxed{\langle v \rangle = \dfrac{\kappa}{\mu L} (\mathcal{P}_1 - \mathcal{P}_2)}$

- κ - bed permiability (=) cm^2

- volumetric flow rate: $Q = \langle v \rangle A \varepsilon$

> Darcy's law $\boxed{Q = \dfrac{\kappa A \varepsilon}{\mu L} (\mathcal{P}_1 - \mathcal{P}_2)}$

- A - empty bed cross section
- ε - porosity, void fraction

→ Bed permiability κ

- $\kappa \propto \dfrac{R_{eff}^2}{8}$ not useful

> Black-Kozeny model $\boxed{\kappa = \dfrac{D_p^2}{150} \left(\dfrac{\varepsilon}{1-\varepsilon}\right)^2}$

- D_p - effective packing particle diameter

  - spheres: $D_p = D$
  - general: $\boxed{D_p = \dfrac{6}{a_v}}$ $a_v = \dfrac{\text{area}}{\text{vol}}$ of packing element

EX1

$$a_v = \frac{2\pi\left(\frac{D^2}{4}\right) + \pi D^2}{\pi\left(\frac{D^2}{4}\right) D} = \frac{\frac{3}{2}\pi D^2}{\frac{\pi D^3}{4}} = \frac{6}{D}$$

→ Bed Reynolds number

- $Re = \dfrac{D_p \langle v \rangle \rho}{\mu} \dfrac{\varepsilon}{1-\varepsilon} = \dfrac{D_p Q \rho}{\mu A (1-\varepsilon)}$ ⇒ $\boxed{Re = \dfrac{D_p Q \rho}{\mu A (1-\varepsilon)}}$

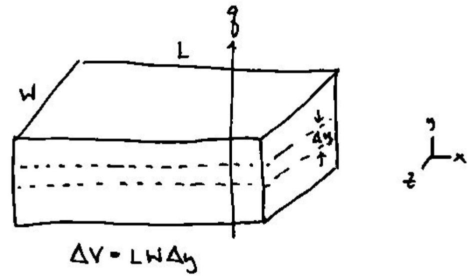
- $Re_{critical} \cong 10$

  - laminar flow if $Re < 10$

I.D.13 Unsteady State Transport

→ Unsteady conduction in a rectangular slab

- edge insulated (1D transport)
- uniform generation S



OIGA

$$\lim_{\Delta y \to 0} \frac{[q]_{y+\Delta y} LW - [q]_y LW}{LW\Delta y} = \frac{S\, LW\Delta y}{LW\Delta y} - \frac{\frac{\partial C_H}{\partial t} LW\Delta y}{LW\Delta y}$$

C_H = Concentration of thermal energy (enthalpy)

$$C_H = \frac{\text{enthalpy}}{\text{vol}} = \frac{\text{enthalpy}}{\text{mass}}\frac{\text{mass}}{\text{vol}} = \rho\hat{H}$$

$$\hat{H} = \hat{H}_0 + \hat{c}_p (T - T_0)$$

$$C_H = \rho(\hat{H}_0 + \hat{c}_p(T-T_0))$$

$$\left(\frac{\partial C_H}{\partial t}\right)_y = \rho\hat{c}_p\left(\frac{\partial T}{\partial t}\right)_y$$

$$\left(\frac{\partial q}{\partial y}\right)_t = S - \left(\frac{\partial C_H}{\partial t}\right)_y$$

$$\left(\frac{\partial q}{\partial y}\right)_t = S - \rho\hat{c}_p\left(\frac{\partial T}{\partial t}\right)_y$$

$$\frac{\partial}{\partial y}\left(-k\frac{\partial T}{\partial y}\right) = S - \rho\hat{c}_p\left(\frac{\partial T}{\partial t}\right)_y \quad \text{Fourier's law}$$

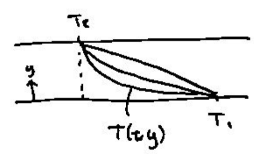
$$-k\frac{\partial^2 T}{\partial y^2} = S - \rho\hat{c}_p\left(\frac{\partial T}{\partial t}\right)_y$$

$$\left(\frac{\partial T}{\partial t}\right)_y = \frac{k}{\rho\hat{c}_p}\frac{\partial^2 T}{\partial y^2} + \frac{S}{\rho\hat{c}_p}$$

$$\boxed{\alpha \equiv \frac{k}{\rho\hat{c}_p}} \quad \text{thermal diffusivity}$$

$$\boxed{\left(\frac{\partial T}{\partial t}\right)_y = \alpha\frac{\partial^2 T}{\partial y^2} + \frac{S}{\rho\hat{c}_p}}$$

- need 2 boundary conditions & 1 initial condition



→ Unsteady State Diffusion & Concentration Profile

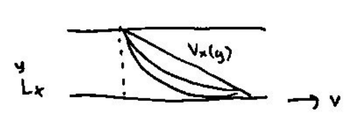
- Analogous to heat transfer, we have

$$\boxed{\left(\frac{\partial C_A}{\partial t}\right)_y = D_{AB}\frac{\partial^2 C_A}{\partial y^2} + R_A}$$

- R_A - rate of species A generation per unit volume

→ Unsteady State Cuette Flow (1D shear flow)

- Cuette flow has no driving force. In general, it may have pressure-gravity driving force $-\frac{dP}{dx} + \rho g\cos\beta = \frac{P_1 - P_2}{L}$
- Concentration of momentum is ρV_x.



$$\left(\frac{\partial \tau_{yx}}{\partial y}\right)_t = [\cancel{\text{driving force}}^{\,0}] - \frac{\partial}{\partial t}(\rho V_x)$$

$$\left(\frac{\partial \tau_{yx}}{\partial y}\right)_t = -\rho\left(\frac{\partial V_x}{\partial t}\right)_y$$

$$\frac{\partial}{\partial y}\left(-\mu\frac{\partial V_x}{\partial y}\right) = -\rho\left(\frac{\partial V_x}{\partial t}\right)_y \quad \text{Newton's law of viscosity}$$

$$\left(\frac{\partial V_x}{\partial t}\right)_y = \frac{\mu}{\rho}\left(\frac{\partial^2 V_x}{\partial y^2}\right)_t$$

$$\boxed{\nu \equiv \frac{\mu}{\rho}} \quad \text{Kinematic viscosity}$$

$$\boxed{\left(\frac{\partial V_x}{\partial t}\right)_y = \nu\left(\frac{\partial^2 V_x}{\partial y^2}\right)_t}$$

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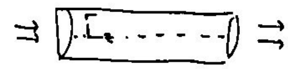
→ Unsteady state flow in cylindrical systems

- OIGA gives the diff. eqn.



$$\frac{1}{r}\frac{\partial(r\tau_{rz})}{\partial r} = [\ \ ] - \rho\left(\frac{\partial v_z}{\partial t}\right)$$

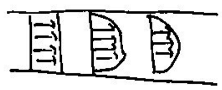
[ ] is pressure-gravity driving force $\frac{\mathcal{P}_1 - \mathcal{P}_2}{L}$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\left[-\mu\frac{\partial v_z}{\partial r}\right]\right) = [\ \ ] - \rho\left(\frac{\partial v_z}{\partial t}\right)$$

Newton's law of viscocity

$$\frac{1}{r}\left[\frac{\partial v_z}{\partial r} + r\frac{\partial^2 v_z}{\partial r^2}\right] = -\frac{1}{\mu}[\ \ ] + \frac{\rho}{\mu}\frac{\partial v_z}{\partial t} \qquad \left(\frac{\rho}{\mu} = \frac{1}{\nu}\right)$$

$$\boxed{\left(\frac{\partial v_z}{\partial t}\right)_r = \nu\left[\frac{\partial^2 v_z}{\partial r^2} + \frac{1}{r}\frac{\partial v_z}{\partial r}\right] + \frac{1}{\rho}\left[\frac{\mathcal{P}_1 - \mathcal{P}_2}{L}\right]}$$



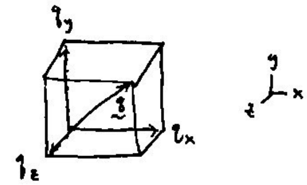
→ t

I.E Generalization of Rate Law to 2D/3D & Curvilinear Coordinates

I.E.1 Fourier's Law & Fick's Law

· 1D cases
$$\begin{cases} q_y = -k \dfrac{dT}{dy} & J^*_{Ay} = -D_{AB} \dfrac{dc_A}{dy} \\ q_r = -k \dfrac{dT}{dr} & J^*_{Ar} = -D_{AB} \dfrac{dc_A}{dr} \end{cases}$$

· 3D case :
$$\underset{\sim}{q} = \underset{\sim}{\delta_x} q_x + \underset{\sim}{\delta_y} q_y + \underset{\sim}{\delta_z} q_z$$
$$= \underset{\sim}{\delta_x}\left(-k\frac{\partial T}{\partial x}\right) + \underset{\sim}{\delta_y}\left(-k\frac{\partial T}{\partial y}\right) + \underset{\sim}{\delta_z}\left(-k\frac{\partial T}{\partial z}\right)$$
$$= -k\left[\underset{\sim}{\delta_x}\frac{\partial}{\partial x} + \underset{\sim}{\delta_y}\frac{\partial}{\partial y} + \underset{\sim}{\delta_z}\frac{\partial}{\partial z}\right] T$$



$\boxed{\underset{\sim}{q} = -k \nabla T}$ Assume isotropy of k (same in all dir). $\left(J^*_A = -D_{AB} \nabla C_A\right)$

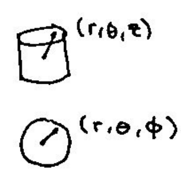
> gradient operator ∇ - operates on scalar to give a vector whose magnitude is the maximum rate of change of the scalar with position, and whose direction points in the direction of that change.

· Cartesian : $\nabla = \underset{\sim}{\delta_x}\frac{\partial}{\partial x} + \underset{\sim}{\delta_y}\frac{\partial}{\partial y} + \underset{\sim}{\delta_z}\frac{\partial}{\partial z}$

· Cylindrical : $\nabla = \underset{\sim}{\delta_r}\frac{\partial}{\partial r} + \underset{\sim}{\delta_\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \underset{\sim}{\delta_z}\frac{\partial}{\partial z}$

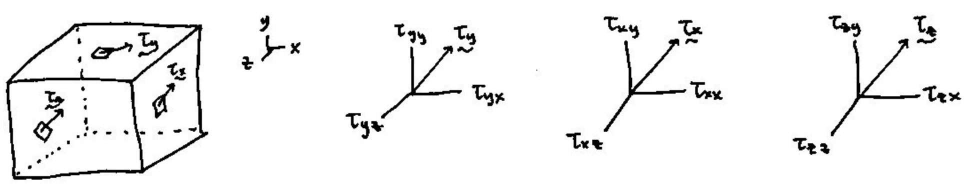
· Spherical : $\nabla = \underset{\sim}{\delta_r}\frac{\partial}{\partial r} + \underset{\sim}{\delta_\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \underset{\sim}{\delta_\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}$

$[=] \frac{1}{L}$



I.E.2 Newton's Law of Viscocity

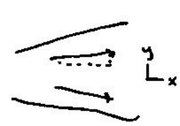
→ Cartesian coordinate



> viscous stress tensor $\underset{\approx}{\tau} = \begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{bmatrix}$

· 1D assumes $v_y = v_z = 0$, so
$$\tau_{yx} = -\mu \frac{dv_x}{dy}$$

· 2D does not have such assumption, so (assume ρ = const.)
$$\tau_{yx} = -\mu\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right)$$



$\boxed{\tau_{ij} = -\mu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)}$, so $\boxed{\tau_{ij} = \tau_{ji}}$

$\Rightarrow$ six independent $\underset{\approx}{\tau}$ component

→ Equation in Tensor Form

> rate of strain tensor - $\underset{\approx}{\Delta} = \begin{bmatrix} \frac{\partial v_x}{\partial x} & \frac{\partial v_x}{\partial y} & \frac{\partial v_x}{\partial z} \\ \frac{\partial v_y}{\partial x} & \frac{\partial v_y}{\partial y} & \frac{\partial v_y}{\partial z} \\ \frac{\partial v_z}{\partial x} & \frac{\partial v_z}{\partial y} & \frac{\partial v_z}{\partial z} \end{bmatrix}$; $\underset{\approx}{\Delta}^{+} = \begin{bmatrix} \frac{\partial v_x}{\partial x} & \frac{\partial v_y}{\partial x} & \frac{\partial v_z}{\partial x} \\ \frac{\partial v_x}{\partial y} & \frac{\partial v_y}{\partial y} & \frac{\partial v_z}{\partial y} \\ \frac{\partial v_x}{\partial z} & \frac{\partial v_y}{\partial z} & \frac{\partial v_z}{\partial z} \end{bmatrix}$ (transpose)

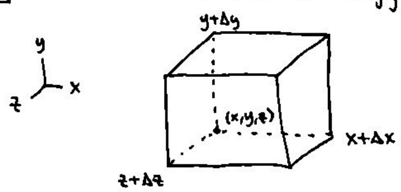
> Newton's law of viscocity

$$\boxed{\underset{\approx}{\tau} = -\mu\left(\underset{\approx}{\Delta} + \underset{\approx}{\Delta}^{+}\right)}$$

- cartesian coord
- constant ρ

$\boxed{\text{I.F}}$ Generalization of conservation equations to 3D and curvilinear coordinates

$\boxed{\text{I.F.1}}$ Generalization of thermal energy in Cartesian Coordinates



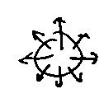
$\Delta V = \Delta x \Delta y \Delta z$

OIGA

$$\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0\\ \Delta z\to 0}} \frac{[q_x]_{x+\Delta x}\Delta y\Delta z - [q_x]_x \Delta y \Delta z}{\Delta x\Delta y\Delta z} + \frac{[q_y]_{y+\Delta y}\Delta x\Delta z - [q_y]_y \Delta x\Delta z}{\Delta x\Delta y\Delta z} + \frac{[q_z]_{z+\Delta z}\Delta x\Delta y - [q_z]_z\Delta x\Delta y}{\Delta x\Delta y\Delta z} = \frac{S\Delta x\Delta y\Delta z}{\Delta x\Delta y\Delta z} - \frac{\overbrace{\rho\hat{c}_p\frac{\partial T}{\partial t}}^{\frac{\partial C_H}{\partial t}}\Delta x\Delta y\Delta z}{\Delta x\Delta y\Delta z}$$

$$\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z} = S - \rho\hat{c}_p\frac{\partial T}{\partial t}$$

$$\boxed{\nabla \cdot \underset{\sim}{q} = S - \rho\hat{c}_p\frac{\partial T}{\partial t}} \quad [=] \ \frac{\text{energy / time}}{\text{volume}}$$



> divergence operator $(\nabla\cdot)$ - operates on a vector to give a scalar

· divergence of a flux vector - rate of efflux (outflow) of the transported quantity per unit volume.

$$\nabla\cdot\underset{\sim}{q} = \left(\underset{\sim}{\delta}_x\frac{\partial}{\partial x} + \underset{\sim}{\delta}_y\frac{\partial}{\partial y} + \underset{\sim}{\delta}_z\frac{\partial}{\partial z}\right)\cdot\left(\underset{\sim}{\delta}_x q_x + \underset{\sim}{\delta}_y q_y + \underset{\sim}{\delta}_z q_z\right) = \frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z}$$

- Cartesian: $\nabla\cdot\underset{\sim}{q} = \frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z}$
- Cylindrical: $\nabla\cdot\underset{\sim}{q} = \frac{1}{r}\frac{\partial}{\partial r}(r q_r) + \frac{1}{r}\frac{\partial q_\theta}{\partial \theta} + \frac{\partial q_z}{\partial z}$
- Spherical: $\nabla\cdot\underset{\sim}{q} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 q_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(q_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial q_\phi}{\partial\phi}$

→ Profile equations

- Substitute relavent rate law to get the profile equation.

$$\nabla \cdot \underset{\sim}{q} = S - \rho \hat{c}_p \frac{\partial T}{\partial t}$$

$$\nabla \cdot (-k \nabla T) = S - \rho \hat{c}_p \frac{\partial T}{\partial t} \qquad \text{Fourier's law} \quad \underset{\sim}{q} = -k \nabla T$$

$$-k \, \nabla \cdot \nabla T = S - \rho \hat{c}_p \frac{\partial T}{\partial t} \qquad \text{isotropic } k$$

$$-k \nabla^2 T = S - \rho \hat{c}_p \frac{\partial T}{\partial t} \qquad \text{Laplacian operator}$$

$$\frac{\partial T}{\partial t} = \frac{k}{\rho \hat{c}_p} \nabla^2 T + \frac{S}{\rho \hat{c}_p}$$

Assume no convection

$$\boxed{\frac{\partial T}{\partial t} = \alpha \nabla^2 T + \frac{S}{\rho \hat{c}_p}} \qquad \text{Conduction equation}$$

$$\boxed{\frac{\partial c_A}{\partial t} = D_{AB} \nabla^2 c_A + R_A} \qquad \text{Molecular diffusion equation}$$

Volumetric generation of species A.

> Laplacian operator $\nabla^2 = \nabla \cdot \nabla$

- Cartesian $\quad \nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}$
- Cylindrical $\quad \nabla^2 T = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2}$
- Spherical $\quad \nabla^2 T = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial T}{\partial \theta}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial^2 T}{\partial \phi^2}$

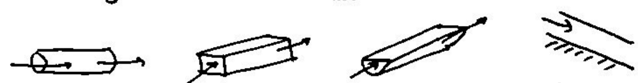
$\boxed{\text{II}}$ Fluid Mechanics

$\boxed{\text{II.A}}$ Introduction

· Objective : obtain velocity profile $\underset{\sim}{v}(t,x,y,z)$ and related quantities Q, $\langle v \rangle$, $p(t,x,y,z)$, τ (stress on surface bounding the flow), etc.

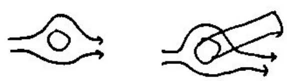
$\boxed{\text{II.A.1}}$ · Two categories of flow

1. Flow in conduits or open channels
   · can identify flow cross sections



falling film

2. Flow around submerged objects
   · assume ∞ medium, no bounds

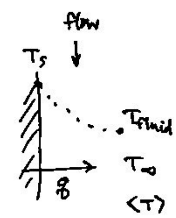


· automobile
· airplane
· submarine

$\boxed{\text{II.A.2}}$ Convective transport of heat or species

· Lumped parameter method

> Newton's law of cooling $\boxed{q = h(T_s - T_{fluid})}$ h - heat transfer coeff



· Species transport $\boxed{N_A = k_m(c_{A,s} - c_{A,fluid})}$ k_m - mass transfer coeff

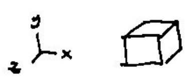
$\boxed{\text{II.B}}$ General Differential Equations of Fluid Mechanics (Navier - Stokes Equation)

· conservation of mass - continuity eqn

· conservation of momentum - momentum eqn / eqn of motion $\Leftrightarrow$ Force balance

$\boxed{\text{II.B.1}}$ Conservation of Total Mass ; Continuity Eqn

· Cartesian coordinates



$\Delta V = \Delta x \Delta y \Delta z$

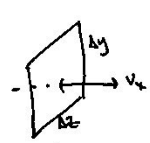
OIGA

$$\begin{bmatrix}\text{rate of mass}\\ \text{flow out}\end{bmatrix} - \begin{bmatrix}\text{rate of mass}\\ \text{flow in}\end{bmatrix} = \cancel{\begin{bmatrix}\text{rate of mass}\\ \text{generation}\end{bmatrix}}^{\,0} - \begin{bmatrix}\text{rate of mass}\\ \text{accumulation}\end{bmatrix}$$

→ convective transport

· mass flow rate $= \left(\frac{\text{mass}}{\text{vol}}\right)\left(\frac{\text{vol}}{\text{time}}\right) = \rho Q_x = \rho v_x A = \rho v_x \Delta y \Delta z$

· convective flux $= \frac{\text{flow rate}}{\text{area}} = \rho v_x$



x-plane

OIGA

$$\begin{bmatrix}\text{rate of mass}\\ \text{flow out}\end{bmatrix} - \begin{bmatrix}\text{rate of mass}\\ \text{flow in}\end{bmatrix} = \begin{bmatrix}\text{rate of } \cancel{\text{mass}}^{\,0}\\ \cancel{\text{generation}}\end{bmatrix} - \begin{bmatrix}\text{rate of mass}\\ \text{accumulation}\end{bmatrix}$$

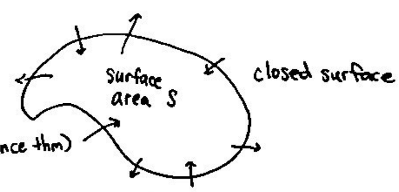
$$\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0\\ \Delta z\to 0}} \frac{\{[\rho v_x]_{x+\Delta x}-[\rho v_x]_x\}\Delta y\Delta z}{\Delta x\Delta y\Delta z} + \frac{\{[\rho v_y]_{y+\Delta y}-[\rho v_y]_y\}\Delta x\Delta z}{\Delta x\Delta y\Delta z} + \frac{\{[\rho v_z]_{z+\Delta z}-[\rho v_z]_z\}\Delta x\Delta y}{\Delta x\Delta y\Delta z} = \frac{-\left(\frac{\partial \rho}{\partial t}\right)\Delta x\Delta y\Delta z}{\Delta x\Delta y\Delta z}$$

$$\frac{\partial(\rho v_x)}{\partial x} + \frac{\partial(\rho v_y)}{\partial y} + \frac{\partial(\rho v_z)}{\partial z} = -\frac{\partial \rho}{\partial t}$$

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho \underset{\sim}{v}) = 0}$$ continuity eqn

$$\boxed{\nabla\cdot\underset{\sim}{v} = 0}$$ continuity eqn of incompressible fluid ($\Delta\rho=0$)

- $\underset{\sim}{v}$ is flux of fluid volume
  - no net total efflux
  - $0 = \iiint_V \nabla\cdot v \, dV = \iint_S v_z \, dS$ (Divergence thm)



- For conduits, incompressible gives ($\Delta\rho=0$)

$$\rho_1 Q_1 = \rho_2 Q_2$$

$$Q_1 = Q_2 \qquad (\Delta\rho=0)$$

$$A_1\langle v\rangle_1 = A_2\langle v\rangle_2$$

$\boxed{\text{II.B.2}}$ Constant property fluids : constant μ, ρ

→ Viscocity dependence on T, P.

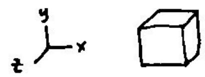
- T $\begin{cases}\text{gas} - \Delta T \lesssim 30K\\ \text{liquid} - \Delta T \lesssim 10K\end{cases}$ no effect within small change
- P $\begin{cases}\text{gas} - \text{no effect at moderate pressure } (\lesssim 20\text{atm})\\ \text{liquid} - \text{no effect by pressure}\end{cases}$
- Newtonian $\begin{cases}\text{gas - all gases}\\ \text{liquid - pure, nonpolymeric}\end{cases}$

→ Density dependence on T, P

- T $\begin{cases}\text{gas} - \Delta T \lesssim 20K\\ \text{liquid} - \Delta T \lesssim 20K\end{cases}$ no effect within small change
  $\beta = -\frac{1}{v}\left(\frac{\partial v}{\partial T}\right)_P \underset{liq}{\approx} 10^{-3} K^{-1}$
- P effect on gas ⇒ v effect on gas
  - > compressibility factor $\kappa = -\frac{1}{v}\left(\frac{\partial v}{\partial P}\right)_T = \frac{1}{\rho}\left(\frac{\partial \rho}{\partial P}\right)_T$
    - ideal gas : $v = \frac{nRT}{P}$, $\left(\frac{\partial v}{\partial P}\right)_T = -\frac{nRT}{P^2}$
    - $\kappa = -\frac{P}{nRT}\left(-\frac{nRT}{P^2}\right) = \frac{1}{P}$ ⇒ P effect neglegible with low v. (high v, high P drop)
  - > Mach number - $\boxed{Ma = \frac{v_{char}}{v_{sound}}}$
    - $Ma \lesssim 0.3$ → assume constant ρ ($v_{char} \approx 100$ m/s)

II.B.3 Conservation of Momentum : Eqn of Motion

- Cartesian coordinates
- x-momentum



$\Delta V = \Delta x \Delta y \Delta z$

OIGA

$$\begin{bmatrix}\text{rate of x-mom}\\ \text{out by diffusion}\end{bmatrix} - \begin{bmatrix}\text{rate of x-mom}\\ \text{in by diffusion}\end{bmatrix} + \begin{bmatrix}\text{rate of x-mom}\\ \text{out by convection}\end{bmatrix} - \begin{bmatrix}\text{rate of x-mom}\\ \text{in by convection}\end{bmatrix} = \overset{\text{(generation)}}{\begin{bmatrix}\text{sum of}\\ \text{external x-forces}\end{bmatrix}} - \begin{bmatrix}\text{rate of x-mom}\\ \text{accumulation}\end{bmatrix}$$

diffusion: $\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0\\ \Delta z\to 0}}$

$$\frac{\{[\tau_{xx}]_{x+\Delta x} - [\tau_{xx}]_x\}\Delta y\Delta z}{\Delta x\Delta y\Delta z} + \frac{\{[\tau_{yx}]_{y+\Delta y} - [\tau_{yx}]_y\}\Delta x\Delta z}{\Delta x\Delta y\Delta z} + \frac{\{[\tau_{zx}]_{z+\Delta z} - [\tau_{zx}]_z\}\Delta x\Delta y}{\Delta x\Delta y\Delta z}$$

Convection:

$$+\frac{\{[\rho v_x v_x]_{x+\Delta x} - [\rho v_x v_x]_x\}\Delta y\Delta z}{\Delta x\Delta y\Delta z} + \frac{\{[\rho v_x v_y]_{y+\Delta y} - [\rho v_x v_y]_y\}\Delta x\Delta z}{\Delta x\Delta y\Delta z} + \frac{\{[\rho v_x v_z]_{z+\Delta z} - [\rho v_x v_z]_{z+\Delta z}\}\Delta x\Delta y}{\Delta x\Delta y\Delta z}$$

& generation accumulation:

$$= \frac{\{[p]_x - [p]_{x+\Delta x}\}\Delta y\Delta z}{\Delta x\Delta y\Delta z} + \frac{\rho g_x \Delta x\Delta y\Delta z}{\Delta x\Delta y\Delta z} - \frac{\rho\frac{\partial v_x}{\partial t}\Delta x\Delta y\Delta z}{\Delta x\Delta y\Delta z}$$

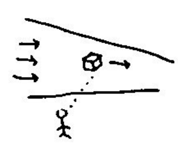
$$\frac{\partial\tau_{xx}}{\partial x} + \frac{\partial\tau_{yx}}{\partial y} + \frac{\partial\tau_{zx}}{\partial z} + \rho\underbrace{\left[\frac{\partial v_x v_x}{\partial x} + \frac{\partial v_x v_y}{\partial y} + \frac{\partial v_x v_z}{\partial z}\right]}_{} = -\frac{\partial p}{\partial x} + \rho g_x - \rho\frac{\partial v_x}{\partial t}$$

$$v_x\frac{\partial v_x}{\partial x} + \boxed{v_x\frac{\partial v_x}{\partial x}} + v_y\frac{\partial v_x}{\partial y} + \boxed{v_x\frac{\partial v_y}{\partial y}} + v_z\frac{\partial v_x}{\partial z} + \boxed{v_x\frac{\partial v_z}{\partial z}}$$

$$\underset{\sim}{v}\cdot\nabla v_x \qquad \boxed{v_x\nabla\cdot\underset{\sim}{v} = 0} \text{ (continuity for } \Delta\rho = 0)$$

$$\frac{\partial\tau_{xx}}{\partial x} + \frac{\partial\tau_{yx}}{\partial y} + \frac{\partial\tau_{zx}}{\partial z} + \rho\underset{\sim}{v}\cdot\nabla v_x = -\frac{\partial p}{\partial x} + \rho g_x - \rho\frac{\partial v_x}{\partial t}$$

$$\boxed{\underbrace{\rho\left[\frac{\partial v_x}{\partial t} + \underset{\sim}{v}\cdot\nabla v_x\right]}_{\left(\frac{\text{mass}}{\text{vol}}\right)(\text{x-acceleration})} = \underbrace{-\frac{\partial p}{\partial x}}_{\frac{\text{x-pressure force}}{\text{vol}}} - \underbrace{\left(\frac{\partial\tau_{xx}}{\partial x} + \frac{\partial\tau_{yx}}{\partial y} + \frac{\partial\tau_{zx}}{\partial z}\right)}_{\frac{\text{x-viscous force}}{\text{vol}}} + \underbrace{\rho g_x}_{\frac{\text{x-gravity force}}{\text{vol}}}}$$



$$\frac{ma_x}{V} = \frac{F_p}{V} + \frac{F_v}{V} + \frac{F_g}{V}$$

$\Rightarrow$ x-component of Newton's second law applied to a control mass moving with the flow

→ Acceleration

- $a_x = \underbrace{\frac{\partial v_x}{\partial t}}_{\text{local}} + \underbrace{\underset{\sim}{v}\cdot\nabla v_x}_{\text{convective}}$ $\Rightarrow$ acceleration experienced by control mass drifting with flow

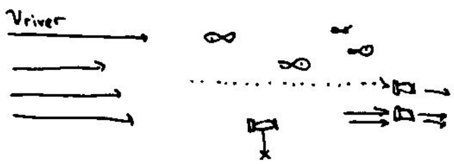
> Substantial derivative (material derivative) – $\boxed{\frac{Dv_x}{Dt} = \frac{\partial v_x}{\partial t} + \underset{\sim}{v}\cdot\nabla v_x}$ rate of change of v_x with t experienced by fluid particle drifting with flow

- e.g. $\frac{DT}{Dt} = \frac{\partial T}{\partial t} + \underset{\sim}{v}\cdot\nabla T$

$$\frac{DC_A}{Dt} = \frac{\partial C_A}{\partial t} + \underset{\sim}{v}\cdot\nabla C_A$$

→ Derivative Interpretations

· Imagine a submarine observing fish concentration C_f.



> partial derivative - $\frac{\partial C_f}{\partial t}$ at fixed location

> substantial derivative - $\frac{DC_f}{Dt} = \frac{\partial C_f}{\partial t} + \underset{\sim}{v}_{river} \cdot \nabla C_f$ drifting with flow

> total derivative - $\frac{dC_f}{dt} = \frac{\partial C_f}{\partial t} + \underset{\sim}{v}_{sub} \cdot \nabla C_f$ from moving location (propelled submarine)

→ Generalization to vector form

$$\rho \frac{Dv_x}{Dt} = -\frac{\partial p}{\partial x} - \left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \right) + \rho g_x$$

$$\rho \frac{Dv_x}{Dt} = -\frac{\partial p}{\partial x} - \left(-\mu \nabla^2 v_x \right) + \rho g_x$$

(substitute rate law $\tau_{ij} = -\mu \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right)$)

· Generalize to y and z component, we have

$$\boxed{\rho \frac{D\underset{\sim}{v}}{Dt} = -\nabla p + \mu \nabla^2 \underset{\sim}{v} + \rho g}$$

Equation of Motion (momentum eqn)

→ Generalization of conduction & diffusion eqn

· $\frac{\partial T}{\partial t} = \alpha \nabla^2 T + \frac{S}{\rho \hat{C}_p}$ ⇒ $\boxed{\frac{DT}{Dt} = \alpha \nabla^2 T + \frac{S}{\rho \hat{C}_p}}$ thermal energy equation

· $\frac{\partial C_A}{\partial t} = D_{AB} \nabla^2 C_A + R_A$ ⇒ $\boxed{\frac{DC_A}{Dt} = D_{AB} \nabla^2 C_A + R_A}$ convective diffusion equation

→ Differential equations of fluid mechanics (constant μ, ρ)

> continuity eqn $\boxed{\nabla \cdot v = 0}$

> eqn of motion

$$\boxed{\rho \frac{D\underset{\sim}{v}}{Dt} = -\nabla p + \mu \nabla^2 v + \rho g}$$

or

$$\boxed{\rho \frac{D\underset{\sim}{v}}{Dt} = -\nabla p - \nabla \cdot \underset{\approx}{\tau} + \rho g}$$ in τ form

Navier-Stokes Eqn

→ Caveat

· may encounter substantial derivative of vector quantity : $\frac{D\underset{\sim}{v}}{Dt} = \frac{\partial \underset{\sim}{v}}{\partial t} + \underset{\sim}{v} \cdot \underbrace{\nabla \underset{\sim}{v}}$ gradient of vector (a tenser)

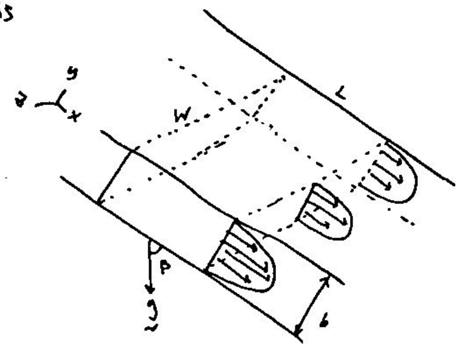
· completely general form is $\frac{D\underset{\sim}{v}}{Dt} = \frac{\partial \underset{\sim}{v}}{\partial t} + \frac{1}{2} \nabla (\underset{\sim}{v} \cdot \underset{\sim}{v}) - \underset{\sim}{v} \times [\nabla \times \underset{\sim}{v}]$

II.B.4 What can we do with the equations?

- no general soln!
- prune down for special, simple cases (top-down method)
  - analytical soln
  - numerical soln
- time-smoothed eqn to describe turbulent flow
- non-dimensionalize
  - identify regions of behavior
  - dynamic similarity or dimensional analysis

II.C Examples of Top-Down Procedure

II.C.1 Flow between parallel plates



- given $\left[\frac{P_0 - P_L}{L}\right] = -\frac{dP}{dx}$

→ Physical Description (Assumptions)

- constant ρ, μ : $\Delta\rho = 0$, $\Delta\mu = 0$
- laminar flow
- steady state : $\frac{\partial}{\partial t} = 0$
- v_x is only velocity component : $v_y = v_z = 0$
- negligible edge effect : $\frac{\partial}{\partial z} = 0$
- negligible end effect : $\frac{\partial v_x}{\partial x} = 0$
- no hydrostatic pressure diff between plate : $b \ll W, L \Rightarrow -\frac{\partial p}{\partial y} + \rho g_y = 0$

→ Top-Down Method

- continuity

$$\cancel{\frac{\partial v_x}{\partial x}} + \cancel{\frac{\partial v_y}{\partial y}} + \cancel{\frac{\partial v_z}{\partial z}} = 0$$

$$0 = 0$$

- x-momentum

$$\rho\left[\cancel{\frac{\partial v_x}{\partial t}} + v_x \cancel{\frac{\partial v_x}{\partial x}} + \cancel{v_y}\frac{\partial v_y}{\partial y} + \cancel{v_z}\frac{\partial v_x}{\partial z}\right] = -\frac{\partial p}{\partial x} + \mu\left[\cancel{\frac{\partial^2 v_x}{\partial x^2}} + \frac{\partial^2 v_x}{\partial y^2} + \cancel{\frac{\partial^2 v_x}{\partial z^2}}\right] + \rho g_x$$

$$0 = -\frac{\partial p}{\partial x} + \rho g_x + \mu \frac{\partial^2 v_x}{\partial y^2}$$

$$\boxed{0 = \frac{P_0 - P_L}{L} + \mu \frac{\partial^2 v_x}{\partial y^2}}$$

- y-momentum

$$\rho\left[\cancel{\frac{\partial v_y}{\partial t}} + v_x \cancel{\frac{\partial v_y}{\partial x}} + v_y \cancel{\frac{\partial v_y}{\partial y}} + v_z \cancel{\frac{\partial v_y}{\partial z}}\right] = -\frac{\partial p}{\partial y} + \mu\left[\cancel{\frac{\partial^2 v_y}{\partial x^2}} + \cancel{\frac{\partial^2 v_y}{\partial y^2}} + \cancel{\frac{\partial^2 v_y}{\partial z^2}}\right] + \rho g_y$$

$$0 = \cancel{-\frac{\partial p}{\partial y} + \rho g_y} \rightarrow 0$$

$$0 = 0$$

- z-momentum

$$\rho\left[\cancel{\frac{\partial v_z}{\partial t}} + v_x \cancel{\frac{\partial v_z}{\partial x}} + v_y \cancel{\frac{\partial v_z}{\partial y}} + \cancel{v_z}\frac{\partial v_z}{\partial z}\right] = -\cancel{\frac{\partial p}{\partial z}} + \mu\left[\cancel{\frac{\partial^2 v_z}{\partial x^2}} + \cancel{\frac{\partial^2 v_z}{\partial y^2}} + \cancel{\frac{\partial^2 v_z}{\partial z^2}}\right] + \cancel{\rho g_z}$$

$$0 = 0$$

→ Velocity profile

$$\frac{d^2 v_x}{dy^2} = -\frac{1}{\mu}\left[\frac{\mathcal{P}_0-\mathcal{P}_L}{L}\right] \qquad \text{b.c.} \begin{cases} 1. & y=0, \quad v_x=0 \\ 2. & y=b, \quad v_x=0 \end{cases}$$

$$d\left(\frac{dv_x}{dy}\right) = -\frac{1}{\mu}\left[\frac{\mathcal{P}_0-\mathcal{P}_L}{L}\right] dy$$

$$\frac{dv_x}{dy} = -\frac{1}{\mu}\left[\frac{\mathcal{P}_0-\mathcal{P}_L}{L}\right] y + c_1$$

$$dv_x = \left\{-\frac{1}{\mu}\left[\frac{\mathcal{P}_0-\mathcal{P}_L}{L}\right] y + c_1\right\} dy$$

$$v_x = -\frac{1}{\mu}\left[\frac{\mathcal{P}_0-\mathcal{P}_L}{L}\right]\frac{y^2}{2} + c_1 y + c_2$$

$$0 = -\frac{1}{\mu}[\;]\frac{0^2}{2} + c_1(0) + c_2 \quad \Rightarrow c_2 = 0 \qquad (\text{b.c. } 1)$$

$$0 = -\frac{1}{\mu}[\;]\frac{b^2}{2} + c_1 b \quad \Rightarrow c_1 = \frac{1}{2\mu}[\;]b \qquad (\text{b.c. } 2)$$

$$\boxed{v_x = +\frac{1}{2\mu}\left[\frac{\mathcal{P}_0-\mathcal{P}_L}{L}\right](-y^2+by)}$$

→ Flow descriptors

- Average velocity

$$\langle v_x\rangle = \frac{Q}{A} = \frac{1}{Wb}\int v_\perp \, dA = \frac{1}{Wb}\int v_x (W\,dy) = \dots$$

$$\boxed{\langle v_x\rangle = \frac{b^2}{12\mu}\left[\frac{\mathcal{P}_0-\mathcal{P}_L}{L}\right]}$$

- Skin friction at bottom plate

$$\tau^\circ = -(\tau_{yx})_{y=0} = -\left(\mp\mu\left(\frac{\partial v_x}{\partial y} + \cancel{\frac{\partial v_y}{\partial x}}^{\,0}\right)\right)_{y=0}$$

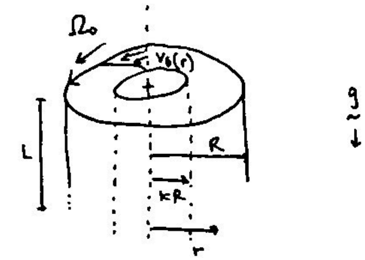
$$= +\mu\left(+\frac{1}{2\mu}\left(\frac{\mathcal{P}_0-\mathcal{P}_L}{L}\right)(-2y+b)\right)\Big|_{y=0}$$

$$\boxed{\tau^\circ = \frac{b}{2}\,\frac{\mathcal{P}_0-\mathcal{P}_L}{L}}$$

$\boxed{\text{II.C.2}}$ Couette Flow between concentric, rotating cylinders

→ Physical Description (Assumptions)

- constant ρ, μ : $\Delta\rho = 0$, $\Delta\mu = 0$
- laminar flow
- steady state : $\frac{\partial}{\partial t} = 0$
- V_θ is only velocity component : $V_r = V_z = 0$
- axial symmetry : $\frac{\partial}{\partial \theta} = 0$
- no end effect : $\frac{\partial V_\theta}{\partial z} = 0$
- vertical : $g_z = -g$, $g_\theta = g_r = 0$



→ Top-Down Method

- continuity

$$\cancel{\frac{1}{r}\frac{\partial}{\partial r}(r v_r)} + \cancel{\frac{1}{r}\frac{\partial v_\theta}{\partial \theta}} + \cancel{\frac{\partial v_z}{\partial z}} = 0$$

$$0 = 0$$

- r-momentum

$$\rho\left[\cancel{\frac{\partial v_r}{\partial t}} + v_r\cancel{\frac{\partial v_r}{\partial r}} + \frac{v_\theta}{r}\cancel{\frac{\partial v_r}{\partial \theta}} - \frac{v_\theta^2}{r} + v_z\cancel{\frac{\partial v_r}{\partial z}}\right] = -\frac{\partial p}{\partial r} + \mu\left[\cancel{\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_r)\right)} + \cancel{\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2}} - \cancel{\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}} + \cancel{\frac{\partial^2 v_r}{\partial z^2}}\right] + \cancel{\rho g_r}$$

$$\boxed{-\rho\frac{v_\theta^2}{r} = -\frac{\partial p}{\partial r}} \qquad ①$$

- θ-momentum

$$\rho\left[\cancel{\frac{\partial v_\theta}{\partial t}} + v_r\cancel{\frac{\partial v_\theta}{\partial r}} + \frac{v_\theta}{r}\cancel{\frac{\partial v_\theta}{\partial \theta}} + \cancel{\frac{v_r v_\theta}{r}} + v_z\cancel{\frac{\partial v_\theta}{\partial z}}\right] = -\frac{1}{r}\cancel{\frac{\partial p}{\partial \theta}} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_\theta)\right) + \frac{1}{r^2}\cancel{\frac{\partial^2 v_\theta}{\partial \theta^2}} + \frac{2}{r^2}\cancel{\frac{\partial v_r}{\partial \theta}} + \cancel{\frac{\partial^2 v_\theta}{\partial z^2}}\right] + \cancel{\rho g_\theta}$$

$$\boxed{0 = \mu\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_\theta)\right)} \qquad ②$$

- z-momentum

$$\rho\left[\cancel{\frac{\partial v_z}{\partial t}} + v_r\cancel{\frac{\partial v_z}{\partial r}} + \frac{v_\theta}{r}\cancel{\frac{\partial v_z}{\partial \theta}} + v_z\cancel{\frac{\partial v_z}{\partial z}}\right] = -\frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\cancel{\frac{\partial v_z}{\partial r}}\right) + \frac{1}{r^2}\cancel{\frac{\partial^2 v_z}{\partial \theta^2}} + \cancel{\frac{\partial^2 v_z}{\partial z^2}}\right] + \rho g_z$$

$$\boxed{0 = -\frac{\partial p}{\partial z} - \rho g} \qquad ③$$

→ Velocity Profile $v_\theta(r)$

From ② : $\frac{1}{r}\frac{\partial}{\partial r}(r V_\theta) = C_1$

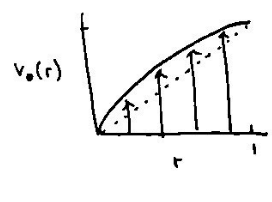
$$r V_\theta = C_1\frac{r^2}{2} + C_2$$

$$\boxed{V_\theta = C_1\frac{r}{2} + \frac{C_2}{r}}$$

b.c. $\begin{cases} 1. & r = \kappa R, \; V_\theta = 0 \\ 2. & r = R, \; V_\theta = \Omega_0 R \end{cases}$

$\Rightarrow C_1 = \frac{2\Omega_0}{1-\kappa^2}$, $C_2 = -\frac{\kappa^2 R^2 \Omega_0}{1-\kappa^2}$

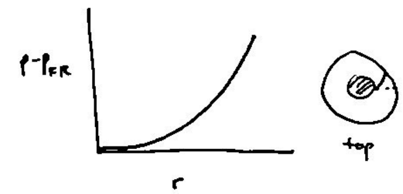
$$\boxed{V_\theta = \frac{\Omega_0}{1-\kappa^2}\left[r - \frac{(\kappa R)^2}{r}\right]}$$



→ pressure profile $p(r)$ at some z

From ①: $\frac{dp}{dr} = \rho \frac{V_\theta^2}{r} = \frac{\rho}{r} \frac{\Omega_o^2}{(1-K^2)^2}\left[r - \frac{(KR)^2}{r}\right]^2$

$$\int_{p_{KR}}^{p} dp = \int_{KR}^{R} \frac{\rho}{r} \frac{\Omega_o^2}{(1-K^2)^2}\left[r - \frac{(KR)^2}{r}\right]^2 dr$$

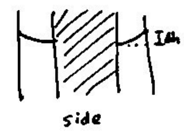


$$p - p_{KR} = \frac{1}{2}\rho\left(\frac{\Omega_o KR}{1-K^2}\right)^2\left[\left(\frac{r}{KR}\right)^2 - \left(\frac{KR}{r}\right)^2 - 4\ln\left(\frac{r}{KR}\right)\right]$$

· We can calculate hight difference caused by pressure difference.

· $\Delta p = \rho g \Delta h$ Given water at 20°C, $\rho = 1\,g/cm^3$,

$\Delta h = \frac{\Delta p}{\rho g}$ $\Delta p = 0.869$ dyne/cm² $g = 981$ cm²/s



$= \frac{0.869 \text{ gcm cm}^3 \text{ s}}{s^2 cm^2 (1) g (981) cm^2}$

$= 8.87 \times 10^{-4}$ cm

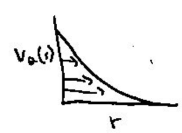
$= 8.87\ \mu m$ ⇒ negligible Δh

· If inner cylinder spins,

b.c. $\begin{cases} 1.\ r = KR,\ V_\theta = \Omega_i KR \\ 2.\ r = R,\ V_\theta = 0 \end{cases}$

$V_\theta = \frac{1}{2}c_1 r + \frac{c_2}{r}$

$V_\theta = \frac{\Omega_i (KR)^2}{(1-K^2)}\left[\frac{1}{r} - \frac{r}{R^2}\right]$



· If fast enough, Taylor vortices may form

→ pressure variation vertically $p(z)$

From ③: $\frac{dp}{dz} = -\rho g$

$p_2 - p_1 = -\rho g (z_2 - z_1)$ ⇒ hydrostatic

→ shear stress distribution & torque

$$\tau_{r\theta} = -\mu\left[r\frac{\partial}{\partial r}\left(\frac{V_\theta}{r}\right) + \frac{1}{r}\cancelto{0}{\left(\frac{\partial V_r}{\partial \theta}\right)}\right]$$

$$= -\mu r \frac{d}{dr}\left\{\frac{\Omega_o}{1-K^2}\left[1 - \frac{(KR)^2}{r^2}\right]\right\}$$

$$= -2\mu K^2\left(\frac{\Omega_o}{1-K^2}\right)\left(\frac{R}{r}\right)^2$$

T = (force)(lever arm)

= (stress)(area)(lever arm)

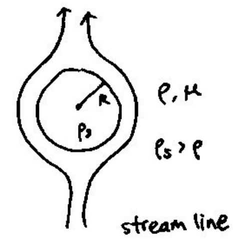
$= -\tau_{r\theta}\big]_{r=R}(2\pi RL)R$

$= 2\mu K^2\left(\frac{\Omega_o}{1-K^2}\right)2\pi R^2 L$

$= 4\pi\mu L \Omega_o R^2 \frac{K^2}{1-K^2}$

$$\Rightarrow \boxed{\mu = \frac{T}{4\pi L \Omega_o R^2}\ \frac{1-K^2}{K}}$$ Couette-Hatschak viscometer

$\boxed{\text{II.C.3}}$ Flow around a sphere : Stoke's Law



- Determine terminal velocity v_∞ at steady state
  - Determine $v_r(r,\theta)$, $v_\theta(r,\theta)$, $p(r,\theta)$
  - > stream line - line drawn in the flow everywhere tangent to $\underset{\sim}{v}$

→ Physical Description (Assumptions)

- constant μ, ρ
- laminar flow
- steady state : $\frac{\partial}{\partial t} = 0$
- axial symmetry : $\frac{\partial}{\partial \phi} = 0$
- no spinning : $v_\phi = 0$
- gravity: $g_r = -g\cos\theta$, $g_\theta = g\sin\theta$, $g_\phi = 0$
- creeping flow : $v^2 \approx 0$
  - > v-components are so small that any $v^2 \approx 0$ $\left(\sim \frac{dv}{dx} = 0\right)$

{ · small ball · viscous liquid · $Re \leq 0.1$ }

→ Top-Down Method

- steady creeping flow cancels LHS of momentum eqn.

continuity : $\frac{1}{r}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial(v_\theta \sin\theta)}{\partial\theta} + \cancel{\frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}} = 0$

r : $\rho\left[\cancel{\frac{\partial v_r}{\partial t}} + \cancel{v_r\frac{\partial v_r}{\partial r}} + \cancel{\frac{v_\theta}{r}\frac{\partial v_r}{\partial\theta}} + \cancel{\frac{v_\phi}{r\sin\theta}\frac{\partial v_r}{\partial\phi}} - \cancel{\frac{v_\theta^2+v_\phi^2}{r}}\right] = -\frac{\partial p}{\partial r} + \mu\left[\frac{1}{r^2}\frac{\partial^2}{\partial r^2}(r^2 v_r) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v_r}{\partial\theta}\right) + \cancel{\frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_r}{\partial\phi^2}}\right] + \rho g_r$

θ : $\rho\left[\cancel{\frac{\partial v_\theta}{\partial t}} + \cancel{v_r\frac{\partial v_\theta}{\partial r}} + \cancel{\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial\theta}} + \cancel{\frac{v_\phi}{r\sin\theta}\frac{\partial v_\theta}{\partial\phi}} - \cancel{\frac{v_r v_\theta - v_\phi^2\cot\theta}{r}}\right] = -\frac{1}{r}\frac{\partial p}{\partial\theta} + \mu\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_\theta}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(v_\theta\sin\theta)\right) + \cancel{\frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_\theta}{\partial\phi^2}} + \frac{2}{r^2}\frac{\partial v_r}{\partial\theta} - \cancel{\frac{2\cot\theta}{r^2\sin\theta}\frac{\partial v_\phi}{\partial\phi}}\right] + \rho g_\theta$

φ : $\rho\left[\cancel{\frac{\partial v_\phi}{\partial t}} + \cancel{v_r\frac{\partial v_\phi}{\partial r}} + \cancel{\frac{v_\theta}{r}\frac{\partial v_\phi}{\partial\theta}} + \cancel{\frac{v_\phi}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}} - \cancel{\frac{v_r v_\phi - v_\theta v_\phi\cot\theta}{r}}\right] = -\cancel{\frac{1}{r\sin\theta}\frac{\partial p}{\partial\phi}}$
$+ \mu\left[\cancel{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_\phi}{\partial r}\right)} + \cancel{\frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(v_\phi\sin\theta)\right)} + \cancel{\frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_\phi}{\partial\phi^2}} + \cancel{\frac{2}{r^2\sin\theta}\frac{\partial v_r}{\partial\phi}} - \cancel{\frac{2\cot\theta}{r^2\sin\theta}\frac{\partial v_\theta}{\partial\phi}}\right] + \cancel{\rho g_\phi}$

$$\Rightarrow \begin{cases} \frac{1}{r}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{\sin\theta}\frac{\partial(v_\theta\sin\theta)}{\partial\theta} = 0 \\ 0 = -\frac{\partial p}{\partial r} + \mu\left[\frac{1}{r^2}\frac{\partial^2}{\partial r^2}(r^2 v_r) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v_r}{\partial\theta}\right)\right] - \rho g\cos\theta \\ 0 = -\frac{1}{r}\frac{\partial p}{\partial\theta} + \mu\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_\theta}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(v_\theta\sin\theta)\right) + \frac{2}{r^2}\frac{\partial v_r}{\partial\theta}\right] + \rho g\sin\theta \end{cases}$$

$$\Rightarrow \begin{cases} v_r = v_\infty\left[1 - \frac{3}{2}\left[\frac{R}{r}\right] + \frac{1}{2}\left(\frac{R}{r}\right)^3\right]\cos\theta \\ v_\theta = -v_\infty\left[1 - \frac{3}{4}\left(\frac{R}{r}\right) - \frac{1}{4}\left(\frac{R}{r}\right)^3\right]\sin\theta \\ p = p_0 - \rho g z - \frac{3}{2}\frac{\mu v_\infty}{R}\left(\frac{R}{r}\right)^2\cos\theta \end{cases}$$

With b.c. $\begin{cases} r = R,\ v_r = 0,\ v_\theta = 0 \\ r = \infty,\ v_r = v_\infty\cos\theta \\ \qquad v_\theta = -v_\infty\sin\theta \\ \text{\& } z = 0,\ r \to \infty,\ p \to p_0 \end{cases}$

→ Calculate forces acting on the sphere

→ Total force

$$\Sigma F = 0 = \text{gravity}\downarrow + \text{viscous drag}\uparrow + \text{pressure force}\uparrow$$

$$0 = -\tfrac{4}{3}\pi R^3 \rho_s g + \int \tau \, dA + \int P_R \uparrow dA$$

→ Gravity force

$$F_g = -mg = -\rho V g = -\rho_s \tfrac{4}{3}\pi R^3 g$$

→ Viscous drag

- $$\tau_{r\theta} = -\mu\left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \cancel{\frac{1}{r}\frac{\partial v_r}{\partial \theta}}^{\,0}\right]$$

$$= -\mu\left[r\frac{\partial}{\partial r}\left[-\frac{1}{r} v_\infty\left[1 - \frac{3}{4}\left(\frac{R}{r}\right) - \frac{1}{4}\left(\frac{R}{r}\right)^3\right]\sin\theta\right]\right]$$

$$= -\mu\left[r\left(-v_\infty\left(-\frac{1}{r^2} + \frac{3R\cdot 2}{4r^3} + \frac{R^3}{r^5}\right)\sin\theta\right)\right]$$

$$= +\mu v_\infty\left[-\frac{1}{r} + \frac{3}{2}\frac{R}{r^2} + \frac{R^3}{r^4}\right]\sin\theta$$

- $$\tau_{r\theta}\Big]_{r=R} = \mu v_\infty\left[-\frac{1}{R} + \frac{3}{2}\frac{1}{R} + \frac{1}{R}\right]\sin\theta$$

$$= \frac{3\mu v_\infty \sin\theta}{2R}$$

- $$dA = 2\pi R^2 \sin\theta \, d\theta$$

- $$\int \tau \, dA$$

$$= \int_0^\pi \tau_{r\theta}\Big]_{r=R} \sin\theta \; 2\pi R^2 \sin\theta \, d\theta$$

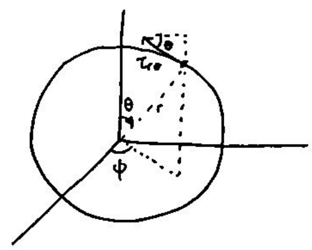
$$= \int_0^\pi \frac{3\mu v_\infty \sin\theta}{2R} \sin\theta \; 2\pi R^2 \sin\theta \, d\theta$$

$$= \frac{3\mu v_\infty 2\pi R^2}{2R}\int_0^\pi \sin^3\theta \, d\theta$$

$$= 3\mu v_\infty R\pi \, \frac{4}{3}$$

$$= 4\pi\mu v_\infty R$$

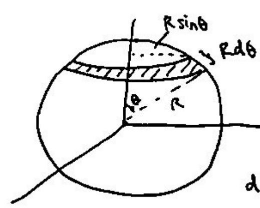
$$\boxed{\text{upward viscous drag} = 4\pi\mu v_\infty R}$$



- upward component of $\tau_{r\theta}$ is

  $+\tau_{r\theta}\Big]_{r=R} \sin\theta$

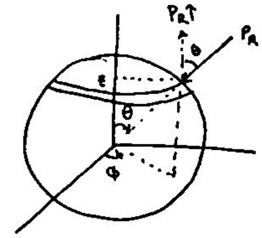
- two neg signs:
  1. negative θ direction
  2. τ is greater r on lesser r (fluid on solid)



$$dA = 2\pi(R\sin\theta)R\,d\theta$$
$$= 2\pi R^2 \sin\theta \, d\theta$$

→ Calculate forces acting on a sphere (cont.)

→ pressure force



$P_R\uparrow = -P_R\cos\theta$

$dA = 2\pi R^2 \underline{\sin\theta\, d\theta}$

$= 2\pi R^2(-d(\cos\theta))$, let $x \equiv \cos\theta$

$= -2\pi R^2\, dx$

$$\int P_R\uparrow dA$$

$$= \int -P_R\cos\theta\,(-2\pi R^2 dx) \qquad \begin{cases} \theta \in [0,\pi] \\ x \in [1,-1] \end{cases}$$

$$= 2\pi R^2 \int_1^{-1} P_R\, x\, dx$$

$$= 2\pi R^2 \int_1^{-1} \left[P_0 - \rho g \cancelto{Rx}{z} - \frac{3}{2}\frac{\mu V_\infty}{R}\left(\cancelto{1}{\frac{R}{r}}\right)^2 \cancelto{x}{\cos\theta}\right]_{r=R} x\, dx \qquad \{ z = R\cos\theta = Rx$$

$$= 2\pi R^2 \int_{+1}^{-1} \left[P_0 - \rho g R x - \frac{3}{2}\frac{\mu V_\infty}{R} x\right] x\, dx$$

$$= \cancelto{0}{2\pi R^2 \int_1^{-1} P_0\, x\, dx} - 2\pi R^3 \rho g \int_{+1}^{-1} x^2\, dx - 2\pi R^2 \frac{3}{2}\frac{\mu V_\infty}{R}\int_1^{-1} x^2\, dx$$

$$= 2\pi R^3 \rho g \left[\frac{x^3}{3}\right]_{-1}^{1} + 3\pi R\mu V_\infty \left[\frac{x^3}{3}\right]_{-1}^{1}$$

$$= 2\pi R^3 \rho g\, \frac{2}{3} + 3\pi R\mu V_\infty \frac{2}{3}$$

$$= \boxed{\underbrace{\frac{4}{3}\pi R^3 \rho g}_{\text{buoyancy}} + \underbrace{2\pi R\mu V_\infty}_{\text{form drag}}}$$

→ Total force (Stokes' law)

$$0 = \text{gravity}\downarrow + \text{viscous drag}\uparrow + \text{pressure force}\uparrow$$

$$0 = -\frac{4}{3}\pi R^3 \rho_s g + 4\pi\mu V_\infty R + \left(\frac{4}{3}\pi R^3 \rho g + 2\pi R\mu V_\infty\right)$$

$$0 = -\frac{4}{3}\pi R^3(\rho_s - \rho) g + 6\pi\mu R V_\infty$$

$$\boxed{V_\infty = \frac{2R^2(\rho_s-\rho)g}{9\mu}} \quad \text{Stoke's law}$$

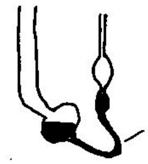
→ Falling ball viscometer & Creeping flow assumption

$$\boxed{\mu = \frac{2R^2(\rho_s-\rho)g}{9V_\infty}} \quad \text{Falling ball viscometer}$$

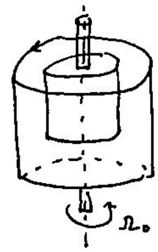
- Requires $Re \le 0.1$
  - $Re = \frac{D V_\infty \rho}{\mu} = \frac{2RV_\infty\rho}{\mu} = \frac{2RV_\infty}{\nu}$
  - small sphere
  - viscous liquid

→ Types of Viscometer

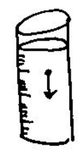
1. Capilary visconeter
   - use Hagen-Poiseiulle law in round tube capillary
   - not suitable for power law fluid



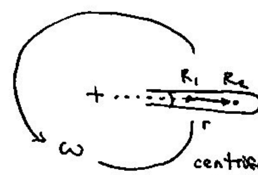
2. Couette viscometer
   - couette flow between concentric, rotating cylinders
   - suitable for power law fluid
   - rotate outer cylinder
     - Taylor vorteces generated if rotate Inner cylinder



3. Falling ball viscometer
   - flow around a sphere (Stoke's law)
   - creeping flow assumption
     - small ball
     - Viscous, liquid



4. Centrifuge viscometer
   - Stokes's law
   - centrifugal acceleration replace gravity
     - $g \to \omega^2 r$
   - $$\boxed{v_\infty = \frac{dr}{dt} = \frac{2R^2(\rho_s-\rho)\omega^2 r}{9\mu}}$$



$$\int \frac{1}{r}\,dr = \int \frac{2R^2(\rho_s-\rho)\omega^2}{9\mu}\,dt$$

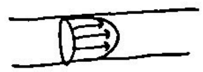
$$\ln\left(\frac{R_2}{R_1}\right) = \frac{2R^2(\rho_s-\rho)\omega^2}{9\mu}\Delta t$$

$$\boxed{\mu = \frac{2R^2(\rho_s-\rho)\omega^2}{9\ln\left(\frac{R_2}{R_1}\right)}\Delta t}$$

I.D Turbulence

II.D.1 Transition to Turbulence

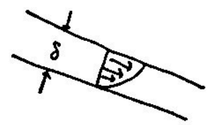
1. tube flow



$$Re = \frac{D\langle v\rangle \rho}{\mu}$$

$Re_c \approx 2100$

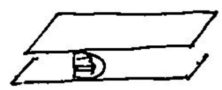
2. falling film



$$Re = \frac{4\delta\langle v\rangle \rho}{\mu}$$

$Re_c = 1500$

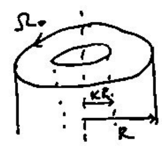
3. flow between parallel plate



$$Re = \frac{2b\langle v\rangle \rho}{\mu}$$

$Re_c = 1780$

4. Tangential flow in an annulus

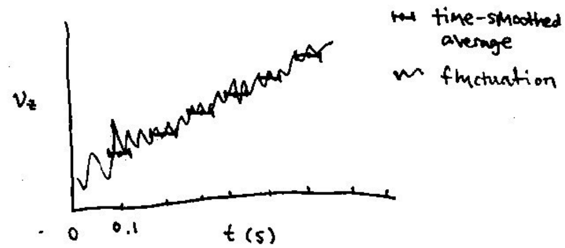


$$Re = \frac{\Omega_0 R^2 \rho}{\mu}$$

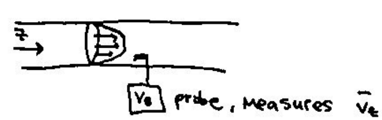
$Re_c \approx 50000$

II.D.2 Nature of Turbulence

- Chaotic eddies superimposed on overall velocity
  - dispersion of dye in Renyolds' expeiment
- properties can be split into time-smoothed component and fluctuation.
  - $v_z = \bar{v}_z + v_z'$
  - $P = \bar{P} + P'$
  - $T = \underbrace{\bar{T}}_{\text{time-smoothed average}} + \underbrace{T'}_{\text{fluctuation}}$

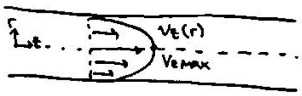
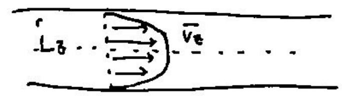


- scales
  - $v_z' \approx 1\text{-}10\%\ \bar{v}_z$
  - freq ≈ mHz



- hotwire antometer can measure v_z'

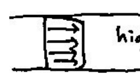
II.D.3 Comparison of laminar and time smoothed turbulent velocity profile in tube

Laminar flow ($Re < 2100$)		Turbulent flow ($10^4 < Re < 10^5$)
	profile schematic	
$\frac{V_z}{V_{zmax}} = 1 - \left(\frac{r}{R}\right)^2$	velocity profile	$\frac{\bar{V}_z}{\bar{V}_{zmax}} \approx \left(1 - \frac{r}{R}\right)^{\frac{1}{7}}$
$\langle V_z \rangle = \frac{1}{2} V_{z max}$	average velocity	$\langle V_z \rangle \approx \frac{4}{5} \bar{V}_{z max}$
$Q = \frac{\pi R^4}{8\mu}\left(\frac{P_0 - P_1}{L}\right)$	volumetric flow rate	$Q \propto \left[\frac{P_0 - P_1}{L}\right]^{\frac{4}{7}}$
$L_e = 0.035\, D\, Re$	entry length	$L_e \approx 40D$
from theory	derivation	from experiment

→ General turbulent flow

$$\frac{\bar{V}_z}{\bar{V}_{zmax}} \approx \left(1 - \frac{r}{R}\right)^{\frac{1}{n}}, \quad \text{where } n = \begin{cases} 6 & Re \in [2\times10^3, 10^4] \\ 7 & Re \in [10^4, 10^5] \\ 8 & Re \in [10^5, 10^6] \end{cases}$$

- As Re increases, velocity profile flattens
- As Re is high, velocity is nearly the same across the cross section
  - "plug flow"



high Re

II.D.4 Time smoothing the Navier-Stokes Equations

> time smoothing – integration over time interval t_0

fluctuation is mean centered

$$(V_z)_{time-smoothed} = \frac{1}{t_0}\int_{t-\frac{1}{2}t_0}^{t+\frac{1}{2}t_0} V_z\, dt = \frac{1}{t_0}\int_{t-\frac{1}{2}t_0}^{t+\frac{1}{2}t_0} \bar{V}_z + V_z'\, dt = \frac{\bar{V}_z}{t_0}\int_{t-\frac{1}{2}t_0}^{t+\frac{1}{2}t_0} dt + \cancelto{0}{\frac{1}{t_0}\int_{t-\frac{1}{2}t_0}^{t+\frac{1}{2}t_0} V_z'\, dt} = \bar{V}_z + 0$$

$$\boxed{(V_z)_{time-smoothed} = \bar{V}_z}$$

→ Continuity Equation

$$\underbrace{\frac{\partial V_x}{\partial x}} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} = 0$$

$$\frac{\partial}{\partial x}\left\{\frac{1}{t_0}\int_{t-\frac{1}{2}t_0}^{t+\frac{1}{2}t_0} (\bar{V}_x + V_x')\,dt\right\} = \frac{\partial}{\partial x}(\bar{V}_x + 0) = \frac{\partial \bar{V}_x}{\partial x}$$

$$\Rightarrow \frac{\partial \bar{V}_x}{\partial x} + \frac{\partial \bar{V}_y}{\partial y} + \frac{\partial \bar{V}_z}{\partial z} = 0 \qquad \boxed{\nabla \cdot \bar{\underset{\sim}{V}} = 0}$$

$$\frac{\partial V_x'}{\partial x} + \frac{\partial V_y'}{\partial y} + \frac{\partial V_z'}{\partial z} = 0 \qquad \boxed{\nabla \cdot \underset{\sim}{V}' = 0}$$

→ Momentum equation (τ-form) x component

$$\rho\left[\frac{\partial v_x}{\partial t} + \underset{\sim}{v}\cdot\nabla v_x\right] = -\frac{\partial p}{\partial x} - \left[\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right] + \rho g_x$$

$$\rho\frac{\partial v_x}{\partial t} + \rho\left[\frac{\partial v_x v_x}{\partial x} + \frac{\partial v_y v_x}{\partial y} + \frac{\partial v_z v_x}{\partial z}\right] = -\frac{\partial p}{\partial x} - \left[\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right] + \rho g_x$$

$$\rho\frac{\partial \bar{v}_x}{\partial t} + \rho\left[\frac{\partial \overline{v_x v_x}}{\partial x} + \underbrace{\frac{\partial \overline{v_y v_x}}{\partial y}} + \frac{\partial \overline{v_z v_x}}{\partial z}\right] = -\frac{\partial \bar{p}}{\partial x} - \left[\frac{\partial \bar{\tau}_{xx}}{\partial x} + \frac{\partial \bar{\tau}_{yx}}{\partial y} + \frac{\partial \bar{\tau}_{zx}}{\partial z}\right] + \rho g_x$$

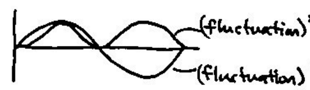
$$\frac{\partial}{\partial y}\left\{\frac{1}{t_0}\int_{t-\frac{1}{2}t_0}^{t+\frac{1}{2}t_0} (\bar{v}_x + v_x')(\bar{v}_y + v_y')\,dt\right\}$$

$$= \frac{\partial}{\partial y}\left\{\frac{1}{t_0}\int_{t-\frac{1}{2}t_0}^{t+\frac{1}{2}t_0} (\bar{v}_x\bar{v}_y + \cancelto{0}{\bar{v}_x v_y'} + \cancelto{0}{v_x'\bar{v}_y} + v_x' v_y')\,dt\right\}$$

$$= \frac{\partial \bar{v}_x\bar{v}_y}{\partial y} + 0 + 0 + \frac{\partial \overline{v_x' v_y'}}{\partial y}$$

Note: (fluctuation)^1 is mean centered ⇒ cancelled

but (fluctuation)^2 is not mean centered → retained



so:

$$\underbrace{\rho\frac{\partial \bar{v}_x}{\partial t} + \rho\left[\frac{\partial \bar{v}_x\bar{v}_x}{\partial x} + \frac{\partial \bar{v}_y\bar{v}_x}{\partial y} + \frac{\partial \bar{v}_z\bar{v}_x}{\partial z}\right]}_{\rho\frac{D\bar{v}_x}{Dt}} + \underbrace{\rho\left[\frac{\partial \overline{v_x' v_x'}}{\partial x} + \frac{\partial \overline{v_y' v_x'}}{\partial y} + \frac{\partial \overline{v_z' v_x'}}{\partial z}\right]}_{\text{turbulent (Reynolds) stress}} = -\frac{\partial \bar{p}}{\partial x} - \left[\frac{\partial \bar{\tau}_{xx}}{\partial x} + \frac{\partial \bar{\tau}_{yx}}{\partial y} + \frac{\partial \bar{\tau}_{zx}}{\partial z}\right] + \rho g_x$$

absorb to τ

$$\rho\frac{D\bar{v}_x}{Dt} = -\frac{\partial \bar{p}}{\partial x} - \left[\frac{\partial \bar{\tau}^{total}_{xx}}{\partial x} + \frac{\partial \bar{\tau}^{total}_{yx}}{\partial y} + \frac{\partial \bar{\tau}^{total}_{zx}}{\partial z}\right] + \rho g_x$$

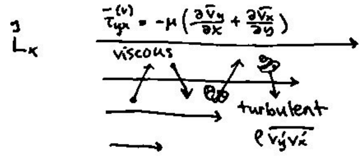
↳ where $\bar{\tau}^{total}_{yx} = \bar{\tau}_{yx} + \rho\overline{v_y' v_x'}$

$= \bar{\tau}^{(v)}_{yx} + \bar{\tau}^{(t)}_{yx}$

= viscous stress + turbulent stress

→ turbulent stress

- turbulent stress is eddies jumping between planes of different velocity, just like viscous stress is molecules jumping.



- N-S equation in terms of τ is understood to be τ^{total}.
- N-S equation in terms of v cannot be used because turbulent flow fluctuation can't be calculated
  - velocity profile has to be measured experimentally

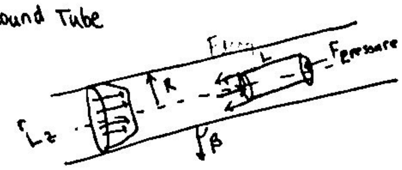
→ summary

- For turbulent flow
  - shear stress distribution - theory-derived
    - use N-S equation (or shell balance) in τ form
  - time-smoothed velocity profile (& related quantity) - experimentally measured
    - no model for fluctuation

II.D.5 Shear Stress Distribution and skin friction for turbulent flow in conduit

- Derive shear stress distribution in conduit $\tau_{rz} \equiv \overline{\tau}_{rz}^{(total)}(r)$

→ Round Tube



- steady state $\frac{\partial}{\partial t} = 0$
- v_z is only comp: $v_r = v_\theta = 0$
- no end effect: $\frac{\partial v_z}{\partial z} = 0$
- τ has no θ, z dependence $\frac{\partial \tau}{\partial \theta} = \frac{\partial \tau}{\partial z} = 0$

z-momentum cylindrical:
$$\rho\left(\cancel{\frac{\partial v_z}{\partial t}} + \cancel{v_r\frac{\partial v_z}{\partial r}} + \cancel{\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}} + \cancel{v_z\frac{\partial v_z}{\partial z}}\right) = -\frac{\partial p}{\partial z} - \left[\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rz}) + \cancel{\frac{1}{r}\frac{\partial}{\partial \theta}\tau_{\theta z}} + \cancel{\frac{\partial}{\partial z}\tau_{zz}}\right] + \rho g_z$$

$$0 = -\frac{dp}{dz} - \frac{1}{r}\frac{d}{dr}(r\tau_{rz}) + \rho g_z$$

$$\frac{1}{r}\frac{d}{dr}(r\tau_{rz}) = \frac{\mathcal{P}_0 - \mathcal{P}_L}{L}$$

$$d(r\tau_{rz}) = \left[\frac{\mathcal{P}_0 - \mathcal{P}_L}{L}\right] r\,dr$$

$$r\tau_{rz} = \left[\frac{\mathcal{P}_0 - \mathcal{P}_L}{L}\right]\frac{r^2}{2} + c_1$$

$$\tau_{rz} = \frac{1}{2}\left[\frac{\mathcal{P}_0 - \mathcal{P}_L}{L}\right] r + \cancel{\frac{c_1}{r}}^{0} \qquad c_1 = 0 \text{ by b.c. } r=0,\ \tau_{rz}=0$$

$$\boxed{\tau_{rz} = \frac{1}{2}\left[\frac{\mathcal{P}_0 - \mathcal{P}_L}{L}\right] r}$$

- form for turbulent flow (here) is the same as laminar flow, but τ includes fluctuations / turbulence

- Alternative method: derive stress distribution in conduit from a force balance
  - consider a cylindrical shell in figure.

$$\Sigma F_z = 0 = F_{z,drag} + F_{z,pressure} + F_{z,gravity}$$

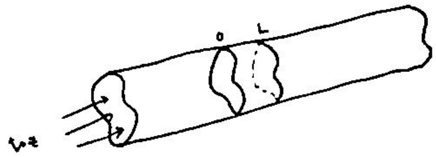
$$0 = \frac{-\tau_{rz}(2\pi r L)}{2\pi r L} + \frac{\pi r^2 (p_0 - p_L)}{2\pi r L} + \frac{\pi r^2 L \rho g \cos\beta}{2\pi r L}$$

$$\tau_{rz} = \frac{1}{2}\left[\left[\frac{p_0 - p_L}{L}\right] r + r\rho g\cos\beta\right]$$

$$\boxed{\tau_{rz} = \frac{1}{2}\left[\frac{\mathcal{P}_0 - \mathcal{P}_L}{L}\right] r} \qquad \text{same as N-S eqn result.}$$

skin friction $\tau^\circ = \frac{1}{2}\left[\frac{\mathcal{P}_0 - \mathcal{P}_1}{L}\right] R$

→ Other cross-sectional shape conduit



$$\Sigma F_z = F_{z,drag} + F_{z,pressure},\ F_{z,gravity} = 0$$

$$0 = -\tau^\circ(\text{wetted area of the wall}) + (\mathcal{P}_0 - \mathcal{P}_L)(\text{cross sectional area})$$

$$0 = -\tau^\circ(\text{wetted perimeter})L + (\mathcal{P}_0 - \mathcal{P}_L)(\text{cross sectional area})$$

$$\tau^\circ = \left[\frac{\mathcal{P}_0 - \mathcal{P}_L}{L}\right]\left(\frac{\text{cross sectional area}}{\text{wetted perimeter}}\right)$$

$\boxed{\tau^\circ = \left[\frac{\mathcal{P}_0 - \mathcal{P}_L}{L}\right] R_H}$, where hydrolic radius $\boxed{R_H \equiv \frac{\text{cross sectional area}}{\text{wetted perimeter}}}$

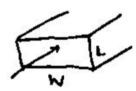
→ Hydrolic radius of different cross sections

- pipes or tube

$$R_H = \frac{\pi R^2}{2\pi R} = \frac{R}{2}$$

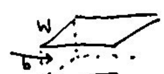
- rectangular duct

$$R_H = \frac{LW}{2(L+W)}$$



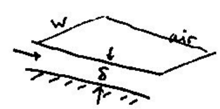
- parallel plates

$$R_H = \frac{Wb}{2W} = \frac{b}{2}$$



- falling film

$$R_H = \frac{W\delta}{W} = \delta$$



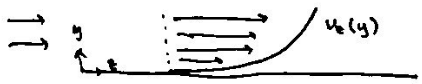
→ Characteristic values

- characteristic values can be used to calculate Re.
- char. length for conduit flow $= 4R_H$
- char. velocity for conduit flow $= \langle v_z \rangle$

$\boxed{\text{Ex1}}$ Find char length for:

- pipe flow : $4R_H = 4\left(\frac{R}{2}\right) = 2R = D$
- falling film : $4R_H = 4\delta$

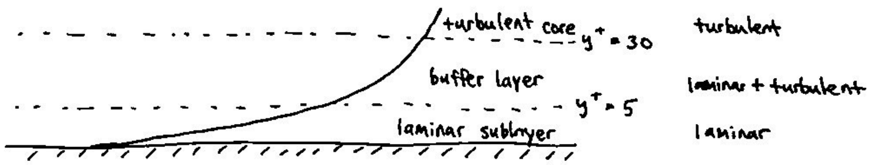
II.D.6 Universal Velocity Profile - smooth, straight conduits



- To build velocity profile that's independent of cross section, we need driving force and from that the skin friction to build characteristic params, then normalize.

- $y^+ = \frac{y}{y_*}$, where char length $\boxed{y_* = \frac{\mu}{v_* \rho}} = \sqrt{\frac{\rho}{\tau^\circ}}\,\frac{\mu}{\rho} = \frac{\mu}{\sqrt{\tau^\circ \rho}}$
- $v^+ = \frac{v}{v_*}$, where char velocity $\boxed{v_* = \sqrt{\frac{\tau^\circ}{\rho}}}$

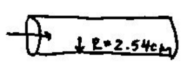
→ Velocity profile in layers



- laminar sublayer
  - $v^+ = y^+$ $\qquad y^+ \in (0,5)$
- buffer layer
  - $v^+ = 5\ln(y^+ + 0.205) - 3.27$ $\qquad y^+ \in (5,30)$
- turbulent core
  - $v^+ = 2.5\ln(y^+) + 5.5$ $\qquad y^+ \in (30,\infty)$

→ Example

EX2 Calculate thickness of laminar sublayer and buffer layer flow in a pipe of 2 inch diameter under a pressure gradient of $\frac{P_0 - P_L}{L} = 0.05$ psi/ft $= 113.1$ dyne/cm^3



Known: water @ 20°C
- $\rho = 1\,g/cm^3$
- $\mu = 0.01\,g/cm \cdot s$

- laminar layer thickness $\quad y = y^+ y_* = 5 y_* \qquad (y^+ = 5$ at boundary$)$

$$\tau^\circ = \left[\frac{P_0 - P_L}{L}\right]\frac{R}{2} = 113.1\left(\frac{2.54}{2}\right) = 143.6 \text{ dyne/cm}^2$$

$$v_* = \sqrt{\frac{\tau^\circ}{\rho}} = \sqrt{\frac{143.6}{1.0}} = 12.0 \text{ cm/s}$$

$$y_* = \frac{\mu}{v_* \rho} = \frac{0.01}{(12.0)(1)} = 8.33\times10^{-4} \text{ cm}$$

$$y = 5y_* = 5(8.33\times10^{-4}\text{cm}) = 4.17\times10^{-3}\text{ cm} \quad \Rightarrow \text{small!}$$

- buffer layer + laminar sublayer $\quad y = y^+ y_* = 30 y_* \qquad (y^+ = 30$ at boundary, laminar contribution negligible$)$

$$y = 30y_* = 30(8.33\times10^{-4}\text{cm}) = 2.50\times10^{-2}\text{ cm} \quad \Rightarrow \text{still small!}$$

- Although laminar flow is small region near the wall and doesn't contribute to overall flow, but there's where the resistance to heat and mass transfer is the most, since in turbulent region it's more efficient.

→ Eddy viscocity (Boussinesq)

- $\bar{\tau}_{yz}^{total} = \bar{\tau}_{yz}^{(v)} + \tau_{yz}^{(t)}$

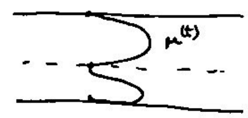
$$= \underbrace{-\mu \frac{dv_z}{dy}}_{\text{dominates near wall}} + \underbrace{\rho \overline{v_y' v_z'}}_{\text{dominates away from wall}}$$

- Boussinesq proposed eddy viscocity, although it depends on flow condition

  - $\rho \overline{v_y' v_z'} = -\mu^{(t)} \frac{dv_z}{dy}$

  - $\bar{\tau}_{yz}^{total} = -\mu \frac{dv_z}{dy} - \mu^{(t)} \frac{dv_z}{dy}$

  $\Rightarrow \mu^{(t)} = -\frac{\bar{\tau}_{yz}^{total}}{\left(\frac{dv_z}{dy}\right)} - \mu = -\frac{\left[\frac{P_0 - P_L}{L}\right]\frac{r}{2}}{\frac{dv_z}{dy}} - \mu$



II.E Dynamic Similarity & Dimensional Analysis

II.E.1 Flow around a sphere outside Stoke's law ($Re \geq 0.1$)

- For large Re, Stoke's law derived from N-S equation using top down approach deviates from reality. We therefore need experiments to solve problems.
- We can scale experiments to use easy systems to model large systems.

→ Nondimensionalizing N-S eqn.

→ characteristic properties

- char length l_0
- char velocity v_0
- char pressure $p_0 = \rho v_0^2$
- char time $t_0 = \frac{l_0}{v_0}$
- char gravity g

→ Nondimensionalized properties

- $\check{x} = \frac{x}{l_0}$ $\qquad x = l_0 \check{x}$
- $\check{v}_x = \frac{v_x}{v_0}$ $\qquad v_x = v_0 \check{v}_x$
- $\check{p} = \frac{p}{\rho v_0^2}$ $\qquad p = \rho v_0^2 \check{p}$
- $\check{t} = \frac{t}{t_0} = \frac{t v_0}{l_0}$ $\qquad t = \frac{\check{t} l_0}{v_0}$
- $\check{g}_x = \frac{g_x}{g}$ $\qquad g_x = g \check{g}_x$
- $\check{\nabla} \cdot = l_0 \nabla \cdot$ $\qquad \nabla \cdot = \frac{1}{l_0} \check{\nabla} \cdot$
- $\check{\nabla}^2 = l_0^2 \nabla^2$ $\qquad \nabla^2 = \frac{1}{l_0^2} \check{\nabla}^2$
- $\check{F}_0 = \frac{F_0}{\frac{1}{2}\rho v_0^2 A} = c_D = f$ $\qquad F_0 = \frac{1}{2}\rho v_0^2 A \; \check{F}_0$

→ Continuity Eqn

$$\nabla \cdot \underset{\sim}{v} = 0$$

$$\frac{v_0}{l_0} \check{\nabla} \cdot \underset{\sim}{\check{v}} = 0$$

$$\boxed{\check{\nabla} \cdot \underset{\sim}{\check{v}} = 0}$$

→ x-component of momentum eqn

$$\rho \frac{Dv_x}{Dt} = -\frac{\partial p}{\partial x} + \mu \nabla^2 v_x + \rho g_x$$

$$\frac{\rho v_0^2}{l_0} \frac{D\check{v}_x}{D\check{t}} = -\frac{\rho v_0^2}{l_0} \frac{\partial \check{p}}{\partial \check{x}} + \mu \frac{v_0}{l_0^2} \check{\nabla}^2 \check{v}_x + \rho g \check{g}_x$$

$$\frac{D\check{v}_x}{D\check{t}} = -\frac{\partial \check{p}}{\partial \check{x}} + \frac{\mu}{l_0 v_0 \rho} \check{\nabla}^2 \check{v}_x + \frac{l_0 g}{v_0^2} \check{g}_x$$

$$\boxed{\frac{D\check{v}_x}{D\check{t}} = -\frac{\partial \check{p}}{\partial \check{x}} + \frac{1}{Re} \check{\nabla}^2 \check{v}_x + \frac{1}{Fr} \check{g}_x}$$

→ Dimensionless groups

> Reynold's number - $\boxed{Re = \frac{l_0 v_0 \rho}{\mu}} = \frac{\text{inertial forces}}{\text{viscous forces}} = \frac{\rho v_0^2 l_0^2}{\mu v_0 l_0}$

> Froude number - $\boxed{Fr = \frac{v_0^2}{g l_0}} = \frac{\text{inertial forces}}{\text{gravitational forces}} = \frac{\rho v_0^2 l_0^2}{\rho g l_0^3}$

> Capillary number - $\boxed{Ca = \frac{\mu v_0}{\sigma}} = \frac{\text{viscous forces}}{\text{surface tension forces}}$

> Weber number - $\boxed{We = \frac{l_0 \rho v_0^2}{\sigma}} = \frac{\text{inertial forces}}{\text{surface tension forces}}$

σ = surface tension

$[\sigma]$ $\frac{\text{Force}}{\text{Length}}$

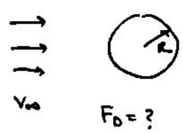
→ Dynamic Similarity

> geometric similarity – length ratio & geometry are the same

> dynamic similarity – two systems which $\begin{cases} \cdot \text{ are geometrically similar} \\ \cdot \text{ have same relavent force ratios} \end{cases}$

→ Example

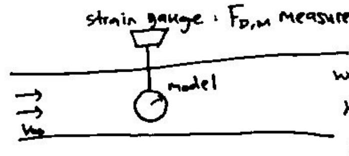
· Design experiment to measure the drag force F_D on a sphere below.



air 25°C, 1atm
$\nu = 15.05\ cm^2/s$
$R = 5m = 500cm$
$V_\infty = 3m/s$

Desired situation $\rho = 1.19\times10^{-3}\ g/cm^3$

strain gauge : $F_{D,M}$ measured : $F_{D,M} = 7400$ dyne



water 20°C
$\nu = 0.01 cm^2/s$
$\rho_M = 1\ g/cm^3$

experiment situation

· Verify stoke's law cannot be used ($Re \geq 0.1$)

$$Re = \frac{DV_\infty}{\nu} = \frac{(1000cm)(300cm/s)}{15.05\ cm^2/s} = 19600 \gg 0.1$$

· Require same force ratio : $Re_M = Re$

$$\frac{2R_M (V_\infty)_M}{\nu_M} = \frac{2RV_\infty}{\nu}$$

$$(V_\infty)_M = V_\infty\left(\frac{R}{R_M}\right)\left(\frac{\nu_M}{\nu}\right) = (300cm/s)\left(\frac{500cm}{10cm}\right)\left(\frac{0.01\ cm^2/s}{15.05\ cm^2/s}\right) = 10\ cm/s$$

· Non-dimensionalize drag force

$$\check{F}_D = c_D = f = \frac{F_D}{(\text{scaling pressure})(\text{approach area})} = \frac{F_D}{(\frac{1}{2}\rho V_\infty^2)(\pi R^2)}$$

(projected area facing flow) ↳ in this case, circle

> drag coefficient c_D (friction factor f) – $\boxed{c_D = \frac{F_D}{\frac{1}{2}\rho V_\infty^2 \pi R^2}}$ of a sphere $\boxed{C_D = \frac{2F_D}{\rho V_\infty^2 A}}$ generally.

· Require same force ratio : $c_D = c_{DM}$

$$\frac{F_D}{\frac{1}{2}\rho V_\infty^2 \pi R^2} = \frac{F_{DM}}{\frac{1}{2}\rho_M (V_{\infty})_M^2 \pi R_M^2}$$

$$F_D = F_{DM}\left(\frac{R}{R_M}\right)^2\left(\frac{\rho}{\rho_M}\right)\left(\frac{V_\infty}{V_{\infty M}}\right)^2$$

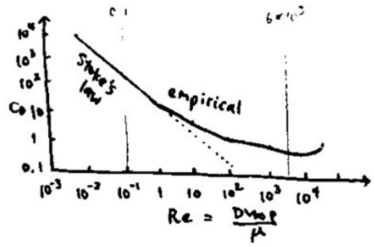
$$= 7400\ \text{dyne}\left(\frac{500cm}{10cm}\right)^2\left(\frac{1.19\times10^{-3}\ g/cm^3}{1\ g/cm^3}\right)\left(\frac{3.0\ m/s}{0.1\ m/s}\right)^2$$

$$= 18682\ \text{dyne}$$

$\boxed{\text{II.E.2}}$ Terminal velocity of falling sphere outside Stoke's law ($Re \geq 0.1$)

- We have experimental data of a sphere's drag coefficient C_D vs. Reynolds number Re.
  - Stoke's law region : $Re < 0.1$

$$C_D = \frac{F_D}{\frac{1}{2}\rho V_\infty^2 \pi R^2} = \frac{6\pi\mu R V_\infty}{\frac{1}{2}\rho V_\infty^2 \pi R^2} = \frac{12\mu}{\rho V_\infty R} = 24\left(\frac{\mu}{D V_\infty \rho}\right) = \frac{24}{Re}$$



  - Empirical rule : $Re \in [0.1, 6\times10^3]$

$$C_D = \left(\sqrt{\frac{24}{Re}} + 0.5407\right)^2$$

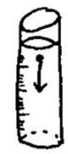
→ Example

► Calculate terminal velocity V_∞ for sphere in water.

$$F\downarrow = F\uparrow$$

gravity − buoyancy = drag force

$$\frac{4}{3}\pi R^3(\rho_s - \rho)\,g = C_D \frac{1}{2}\pi R^2 \rho V_\infty^2$$



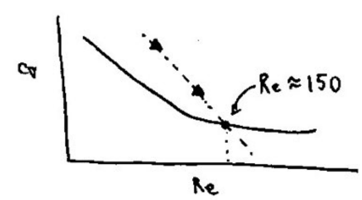
$R = 0.05\,cm$
$\rho = 1.0\,g/cm^3$
$\rho_s = 2.5\,g/cm^3$
$\mu = 0.01\,g/cm\cdot s$
$g = 980\,cm/s^2$

- Rearrange C_D in terms of Re

$$C_D = \frac{8R(\rho_s-\rho)g}{3\rho V_\infty^2} = \frac{32R^3\rho(\rho_s-\rho)g}{3\mu^2 Re^2} = \frac{const}{Re^2} = \frac{32(0.05)^3(1.0)(2.5-1.0)(980)}{3(0.01)^2 Re^2} = \frac{19600}{Re^2}$$

where $Re = \frac{2RV_\infty\rho}{\mu}$

- Plot $C_D(Re)$ in the experimental data plot. Find Re at intersection, then solve for V_∞. Alternatively, use the empirical rule to solve for Re.
  - $Re \approx 150 \Rightarrow V_\infty = \frac{\mu Re}{2R\rho} = 15\,cm/s$

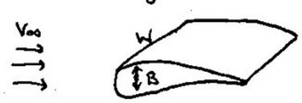


  - $\boxed{\text{OR}}$ $C_D = \frac{19600}{Re^2} = \left(\sqrt{\frac{24}{Re}} + 0.5407\right)^2$

root $Re = 148.9 \Rightarrow V_\infty = 14.9\,cm/s$

$\boxed{\text{II.E.3}}$ Drag forces on other submerged objects

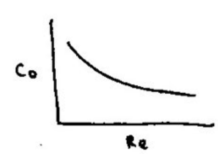
- We measure drag force F_D and plot drag coeff. vs. Reynolds number (C_D vs. Re)

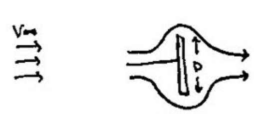


air foil

$C_D = C_D(Re)$

$$C_D = \frac{F_D}{\frac{1}{2}\rho V_\infty^2 (BW)}$$





disk

cylinder

$\boxed{\text{II.E.4}}$ Classification of flow regimes

· Recognize momentum eqn (x-comp) of NS eqns.

$$\frac{D\check{v}_x}{Dt} = \frac{\partial \check{v}_x}{\partial \check{t}} + \underbrace{\check{v}\cdot\check{\nabla}\check{v}_x}_{\substack{\text{inertial}\\\text{term}}} = -\frac{\partial \check{p}}{\partial \check{x}} + \underbrace{\frac{1}{Re}\check{\nabla}^2\check{v}_x}_{\substack{\text{viscous}\\\text{term}}} + \frac{1}{Fr}\check{g}_x$$

· Creeping flow : $Re \to 0$

- · inertial term $\to 0$
- · linear differential eqn

$$\frac{\partial \check{v}_x}{\partial \check{t}} = -\frac{\partial \check{p}}{\partial \check{x}} + \frac{1}{Re}\check{\nabla}^2\check{v}_x + \frac{1}{Fr}\check{g}_x$$

· Ideal / Inviscid flow ($\mu = 0$) : $Re \to \infty$

- · viscous term $\to 0$
- · nonlinear diff. eqn : turbulence

$$\frac{\partial \check{v}_x}{\partial \check{t}} + \check{v}\cdot\check{\nabla}\check{v}_x = -\frac{\partial \check{p}}{\partial \check{x}} + \frac{1}{Fr}\check{g}_x$$

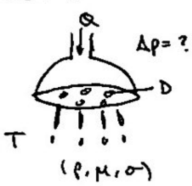
$\boxed{\text{II.E.5}}$ Dimensional analysis

→ Buckingham π theorem

> A function $f(X_1, X_2, \ldots, X_k)$ with dimensional variables X_i can be rewritten in a function $\Phi(\Pi_1, \Pi_2, \ldots, \Pi_{k-n})$ with dimensionless variables Π_j by enforcing dimensional consistency using n fundamental dimensions.

→ Example

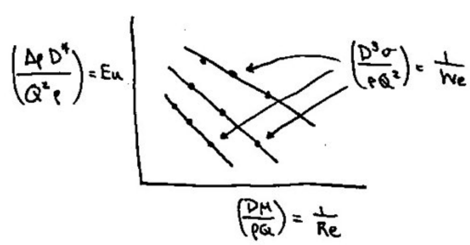
► Determine pressure drop Δp required to operate spray nozzles of different office diameters D at different volumetric flow rates Q for different liquids (ρ, μ, σ).



$$\Delta p = \Delta p(D, Q, \rho, \mu, \sigma)$$ – too many variables & experiments!

· Express variables in terms of fundamental dimensions : mass M, length L, time T

- · $\Delta p \;[=]\; \frac{\text{force}}{\text{area}} = \frac{ML}{T^2}\cdot\frac{1}{L^2} = \frac{M}{LT^2}$
- · $D \;[=]\; L$
- · $Q \;[=]\; \frac{L^3}{T}$
- · $\rho \;[=]\; \frac{M}{L^3}$
- · $\mu \;[=]\; \frac{M}{L\cdot T}$
- · $\sigma \;[=]\; \frac{\text{force}}{\text{length}} = \frac{ML}{T^2}\frac{1}{L} = \frac{M}{T^2}$



· Assume result can put in the form and enforce dim. consistency

$$\Delta p \;[=]\; K D^a Q^b \rho^c \mu^d \sigma^e$$

$$\frac{M}{LT^2} \;[=]\; L^a \left(\frac{L^3}{T}\right)^b \left(\frac{M}{L^3}\right)^c \left(\frac{M}{LT}\right)^d \left(\frac{M}{T^2}\right)^e$$

· Set up systems of eqn, solve in terms of d, e

- · M $\quad 1 = c + d + e$
- · L $\quad -1 = a + 3b - 3c - d$
- · T $\quad -2 = -b - d - 2e$

$$\Rightarrow \begin{cases} c = 1 - d - e \\ b = 2 - d - 2e \\ a = -4 + d + 3e \end{cases}$$

$$\Delta p = K\frac{Q^2\rho}{D^4}\left(\frac{D\mu}{Q\rho}\right)^d\left(\frac{D^3\sigma}{Q^2\rho}\right)^e$$

$$\left(\frac{\Delta p D^4}{Q^2\rho}\right) = K\left(\frac{D\mu}{Q\rho}\right)^d\left(\frac{D^3\sigma}{Q^2\rho}\right)^e$$

measure dimensionless groups (less variables!)

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→ Stand in variables

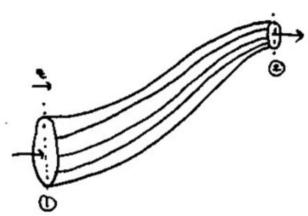
- Solving for systems of eqn is tedious for dim analysis. We can instead choose stand in variables for fundamental dimensions and assemble dimensionless groups. in terms of stand in.
- fundamental dim : mass M, length L, temperature T
  - Δp [=] $\frac{M}{LT^2}$ ← $\Delta p \cdot \frac{1}{\rho} \cdot \frac{1}{Q^2} \cdot D^4 = \left(\frac{(\Delta p) D^4}{\rho Q^2}\right)$; $\frac{\cancel{M}}{\cancel{L}\cancel{T^2}} \frac{L^{\cancel{3}}}{\cancel{M}} \frac{\cancel{T^2}}{L^6} \cdot L^4$
  - D [=] L ← stand in for L
  - Q [=] $\frac{L^3}{T}$ ← stand in for T
  - ρ [=] $\frac{M}{L^3}$ ← stand in for M
  - μ [=] $\frac{M}{LT}$ ← $\mu \cdot \frac{1}{\rho} \cdot \frac{1}{Q} \cdot D = \left(\frac{\mu D}{\rho Q}\right)$; $\frac{\cancel{M}}{\cancel{L}\cancel{T}} \cdot \frac{L^{\cancel{3}}}{\cancel{M}} \cdot \frac{\cancel{T}}{L^3} \cdot L$
  - σ [=] $\frac{M}{T^2}$ ← $\sigma \cdot \frac{1}{\rho} \cdot \frac{1}{Q^2} \cdot D^3 = \left(\frac{D^3 \sigma}{\rho Q^2}\right)$; $\frac{\cancel{M}}{\cancel{T^2}} \cdot \frac{\cancel{L^3}}{\cancel{M}} \cdot \frac{\cancel{T^2}}{L^{6}} \cdot L^3$
- We therefore have dimless groups to construct equation :

$$\left(\frac{(\Delta p) D^4}{\rho Q^2}\right) = K \left(\frac{\mu D}{\rho Q}\right)^a \left(\frac{D^3 \sigma}{\rho Q^2}\right)^b$$

$$Eu = K \left(\frac{1}{Re}\right)^a \left(\frac{1}{We}\right)^b \quad , \quad \boxed{Eu \equiv \frac{(\Delta p) D^4}{\rho Q^2}} \text{ Euler's number}$$

II.F Bernoulli Analysis & Applications

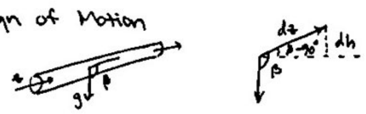
II.F.1 N-S eqn for steady flow in "stream tubes"



> stream tube – conceptual tube bounded by streamlines
  > streamline – lines drawn in the flow everywhere tangent to $\underset{\sim}{v}$
  - no flow crosses streamline by definition ⇒ inpenetrable!

→ Continuity eqn
- Assume constant ρ fluid
- $Q_1 = Q_2$
  $A_1 \langle v \rangle_1 = A_2 \langle v \rangle_2$

→ Eqn of Motion



$$dh = dz \sin(\beta - 90°) = dz(-\cos\beta) = -dz\cos\beta \Rightarrow \cos\beta = -\frac{dh}{dz}$$

$$\rho\left(\cancel{\frac{\partial v_z}{\partial t}} + v_r \cancel{\frac{\partial v_z}{\partial r}} + \frac{v_\theta}{r}\cancel{\frac{\partial v_z}{\partial \theta}} + v_z \frac{\partial v_z}{\partial z}\right) = -\frac{\partial p}{\partial z} - \left[\cancel{\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rz})} + \cancel{\frac{1}{r}\frac{\partial}{\partial \theta}\tau_{\theta z}} + \cancel{\frac{\partial}{\partial z}\tau_{zz}}\right] + \rho g_z$$

$$\rho v \frac{dv}{dz} = -\frac{dp}{dz} + \rho g \cos\beta$$

$$\rho v \frac{dv}{dz} = -\frac{dp}{dz} - \rho g \frac{dh}{dz} \quad \leftarrow \frac{\text{momentum}}{\text{volume}}$$

← Assumptions
- 1D flow in z direction : $v_r = v_\theta = 0$
- no sharp bend
- plug flow : uniform v_z across cross section : $v_z = v_z(z)$
- inviscid flow : $\mu \approx 0$, $Re \geq 10000$

II.F.2 Bernoulli Equation

- Bernoulli eqn is an energy eqn (can also be in head form)
- From eqn of motion simplified by Bernoulli analysis,

$$\int_{v_1}^{v_2} \rho v \, dv = -\int_{p_1}^{p_2} dp - \int_{h_1}^{h_2} \rho g \, dh$$

$$\frac{1}{2}\rho(v_2^2 - v_1^2) + (p_2 - p_1) + \rho g (h_2 - h_1) = 0 \quad \rightarrow \quad \boxed{p_1 + \frac{1}{2}\rho v_1^2 + \rho g h_1 = p_2 + \frac{1}{2}\rho v_2^2 + \rho g h_2}$$

$$\frac{v_2^2 - v_1^2}{2g} + \frac{p_2 - p_1}{\rho g} + (h_2 - h_1) = 0 \quad \Rightarrow \quad \boxed{\frac{v_1^2}{2g} + \frac{p_1}{\rho g} + h_1 = \frac{v_2^2}{2g} + \frac{p_2}{\rho g} + h_2}$$

Bernoulli Equation head form

- energy per unit weight = length

- head terms add to constant

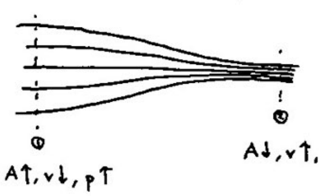
$$\underbrace{\frac{v^2}{2g}}_{\text{kinetic head}} + \underbrace{\frac{p}{\rho g}}_{\text{pressure head}} + \underbrace{h}_{\text{gravity head}} = \underbrace{\text{constant } B}_{\text{Bernoulli head}}$$

II.F.3 The Bernoulli Effect

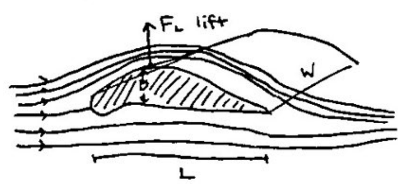
> At same height, increase in speed of fluid occurs with decrease in fluid pressure.

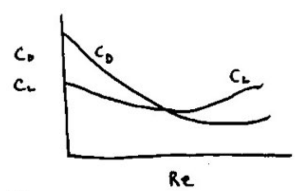
$$\frac{v^2 \uparrow}{2g} + \frac{p \downarrow}{\rho g} + h = \text{constant}$$

- Note by continuity eqn, $A \downarrow$, $v \uparrow$
- Streamline is more dense in higher speed region.
- e.g. atomizer



→ Aerodynamic Lift





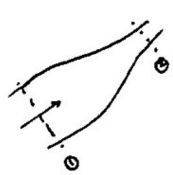
$$C_D = \frac{F_D}{\frac{1}{2}\rho v_\infty^2 A_{approach}}$$

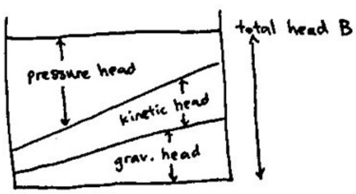
$$C_L = \frac{F_L}{\frac{1}{2}\rho v_\infty^2 A_{planform}}$$

> lift coefficient – $\boxed{C_L = \frac{F_L}{\frac{1}{2}\rho v_\infty^2 A_{planform}}}$ is function of Re

- here, $C_L = \frac{F_L}{\frac{1}{2}\rho v_\infty^2 LW}$

→ Slanted, converging conduit

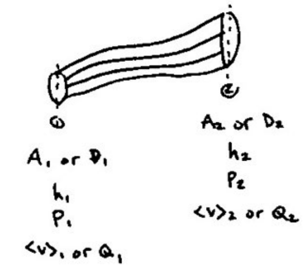




$$B = \frac{v^2}{2g} + \frac{p}{\rho g} + h$$

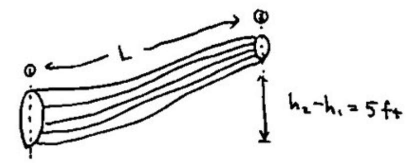
II.F.4 Applications of Simple Bernoulli Analysis

- Define a stream tube
  - end sections located where streamlines are straight & parallel
  - simple analysis does not concern anything in between
    - only end sections!
  - can solve any 2 variables using continuity eqn & Bernoulli eqn



II.F.4.a Flow in a contracting conduit

► Find $\Delta p = p_1 - p_2$



$$\frac{\langle v\rangle_1^2}{2g} + \frac{p_1}{\rho g} + h_1 = \frac{\langle v\rangle_2^2}{2g} + \frac{p_2}{\rho g} + h_2$$

$$\Delta p = \tfrac{1}{2}\rho\left[\langle v\rangle_2^2 - \langle v\rangle_1^2\right] + \rho g(h_2 - h_1)$$

$$= \tfrac{1}{2}\rho\left[\left(\frac{4Q}{\pi D_2^2}\right)^2 - \left(\frac{4Q}{\pi D_1^2}\right)^2\right] + \rho g(h_2 - h_1)$$

$$\boxed{\Delta p = \frac{8\rho Q^2}{\pi^2 D_1^4}\left[\left(\frac{D_1}{D_2}\right)^4 - 1\right] + \rho g(h_2 - h_1)}$$

$\rho = 1\,g/cm^3 = 62.4\ lbm/ft^3$

$D_1 = 8'' = 8in$

$D_2 = 4'' = 4in$

$L = 10' = 10ft$ (not needed for simple analysis)

$Q = 1.0\ ft^3/s$

$g = 32.2\ ft/s^2$

$$\Delta p = \frac{8(62.4\ lbm/ft^3)(1.0\ ft^3/s)^2}{\pi^2(8in)^4\left(\frac{1ft}{12in}\right)^4}\left[\left(\frac{8in}{4in}\right)^4 - 1\right] + (62.4\ lbm/ft^3)(32.2\ ft/s^2)(5ft)$$

$$= (3841 + 10046)\ \frac{lbm}{ft\cdot s^2}\cdot\frac{ft}{ft}$$

$$= 13887\ \frac{pd}{ft^2}\cdot\frac{lb_f}{32.2pd}\cdot\frac{1ft^2}{144in^2}$$

$$= 3\ psi$$

→ British units

- $g = 32.2\ ft/s^2$
- Force [=] $1\ \frac{lbm\cdot ft}{s^2} = 1$ poundal (pd)

$$\boxed{\begin{array}{l} 1\ lb_f = 32.2\ pd \\ g_c = 32.2\ \frac{pd}{lb_f} \end{array}}$$

- Pressure [=] $\frac{pd}{ft^2}$

$$\boxed{1\ psi = \frac{pd}{ft^2}\ \frac{lb_f}{32.2\ pd}\ \frac{1\ ft^2}{144\ in^2}}$$

$1\ atm = 14.696\ psi$

- power [=] $\frac{ft\cdot pd}{s}$

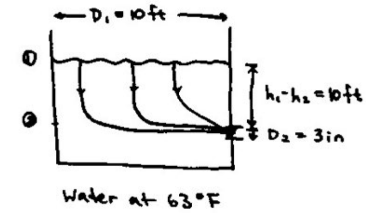
$$\boxed{1\ hp = 17696\ \frac{ft\cdot pd}{s}}$$

- volumetric flow rate [=] $\frac{ft^3}{s}$

$$\boxed{1\ ft^3 = 7.48\ gal}$$

II.F.4.b Efflux from large vessel due to gravity

- Find efflux rate Q.



$$\cancel{\frac{\langle v\rangle_1^2}{2g}} + \cancel{\frac{p_1}{\rho g}} + h_1 = \frac{\langle v\rangle_2^2}{2g} + \cancel{\frac{p_2}{\rho g}} + h_2$$

$\langle v\rangle_1 \approx 0$ since $D_1 \gg D_2$.

atmospheric pressure at both ends cancel

$$\boxed{\langle v\rangle_2 = \sqrt{2g(h_1-h_2)}}$$ Torricelli's law

$$Q = A_2\langle v\rangle_2 = \frac{\pi D_2^2}{4}\sqrt{2g(h_1-h_2)}$$

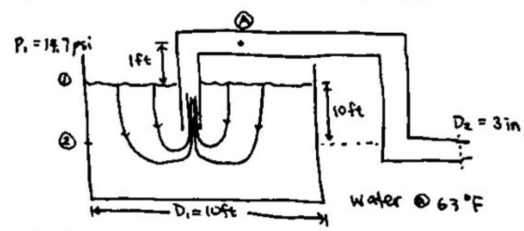
$$= \frac{\pi}{4}(3in)^2\left(\frac{1ft}{12in}\right)^2\sqrt{2(32.2\,ft/s^2)(10ft)} = 1.25\ ft^3/s$$

II.F.4.c Flow through a siphon from large vessel

- Find efflux rate Q

  - Simple Bernoulli analysis does not care what happens in between, it only cares about end sections. Since end sections are the same as previous, we have $Q = 1.25\,ft^3/s$.

- Find pressure at A p_A in siphon



$$p_A = p_1 - \frac{1}{2}\rho\left(\langle v\rangle_A^2 - \cancel{\langle v\rangle_1^2}^{\,0}\right) - \rho g(h_A - h_1)$$

$$= p_1 - \frac{1}{2}\rho\left(\frac{4Q}{\pi D_2^2}\right)^2 - \rho g(h_A - h_1)$$

$$= p_1 - \frac{1}{2}\left(62.4\frac{lbm}{ft^3}\right)\left(\frac{4(1.25\,ft^3/s)}{\pi(3in)^2\left(\frac{1ft}{12in}\right)^2}\right)^2 - (62.4\,lbm/ft^3)(32.2\,ft/s^2)(1ft)$$

$$= p_1 - (20232 + 2009)\frac{pd}{ft^2}\cdot\frac{lb_f}{32.2\,pd}\times\left(\frac{ft}{12in}\right)^2$$

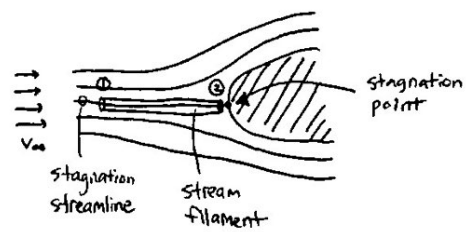
$$= p_1 - 4.80\ psi$$

$$= (14.7 - 4.80)psi$$

$$= 9.90\ psia$$

II.F.4.d Pressure at stagnation point

- Find pressure at stagnation point ② p_2.

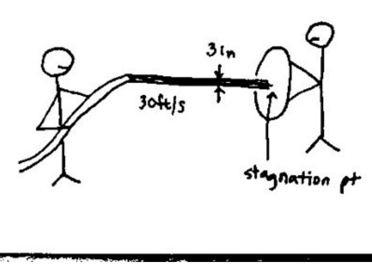


$$\frac{v_1^2}{2g} + \frac{p_1}{\rho g} + \cancel{h_1} = \cancel{\frac{v_2^2}{2g}}^{\,0} + \frac{p_2}{\rho g} + \cancel{h_2}$$

$$p_2 = p_1 + \frac{1}{2}\rho v_1^2 = \underbrace{p_1}_{\text{static pressure}} + \underbrace{\frac{1}{2}\rho v_\infty^2}_{\text{dynamic pressure}}$$

→ Example

- Calculate force on the plate



$$F = pA = \frac{1}{2}\rho V_\infty^2\frac{\pi D^2}{4}$$

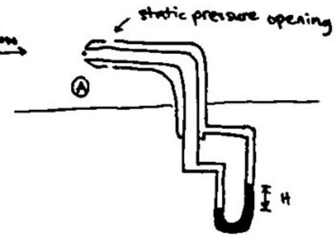
$$= \frac{1}{2}(62.4\,lbm/ft^3)(30ft/s)^2\frac{\pi}{4}(3in)^2\left(\frac{ft}{12in}\right)^2$$

$$= 1378\,pd\cdot\frac{1\,lb_f}{32.2\,pd}$$

$$= 42.8\ lb_f$$

I.F.5 Measurement of Flow Rate : Analysis of Flow-Metering Device

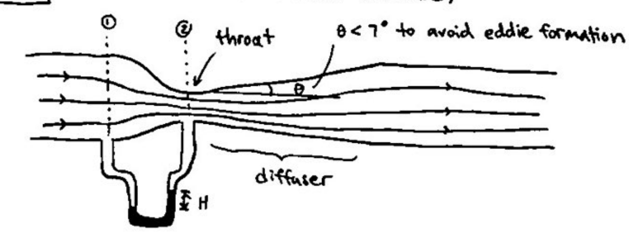
II.F.5.a Pitot Tube (local velocity)



- Measures $\Delta p = p_A - p_{static}$
  - $\Delta p = \frac{1}{2}\rho v^2$ dynamic pressure
  - $\Delta p = (\rho_m - \rho) g H$ manometer eqn
- Local velocity $\boxed{v = \sqrt{\frac{2\Delta p}{\rho}} = \sqrt{\frac{2(\rho_m - \rho) g H}{\rho}}}$

> Pitot traverse – measure velocity profile point by point locally in conduit

II.F.5.b Venturi Meter (volumetric flow rate)



$$\frac{\langle v\rangle_1^2}{2g} + \frac{p_1}{\rho g} + \cancel{h_1} = \frac{\langle v\rangle_2^2}{2g} + \frac{p_2}{\rho g} + \cancel{h_2}$$

$$\frac{8Q^2}{\pi^2 D_1^2} + \frac{p_1}{\rho} = \frac{8Q^2}{\pi^2 D_2^2} + \frac{p_2}{\rho}$$

$Q = A\langle v\rangle = \pi R^2 \langle v\rangle = \frac{\pi D^2 \langle v\rangle}{4}$

$\langle v\rangle = \frac{4Q}{\pi D^2}$

$D_0 \equiv D_2$ throat diameter

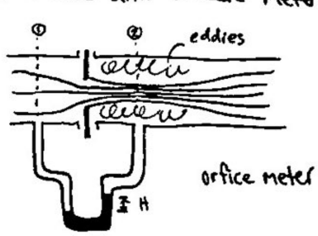
$D \equiv D_1$ pipe diameter

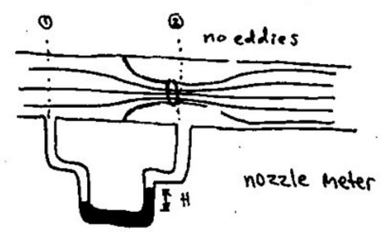
$$\boxed{Q = C_0 \pi D_0^2 \sqrt{\frac{\Delta p}{8\rho\left[1 - \left(\frac{D_0}{D}\right)^4\right]}}}$$

- discharge coefficient C_0
  - $0.96 \le C_0 \le 0.98$
- Δp from manometer eqn
  - $\Delta p = (\rho_m - \rho) g H$

- Drawback : expensive, long (take up space)

II.F.5.c Orfice and Nozzle Meters



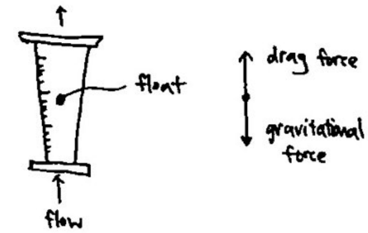


$$\boxed{Q = C_0 \pi D_0^2 \sqrt{\frac{\Delta p}{8\rho\left[1 - \left(\frac{D_0}{D}\right)^4\right]}}}$$

- orficemeter $C_0 \approx 0.4 - 0.8$
- nozzle meter $C_0 \approx 1$

- orficemeter is inexpensive and takes little space
  - it generates eddies so need calibration of C_0
- nozzle meter is expensive but takes less space

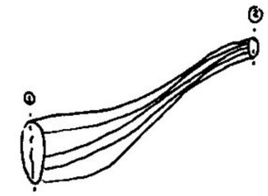
II.F.5.d Rotameter



- need vertical flow from bottom up
- have a float (usually a sphere) floating
- rotameter is calibrated specific to a fluid

I.F.6 Full Bernoulli Analysis for flow in Conduit

- Need to account for assumptions in simple Bernoulli analysis:
  1. inviscid flow : $\mu \approx 0$
  2. no sharp bend : straight streamline



- Head loss between ① and ② H_{L12} takes them into account:

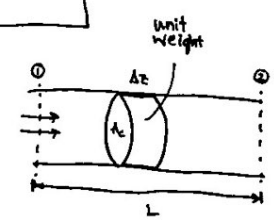
$$\frac{\langle v\rangle_1^2}{2g} + \frac{p_1}{\rho g} + h_1 = \frac{\langle v\rangle_2^2}{2g} + \frac{p_2}{\rho g} + h_2 + H_{L12}$$

- $H_{L12} = H_{L12f} + H_{L12c}$
  - > skin friction loss H_{L12f} - loss due to friction with pipe of fluid with viscocity
  - > configurational loss H_{L12c} - due to curved streamlines (tube bend, valves, fittings)

  } energy per unit weight

I.F.6.a Skin Friction Loss

> skin friction loss H_{L12f} - viscous work done per unit weight by fluid on the walls of the conduit in moving from ① to ②.



- $$H_{L12f} = \frac{(\text{viscous force})(\text{distance traveled})}{\text{weight}} = \frac{\tau_0 (\text{wetted area}) L}{\rho g (\text{volume})}$$

$$= \frac{\tau_0 (\text{wetted perimeter}) \Delta z L}{\rho g A_c \Delta z} \qquad \left(R_H \equiv \frac{\text{cross sectional area}}{\text{wetted perimeter}}\right)$$

- $$\boxed{H_{L12f} = \frac{\tau^\circ L}{\rho g R_H}}$$

- $$\boxed{H_{L12f} = \frac{4\tau^\circ L}{\rho g D}}$$ for tube where $R_H = \frac{D}{4}$

- Nondimensionalize skin friction as a function of Re.
  - > Fanning friction factor f - dimensionless skin friction, a drag coefficient inside conduit wall

$$\boxed{f = \frac{\tau^\circ}{\frac{1}{2}\rho\langle v\rangle^2}} \quad , \text{ note } \quad f = c_D = \frac{F_D}{\frac{1}{2}\rho\langle v\rangle^2 A} = \frac{\tau_0}{\frac{1}{2}\rho\langle v\rangle^2}$$

$$\tau^\circ = \frac{1}{2}\rho\langle v\rangle^2 f$$

- Substitute τ° into H_{L12f},

$$\boxed{H_{L12f} = \frac{2\langle v\rangle^2 L}{gD} f = \frac{32 Q^2 L}{\pi^2 D^5 g} f} \quad , \quad Q = \frac{\pi D^2 \langle v\rangle}{4} \text{ for circular pipes}$$

II.F.6.b Analytic Correlation of Friction Factor f in Conduits

→ Hydraulically smooth piping

- Blasius - $\boxed{f = \frac{0.0791}{Re^{0.25}}}$ $Re \in [2100, 10^5]$

- Koo - $\boxed{f = 0.0014 + \frac{0.125}{Re^{0.32}}}$ $Re \in [10^4, 10^7]$

→ Pipes of general roughness

- Haaland - $\boxed{\frac{1}{\sqrt{f}} = -3.6 \log_{10}\left[\frac{6.9}{Re} + \left(\frac{k/D}{3.7}\right)^{10/9}\right]}$ $Re \in [4\times10^4, 10^7]$, $k/D < 0.05$

→ Commercial standard piping

- Drew - $\boxed{f = 0.0014 + \frac{0.090}{Re^{0.27}}}$ $Re \in [10^4, 10^7]$, $k/D \geq 0.00015$

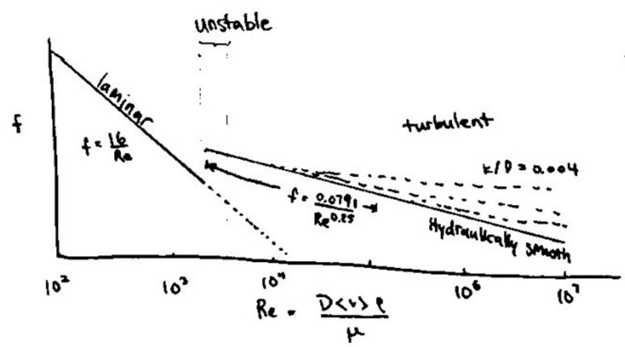
→ Full Rough conduit

- k extends into turbulent core

- Nikuradse & Reichert $\boxed{\frac{1}{\sqrt{f}} = 2.28 - 4.0 \log_{10}\left(\frac{k}{D}\right)}$ $Re > 10^4$, $k/D > 0.01$

  not dependent on Re

→ Graph



- Avoid unstable region at transition between laminar & turbulent

II.F.6.c Skin Friction Loss of Noncircular Cross Sections

- Substitute $D = 4R_H$ to $H_{L,sf}$

$$\boxed{H_{L,sf} = \frac{\langle v\rangle^2 L}{2g R_H} f = \frac{Q^2 L f}{2g A_c^2 R_H}}, \quad \langle v\rangle = \frac{Q}{A_c}$$

- f is estimated from round tube correlation above, with

$$\boxed{Re = \frac{4R_H \langle v\rangle \rho}{\mu}}$$

II.F.6.d Configurational Loss

- $$\boxed{H_{Lc} = e_v \frac{\langle v \rangle^2_{downstream}}{2g}}$$

> kinetic energy factor for fittings e_v
  - experimentally determined
  - dimensionless

- $$\boxed{H_{L12c} = \frac{\langle v \rangle^2_{down}}{2g}\left(\sum_i e_{v,i}\right) = \frac{8Q^2}{\pi^2 D^4 g}\left(\sum_i e_{v,i}\right)}$$

II.F.6.e Total Head Loss

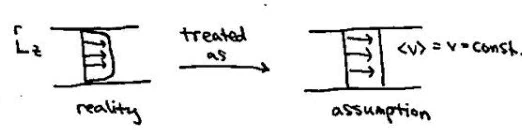
- $H_{L12} = H_{L12f} + H_{L12c}$

- $$\boxed{H_{L12} = \begin{cases} \dfrac{2\langle v \rangle^2}{Dg}\left[(\Sigma L_i)f + \dfrac{D}{4}(\Sigma e_{v,i})\right] \\[2ex] \dfrac{32Q^2}{\pi^2 D^5 g}\left[(\Sigma L_i)f + \dfrac{D}{4}(\Sigma e_{v,i})\right] \end{cases}}$$

- For piping system with diff. diameters
  - Assign fitting to downstream
  - use continuity eqn $Q_A = Q_B$ at end

II.F.7 Kinetic Head Correction Factor

- Need to account for assumption of plug flow : uniform $v_z = v_z(z)$, $\langle v \rangle = v = \text{const}$.
- Our appoximation was $\frac{\langle v \rangle^2}{2g}$
- Reality is $\frac{\text{Kinetic E}}{\text{weight}} = \frac{\text{rate of KE flow}}{\text{rate of weight flow}}$



- rate of KE flow $= \int \frac{v^2}{2g}\rho g v\, dA = \int \frac{v^2}{2g}\rho g v\, 2\pi r\, dr = \pi\rho\int_0^R v^3 r\, dr = \frac{1}{2}\rho\langle v^3\rangle \pi R^2$
- rate of weight flow $= \rho g Q = \rho g \pi R^2 \langle v \rangle$
- $\frac{\text{kinetic E}}{\text{weight}} = \frac{\frac{1}{2}\rho\langle v^3\rangle\pi R^2}{\rho g \pi R^2 \langle v \rangle} = \frac{\langle v^3 \rangle}{2g\langle v \rangle}$

- Derive correction factor α :

$$\alpha \frac{\langle v \rangle^2}{2g} = \frac{\langle v^3 \rangle}{2g\langle v \rangle}$$

$$\boxed{\alpha = \frac{\langle v^3 \rangle}{\langle v \rangle^3}}$$ depends on v profile

Ex1 For turbulent v profile $v(r) = v_{max}\left[1 - \frac{r}{R}\right]^{\frac{1}{n}}$,

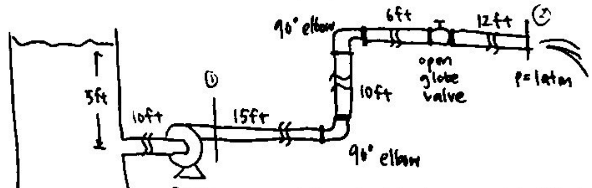
$$\alpha = \frac{\langle v^3 \rangle}{\langle v \rangle^3} = \frac{\left[\frac{1}{n+3} - \frac{1}{2n+3}\right]}{4n^2\left[\frac{1}{n+1} - \frac{1}{2n+1}\right]^3}$$

Re	n	α
$2\times10^3 - 10^4$	6	1.08
$10^4 - 10^5$	7	1.06
$10^5 - 10^7$	8	1.05

$\boxed{\text{II.F.7}}$ Evaluation of pressure drop, flow rate, piping requirements etc. using full Bernoulli analysis

$\boxed{\text{II.F.7.a}}$ Pressure drop required for specific flow rate

- Calculate gauge pressure at pump outlet to sustain a flow of $Q = 135$ gal/min



2-in schedule 40 wrought steel

$$\cancel{\alpha \frac{\langle v\rangle_1^2}{2g}} + \frac{p_1}{\rho g} + h_1 = \cancel{\alpha \frac{\langle v\rangle_2^2}{2g}} + \frac{p_2}{\rho g} + h_2 + H_{L12} \qquad (D_1 = D_2 \Rightarrow v_1 = v_2)$$

$$p_1 - p_2 = \rho g (h_2 - h_1) + \rho g H_{L12}$$

$$p_{1-g} = p_1 - p_2 = \rho g (h_2 - h_1) + \frac{32 \rho Q^2}{\pi^2 D^5} \left[(\Sigma L_i) f + \frac{D}{4} (\Sigma e_j) \right]$$

$g = 32.2$ pd/lbm

$\rho = 62.4$ lbm/ft^3

$\mu = 6.72 \times 10^{-4}$ lbm/(ft·s)

$Q = 135 \frac{\text{gal}}{\text{min}} \times \frac{1 \text{ ft}^3}{7.48 \text{ gal}} \times \frac{1 \text{ min}}{60 \text{ s}} = 0.30$ ft^3/s

$D = 2.067 \text{ in} \times \frac{1 \text{ ft}}{12 \text{ in}}$

$\Sigma L_i = (15 + 10 + 12 + 6)$ ft $= 43$ ft

$\Sigma e_j = 2(90°\text{ elbow}) + 1(\text{open globe valve})$

$= 2(0.75) + 1(6.0) = 7.5$

$$Re = \frac{D\langle v\rangle\rho}{\mu} = \frac{4Q\rho}{\pi D \mu} = \frac{4(0.30)(62.4)}{\pi \left(\frac{2.067}{12}\right)(6.72 \times 10^{-4})} = 2.06 \times 10^5$$

$$f = 0.0014 + \frac{0.0090}{Re^{0.27}} = 0.00471 \qquad \text{(Drew)}$$

$h_2 - h_1 = 10$ ft

British Units
M [=] lbm
L [=] ft
T [=] s
F [=] pd
p [=] pd/ft^2

$$p_1 - p_2 = (62.4)(32.2)(10) + \frac{32(62.4)(0.30)^2}{\pi^2 \left(\frac{2.067}{12}\right)^5} \left[(43)(0.00471) + \frac{1}{4}\left(\frac{2.067}{12}\right)(7.5) \right]$$

$$= 20092 + 62968 = 83060 \frac{\text{pd}}{\text{ft}^2} \times \frac{1 \text{ lb}_f}{32.2 \text{ pd}} \times \left(\frac{1 \text{ ft}}{12 \text{ in}}\right)^2$$

$$= 17.9 \text{ psig}$$

II.F.7.b Pump horse power required for specific flow rate

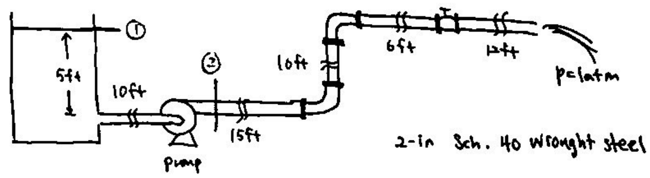
- Calculate pumphorsepower required to sustain $Q = 0.30\ ft^3/s$

  > pump head H_p – increase in head done by pump, at left side of Bernoulli eqn (developed head)

  > brake horsepower – $bhp = \dfrac{\text{power delivered to fluid by pump}}{\text{pump efficiency}} = \dfrac{P}{\eta}$

$$= \frac{1}{\eta}\left(\frac{\text{work done}}{\text{weight of fluid}}\right)\left(\frac{\text{weight of fluid}}{\text{time}}\right) = \frac{1}{\eta}(H_p)(\rho g Q)$$

$$\boxed{bhp = \frac{H_p \rho g Q}{\eta}} \qquad \text{unit : } \frac{ft\cdot pd}{s} \text{ or } hp$$



Assume $\eta = 0.7$

$$\underbrace{\alpha \frac{\langle v\rangle_1^2}{2g}}_{0 \text{ steady state}} + \frac{p_1}{\rho g} + h_1 + H_p = \underbrace{\alpha \frac{\langle v\rangle_2^2}{2g}}_{\alpha \frac{8Q^2}{\pi^2 D^4 g}} + \frac{p_2}{\rho g} + h_2 + \underbrace{H_{L12}}_{\frac{32Q^2}{\pi^2 D^5 g}\left[(\Sigma L_i)f + (\Sigma e_j)\frac{D}{4}\right]}$$

$\rightarrow$ combine $\leftarrow$

$$\downarrow$$

$$\frac{32Q^2}{\pi^2 D^5 g}\left[(\Sigma L_i)f + \frac{D}{4}(\alpha + \Sigma e_j)\right]$$

Rearrange for H_p,

$$H_p = \frac{p_2 - p_1}{\rho g} + (h_2 - h_1) + \frac{32Q^2}{\pi^2 D^5 g}\left[(\Sigma L_i)f + \frac{D}{4}(\alpha + \Sigma e_j)\right]$$

$p_2 - p_1 = 83060\ pd/ft^2$

$\rho = 62.4\ lbm/ft^3$

$g = 32.2\ pd/lbm$

$h_2 - h_1 = -5\ ft$

$Q = 0.30\ ft^3/s$

$D = \frac{2.064}{12}\ ft$

$\Sigma L_i = 10\ ft$

$Re = 2.06\times10^5 \Rightarrow f = 0.00471$ (last example)

$\alpha = 1.06$

$\Sigma e_j = 0.05$ (entrance)

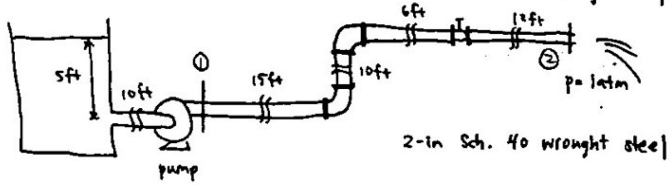
$$H_p = \frac{83060}{(62.4)(32.2)} - 5 + \frac{32(0.30)^2}{\pi^2\left(\frac{2.064}{12}\right)^5(32.2)}\left[(10)(0.00471) + \left(\frac{2.067/12}{4}\right)(1.06 + 0.05)\right]$$

$$H_p = 41.40 - 5 + 5.45 = 42.1\ ft$$

$$bhp = \frac{H_p \rho g Q}{\eta} = \frac{(42.1)(62.4)(32.2)(0.30)}{0.7} = 36252\ \frac{ft\cdot pd}{s} \times \frac{1\ hp}{17696\ \frac{ft\,pd}{s}} = 2.09\ hp$$

$\boxed{\text{II.F.7.c}}$ Flow rate for a given pressure drop

- Calculate volumetric flow rate required given $P_1 - P_2 = P_{1-g} = 10\,psig \times \frac{32.2\,pd}{1\,lb_f} \times \left(\frac{12\,in}{1\,ft}\right)^2 = 46368\,pd/ft^2$



From example II.F.7.a, we have

$$P_{1-g} = P_1 - P_2 = \rho g (h_2 - h_1) + \frac{32 \rho Q^2}{\pi^2 D^5}\left[(\Sigma L_i) f + \frac{D}{4}(\Sigma e_j)\right]$$

$$46368 = (62.4)(32.2)(10) + \frac{(32)(62.4)\,Q^2}{\pi^2 \left(\frac{2.067}{12}\right)^5}\left[43f + \left(\frac{2.067/12}{4}\right)(7.5)\right]$$

$$46368 = 20092 + 1.334 \times 10^6\, Q^2\,[43f + 0.323]$$

$$0.01970 = Q^2\,[43f + 0.323]$$

Write f interms of Q:

$$Re = \frac{4Q\rho}{\pi D \mu} = \frac{4Q(62.4)}{\pi\left(\frac{2.067}{12}\right)(6.72\times10^{-4})} = 6.864\times10^5\, Q$$

$$f = 0.00140 + \frac{0.09}{Re^{0.27}} \qquad \text{(Drew)}$$

$$= 0.00140 + \frac{0.090}{37.66\, Q^{0.27}}$$

$$= 0.00140 + \frac{0.00239}{Q^{0.27}}$$

Substitute f to above equation and solve for Q:

$$0.01970 = Q^2\left[43\left(0.00140 + \frac{0.00239}{Q^{0.27}}\right) + 0.323\right]$$

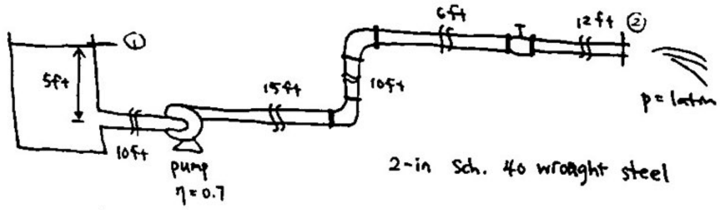
$$0.01970 = 0.3832\, Q^2 + 0.1028\, Q^{1.73}$$

$$Q = 0.19\, ft^3/s \times \frac{60\,s}{1\,min} \times \frac{7.48\,gal}{1\,ft^3}$$

$$Q = 85.3\ gal/min$$

$\boxed{\text{II.F.7.d}}$ Flow rate for a given horsepower

- Calculate volumetric flow rate that can be sustained by bhp = 1.0 hp.



2-in Sch. 40 wrought steel

$$\alpha \cancel{\frac{\langle v\rangle_1^2}{2g}} + \cancel{\frac{p}{\rho g}} + h_1 + H_p = \alpha \frac{\langle v\rangle_2^2}{2g} + \cancel{\frac{p}{\rho g}} + h_2 + H_{L12} \qquad \text{(atm pressure)}$$

0 large resevoir

$$H_p + (h_1 - h_2) = \alpha \frac{\langle v\rangle_2^2}{2g} + H_{L12}$$

$$\frac{(bhp)\eta}{\rho g Q} + (h_1 - h_2) = \frac{32Q^2}{\pi^2 D^5 g}\left[(\Sigma L_i) f + \frac{D}{4}\left(\Sigma e_j + \alpha\right)\right]$$

$$bhp = 1\,hp \times \frac{17696\ pd\,ft/s}{1hp} = 17696\ \frac{ft\,pd}{s}$$

$$\Sigma L_i = 10 + 15 + 10 + 6 + 12 = 53\ ft$$

$$\Sigma e_j = 1(\text{entrance}) + 2(90^\circ\ \text{elbow}) + 1(\text{valve})$$

$$= 0.05 + 2(0.75) + 6.00 = 7.55$$

$$\alpha = 1.06$$

$$\frac{(17696)(0.7)}{(62.4)(32.2)Q} - 5 = \frac{32Q^2}{\pi^2 \left(\frac{2.067}{12}\right)^5 (32.2)}\left[53f + \frac{2.067/12}{4}(7.55 + 1.06)\right]$$

$$\frac{6.165}{Q} - 5 = 664.05\, Q^2 \left[53f + 0.3708\right]$$

Write f in terms of Q (already had in Example II.F.7.c):

$$f = 0.00140 + \frac{0.00239}{Q^{0.27}} \qquad \text{(Drew)}$$

Substitute f into above equation and solve for Q:

$$\frac{6.165}{Q} - 5 = 664.05\, Q^2 \left[53\left(0.00140 + \frac{0.00239}{Q^{0.27}}\right) + 0.3708\right]$$

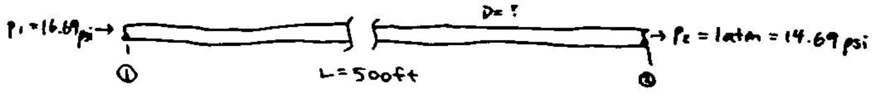
$$295.48 Q^3 + 84.135 Q^{2.73} + 5Q - 6.165 = 0$$

$$Q = 0.229\ ft^3/s \times \frac{7.48\ gal}{1\ ft^3} \times \frac{60s}{1min}$$

$$Q = 102.8\ gal/min$$

$\boxed{\text{I.F.7.e}}$ Diameter required for pressure and flow rate

- Calculate required diameter of a smooth, level pipe of $L = 500\,ft$ to carry air at 68°F at $Q = 0.40\,ft^3/s$ with $\Delta p = 2\,psi$.



$$\alpha \cancel{\frac{\langle v_1\rangle^2}{2g}} + \frac{p_1}{\rho g} + \cancel{h_1} = \alpha \cancel{\frac{\langle v_2\rangle^2}{2g}} + \frac{p_2}{\rho g} + \cancel{h_2} + H_{L12} \qquad (D_1 = D_2,\ h_1 = h_2)$$

$$p_1 - p_2 = \rho g H_{L12} = \rho g (H_{L12f} + \cancel{H_{L12c}}^{\,0}) = \rho g H_{L12f}$$

$$p_1 - p_2 = \frac{32 Q^2 \rho}{\pi^2 D^5} L f$$

$Q = 0.40\,ft^3/s$

$L = 500\,ft$

use μ of H_2O as conversion factor

$$\mu = 0.01813\,cP \times \frac{6.72\times10^{-4}\,lbm/ft\cdot s}{1\,cP} = 1.22\times10^{-5}\,lbm/ft\cdot s$$

For density, use ideal gas law:

$$p = \bar{p} = \tfrac{1}{2}(p_1 + p_2) = 15.696\,psi \times \frac{1\,atm}{14.696\,psi} = 1.068\,atm$$

$$\rho = \frac{m}{V} = \frac{n\mathcal{M}}{V} = \frac{p}{RT}\mathcal{M} = \frac{(1.068\,atm)(28.9\,g/mol)}{(82.06\,cm^3\,atm/mol\,K)(293.2\,K)} = 0.00128\,g/cm^3$$

$$\rho = 0.00128\,\frac{g}{cm^3} \times \frac{62.4\,lbm/ft^3}{1\,g/cm^3} = 0.080\,lbm/ft^3$$

$$\Delta p = 2\,psi \times \frac{144\,in^2}{ft^2} \times \frac{32.2\,pd}{lbf} = 9274\,pd/ft^2$$

$$9274 = \frac{32(0.4)^2(0.08)}{\pi^2 D^5}(500) f$$

$$9274 = 20.75\,\frac{f}{D^5}$$

Write f in terms of D:

$$Re = \frac{4Q\rho}{\pi D \mu} = \frac{4(0.4)(0.08)}{\pi D (1.22\times10^{-5})} = \frac{3340}{D}$$

$$f = 0.0014 + \frac{0.125}{Re^{0.32}} \qquad \text{(Koo, smooth pipe)}$$

$$= 0.0014 + 0.00932\,D^{0.32}$$

Substitute f into the above equation and solve for D:

$$9274 = 20.75\,\frac{0.0014 + 0.00932\,D^{0.32}}{D^5}$$

$$446.905 - 0.00932\,D^{0.32} - 0.00140 = 0$$

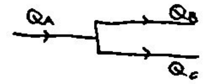
$$D = 0.1057\,ft = 1.27\,in$$

Verify $Re = \frac{3340}{D} = \frac{3340}{0.1057} = 3.16\times10^4$ is in Koo correlation's range.

II.F.8 Piping system in series & parallel

- Series : $Q_A = Q_B$

- parallel : $Q_A = Q_B + Q_C$

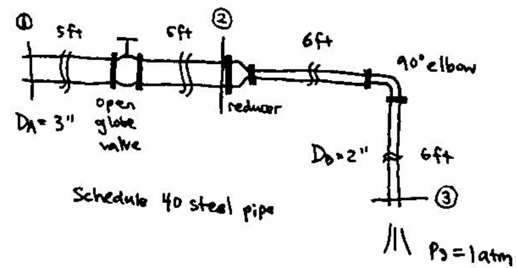


II.F.8.a Piping system with different diameters (in series)

► Calculate pressure drop $P_{gauge} = p_1 - p_3$ for flow rate $Q_A = Q_B = 0.30\ ft^3/s$

- First calculate pressure drop for each section, then sum them up:

$$p_1 - p_3 = (p_1 - p_2) + (p_2 - p_3)$$



$\boxed{A}$ Find $p_1 - p_2$

(level, $D_1 = D_2 = D_A$)

$$\alpha \cancel{\frac{\langle v \rangle_1^2}{2g}} + \frac{p_1}{\rho g} + \cancel{h_1} = \alpha \cancel{\frac{\langle v \rangle_2^2}{2g}} + \frac{p_2}{\rho g} + \cancel{h_2} + H_{L12}$$

$$p_1 - p_2 = \rho g H_{L12}$$

$$p_1 - p_2 = \frac{32 Q^2 \rho}{\pi^2 D^5}\left[(\Sigma L_i)_A f_A + \frac{D_A}{4}(\Sigma e_j)_A\right]$$

$$Q_A = 0.30\ ft^3/s$$

$$D_A = \frac{3.068}{12}\ ft \quad (3''\ \text{schedule } 40)$$

$$(\Sigma L_i)_A = 5 + 5 = 10\ ft$$

$$(\Sigma e_j)_A = 1\ (\text{open globe valve}) = 6.0$$

$$Re_A = \frac{4Q\rho}{\pi D_A \mu} = \frac{4(0.3)(62.4)}{\pi\left(\frac{3.068}{12}\right)(6.72\times10^{-4})} = 1.387\times10^5$$

$$f = 0.0014 + \frac{0.090}{Re^{0.32}} \qquad \text{(Drew)}$$

$$= 0.0014 + \frac{0.090}{(1.387\times10^5)^{0.32}} = 0.00508$$

$$p_1 - p_2 = \frac{32(62.4)(0.30)^2}{\pi^2\left(\frac{3.068}{12}\right)^5}\left[(10)(0.00508) + \frac{3.068/12}{4}(6.0)\right]$$

$$= 7239\ pd/ft^2$$

· Piping system with different diameters (cont.)

$\boxed{B}$ Find $p_2 - p_3$

$$\alpha \cancel{\frac{\langle v \rangle_2^2}{2g}} + \frac{p_2}{\rho g} + h_2 = \alpha \cancel{\frac{\langle v \rangle_3^2}{2g}} + \frac{p_3}{\rho g} + h_3 + H_{L23} \qquad (D_2 = D_3 = D_B)$$

$$p_2 - p_3 = \rho g (h_3 - h_2) + \frac{32 \rho Q^2}{\pi^2 D_B^5} \left[(\Sigma L_i)_B f_B + \frac{D_B}{4} (\Sigma e_j)_B \right]$$

$h_3 - h_2 = -6\text{ft}$

$D_B = \frac{2.067}{12}$ ft

$(\Sigma L_i)_B = 6 + 6 = 12\text{ft}$

$(\Sigma e_j)_B = 1(\text{reducer}) + 1(90° \text{ elbow})$

$$\beta = \frac{A_B}{A_A} = \left(\frac{D_B}{D_A}\right)^2 = \left(\frac{2.067}{3.068}\right)^2 = 0.454$$

$= 0.45(1-\beta) + 0.75$

$= 0.45(1-0.454) + 0.75$

$= 0.996$

$$Re = \frac{4Q\rho}{\pi D_B \mu} = Re_A\left(\frac{D_A}{D_B}\right) = 1.387 \times 10^5 \left(\frac{3.068}{2.067}\right) = 2.06 \times 10^5$$

$$f = 0.00140 + \frac{0.090}{Re^{0.32}} \qquad (\text{Drew})$$

$$= 0.00140 + \frac{0.090}{(2.06 \times 10^5)^{0.32}} = 0.00471$$

$$p_2 - p_3 = (62.4)(32.2)(-6) + \frac{32(62.4)(0.3)^2}{\pi^2 \left(\frac{2.067}{12}\right)^5} \left[(12)(0.00471) + \left(\frac{2.067/12}{4}\right)(0.996) \right]$$

$= -12056 + 11838$

$= -218$ pd/ft³

$\boxed{A} + \boxed{B}$ Find $p_1 - p_3$

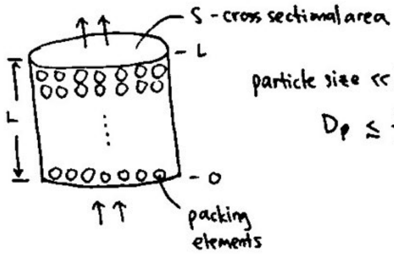
$p_1 - p_3 = (p_1 - p_2) + (p_2 - p_3) = 7239 - 218$

$$= 7020 \frac{\text{pd}}{\text{ft}^2} \times \frac{1\,\text{lbf}}{32.2\,\text{pd}} \times \left(\frac{1\text{ft}}{12\text{in}}\right)^2$$

$= 1.51$ psig

2 Dec 2021 L34 Flow through packed bed (Laminar) CHEME 330

II.F.9 Flow through packed bed



particle size << bed diameter

$D_p \lesssim \frac{D}{20}$

→ Applications

- gas absorption (stripping)
- gas adsorption
- distillation
- chromatography
- filtration
- catalytic reaction

► Objective: Relate driving force $\left[\frac{\mathcal{P}_0 - \mathcal{P}_L}{L}\right]$ to

- bed properties: S, L, particle properties
- fluid properties $\begin{cases} \text{liquid: constant } \rho, \mu \\ \text{gas: } \mu, \ \bar{\rho} = \frac{1}{2}(\rho_0 + \rho_L) \end{cases}$
- flow rate: Q, $\langle v \rangle$

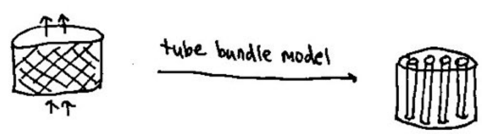
→ Packing Element Property

> specific area - $\boxed{a_v = \frac{\text{area of packing element}}{\text{volume of packing element}}}$

- sphere: $a_v = \frac{4\pi R^2}{\frac{4}{3}\pi R^3} = \frac{3}{R} = \frac{6}{D}$

> effective particle diameter - $\boxed{D_p = \frac{6}{a_v}}$

→ Tube Bundle Model



- Assume a bundle of straight circular cross section capilary tube
- Assume slow (creeping) flow
  - laminar flow
  - Hagen - Poiseiulle flow
- velocity through bed - $\langle v \rangle = \frac{R_{eff}^2}{8\mu}\left[\frac{\mathcal{P}_0 - \mathcal{P}_L}{L}\right]$ R_{eff} = effective tube radius

> Darcy's law - $\boxed{\langle v \rangle = \frac{K}{\mu}\left[\frac{\mathcal{P}_0 - \mathcal{P}_L}{L}\right]}$
- K = bed permiability $[=]$ cm^2
- linear relationship
- $Re < 10$

- volumetric flow rate - $\boxed{Q = \langle v \rangle \varepsilon S = v_0 S}$
  > porosity ε - fraction of void volume in bed
  > superficial velocity v_0 - linear velocity if bed were empty $v_0 = \langle v \rangle \varepsilon$
  > cross sectional area S

→ Bed Reynolds Number

$$\cdot\ Re_{bed} = \frac{D_p \langle v\rangle \rho}{\mu}\,\frac{\varepsilon}{1-\varepsilon} = \boxed{\frac{D_p V_o \rho}{\mu}\,\frac{1}{1-\varepsilon}} = \frac{D_p Q \rho}{\mu S}\,\frac{1}{1-\varepsilon}$$

· Darcy's law valid for $Re_{bed} \le 10$.

→ Hydrolic Radius

$$\cdot\ \left[\frac{P_o - P_L}{L}\right] = \frac{\tau_o}{R_H} = \frac{\frac{1}{2}\rho\langle v\rangle^2 f_{tube}}{R_H}$$

$$\cdot\ R_H = \frac{\text{cross section area}}{\text{wetted perimeter}}\cdot\frac{\text{length}}{\text{length}}$$

$$= \frac{\text{volume of bed available for flow}}{\text{wetted area of packing elements}}$$

$$= \frac{\varepsilon\ (\text{volume}\ \cancel{\text{of bed}})}{\left(\frac{\text{area of packing}}{\text{volume of packing}}\right)\left(\frac{\text{volume of packing}}{\text{volume of bed}}\right)(\cancel{\text{volume of}}\ \text{bed})}$$

$$= \frac{\varepsilon}{a_v(1-\varepsilon)}$$

$$\cdot\ \boxed{R_H = \frac{\varepsilon D_p}{6(1-\varepsilon)}}$$

$$\cdot\ \left[\frac{P_o - P_L}{L}\right] = \frac{\frac{1}{2}\rho\langle v\rangle^2 f_{tube}}{R_H} = \frac{\frac{1}{2}\rho\langle v\rangle^2 f_{tube}}{\left(\frac{\varepsilon D_p}{6(1-\varepsilon)}\right)} = \frac{3\rho\langle v\rangle^2(1-\varepsilon)}{D_p\ \varepsilon}f_{tube} = \frac{3\rho V_o^2(1-\varepsilon)}{D_p\ \varepsilon^3}f_{tube}$$

→ Friction factor of Tube ($Re_{bed} \le 10$)

$$\cdot\ \left[\frac{P_o - P_L}{L}\right] = \frac{3\rho V_o^2(1-\varepsilon)}{D_p\ \varepsilon^3}f_{tube}$$

$$\cdot\ Re_{tube} = \frac{4R_H\langle v\rangle\rho}{\mu} = \frac{4\varepsilon D_p}{6(1-\varepsilon)}\frac{\langle v\rangle\rho}{\mu} = \frac{2D_pV_o\rho}{3(1-\varepsilon)\mu} = \frac{2}{3}Re_{bed}$$

$$\cdot\ \boxed{Re_{tube} = \frac{2}{3}Re_{bed}}$$

$$\cdot\ f_{tube} = \frac{16}{Re_{tube}} = \frac{16}{\frac{2}{3}\frac{D_pV_o\rho}{(1-\varepsilon)\mu}} = \frac{24(1-\varepsilon)\mu}{D_p V_o \rho} \qquad (Re_{bed} \le 10)$$

$$\cdot\ \boxed{f_{tube} = \frac{24(1-\varepsilon)\mu}{D_pV_o\rho}}$$

$$\cdot\ \left[\frac{P_o - P_L}{L}\right] = \frac{3\rho V_o^2(1-\varepsilon)}{D_p\ \varepsilon^3}\,\frac{24(1-\varepsilon)\mu}{D_pV_o\rho} = 72\,\frac{\mu V_o}{D_p^2}\,\frac{(1-\varepsilon)^2}{\varepsilon^3}$$

> Blake-Kozeny equation - $\boxed{\left[\frac{P_o - P_L}{L}\right] = 150\,\frac{\mu V_o}{D_p^2}\,\frac{(1-\varepsilon)^2}{\varepsilon^3}}$ $Re_{bed} \le 10$

$$\cdot\ \text{bed permiability}:\quad K = \frac{\mu\langle v\rangle}{\left[\frac{P_o-P_L}{L}\right]} = \frac{\mu V_o}{\varepsilon\left[\frac{P_o-P_L}{L}\right]} = \frac{\mu V_o}{150\,\varepsilon\left(\frac{\mu V_o}{D_p^2}\right)\frac{(1-\varepsilon)^2}{\varepsilon^3}} = \boxed{\frac{D_p^2}{150}\left(\frac{\varepsilon}{1-\varepsilon}\right)^2 = K}$$

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→ Friction Factor of Tube ($Re_{bed} > 1000$)

· $\left[\frac{P_0 - P_L}{L}\right] = \frac{3\rho v_0^2}{D_p} \frac{1-\varepsilon}{\varepsilon^3} f_{tube}$

· $f_{tube} = \frac{7}{12}$ ($Re_{bed} > 1000$)

· $\left[\frac{P_0 - P_L}{L}\right] = \frac{3\rho v_0^2}{D_p} \frac{1-\varepsilon}{\varepsilon^3} \frac{7}{12}$

> Burke - Plummer equation - $\boxed{\left[\frac{P_0 - P_L}{L}\right] = \frac{7\rho v_0^2}{4 D_p} \frac{1-\varepsilon}{\varepsilon^2}}$ ($Re_{bed} > 1000$)

→ Intermediate Range ($10 < Re_{bed} < 1000$)

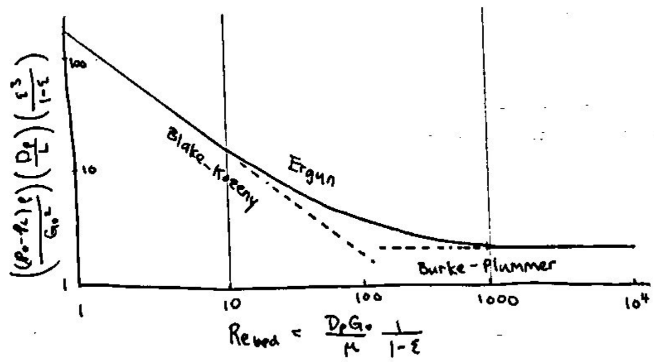
· Add the expression at two asymptotic ends for intermediate range

· $\left[\frac{P_0 - P_L}{L}\right] = 150 \frac{\mu v_0}{D_p^2} \frac{1-\varepsilon}{\varepsilon^3} + \frac{7}{4} \frac{\rho v_0^2}{D_p} \frac{1-\varepsilon}{\varepsilon^3}$

> superficial mass flux - $\boxed{G_0 = \rho v_0}$ $[=] \frac{\text{mass}}{\text{area} \cdot \text{time}} [=]$ $\boxed{G_0 = \frac{\dot{m}}{S}}$

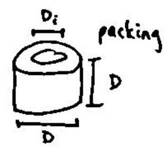
· $\left[\frac{P_0 - P_L}{L}\right] = 150 \frac{\mu G_0}{\rho D_p^2} \frac{1-\varepsilon}{\varepsilon^3} + \frac{7}{4} \frac{G_0^2}{\rho D_p} \frac{1-\varepsilon}{\varepsilon^3}$

> Ergun equation - $\boxed{\left[\frac{(P_0 - P_L)\rho}{G_0^2}\right]\left(\frac{D_p}{L}\right)\left(\frac{\varepsilon^3}{1-\varepsilon}\right) = 150\underbrace{\left[\frac{1-\varepsilon}{\left(\frac{D_p G_0}{\mu}\right)}\right]}_{\frac{1}{Re_{bed}}} + \frac{7}{4}}$

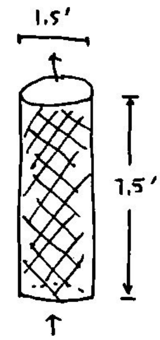


→ Example

- Calculate the pressure difference $p_0 - p_L$.
  - porosity $\varepsilon = 0.42$
  - packed with Raschig rings



    - $D = 6\,mm$, $D_i = 3\,mm$
  - Mass flow rate $\dot{m} = 0.50\ lbm/s$



air 300K, 1atm

$\mu = 1.85\times10^{-2}\,cP$

→ Neglect gravitational contribution

- $\rho g \Delta h = (0.0012)(62.4)(32.2)(7.5)$

$$= 18.1\ pd/ft^2 \times \frac{lbf}{32.2\,pd} \times \left(\frac{1\,ft}{12\,in}\right)^2$$

$$= 0.004\ psi \quad \Rightarrow \text{neglegible}$$

- $p_0 - p_L \approx \mathcal{P}_0 - \mathcal{P}_L$

→ Fluid Properties

- $\mu = 1.85\times10^{-2}\,cP \times \dfrac{6.72\times10^{-4}\ lbm/ft\,s}{1\,cP} = 1.243\times10^{-5}\ \dfrac{lbm}{ft\,s}$

- Assume ideal gas, $\bar{p} = 1atm$

$$\bar{\rho} = \frac{m}{V} = \frac{n}{V}\mathcal{M} = \frac{p}{RT}\mathcal{M} = \frac{(1atm)\ (28.9\,g/mol)}{(82.06\,cm^3\,atm/mol\,K)\ (300K)} = 1.17\times10^{-3}\ g/cm^3 = 0.0733\ \frac{lbm}{ft^3}$$

→ Bed properties

- $a_v = \dfrac{\text{area}}{\text{volume}} = \dfrac{(\pi D + \pi D_i)D + \frac{2\pi}{4}(D^2 - D_i^2)}{\frac{\pi}{4}(D^2 - D_i)D} = \dfrac{4}{D-D_i} + \dfrac{2}{D} = \dfrac{4}{0.6-0.3} + \dfrac{2}{0.6} = 16.67\ cm^{-1}$

- $D_p = \dfrac{6}{a_v} = \dfrac{6}{16.67} = 0.360\,cm \times \dfrac{ft}{30.48\,cm} = 0.0118\,ft$

- $\dfrac{D_p G_0}{\mu} = \dfrac{D_p \dot{m}}{\mu S} = \dfrac{(0.0118\,ft)(0.50\,lbm/s)}{(1.243\times10^{-5}\,lbm/fts)\ (\pi(0.75\,ft)^2)} = 268.9$

→ Iterate for pressure diff.

- Ergun eqn = $\dfrac{(\mathcal{P}_0 - \mathcal{P}_L)\bar{\rho}}{G_0^2}\left(\dfrac{D_p}{L}\right)\left(\dfrac{\varepsilon^3}{1-\varepsilon}\right) = 150\left[\dfrac{1-\varepsilon}{\frac{D_p G_0}{\mu}}\right] + \dfrac{7}{4}$

$$\frac{(\mathcal{P}_0 - \mathcal{P}_L)(0.0733)}{(0.283)^2}\left(\frac{0.0118}{7.5}\right)\left(\frac{0.42^3}{1-0.42}\right) = 150\left[\frac{1-0.42}{268.9}\right] + \frac{7}{4} \Rightarrow (\mathcal{P}_0 - \mathcal{P}_L)(1.839\times10^{-4}) = 2.074$$

(used later for $\bar{\rho}$ scaling)

$$\mathcal{P}_0 - \mathcal{P}_L = 1.128\times10^4\ \frac{pd}{ft^2} \times \left(\frac{1\,ft}{12\,in}\right)^2 \times \frac{1\,lbf}{32.2\,pd} = 2.43\ psi$$

- Update pressure used for calculating $\bar{\rho}$:

$$\rho \text{ scaling} = \frac{\rho_{update}}{\rho_{original}} = \frac{P_{update}}{P_{original}} = \frac{\frac{1}{2}\left[14.696 + (14.696 + 2.43)\right]psi}{14.696\ psi} = 1.083$$

$$(\mathcal{P}_0 - \mathcal{P}_L)(1.839\times10^{-4})(\rho\,\text{scaling}) = 2.074 \Rightarrow \mathcal{P}_0 - \mathcal{P}_L = 2.25\ psi$$

- Further iteration converges, giving

$$\mathcal{P}_0 - \mathcal{P}_L = 2.26\,psi = p_0 - p_L$$

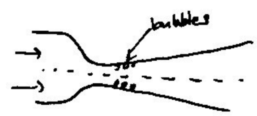
$\boxed{\text{I.F.10}}$ Cavitation 气穴现象

> cavitation - a phenomenon in which vapor bubbles form and collapse due to pressure variations in flowing liquid

- Cavitation is initiated at solid surfaces by the nucleation and growth of new vapor bubbles (when $p < p^*$), and/or by the growth of pre-existing micro vapor or gas pockets at solid surfaces.

→ When does it occur?

- closed flow channel contraction
    - venturi tube



    - increase in speed cause pressure to decrease to vapor pressure, forming bubbles
- pump intakes
- flow over submerged objects - hydrofoils

- trailing edges of propeller, mixer

→ What does it do?

→ Bad
- reduce flow efficiency
- reduce performance of pump
- reduce performance of propeller
- causes surface damage

→ Good
- enhances mixing
- useful for ultrasonic cleaning
- enables hydrofoil operation

→ Cavitation number

$$\boxed{\sigma \equiv \frac{P_A - P_c}{\frac{1}{2}\rho V_\infty^2}}$$

$\sigma > \sigma_{cr}$ no cavitation

$\sigma < \sigma_{cr}$ cavitation occurs

II.F.11 Vortex Motion

› vortex motion – circular fluid motion about an axis of rotation.

- blackhole, eye of hurricane, tornados, draining

→ Angular momentum

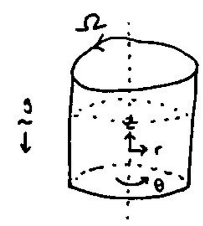
- Angular momentum of a fluid particle of mass m about an axis of rotation a distance r away:

$$L = I\omega = mr^2\omega = mrV_\theta$$

- Two kinds of Angular momentum
  1. forced vortex flow (application of a torque)
  2. free vortex flow (application of small disturbance in unstable system)

→ Forced Vortex motion

→ Forced vortex motion in cylindrical tank rotating



* Setup similar to couette flow in L20, but with only 1 rotating cylinder.

→ Assumptions

- steady state $\frac{\partial}{\partial t} = 0$
- V_θ is only v component : $V_z = V_r = 0$
- axial symmetry : $\frac{\partial}{\partial \theta} = 0$
- no end effect : $\frac{\partial V_\theta}{\partial z} = 0$
- vertical : $g_z = -g$, $g_\theta = g_r = 0$

→ Simplify N-S eqns (top-down method)

- r-Mom $-\rho \frac{V_\theta^2}{r} = -\frac{\partial p}{\partial r}$
- θ-Mom $0 = \mu \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rV_\theta)\right)$
- z-Mom $0 = -\frac{\partial p}{\partial z} - \rho g$

→ velocity profile

- θ-mom gives $\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rV_\theta)\right) = 0$

$$\frac{\partial}{\partial r}(rV_\theta) = c_1 r$$

- general form $\boxed{V_\theta = c_1 \frac{r}{2} + \frac{c_2}{r}}$

b.c. $\begin{cases} 1.\ r=0,\ V_\theta = 0 \\ 2.\ r=R,\ V_\theta = R\Omega \end{cases}$

$$0 = c_1 \frac{0}{2} + \frac{c_2}{0} \rightarrow c_2 = 0$$

$$R\Omega = \frac{c_1}{2} R \rightarrow c_1 = 2\Omega$$

$$\boxed{V_\theta = r\Omega}$$

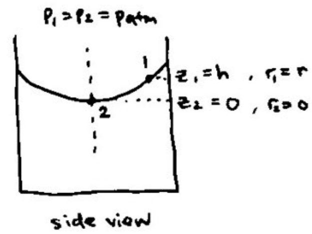
→ Forced Vortex Motion

→ Forced vortex motion in rotating cylinder (cont.)

→ pressure difference

· simplify and integrate r- and z-Mom eqn,

$$\begin{cases} \frac{\partial p}{\partial r} = \rho \frac{V_\theta^2}{r} = \rho \frac{(r\Omega)^2}{r} = \rho r \Omega^2 \\ \frac{\partial p}{\partial z} = -\rho g \end{cases}$$



· $$\boxed{P_2 - P_1 = \frac{1}{2}\rho\Omega^2(r_2^2 - r_1^2) + \rho g(z_1 - z_2)}$$

$$0 = \frac{1}{2}\rho\Omega^2(0 - r^2) + \rho g h$$

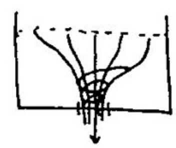
· $$\boxed{h = \frac{\Omega^2}{2g} r^2}$$

→ Other forced vortex motion

· flow around bluff body

· flow around sharp edge

→ Free Vortex Flow



· occur when we have converging streamline, or during drainage.

· conservation of angular momentum gives

$$\frac{\partial L}{\partial t} = 0$$

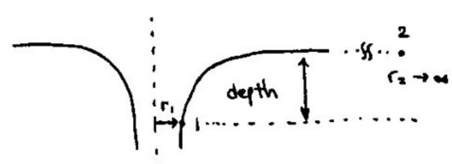
$$\frac{\partial}{\partial t}(m r V_\theta) = 0$$

$$r V_\theta = c$$

$$V_\theta = \frac{c}{r}$$

· simplify and rearrange r- and z-Mom eqn:

$$\begin{cases} \frac{\partial p}{\partial r} = \rho \frac{V_\theta^2}{r} = \rho \frac{c^2}{r^3} \\ \frac{\partial p}{\partial z} = -\rho g \end{cases}$$



· $$\boxed{P_2 - P_1 = \frac{1}{2}\rho c^2\left(\frac{1}{r_1^2} - \frac{1}{r_2^2}\right) + \rho g(z_1 - z_2)}$$

$$0 = \frac{1}{2}\rho c^2\left(\frac{1}{r_1^2}\right) + \rho g h$$

· depth $\boxed{h = \frac{c^2}{2g}\frac{1}{r^2}}$

· vortex direction does not depend on hemispheric location, but initial direction of dominant disturbance.

· baffles can surpress free vortex formation