28 August 2018 ch 1.3 Probability Day 1

1.(a)

$$\langle j \rangle = \frac{\sum j P(j)}{N} = \sum_{j=0}^{\infty} j P(j) = 21$$

$$\langle j \rangle^2 = \left(\frac{\sum j P(j)}{N}\right)^2$$

$$= \left(\frac{14(\frac{1}{14}) + 15(\frac{1}{14}) + 16(\frac{3}{14}) + 22(\frac{2}{14}) + 24(\frac{2}{14}) + 25(\frac{5}{14})}{\cancel{14}}\right)^2$$

$$= \cancel{(1.5)} \ 21^2$$

$$= \cancel{\#} \quad 441$$

$$\langle j^2 \rangle = \sum_{j=0}^{\infty} j^2 P(j)$$

$$= 14^2 \cdot (\tfrac{1}{14}) + 15^2 \cdot (\tfrac{1}{14}) + 16^2 \cdot (\tfrac{3}{14}) + 22^2 (\tfrac{2}{14}) + 24^2 (\tfrac{2}{14}) + 25^2 (\tfrac{5}{14})$$

$$= \frac{3217}{7} \approx 459.57$$

(b)

$14 - 21 = -7$

$15 - 21 = -6$

$16 - 21 = -5$

$22 - 21 = 1$

$24 - 21 = 3$

$25 - 21 = 4$

$$\sigma^2 = \langle (\Delta j)^2 \rangle$$

$$= \sum (\Delta j)^2 P(j)$$

$$= (-7)^2 (\tfrac{1}{14}) + (-6)^2 (\tfrac{1}{14}) + (-5)^2 (\tfrac{3}{14}) + 1^2 (\tfrac{2}{14}) + 3^2 (\tfrac{2}{14}) + 4^2 (\tfrac{5}{14})$$

$$= \frac{130}{7}$$

$$\sigma = \sqrt{\sigma^2} = \sqrt{\frac{130}{7}} = 4.31$$

(c)

$$\sigma = \sqrt{\langle j^2 \rangle - \langle j \rangle^2}$$

$$= \sqrt{459.57 - 441}$$

$$= 4.31$$

2.(a) $p(x) = \frac{1}{2\sqrt{hx}}$

$$P_{ab} = \int_a^b p(x)\,dx$$

$$1 = \int_{-\infty}^{+\infty} p(x)\,dx$$

$$\langle x \rangle = \int_{-\infty}^{+\infty} x\, p(x)\,dx$$

$$\langle f(x) \rangle = \int_{-\infty}^{+\infty} f(x)\, p(x)\,dx$$

$$\sigma^2 = \langle (\Delta x)^2 \rangle = \langle x^2 \rangle - \langle x \rangle^2$$

$$= \tfrac{1}{5} h^2 - \tfrac{1}{9} h^2$$

$$= \tfrac{4}{45} h^2$$

$$\sigma = \sqrt{\sigma^2} = 0.298\,h$$

$$\langle x \rangle = \int_{-\infty}^{+\infty} x\, p(x)\,dx$$

$$= \int_{-\infty}^{+\infty} x \frac{1}{2\sqrt{hx}}\,dx$$

$$= \frac{1}{2} \int_{-\infty}^{+\infty} \frac{x}{\sqrt{hx}}\,dx$$

$$= \frac{1}{2\sqrt{h}} \int_{-\infty}^{+\infty} \sqrt{x}\,dx$$

$$= \frac{1}{2\sqrt{h}} \left[\int_{-\infty}^{0} \sqrt{x}\,dx + \int_0^h \sqrt{x}\,dx + \int_h^{\infty} \sqrt{x}\,dx \right]$$

$$= \frac{1}{2\sqrt{h}} \int_0^h \sqrt{x}\,dx$$

$$= \frac{1}{2\sqrt{h}} \left[\tfrac{2}{3} x^{\frac{3}{2}} \right]_0^h$$

$$= \frac{1}{2\sqrt{h}} \left(\tfrac{2}{3} h^{\frac{3}{2}} - 0 \right)$$

$$= \tfrac{1}{3} h$$

$$\langle x \rangle^2 = \tfrac{1}{9} h^2$$

$$\langle x^2 \rangle = \int_{-\infty}^{+\infty} x^2\, p(x)\,dx$$

$$= \int_{-\infty}^{+\infty} x^2 \frac{1}{2\sqrt{hx}}\,dx$$

$$= \frac{1}{2\sqrt{h}} \int_{-\infty}^{+\infty} x^{\frac{3}{2}}\,dx$$

$$= \frac{1}{2\sqrt{h}} \int_0^h x^{\frac{3}{2}}\,dx$$

$$= \frac{1}{2\sqrt{h}} \left[\tfrac{2}{5} x^{\frac{5}{2}} \right]_0^h$$

$$= \frac{1}{2\sqrt{h}} \left(\tfrac{2}{5} h^{\frac{5}{2}} - 0 \right)$$

$$= \tfrac{1}{5} h^2$$

(b) $\sigma = 0.298h$ $\langle x\rangle - \sigma = 0.0353h$

$\langle x\rangle = \frac{1}{3}h$ $\langle x\rangle + \sigma = 0.6313h$

$p(x) = \frac{1}{2\sqrt{hx}}$

$$\int_0^{\langle x\rangle-\sigma} p(x)\,dx + \int_{\langle x\rangle+\sigma}^{h} p(x)\,dx$$

$$= \int_0^{0.0353h} p(x)\,dx + \int_{0.6313h}^{h} p(x)\,dx$$

$$= \frac{1}{2\sqrt{h}}\int_0^{0.0353h} x^{-\frac{1}{2}}\,dx + \frac{1}{2\sqrt{h}}\int_{0.6313h}^{h} x^{-\frac{1}{2}}\,dx$$

$$= \frac{1}{2\sqrt{h}}\left[2x^{\frac{1}{2}}\right]_0^{0.0353h} + \frac{1}{2\sqrt{h}}\left[2x^{+\frac{1}{2}}\right]_{0.6313h}^{h}$$

$$= \frac{1}{2\sqrt{h}}\left(2(0.0353h)^{\frac{1}{2}} + 2h^{+\frac{1}{2}} - 2(0.6313h)^{+\frac{1}{2}}\right)$$

$$= 0.404$$

- indeterminacy - can't predict the particle's position.
- realist
- hidden variable
- orthodox
- agnostic
- $N(j)$ is the number of people of age j
- N is the total number of people in the room
- $N = \sum_{j=0}^{\infty} N(j)$
- probability
- $P(j)$ is the probability of getting age j.
- $P(j) = \frac{N(j)}{N}$
- $\sum_{j=0}^{\infty} P(j) = 1$
- most probable
- median
- mean - average
- $\langle j\rangle$ is average
- $\langle j\rangle = \frac{\sum jN(j)}{N} = \sum_{j=0}^{\infty} j\,P(j)$
- expectation value - average
- $\langle j^2\rangle$ is the average of square
- $\langle j^2\rangle = \sum_{j=0}^{\infty} j^2 P(j)$
- average value of function of j: $\langle f(j)\rangle = \sum_{j=0}^{\infty} f(j)\,P(j)$

30 August 2018

~~Ch 1.4 Normalization~~
Ch 1.3 Probability

1.77

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = A \quad \boxed{1}$$

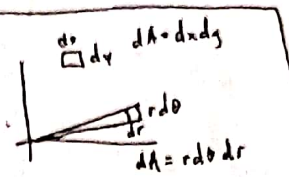
$$\int_{-\infty}^{0} e^{-x^2}\,dx = \int_{0}^{\infty} e^{-x^2}\,dx = \frac{A}{2} \quad \boxed{2}$$

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx \cdot \int_{-\infty}^{\infty} e^{-y^2}\,dy = A^2$$

$$\Rightarrow \iint e^{-x^2-y^2}\,dx\,dy$$

$$= \iint e^{-r^2}\,r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\infty} e^{-r^2}\,r\,dr\,d\theta \quad \boxed{3}$$



$dx\,dy = r\,d\theta\,dr$

$r^2 = x^2 + y^2$

u substitution: $u = r^2$, $du = 2r\,dr$

$$\boxed{3} = \int_0^{2\pi}\int_0^{\infty} \frac{1}{2} e^{-u}\,du\,d\theta$$

$$= \cancel{\tfrac{1}{2}\int_0^{2\pi} [-e^{-u}]_0^\infty} \quad \frac{1}{2}\int_0^{2\pi} \lim_{a\to\infty} \int_0^{a} \frac{1}{2} e^{-u}\,du\,d\theta$$

$$= \cancel{\tfrac{1}{2}\int_0^{2\pi} (-e^{-\infty}} + \quad \frac{1}{2}\int_0^{2\pi} \lim_{a\to\infty} [-e^{-u}]_0^a\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} (-e^{-a}+1)\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} 1\,d\theta$$

$$= \frac{1}{2}(\theta)_0^{2\pi}$$

$$= \pi$$

that is, $A^2 = \pi$, thus, $A = \sqrt{\pi}$

3. (a) $1 = \int_{-\infty}^{\infty} \rho(x)\,dx \quad \boxed{1}$

$$\rho(x) = A e^{-\lambda(x-a)^2} \quad \boxed{2}$$

$$\int_{-\infty}^{\infty} \rho(x)\,dx = \int_{-\infty}^{\infty} A e^{-\lambda(x-a)^2}\,dx = 1$$

$$A\int_{-\infty}^{\infty} e^{-\lambda(x-a)^2}\,dx = 1$$

$$\int_{-\infty}^{\infty} e^{-\lambda(x-a)^2}\,dx = \frac{1}{A} \quad \boxed{3}$$

$$\cancel{\int_{-\infty}^{\infty} e^{-\lambda(x-a)^2}\,dx \cdot \int_{-\infty}^{\infty} e^{-\lambda(y-a)^2}\,dy = \frac{1}{A^2}}$$

$$\Rightarrow \iint e$$

Use u substitution,

$u = x - a$, $\frac{du}{dx} = 1$, $du = dx$

$$\boxed{3}: \int_{-\infty}^{\infty} e^{-\lambda u^2}\,du = \frac{1}{A} \quad \boxed{4}$$

$v = \sqrt{\lambda}\,u$, $\frac{dv}{du} = \sqrt{\lambda}$, $\frac{dv}{\sqrt{\lambda}} = du$

$$\boxed{4}: \frac{1}{\sqrt{\lambda}}\int_{-\infty}^{\infty} e^{-v^2}\,dv = \frac{1}{A}$$

$$\frac{1}{\sqrt{\lambda}}\sqrt{\pi} = \frac{1}{A}$$

$$A = \sqrt{\frac{\lambda}{\pi}} \quad \boxed{5}$$

(b) Substitute $A = \sqrt{\frac{\lambda}{\pi}}$ to $\boxed{2}$,

$$p(x) = \sqrt{\frac{\lambda}{\pi}}\, e^{-\lambda(x-a)^2} \quad \boxed{6}$$

#1 $\langle x \rangle = \int_{-\infty}^{\infty} x\, p(x)\, dx$

$$\langle x \rangle = \int_{-\infty}^{\infty} x \sqrt{\frac{\lambda}{\pi}}\, e^{-\lambda(x-a)^2}\, dx$$

$$\langle x \rangle = \sqrt{\frac{\lambda}{\pi}} \int_{-\infty}^{\infty} x e^{-\lambda(x-a)^2}\, dx \quad \boxed{7}$$

Use integration by parts (integration chain rule)

$$\left[\int f(x) g'(x)\, dx = f(x) g(x) - \int g(x) f'(x)\, dx \right]$$

$f(x) = x \qquad g'(x) = e^{-\lambda(x-a)^2}$

$f'(x) = 1 \qquad g(x) = \int e^{-\lambda(x-a)^2}\, dx$

$u = x - a, \quad du = dx$

$\int e^{-\lambda u^2}\, du$

$v = \sqrt{\lambda}\, u, \quad \frac{dv}{du} = \sqrt{\lambda}, \quad du = \frac{dv}{\sqrt{\lambda}}$

$\frac{1}{\sqrt{\lambda}} \int e^{-v^2}\, dv$

Use u substitution.

$u = x - a \Rightarrow u + a = x$

$\frac{du}{dx} = 1 \Rightarrow du = dx$

$\boxed{7}$: $\langle x \rangle = \sqrt{\frac{\lambda}{\pi}} \int_{-\infty}^{\infty} (u+a)\, e^{-\lambda u^2}\, du \quad \boxed{8}$

$v = \sqrt{\lambda}\, u \Rightarrow u = \frac{v}{\sqrt{\lambda}}$

$\frac{dv}{du} = \sqrt{\lambda} \Rightarrow du = \frac{dv}{\sqrt{\lambda}}$

$\langle x \rangle = \sqrt{\frac{\lambda}{\pi}} \int_{-\infty}^{\infty} \left(\frac{v}{\sqrt{\lambda}} + a \right) e^{-\lambda v^2}\, dv$

Use distributive property;

$\boxed{8}$: $\langle x \rangle = \sqrt{\frac{\lambda}{\pi}} \int_{-\infty}^{\infty} u e^{-\lambda u^2} + a e^{-\lambda u^2}\, du$

$\boxed{\int_{-\infty}^{\infty} u e^{-\lambda u^2}\, du = 0}$

$\langle x \rangle = \sqrt{\frac{\lambda}{\pi}} \left[\int_{-\infty}^{\infty} u e^{-\lambda u^2}\, du + a \int_{-\infty}^{\infty} e^{-\lambda u^2}\, du \right]$

$\langle x \rangle = a \sqrt{\frac{\lambda}{\pi}} \int_{-\infty}^{\infty} e^{-\lambda u^2}\, du$

$\boxed{\int_{-\infty}^{\infty} e^{-\lambda u^2}\, du = \sqrt{\frac{\pi}{\lambda}}}$

$\langle x \rangle = a \sqrt{\frac{\lambda}{\pi}} \sqrt{\frac{\pi}{\lambda}}$

$\langle x \rangle = a \quad \boxed{9}$

Day 2

30 August 2018

~~Ch 1.4 Normalization~~
Ch 1.3 Probability

3. (b) #2 $\langle x^2 \rangle = \int_{-\infty}^{\infty} x^2 \rho(x)\,dx$ $\boxed{10}$

$\langle x^2 \rangle = \int_{-\infty}^{\infty} x^2 \sqrt{\frac{\lambda}{\pi}} e^{-\lambda(x-a)^2} dx$

$\langle x^2 \rangle = \sqrt{\frac{\lambda}{\pi}} \int_{-\infty}^{\infty} x^2 e^{-\lambda(x-a)^2} dx$ $\boxed{11}$

Use u substitution,

$u = x - a \Rightarrow x = u + a$

$\frac{du}{dx} = 1 \Rightarrow du = dx$

$\langle x^2 \rangle = \sqrt{\frac{\lambda}{\pi}} \int_{-\infty}^{\infty} (u+a)^2 e^{-\lambda u^2} du$ $\boxed{12}$

$\boxed{\int_{-\infty}^{\infty} u e^{-\lambda u^2} du = 0 \quad \int_{-\infty}^{\infty} e^{-\lambda u^2} du = \sqrt{\frac{\pi}{\lambda}}}$

Use distributive property,

$\langle x^2 \rangle = \sqrt{\frac{\lambda}{\pi}} \left[\int_{-\infty}^{\infty} u^2 e^{-\lambda u^2} + 2au e^{-\lambda u^2} + a^2 e^{-\lambda u^2} du \right]$

$\langle x^2 \rangle = \sqrt{\frac{\lambda}{\pi}} \left[\int_{-\infty}^{\infty} u^2 e^{-\lambda u^2} du + 2a \int_{-\infty}^{\infty} u e^{-\lambda u^2} du + a^2 \int_{-\infty}^{\infty} e^{-\lambda u^2} du \right]$

$\langle x^2 \rangle = \sqrt{\frac{\lambda}{\pi}} \left[a^2 \sqrt{\frac{\pi}{\lambda}} + \int_{-\infty}^{\infty} u^2 e^{-\lambda u^2} du \right]$

$\langle x^2 \rangle = a^2 + \sqrt{\frac{\lambda}{\pi}} \int_{-\infty}^{\infty} u^2 e^{-\lambda u^2} du$ $\boxed{13}$

Use integration by parts,

$f(x) = u^2$	$g'(x) = e^{-\lambda u^2}$
$f'(x) = 2u$	$g(x) = \int e^{-\lambda u^2} du$

$f(x) = u$ $\quad g'(x) = u e^{-\lambda u^2}$

$f'(x) = 1$ $\quad g(x) = \int u e^{-\lambda u^2} du$

$v = -\lambda u^2$

$\frac{dv}{du} = -2\lambda u \Rightarrow u\,du = -\frac{dv}{2\lambda}$

$= -\frac{1}{2\lambda} \int e^v dv$

$= -\frac{1}{2\lambda} e^v$

$= -\frac{1}{2\lambda} e^{-\lambda u^2}$

$\boxed{13}$: $\langle x^2 \rangle = a^2 + \sqrt{\frac{\lambda}{\pi}} \left(\left[-u \frac{1}{2\lambda} e^{-\lambda u^2} \right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} -\frac{1}{2\lambda} e^{-\lambda u^2} du \right)$

$\langle x^2 \rangle = a^2 + \sqrt{\frac{\lambda}{\pi}} \left(\left[-\frac{u}{2\lambda} e^{-\lambda u^2} \right]_{-\infty}^{\infty} + \frac{1}{2\lambda} \int_{-\infty}^{\infty} e^{-\lambda u^2} du \right)$

$\boxed{\int_{-\infty}^{\infty} e^{-\lambda u^2} du = \sqrt{\frac{\pi}{\lambda}}}$

$\langle x^2 \rangle = a^2 + \sqrt{\frac{\lambda}{\pi}} \left(\left[-\frac{u}{2\lambda} e^{-\lambda u^2} \right]_{-\infty}^{\infty} + \frac{1}{2\lambda} \sqrt{\frac{\pi}{\lambda}} \right)$

$\langle x^2 \rangle = a^2 + \frac{1}{2\lambda} - \left[\frac{u}{2\lambda} e^{-\lambda u^2} \right]_{-\infty}^{\infty}$

$\langle x^2 \rangle = a^2 + \frac{1}{2\lambda}$

3. (b) #3 $\sigma = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$

$$= \sqrt{a^2 + \frac{1}{2\lambda} - a^2}$$

$$= \sqrt{\frac{1}{2\lambda}} = \frac{1}{\sqrt{2\lambda}}$$

$\boxed{pg14}$ 4.

$$\Psi(x,0) = \begin{cases} A\frac{x}{a} & \text{if } 0 \le x \le a, \\ A\frac{(b-x)}{(b-a)} & \text{if } a \le x \le b, \\ 0 & \text{otherwise,} \end{cases}$$

known $|\Psi|^2 = \Psi^* \Psi$.
Ψ^* is the complex conjugate:
$\Psi = a + bi$
$\Psi^* = a - bi$

(a) $\int_0^a |\Psi|^2 \, dx$

$= \int_0^a A^2 \frac{x^2}{a^2} \, dx$

$= \frac{A^2}{a^2} \int_0^a x^2 \, dx$

$= \frac{A^2}{a^2} \left[\frac{1}{3}x^3\right]_0^a$

$= \frac{A^2}{a^2} \cdot \frac{1}{3} a^3$

$= \frac{1}{3} a A^2$

$\int_a^b |\Psi|^2 \, dx$ $\quad u = b - x$, $du = -dx$

$= \int_a^b A^2 \frac{(b-x)^2}{(b-a)^2} dx$ $\quad -\int_a^b u^2 \, du$

$= \frac{A^2}{(b-a)^2} \int_a^b (b-x)^2 \, dx$ $\quad \left[-\frac{1}{3}u^3\right]_a^b$

$= \frac{A^2}{(b-a)^2} \left[\int_a^b b^2 dx - 2\int_a^b bx \, dx + \int_a^b x^2 dx \right]$

$= \frac{A}{b-a} \left[[bx]_a^b - \left[\frac{1}{2}x^2\right]_a^b \right]$

$= \frac{A}{b-a} \left[b^2 - ab - \left(\frac{1}{2}b^2 - \frac{1}{2}a^2\right) \right]$

$= \frac{A}{b-a} \left[\frac{1}{2}b^2 + \frac{1}{2}a^2 - ab \right]$

$= \frac{A}{b-a} \cdot \frac{1}{2}(a-b)^2$

$= \frac{A^2}{(b-a)^2} \left(\left[b^2 x\right]_a^b - 2b\left[\frac{1}{2}x^2\right]_a^b + \left[\frac{1}{3}x^3\right]_a^b \right)$

$= \frac{A^2}{(b-a)^2} \left(b^3 - ab^2 - 2b\left[\frac{1}{2}b^2 - \frac{1}{2}a^2\right] + \left[\frac{1}{3}b^2 - \frac{1}{3}a^2\right] \right)$

$= \frac{A^2}{(b-a)^2} \left(\frac{1}{3}b^2 - \frac{1}{3}a^2 \right)$

$= \frac{1}{3} \frac{A^2}{(b-a)^2} (b+a)(b-a)$

$= \frac{1}{3} \frac{A^2(b+a)}{(b-a)}$

$\int_{-\infty}^{0} |\Psi^*|^2 \, dx$

$= \int_{-\infty}^{0} a^2 \, dx$

$= \lim_{\alpha \to -\infty} [a^2 x]_\alpha^0$

$= \infty$

$\int_b^\infty |\Psi|^2 \, dx$

$= \int_b^\infty a^2 \, dx$

$= \lim_{\beta \to \infty} [a^2 x]_b^\beta$

$= \infty$

$\int_{-\infty}^{0} |\Psi|^2 \, dx$

$= \int_{-\infty}^{0} 0 \, dx$

$= 0$

$\int_b^\infty |\Psi|^2 \, dx$

$= \int_b^\infty 0 \, dx$

$= 0$

$\int_a^b |\Psi|^2 \, dx$

$= \frac{A^2}{(b-a)^2} \int_a^b (b-x)^2 \, dx$

Use u substitution, $u = b - x$, $dx = -du$

$= \frac{A^2}{(b-a)^2} \left(-\int_a^b u^2 \, du \right)$

$= \frac{A^2}{(b-a)^2} \left(-\left[\frac{1}{3}u^3\right]_a^b \right)$

$= -\frac{A^2}{(b-a)^2} \left(\frac{1}{3}(b-b)^3 - \frac{1}{3}(b-a)^3 \right)$

$= \frac{1}{3} A^2 (b-a)$

$\int_{-\infty}^{\infty} |\Psi|^2 dx = \int_{-\infty}^{0} |\Psi|^2 dx + \int_0^a |\Psi|^2 dx + \int_a^b |\Psi|^2 dx + \int_b^\infty |\Psi|^2 dx$

$1 = 0 + \cancel{\frac{1}{3}\frac{A^2(b+a)}{b-a}} + 0$

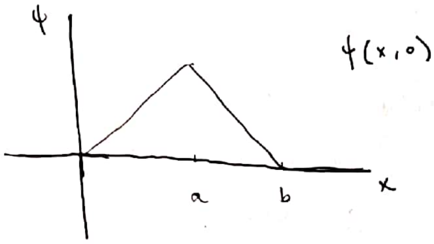
$\frac{1}{3} a A^2 + \frac{1}{3} A^2 (b-a)$

$1 = \frac{1}{3} A^2 \left(a + \cancel{\frac{b+a}{b-a}} \right)$ $\quad b-a$

$\frac{3}{a + \cancel{\frac{b+a}{b-a}}\ b-a} = A^2$

$\cancel{A = \pm\sqrt{\frac{3}{a + \frac{b+a}{b-a}}}}$ $\quad A = \pm\sqrt{\frac{3}{b}}$

(b)



(c) The particle is most likely to be found at a, at $t = 0$.

(d) $\int_{-\infty}^{a} |\psi|^2 dx$

$= \frac{1}{3} a A^2 + 0$

$= \frac{1}{3} a \frac{3}{b}$

$= \frac{a}{b}$

if $b = a$, then,

$\int_{-\infty}^{a} |\psi|^2 dx = \frac{a}{a} = 1$

if $b = 2a$, then,

$\int_{-\infty}^{a} |\psi|^2 dx = \frac{a}{2a} = \frac{1}{2}$

(e) $\langle x \rangle = \int_{-\infty}^{\infty} x |\psi|^2 dx$

$\int_0^a x|\psi|^2 dx$

$= \int_0^a x\left(A^2 \frac{x^2}{a^2}\right) dx$

$= \frac{A^2}{a^2} \int_0^a x^3 dx$

$= \frac{A^2}{a^2} \left[\frac{1}{4} x^4\right]_0^a$

$= \frac{A^2}{a^2} \frac{1}{4} a^4$

$= \frac{1}{4} a^2 A^2$

$\int_a^b x|\psi|^2 dx$

$= \int_a^b x \, A^2 \frac{(b-x)^2}{(b-a)^2} dx$

$= \frac{A^2}{(b-a)^2} \int_a^b xb^2 - 2bx^2 + x^3 dx$

$= \frac{A^2}{(b-a)^2} \left[\left(\frac{1}{2} b^2 x^2\right)_a^b - \left(2b \frac{1}{3} x^3\right)_a^b + \left(\frac{1}{4} x^4\right)_a^b \right]$

$= \frac{A^2}{(b-a)^2} \left(\frac{1}{2} b^4 - \frac{1}{2} a^2 b^2 - \frac{2}{3} b^4 + \frac{2}{3} a^3 b + \frac{1}{4} b^4 - \frac{1}{4} a^4\right)$

$= \frac{A^2}{(b-a)^2} \left(\frac{1}{12} b^4 - \frac{1}{2} a^2 b^2 + \frac{2}{3} a^3 b - \frac{1}{4} a^4\right)$

$\frac{6}{12} + \frac{3}{12} - \frac{8}{12}$

3 September 2018 ch 1.4 Normalization

5. $\Psi(x,t) = Ae^{-\lambda|x|}e^{-i\omega t}$

(a) $\int_{-\infty}^{\infty} |\Psi|^2\,dx$

$= \int_{-\infty}^{\infty} \Psi(x,t)\cdot\Psi^*(x,t)\,dx$

$= \int_{-\infty}^{\infty} Ae^{-\lambda|x|}e^{-i\omega t}\cdot Ae^{-\lambda|x|}e^{i\omega t}\,dx$

$= \int_{-\infty}^{\infty} A^2 e^{-2\lambda|x|}\,dx$

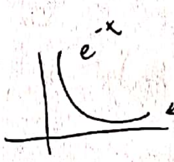
$= 2\int_0^{\infty} A^2 e^{-2\lambda x}\,dx$

$= 2A^2 \int_0^{\infty} e^{-2\lambda x}\,dx$

Use u substitution : $u = -2\lambda x$, $\frac{du}{dx} = -2\lambda$, $dx = -\frac{du}{2\lambda}$

$= 2A^2\int_0^{\infty} e^u \cdot -\frac{du}{2\lambda}$

$= -\frac{A^2}{\lambda}\int_0^{\infty} e^u\,du$



$= -\frac{A^2}{\lambda}\left[e^{-2\lambda x}\right]_0^{\infty}$

$= \frac{A^2}{\lambda}$

$\int_{-\infty}^{\infty} |\Psi|^2\,dx = 1 = \frac{A^2}{\lambda}$

$\cancel{\lambda} = A^2$

$A = \cancel{\sqrt{\lambda}}\ \pm\sqrt{\lambda}$

$\boxed{\int_{-a}^{a} g(x)f(x)\,dx = 0 \quad \text{if } g(x) \text{ is odd, } f(x) \text{ is even}}$

$g(x) = g(-x)$

$-f(x) = f(-x)$

(b) $\langle x\rangle = \int_{-\infty}^{\infty} x|\Psi|^2\,dx$

$= \int_{-\infty}^{\infty} x\cdot A^2 e^{-2\lambda|x|}\,dx$

$= 2\int_0^{\infty} \underbrace{x}_{\text{odd}}\, \underbrace{Ae^{-2\lambda x}}_{\text{even}}\,dx$

$= 0$

$\int_{-a}^{a} g(x)f(x)\,dx$

$= \int_{-a}^{0} g(x)f(x)\,dx + \int_0^a g(x)f(x)\,dx$

Use u substitution, $u = -x$, $-du = dx$

change upper/lower bound in u substitution

$= -\int_{a}^{0} g(-u)f(-u)\,du + \int_0^a g(x)f(x)\,dx$

$= \int_0^a g(-u)f(-u)\,du + \int_0^a g(x)f(x)\,dx$

$= -\int_0^a g(u)f(u)\,du + \int_0^a g(x)f(x)\,dx$

$= 0$

QED

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} x^2 \, |\psi|^2 \, dx$$

$$= \int_{-\infty}^{\infty} x^2 \, A^2 \, e^{2-\lambda|x|} \, dx$$

$$= 2\int_0^{\infty} x^2 A^2 e^{-2\lambda x} \, dx$$

$$= 2A^2 \int_0^{\infty} x^2 e^{-2\lambda x} \, dx$$

Use integration by parts,

$f(x) = x^2 \qquad g'(x) = e^{-2\lambda x}$

$f'(x) = 2x \qquad g(x) = \int e^{-2\lambda x} \, dx$

$$= \int e^{u} \frac{du}{-2\lambda}$$

$$= -\frac{1}{2\lambda} e^{-2\lambda x}$$

$$= 2A^2 \left[\left[-x^2 \frac{1}{2\lambda} e^{-2\lambda x} \right]_0^{\infty} - \int_0^{\infty} 2x e^{-2\lambda x} \, dx \right]$$

constant $-\frac{1}{2\lambda}$

$$= -2A^2 \cdot 2 \int x e^{-2\lambda x} \, dx$$

$$= -4A^2 \left[\left[x \cdot \frac{-1}{2\lambda} e^{-2\lambda x} \right]_0^{\infty} - \int_0^{\infty} e^{-2\lambda x} \, dx \right]$$

$f(x) = x \qquad g'(x) = e^{-2\lambda x}$

$f'(x) = 1 \qquad g(x) = -\frac{1}{2\lambda} e^{-2\lambda x}$

$$= -4A^2 \cdot \left(-\left[-\frac{1}{2\lambda} e^{-2\lambda x} \right]_0^{\infty} \right)$$

$$= -4A^2 \, \cancel{(\ )} \left(\frac{1}{2\lambda} \right)$$

$$= -4A^2 \left(\frac{1}{2\lambda} \right)$$

$$= -\frac{2A^2}{\lambda} = -2$$

$$= 2A^2 \left[\left[-x^2 \frac{1}{2\lambda} e^{-2\lambda x} \right]_0^{\infty} - \int_0^{\infty} 2x \frac{1}{2\lambda} e^{-2\lambda x} \, dx \right]$$

$$= -2A^2 \frac{1}{\lambda} \int_0^{\infty} x e^{-2\lambda x} \, dx$$

$$= -2 \int_0^{\infty} x e^{-2\lambda x} \, dx$$

$f(x) = x \qquad g'(x) = e^{-2\lambda x}$

$f'(x) = 1 \qquad g(x) = -\frac{1}{2\lambda} e^{-2\lambda x}$

$$= -2 \left[\left[-x \frac{1}{2\lambda} e^{-2\lambda x} \right]_0^{\infty} - \int_0^{\infty} -\frac{1}{2\lambda} e^{-2\lambda x} \, dx \right]$$

$$= -2 \frac{1}{2\lambda} \int_0^{\infty} e^{-2\lambda x} \, dx$$

$u = 2\lambda x, \quad dx = \frac{du}{2\lambda}$

$$= \left(-\frac{1}{\lambda} \right) \left(\frac{1}{2\lambda} \right) \int_0^{\infty} e^{u} \, du$$

$$= -\frac{1}{2\lambda^2} [0 - 1]$$

$$= \frac{1}{2\lambda^2}$$

P1.8 7 Prove $\frac{d\langle p\rangle}{dt} = \left\langle -\frac{\partial V}{\partial x}\right\rangle$

$$\boxed{i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi}$$

$$\frac{\partial\Psi}{\partial t} = \frac{i\hbar}{2m}\frac{\partial^2\Psi}{\partial x^2} - \frac{i}{\hbar}V\Psi$$

$$\frac{\partial\Psi^*}{\partial t} = -\frac{i\hbar}{2m}\frac{\partial^2\Psi^*}{\partial x^2} + \frac{i}{\hbar}V\Psi^*$$

$$\langle p\rangle = m\frac{d\langle x\rangle}{dt} = -i\hbar\int\left(\Psi^*\frac{\partial\Psi}{\partial x}\right)dx$$

$$\frac{d\langle p\rangle}{dt} = -i\hbar\frac{d}{dt}\int\left(\Psi^*\frac{\partial\Psi}{\partial x}\right)dx$$

$$= -i\hbar\left(\Psi^*\frac{\partial\Psi}{\partial x}\right)$$

$$= -i\hbar\left(\Psi^*\left(\frac{i\hbar}{2m}\frac{\partial^2\Psi}{\partial x^2} - \frac{i}{\hbar}V\Psi\right)\right)$$

$$= -i\hbar\left(\Psi^*\frac{i\hbar}{2m}\frac{\partial^2\Psi}{\partial x^2} - \frac{i}{\hbar}V\Psi^*\Psi\right)$$

$\Psi^*\,]^{\infty}$ and $\Psi^*\,]_{-\infty} = 0$ because probability $-\infty$ and ∞ is

$f(x) = \Psi^*$ $\quad g'(x) = \frac{\partial}{\partial x}\left(\frac{i\hbar}{2m}\frac{\partial^2\Psi}{\partial x^2}\right.$

$f'(x) = \frac{\partial\Psi^*}{\partial x}$ $\quad g(x) = \frac{i\hbar}{2m}\frac{\partial^2\Psi}{\partial x^2} -$

$$\left[\Psi^*\cdot\left(\frac{i\hbar}{2m}\frac{\partial^2\Psi}{\partial x^2} - \frac{i}{\hbar}V\Psi\right)\right]_{-\infty}^{\infty} - \int\left(\frac{\partial\Psi^*}{\partial x}\right)\left(\frac{i\hbar}{2m}\frac{\partial^2\Psi}{\partial x^2} - \frac{i}{\hbar}V\Psi\right)dx$$

$$= -\int\left(\frac{\partial\Psi^*}{\partial x}\right)\left(\frac{i\hbar}{2m}\frac{\partial^2\Psi}{\partial x^2} - \frac{i}{\hbar}V\Psi\right)dx$$

$$= -i\hbar\int\frac{\partial}{\partial t}\left(\Psi^*\frac{\partial\Psi}{\partial x}\right)dx$$

$$= -i\hbar\int\left(\frac{\partial\Psi^*}{\partial t}\frac{\partial\Psi}{\partial x} + \left(\frac{\partial}{\partial t}\frac{\partial\Psi}{\partial x}\right)\Psi^*\right)dx$$

$$= -i\hbar\left[\int\frac{\partial\Psi^*}{\partial t}\frac{\partial\Psi}{\partial x}dx + \int\left(\frac{\partial}{\partial t}\frac{\partial\Psi}{\partial x}\right)\Psi^*dx\right]$$

$$= -i\hbar\left[\int\left(-\frac{i\hbar}{2m}\frac{\partial^2\Psi^*}{\partial x^2} + \frac{i}{\hbar}V\Psi^*\right)\frac{\partial\Psi}{\partial x}dx + \int\left(\frac{\partial}{\partial x}\left(\frac{i\hbar}{2m}\frac{\partial^2\Psi}{\partial x^2} - \frac{i}{\hbar}V\Psi\right)\right)\cdot\Psi^*dx\right]$$

product rule

$$\cancel{-i\hbar\left[\int -\frac{i\hbar}{2m}\left(\frac{\partial}{\partial x}\frac{\partial}{\partial x}\Psi^*\right)\left(\frac{\partial\Psi}{\partial x}\right) + \frac{i}{\hbar}V\Psi^*\left(\frac{\partial\Psi}{\partial x}\right)dx\right.}$$

$$= -i\hbar\left[\int\left(-\frac{i\hbar}{2m}\frac{\partial^2\Psi^*}{\partial x^2} + \frac{i}{\hbar}V\Psi^*\right)\left(\frac{\partial\Psi}{\partial x}\right)dx - \int\left(\cancel{\frac{i\hbar}{2m} - \frac{i}{\hbar}V\Psi^*}\right)\cdot\left(\frac{\partial\Psi^*}{\partial x}\right)dx\right.$$

$$\left(\frac{i\hbar}{2m}\frac{\partial^2\Psi}{\partial x^2} - \frac{i}{\hbar}V\Psi\right)$$

$$= -i\hbar\int\left(-\frac{i\hbar}{2m}\frac{\partial^2\Psi^*}{\partial x^2} + \frac{i}{\hbar}V\Psi^*\right)\left(\frac{\partial\Psi}{\partial x}\right) - \left(\frac{i\hbar}{2m}\frac{\partial^2\Psi}{\partial x^2} - \frac{i}{\hbar}V\Psi\right)\left(\frac{\partial\Psi^*}{\partial x}\right)dx$$

$$= -i\hbar\int -\frac{i\hbar}{2m}\frac{\partial^2\Psi^*}{\partial x^2}\frac{\partial\Psi}{\partial x} + \frac{i}{\hbar}V\Psi^* - \frac{\partial\Psi^*}{\partial x}\frac{i\hbar}{2m}\frac{\partial^2\Psi}{\partial x^2} + \frac{i}{\hbar}V\Psi$$

$$= -i\hbar\int -\frac{i\hbar}{2m}\frac{\partial^2\Psi^*}{\partial x^2}\frac{\partial\Psi}{\partial x} + \frac{i}{\hbar}V\Psi^*\frac{\partial\Psi}{\partial x} + \Psi^*\frac{i\hbar}{2m}\frac{\partial^3\Psi}{\partial x^3} - \Psi^*\frac{i}{\hbar}\frac{\partial V}{\partial x}\Psi - \Psi^*\cancel{\frac{i}{\hbar}}\frac{\partial\Psi}{\partial x}V$$

$$= -i\hbar\int -\frac{i\hbar}{2m}\frac{\partial^2\Psi^*}{\partial x^2}\frac{\partial\Psi}{\partial x} + \Psi^*\frac{i\hbar}{2m}\frac{\partial^3\Psi}{\partial x^3}dx - \left\langle\frac{\partial V}{\partial x}\right\rangle$$

$$= -i\hbar\left[\int -\frac{i\hbar}{2m}\frac{\partial^2\Psi^*}{\partial x^2}\frac{\partial\Psi}{\partial x}dx + \int\Psi^*\frac{i\hbar}{2m}\frac{\partial^3\Psi}{\partial x^3}dx\right] - \left\langle\frac{\partial V}{\partial x}\right\rangle$$

$f(x) = \Psi^*$ $\quad g'(x) = \frac{\partial^3\Psi}{\partial x^3}$

$f'(x) = \frac{\partial\Psi^*}{\partial x}$ $\quad g(x) = \frac{\partial^2\Psi}{\partial x^2}$

$$= -i\hbar\left[\int -\frac{i\hbar}{2m}\frac{\partial^2\Psi^*}{\partial x^2}\frac{\partial\Psi}{\partial x}dx + \cancel{\left[\Psi^*\frac{\partial^2\Psi}{\partial x^2}\right]_{-\infty}^{\infty}} - \int\frac{\partial\Psi^*}{\partial x}\frac{\partial^2\Psi}{\partial x^2}\frac{i\hbar}{2m}dx\right] - \left\langle\frac{\partial V}{\partial x}\right\rangle$$

$f(x) =$

$f'(x) =$

$$= -i\hbar\left[\int -\frac{i\hbar}{2m}\frac{\partial^2\Psi^*}{\partial x^2}\frac{\partial\Psi}{\partial x}dx - \left[\left(\cancel{\frac{\partial\Psi^*}{\partial x}}\frac{\partial\Psi}{\partial x}\right)_{-\infty}^{\infty} - \int\frac{\partial^2\Psi^*}{\partial x^2}\frac{\partial\Psi}{\partial x}\frac{i\hbar}{2m}dx\right]\right] - \left\langle\frac{\partial V}{\partial x}\right\rangle$$

$$= -\left\langle\frac{\partial V}{\partial x}\right\rangle$$

9 $\Psi(x,t) = Ae^{-a[mx^2/\hbar + it]} = Ae^{-\frac{amx^2}{\hbar}} e^{-ait}$

(a) $\int_{-\infty}^{\infty} |\Psi|^2 dx$

$= \int_{-\infty}^{\infty} \Psi^* \Psi \, dx$

$= \int_{-\infty}^{\infty} A^2 e^{-\frac{2amx^2}{\hbar}} e^{-ait} e^{ait} dx$

$= A^2 \int_{-\infty}^{\infty} e^{-\frac{2amx^2}{\hbar}} dx$

~~u-substitution: $u = \frac{-2amx^2}{\hbar}$, $dx = \frac{du}{-\frac{4amx}{\hbar}} = -\frac{\hbar}{4amx} du$~~

$\cancel{= A^2 e^{-\frac{2am}{\hbar}} \int_{-\infty}^{\infty} e^{x^2} dx}$

$\cancel{= A^2 e^{-\frac{2am}{\hbar}} \sqrt{\pi}}$

$\cancel{A^2 e^{-\frac{2am}{\hbar}} \sqrt{\pi} = 1}$

$\cancel{A^2 = \frac{1}{e^{-\frac{2am}{\hbar}} \sqrt{\pi}}}$

$\cancel{A = \sqrt{e^{\frac{2am}{\hbar}} \frac{1}{\sqrt{\pi}}}}$

$\cancel{A^2 \int_{-\infty}^{\infty} e^{u} \cdot -\frac{\hbar}{4amx} du}$

$u = \frac{-2amx}{\hbar}$, $dx = \frac{du}{-\frac{2am}{\hbar}} = -\frac{\hbar}{2am} du$

$\cancel{= A^2 \int_{-\infty}^{\infty} e^{ux} \cdot -\frac{\hbar}{2am} du}$

$\cancel{= -A^2 \frac{\hbar}{2am} \int_{-\infty}^{\infty} e^{ux} du}$

$v = ux$, $du = \frac{dv}{x}$

$\cancel{= -A^2 \frac{\hbar}{2am} \frac{1}{x} \int_{-\infty}^{\infty} e^{v} dv}$

$\cancel{= -A^2 \frac{\hbar}{2am}\left(-\frac{\hbar}{2amx}\right) \cdot \left[e^{v}\right]_{-\infty}^{\infty}}$

$\cancel{= -A^2 \frac{\hbar}{2am}\left(-\frac{\hbar}{2amx}\right) \cdot \left[e^{-\frac{2amx^2}{\hbar}}\right]_{-\infty}^{\infty}}$

$u = \sqrt{\frac{2am}{\hbar}}\, x$

$\frac{du}{dx} = \sqrt{\frac{2am}{\hbar}} \Rightarrow dx = \frac{1}{\sqrt{\frac{2am}{\hbar}}} du$

$= A^2 \int_{-\infty}^{\infty} e^{-u^2} \sqrt{\frac{\hbar}{2am}} du$

$= A^2 \sqrt{\frac{\hbar}{2am}} \int_{-\infty}^{\infty} e^{-u^2} du$

$= A^2 \sqrt{\frac{\hbar}{2am}} \sqrt{\pi}$

$A^2 \sqrt{\frac{\hbar}{2am}} \sqrt{\pi} = 1$

$A^2 = \sqrt{\frac{2am}{\hbar\pi}}$

$A = \left(\frac{2am}{\hbar\pi}\right)^{\frac{1}{4}}$

(b) $i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2} + V\Psi$ Schrödinger Equation

$\frac{\partial \Psi}{\partial t} = -\frac{i\hbar}{2m} \frac{\partial^2 \Psi}{\partial x^2} - \frac{i}{\hbar} V\Psi$

$\Psi(x,t) = Ae^{-a[mx^2/\hbar + it]}$

#1 $\frac{\partial \Psi}{\partial t} = \frac{\partial}{\partial t} Ae^{-\frac{amx^2}{\hbar}} e^{-ait}$

$= Ae^{-\frac{amx^2}{\hbar}} \frac{\partial}{\partial t} e^{-ait}$

$= Ae^{-\frac{amx^2}{\hbar}} e^{-ait} (-ai\cancel{t})$

$\frac{\partial^2 \Psi}{\partial t^2} = \frac{\partial}{\partial t} -Aait\, e^{-\frac{amx^2}{\hbar}} e^{-ait}$

$= \left(-Aai e^{-\frac{amx^2}{\hbar}}\right) \frac{\partial}{\partial t} t \cdot e^{-ait}$

$= \left(-Aai e^{-\frac{amx^2}{\hbar}}\right) \left[e^{-ait}(-ai\cancel{t}) + e^{-ait}\right]$

$= \left(Aa^2\cancel{t} e^{-\frac{amx^2}{\hbar}} e^{-ait}\right) - Aai e^{-\frac{amx^2}{\hbar}} e^{-ait}$

$= -Aa e^{-\frac{amx^2}{\hbar}} e^{-ait} \left(a\cancel{t} + i\right)$

(b) $i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} + V\psi$ Endless

(a)

$\frac{i}{\hbar} V\psi = \frac{\partial \psi}{\partial t} - \frac{i\hbar}{2m}\frac{\partial^2 \psi}{\partial x^2}$

$V = -\frac{\hbar}{i}\frac{1}{\psi}\left(\frac{\partial \psi}{\partial t} - \frac{i\hbar}{2m}\frac{\partial^2 \psi}{\partial x^2}\right)$

$V = -\frac{\hbar}{i}\frac{1}{\psi}\frac{\partial \psi}{\partial t} + \frac{\hbar^2}{2m}\frac{1}{\psi}\frac{\partial^2 \psi}{\partial x^2}$

$V = -\frac{\hbar}{i}\frac{1}{A e^{-a(mx^2/\hbar + it)}} \cdot A e^{-\frac{amx^2}{\hbar}} e^{-ait}(-ai) + \frac{\hbar^2}{2m}\frac{1}{A e^{-a(mx^2/\hbar + it)}}\left(A\frac{2am}{\hbar} e^{-\frac{amx^2}{\hbar}} e^{-ait}\left(\frac{2am}{\hbar}x^2 - 1\right)\right)$

$$\boxed{\begin{aligned} \cancel{V} &= -\frac{\hbar}{i}(-ai) + \frac{\hbar^2}{2m}(-a)(a\hbar + i) \\ &= \hbar a - \frac{\hbar^2 a^2}{2m} - \frac{\hbar^2 i a}{2m} \\ &= \hbar a\left(1 - \frac{\hbar a}{2m} - \frac{\hbar i}{2m}\right) \\ &= \hbar a\left(1 - \frac{\hbar(a+i)}{2m}\right) \end{aligned}}$$

$V = -\frac{\hbar}{i}(-ai) + \hbar a\left(\frac{2am}{\hbar}x^2 - 1\right)$

$= \hbar a + 2a^2 m x^2 - a\hbar$

$\boxed{= 2a^2 m x^2}$

(a)

$\frac{\partial \psi}{\partial x} = \frac{\partial}{\partial x} A e^{-\frac{amx^2}{\hbar}} e^{-ait}$

$= A e^{-ait}\frac{\partial}{\partial x} e^{-\frac{amx^2}{\hbar}}$

$= A e^{-ait} e^{-\frac{amx^2}{\hbar}}\left(-\frac{2amx}{\hbar}\right)$

$\frac{\partial^2 \psi}{\partial x^2} = \frac{\partial}{\partial x}\, -\frac{2am}{\hbar} A e^{-ait} e^{-\frac{amx^2}{\hbar}} \cdot x$

$= -\frac{2am}{\hbar} A e^{-ait}\frac{\partial}{\partial x} e^{-\frac{amx^2}{\hbar}} \cdot x$

$= -\frac{2am}{\hbar} A e^{-ait}\left(e^{-\frac{amx^2}{\hbar}} \cdot \left(-\frac{2amx}{\hbar}\right) \cdot x + e^{-\frac{amx^2}{\hbar}}\right)$

$= A\left(\frac{2am}{\hbar}\right)^2 x^2 e^{-ait} e^{-\frac{amx^2}{\hbar}} - A\frac{2am}{\hbar} e^{-ait} e^{-\frac{amx^2}{\hbar}}$

$= A\frac{2am}{\hbar} e^{-ait} e^{-\frac{amx^2}{\hbar}}\left(\frac{2am}{\hbar}x^2 - 1\right)$

pg 20 9. (a) $\langle x \rangle = \int_{-\infty}^{\infty} x\,|\Psi|^2\,dx$

#1

$= A^2 \int_{-\infty}^{\infty} x\, e^{-\frac{2amx^2}{\hbar}}\,dx$

$u = \sqrt{\frac{2am}{\hbar}}\,x \Rightarrow x = u\sqrt{\frac{\hbar}{2am}}$

$\frac{du}{dx} = \sqrt{\frac{2am}{\hbar}} \Rightarrow dx = du\sqrt{\frac{\hbar}{2am}}$

$= A^2 \int_{-\infty}^{\infty} u\sqrt{\frac{\hbar}{2am}}\, e^{-u^2}\left(du\cdot\sqrt{\frac{\hbar}{2am}}\right)$

$= A^2 \frac{\hbar}{2am} \int_{-\infty}^{\infty} u\,e^{-u^2}\,du$

$\boxed{\int_{-a}^{a} g(x) f(x)\,dx = 0 \quad \text{if } g(x) \text{ is odd, } f(x) \text{ is even}}$

$= 0$

#3 $\langle p \rangle = m\frac{d\langle x \rangle}{dt} = m(0) = 0$

#2 $\langle x^2 \rangle = \int_{-\infty}^{\infty} x^2\,|\Psi|^2\,dx$

$= A^2 \int_{-\infty}^{\infty} x^2\, e^{-\frac{2amx^2}{\hbar}}\,dx$

u-substitution,

$= A^2 \int_{-\infty}^{\infty} u^2 \frac{\hbar}{2am}\, e^{-u^2}\left(du\sqrt{\frac{\hbar}{2am}}\right)$

$= A^2 \frac{\hbar}{2am}\sqrt{\frac{\hbar}{2am}} \int_{-\infty}^{\infty} u^2\, e^{-u^2}\,du$

$f(x) = u \qquad g'(x) = u e^{-u^2}$

$f'(x) = 1 \qquad g(x) = \int u e^{-u^2}\,du$

$v = -u^2$

$\frac{dv}{du} = -2u \Rightarrow u\,du = -\frac{dv}{2}$

$= \int e^{v} \cdot -\frac{dv}{2}$

$= -\frac{1}{2} e^{-u^2}$

$= A^2 \frac{\hbar}{2am}\sqrt{\frac{\hbar}{2am}} \left[\left[-\frac{1}{2} u e^{-u^2}\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} -\frac{1}{2} e^{-u^2}\,du\right]$

$= A^2 \frac{\hbar}{2am}\sqrt{\frac{\hbar}{2am}}\left(\frac{1}{2}\sqrt{\pi}\right)$

$= \sqrt{\frac{2am}{\hbar\pi}}\,\frac{\hbar}{2am}\sqrt{\frac{\hbar}{2am}}\,\frac{1}{2}\sqrt{\pi}$

$= \frac{\hbar}{4am}$

#4-1) $\langle p^2 \rangle = \int_{-\infty}^{\infty} p^2 |\Psi|^2 \, dx$

$\cancel{= \hbar^2 \int_{-\infty}^{\infty} p^2 e^{-\frac{2amx^2}{\hbar}} dx}$

$= \int_{-\infty}^{\infty} \Psi^* \left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)^2 \Psi \, dx$

$= -\hbar^2 \int_{-\infty}^{\infty} \Psi^* \frac{\partial^2 \Psi}{\partial x^2} dx$

$= -\hbar^2 \int_{-\infty}^{\infty} A e^{-\frac{amx^2}{\hbar}} e^{iat} \frac{\partial^2}{\partial x^2}\left(A e^{-\frac{amx^2}{\hbar}} e^{-iat}\right) dx$

$= -A^2 \hbar^2 \int_{-\infty}^{\infty} e^{-\frac{amx^2}{\hbar}} \frac{\partial^2}{\partial x^2}\left(e^{-\frac{amx^2}{\hbar}}\right) dx$

$$\frac{\partial^2}{\partial x^2}\left(e^{-\frac{amx^2}{\hbar}}\right)$$

$$= \frac{\partial}{\partial x}\left(-\frac{2amx}{\hbar} e^{-\frac{amx^2}{\hbar}}\right)$$

$$= -\frac{2am}{\hbar} \frac{\partial}{\partial x}\left(x \cdot e^{-\frac{amx^2}{\hbar}}\right)$$

$$= -\frac{2am}{\hbar}\left[e^{-\frac{amx^2}{\hbar}} + x\left(-\frac{2amx}{\hbar} e^{-\frac{amx^2}{\hbar}}\right)\right]$$

$$= -\frac{2am}{\hbar}\left[\left(e^{-\frac{amx^2}{\hbar}}\right)\left(1 - \frac{2amx^2}{\hbar}\right)\right]$$

$$= -\frac{2am}{\hbar} e^{-\frac{amx^2}{\hbar}} + \frac{4a^2m^2x^2}{\hbar^2} e^{-\frac{amx^2}{\hbar}}$$

$= -A^2 \hbar^2 \int_{-\infty}^{\infty} e^{-\frac{amx^2}{\hbar}}\left(-\frac{2am}{\hbar} e^{-\frac{amx^2}{\hbar}}\right) + e^{-\frac{amx^2}{\hbar}}\left(\frac{4a^2m^2x^2}{\hbar^2} e^{-\frac{amx^2}{\hbar}}\right) dx$

$= -A^2 \hbar^2 \left[-\frac{2am}{\hbar}\int_{-\infty}^{\infty} e^{-\frac{2amx^2}{\hbar}} dx + \frac{4a^2m^2}{\hbar^2}\int_{-\infty}^{\infty} x^2 e^{-\frac{2amx^2}{\hbar}} dx\right]$

$= -A^2 \hbar^2 \left[-\frac{2am}{\hbar}\sqrt{\frac{\pi\hbar}{2am}} + \frac{4a^2m^2}{\hbar^2}\frac{1}{2}\frac{\hbar}{2am}\sqrt{\pi}\sqrt{\frac{\hbar}{2am}}\right]$

$= \sqrt{\frac{2am}{\pi\hbar}}\,\hbar^2 \frac{2am}{\hbar}\sqrt{\frac{\pi\hbar}{2am}} - \frac{1}{2}\sqrt{\frac{2am}{\pi\hbar}}\,\hbar^2 \frac{4a^2m^2}{\hbar^2}\frac{\hbar}{2am}\sqrt{\pi}\sqrt{\frac{\hbar}{2am}}$

$= 2am\sqrt{2}\hbar - \cancel{4}\, am \cancel{\sqrt{\frac{2am}{\pi\hbar}}}\, \hbar\sqrt{2}$

$= 2am\hbar\left(\sqrt{2} - \sqrt{\frac{2am}{\pi}}\right)$

$\int_{-\infty}^{\infty} e^{-\frac{2amx^2}{\hbar}} dx$

$u = \sqrt{\frac{2am}{\hbar}}\,x \Rightarrow x = u\sqrt{\frac{\hbar}{2am}}$

$\frac{du}{dx} = \sqrt{\frac{2am}{\hbar}} \Rightarrow dx = du\sqrt{\frac{\hbar}{2am}}$

$= \int_{-\infty}^{\infty} e^{-u^2}\left(du\sqrt{\frac{\hbar}{2am}}\right)$

$= \sqrt{\frac{\hbar}{2am}}\int_{-\infty}^{\infty} e^{-u^2} du$

$= \sqrt{\frac{\pi\hbar}{2am}}$

$\int_{-\infty}^{\infty} x^2 e^{-\frac{2amx^2}{\hbar}} dx$

$= \int_{-\infty}^{\infty} u^2 \frac{\hbar}{2am} e^{-u^2} du\left(\sqrt{\frac{\hbar}{2am}}\right)$

$= \frac{\hbar}{2am}\sqrt{\frac{\hbar}{2am}}\int_{-\infty}^{\infty} u^2 e^{-u^2} du$

$f(x) = u \qquad g'(x) = u e^{-u^2}$

$f'(x) = 1 \qquad g(x) = -\frac{1}{2}e^{-u^2}$

$= \frac{\hbar}{2am}\sqrt{\frac{\hbar}{2am}}\left[\left(-\frac{1}{2}u e^{-u^2}\right)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} -\frac{1}{2}e^{-u^2} du\right]$

$\cancel{= \frac{1}{2}\frac{\hbar}{2am} u e^{-u^2} + \frac{1}{2}}$

$= \frac{1}{2}\frac{\hbar}{2am}\sqrt{\pi}\sqrt{\frac{\hbar}{2am}}$

sometimes, you got to have some crappy algebra and a pair of eyes that fail to pass the Kinglee eye test

(4.1) $\langle p^2 \rangle = \int_{-\infty}^{\infty} p^2 \, |\Psi|^2 \, dx$

$\cancel{= \int_{-\infty}^{\infty} p^2 \, e^{-\frac{2amx^2}{\hbar}} \, dx}$

$= \int_{-\infty}^{\infty} \Psi^* \left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)^2 \Psi \, dx$

$= -\hbar^2 \int_{-\infty}^{\infty} \Psi^* \frac{\partial^2 \Psi}{\partial x^2} \, dx$

$= -\hbar^2 \int_{-\infty}^{\infty} A e^{-\frac{amx^2}{\hbar}} e^{ait} \cdot \frac{\partial^2}{\partial x^2}\left(A e^{-\frac{amx^2}{\hbar}} e^{-ait}\right) dx$

$= -A^2 \hbar^2 \int_{-\infty}^{\infty} e^{-\frac{amx^2}{\hbar}} \cdot \frac{\partial^2}{\partial x^2}\left(e^{-\frac{amx^2}{\hbar}}\right) dx$

$$\frac{\partial^2}{\partial x^2}\left(e^{-\frac{amx^2}{\hbar}}\right)$$
$$= \frac{\partial}{\partial x}\left(-\frac{2amx}{\hbar} e^{-\frac{amx^2}{\hbar}}\right)$$
$$= -\frac{2am}{\hbar} \frac{\partial}{\partial x}\left(x \cdot e^{-\frac{amx^2}{\hbar}}\right)$$
$$= -\frac{2am}{\hbar}\left[e^{-\frac{amx^2}{\hbar}} + x\left(-\frac{2amx}{\hbar} e^{-\frac{amx^2}{\hbar}}\right)\right]$$
$$= -\frac{2am}{\hbar}\left[\left(e^{-\frac{amx^2}{\hbar}}\right) \cdot \left(1 - \frac{2amx^2}{\hbar}\right)\right]$$
$$= -\frac{2am}{\hbar} e^{-\frac{amx^2}{\hbar}} + \frac{4a^2m^2x^2}{\hbar^2} e^{-\frac{amx^2}{\hbar}}$$

$= -A^2 \hbar^2 \int_{-\infty}^{\infty} e^{-\frac{amx^2}{\hbar}}\left(-\frac{2am}{\hbar} e^{-\frac{amx^2}{\hbar}}\right) + e^{-\frac{amx^2}{\hbar}}\left(\frac{4a^2m^2x^2}{\hbar^2} e^{-\frac{amx^2}{\hbar}}\right) dx$

$= -A^2 \hbar^2 \left[-\frac{2am}{\hbar} \int_{-\infty}^{\infty} e^{-\frac{2amx^2}{\hbar}} dx + \frac{4a^2m^2}{\hbar^2} \int_{-\infty}^{\infty} x^2 e^{-\frac{2amx^2}{\hbar}} dx \right]$

$= -A^2 \hbar^2 \left[-\frac{2am}{\hbar}\sqrt{\frac{\hbar\pi}{2am}} + \frac{4a^2m^2}{\hbar^2} \frac{1}{2} \frac{\hbar}{2am} \sqrt{\pi} \sqrt{\frac{\hbar}{2am}} \right]$

$= +\sqrt{\frac{2am}{\hbar\pi}} \hbar^2 \cdot \frac{2am}{\hbar} \sqrt{\frac{\hbar\pi}{2am}} - \frac{1}{2}\sqrt{\frac{2am}{\hbar\pi}} \hbar^2 \frac{4a^2m^2}{\hbar^2} \frac{\hbar}{2am} \sqrt{\pi} \sqrt{\frac{\hbar}{2am}}$

$= 2am\sqrt{2}\hbar - 2am \hbar \sqrt{2}$

$= 2am\hbar\left(\sqrt{2} - \sqrt{\frac{2am}{\hbar}}\right)$

$\int_{-\infty}^{\infty} e^{-\frac{2amx^2}{\hbar}} dx$

$u = \sqrt{\frac{2am}{\hbar}}x \Rightarrow x = u\sqrt{\frac{\hbar}{2am}}$

$\frac{du}{dx} = \sqrt{\frac{2am}{\hbar}} \Rightarrow dx = du\sqrt{\frac{\hbar}{2am}}$

$= \int_{-\infty}^{\infty} e^{-u^2}\left(du\sqrt{\frac{\hbar}{2am}}\right)$

$= \sqrt{\frac{\hbar}{2am}} \int_{-\infty}^{\infty} e^{-u^2} du$

$= \sqrt{\frac{\hbar\pi}{2am}}$

$\int_{-\infty}^{\infty} x^2 e^{-\frac{2amx^2}{\hbar}} dx$

$= \int_{-\infty}^{\infty} u^2 \frac{\hbar}{2am} e^{-u^2} du\left(\sqrt{\frac{\hbar}{2am}}\right)$

$= \frac{\hbar}{2am}\sqrt{\frac{\hbar}{2am}} \int_{-\infty}^{\infty} u^2 e^{-u^2} du$

$f(u) = u \qquad g'(u) = u e^{-u^2}$

$f'(u) = du \qquad g(u) = -\frac{1}{2} e^{-u^2}$

$= \frac{\hbar}{2am}\sqrt{\frac{\hbar}{2am}}\left[\left(-\frac{1}{2} u e^{-u^2}\right)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} -\frac{1}{2} e^{-u^2} du\right]$

$\cancel{= -\frac{1}{2}\frac{\hbar}{2am} u e^{-u^2} + \frac{1}{2}}$

$= \frac{1}{2}\frac{\hbar}{2am}\sqrt{\pi}\sqrt{\frac{\hbar}{2am}}$

sometimes, you got to have some ~~crappy~~ algebra and a pair of eyes that fail to pass the Kinglee eye test

11 September 2018 — 1.6 The Uncertainty Principle — Day 6

#4-c

9 $\langle p^2 \rangle = \int_{-\infty}^{\infty} p^2 |\Psi|^2 \, dx$

$$= \int_{-\infty}^{\infty} \Psi^* \left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)^2 \Psi \, dx$$

$$= -\hbar^2 A^2 \int e^{-\frac{amx^2}{\hbar}} \frac{\partial^2}{\partial x^2}\left(e^{-\frac{amx^2}{\hbar}}\right) dx$$

$f(x) = e^{-\frac{amx^2}{\hbar}}$ $\qquad g'(x) = \frac{\partial^2}{\partial x^2}\left(e^{-\frac{amx^2}{\hbar}}\right)$

$f'(x) = -\frac{2amx}{\hbar} e^{-\frac{amx^2}{\hbar}}$ $\qquad g(x) = \frac{\partial}{\partial x}\left(e^{-\frac{amx^2}{\hbar}}\right)$

$$= -\hbar^2 A^2 \left[\left(e^{-\frac{amx^2}{\hbar}} \frac{\partial}{\partial x}\left(e^{-\frac{amx^2}{\hbar}}\right)\right)_{-\infty}^{\infty} - \int_{-\infty}^{\infty} -\frac{2amx}{\hbar} e^{-\frac{amx^2}{\hbar}} \frac{\partial}{\partial x} e^{-\frac{amx^2}{\hbar}} \, dx\right]$$

$$= -\hbar^2 A^2 \cdot -\int_{-\infty}^{\infty} -\frac{2amx}{\hbar} e^{-\frac{amx^2}{\hbar}} \left(-2\frac{amx}{\hbar} e^{-\frac{amx^2}{\hbar}}\right) dx$$

$$= +\hbar^2 A^2 \int_{-\infty}^{\infty} \frac{4a^2m^2x^2}{\hbar^2} e^{-\frac{2amx^2}{\hbar}} \, dx$$

$$\boxed{A^2 \int_{-\infty}^{\infty} x^2 e^{-\frac{2amx^2}{\hbar}} \, dx = \frac{\hbar}{4am}}$$

$$= +\hbar^2 \frac{4a^2m^2}{\hbar^2} A^2 \int_{-\infty}^{\infty} x^2 e^{-\frac{2amx^2}{\hbar}} \, dx$$

$$= + 4a^2m^2 \frac{\hbar}{4am}$$

$$= + am\hbar$$

(d) $\sigma_x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$ $\qquad \sigma_p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2}$

$= \sqrt{\frac{\hbar}{4am} - 0}$ $\qquad = \sqrt{am\hbar - 0}$

$= \sqrt{\frac{\hbar}{4am}}$ $\qquad = \sqrt{am\hbar}$

Uncertainty principle: $\boxed{\sigma_x \sigma_p \geq \frac{\hbar}{2}}$, hence,

$$\sqrt{\frac{\hbar}{4am}} \cdot \sqrt{am\hbar} = \sqrt{\frac{\hbar^2}{4}} = \frac{\hbar}{2}$$

Ψ is the Harmonic oscillator, since it meets the optimum condition of uncertainty principle.

$\boxed{\text{Pg 34}}$ 1 (a)

$$i\hbar \frac{1}{\Psi}\frac{d\Psi}{dt} = E$$

Assume energy is complex,

$$i\hbar \frac{1}{\Psi}\frac{d\Psi}{dt} = (E_1 + E_0 i)$$

$$i\hbar \frac{1}{E_1 + E_0 i}\frac{1}{\Psi}\frac{d\Psi}{dt} = 1$$

$$i\hbar \frac{1}{E_1 + E_0 i}\frac{1}{\Psi}\,d\Psi = dt$$

$$\int i\hbar \frac{1}{E_1 + E_0 i}\frac{1}{\Psi}\,d\Psi = \int dt$$

$$i\hbar \frac{1}{E_1 + E_0 i}\int \frac{1}{\Psi}\,d\Psi = t + C$$

$$i\hbar \frac{1}{E_1 + E_0 i}\ln|\Psi| = t + C$$

$$\ln|\Psi| = \frac{1}{i\hbar}(E_1 + E_0 i)t + C$$

$$|\Psi| = e^{\frac{1}{i\hbar}(E_1 + E_0 i)t + C}$$

$$|\Psi| = e^{\frac{1}{i\hbar}E_1 t + \frac{E_0}{i\hbar} i t + C}$$

$$|\Psi| = e^{\frac{1}{i\hbar}E_1 t + \frac{E_0}{\hbar}t + C} \quad \boxed{1}$$

$\boxed{1}$: $|\Psi|^2 = \Psi^*\Psi = e^{\frac{1}{i\hbar}E_1 t}\cdot e^{\frac{E_0}{\hbar}t}\cdot e^{c}\cdot e^{-\frac{1}{i\hbar}E_1 t}\cdot e^{\frac{E_0}{\hbar}t}\cdot e^{c}$

$$= e^{\frac{2E_0}{\hbar}t}\cdot e^{c}$$

~~$\int |\Psi|^2 dx$~~

$$\Psi(x,t) = \psi(x)\, e^{-\frac{iEt}{\hbar}}$$

Assume energy is complex,

$$\Psi(x,t) = \psi(x)\, e^{-i(E_1 + E_0 i)t/\hbar}$$

$$= \psi(x)\, e^{-E_1 it/\hbar} e^{E_0 t/\hbar}$$

$$|\Psi(x,t)|^2 = \psi(x)\, e^{-E_1 it/\hbar} e^{E_0 t/\hbar}\,\psi(x)\, e^{E_1 it/\hbar} e^{E_0 t/\hbar}$$

$$= (\psi(x))^2 e^{2E_0 t/\hbar}$$

$$\int_{-\infty}^{\infty}|\Psi|^2 dx = \int_{-\infty}^{\infty}(\psi(x))^2 e^{2E_0 t/\hbar}\,dx$$

$$1 = e^{2E_0 t/\hbar}\int_{-\infty}^{\infty}(\psi(x))^2 dx$$

To make $\int_{-\infty}^{\infty}(\psi(x))^2 dx = 1$,

$e^{2E_0 t/\hbar} = 1$, that is,

$2E_0 t/\hbar = 0$, hence,

$E_0 = 0$, therefore

E has to be real.

$\boxed{1.9.1}$ 1 (b) $z = a + ib$

$z^* = a - ib$

$\frac{z + z^*}{2} = a$ $\boxed{0.1}$

$\frac{z - z^*}{2i} = b$ $\boxed{0.2}$

$f(x) = a(x) + ib(x)$

If ψ_1 and ψ_2 are solutions then $A\psi_1 + B\psi_2$ is a solution.

$$-\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} + V\psi = E\psi \quad \boxed{1}$$

Take complex conjugate.

$$-\frac{\hbar^2}{2m}\frac{\partial^2 \psi^*}{\partial x^2} + V\psi^* = E\psi^* \quad \boxed{2}$$

Add $\boxed{1}$ and $\boxed{2}$:

$$-\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} + V\psi - \frac{\hbar^2}{2m}\frac{\partial^2 \psi^*}{\partial x^2} + V\psi^* = E\psi + E\psi^*$$

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi^*}{\partial x^2}\right) + V(\psi + \psi^*) = E(\psi + \psi^*) \quad \boxed{3}$$

ψ is real: Since ~~a~~ ψ in $\boxed{1}$ is real, its complex conjugate ~~a~~ ψ^* is real too in $\boxed{2}$. Linearly combine $\boxed{1}$ and $\boxed{2}$, the resulting $(\psi + \psi^*)$ is real as well.

ψ is complex: The linear combination of $\boxed{1}$ and $\boxed{2}$ creates $(\psi + \psi^*)$, which is always real despite if ψ is imagenary or not. ~~QED~~

$\boxed{1} - \boxed{2}$:

$$-\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} + V\psi + \frac{\hbar^2}{2m}\frac{\partial^2 \psi^*}{\partial x^2} - V\psi^* = E\psi - E\psi^* \quad \boxed{\times}$$

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi^*}{\partial x^2}\right) + V(\psi - \psi^*) = E(\psi - \psi^*) \quad \boxed{4}$$

The linear combination of $\boxed{1}$ and $\boxed{2}$ also creates $(\psi - \psi^*)$, which is always real despite if ψ is imagenary or not, according to $\boxed{0.2}$.

Thus, with two imaginary ψ solutions, there are two real ψ solutions that can be derived by linear combination. QED

(c) ① if $\psi(x)$ is an even function,

$$\psi(x) = \psi(-x) \quad \boxed{1}$$

$$-\frac{\hbar^2}{2m}\frac{\partial^2 \psi(x)}{\partial x^2} + V(x)\psi(x) = E\psi(x), \quad \boxed{2}$$

$$-\frac{\hbar}{2m}\frac{\partial^2 \psi(-x)}{\partial x^2} + V(-x)\psi(-x) = E\psi(-x) \quad \boxed{3}$$

If $\psi(x)$ is an even function, and $V(x)$ is an even function, then $\boxed{3}$ satisfies $\boxed{2}$ since $\boxed{1}$.

② if $\psi(x)$ is an odd function,

$$\psi(x) = -\psi(-x) \quad \boxed{4}$$

$$+\frac{\hbar}{2m}\frac{\partial^2 \psi(-x)}{\partial x^2} \mp V(-x)\psi(-x) = -E\psi(-x) \quad \boxed{5}$$

Since $\boxed{4}$, $-\frac{\hbar}{2m}\frac{\partial^2 \psi(x)}{\partial x^2} + V(x)\psi(x) = E\psi(x) \quad \boxed{6}$

If $\psi(x)$ is an odd function, and $V(x)$ is an even function, then $\boxed{5}$ satisfies $\boxed{6} = \boxed{2}$ since $\boxed{4}$

(c) (#2)

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi = E\psi \quad \boxed{1}$$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(-x)}{dx^2} + V(-x)\psi(-x) = E\psi(-x) \quad \boxed{2}$$

$$\psi(x) = -\psi(-x)$$

$\phi(x) = \psi(x) + \psi(-x)$ is even

$\varepsilon(x) = \psi(x) - \psi(-x)$ is odd

$\boxed{1} - \boxed{2}$:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi + \frac{\hbar^2}{2m}\frac{d^2\psi(-x)}{dx^2} - V(-x)\psi(-x) = E\psi - E\psi(-x)$$

$$-\frac{\hbar^2}{2m}\left(\frac{d^2\psi}{dx^2} - \frac{d^2\psi(-x)}{dx^2}\right) + V(\psi(x) - \psi(-x)) = E(\psi(x) - \psi(-x))$$

$$-\frac{\hbar^2}{2m}\left(\frac{d^2}{dx^2}(\psi(x) - \psi(x))\right) + V(\psi(x) - \psi(-x)) = E(\psi(x) - \psi(-x))$$

Let $u = \psi(x) - \psi(-x)$, then

$$-\frac{\hbar^2}{2m}\frac{d^2 u}{dx^2} + V(u) = E(u)$$

$$\cancel{-\frac{\hbar^2}{2m}\frac{du^2}{dx^2}\frac{d^2}{du^2}u + Vu = Eu}$$

$$\cancel{\frac{d^2u}{du^2}\frac{dx^2}{dx^2}}$$

$\Rightarrow$

$$\frac{du}{dx} = \frac{d}{dx}\psi(x) - \frac{d}{dx}\psi(-x)$$
$$= \frac{d}{dx}\psi(x) + \frac{d}{dx}\psi(x)$$
$$= \frac{d}{dx}(\psi(x) + \psi(-x))$$
$$\frac{d^2u}{dx^2} = \frac{d}{dx}\frac{d}{dx}(\psi(x) + \psi(-x))$$
$$\cancel{= \frac{d^2}{dx^2}\psi(x)}$$
$$= \frac{d^2}{dx^2}\psi(x) + \frac{d^2}{dx^2}\psi(-x)$$
$$= \frac{d^2}{dx^2}\psi(x) - \frac{d^2}{dx^2}\psi(-x)$$
$$= \frac{d}{dx^2}(\psi(x) - \psi(-x))$$

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

$\frac{d}{dx}\psi(u) \rightarrow \frac{du}{dx} = -1$, $u = -x$

$\frac{du}{dx}\frac{d}{du}\psi(u)$

$-\frac{d}{du}\psi(u)$

$\frac{d^2}{dx^2}\psi(-x)$

$x(u)$, $u = -x$

$\frac{du}{dx^2}\frac{d^2}{du^2}x(u)$

$$\cancel{-\frac{\hbar^2}{2m}\left[\frac{d}{dx^2}(\psi(x) - \psi(-x))\right]}$$

$$-\frac{\hbar^2}{2m}(\psi''(x) - \psi''(-x)) + Vu = Eu$$

$$-\frac{\hbar^2}{2m}(\psi''(x) - \psi''(-x)) + V(\psi(x) - \psi(-x)) = E(\psi(x) - \psi(-x)) \quad \boxed{3}$$

$$\frac{du}{dx} = \frac{d}{dx}(\psi(x) - \psi(-x))$$
$$= \psi'(x) - \frac{d}{dx}\psi(-x)$$
$$= \psi'(x) - \frac{du}{dx}\frac{d}{du}\psi(u) \qquad \text{for } u = -x,\ \frac{du}{dx} = -1$$
$$= \psi'(x) + \psi'(-x)$$

$$\frac{d^2u}{dx^2} = \frac{d}{dx}(\psi'(x) + \psi'(-x))$$
$$= \psi''(x) + \frac{du}{dx}\frac{d}{du}\psi'(u)$$
$$= \psi''(x) - \psi''(-x)$$

$\boxed{3} - \boxed{1}$:

$$-\frac{\hbar^2}{2m}(\psi''(x) - \psi''(-x)) + V(\psi(x) - \psi(-x)) + \frac{\hbar^2}{2m}\psi''(x) - V\psi(x) = E(\psi(x) - \psi(x)) - E\psi(x)$$

$$-\frac{\hbar^2}{2m}(-\psi''(-x)) - V\psi(-x) = -E\psi(-x)$$

$$\frac{\hbar^2}{2m}\psi''(-x) + V\psi(-x) = E\psi(-x)$$

Since $V(x) = V(-x)$, thus

$$\frac{\hbar^2}{2m}\psi''(-x) + V(-x)\psi(-x) = E\psi(-x)$$

satisfies the schrodinger equation.

[P] 51 | 2. $\frac{d^2\psi}{dx^2} = \frac{2m}{\hbar^2}[V(x)-E]\psi$

→ If $E < V_{min}$, then $[V(x)-E] \geq 0$, thus $\frac{d^2\psi}{dx^2}$ and ψ have the same sign.

If $\psi > 0$, then $\frac{d^2\psi}{dx^2} > 0$, thus ψ is a positive function that is concave up. As $x \to \infty$, $\psi \to \infty$, hence ψ is not normalizable. If $\psi < 0$, then $\frac{d^2\psi}{dx^2} < 0$, thus ψ is a negative function that is concave down. As $x \to -\infty$, $\psi \to -\infty$, hence ψ is not normalizable. QED.

→ If $E < V_{min}$, then $[V(x)-E] > 0$.

Assume $V(x)$ is constant:

$$\frac{d^2\psi}{dx^2} = \frac{2m}{\hbar^2}[V(x)-E]\psi \Rightarrow \psi'' = \frac{2m}{\hbar^2}[V(x)-E]\psi$$

$$\psi'\psi'' = \frac{2m}{\hbar^2}[V(x)-E]\psi\psi'$$

$$\frac{d}{dx}\left[\tfrac{1}{2}\psi'^2\right] = \frac{2m}{\hbar^2}[V(x)-E]\frac{d}{dx}\left[\tfrac{1}{2}\psi^2\right]$$ chain rule

$$\psi'^2 = \frac{2m}{\hbar^2}[V(x)-E]\psi^2$$

$$\psi' = \pm\sqrt{\frac{2m}{\hbar^2}[V(x)-E]}\,\psi \quad \boxed{1}$$

~~If $E < V_{min}$, then $\sqrt{\frac{2m}{\hbar^2}[V(x)-E]} > 0$~~

$$\frac{d\psi}{dx} = \pm\sqrt{\frac{2m}{\hbar^2}[V(x)-E]}\,\psi$$

$$\int\frac{d\psi}{\psi} = \int \pm\sqrt{\frac{2m}{\hbar^2}[V(x)-E]}\,dx$$

$$\ln(\psi) = \pm\sqrt{\frac{2m}{\hbar^2}[V(x)-E]}\,x + C$$

$$\psi = e^{\pm\sqrt{\frac{2m}{\hbar^2}[V(x)-E]}\,x + C}$$

$$\psi = e^{C}\cdot e^{\pm\sqrt{\frac{2m}{\hbar^2}[V(x)-E]}\,x}$$

ψ is an exponential function that behaves either as $x \to -\infty$, $\psi \to \infty$, or as $x \to \infty$, $\psi \to \infty$, thus ψ is not normalizable if $\left(\frac{2m}{\hbar^2}[V(x)-E]\right) > 0$, that is, if $E < V_{min}$.

→ If $E > V_{min}$, then $[V(x)-E] < 0$.

$\boxed{1}$: $$\psi' = \pm i\sqrt{\frac{2m}{\hbar^2}[V(x)-E]}\,\psi$$

$$\int\frac{d\psi}{\psi} = \int \pm i\sqrt{\frac{2m}{\hbar^2}[V(x)-E]}\,dx$$

$$\ln(\psi) = \pm i\sqrt{\frac{2m}{\hbar^2}[V(x)-E]}\,x + C$$

$$\psi = e^{\pm i\sqrt{\frac{2m}{\hbar^2}[V(x)-E]}\,x + C}$$

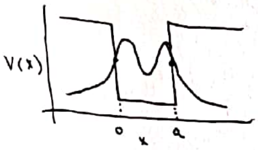
$$\psi = e^{C}\cdot e^{\pm i\sqrt{\frac{2m}{\hbar^2}[V(x)-E]}\,x}$$

Since $e^{ix} = \cos(x) + i\sin(x)$, ψ is a linear combination of $\cos(x)$ and $\sin(x)$, two normalizable solutions, hence, ψ is normalizable.

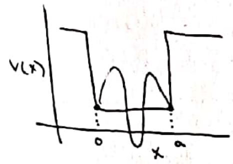
Day 10

19 September 2018   2.2 The Infinite Square Well   Day 9

$$V(x) = \begin{cases} V & x<0,\ x>a \\ 0 & \text{elsewhere} \end{cases}$$



If $V \to \infty$, then



Notice when

$\psi(0) = \psi(a) = 0$

$$-\frac{\hbar^2}{2m}\psi'' + V\psi = E\psi \quad \boxed{1}$$

$$\psi'' = -\frac{E-V}{\hbar^2}2m\psi$$

$$\psi'' = \omega^2\psi$$

$$\psi = Ae^{\omega x} - Be^{-\omega x}$$

$$\psi'' = -k^2\psi$$

$$\psi = A\sin(kx) + B\cos(kx)$$

When $\psi = 0$ at $x<0,\ x>a$,

$$\boxed{1} \quad -\frac{\hbar^2}{2m}\psi'' + 0\psi = E\psi$$

$$\psi'' = -\frac{2mE}{\hbar^2}\psi$$

$$\psi'\psi'' = -\frac{2mE}{\hbar^2}\psi\psi'$$

$$\frac{1}{2}\frac{d}{dx}\psi'^2 = -\frac{2mE}{\hbar^2}\frac{1}{2}\frac{d}{dx}\psi^2$$

$$\psi' = \pm\sqrt{-\frac{2mE}{\hbar^2}\cancel{\psi}}\ \psi$$

$$\frac{d\psi}{dx} = \pm i\sqrt{\frac{2mE}{\hbar^2}}\psi$$

$$\int\frac{d\psi}{\psi} = \pm\int i\sqrt{\frac{2mE}{\hbar^2}}dx$$

$$\ln(\psi) = \pm i\sqrt{\frac{2mE}{\hbar^2}}x + C$$

$$\psi = Ae^{\pm i\sqrt{\frac{2mE}{\hbar^2}}x} \quad \boxed{2}$$

Since the linear combination of two solutions is also a solution, then,

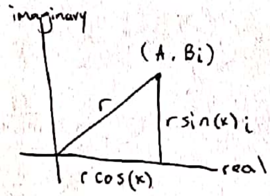
$$\psi = Ae^{i\sqrt{\frac{2mE}{\hbar^2}}x} + Be^{-i\sqrt{\frac{2mE}{\hbar^2}}x} \quad \boxed{3}.$$

Consider Euler's equation

$$re^{ix} = r\cos(x) + ir\sin(x) \quad \boxed{4}$$

and its complex conjugate

$$re^{-ix} = r\cos(x) - ir\sin(x) \quad \boxed{5}$$



$$\frac{\boxed{4}+\boxed{5}}{2}: \quad r\frac{e^{ix}+e^{-ix}}{2} = r\cos(x) \quad \boxed{6}$$

$$\frac{\boxed{4}-\boxed{5}}{2i}: \quad r\frac{e^{ix}-e^{-ix}}{2i} = r\sin(x) \quad \boxed{7}$$

Since $\boxed{6}$ and $\boxed{7}$, $\boxed{3}$ can be written as

$$\psi = A\cos\left(\sqrt{\frac{2mE}{\hbar^2}}x\right) + B\sin\left(\sqrt{\frac{2mE}{\hbar^2}}x\right)$$

Because $\psi(0) = 0$, thus

$$\psi(0) = A = 0$$

Because $\psi(a) = 0$, thus

$$\psi(a) = 0 = B\sin\left(\sqrt{\frac{2mE}{\hbar^2}}a\right)$$

B cannot be zero, else ψ will be unnormalizeable, hence

$$\sin\left(\sqrt{\frac{2mE}{\hbar^2}}a\right) = 0$$

$$\sqrt{\frac{2mE}{\hbar^2}}a = n\pi \quad (n \text{ is an integer})$$

$$\frac{2mE}{\hbar^2} = \frac{n^2\pi^2}{a^2}$$

$$E = \frac{n^2\pi^2\hbar^2}{a^2 2m} \quad \boxed{8}$$

Thus, energy in the well is the multiple of a constant.

Find constant B.

Known that

$$\int_0^a |\psi|^2 \, dx = 1 \qquad \boxed{9}$$

$$\int_0^a \psi^* \psi \, dx$$

$$= \int_0^a |B|^2 \sin^2\left(\sqrt{\frac{2mE}{\hbar^2}}\, x\right) dx$$

$$= |B|^2 \int_0^a \sin^2\left(\sqrt{\frac{2mE}{\hbar^2}}\, x\right) dx$$

$$u = \sqrt{\frac{2mE}{\hbar^2}}\, x, \quad \frac{du}{dx} = \sqrt{\frac{2mE}{\hbar^2}} \Rightarrow dx = du\sqrt{\frac{\hbar^2}{2mE}}$$

$$= |B|^2 \sqrt{\frac{\hbar^2}{2mE}} \int_0^a \sin^2(u)\, du$$

$$v = \sin u, \quad \frac{dv}{du} = \cos u \Rightarrow du = \frac{dv}{\cos u}$$

$$\hookrightarrow u = \arcsin(v) \qquad du = \frac{dv}{\cos(\arcsin(v))}$$

$$= |B|^2 \sqrt{\frac{\hbar^2}{2mE}} \int_0^a v^2 \frac{1}{\cos(\arcsin(v))}\, dv$$

Since $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos(2x)$ from half-angle formulas,

$$= |B|^2 \int_0^a \frac{1}{2} - \frac{1}{2}\cos\left(2\sqrt{\frac{2mE}{\hbar^2}}\, x\right) dx$$

$$= |B|^2 \left[\int_0^a \frac{1}{2}\, dx - \frac{1}{2}\int_0^a \cos\left(2\sqrt{\frac{2mE}{\hbar^2}}\, x\right) dx \right]$$

$$u = 2\sqrt{\frac{2mE}{\hbar^2}}\, x \qquad dx = \frac{1}{2}\sqrt{\frac{\hbar^2}{2mE}}\, dx$$

$$= |B|^2 \left[\frac{1}{2}a - \frac{1}{2}\int_0^a \cos(u)\, \frac{1}{2}\sqrt{\frac{\hbar^2}{2mE}}\, dx \right]$$

$$= |B|^2 \left[\frac{1}{2}a - \frac{1}{4}\sqrt{\frac{\hbar^2}{2mE}} \Big[\sin u\Big]_0^{2\sqrt{\frac{2mE}{\hbar^2}}a} \right]$$

$$= |B|^2 \left[\frac{1}{2}a - \frac{1}{4}\sqrt{\frac{\hbar^2}{2mE}} \cdot \sin\left(2\sqrt{\frac{2mE}{\hbar^2}}\, a\right) \right] = 1$$

$$B = \sqrt{\frac{1}{\frac{1}{2}a - \frac{1}{4}\sqrt{\frac{\hbar^2}{2mE}} \sin\left(2\sqrt{\frac{2mE}{\hbar^2}}\, a\right)}}$$

Since $\sqrt{\frac{2mE}{\hbar^2}} = \frac{n\pi}{a}$,

$$B = \sqrt{\frac{1}{\frac{1}{2}a - \frac{1}{4}\frac{a}{n\pi}\sin(n\pi)}}$$

$$B = \sqrt{\frac{2}{a}}$$

$\boxed{pg\,39}$ 4. $\psi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a}x\right)$

$\langle x \rangle = \int_{-\infty}^{\infty} \psi^* \psi\, x \, dx$

$= \int_0^a \frac{2}{a} \sin^2\left(\frac{n\pi}{a}x\right) x \, dx$

$= \frac{2}{a} \int_0^a x \sin^2\left(\frac{n\pi}{a}x\right) dx$

$f(x) = x \qquad g'(x) = \sin^2\left(\frac{n\pi}{a}x\right)$

$f'(x) = 1 \qquad g(x) = \int \frac{1}{2} - \frac{1}{2}\cos\left(\frac{2n\pi}{a}x\right) dx$

$= \frac{x}{2} - \frac{1}{2} \cdot \frac{a}{2n\pi} \cdot \sin\left(\frac{2n\pi}{a}x\right)$

$= \frac{2}{a}\left[\left[x\left(\frac{x}{2} - \frac{1}{2}\frac{a}{2n\pi}\sin\left(\frac{2n\pi}{a}x\right)\right)\right]_0^a - \int_0^a \frac{x}{2} - \frac{1}{2}\frac{a}{2n\pi}\sin\left(\frac{2n\pi}{a}x\right) dx \right]$

$= \frac{2}{a}\left[\frac{a^2}{2} - \left[\frac{1}{2}\left[\frac{1}{2}x^2\right]_0^a - \frac{1}{2}\frac{a}{2n\pi}\frac{a}{2n\pi}\left(-\cos\left(\frac{2n\pi}{a}x\right)\right)_0^a \right]\right]$

$= \frac{2}{a}\left[\frac{a^2}{2} - \left[\frac{1}{4}a^2 \cancel{+ \frac{a^2}{8\pi^2 n^2}}\right]\right]$

$= \frac{2}{a}\left(\frac{1}{4}a^2 \cancel{- \frac{a^2}{8n^2\pi^2}} \right)$

$= \frac{1}{2}a \cancel{- \frac{1}{4}\frac{a}{n^2\pi^2}}$

$\langle x^2 \rangle = \int_{-\infty}^{\infty} \psi^* \psi\, x^2 \, dx$

$= \frac{2}{a} \int_0^a x^2 \sin^2\left(\frac{n\pi}{a}x\right) dx$

$f(x) = x^2 \qquad g'(x) = \sin^2\left(\frac{n\pi}{a}x\right)$

$f'(x) = 2x \qquad g(x) = \frac{x}{2} - \frac{1}{2}\frac{a}{2n\pi}\sin\left(\frac{2n\pi}{a}x\right)$

$= \frac{2}{a}\left[\left[x^2\left(\frac{x}{2} - \frac{1}{2}\frac{a}{2n\pi}\sin\left(\frac{2n\pi}{a}x\right)\right)\right]_0^a - \int_0^a x^2 - x\frac{a}{2n\pi}\sin\left(\frac{2n\pi}{a}x\right) dx \right]$

$= \frac{2}{a}\left[\frac{a^3}{2} - \left[\frac{1}{3}a^3 - \int_0^a \frac{a}{2n\pi} x \sin\left(\frac{2n\pi}{a}x\right) dx \right]\right]$

$f(x) = x \qquad g'(x) = \sin\left(\frac{2n\pi}{a}x\right)$

$f'(x) = 1 \qquad g(x) = \frac{-a}{2n\pi}\cos\left(\frac{2n\pi}{a}x\right)$

$= \frac{2}{a}\left[\frac{a^3}{2} - \frac{1}{3}a^3 + \frac{a}{2n\pi}\left[\left[-x\frac{a}{2n\pi}\cos\left(\frac{2n\pi}{a}x\right)\right]_0^a - \int_0^a -\frac{a}{2n\pi}\cos\left(\frac{2n\pi}{a}x\right) dx\right]\right]$

$= \frac{2}{a}\left[\frac{a^3}{2} - \frac{1}{3}a^3 + \frac{a}{2n\pi}\left[-\frac{a^2}{2n\pi} + \frac{a}{2n\pi}\frac{a}{2n\pi}\left[\sin\left(\frac{2n\pi}{a}x\right)\right]_0^a\right]\right]$

$= \frac{2}{a}\left[\frac{a^3}{2} - \frac{1}{3}a^3 + \frac{a}{2n\pi}\left[-\frac{a^2}{2n\pi}\right]\right]$

$= a^2 - \frac{2}{3}a^2 - \frac{a^2}{2n^2\pi^2}$

$= \frac{1}{3}a^2 - \frac{a^2}{2n^2\pi^2}$

$$\langle p \rangle = m\frac{d\langle x \rangle}{dt} = 0 \qquad \int_0^{8n\pi} \sin^2(x)\,dx = \frac{1}{2}n\pi$$

$$\langle p^2 \rangle = \int_{-\infty}^{\infty} \psi^* \left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)^2 \psi \, dx$$

$$= \int_0^a \frac{a}{2} \sin\left(\frac{n\pi}{a}x\right) \cdot \frac{\hbar^2}{i^2} \cdot \frac{\partial^2}{\partial x^2}\left(\sin\left(\frac{n\pi}{a}x\right)\right) dx$$

$$= \frac{a}{2}\frac{\hbar^2}{i^2} \int_0^a \sin\left(\frac{n\pi}{a}x\right) \cdot \frac{\partial^2}{\partial x^2}\left(\sin\left(\frac{n\pi}{a}x\right)\right) dx$$

$$f(x) = \cancel{\cos}\ \sin\left(\frac{n\pi}{a}x\right) \qquad g'(x) = \frac{\partial^2}{\partial x^2}\left(\sin\left(\frac{n\pi}{a}x\right)\right)$$

$$f'(x) = \frac{n\pi}{a} \cdot \cos\left(\frac{n\pi}{a}x\right) \qquad g(x) = \frac{\partial}{\partial x}\left(\sin\left(\frac{n\pi}{a}x\right)\right)$$

$$= \frac{a}{2}\frac{\hbar^2}{i^2}\left[\left[\sin\left(\frac{n\pi}{a}x\right) \cdot \frac{\partial}{\partial x}\left(\sin\left(\frac{n\pi}{a}x\right)\right)\right]_0^a - \int_0^a \frac{n\pi}{a} \cos\left(\frac{n\pi}{a}x\right) \cdot \frac{\partial}{\partial x}\left(\sin\left(\frac{n\pi}{a}x\right)\right) dx \right]$$

$$= \frac{a}{2}\frac{\hbar^2}{i^2}\left[-\frac{n\pi}{a} \int_0^a \cos\left(\frac{n\pi}{a}x\right) \cdot \frac{\partial}{\partial x}\left(\sin\left(\frac{n\pi}{a}x\right)\right) dx \right]$$

$$f(x) = \cos\left(\frac{n\pi}{a}x\right) \qquad g'(x) = \frac{\partial}{\partial x}\sin\left(\frac{n\pi}{a}x\right)$$

$$f'(x) = -\frac{n\pi}{a}\sin\left(\frac{n\pi}{a}x\right) \qquad g(x) = \sin\left(\frac{n\pi}{a}x\right)$$

$$= -\frac{\hbar^2 n\pi}{2i^2}\left[\left[\cos\left(\frac{n\pi}{a}x\right) \cdot \sin\left(\frac{n\pi}{a}x\right)\right]_0^a - \int_0^a -\frac{n\pi}{a}\sin\left(\frac{n\pi}{a}x\right) \cdot \sin\left(\frac{n\pi}{a}x\right) dx \right]$$

$$= -\frac{\hbar^2 n\pi}{2i^2}\left[+\frac{n\pi}{a} \int_0^a \sin^2\left(\frac{n\pi}{a}x\right) dx \right]$$

$$= +\frac{\hbar^2 n^2 \pi^2}{2a} \int_0^a \frac{1}{2} - \frac{1}{2}\cos\left(\frac{2n\pi}{a}x\right) dx$$

$$= +\frac{\hbar^2 n^2 \pi^2}{2a}\left[\left[\frac{1}{2}x\right]_0^a - \frac{1}{2}\left[\frac{a}{2n\pi}\sin\left(\frac{2n\pi}{a}x\right)\right]_0^a \right]$$

$$= -\frac{a^3\hbar^2}{2i^2 n^2 \pi^2}\left[\frac{a}{2} \right]$$

$$= -\frac{a^4 \hbar^2}{4i^2 n^2 \pi^2}$$

$$= \hbar^2 \frac{a^4}{4n^2\pi^2}$$

$$\sigma_x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \sqrt{\frac{1}{3}a^2 - \frac{a^2}{2n^2\pi^2} - \frac{a^2}{4}} = \sqrt{a^2 \frac{4n^2\pi^2 - 6 - 3n^2\pi^2}{12n^2\pi^2}}$$

$$\sigma_p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2} = \sqrt{\frac{\hbar^2 n^2 \pi^2}{a^2}} = \frac{\hbar n\pi}{a}$$

$$\sigma_x \sigma_p = \hbar n\pi \sqrt{\frac{4n^2\pi^2 - 6 - 3n^2\pi^2}{12n^2\pi^2}} = \hbar n\pi \sqrt{\frac{n^2\pi^2 - 6}{12n^2\pi^2}} = \hbar n\pi \frac{\sqrt{n^2\pi^2 - 6}}{2\sqrt{3}\,n\pi} = \frac{\hbar}{2}\sqrt{\frac{n^2\pi^2 - 6}{3}} \geq \frac{\hbar}{2}$$

Day 12

28 September 2018 — 2.1 The Infinite Square Well

$\boxed{\text{pg } 40}$ 2.7

$$\Psi(x,0) = \begin{cases} Ax & 0 \le x \le \frac{a}{2} \\ A(a-x) & \frac{a}{2} \le x \le a \end{cases}$$

(a) $\int_{-\infty}^{\infty} |\Psi|^2 \, dx = 1$

$$= \int_0^{\frac{a}{2}} A^2x^2 \, dx + \int_{\frac{a}{2}}^{a} A^2a^2 - 2A^2ax + A^2x^2 \, dx$$

$$= A^2\left[\tfrac{1}{3}x^3\right]_0^{\frac{a}{2}} + \left[A^2a^2x\right]_{\frac{a}{2}}^{a} - 2A^2a\left[\tfrac{1}{2}x^2\right]_{\frac{a}{2}}^{a} + A^2\left[\tfrac{1}{3}x^3\right]_{\frac{a}{2}}^{a}$$

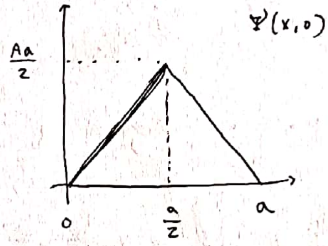
$$= \tfrac{1}{3}A^2\frac{a^3}{8} + \left[A^2a^3 - \tfrac{1}{2}A^2a^3\right] - 2A^2a\left[\tfrac{1}{2}a^2 - \tfrac{1}{8}a^2\right] + A^2\left[\tfrac{1}{3}a^3 - \tfrac{1}{3}\frac{a^3}{8}\right]$$

$$= \frac{A^2a^3}{24} + \tfrac{1}{2}A^2a^3 - \tfrac{3}{4}A^2a^3 + \tfrac{7}{24}A^2a^3$$

$$= \tfrac{1}{12}A^2a^3 = 1$$

$$A^2 = \frac{12}{a^3}$$

$$A = \frac{2\sqrt{3}}{\sqrt{a^3}}$$



(b) Known $f(x) = \sum_{n=1}^{\infty} \sin\left(\frac{\pi n x}{a}\right) \cdot \int_0^a f(x)\sin\left(\frac{\pi n x}{a}\right) dx$

$$= \sum_{n=1}^{\infty} A_n \sin\left(\frac{\pi n x}{a}\right)$$

then $\Psi(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{a}\right) e^{\frac{-iEt}{\hbar}}$

$$\Psi(x,0) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{a}\right) \cdot \left[\int_0^{\frac{a}{2}} Ax \sin\left(\frac{\pi n x}{a}\right) dx + \int_{\frac{a}{2}}^{a} A(a-x)\sin\left(\frac{\pi n x}{a}\right) dx\right]$$

$f(x) = x$ $\quad g(x) = \sin\left(\frac{n\pi x}{a}\right)$

$f'(x) = 1$ $\quad g'(x) = \frac{-a}{n\pi}\cos\left(\frac{n\pi}{a}x\right)$

$$= \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{a}\right) \cdot \left[A\left[\left[-\frac{ax}{n\pi}\cos\left(\frac{n\pi x}{a}\right)\right]_0^{a/2} - \int_0^{a/2} -\frac{a}{n\pi}\cos\left(\frac{n\pi x}{a}\right) dx\right] + \left[Aax\right]_{\frac{a}{2}}^{a} - A\left[\left[-\frac{ax}{n\pi}\cos\left(\frac{n\pi x}{a}\right)\right]_{a/2}^{a} - \int_{a/2}^{a} -\frac{a}{n\pi}\cos\left(\frac{n\pi x}{a}\right) dx\right]\right]$$

$$= \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{a}\right) \cdot \left[\left(+\overset{0}{\cancel{\frac{Aax}{n\pi}}}\right) + \frac{a}{n\pi}\frac{a}{n\pi}\left[\sin\left(\frac{n\pi x}{a}\right)\right]_0^{a/2} + Aa^2 - \tfrac{1}{2}Aa^2 + \underset{\frac{a^2}{n\pi}\cos(n\pi)}{\cancel{\frac{Aax}{n\pi}}} + \frac{a}{n\pi}\frac{a}{n\pi}\left[\sin\left(\frac{n\pi x}{a}\right)\right]_{a/2}^{a}\right]$$

$$= \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{a}\right)\left[\cancel{\frac{Aax}{n\pi}} + \cancel{\frac{a^2}{n^2\pi^2}}\ \tfrac{1}{2}Aa^2 + \overset{\frac{a^2}{n\pi}\cos(n\pi)}{\cancel{\frac{Aax}{n\pi}}}\right]$$

$$= \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{a}\right)\left[\underset{\frac{a^2}{n\pi}\cos(n\pi)}{\cancel{2\frac{Aax}{n\pi}}} + \tfrac{1}{2}Aa^2\right]$$

$$= \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{a}\right)\left[(-1)^n\frac{a^2}{n\pi} + \tfrac{1}{2}Aa^2\right]$$

$$\sin\left(\frac{n\pi}{2}\right) = -(-1)^{\frac{n+1}{2}} \cdot \frac{1-(-1)^n}{2}$$

$$\cos(n\pi) = (-1)^n$$

$$\sin\left(\frac{2n+1}{2}\pi\right) = (-1)^n$$

$$C_n = \int \psi_n(x)^* f(x)\, dx$$

$$= \int_0^a \sin\left(\frac{n\pi x}{a}\right) f(x)\, dx$$

$$= \int_0^{a/2} \sin\left(\frac{n\pi x}{a}\right) Ax\, dx + \int_{a/2}^a \sin\left(\frac{n\pi x}{a}\right) \cdot Aa \cancel{\pi} \sin\left(\frac{n\pi x}{a}\right) \cdot Ax\, dx$$

$$= A \int_0^{a/2} \sin\left(\frac{n\pi x}{a}\right) x\, dx + Aa \int_{a/2}^a \sin\left(\frac{n\pi x}{a}\right) dx - A \int_{a/2}^a \sin\left(\frac{n\pi x}{a}\right) x\, dx$$

$f(x) = x$ $\quad g'(x) = \sin\left(\frac{n\pi x}{a}\right)$

$f'(x) = 1$ $\quad g(x) = \frac{-a}{n\pi} \cos\left(\frac{n\pi x}{a}\right)$

$u = \frac{n\pi x}{a}$

$\frac{du}{dx} = \frac{n\pi}{a} \Rightarrow dx = \frac{a}{n\pi}\, du$

$$= A\left[\left[-\frac{ax}{n\pi}\cos\left(\frac{n\pi x}{a}\right)\right]_0^{a/2} - \int_0^{a/2} -\frac{a}{n\pi}\cos\left(\frac{n\pi x}{a}\right) dx\right] - A\left[\left[-\frac{ax}{n\pi}\cos\left(\frac{n\pi x}{a}\right)\right]_{a/2}^{a} - \int_{a/2}^{a} -\frac{a}{n\pi}\cos\left(\frac{n\pi x}{a}\right) dx\right]$$

$$+ Aa\left[\frac{-a}{n\pi}\cos\left(\frac{n\pi x}{a}\right)\right]_{a/2}^{a}$$

$$= \frac{Aa}{n\pi}\left[\frac{a}{n\pi}\sin\left(\frac{n\pi x}{a}\right)\right]_a^{a/2} - A\left[-\frac{a^2}{n\pi}(-1)^n + \frac{a}{n\pi}\left[\frac{a}{n\pi}\sin\left(\frac{n\pi x}{a}\right)\right]_a^{a/2}\right] + Aa\left(-\frac{a}{n\pi}(-1)^n\right)$$

$$= \cancel{\frac{Aa^2}{n^2\pi^2}} \frac{Aa}{n\pi}\left[\frac{-a}{n\pi}(-1)^{\frac{n+1}{2}} \cdot \frac{1-(-1)^n}{2}\right] + \cancel{\frac{Aa^2}{n\pi}(-1)^n} - \frac{Aa}{n\pi}\left[\frac{-a}{n\pi}(-1)^{\frac{n+1}{2}} \cdot \frac{1-(-1)^n}{2}\right] - \cancel{\frac{Aa^2}{n\pi}(-1)^n}$$

$\boxed{pg 40}$ 2.7. (c) $\Psi(x,0) = \sum_{n=1}^{\infty} 2\frac{\sqrt{3}}{a^2}(1-(-1)^n)\frac{a}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{n\pi x}{a}\right)e^{-\frac{iEt}{\hbar}}$

$P = |c|^2 = c^* c = \left(\sum_{n=1}^{\infty} 2\frac{\sqrt{3}}{a^2}(1-(-1)^n)\frac{a}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\right)^2$

$c_n = 2\frac{\sqrt{3}}{a^2}\left(1-(-1)^n\right)\frac{a}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)$

$c_1 = \frac{4\sqrt{3}}{a\pi^2}$

$P(E_1) = |c_1|^2 = \left|\frac{4\sqrt{3}}{a\pi^2}\right|^2 = \frac{48}{a^2\pi^4}$

WOW!!

11 October 2018

(b) $c_1 = \int_0^a f(x)\sqrt{\frac{2}{a}}\sin\left(\frac{\pi x}{a}\right)dx$

$= \int_0^{a/2} Ax\sqrt{\frac{2}{a}}\sin\left(\frac{\pi x}{a}\right)dx + \int_{a/2}^{a} A(a-x)\sqrt{\frac{2}{a}}\sin\left(\frac{\pi x}{a}\right)dx$

$\int_0^{a/2} Ax\sqrt{\frac{2}{a}}\sin\left(\frac{\pi x}{a}\right)dx$

$= A\sqrt{\frac{2}{a}}\int_0^{a/2} x\sin\left(\frac{\pi x}{a}\right)dx$

$f(x)=x \qquad g'(x)=\sin\left(\frac{\pi x}{a}\right)$

$f'(x)=1 \qquad g(x)=\int\sin\left(\frac{\pi x}{a}\right)dx \qquad u=\frac{\pi x}{a}$

$du=\frac{\pi}{a}dx$

$= -\frac{a}{\pi}\cos\left(\frac{\pi x}{a}\right)$

$= \left(-\frac{a}{\pi}x\cos\left(\frac{\pi x}{a}\right) - \int -\frac{a}{\pi}\cos\left(\frac{\pi x}{a}\right)dx\right)\cdot A\sqrt{\frac{2}{a}}$

$= A\sqrt{\frac{2}{a}}\left[-\frac{a}{\pi}x\cos\left(\frac{\pi x}{a}\right) + \frac{a}{\pi}\int\cos\left(\frac{\pi x}{a}\right)dx\right]$

$= A\sqrt{\frac{2}{a}}\frac{a}{\pi}\left[\left[-\cancel{x\cos\left(\frac{\pi x}{a}\right)}\right]_0^{a/2} + \left[\frac{a}{\pi}\sin\left(\frac{\pi x}{a}\right)\right]_0^{a/2}\right]$

$= A\sqrt{\frac{2}{a}}\frac{a}{\pi}\left[\cancel{[0-(-x)]} + \frac{a}{\pi}\right]$

$= A\sqrt{\frac{2}{a}}\frac{a}{\pi}\left[\cdot\frac{a}{\pi}\right]$

$= \frac{2\sqrt{3}}{\sqrt{a^3}}\frac{a^2}{\pi^2}\sqrt{\frac{2}{a}}$

$= \cancel{\frac{2\sqrt{3}a^{\frac{7}{2}}}{\pi^2}\sqrt{\frac{2}{a}}} \quad \frac{2\sqrt{6}}{\pi^2}$

$\int_{a/2}^{a} A(a-x)\sqrt{\frac{2}{a}}\sin\left(\frac{\pi x}{a}\right)dx$

$= A\sqrt{\frac{2}{a}}\left[\int_{a/2}^{a} a\sin\left(\frac{\pi x}{a}\right)dx - \int_{a/2}^{a} x\sin\left(\frac{\pi x}{a}\right)dx\right]$

$= A\sqrt{\frac{2}{a}}\left[a\left[-\frac{a}{\pi}\cos\left(\frac{\pi x}{a}\right)\right]_{a/2}^{a} - \frac{a}{\pi}\left[\left(-x\cos\left(\frac{\pi x}{a}\right)\right)_{a/2}^{a} + \left(\frac{a}{\pi}\sin\left(\frac{\pi x}{a}\right)\right)_{a/2}^{a}\right]\right]$

$= A\sqrt{\frac{2}{a}}\left[a\left(\frac{a}{\pi}-0\right) - \frac{a}{\pi}\left[(a)+\left(\frac{-a}{\pi}\right)\right]\right]$

$= A\sqrt{\frac{2}{a}}\left[\frac{a^2}{\pi} - \frac{a^2}{\pi} + \frac{a^2}{\pi^2}\right]$

$= +\frac{2\sqrt{3}}{\sqrt{a^3}}\sqrt{\frac{2}{a}}\frac{a^2}{\pi^2}$

$= \frac{2\sqrt{6}}{\pi^2}$

$\cancel{\frac{2\sqrt{3}a^{\frac{7}{2}}}{\pi^2} - \frac{2\sqrt{3}a^{\frac{7}{2}}}{\pi^2}\sqrt{\frac{2}{a}}}$

$\cancel{= \left(1-\sqrt{\frac{2}{a}}\right)\cdot\frac{2\sqrt{3}a^{\frac{7}{2}}}{\pi^2}}$

(c) $c_1 = \frac{2\sqrt{6}}{\pi^2} + \frac{2\sqrt{6}}{\pi^2} = \cancel{\frac{4\sqrt{6}}{\pi^4}} \quad \frac{4\sqrt{6}}{\pi^2}$

$P(E_1) = |c_1|^2 = \left(\frac{4\sqrt{6}}{\pi^2}\right)^2 = \frac{96}{\pi^4}$

(d) Because $-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi = E\psi$, $\psi = \underbrace{\sqrt{\frac{2}{a}}}_{\text{normalization constant}}\sin\left(\frac{n\pi x}{a}\right)$, $V=0$ in the well,

so $\frac{\hbar^2}{2m}\frac{n\pi^2}{a^2} = E$

$$\cancel{\langle E\rangle = \int_{-\infty}^{\infty}\psi^*\left(\frac{\hbar^2}{2m}\frac{n\pi^2}{a}\right)\psi\,dx}$$

$$\int_{-\infty}^{\infty}\frac{\hbar^2}{2m}\frac{n\pi^2}{a}\frac{2}{a}\sin^2\left(\frac{n\pi x}{a}\right)dx$$

$$=\frac{\hbar^2}{2m}\frac{n\pi^2}{a}\frac{2}{a}\int_0^a \sin^2\left(\frac{n\pi x}{a}\right)dx$$

$$=\frac{\hbar^2 n\pi^2}{ma^2}\int_0^a \frac12 - \frac12\cos\left(\frac{2n\pi x}{a}\right)dx$$

$$=\frac{\hbar n\pi^2}{ma^2}\left[\int_0^a\frac12 dx - \frac12\int_0^a\cos\left(\frac{2n\pi x}{a}\right)dx\right]$$

$$=\frac{\hbar n\pi^2}{ma^2}\left[\left[\frac12 x\right]_0^a - \frac12\left[\frac{a}{2n\pi}\sin\left(\frac{2n\pi x}{a}\right)\right]_0^a\right]$$

$$=\frac{\hbar n\pi^2}{ma^2}\left[\frac12 a - \frac12[0]\right]$$

$$=\frac{\hbar n\pi^2}{2ma}$$

$2n-1 \Rightarrow \sum_{n=1}^{\infty}$

$2n+1 \Rightarrow \sum_{n=0}^{\infty}$

$$c_n = \int_0^l \sum \frac{4\sqrt6}{n^2\pi^2}\sin\left(\frac{\pi n}{2}\right)\sqrt{\frac{2}{l}}\sin\left(\frac{n\pi x}{l}\right)\hat H\sum\frac{4\sqrt6}{m^2\pi^2}\sqrt{\frac2l}\sin\left(\frac{m\pi}{2}\right)\sin\left(\frac{m\pi x}{l}\right)dx$$

$$=\int_0^l\sum\frac{4\sqrt6}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\sqrt{\frac2l}\sin\left(\frac{n\pi x}{l}\right)\sum\frac{4\sqrt6}{m^2\pi^2}\frac{\hbar^2 m^2\pi^2}{2\mu l^2}\sqrt{\frac2l}\sin\left(\frac{m\pi}{2}\right)\sin\left(\frac{m\pi x}{l}\right)dx$$

$$=\sum_n\sum_m \frac{4\sqrt6}{n^2\pi^2}\frac{4\sqrt6}{m^2\pi^2}\frac{\hbar^2m^2\pi^2}{2\mu l^2}\delta_{mn}\sin\left(\frac{m\pi}{2}\right)\sin\left(\frac{n\pi}{2}\right)$$

$$=\sum_{n=1}^{\infty}\frac{1-(-1)^n}{2}\frac{96}{\pi^4}$$

$\langle E\rangle = \sum P(E_n)\cdot E_n$

Known $E_n = \frac{n^2\hbar^2\pi^2}{2ml^2}$

$c_n = \frac{4\sqrt6}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)$

$$=\sum |c_n|^2\cdot E_n$$

Add them together so the terms of diff. signs cancel, then time $\frac12$ to reduce the redundant terms

$$=\sum_{n=1}^\infty\left|\frac{4\sqrt6}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\right|^2\cdot\frac{n^2\hbar^2\pi^2}{2ml^2}$$

$$=\sum_{n=1}^\infty \frac{96}{n^4\pi^4}\sin^2\left(\frac{n\pi}{2}\right)\cdot\frac{n^2\hbar^2\pi^2}{2ml^2}$$

$$=\sum_{n=1,3,5}^\infty\frac{96}{n^4\pi^4}\frac{n^2\hbar^2\pi^2}{2ml^2}$$

Riemann zeta function

$\zeta(s)=\sum_{n=1}^\infty\frac{1}{n^s}$

$\zeta(2)=\frac{\pi^2}{6}$

$\sum_{n=0}^\infty\frac{1}{(2n+1)^2}=\frac12(\zeta(2)+\eta(2))$

Dirichlet eta function (alternating zeta function)

$\eta(s)=\sum_{n=1}^\infty\frac{(-1)^{n+1}}{n^s}$

$=\frac{1}{1^s}-\frac{1}{2^s}+\frac{1}{3^s}\dots$

$=(1-2^{1-s})\zeta(s)$

$$=\sum_{n=1,3,5}^\infty\frac{48\hbar^2}{n^2\pi^2ml^2} = \sum_{n=0}^\infty\frac{48\hbar^2}{(2n+1)^2\pi^2ml^2} = \frac{48\hbar^2}{\pi^2ml^2}\sum_{n=0}^\infty\frac{1}{(2n+1)^2}$$

$$=\frac{48\hbar^2}{\pi^2ml^2}\frac12(\zeta(2)+\eta(s)) = \frac{48\hbar^2}{\pi^2ml^2}\frac12\left(\frac{\pi^2}{6}+\frac12\frac{\pi^2}{6}\right) = \frac{48\hbar^2}{\pi^2ml^2}\frac{\pi^2}{8} = \frac{6\hbar^2}{ml^2}$$

16 October 2018 2.3 The Harmonic Oscillator

2.10 (a) $\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2}$

$\psi_n(x) = A_n (a_+)^n \psi_0(x)$

$\psi_2(x) = A_2 (a_+)^2 \psi_0(x)$

$= A_2 \frac{1}{2\hbar m\omega} (-ip + m\omega x)^2 \psi_0(x)$ ∴ x and p do NOT commute

$= A_2 \frac{1}{2\hbar m\omega} \left(-p^2 \cancel{+} - \cancel{2}ipm\omega x + m^2\omega^2x^2\right) \psi_0(x)$
$- m\omega x i p$

$$\boxed{\begin{aligned}[x,p] &= i\hbar \\ xp - px &= i\hbar \\ xp + px &= xp - px + 2px \\ &= i\hbar + 2px\end{aligned}}$$

$= A_2 \frac{1}{2\hbar m\omega} \left(-p^2 - m\omega i px - m\omega i xp + m^2\omega^2x^2\right) \psi_0(x)$

$= A_2 \frac{1}{2\hbar m\omega} \left(-p^2 - m\omega i(i\hbar + 2px) + m^2\omega^2x^2\right) \psi_0(x)$

$\frac{d}{dx} e^{-\frac{m\omega}{2\hbar}x^2}$

$= e^{-\frac{m\omega}{2\hbar}x^2} \cdot \left(-\frac{2m\omega}{2\hbar}x\right)^2$

$= A_2 \frac{1}{2\hbar m\omega} \left(-\left(-i\hbar\frac{d}{dx}\right)^2 - m\omega i^2\hbar - 2m\omega i\left(-i\hbar\frac{d}{dx}\right)x + m^2\omega^2x^2\right) \psi_0(x)$

$= A_2 \left(+\frac{\hbar^2}{2\hbar m\omega}\frac{d^2}{dx^2}\psi_0(x) + \frac{m\omega\hbar}{2\hbar m\omega}\psi_0(x) \cancel{+} - \frac{2m\omega\hbar}{2\hbar m\omega}\frac{d}{dx}(x\psi_0(x)) + \frac{m^2\omega^2x^2}{2\hbar m\omega}\psi_0(x)\right)$

$= A_2 \left[\frac{\hbar}{2m\omega}\left[\frac{d^2}{dx^2}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2}\right] + \frac{1}{2}\psi_0(x) - \left[\frac{d}{dx}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2} x\right] + \frac{m\omega x^2}{2\hbar}\psi_0(x)\right)$

$= A_2 \left[\frac{\hbar}{2m\omega}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\left(\left(\frac{m\omega x}{\hbar}\right)^2 e^{-\frac{m\omega}{2\hbar}x^2} + e^{-\frac{m\omega}{2\hbar}x^2}\cdot\left(-\frac{m\omega}{\hbar}\right)\right) + \frac{1}{2}\psi_0(x) \cancel{-} - \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\left(-\frac{m\omega}{\hbar}x\right) e^{-\frac{m\omega}{2\hbar}x^2} x + \frac{m\omega x^2}{2\hbar}\psi_0(x)\right.$
$- e^{-\frac{m\omega}{2\hbar}x^2}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}$

$$\boxed{\begin{aligned}
&\frac{d}{dx} e^{-\frac{m\omega}{2\hbar}x^2} \cdot x \\
&= e^{-\frac{m\omega}{2\hbar}x^2}\left(-\frac{m\omega}{\hbar}x\right)\cdot x + e^{-\frac{m\omega}{2\hbar}x^2} \\
&\frac{d^2}{dx^2} e^{-\frac{m\omega}{2\hbar}x^2} \\
&= -\frac{m\omega}{\hbar}\frac{d}{dx}\left(e^{-\frac{m\omega}{2\hbar}x^2} x\right) \\
&= -\frac{m\omega}{\hbar}\left(x e^{-\frac{m\omega}{2\hbar}x^2}\left(-\frac{m\omega}{\hbar}x\right) + e^{-\frac{m\omega}{2\hbar}x^2}\right) \\
&= +\left(\frac{m\omega x}{\hbar}\right)^2 e^{-\frac{m\omega}{2\hbar}x^2} + e^{-\frac{m\omega}{2\hbar}x^2}\left(-\frac{m\omega}{\hbar}\right)
\end{aligned}}$$

$= A_2$

2.10 (a) $\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2}$

$[x, p] = i\hbar$

$a_\pm \equiv \frac{1}{\sqrt{2\hbar m\omega}}(\mp ip + m\omega x)$

$\psi_n(x) = A_n (a_+)^n \psi_0(x)$ with $E_n = (n+\frac{1}{2})\hbar\omega$

$a_+\psi_n = \sqrt{n+1}\,\psi_{n+1}$, $a_-\psi_n = \sqrt{n}\,\psi_{n-1}$

$\psi_1(x) = A_1 a_+ \psi_0 = A_1 \frac{1}{\sqrt{2\hbar m\omega}}(-ip + m\omega x)\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2}$

$= A_1 \frac{1}{\sqrt{2\hbar m\omega}}\left(-\hbar\frac{d}{dx} + m\omega x\right)\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2}$

$= A_1 \frac{1}{\sqrt{2\hbar m\omega}}\left[-\hbar\frac{d}{dx}\left[\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2}\right] + m\omega x\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{m\omega}{2\hbar}x^2}\right]$

$= A_1 \frac{1}{\sqrt{2\hbar m\omega}}\left[\cancel{-}\hbar\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2}\left(\cancel{-}\cancel{2}x\frac{m\omega}{\cancel{2}\hbar}\right) + m\omega x\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{m\omega}{2\hbar}x^2}\right]$

$= A_1 \frac{1}{\sqrt{2\hbar m\omega}}\left(2\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{m\omega}{2\hbar}x^2} m\omega x\right)$

$= A_1 \sqrt{\frac{2m\omega}{\hbar}}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{m\omega}{2\hbar}x^2} \cdot x$

$\int_{-\infty}^{\infty}|\psi_1(x)|^2 dx = A_1^2 \frac{2m\omega}{\hbar}\sqrt{\frac{m\omega}{\pi\hbar}}\int_{-\infty}^{\infty} e^{-2\frac{m\omega}{2\hbar}x^2} \cdot x^2 dx$

$\boxed{u = \sqrt{\frac{m\omega}{\hbar}}\,x \rightarrow dx = \sqrt{\frac{\hbar}{m\omega}}\,du \qquad x = \sqrt{\frac{\hbar}{m\omega}}\,u}$

$= A_1^2 \frac{2m\omega}{\hbar}\sqrt{\frac{m\omega}{\pi\hbar}}\int_{-\infty}^{\infty} e^{-u^2} \cdot \frac{\hbar}{m\omega}u^2 \cdot \sqrt{\frac{\hbar}{m\omega}}\,du$

$= A_1^2 \frac{2m\omega}{\hbar}\sqrt{\frac{m\omega}{\pi\hbar}} \cdot \frac{\hbar}{m\omega}\sqrt{\frac{\hbar}{m\omega}}\int_{-\infty}^{\infty} e^{-u^2} u^2\,du$

$f(x) = u \qquad g'(x) = e^{-u^2}u$

$f'(x) = 1 \qquad g(x) = \int e^{-u^2}u\,du$

$f(x) = u \qquad g'(x) = e^{-u^2}$ $\qquad v = u^2 \quad \frac{dv}{du} = 2u$

$f'(x) = 1 \qquad g(x) = \int e^{-u^2}du = \int_{-\infty}^{\infty} e^{-v}\frac{dv}{2u}$ $\qquad du = \frac{dv}{2u}$

$= u\sqrt{\pi} - \sqrt{\pi}$

$= A_1^2 \frac{2m\omega}{\hbar}\sqrt{\frac{m\omega}{\pi\hbar}}\,\frac{\hbar}{m\omega}\sqrt{\frac{\hbar}{m\omega}}$

22 October 2018

2.10 (a) ~~$\psi_2(x)=$~~ $a_+ \psi_1(x) = \sqrt{2}\,\psi_2(x)$

$$= \frac{1}{\sqrt{2\hbar m\omega}}(-ip+m\omega x)\left(\sqrt{\frac{2m\omega}{\hbar}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2}\cdot x\right)$$

$$= \frac{1}{\sqrt{2\hbar m\omega}}\sqrt{\frac{2m\omega}{\hbar}}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}}\left(-\hbar\frac{d}{dx}+m\omega x\right)\left(e^{-\frac{m\omega}{2\hbar}x^2}\cdot x\right)$$

$$= \frac{1}{\sqrt{2\hbar m\omega}}\sqrt{\frac{2m\omega}{\hbar}}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}}\left[-\hbar\frac{d}{dx}\left(e^{-\frac{m\omega}{2\hbar}x^2}\cdot x\right)+m\omega x^2 e^{-\frac{m\omega}{2\hbar}x^2}\right]$$

$$= \frac{1}{\sqrt{2\hbar m\omega}}\sqrt{\frac{2m\omega}{\hbar}}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}}\left[-\hbar\left[e^{-\frac{m\omega}{2\hbar}x^2}+-\frac{m\omega}{2\hbar}2x^2 e^{-\frac{m\omega}{2\hbar}x^2}\right]+m\omega x^2 e^{-\frac{m\omega}{2\hbar}x^2}\right]$$

$$= \frac{1}{\hbar}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}}\left[-\hbar e^{-\frac{m\omega}{2\hbar}x^2}+m\omega x^2 e^{-\frac{m\omega}{2\hbar}x^2}+m\omega x^2 e^{-\frac{m\omega}{2\hbar}x^2}\right]$$

$$= -\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{m\omega}{2\hbar}x^2}+\frac{2}{\hbar}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} m\omega x^2 e^{-\frac{m\omega}{2\hbar}x^2}$$

$$\psi_2(x) = \frac{1}{\sqrt{2}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2}\left(\frac{2}{\hbar}m\omega x^2-1\right)$$

(c) $$\delta_{00}=\int_{-\infty}^{\infty}\psi_0^*(x)\cdot\psi_0(x)\,dx$$

$$=\int_{-\infty}^{\infty}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4\cdot 2}\cdot e^{-\frac{m\omega}{\hbar}x^2}dx$$

$$=\sqrt{\frac{m\omega}{\pi\hbar}}\int_{-\infty}^{\infty}e^{-\frac{m\omega}{\hbar}x^2}dx$$

$$u=\sqrt{\frac{m\omega}{\hbar}}x \Rightarrow \frac{du}{dx}=\sqrt{\frac{m\omega}{\hbar}},\quad dx=\sqrt{\frac{\hbar}{m\omega}}du$$

$$=\sqrt{\frac{m\omega}{\pi\hbar}}\int_{-\infty}^{\infty}e^{-u^2}\sqrt{\frac{\hbar}{m\omega}}du$$

~~$v=-u^2 \Rightarrow \frac{dv}{du}=-2u,\quad du=-\frac{1}{2u}dv$~~

$$=\sqrt{\frac{m\omega}{\pi\hbar}}\sqrt{\frac{\hbar}{m\omega}}\cdot\sqrt{\pi}$$

$$=1$$

$$\delta_{11}=\int_{-\infty}^{\infty}\psi_1^*(x)\,\psi_1(x)\,dx$$

$$=\int_{-\infty}^{\infty}\frac{2m\omega}{\hbar}\sqrt{\frac{m\omega}{\pi\hbar}}x^2 e^{-\frac{m\omega}{\hbar}x^2}dx$$

$$=\frac{2m\omega}{\hbar}\sqrt{\frac{m\omega}{\pi\hbar}}\int_{-\infty}^{\infty}x^2 e^{-\frac{m\omega}{\hbar}x^2}dx$$

$f(x)=x \qquad g'(x)=xe^{-\frac{m\omega}{\hbar}x^2}$

$f'(x)=1 \qquad g(x)=\int xe^{-\frac{m\omega}{\hbar}x^2}dx$

$$u=-\frac{m\omega}{\hbar}x^2$$

$$\frac{du}{dx}=-\frac{2m\omega}{\hbar}x \Rightarrow dx\cdot x=-\frac{\hbar}{2m\omega}du$$

$$=-\frac{\hbar}{2m\omega}\int e^u du$$

$$=-\frac{\hbar}{2m\omega}e^{-\frac{m\omega}{\hbar}x^2}$$

$$= \frac{2m\omega}{\hbar}\sqrt{\frac{m\omega}{\pi\hbar}}\left[\left[-x\,\frac{\hbar}{2m\omega}\,e^{-\frac{m\omega}{\hbar}x^2}\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} -\frac{\hbar}{2m\omega}\,e^{-\frac{m\omega}{\hbar}x^2}\,dx\right]$$

$$= \frac{2m\omega}{\hbar}\sqrt{\frac{m\omega}{\pi\hbar}}\left[\frac{\hbar}{2m\omega}\int_{-\infty}^{\infty} e^{-\frac{m\omega}{\hbar}x^2}\,dx\right]$$

$$u = \sqrt{\frac{m\omega}{\hbar}}\,x \Rightarrow \frac{du}{dx} = \sqrt{\frac{m\omega}{\hbar}}, \quad dx = \sqrt{\frac{\hbar}{m\omega}}\,du$$

$$= \frac{2m\omega}{\hbar}\sqrt{\frac{m\omega}{\pi\hbar}}\,\frac{\hbar}{2m\omega}\int_{-\infty}^{\infty} e^{-u^2}\sqrt{\frac{\hbar}{m\omega}}\,du$$

$$= 1$$

$$\delta_{22} = \int_{-\infty}^{\infty} \psi_2^*(x)\,\psi_2(x)\,dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{2}\sqrt{\frac{m\omega}{\pi\hbar}}\,e^{-\frac{m\omega}{\hbar}x^2}\left(\frac{2}{\hbar}m\omega x^2 - 1\right)^2 dx$$

$$= \frac{1}{2}\sqrt{\frac{m\omega}{\pi\hbar}}\int_{-\infty}^{\infty} e^{-\frac{m\omega}{\hbar}x^2}\left(\frac{4}{\hbar^2}m^2\omega^2x^4 - \frac{4}{\hbar}m\omega x^2 + 1\right)dx$$

$$= \frac{1}{2}\sqrt{\frac{m\omega}{\pi\hbar}}\left[\int_{-\infty}^{\infty}\frac{4}{\hbar^2}m^2\omega^2x^4 e^{-\frac{m\omega}{\hbar}x^2}dx - \int_{-\infty}^{\infty}\frac{4}{\hbar}m\omega x^2 e^{-\frac{m\omega}{\hbar}x^2}dx + \int_{-\infty}^{\infty} e^{-\frac{m\omega}{\hbar}x^2}dx\right]$$

$$= \frac{1}{2}\sqrt{\frac{m\omega}{\pi\hbar}}\left[\frac{4}{\hbar^2}m^2\omega^2\int_{-\infty}^{\infty}x^4 e^{-\frac{m\omega}{\hbar}x^2}dx - \frac{4}{\hbar}m\omega\int_{-\infty}^{\infty}x^2 e^{-\frac{m\omega}{\hbar}x^2}dx + \int_{-\infty}^{\infty}e^{-\frac{m\omega}{\hbar}x^2}dx\right]$$

$$f(x) = x^3 \qquad g'(x) = x^4 e^{-\frac{m\omega}{\hbar}x^2}$$

$$f'(x) = 3x^2 \qquad g(x) = \int x^4 e^{-\frac{m\omega}{\hbar}x^2}$$

$$= -\frac{\hbar}{2m\omega}e^{-\frac{m\omega}{\hbar}x^2}$$

$$= \frac{1}{2}\sqrt{\frac{m\omega}{\pi\hbar}}\left[\frac{4}{\hbar^2}m^2\omega^2\left\{\left[-\frac{\hbar}{2m\omega}x^3e^{-\frac{m\omega}{\hbar}x^2}\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty}-\frac{\hbar}{2m\omega}3x^2e^{-\frac{m\omega}{\hbar}x^2}dx\right] - \frac{4}{\hbar}m\omega\cdot\frac{\hbar}{2m\omega}\sqrt{\frac{\hbar\pi}{m\omega}} + \sqrt{\frac{\hbar\pi}{m\omega}}\right]$$

$$= \frac{1}{2}\sqrt{\frac{m\omega}{\pi\hbar}}\left[\frac{4}{\hbar^2}m^2\omega^2\left(\frac{3\hbar}{2m\omega}\int_{-\infty}^{\infty}x^2e^{-\frac{m\omega}{\hbar}x^2}dx\right) - \frac{4m\omega}{\hbar}\cdot\frac{\hbar}{2m\omega}\sqrt{\frac{\hbar\pi}{m\omega}} + \sqrt{\frac{\hbar\pi}{m\omega}}\right]$$

$$= \frac{1}{2}\sqrt{\frac{m\omega}{\pi\hbar}}\,\frac{4}{\hbar^2}m^2\omega^2\,\frac{3\hbar}{2m\omega}\,\frac{\hbar}{2m\omega}\sqrt{\frac{\hbar\pi}{m\omega}} - \frac{1}{2}\sqrt{\frac{m\omega}{\pi\hbar}}\,\frac{4m\omega}{\hbar}\cdot\frac{\hbar}{2m\omega}\sqrt{\frac{\hbar\pi}{m\omega}} + \frac{1}{2}\sqrt{\frac{m\omega}{\pi\hbar}}\sqrt{\frac{\hbar\pi}{m\omega}}$$

$$= \frac{3}{2} - 1 + \frac{1}{2}$$

$$= 1$$

$$\delta_{01} = \int_{-\infty}^{\infty}\psi_0(x)^*\,\psi_1(x)\,dx$$

$$= \int_{-\infty}^{\infty}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}}\sqrt{\frac{2m\omega}{\hbar}}\,e^{-\frac{m\omega}{\hbar}x^2}\cdot x\,dx$$

$$= \sqrt{\frac{m\omega}{\pi\hbar}}\sqrt{\frac{2m\omega}{\hbar}}\int_{-\infty}^{\infty}e^{-\frac{m\omega}{\hbar}x^2}\cdot x\,dx$$

$$= \sqrt{\frac{2m^2\omega^2}{\pi\hbar^2}}\,\cancel{\sqrt{\frac{\hbar\pi}{m\omega}}}\left(-\frac{\hbar}{2m\omega}e^{-\frac{m\omega}{\hbar}x^2}\right)_{-\infty}^{\infty}$$

$$= \cancel{\sqrt{\frac{2m\omega}{\hbar}}}\;0$$

22 October 2018

2.10

$$\delta_{12} = \int_{-\infty}^{\infty} \psi_1^*(x)\psi_2(x)\,dx$$

$$= \int_{-\infty}^{\infty} \sqrt{\frac{2m\omega}{\hbar}} \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2} \cdot x \cdot \frac{1}{\sqrt{2}} \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{m\omega}{2\hbar}x^2}\left(\frac{2}{\hbar}m\omega x^2 - 1\right) dx$$

$$= \sqrt{\frac{2m\omega}{\hbar}} \sqrt{\frac{m\omega}{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-\frac{m\omega}{\hbar}x^2} \cdot x\left(\frac{2}{\hbar}m\omega x^2 - 1\right) dx$$

$$= \sqrt{\frac{2m\omega}{\hbar}} \sqrt{\frac{m\omega}{2\pi\hbar}} \left[\frac{2}{\hbar}m\omega \int_{-\infty}^{\infty} x^3 e^{-\frac{m\omega}{\hbar}x^2} dx - \int_{-\infty}^{\infty} e^{-\frac{m\omega}{\hbar}x^2} \cdot x \, dx\right]$$

$f(x) = x^2$ $\quad g'(x) = x e^{-\frac{m\omega}{\hbar}x^2}$

$f'(x) = 2x$ $\quad g(x) = \int x e^{-\frac{m\omega}{\hbar}x^2}$

$\qquad = -\frac{\hbar}{2m\omega} e^{-\frac{m\omega}{\hbar}x^2}$

$$= \sqrt{\frac{2m\omega}{\hbar}} \sqrt{\frac{m\omega}{2\pi\hbar}} \left[\frac{2}{\hbar}m\omega\left[\cancel{\left[-x^2\frac{\hbar}{2m\omega}e^{-\frac{m\omega}{\hbar}x^2}\right]_{-\infty}^{\infty}} - \int_{-\infty}^{\infty} -2x\frac{\hbar}{2m\omega}e^{-\frac{m\omega}{\hbar}x^2}dx\right] + \cancel{\left[\frac{\hbar}{2m\omega}e^{-\frac{m\omega}{\hbar}x^2}\right]_{-\infty}^{\infty}}\right]$$

$$= \sqrt{\frac{2m\omega}{\hbar}} \sqrt{\frac{m\omega}{2\pi\hbar}} \left[\frac{2}{\hbar}m\omega \cdot 2\frac{\hbar}{2m\omega} \int_{-\infty}^{\infty} x e^{-\frac{m\omega}{\hbar}x^2} dx\right]$$

$$= \sqrt{\frac{2m\omega}{\hbar}} \sqrt{\frac{m\omega}{2\pi\hbar}} \left[2\left[-\frac{\hbar}{2m\omega}e^{-\frac{m\omega}{\hbar}x^2}\right]_{-\infty}^{\infty}\right]$$

$$= 0$$

$$\delta_{02} = \int_{-\infty}^{\infty} \psi_0^*(x)\psi_2(x)\,dx$$

$$= \int_{-\infty}^{\infty} \sqrt{\frac{m\omega}{\pi\hbar}} \frac{1}{\sqrt{2}} e^{-\frac{m\omega}{\hbar}x^2}\left(\frac{2}{\hbar}m\omega x^2 - 1\right) dx$$

$$= \sqrt{\frac{m\omega}{2\pi\hbar}} \left[\int_{-\infty}^{\infty} \frac{2}{\hbar}m\omega x^2 e^{-\frac{m\omega}{\hbar}x^2} dx - \int_{-\infty}^{\infty} e^{-\frac{m\omega}{\hbar}x^2} dx\right]$$

$$= \sqrt{\frac{m\omega}{2\pi\hbar}} \left[\frac{2}{\hbar}m\omega\left[\frac{\hbar}{2m\omega}\sqrt{\frac{\hbar\pi}{m\omega}}\right] - \sqrt{\frac{\hbar\pi}{m\omega}}\right]$$

$$= 0$$

2.18

(#1) Solve the harmonic oscillater equation,

$$\frac{d^2}{dx^2}\psi = -k^2\psi$$

$$\psi'\psi'' = -k^2\psi\psi'$$

$$\frac{d}{dx}\left(\frac{1}{2}\psi'^2\right) = -k^2\frac{d}{dx}\left(\frac{1}{2}\psi^2\right)$$

$$\frac{1}{2}\psi'^2 = -k^2\left(\frac{1}{2}\psi^2\right)$$

$$\psi' = \pm ik\psi$$

$$\int\frac{d\psi}{\psi} = \pm ik\int dx$$

$$\ln|\psi| = \pm ikx + C$$

$$\psi = e^{\pm ikx + C}$$

$$\psi = Ae^{+ikx} + Be^{-ikx} \quad (1)$$

Notice Euler's formula,

$$e^{ix} = i\sin x + \cos x \quad (2)$$

and its complex conjugate

$$e^{-ix} = -i\sin x + \cos x. \quad (3)$$

To get sine, $\frac{(2)-(3)}{2i}$. (4)

To get cosine, $\frac{(2)+(3)}{2}$. (5)

Apply (4) to (1),

$$C\sin(kx) = C\,\frac{e^{ikx} - e^{-ikx}}{2i} \quad (6)$$

Apply (5) to (1),

$$D\cos(kx) = D\,\frac{e^{ikx} + e^{-ikx}}{2} \quad (7)$$

Thus, $[Ae^{ikx} + Be^{-ikx}]$ is equivalent to $[$~~Csink~~ $C\sin(kx) + D\cos(kx)]$.

(#2) Find C and D in terms of A and B.

(2) & (3) → (1)

$$\psi = A[i\sin(kx) + \cos(kx)] + B[-i\sin(kx) + \cos(kx)]$$

$$= Ai\sin(kx) - Bi\sin(kx) + A\cos(kx) + B\cos(kx)$$

$$= (Ai - Bi)\sin(kx) + (A+B)\cos(kx)$$

Thus, $C = A_i - B_i$, $D = A+B$

2.19 $\Psi_k(x,t) = A e^{i(kx - \frac{\hbar k^2}{2m}t)}$

- $P_{ab}(t)$ is the probability of finding a particle in range $(a < x < b)$ at time t.
- J is the probability current

$$\frac{dP_{ab}}{dt} = J(a,t) - J(b,t)$$

$$J(x,t) = \frac{i\hbar}{2M}\left(\Psi \frac{\partial \Psi^*}{\partial x} - \Psi^* \frac{\partial \Psi^*}{\partial x}\right)$$

$$J(x,t) = \frac{i\hbar}{2m}\left(\Psi_k \frac{\partial \Psi_k^*}{\partial x} - \Psi_k^* \frac{\partial \Psi_k}{\partial x}\right)$$

$$= \frac{i\hbar}{2m}\left(A e^{i(kx - \frac{\hbar k^2}{2m}t)} \cdot \frac{\partial}{\partial x} A e^{-i(kx - \frac{\hbar k^2}{2m}t)} - A e^{-i(kx - \frac{\hbar k^2}{2m}t)} \cdot \frac{\partial}{\partial x} A e^{i(kx - \frac{\hbar k^2}{2m}t)}\right)$$

$$= \frac{i\hbar}{2m}\left[A^2 e^{ikx} \cancel{e^{-\frac{\hbar k^2}{2m}t} e^{\frac{\hbar k^2}{2m}t}} \frac{\partial}{\partial x} e^{-ikx} - A^2 e^{-ikx} \cancel{e^{\frac{\hbar k^2}{2m}t} e^{-\frac{\hbar k^2}{2m}t}} \frac{\partial}{\partial x} e^{ikx}\right)$$

$$= \frac{i\hbar}{2m}\left[A^2 e^{ikx}\left(e^{-ikx}(-ik)\right) - A^2 e^{-ikx}\left(e^{ikx}(ik)\right)\right]$$

$$= \frac{i\hbar}{2m}\left[-A^2 ik - A^2 ik\right]$$

$$\cancel{+\frac{i\hbar}{2m}(2A^2k^2)} \quad = \frac{i\hbar}{2m}\left[-2A^2 ik\right]$$

$$\cancel{\frac{i\hbar A^2 k^2}{m}} \quad = \frac{\hbar A^2 k}{m}$$

$$\int_{-\infty}^{\infty} |\Psi_k|^2 dx = 1 = \int_{-\infty}^{\infty} A^2 e^{ikx} e^{-\frac{\hbar k^2}{2m}t} e^{-ikx} e^{\frac{\hbar k^2}{2m}t} dx$$

$$= \int_{-\infty}^{\infty} A^2 dx$$

$$= A^2[\infty]$$

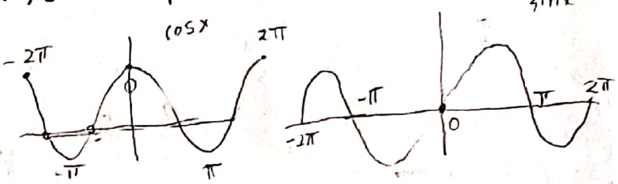
Ψ_k cannot be normalized.

Day 21

26 October 2018 2.4 The Free Particle

2.20 (a) Proof of $f(x) = \sum_{n=0}^{\infty} \left[a_n \sin\left(\frac{n\pi x}{a}\right) + b_n \cos\left(\frac{n\pi x}{a}\right)\right]$ is equivalent to

$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/a}$.



Known Euler's formula and its complex conjugate

$$e^{ix} = i \sin x + \cos x \quad \longrightarrow \quad e^{\frac{in\pi}{a}x} = i \sin\left(\frac{n\pi x}{a}\right) + \cos\left(\frac{n\pi x}{a}\right)$$

$$e^{-ix} = -i \sin x + \cos x \quad \longrightarrow \quad e^{-\frac{in\pi}{a}x} = -i \sin\left(\frac{n\pi x}{a}\right) + \cos\left(\frac{n\pi x}{a}\right)$$

and

$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i} \quad \longrightarrow \quad \sin\left(\frac{n\pi x}{a}\right) = \frac{e^{\frac{in\pi}{a}x} - e^{-\frac{in\pi}{a}x}}{2i}$$

$$\cos(x) = \frac{e^{ix} + e^{-ix}}{2} \quad \longrightarrow \quad \cos\left(\frac{n\pi x}{a}\right) = \frac{e^{\frac{in\pi}{a}x} - e^{-\frac{in\pi}{a}x}}{2}$$

$$\sum_{n=0}^{\infty} \left[a_n \sin\left(\frac{n\pi x}{a}\right) + b_n \cos\left(\frac{n\pi x}{a}\right)\right]$$

$$= \sum_{n=-\infty}^{n=0} \left[a_{-n} \sin\left(-\frac{n\pi x}{a}\right) + b_{-n} \cos\left(-\frac{n\pi x}{a}\right)\right]$$

$$\sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{a}}$$

$$= \sum_{n=-\infty}^{\infty} c_n \left(i \sin\left(\frac{n\pi x}{a}\right) + \cos\left(\frac{n\pi x}{a}\right)\right)$$

$$= \sum_{n=-\infty}^{-1} c_n \left(i \sin\left(\frac{n\pi x}{a}\right) + \cos\left(\frac{n\pi x}{a}\right)\right) + \sum_{n=0}^{0} c_n \left(i \sin\left(\frac{n\pi x}{a}\right) + \cos\left(\frac{n\pi x}{a}\right)\right) + \sum_{n=1}^{\infty} c_n \left(i \sin\left(\frac{n\pi x}{a}\right) + \cos\left(\frac{n\pi x}{a}\right)\right)$$

$$= \sum_{n=1}^{n=\infty} c_{-n} \left(-i \sin\left(\frac{n\pi x}{a}\right) + \cos\left(\frac{n\pi x}{a}\right)\right) + c_0 + \sum_{n=1}^{\infty} c_n \left(i \sin\left(\frac{n\pi x}{a}\right) + \cos\left(\frac{n\pi x}{a}\right)\right)$$

$$= c_0 + \sum_{n=1}^{\infty} c_{-n} \left(-i \sin\left(\frac{n\pi x}{a}\right) + \cos\left(\frac{n\pi x}{a}\right)\right) + c_n \left(i \sin\left(\frac{n\pi x}{a}\right) + \cos\left(\frac{n\pi x}{a}\right)\right)$$

$$= c_0 + \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{a}\right) \cdot (-i c_{-n} + i c_n) + \cos\left(\frac{n\pi x}{a}\right) \cdot (c_{-n} + c_n)$$

$$= \sum_{n=0}^{\infty} \sin\left(\frac{n\pi x}{a}\right) \cdot (-i c_{-n} + i c_n) + \cos\left(\frac{n\pi x}{a}\right) (c_{-n} + c_n)$$, where $c_0 = c_{-n} + c_n$.

~~let $a_n \equiv c_n$, $b_n \equiv c_{-n}$,~~

$a_n \equiv i c_n - i c_{-n}$

$b_n = c_n + c_{-n}$, $n \neq 0$

$b_0 = c_0$

29 October 2018

2.4 The Free Particle

2.20 (b) Show that $c_n = \frac{1}{2a}\int_{-a}^{a} f(x)\, e^{in\pi x/a}\, dx$ $\boxed{1}$

Known $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/a}$ $\boxed{2}$

$$f(x)\, e^{-\frac{im\pi x}{a}} = \sum_{n=-\infty}^{\infty} c_n\, e^{\frac{in\pi x}{a}}\, e^{-\frac{im\pi x}{a}} \quad \boxed{3}$$

$$\frac{1}{2a}\int_{-a}^{a} f(x)\, e^{-\frac{im\pi x}{a}}\, dx = \frac{1}{2a}\int_{-a}^{a} \sum_{n=-\infty}^{\infty} c_n\, e^{\frac{in\pi x}{a}}\, e^{-\frac{im\pi x}{a}}\, dx \quad \boxed{4}$$

$$= \frac{1}{2a}\sum_{n=-\infty}^{\infty} c_n \int_{-a}^{a} e^{\frac{in\pi x}{a}}\, e^{-\frac{im\pi x}{a}}\, dx$$

$$= \frac{1}{2a}\sum_{n=-\infty}^{\infty} c_n \int_{-a}^{a} e^{\frac{i\pi x}{a}(n-m)}\, dx \quad \boxed{5}$$

$$\cancel{\frac{1}{2a}\sum_{n=-\infty}^{\infty} c_n\, e^{\frac{i\pi}{a}} \int_{-a}^{a} e^{(n-m)x}\, dx}$$

$$= \frac{1}{2a}\sum_{-\infty}^{\infty} c_n \left[\frac{a}{i\pi(n-m)}\, e^{\frac{i\pi x}{a}(n-m)}\right]_{-a}^{a}$$

$$= \frac{1}{2a}\sum_{-\infty}^{\infty} c_n \left[\frac{a}{i\pi(n-m)}\, e^{i\pi(n-m)} - \frac{a}{i\pi(n-m)}\, e^{-i\pi(n-m)}\right]$$

when $(n \neq m)$:

$$= \sum_{-\infty}^{\infty} c_n \frac{1}{2i\pi(n-m)}\left(e^{i\pi(n-m)} - e^{-i\pi(n-m)}\right)$$

$$= \sum_{-\infty}^{\infty} c_n \frac{1}{\pi(n-m)}\, \frac{e^{i\pi(n-m)} - e^{-i\pi(n-m)}}{2i}$$

$$= \sum_{-\infty}^{\infty} c_n \frac{1}{\pi(n-m)} \sin((n-m)\pi)$$

$$= 0$$

when $(n=m)$:

$\boxed{5}$:

$$= \frac{1}{2a}\sum_{n=-\infty}^{\infty} c_n \int_{-a}^{a} e^{0}\, dx$$

$$= \sum_{n=-\infty}^{\infty} c_n \frac{1}{2a}\cdot 2a$$

$$= \sum_{n=-\infty}^{\infty} c_n$$

(since $n=m$ ...)

$$= c_n$$

When $n \neq m$, $\frac{1}{2a}\int_{-a}^{a} f(x)\, e^{-\frac{im\pi x}{a}} dx = 0$,

When $n=0$, $\frac{1}{2a}\int_{-a}^{a} f(x)\, e^{-\frac{im\pi x}{a}} dx = c_n$,

thus, $\frac{1}{2a}\int_{-a}^{a} f(x)\, e^{-\frac{in\pi x}{a}} dx = c_n$.

2.20 (c) Let $k = \frac{n\pi}{a}$, $F(k) = \sqrt{\frac{2}{\pi}}\, a c_n$

From part (a), $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{a}}$

$$= \sum_{n=-\infty}^{\infty} F(k) \cdot \frac{1}{a}\sqrt{\frac{\pi}{2}}\, e^{ikx}$$

Since $F(k) = \sqrt{\frac{2}{\pi}}\, a c_n$,

from part (b), $= \sqrt{\frac{2}{\pi}}\, a \frac{1}{2a} \int_{-a}^{a} f(x) e^{-\frac{in\pi x}{a}} dx$

$$= \sqrt{\frac{2}{\pi}} \frac{1}{2} \int_{-a}^{a} f(x) e^{-\frac{in\pi x}{a}} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} f(x) e^{-ikx} dx$$

31 October 2018 — 2.4 Free Particle — Day 23

$$\delta(x) = \begin{cases} h & |x| < \frac{a}{2} \\ 0 & \text{otherwise} \end{cases}$$ where $h \cdot a = 1$ ← infentesimally small

$|x| < \frac{a}{2}$

$2 - \frac{a}{2} < x < \frac{a}{2} + 2$

Solve $\int_{-\infty}^{\infty} f(x)\,\delta(x)\,dx$

$= \int_{-\infty}^{-\frac{a}{2}} f(x)\,\delta(x)\,dx + \int_{-\frac{a}{2}}^{\frac{a}{2}} f(x)\,\delta(x)\,dx + \int_{\frac{a}{2}}^{\infty} f(x)\,\delta(x)\,dx$

$= \int_{-\frac{a}{2}}^{\frac{a}{2}} f(x)\,\delta(x)\,dx$

If $f(x) = x$, then

$\int_{-\frac{a}{2}}^{\frac{a}{2}} x \cdot h\,dx$

$= \left[\frac{1}{2}hx^2\right]_{-\frac{a}{2}}^{\frac{a}{2}}$

$= \frac{1}{2}h\left(\frac{a}{2}\right)^2 - \frac{1}{2}h\left(-\frac{a}{2}\right)^2$

$= \frac{h}{2}\frac{a^2}{4} - \frac{h}{2}\left(\frac{a^2}{4}\right)$

$= 0$

If $f(x) = x^2$, then

$\int_{-\frac{a}{2}}^{\frac{a}{2}} x^2 \cdot h\,dx$

$= \left[\frac{1}{3}hx^3\right]_{-\frac{a}{2}}^{\frac{a}{2}}$

$= \frac{1}{3}h\frac{a^3}{8} - \frac{1}{3}h\frac{-a^3}{8}$

$= \frac{ha^3}{12}$

$= \frac{a^2}{12} = 0$

If $f(x) = x^3$, then

$\int_{-\frac{a}{2}}^{\frac{a}{2}} x^3 \cdot h\,dx$

$= \left[\frac{1}{4}hx^4\right]_{-\frac{a}{2}}^{\frac{a}{2}}$

$= \frac{1}{4}h\frac{a^4}{16} - \frac{1}{4}h\frac{a^4}{16}$

$= 0$

If $f(x) = 2$, then

$\int_{-\frac{a}{2}}^{\frac{a}{2}} 2h\,dx$

$= \left[2hx\right]_{-\frac{a}{2}}^{\frac{a}{2}}$

$= 2h\frac{a}{2} - 2h\left(-\frac{a}{2}\right)$

$= 2$

If $f(x) = x$, $\delta(x) = \delta(x-2)$

$\int_{2-\frac{a}{2}}^{\frac{a}{2}+2} hx\,dx$

$= \left[\frac{1}{2}hx^2\right]_{2-\frac{a}{2}}^{\frac{a}{2}+2}$

$= \frac{1}{2}h\left(\frac{a}{2}+2\right)^2 - \frac{1}{2}h\left(2-\frac{a}{2}\right)^2$

$= \frac{1}{2}h(2a) - \frac{1}{2}h(-2a)$

$= 2ah$

$= 2$

$$\delta(x) = \delta(x-2) = \begin{cases} h & \left(2-\frac{a}{2}\right) < x < \left(\frac{a}{2}+2\right) \\ 0 & \text{otherwise} \end{cases}$$

If $f(x) = \pi$, then

$\int_{-\frac{a}{2}}^{\frac{a}{2}} \pi h\,dx$

$= \left[\pi h x\right]_{-\frac{a}{2}}^{\frac{a}{2}}$

$= \frac{\pi h a}{2} - \frac{-\pi h a}{2}$

$= \pi h a$

$= \pi$

If $f(x) = x^4$, then

$\int_{-\frac{a}{2}}^{\frac{a}{2}} x^4 h\,dx$

$= \left[\frac{1}{5}hx^5\right]_{-\frac{a}{2}}^{\frac{a}{2}}$

$= \frac{1}{5}h\frac{a^5}{2^5} + \frac{1}{5}h\frac{a^5}{2^5}$

$= \frac{ha^5}{5 \cdot 2^4}$

$= \frac{a^4}{90} = 0$

If $f(x) = \sin x$, then

$\int_{-\frac{a}{2}}^{\frac{a}{2}} \sin x\, h\,dx$

$= \left[-h\cos(x)\right]_{-\frac{a}{2}}^{\frac{a}{2}}$

$= -h\cos\left(\frac{a}{2}\right) + h\cos\left(-\frac{a}{2}\right)$

$= h\left(\cos\left(-\frac{a}{2}\right) - \cos\left(\frac{a}{2}\right)\right)$

$= h\left(\cos\left(\frac{a}{2}\right) - \cos\left(\frac{a}{2}\right)\right)$

$= h \cdot 0$

$= 0$

If $f(x) = \cos x$, then

$\int_{-\frac{a}{2}}^{\frac{a}{2}} \cos x\, h\,dx$

$= \left(h\sin x\right)_{-\frac{a}{2}}^{\frac{a}{2}}$

$= h\sin\left(\frac{a}{2}\right) - h\sin\left(-\frac{a}{2}\right)$

$= h\left(\sin\left(\frac{a}{2}\right) - \sin\left(-\frac{a}{2}\right)\right)$

$= h\left(\sin\left(\frac{a}{2}\right) + \sin\left(\frac{a}{2}\right)\right)$

$= 2h\sin\left(\frac{a}{2}\right)$

$= \underline{\frac{2\sin\left(\frac{a}{2}\right)}{a}}$

$= 1$

$\frac{2\sin\left(\frac{a}{2}\right)}{a}$

If $f(x) = \cos x$, $\delta(x) = \delta(x-2)$

$\int_{2-\frac{a}{2}}^{\frac{a}{2}+2} \cos x\,\delta(x-2)\,dx$

$= \int_{2-\frac{a}{2}}^{\frac{a}{2}+2} h\cos x\,dx$

$= \left[h\sin x\right]_{2-\frac{a}{2}}^{\frac{a}{2}+2}$

$= h\sin\left(\frac{a}{2}+2\right) - h\sin\left(2-\frac{a}{2}\right)$

$= h\left[\sin\left(\frac{a}{2}+2\right) - \sin\left(2-\frac{a}{2}\right)\right]$

$= \left\{\begin{array}{l}\sin\left(\frac{a}{2}\right)\cos(2) + \cos\left(\frac{a}{2}\right)\sin(2) \\ -\left[\sin(2)\cos\left(\frac{a}{2}\right) - \cos(2)\sin\left(\frac{a}{2}\right)\right]\end{array}\right\} \cdot h$

$= \underline{\frac{2\cos(2)\sin\left(\frac{a}{2}\right)}{a}}$

$= \cos(2)$

31 October 2018   2.4 The Free Particle

If $f(x) = x^2$, $\delta(x) = \delta(x-2)$,

$$\int_{2-\frac{a}{2}}^{\frac{a}{2}+2} x^2 h \, dx$$

$$= \left[\tfrac{1}{3} h x^3\right]_{2-\frac{a}{2}}^{\frac{a}{2}+2}$$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

$(a - b)^3$

$= (a^2 - 2ab + b^2)(a - b)$

$= a^3 - a^2 b - 2a^2 b - 2ab^2 + ab^2 - b^3$

$$= \tfrac{1}{3} h \left(\tfrac{a}{2}+2\right)^3 - \tfrac{1}{3} h \left(2 - \tfrac{a}{2}\right)^3$$

$$= \tfrac{1}{3} h \left(\frac{a^3}{2^3} + 3\frac{a^2}{\cancel{2^2}}\cdot \cancel{2} + 3\,\frac{a}{2}\cdot 2^2 + \cancel{2^3}\right) - \tfrac{1}{3} h \left(\cancel{2^3} - 3\cdot 2^2 \cdot \frac{a}{2} + \cancel{3\cdot 2 \cdot \frac{a^2}{2^2}} - \frac{a^3}{2^3}\right)$$

$$= \tfrac{1}{3} h \left(2\frac{a^3}{2^3} + 2\cdot 3 \cdot \frac{a}{2}\cdot 2^2\right)$$

$$= \tfrac{1}{3} h \left(\frac{a^3}{4} + 12a\right)$$

$$= \frac{a^3 h}{12} + \cancel{4h}\ 4ah$$

$$= \frac{a^2}{12} + 4$$

$$= 4$$

Based on observation, if we have an arbitrary function $f(x)$ and a $\delta(x)$ function evaluated at $\delta(x-c)$, then

$$\int_{-\infty}^{\infty} f(x)\, \delta(x)\, dx = f(c)$$

If $f(x) = x^3$, $\delta(x) = \delta(x-2)$

$$\int_{2-\frac{a}{2}}^{\frac{a}{2}+2} x^3 h \, dx$$

$$= \left[\tfrac{1}{4} h x^4\right]_{2-\frac{a}{2}}^{\frac{a}{2}+2}$$

$$= \tfrac{1}{4} h \left(\tfrac{a}{2}+2\right)^4 - \tfrac{1}{4} h \left(\tfrac{2}{2} - \tfrac{a}{2}\right)^4$$

$$= \tfrac{1}{4} h \left(\cancel{2^4} + 4\cdot 2^3 \cdot \frac{a}{2} + \cancel{6\cdot 2^2 \cdot \frac{a^2}{2^2}} + 4\cdot 2 \cdot \frac{a^3}{2^3} + \cancel{\frac{a^4}{2^4}}\right)$$

$$- \tfrac{1}{4} h \left(\cancel{2^4} - 4\cdot 2^3 \frac{a}{2} + \cancel{6\cdot 2^2 \cdot \frac{a^2}{2^2}} - 4\cdot 2\cdot \frac{a^3}{2^3} + \cancel{\frac{a^4}{2^4}}\right)$$

$$= \tfrac{1}{4} h \left(2\cdot 4 \cdot 2^3 \cdot \frac{a}{2} + 2\cdot 4\cdot 2\cdot \frac{a^3}{2^3}\right)$$

$$= \tfrac{1}{4} h \left(32a + 2a^3\right)$$

$ah = 1$

$h = \frac{1}{a}$

$$= 8ah + \tfrac{1}{2}\frac{a^3}{h}$$

$$= 8$$

2.4 The Free Particle

2.22 (a) $\Psi(x,0) = Ae^{-ax^2}$

$$\int_{-\infty}^{\infty} |\Psi|^2 dx = 1 = A^2 \int_{-\infty}^{\infty} e^{-2ax^2} dx$$

$u = \sqrt{2a}\,x$

$du = \sqrt{2a}\,dx$

$dx = \frac{du}{\sqrt{2a}}$

$$= A^2 \int_{-\infty}^{\infty} e^{-u^2} du \frac{1}{\sqrt{2a}}$$

$$1 = \frac{1}{\sqrt{2a}} A^2 \sqrt{\pi}$$

$$\sqrt{\frac{2a}{\pi}} = A^2$$

$$\left(\frac{2a}{\pi}\right)^{\frac{1}{4}} = A$$

(b) Find $\Psi(x,t)$

First, find the Fourier transform of the wave function $\Psi(x,0)$

$$\boxed{f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(k) e^{ikx} dk \iff F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx}$$

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\frac{2a}{\pi}\right)^{\frac{1}{4}} e^{-ax^2} e^{-ikx} dx,$$

and $\boxed{\phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Psi(x,0) e^{-ikx} dx}$

$$\phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\frac{2a}{\pi}\right)^{\frac{1}{4}} e^{-ax^2} e^{-ikx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{2a}{\pi}\right)^{\frac{1}{4}} \int_{-\infty}^{\infty} e^{-ax^2 - ikx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{2a}{\pi}\right)^{\frac{1}{4}} \int_{-\infty}^{\infty} e^{-a\left(x^2 + \frac{ik}{a}x + \left(\frac{ik}{2a}\right)^2\right) + \left(\frac{ik}{2a}\right)^2 a} dx$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{2a}{\pi}\right)^{\frac{1}{4}} \int_{-\infty}^{\infty} e^{-a\left(x + \frac{ik}{2a}\right)^2} e^{\left(\frac{ik}{2a}\right)^2 a} dx$$

$u = \sqrt{a}\left(x + \frac{ik}{2a}\right)$

$\frac{du}{dx} = \sqrt{a}$

$dx = \frac{du}{\sqrt{a}}$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{2a}{\pi}\right)^{\frac{1}{4}} e^{\left(\frac{ik}{2a}\right)^2 a} \int_{-\infty}^{\infty} e^{-u^2} \frac{du}{\sqrt{a}}$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{2a}{\pi}\right)^{\frac{1}{4}} e^{\left(\frac{ik}{2a}\right)^2 a} \frac{1}{\sqrt{a}} \sqrt{\pi}$$

$$= \frac{1}{\sqrt{2a}} \left(\frac{2a}{\pi}\right)^{\frac{1}{4}} e^{-\frac{k^2}{4a}} \leftarrow \text{Momentum distribution}$$

Since $\boxed{\Psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k) e^{i\left(kx - \frac{\hbar k^2}{2m}t\right)} dk}$

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2a}} \left(\frac{2a}{\pi}\right)^{\frac{1}{4}} e^{-\frac{k^2}{4a}} e^{i\left(kx - \frac{\hbar k^2}{2m}t\right)} dk$$

$$=$$

2.22 b) Continued...

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{2a}}\left(\frac{2a}{\pi}\right)^{\frac{1}{4}}\int_{-\infty}^{\infty} e^{-\frac{k^2}{4a^2}} e^{i(kx - \frac{\hbar k^2}{2m}t)}\,dk$$

$$= \frac{1}{2\sqrt{\pi a}}\left(\frac{2a}{\pi}\right)^{\frac{1}{4}}\int_{-\infty}^{\infty} e^{-\frac{1}{4a^2}k^2 + ixk - \frac{i\hbar t}{2m}k^2}\,dk$$

$$= \frac{1}{2\sqrt{\pi a}}\left(\frac{2a}{\pi}\right)^{\frac{1}{4}}\int_{-\infty}^{\infty} e^{-\left[\left(\frac{1}{4a^2} + \frac{i\hbar t}{2m}\right)k^2 - ixk\right]}\,dk$$

Let $\alpha \equiv \left(\frac{1}{4a} + \frac{i\hbar^2}{2m}t\right)$ and $\beta = +ix$,

$$= \frac{1}{2\sqrt{\pi a}}\left(\frac{2a}{\pi}\right)^{\frac{1}{4}}\int_{-\infty}^{\infty} e^{-\alpha k^2 + \beta k}\,dk$$

$$= \frac{1}{\sqrt{2\pi a}}\left(\frac{a}{2\pi}\right)^{\frac{1}{4}}\int_{-\infty}^{\infty} e^{-\alpha\left(k^2 - \frac{\beta}{a}k + \frac{\beta^2}{4a^2}\right) + \frac{\beta^2}{4\alpha}}\,dk$$

$$= \frac{1}{\sqrt{2\pi a}}\left(\frac{a}{2\pi}\right)^{\frac{1}{4}} e^{\frac{\beta^2}{4a}}\int_{-\infty}^{\infty} e^{-\alpha\left(k - \frac{\beta}{2}\right)^2}\,dk$$

$$u = \sqrt{\alpha}\left(k - \frac{\beta}{2}\right)$$

$$\frac{du}{\sqrt{\alpha}} = dk$$

$$= \frac{1}{\sqrt{2\pi a}}\left(\frac{a}{2\pi}\right)^{\frac{1}{4}} e^{\frac{\beta^2}{4a}}\frac{1}{\sqrt{\alpha}}\int_{-\infty}^{\infty} e^{-u^2}\,du$$

$$= \frac{1}{\sqrt{2\pi a}}\left(\frac{a}{2\pi}\right)^{\frac{1}{4}} e^{\frac{\beta^2}{4a}}\sqrt{\frac{\pi}{\alpha}}$$

$$= \left(\frac{a}{2\pi}\right)^{\frac{1}{4}} e^{\frac{-x^2}{\frac{1}{a} + \frac{2i\hbar^2 t}{m}}}\frac{1}{\sqrt{2a}}\frac{1}{\sqrt{\alpha}}$$

$$= \left(\frac{a}{2\pi}\right)^{\frac{1}{4}}\frac{1}{\sqrt{2a}} e^{\frac{-ax^2}{1 + \frac{2ai\hbar^2 t}{m}}}\frac{1}{\sqrt{\frac{1}{4a} + \frac{i\hbar^2}{2m}t}}$$

$$= \left(\frac{a}{2\pi}\right)^{\frac{1}{4}}\frac{1}{\sqrt{2a}} e^{-\frac{ax^2}{1 + \frac{2ai\hbar^2 t}{m}}}\frac{\sqrt{2a}}{\sqrt{\frac{1}{2} + \frac{ai\hbar^2 t}{m}}}$$

$$= \left(\frac{a}{2\pi}\right)^{\frac{1}{4}}\frac{\sqrt{2}\, e^{-\frac{ax^2}{1 + \frac{2ai\hbar^2 t}{m}}}}{\sqrt{1 + \frac{2ai\hbar^2 t}{m}}}$$

$$= \left(\frac{2a}{\pi}\right)^{\frac{1}{4}}\frac{e^{\frac{-ax^2}{1 + \frac{2ai\hbar^2 t}{m}}}}{\sqrt{1 + \frac{2ai\hbar^2 t}{m}}}$$

(c) Find $|\Psi(x,t)|^2$

$$= \left| \left(\frac{2a}{\pi}\right)^{\frac{1}{4}} \frac{e^{-ax^2/[1+(2i\hbar at/m)]}}{\sqrt{1+(2i\hbar at/m)}} \right|^2$$

$$= \sqrt{\frac{2a}{\pi}} \frac{e^{-ax^2/[1+(2i\hbar at/m)]} \quad e^{-ax^2/[1-(2i\hbar at/m)]}}{\sqrt{1+(2i\hbar at/m)} \cdot \sqrt{1-(2i\hbar at/m)}}$$

Let $w = \sqrt{\dfrac{a}{1+(2\hbar at/m)^2}}$

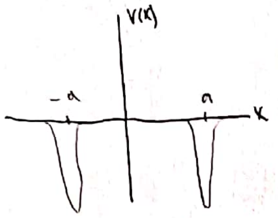
$$= \sqrt{\frac{2a}{\pi}} \frac{e^{-\frac{ax^2}{1+\frac{2i\hbar at}{m}} - \frac{ax^2}{1-\frac{2i\hbar at}{m}}}}{\left(1+\frac{2i\hbar at}{m}\right)^{\frac{1}{2}} \cdot \left(1-\frac{2i\hbar at}{m}\right)^{\frac{1}{2}}}$$

$$= \sqrt{\frac{2a}{\pi}} \frac{e^{-2ax^2\left\{\left(1+\frac{2i\hbar at}{m}\right)^{-1} - \left(1-\frac{2i\hbar at}{m}\right)^{-1}\right\}}}{\left(1+\frac{2i\hbar at}{m}\right)\left(1-\frac{2i\hbar at}{m}\right)}$$

$$= \sqrt{\frac{2a}{\pi}} \frac{e^{-2ax^2 \cdot \frac{2 \cdot \frac{2\hbar at}{m} i}{1+\left(\frac{2\hbar at}{m}\right)^2}}}{1+\left(\frac{2\hbar at}{m}\right)^2}$$

$$= \sqrt{\frac{2a}{\pi}} \frac{e^{\frac{-\frac{8a^2\hbar tix^2}{m}}{1+\left(\frac{2\hbar at}{m}\right)^2}}}{1+\left(\frac{2\hbar at}{m}\right)^2}$$

2.27 (a) $V(x) = -\alpha\,[\,\delta(x+a) + \delta(x-a)\,]$



(b)

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} - \alpha[\delta(x+a)+\delta(x-a)] = E\psi$$

For even $\psi(x)$:

In bound states, $E<0$.

In $(-\infty,-a)$, $V(x)=0$, so

$$\frac{d^2\psi}{dx^2} = -\frac{2mE}{\hbar^2}\psi = K^2\psi \quad \text{where } K = \frac{\sqrt{-2mE}}{\hbar};$$

the solution is

$$\psi(x) = Ae^{-Kx} + Be^{Kx}$$

choose $A=0$, since first term $\to\infty$ as $x\to-\infty$, then

$$\psi(x) = Be^{Kx}, \quad (x<-a)$$

In $(-a,a)$, $V(x)=0$, so the solution is

$$\psi(x) = Ce^{-Kx} + De^{Kx} \quad (-a<x<a)$$

In (a,∞), $V(x)=0$, so the solution is

$$\psi(x) = Fe^{-Kx} + Ge^{Kx}$$

choose $G=0$, since second term $\to\infty$ as $x\to\infty$, then

$$\psi(x) = Fe^{-Kx} \quad (x>a)$$

Since in boundary condition, ψ is always continuous, then $F = B$, $C=D$,

$$\psi(x) = \begin{cases} Be^{Kx} & (x<-a) \\ Ce^{-Kx} + Ce^{Kx} & (-a<x<a) \\ Be^{-Kx} & (x>a) \end{cases} \qquad \boxed{1}$$

Integrate the Schrödinger Equation from $(a-\epsilon)$ to $(a+\epsilon)$, take the limit as $\epsilon\to0$:

$$-\frac{\hbar^2}{2m}\int_{a-\epsilon}^{a+\epsilon}\frac{d^2\psi}{dx^2}dx + \int_{a-\epsilon}^{a+\epsilon}V(x)\psi(x)\,dx = E\cancel{\int_{a-\epsilon}^{a+\epsilon}\psi(x)\,dx}^{\,0}$$

$$-\frac{\hbar^2}{2m}\left[\left.\frac{\partial\psi}{\partial x}\right|_{a+\epsilon} - \left.\frac{\partial\psi}{\partial x}\right|_{a-\epsilon}\right] + \int_{a-\epsilon}^{a+\epsilon}V(x)\psi(x)\,dx = 0 \quad , \text{ let } \Delta\left(\frac{\partial\psi}{\partial x}\right) \equiv \left.\frac{\partial\psi}{\partial x}\right|_{a+\epsilon} - \left.\frac{\partial\psi}{\partial x}\right|_{a-\epsilon}$$

$$\Delta\left(\frac{\partial\psi}{\partial x}\right) = +\frac{2m}{\hbar^2}\lim_{\epsilon\to0}\int_{a-\epsilon}^{a+\epsilon}V(x)\psi(x)\,dx$$

$$\Delta\left(\frac{\partial\psi}{\partial x}\right) = \frac{2m}{\hbar^2}\lim_{\epsilon\to0}\int_{a-\epsilon}^{a+\epsilon} -\alpha[\delta(x+a)+\delta(x-a)]\psi(x)\,dx$$

$$\Delta\left(\frac{\partial\psi}{\partial x}\right) = -\frac{2m\alpha}{\hbar^2}\left[\lim_{\epsilon\to 0}\int_{-a-\epsilon}^{-a+\epsilon}\delta(x+a)\psi(x)\,dx + \lim_{\epsilon\to 0}\int_{a-\epsilon}^{a+\epsilon}\delta(x-a)\psi(x)\,dx\right]$$

$$\Delta\left(\frac{\partial\psi}{\partial x}\right) = -\frac{2m\alpha}{\hbar^2}\left[\cancel{\psi(-a)+}\ \psi(a)\right] \qquad \boxed{2}$$

Take the derivative of $\boxed{1}$, then

$$\frac{d\psi}{dx} = \begin{cases} B\kappa e^{\kappa x} & (x<-a) \quad \text{so } \frac{d\psi}{dx}\Big|_- = B\kappa \\ -C\kappa e^{-\kappa x} + C\kappa e^{\kappa x} & (-a<x<a) \quad \text{so} \\ -B\kappa e^{-\kappa x} & (x>a) \quad \text{so } \frac{d\psi}{dx}\Big|_+ = -B\kappa \end{cases} \qquad \boxed{3}$$

hence $\Delta\left(\frac{d\psi}{dx}\right) = -2B\kappa$ $\boxed{4}$

Since $\frac{d\psi}{dx}$ is continuous except at points where potential is infinite,

at $x=a$, $\frac{d\psi}{dx} = B\kappa e^{\kappa x} = -C\kappa e^{-\kappa x} + C\kappa e^{\kappa x}$

$$\cancel{(B-C)e^{\kappa x} = -Ce^{-\kappa x}}$$

$$B = \frac{-Ce^{-\kappa x} + Ce^{\kappa x}}{e^{\kappa x}}$$

$$B = -Ce^{-2\kappa x} + C \qquad \boxed{3}$$

at $x=-a$, $\frac{d\psi}{dx} = -B\kappa e^{-\kappa x} = -C\kappa e^{-\kappa x} + C\kappa e^{\kappa x}$

$$-B = -C + Ce^{2\kappa x} \qquad \boxed{4}$$

$$B = C - Ce^{2\kappa x} \qquad \boxed{4}$$

$\boxed{3}\to\boxed{4}$:

$$-Ce^{-2\kappa x} + C = C - Ce^{2\kappa x}$$

$$x = 0$$

Since $\psi(x)$ is continuous,

at $x=-a$, $\psi(-a) = Be^{-\kappa a} = Ce^{+\kappa a} + Ce^{-\kappa a}$ $\boxed{5}$

at $x=a$, $\psi(a) = Be^{-\kappa a} = Ce^{-\kappa a} + Ce^{\kappa a}$ $\boxed{6}$

$\boxed{1}\boxed{3}\boxed{6}\to\boxed{2}$:

$$\cancel{-2B\kappa = -\frac{2m\alpha}{\hbar^2}}\left[\ \frac{\partial\psi}{\partial x}\Big|_{a-\epsilon}^{a+\epsilon} = -\frac{2m\alpha}{\hbar^2}Be^{-\kappa a}\right.$$

$$-B\kappa e^{-\kappa a} + C\kappa e^{-\kappa a} - C\kappa e^{\kappa a} = -\frac{2m\alpha}{\hbar^2}Be^{-\kappa a}$$

$$\to -C\kappa e^{\kappa a} + C\kappa e^{\kappa a} + C(\kappa e^{-\kappa a} - \kappa e^{\kappa a}) = -\frac{2m\alpha}{\hbar^2}Be^{-\kappa a} + B\kappa e^{-\kappa a}$$

$$\cancel{-}2C\kappa e^{\kappa a} = \cancel{-}\frac{2m\alpha}{\hbar^2}Be^{-\kappa a}$$

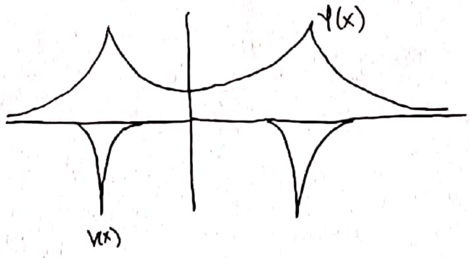
$$\frac{C}{B}\kappa e^{\kappa a} = \frac{m\alpha}{\hbar^2}e^{-\kappa a}$$

$$\frac{C}{B} = \frac{m\alpha}{\hbar^2\kappa}e^{-2\kappa a} \qquad \boxed{7}$$

$\boxed{6}$:

$$\frac{C}{B} = \frac{e^{-ka}}{e^{-ka}+e^{ka}} \qquad \boxed{8}$$

Even $\psi(x)$.



$\boxed{7} = \boxed{8}$:

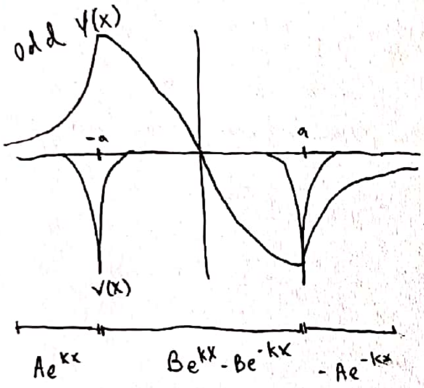
$$\frac{e^{-ka}}{e^{-ka}+e^{ka}} = \frac{m\alpha}{\hbar^2 k} e^{-2ka}$$

$$e^{-ka} = \frac{m\alpha}{\hbar^2 k}\left(e^{-ka}+e^{-3ka}\right)$$

$$1 = \frac{m\alpha}{\hbar^2 k}\left(1+e^{-2ka}\right) \qquad \boxed{9}$$

If α and a are given, then k can be solved, and $\psi(x)$ can be normalized by finding B and C.

for odd $\psi(x)$:



$$\psi(x) = \begin{cases} Ae^{kx} & x<-a \\ Be^{kx} - Be^{-kx} & -a<x<a \\ -Ae^{-kx} & x>a \end{cases} \qquad \boxed{10}$$

Since $\psi(x)$ is odd, then

$$-Ae^{-kx} = \cancel{Ae^{kx}}\ Ae^{k(-x)} \qquad |x|>a \qquad \boxed{12}$$

$$Be^{kx}-Be^{-kx} = -Be^{kx}+Be^{-kx} \qquad |x|<a \qquad \boxed{13}$$

$$\frac{d\psi}{dx} = \begin{cases} Ake^{kx} & x<-a \\ Bke^{kx}+Bke^{-kx} & -a<x<a \\ Ake^{-kx} & x>a \end{cases} \qquad \boxed{11}$$

Since $\psi(x)$ is continuous, then

at $x=-a$, $\psi(-a) = Ae^{-ka} = Be^{-ka} - Be^{+ka}$ $\boxed{14}$

at $x=a$, $\psi(a) = -Ae^{-ka} = Be^{ka} - Be^{-ka}$ $\boxed{15}$

Because $\boxed{2}$,

$$\left.\frac{\partial\psi}{\partial x}\right|_{a-\epsilon}^{a+\epsilon} = -\frac{2m\alpha}{\hbar^2}\left(-Ae^{-ka}\right)$$

$$Ake^{-ka} - Bke^{ka} - Bke^{-ka} = -\frac{2m\alpha}{\hbar^2}\left(-Ae^{-ka}\right)$$

$$\cancel{Ake^{-ka}} - Bke^{ka} - Bke^{ka} + \underline{Bke^{ka}\ \cancel{Bke^{-ka}}} = -\frac{2m\alpha}{\hbar^2}\left(-Ae^{-ka}\right) - \cancel{Ake^{-ka}}$$

$$\cancel{Ake^{-ka}} - 2Bke^{ka} = -\frac{2m\alpha}{\hbar^2}\left(-Ae^{-ka}\right)$$

$$-Bke^{ka} = \frac{m\alpha}{\hbar^2} Ae^{-ka}$$

$$\frac{B}{A} = -\frac{m\alpha}{\hbar^2 k}\frac{e^{-ka}}{e^{ka}}$$

$$\frac{B}{A} = -\frac{m\alpha}{\hbar^2 k} e^{-2ka} \qquad \boxed{16}$$

$\boxed{15}$:

$$\frac{B}{A} = -\frac{e^{-ka}}{e^{ka} - e^{-ka}} \qquad \boxed{17}$$

$\boxed{16} = \boxed{17}$:

$$-\frac{e^{-ka}}{e^{ka} - e^{-ka}} = -\frac{m\alpha}{\hbar^2 K} e^{-2ka}$$

$$e^{-ka} = \frac{m\alpha}{\hbar^2 K}\left(e^{-ka} - e^{-3ka}\right)$$

$$1 = \frac{m\alpha}{\hbar^2 K}\left(1 - e^{-2ka}\right) \qquad \boxed{18}$$

If α and a are given, then K can be solved, and $\psi(x)$ can be normalized by finding A and B.

2.29.

$$V(x) = \begin{cases} -V_0 & \text{for } -a < x < a \\ 0 & \text{for } |x| > a \end{cases} \qquad \boxed{1}$$

Find the odd bound state (with $E < 0$).

- In $(-\infty, -a)$, $V(x) = 0$, thus

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + 0\psi = E\psi$$

$$\frac{d^2\psi}{dx^2} = -\frac{2m}{\hbar^2}E\psi$$

$$\frac{d^2\psi}{dx^2} = \kappa^2\psi \quad , \text{ where } \kappa \equiv \frac{\sqrt{-2mE}}{\hbar}$$

so the solution

$$\psi(x) = Ae^{-\kappa x} + Be^{\kappa x}$$

As $x \to -\infty$, $Ae^{-\kappa x} \to \infty$, so

$$\psi(x) = Be^{\kappa x} \quad , \text{ for } x < -a.$$

- In $(-a, a)$, $V(x) = -V_0$, thus

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} - V_0\psi = E\psi$$

$$\frac{d^2\psi}{dx^2} = -\frac{2m}{\hbar^2}(E + V_0)\psi$$

$$\frac{d^2\psi}{dx^2} = -l^2\psi \quad , \text{ where } l \equiv \frac{\sqrt{2m(E+V_0)}}{\hbar}$$

so the solution

$$\psi(x) = Ce^{ilx} + De^{-ilx} \quad , \text{ where } C = -D \text{ (to make it odd)}$$

$$= C(i\sin(lx) + \cos(lx)) + D(-i\sin(lx) + \cos(lx))$$

$$= (C - D)i\sin(lx) + \cancel{(C+D)\cos(lx)}$$

$$= 2Ci\sin(lx) \quad \text{for } -a < x < a$$

- In (a, ∞), $V(x) = 0$, thus similarly,

$$\psi(x) = Fe^{-\kappa x} + Ge^{\kappa x}$$

As $x \to \infty$, $Ge^{\kappa x} \to \infty$, so

$$\psi(x) = Fe^{-\kappa x} \quad , \text{ for } x > a$$

- Since the wave function is odd, $B = -F$, we have

$$\psi(x) \begin{cases} Be^{\kappa x} & , \text{ for } x < -a \\ 2Ci\sin(lx) & , \text{ for } -a < x < a \\ -Be^{-\kappa x} & , \text{ for } x > a \end{cases}$$

Rename the constants so that $\alpha \equiv B$ and $\beta \equiv 2Ci$:

$$\psi(x) \begin{cases} \alpha e^{\kappa x} & , \text{ for } x < -a \\ \beta\sin(lx) & , \text{ for } -a < x < a \\ -\alpha e^{-\kappa x} & , \text{ for } x > a \end{cases} \qquad \boxed{2}$$

Impose boundary condition 1 : ψ is continuous at $\pm a$, so

$$\psi(-a) = \alpha e^{-Ka} = \beta \sin(-la) \quad \boxed{3}$$

$$\psi(a) = -\alpha e^{-Ka} = \beta \sin(la) \quad \boxed{4}$$

Notice the odd property

Impose boundary condition 2 : $\frac{d\psi}{dx}$ is continuous at $\pm a$, so

$$\boxed{2} \quad \frac{d\psi}{dx} = \begin{cases} \alpha K e^{Kx} & , \text{for } x < -a \\ \beta l \cos(lx) & , \text{for } -a < x < a \\ \alpha K e^{-Kx} & , \text{for } x > a \end{cases} \quad \boxed{5}$$

$$\left.\frac{d\psi}{dx}\right|_{x=-a} = \alpha K e^{-Ka} = \beta l \cos(-la) \quad \boxed{6}$$

$$\left.\frac{d\psi}{dx}\right|_{x=a} = \alpha K e^{-Ka} = \beta l \cos(la) \quad \boxed{7}$$

Notice the even property

Find the relationship between K and l :

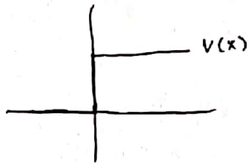
$$\frac{\boxed{7}}{\boxed{4}} : \quad \frac{\cancel{\alpha} K \cancel{e^{-Ka}}}{-\cancel{\alpha} \cancel{e^{-Ka}}} = \frac{\cancel{\beta} l \cos(la)}{\cancel{\beta} \sin(la)}$$

$$-K = l \cot(la)$$

$$K = -l \cot(la)$$

19 November 2018 2.5 The Finite ~~Potential~~ Square Well

2.34 $V(x) = \begin{cases} 0 & \text{if } x \le 0 \\ V_0 & \text{if } x > 0 \end{cases}$



(a) Find the reflection coefficient for $E < V_0$

Known $R \equiv \frac{|B|^2}{|A|^2} = \frac{\beta^2}{1+\beta^2}$, where $\beta = \frac{m\alpha}{\hbar^2 k}$ $\boxed{0}$

$\cancel{R = \frac{+}{1+\frac{2\hbar^2 E}{m\alpha^2}}}$

- In $(-\infty, 0)$, $V(x) = 0$, thus

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + 0\psi = E\psi$$

$$\frac{d^2\psi}{dx^2} = -\frac{2mE}{\hbar^2}\psi$$

$$\frac{d^2\psi}{dx^2} = mK^2\psi \quad \text{, where } K = \frac{\sqrt{-2mE}}{\hbar}$$

So it has solution

$$\psi(x) = \cancel{A\sin}\ Ae^{Kx} + Be^{-Kx} \quad \text{, for } x < 0$$

~~As $x \to \infty$, $Be^{-Kx} \to \infty$, so~~ since K is complex.

$$\cancel{\psi(x) = Ae^{Kx}, \text{ for } x < 0}$$

- In $(0, \infty)$, $V(x) = V_0$, thus

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V_0\psi = E\psi$$

$$\frac{d^2\psi}{dx^2} = -\frac{2m(E-V_0)}{\hbar^2}\psi$$

$$\frac{d^2\psi}{dx^2} = l^2\psi \quad \text{, where } l = \frac{\sqrt{2m(E-V_0)}}{\hbar}$$

So it has solution

$$\psi(x) = Ce^{lx} + De^{-lx} \quad \cancel{\text{for } x > 0}$$

As $x \to \infty$, $Ce^{lx} \to \infty$, so

$$\psi(x) = De^{-lx} \quad \text{, for } x > 0$$

- So we have wave function

$$\psi(x) = \begin{cases} Ae^{Kx} + Be^{-Kx} & x < 0 \\ De^{-lx} & x > 0 \end{cases} \quad \boxed{1}$$

- Impose boundary condition 1: continuous $\psi(x)$:

$$\psi(0) = B + A = D \quad \boxed{2}$$

- Impose boundary condition 2: continuous $\psi'(x)$:

$$\psi'(x) = \begin{cases} AKe^{Kx} - BKe^{-Kx} & x < 0 \\ -Dle^{-lx} & x > 0 \end{cases} \quad \boxed{3}$$

$$\psi'(0) = \cancel{AK}\ -Dl = AK - BK \quad \boxed{4}$$

$\boxed{2} \to \boxed{4}$

$$-(B+A)l = k(A-B)$$

$$-Bl - Al = Ak - Bk$$

$$-Bl + Bk = Ak + Al$$

$$B(k-l) = A(k+l)$$

$$\frac{B}{A} = \frac{k+l}{k-l}$$

$\boxed{5}$: $$R = \left|\frac{B}{A}\right|^2 = \left|\frac{k+l}{k-l}\right|^2 = \frac{k+l}{k-l} \cdot \frac{-k+l}{-k-l} = \frac{k+l}{k-l} \cdot \frac{-(k-l)}{-(k+l)} = 1 \quad \boxed{5}$$

(b) ▸ In $(-\infty, 0)$, $V(x) = 0$, so

Find the reflection coefficient when $E > V_0$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + 0\psi = E\psi$$

$$\frac{d^2\psi}{dx^2} = -\frac{2mE}{\hbar^2}\psi$$

$$\frac{d^2\psi}{dx^2} = k^2\psi \quad , \text{ where } k = \frac{\sqrt{-2mE}}{\hbar}$$

So it has solution

$$\psi(x) = Ae^{kx} + Be^{-kx} \quad , \text{ for } x < 0 \text{, since } k \text{ is complex.}$$

▸ In $(0, \infty)$, $V(x) = V_0$, so

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V_0\psi = E\psi$$

$$\frac{d^2\psi}{dx^2} = -\frac{2m(E-V_0)}{\hbar^2}\psi$$

$$\frac{d^2\psi}{dx^2} = l^2\psi \quad , \text{ where } l = \frac{\sqrt{-2m(E-V_0)}}{\hbar}$$

So it has solution

$$\psi(x) = Ce^{lx} + De^{-lx} \quad , \text{ for } x > 0 \text{, since } l \text{ is complex}$$

Since we are only concerned of the reflected wave and transmitted wave,

$$\psi(x) = Ce^{lx} \quad \text{and} \quad D = 0 \quad \text{since no wave is coming from the right.}$$

Thus, we have wave function

$$\psi(x) = \begin{cases} Ae^{kx} + Be^{-kx} & \text{for } x < 0, \\ Ce^{lx} & \text{for } x > 0. \end{cases} \quad \boxed{6}$$

$$\psi(0) = A + B = C \quad \boxed{7}$$

Its derivative is also continuous

$$\psi'(x) = \begin{cases} Ake^{kx} - Bke^{-kx} & \text{for } x < 0, \\ Cle^{lx} & \text{for } x > 0. \end{cases} \quad 8$$

$$\psi'(0) = Ak - Bk = Cl \quad \boxed{9}$$

$\boxed{I} \to \boxed{II}$:

$$(A+B)\,l = Ak - Bk$$

$$Al + Bl = Ak - Bk$$

$$Al - Ak = -Bl - Bk$$

$$A(l-k) = -B(l+k)$$

$$\frac{B}{A} = -\frac{l-k}{l+k}$$

$\boxed{b}$:

$$R = \left|\frac{B}{A}\right|^2 = \left(-\frac{l-k}{l+k}\right)\cdot\left(-\frac{k-l}{-(l+k)}\right) = \frac{l-k}{l+k}\cdot\frac{-(l-k)}{-(l+k)} = \frac{(l-k)^2}{(l+k)^2}$$

$$= \frac{\left(\frac{\sqrt{-2m(E-V_0)}}{\hbar} - \frac{\sqrt{-2mE}}{\hbar}\right)^2}{\left(\frac{\sqrt{-2m(E-V_0)}}{\hbar} + \frac{\sqrt{-2mE}}{\hbar}\right)^2} = \frac{\frac{-2m(E-V_0)}{\hbar^2} - 2\frac{\sqrt{4m^2E(E-V_0)}}{\hbar^2} + \frac{-2mE}{\hbar^2}}{\frac{-2m(E-V_0)}{\hbar^2} + 2\frac{\sqrt{4m^2E(E-V_0)}}{\hbar^2} + \frac{-2mE}{\hbar^2}}$$

$$= \frac{-2m(E-V_0) - 2mE - 2\sqrt{4m^2E(E-V_0)}}{-2m(E-V_0) - 2mE + 2\sqrt{4m^2E(E-V_0)}}$$

$$= \frac{(l-k)^2\,(l-k)^2}{(l+k)^2(l-k)^2} = \frac{(l-k)^4}{(l^2-k^2)^2}$$

$$l-k = \frac{\sqrt{-2m(E-V_0)}}{\hbar} - \frac{\sqrt{-2mE}}{\hbar} = \frac{\sqrt{-2mE}\sqrt{E-V_0} - \sqrt{2mE}}{\hbar} = \frac{\sqrt{-2mE}\left(\sqrt{E-V_0}-1\right)}{\hbar}$$

$$l^2-k^2 = \frac{-2m(E-V_0)}{\hbar^2} - \frac{-2mE}{\hbar^2} = \frac{-2m(E-V_0)+2mE}{\hbar^2} = \frac{2mV_0}{\hbar^2}$$

$$= \frac{\left(\frac{\sqrt{-2mE}\left(\sqrt{E-V_0}-1\right)}{\hbar}\right)^4}{\left(\frac{2mV_0}{\hbar^2}\right)^2} = \frac{(2mE)^2\left(\sqrt{E-V_0}-1\right)^4}{(2mV_0)^2} = \frac{E^2\left(\sqrt{E-V_0}-1\right)^4}{V_0^2}$$

3.1 a) ▸ Associativity of addition

$u + (v + w) = (u + v) + w$

$f(x) + (g(x) + h(x)) \stackrel{?}{=} (f(x) + g(x)) + h(x)$

$$- \sqrt{\int_a^b |f(x)|^2 dx \int_a^b |g(x)|^2 dx} \leftarrow \text{a constant}$$

$$\leq \int_a^b f(x)^* g(x)\, dx$$

$$\leq \sqrt{\int_a^b |f(x)|^2 dx \int_a^b |g(x)|^2 dx} \leftarrow \text{a constant}$$

$$\int_a^b |f(x) + (g(x) + h(x))|^2 dx$$

$$= \int_a^b (f^*(x) + (g^*(x) + h^*(x)))(f(x) + (g(x) + h(x)))\, dx$$

$$= \int_a^b f^*(x) f(x) + f^*(x)(g(x) + h(x)) + (g^*(x) + h^*(x)) f(x) + (g^*(x) + h^*(x))(g(x) + h(x))\, dx$$

$$= \int_a^b |f(x)|^2 dx + \int_a^b f^*(x) g(x) + f^*(x) h(x) + g^*(x) f(x) + h^*(x) f(x) + g^*(x) g(x)$$

$$+ g^*(x) h(x) + h^*(x) g(x) + h^*(x) h(x)\, dx$$ is square-integratable

Because Schwarz inequality:

$$\left| \int_a^b f(x)^* g(x)\, dx \right| \leq \sqrt{\int_a^b |f(x)|^2 dx \int_a^b |g(x)|^2 dx},$$

the integral of a function's complex conjugate times a function is smaller or equal to a finite number, satisfying the definition of square-integratable function that

$$\int_a^b f(x)^* g(x)\, dx < \infty.$$

$$\int_a^b |(f(x) + g(x)) + h(x)|^2 dx$$

$$= \int_a^b (f^*(x) + g^*(x))(f(x) + g(x)) + (f^*(x) + g^*(x)) h(x) +$$

$$h^*(x)(f(x) + g(x)) + h^*(x) \cdot h(x)\, dx$$

$$= \int_a^b f^*(x) f(x) + f^*(x) g(x) + g^*(x) f(x) + g^*(x) g(x) + f^*(x) h(x) +$$

$$g^*(x) h(x) + h^*(x) f(x) + h^*(x) g(x) + h^*(x) h(x)\, dx$$

is square-integratable.

▸ Commutativity of addition

$u + v = v + u$

$$\int_a^b |f(x) + g(x)|^2 dx = \int_a^b f^*(x) f(x) + f^*(x) g(x) + g^*(x) f(x) + g^*(x) g(x)\, dx$$ and

$$\int_a^b |g(x) + f(x)|^2 dx = \int_a^b g^*(x) f(x) + g^*(x) g(x) + f^*(x) g(x) + f^*(x) f(x)\, dx$$

are square-integratable.

▸ Identity element of addition

$v + 0 = v$

$$\int_a^b |v + 0|^2 dx = \int_a^b v^* v + v^* 0 + 0^* v + 0^* 0\, dx$$ and

$$\int_a^b v^* v\, dx$$ are square-integrable.

▸ Inverse elements of addition

$v + (-v) = 0$

$$\int_a^b v^* v\, dx$$ and

$$\int_a^b |-v|^2 dx = \int_a^b v^* v\, dx$$ are square-integrable.

23 Nov 2018

3.1 Hilbert Space

3.1 (a)

- Compatibility of scalar m with field multiplication $a(bv) = (ab)v$

$$\int_a^b a^*(b^*v^*) \cdot a(bv)\, dx = \int_a^b a^* a \cdot b^* b \cdot v^* v\, dx$$ and

$$\int_a^b (a^*b^*)v^* \cdot (ab)v\, dx = \int_a^b a^* a \cdot b^* b \cdot v^* v\, dx$$ are square integratable.

- Identity element of Scalar multiplication $1v = v$

$$\int_a^b 1^* v^* \cdot 1 v\, dx = \int_a^b 1^* 1 \cdot v^* v\, dx$$ and

$$\int_a^b v^* v\, dx$$ are square-integratable.

- Distoibutivity of scalar multiplication with respect to vector addition $a(u+v) = au + av$

$$\int_a^b |a(u+v)|^2 dx = \int_a^b a^*(u^*+v^*)\, a(u+v)\, dx$$
$$= \int_a^b a^* a\,(u^* u + u^* v + v^* u + v^* v)\, dx$$ and

$$\int_a^b (a^* u^* + a^* v^*)(au + av)\, dx$$
$$= \int_a^b a^* a \cdot u^* u + a^* a \cdot u^* v + a^* a v^* u + a^* a \cdot v^* v\, dx$$ are square-integratable

- Distributivity of scalar multiplication with respect to field addition $(a+b)v = av + bv$

$$\int_a^b (a^* + b^*) v^* (a+b) v\, dx$$
$$= \int_a^b v^* v\,(a^* a + a^* b + b^* a + b^* b)\, dx$$ and

$$\int_a^b (a^* v^* + b^* v^*)(av + bv)\, dx$$
$$= \int_a^b a^* a v^* v + a^* b \cdot v^* v + b^* b \cdot v^* v + b^* a \cdot v^* v\, dx$$ are square-integratable

(b) An inner product of two functions $f(x)$ and $g(x)$ are

$$\langle f | g \rangle = \int_a^b f(x)^* g(x)\, dx$$

- Conjugate symmetry :

$$\langle x, y \rangle = \overline{\langle y, x \rangle}$$

$$\langle f | g \rangle = \int_a^b f(x)^* g(x)\, dx \quad \boxed{1}$$

$$\langle \overline{g | f} \rangle = \int_a^b g(x)^{**} f(x)^*\, dx = \int_a^b f(x)^* g(x)\, dx \quad \boxed{2}$$

$\boxed{1} = \boxed{2}$ ✓

▸ Linearity in the first argument :

$$\langle ax, y\rangle = a\langle x, y\rangle$$

$$\langle x+y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$$

$$\langle af | g\rangle = \int_a^b a^* f(x)^* g(x)\, dx \qquad \boxed{3}$$

$$a\langle f|g\rangle = a\int_a^b f(x)^* g(x)\, dx$$

Since the coefficients are real numbers in the Hilbert Space,

$$= \int_a^b a^* f(x)^* g(x)\, dx \qquad \boxed{4}$$

$\boxed{3} = \boxed{4}$ ✓

$$\langle f+g | h\rangle = \int_a^b (f(x)^* + g(x)^*) \cdot h(x)\, dx = \int_a^b f(x)^* h(x)\, dx + \int_a^b g(x)^* h(x)\, dx \qquad \boxed{5}$$

$$\langle f | h\rangle + \langle g|h\rangle = \int_a^b f(x)^* h(x)\, dx + \int_a^b g(x)^* h(x)\, dx \qquad \boxed{6}$$

$\boxed{5} = \boxed{6}$ ✓

▸ Positive - definiteness :

$$\langle x, x\rangle \geq 0$$

$$\langle x, x\rangle = 0 \iff x = 0$$

$$\langle f|f\rangle = \int_a^b f(x)^* f(x)\, dx = \int_a^b |f(x)|^2\, dx \geq 0$$

~~if $\langle f|f\rangle = \int_a^b |f(x)|^2 dx = 0$~~

Suppose a closed interval $[c, d]$ in (a, b),

if $|f(x)|^2 > 0$,

then $\min\{\int_c^d |f(x)|^2 dx\} = \min\{|f(x)|^2\}(d-c) > 0$.

Use proof by contradiction, if $\langle f|f\rangle = 0$, $|f(x)|^2 = 0$, thus $f(x) = 0$. ✓

3.3 $\langle h | \hat{Q} h\rangle = \int_{-\infty}^{\infty} h^* \hat{Q} h\, dx \qquad \boxed{1}$

$\langle \hat{Q} h | h\rangle = \int_{-\infty}^{\infty} \hat{Q}^* h^* h\, dx = \int_{-\infty}^{\infty} h^* \hat{Q} h\, dx \quad \boxed{2}$ (since $\langle Q\rangle = \langle Q\rangle^*$)

$\boxed{1} = \boxed{2}$, so $\langle h | \hat{Q} h\rangle = \langle \hat{Q} h | h\rangle$

if $\langle h | \hat{Q} h\rangle = \langle \hat{Q} h | h\rangle$, 3.3 (Contined) Let $h = f + g$, then

$$\langle h | \hat{Q} h \rangle = \langle (f+g) | \hat{Q}(f+g) \rangle$$
$$= \int_{-\infty}^{\infty} (f^* + g^*) \hat{Q}(f+g)\, dx$$
$$= \int_{-\infty}^{\infty} (f^* + g^*)(\hat{Q} f + \hat{Q} g)\, dx$$
$$= \int_{-\infty}^{\infty} f^* \hat{Q} f + f^* \hat{Q} g + g^* \hat{Q} f + g^* \hat{Q} g \, dx$$
$$= \langle f | \hat{Q} f \rangle + \langle f | \hat{Q} g \rangle + \langle g | \hat{Q} f \rangle + \langle g | \hat{Q} g \rangle \qquad \boxed{1}$$

$$\langle \hat{Q} h | h \rangle = \langle \hat{Q}(f+g) | (f+g) \rangle$$
$$= \int_{-\infty}^{\infty} \hat{Q}^*(f^* + g^*)(f+g)\, dx$$
$$= \int_{-\infty}^{\infty} (\hat{Q}^* f^* + \hat{Q}^* g^*)(f+g)\, dx$$
$$= \int_{-\infty}^{\infty} \hat{Q}^* f^* f + \hat{Q}^* f^* g + \hat{Q}^* g^* f + \hat{Q}^* g^* g \, dx$$
$$= \langle \hat{Q} f | f \rangle + \langle \hat{Q} f | g \rangle + \langle \hat{Q} g | f \rangle + \langle \hat{Q} g | g \rangle \qquad \boxed{2}$$

Since $\langle h | \hat{Q} h \rangle = \langle \hat{Q} h | h \rangle$, thus $\boxed{1} = \boxed{2}$:

$$\langle f | \hat{Q} g \rangle + \langle g | \hat{Q} f \rangle = \langle \hat{Q} f | g \rangle + \langle \hat{Q} g | f \rangle \qquad \boxed{3}$$

Let $h = f + ig$, then

$$\langle h | \hat{Q} h \rangle = \langle (f+ig) | \hat{Q}(f+ig) \rangle$$
$$= \int_{-\infty}^{\infty} (f^* - ig^*) \hat{Q}(f+ig)\, dx$$
$$= \int_{-\infty}^{\infty} (f^* - ig^*)(\hat{Q} f + \hat{Q} ig)\, dx$$
$$= \int_{-\infty}^{\infty} f^* \hat{Q} f + f^* \hat{Q} ig - ig^* \hat{Q} f - ig^* \hat{Q} ig \, dx$$
$$= \langle f | \hat{Q} f \rangle + \langle f | \hat{Q} ig \rangle + \langle ig | \hat{Q} f \rangle + \langle ig | \hat{Q} g \rangle \qquad \boxed{4}$$

$$\langle \hat{Q} h | h \rangle = \langle \hat{Q}(f+ig) | (f+ig) \rangle$$
$$= \int_{-\infty}^{\infty} (\hat{Q}^* f^* - \hat{Q}^* ig^*)(f+ig)\, dx$$
$$= \int_{-\infty}^{\infty} \hat{Q}^* f^* f + \hat{Q}^* f^* ig - \hat{Q}^* ig^* f - \hat{Q}^* ig^* ig \, dx$$
$$= \langle \hat{Q} f | f \rangle + \langle \hat{Q} f | ig \rangle + \langle \hat{Q} ig | f \rangle + \langle \hat{Q} ig | g \rangle \qquad \boxed{5}$$

$\boxed{4} = \boxed{5}$:

$$\langle f | \hat{Q} ig \rangle + \langle ig | \hat{Q} f \rangle = \langle \hat{Q} f | ig \rangle + \langle \hat{Q} ig | f \rangle \qquad \boxed{6}$$
$$i \langle f | \hat{Q} g \rangle - i \langle g | \hat{Q} f \rangle = i \langle \hat{Q} f | g \rangle - i \langle \hat{Q} g | f \rangle$$
$$\langle f | \hat{Q} g \rangle - \langle g | \hat{Q} f \rangle = \langle \hat{Q} f | g \rangle - \langle \hat{Q} g | f \rangle \qquad \boxed{7}$$

$\boxed{3} - \boxed{7}$

$$2\langle g|\hat{Q}f\rangle = 2\langle \hat{Q}g|f\rangle$$

$$\langle g|\hat{Q}f\rangle = \langle \hat{Q}g|f\rangle$$

$\boxed{3} + \boxed{7}$

$$2\langle f|\hat{Q}g\rangle = 2\langle \hat{Q}f|g\rangle$$

$$\langle f|\hat{Q}g\rangle = \langle \hat{Q}f|g\rangle$$

3.5 The hermitian conjugate of an operator $\hat{Q}$ is the operator $\hat{Q}^{\dagger}$ such that

$$\langle f|\hat{Q}g\rangle = \langle \hat{Q}^{\dagger} f|g\rangle \quad \text{(for all } f \text{ and } g\text{)}$$

(a)

$$\langle x\rangle = \langle f|\hat{x}g\rangle = \int_{-\infty}^{\infty} f^* \hat{x} g\, dx = \int_{-\infty}^{\infty} \hat{x}^* f^* g\, dx = \langle \hat{x} f|g\rangle$$

$$x^{\dagger} = x$$

$$\langle i\rangle = \langle f|ig\rangle = \int_{-\infty}^{\infty} f^* i g\, dx = \int_{-\infty}^{\infty} -i^* f^* g\, dx = \langle -if|g\rangle$$

$$i^{\dagger} = -i$$

$$\langle \tfrac{d}{dx}\rangle = \langle f|\tfrac{d}{dx}g\rangle = \int_{-\infty}^{\infty} f^* \frac{d}{dx} g\, dx = \left[f^* g\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{df^*}{dx}\cdot g\, dx = -\langle \tfrac{d}{dx} f^*|g\rangle$$

$f(x) = f^*$ $\quad g'(x) = \frac{dg}{dx}$

$f'(x) = \frac{df^*}{dx}$ $\quad g(x) = g$

$$\frac{d}{dx}^{\dagger} = -\frac{d}{dx}$$

(b)

$$\langle a_+\rangle = \langle f|a_+ g\rangle = \int_{-\infty}^{\infty} f^* a_+ g\, dx$$

$$\cancel{= \int_{-\infty}^{\infty} f^* \frac{1}{\sqrt{2\hbar m\omega}}(-ip + m\omega x)\, g\, dx}$$

$$\cancel{= \int_{-\infty}^{\infty} \frac{-2ip}{\sqrt{2\hbar m\omega}} a_+^* f^* g\, dx} = \int_{-\infty}^{\infty} \left(\frac{-2i\frac{\hbar}{i}\frac{\partial}{\partial x}}{\sqrt{2\hbar m\omega}}\right)^* a_+^* f^* g\, dx$$

$$\cancel{= \langle \frac{-2ip}{\sqrt{2\hbar m\omega}} a_+ f|g\rangle}$$

$$\cancel{a_+^{\dagger} = \frac{-2ip}{\sqrt{2\hbar m\omega}} a_+ = \frac{-2ip}{\sqrt{2\hbar m\omega}}\cdot\frac{1}{\sqrt{2\hbar m\omega}}(-ip + m\omega x)}$$

$$= \frac{}{\sqrt{2\hbar m\omega}}$$

$$a_{\pm} = \frac{1}{\sqrt{2\hbar m\omega}}(\mp ip + m\omega x)$$

$$a_+ = \frac{1}{\sqrt{2\hbar m\omega}}\left(-i\frac{\hbar}{i}\frac{d}{dx} + m\omega x\right)$$

$$a_+^{\dagger} = \frac{1}{\sqrt{2\hbar m\omega}}\left(-(-i)\frac{\hbar}{(i)}\frac{-d}{dx} + m\omega x\right) = \frac{1}{\sqrt{2\hbar m\omega}}\left(+i\frac{\hbar}{i}\frac{d}{dx} + m\omega x\right) = a_-$$

(b) $\langle a_+ \rangle = \langle f | a_+ g \rangle$ $\qquad a_\pm = \frac{1}{\sqrt{2\hbar m\omega}} (\mp ip + m\omega x)$

$= \int_{-\infty}^{\infty} f^* a_+ g \, dx$

$= \int_{-\infty}^{\infty} f^* \frac{1}{\sqrt{2\hbar m\omega}} (-i \frac{\hbar}{i} \frac{d}{dx} + m\omega x) g \, dx$

$= \int_{-\infty}^{\infty} f^* \frac{1}{\sqrt{2\hbar m\omega}} (-i \frac{\hbar}{i} \frac{dg}{dx}) dx + \int_{-\infty}^{\infty} f^* \frac{1}{\sqrt{2\hbar m\omega}} (m\omega x g) \, dx$

$= -\frac{\hbar}{\sqrt{2\hbar m\omega}} \int_{-\infty}^{\infty} f^* \frac{dg}{dx} dx + \frac{m\omega}{\sqrt{2\hbar m\omega}} \int_{-\infty}^{\infty} f^* x g \, dx$

$= -\frac{\hbar}{\sqrt{2\hbar m\omega}} \left(\cancel{[f^* g]_{-\infty}^{\infty}} - \int_{-\infty}^{\infty} \frac{df^*}{dx} \cdot g \, dx \right) + \frac{m\omega}{\sqrt{2\hbar m\omega}} \int_{-\infty}^{\infty} x^* f^* g \, dx$

$= \int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{2\hbar m\omega}} i \frac{\hbar}{i} \frac{d}{dx} \right)^* f^* g \, dx + \int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{2\hbar m\omega}} m\omega \right)^* x^* f^* g \, dx$

$= \int_{-\infty}^{\infty} \left[\frac{1}{\sqrt{2\hbar m\omega}} (ip + m\omega x) \right]^* f^* g \, dx$

$= \int_{-\infty}^{\infty} a_-^* f^* g \, dx$

$= \langle a_- f | g \rangle$

$a_+^\dagger = a_-$

(c) Show that $(\hat{Q}\hat{R})^\dagger = \hat{R}^\dagger \hat{Q}^\dagger$

$\langle \hat{Q}\hat{R} \rangle = \langle f | \hat{Q}\hat{R} g \rangle$

$= \int_{-\infty}^{\infty} f^* \hat{Q}\hat{R} g \, dx = \int_{-\infty}^{\infty} \hat{Q}^* f^* \hat{R} g \, dx$

$\cancel{= \int_{-\infty}^{\infty} (\hat{Q}\hat{R})^* f^* g \, dx} = \int_{-\infty}^{\infty} \hat{R}^* \hat{Q}^* f^* g \, dx$

$\cancel{= \langle \hat{Q}\hat{R} f | g \rangle} = \langle \hat{R}\hat{Q} f | g \rangle$

$(\hat{Q}\hat{R})^\dagger = \hat{R}\hat{Q} = \hat{R}^\dagger \hat{Q}^\dagger$

$\langle \hat{R} \rangle = \langle f | \hat{R} g \rangle$ $\qquad \langle \hat{Q} \rangle = \langle f | \hat{Q} g \rangle$

$= \int_{-\infty}^{\infty} f^* \hat{R} g \, dx$ $\qquad = \int_{-\infty}^{\infty} f^* \hat{Q} g \, dx$

$= \int_{-\infty}^{\infty} \hat{R}^* f^* g \, dx$ $\qquad = \int_{-\infty}^{\infty} \hat{Q}^* f^* g \, dx$

$= \langle \hat{R} f | g \rangle$ $\qquad = \langle \hat{Q} f | g \rangle$

$\hat{R}^\dagger = \hat{R}$ $\qquad \hat{Q}^\dagger = \hat{Q}$

Since $\hat{R}$ and $\hat{Q}$ are Hermitian Operators

3.6

$$\hat{Q} \equiv \frac{d^2}{d\phi^2}$$

$$f(\phi + 2\pi) = f(\phi) \quad \boxed{1}$$

$$\langle f | \hat{Q} g \rangle = \int_0^{2\pi} f^* \left(\frac{d^2 g}{d\phi^2}\right) d\phi$$

$f(\phi) = f^*$ $\quad g'(\phi) = \frac{d^2 g}{d\phi^2}$

$f'(\phi) = \frac{df^*}{d\phi}$ $\quad g(\phi) = \frac{dg}{d\phi}$

$$= \left[f^* \frac{dg}{d\phi} \right]_0^{2\pi} - \int_0^{2\pi} \frac{df^*}{d\phi} \frac{dg}{d\phi} d\phi$$

$f(\phi) = \frac{df^*}{d\phi}$ $\quad g'(\phi) = \frac{dg}{d\phi}$

$f'(\phi) = \frac{d^2 f^*}{d\phi^2}$ $\quad g(\phi) = g$

$$= -\left[\frac{df^*}{d\phi} g \right]_0^{2\pi} + \int_0^{2\pi} \frac{d^2 f^*}{d\phi^2} g \, d\phi$$

$$= \langle \hat{Q} f | g \rangle$$

$\because$ $\langle f | \hat{Q} g \rangle = \langle \hat{Q} f | g \rangle$ $\quad \boxed{2}$

$\therefore$ $\hat{Q}$ is hermitian.

The eigenvalue function

$$\frac{d^2}{d\phi^2} f(\phi) = q f(\phi) \quad \boxed{3}$$

~~has solution of~~

$$\cancel{f(\phi) = A\sin(kx) + B\cos(kx), \text{ where } k = \sqrt{q}} \quad \boxed{4}$$

~~Since 4~~,

$$\cancel{f(\phi) = A\sin(k\phi) + B\cos(k\phi) = A\sin(k\phi + 2\pi k) + B\cos(k\phi + 2\pi k)}$$

" defense position " ambiguous

$$\frac{d^2}{d\phi^2} f(\phi) = -k^2 f(\phi), \quad \text{where } k = \pm\sqrt{-q} \quad \boxed{4}$$

$$\frac{d^2 f(\phi)}{d\phi^2} \frac{df(\phi)}{d\phi} = -k^2 \frac{df(\phi)}{d\phi} f(\phi)$$

$$\frac{1}{2}\left(\frac{df(\phi)}{d\phi}\right)^2 = -\frac{1}{2}\left(k f(\phi)\right)^2$$

$$\frac{df(\phi)}{d\phi} = ik f(\phi)$$

$$\frac{df(\phi)}{d\phi} = \pm\sqrt{q} f(\phi)$$

$$\int \frac{df(\phi)}{f(\phi)} = \int \pm\sqrt{q} \, d\phi$$

$$\ln(f(\phi)) = \pm\sqrt{q}\phi + c$$

$$f(\phi) = e^{\pm\sqrt{q}\phi + c}$$

$$f(\phi) = e^c \cdot e^{\pm\sqrt{q}\phi} \quad \boxed{5}$$

Since $\boxed{1}$, $$e^{c}e^{\pm\sqrt{q}\phi} = e^{c}e^{\pm\sqrt{q}\phi + 2\pi\sqrt{q}}$$

$$e^{\pm\sqrt{q}\phi} = e^{\pm\sqrt{q}\phi}e^{2\pi\pm\sqrt{q}}$$

$$1 = e^{2\pi\pm\sqrt{q}}$$

Because $e^{2\pi i n} = 1$, so

$$in = \pm\sqrt{q}$$

$$\sqrt{-n^2} = \pm\sqrt{q}$$

$$-n^2 = q \quad , \text{ where } n \text{ is integer}$$

The spectrum, or collection of all eigenvalues of $\hat{Q}$, of $\hat{Q}$ is $(-n^2)$, where n is integer.

The spectrum is degenerate, since every eigenvalue has two eigenfunctions:

$$e^{c}\cdot e^{\sqrt{q}\phi}$$

and

$$e^{c}\cdot e^{-\sqrt{q}\phi}.$$

3.7 (a) Given $f(x)$ and $g(x)$ are eigenfunctions of operator $\hat{Q}$, with the same eigenvalue q, that is,

$$\hat{Q}f = qf \qquad \hat{Q}f = \langle f|\hat{Q}f\rangle = \langle\hat{Q}f|f\rangle = \int_{-\infty}^{\infty} f^*\hat{Q}f\,dx$$

$$\hat{Q}g = qg \qquad qf = q\langle f|f\rangle = q\int_{-\infty}^{\infty} f^* f\,dx = \int_{-\infty}^{\infty} f^* q f\,dx$$

Show the linear combination of f and g is itself an eigenfunction of $\hat{Q}$, with eigenvalue q.

$$\hat{Q}(f+g) = q(f+g)$$

$$\hat{Q}(f+g) = \hat{Q}f + \hat{Q}g = qf + qg = q(f+g)$$

(b) $f(x) = e^{x}$

$g(x) = e^{-x}$

$\hat{Q} = \frac{d^2}{dx^2}$

$$\hat{Q}f \stackrel{!}{=} qf \qquad \hat{Q}g = qg$$

$$\frac{d^2}{dx^2}e^{x} = e^{x} \qquad \frac{d^2}{dx^2}e^{-x} = \frac{d}{dx}(-e^{-x}) = e^{-x}$$

$$q = 1 \qquad q = 1$$

f and g are eigenfunction of $\hat{Q}$, with the same eigenvalue.

We have linear combination

$$ae^{x} + be^{-x} \qquad \boxed{1}$$

and

$$ce^{x} + de^{-x}, \qquad \boxed{2}$$

so we have

$$a(e^{x} + \tfrac{b}{a}e^{-x})$$

and

$$c(e^{x} + \tfrac{c}{d}e^{-x})$$

Make them orthonormal in $[-1,1]$,

$$\cancel{ac}\int_{-1}^{1}(e^{x} + \tfrac{b}{a}e^{-x})(e^{x} + \tfrac{c}{d}e^{-x})\,dx = 0$$

$$\int_{-1}^{1} e^{2x} + \frac{c}{d} + \frac{b}{a} + \frac{bc}{ad}e^{-2x}\,dx = 0$$

Make $\boxed{1}$ and $\boxed{2}$ orthonormal in $[-1,1]$,

$$\int_{-1}^{1} (ae^{x} + be^{-x})(ce^{x} + de^{-x})\,dx = 0 \qquad (a+c)e' + (b+d)e^{x}$$

$$\int_{-1}^{1} ace^{2x} + ad + bc + bde^{-2x}\,dx = 0$$

$$ac(e^{2}-e^{-2}) + 2ad + 2bc + bd(-e^{-2}+e^{2}) = 0$$

$$(ac+bd)(e^{2}-e^{-2}) + 2(ad+bc) = 0$$

Tricks: if $A+B=0$, where A has irrational part, and B has rational part, then $A=0$ and $B=0$.

If a, b, c, and d are rational, let $a \equiv \frac{1}{2}$ and $c \equiv \frac{1}{2}$,

$$(ac+bd)(e^{2}-e^{-2}) = 0$$

$$ac+bd = 0$$

$$\tfrac{1}{4} + bd = 0$$

$$\tfrac{1}{4} = -bd \qquad \boxed{3}$$

$$2(ad+bc) = 0$$

$$ad+bc = 0$$

$$\tfrac{1}{2}d = -\tfrac{1}{2}b$$

$$d = -b \qquad \boxed{4}$$

$\boxed{4} \to \boxed{3}$ $\quad \frac{1}{4} = -b(-b)$

$$\tfrac{1}{4} = +b^{2}$$

$$\pm\tfrac{1}{2} = b$$

$$d = \mp\tfrac{1}{2}$$

Thus, $\underbrace{(\frac{1}{2}e^{x} + \frac{1}{2}e^{-x})}_{\sinh x =}$ and $\underbrace{(\frac{1}{2}e^{x} - \frac{1}{2}e^{-x})}_{\cosh x =}$ are linear combinations that is orthogonal.

3.9 (a) Discrete spectrum: Stationary states of infinite square well

$$E_n = \frac{\hbar^2 k_n^2}{2m} = \frac{n^2\pi^2\hbar^2}{2ma^2} \qquad [2.27]$$

$$\langle H \rangle = \sum_{n=1}^{\infty} |c_n|^2 E_n \qquad [2.39]$$

· Wide, deep finite square well

$$E_n + V_0 \cong \frac{n^2\pi^2\hbar^2}{2m(2a)^2} \qquad [157]$$

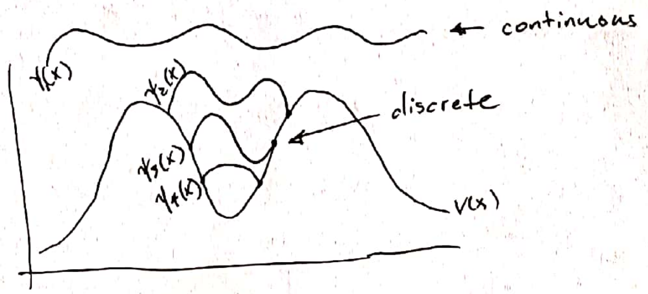
· Shallow, narrow well

$$E_n + V_0 = \frac{n^2\pi^2\hbar^2}{2m(2a)^2} \qquad [171]$$

(b) Continuous spectrum: Bound states of delta-function well

$$E = \frac{m\alpha^2}{2\hbar^2}$$

(c) Both discrete and continuous spectrum



3.10 $\psi(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{\pi}{a}x\right)$, $E = \frac{\pi^2\hbar^2}{2ma^2}$

$\hat{p}\,\psi(x) \overset{?}{=} p\,\psi(x)$

$\frac{\hbar}{i}\frac{d\psi}{dx} = \frac{\hbar}{i}\sqrt{\frac{2}{a}} \cos\left(\frac{\pi}{a}x\right)\cdot\frac{\pi}{a}$

No, the ground state of the infinite square well is not an eigenfunction of momentum, since sin fuction $\neq$ cos function.

3.11 The position-space wave function for a particle in the ground state of Harmonic oscillator is

$$\psi_0(x) = \cancel{\sqrt{\frac{2}{a}}\sin\left(\frac{\pi}{a}x\right)} \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{\frac{m\omega}{2\hbar}x^2}$$

Since $\Phi(p,t) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} e^{-ipx/\hbar}\,\Psi(x,t)\,dx$ by Plancherel's theorem,

$$\Phi(p,t) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} e^{-ipx/\hbar} \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{\frac{m\omega}{2\hbar}x^2} \cancel{\sqrt{\frac{2}{a}}\sin\left(\frac{\pi}{a}x\right)}\, e^{-iEt/\hbar}\,dx$$

$$= \frac{1}{\sqrt{2\pi\hbar}}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{\frac{-iEt}{\hbar}}\int_{-\infty}^{\infty} e^{-\frac{ipx}{\hbar}}\, e^{\frac{m\omega}{2\hbar}x^2}\,dx$$

$$\cancel{\int_{-\infty}^{\infty} e^{-\frac{ipxm\omega}{2\hbar^2}x^2}\,dx = \int_{-\infty}^{\infty} e^{-u^2}\,dx = \int_{-\infty}^{\infty} e^{-u^2}\,du\sqrt{\frac{2\hbar^2}{ipxm\omega}} = \sqrt{\pi}\sqrt{\frac{2\hbar^2}{ipxm\omega}}}$$

$$\cancel{u = \sqrt{\frac{ipxm\omega}{2\hbar^2}}\,x \qquad dx = du\cdot\sqrt{\frac{2\hbar^2}{ipxm\omega}}}$$

$$\cancel{= \frac{1}{\sqrt{2\pi\hbar}}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{\frac{-iEt}{\hbar}}\sqrt{\frac{2\hbar^2\pi}{ipxm\omega}}}$$

$$\cancel{= \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{\frac{-iEt}{\hbar}}\sqrt{\frac{\hbar}{ip^2\omega}}}$$

$$\cancel{= \sqrt{\frac{\hbar}{i\omega}}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}}\frac{1}{p^2}\, e^{\frac{-iEt}{\hbar}}}$$

$\frac{m\omega}{2\hbar}\left(x^2 - \frac{ip2}{m\omega}x\right)$

$-\frac{m\omega}{2\hbar}\left(x^2 + \frac{2ip}{m\omega} + \frac{ip}{m\omega}\right)$

$-\frac{m\omega}{2\hbar}\left(x + \frac{ip}{m\omega}\right)^2 - \frac{ip}{2\hbar}$

$$= \frac{1}{\sqrt{2\pi\hbar}}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{\frac{-iEt}{\hbar}}\int_{-\infty}^{\infty} e^{-\frac{ipx}{\hbar}+\frac{m\omega}{2\hbar}x^2}\,dx$$

$$= \frac{1}{\sqrt{2\pi\hbar}}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{\frac{-iEt}{\hbar}}\int_{-\infty}^{\infty} e^{-\frac{m\omega}{2\hbar}\left(x+\frac{ip}{m\omega}\right)^2 - \frac{ip}{2\hbar}}\,dx$$

$$= \frac{1}{\sqrt{2\pi\hbar}}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{ip}{2\hbar}}\, e^{\frac{-iEt}{\hbar}}\int e^{-\frac{m\omega}{2\hbar}\left(x+\frac{ip}{m\omega}\right)^2}\,dx \qquad u = \sqrt{\frac{m\omega}{2\hbar}}\left(x+\frac{ip}{m\omega}\right)$$

$$= \frac{1}{\sqrt{2\pi\hbar}}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{ip}{2\hbar}}\, e^{\frac{-iEt}{\hbar}}\int e^{-u^2}\,du\cdot\sqrt{\frac{2\hbar}{m\omega}} \qquad dx = du\sqrt{\frac{2\hbar}{m\omega}}$$

Momentum-space wave function $\Phi(x,t)$ → $= \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{ip}{2\hbar}}\, e^{\frac{-iEt}{\hbar}}\frac{1}{\sqrt{m\omega}}$

$E = \frac{1}{2}mv^2 = \frac{1}{2}\frac{m^2v^2}{m} = \frac{p^2}{2m}$

$E = \frac{\hbar\omega}{2} = \frac{p^2}{2m}$

$|p| = \sqrt{\hbar m \omega}$

Day 40

14 December 2018 — 3.5 The Uncertainty Principle

3.14 Prove $\sigma_x \sigma_H \geq \frac{\hbar}{2m}|\langle p\rangle|$, where $A \equiv x$, $B \equiv p^2/2m + V$.

Known generalized uncertainty principle

$$\sigma_A^2 \sigma_B^2 \geq \left(\frac{1}{2i}\langle[\hat{A},\hat{B}]\rangle\right)^2$$

Find commutator of A and B ($\hat{x}$ and $\hat{H}$):

$$[\hat{A},\hat{B}] \equiv \hat{A}\hat{B} - \hat{B}\hat{A}$$

$$[\hat{x},\hat{H}] = \hat{x}\hat{H} - \hat{H}\hat{x}$$

$$[\hat{x},\hat{H}] = x\left(\frac{p^2}{2m}+V\right) - \left(\frac{p^2}{2m}+V\right)x$$

$$[\hat{x},\hat{H}]f(x) = x\left(\frac{1}{2m}\left(\frac{\hbar}{i}\frac{d}{dx}\right)^2+V\right)f - \left(\frac{1}{2m}\left(\frac{\hbar}{i}\frac{d}{dx}\right)^2+V\right)(xf)$$

$$= \frac{x}{2m}\frac{\hbar^2}{i^2}\frac{d^2f}{dx^2} + xVf - \left[\cancel{\frac{1}{2m}\frac{\hbar}{i}f + \frac{x}{2m}\frac{\hbar}{i}\frac{df}{dx}} + Vxf\right]$$

$$\cancel{= -\frac{1}{2m}\frac{\hbar}{i}f} \qquad \frac{1}{2m}\frac{\hbar^2}{i^2}\frac{d}{dx}\left(f + x\frac{df}{dx}\right)$$

$$\cancel{[\hat{x},\hat{H}] = -\frac{1}{2m}\frac{\hbar}{i} = \frac{i\hbar}{2m}} \qquad \frac{1}{2m}\frac{\hbar^2}{i^2}\left(\frac{df}{dx} + \frac{df}{dx} + \frac{d^2f}{dx^2}x\right)$$

$$[\hat{x},\hat{H}] = -\frac{\hbar^2}{m}\frac{d}{dx}$$

so

$$\sigma_x^2\sigma_H^2 \geq \cancel{\left(\frac{1}{2i}\frac{i\hbar}{2m}\right)^2}\left(-\frac{1}{2i}\frac{\hbar^2}{m}\frac{d}{dx}\right)^2$$

$$\sigma_x^2\sigma_H^2 \geq \cancel{\left(\frac{\hbar}{4m}\right)^2} - \frac{1}{2i}\frac{\hbar^2}{m}\frac{d}{dx}$$

$$\cancel{\sigma_x\sigma_H \geq \frac{\hbar}{4m}}$$

$$\sigma_x\sigma_H \geq -\frac{\hbar}{2m}\left(\frac{\hbar}{i}\frac{d}{dx}\right)$$

$$\sigma_x\sigma_H \geq \frac{\hbar}{2m}|-\hat{p}|$$

$$\sigma_x\sigma_H \geq \frac{\hbar}{2m}|\langle p\rangle|$$

$|1\rangle = \begin{pmatrix}1\\0\\0\end{pmatrix}$ $|2\rangle = \begin{pmatrix}0\\1\\0\end{pmatrix}$ $|3\rangle = \begin{pmatrix}0\\0\\1\end{pmatrix}$

$\langle 1| = (1\ 0\ 0)$ $\langle 2| = (0\ 1\ 0)$ $\langle 3| = (0\ 0\ 1)$

3.22 (a) $|\alpha\rangle = i|1\rangle - 2|2\rangle - i|3\rangle$

$|\beta\rangle = i|1\rangle + 2|3\rangle$

$|\alpha\rangle = \begin{pmatrix}i\\-2\\-i\end{pmatrix}$ $\langle\alpha| = \cancel{\begin{pmatrix}i^*\\-2^*\\-i^*\end{pmatrix}}\ (i^*\ \ -2^*\ \ -i^*) = (-i\ \ -2\ \ +i)$

$= -i|1\rangle^T - 2|2\rangle^T + i|3\rangle^T = -i\langle 1| - 2\langle 2| + i\langle 3|$

$|\beta\rangle = \begin{pmatrix}i\\0\\2\end{pmatrix}$ $\langle\beta| = (i^*\ \ 0^*\ \ 2^*) = (-i\ \ 0\ \ 2)$

$= -i|1\rangle^T + 2|3\rangle^T = -i\langle 1| + 2\langle 3|$

(b) $\langle\alpha|\beta\rangle = \cancel{\alpha^*\beta}\ (-i\langle 1| - 2\langle 2| + i\langle 3|)\cdot(i|1\rangle + 2|3\rangle)$

$= 1 + 2i$

$\langle\beta|\alpha\rangle = (-i\langle 1| + 2\langle 3|)\cdot(i|1\rangle - 2|2\rangle - i|3\rangle)$

$= 1 - 2i$

$\langle\beta|\alpha\rangle = \langle\alpha|\beta\rangle^* = 1 - 2i$

(c) $\hat{A} = |\alpha\rangle\langle\beta|$

$= (i|1\rangle - 2|2\rangle - i|3\rangle)\cdot(-i\langle 1| + 2\langle 3|)$

$= \begin{pmatrix}i\\-2\\-i\end{pmatrix}\cdot(-i\ \ 0\ \ 2)$

$= \begin{pmatrix}-1 & 0 & 2i\\ 2i & 0 & -4\\ -1 & 0 & -2i\end{pmatrix}$

$A = \begin{pmatrix}-1 & 0 & 2i\\ 2i & 0 & -4\\ -1 & 0 & -2i\end{pmatrix}$

$A^\dagger = \begin{pmatrix}-1 & -2i & -1\\ 0 & 0 & 0\\ -2i & -4 & 2i\end{pmatrix}$

$A \neq A^\dagger$, so A is not hermitian

$A^* + A^T \rightarrow A^\dagger$

The complex conjugate and transpose of a matrix is the dagger.

If a matrix is hermitian,

$A = A^\dagger$

HW: Find the eigenfunction $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ given operator $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 1 & 0 & 0 \end{bmatrix}$.

Let $\lambda = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$,

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \lambda \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$$

$$\begin{bmatrix} 1-\lambda & 0 & 2 \\ 0 & 1-\lambda & 3 \\ 1 & 0 & -\lambda \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

since the determinant should be zero,

$$(1-\lambda)\left[(-1)^{1+1}(1-\lambda)(-\lambda)\right] + 2\left[(-1)^{1+3}(1-\lambda)\right] = 0$$

$$-\lambda(1-2\lambda+\lambda^2) + 2 - 2\lambda = 0$$

$$-\lambda + 2\lambda^2 - \lambda^3 + 2 - 2\lambda = 0$$

$$-\lambda^3 + 2\lambda^2 - 3\lambda + 2 = 0$$

$$\begin{array}{r|rrrr} 1 & 1 & -2 & 3 & -2 \\ & & 1 & -1 & 2 \\ \hline & 1 & -1 & 2 & 0 \end{array}$$

$$-(\lambda-1)(\lambda^2-\lambda+2) = 0$$

$$-(\lambda-1)\left(\lambda-\tfrac{1}{2}-\tfrac{\sqrt{7}}{2}i\right)\left(\lambda-\tfrac{1}{2}+\tfrac{\sqrt{7}}{2}i\right) = 0 \qquad \frac{-(-1)\pm\sqrt{1-8}}{2} = \frac{1\pm\sqrt{7}}{2} = \frac{1}{2}\pm\frac{\sqrt{7}}{2}i$$

$\lambda = 1$

$\lambda = \frac{1}{2} + \frac{\sqrt{7}}{2}i$ eigenvalues

$\lambda = \frac{1}{2} - \frac{\sqrt{7}}{2}i$

When $\lambda = 1$,

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\begin{cases} a + 2c = a \\ b + 3c = b \\ a = c \end{cases}$$

So a must be zero, thus

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ b \\ 0 \end{bmatrix}$$

b can be any number.

When $\lambda = \frac{1}{2} \pm \frac{\sqrt{7}}{2}i$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{2}\pm\frac{\sqrt{7}}{2}i\right)a \\ \left(\frac{1}{2}\pm\frac{\sqrt{7}}{2}i\right)b \\ \left(\frac{1}{2}\pm\frac{\sqrt{7}}{2}i\right)c \end{bmatrix}$$

$$\begin{cases} a + 2c = \left(\frac{1}{2}\pm\frac{\sqrt{7}}{2}i\right)a \\ b + 3c = \left(\frac{1}{2}\pm\frac{\sqrt{7}}{2}i\right)b \\ a = \left(\frac{1}{2}\pm\frac{\sqrt{7}}{2}i\right)c \end{cases}$$

~~test~~ Let $a = 1$,

if $\lambda = \frac{1}{2} + \frac{\sqrt{7}}{2}i$; if $\lambda = \frac{1}{2} - \frac{\sqrt{7}}{2}i$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ -\frac{3}{2} \\ \frac{2}{1+\sqrt{7}i} \end{bmatrix} \qquad \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ -\frac{3}{2} \\ \frac{2}{1-\sqrt{7}i} \end{bmatrix}$$

3.23 $\hat{H} = E\left(|1\rangle\langle 1| - |2\rangle\langle 2| + |1\rangle\langle 2| + |2\rangle\langle 1| \right)$

Known $|1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\langle 1| = [1\ 0]$, $|2\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\langle 2| = [0\ 1]$

so $\hat{H} = E\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right)$

$$\hat{H} = E\left(\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \right)$$

$$\hat{H} = E \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\hat{H}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} E & \cancel{0}E \\ \cancel{0}E & E \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}$$

$$\hat{H}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} E & \cancel{0}E \\ \cancel{0}E & -E \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \lambda\begin{bmatrix} a \\ b \end{bmatrix}$$

$$\begin{bmatrix} E & E \\ E & -E \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = 0$$

$$\begin{bmatrix} E-\lambda & E \\ E & -E-\lambda \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = 0$$

determinant should be zero

$(E-\lambda)(-E-\lambda) \cancel{+} - E^2 = 0$

$-E^2 - E\lambda + E\lambda + \lambda^2 \cancel{+} - E^2 = 0$

$$-2E^2 + \lambda^2 = 0$$

$$\lambda = \pm\sqrt{2}\,E$$

eigenvalues

when $\lambda = \sqrt{2}E$

$$\begin{bmatrix} E & E \\ E & -E \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \sqrt{2}Ea \\ \sqrt{2}Eb \end{bmatrix}$$

$$\begin{cases} Ea + Eb = \sqrt{2}Ea \\ Ea - Eb = \sqrt{2}Eb \end{cases} \Rightarrow \begin{cases} a + b = a\sqrt{2} \\ a - b = b\sqrt{2} \end{cases} \rightarrow b = (\sqrt{2}-1)a$$

$\cancel{\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}}$ $\cancel{\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}}$

$\cancel{a - (\sqrt{2}-1)a = (\sqrt{2}-1)a\sqrt{2}}$

$\cancel{(2-\sqrt{2})a = (2-\sqrt{2})a}$

let $a = 1$,

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ \sqrt{2}-1 \end{bmatrix}$$

eigenvectors

when $\lambda = -\sqrt{2}E$

$$\begin{bmatrix} E & E \\ E & -E \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -\sqrt{2}Ea \\ -\sqrt{2}Eb \end{bmatrix}$$

$$\begin{cases} Ea + Eb = -\sqrt{2}Ea \\ Ea - Eb = -\sqrt{2}Eb \end{cases} \Rightarrow \begin{cases} a + b = -\sqrt{2}a \\ a - b = -\sqrt{2}b \end{cases} \rightarrow b = -\sqrt{2}a - a$$

let $a = 1$

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ -\sqrt{2}-1 \end{bmatrix}$$

3.25

$\int_{-1}^{1} |1|^2 \, dx = 2$

$A^2 \cdot 2 = 1$

$A = \sqrt{2}$

$f(x) = \sqrt{2}$.

$g(x) = B^2 \left(x - \sqrt{2} \int_{-1}^{1} \sqrt{2}^* \, x \, dx \right)$

$= B^2 \left(x - \sqrt{2} \left[\frac{\sqrt{2}}{2} x^2 \right]_{-1}^{1} \right)$

$= B^2 \left(x - \sqrt{2} \left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} \right) \right)$

$= B^2 (x \ldots)$

$\cancel{B^2 \quad x - 2}$

$B^2 \int_{-1}^{1} (x \ldots) \, dx = 1$

$B^2 \left[\frac{1}{2} x^2 - 2x \right]_{-1}^{1} = 1$

$B^2 \left(-\frac{3}{2} - \frac{3}{2} \right) = 1$

$B^2 = -\frac{1}{3}$

B

(a) $f(x) = A \cdot 1$

$|A|^2 \int_{-1}^{1} dx = 1$

$2|A|^2 = 1$

$A = \frac{1}{\sqrt{2}}$

$f(x) = \frac{1}{\sqrt{2}}$

(b) $g(x) = B \left(x - \frac{1}{\sqrt{2}} \cancel{\int_{-1}^{1} \frac{1}{\sqrt{2}}^* x \, dx} \right)$

$= Bx$

$|B|^2 \int_{-1}^{1} x^* x \, dx = 1$

$|B|^2 \left[\frac{1}{3} x^3 \right]_{-1}^{1} = 1$

$\left(\frac{1}{3} + \frac{1}{3} \right) |B|^2 = 1$

$|B|^2 = \frac{3}{2}$

$B = \frac{\sqrt{3}}{\sqrt{2}}$

$g(x) = \frac{\sqrt{3}}{\sqrt{2}} x$

(c) $h(x) = C \left(x^2 - \frac{1}{\sqrt{2}} \int_{-1}^{1} \frac{1}{\sqrt{2}} x^2 \, dx - \frac{\sqrt{3}}{\sqrt{2}} x \int_{-1}^{1} \frac{\sqrt{3}}{\sqrt{2}} x \cdot x^2 \, dx \right)$

$= C \left(x^2 - \frac{1}{\sqrt{2}} \left[\frac{1}{3\sqrt{2}} x^3 \right]_{-1}^{1} - \frac{\sqrt{3}}{\sqrt{2}} x \left[\frac{\sqrt{3}}{4\sqrt{2}} x^4 \right]_{-1}^{+1} \right)$

$= C \left(x^2 - \frac{1}{\sqrt{2}} \left(\frac{1}{3\sqrt{2}} + \frac{1}{3\sqrt{2}} \right) - \frac{\sqrt{3}}{\sqrt{2}} x \cancel{\left(\frac{\sqrt{3}}{4\sqrt{2}} - \frac{\sqrt{3}}{4\sqrt{2}} \right)} \right)$

$= C \left(x^2 - \frac{2}{6} \right)$

$u = x^2 - \frac{1}{3}$

$\frac{du}{dx} = 2x$

$dx = \frac{du}{2x}$

$|C|^2 \int_{-1}^{1} \left(x^2 - \frac{2}{6} \right)^2 dx = 1$

$|C|^2 \int_{-1}^{1} x^4 - \frac{2}{3} x^2 + \frac{1}{9} \, dx = 1$

$|C|^2 \left[\frac{1}{5} x^5 - \frac{2}{9} x^3 + \frac{1}{9} x \right]_{-1}^{1} = 1$

$|C|^2 \left[\left(\frac{1}{5} - \frac{2}{9} + \frac{1}{9} \right) - \left(-\frac{1}{5} + \frac{2}{9} - \frac{1}{9} \right) \right] = 1$

$|C|^2 \left(\frac{2}{5} - \frac{2}{9} \right) = 1$

$|C|^2 = \frac{45}{8}$

$|C| = \sqrt{\frac{45}{8}}$

$h(x) = \sqrt{\frac{45}{8}} \left(x^2 - \frac{1}{3} \right)$

3.25 d)

$k(x) = D\left(\frac{1}{\sqrt{2}}\int_{-1}^{1}\frac{1}{\sqrt{2}}\right.$

$k(x) = D\left(x^3 - \frac{1}{\sqrt{2}}\cancel{\int_{-1}^{1}\frac{1}{\sqrt{2}}^* x^3\,dx} - \sqrt{\frac{3}{2}}x\int_{-1}^{1}\sqrt{\frac{3}{2}}x^* x^3\,dx - \sqrt{\frac{45}{8}}\left(x^2-\frac{1}{3}\right)\cancel{\int_{-1}^{1}\sqrt{\frac{45}{8}}\left(x^2-\frac{1}{3}\right)x^3\,dx}\right)$

$= D\left(x^3 - \sqrt{\frac{3}{2}}x\sqrt{\frac{3}{2}}\int_{-1}^{1}x^4\,dx\right)$

$= D\left(x^3 - \frac{3}{2}x\left[\frac{1}{5}x^5\right]_{-1}^{1}\right)$

$= D\left(x^3 - \frac{3}{2}x\left(\frac{2}{5}\right)\right)$

$= D\left(x^3 - \frac{3}{5}x\right)$

$|D|^2\int_{-1}^{1}\left(x^3-\frac{3}{5}x\right)^2 dx = 1$

$|D|^2\int_{-1}^{1}x^6 - \frac{6}{5}x^4 + \frac{9}{25}x^2\,dx = 1$

$|D|^2\left[\frac{1}{7}x^7 - \frac{6}{25}x^5 + \frac{3}{25}x^3\right]_{-1}^{1} = 1$

$|D|^2\left(\left(\frac{1}{7}-\frac{6}{25}+\frac{3}{25}\right)-\left(-\frac{1}{7}+\frac{6}{25}-\frac{3}{25}\right)\right) = 1$

$|D|^2\left(\frac{2}{7}-\frac{6}{25}\right) = 1$

$|D|^2 = \boxed{\frac{175}{8}}$

$D = \sqrt{\frac{175}{8}}$

$k(x) = \sqrt{\frac{175}{8}}\left(x^3-\frac{3}{5}x\right)$

4.1 (a) Canonical Commutation Relation

(1) $[r_i, r_j] = r_i r_j - r_j r_i = r_i r_j - r_j r_i = 0$

(2) $[p_i, p_j]f = \frac{\hbar}{i}\frac{\partial f}{\partial r_i}\frac{\hbar}{i}\frac{\partial f}{\partial r_j} - \frac{\hbar}{i}\frac{\partial f}{\partial r_j}\frac{\hbar}{i}\frac{\partial f}{\partial r_i} = 0$

$[r_i, p_j]f = r_i \frac{\hbar}{i}\frac{\partial f}{\partial r_j} - \frac{\hbar}{i}\frac{\partial}{\partial r_j}(r_i f)$

$= r_i \frac{\hbar}{i}\frac{\partial f}{\partial r_j} - \frac{\hbar}{i}\left(\frac{\partial r_i}{\partial r_j} f + \frac{\partial f}{\partial r_j} r_i\right)$

$= - \frac{\hbar}{i}\frac{\partial r_i}{\partial r_j} f$

$\begin{cases} \text{if } i = j, & \frac{\partial r_j}{\partial r_j} = 1 \\ \text{if } i \neq j, & \frac{\partial r_i}{\partial r_j} = 0 \end{cases}$

(3) $[r_i, p_j] = -\frac{\hbar}{i}\frac{\partial r_i}{\partial r_j} = i\hbar \frac{\partial r_i}{\partial r_j} = i\hbar\delta_{ij}$

$[p_i, r_j]f = \frac{\hbar}{i}\frac{\partial}{\partial r_i}(r_j f) - r_j \frac{\hbar}{i}\frac{\partial f}{\partial r_i}$

$= \frac{\hbar}{i}\left(\frac{\partial r_j}{\partial r_i} f + \frac{\partial f}{\partial r_i} r_j\right) - r_j \frac{\hbar}{i}\frac{\partial f}{\partial r_i}$

$= \frac{\hbar}{i}\frac{\partial r_j}{\partial r_i} f$

(4) $[p_i, r_j] = \frac{\hbar}{i}\frac{\partial r_j}{\partial r_i} = -i\hbar\frac{\partial r_j}{\partial r_i} = -i\hbar\delta_{ij}$

So we have $[r_i, p_j] = -[p_i, r_j] = i\hbar\delta_{ij}$

and $[r_i, r_j] = [p_i, p_j] = 0$

(b) Show that $\frac{d}{dt}\langle\vec{r}\rangle = \frac{1}{m}\langle\vec{p}\rangle$ and $\frac{d}{dt}\langle\vec{p}\rangle = \langle -\nabla V\rangle$

First show that it's generally true in any dimention, so show that

$\frac{d}{dt}\langle r_i\rangle = \frac{1}{m}\langle p_i\rangle$ and $\frac{d}{dt}\langle p_i\rangle = \langle -\frac{d}{dr_i}V_i\rangle$

(1) $\frac{d}{dt}\langle r_i\rangle = \frac{d}{dt}\int \psi^* r_i \psi \, dr_i$

$\psi(r_i, t)$

$r_i(r_i)$

$= \int \frac{\partial}{\partial t}(\psi^* r_i \psi)\, dr_i$

$= \int \left[\frac{\partial}{\partial t}(\psi^* r_i)\right]\psi + \frac{\partial \psi}{\partial t}\psi^* r_i \, dr_i$

$= \int \frac{\partial \psi^*}{\partial t} r_i \psi + \cancel{\frac{\partial r_i}{\partial t}\psi^*\psi} + \frac{\partial \psi}{\partial t}\psi^* r_i \, dr_i$

One Dimention:

$$\boxed{i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V\psi}$$

$$\frac{\partial\psi}{\partial t} = \frac{i\hbar}{2m}\frac{\partial^2\psi}{\partial x^2} - \frac{i}{\hbar}V\psi$$

$$\frac{\partial\psi^*}{\partial t} = -\frac{i\hbar}{2m}\frac{\partial^2\psi^*}{\partial x^2} + \frac{i}{\hbar}V\psi^*$$

Three Dimention:

$$\boxed{i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V\psi}$$

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

$$i\hbar \frac{\partial\psi}{\partial t} = \left[\sum_j -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial r_j^2}\psi\right] + V\psi$$

$$\frac{\partial\psi}{\partial t} = \left[\sum_j + \frac{i\hbar}{2m}\frac{\partial^2}{\partial r_j^2}\psi\right] - \frac{i}{\hbar}V\psi$$

$$\frac{\partial\psi^*}{\partial t} = \left[\sum_j - \frac{i\hbar}{2m}\frac{\partial^2}{\partial r_j^2}\psi^*\right] + \frac{i}{\hbar}V\psi^*$$

$$\cancel{= \int r_i\psi\left(-\frac{i\hbar}{2m}\frac{\partial^2\psi^*}{\partial r_i^2} + \frac{i}{\hbar}V\psi^*\right) + \psi^* r_i\left(\frac{i\hbar}{2m}\frac{\partial^2\psi}{\partial r_i^2} - \frac{i}{\hbar}V\psi\right) dr_i}$$

$$\cancel{= \int r_i\psi \frac{i\hbar}{2m}\frac{\partial^2\psi^*}{\partial r_i^2} + \psi^* r_i \frac{i\hbar}{2m}\frac{\partial^2\psi}{\partial r_i^2} dr_i}$$

$$= \int + r_i\psi\left[\frac{i\hbar}{2m}\left(\sum_j \frac{\partial^2\psi^*}{\partial r_j^2}\right) + \cancel{\frac{i}{\hbar}V\psi^*}\right] + \psi^* r_i\left[\frac{i\hbar}{2m}\left(\sum_j\frac{\partial^2\psi}{\partial r_j^2}\right) - \cancel{\frac{i}{\hbar}V\psi}\right] dr_i$$

$$= \int -r_i\psi\frac{i\hbar}{2m}\left(\sum_j \frac{\partial^2\psi^*}{\partial r_j^2}\right) + \psi^* r_i\frac{i\hbar}{2m}\left(\sum_j\frac{\partial^2\psi}{\partial r_j^2}\right) dr_i$$

(Both term) Integration by Parts, where the number evaluation part is evaluated at $-\infty$ and ∞, going to 0. →

$$= \int +\frac{i\hbar}{2m}\left(\sum_j \frac{\partial\psi^*}{\partial r_j}\cdot\frac{\partial}{\partial r_j}(r_i\psi)\right) - \frac{i\hbar}{2m}\left(\sum_j\frac{\partial\psi}{\partial r_j}\cdot\frac{\partial}{\partial r_j}(\psi^* r_i)\right) dr_i$$

$$= \int +\frac{i\hbar}{2m}\left(\sum_j\frac{\partial\psi^*}{\partial r_j}\left(\frac{\partial r_i}{\partial r_j}\psi + \frac{\partial\psi}{\partial r_j}r_i\right)\right) - \frac{i\hbar}{2m}\left(\sum_j\frac{\partial\psi}{\partial r_j}\left(\frac{\partial\psi^*}{\partial r_j}r_i + \frac{\partial r_i}{\partial r_j}\psi^*\right)\right) dr_i$$

$$= \int +\frac{i\hbar}{2m}\left(\sum_j\frac{\partial\psi^*}{\partial r_j}\delta_{ij}\psi + \cancel{\frac{\partial\psi^*}{\partial r_j}\frac{\partial\psi}{\partial r_j}r_i}\right) - \frac{i\hbar}{2m}\left(\sum_j \cancel{\frac{\partial\psi}{\partial r_j}\frac{\partial\psi^*}{\partial r_j}r_i} + \frac{\partial\psi}{\partial r_j}\delta_{ij}\psi^*\right) dr_i$$

$\boxed{\frac{\partial\psi^*}{\partial r_j}\delta_{ij} = \frac{\partial\psi^*}{r_i}}$ →

$$= \int +\frac{i\hbar}{2m}\left(\sum_j\frac{\partial\psi^*}{\partial r_i}\psi\right) - \frac{i\hbar}{2m}\left(\sum_j\frac{\partial\psi}{\partial r_i}\psi^*\right) dr_i$$

$$\cancel{= \int \frac{i\hbar}{2m}\left(\sum_j\frac{\partial\psi^*}{\partial r_i}\psi\right) + \frac{i\hbar}{2m}\left(\sum_j \psi\frac{\partial}{\partial r_i}\right)}$$

$$\cancel{= \int -\frac{i\hbar}{2m}\left(\sum_j \psi^*\frac{\partial\psi}{\partial r_i}\right)}$$

$$\cancel{= \int\frac{i\hbar}{2m}\left(\sum_j\frac{\partial\psi^*}{\partial r_i}\psi\right) + \frac{i\hbar}{2m}\left(\sum_j\psi\frac{\partial\psi^*}{\partial r_i}\right) dr_i}$$

$$\cancel{\int \frac{i\hbar}{m}\left(\sum_j\frac{\partial\psi^*}{\partial r_i}\psi\right) dr_i}$$

$$\cancel{= \int\frac{i\hbar}{2m}\left(\sum_j\frac{\partial\psi^*}{\partial r_i}\psi\right) + \frac{i\hbar}{2m}\left(\sum_j\psi\frac{\partial\psi^*}{\partial r_i}\right) dr_i}$$

$$\int -\frac{i\hbar}{2m}\left(\sum_j\psi^*\frac{\partial\psi}{\partial r_i}\right) - \frac{i\hbar}{2m}\left(\sum_j\frac{\partial\psi}{\partial r_i}\psi^*\right) dr_i$$

$$\int -\frac{i\hbar}{m}\left(\sum_j\psi^*\frac{\partial\psi}{\partial r_i}\right) dr_i$$

$$\frac{1}{m}\int\sum_j\psi^*\frac{\hbar}{i}\frac{\partial\psi}{\partial r_i}\, dr_i$$

Integration by parts for 1st term with some argument to obtain the REAL part

$$= \int -\frac{i\hbar}{2m}\left(\sum_j\psi^*\frac{\partial\psi}{\partial r_i}\right) + \frac{i\hbar}{2m}\left(\sum_j\frac{\partial\psi}{\partial r_i}\psi^*\right) dr_i$$

$$= \frac{1}{m}\int\sum_j\psi^*(-i\hbar)\frac{\partial\psi}{\partial r_i}\, dr_i$$

$$= \frac{1}{m}\int\sum_j\psi^*\left(\frac{\hbar}{i}\frac{\partial}{\partial r_i}\right)\psi\, dr_i$$

$$= \frac{1}{m}\langle p_i\rangle$$

4.1 (b) (2)

$$\frac{d}{dt}\langle p_i \rangle = \frac{d}{dt}\int \psi^* \left(\frac{\hbar}{i}\frac{\partial}{\partial r_i}\right)\psi \, dr_i$$

$$= \frac{\hbar}{i}\int \frac{\partial}{\partial t}\left(\psi^* \frac{\partial}{\partial r_i}\psi\right) dr_i$$

$$= \frac{\hbar}{i}\int \frac{\partial \psi^*}{\partial t}\cdot\frac{\partial \psi}{\partial r_i} + \psi^* \frac{\partial}{\partial t}\frac{\partial \psi}{\partial r_i} \, dr_i$$

$$= \frac{\hbar}{i}\int \left[\left(\sum_j -\frac{i\hbar}{2m}\frac{\partial^2 \psi^*}{\partial r_j^2}\right)\frac{\partial \psi}{\partial r_i} + \psi^* \frac{\partial}{\partial r_i}\left[\left(\sum_j \frac{i\hbar}{2m}\frac{\partial^2}{\partial r_j^2}\psi\right) - \frac{i}{\hbar}V\psi\right] dr_i$$

$$+\frac{i}{\hbar}V\psi^*$$

$$\boxed{\int f(x)\sum_{i=1}^{n} x_i \, dx = \sum_{i=1}^{n}\int f(x)\, x_i \, dx}$$

$$= \frac{\hbar}{i}\int \left(-\frac{i\hbar}{2m}\left(\sum_j \frac{\partial^2 \psi^*}{\partial r_j^2}\right) + \frac{i}{\hbar}V\psi^*\right)\frac{\partial \psi}{\partial r_i} + \psi^* \frac{\partial}{\partial r_i}\left[\left(\frac{i\hbar}{2m}\sum_j \frac{\partial^2 \psi}{\partial r_j^2} - \frac{i}{\hbar}V\psi\right)\right] dr_i$$

$$= -i\hbar\int -\frac{i\hbar}{2m}\left(\sum_j \frac{\partial^2 \psi^*}{\partial r_j^2}\right)\frac{\partial \psi}{\partial r_i} + \frac{i}{\hbar}V\psi^*\frac{\partial \psi}{\partial r_i} + \frac{i\hbar}{2m}\psi^*\frac{\partial}{\partial r_i}\left(\sum_j \frac{\partial^2 \psi}{\partial r_j^2}\right) - \frac{i}{\hbar}\psi^*\frac{\partial}{\partial r_i}V\psi \, dr_i$$

For 1st term:

$$u = \frac{\partial \psi}{\partial r_i} \qquad \frac{\partial v}{\partial r_j} = \sum_j \frac{\partial^2 \psi^*}{\partial r_j^2}$$

$$\frac{\partial u}{\partial r_j} = \frac{\partial^2 \psi}{\partial r_i \partial r_j} \qquad v = \sum_j \frac{\partial \psi^*}{\partial r_j}$$

boundary evaluated at $-\infty$ and ∞ remember the "$-$" sign

$$= -i\hbar\int -\frac{i\hbar}{2m}\frac{\partial \psi}{\partial r_i}\left(\sum_j \frac{\partial^2 \psi^*}{\partial r_j^2}\right) + \cancel{\frac{i}{\hbar}V\psi^*\frac{\partial \psi}{\partial r_i}} + \frac{i\hbar}{2m}\psi^*\left(\sum_j \frac{\partial^3 \psi}{\partial r_i \partial r_j^2}\right) - \frac{i}{\hbar}\psi^*\frac{\partial V}{\partial r_i}\psi - \cancel{\frac{i}{\hbar}\psi^* V\frac{\partial \psi}{\partial r_i}} \, dr_i$$

$$= -i\hbar\int +\frac{i\hbar}{2m}\frac{\partial^2 \psi}{\partial r_i \partial r_j}\left(\sum_j \frac{\partial \psi^*}{\partial r_j}\right) + \frac{i\hbar}{2m}\psi^*\left(\sum_j \frac{\partial^3 \psi}{\partial r_i \partial r_j^2}\right) dr_i - \left\langle \frac{\partial V}{\partial r_i}\right\rangle$$

$$u = \frac{\partial^2 \psi}{\partial r_i \partial r_j} \qquad \frac{\partial v}{\partial r_j} = \sum_j \frac{\partial \psi^*}{\partial r_j}$$

$$\frac{\partial u}{\partial r_j} = \frac{\partial^3 \psi}{\partial r_i \partial r_j^2} \qquad v = \sum_j \psi^*$$

$$= -i\hbar\int -\frac{i\hbar}{2m}\frac{\partial^2 \psi}{\partial r_i \partial r_j^2}\left(\sum_j \psi^*\right) + \frac{i\hbar}{2m}\psi^*\left(\sum_j \frac{\partial^3 \psi}{\partial r_i \partial r_j^2}\right) dr_i - \left\langle \frac{\partial V}{\partial r_i}\right\rangle$$

$$= -i\hbar\sum_j\int \cancel{-\frac{i\hbar}{2m}\psi^*\frac{\partial^3 \psi}{\partial r_i \partial r_j^2}} + \cancel{\frac{i\hbar}{2m}\psi^*\frac{\partial^3 \psi}{\partial r_i \partial r_j^2}} \, dr_i - \left\langle \frac{\partial V}{\partial r_i}\right\rangle$$

$$= -\left\langle \frac{\partial V}{\partial r_i}\right\rangle$$

$$V(x) = \begin{cases} 0 & [0,a] \\ \infty & \text{otherwise} \end{cases}$$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi$$

$$\frac{d^2\psi}{dx^2} = -k^2\psi \quad , \quad k \equiv \frac{\sqrt{2mE}}{\hbar}$$

$$\psi = A\sin kx + B\sin kx$$

$$\psi(0) = \psi(a) = 0$$

$$\psi = A\sin ka = 0$$

$$ka = 0, \pm\pi, \pm 2\pi \ldots$$

$$k = \frac{n\pi}{a}$$

$$\frac{\sqrt{2mE}}{\hbar} = \frac{n\pi}{a}$$

$$E = \frac{n^2\pi^2}{a^2}\frac{\hbar}{2m}$$

$$\boxed{\psi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi}{a}x\right)}$$

orthonormality:

$$\int \psi_m(x)^* \psi_n(x)\,dx = \delta_{mn}$$

$$\delta_{mn} = \begin{cases} 0 & m \neq 0 \\ 1 & m = n \end{cases}$$

completeness:

$$f(x) = \sum_{n=1}^{\infty} c_n \psi_n(x)$$

$$= \sqrt{\frac{2}{a}}\sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{a}x\right)$$

Fourier series

Dirichlet theorem

· Fourier trick

$$\int \psi_m(x)^* f(x)\,dx$$

$$= \int \psi_m(x)^* \sum_{n=1}^{\infty} c_n \psi_n(x)\,dx$$

$$= \sum_{n=1}^{\infty} c_n \int \cancel{\psi_m(x)^* \psi_n(x)}\,dx$$

$$= \sum_{n=1}^{\infty} c_n \delta_{mn}$$

$$= \cancel{\sum_{n=1}^{\infty}} c_m$$

$$c_n = \int \psi_m(x)^* f(x)\,dx$$

$$\sqrt{\frac{2}{a}}\sin\left(\frac{n\pi}{a}x\right)$$

$$\Psi(x,0) = Ax(a-x) \qquad [0,a]$$

$$V(x) = \begin{cases} 0 & [0,a] \\ \infty & \text{otherwise} \end{cases} \qquad \text{Find } \Psi(x,t)$$

$$A^2 \int_{-\infty}^{\infty} x^2(a-x)^2\,dx = 1$$

$$\int_0^a x(a-x)\,dx = \frac{1}{A^2}$$

$$\int_0^a ax - x^2\,dx = \frac{1}{A^2}$$

$$\frac{a}{2}x^2 - \frac{x^3}{3}\Big|_0^a = \frac{1}{A^2}$$

10-15+6

$$\frac{a^3}{2} - \frac{a^3}{3} = \frac{1}{A^2}$$

$$\frac{a^3}{6} = \frac{1}{A^2}$$

$$A^2 = \frac{6}{a^3}$$

$$A = \sqrt{\frac{6}{a^3}}$$

$$A^2\int_0^a x^2(a-x)^2\,dx = 1$$

$$A^2\int_0^a a^2x^2 - 2ax^3 + x^4\,dx = 1$$

$$A^2\left[\frac{a^2x^3}{3} - \frac{2ax^4}{4} + \frac{x^5}{5}\right]_0^a = 1$$

$$A^2\left[\frac{a^5}{3} - \frac{a^5}{2} + \frac{a^5}{5}\right] = 1$$

$$A^2 = \frac{30}{a^5}$$

$$A = \sqrt{\frac{30}{a^5}}$$

$$\Psi(x,0) = \sqrt{\frac{30}{a^5}}A(x(a-x))$$

$$c_n = \int_0^a \psi_m(x)^* f(x)\,dx$$

$$= \int_0^a \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi}{a}x\right)\sqrt{\frac{30}{a^5}}Ax(a-x)\,dx$$

2.5

$$\Psi(x,0) = A\left[\psi_1(x) + \psi_2(x)\right]$$

(a) $$A^2\int_0^a (\psi_1(x) + \psi_2(x))^2\,dx = 1$$

$$= A^2\int_0^a \cancel{\psi_1^2(x)} + 2\psi_1\psi_2 + \cancel{\psi_2^2}\,dx = 1$$

$$= A^2\, 2\int_0^a \psi_1\psi_2\,dx = 1$$

$$2A^2 = 1$$

$$A = \sqrt{\frac{1}{2}}$$

$$\psi(t) = e^{-\frac{iEt}{\hbar}}$$

(b) Find $\Psi(x,t)$

$$c_n = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi}{a}x\right)\sqrt{\frac{1}{2}}\left[\psi_1(x) + \psi_2(x)\right]dx$$

$$= \frac{1}{\sqrt{a}}\left[\int \sin\left(\frac{n\pi}{a}x\right)\psi_1(x)\,dx + \int \sin\left(\frac{n\pi}{a}x\right)\psi_2(x)\,dx\right]$$

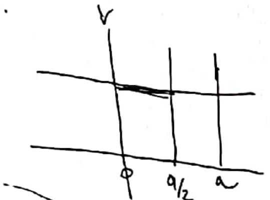
$$=$$

$$\Psi(x,t) = \frac{1}{\sqrt{2}}\left[\psi_1(x)e^{-\frac{iE_1t}{\hbar}} + \psi_2(x)e^{-\frac{iE_2t}{\hbar}}\right]$$

$$\omega = \frac{\pi^2\hbar}{2ma^2} \qquad = \sqrt{\frac{1}{2}}\left[\psi_1(x)e^{-i\frac{\pi^2\hbar}{2ma^2}t} + \psi_2(x)e^{-i\frac{n^2\pi^2}{2ma^2}t}\right]$$

$$= \sqrt{\frac{1}{2}}\left[\psi_1 e^{-in^2\omega} + \psi_2 e^{-in^2\omega}\right]$$

2.8



$V = \begin{cases} 0 & [0,a] \\ \infty & \text{otherwise} \end{cases}$

$\Psi(x,0) = \begin{cases} A & [0,a] \\ 0 & \text{otherwise} \end{cases}$

$\int |\Psi|^2 dx$

$= \int_0^{a/2} A^2 dx = 1$

$A^2 x \Big|_0^{a/2} = 1$

$A^2 \frac{a}{2} = 1$

$A = \sqrt{\frac{2}{a}}$

$E = \frac{\pi^2 \hbar^2}{2ma^2}, \ n=1$

$\sum_{n=1}^{\infty} |c_n|^2 E$

$c_n = \int \psi_n(x) f(x) dx$

$= \int \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a}x\right) \sqrt{\frac{2}{a}} dx$

$= \frac{2}{a} \int_0^{a/2} \sin\left(\frac{n\pi x}{a}\right) dx$

$u = \frac{n\pi x}{a}$

$\frac{du}{dx} = \frac{n\pi}{a} \rightarrow dx = du \cdot \frac{a}{n\pi}$

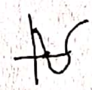
$= \frac{2}{\cancel{a}} \frac{\cancel{a}}{n\pi} \left[\cos\left(\frac{n\pi x}{a}\right)\right]_0^{a/2}$

$= -\frac{2}{n\pi}\left[\cos\left(\frac{1}{2}n\pi\right) - \cos(0)\right]$ $(n=1)$

$= -\frac{2}{\pi}\left[\cos\left(\frac{\pi}{2}\right) - \cos(0)\right]$

$= +\frac{2}{\pi}$

$|c_1|^2 = \left(+\frac{2}{\pi}\right)^2 = \frac{4}{\pi^2}$ ✓



$\Psi(x,0) = Ae^{-a|x|} = \begin{cases} Ae^{-ax} & x \geq 0 \\ Ae^{ax} & x < 0 \end{cases}$

$\int |\Psi|^2 dx = 1$

$A^2 \int_{-\infty}^{\infty} e^{-2a|x|} dx = 1$

$A^2\left[\int_{-\infty}^{0} e^{+2ax} dx + \int_0^{\infty} e^{-2ax}\right] = 1$

$A^2\left[\frac{1}{2a}\left[e^{+2ax}\right]_{-\infty}^{0} + \frac{1}{2a}\left[e^{-2ax}\right]_0^{\infty}\right] = 1$

$A^2\left[-\frac{1}{2a} + \left(-\frac{1}{2a}\right)\right] = 1$

$A^2 = a \qquad A = \sqrt{a}$

free particle:

$V(x) = 0$ (V).

$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi$

$\frac{d^2\psi}{dx^2} = -k^2\psi, \ k = \frac{\sqrt{2mE}}{\hbar}$

$\psi(x) = Ae^{ikx} + Be^{-ikx}$ $\rightarrow E = \frac{k^2\hbar^2}{2m}$

$\Psi(x,t) = Ae^{ikx}e^{-\frac{iEt}{\hbar}} + Be^{-ikx}e^{-\frac{iEt}{\hbar}}$

$= Ae^{ikx - i\frac{k^2\hbar}{2m}} + Be^{-ikx - i\frac{k^2\hbar}{2m}}$ $\qquad \frac{k\hbar}{2m} = v.$

$= Ae^{ik\left(x - \frac{k\hbar}{2m}t\right)} + Be^{-ik\left(x + \frac{k\hbar}{2m}t\right)}$

to right $\qquad$ to left

$\Psi_k(x,t) = Ae^{i\left(kx - \frac{k^2\hbar}{2m}t\right)}$

where $k = \pm\frac{\sqrt{2mE}}{\hbar}$ $\begin{cases} k > 0 & \text{right} \rightarrow \\ k < 0 & \text{left} \leftarrow \end{cases}$

$\lambda = \frac{2\pi}{|k|} \quad p = \hbar k \quad v = \frac{\hbar k}{2m}$

$\frac{p}{2m} = \frac{mv}{2m} = \frac{v}{2}$

~~$\frac{1}{2}mv^2 = \frac{1}{2}pv = \hbar E$~~

wave packet

$$\boxed{\Psi(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \phi(k)\, e^{i\left(kx - \frac{\hbar k^2}{2m}t\right)} dk}$$

$\Psi(x,0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \phi(k)\, e^{ikx} dk$

Plancherel's Theorem:

Fourier transform

$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(k)e^{ikx} dk$

$\Leftrightarrow$

$F(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-ikx} dx$

inverse Fourier transform

$\phi(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \Psi(x,0)\, e^{-ikx} dx$

$[-a,a] \quad t=0$

$\Psi(x,0) = \begin{cases} A & [-a,a] \\ 0 & \text{otherwise} \end{cases}$

$\int |\Psi|^2 dx = 1$

$\int A^2 dx = 1$

$\left[A^2 x\right]_{-a}^{a} = 1$

$2A^2 a = 1$

$A = \frac{1}{\sqrt{2a}}$

$\Psi(x,0) = \begin{cases} \frac{1}{\sqrt{2a}} & [-a,a] \\ 0 & \text{otherwise} \end{cases}$

#22

$\Psi(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} -\frac{1}{\sqrt{\pi a}}\frac{\sin(ka)}{k} e^{i\left(kx - \frac{\hbar k^2}{2m}t\right)} dk$

$\phi(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \Psi(x,0)\, e^{-ikx} dx$

$= \frac{1}{\sqrt{2\pi}}\int_{-a}^{a} \frac{1}{\sqrt{2a}} e^{-ikx} dx$

$= \frac{1}{\sqrt{4\pi a}}\int_{-a}^{a} e^{-ikx} dx \qquad u = -ikx, \ dx = du\left(-\frac{1}{ik}\right)$

$= \frac{1}{\sqrt{4\pi a}}\frac{i}{k}\left[e^{-ikx}\right]_{-a}^{a}$

$= \frac{1}{\sqrt{4\pi a}}\frac{i}{k}\left[e^{-ika} - e^{ika}\right]$

$= -\frac{1}{\sqrt{\pi a}}\frac{e^{-ika} - e^{ika}}{2ik}$

$= -\frac{1}{\sqrt{\pi a}}\frac{\sin(ka)}{k}$

10 January 2019 Owen Lin $12^{\underline{th}}$ Applied Math

$\boxed{2.21}$

$$\Psi(x,0) = Ae^{-a|x|} = \begin{cases} Ae^{-ax} & x \geq 0 \\ Ae^{ax} & x < 0 \end{cases}$$

(a) $\int_{-\infty}^{\infty} |\Psi|^2 \, dx = 1$

$$2A^2 \int_0^{\infty} e^{-2ax} \, dx = 1$$

$$2A^2 \left[-\frac{1}{2a} e^{-2ax} \right]_0^{\infty} = 1$$

$$A^2 = a$$

$$A = \sqrt{a}$$

(b) $\phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Psi(x,0)\, e^{-ikx} \, dx$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{0} \sqrt{a}\, e^{ax} e^{-ikx} \, dx + \int_0^{\infty} \sqrt{a}\, e^{-ax} e^{-ikx} \, dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\sqrt{a} \left[\frac{1}{a-ik} e^{(a-ik)x} \right]_{-\infty}^{0} + \sqrt{a} \left[\frac{1}{-a-ik} e^{(-a-ik)x} \right]_0^{\infty} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\sqrt{a} \left[\frac{1}{a-ik} \right] + \sqrt{a} \left[-\frac{1}{-a-ik} \right] \right]$$

$$= \sqrt{\frac{a}{2\pi}} \left(\frac{1}{a-ik} + \frac{1}{a+ik} \right)$$

$$= \sqrt{\frac{a}{2\pi}} \; \frac{2a}{a^2+k^2}$$

(c) $\Psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k)\, e^{i(kx - \frac{\hbar k^2}{2m} t)} \, dk$

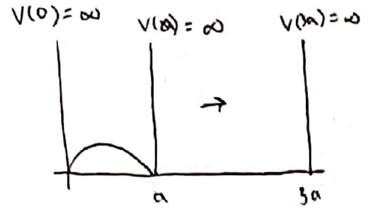
$$= \frac{\sqrt{a}}{2\pi} \int_{-\infty}^{\infty} \frac{2a}{a^2+k^2} e^{i(kx - \frac{\hbar k^2}{2m} t)} \, dk$$

$$= \frac{a^{\frac{3}{2}}}{\pi} \int_{-\infty}^{\infty} \frac{1}{a^2+k^2} e^{i(kx - \frac{\hbar k^2}{2m} t)} \, dk$$

(d)

12 December 2018 Owen Lin 12^{th} Applied Math

Test 1



$$V(x) = \begin{cases} 0 & \text{in } [0, a] \\ \infty & \text{otherwise} \end{cases} \rightarrow V(x) = \begin{cases} 0 & \text{in } [0, 3a] \\ \infty & \text{otherwise} \end{cases}$$

Inside the well, $V = 0$, so

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi$$

$$\frac{d^2\psi}{dx^2} = -k^2\psi \quad , \text{ where } k \equiv \frac{\sqrt{2mE}}{\hbar}$$

The energy for boundary condition $x = a$ is

$$E_n = \frac{\hbar^2 k_n^2}{2m} = \frac{n^2\pi^2\hbar^2}{2ma^2}$$

when the boundary condition is $x = 3a$, the energy

$$E_n = \frac{\hbar^2 k_n^2}{2m} = \frac{n^2\pi^2\hbar^2}{18ma^2}$$

The wave function is

$$\psi_n(x) = \sqrt{\frac{2}{3a}} \sin\left(\frac{n\pi x}{3a}\right)$$

$$E_n = \frac{\hbar^2 k_a^2}{2m} = \frac{n^2\pi^2\hbar^2}{2ma^2}$$

$$E_1 = \frac{\hbar^2 k_1^2}{2m} = \frac{\pi^2\hbar^2}{2ma^2}$$

$$E_2 = \frac{\hbar^2 k_2^2}{2m} = \frac{2\pi^2\hbar^2}{ma^2}$$

#1) find the wave function:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi$$

$$\frac{d^2\psi}{dx^2} = -k^2\psi, \quad k \equiv \frac{\sqrt{2mE}}{\hbar}$$

$$\psi(x) = A\sin(kx) + B\cos(kx)$$

$$\psi(0) = 0 = \psi(a)$$

At ground state, $\psi_1(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{\pi x}{a}\right)$

#2) $C_n = \int \psi_n(x)^* f(x)\, dx \quad , \psi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right)$

$$C_n = \int_0^{3a} \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right) \cdot \sqrt{\frac{2}{a}} \sin\left(\frac{\pi x}{a}\right) dx$$

$$= \frac{2}{a} \int_0^{3a} \sin\left(\frac{n\pi x}{a}\right)^* \sin\left(\frac{\pi x}{a}\right) dx$$

$$= \frac{2}{a} \int_0^{3a} \frac{1}{2}\left[\cos\left(\frac{n\pi x}{a} - \frac{\pi x}{a}\right) - \cos\left(\frac{n\pi x}{a} + \frac{\pi x}{a}\right)\right] dx$$

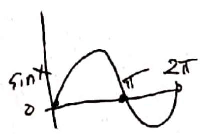
$$= \frac{2}{a} \int_0^{3a} \frac{1}{2}\cos\left(\frac{\pi x(n-1)}{a}\right) - \frac{1}{2}\cos\left(\frac{\pi x(n+1)}{a}\right) dx$$

$$= \frac{2}{a}\left[\frac{1}{2}\int_0^{3a} \cos\left(\frac{\pi x(n-1)}{a}\right) dx - \frac{1}{2}\int_0^{3a}\cos\left(\frac{\pi x(n+1)}{a}\right) dx\right]$$

$$u = \frac{\pi x(n-1)}{a}$$

$$dx = \frac{du \cdot a}{\pi(n-1)}$$

$$= \frac{1}{a}\int_0^{3\pi(n-1)} \cos u \frac{du \cdot a}{\pi(n-1)} - \frac{1}{a}\int_0^{3\pi(n+1)} \cos u \frac{du\, a}{\pi(n+1)}$$



$$= \frac{1}{\pi(n-1)}\left[\sin u\right]_0^{3\pi(n-1)} - \frac{1}{\pi(n+1)}\left[\sin u\right]_0^{3\pi(n+1)}$$

$$= \frac{1}{\pi(n-1)}\sin(3\pi(n-1)) - \frac{1}{\pi(n+1)}\sin(3\pi(n+1))$$

$$= 0$$

If $x = a$, $\quad E_n = \dfrac{\hbar^2 k_n^2}{2m} = \dfrac{n^2\pi^2\hbar^2}{2ma^2}$, $\quad \psi_n(x) = \sqrt{\dfrac{2}{a}}\sin\left(\dfrac{n\pi x}{a}\right)$

if $x = 3a$, $\quad E_n = \dfrac{\hbar^2 k_n^2}{2m} = \dfrac{n^2\pi^2\hbar^2}{18ma^2}$, $\quad \psi_n(x) = \sqrt{\dfrac{2}{3a}}\sin\left(\dfrac{n\pi x}{3a}\right)$

$$\boxed{c_n = \int \psi_n(x)^* f(x)\,dx}$$

The original wave function at bound state : $\psi_1(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{\pi x}{a}\right) = f(x)$

$$c_n = \int_0^a \sqrt{\frac{2}{3a}}\sin\left(\frac{n\pi x}{3a}\right)\cdot\sqrt{\frac{2}{a}}\sin\left(\frac{\pi x}{a}\right)dx$$

$$= \frac{2}{\sqrt{3}a}\int_0^a \sin\left(\frac{n\pi x}{3a}\right)\sin\left(\frac{\pi x}{a}\right)dx$$

$$= \frac{2}{\sqrt{3}a}\int_0^a \frac{1}{2}\left[\cos\left(\frac{n\pi x}{3a} - \frac{\pi x}{a}\right) - \cos\left(\frac{n\pi x}{3a} + \frac{\pi x}{a}\right)\right]dx \qquad u = \frac{(n-3)\pi x}{3a}$$

$$= \frac{2}{\sqrt{3}a}\left[\int_0^a \frac{1}{2}\cos\left(\frac{(n-3)\pi x}{3a}\right)dx - \int_0^a \frac{1}{2}\cos\left(\frac{(n+3)\pi x}{3a}\right)dx\right] \qquad dx = \frac{du\cdot 3a}{(n-3)\pi}$$

$$= \frac{1}{\sqrt{3}a}\int_0^{\frac{(n-3)\pi}{3}}\cos u\,\frac{du\cdot 3a}{(n-3)\pi} - \frac{1}{\sqrt{3}a}\int_0^{\frac{(n+3)\pi}{3}}\cos u\,\frac{du\cdot 3a}{(n+3)\pi} \qquad n \neq \pm 3$$

$$= \frac{3a}{\sqrt{3}a(n-3)\pi}\cdot\sin\left(\frac{(n-3)\pi}{3}\right) - \frac{3a}{\sqrt{3}a(n+3)\pi}\cdot\sin\left(\frac{(n+3)\pi}{3}\right)$$

$$= \frac{3}{\sqrt{3}(n-3)\pi}\sin\left(\frac{(n-3)\pi}{3}\right) - \frac{3}{\sqrt{3}(n+3)\pi}\sin\left(\frac{(n+3)\pi}{3}\right)$$

$$= \frac{3}{\sqrt{3}(n-3)\pi}\sin\left(\frac{n\pi}{3}\right) - \frac{3}{\sqrt{3}(n+3)\pi}\sin\left(\frac{n\pi}{3}\right)$$

$$= \left(\frac{3}{\sqrt{3}(n-3)\pi} - \frac{3}{\sqrt{3}(n+3)\pi}\right)\sin\left(\frac{n\pi}{3}\right)$$

$$= \left(\frac{(n+3)-(n-3)}{(n-3)(n+3)}\right)\frac{3}{\sqrt{3}\pi}\sin\left(\frac{n\pi}{3}\right)$$

$$= \frac{6}{(n-3)(n+3)}\,\frac{3}{\sqrt{3}\pi}\sin\left(\frac{n\pi}{3}\right)$$

$$= \frac{18}{(n^2-9)\sqrt{3}\pi}\sin\left(\frac{n\pi}{3}\right)$$

$$= \begin{cases} 0, & \text{if } n \text{ is the multiple of } 3 \\ \dfrac{18}{(n^2-9)\sqrt{3}\pi}\sin\left(\dfrac{n\pi}{3}\right) & \text{otherwise} \end{cases}$$

12 December 2018

Owen Lin 12^{th}

$$C_3 = \int_0^a \sqrt{\frac{2}{3a}} \sin\left(\frac{3\pi x}{3a}\right) \cdot \sqrt{\frac{2}{a}} \sin\left(\frac{\pi x}{a}\right) dx$$

$$= \frac{2}{\sqrt{3}a} \int_0^a \sin^2\left(\frac{\pi x}{a}\right) dx$$

$\cancel{u = \frac{\pi x}{a}}$

$\cancel{dx = \frac{du \cdot a}{\pi}}$

$$\cancel{= \frac{2}{\sqrt{3}a} \int_0^\pi \sin^2(u) \frac{du \cdot a}{\pi}}$$

$$\cancel{= \frac{2}{\sqrt{3}\pi} \int_0^\pi \sin^2(u)\, du}$$

$\cancel{v = \sin u}$

$$\cancel{= \frac{2}{\sqrt{3}\pi} \int_0^0}$$

$u = \sin\left(\frac{\pi x}{a}\right)$

$$= \frac{2}{\sqrt{3}\pi} \int_0^0 \quad .. \quad dx$$

$$= 0$$

$$|c_n|^2 = \frac{18^2}{(n^2-9)^2(\sqrt{3})^2\pi^2} \sin^2\left(\frac{n\pi}{3}\right)$$

$$= \frac{108}{(n^2-9)^2\pi^2} \sin^2\left(\frac{n\pi}{3}\right)$$

$$P_n = |c_n|^2 = \begin{cases} 0, & \text{if } n \text{ is multiple of } 3 \\ \frac{108}{(n^2-9)^2\pi^2} \sin^2\left(\frac{n\pi}{3}\right), & \text{otherwise} \end{cases}$$

$$P_{n,max} = \frac{108}{25\pi^2} \sin^2\left(\frac{2\pi}{3}\right)$$

$n = 2$

$$P_{n \text{ second max}} = \frac{108}{49\pi^2} \sin^2\left(\frac{4\pi}{3}\right)$$

$n = 4$