

<< Intro to Waves

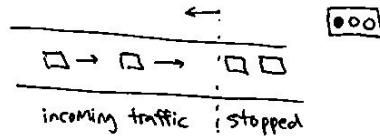
→ Stadium Waves

- medium (fans) moves up & down
- wave moves left & right

→ transverse wave (traveling pulse)

→ Traffic density waves

- car moves forward
- red light signal moves backward



→ Water Waves

- > frequency - # of peaks/troughs that pass a fixed point per second
- > period - reciprocal of frequency
• $T = f^{-1}$
- > wavelength - distance between two successive waves
- diff waves are formed due to diff disturbing forces, restoring forces on diff. length & time scales
- > progressive (traveling) waves - moves horizontally
- > standing waves - no horizontal movement, oscillates vertically.

<< Intro to Waves

→ Water Waves (Mathematical Description)

• properties of water waves

- > period T
- > frequency $f = \frac{1}{T}$
- > angular frequency $\omega = 2\pi f$
- > wavelength λ
- > wave number $k = \frac{2\pi}{\lambda}$
- > phase speed $c = \frac{\lambda}{T}$
(velocity of propagation)
- > depth of water

• speed of water waves is depth-dependent

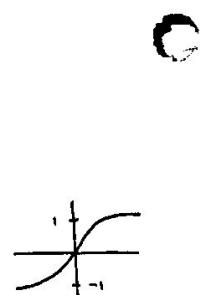
$$\cdot c = \sqrt{\left(\frac{g\lambda}{2\pi} + \frac{T^2\pi}{\lambda g}\right) \tanh\left(\frac{2\pi}{\lambda} d\right)}$$

• hyperbolic functions

$$\cdot \sinh x = \frac{e^x - e^{-x}}{2}$$

$$\cdot \cosh x = \frac{e^x + e^{-x}}{2}$$

$$\cdot \tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$



• small x approx : $\tanh x \approx x$ if $|x| < 0.05$

• large x approx : $\tanh x \approx 1$ if $x > \pi$

• neg. large x approx : $\tanh x \approx -1$ if $x < -\pi$

gravity & surface tension

$$c \approx \sqrt{\frac{g\lambda}{2\pi} \tanh\left(\frac{2\pi}{\lambda} d\right)}$$

negligible surface tension : $\frac{T^2\pi}{\lambda g} \rightarrow 0$ as $\lambda \uparrow$

→ Deep Water Waves ($d > \frac{\lambda}{2}$)

$$\begin{aligned} \cdot c &= \sqrt{\frac{g\lambda}{2\pi}} \\ \cdot c &= \frac{\lambda}{T} \end{aligned} \quad \left. \begin{aligned} \frac{g\lambda}{2\pi} &= \frac{\lambda^2}{T} \\ \lambda &= \frac{gT^2}{2\pi} \\ c &= \frac{gT}{2\pi} \end{aligned} \right.$$

> dispersion - phase velocity depend on frequency (period)

→ Shallow Water Waves ($d \ll \frac{\lambda}{2}$)

$$\cdot c = \sqrt{gd} \quad \rightarrow \text{no dispersion}$$

$$\cdot \lambda = \sqrt{gd} T$$

• tsunami formation and danger

• tsunami initially moves fast in deep water (large T)

• deep \rightarrow shallow

• fast \rightarrow slow (loss of T dependence)

• E flux is constant

• $c \downarrow$, E density \uparrow , mech. E dissipates thru height \uparrow

→ Seismic Waves

> P Wave - longitudinal, body wave, Spherical dissipation $\propto \frac{1}{r^2}$

> S wave - transverse, body wave, Spherical dissipation $\propto \frac{1}{r^2}$, $1/2$ speed of P wave

> Love wave - surface wave, dissipation $\propto \frac{1}{r}$

> Rayleigh wave - surface wave, dissipation $\propto \frac{1}{r}$

<< Intro to Waves

→ Waves of Biological Invasion

- medium moves with waves
- bird migration, disease spread

→ Other Examples of Waves

- light, sound, brain waves

<< Definition of Waves

1. Disturbance \Rightarrow signal

- measurable diagnostic signal
- e.g. height, density

2. Speed of propagation

> Wave - any recognizable signal that is transferred from one part of a medium to another with recognizable velocity of propagation.

<< Mathematical Representation of Waves

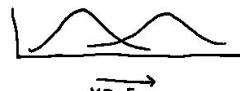
[Ex1] $u(x,t) = f(x - ct)$

- moves to the right at speed c
- at $t=0$, $u(x,0) = f(x)$
- track $f(x=0)$ at later t ,

$$x - ct = 0$$

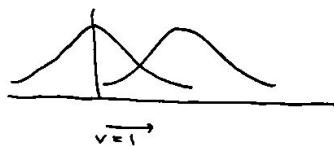
$$x = ct$$

$$\frac{dx}{dt} = c$$



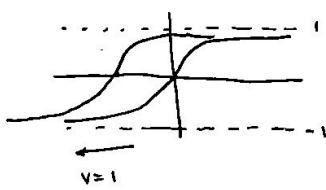
[Ex2] $u(x,t) = e^{-(x-t)^2}$

- moves to the right at speed 1
- $u(x,0) = e^{-x^2}$



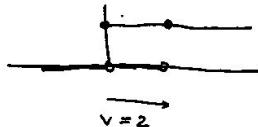
[Ex3] $u(x,t) = \tanh(x+vt)$

- moves to the left at speed 1
- $u(x,0) = \tanh(x)$



[Ex4] $u(x,t) = H(x-2t)$

- moves to the right at speed 2
- $u(x,0) = H(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$

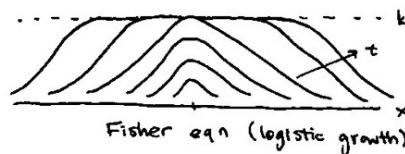
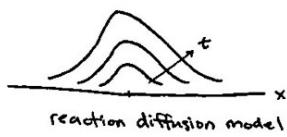


→ Visualization

- Snapshots
- Animations
- Slice Plot
- Surface Plot
- Density Plot ($x-t$)

<< Examples of PDE

- advection eqn - $u_t + c u_x = 0$ linear 1st order homogeneous PDE
- diffusion eqn - $u_t = D u_{xx}$ linear 2nd order homogeneous PDE
- convection eqn - $u_t + c u_x = D u_{xx}$ linear 2nd order homogeneous PDE
- Burger's eqn - $u_t + u u_x = D u_{xx}$ nonlinear 2nd order PDE
- reaction-diffusion model - $u_t = r u + D u_{xx}$ linear 2nd order homogeneous PDE



- Fisher eqn - $u_t = r u \left(1 - \frac{u}{k}\right) + u_{xx}$ nonlinear 2nd order PDE
- Nagumo eqn - $u_t = u(u-a)(1-u) + u_{xx}$ nonlinear 2nd order PDE
- wave eqn - $u_{tt} = c^2 u_{xx}$ linear 2nd order homogeneous PDE
- PDEs are derived from first principles
 - mass balance eqn
 - conservation laws
 - other physical laws (e.g. Newton's 2nd law)

<< Classification of PDE

- order - order of the highest partial derivative

- $u_t + c u_x = 0$ - 1st order
- $u_t = D u_{xx}$ - 2nd order
- $u_t + u u_x + u_{xxx} = 0$ - 3rd order

- number of variables

- $u_t = D u_{xx}$ - 2 var
- $u_t = D(u_{xx} + u_{yy})$ - 3 var

- linearity - dependent variable u are not multiplied together or in function arguments

- 1st order linear PDE : $a u_t + b u_x + c u = h$

- 2nd order linear PDE : $a u_{tt} + b u_{tx} + c u_{xx} + d u_t + e u_x + f u + h$

→ Linear PDE Classification

- homogeneity - homogeneous if $h=0$

- Coefficient - constant or variable (does not depend on u either way!)

- basic type - classifies 2nd order PDE

- hyperbolic - $b^2 - 4ac > 0$ disturbing-preserving waves
- parabolic - $b^2 - 4ac = 0$ smoothing, spreading flows
- elliptic - $b^2 - 4ac < 0$ equilibrium, energy minimizing surfaces

*** Example of Basic Types

Ex1 $u_t = u_{xx}$ $u_t - u_{xx} = 0$

$$b^2 - 4ac = 0 - 4(1)(0) = 0 \Rightarrow \text{parabolic}$$

Ex2 $u_{tt} = u_{xx}$ $u_{tt} - u_{xx} = 0$

$$b^2 - 4ac = 0 - 4(1)(-1) = 4 > 0 \Rightarrow \text{hyperbolic}$$

Ex3 $u_{tx} = 0$

$$b^2 - 4ac = 1^2 - 4(1)(0) = 1 > 0 \Rightarrow \text{hyperbolic}$$

Ex4 $u_{xx} + u_{yy} = 0$

$$b^2 - 4ac = 0 - 4(1)(1) = -4 < 0 \Rightarrow \text{elliptic}$$

Ex5 $y u_{xx} + u_{yy} = 0$

$$b^2 - 4ac = 0 - 4(y)(1) = -4y$$

$$-4y = 0$$

$$y = 0 \Rightarrow \begin{cases} y > 0 & \text{hyperbolic} \\ y = 0 & \text{parabolic} \\ y < 0 & \text{elliptic} \end{cases}$$

*** Physical Interpretation of Derivatives

- u_t - rate of change (ROC) of u (e.g. velocity)

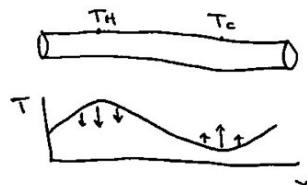
- u_{tt} - ROC of ROC of u (e.g. acceleration)

- u_x - slope

- u_{xx} - concavity

Ex6 Interpret $u_t = u_{xx}$ at Temperature of a 1D rod

- Rate of change of Temp is proportional to concavity of Temp curve.



- at max, concave down, T decrease

- at min, concave up, T increase

<< Traveling Waves

> traveling wave solution - $u(x,t) = f(x-ct)$

- depending on PDE, may be restriction on f and c .

Ex Formulate advection eqn using traveling wave soln.

- Let $z \equiv x-ct$, so $u(x,t) = f(x-ct) = f(z)$

$$\left. \begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial f}{\partial z} \frac{\partial z}{\partial t} = -c \frac{df}{dz} = -c f'(z) \\ \frac{\partial u}{\partial x} &= \frac{\partial f}{\partial z} \frac{\partial z}{\partial x} = \frac{df}{dz} = f'(z) \end{aligned} \right\} \quad \left. \begin{aligned} -c f'(z) + c f'(z) &= 0 \\ \frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} &= 0 \end{aligned} \right\}$$

- Function-independent

Ex Verify traveling wave is soln to wave eqn $u_{tt} = a^2 u_{xx}$

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2}{\partial z^2} [-c f'(z)] = -c \cdot \frac{\partial^2 f}{\partial z^2} \frac{\partial z}{\partial t} = +c^2 f''(z)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2}{\partial z^2} f'(z) = \frac{\partial^2 f}{\partial z^2} \frac{\partial z}{\partial x} = f''(z)$$

$$\text{Substitute, } +c^2 f''(z) = a^2 f''(z)$$

$$(c^2 - a^2) f''(z) = 0$$

$$c^2 = a^2 \quad \text{or} \quad f''(z) = 0$$

$$c = \pm a$$

$$f(z) = A + Bz$$

$$u(x,t) = f(x \pm at)$$

$$\text{or} \quad u(x,z) = A + B(x - ct)$$

- restriction on $c = \pm a$

- restriction on f

- no restriction on f

- no restriction on c

$$\text{general soln: } u(x,t) = c_1 f_1(x-at) + c_2 f_2(x+at) + c_3 [A + B(x-ct)]$$

- does not have to be traveling wave after linear combination.

- could have diff speed and/or direction.

Example of Wave Train and Dispersion (cont.)

EX Find dispersion relation of Klein-Gordon eqn $U_{tt} = a^2 U_{xx} - bu$

- $u(x,t) = A \cos(kx - \omega t)$
 - $U_{tt} = -A \sin(kx - \omega t)(-\omega)$
 - $U_{tt} = -A\omega^2 \cos(kx - \omega t)$
 - $U_x = -A k \sin(kx - \omega t)$
 - $U_{xx} = -Ak^2 \cos(kx - \omega t)$
- $-A\omega^2 \cos(kx - \omega t) = -a^2 A k^2 \cos(kx - \omega t) - b A \cos(kx - \omega t)$
- $A(\omega^2 - a^2 k^2 - b) \cos(kx - \omega t) = 0$
- $\omega^2 - a^2 k^2 - b = 0$
- $\omega = \pm \sqrt{a^2 k^2 + b}$
- phase speed $c = \frac{\omega}{k} = \pm \sqrt{\frac{a^2 k^2 + b}{k^2}} = \pm \sqrt{a^2 + \frac{b}{k^2}} = \pm \sqrt{a^2 + \frac{a^2 b}{\omega^2 - b}}$
 - PDE is dispersive $\therefore c(k)$ or $c(\omega)$
 - normal dispersion - $\omega \uparrow, c \downarrow$

EX Find dispersion relation of beam eqn $U_{tt} + a^2 U_{xxxx} = 0$

- $U_{xxxx} = +A k^3 \sin(kx - \omega t)$
 - $U_{xxxx} = A k^4 \cos(kx - \omega t)$
 - Substitute, $-A\omega^2 \cos(kx - \omega t) + a^2 A k^4 \cos(kx - \omega t) = 0$
- $\Lambda(-\omega^2 + a^2 k^4) \cos(kx - \omega t) = 0$
- $\omega^2 = a^2 k^4$
- $\omega = \pm a k^2$
- $c = \frac{\omega}{k} = \pm a k = \pm \sqrt{a \omega}$
 - PDE is dispersive

→ Dispersion

- dispersion relation can be complex
- dispersive only for phase speed that's real and nonconstant
- In dispersion, each component move in its own phase speed without preserving shape.
- certain wave packets preserve shape and move in group velocity.
- beats - wave packets

<< Dispersion & Group Velocity

EX Derive expression of group velocity of dispersive PDE traveling wave soln.

- Assume $u(x,t) = u_1(x,t) + u_2(x,t)$

- $u_1(x,t) = a \cos(k_1 x - \omega_1 t)$

$$k_1 = k + \Delta k \quad \omega_1 = \omega + \Delta \omega$$

- $u_2(x,t) = a \cos(k_2 x - \omega_2 t)$

$$k_2 = k - \Delta k \quad \omega_2 = \omega - \Delta \omega$$

- By trig identity $\cos A + \cos B = 2 \cos[\frac{1}{2}(A-B)] \cos[\frac{1}{2}(A+B)]$

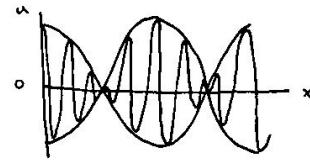
$$u(x,t) = 2a \cos[\frac{1}{2}(k_1 x - \omega_1 t - k_2 x + \omega_2 t)] \cos[\frac{1}{2}(k_1 x - \omega_1 t + k_2 x - \omega_2 t)]$$

$$= 2a \cos[\frac{1}{2}(2\Delta k x - 2\Delta \omega t)] \cos[\frac{1}{2}(2kx - 2\omega t)]$$

$$= 2a \underbrace{\cos(\Delta k x - \Delta \omega t)}_{\text{Amplitude (envelope)}} \underbrace{\cos(kx - \omega t)}_{\text{traveling wave}}$$

- phase speed $c = \frac{\omega}{k}$

- group speed $c_g = \frac{\Delta \omega}{\Delta k} \rightarrow c_g = \boxed{\frac{d\omega}{dk}}$



EX Find group speed of dispersion relation $\omega = ak^2$ of beam eqn $u_{ttt} + \alpha^2 u_{xxxx} = 0$

- $c = \frac{\omega}{k} = \frac{ak^2}{k} = ak$

- $c_g = \frac{d\omega}{dk} = \frac{d}{dk}(ak^2) = 2ak$

Derivation of the Sine-Gordon Eqn

- crystal defect, coupled torsion pendula

- Let $u_i(t)$ be angle of rotation of i th pendulum

$$\sum \tau = I \frac{d^2 u}{dt^2} = m l^2 \frac{d^2 u}{dt^2} \quad (\text{Newton's second law})$$

- torque of gravity = $-mgL \sin(u_i(t))$

- torque of $i+1$ th pendulum = $K \frac{u_{i+1}(t) - u_i(t)}{\Delta x}$

- torque of $i-1$ th pendulum = $K \frac{u_{i-1}(t) - u_i(t)}{\Delta x}$

$$m l^2 \frac{d^2 u}{dt^2} = -mgL \sin(u_i(t)) + K \frac{u_{i+1} - 2u_i + u_{i-1}}{(\Delta x)^2}$$

- Let $\lim_{x \rightarrow 0} \frac{m}{\Delta x} = M$ be mass density

- Note the second order central difference term

$$u(x + \Delta x) = u(x) + u'(x) \Delta x + \frac{1}{2} u''(x) (\Delta x)^2 + \dots$$

$$+ u(x - \Delta x) = u(x) - u'(x) \Delta x + \frac{1}{2} u''(x) (\Delta x)^2 + \dots$$

$$u(x + \Delta x) + u(x - \Delta x) = 2u(x) + u''(x) (\Delta x)^2$$

$$\Rightarrow u''(x) = \frac{u(x + \Delta x) - 2u(x) + u(x - \Delta x)}{(\Delta x)^2}$$

$$Ml^2 u_{tt} = -MgL \sin(u) + Ku_{xx}$$

- Let $A \approx Ml^2$

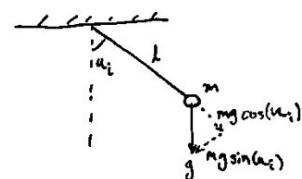
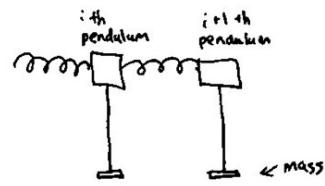
- Let $T = mgL$

$$A u_{tt} - Ku_{xx} + T \sin(u) = 0$$

$$u_{tt} - u_{xx} + \sin(u) = 0$$

change of variables

sine-Gordon equation



• Traveling Wave Solution of Sine-Gordon eqn

$$\cdot u(x,t) = f(z) = f(x-\alpha t)$$

$$\cdot u_t = -c f'(z)$$

$$\cdot u_{tt} = c^2 f''(z)$$

$$\cdot u_x = f'(z)$$

$$\cdot u_{xx} = f''(z)$$

$$\cdot u_{tt} - u_{xx} + \sin(u) = 0$$

$$c^2 f'' - f'' + \sin(f) = 0$$

$$(c^2 - 1) f'' + \sin(f) = 0$$

$$\int (c^2 - 1) f'' f' + \sin(f) f' = \int 0$$

$$\frac{1}{2} (c^2 - 1) (f')^2 - \cos(f) = a = -1$$

$$(f')^2 = \frac{2}{1-c^2} (1 - \cos(f))$$

$$(f')^2 = \frac{4}{1-c^2} \sin^2\left(\frac{x}{2}\right)$$

$$f' = -\frac{2}{\sqrt{1-c^2}} \sin\left(\frac{x}{2}\right)$$

$$g' = -\frac{1}{\sqrt{1-c^2}} \sin(g)$$

$$\int \frac{1}{\sin(g)} dg = \int -\frac{1}{\sqrt{1-c^2}} dz$$

$$\ln(\tan(\frac{g}{2})) = -\frac{1}{\sqrt{1-c^2}} z$$

$$g = 2 \tan^{-1}(\exp(-\frac{z}{\sqrt{1-c^2}}))$$

$$f = 4 \tan^{-1}(\exp(-\frac{z}{\sqrt{1-c^2}}))$$

$$u(x,t) = 4 \tan^{-1}(\exp(-\frac{x-ct}{\sqrt{1-c^2}}))$$

$$\text{b.c. } \begin{cases} \lim_{z \rightarrow \infty} f(z) = 0 \\ \lim_{z \rightarrow -\infty} f'(z) = 0 \end{cases} \text{ for unperturbed at } \infty$$

$$2 \sin^2\left(\frac{x}{2}\right) = 1 - \cos(x) \quad \text{half angle formula}$$

choose negative \therefore expect decrease to zero

$$\text{Let } g = \frac{x}{2}, \quad g' = \frac{1}{2} f'$$

$$\int \frac{1}{\cos x} dx = \int \sec x dx = \ln(\sec x + \tan x) = \ln(\tan(\frac{x}{2} + \frac{\pi}{4}))$$

$$\int \frac{1}{\sin x} dx = \int \csc x dx = \ln(\csc x - \cot x) = \ln(\tan(\frac{x}{2}))$$

• physical interpretation

• Far ahead the front, pendula are unperturbed

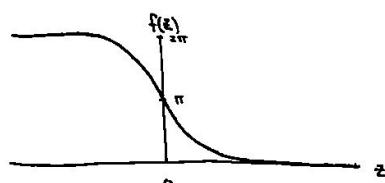
• Far behind the front, pendula are at angle of 2π

• Pendula in the back has rotated counterclockwise exactly one revolution

• antikink traveling wave

• Steepness of the front determined by speed c

• $\uparrow c, \uparrow$ steepness



<< Qualitative Understanding of Nagumo Eqn

- reaction-diffusion model for modeling spread of invading organisms.

- $u(x,t)$ is population density

- $$u_t = u(u-a)(1-u) + u_{xx}$$

reaction
diffusion

- ODE - $\frac{du}{dt} = u(u-a)(1-u)$

- threshold at $u=a$ called strong Allee effect

<< Traveling Wave Soln of Nagumo eqn

- $u(x,t) = f(z) = f(x-ct)$

- $u_t = -cf'(z)$
- $u_x = f'(z)$
- $u_{xx} = f''(z)$

$$\left. \begin{array}{l} u_t = u(u-a)(1-u) + u_{xx} \\ -cf' = f(f-a)(1-f) + f'' \\ f'' + cf' + f(f-a)(1-f) = 0 \end{array} \right\}$$

→ Parabolic heteroclinic orbit

> heteroclinic orbit - connection between eqm pts in phase plot

- connects saddle eqm pts $(0,0), (1,0)$

- Verify parabolic guess $f' = bf(f-1)$

$$\begin{aligned} f'' &= bff' + bf'(f-1) \\ &= bf'(2f-1) \\ &= b^2 f(f-1)(2f-1) \end{aligned}$$

- Substitute in PDE,

$$b^2 f(f-1)(2f-1) + cbf(f-1) + f(f-a)(1-f) = 0$$

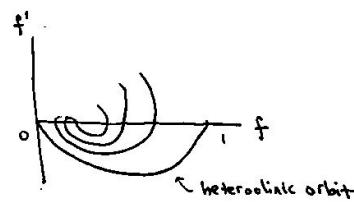
$$f(f-1) [b^2(2f-1) + cb - (f-a)] = 0$$

$$(-b^2 + cb + a) + (2b^2 - 1)f = 0$$

$$-b^2 + cb + a = 0 \quad \text{or} \quad 2b^2 - 1 = 0$$

$$c = \sqrt{\frac{1}{2} - a} \quad b = \frac{1}{\sqrt{2}}$$

b.c. $\left\{ \begin{array}{l} \lim_{z \rightarrow -\infty} f(z) = 0, \quad \lim_{z \rightarrow \infty} f'(z) = 0 \\ \lim_{z \rightarrow -\infty} f(z) = 1, \quad \lim_{z \rightarrow \infty} f'(z) = 0 \end{array} \right.$



$\left\{ \begin{array}{l} c < 0, \text{ move fwd, } a < \frac{1}{2} \\ c > 0, \text{ move backward, } a > \frac{1}{2} \end{array} \right.$

→ Traveling Wave Soln

- The parabolic guess ODE $f' = bf(f-1)$

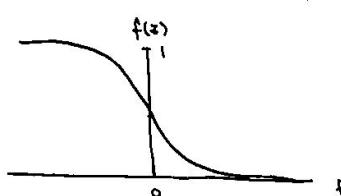
$$\int \frac{df}{f(f-1)} = \int \frac{1}{\sqrt{2}} dz$$

$$\int \left(\frac{1}{f-1} - \frac{1}{f} \right) df = \int \frac{1}{\sqrt{2}} dz$$

$$\ln \left| \frac{f-1}{f} \right| = \frac{1}{\sqrt{2}} z + C^0$$

$$\frac{1-f}{f} = \exp \left(\frac{1}{\sqrt{2}} z \right)$$

$$f = \frac{1}{1 + e^{-z/\sqrt{2}}}$$



choose flip sign for continuous value at 0.

<< Observation of Traveling Pulse

- Russell observed solitary wave (soliton)
- shape of hyperbolic secant
- large enough water can produce 2^k independent solitary waves
- solitary waves cross each other without shape change
- speed given by $c^2 = g(d + a)$, shallow water wave (d , depth; a , amplitude)
- Korteweg-de Vries eqn (KdV) describes long wavelength, small amplitude waves in shallow water

$$u_t + uu_x + u_{xxx} = 0$$

$u(x,t)$ is height of water

<< Traveling Wave Solution of KdV eqn

$$u(x,t) = f(z) = f(x-ct)$$

$$u_t = -cf'(z)$$

$$u_x = f'(z)$$

$$u_{xx} = f''(z)$$

$$u_{xxx} = f'''(z)$$

$$\left. \begin{array}{l} u_t + uu_x + u_{xxx} = 0 \\ -cf' + ff' + f''' = 0 \\ -cf + \frac{1}{2}f^2 + f'' = a = 0 \\ -cff' + \frac{1}{2}f^2f' + f''f' = 0 \\ -\frac{1}{2}cf^2 + \frac{1}{6}f^3 + \frac{1}{2}(f')^2 = b = 0 \end{array} \right\}$$

$$\left. \begin{array}{l} \text{b.c.} \\ \lim_{z \rightarrow \pm\infty} f(z) = 0 \\ \lim_{z \rightarrow \pm\infty} f'(z) = 0 \\ \lim_{z \rightarrow \pm\infty} f''(z) = 0 \end{array} \right\}$$

$$3(f')^2 = (3c-f)f^2 \Rightarrow f(z) \in [0, 3c]$$

$$f' = \pm \sqrt{\frac{3c-f}{3}} f^2$$

$$\frac{df}{dz} = \pm \sqrt{\frac{3c-f}{3}} f$$

$$\int \frac{\sqrt{3}}{\sqrt{3c-f} f} df = \pm \int dz$$

$$\left. \begin{array}{l} g = \sqrt{3c-f} \\ f = 3c-g^2 \\ df = -2g dg \end{array} \right\}$$

$$2\sqrt{3} \int \frac{1}{3c-g^2} dg = -z$$

$$\int \left(\frac{1}{3c+g} + \frac{1}{3c-g} \right) dg = -\sqrt{3} z$$

$$\ln \left| \frac{\sqrt{3c+g}}{\sqrt{3c-g}} \right| = -\sqrt{3} z$$

$$\left. \begin{array}{l} \frac{1}{3c-g^2} = \frac{A}{3c+g} + \frac{B}{3c-g} \\ A(\sqrt{3c-g}) + B(\sqrt{3c+g}) = 1 \\ (A+B)\sqrt{3c} + (B-A)g = 1 \\ (A+B)\sqrt{3c} = 1 \Rightarrow 2A = \frac{1}{\sqrt{3c}} \\ B-A=0 \end{array} \right\}$$

$$\frac{1}{3c-g^2} = \frac{A}{3c+g} + \frac{B}{3c-g}$$

$$A(\sqrt{3c-g}) + B(\sqrt{3c+g}) = 1$$

$$(A+B)\sqrt{3c} + (B-A)g = 1$$

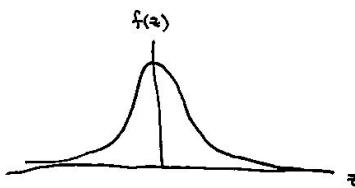
$$\left\{ \begin{array}{l} (A+B)\sqrt{3c} = 1 \Rightarrow 2A = \frac{1}{\sqrt{3c}} \\ B-A=0 \end{array} \right.$$

$$A = \frac{1}{2\sqrt{3c}}$$

$$B = \frac{1}{2\sqrt{3c}}$$

Amplitude depends on speed

$$A = 3c$$



$$f(z) = 3c \operatorname{sech}^2 \left(\frac{\sqrt{3}z}{2} \right)$$

Amplitude

$$\operatorname{sech}^2 x = 1 - \tanh^2 x$$

<< Derivation of the Wave Equation

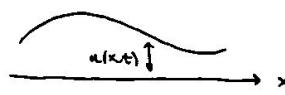
- Context: tightly stretched vibrating string

- $u(x,t)$ is vertical displacement

- u_t is vertical velocity

- u_{tt} is vertical acceleration

- u_x is slope of string



- Simplifying assumptions

- uniform string - constant density ρ

- planar vibration - string remains in vertical plane

- uniform tension - tension is same magnitude (but diff direction) for any piece of string

- no other force - tension only. no gravity, friction, restoring force, etc.

- small vibration - ignore nonlinear effects, $u(x,t)$, $u_x(x,t)$ are small

- Newton's second law on small segment

$$(u(x,t) \ll 1, u_x(x,t) \ll 1)$$

- $F = ma$

- $F = \text{vertical tension at two ends}$

- left end = $-T u_x(x,t)$

- right end = $T u_x(x+\Delta x, t)$

- $m = \rho \Delta x$

- $a = u_{tt}$

$$\begin{aligned} T &\quad u_x(x+\Delta x, t) \\ T &\quad u_x(x, t) \\ -T & \frac{\langle 1, u_x(x, t) \rangle}{\| \langle 1, u_x(x, t) \rangle \|} \\ y\text{-comp} &= -T \frac{u_x(x, t)}{\sqrt{1 + u_x^2(x, t)}} \\ &= -T u_x(x, t) \end{aligned}$$

- $\rho \Delta x u_{tt} = -T u_x(x, t) + T u_x(x+\Delta x, t)$

$$\rho u_{tt} = \frac{-T u_x(x, t) + T u_x(x+\Delta x, t)}{\Delta x}$$

$$\rho u_{tt} = T u_{xx}$$

$$u_{tt} = c^2 u_{xx}$$

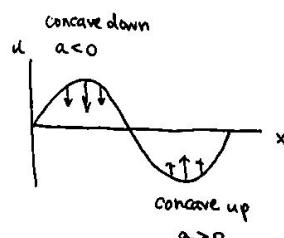
$$\text{let } c = \sqrt{\frac{T}{\rho}}$$

- Physical significance

- Acceleration proportional to concavity

- concave up accelerates upward

- concave down accelerates downward



← Derivation of the d'Alembert's solution

→ Change of variables

$$\cdot \xi = x - ct$$

$$\cdot \eta = x + ct$$

$$\cdot \frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial x} = u_{\xi} + u_{\eta}$$

$$\cdot \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x} (u_{\xi} + u_{\eta}) = \frac{\partial}{\partial \xi} (u_{\xi} + u_{\eta}) \frac{\partial \xi}{\partial x} + \frac{\partial}{\partial \eta} (u_{\xi} + u_{\eta}) \frac{\partial \eta}{\partial x}$$

$$= u_{\xi\xi} + u_{\eta\xi} + u_{\xi\eta} + u_{\eta\eta}$$

$$= u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$$

{ smooth for $u_{\eta\xi} = u_{\xi\eta}$
twice continuously differentiable

$$\cdot \frac{\partial u}{\partial t} = \frac{\partial u}{\partial \xi} \frac{\partial \xi}{\partial t} + \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial t}$$

$$= -c u_{\xi} + c u_{\eta}$$

$$\cdot \frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial t} (-c u_{\xi} + c u_{\eta})$$

$$= \frac{\partial}{\partial \xi} (-c u_{\xi} + c u_{\eta}) \frac{\partial \xi}{\partial t} + \frac{\partial}{\partial \eta} (-c u_{\xi} + c u_{\eta}) \frac{\partial \eta}{\partial t}$$

$$= (-c u_{\xi\xi} + c u_{\eta\xi})(-c) + (-c u_{\xi\eta} + c u_{\eta\eta})(c)$$

$$= c^2 u_{\xi\xi} - 2c^2 u_{\xi\eta} + c^2 u_{\eta\eta}$$

→ Substitution into the wave equation

$$u_{tt} = c^2 u_{xx}$$

$$c^2 u_{\xi\xi} - 2c^2 u_{\xi\eta} + c^2 u_{\eta\eta} = c^2 (u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta})$$

$$4u_{\xi\eta} = 0$$

$$\int u_{\xi\eta} = \int 0$$

$$\int u_{\xi} = \int \phi(\xi)$$

$$u(\xi, \eta) = \int \phi(\xi) d\xi + G(\eta)$$

$$u(\xi, \eta) = F(\xi) + G(\eta)$$

$$u(x, t) = F(x - ct) + G(x + ct)$$

• One wave moves to right, one moves to left

• both with speed c

• F and G may be any twice differentiable function.

• need initial condition on ∞ domain

• need boundary condition on finite or semi- ∞ domain

<< d'Alembert's Solution for Initial Value Problem for Wave Eqn

• Cauchy problem: wave eqn on an infinite domain

• Solve the wave eqn $u_{tt} = c^2 u_{xx}$ for $t > 0$ and $x \in (-\infty, \infty)$ with initial conditions

$$u(x, 0) = f(x)$$

$$u_t(x, 0) = g(x)$$

• At $t=0$, $u(x, 0) = f(x) = F(x) + G(x)$

• Differentiate general soln.

$$\frac{\partial u}{\partial t} = \frac{dF}{dx} \frac{\partial x}{\partial t} + \frac{dG}{dx} \frac{\partial x}{\partial t}$$

$$= -c \frac{dF}{dx} + c \frac{dG}{dx}$$

$$= -c F'(x) + c G'(x)$$

$$\begin{cases} F'(x) = \frac{dF}{dx} \\ G'(x) = \frac{dG}{dx} \end{cases}$$

System of eqns
of F and G

• At $t=0$, $u_t(x, 0) = g(x) = -c F'(x) + c G'(x)$

$$\frac{1}{c} g(x) = -F'(x) + G'(x)$$

• Solve the system $\begin{cases} F(x) + G(x) = f(x) \\ -F'(x) + G'(x) = \frac{1}{c} g(x) \end{cases} \Rightarrow F'(x) + G'(x) = f'(x)$

$$\text{add \& subtract, } \Rightarrow \begin{cases} F'(x) = \frac{1}{2} f'(x) - \frac{1}{2c} g(x) \\ G'(x) = \frac{1}{2} f'(x) + \frac{1}{2c} g(x) \end{cases}$$

$$\text{integrate, } \Rightarrow \begin{cases} F(x) = \frac{1}{2} f(x) - \frac{1}{2c} \int_0^x g(y) dy + k_1 \\ G(x) = \frac{1}{2} f(x) + \frac{1}{2c} \int_0^x g(y) dy + k_2 \end{cases}$$

$$\begin{cases} F(x) + G(x) = f(x), \text{ so} \\ k_1 + k_2 = 0 \\ k_1 = k_2 = -k_2 \end{cases}$$

$$\cdot \text{ change variables } \Rightarrow \begin{cases} F(x-ct) = \frac{1}{2} f(x-ct) + \int_0^0 g(y) dy + k \\ G(x+ct) = \frac{1}{2} f(x+ct) + \int_0^{x+ct} g(y) dy - k \end{cases}$$

add the eqns

$$u(x, t) = F(x-ct) + G(x+ct)$$

$$u(x, t) = \frac{1}{2} [f(x-ct) + f(x+ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} g(y) dy$$

→ Restriction on f and g

• thrice continuously differentiable (classical soln)

• discontinuity and oscillations can be introduced.

<< Examples of d'Alembert's solution

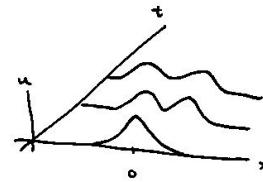
Ex For $u_{tt} = c^2 u_{xx}$ with initial conditions $u(x,0) = e^{-x^2}$, $u_t(x,0) = 0$ for $t > 0$, $x \in (-\infty, \infty)$, find the d'Alembert's solution.

- $f(x) = e^{-x^2}$

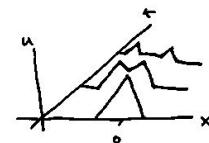
- $g(x) = 0$

- $u(x,t) = \frac{1}{2} [f(x-ct) + f(x+ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} g(y) dy$

$$u(x,t) = \frac{1}{2} [e^{-(x-ct)^2} + e^{-(x+ct)^2}]$$



Ex For $u_{tt} = c^2 u_{xx}$ with $u(x,0) = f(x) = \begin{cases} 1 - |x| & |x| \leq 1 \\ 0 & |x| > 1 \end{cases}$ and $u_t(x,0) = g(x) = 0$, the d'Alembert's soln gives two triangular pulse, each 1/2 as high as initial.



Ex Verify $u(x,t) = \cos(ct) \sin(x)$ is a d'Alembert's solution to $u_{tt} = c^2 u_{xx}$.

- $u(x,0) = \sin(x)$

- $u_t(x,0) = [-c \sin(ct) \sin(x)]_{t=0} = 0$

- $u(x,t) = \frac{1}{2} [\sin(x-ct) + \sin(x+ct)]$

- $\sin(x-ct) = \sin(x) \cos(ct) - \cos(x) \sin(ct)$

- $\sin(x+ct) = \sin(x) \cos(ct) + \cos(x) \sin(ct)$

trig addition formula

- $u(x,t) = \cos(ct) \sin(x)$

Ex Find d'Alembert's soln of $u_{tt} = c^2 u_{xx}$ with init condition $u(x,0) = 0$, $u_t(x,0) = \sin x$ for $t > 0$, $x \in (-\infty, \infty)$

- $f(x) = 0$

- $g(x) = \sin x$

- $u(x,t) = \frac{1}{2c} \int_{x-ct}^{x+ct} \sin y dy$

$$= \frac{1}{2c} [\cos(x+ct) - \cos(x-ct)]$$

<> The Characteristic Triangle

→ zero initial velocity

- $u_{tt} = c^2 u_{xx}$, $t > 0$, $x \in (-\infty, \infty)$
- $u(x, 0) = f(x)$
- $u_t(x, 0) = 0$
- need two values at initial time to know value at current time
- solution is two waves one to left, one to right

trace back the waves to $t=0$

> characteristics - lines connecting $u(x, t)$ to $u(x_1, 0)$ and $u(x_2, 0)$

$$\begin{aligned} x - ct &= x_0 - ct_0 \\ x + ct &= x_0 + ct_0 \end{aligned}$$

- Average of initial values $u(x, 0)$ following the characteristic lines is the current value.

Ex: $u_{tt} = c^2 u_{xx}$, $t > 0$, $x \in (-\infty, \infty)$

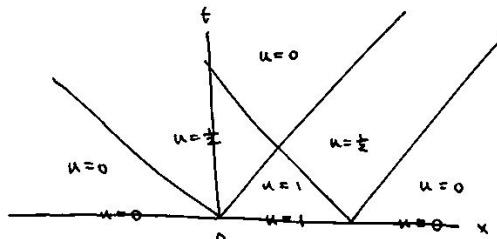
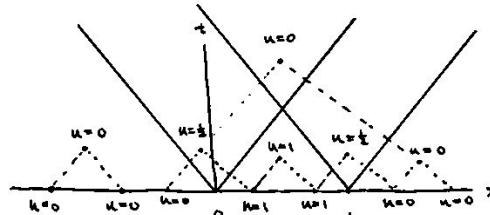
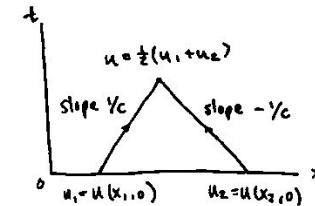
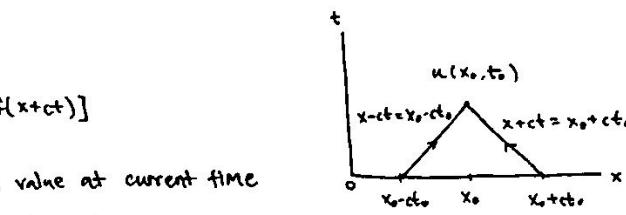
$$u(x, 0) = f(x) = \begin{cases} 1 & x \in [0, 1] \\ 0 & \text{otherwise} \end{cases}$$

$$u_t(x, 0) = g(x) = 0$$

- At initial time $t=0$, identify boundaries that have diff. $u(x, 0)$ values

Draw left and right characteristic lines to identify range of influence.

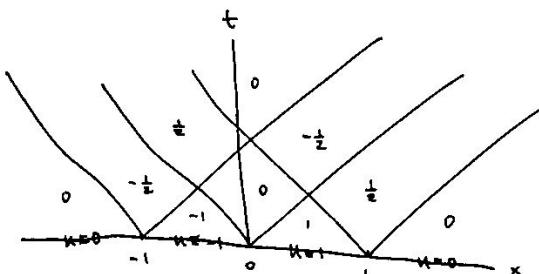
Value of arbitrary pt $u(x, t)$ is average of $u(x, 0)$ following the two characteristic lines.



Ex: $u_{tt} = c^2 u_{xx}$, $t > 0$, $x \in (-\infty, \infty)$

$$u(x, 0) = f(x) = \begin{cases} -1 & x \in [-1, 0] \\ 1 & x \in [0, 1] \\ 0 & \text{otherwise} \end{cases}$$

$$u_t(x, 0) = g(x) = 0$$



ccc The Characteristic Triangle

→ Zero initial function value

$$\cdot u_{tt} = c^2 u_{xx}, \quad t > 0, \quad x \in (-\infty, \infty)$$

$$\cdot u(x, 0) = 0$$

$$\cdot u_t(x, 0) = g(x) \quad \left\{ \begin{array}{l} u(x, t) = \frac{1}{2c} \int_{x-ct}^{x+ct} g(y) dy \\ \end{array} \right.$$

• value at current time ($u(x, t)$) is the integral of initial velocity between $x-ct$ and $x+ct$ (domain of dependence)

> domain of dependence - $[x_0 - ct_0, x_0 + ct_0]$

• only initial values in domain of dependence affect solution at (x_0, t_0) .

> range of influence - disturbance in interval I at $t=0$ influences all points whose domain of dependence includes I.

• bounded by

• $-1/c$ characteristic line at left bound

• $1/c$ characteristic line at right bound

EX $u_{tt} = c^2 u_{xx}, \quad t > 0, \quad x \in (-\infty, \infty)$

$$\cdot u(x, 0) = f(x) = 0$$

$$\cdot u_t(x, 0) = g(x) = \begin{cases} 1 & x \in [0, 1] \\ 0 & \text{otherwise} \end{cases}$$

• Consider 6 diff regions :

$$\textcircled{1} \quad u(x, t) = \frac{1}{2c} \int_{x-ct}^{x+ct} 0 dy = 0$$

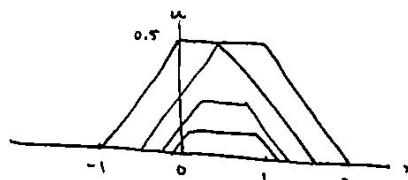
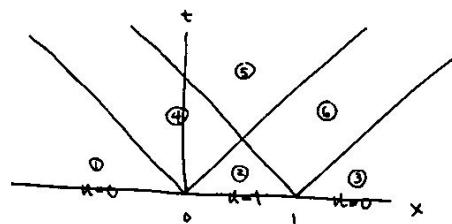
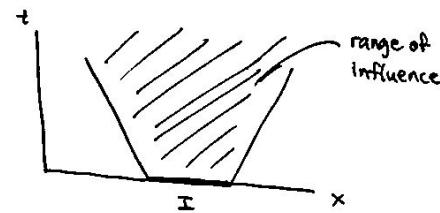
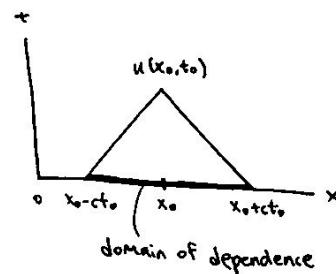
$$\textcircled{2} \quad u(x, t) = \frac{1}{2c} \int_{x-ct}^{x+ct} 1 dy = t$$

$$\textcircled{3} \quad u(x, t) = \frac{1}{2c} \int_{x-ct}^{x+ct} 0 dy = 0$$

$$\textcircled{4} \quad u(x, t) = \frac{1}{2c} \int_0^{x+ct} 1 dy = \frac{x+ct}{2c}$$

$$\textcircled{5} \quad u(x, t) = \frac{1}{2c} \int_0^1 1 dy = \frac{1}{2c}$$

$$\textcircled{6} \quad u(x, t) = \frac{1}{2c} \int_{x-ct}^1 1 dy = \frac{1-x+ct}{2c}$$

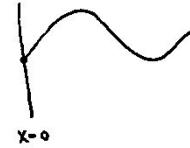


} bounds changes to ignore part where $g(x) > 0$, since its integral is zero anyways

<< Semi-infinite string with one fixed end

→ Problem

- $u_{tt} = c^2 u_{xx}$, $t > 0$, $x \in [0, \infty)$
- initial conditions - $u(x,0) = f(x)$, $u_t(x,0) = g(x)$
- boundary condition - $u(0,t) = 0$
- $u(0,0) = f(0) = 0 \Rightarrow$ fixed end



→ $x \geq ct$

- d'Alembert's soln - $u(x,t) = F(x-ct) + G(x+ct)$
- substitute init cond., $\begin{cases} F(x) = \frac{1}{2} f(x) - \frac{1}{2c} \int_0^x g(s) ds + k_1 \\ G(x) = \frac{1}{2} f(x) + \frac{1}{2c} \int_0^x g(s) ds + k_2 \end{cases}$ (see derivation of d'Alembert's soln for init value problem)
- restriction: $x > 0$
- substitute $x-ct$ restricts $x-ct > 0 \Rightarrow x > ct$
- same soln as d'Alembert's soln for init value problem ($x > ct$)
- $u(x,t) = \frac{1}{2} [f(x-ct) + f(x+ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s) ds$

→ $x < ct$

- substitute d'Alembert's soln into boundary cond.

$$u(0,t) = 0 = F(-ct) + G(ct)$$

$$F(-ct) = -G(ct)$$

$$F(z) = -G(-z)$$

$$z = -ct$$

right moving wave is negative
of left moving wave

- For positive argument of G ($-z > 0, z < 0$), we can determine F for negative argument ($z < 0$)

$$\begin{aligned} F(x-ct) &= -G(-(x-ct)) \\ &= -G(ct-x) \\ &= -\frac{1}{2} f(ct-x) - \frac{1}{2c} \int_0^{ct-x} g(s) ds - k_2 \quad \text{reflected wave} \end{aligned}$$

- total soln for $x < ct$ is sum of left-traveling unreflected wave and right-traveling reflected wave

$$\begin{aligned} u(x,t) &= F(x-ct) + G(x+ct) \\ &= -G(ct-x) + G(x+ct) \\ &= \frac{1}{2} [-f(ct-x) + f(x+ct)] + \frac{1}{2c} \int_{ct-x}^{x+ct} g(s) ds \end{aligned}$$

→ Solution (full range)

$$u(x,t) = \begin{cases} \frac{1}{2} [f(x-ct) + f(x+ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s) ds & x \geq ct \\ \frac{1}{2} [-f(ct-x) + f(x+ct)] + \frac{1}{2c} \int_{ct-x}^{x+ct} g(s) ds & x < ct \end{cases}$$

<< Example of fixed end semi-infinite string

Ex $u_{tt} = c^2 u_{xx}$, $t > 0$, $x \in [0, \infty)$

- Init cond - $u(x,0) = f(x) = x e^{-x^2}$

$$u_t(x,0) = g(x) = 0$$

- bound. cond - $u(0,t) = 0$

- $x > ct$

$$\begin{aligned} u(x,t) &= \frac{1}{2} [f(x-ct) + f(x+ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s) ds \\ &= \frac{1}{2} [(x-ct) e^{-(x-ct)^2} + (x+ct) e^{-(x+ct)^2}] \end{aligned}$$

- $x < ct$

$$\begin{aligned} u(x,t) &= \frac{1}{2} [-f(ct-x) + f(x+ct)] + \frac{1}{2c} \int_{ct-x}^{x+ct} g(s) ds \\ &= \frac{1}{2} [-(ct-x) e^{-(ct-x)^2} + (x+ct) e^{-(x+ct)^2}] \\ &= \frac{1}{2} [(x-ct) e^{-(x-ct)^2} + (x+ct) e^{-(x+ct)^2}] \end{aligned}$$

- soln

$$u(x,t) = \frac{1}{2} [(x-ct) e^{-(x-ct)^2} + (x+ct) e^{-(x+ct)^2}] \quad \text{in all } x \in [0, \infty)$$

same functional form
but have diff sign of $x-ct$

<< Graphical Interpretation with Characteristics

- characteristic $x=ct$ from origin separates the two regions

- $x > ct$ - same as init value problem

- $x < ct$ - reflection (sign change) at boundary $x=0$

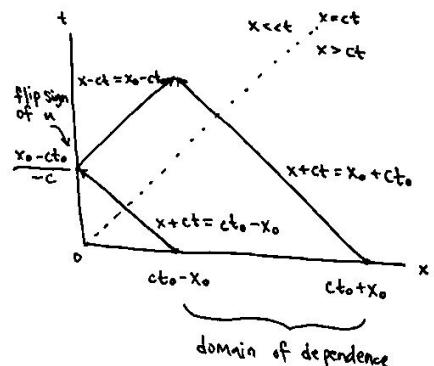
- right characteristic - $x+ct = x_0 + ct_0$

- left characteristic - $x-ct = x_0 - ct_0$

- sign change at boundary, where $x=0$, $t = \frac{x_0 - ct_0}{-c}$

$$x+ct = ct_0 - x_0$$

- domain of dependence - $[ct_0 - x_0, ct_0 + x_0]$



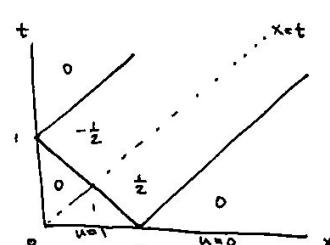
Ex $u_{tt} = u_{xx}$, $t > 0$, $x \in [0, \infty)$

$$u(x,0) = \begin{cases} 1 & x \in (0,1] \\ 0 & \text{otherwise} \end{cases}$$

$$u_t(x,0) = 0$$

$$u(0,t) = 0$$

- Note that u changes sign after bouncing off $x=0$



Alternative Graphical Interpretation

- Since sign of u flips at $x=0$, we can assign values to negative x at $t=0$

$$F(x-ct) = -G(-(x-ct)) = -G(ct-x)$$

$$F(x) = -G(-x)$$

$$\frac{1}{2} \int_0^x f(s) - \frac{1}{2c} \int_0^s g(s) ds = -\frac{1}{2} f(-x) - \frac{1}{2c} \int_0^{-x} g(s) ds$$

$$\frac{1}{2} \int_0^x f(s) - \frac{1}{2c} \int_0^s g(s) ds = -\frac{1}{2} f(-x) - \frac{1}{2c} \int_0^x -g(-s) ds$$

- Use odd extension of original functions.

$$f_o(x) = \begin{cases} f(x) & x \geq 0 \\ -f(-x) & x < 0 \end{cases}$$

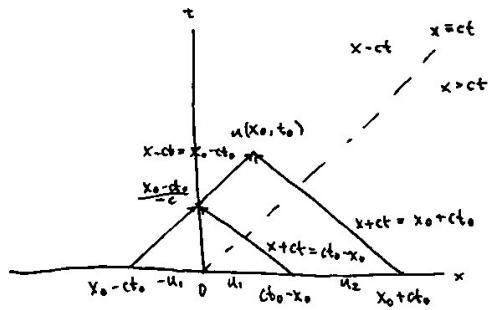
$$g_o(x) = \begin{cases} g(x) & x \geq 0 \\ -g(-x) & x < 0 \end{cases}$$

Ex: $u_{tt} = c^2 u_{xx}$

$$\text{IC} \left\{ \begin{array}{l} u(x,0) = f(x) = xe^{-x^2} \\ u_t(x,0) = g(x) = 0 \end{array} \right. \xrightarrow{\substack{\text{odd} \\ \text{extension}}} \quad f_o = \begin{cases} f(x) = xe^{-x^2} & x \geq 0 \\ -f(-x) = xe^{-x^2} & x < 0 \end{cases}$$

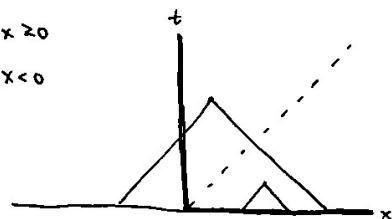
$$\text{fixed BC: } u(0,t) = 0$$

$$u(x,t) = \frac{1}{2} [f_o(x-ct) + f_o(x+ct)]$$



two ways to think

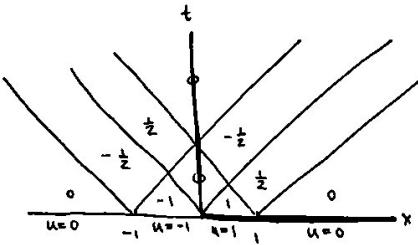
- functional - use odd extension f_o, g_o
- graphical - use negative values of u in $x < 0$
given u in $x > 0$
only consider 1st quadrant
solve like init value prob.



Ex: $u_{tt} = c^2 u_{xx}$

$$\text{IC} \left\{ \begin{array}{l} u(x,0) = \begin{cases} 1 & x \in (0,1) \\ 0 & \text{otherwise} \end{cases} \\ u_t(x,0) = 0 \end{array} \right.$$

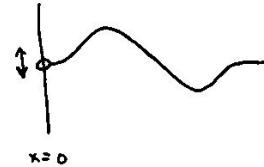
$$\text{fixed BC: } u(0,t) = 0$$



<< Semi-infinite string with a free end

→ Problem

- $u_{tt} = c^2 u_{xx}$, $t > 0$, $x \in (0, \infty)$
- initial conditions: $u(x, 0) = f(x)$, $u_t(x, 0) = g(x)$
- free boundary conditions: $u_x(0, t) = 0$
 - semi-infinite string attached to a ring moves vertically
 - end of string is horizontal at $x=0$ (no tensile force)



→ $x \geq ct$

$$\text{d'Alembert's soln} - u(x, t) = F(x-ct) + G(x+ct)$$

$$u(x, t) = \frac{1}{2} [f(x-ct) + f(x+ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s) ds$$

→ $x < ct$

- substitute d'Alembert's soln to free B.C.

$$u_x(0, t) = 0 = F'(ct) + G'(ct)$$

$$F'(ct) = -G'(ct)$$

$$F'(-z) = -G'(-z) \quad z = -ct$$

- calculate $F'(z)$ from G

$$G(z) = \frac{1}{2} F(z) + \frac{1}{2c} \int_0^z g(s) ds + k_2$$

$$F'(-z) = -G'(-z)$$

$$\int F'(z) dz = -\frac{1}{2} F(-z) - \frac{1}{2c} g(-z)$$

$$F(z) = \frac{1}{2} f(-z) + \frac{1}{2c} \int_0^{-z} g(s) ds - k_2$$

$$F(x-ct) = \frac{1}{2} f(ct-x) + \frac{1}{2c} \int_0^{ct-x} g(s) ds - k_2$$

total soln

$$u(x, t) = F(x-ct) + G(x+ct)$$

$$= \frac{1}{2} [f(x-ct) + f(x+ct)] + \frac{1}{2c} \left[\int_0^{ct-x} g(s) ds + \int_0^{x+ct} g(s) ds \right]$$

→ soln (full range)

$$u(x, t) = \begin{cases} \frac{1}{2} [f(x-ct) + f(x+ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s) ds & x \geq ct \\ \frac{1}{2} [f(ct-x) + f(x+ct)] + \frac{1}{2c} \left[\int_0^{ct-x} g(s) ds + \int_0^{x+ct} g(s) ds \right] & x < ct \end{cases}$$

∞ Finite String (1st try)

- solve finite string with d'Alembert's soln is challenging

→ Problem

- $u_{tt} = c^2 u_{xx}$, $x \in (0, L)$, $t > 0$

- initial conditions: $u(x, 0) = f(x)$, $u_t(x, 0) = g(x)$

- fixed boundaries: $u(0, t) = 0$, $u(L, t) = 0$

- $f(0) = 0$, $f(L) = 0$

→ d'Alembert soln

- d'Alembert soln - $u(x, t) = F(x - ct) + G(x + ct)$

$$F(x) = \frac{1}{2} f(x) - \frac{1}{2c} \int_0^x g(s) ds + k_1 \quad \left. \begin{array}{l} \\ \end{array} \right\} x \in (0, L)$$

$$G(x) = \frac{1}{2} f(x) + \frac{1}{2c} \int_0^x g(s) ds + k_2$$

- restriction: $0 < x < L$

- $0 < x - ct < L$

- $0 < x + ct < L$

→ non-d'Alembert region

- substitute d'Alembert soln into B.C.

- $u(0, t) = 0 = F(-ct) + G(ct)$

- $u(L, t) = 0 = F(L - ct) + G(L + ct)$

$\xrightarrow{\text{reflection}}_{\text{at boundary}}$

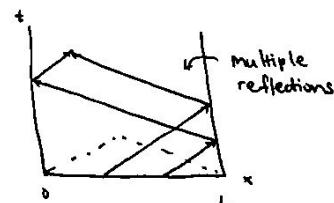
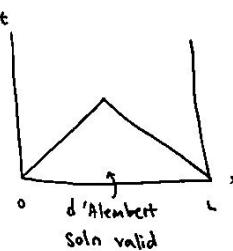
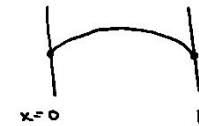
$$F(-ct) = -G(ct)$$

$$G(L + ct) = -F(L - ct)$$

- multiple reflection can occur

- hard to keep track of

- need alternative method of soln



<<< Standing Waves

- d'Alembert used standing waves to better solve finite string problem
- Motivate standing waves from d'Alembert soln

$$\begin{aligned}
 u(x,t) &= \sin(x-t) + \sin(x+t) && \text{sum of left and right moving waves} \\
 &= [\sin(x)\cos(t)] - [\cos(x)\sin(t)] \\
 &\quad + [\sin(x)\cos(t)] + [\cos(x)\sin(t)] && \text{trig identity} \\
 &= 2 \sin(x) \cos(t) \\
 &= 2 \underbrace{\cos(t)}_{\substack{\text{time-varying} \\ \text{amplitude}}} \underbrace{\sin(x)}_{\substack{\text{stationary} \\ \text{sine}}}
 \end{aligned}$$

> standing waves - nonconstant function of the form

$$u(x,t) = w(t) v(x)$$

- basic shape $v(x)$ scaled vertically by $w(t)$

<<< Separation of variables

- not every standing wave is a soln of wave eqn
- focus on PDE first without B.C.
- standing wave - $u(x,t) = w(t)v(x)$

$$u_{tt} = v(x) \ddot{w}(t) \quad (\text{dot denote } \frac{d}{dt})$$

$$u_{xx} = w(t) v''(x) \quad (\text{prime denote } \frac{d}{dx})$$

$$\text{wave eqn} - u_{tt} = c^2 u_{xx}$$

$$v(x) \ddot{w}(t) = c^2 w(t) v''(x)$$

$$\frac{\ddot{w}(t)}{w(t)} = c^2 \frac{v''(x)}{v(x)} = \lambda \quad \leftarrow \text{for } f(t) = g(x) \text{ of diff variable}$$

$$\frac{\ddot{w}(t)}{w(t)} = \lambda$$

$$\frac{v''(x)}{v(x)} = \lambda$$

$$\ddot{w}(t) = \lambda w(t)$$

$$v''(x) = \frac{\lambda}{c^2} v(x)$$

- diff soln for $\lambda > 0, \lambda = 0, \lambda < 0$.

<< Separation of Variables (cont.)

$\rightarrow \lambda = 0$

- $\ddot{w}(t) = \lambda w(t)$ • $v''(x) = \frac{\lambda}{c^2} v(x)$
- $\ddot{w}(t) = 0$ • $v''(x) = 0$
- $w(t) = A + Bt$ • $v(x) = C + Dx$

• standing wave - $u(x,t) = w(t)v(x)$

$$u(x,t) = (A+Bt)(C+Dx)$$

$\rightarrow \lambda > 0$

- $\lambda = r^2 > 0$
- $\ddot{w}(t) = \lambda w(t)$ • $v''(x) = \frac{\lambda}{c^2} v(x)$
- $\ddot{w}(t) = r^2 w(t)$ • $v''(x) = \frac{r^2}{c^2} v(x)$
- $w(t) = Ae^{rt} + Be^{-rt}$ • $v(x) = Ce^{\frac{rx}{c}} + De^{-\frac{rx}{c}}$

← linear, 2nd order
const-coeff ODE

• standing wave - $u(x,t) = w(t)v(x)$

$$u(x,t) = (Ae^{rt} + Be^{-rt})(Ce^{\frac{rx}{c}} + De^{-\frac{rx}{c}})$$

$\rightarrow \lambda < 0$

- $\lambda = -r^2 < 0$
- $\ddot{w}(t) = \lambda w(t)$ • $v''(x) = \frac{\lambda}{c^2} v(x)$
- $\ddot{w}(t) = -r^2 w(t)$ • $v''(x) = -\frac{r^2}{c^2} v(x)$
- $\ddot{w}(t) + r^2 w(t) = 0$ • $v''(x) + \left(\frac{r^2}{c^2}\right)v(x) = 0$ ← harmonic oscillator eqn
- $w(t) = A\cos(rt) + B\sin(rt)$ • $v(x) = C\cos\left(\frac{r}{c}x\right) + D\sin\left(\frac{r}{c}x\right)$

• standing wave - $u(x,t) = w(t)v(x)$

$$u(x,t) = (A\cos(rt) + B\sin(rt))(C\cos\left(\frac{r}{c}x\right) + D\sin\left(\frac{r}{c}x\right))$$

<< Standing Waves for Finite String

- impose fixed boundary conditions on standing waves
- Problem

- $u_{tt} = c^2 u_{xx}$, $t > 0$, $x \in (0, L)$

- initial conditions : arbitrary

- fixed boundaries : $u(0,t) = 0$, $u(L,t) = 0$

→ Standing Wave Soln

- standing wave - $u(x,t) = w(t) v(x)$

- Sub into B.C. - $u(0,t) = 0 = w(t) v(0)$

$$u(L,t) = 0 = w(t) v(L)$$

$$\rightarrow \lambda = 0$$

- $v(x) = C + Dx$

- $v(0) = 0 = C \Rightarrow C = 0$

- $v(L) = C + DL = 0 \Rightarrow D = 0$

- $u(x,t) = w(t) v(x) = 0$

- trivial soln

$$\rightarrow \lambda > 0$$

- $v(x) = Ce^{rx/c} + De^{-rx/c}$

- $v(0) = 0 = C + D \Rightarrow D = -C$

$$v(L) = Ce^{rL/c} + De^{-rL/c} \quad \leftarrow$$

$$0 = Ce^{rL/c} - Ce^{-rL/c}$$

$$0 = C(e^{rL/c} - e^{-rL/c})$$

non-zero

$$C = 0, D = 0$$

- $v(x) = 0$

- $u(x,t) = w(t) v(x) = 0$

- trivial soln

$$\rightarrow \lambda < 0$$

- $v(x) = C \cos\left(\frac{\pi}{c}x\right) + D \sin\left(\frac{\pi}{c}x\right)$

- $v(0) = 0 = C$

- $v(L) = 0 = D \sin\left(\frac{\pi L}{c}\right)$

$$D = 0, \sin\left(\frac{\pi L}{c}\right) = 0$$

$$\frac{\pi L}{c} = n\pi$$

$r = \frac{n\pi c}{L}$

- $v(x) = D \sin\left(\frac{n\pi x}{L}\right)$

- $w(t) = A \cos\left(\frac{n\pi ct}{L}\right) + B \sin\left(\frac{n\pi ct}{L}\right)$

- $u(x,t) = w(t) v(x)$

$$u(x,t) = \left[A \cos\left(\frac{n\pi ct}{L}\right) + B \sin\left(\frac{n\pi ct}{L}\right) \right] D \sin\left(\frac{n\pi x}{L}\right)$$

$u_n(x,t) = \left[A_n \cos\left(\frac{n\pi ct}{L}\right) + B_n \sin\left(\frac{n\pi ct}{L}\right) \right] \sin\left(\frac{n\pi x}{L}\right)$

• nth mode of vibration ↑
 • nth harmonic amplitude shape

<< Standing Waves for Finite String

→ Modes of vibration

$$\begin{aligned} u_n(x,t) &= \left[A_n \cos\left(\frac{n\pi ct}{L}\right) + B_n \sin\left(\frac{n\pi ct}{L}\right) \right] \sin\left(\frac{n\pi x}{L}\right) \\ &= \sqrt{A_n^2 + B_n^2} \left[\frac{A_n}{\sqrt{A_n^2 + B_n^2}} \cos\left(\frac{n\pi ct}{L}\right) + \frac{B_n}{\sqrt{A_n^2 + B_n^2}} \sin\left(\frac{n\pi ct}{L}\right) \right] \sin\left(\frac{n\pi x}{L}\right) \\ &= R_n \left[\cos(\delta_n) \cos\left(\frac{n\pi ct}{L}\right) + \sin(\delta_n) \sin\left(\frac{n\pi ct}{L}\right) \right] \sin\left(\frac{n\pi x}{L}\right) \end{aligned}$$

$$u_n(x,t) = R_n \underbrace{\cos\left(\frac{n\pi ct}{L} - \delta_n\right)}_{\text{Amplitude}} \underbrace{\sin\left(\frac{n\pi x}{L}\right)}_{\text{shape}}$$

• angular frequency - $\omega_n = \frac{n\pi c}{L}$

• period - $T_n = \frac{2\pi}{\omega_n} = \frac{2L}{nc}$

• frequency - $f_n = \frac{1}{T_n} = \frac{nc}{2L} = \frac{n}{2L} \sqrt{\frac{c}{\rho}}$

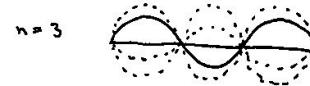
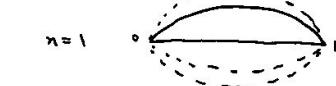
> fundamental frequency - $n=1, f_1$

> overtones - $n>1, f_n$

$$R_n = \sqrt{A_n^2 + B_n^2}$$

$$\cos(\delta_n) = \frac{A_n}{R_n} = \frac{A_n}{\sqrt{A_n^2 + B_n^2}}$$

$$\sin(\delta_n) = \frac{B_n}{R_n} = \frac{B_n}{\sqrt{A_n^2 + B_n^2}}$$



<< Superposition

> principle of superposition - linear combination of solutions is also a solution if

1. linear PDE

2. homogeneous PDE

3. homogeneous boundary condition - boundary condition is 0.

> Verify wave eqn with fixed end satisfies condition

• $U_{ttt} = c^2 U_{xxx}$ is linear and homogeneous PDE

• $u(0,t) = u(L,t) = 0$ is homogeneous B.C.

> Verify superposition of two known soln u_1 and u_2 gives another soln $u = c_1 u_1 + c_2 u_2$

• PDE: $U_{ttt} = (c_1 u_1 + c_2 u_2)_{ttt}$

$$= c_1 U_{1ttt} + c_2 U_{2ttt}$$

$$= c_1 (c^2 U_{1xxx}) + c_2 (c^2 U_{2xxx})$$

$$= c^2 (c_1 U_{1xxx} + c_2 U_{2xxx})$$

$$= c^2 U_{xxx} \quad \checkmark$$

• B.C.1: $u(0,t) = c_1 u_1(0,t) + c_2 u_2(0,t)$

$$= c_1(0) + c_2(0)$$

$$= 0 \quad \checkmark$$

• B.C.2: $u(L,t) = c_1 u_1(L,t) + c_2 u_2(L,t)$

$$= c_1(0) + c_2(0)$$

$$= 0 \quad \checkmark$$

> compound wave - superposition of N harmonics

$$u(x,t) = u_1(x,t) + u_2(x,t) + \dots + u_N(x,t)$$

$$= \sum_{n=1}^N u_n(x,t)$$

$$u(x,t) = \sum_{n=1}^N \left[A_n \cos\left(\frac{n\pi ct}{L}\right) + B_n \sin\left(\frac{n\pi ct}{L}\right) \right] \sin\left(\frac{n\pi x}{L}\right)$$

for standing wave on finite string
 $U_{ttt} = c^2 U_{xxx}$

$N \rightarrow \infty$

<< Superposition Example

Ex • $u_{tt} = u_{xx}$, $c=1$, $x \in (0,1)$, $t > 0$, $L=1$

- $u(0,t) = 0$, $u(1,t) = 0$

- $u(x,0) = f(x) = 0$

- $u_t(x,0) = g(x) = 2\sin(\pi x) - 3\sin(2\pi x)$

- general soln - $u(x,t) = \sum_{n=1}^N [A_n \cos(n\pi t) + B_n \overset{\circ}{\sin}(n\pi t)] \sin(n\pi x)$ $(L=1)$
 $(c=1)$

$$u(x,0) = \sum_{n=1}^N A_n \sin(n\pi x) = f(x) = 0$$

$$A_n = 0$$

- time derivative - $u_t(x,t) = \sum_{n=1}^N [-A_n n\pi \overset{\circ}{\sin}(n\pi t) + B_n n\pi \cos(n\pi t)] \sin(n\pi x)$

$$u_t(x,0) = \sum_{n=1}^N B_n n\pi \sin(n\pi x) = g(x) = 2\sin(\pi x) - 3\sin(2\pi x)$$

$$B_1 \pi \sin(\pi x) + B_2 2\pi \sin(2\pi x) = 2\sin(\pi x) - 3\sin(2\pi x)$$

$$B_1 \pi = 2 \quad B_2 2\pi = -3$$

$$B_1 = \frac{2}{\pi} \quad B_2 = -\frac{3}{2\pi}$$

- soln - $u(x,t) = \sum_{n=1}^N [A_n \cos(n\pi t) + B_n \overset{\circ}{\sin}(n\pi t)] \sin(n\pi x)$

$$= \frac{2}{\pi} \sin(\pi t) \overset{\circ}{\sin}(\pi) - \frac{3}{2\pi} \sin(2\pi t) \overset{\circ}{\sin}(2\pi x)$$

→ Infinite series

- generally, infinite series allows more complicated functions

- $u(x,t) = \sum_{n=1}^{\infty} [A_n \cos\left(\frac{n\pi ct}{L}\right) + B_n \overset{\circ}{\sin}\left(\frac{n\pi ct}{L}\right)] \sin\left(\frac{n\pi x}{L}\right)$

- A_n, B_n decreases rapidly → series converges

Ex • $u_{tt} = u_{xx}$, $c=1$, $x \in (0,1)$, $t > 0$, $L=1$

- $u(x,0) = f(x) = \sin(2\pi x)$

- $u_t(x,0) = g(x) = -3 \sin(2\pi x)$

- general soln - $u(x,0) = \sum_{n=1}^N A_n \sin(n\pi x) = f(x) = \sin(2\pi x)$

$$A_2 = 1$$

- time derivative - $u_t(x,0) = \sum_{n=1}^N B_n n\pi \sin(n\pi x) = g(x) = -3 \sin(2\pi x)$

$$B_2 2\pi = -3$$

$$B_2 = -\frac{3}{2\pi}$$

- soln - $u(x,t) = \sum_{n=1}^N [A_n \cos(n\pi t) + B_n \overset{\circ}{\sin}(n\pi t)] \sin(n\pi x)$

$$= [\cos(2\pi t) - \frac{3}{2\pi} \sin(2\pi t)] \sin(2\pi x)$$

<< Fourier Series

- Solve the wave eqn with fixed BC with general initial conditions.

$u_{tt} = c^2 u_{xx}, \quad x \in (0, L), \quad t > 0$

BC : $u(0, t) = 0, \quad u(L, t) = 0$

IC $\begin{cases} u(x, 0) = f(x) \\ u_t(x, 0) = g(x) \end{cases}$

general soln - $u(x, t) = \sum_{n=1}^{\infty} \left[A_n \cos\left(\frac{n\pi ct}{L}\right) + B_n \sin\left(\frac{n\pi ct}{L}\right) \right] \sin\left(\frac{n\pi x}{L}\right)$

$$u(x, 0) = \boxed{\sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)} = f(x)$$

Fourier sine series

Fourier sine series requires $\begin{cases} f(0) = f(L) = 0 \\ f(x) \text{ continuous} \end{cases}$

time derivative - $u_t(x, t) = \sum_{n=1}^{\infty} \left[-\frac{n\pi c}{L} A_n \sin\left(\frac{n\pi ct}{L}\right) + \frac{n\pi c}{L} B_n \cos\left(\frac{n\pi ct}{L}\right) \right] \sin\left(\frac{n\pi x}{L}\right)$

$$u_t(x, 0) = \sum_{n=1}^{\infty} \frac{n\pi c}{L} B_n \cos\left(\frac{n\pi ct}{L}\right) = g(x)$$

$$\sum_{n=1}^{\infty} b_n \cos\left(\frac{n\pi ct}{L}\right) = g(x) \quad \text{Fourier sine series}$$

$$b_n \equiv \frac{n\pi c}{L} B_n$$

→ Solving coefficients of Fourier Sine Series

orthogonality - $\int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx$

$$= \int_0^L \frac{1}{2} \left[\cos\left(\frac{(n-m)\pi x}{L}\right) - \cos\left(\frac{(n+m)\pi x}{L}\right) \right] dx$$

$$= \begin{cases} 0 & m \neq n \\ L/2 & m = n \end{cases}$$

solve for A_n : $\sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) = f(x)$

$$\sum_{n=1}^{\infty} A_n \int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx = \int_0^L f(x) \sin\left(\frac{m\pi x}{L}\right) dx$$

$$A_n \frac{1}{2} = \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

orthogonality, $m = n$

$$A_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

solve for B_n : $b_n = \frac{2}{L} \int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) dx$

$$B_n = \frac{L}{n\pi c} b_n = \frac{2}{n\pi c} \int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

→ General soln

$u(x, t) = \sum_{n=1}^{\infty} \left[A_n \cos\left(\frac{n\pi ct}{L}\right) + B_n \sin\left(\frac{n\pi ct}{L}\right) \right] \sin\left(\frac{n\pi x}{L}\right)$

$A_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$

$B_n = \frac{2}{n\pi c} \int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) dx$

<<< Fourier Series Example

- $u_{tt} = u_{xx}$, $c=1$, $x \in (0,2)$, $t > 0$, $L=2$

- BC - $u(0,t) = 0$, $u(2,t) = 0$

- IC $\begin{cases} u(x,0) = f(x) = 1 - |x-1| \\ u_t(x,0) = g(x) = 0 \end{cases}$

$$\cdot A_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

$$= \int_0^2 [1 - |x-1|] \sin\left(\frac{n\pi x}{2}\right) dx$$

$$= \int_0^1 x \sin\left(\frac{n\pi x}{2}\right) dx + \int_1^2 (2-x) \sin\left(\frac{n\pi x}{2}\right) dx$$

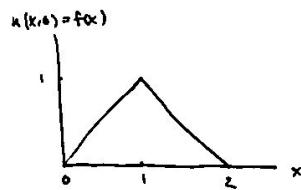
:

$$= \frac{8}{(n\pi)^2} \sin\left(\frac{n\pi}{2}\right)$$

$$\cdot B_n = \frac{2}{nc} \int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) dx = 0$$

$$\cdot f(x) = 1 - |x-1| = \sum_{n=1}^{\infty} \frac{8}{(n\pi)^2} \sin\left(\frac{n\pi}{2}\right) \sin\left(\frac{n\pi x}{2}\right)$$

$$\cdot u(x,t) = \sum_{n=1}^{\infty} \frac{8}{(n\pi)^2} \sin\left(\frac{n\pi}{2}\right) \cos\left(\frac{n\pi t}{2}\right) \sin\left(\frac{n\pi x}{2}\right)$$



<<< Other Boundary Conditions

- homogeneous boundary conditions of wave eqn (finite string)

- fixed endpoints

- $u(0,t) = 0$, $u(L,t) = 0$

- BC of 1st kind, homogeneous Dirichlet BC

- free endpoints

- $u_x(0,t) = 0$, $u_x(L,t) = 0$

- BC of 2nd kind, homogeneous Neumann BC

- elastic attachment

- $-T u_x(0,t) = -k u(0,t)$, $T u_x(L,t) = -k u(L,t)$

$$u_x(0,t) = \alpha u(0,t) \quad u_x(L,t) = -\alpha u(L,t), \quad \alpha \in \frac{k}{T}$$

- BC of 3rd kind, homogeneous Robin BC

- Mixed boundary conditions

- mixing fixed & free BC.

<< Example of two free ends

- $u_{tt} = c^2 u_{xx}, \quad x \in (0, L), \quad t > 0$

- IC : $u(x, 0) = f(x), \quad u_x(x, 0) = g(x)$

- BC : $u_x(0, t) = u_x(L, t) = 0 \quad (\text{free ends, homogeneous Neumann})$

- Standing wave - $u(x, t) = w(t) v(x)$

- Sub into PDE : $u_{tt} = c^2 u_{xx}$

$$\begin{aligned} \text{BC: } u_x(0, t) = 0 &= w(t)v'(0) \Rightarrow v'(0) = 0 \\ u_x(L, t) = 0 &= w(t)v'(L) \end{aligned}$$

$$w(t)v(x) = c^2 w(t)v''(x)$$

$$\frac{\ddot{w}(t)}{w(t)} = c^2 \frac{v''(x)}{v(x)} = \lambda$$

for diff variable of x and t to equal,
they must equal to a constant.

$$\ddot{w}(t) = \lambda w(t)$$

$$v''(x) = \frac{\lambda}{c^2} v(x)$$

$$\rightarrow \lambda = 0$$

- $\ddot{w}(t) = 0$

- $u(x, t) = w(t)v(x)$

- $w(t) = A + Bt$

- $= (A + Bt)C$

- $v''(x) = 0$

- $= A_0 + B_0 t$

- $v(x) = C + Dx \quad \leftarrow$

- $v'(x) = D$

- $v'(0) = D = 0 \quad \rightarrow$

$$\rightarrow \lambda > 0$$

- $\ddot{w}(t) = r^2 w(t)$

- $w(t) = Ae^{rt} + Be^{-rt}$

- $v''(x) = \frac{\lambda}{c^2} v(x)$

- $v(x) = Ce^{rx/c} + De^{-rx/c}$

- $v'(x) = C\frac{r}{c}e^{rx/c} + D\left(-\frac{r}{c}\right)e^{-rx/c}$

- $v'(0) = 0 = C\frac{r}{c} - D\frac{r}{c}$

- $0 = \frac{r}{c}(C - D)$

- $C = D$

- $v'(L) = 0 = C\frac{r}{c}e^{rL/c} + D\left(-\frac{r}{c}\right)e^{-rL/c}$

- $0 = \frac{r}{c}C\left(e^{rL/c} - e^{-rL/c}\right) \quad > 0$

- $C = 0 \Rightarrow D = 0$

- $v(x) = 0$

- $w(x, t) = w(t)v(x) = 0 \Rightarrow \text{trivial soln}$

Example of two free ends (cont.)

$$\rightarrow \lambda < 0$$

$$\cdot \ddot{w}(t) = -r^2 w(t)$$

$$w(t) = A \cos(rt) + B \sin(rt)$$

$$\cdot v''(x) = -\frac{r^2}{c^2} v(x)$$

$$v(x) = C \cos\left(\frac{r}{c}x\right) + D \sin\left(\frac{r}{c}x\right)$$

$$\cdot v'(x) = -C \frac{r}{c} \sin\left(\frac{r}{c}x\right) + D \frac{r}{c} \cos\left(\frac{r}{c}x\right)$$

$$\cdot v'(0) = 0 = D \frac{r}{c} \Rightarrow D = 0$$

$$\cdot v'(L) = 0 = -C \frac{r}{c} \sin\left(\frac{r}{c}L\right) + D \frac{r}{c} \cos\left(\frac{r}{c}L\right)$$

$$C = 0, \quad \sin\left(\frac{r}{c}L\right) = 0$$

$$\frac{rL}{c} = n\pi$$

$$r = \frac{n\pi c}{L}$$

$$\cdot v_n(x) = C \cos\left(\frac{n\pi x}{L}\right)$$

$$\cdot w_n(t) = A \cos\left(\frac{n\pi ct}{L}\right) + B \sin\left(\frac{n\pi ct}{L}\right)$$

$$\cdot u_n(x,t) = w_n(t)v_n(x)$$

$$= \left[A_n \cos\left(\frac{n\pi ct}{L}\right) + B_n \sin\left(\frac{n\pi ct}{L}\right) \right] \cos\left(\frac{n\pi x}{L}\right)$$

→ General solution

• Superposition principle

$$\cdot u_n(x,t) = A_0 + B_0 t + \sum_{n=1}^{\infty} \left[A_n \cos\left(\frac{n\pi ct}{L}\right) + B_n \sin\left(\frac{n\pi ct}{L}\right) \right] \cos\left(\frac{n\pi x}{L}\right)$$

$$\boxed{u_n(x,t) = A_0 + B_0 t + \sum_{n=1}^{\infty} [A_n \cos(\omega_n t) + B_n \sin(\omega_n t)] \cos(k_n x)}$$

$$\omega_n = \frac{n\pi c}{L}, \quad k_n = \frac{n\pi}{L}$$

• Initial conditions

$$\cdot u(x,0) = f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos(k_n x)$$

Fourier cosine series

$$\cdot u_t(x,0) = g(x) = B_0 + \sum_{n=1}^{\infty} B_n \omega_n \cos(k_n x)$$

→ Solve for constants of Fourier cosine series

$$\cdot n=0: \quad f(x) = A_0$$

$$\int_0^L f(x) dx = \int_0^L A_0 dx$$

$$\boxed{A_0 = \frac{1}{L} \int_0^L f(x) dx}$$

$$g(x) = B_0$$

$$\int_0^L g(x) dx = \int_0^L B_0 dx$$

$$\boxed{B_0 = \frac{1}{L} \int_0^L g(x) dx}$$

$$\cdot n \geq 1: \quad f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos(k_n x)$$

$$\int_0^L f(x) \cos(k_n x) dx = \int_0^L [A_0 + \sum_{n=1}^{\infty} A_n \cos(k_n x)] \cos(k_n x) dx$$

$$\int_0^L f(x) \cos(k_n x) dx = A_n \frac{L}{2}$$

Orthogonality

$$\boxed{A_n = \frac{2}{L} \int_0^L f(x) \cos(k_n x) dx}$$

$$B_n = \frac{2}{L \omega_n} \int_0^L g(x) \cos(k_n x) dx$$

$$\boxed{B_n = \frac{2}{n\pi c} \int_0^L g(x) \cos(k_n x) dx}$$

• The Heat Equation

• $u_t = Du_{xx}, \quad x \in (0, L)$

• IC : $u(x, 0) = f(x)$

• BC : $u(0, t) = u(L, t) = 0$

diffusion eqn - describes temperature distribution

- parabolic PDE

• Standing wave soln - $u(x, t) = w(t)v(x)$

• Sub into PDE : $u_t = Du_{xx}$

$$w(t)v(x) = D w(t)v''(x)$$

$$\frac{1}{D} \frac{w'(t)}{w(t)} = \frac{v''(x)}{v(x)} = \lambda = -r^2$$

$\rightarrow \lambda < 0$

$w(t) = -Dr^2 w(t)$

$$w(t) = C e^{-Dr^2 t}$$

• temp decrease with time $w(t) < 0$

• temp always positive $w(t) > 0$

• choose $\lambda = -r^2 < 0$

• $\lambda \geq 0$ gives trivial soln

$v''(x) = -r^2 v(x)$

$$BC: v(0) = 0 = B \cos(rx) \Rightarrow B = 0$$

$$v(x) = A \sin(rx)$$

$$v(x) = A \sin(rx)$$

$u(x, t) = w(t)v(x)$

$$v(L) = 0 = A \sin(rL)$$

$$= C e^{-Dr^2 t} [A \sin(rx) + B \cos(rx)]^0$$

$$rL = n\pi$$

$$= A e^{-Dr^2 t} \sin(rx)$$

$$r = \frac{n\pi}{L}$$

$$= A_n e^{-D(n\pi/L)^2 t} \sin\left(\frac{n\pi}{L} x\right)$$

\rightarrow General Soln

• Superposition principle

$$u(x, t) = \sum_{n=1}^{\infty} A_n e^{-D(n\pi/L)^2 t} \sin\left(\frac{n\pi}{L} x\right)$$

\leftarrow independent spatial modes with their own characteristic decay const.

$$\lim_{t \rightarrow \infty} u(x, t) \approx A_1 \sin\left(\frac{\pi x}{L}\right)$$

\rightarrow Calculate constant A_n

• IC : $u(x, 0) = f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)$

Fourier sine series

$$\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx = \sum_{n=1}^{\infty} A_n \int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx$$

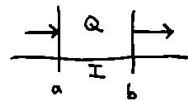
$$\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx = A_n \frac{L}{2}$$

orthogonality

$$A_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

<< Scalar Conservation Laws

$$\left(\begin{array}{l} \text{rate of change} \\ \text{of total quantity} \end{array} \right) = \left(\begin{array}{l} \text{rate of production} \\ \text{in I} \end{array} \right) + \left(\begin{array}{l} \text{rate of entry} \\ \text{at a} \end{array} \right) - \left(\begin{array}{l} \text{rate of departure} \\ \text{at b} \end{array} \right)$$



$$\frac{dQ}{dt} = \int_a^b f(x,t) dx + \phi(a,t) - \phi(b,t)$$

$$\frac{d}{dt} \int_a^b u(x,t) dx = \int_a^b f(x,t) dx + \phi(a,t) - \phi(b,t)$$

general balance eqn
(conservation law with sources)
integral form

- Interval - $I \subset [a,b]$
- Density / concentration of quantity - $u(x,t)$
- Total quantity - $Q = \int_a^b u(x,t) dx$
- Rate of production of quantity per unit length - $f(x,t)$
- Left-to-right flux of quantity - $\phi(x,t)$
- If continuously differentiable $u(x,t)$ and $\phi(x,t)$,

 - $\phi(a,t) - \phi(b,t) = - \int_a^b \frac{\partial \phi}{\partial x} dx$
 - $\frac{d}{dt} \int_a^b u(x,t) dx = \int_a^b \frac{\partial u}{\partial t} dx$

- If continuous $f(x,t)$, then
 - $\int_a^b \left[\frac{\partial u}{\partial t} - f(x,t) + \frac{\partial \phi}{\partial x} \right] dx = 0$
 - $\frac{\partial u}{\partial t} - f(x,t) + \frac{\partial \phi}{\partial x} = 0$
 - $\frac{\partial u}{\partial t} + \frac{\partial \phi}{\partial x} = f(x,t)$

continuous integrand
arbitrary interval \Rightarrow integrand is 0.

$$u_t + \phi_x = f$$

general balance eqn
differential form

<< Source Terms

- Source / reaction / generation
- no production : $f = 0$
- constant production rate : $f = r$
- exponential growth : $f = ru$
- logistic growth : $f = ru\left(1 - \frac{u}{K}\right)$
- logistic growth (with spatial dependence) : $f = r(x)u(x,t)\left[1 - \frac{u(x,t)}{K(x)}\right]$

Constitutive Eqn for Flux

> constitutive eqn - flux & dependence on u .

→ Advection and/or convection

- quantity moves horizontally (advection) or vertically (convection) with velocity $v(x,t)$

- $\phi = v(x,t) u(x,t)$

- general balance : $\frac{\partial u}{\partial t} + \frac{\partial \phi}{\partial x} = f(x,t)$

$$\frac{\partial u}{\partial t} + v \frac{\partial u}{\partial x} + u \frac{\partial v}{\partial x} = f(x,t)$$

Ex If no source term $f=0$

If constant velocity $v(x)=c \Rightarrow \frac{\partial v}{\partial x}=0$

then balance is $\frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} = 0$

→ Density-dependent advection

- advection speed depends on density $v(x,t) = v(u(x,t))$

- $\phi = v(u(x,t)) u(x,t)$

Ex If $u(x,t)$ is density of cars, then the unit of flux is

$$\phi [\text{ }] \frac{\text{km}}{\text{hr}} \times \frac{\text{cars}}{\text{km}} = \frac{\text{cars}}{\text{hr}}$$

Ex If $v=u^2$, then $\phi = v(u)u = u^3$

Balance eqn is $ut + 2auu_t = 0 \Rightarrow$ nonlinear, but quasilinear

> quasilinear PDE - PDE that's linear with respect to highest order derivative

Ex If $v(u) = v_1 \left(1 - \frac{u}{u_1}\right)$, $u \in [0, u_1]$

then $\phi = v(u)u = v_1 \left(1 - \frac{u}{u_1}\right)u$

Balance eqn is $u_t + v_1 \left(1 - \frac{2u}{u_1}\right)u_x = 0$

→ Diffusion

- Fickian diffusion - flux proportional to negative gradient of density

- $\phi = -D \frac{\partial u}{\partial x}$

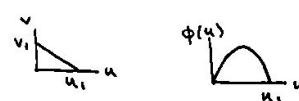
- general balance : $\frac{\partial u}{\partial t} - D \frac{\partial^2 u}{\partial x^2} = f(x,t)$

Ex Heat eqn - $\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2}$ $f=0$

Ex KISS model - $\frac{\partial u}{\partial t} = ru + D \frac{\partial^2 u}{\partial x^2}$ $f=ru$

Ex Fisher model - $\frac{\partial u}{\partial t} = ru \left(1 - \frac{u}{K}\right) + D \frac{\partial^2 u}{\partial x^2}$ $f=ru \left(1 - \frac{u}{K}\right)$

Ex Nagumo eqn (bistable) - $\frac{\partial u}{\partial t} = u(u-a)(1-u) + D \frac{\partial^2 u}{\partial x^2}$ $f=u(u-a)(1-u)$



Method of Characteristics

- reduce PDE into ODE by changing variables

let $x(s)$, $t(s)$, so $u(x,t) = u(x(s), t(s)) = u(s)$

EX If $v=c$, then $\phi = cu$

$$\text{PDE : } ut + cu_x = 0$$

$$\text{IC : } u(x,0) = u_0(x)$$

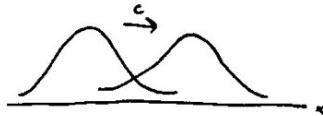
$$\begin{aligned} \text{Chain rule : } \frac{\partial u}{\partial s} &= \underbrace{\frac{\partial u}{\partial t}}_0 \underbrace{\frac{\partial t}{\partial s}}_1 + \underbrace{\frac{\partial u}{\partial x}}_c \underbrace{\frac{\partial x}{\partial s}}_0 \\ 0 &= ut + cu_x \end{aligned}$$

$$\Rightarrow \begin{cases} \frac{\partial t}{\partial s} = 1 & t(s=0) = 0 \\ \frac{\partial x}{\partial s} = c & x(s=0) = x_0 \\ \frac{\partial u}{\partial s} = 0 & u(s=0) = u_0(x_0) \end{cases}$$

new variables s, x

integrate ODE :

- $t = s \rightarrow s = t$
- $x = cs + x_0 \Rightarrow x_0 = x - ct$
- $u = u_0(x_0) \Rightarrow u = u_0(x - ct)$

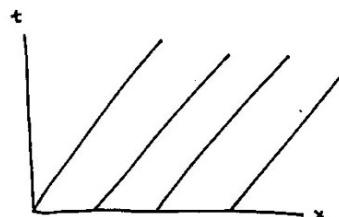


right-moving traveling soln - initial distribution moves to the right at constant speed c .

u is constant for fixed x .

$x - ct = x_0$ is leveling curve of $u(x-ct)$ in the $x-t$ plane

> characteristic - line at which u has the same value



Method linear homogeneous first-order PDE

$$a(x,t)ut + b(x,t)ux + c(x,t)u = 0$$

$$\text{IC : } u(x,0) = u_0(x)$$

Method of
characteristic

$$\begin{cases} \frac{dt}{ds} = a(x,t) & t(0) = 0 \\ \frac{dx}{ds} = b(x,t) & x(0) = x_0 \\ \frac{du}{ds} + c(x,t)u = 0 & u(0) = u_0(x_0) \end{cases}$$

<< Examples of Method of Characteristics

Ex

$$ut + ux + 2u = 0$$

$$u(x,0) = u_0(x) = \sin(x)$$

$$\begin{cases} \frac{dt}{ds} = 1 & t(0) = 0 \\ \frac{dx}{ds} = 1 & x(0) = x_0 \\ \frac{du}{ds} = -2u & u(0) = u_0(x_0) = \sin(x_0) \end{cases}$$

$$\Rightarrow t = s$$

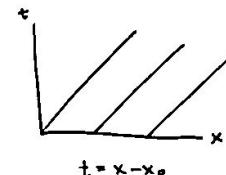
$$\Rightarrow x = s + x_0$$

$$\Rightarrow x_0 = x - s = x - t$$

$$\int \frac{du}{u} = \int -2 ds$$

$$\ln(u) = -2s + C$$

$$u = A e^{-2s} = \sin(x_0) e^{-2t} \Rightarrow u = \sin(x-t) e^{-2t}$$



* u does not stay const along the characteristics
∴ decaying term of e^{-2t}

Ex

$$ut + xu_x + tu = 0$$

$$u(x,0) = u_0(x)$$

$$\begin{cases} \frac{dt}{ds} = 1 & t(0) = 0 \\ \frac{dx}{ds} = x & x(0) = x_0 \\ \frac{du}{ds} = -tu & u(0) = u_0(x_0) \end{cases}$$

$$\Rightarrow t = s$$

$$\Rightarrow s = t$$

$$\Rightarrow \int \frac{dx}{x} = \int ds$$

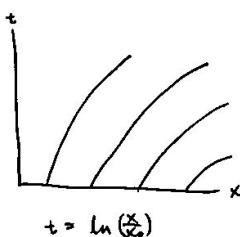
$$\ln(x) = s + C$$

$$x = A e^s = x_0 e^s \Rightarrow x_0 = x e^{-s} = x e^{-t}$$

$$\Rightarrow \int \frac{du}{u} = \int -t dt$$

$$\ln(u) = -\frac{1}{2}t^2 + C$$

$$u = A e^{-\frac{1}{2}t^2} = u_0(x_0) e^{-\frac{1}{2}t^2} \Rightarrow u(x,t) = u_0(x e^{-t}) e^{-\frac{1}{2}t^2}$$



• characteristics is not straight line

• u decreases along characteristics due to decay term $e^{-\frac{1}{2}t^2}$

<< Linear, Nonhomogeneous Problems

- $a(x,t) u_t + b(x,t) u_x + c(x,t) u = f(x,t)$

- $u(x,0) = u_0(x)$

Ex $ut + cu_x = e^{-3x}$

$$u(x,0) = u_0(x)$$

$$\begin{cases} \frac{dt}{ds} = 1 & t(0) = 0 \\ \frac{dx}{ds} = c & x(0) = x_0 \\ \frac{du}{ds} = e^{-3x} & u(0) = u_0(x_0) \end{cases} \Rightarrow \begin{aligned} t &= s \\ x &= cs + x_0 \Rightarrow x_0 = x - cs = x - ct \\ \frac{du}{ds} &= e^{-3(cs+x_0)} \end{aligned} \Rightarrow s = t$$

$$\int du = \int e^{-3(cs+x_0)} ds$$

$$u = -\frac{1}{3c} e^{-3(cs+x_0)} + g(x_0)$$

$$u(0) = u_0(x_0) = -\frac{1}{3c} e^{-3x_0} + g(x_0)$$

$$g(x_0) = u_0(x_0) + \frac{1}{3c} e^{-3x_0}$$

$$u = -\frac{1}{3c} e^{-3(cs+x_0)} + u_0(x_0) + \frac{1}{3c} e^{-3x_0}$$

$$u(x,t) = \frac{1}{3c} e^{-3(x-ct)} (1 - e^{-3ct}) + u_0(x-ct)$$

<< Linear, Nonhomogeneous Problem

$$\boxed{\text{Ex}} \quad u_t + x u_x = t$$

$$u(x, 0) = u_0(x)$$

$$\begin{cases} \frac{dt}{ds} = 1 & t(0) = 0 \Rightarrow t = s \Rightarrow s = t \\ \frac{dx}{ds} = x & x(0) = x_0 \Rightarrow x = x_0 e^s = x_0 e^t \Rightarrow x_0 = x e^{-t} \\ \frac{du}{ds} = t & u(0) = u_0(x_0) \Rightarrow \frac{du}{ds} = s \end{cases}$$

$u = \frac{1}{2} s^2 + g(x_0) \quad \leftarrow$

$$u(x, 0) = u_0(x) = g(x_0) \quad \leftarrow$$

$$u = \frac{1}{2} s^2 + g(x_0)$$

$$u(x, t) = \frac{1}{2} t^2 + g(x e^{-t})$$

<< Nonlinear Problems

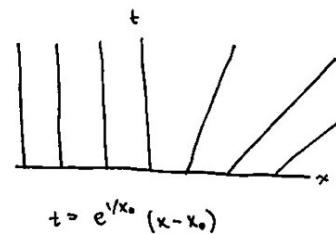
- $u_t + c(u) u_x = 0 \rightarrow$ nonlinear, but quasilinear
- $u(x, 0) = u_0(x)$

$$\begin{cases} \frac{dt}{ds} = 1 & t(0) = 0 \Rightarrow t = s \Rightarrow s = t \\ \frac{dx}{ds} = c(u) & x(0) = x_0 \Rightarrow \frac{dx}{dt} = c(u_0(x_0)) \Rightarrow x = x_0 + c(u_0(x_0))t \\ \frac{du}{ds} = 0 & u(0) = u_0(x_0) \Rightarrow u = u_0(x_0) \end{cases} \quad \leftarrow u \text{ is const along characteristics, varying slope } c(u_0(x_0)) \text{ depending on } x_0, u_0(x_0)$$

$$\boxed{\text{Ex}} \quad u_t + u u_x = 0, \quad x \in (-\infty, \infty), t > 0$$

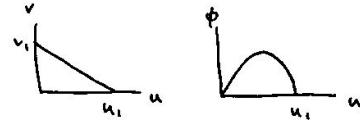
$$u(x, 0) = \begin{cases} 0 & x \leq 0 \\ e^{-1/x} & x > 0 \end{cases}$$

$$\begin{cases} \frac{dt}{ds} = 1 & t(0) = 0 \Rightarrow t = s \Rightarrow s = t \\ \frac{dx}{ds} = u & x(0) = x_0 \Rightarrow \frac{dx}{dt} = u_0(x_0) \Rightarrow x = \begin{cases} x_0 & x_0 \leq 0 \\ x_0 + e^{-1/x_0} t & x_0 > 0 \end{cases} \\ \frac{du}{ds} = 0 & u(0) = u_0(x_0) \Rightarrow u = u_0(x_0) = \begin{cases} 0 & x_0 \leq 0 \\ e^{-1/x_0} & x_0 > 0 \end{cases} \end{cases}$$



<-- The Traffic Problem

- car speed - $v(u) = v_i(1 - \frac{u}{u_i})$, $u \in [0, u_i]$



- flux - $\phi(u) = v(u)u = v_i(1 - \frac{u}{u_i})u$

$$\cdot \phi_u = v_i(1 - \frac{2u}{u_i})u_x$$

- PDE - $u_t + \phi_x = f$ ($f=0$)

$$u_t + v_i(1 - \frac{2u}{u_i})u_x = 0$$

$\underbrace{\phantom{v_i(1 - \frac{2u}{u_i})}}$
 $c(u)$

- Speed of density wave - $c(u) = v_i(1 - \frac{2u}{u_i})$

→ Problem

- $u_t + c(u)u_x = 0$

- $u(x, 0) = u_0(x)$

→ Method of Characteristics

$$\left\{ \begin{array}{l} \frac{dt}{ds} = 1 \quad t(0) = 0 \quad \Rightarrow \quad t = s \quad \Rightarrow \quad s = t \\ \frac{dx}{ds} = c(u) \quad x(0) = x_0 \quad \Rightarrow \quad x = c(u_0(x_0))s + x_0 \quad \Rightarrow \quad x = \left[v_i(1 - \frac{2u_0(x_0)}{u_i}) \right] t + x_0 \\ \frac{du}{ds} = 0 \quad u(0) = u_0(x_0) \quad \Rightarrow \quad u = u_0(x_0) \end{array} \right.$$

• u const along characteristics

$$x_0 = x - \left[v_i(1 - \frac{2u_0(x_0)}{u_i}) \right] t$$

$$\Rightarrow u = u_0 \left(x - \left[v_i(1 - \frac{2u_0(x_0)}{u_i}) \right] t \right)$$

→ speeds of interest

- car speed - $v(u) = v_i(1 - \frac{u}{u_i})$

- density wave speed - $c(u) = v_i(1 - \frac{2u_0(x_0)}{u_i})$

→ traffic regions

→ Light traffic (LT)

- $u_0(x_0) < \frac{1}{2}u_i \Rightarrow c(u) = v_i(1 - \frac{2u_0(x_0)}{u_i}) > 0$

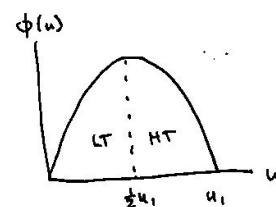
- cars forward, density wave forward

→ Heavy traffic (HT)

- $u_0(x_0) > \frac{1}{2}u_i \Rightarrow c(u) = v_i(1 - \frac{2u_0(x_0)}{u_i}) < 0$

- cars forward, density wave backward

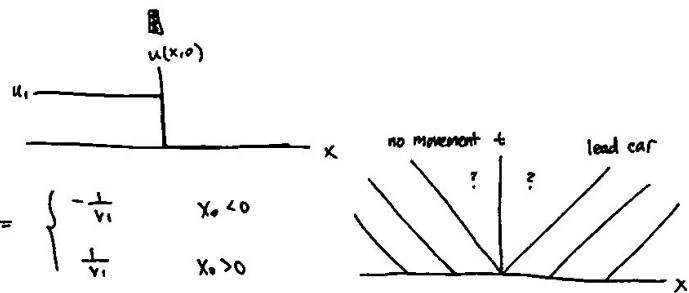
- info of bumper to bumper signal moves back.



<< Zone of Rarefaction

- traffic at red light

$$\cdot u(x,0) = u_0(x) = \begin{cases} u_1 & x < 0 \\ 0 & x > 0 \end{cases}$$



$$\cdot \text{slope of characteristics } \frac{dt}{dx} = \frac{1}{c(u)} = \frac{1}{v_1 \left(1 - \frac{2u_0(x_0)}{u_1}\right)} = \begin{cases} -\frac{1}{v_1} & x_0 < 0 \\ \frac{1}{v_1} & x_0 > 0 \end{cases}$$

• left - max density u_1 , zero car velocity, negative slope of characteristics

• right - zero density $u=0$, no car, positive slope of characteristics

• middle - cone-shaped zone of rarefaction

• density u slowly decrease

> Zone of rarefaction

$$\cdot x = \alpha t \Rightarrow \alpha = \frac{x}{t}$$

$$\cdot u(x,t) = u(\alpha t, t) = g(\alpha) = g\left(\frac{x}{t}\right)$$

$$\cdot u_t + c(u) u_x = 0$$

$$-\frac{x}{t^2} g'(\alpha) + c(g(\alpha)) \frac{1}{t} g'(\alpha) = 0$$

$$\frac{1}{t} g'(\alpha) \left[c(g(\alpha)) - \frac{x}{t} \right] = 0$$

$$c(g(\alpha)) - \frac{x}{t} = 0$$

$$\boxed{c(u) = \frac{x}{t}}$$

$$\cdot c(u) = v_1 \left(1 - \frac{2u}{u_1}\right) = \frac{x}{t}$$

$$u = \frac{u_1}{2} \left(1 - \frac{x}{v_1 t}\right)$$

$$\cdot \frac{dx}{dt} = v(u) = v_1 \left(1 - \frac{u}{u_1}\right) \quad \leftarrow$$

$$\frac{dx}{dt} = \frac{v_1}{2} + \frac{x}{2t}$$

$$\cdot \text{at } v=0, u=u_1, x\left(t=\frac{x_0}{v_1}\right) = -x_0.$$

$$\frac{dx}{dt} - \frac{1}{2t} x = \frac{v_1}{2}$$

$$\cdot \mu = e^{-\int \frac{1}{2t} dt} = \frac{1}{\sqrt{t}}$$

$$\frac{d}{dt} \left(\frac{1}{\sqrt{t}} x \right) = \frac{v_1}{2} \frac{1}{\sqrt{t}}$$

$$\frac{d}{dt} \left(\frac{1}{\sqrt{t}} x \right) = \frac{v_1}{2} \frac{1}{\sqrt{t}}$$

$$\cdot x(t) = v_1 t + k \sqrt{t}$$

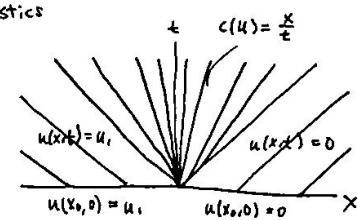
$$\cdot x\left(t=\frac{x_0}{v_1}\right) = -x_0 = x_0 + k \sqrt{\frac{x_0}{v_1}}$$

$$k = -2\sqrt{v_1 x_0}$$

$$\boxed{x(t) = v_1 t - 2\sqrt{v_1 x_0 t}}$$

$$\boxed{v = \frac{dx}{dt} = v_1 - \frac{\sqrt{v_1 x_0}}{t}}$$

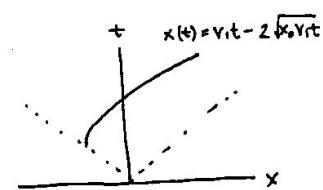
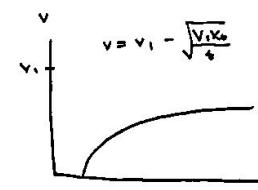
u is const and indep. on t
along the characteristics



• time for car to pass the light is 4x the max speed:

$$x=0 = v_1 t - 2\sqrt{v_1 x_0 t}$$

$$\boxed{t = 4 \frac{x_0}{v_1}}$$



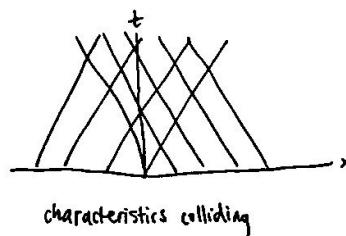
<< Gradient Catastrophe

- Foggy day with traffic ahead

$$\cdot u(x,0) = u_0(x_0) = \begin{cases} \frac{u_1}{4} & x_0 < 0 \\ u_1 & x_0 > 0 \end{cases}$$

$$\cdot \frac{dx}{dt} = c(u) = v_1 \left(1 - \frac{2u_0(x_0)}{u_1} \right) = \begin{cases} \frac{v_1}{2} & x_0 < 0 \\ -v_1 & x_0 > 0 \end{cases}$$

$$\cdot \frac{dt}{dx} = \left(\frac{dx}{dt} \right)^{-1} = \begin{cases} \frac{2}{v_1} & x_0 < 0 \\ -\frac{1}{v_1} & x_0 > 0 \end{cases}$$



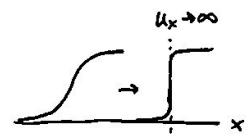
> gradient catastrophe - sudden change (discontinuity) in u that makes $u_x \rightarrow \infty$

- approach 1 - shock wave soln

- acknowledge discontinuity, let it be piecewise continuous
- use integral form of conservation law

- approach 2 - viscosity soln

- smooth out by adding "look ahead" term that's proportional to gradient of density



<< Shock Wave

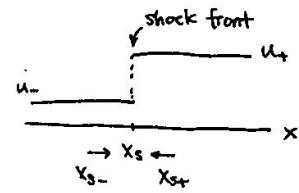
> shock wave - discontinuity in soln $u(x,t)$

- integral form of conservation law

$$\frac{d}{dt} \int_a^b u(x,t) dx = \phi(a,t) - \phi(b,t)$$

$$\frac{d}{dt} \left[\int_a^{x_s(t)} u(x,t) dx + \int_{x_s(t)}^b u(x,t) dx \right] = \phi(a,t) - \phi(b,t)$$

$$\int_a^{x_s(t)} \frac{\partial u}{\partial t} dx + u(x_s^-, t) \frac{dx_s}{dt} + \int_{x_s^-(t)}^b \frac{\partial u}{\partial t} dx + u(x_s^+, t) \frac{dx_s}{dt} = \phi(a,t) - \phi(b,t) \quad \text{Leibnitz's rule}$$



$$\lim_{a \rightarrow x_s^-(t)} \lim_{b \rightarrow x_s^+(t)} : u(x_s^-, t) \frac{dx_s}{dt} + u(x_s^+, t) \frac{dx_s}{dt} = \phi(x_s^-, t) - \phi(x_s^+, t)$$

$$\boxed{\frac{dx_s}{dt} = \frac{[\phi]}{[u]} = \frac{\phi(x_s^-, t) - \phi(x_s^+, t)}{u(x_s^-, t) - u(x_s^+, t)}}$$

Rankine-Hugoniot jump condition

> shock wave - piecewise smooth soln with a jump along a curve $x_s(t)$ satisfying the Rankine-Hugoniot jump condition

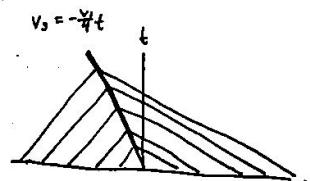
> shock path - curve $x_s(t)$

$u(x,0) = u_0(x_0) = \begin{cases} \frac{u_1}{4} & x_0 < 0 \\ u_1 & x_0 > 0 \end{cases} \Rightarrow u^- = \frac{u_1}{4}$

$$\phi = v_1 \left(1 - \frac{u_1}{4u_1} \right) u$$

$$\frac{dx_s}{dt} = \frac{v_1 \left(1 - \frac{u_1}{4u_1} \right) u^+ - v_1 \left(1 - \frac{u_1}{4u_1} \right) u^-}{u^+ - u^-} = \frac{v_1 \left(1 - \frac{u_1}{4u_1} \right) u_1 - v_1 \left(1 - \frac{u_1}{4u_1} \right) \frac{u_1}{4}}{u_1 - \frac{u_1}{4}} = -\frac{v_1}{4}$$

$$x_s = -\frac{v_1}{4} t$$



ccs Viscosity Soln

- allow drivers to look ahead and respond to relative changes in u (gradient of u).

$$\cdot v = v_i \left(1 - \frac{u}{u_i}\right) - r \frac{1}{u} \frac{\partial u}{\partial x}$$

$$\cdot \phi = v(u)u = v_i \left(1 - \frac{u}{u_i}\right)u - r \frac{\partial u}{\partial x}$$

$$\cdot \phi_x = v_i \left(1 - \frac{2u}{u_i}\right)u_x - r u_{xx}$$

$$\cdot \text{PDE} - u_t + v_i \left(1 - \frac{2u}{u_i}\right)u_x = r u_{xx}$$

$$\cdot \text{traveling wave soln} - u(x,t) = f(x-ct) = f(z)$$

$$\cdot u_t = -cf'$$

$$\cdot u_x = f'$$

$$\cdot \text{sub in PDE: } \int -cf' + v_i \left(1 - \frac{2f}{u_i}\right) f' = \int r f''$$

$$-cf + v_i \left(1 - \frac{f}{u_i}\right)f = rf'' + k$$

$$z \rightarrow \infty: -cu_i + v_i \left(1 - \frac{f}{u_i}\right)u_i = 0 + k$$

$$z \rightarrow -\infty:$$

$$\begin{aligned} -cu_0 + v_i \left(1 - \frac{f}{u_i}\right)u_0 &= 0 + k \\ -cu_0 + v_i \left(1 - \frac{f}{u_i}\right)u_0 &= k \end{aligned} \quad \left. \begin{array}{l} c = -v_i \frac{u_0}{u_i} \\ k = v_i u_0 \end{array} \right\}$$

$$v_i \frac{u_0}{u_i} f + v_i f \left(1 - \frac{f}{u_i}\right) = rf'' + v_i u_0$$

$$u_0 f + f(u_i - f) = \frac{cu_i}{v_i} f' + u_0 u_i$$

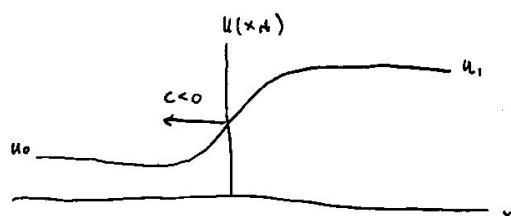
$$-\frac{cu_i}{v_i} f' = f^2 - (u_i + u_0) f + u_0 u_i$$

$$-\frac{cu_i}{v_i} f' = (f - u_i)(f - u_0)$$

$$\int \frac{df}{(f - u_i)(f - u_0)} = \int -\frac{v_i}{cu_i} dz$$

: partial fraction

$$u(x,t) = f(z) = u_i + \frac{u_0 - u_i}{1 + \exp \left[\frac{v_i(u_i - u_0)}{cu_i} z \right]}, \quad z \equiv x - ct = x + v_i \frac{u_0}{u_i} t$$



$r \rightarrow 0$, steeper wave front,
recover shock wave

<<< Breaking Times and Earliest Shock Wave

- smooth initial conditions can still cause shock waves later

Ex antisocial speed maniac

- $v(u) = au$
- $\phi = v(u)u = au^2$
- $\phi_x = 2auux$

- PDE - $u_t + \phi_x = 0$

$$u_t + 2auux = 0 \quad (a = \frac{1}{2})$$

$$u_t + uux = 0$$

- IC: $u(x, 0) = u_0(x) = e^{-x^2}$

> breaking time - earliest time $t_b > 0$ at which
characteristics intersect corresponds
to a gradient catastrophe ($u_x \rightarrow \infty$)

→ Break time calculation

- $u_t + c(u)u_x = 0$
- $u(x, 0) = u_0(x)$

$$\begin{cases} \frac{dt}{ds} = 1 & t(0) = 0 \Rightarrow t = s \\ \frac{dx}{ds} = c(u) & x(0) = x_0 \Rightarrow x = x_0 + c(u_0(x_0))t \\ \frac{du}{ds} = 0 & u(s) = u_0(x_0) \Rightarrow u = u_0(x_0) \end{cases}$$

$$u_x = u'_0(x_0) \frac{\partial x_0}{\partial x}$$

$$\begin{aligned} \frac{\partial}{\partial x} x &= \frac{\partial}{\partial x} (x_0 + c(u_0(x_0))t) \\ 1 &= \frac{\partial x_0}{\partial x} + t \frac{\partial x_0}{\partial x_0} c(u_0(x_0)) \\ \frac{\partial x_0}{\partial x} &= \frac{1}{1 + t \frac{d}{dx_0} c(u_0(x_0))} \end{aligned}$$

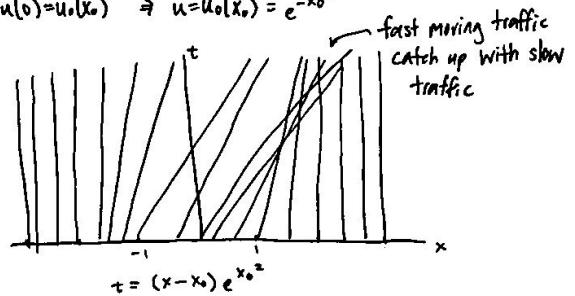
$$u_x = \frac{u'_0(x_0)}{1 + t \frac{d}{dx_0} c(u_0(x_0))} \rightarrow \infty$$

$$1 + t \frac{d}{dx_0} c(u_0(x_0)) = 0$$

$$t_b = -\frac{1}{\frac{d}{dx_0} c(u_0(x_0))}$$

- $x_0(t_b)$ is x_0 that makes $\frac{d}{dx_0} c(u_0(x_0))$ most negative.

$$\begin{cases} \frac{dt}{ds} = 1 & t(0) = 1 \Rightarrow t = s \\ \frac{dx}{ds} = u & x(0) = x_0 \Rightarrow x = u_0(x_0)t + x_0 = e^{-x_0^2}t + x_0 \\ \frac{du}{ds} = 0 & u(s) = u_0(x_0) \Rightarrow u = u_0(x_0) = e^{-x_0^2} \end{cases}$$

**Ex** antisocial speed maniac

$$\begin{cases} c(u) = u \\ u_0(x) = e^{-x^2} \end{cases} \Rightarrow c(u_0(x_0)) = e^{-x_0^2}$$

$$F(x) = \frac{d}{dx_0} e^{-x_0^2} = -2x_0 e^{-x_0^2}$$

$$F'(x_0) = (-2 + 4x_0^2) e^{-x_0^2} = 0$$

$$x_0 = \pm \frac{1}{\sqrt{2}}$$

choose $x_0 = \frac{1}{\sqrt{2}}$ that makes $F(x_0)$ most negative.

$$t_b = \frac{-1}{F(x_0)} = \frac{-1}{-2x_0 e^{-x_0^2}} = \frac{-1}{-2 \cdot \frac{1}{\sqrt{2}} e^{-1/2}} = \sqrt{\frac{e}{2}} \approx 1.16$$

<< Combination of Rarefaction & Shock Waves

EX . $ut + uu_x = 0$, $x \in (-\infty, \infty)$, $t > 0$

$$\cdot u(x, 0) = u_0(x) = \begin{cases} 0 & x \leq 0 \\ 1 & 0 < x < 1 \\ 0 & x \geq 1 \end{cases}$$

$$\left\{ \begin{array}{l} \frac{dt}{ds} = 1 \quad t(0) = 0 \quad \Rightarrow \quad t = s \\ \frac{dx}{ds} = u \quad x(0) = x_0 \quad \Rightarrow \quad x = u_0(x_0)s + x_0 = \begin{cases} x_0 + t & 0 < x_0 < 1 \\ x_0 & \text{otherwise} \end{cases} \\ \frac{du}{ds} = 0 \quad u(0) = u_0(x_0) \quad \Rightarrow \quad u = u_0(x_0) \end{array} \right.$$

• Zone of rarefaction : $c(u) = u(x,t) = \frac{x}{t}$
 $x = ut$

• Shock path 1 : $u_t + uu_x = 0$

$\Phi_x = uu_x$

$\Phi = \frac{1}{2}u^2$

$\frac{dx_s}{dt} = \frac{[\Phi]}{[u]} = \frac{\frac{1}{2}(u^+)^2 - \frac{1}{2}(u^-)^2}{u^+ - u^-} = \frac{1}{2}(u^+ + u^-) = \frac{1}{2}(0+1) = \frac{1}{2}$

$x_s = \frac{1}{2}t + c_1, \quad \text{shock starts at } x_s(0) = 1 \Rightarrow c_1 = 1$

$x_s = \frac{1}{2}t + 1, \quad x \in [0, 2]$

• shock path 2 :

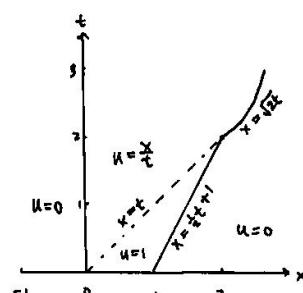
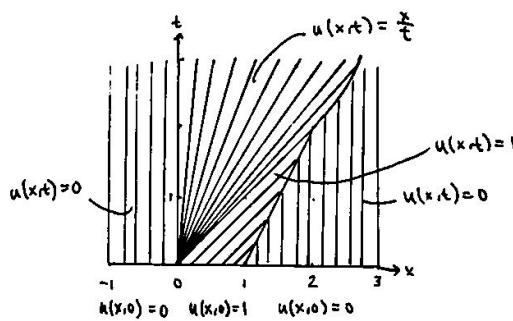
$\frac{dx_s}{dt} = \frac{[\Phi]}{[u]} = \frac{\frac{1}{2}(u^+)^2 - \frac{1}{2}(u^-)^2}{u^+ - u^-} = \frac{1}{2}(u^+ + u^-) = \frac{1}{2}(0 + \frac{x_s}{t}) = \frac{x_s}{2t}$

$\int \frac{dx_s}{x_s} = \int \frac{dt}{2t}$

$\ln(x_s) = \ln(\sqrt{t}) + c_2$

$x_s = c_2 \sqrt{t} \quad \text{shock starts at } x_s(2) = 2 \Rightarrow c_2 = \sqrt{2}$

$x_s = \sqrt{2t} \quad t \geq 2$



For $t < 2$:

$$u(x,t) = \begin{cases} 0 & x < 0 \\ \frac{x}{t} & 0 < x < t \\ 1 & t < x < (\frac{1}{2}t + 1) \\ 0 & \frac{1}{2}t + 1 < x \end{cases}$$

For $t > 2$:

$$u(x,t) = \begin{cases} 0 & x < 0 \\ \frac{x}{t} & 0 < x < \sqrt{2t} \\ 0 & \sqrt{2t} < x \end{cases}$$