Time-independent Schrödinger Equation in spherical coordinates:

$$-\frac{\hbar^2}{2m}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\left(\frac{\partial^2\psi}{\partial\phi^2}\right)\right]+V\psi=E\psi \qquad \boxed{1}$$

Assume solution ψ is separable:

$$\psi(r,\theta,\phi)=R(r)\,Y(\theta,\phi) \qquad \boxed{2}$$

$\boxed{2}\rightarrow\boxed{1}$:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right)=\frac{1}{r^2}\left[\left(\frac{\partial}{\partial r}r^2\right)\frac{\partial\psi}{\partial r}+\frac{\partial^2\psi}{\partial r^2}\cdot r^2\right]=\frac{1}{r^2}\left[2r\frac{\partial\psi}{\partial r}+r^2\frac{\partial^2\psi}{\partial r^2}\right]=\frac{2}{r}\frac{\partial\psi}{\partial r}+\frac{\partial^2\psi}{\partial r^2}$$

$$=\frac{2}{r}\frac{\partial}{\partial r}RY+\frac{\partial}{\partial r}\frac{\partial}{\partial r}RY=\frac{2}{r}Y\frac{\partial R}{\partial r}+Y\frac{\partial^2 R}{\partial r^2}$$

$$\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial\psi}{\partial\theta}\right]=\frac{1}{r^2\sin\theta}\left[\cos\theta\frac{\partial\psi}{\partial\theta}+\sin\theta\frac{\partial^2\psi}{\partial\theta^2}\right]=\frac{\cos\theta}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}+\frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}$$

$=$

$$-\frac{\hbar^2}{2m}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\cdot R\right)+\frac{1}{r^2\sin\theta}\left(\frac{\partial^2 Y}{\partial\phi^2}\cdot R\right)+VRY=ERY\right.$$

$$-\frac{\hbar^2}{2m}\left[\frac{Y}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)+\frac{R}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right)+\frac{R}{r^2\sin\theta}\frac{\partial^2 Y}{\partial\phi^2}+VRY=ERY\right.$$

4.2 (a) $V(x,y,z)=\begin{cases}0 & \text{if } x,y,z \text{ are all between } 0 \text{ and } a\\ \infty & \text{otherwise}\end{cases}$

$$\boxed{-\frac{\hbar^2}{2m}\nabla^2\psi+V\psi=E\psi}$$

Inside the well, $V=0$, so

$$\cancel{\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\frac{2mr^2}{\hbar^2}E=l(l+1)}$$

$$-\frac{\hbar^2}{2m}\nabla^2\psi=E\psi$$

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2}\right)=E\psi \qquad \boxed{1}$$

Assume the solution ψ is separable:

$$\psi(x,y,z)=X(x)\,Y(y)\,Z(z) \qquad \boxed{2}$$

$\boxed{2}\rightarrow\boxed{1}$:

$$-\frac{\hbar^2}{2m}\left[YZ\frac{\partial^2 X}{\partial x^2}+XZ\frac{\partial^2 Y}{\partial y^2}+YX\frac{\partial^2 Z}{\partial z^2}\right]=E\,XYZ$$

$$-\left(\frac{1}{X}\frac{\partial^2 X}{\partial x^2}+\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2}+\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2}\right)=+\frac{2m}{\hbar^2}E \qquad \boxed{3}$$

Since X, Y, and Z are on the left, they must be constant, so

$$-\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = k_x^2 \rightarrow \frac{\partial^2 X}{\partial x^2} = -k_x^2 X \quad \boxed{4}$$

$$-\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = k_y^2 \rightarrow \frac{\partial^2 Y}{\partial y^2} = -k_y^2 Y \quad \boxed{5}$$

$$-\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = k_z^2 \rightarrow \frac{\partial^2 Z}{\partial z^2} = -k_z^2 Z \quad \boxed{6}$$

We can calculate E by $\boxed{4}\boxed{5}\boxed{6} \rightarrow \boxed{3}$.

$$-\frac{1}{X}k_x^2 X - \frac{1}{Y}k_y^2 Y - \frac{1}{Z}k_z^2 Z = \frac{2m}{\hbar^2}E$$

$$-\frac{\hbar^2}{2m}(k_x^2 + k_y^2 + k_z)^2 = E \quad \boxed{7}$$

Solve for $\boxed{4}\boxed{5}\boxed{6}$,

$$X = A_x \sin(k_x x) + B_x \cos(k_x x) \quad \boxed{8}$$

$$Y = A_y \sin(k_y y) + B_y \cos(k_y y) \quad \boxed{9}$$

$$Z = A_z \sin(k_z z) + B_z \cos(k_z z) \quad \boxed{10}$$

Since at boundary conditions, $\psi(0) = \psi(a) = 0$,

$$X(0) = B_x = 0 \qquad\qquad X = A_x \sin(k_x X) \quad \boxed{11}$$

$$Y(0) = B_y = 0 \quad \Rightarrow \quad Y = A_y \sin(k_y y) \quad \boxed{12}$$

$$Z(0) = B_z = 0 \qquad\qquad Z = A_z \sin(k_z z) \quad \boxed{13}$$

Solve for k_x, ~~k_y, k_z~~:

$$X(a) = 0 = A_x \sin(k_x a)$$

$$0 = \sin(k_x a)$$

$$n\pi = k_x a$$

$$\frac{n\pi}{a} = k_x$$

Similarly, $k_y = \frac{n_y \pi}{a}$, $k_z = \frac{n_z \pi}{a}$. $\boxed{15}$

Normalize X to find A_x:

$$1 = A_x^2 \int_0^a \sin^2(k_x x)\, dx \quad \cancel{= A_x^2 \int_0^{k_x a} (\sin u)^2 \frac{du}{k_x} =}$$

$u = k_x x$ $\qquad$ ~~$v = \sin u$~~

$\frac{du}{dx} = k_x$ $\qquad$ ~~$\frac{dv}{du} = \cos u$~~

~~$\frac{du}{k_x} = dx$~~ $\qquad$ $\frac{dv}{\cos u} = du$

From half angle formula, $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos(2x)$, so

$$1 = A_x^2 \int_0^a \sin^2(k_x x)\, dx$$

$$= A_x^2 \int_0^a \frac{1}{2} - \cos(2 k_x x)\, dx$$

$$= A_x^2 \left[\int_0^a \frac{1}{2}\, dx - \int_0^a \cos(2k_x x)\, dx \right]$$

$u = 2k_x x$

$\frac{du}{dx} = 2k_x$

$\frac{du}{2k_x} = dx$

$$= A_x^2 \left[\frac{1}{2} a - \int_0^{2k_x a} \cos(u)\, \frac{du}{2k_x} \right]$$

$$\frac{1}{A_x^2} = \frac{a}{2} - \frac{1}{2k_x}\left[\cancel{\overset{1}{\cos(2k_x a)} - \overset{1}{\cos(0)}} \right]$$

$$\frac{1}{A_x^2} = \frac{a}{2}$$

$$A_x = \sqrt{\frac{2}{a}}$$

Similarly, $A_y = \sqrt{\frac{2}{a}}$, $A_z = \sqrt{\frac{2}{a}}$.

Organize the functions, we have

$$X = \sqrt{\frac{2}{a}} \sin\left(\frac{n_x \pi}{a} x\right) \qquad \boxed{16}$$

$$Y = \sqrt{\frac{2}{a}} \sin\left(\frac{n_y \pi}{a} y\right) \qquad \boxed{17}$$

$$Z = \sqrt{\frac{2}{a}} \sin\left(\frac{n_z \pi}{a} z\right) \qquad \boxed{18}$$

$\boxed{16}\boxed{17}\boxed{18} \to \boxed{2}$

$$\psi(x,y,z) = \left(\frac{2}{a}\right)^{\frac{3}{2}} \sin\left(\frac{n_x \pi}{a} x\right) \sin\left(\frac{n_y \pi}{a} y\right) \sin\left(\frac{n_z \pi}{a} z\right) \qquad \boxed{19}$$

$\boxed{15} \to \boxed{7}$

$$E = -\frac{\hbar^2}{2m}\left(\frac{n_x^2 \pi^2}{a^2} + \frac{n_y^2 \pi^2}{a^2} + \frac{n_z^2 \pi^2}{a^2} \right)$$

$$= -\frac{\hbar^2 \pi^2}{2ma^2}\left(n_x^2 + n_y^2 + n_z^2 \right), \quad n_x, n_y, n_z \in \mathbb{N} \qquad \boxed{20}$$

n/m

16) $E_{111} = -\frac{3\hbar^2\pi^2}{2ma^2}$

$E_{211} = -\frac{6\hbar^2\pi^2}{2ma^2}$

$E_{121} = -\frac{6\hbar^2\pi^2}{2ma^2}$

$E_{112} = -\frac{6\hbar^2\pi^2}{2ma^2}$

$E_{221} = \frac{9\hbar^2\pi^2}{2ma^2}$

$E_{212} = \frac{9\hbar^2\pi^2}{2ma^2}$

$E_{122} = \frac{9\hbar^2\pi^2}{2ma^2}$

$E_{222} = \frac{12\hbar^2\pi^2}{2ma^2}$

$E_{123} = \frac{14\hbar^2\pi^2}{2ma^2}$

$E_{132} = \frac{14\hbar^2\pi^2}{2ma^2}$

$E_{213} = \frac{14\hbar^2\pi^2}{2ma^2}$

$E_{231} = \frac{14\hbar^2\pi^2}{2ma^2}$

$E_{312} = \frac{14\hbar^2\pi^2}{2ma^2}$

$E_{321} = \frac{14\hbar^2\pi^2}{2ma^2}$

n	E	degeneracy
1	$\frac{3\hbar^2\pi^2}{2ma^2}$	1
2	$\frac{6\hbar^2\pi^2}{2ma^2}$	3
3	$\frac{9\hbar^2\pi^2}{2ma^2}$	3
4	$\frac{12\hbar^2\pi^2}{2ma^2}$	4
5	$\frac{14\hbar^2\pi^2}{2ma^2}$	6

4.3 Construct Y_0^0 and Y_2^1 and check they are normalizable and orthogonal.

Objective: $Y_0^0 = \left(\frac{1}{4\pi}\right)^{\frac{1}{2}}$, $Y_2^1 = -\left(\frac{15}{8\pi}\right)^{\frac{1}{2}} \sin\theta \cos\theta \, e^{+i\phi}$

Known:

$$P_l^m(x) \equiv (1-x^2)^{\frac{|m|}{2}} \left(\frac{d}{dx}\right)^{|m|} P_l(x)$$

$$P_l(x) \equiv \frac{1}{2^l l!} \left(\frac{d}{dx}\right)^l (x^2-1)^l$$

$$Y_l^m(\theta,\phi) = \epsilon \sqrt{\frac{(2l+1)}{4\pi}\frac{(l-|m|)!}{(l+|m|)!}} \, e^{im\phi} P_l^m(\cos\theta)$$

$\epsilon = (-1)^m$ for $m \geq 0$, $\epsilon = 1$ for $m \leq 0$.

~~$Y_0^0(\theta,\phi) = \epsilon$~~

$l = 0$, $m = 0$, so $\epsilon = 1$.

$$Y_0^0(\theta,\phi) = \sqrt{\frac{(1)}{4\pi}\cdot\frac{\cancel{0!}}{\cancel{0!}}} \, \cancel{e^{i\phi\cdot 0}} P_0^0(\cancel{\cos\theta}) = \sqrt{\frac{1}{4\pi}}$$

$$P_0^0(x) = (1-\cancel{x^2})^0 \left(\cancel{\frac{d}{dx}}\right)^0 P_0(x) = 1$$

$$P_0(x) = \frac{1}{2^0 \cdot 0!} \left(\frac{d}{dx}\right)^0 (x^2-1)^0 = 1$$

$$P_0^0(\cos\theta) = 1$$

$$\int_0^{2\pi}\int_0^{\pi} |Y|^2 \sin\theta \, d\theta \, d\phi = 1$$

$$= \int_0^{2\pi}\int_0^{\pi} \frac{1}{4\pi} \sin\theta \, d\theta \, d\phi$$

~~$= \frac{1}{4\pi}\int_0^{\pi} (-\cos(\pi) + \cos(0)) \, d\phi$~~

~~$= \frac{1}{4\pi}$~~

$$= \frac{1}{4\pi}\int_0^{2\pi} (-\cos(\pi) + \cos(0)) \, d\phi$$

$$= \frac{1}{4\pi}\int_0^{2\pi} 2 \, d\phi$$

$$= \frac{1}{4\pi}[2(2\pi) - 2(0)]$$

$$= 1$$

Normality ✓

~~$\int_0^{2\pi}\int_0^{\pi} [Y_0^0(\theta,\phi)]^* [Y_0^0(\theta,\phi)] \sin\theta \, d\theta \, d\phi$~~

~~$= \int_0^{2\pi}\int_0^{\pi} \frac{1}{4\pi} \sin\theta \, d\theta \, d\phi$~~

~~$= 1$~~

$l = 2, \; m = 1, \; \text{so } \epsilon = -1$

$$Y_2^1 = -\sqrt{\frac{5 \cdot 1}{4\pi \cdot 6}} \, e^{i\phi} \, P_2^1(\cos\theta)$$

$$P_2(x) = \frac{1}{2^2 \cdot 2!} \frac{d^2}{dx^2}(x^2-1)^2 = \frac{1}{8}\frac{d}{dx}(4x^3 - 4x + 0) = \frac{1}{8}(12x^2 - 4) = \frac{1}{2}(3x^2-1)$$

$$P_2^1(x) = (1-x^2)^{\frac{1}{2}} \frac{d}{dx} P_2(x)$$

$$= \sqrt{1-x^2}\,\frac{1}{2}\frac{d}{dx}(3x^2-1)$$

$$= \frac{1}{2}\sqrt{1-x^2}\,(6x)$$

$$= 3x\sqrt{1-x^2}$$

$$Y_2^1 = -\sqrt{\frac{5}{24\pi}}\, e^{i\phi}\, 3\cos\theta \sqrt{1-\cos^2\theta} = -\sqrt{\frac{5}{24\pi}}\, e^{i\phi}\, 3\cos\theta\sin\theta = -\sqrt{\frac{15}{8\pi}}\, e^{i\phi}\cos\theta\sin\theta$$

$$\int_0^{2\pi}\int_0^{\pi} |Y|^2 \sin\theta \, d\theta\, d\phi = 1$$

complex conjugate

$e^{i\phi - i\phi}$

$$= \int_0^{2\pi}\int_0^{\pi} + \frac{15}{8\pi} \cancel{e^{2i\phi}} \cos^2\theta \sin^3\theta \, d\phi \, d\theta$$

~~$= \int_0^{2\pi}\int_0^{\pi} -\frac{5}{24\pi}\cos^2\theta\sin^3\theta \; (\cos^2$~~

$u = \cos\theta \quad \rightarrow \quad u^2 = \cos^2\theta$

$\frac{du}{d\theta} = -\sin\theta \qquad \sin^2\theta = 1 - \cos^2\theta$

$\frac{du}{-\sin\theta} = d\theta \qquad\quad = 1 - u^2$

$$= \int_0^{2\pi}\int_1^{-1} + \frac{15}{8\pi} \cancel{e^{2i\phi}}\, u^2 \sin^3\theta \, d\phi \, \frac{du}{-\sin\theta}$$

$$= -\int_0^{2\pi}\int_1^{-1} + \frac{15}{8\pi} \cancel{e^{2i\phi}}\, u^2(1-u^2) \, d\phi \, du$$

$$= -\int_0^{2\pi}\int_1^{-1} + \frac{15}{8\pi} \cancel{e^{2i\phi}}(u^2 - u^4) \, d\phi \, du$$

$$= -+\frac{15}{8\pi}\int_0^{2\pi} \cancel{e^{2i\phi}}\, d\phi \cdot \left[\frac{1}{3}u^3 - \frac{1}{5}u^5\right]_1^{-1}$$

$-\frac{1}{3} + \frac{1}{5} - \frac{1}{3} + \frac{1}{5}$

$-\frac{2}{3} + \frac{2}{5}$

$-\frac{4}{15}$

$$= -+\frac{15}{8\pi}\left(-\frac{4}{15}\right)\int_0^{2\pi} \cancel{e^{2i\phi}}\, d\phi$$

$$= \cancel{\frac{15}{8}}\; 1$$

~~$= \frac{5}{36}\int_0^{2\pi} e^{2i\phi}\, d\phi$~~

~~$u = 2i\phi$~~

~~$\frac{du}{d\phi} = 2i \Rightarrow d\phi = \frac{du}{2i}$~~

~~$= \frac{5}{36}\int_0^{4\pi i} e^u \frac{du}{2i}$~~

~~$= \frac{5}{36}\frac{1}{2i}\left(e^{4\pi i} - e^0\right)$~~

~~$= \frac{5}{72i}\left(e^{4\pi i} - 1\right)$~~

▸ Orthogonal : $\int_0^{2\pi}\int_0^{\pi} [Y_l^m(\theta,\phi)]^* [Y_{l'}^{m'}(\theta,\phi)] \sin\theta \, d\theta \, d\phi = \delta_{ll'}\delta_{mm'}$ $\boxed{1}$

$$\int_0^{2\pi}\int_0^{\pi} Y_0^{0*} \, Y_2^1 \sin\theta \, d\theta \, d\phi$$

$$= \int_0^{2\pi}\int_0^{\pi} -\sqrt{\frac{1}{4\pi}}\sqrt{\frac{15}{8\pi}} \sin^2\theta \cos\theta \, e^{i\phi} \, d\theta \, d\phi$$

$$= -\sqrt{\frac{15}{32\pi^2}} \int_0^{2\pi}\int_0^{\pi} \sin^2\theta \cos\theta \, e^{i\phi} \, d\theta \, d\phi \quad \longleftarrow \quad u = \sin\theta, \quad \frac{du}{\cos\theta} = d\theta$$

$$= -\sqrt{\frac{15}{32\pi^2}} \int_0^{2\pi}\int_0^{0} u^2 \, du \, \cancel{d\phi} \, e^{i\phi} \, d\phi$$

$$= 0$$

When $m \neq m'$ and $l \neq l'$, $\boxed{1}$ $= 0$

When $m = m'$ and $l = l'$, $\boxed{1}$ is a normalization of the function $= 1$.

8 March 2019 4.1.3 The Radial Equation Day 5

4.7 Known $n_l(x) \equiv -(-x)^l \left(\frac{1}{x}\frac{d}{dx}\right)^l \frac{\cos x}{x}$

(a) $n_1(x) = -(-x)\left(\frac{1}{x}\frac{d}{dx}\right)\frac{\cos x}{x} = \frac{d}{dx}\frac{\cos x}{x} = \frac{-x\sin x - \cancel{x/x}\cos x}{x^2} = -\frac{\cos x}{x^2} - \frac{\sin x}{x}$

$$n_2(x) = -(-x)^2\left(\frac{1}{x}\frac{d}{dx}\right)^2\frac{\cos x}{x} = -\frac{d^2}{dx^2}\frac{\cos x}{x} = -\frac{d}{dx}\left(-\frac{\cos x}{x^2} - \frac{\sin x}{x}\right)$$

$$\cancel{= \frac{-x^2\sin x - 2x\cos x}{x^4} + \frac{x\cos x - \sin x}{x^2} = \frac{\sin x}{x^2} \quad \frac{2\cos x}{x^4} + \frac{\cos x}{x} \quad \frac{\sin x}{x^2}}$$

$$= -x^2 \cdot \frac{1}{x}\frac{d}{dx}\frac{1}{x}\frac{d}{dx}\frac{\cos x}{x}$$

$$= -x\frac{d}{dx}\frac{1}{x}\cdot\left(-\frac{\cos x}{x^2} - \frac{\sin x}{x}\right)$$

$$= + x\frac{d}{dx}\frac{\cos x}{x^3} + \frac{\sin x}{x^2}$$

$$= x\left[\frac{-x^3\sin x - 3x^2\cos x}{x^6} + \frac{x^2\cos x - 2x\sin x}{x^4}\right]$$

$$= \cancel{4} \; -\frac{\sin x}{x^2} - \frac{3\cos x}{x^3} + \frac{\cos x}{x} - \frac{2\sin x}{x^2}$$

$$= -\frac{3\sin x}{x^2} - \left(\frac{3}{x^3} - \frac{1}{x}\right)\cos x$$

4.7 (b) Known

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$n_1(x) = -\frac{\cos x}{x^2} - \frac{\sin x}{x}$$

$$n_2(x) = -\left(\frac{3}{x^3} - \frac{1}{x}\right)\cos x - \frac{3}{x^2}\sin x$$

$$\frac{\sin x}{x} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n+1)!}$$

$$\frac{\sin x}{x^2} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n-1}}{(2n+1)!}$$

$$\frac{\cos x}{x} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n-1}}{(2n)!}$$

$$\frac{\cos x}{x^2} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n-2}}{(2n)!}$$

$$\frac{\cos x}{x^3} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n-3}}{(2n)!}$$

$$n_1(x) = -\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n-2}}{(2n)!} - \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n+1)!}$$

$$= -\sum_{n=0}^{\infty} (-1)^n \left[\frac{x^{2n-2}}{(2n)!} + \frac{x^{2n}}{(2n+1)!}\right]$$

$$= -\sum_{n=0}^{\infty} (-1)^n \frac{(2n+1)\, x^{2n-2} + x^{2n}}{(2n+1)!}$$

$$\lim_{x\to 0} n_1(x) = \lim_{x\to 0} -\sum_{n=0}^{\infty} (-1)^n \frac{(2n+1)\, x^{2n-2} + x^{2n}}{(2n+1)!} \approx \lim_{x\to 0} -\frac{x^{-2} + 1}{1} + \frac{3 + x^2}{6} - \frac{5x^2 + x^4}{120}$$

$$= \lim_{x\to 0} -\frac{1}{x^2} - 1 + \frac{1}{2} + \frac{x^2}{6} - \frac{x^2}{24} + \frac{x^4}{120}$$

$$= \lim_{x\to 0} -\frac{1}{x^2} - \frac{1}{2} + \frac{x^2}{8} + \frac{x^4}{120}$$

$$= -\infty$$

~~Use ratio test,~~ $\left[\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|\right]$

$$\lim_{x\to 0} n_1(x) = -\lim_{x\to 0}\lim_{n\to\infty} \frac{(-1)^{n+1} \dfrac{(2(n+1)+1)\, x^{2(n+1)-2} + x^{2(n+1)}}{(2(n+1)+1)!}}{(-1)^n \dfrac{(2n+1)\, x^{2n-2} + x^{2n}}{(2n+1)!}}$$

$$= \lim_{x\to 0}\lim_{n\to\infty} \frac{(-1)^{n+1}\left[(2n+3)\, x^{2n-1} + x^{2n+2}\right]}{(2n+3)!} \cdot \frac{(2n+1)!}{(-1)^n\left[(2n+1)\, x^{2n-2} + x^{2n}\right]}$$

$$= \lim_{x\to 0}\lim_{n\to\infty} \frac{(2n+3)\, x^{2n-1} + x^{2n+2}}{(2n+3)(2n+2)\left[(2n+1)\, x^{2n-2} + x^{2n}\right]}$$

$$n_2(x) = -\left(\frac{3}{x^3}-\frac{1}{x}\right)\cdot\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} - \frac{3}{x^2}\cdot\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}$$

$$= \sum_{n=0}^{\infty}\left(-\frac{3x^{2n-3}}{(2n)!}+\frac{x^{2n-1}}{(2n)!}\right)(-1)^n - \sum_{n=0}^{\infty}(-1)^n\frac{3x^{2n-1}}{(2n+1)!}$$

$$= \sum_{n=0}^{\infty}(-1)^n\left[\frac{x^{2n-1}-3x^{2n-3}}{(2n)!} - \frac{3x^{2n-1}}{(2n+1)!}\right]$$

$$= \sum_{n=0}^{\infty}(-1)^n\frac{(2n+1)x^{2n-1}-(2n+1)3x^{2n-3}-3x^{2n-1}}{(2n+1)!}$$

$$\approx \frac{x^{-1}-3x^{-3}-3x^{-1}}{1} - \frac{3x-9x^{-1}-3x}{6} + \frac{5x^3-15x-3x^3}{120}$$

$$= -2x^{-1}-3x^{-3}-\frac{3}{2}x^{-1}+\frac{x^3}{60}-\frac{x}{8}$$

$$= -3x^{-3}-\frac{7}{2}x^{-1}+\frac{x^3}{60}-\frac{x}{8}$$

$$\lim_{x\to 0} n_2(x) = \lim_{x\to 0} -3x^{-3}-\frac{7}{2}x^{-1}+\frac{x^3}{60}-\frac{x}{8} = -\infty$$

even → even, otherwise divergent

$n_2 = 0$

$k=0,\ n=0,$

Day 7

14 March 2019   4.2 The Hydrogen Atom

- Known Legendre's differential equation   These "n"'s are not the same.

$$\frac{d}{dx}\left[(1-x^2)\frac{dP_n(x)}{dx}\right] + n(n+1)P_n(x) = 0 \qquad \boxed{0}$$

Solve for Legendre polynomials $P_n(x)$

Assume Legendre polynomials

$$P_n(x) = \sum_{n=0}^{\infty} a_n x^n \qquad \boxed{1}$$

Its derivative is

$$P_n'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1} = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n \qquad \boxed{2}$$

and its second derivative is

$$P_n''(x) = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n \qquad \boxed{3}$$

Simplify $\boxed{0}$:

$$-2x\frac{dP_n(x)}{dx} + (1-x^2)\frac{d^2P_n(x)}{dx^2} + k(k+1)P_n(x) = 0 \qquad \boxed{4}$$

Substitute $\boxed{1}\boxed{2}\boxed{3}$ to $\boxed{4}$:

$$-2x\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n + (1-x^2)\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + k(k+1)\sum_{n=0}^{\infty}a_n x^n = 0$$

$$\cancel{\sum_{n=0}^{\infty}\left(-2x(n+1)a_{n+1} + (1-x^2)(n+2)(n+1)a_{n+2} + n(n+1)a_n\right)x^n = 0}$$

$$\cancel{\sum_{n=0}^{\infty} -2x a_{n+1} + (1-x^2)(n+2)a_{n+2} + n a_n = 0}$$

Since the Taylor Expasion of 0 is

$$0 = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} 0 \cdot x^n \qquad (\text{where } a_n = 0)$$

So

$$-2x a_{n+1} + (1-x^2)(n+2)a_{n+2} + n a_n = 0$$

$$a_{n+2} = \frac{2x a_{n+1} - n a_n}{(1-x^2)(n+2)}$$

Assume that $a_1 = 1$, then

$$a_2 = \frac{2x}{2(1-x^2)} = \frac{x}{1-x^2}$$

$$a_3 = \frac{\frac{2x^2}{1-x^2} - 1}{3(1-x^2)} = \frac{1}{3(1-x^2)}\cdot\frac{1-x^2}{2x^2-(1-x^2)} = \cancel{6x^2+6x^4}\ \frac{1}{9x^2-3}$$

$(n+1)\, a_{n+1} = 0$ $\qquad a_1 = 1$

$(n+2)(n+1)\, a_{n+2} = 0$ $\qquad a_2 =$

$n(n+1)\, a_n = 0$ $\qquad k$: order of polyminal

$= -2a_1 x^1 - 2(2) a_2 x^2 + \cdots$

$$\sum_{n=0}^{\infty} -2(n+1) a_{n+1} x^{n+1} + \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} -(n+2)(n+1) a_{n+2} x^{n+2} + \sum_{n=0}^{\infty} \overset{k(k+1)}{\cancel{n(n+1)}} a_n x^n = 0$$

$$\sum_{n=1}^{\infty} -2n a_n x^n + \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=2}^{\infty} -n(n-1) a_n x^n + \sum_{n=0}^{\infty} \overset{k(k+1)}{\cancel{n(n+1)}} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} -2n a_n x^n + \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} -n(n-1) a_n x^n + \sum_{n=0}^{\infty} \overset{k(k+1)}{\cancel{n(n+1)}} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left(-2n a_n + (n+2)(n+1) a_{n+2} - n(n-1) a_n + k(k+1) a_n\right) x^n = 0$$

$$a_{n+2} = \frac{\{(n+n^2) - k^2\} a_n}{-(n^2+3n-2)}$$

$$-2n a_n + (n^2+3n+2) a_{n+2} - (n^2-n) a_n + (k^2+k) a_n = 0$$

$$(k^2+k) a_n - n^2 a_n - n a_n + n^2 a_{n+2} + 3n a_{n+2} + 2a_{n+2} = 0$$

$$a_{n+2} = \frac{-\{(k^2+k) - n^2 - n\}}{n^2+3n+2} a_n \qquad a_n = \frac{(n^2+3n+2) a_{n+2}}{\cancel{2n(n+1)}\,(k^2+k) - n^2 - n}$$

$$\frac{(-9-3)+2}{6} = -\frac{5}{3}$$

~~If $k=0$,~~

~~Assume $a_1 = 3$,~~ $\quad a_1 = 3a_3$

~~$a_2 = 1$~~

$a_3 = 1$

Assume

If $k=0$, $\quad a_0 = 1$, $\quad a_2 = 0$. $\rightarrow P_0 = C_0 \cdot 1 \cdot x^0 = C_0$

If $k=1$, $\quad a_1 = 1$, $\quad a_3 = 0$. $\rightarrow P_1 = C_1 \cdot 1 \cdot x^1 = C_1 x$

If $k=2$, $\quad a_0 = 1$, $\quad a_2 = -3$, $a_4 = 0 \Rightarrow P_2 = C_2 \cdot (1x^0 + 3x^2) = C_n(1+3x^2)$

If $k=3$, $\quad a_1 = 1$, $\quad a_3 = -\frac{5}{3}$, $a_5 = 0$

If $k=4$, $\quad a_0 = 1$, $\quad a_2 = -6$, $a_4 = 5$

$$-\frac{14}{12} \cdot (-10) = \frac{70}{6}$$

$$\frac{-16+4+}{2} = \frac{-16+4+2}{12}(-6) = \frac{-60}{12} = +5$$

0

2

6

12

20

30

36

$$a_{n+2} = \frac{n(n+1)-k(k+1)}{(n+2)(n+1)} a_n$$

If $k=0$, $a_0 = 1$, $a_2 = \frac{0-0}{2} a_0 = 0$

If $k=1$, $a_1 = 1$, $a_3 = \frac{2-2}{6} a_1 = 0$

If $k=2$, $a_0 = 1$, $a_2 = \frac{0-6}{2} a_0 = -3$, $a_4 = \frac{6-6}{12} a_2 = 0$

If $k=3$, $a_1 = 1$, $a_3 = \frac{2-12}{6} a_1 = -\frac{5}{3}$, $a_5 = \frac{12-12}{20} a_3 = 0$

If $k=4$, $a_0 = 1$, $a_2 = \frac{0-20}{2} a_0 = -10$, $a_4 = \frac{6-20}{12} a_2 = \frac{35}{3}$, $a_6 = \frac{20-20}{30} a_4 = 0$

If $k=5$, $a_1 = 1$, $a_3 = \frac{2-30}{6} a_1 = -\frac{14}{3}$, $a_5 = \frac{12-30}{20} a_3 = \frac{21}{5}$, $a_7 = \frac{30-30}{36} a_5 = 0$

Normalize the coefficients using

$$\int_{-1}^{1} |P_n(x)|^2 \, dx = \frac{2}{2n+1}$$

$k=0$:

$$2 = \int_{-1}^{1} A_0^2 \, dx$$

$$2 = 2A_0^2$$

$$A_0 = 1$$

$k=1$:

$$\frac{2}{3} = \int_{-1}^{1} A_1^2 x^2 \, dx$$

$$\frac{2}{3} = A_1^2 \left[\frac{1}{3}x^3\right]_{-1}^{1}$$

$$\frac{2}{3} = A_1^2 \left(\frac{1}{3} + \frac{1}{3}\right)$$

$$A_1 = 1$$

$k=2$:

$$\frac{2}{5} = \int_{-1}^{1} A_2^2 (-3x^2+1)^2 \, dx$$

$$\frac{2}{5} = A_2^2 \int_{-1}^{1} (9x^4 - 6x^2 + 1) \, dx$$

$$\frac{2}{5} = A_2^2 \left[\frac{9}{5}x^5 - 2x^3 + x\right]_{-1}^{1}$$

$$\frac{2}{5} = A_2^2 \left[\left(\frac{9}{5} - 2 + 1\right) - \left(-\frac{9}{5} + 2 - 1\right)\right]$$

$$\frac{2}{5} = A_2^2 \frac{8}{5}$$

$$A_2 = \frac{1}{2}$$

$k=3$:

$$\frac{2}{7} = \int_{-1}^{1} A_3^2 \left(-\frac{5}{3}x^3 + x\right)^2 dx$$

$$\frac{2}{7} = A_3^2 \int_{-1}^{1} \left(\frac{25}{9}x^6 - \frac{10}{3}x^4 + x^2\right) dx$$

$$\frac{2}{7} = A_3^2 \left[\frac{25}{63}x^7 - \frac{2}{3}x^5 + \frac{1}{3}x^2\right]_{-1}^{1}$$

$$\frac{2}{7} = A_3^2 \left[\left(\frac{25}{63} - \frac{2}{3} + \frac{1}{3}\right) - \left(-\frac{25}{63} + \frac{2}{3} - \frac{1}{3}\right)\right]$$

$$\frac{2}{7} = A_3^2 \left(\frac{8}{63}\right)$$

$$\frac{9}{4} = A_3^2$$

$$A_3 = \frac{3}{2}$$

$k=4$:

$$\frac{2}{9} = \int_{-1}^{1} A_4^2 \left(\frac{35}{3}x^4 - 10x^2 + 1\right)^2 dx$$

$$\frac{2}{9} = A_4^2 \int_{-1}^{1} \left(\frac{1225}{9}x^8 - \frac{350}{3}x^6 + \frac{35}{3}x^4 - \frac{350}{3}x^6 + 100x^4 - 10x^2 + \frac{35}{3}x^4 - 10x^2 + 1\right) dx$$

$$\frac{2}{9} = A_4^2 \int_{-1}^{1} \left(\frac{1225}{9}x^8 - \frac{700}{3}x^6 + \frac{370}{3}x^4 - 20x^2 + 1\right) dx$$

$$\frac{2}{9} = A_4^2 \left[\frac{1225}{81}x^9 - \frac{100}{3}x^7 + \frac{74}{3}x^5 - \frac{20}{3}x^3 + x\right]_{-1}^{1}$$

$$\frac{2}{9} = A_4^2 \cdot 2\left(\frac{1225}{81} - \frac{100}{3} + \frac{74}{3} - \frac{20}{3} + 1\right)$$

$$\frac{2}{9} = A_4^2 \left(\frac{128}{81}\right)$$

$$\frac{9}{64} = A_4^2$$

$$A_4 = \frac{3}{8}$$

$k=5$:

$$\frac{2}{11} = \int_{-1}^{1} A_5^2 \left(\frac{21}{5}x^5 - \frac{14}{3}x^3 + x\right)^2 dx$$

$$\frac{2}{11} = A_5^2 \int_{-1}^{1} \left(\frac{441}{25}x^{10} - \frac{98}{5}x^8 + \frac{21}{5}x^6 - \frac{98}{5}x^8 + \frac{196}{9}x^6 - \frac{14}{3}x^4 + \frac{21}{5}x^6 - \frac{14}{3}x^4 + x^2\right) dx$$

$$\frac{2}{11} = A_5^2 \cancel{\left[\frac{441}{125}x^5\right.} \int_{-1}^{1} \left(\frac{441}{25}x^{10} - \frac{196}{5}x^8 + \frac{1358}{45}x^6 - \frac{28}{3}x^4 + x^2\right) dx$$

$$\frac{2}{11} = A_5^2 \left[\frac{441}{275}x^{11} - \frac{196}{45}x^9 + \frac{194}{45}x^7 \cancel{+ \frac{3}{5}x^6} - \frac{28}{15}x^5 + \frac{1}{3}x^3\right]_{-1}^{1}$$

$$\frac{2}{11} = A_5^2 \cdot \cancel{2\left(\left[\frac{441}{275} - \frac{196}{45} + \frac{28}{9} + \frac{7}{5} - \frac{7}{3} + 1\right] - \left[-\frac{441}{275} + \frac{196}{45} - \frac{28}{9} + \frac{7}{5} - \frac{7}{3} - 1\right]\right)} \quad 2\left(\frac{441}{275} - \frac{196}{45} + \frac{194}{45} - \frac{28}{15} + \frac{1}{3}\right.$$

$$\frac{2}{11} = A_5^2 \cdot \cancel{\frac{2198}{2475}} \quad \cancel{\frac{4728}{2475}} \quad \frac{128}{2475}$$

$$\frac{882}{275} - \frac{142}{45}$$

$$\frac{225}{64} \quad \cancel{\frac{225}{3364}} \quad \cancel{\frac{225}{1054}} = A_5^2$$

$$\frac{640}{12375}$$

$$A_5 = \cancel{\frac{45}{454}} \quad \cancel{\frac{15}{58}} \quad \frac{15}{8}$$

$$\frac{128}{2475}$$

20 March 2019 Owen Lin 12^{th} Schrodinger Equation in Spherical Coordinates Day 10

4.6 Derive orthonormality condition for Legendre polynomials.

$$\int_{-1}^{1} P_l(x)\, P_{l'}(x)\, dx = \left(\frac{2}{2l+1}\right)\delta_{ll'}$$

Known Rodrigues formula:

$$P_l(x) \equiv \frac{1}{2^l l!}\left(\frac{d}{dx}\right)^l (x^2-1)^l$$

when $l \neq l'$: $l > l'$ wlog.

$$\int_{-1}^{1} P_l(x) P_{l'}(x)\, dx$$

$$= \int_{-1}^{1} \frac{1}{2^l l!}\left(\frac{d}{dx}\right)^l (x^2-1)^l \cdot \frac{1}{2^{l'} l'!}\left(\frac{d}{dx}\right)^{l'} (x^2-1)^{l'}\, dx$$

$f(x) = \left(\frac{d}{dx}\right)^l (x^2-1)^l$

$f'(x) = \left(\frac{d}{dx}\right)^{l+1} (x^2-1)^l$

$g'(x) = \left(\frac{d}{dx}\right)^{l'} (x^2-1)^{l'}$

$g(x) = \left(\frac{d}{dx}\right)^{l'-1} (x^2-1)^{l'}$

$$= \frac{1}{2^{l+l'} \cdot l!\, l'!}\int_{-1}^{1} \left(\frac{d}{dx}\right)^l (x^2-1)^l \left(\frac{d}{dx}\right)^{l'} (x^2-1)^{l'}\, dx$$

$f(x) = \left(\frac{d}{dx}\right)^{l'} (x^2-1)^{l'}$ $\quad g'(x) = \left(\frac{d}{dx}\right)^{l} (x^2-1)^{l}$

$f'(x) = \left(\frac{d}{dx}\right)^{l'+1} (x^2-1)^{l'}$ $\quad g(x) = \left(\frac{d}{dx}\right)^{l-1} (x^2-1)^{l}$

wlog = without loss of generality

$$= \frac{1}{2^{l+l'}\, l!\, l'!}\cancel{\left[\left(\frac{d}{dx}\right)^{l'}(x^2-1)^{l'}\right]\cdot\left[\left(\frac{d}{dx}\right)^{l-1}(x^2-1)^{l}\right]\Bigg|_{-1}^{1}} - \int_{-1}^{1}\left(\frac{d}{dx}\right)^{l'+1}(x^2-1)^{l'}\cdot\left(\frac{d}{dx}\right)^{l-1}(x^2-1)^l\, dx$$

do IBP l' times, $\rightarrow$
$$= \frac{1}{2^{l+l'}\, l!\, l'!}(-1)^{l'}\int_{-1}^{1}\left(\frac{d}{dx}\right)^{l-l'}(x^2-1)^l \underbrace{\left(\frac{d}{dx}\right)^{l'+l}(x^2-1)^{l'}}_{0}\, dx$$

$= 0.$

$2x^3 - 2x$ $\quad = 2 \cdot 2 \ldots$

$\left(\frac{d}{dx}\right)^{2\cdot 2}(x^2-1)^2 = 2\left(\frac{d}{dx}\right)^3(x^2-1)\cdot 2x$ $\quad x^2$

$= 4\left(\frac{d}{dx}\right)^2 x$

when $l = l'$:

$$\int_{-1}^{1} P_l^2(x)\, dx$$

(#1) $$= \int_{-1}^{1}\frac{1}{2^{2l}(l!)^2}\left[\left(\frac{d}{dx}\right)^{l}(x^2-1)^{l}\right]^2 dx$$

$f(x) = \left(\frac{d}{dx}\right)^l (x^2-1)^l$ $\quad g'(x) = \left(\frac{d}{dx}\right)^l (x^2-1)^l$

$f'(x) = \left(\frac{d}{dx}\right)^{l+1} (x^2-1)^l$ $\quad g(x) = \left(\frac{d}{dx}\right)^{l-1} (x^2-1)^l$

$$= \frac{1}{2^{2l}(l!)^2}\left\{\cancel{\left(\left(\frac{d}{dx}\right)^l(x^2-1)^l\left(\frac{d}{dx}\right)^{l-1}(x^2-1)^l\right)\Bigg|_{-1}^{1}} - \int_{-1}^{1}\left(\frac{d}{dx}\right)^{l+1}(x^2-1)^l\cdot\left(\frac{d}{dx}\right)^{l-1}(x^2-1)^l\, dx\right\}$$

$$= + \frac{1}{2^{2l}(l!)^2}(-1)^l\int_{-1}^{1}\left(\frac{d}{dx}\right)^{l-l}(x^2-1)^l \cdot \left(\frac{d}{dx}\right)^{2l}(x^2-1)^l\, dx$$

highest order: x^{2l}

$$= \frac{1}{2^{2l}(l!)^2}(-1)^l\int_{-1}^{1}(x^2-1)^l\left(\frac{d}{dx}\right)^{2l}(x^2-1)^l\, dx$$

$\left(\frac{d}{dx}\right)^n c x^n = c(n!)$

(#2) $$= \frac{1}{2^{2l}(l!)^2}(-1)^l (2l)!\int_{-1}^{1}(x^2-1)^l\, dx \quad \boxed{1}$$

$(2l)!\,\alpha\,(2l+1) = 2^{2l+1}(l!)^2$

$$\alpha = \frac{2^{2l+1}(l!)^2}{(2l+1)(2l)!}$$

$$\cancel{= \frac{1}{2^{2l}(l!)^2}(2l)!\int_{-1}^{1}(1-x^2)^l\, dx}\ \boxed{1}\ \frac{(2l)!}{2^{2l}(l!)^2}\ \alpha = \frac{2}{2l+1}$$

$$= \frac{1}{2^{2l}(l!)^2} 2(2l)! \int_0^{\frac{\pi}{2}} \cancel{\cos^{2l+1}\theta \, d\theta}$$

$$= \cancel{\frac{1}{2^{2l}(l!)^2}} 2(2l)! \int_1^0 \frac{u^{2l+1}}{-\sqrt{1-u^2}} du$$

$\sin\left(\frac{\pi}{2} - u\right) = \cos u$ $\quad$ ~~$\cos\theta = \sin\theta$~~

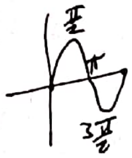
~~$u = 1 - x^2$~~ $\quad x = \sin\theta$ $\quad \frac{dx}{d\theta} = \cos\theta$

$\frac{du}{-2x} = dx$ $\quad 1 - x^2 = \cos^2\theta$ $\quad dx = \cos\theta \, d\theta$

$\frac{dx}{d\theta} = \cos\theta$

$x = \sin\theta$

$dx = \cos\theta \, d\theta$ $\quad \sqrt{1-x^2} = \cos\theta$



$$\int_{-1}^{1} \left(\sqrt{1-x^2}\right)^{2l} dx$$

$v = \cos\theta$ $\quad u = \cos\theta$

$\frac{dv}{d\theta} = -\sin\theta$ $\quad \frac{du}{d\theta} = -\sin\theta$

$dv = -\sin\theta \, d\theta$ $\quad \frac{du}{-\sin\theta} = d\theta$

$$\frac{du}{-\sqrt{1-u^2}} = d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos\theta)^{2l+1} d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

▶ Proof by Induction : Proof $\sum_{n=1}^{N} = \frac{N(N+1)}{2}$

#1 show $N = 1$:

$$\sum_{n=1}^{1} n = 1 \quad ; \quad \frac{1 \cdot (2)}{2} = 1$$

#2 Assume $\sum_{n=1}^{N-1} n = \frac{N(N-1)}{2}$,

show $\sum_{n=1}^{N} n = \frac{N(N+1)}{2}$

$$\frac{(N-1)N}{2} + N = \frac{(N-1)N + 2N}{2} = \frac{(N+1)N}{2}$$

Show $\int_{-1}^{1} (1-x^2)^l dx = \frac{2^{2l+1}(l!)^2}{(2l+1)(2l)!}$

#1 $l = 0$: $\int_{-1}^{1} dx = 2 = \frac{2}{1} = 2$

$$= \int_{-1}^{1} (1-x^2)^{l-1} dx - \int x^2 (1-x^2)^{l-1} dx$$

$u = x$ $\quad dv = x(1-x^2)^{l-1}$

$du = dx$ $\quad v = \frac{-1}{2x}(1-x^2)^l$

$$= \left[-\frac{1}{2l} \cancel{x(1-x^2)^l}\right]_{-1}^{1} + \int_{-1}^{1} \frac{1}{2l}(1-x^2)^l dx$$

$$\int_{-1}^{1} (1-x^2)^{l-1} dx = \left(1 + \frac{1}{2l}\right) \int_{-1}^{1} (1-x^2)^l dx$$

#2 : $\cancel{\int_{-1}^{1} (1-x^2)^{(l+1)-1} dx = \int_{-1}^{1} (1-x^2)^l dx}$

$$\int_{-1}^{1} (1-x^2)^{l-1} dx = \left(1 + \frac{1}{2l}\right) \frac{2^{2l+1}(l!)^2}{(2l+1)(2l)!}$$

26 March 2019

(#13) $= \dfrac{(-1)^l}{2^{2l}(l!)^2}(2l)! \left\{ -\int_{-1}^{1} (x^2-1)^{l-1}\,dx + \int_{-1}^{1} x^2(x^2-1)^{l-1}\,dx \right\}$

$u = x \qquad dv = x(x^2-1)^{l-1}$

$du = dx \qquad v = \frac{1}{2l}(x^2-1)^l$

(#14) $= \dfrac{(-1)^l}{2^{2l}(l!)^2}(2l)! \left[-\int_{-1}^{1} (x^2-1)^{l-1}\,dx + \overset{0}{\cancel{\left[\tfrac{1}{2l}x(x^2-1)^l\right]_{-1}^{1}}} - \int_{-1}^{1} \frac{1}{2l}(x^2-1)^l\,dx \right]$ $\boxed{2}$

$\boxed{1} = \boxed{2}$, so

$$\int_{-1}^{1}(x^2-1)^l\,dx = \int_{-1}^{1}(x^2-1)^{l-1}\,dx - \int_{-1}^{1}\frac{1}{2l}(x^2-1)^l\,dx$$

$$\left(1+\frac{1}{2l}\right)\int_{-1}^{1}(x^2-1)^l\,dx = \int_{-1}^{1}(x^2-1)^{l-1}\,dx \Rightarrow \int_{-1}^{1}(x^2-1)^l\,dx = \int_{-1}^{1}(x^2-1)^{l-1}\,dx\cdot\frac{2l}{2l+1}$$

ASSUME ... is true

Goal: proof $\int_{-1}^{1}(x^2-1)^{l-1}\,dx \approx \int_{-1}^{1}\left(\frac{d}{dx}(x^2-1)^{l-1}\right)^2 dx$

$\int_{-1}^{1}\left(\frac{d}{dx}(x^2-1)^{l-1}\right)^2 dx$

$= \int_{-1}^{1}\left(2x(l-1)(x^2-1)^{l-2}\right)^2 dx$

$= (l-1)^2 \int_{-1}^{1}(x^2-1)^{2l-4}\cdot 4x^2\,dx$

$= 4(l-1)^2\int_{-1}^{1} u^{2l-4}\sqrt{u+1}\; x\cdot\frac{du}{2x}$

$\dfrac{1}{2^{2l}(l!)^2}(-1)^l(2l)!\dfrac{2l}{2l+1}\int_{-1}^{1}(x^2-1)^{l-1}\,dx$

$= \dfrac{1}{2^{2l}(l!)^2}(-1)^l(2l)!\dfrac{2l}{2l+1}\dfrac{1}{(2l-2)!}\int_{-1}^{1}|P_{l-1}|^2\,dx$

$u = x^2-1 \Rightarrow x = \pm\sqrt{u+1}$

$\frac{du}{dx} = 2x$

$\frac{du}{2x} = dx$

★ $\int_{-1}^{1}(x^2-1)^{l-1}\,dx$

$= \dfrac{(2l-2)!}{(2l-2)!}\int_{-1}^{1}(x^2-1)^{l-1}\,dx$

$= \dfrac{1}{(2l-2)!}\int_{-1}^{1}\left[\left(\frac{d}{dx}\right)^{2l-2}(x^2-1)^{l-1}\right](x^2-1)^{l-1}\,dx$ by $\left(\frac{d}{dx}\right)^n = cx^n = c(n!)$

$= \dfrac{1}{(2l-2)!}\int_{-1}^{1}\left(\frac{d}{dx}\right)^{l-1}(x^2-1)^{l-1}\left(\frac{d}{dx}\right)^{l-1}(x^2-1)^{l-1}\,dx$

$= \dfrac{1}{(2l-2)!}\int_{-1}^{1}|P_{l-1}(x)|^2\,dx$

If $l=1$, $\int_{-1}^{1}|P_0(x)|^2\,dx = \int_{-1}^{1}\frac{1}{2^0 0!}\left(\frac{d}{dx}\right)^0(x^2-1)^0\,dx = \cancel{\frac{1}{2}\int_{-1}^{1}2x\,dx = \frac{1}{2}\left[x^2\right]_{-1}^{1} = 1}$

$\int_{-1}^{1}dx = 2$

If $l=2$, $\int_{-1}^{1}|P_1(x)|^2\,dx = \int_{-1}^{1}\frac{1}{2}2x\,dx = \left[\frac{1}{2}x^2\right]_{-1}^{1} = 1$

$$\left(\frac{1}{2^l l!}\right)^2 (2l!) \int_{-1}^{1} (x^2-1)^{l-1}\,dx \qquad l! = l(l-1)!$$

$$= \frac{2\cancel{l!}}{2^l 2^l\, l!\cancel{l!}} \int_{-1}^{1} (x^2-1)^{l-1}\,dx \qquad \frac{2\cancel{l!}}{2^{2l}\, l!\cancel{l!}}$$

$$= \frac{2}{2^l 2^l\, l(l-1)!} \int_{-1}^{1} (x^2-1)^{l-1}\,dx \qquad = \frac{2}{2^{2l}\, l(l-1)!}$$

$$\frac{2l(2l-1)}{2\cdot 2\cdot l l} \frac{(2l-2)!}{(2^{l-1})^2 (l-1)!^2}$$

$$= -\frac{(2l-1)(2(l-1))!}{2l(2^{2l-1})(l-1)!(2l-1)!}$$

When $l = l'$:

$$\int_{-1}^{1} P_l^2(x)\, dx$$

$$= \left(\frac{1}{2^l l!}\right)^2 \int_{-1}^{1} \left(\left(\frac{d}{dx}\right)^l (x^2-1)^l\right)^2 dx$$

$$= \left(\frac{1}{2^l l!}\right)^2 (2l)! \int_{-1}^{1} (x^2-1)^l \, dx \qquad \boxed{1}$$

$$= \left(\frac{1}{2^l l!}\right)^2 (2l)! \left[\int_{-1}^{1} x^2 (x^2-1)^{l-1} dx - \int_{-1}^{1} (x^2-1)^{l-1} dx \right] \qquad \begin{matrix} u = x & dv = x(1-x^2)^{l-1} \\ \underline{du} = dx & v = \frac{1}{2l}(1-x^2)^l \end{matrix}$$

$$= \left(\frac{1}{2^l l!}\right)^2 (2l)! \left[\left[-\frac{1}{2l} x \cancel{(x^2-1)^l}^{\,0}\right]_{-1}^{1} - \int_{-1}^{1} \frac{1}{2l} (x^2-1)^l dx - \int_{-1}^{1} (x^2-1)^{l-1} dx \right] \qquad \boxed{2}$$

$\boxed{1} = \boxed{2}$:

$$\left(\frac{1}{2^l l!}\right)^2 (2l)! \left(1 + \frac{1}{2l}\right) \int_{-1}^{1} (x^2-1)^l dx = \left(\frac{1}{2^l l!}\right)^2 (2l)! \int_{-1}^{1} (x^2-1)^{l-1} dx$$

$$'' \quad = \left(\frac{1}{2 \cdot 2^{l-1} \cdot l((l-1)!)}\right)^2 \left(2l \cdot (2l-1) \cdot (2l-2)!\right) \int_{-1}^{1} (x^2-1)^{l-1} dx$$

$$'' \quad = \frac{2l(2l-1)}{\cancel{4}(l)^2} \left(\frac{1}{2^{l-1}(l-1)!}\right)^2 \int_{-1}^{1} (2l-2)!\,(x^2-1)^{l-1} dx$$

$$'' \quad = \frac{(2l-1)}{2l} \left(\frac{1}{2^{l-1}(l-1)!}\right)^2 \int_{-1}^{1} \left(\left(\frac{d}{dx}\right)^{l-1} (x^2-1)^{l-1}\right)^2 dx$$

$$\left(\frac{1}{2^l l!}\right)^2 (2l)! \int_{-1}^{1} (x^2-1)^l dx = \frac{\cancel{2l}}{2l+1} \frac{2l-1}{\cancel{2l}} \underbrace{\left(\frac{1}{2^{l-1}(l-1)!}\right)^2 \int_{-1}^{1} \left(\left(\frac{d}{dx}\right)^{l-1} (x^2-1)^{l-1}\right)^2 dx}_{\text{Assume } = \frac{2}{2(l-1)+1}}$$

$$\frac{2}{2l+1} = \frac{\cancel{2l-1}}{2l+1} \cancel{\left(\frac{1}{2^{l-1}(l-1)!}\right)^2} \frac{2}{\cancel{2l-1}}$$

$$= \frac{2}{2l+1} \qquad \text{Q.E.D.}$$

$u(r) \equiv rR(r)$

Radial equation: $-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + \left[V + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}\right]u = Eu$ $\boxed{1}$

Coulomb's law: $V(r) = -\frac{e^2}{4\pi\epsilon_0}\frac{1}{r}$ $\boxed{2}$

$\boxed{2} \to \boxed{1}$

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + \left[-\frac{e^2}{4\pi\epsilon_0}\frac{1}{r} + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}\right]u = Eu \quad \boxed{3}$$

$k^2 = \frac{-2mE}{\hbar^2}$ $\frac{1}{k^2} = \frac{\hbar^2}{-2mE}$

Let $k \equiv \frac{\sqrt{-2mE}}{\hbar}$, bound states $E<0$, k is real.

$\frac{\boxed{3}}{E}$: $\frac{1}{k^2}\frac{d^2u}{dr^2} = \left[1 - \frac{me^2}{2\pi\epsilon_0\hbar^2 k}\frac{1}{kr} + \frac{l(l+1)}{k^2r^2}\right]u$ $\boxed{4}$

Let $\rho \equiv kr$, $\rho_0 \equiv \frac{me^2}{2\pi\epsilon_0\hbar^2 k}$, so $\frac{d\rho}{dr} = k$

$\boxed{4}$: $\cancel{\frac{1}{k^2}\frac{d^2u}{d\rho^2}\frac{d\rho^2}{dr^2}}$ $\frac{1}{k^2}\frac{d\rho}{dr}\frac{d}{d\rho}\frac{d\rho}{dr}\frac{d}{d\rho}u = \frac{1}{k^2}\left(\frac{d\rho}{dr}\right)^2\frac{d^2u}{d\rho^2} = \frac{d^2u}{d\rho^2} = \left[1 - \frac{\rho_0}{\rho} + \frac{l(l+1)}{\rho^2}\right]u$ $\boxed{5}$

As $\rho \to \infty$, $\frac{d^2u}{d\rho^2} = u$

The general solution $u(\rho) = Ae^{-\rho} + Be^{\rho}$, where $B=0$, since $e^{\rho} \to \infty$ as $\rho \to \infty$.

so $u(\rho) \sim Ae^{-\rho}$ for ~~[illegible]~~ large ρ.

As $\rho \to 0$, $\frac{d^2u}{d\rho^2} = \frac{l(l+1)}{\rho^2}u$

The general solution $u(\rho) = C\rho^{l+1} + D\rho^{-l}$, where $D=0$, since $\rho^{-l} \to \infty$ as $\rho \to 0$.

So $u(\rho) \sim C\rho^{l+1}$ for small ρ

To peal off asymptotic behavior, introduce $v(\rho)$:

$$u(\rho) = \rho^{l+1}e^{-\rho}v(\rho) \quad \boxed{6}$$

~~$\boxed{6} \to \boxed{5}$~~ $\cancel{\frac{d^2u}{d\rho^2} = \left[1 - \frac{\rho_0}{\rho} + \frac{l(l+1)}{\rho^2}\right]\rho^{l+1}e^{-\rho}v(\rho)}$

take its derivative,

$$\frac{du}{d\rho} = (l+1)\rho^{l}e^{-\rho}v(\rho) + \rho^{l+1}\left[-e^{-\rho}v(\rho) + e^{-\rho}\frac{dv}{d\rho}\right]$$
$$= \rho^{l}e^{-\rho}\left[(l+1-\rho)v(\rho) + \rho\frac{dv}{d\rho}\right] \quad \boxed{7}$$

then its second derivative is

$$\frac{d^2u}{d\rho^2} = \left[(l+1-\rho)v + \rho\frac{dv}{d\rho}\right]\left[l\rho^{l-1}e^{-\rho} - e^{-\rho}\rho^l\right] + \left[\rho^l e^{-\rho}\right]\left[(l+1)\frac{dv}{d\rho} - \left[v+\rho\frac{dv}{d\rho}\right] + \left[\rho\frac{d^2v}{d\rho^2} + \frac{dv}{d\rho}\right]\right]$$

$$= \left[(l+1-\rho)v + \rho\frac{dv}{d\rho}\right]\left(\frac{l}{\rho}-1\right)\left(\rho^l e^{-\rho}\right) + \left[\rho^l e^{-\rho}\right]\left[-v + (l+2-\rho)\frac{dv}{d\rho} + \rho\frac{d^2v}{d\rho^2}\right]$$

$$= \left[\rho^l e^{-\rho}\right]\cdot\left[\left(\frac{l^2}{\rho} - l + \frac{l}{\rho} - 1 - l + \rho\right)v + (l-\rho)\frac{dv}{d\rho} - v + (l+2-\rho)\frac{dv}{d\rho} + \rho\frac{d^2v}{d\rho^2}\right]$$

$$= \rho^l e^{-\rho}\left[\left[-2l-2+\rho+\frac{l(l+1)}{\rho}\right]v + 2(l+1-\rho)\frac{dv}{d\rho} + \rho\frac{d^2v}{d\rho^2}\right] \qquad \boxed{8}$$

$\boxed{8} \to \boxed{5}$:

$$\left[1 - \frac{\rho_0}{\rho} + \frac{l(l-1)}{\rho^2}\right]u = \rho^l e^{-\rho}\left[\left[-2l-2+\rho+\frac{l(l+1)}{\rho}\right]v + 2(l+1-\rho)\frac{dv}{d\rho} + \rho\frac{d^2v}{d\rho^2}\right]$$

$$\left[1 - \frac{\rho_0}{\rho} + \frac{l(l-1)}{\rho^2}\right]\rho\cdot\rho^l e^{-\rho}v = \rho^l e^{-\rho}\left[\left[-2l-2+\rho+\frac{l(l+1)}{\rho}\right]v + 2(l+1-\rho)\frac{dv}{d\rho} + \rho\frac{d^2v}{d\rho^2}\right]$$

$$\left[\rho - \rho_0 + \frac{l(l-1)}{\rho}\right]\rho^l e^{-\rho}v = \ldots$$

$$0 = \cancel{\rho^l e^{-\rho}}\left[\left[\rho_0 - 2l - 2\right]v + 2(l+1-\rho)\frac{dv}{d\rho} + \rho\frac{d^2v}{d\rho^2}\right] \qquad \boxed{9}$$

Assume $v(\rho)$ is a polynimial :

$$v(\rho) = \sum_{j=0}^{\infty} c_j \rho^j \qquad \boxed{10}$$

its derivative is

$$\frac{dv}{d\rho} = \sum_{j=0}^{\infty} c_j \cdot j\,\rho^{j-1} = \underbrace{c_0\cdot 0\cdot\rho^{-1}}_{0} + c_1\cdot 1\cdot\rho^0 + c_2\cdot 2\cdot\rho^1 + \ldots$$

$$= \sum_{j=0}^{\infty} c_{j+1}(j+1)\rho^j \qquad \boxed{11}$$

and its second derivative is

$$\frac{d^2v}{d\rho^2} = \sum_{j=0}^{\infty} c_{j+1}(j+1)\,j\,\rho^{j-1} \; \cancel{= c_1\cdot 1\cdot 0\cdot\rho^{-1} + c_2\cdot 2\cdot 1\,\rho^0 + \ldots} \qquad \boxed{12}$$

$\boxed{10}\boxed{11}\boxed{12} \to \boxed{9}$:

$$0 = \left[\rho_0 - 2l - 2\right]\sum_{j=0}^{\infty} c_j\rho^j + 2(l+1-\rho)\sum_{j=0}^{\infty} c_{j+1}(j+1)\rho^j + \rho\sum_{j=0}^{\infty} c_{j+1}(j+1)j\,\rho^{j-1}$$

$$0 = \left[\rho_0 - 2(l+1)\right]\sum_{j=0}^{\infty} c_j\rho^j + 2(l+1)\sum_{j=0}^{\infty} c_{j+1}(j+1)\rho^j - 2\sum_{j=0}^{\infty} c_{j+1}(j+1)\rho^{j+1} + \sum_{j=0}^{\infty} c_{j+1}(j+1)j\,\rho^j$$

$$0 = \left[\rho_0 - 2(l+1)\right]\sum_{j=0}^{\infty} c_j\rho^j + 2(l+1)\sum_{j=0}^{\infty} c_{j+1}(j+1)\rho^j - 2\sum_{j=0}^{\infty} j\,c_j\rho^j + \sum_{j=0}^{\infty} c_{j+1}(j+1)j\,\rho^j$$

1 April 2019 4.2.1 The Radial Wave function Day 14

Make use of the property that $\sum_{j=0}^{\infty} c_j \rho^j = 0$, where $c_j = 0$,

$$[\rho_0 - 2(l+1)]c_j + 2(l+1)c_{j+1}(j+1) - 2jc_j + c_{j+1}(j+1)j = 0$$

Isolate c_{j+1}:

$$c_{j+1} = \frac{2(l+1+j)-\rho_0}{2(l+1)(j+1)+(j+1)j} c_j = \frac{2(l+1+j)-\rho_0}{(j+1)(2l+2+j)} c_j \quad \boxed{13}$$

As $j \to \infty$ ($\rho \to \infty$):

$$c_{j+1} \cong \frac{2j}{(j+1)j} c_j = \frac{2}{j+1} c_j \quad \boxed{14}$$

$$\cancel{c_j = \frac{j+1}{2} c_{j+1}}$$

so

$$c_1 = \frac{2}{0+1} c_0 = \frac{2}{1} c_0$$

$$c_2 = \frac{2}{1+1} c_1 = \frac{2}{2} c_1 = \frac{2}{2}\frac{2}{1} c_0$$

$$c_3 = \frac{2}{2+1} c_2 = \frac{2}{3} c_2 = \frac{2}{3}\frac{2}{2}\frac{2}{1} c_0$$

to generalize,

$$c_j = \frac{2^j}{j!} c_0 \quad \boxed{15}$$

$\boxed{15} \to \boxed{10}$:

$$v(\rho) = \sum_{j=0}^{\infty} \frac{2^j}{j!} c_0 \rho^j = c_0 \sum_{j=0}^{\infty} \frac{(2\rho)^j}{j!} = c_0 e^{2\rho} \quad \boxed{16}$$

(since $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$)

$\boxed{16} \to \boxed{6}$:

$$u(\rho) = \rho^{l+1} e^{-\rho} (c_0 e^{2\rho})$$

$$= c_0 \rho^{l+1} e^{\rho} \quad \boxed{17}$$

Since as $\rho \to \infty$, $u(\rho) \to \infty$, the series have to terminate. Let j_{max} be maximum integer:

$$c_{(j_{max}+1)} = 0 \quad \boxed{18}$$

$\boxed{13}$:

$$0 = c_{(j_{max}+1)} = \frac{2(l+1+j_{max}) - \rho_0}{(j_{max}+1)(2l+2+j_{max})} c_{j_{max}}$$

$$2(l+1+j_{max}) - \rho_0 = 0 \quad \boxed{19}$$

Define $n \equiv j_{max} + l + 1$, $\boxed{19}$:

$$2n - \rho_0 = 0 \quad \Rightarrow \quad 2n = \rho_0 \qquad \boxed{20}$$

To determine the Energy E, ~~observe boundary conditions~~ by definition of K and ρ_0,

$$\left(\begin{aligned} K &\equiv \frac{\sqrt{-2mE}}{\hbar} \\ K^2 &= \frac{-2mE}{\hbar^2} \\ E &= -\frac{\hbar^2 K^2}{2m} \end{aligned} \right)$$

$$\rho_0 = \frac{me^2}{2\pi \epsilon_0 \hbar^2 K} = \frac{me^2}{2\pi \epsilon_0 \hbar^2} \frac{\hbar}{\sqrt{-2mE}}$$

$$\rho_0^2 = \frac{m^2 e^4}{4\pi^2 \epsilon_0^2 \hbar^2} \cdot \frac{1}{-2mE}$$

$$E = \frac{m^{\cancel{2}} e^4}{-8 \cancel{m} \pi^2 \epsilon_0^2 \hbar^2 \rho_0^2} = -\frac{me^4}{8\pi^2 \epsilon_0^2 \hbar^2 \rho_0^2} \qquad \boxed{21}$$

So the allowed energies are: $\boxed{20} \rightarrow \boxed{21}$:

$$E_n = -\frac{me^4}{8\pi^2 \epsilon_0^2 \hbar^2} \frac{1}{4n^2} \qquad (n = 1, 2, 3, \ldots)$$

$$= -\left[\frac{m}{2\hbar^2} \left(\frac{e^2}{4\pi \epsilon_0 \cancel{\hbar}} \right)^2 \right] \frac{1}{n^2} \qquad \boxed{22}$$

when $n = 1$,

$$E_1 = -\left[\frac{m}{2\hbar^2} \left(\frac{e^2}{4\pi \epsilon_0} \right)^2 \right] \qquad \boxed{23}$$

$\boxed{23} \rightarrow \boxed{22}$

$$E_n = \frac{E_1}{n^2}, \quad (n = 1, 2, 3, \ldots) \qquad \boxed{24}$$

4.10 Known recursion formula $c_{j+1} = \frac{2(j+l+1-n)}{(j+1)(j+2l+2)} c_j$

$R_{nl}(r) = \frac{1}{r}\rho^{l+1} e^{-\rho} v(\rho)$, where $\rho = \frac{r}{an}$, $v(\rho) = \sum_{j=0}^{\infty} c_j \rho^j$, $j_{max} = n-l-1$

▸ $R_{30}(r) = \frac{1}{r}\left(\frac{r}{3a}\right)^1 e^{-\frac{r}{3a}} \sum_{j=0}^{\infty} c_j \rho^j$ $\qquad j_{max} = 3-0-1 = 2$.

When $j=0$, $c_1 = \frac{2(0+0+1-3)}{2} c_0 = -2c_0$

When $j=1$, $c_2 = \frac{2(1+0+1-3)}{2(3)} c_1 = -\frac{1}{3}c_1$

When $j=2$, $c_3 = \frac{2(2+0+1-3)}{3(4)} c_2 = 0$, so $v(\rho) = c_0\rho^0 + c_1\rho^1 + c_2\rho^2$

$= c_0 - 2c_0\rho + \frac{2}{3}c_0\rho^2$

$R_{30}(r) = \frac{1}{r}\frac{r}{3a} e^{-\frac{r}{3a}} (c_0 - 2c_0\rho + \frac{2}{3}c_0\rho^2)$

$R_{30}(r) = \frac{1}{3a} e^{-\frac{r}{3a}} \left(c_0 - 2c_0\frac{r}{an} + \frac{2}{3}c_0\frac{r^2}{a^2n^2}\right)$

$$\boxed{R_{30}(r) = \frac{1}{3a} e^{-\frac{r}{3a}} \left(c_0 - 2c_0\frac{r}{3a} + \frac{2}{27}c_0\frac{r^2}{a^2}\right)}$$

▸ $R_{31}(r) = \frac{1}{r}\left(\frac{r}{3a}\right)^2 e^{-\frac{r}{3a}} \sum_{j=0}^{\infty} c_j \rho^j$ $\qquad j_{max} = 3-1-1 = 1$

When $j=0$, $c_1 = \frac{2(0+1+1-3)}{4} c_0 = -\frac{1}{2}c_0$

When $j=1$, $c_2 = \frac{2(1+1+1-3)}{2(5)} c_1 = 0$

so $v(\rho) = c_0\rho^0 + c_1\rho^1 = c_0 - \frac{1}{2}c_0\rho$

$$\boxed{R_{31}(r) = \frac{r}{9a^2} e^{-\frac{r}{3a}} \left(c_0 - \frac{1}{6}c_0\frac{r}{a}\right)}$$

▸ $R_{32}(r) = \frac{1}{r}\left(\frac{r}{3a}\right)^3 e^{-\frac{r}{3a}} \sum_{j=0}^{\infty} c_j \rho^j$ $\qquad j_{max} = 3-2-1 = 0$

When $j=0$, $c_1 = \frac{2(0+2+1-3)}{6} c_0 = 0$

so $v(\rho) = c_0\rho^0 = c_0$

$$\boxed{R_{32} = \frac{r^2}{27a^3} e^{-\frac{r}{3a}} c_0}$$

4.11 (a) Normalize $R_{20}(r) = \frac{C_0}{2a}\left(1 - \frac{r}{2a}\right)e^{-\frac{r}{2a}}$, and construct ψ_{200}.

$$\int_0^\infty |R|^2 r^2\, dr = 1$$

$$\int_0^\infty \frac{C_0^2}{4a^2}\left(1-\frac{r}{2a}\right)^2 e^{-\frac{r}{a}} r^2\, dr = 1$$

$$\frac{C_0^2}{4a^2}\int_0^\infty \left(1 - \frac{r}{a} + \frac{r^2}{4a^2}\right) e^{-\frac{r}{a}} r^2\, dr = 1$$

$$\frac{C_0^2}{4a^2}\left[\int_0^\infty e^{-\frac{r}{a}} r^2\, dr - \int_0^\infty e^{-\frac{r}{a}} \frac{r^3}{a}\, dr + \int_0^\infty e^{-\frac{r}{a}} \frac{r^4}{4a^2}\, dr\right] = 1$$

$$\boxed{\begin{aligned} &\int_0^\infty e^{-x}\, dx = 1 \\ &\int_0^\infty e^{-x} x\, dx = 1 \\ &\int_0^\infty e^{-x} x^2\, dx = 2 \\ &\int_0^\infty e^{-x} x^3\, dx = 6 \\ &\int_0^\infty e^{-x} x^4\, dx = 24 \\ &\int_0^\infty e^{-x} x^n\, dx = n! \end{aligned}}$$

$u = +\frac{r}{a}$ $\Rightarrow r = +ua$

$\frac{du}{dr} = +\frac{1}{a}$ $r^2 = u^2 a^2$

$dr = +a\, du$

$$\frac{C_0^2}{4a^2}\left[\int_0^\infty e^{-u} u^2 a^2 \cdot a\, du - \int_0^\infty e^{-u} \frac{u^3 a^3}{a} a\, du + \int_0^\infty e^{-u} \frac{u^4 a^4}{4a^2}\cdot a\, du\right] = 1$$

$f(x) = u^2$ $g'(x) = e^{-u}$ $f(x) = u^3$ $g'(x) = e^{-u}$ $f(x) = u^4$ $g'(x) = e^{-u}$

$f'(x) = 2u$ $g(x) = -e^{-u}$ $f'(x) = 3u^2$ $g(x) = -e^{-u}$ $f'(x) = 4u^3$ $g(x) = -e^{-u}$

$$\frac{C_0^2}{4a^2}\left[a^3\left[\left[-u^2 \cancel{e^{-u}}\right]_0^\infty - \int_0^\infty -2ue^{-u}\, du\right] - a^3\left[\left[-u^3 \cancel{e^{-u}}\right]_0^\infty - \int_0^\infty -3u^2 e^{-u}\, du\right] + \frac{a^3}{4}\left[\left[-u^4 \cancel{e^{-u}}\right]_0^\infty - \int_0^\infty -4u^3 e^{-u}\, du\right]\right]$$

$f(x) = u$ $g'(x) = e^{-u}$

$f'(x) = 1$ $g(x) = -e^{-u}$

$$\frac{C_0^2 a}{4}\left[2\left[\left[-ue^{-u}\right]_0^\infty - \int_0^\infty -e^{-u}\, du\right] - 6 + \frac{24}{4}\right] = 1$$

$$\frac{C_0^2 a}{4}\left[2 \cdot 1 - 6 + \frac{24}{4}\right] = 1$$

$$\frac{a C_0^2}{2} = 1$$

$$C_0 = \sqrt{\frac{2}{a}}$$

$$R_{20}(r) = \frac{1}{\sqrt{2}} a^{\frac{3}{2}} \left(1 - \frac{r}{2a}\right) e^{-\frac{r}{2a}}$$

$$\psi_{nlm} = R_{nl}(r)\, Y_l^m(\theta,\phi)$$

$$\psi_{200} = R_{20}(r)\, Y_0^0(\theta,\phi) \qquad \left(\text{By chart, } Y_0^0(\theta,\phi) = \left(\frac{1}{4\pi}\right)^{\frac{1}{2}}\right)$$

$$= \left(\frac{1}{4\pi}\right)^{\frac{1}{2}} \frac{1}{\sqrt{2}} a^{\frac{3}{2}} \left(1 - \frac{r}{2a}\right) e^{-\frac{r}{2a}}$$

4.11 (b) Normalize $R_{21} = \frac{c_0}{4a^2} r e^{\frac{-r}{2a}}$, and construct ψ_{210}, ψ_{211}, ψ_{21-1}.

$$1 = \int_0^\infty \frac{c_0^2}{16a^4} r^2 e^{-\frac{r}{a}} r^2 \, dr$$

$u = +\frac{r}{a} \rightarrow r = ua$

$\frac{du}{dr} = \frac{1}{a}$

$dr = a\, du$

$$1 = \frac{c_0^2}{16a^4} \int_0^\infty e^{-\frac{r}{a}} r^4 \, dr$$

$$1 = \frac{c_0^2}{16a^4} \int_0^\infty e^{-u} (ua)^4 \cdot a \, du$$

$$1 = \frac{c_0^2 a}{16} \int_0^\infty e^{-u} u^4 du$$

$$1 = \frac{c_0^2 a}{16} \cdot 24$$

$$c_0^2 = \frac{2}{3a}$$

so $R_{21} = \sqrt{\frac{2}{3a}} \frac{1}{4a^2} r e^{-\frac{r}{2a}} = \frac{1}{\sqrt{24}} a^{\frac{3}{2}} \frac{r}{a} e^{-\frac{r}{2a}}$

$$c_0 = \sqrt{\frac{2}{3a}}$$

By chart, $Y_1^0 = \left(\frac{3}{4\pi}\right)^{\frac{1}{2}} \cos\theta$

$Y_1^1 = -\left(\frac{3}{8\pi}\right)^{\frac{1}{2}} \sin\theta \, e^{i\phi}$

$Y_1^{-1} = \left(\frac{3}{8\pi}\right)^{\frac{1}{2}} \sin\theta \, e^{-i\phi}$

$$\psi_{200} = R_{21} Y_1^0 = \left(\frac{3}{4\pi}\right)^{\frac{1}{2}} \frac{1}{\sqrt{24}} a^{\frac{3}{2}} \cos\theta \frac{r}{a} e^{-\frac{r}{2a}}$$

$$\psi_{211} = R_{21} Y_1^1 = -\left(\frac{3}{8\pi}\right)^{\frac{1}{2}} \frac{1}{\sqrt{24}} a^{\frac{3}{2}} \sin\theta \, e^{i\phi} \frac{r}{a} e^{-\frac{r}{2a}}$$

$$\psi_{21-1} = R_{21} Y_1^{-1} = \left(\frac{3}{8\pi}\right)^{\frac{1}{2}} \frac{1}{\sqrt{24}} a^{\frac{3}{2}} \sin\theta \, e^{-i\phi} \frac{r}{a} e^{-\frac{r}{2a}}$$

4.13 (a) The electron in ground state in hydrogen atom has wave function of

$$\psi_{100}(r,\theta,\phi) = \frac{1}{\sqrt{\pi a^3}} e^{-\frac{r}{a}}$$

$$\langle r \rangle = \int_0^\infty \psi^* r \psi \, dr = \cancel{\int_0^\infty \left(\frac{1}{\sqrt{\pi a^3}}\right)^2 r \, dr = \frac{1}{\pi a^3}\int_0^\infty r \, dr = \frac{1}{\pi a^3}\left[\frac{1}{2} r^2\right]_0^\infty}$$

$$= \int_0^\infty \frac{1}{\pi a^3} e^{-\frac{2r}{a}} r \, dr$$

$f(x) = r$ $\quad g'(x) = e^{-\frac{2r}{a}}$ $\quad u = -\frac{2r}{a}$

$f'(x) = 1$ $\quad g(x) = \int e^{-\frac{2r}{a}} dr$ $\quad \frac{du}{dr} = -\frac{2}{a}$

$= -\frac{a}{2} e^{-\frac{2r}{a}}$ $\quad dr = -\frac{a\,du}{2}$

$$= \frac{1}{\pi a^3}\left[\left[-\frac{a}{2} r e^{-\frac{2r}{a}}\right]_0^\infty - \int_0^\infty -\frac{a}{2} e^{-\frac{2r}{a}} dr\right]$$

$$= \frac{1}{\pi a^3} \frac{a}{2} \int_0^\infty e^{-\frac{2r}{a}} dr$$

$$= \frac{1}{2\pi a^2}\left[-\frac{a}{2} e^{-\frac{2r}{a}}\right]_0^\infty$$

$$= \frac{1}{2\pi a^2}\left[\frac{a}{2}\right]$$

$$= \frac{1}{4\pi a}$$

The Bohr radius is $a \equiv \frac{4\pi\epsilon_0 \hbar^2}{me^2}$

$$\langle r^n \rangle = \int \psi^* r^n \psi \, r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$= \int \psi^2 r^{n+2} \sin\theta \, dr \, d\theta \, d\phi$$

$$= \int_0^\infty \frac{1}{\pi a^3} e^{-\frac{2r}{a}} r^{n+1} dr \int_0^\pi \sin^2\theta \, d\theta \int_0^{2\pi} d\phi$$

$\cos(2x) = 1 - 2\sin^2 x$

$\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos(2x)$

$$= \int_0^\infty \frac{1}{\pi a^3} e^{-\frac{2r}{a}} r^{n+1} dr \cdot \int_0^\pi \frac{1}{2} - \frac{1}{2}\cos(2\theta) \, d\theta \cdot [\phi]_0^{2\pi}$$

$$= \frac{2\pi}{\pi a^3} \int_0^\infty e^{-\frac{2r}{a}} r^{n+1} dr \left[\frac{\pi}{2} - \frac{1}{2}\int_0^\pi \cos(2\theta) \, d\theta\right]$$

$u = 2\theta$

$\frac{du}{2} = d\theta$

$$= \frac{2\pi}{\pi a^3} \int_0^\infty e^{-\frac{2r}{a}} r^{n+1} dr \left[\frac{\pi}{2} - \frac{1}{4}\left[\sin u\right]_0^{2\pi}\right]$$

$$= \frac{\pi}{\pi a^3}$$

10 April 2019

4.2 The Hydrogen Atom

Day 17

4.13 (a) cont.

$$\langle r^n\rangle = \int \psi^* r^n \psi \, r^2 \sin\theta \, dr\, d\theta\, d\phi$$

$$= \frac{1}{\pi a^3}\int e^{-\frac{2r}{a}} r^{n+2} \sin\theta \, dr\, d\theta\, d\phi$$

$$= \frac{1}{\pi a^3}\int_0^\infty e^{-\frac{2r}{a}} r^{n+2}\, dr \int_0^\pi \sin\theta\, d\theta \int_0^{2\pi} d\phi$$

$$= \frac{1}{\pi a^3}\int_0^\infty e^{-\frac{2r}{a}} r^{n+2}\, dr\,(1+1)(2\pi-0)$$

$$= \frac{4\pi}{\pi a^3}\int_0^\infty e^{-\frac{2r}{a}} r^{n+2}\, dr$$

$$\int_0^\infty e^{-\frac{2r}{a}}\, dr = +\frac{a}{2} = \frac{a}{2}\cdot 0! = \left(\frac{a}{2}\right)^1\cdot 0!$$

$$\int_0^\infty e^{-\frac{2r}{a}} r\, dr = +\frac{a^2}{4} = \frac{a}{2}\cdot 1!\cdot\frac{a}{2} = \left(\frac{a}{2}\right)^2 1!$$

$$\int_0^\infty e^{-\frac{2r}{a}} r^2\, dr = \frac{a^3}{4} = \frac{a^2}{4}\cdot 2!\cdot\frac{a}{2} = \left(\frac{a}{2}\right)^3 2!$$

$$\int_0^\infty e^{-\frac{2r}{a}} r^3\, dr = \frac{3a^4}{4} = \frac{a^3}{4}\cdot 3!\cdot\frac{a}{2} = \left(\frac{a}{2}\right)^4 3!$$

$$\int_0^\infty e^{-\frac{2r}{a}} r^n\, dr \cancel{= \int_0^\infty e^{-\frac{2r}{a}} r\, dr\, \frac{a}{2}\, n!} = \left(\frac{a}{2}\right)^{n+1} n!$$

$n=0$: $\langle r^0\rangle = \frac{4\pi}{\pi a^3}\int_0^\infty e^{-\frac{2r}{a}} r^2\, dr$

$f(x) = r^2$, $f'(x) = 2r$; $g'(x) = e^{-\frac{2r}{a}}$, $g(x) = \int e^{-\frac{2r}{a}} dr = -\frac{a}{2}e^{-\frac{2r}{a}}$

$u = -\frac{2r}{a}$, $\frac{du}{dr} = -\frac{2}{a}$, $dr = -\frac{a\,du}{2}$

$$= \frac{4\pi}{\pi a^3}\left[\left[-\frac{a}{2}r^2 e^{-\frac{2r}{a}}\right]_0^\infty - \int_0^\infty -2r\frac{a}{2}e^{-\frac{2r}{a}}\, dr\right]$$

$$= \frac{4\pi a}{\pi a^3}\int_0^\infty r e^{-\frac{2r}{a}}\, dr$$

$f(x) = r$, $f'(x) = 1$; $g'(x) = e^{-\frac{2r}{a}}$, $g(x) = -\frac{a}{2}e^{-\frac{2r}{a}}$

$$= \frac{4}{a^2}\left[\left[-\frac{a}{2}re^{-\frac{2r}{a}}\right]_0^\infty - \int_0^\infty -\frac{a}{2}e^{-\frac{2r}{a}}\, dr\right]$$

$$= \frac{2}{a}\int_0^\infty e^{-\frac{2r}{a}}\, dr$$

$$= \frac{2}{a}\left[-\frac{a}{2}e^{-\frac{2r}{a}}\right]_0^\infty$$

$$= -[0-1]$$

$$= 1$$

$n=1$: $\langle r\rangle = \frac{4\pi}{\pi a^3}\int_0^\infty e^{-\frac{2r}{a}} r^3\, dr$

$f(x) = r^3$, $f'(x) = 3r^2$; $g'(x) = e^{-\frac{2r}{a}}$, $g(x) = -\frac{a}{2}e^{-\frac{2r}{a}}$

$$= \frac{4}{a^3}\left[\left[r^3\frac{a}{2}e^{-\frac{2r}{a}}\right]_0^\infty - \int -\frac{a}{2}e^{-\frac{2r}{a}} 3r^2 dr\right]$$

$$= \frac{2\cdot 3}{a^2}\int e^{-\frac{2r}{a}} r^2\, dr$$

$$= \frac{6}{a^2}\cdot\frac{a^3}{4}$$

$$= \frac{3a}{2}$$

$n=2$: $\langle r^2\rangle = \frac{4\pi}{\pi a^3}\int_0^\infty e^{-\frac{2r}{a}} r^4\, dr$

$$= \frac{4}{a^3}\cdot\cancel{\left(\frac{3a^4}{4}\cdot\frac{24a}{2}\right)}\left(\left(\frac{a}{2}\right)^5 4!\right)$$

$$\Rightarrow \cancel{9a^2},\ \cancel{36a^2}\ 3a^2$$

(b) $r^2 = x^2 + y^2 + z^2 \Rightarrow r^2 = 3x^2$

$x = y = z \qquad \frac{r^2}{3} = x^2 \qquad \langle x^2 \rangle = \frac{\langle r^2 \rangle}{3} = \frac{3a^2}{3} = a^2$

$\cancel{S = r = \sqrt{3x^2}} \qquad \frac{r}{\sqrt{3}} = \cancel{x} \qquad \cancel{\langle x \rangle = \frac{\langle r \rangle}{\sqrt{3}} = \frac{3a}{2}\frac{1}{\sqrt{3}} =}$

$x = r \sin\theta \cos\phi$

$\langle x \rangle = \int \psi^* \, r\sin\theta\cos\phi \, \psi^* \, dr\,d\theta\,d\phi \, r^2 \sin\theta$

$= \frac{1}{\pi a^3} \int e^{-\frac{2r}{a}} r^2 \sin^2\theta \cos\phi \, dr\,d\theta\,d\phi$

$= \frac{1}{\pi a^3} \int_0^\infty e^{-\frac{2r}{a}} r^2 \, dr \int_0^{\pi} \sin^2\theta \, d\theta \int_0^{2\pi} \cos\phi \, d\phi$

$= \frac{1}{\pi a^3} \frac{a^3}{4} \frac{\pi}{2} (0-0)$

$= 0$

$\langle x^2 \rangle = \int \psi^* \, r \sin^2$

(c) $\psi_{211} = R_{21} Y_1^1 = \frac{1}{\sqrt{24}} a^{-\frac{3}{2}} \frac{r}{a} e^{-\frac{r}{2a}} \left(-\left(\frac{3}{8\pi}\right)^{\frac{1}{2}} \sin\theta e^{i\phi}\right)$

$= -\frac{1}{\sqrt{24}} \left(\frac{3}{8\pi}\right)^{\frac{1}{2}} a^{-\frac{3}{2}} \frac{r}{a} e^{-\frac{r}{2a}} \sin\theta e^{i\phi}$

$\langle x^2 \rangle = \int \psi^* \, r^2 \sin^2\theta \cos^2\phi \, \psi \, (r^2 \sin\theta \, dr\,d\theta\,d\phi)$

$= \int \frac{1}{24} \frac{3}{8\pi} a^{-3} \frac{r^2}{a^2} e^{-\frac{r}{a}} \sin^2\theta \; r^2 \sin^2\theta \cos^2\phi \, r^2 \sin\theta \, dr\,d\theta\,d\phi$

$= \frac{1}{64\pi a^5} \int r^6 e^{-\frac{r}{a}} \sin^5\theta \cos^2\phi \, dr\,d\theta\,d\phi$

$= \frac{1}{64\pi a^5} \int_0^\infty r^6 e^{-\frac{r}{a}} dr \int_0^{\pi} \sin^5\theta \, d\theta \int_0^{2\pi} \cos^2\phi \, d\phi$

$= \frac{1}{64\pi a^5} (6!\, a^7) \cancel{\int_0^\pi \sin\theta (1-\cos^2\theta) d\theta} \left(\frac{16}{15}\right)(\pi) = 12a^2$

$\int_0^\pi \sin^5\theta \, d\theta = \int_0^\pi \sin\theta (1-\cos^2\theta)^2 d\theta \qquad u = \cos\theta \qquad u(0) = 1$

$\frac{du}{-\sin\theta} = d\theta \qquad u(\pi) = -1$

$= \int_1^{-1} \cancel{\sin\theta} (1-u^2)^2 \frac{du}{-\cancel{\sin\theta}}$

$= -\int_{+1}^{-1} 1 - 2u^2 + u^4 \, du$

$= -\left[u - \frac{2}{3}u^3 + \frac{1}{5}u^5\right]_{+1}^{-1}$

$= -2\left(-1 + \frac{2}{3} - \frac{1}{5}\right)$

$= \frac{16}{15}$

$\int_0^{2\pi} \cos^2\phi \, d\phi = \int_0^{2\pi} 1 - \sin^2\phi \, d\phi$

$= 2\pi - \int_0^{2\pi} \sin^2\phi \, d\phi$

$f(x) = \sin\phi \qquad g'(x) = \sin\phi$

$f'(x) = \cos\phi \qquad g(x) = -\cos\phi$

$= 2\pi - \left[\left[-\sin\phi \cancel{\cos\phi}\right]_0^\pi - \int_0^\pi -\cos^2\phi \, d\phi\right]$

$= 2\pi - \int_0^\pi \cos^2\phi \, d\phi$

$\Rightarrow 2\int_0^{2\pi} \cos^2\phi \, d\phi = 2\pi$

$\int_0^{2\pi} \cos^2\phi \, d\phi = \pi$

4.19 Known $[r_i, p_i] = -[p_i, r_j] = i\hbar\delta_{ij}$, $[r_i, r_j] = [p_i, p_j] = 0$

[a] (a) $[L_z, x] = L_z x - x L_z = \left(x\frac{\hbar}{i}\frac{\partial}{\partial y} - y\frac{\hbar}{i}\frac{\partial}{\partial x}\right)x - x\left(x\frac{\hbar}{i}\frac{\partial}{\partial y} - y\frac{\hbar}{i}\frac{\partial}{\partial x}\right)$

$$\cancel{x\frac{\hbar}{i}\frac{\partial (fx)}{\partial y} - y\frac{\hbar}{i}\frac{\partial f}{\partial y}} = x\frac{\hbar}{i}\frac{\partial}{\partial y}(f\cdot x) - y\frac{\hbar}{i}\frac{\partial}{\partial x}(f\cdot x) - x^2\frac{\hbar}{i}\frac{\partial f}{\partial y} + xy\frac{\hbar}{i}\frac{\partial f}{\partial x}$$

$$= x\frac{\hbar}{i}\cancel{\frac{\partial f}{\partial y}}x + x\frac{\hbar}{i}\cancel{f\frac{\partial x}{\partial y}}^{0} - y\frac{\hbar}{i}\cancel{\frac{\partial f}{\partial x}}\cdot x - y\frac{\hbar}{i}f\cdot\frac{\partial x}{\partial x} - x^2\cancel{\frac{\hbar}{i}\frac{\partial f}{\partial y}} + xy\cancel{\frac{\hbar}{i}\frac{\partial f}{\partial x}}$$

$$= -\frac{\hbar}{i}\left(\cancel{xy\frac{\partial f}{\partial x}} + yf \cancel{x^2\frac{\partial f}{\partial y}} \cancel{xy\frac{\partial f}{\partial x}}\right)$$

$$= i\hbar yf$$

$$= i\hbar y$$

(b) $[L_z, y] = L_z y - y L_z = \left(x\frac{\hbar}{i}\frac{\partial}{\partial y} - y\frac{\hbar}{i}\frac{\partial}{\partial x}\right)y - y\left(x\frac{\hbar}{i}\frac{\partial}{\partial y} - y\frac{\hbar}{i}\frac{\partial}{\partial x}\right)$

$$= x\frac{\hbar}{i}\cancel{\frac{\partial f}{\partial y}}\cdot y + x\frac{\hbar}{i}f - y\frac{\hbar}{i}\cancel{\frac{\partial f}{\partial x}}y - y\frac{\hbar}{i}\cancel{f\frac{\partial y}{\partial x}}^{0} - xy\frac{\hbar}{i}\cancel{\frac{\partial f}{\partial y}} + y^2\cancel{\frac{\hbar}{i}\frac{\partial f}{\partial x}}$$

$$= x\frac{\hbar}{i}f$$

$$= -i\hbar x$$

(c) $[L_z, z] = L_z z - z L_z = \left(x\frac{\hbar}{i}\frac{\partial}{\partial y} - y\frac{\hbar}{i}\frac{\partial}{\partial x}\right)z - z\left(x\frac{\hbar}{i}\frac{\partial}{\partial y} - y\frac{\hbar}{i}\frac{\partial}{\partial x}\right)$

$$= x\frac{\hbar}{i}\cancel{\frac{\partial f}{\partial y}}z + x\frac{\hbar}{i}\cancel{f\frac{\partial z}{\partial y}}^{0} - y\frac{\hbar}{i}\cancel{\frac{\partial f}{\partial x}}z - y\frac{\hbar}{i}\cancel{f\frac{\partial z}{\partial x}}^{0} - zx\cancel{\frac{\hbar}{i}\frac{\partial f}{\partial y}} + zy\cancel{\frac{\hbar}{i}\frac{\partial f}{\partial x}}$$

$$= 0$$

(d) $[L_z, p_x] = L_z p_x - p_x L_z = \left(x\frac{\hbar}{i}\frac{\partial}{\partial y} - y\frac{\hbar}{i}\frac{\partial}{\partial x}\right)\frac{\hbar}{i}\frac{\partial}{\partial x} - \frac{\hbar}{i}\frac{\partial}{\partial x}\left(x\frac{\hbar}{i}\frac{\partial}{\partial y} - y\frac{\hbar}{i}\frac{\partial}{\partial x}\right)$

$$= x\left(\frac{\hbar}{i}\right)^2\frac{\partial}{\partial y}\left(f\cdot\frac{\partial f}{\partial x}\right) - y\left(\frac{\hbar}{i}\right)^2\frac{\partial}{\partial x}\left(f\cdot\frac{\partial f}{\partial x}\right) - \left(\frac{\hbar}{i}\right)^2\frac{\partial}{\partial x}\left(fx\frac{\partial f}{\partial y}\right) + \left(\frac{\hbar}{i}\right)^2\frac{\partial}{\partial x}\left(fy\frac{\partial f}{\partial x}\right)$$

$$= \left(\frac{\hbar}{i}\right)^2\Big[x\frac{\partial f}{\partial y}\cancel{\frac{\partial f}{\partial x}} + xf\cancel{\frac{\partial^2 f}{\partial y\partial x}} - y\cancel{\frac{\partial f}{\partial x}}\cancel{\frac{\partial f}{\partial x}} - yf\cancel{\frac{\partial^2 f}{\partial x^2}} - \frac{\partial f}{\partial x}\cancel{x\frac{\partial f}{\partial y}} - f\frac{\partial f}{\partial y} - fx\cancel{\frac{\partial^2 f}{\partial x\partial y}}$$
$$+ \frac{\partial f}{\partial x}\cancel{y\frac{\partial f}{\partial x}} + f\cancel{\frac{\partial y}{\partial x}}^{0}\frac{\partial f}{\partial x} + fy\cancel{\frac{\partial^2 f}{\partial x^2}}\Big]$$

$$= -\left(\frac{\hbar}{i}\right)^2 f\frac{\partial f}{\partial y}$$

$$= -\frac{\hbar}{i}\left(\frac{\hbar}{i}\frac{\partial}{\partial y}\right) = i\hbar p_y$$

(e) $[L_z, p_y] = L_z p_y - p_y L_z = \left(x\frac{\hbar}{i}\frac{\partial}{\partial y} - y\frac{\hbar}{i}\frac{\partial}{\partial x}\right)\frac{\hbar}{i}\frac{\partial}{\partial y} - \frac{\hbar}{i}\frac{\partial}{\partial y}\left(x\frac{\hbar}{i}\frac{\partial}{\partial y} - y\frac{\hbar}{i}\frac{\partial}{\partial x}\right)$

$= x\left(\frac{\hbar}{i}\right)^2\frac{\partial}{\partial y}\left(f\cdot\frac{\partial f}{\partial y}\right) - y\left(\frac{\hbar}{i}\right)^2\frac{\partial}{\partial x}\left(f\cdot\frac{\partial f}{\partial y}\right) - \left(\frac{\hbar}{i}\right)^2\frac{\partial}{\partial y}\left(f x\frac{\partial f}{\partial y}\right) + \left(\frac{\hbar}{i}\right)^2\frac{\partial}{\partial y}\left(f y\frac{\partial f}{\partial x}\right)$

$= \left(\frac{\hbar}{i}\right)^2\Big[\cancel{x\frac{\partial f}{\partial y}\frac{\partial f}{\partial y}} + \cancel{x f\frac{\partial^2 f}{\partial y^2}} - \cancel{y\frac{\partial f}{\partial x}\frac{\partial f}{\partial y}} - \cancel{y f\frac{\partial^2 f}{\partial x\partial y}} - \cancel{\frac{\partial f}{\partial y}x\frac{\partial f}{\partial y}} - f\cancel{\frac{\partial x}{\partial y}}^{0}\frac{\partial f}{\partial y} - \cancel{f x\frac{\partial^2 f}{\partial y^2}}$

$+ \cancel{\frac{\partial f}{\partial y}y\frac{\partial f}{\partial x}} + f\frac{\partial f}{\partial x} + \cancel{f y\frac{\partial^2 f}{\partial y\partial x}}\Big]$

$= \left(\frac{\hbar}{i}\right)^2 f\frac{\partial f}{\partial x}$

$= \frac{\hbar}{i}\left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)$

$= -i\hbar\, p_x$

(f) $[L_z, p_z] = L_z p_z - p_z L_z = \left(x\frac{\hbar}{i}\frac{\partial}{\partial y} - y\frac{\hbar}{i}\frac{\partial}{\partial x}\right)\left(\frac{\hbar}{i}\frac{\partial}{\partial z}\right) - \left(\frac{\hbar}{i}\frac{\partial}{\partial z}\right)\left(x\frac{\hbar}{i}\frac{\partial}{\partial y} - y\frac{\hbar}{i}\frac{\partial}{\partial x}\right)$

$= x\left(\frac{\hbar}{i}\right)^2\frac{\partial}{\partial y}\left(f\cdot\frac{\partial f}{\partial z}\right) - y\left(\frac{\hbar}{i}\right)^2\frac{\partial}{\partial x}\left(f\cdot\frac{\partial f}{\partial z}\right) - \left(\frac{\hbar}{i}\right)^2\frac{\partial}{\partial z}\left(f x\frac{\partial f}{\partial y}\right) + \left(\frac{\hbar}{i}\right)^2\frac{\partial}{\partial z}\left(f y\frac{\partial f}{\partial x}\right)$

$= \left(\frac{\hbar}{i}\right)^2\Big[\cancel{x\frac{\partial f}{\partial y}\frac{\partial f}{\partial z}} + \cancel{x f\frac{\partial^2 f}{\partial y\partial z}} - \cancel{y\frac{\partial f}{\partial x}\frac{\partial f}{\partial z}} - \cancel{y f\frac{\partial^2 f}{\partial x\partial z}} - \cancel{\frac{\partial f}{\partial z}x\frac{\partial f}{\partial y}} - f\cancel{\frac{\partial x}{\partial z}}^{0}\frac{\partial f}{\partial y} - \cancel{f x\frac{\partial^2 f}{\partial z\partial y}}$

$+ \cancel{\frac{\partial f}{\partial z}y\frac{\partial f}{\partial x}} + f\cancel{\frac{\partial y}{\partial z}}\frac{\partial f}{\partial x} + \cancel{f y\frac{\partial^2 f}{\partial z\partial x}}\Big]$

$= 0$

(b) $[L_z, L_x] = L_z L_x - L_x L_z = \left(x\frac{\hbar}{i}\frac{\partial}{\partial y} - y\frac{\hbar}{i}\frac{\partial}{\partial x}\right)\left(y\frac{\hbar}{i}\frac{\partial}{\partial z} - z\frac{\hbar}{i}\frac{\partial}{\partial y}\right) - \left(y\frac{\hbar}{i}\frac{\partial}{\partial z} - z\frac{\hbar}{i}\frac{\partial}{\partial y}\right)\left(x\frac{\hbar}{i}\frac{\partial}{\partial y} - y\frac{\hbar}{i}\frac{\partial}{\partial x}\right)$

$= x\left(\frac{\hbar}{i}\right)^2\frac{\partial}{\partial y}\left(f y\frac{\partial f}{\partial z}\right) - x\left(\frac{\hbar}{i}\right)^2\frac{\partial}{\partial y}\left(f z\frac{\partial f}{\partial y}\right) - y\left(\frac{\hbar}{i}\right)^2\frac{\partial}{\partial x}\left(f y\frac{\partial f}{\partial z}\right) + y\left(\frac{\hbar}{i}\right)^2\frac{\partial}{\partial x}\left(f z\frac{\partial f}{\partial y}\right)$

$- y\left(\frac{\hbar}{i}\right)^2\frac{\partial}{\partial z}\left(f x\frac{\partial f}{\partial y}\right) + y\left(\frac{\hbar}{i}\right)^2\frac{\partial}{\partial z}\left(f y\frac{\partial f}{\partial x}\right) + z\left(\frac{\hbar}{i}\right)^2\frac{\partial}{\partial y}\left(f x\frac{\partial f}{\partial y}\right) - z\left(\frac{\hbar}{i}\right)^2\frac{\partial}{\partial y}\left(f y\frac{\partial f}{\partial x}\right)$

$= \left(\frac{\hbar}{i}\right)^2\Big[\cancel{x\frac{\partial f}{\partial y}y\frac{\partial f}{\partial z}} + x f\frac{\partial y}{\partial y}^{1}\frac{\partial f}{\partial z} + \cancel{x f y\frac{\partial^2 f}{\partial y\partial z}} - \cancel{x\frac{\partial f}{\partial y}z\frac{\partial f}{\partial y}} - x f\cancel{\frac{\partial z}{\partial y}}\frac{\partial f}{\partial y} - \cancel{x f z\frac{\partial^2 f}{\partial y^2}}$

$- \cancel{y\frac{\partial f}{\partial x}y\frac{\partial f}{\partial z}} - y f\cancel{\frac{\partial y}{\partial x}}^{0}\frac{\partial f}{\partial z} - \cancel{y f y\frac{\partial^2 f}{\partial x\partial z}} + \cancel{y\frac{\partial f}{\partial x}z\frac{\partial f}{\partial y}} + y f\cancel{\frac{\partial z}{\partial x}}^{0}\frac{\partial f}{\partial y} + \cancel{y f z\frac{\partial^2 f}{\partial x\partial y}}$

$- \cancel{y\frac{\partial f}{\partial z}x\frac{\partial f}{\partial y}} - y f\cancel{\frac{\partial x}{\partial z}}^{0}\frac{\partial f}{\partial y} - \cancel{y f x\frac{\partial^2 f}{\partial z\partial y}} + \cancel{y\frac{\partial f}{\partial z}y\frac{\partial f}{\partial x}} + y f\frac{\partial y}{\partial z}\cancel{\frac{\partial f}{\partial x}}^{0} + \cancel{y f y\frac{\partial^2 f}{\partial z\partial x}}$

$- \cancel{z\frac{\partial f}{\partial y}y\frac{\partial f}{\partial x}} - z f\frac{\partial y}{\partial y}^{1}\frac{\partial f}{\partial x} - \cancel{z f y\frac{\partial^2 f}{\partial y\partial x}} + \cancel{z\frac{\partial f}{\partial y}x\frac{\partial f}{\partial y}} + z f\cancel{\frac{\partial x}{\partial y}}^{0}\frac{\partial f}{\partial y} + \cancel{z f x\frac{\partial^2 f}{\partial y^2}}$

$\cancel{= \left(\frac{\hbar}{i}\right)^2\left[x f\frac{\partial f}{\partial z} - x z\left(\frac{\partial f}{\partial y}\right)^2 + z f\frac{\partial^2 f}{\partial y\partial x} - z x\left(\frac{\partial f}{\partial y}\right)^2\right]}$ $\quad z\frac{\hbar}{i}\frac{\partial}{\partial x} - x\frac{\hbar}{i}\frac{\partial}{\partial z}$

$\cancel{= \frac{\hbar}{i}\left[x\left(\frac{\hbar}{i}\frac{\partial}{\partial z}\right) + 2xz\frac{\hbar}{i}\left(\frac{\partial}{\partial y}\right)^2 - z\frac{\hbar}{i}\frac{\partial}{\partial y}\frac{\partial}{\partial x}\right]}$

$\cancel{= \left(\frac{\hbar}{i}\right)^2\left[x f\frac{\partial f}{\partial z} - zy\frac{\partial f}{\partial y}\frac{\partial f}{\partial x} - z f\frac{\partial f}{\partial x} + zx f\frac{\partial^2 f}{\partial y^2}\right]}$

$= -\left(\frac{\hbar}{i}\right)^2\left[-x f\frac{\partial f}{\partial z} + z f\frac{\partial f}{\partial x}\right] = -i\hbar\left[x\left(\frac{\hbar}{i}\frac{\partial}{\partial z}\right) + z\left(\frac{\hbar}{i}\right)\frac{\partial}{\partial x}\right] = -i\hbar\, L_y$

16 April 2019 Day 26

4.19 [c]

(a) $[L_z, r^2] = L_z r^2 - r^2 L_z = \left(x\frac{\hbar}{i}\frac{\partial}{\partial y} - y\frac{\hbar}{i}\frac{\partial}{\partial x}\right)(x^2+y^2+z^2) - (x^2+y^2+z^2)\left(x\frac{\hbar}{i}\frac{\partial}{\partial y} - y\frac{\hbar}{i}\frac{\partial}{\partial x}\right)$

$$\cancel{= x^3\frac{\hbar}{i}\frac{\partial}{\partial y} + xy^2\frac{\hbar}{i}\frac{\partial}{\partial y} + xz^2\frac{\hbar}{i}\frac{\partial}{\partial y} - x^2 y\frac{\hbar}{i}}$$

$$\cancel{= x\frac{\hbar}{i}\frac{\partial}{\partial y}x^2 + x\frac{\hbar}{i}\frac{\partial}{\partial y}y^2 + x\frac{\hbar}{i}\frac{\partial}{\partial y}z^2 - y\frac{\hbar}{i}\frac{\partial}{\partial x}x^2 - y\frac{\hbar}{i}\frac{\partial}{\partial x}y^2 - y\frac{\hbar}{i}\frac{\partial}{\partial x}z}$$

$$= x\frac{\hbar}{i}\frac{\partial}{\partial y}(f\cdot x^2) + x\frac{\hbar}{i}\frac{\partial}{\partial y}(f\cdot y^2) + x\frac{\hbar}{i}\frac{\partial}{\partial y}(f\cdot z^2)$$

$$- y\frac{\hbar}{i}\frac{\partial}{\partial x}(f\cdot x^2) - y\frac{\hbar}{i}\frac{\partial}{\partial x}(f\cdot y^2) - y\frac{\hbar}{i}\frac{\partial}{\partial x}(f\cdot z^2)$$

$$- x^3\frac{\hbar}{i}\frac{\partial f}{\partial y} - y^2 x\frac{\hbar}{i}\frac{\partial f}{\partial y} - z^2 x\frac{\hbar}{i}\frac{\partial f}{\partial y}$$

$$+ x^2 y\frac{\hbar}{i}\frac{\partial f}{\partial x} + y^3\frac{\hbar}{i}\frac{\partial f}{\partial x} + z^2 y\frac{\hbar}{i}\frac{\partial f}{\partial x}$$

$$= \frac{\hbar}{i}\left(x^3\frac{\partial f}{\partial y} + xy^2\frac{\partial f}{\partial y} + 2xyf + xz^2\frac{\partial f}{\partial y} - x^2 y\frac{\partial f}{\partial x} - y\,2xf - y^3\frac{\partial f}{\partial x} - z^2 y\frac{\partial f}{\partial x}\right.$$

$$\left. - x^3\frac{\partial f}{\partial y} - xy^2\frac{\partial f}{\partial y} - xz^2\frac{\partial f}{\partial y} + x^2 y\frac{\partial f}{\partial x} + y^3\frac{\partial f}{\partial x} + z^2 y\frac{\partial f}{\partial x}\right)$$

$$\cancel{= \frac{\hbar}{i}\left(x^2 y\frac{\partial f}{\partial x} + z^2 y\frac{\partial f}{\partial x} - xy^2\frac{\partial f}{\partial y} - xz^2\frac{\partial f}{\partial y}\right)} \quad = 0$$

(b) $[L_z, p^2] = L_z p^2 - p^2 L_z = \left(x\frac{\hbar}{i}\frac{\partial}{\partial y} - y\frac{\hbar}{i}\frac{\partial}{\partial x}\right)\left[\left(\frac{\hbar}{i}\right)^2\left(\frac{\partial}{\partial x}\right)^2 + \left(\frac{\hbar}{i}\frac{\partial}{\partial y}\right)^2 + \left(\frac{\hbar}{i}\frac{\partial}{\partial z}\right)^2\right] f$

$$- \left[\left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)^2 + \left(\frac{\hbar}{i}\frac{\partial}{\partial y}\right)^2 + \left(\frac{\hbar}{i}\frac{\partial}{\partial z}\right)^2\right]\left(x\frac{\hbar}{i}\frac{\partial}{\partial y} - y\frac{\hbar}{i}\frac{\partial}{\partial x}\right) f$$

$$= x\left(\frac{\hbar}{i}\right)^3\frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial x^2}\right) + x\left(\frac{\hbar}{i}\right)^3\frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial y^2}\right) + x\left(\frac{\hbar}{i}\right)^3\frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial z^2}\right)$$

$$- y\left(\frac{\hbar}{i}\right)^3\frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x^2}\right) - y\left(\frac{\hbar}{i}\right)^3\frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial y^2}\right) - y\left(\frac{\hbar}{i}\right)^3\frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial z^2}\right)$$

$$- \left(\frac{\hbar}{i}\right)^3\left(\frac{\partial}{\partial x}\right)^2\left(x\cdot\frac{\partial f}{\partial y}\right) - \left(\frac{\hbar}{i}\right)^3\left(\frac{\partial}{\partial y}\right)^2\left(x\frac{\partial f}{\partial y}\right) - \left(\frac{\hbar}{i}\right)^3\left(\frac{\partial}{\partial z}\right)^2\left(x\frac{\partial f}{\partial y}\right)$$

$$+ \left(\frac{\hbar}{i}\right)^3\left(\frac{\partial}{\partial x}\right)^2\left(y\frac{\partial f}{\partial x}\right) + \left(\frac{\hbar}{i}\right)^2\left(\frac{\partial}{\partial y}\right)^2\left(y\frac{\partial f}{\partial x}\right) + \left(\frac{\hbar}{i}\right)^3\left(\frac{\partial}{\partial z}\right)^2\left(y\frac{\partial f}{\partial x}\right)$$

$$\cancel{= \left(\frac{\hbar}{i}\right)^3\left(x\frac{\partial f}{\partial y}\frac{\partial^2 f}{\partial x^2} + x\frac{\partial^3 f}{\partial y\partial x^2}f + x\frac{\partial f}{\partial y}\frac{\partial^2 f}{\partial y^2} + x\frac{\partial^3 f}{\partial y^3}f + x\frac{\partial f}{\partial y}\frac{\partial^2 f}{\partial z^2} + x\frac{\partial^3 f}{\partial y\partial z^2}f\right.}$$

$$\cancel{- y\frac{\partial f}{\partial x}\frac{\partial^2 f}{\partial x^2} - y\frac{\partial^3 f}{\partial x^3}f - y\frac{\partial f}{\partial x}\frac{\partial^2 f}{\partial y^2} - y\frac{\partial^3 f}{\partial x\partial y^2}f - y\frac{\partial f}{\partial x}\frac{\partial^2 f}{\partial z^2} - y\frac{\partial^3 f}{\partial x\partial z^2}f}$$

$$= \left(\frac{\hbar}{i}\right)^3\left(x\frac{\partial^3 f}{\partial x^2\partial y} + x\frac{\partial^3 f}{\partial y^3} + x\frac{\partial^3 f}{\partial y\partial z^2} - y\frac{\partial^3 f}{\partial x^3} - y\frac{\partial^3 f}{\partial x\partial y^2} - y\frac{\partial^3 f}{\partial x\partial z^2}\right.$$

$$- \frac{\partial^2 f}{\partial x\partial y} - \frac{\partial}{\partial x}\frac{\partial^2 f}{\partial x\partial y}x - \frac{\partial}{\partial y}\frac{\partial^2 f}{\partial y^2}x - \frac{\partial}{\partial z}\frac{\partial^2 f}{\partial y\partial z}x$$

$$\left. + \frac{\partial}{\partial x}\frac{\partial^2 f}{\partial x^2}y + \frac{\partial}{\partial y}\frac{\partial f}{\partial x} + \frac{\partial}{\partial y}\frac{\partial^2 f}{\partial x\partial y}y + \frac{\partial}{\partial z}\frac{\partial f}{\partial x\partial z}y\right)$$

$$\cancel{+\frac{\partial}{\partial y}\frac{\partial^2 f}{\partial y\partial y}y}$$

$$= \left(\frac{\hbar}{i}\right)^3 \Big(x \frac{\partial^3 f}{\partial x^2 \partial y} + x \frac{\partial^3 f}{\partial y^3} + x \frac{\partial^3 f}{\partial y \partial z^2} - y \frac{\partial^3 f}{\partial x^3} - y \frac{\partial^3 f}{\partial x \partial y^2} - y \frac{\partial^3 f}{\partial x \partial z^2}$$

$$- \frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^3 f}{\partial x^2 \partial y} x - \frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^3 f}{\partial y^3} x - \frac{\partial^3 f}{\partial y \partial z^2} x$$

$$+ \frac{\partial^3 f}{\partial x^3} y + \frac{\partial^2 f}{\partial x \partial y} + \frac{\partial^3 f}{\partial x \partial y^2} y + \frac{\partial^2 f}{\partial x \partial y} + \frac{\partial^3 f}{\partial x \partial z^2} y \Big)$$

$$= 0$$

[d] $H = \frac{p^2}{2m} + V(r)$

$[L_x, H]$ Since $[L_x, p^2] = [L_y, p^2] = [L_z, p^2] = 0$,

$[L_y, H]$ We only need to know the commutation relationship of

$[L_z, H]$ the components of $\vec{L}$ and r.

$$[L_z, r] = L_z r - r L_z = \left(x \frac{\hbar}{i}\frac{\partial}{\partial y} - y \frac{\hbar}{i}\frac{\partial}{\partial x}\right)\left(\sqrt{x^2+y^2+z^2}\right) - \left(\sqrt{x^2+y^2+z^2}\right)\left(x\frac{\hbar}{i}\frac{\partial}{\partial y} - y\frac{\hbar}{i}\frac{\partial}{\partial x}\right)$$

$$= x\frac{\hbar}{i}\frac{\partial}{\partial y}\left(\sqrt{x^2+y^2+z^2}\, f\right) - y\frac{\hbar}{i}\frac{\partial}{\partial x}\left(\sqrt{x^2+y^2+z^2}\, f\right) - \sqrt{x^2+y^2+z^2}\, x \frac{\hbar}{i}\frac{\partial f}{\partial y} + \sqrt{x^2+y^2+z^2}\, y \frac{\hbar}{i}\frac{\partial f}{\partial x}$$

$$= \frac{\hbar}{i}\Big(x \cdot \frac{y}{\sqrt{x^2+y^2+z^2}} f + x\sqrt{x^2+y^2+z^2}\frac{\partial f}{\partial y} - y \cdot x \sqrt{x^2+y^2+z^2}\, f - y\sqrt{x^2+y^2+z^2}\frac{\partial f}{\partial x}$$

$$- \sqrt{x^2+y^2+z^2}\, x \frac{\partial f}{\partial y} + \sqrt{x^2+y^2+z^2}\, y \frac{\partial f}{\partial x}\Big)$$

$$= 0$$

Similarly, $[L_x, r] = [L_y, r] = 0$

Thus, $[L_x, H] = [L_y, H] = [L_z, H] = 0$, since $[L_i, r]$ and $[L_i, p^2]$ are 0.

4.22 (a) $L_+ Y_l^l = 0$, since ~~t~~ $m = l$, the maximum number that it can have. As L_+ makes Y to exceed the maximum, it gives out 0

(Also see pg 195 Eq [110])

(b) Known $L_+ Y_l^l = 0$

$L_z Y_l^l = \hbar l Y_l^l$

$L_\pm = \pm \hbar e^{\pm i\phi}\left(\frac{\partial}{\partial\theta} \pm i\cot\theta \frac{\partial}{\partial\phi}\right)$

$L_z Y_l^l = \hbar l Y_l^l$

$\frac{\hbar}{i}\frac{\partial}{\partial\phi} Y_l^l = \hbar l Y_l^l$

$\frac{\partial Y_l^l}{\partial\phi} = il Y_l^l$

$\frac{\partial Y_l^l}{Y_l^l} = il\,\partial\phi$

$L_z Y_l^m = \hbar m Y_l^m$

$Y_l^m(\theta,\phi) = \Theta_l^m(\theta)\,\Phi_l^m(\phi)$

$\frac{\hbar}{i}\frac{\partial}{\partial\phi} Y_l^m = \hbar m\, \Theta_l^m(\theta)\,\Phi_l^m(\phi)$

$\frac{\hbar}{i}\Theta_l^m(\theta)\frac{\partial \Phi_l^m}{\partial\phi} = \hbar m\, \Theta_l^m(\theta)\,\Phi_l^m(\phi)$

$\frac{\partial \Phi_l^m}{\partial\phi} = im\,\Phi_l^m(\phi)$

$\int \frac{\partial \Phi_l^m}{\Phi_l^m(\phi)} = \int im\,\partial\phi$

$\ln(\Phi_l^m(\phi)) = im\phi + C$

$\Phi_l^m(\phi) = \cancel{A} e^{im\phi} e^{C}$ $(m = l)$

Since $L_+ Y_l^l = 0$

$\hbar e^{i\phi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right)\left(\cancel{\Theta(\theta)}\ \Theta_l^l(\theta)\, e^{il\phi}\right) = 0$

$\hbar e^{i\phi}\left(e^{il\phi}\frac{\partial\Theta_l^l(\theta)}{\partial\theta} + i\cot\theta\cdot\Theta_l^l(\theta)\frac{\partial}{\partial\phi}e^{il\phi}\right) = 0$

$\cancel{\hbar e^{i\phi}}$ $e^{im\phi}\frac{\partial\Theta_l^l}{\partial\theta} + i\cot\theta\,\Theta_l^l\, il\, e^{il\phi} = 0$

$\frac{\partial\Theta_l^m}{\partial\theta} = \cot\theta\cdot\Theta_l^m\, l$

$\int\frac{\partial\Theta_l^m}{\Theta_l^m} = \int l\cot\theta\,\partial\theta$

$\ln(\Theta_l^m) = l\int\frac{\cos\theta}{\sin\theta}\partial\theta$

$u = \sin\theta$

$\frac{du}{\cos\theta} = d\theta$

$\ln(\Theta_l^m) = l\int\frac{\cancel{\cos\theta}}{u}\frac{du}{\cancel{\cos\theta}}$

$\ln(\Theta_l^m) = l\ln(u) + C$

$$\ln(\Theta_l^m) = l \cdot \ln(\sin\theta) + c$$

$$\cancel{\frac{\ln(\Theta_l^m)}{\ln(\sin^l\theta)}}$$

$$\cancel{\ln(\Theta_l^m) - \ln(\sin^l\theta) = c}$$

$$\cancel{\ln\left(\frac{\Theta_l^m}{\sin^l\theta}\right) = c}$$

$$\Theta_l^m = e^{\ln(\sin^l\theta)} e^c$$

$$\Theta_l^m = A \cancel{\sin^l\theta} \sin^l\theta$$

$$Y_l^l = \Theta_l^m \Phi_l^m = A e^{il\phi} \sin^l\theta$$

(c) $1 = \int |Y_l^l|^2 \sin\theta \, d\theta \, d\phi = A^2 \int \sin^{2l}\theta \sin\theta \, d\theta \, d\phi$

$$= A^2 \int_0^\pi \sin^{2l}\theta \sin\theta \, d\theta \int_0^{2\pi} d\phi$$

$$= 2\pi A^2 \int_0^{\pi} \sin^{(2l+1)}\theta \, d\theta$$

$$= 2\pi A^2 \int_0^{\pi} \sin\theta \, (1-\cos^2\theta)^l \, d\theta$$

$u = \cos\theta \qquad u(0) = 1$

$\frac{du}{-\sin\theta} = d\theta, \qquad u(\pi) = -1$

$$= 2\pi A^2 \int_1^{-1} \cancel{\sin\theta} (1-u^2)^l \frac{du}{-\cancel{\sin\theta}}$$

$$= +2\pi A^2 \int_{-1}^{1} (1-u^2)^l \, du$$

$\cancel{v = 1-u^2 \Rightarrow u = \sqrt{1-v}}$

$\cancel{\frac{dv}{-2u} = du}$

$$= 2\pi A^2 \int_{-1}^{1} \sum_{k=0}^{l} \binom{k}{l} (-x^2)^k \, dx$$

$$= 2\pi A^2 \left[\sum_{k=0}^{l} \binom{k}{l} \frac{(-1)^k}{2k+1} x^{2k+1} \right]_{-1}^{1}$$

$$= 2\pi A^2 \sum_{k=0}^{l} \binom{k}{l} \frac{2(-1)^k}{2k+1}$$

$$= 2\pi A^2 \sum_{k=0}^{l} \frac{l!}{k!(l-k)!} \frac{2(-1)^k}{2k+1}$$

$$A = \left(2\pi \sum_{k=0}^{l} \frac{l!}{k!(l-k)!} \frac{2(-1)^k}{2k+1} \right)^{-\frac{1}{2}}$$

4.26 (a)

$S_x = \frac{\hbar}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix}$

$S_y = \frac{\hbar}{2}\begin{pmatrix}0&-i\\i&0\end{pmatrix}$

$S_z = \frac{\hbar}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}$

$$[S_x, S_y] = S_x S_y - S_y S_x$$
$$= \left(\frac{\hbar}{2}\right)^2\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0&-i\\i&0\end{pmatrix} - \left(\frac{\hbar}{2}\right)^2\begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}$$
$$= \left(\frac{\hbar}{2}\right)^2\left[\begin{pmatrix}i&0\\0&-i\end{pmatrix} - \begin{pmatrix}-i&0\\0&i\end{pmatrix}\right]$$
$$= \left(\frac{\hbar}{2}\right)^2\begin{pmatrix}2i&0\\0&-2i\end{pmatrix}$$
$$= 2i\frac{\hbar^2}{4}\begin{pmatrix}1&0\\0&-1\end{pmatrix} = i\hbar\frac{\hbar}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix} = i\hbar S_z$$

$$[S_y, S_z] = S_y S_z - S_z S_y$$
$$= \frac{\hbar^2}{4}\begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix} - \frac{\hbar^2}{4}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}0&-i\\i&0\end{pmatrix}$$
$$= \frac{\hbar^2}{4}\left[\begin{pmatrix}0&i\\i&0\end{pmatrix} - \begin{pmatrix}0&-i\\-i&0\end{pmatrix}\right]$$
$$= \frac{\hbar^2}{4}\begin{pmatrix}0&2i\\2i&0\end{pmatrix} = 2i\frac{\hbar^2}{4}\begin{pmatrix}0&1\\1&0\end{pmatrix} = i\hbar\frac{\hbar}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix} = i\hbar S_x$$

$$[S_z, S_x] = S_z S_x - S_x S_z$$
$$= \frac{\hbar^2}{4}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix} - \frac{\hbar^2}{4}\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$
$$= \frac{\hbar^2}{4}\left[\begin{pmatrix}0&1\\-1&0\end{pmatrix} - \begin{pmatrix}0&-1\\1&0\end{pmatrix}\right]$$
$$= \frac{\hbar^2}{4}\begin{pmatrix}0&2\\-2&0\end{pmatrix} = 2i\frac{\hbar^2}{4}\begin{pmatrix}0&-i\\i&0\end{pmatrix} = i\hbar\frac{\hbar}{2}\begin{pmatrix}0&-i\\i&0\end{pmatrix} = i\hbar S_y$$

(b) Show Pauli spin matrix satisfy $\sigma_j\sigma_k = \delta_{jk} + i\sum_l \epsilon_{jkl}\sigma_l$, where j stands for x, y, z,

ϵ_{jkl} is Levi-Civita symbol
$$\begin{cases}\text{if } jkl = 123, 231, 312 & \epsilon_{jkl} = +1\\ \text{if } jkl = 132, 213, 321 & \epsilon_{jkl} = -1\\ \text{if } jkl = \text{otherwise} & \epsilon_{jkl} = 0\end{cases}$$

When ~~j=k~~ k=j :

$$\sigma_x\sigma_x = \begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix} = \begin{pmatrix}1&0\\0&1\end{pmatrix}$$
$$\sigma_y\sigma_y = \begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}0&-i\\i&0\end{pmatrix} = \begin{pmatrix}+1&0\\0&+1\end{pmatrix}$$
$$\sigma_z\sigma_z = \begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix} = \begin{pmatrix}1&0\\0&1\end{pmatrix}$$

so $\sigma_j\sigma_j = \delta_{jj} + i\sum_l \epsilon_{jjl}\sigma_l = \delta_{jj} + 0 = 1$

When $j \neq k$:

$$\sigma_x \sigma_y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = i\sigma_z$$

$$= \cancel{\delta_{xy} + i\sum_z} \; \delta_{12} + i\sum \epsilon_{123}\sigma_3 = 0 + i\sum_3 1 \cdot \sigma_z = i\sigma_z$$

$$\sigma_y \sigma_x = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = -i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = -i\sigma_z$$

$$= \delta_{yx} + i\sum \epsilon_{yxz}\sigma_z = 0 + i\sum -\sigma_z = -i\sigma_z$$

$$\sigma_y \sigma_z = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = i\sigma_x$$

$$= \delta_{yz} + i\sum \epsilon_{yzx}\sigma_x = 0 + i\sum 1 \cdot \sigma_x = +i\sigma_x$$

$$\sigma_z \sigma_y = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} = -i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -i\sigma_x$$

$$= \delta_{zy} + i\sum \epsilon_{zyx}\sigma_x = 0 + i\sum -\sigma_x = -i\sigma_x$$

$$\sigma_x \sigma_z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -i\begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix} = -i\sigma_y$$

$$= \delta_{xz} + i\sum \epsilon_{xzy}\sigma_y = 0 + i\sum -\sigma_y = -i\sigma_y$$

$$\sigma_z \sigma_x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = i\sigma_y$$

$$= \delta_{zx} + i\sum \epsilon_{zxy}\sigma_y = 0 + i\sum 1 \cdot \sigma_y = i\sigma_y$$

4.27 (a) $\chi = A\begin{pmatrix} 3i \\ 4 \end{pmatrix} = \begin{pmatrix} 3iA \\ 4A \end{pmatrix}$, since $|a|^2 + |b|^2 = 1$

$$|3iA|^2 + |4A|^2 = 1$$

$$9A^2 + 16A^2 = 1$$

$$25A^2 = 1$$

$$A = \pm\sqrt{\tfrac{1}{25}} = \pm\tfrac{1}{5} = \tfrac{1}{5}$$

(b)
$$\langle S_x \rangle = \chi^\dagger S_x \chi = \begin{pmatrix} -\frac{3i}{5} & \frac{4}{5} \end{pmatrix}\begin{pmatrix} 0 & \frac{\hbar}{2} \\ \frac{\hbar}{2} & 0 \end{pmatrix}\begin{pmatrix} +\frac{3i}{5} \\ \frac{4}{5} \end{pmatrix} = \begin{pmatrix} \frac{2\hbar}{5} & -\frac{3i\hbar}{10} \end{pmatrix}\begin{pmatrix} +\frac{3i}{5} \\ \frac{4}{5} \end{pmatrix} = +\frac{6\hbar i}{25} - \frac{12i\hbar}{50} = 0$$

$$\langle S_y \rangle = \chi^\dagger S_y \chi = \begin{pmatrix} -\frac{3i}{5} & \frac{4}{5} \end{pmatrix}\begin{pmatrix} 0 & -\frac{\hbar i}{2} \\ \frac{\hbar i}{2} & 0 \end{pmatrix}\begin{pmatrix} \frac{3i}{5} \\ \frac{4}{5} \end{pmatrix} = \begin{pmatrix} \frac{2\hbar i}{5} & -\frac{3\hbar}{10} \end{pmatrix}\begin{pmatrix} \frac{3i}{5} \\ \frac{4}{5} \end{pmatrix} = -\frac{6\hbar}{25} - \frac{12\hbar}{50} = -\frac{24\hbar}{50}$$

$$= -\frac{12\hbar}{25}$$

$$\langle S_z \rangle = \chi^\dagger S_z \chi = \begin{pmatrix} -\frac{3i}{5} & \frac{4}{5} \end{pmatrix}\begin{pmatrix} \frac{\hbar}{2} & 0 \\ 0 & -\frac{\hbar}{2} \end{pmatrix}\begin{pmatrix} \frac{3i}{5} \\ \frac{4}{5} \end{pmatrix} = \begin{pmatrix} -\frac{3i\hbar}{10} & -\frac{2\hbar}{5} \end{pmatrix}\begin{pmatrix} \frac{3i}{5} \\ \frac{4}{5} \end{pmatrix} = \frac{9\hbar}{50} - \frac{8\hbar}{25}$$

$$= -\frac{7\hbar}{50}$$

4.27 (c) $\langle S_x \rangle^2 = 0$

$\langle S_y \rangle^2 = \frac{144\hbar^2}{625}$

$\langle S_z \rangle^2 = \frac{49\hbar^2}{2500}$

$$\langle S_x^2 \rangle = \chi^\dagger S_x^2 \chi = \begin{pmatrix} -\frac{3i}{5} & \frac{4}{5} \end{pmatrix} \begin{pmatrix} 0 & \frac{\hbar^2}{4} \\ \frac{\hbar^2}{4} & 0 \end{pmatrix} \begin{pmatrix} \frac{3i}{5} \\ \frac{4}{5} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\hbar^2}{5} & -\frac{3i\hbar^2}{20} \end{pmatrix} \begin{pmatrix} \frac{3i}{5} \\ \frac{4}{5} \end{pmatrix}$$

$$= \frac{3i\hbar^2}{25} - \frac{3i\hbar^2}{25} = 0$$

$$\langle S_y^2 \rangle = \chi^\dagger S_y^2 \chi = \begin{pmatrix} -\frac{3i}{5} & \frac{4}{5} \end{pmatrix} \begin{pmatrix} 0 & -\frac{\hbar^2}{4} \\ -\frac{\hbar^2}{4} & 0 \end{pmatrix} \begin{pmatrix} \frac{3i}{5} \\ \frac{4}{5} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{\hbar^2}{5} & -\frac{3i\hbar^2}{20} \end{pmatrix} \begin{pmatrix} \frac{3i}{5} \\ \frac{4}{5} \end{pmatrix} = -\frac{3i\hbar^2}{25} - \frac{3i\hbar^2}{25} = \frac{6i\hbar^2}{25}$$

$$\langle S_z^2 \rangle = \chi^\dagger S_z^2 \chi = \begin{pmatrix} -\frac{3i}{5} & \frac{4}{5} \end{pmatrix} \begin{pmatrix} \frac{\hbar^2}{4} & 0 \\ 0 & \frac{\hbar^2}{4} \end{pmatrix} \begin{pmatrix} \frac{3i}{5} \\ \frac{4}{5} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{3i\hbar^2}{20} & \frac{\hbar^2}{5} \end{pmatrix} \begin{pmatrix} \frac{3i}{5} \\ \frac{4}{5} \end{pmatrix} = \frac{9\hbar^2}{100} + \frac{4\hbar^2}{25} = \frac{25\hbar^2}{100} = \frac{\hbar^2}{4}$$

$$\hat{S}_x^2 = \begin{pmatrix} 0 & \frac{\hbar}{2} \\ \frac{\hbar}{2} & 0 \end{pmatrix} \begin{pmatrix} 0 & \frac{\hbar}{2} \\ \frac{\hbar}{2} & 0 \end{pmatrix} = \begin{pmatrix} \frac{\hbar^2}{4} & 0 \\ 0 & \frac{\hbar^2}{4} \end{pmatrix} = \frac{\hbar^2}{4}$$

$$\langle S_x^2 \rangle = \begin{pmatrix} -\frac{3i}{5} & \frac{4}{5} \end{pmatrix} \frac{\hbar^2}{4} \begin{pmatrix} \frac{3i}{5} \\ \frac{4}{5} \end{pmatrix} = \frac{\hbar^2}{4}$$

$$S_y^2 = \cancel{\begin{pmatrix} 0 & \frac{\hbar^2}{4} \\ -\frac{\hbar^2}{4} & 0 \end{pmatrix}} \begin{pmatrix} 0 & -\frac{\hbar i}{2} \\ \frac{\hbar i}{2} & 0 \end{pmatrix} \begin{pmatrix} 0 & -\frac{\hbar i}{2} \\ \frac{\hbar i}{2} & 0 \end{pmatrix} = \begin{pmatrix} \frac{\hbar^2}{4} & 0 \\ 0 & \frac{\hbar^2}{4} \end{pmatrix} = \frac{\hbar^2}{4}$$

$$\langle S_y^2 \rangle = \begin{pmatrix} -\frac{3i}{5} & \frac{4}{5} \end{pmatrix} \frac{\hbar^2}{4} \begin{pmatrix} \frac{3i}{5} \\ \frac{4}{5} \end{pmatrix} = \frac{\hbar^2}{4}$$

$$S_z^2 = \begin{pmatrix} \frac{\hbar}{2} & 0 \\ 0 & -\frac{\hbar}{2} \end{pmatrix} \begin{pmatrix} \frac{\hbar}{2} & 0 \\ 0 & -\frac{\hbar}{2} \end{pmatrix} = \begin{pmatrix} \frac{\hbar^2}{4} & 0 \\ 0 & \frac{\hbar^2}{4} \end{pmatrix} = \frac{\hbar^2}{4}$$

$$\langle S_z^2 \rangle = \begin{pmatrix} -\frac{3i}{5} & \frac{4}{5} \end{pmatrix} \frac{\hbar^2}{4} \begin{pmatrix} \frac{3i}{5} \\ \frac{4}{5} \end{pmatrix} = \frac{\hbar^2}{4}$$

$$\sigma_x^2 = \langle S_x^2 \rangle - \langle S_x \rangle^2 = \frac{\hbar^2}{4} - 0 = \frac{\hbar^2}{4} \Rightarrow \sigma_x = \frac{\hbar}{2}$$

$$\sigma_y^2 = \langle S_y^2 \rangle - \langle S_y \rangle^2 = \frac{\hbar^2}{4} - \frac{144\hbar^2}{625} = \frac{12.25\hbar^2}{625} = 0.0196\hbar^2 \Rightarrow \sigma_y = 0.14\hbar = \frac{7\hbar}{50}$$

$$\sigma_z = \langle S_z^2 \rangle - \langle S_z \rangle^2 = \frac{\hbar^2}{4} - \frac{49\hbar^2}{2500} = \frac{576\hbar^2}{2500} = \frac{144\hbar^2}{625} \Rightarrow \sigma_z = \frac{12\hbar}{25}$$

(d) Verify that it follows $\sigma_{S_x}\sigma_{S_y} \geq \frac{\hbar}{2}|\langle S_z \rangle|$

$$\sigma_{S_x}\sigma_{S_y} = \frac{\hbar}{2}\frac{7\hbar}{50} = \frac{7\hbar^2}{100}$$

$$\frac{\hbar}{2}|\langle S_z \rangle| = \frac{\hbar}{2}\left|-\frac{7\hbar}{50}\right| = \frac{7\hbar^2}{100}$$

$$\frac{7\hbar^2}{100} = \frac{7\hbar^2}{100} \quad ✓$$

$$\sigma_{S_x}\sigma_{S_y} = \frac{\hbar}{2}|\langle S_z \rangle|$$

(d) (cont.)

$$\sigma_{S_y}\sigma_{S_z} = \frac{7\hbar}{50}\cdot\frac{12\hbar}{25} = \frac{42\hbar^2}{625} \quad \Rightarrow \quad \frac{42\hbar^2}{625} > 0$$

$$\frac{\hbar}{2}\left|\langle S_x\rangle\right| = \frac{\hbar}{2}\left|0\right| = 0 \qquad \sigma_{S_y}\sigma_{S_z} > \frac{\hbar}{2}\left|\langle S_x\rangle\right| \checkmark$$

$$\sigma_{S_x}\sigma_{S_z} = \frac{\hbar}{2}\cdot\frac{12\hbar}{25} = \frac{6\hbar^2}{25} \quad \Rightarrow \quad \frac{6\hbar^2}{25} = \frac{6\hbar^2}{25}$$

$$\frac{\hbar}{2}\left|\langle S_y\rangle\right| = \frac{\hbar}{2}\left|-\frac{12\hbar}{25}\right| = \frac{6\hbar^2}{25} \qquad \sigma_{S_x}\sigma_{S_z} > \frac{\hbar}{2}\left|\langle S_y\rangle\right| \checkmark$$

4.29 (a) Find eigenvalues and eigenspinors of S_y.

(i) $S_y = \frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} = \begin{pmatrix}0 & -\frac{\hbar}{2}i\\ \frac{\hbar}{2}i & 0\end{pmatrix}$

$$\cancel{\begin{vmatrix}0\lambda & -\frac{\hbar i}{2}\\ \frac{\hbar i}{2} & 0\lambda\end{vmatrix} = 0}$$

$$\begin{bmatrix}0 & -\frac{\hbar i}{2}\\ \frac{\hbar i}{2} & 0\end{bmatrix}\begin{bmatrix}a\\ b\end{bmatrix} = \lambda\begin{bmatrix}a\\ b\end{bmatrix}$$

$$\left(\begin{bmatrix}0 & -\frac{\hbar i}{2}\\ \frac{\hbar i}{2} & 0\end{bmatrix} - \lambda\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right)\begin{bmatrix}a\\ b\end{bmatrix} = 0$$

$$\begin{bmatrix}-\lambda & -\frac{\hbar i}{2}\\ \frac{\hbar i}{2} & -\lambda\end{bmatrix}\begin{bmatrix}a\\ b\end{bmatrix} = 0$$

$$\cancel{\begin{vmatrix}-\lambda & -\frac{\hbar i}{2}\\ \frac{\hbar i}{2} & -\lambda\end{vmatrix} = 0}$$

$\cancel{\lambda a}$

$$\begin{vmatrix}-\lambda & -\frac{\hbar i}{2}\\ \frac{\hbar i}{2} & -\lambda\end{vmatrix} = 0$$

$$(-\lambda)(-\lambda) - \left(-\frac{\hbar i}{2}\right)\left(\frac{\hbar i}{2}\right) = 0$$

$$\lambda^2 + \left(-\frac{\hbar^2}{4}\right) = 0$$

$$\lambda^2 = \frac{\hbar^2}{4}$$

$$\lambda = \pm\frac{\hbar}{2}$$

$$\lambda = \begin{bmatrix}\frac{\hbar}{2} & 0\\ 0 & \frac{\hbar}{2}\end{bmatrix}$$

$$\lambda = \begin{bmatrix}-\frac{\hbar}{2} & 0\\ 0 & -\frac{\hbar}{2}\end{bmatrix}$$

(ii) $\frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \pm\frac{\hbar}{2}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$

$$\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \pm\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$$

$$\begin{pmatrix}-i\beta\\ i\alpha\end{pmatrix} = \pm\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$$

Thus, $-i\beta = \pm\alpha$.

Normalize by $|a|^2 + |b|^2 = 1$

$$|-i\beta|^2 + |i\alpha|^2 = 1$$

$$\alpha^2 + \alpha^2 = 1$$

$$2\alpha^2 = 1$$

$$\alpha^2 = \frac{1}{2}$$

$$\alpha = \pm\frac{1}{\sqrt{2}}$$

$\cancel{\text{So } \beta = i\alpha = i\frac{1}{\sqrt{2}}}$

$$\cancel{\chi_+^{(y)} = \begin{pmatrix}\frac{1}{\sqrt{2}}\\ i\frac{1}{\sqrt{2}}\end{pmatrix}}$$

$$\cancel{\chi_-^{(y)} = (}$$

For positive eigenvalue, we have

$\alpha = \frac{1}{\sqrt{2}}$, $\beta = i\alpha = \frac{i}{\sqrt{2}}$

so $\chi_+^{(y)} = \begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{i}{\sqrt{2}}\end{pmatrix}$

For negative eigenvalue, we have

$\cancel{\alpha = -\frac{1}{\sqrt{2}},\ -\beta = i\alpha = -\frac{i}{\sqrt{2}}}$

$\cancel{\beta =}$

$-\alpha = \frac{1}{\sqrt{2}}$, $-\beta = i\alpha = -\frac{i}{\sqrt{2}}$

so $\chi_-^{(y)} = \begin{pmatrix}\frac{1}{\sqrt{2}}\\ -\frac{i}{\sqrt{2}}\end{pmatrix}$

$= \begin{pmatrix} a \\ b \end{pmatrix}$

(b) Known $\chi = A\chi_+ + B\chi_-$ in general,

$$\begin{pmatrix} a \\ b \end{pmatrix} = A\chi_+ + B\chi_- = A\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix} + B\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{A+B}{\sqrt{2}} \\ \frac{Ai-Bi}{\sqrt{2}} \end{pmatrix}$$

so $\begin{cases} a = \frac{A+B}{\sqrt{2}} & \Rightarrow \quad A = \sqrt{2}a - B \\ b = \frac{Ai-Bi}{\sqrt{2}} \end{cases}$

$$b = \frac{(\sqrt{2}a - B - B)i}{\sqrt{2}} = \frac{(\sqrt{2}a - 2B)i}{\sqrt{2}} \Rightarrow \frac{\sqrt{2}b}{i} = \sqrt{2}a - 2B$$

$$B = -\frac{\sqrt{2}b - \sqrt{2}ai}{2i}$$

$$B = \frac{\sqrt{2}bi + \sqrt{2}a}{2}$$

Thus, $A = \sqrt{2}a - B = \sqrt{2}a - \frac{\sqrt{2}bi + \sqrt{2}a}{2} = \frac{-\sqrt{2}bi + \sqrt{2}a}{2}$

The spinor $\chi = \left(\frac{-\sqrt{2}bi + \sqrt{2}a}{2}\right)\chi_+^{(y)} + \left(\frac{\sqrt{2}bi + \sqrt{2}a}{2}\right)\chi_-^{(y)}$

If you measure S_y, the probability of getting $+\frac{\hbar}{2}$ is $\cancel{\left|\frac{-\sqrt{2}bi + \sqrt{2}a}{2}\right|^2 = \left|\frac{b^2 + a^2}{2}\right|}$.

~~and the probability of getting~~ $-\frac{\hbar}{2}$ is $\cancel{\left|\frac{\sqrt{2}bi + \sqrt{2}a}{2}\right|^2 = \left|\frac{a^2 - b^2}{2}\right|}$.

$$\left|\chi_+^{(y)\dagger}\chi\right|^2 = \left|\left[\tfrac{1}{\sqrt{2}} \ \ -\tfrac{i}{\sqrt{2}}\right]\begin{bmatrix} a \\ b \end{bmatrix}\right|^2 = \left|\frac{a}{\sqrt{2}} - \frac{bi}{\sqrt{2}}\right|^2 = \frac{|a-bi|^2}{2}$$

the probability of getting $-\frac{\hbar}{2}$ is

$$\left|\chi_-^{(y)\dagger}\chi\right|^2 = \left|\left[\tfrac{1}{\sqrt{2}} \ \ \tfrac{i}{\sqrt{2}}\right]\begin{bmatrix} a \\ b \end{bmatrix}\right|^2 = \left|\frac{a}{\sqrt{2}} + \frac{ib}{\sqrt{2}}\right|^2 = \frac{|a+ib|^2}{2}.$$

Check for nomalization:

$$\frac{|a-bi|^2}{2} + \frac{|a+bi|^2}{2} = \frac{1}{2}\left[(a-bi)^*(a-bi) + (a+bi)^*(a+bi)\right]$$

$$= \frac{1}{2}\left[(a^* + b^*i)(a-bi) + (a^* - b^*i)(a+bi)\right]$$

$$= \cancel{a^2 - (bi)^2} \ \frac{1}{2}\left[a^*a - a^*bi + b^*ai + b^*b + a^*a + a^*bi - b^*ai + b^*b\right]$$

$$\cancel{= a^2}$$

$$= a^*a + b^*b = |a|^2 + |b|^2 = 1$$

4.29 (c) $S_y^2 = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \cdot \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

$= \frac{\hbar^2}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$= \frac{\hbar^2}{4}$

$\lambda = \frac{\hbar^2}{4}$, probability of getting $+\frac{\hbar^2}{4}$ by measuring S_y^2 is 1.

4.31

$[S_x, S_y] = i\hbar S_z$ $\qquad S^2 |sm\rangle = \hbar^2 s(s+1)|sm\rangle$ $\boxed{1}$

$[S_y, S_z] = i\hbar S_x$ $\qquad S_z |sm\rangle = \hbar m |sm\rangle$ $\boxed{2}$

$[S_z, S_x] = i\hbar S_y$ $\qquad S_\pm |sm\rangle = \hbar\sqrt{s(s+1) - m(m\pm 1)}\ |s(m\pm1)\rangle$ where $S_\pm \equiv S_x \pm i S_y$ $\boxed{3}$

$S = 1$

$$\chi = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = a\underset{m=1}{\chi_+} + b\underset{m=0}{\chi_0} + c\underset{m=-1}{\chi_-}$$

where $\underset{|1,1\rangle}{\chi_+ = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}$, $\underset{|1,0\rangle}{\chi_0 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}$, $\underset{|1,-1\rangle}{\chi_- = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}$ $\boxed{4}$

By $\boxed{1}$, $S^2\chi_+ = 2\hbar^2\chi_+$, $S^2\chi_0 = 2\hbar^2\chi_0$, $S^2\chi_- = 2\hbar^2\chi_-$ $\boxed{5}$

Write S^2 as an undetermined matrix,

$$S^2 = \begin{pmatrix} d & e & f \\ g & h & j \\ k & l & m \end{pmatrix} \quad \boxed{6}$$

$\boxed{6}\boxed{4}$:
by $\boxed{5}$

$$\begin{pmatrix} d & e & f \\ g & h & j \\ k & l & m \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 2\hbar^2\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} d \\ g \\ k \end{pmatrix} = \begin{pmatrix} 2\hbar^2 \\ 0 \\ 0 \end{pmatrix} \quad \boxed{7} \quad \Rightarrow d = 2\hbar^2,\ g = 0,\ k = 0$$

$$\begin{pmatrix} d & e & f \\ g & h & j \\ k & l & m \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 2\hbar^2\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} e \\ h \\ l \end{pmatrix} = \begin{pmatrix} 0 \\ 2\hbar^2 \\ 0 \end{pmatrix} \quad \boxed{8} \quad \Rightarrow e = 0,\ h = 2\hbar^2,\ l = 0$$

$$\begin{pmatrix} d & e & f \\ g & h & j \\ k & l & m \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 2\hbar^2\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} f \\ j \\ m \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2\hbar^2 \end{pmatrix} \quad \boxed{9} \quad \Rightarrow f = 0,\ j = 0,\ m = 2\hbar^2$$

$\boxed{7}\boxed{8}\boxed{9} \to \boxed{6}$:

$$S^2 = \begin{pmatrix} 2\hbar^2 & 0 & 0 \\ 0 & 2\hbar^2 & 0 \\ 0 & 0 & 2\hbar^2 \end{pmatrix} = 2\hbar^2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \boxed{10}$$

By $\boxed{2}$ $\quad S_z \chi_+ = \hbar \chi_+ \quad , \quad S_z \chi_0 = 0 \quad , \quad S_z \chi_- = -\hbar \chi_-$ $\boxed{11}$

Write S_z as an undetermined matrix:

$$S_z = \begin{pmatrix} d & e & f \\ g & h & j \\ k & l & m \end{pmatrix} \qquad \boxed{12}$$

$\boxed{12} \to \boxed{11}$:

$$\begin{pmatrix} d & e & f \\ g & h & j \\ k & l & m \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \hbar \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} d \\ g \\ k \end{pmatrix} = \begin{pmatrix} \hbar \\ 0 \\ 0 \end{pmatrix} \qquad \boxed{13}$$

$$\begin{pmatrix} d & e & f \\ g & h & j \\ k & l & m \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 0 \quad \Rightarrow \quad \begin{pmatrix} e \\ h \\ l \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \boxed{14}$$

$$\begin{pmatrix} d & e & f \\ g & h & j \\ k & l & m \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = -\hbar \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} f \\ j \\ m \end{pmatrix} = \begin{pmatrix} -\hbar \\ 0 \\ 0 \end{pmatrix} \qquad \boxed{15}$$

$\boxed{13}\boxed{14}\boxed{15} \to \boxed{12}$:

$$* \quad S_z = \begin{pmatrix} \hbar & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\hbar \end{pmatrix} \qquad \boxed{16}$$

By $\boxed{3}$

$$\begin{aligned} S_+ \chi_+ &= 0 & S_- \chi_+ &= \sqrt{2}\hbar \chi_0 \\ S_+ \chi_0 &= \sqrt{2}\hbar \chi_+ & S_- \chi_0 &= \sqrt{2}\hbar \chi_- \\ S_+ \chi_- &= \sqrt{2}\hbar \chi_0 & S_- \chi_- &= 0 \end{aligned} \qquad \boxed{17}$$

Write S_+ as an undetermined matrix:

$$S_+ = \begin{pmatrix} d & e & f \\ g & h & j \\ k & l & m \end{pmatrix} \qquad \boxed{18}$$

$\boxed{18} \to \boxed{17}$:

$$\begin{pmatrix} d & e & f \\ g & h & j \\ k & l & m \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \sqrt{2}\hbar \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} e \\ h \\ l \end{pmatrix} = \begin{pmatrix} \sqrt{2}\hbar \\ 0 \\ 0 \end{pmatrix}$$

and

$$\begin{pmatrix} d & e & f \\ g & h & j \\ k & l & m \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \sqrt{2}\hbar \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} f \\ j \\ m \end{pmatrix} = \begin{pmatrix} 0 \\ \sqrt{2}\hbar \\ 0 \end{pmatrix}$$

so

$$S_+ = \begin{pmatrix} 0 & \sqrt{2}\hbar & 0 \\ 0 & 0 & \sqrt{2}\hbar \\ 0 & 0 & 0 \end{pmatrix} \qquad \boxed{19}$$

Write S_- as an undetermined matrix:

$$S_- = \begin{pmatrix} d & e & f \\ g & h & j \\ k & l & m \end{pmatrix} \qquad \boxed{20}$$

$\boxed{20} \to \boxed{17}$:

$$\begin{pmatrix} d & e & f \\ g & h & j \\ k & l & m \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \sqrt{2}\hbar \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} d \\ g \\ k \end{pmatrix} = \begin{pmatrix} 0 \\ \sqrt{2}\hbar \\ 0 \end{pmatrix} \qquad \boxed{21}$$

$$\begin{pmatrix} d & e & f \\ g & h & j \\ k & l & m \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \sqrt{2}\hbar \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} e \\ h \\ l \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \sqrt{2}\hbar \end{pmatrix} \qquad \boxed{22}$$

so

$$S_- = \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2}\hbar & 0 & 0 \\ 0 & \sqrt{2}\hbar & 0 \end{pmatrix} \qquad \boxed{23}$$

4.31 (cont.) Known $\boxed{3}$: $S_\pm \equiv S_x \pm i S_y$

Then $S_+ + S_- = (S_x + iS_y) + (S_x - iS_y) = 2S_x$

So $S_x = \frac{1}{2}(S_+ + S_-)$ $\boxed{24}$

And $S_+ - S_- = (S_x + iS_y) - (S_x - iS_y) = 2iS_y$

So $S_y = \frac{1}{2i}(S_+ - S_-)$ $\boxed{25}$

Hence, ★ $S_x = \frac{1}{2}\left[\begin{pmatrix} 0 & \sqrt{2}\hbar & 0 \\ 0 & 0 & \sqrt{2}\hbar \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2}\hbar & 0 & 0 \\ 0 & \sqrt{2}\hbar & 0 \end{pmatrix}\right] = \frac{1}{2}\begin{pmatrix} 0 & \sqrt{2}\hbar & 0 \\ \sqrt{2}\hbar & 0 & \sqrt{2}\hbar \\ 0 & \sqrt{2}\hbar & 0 \end{pmatrix}$ $\boxed{26}$

★ $S_y = \frac{1}{2i}\left[\begin{pmatrix} 0 & \sqrt{2}\hbar & 0 \\ 0 & 0 & \sqrt{2}\hbar \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2}\hbar & 0 & 0 \\ 0 & \sqrt{2}\hbar & 0 \end{pmatrix}\right] = \begin{pmatrix} 0 & \frac{\sqrt{2}\hbar}{2i} & 0 \\ -\frac{\sqrt{2}\hbar}{2i} & 0 & \frac{\sqrt{2}\hbar}{2i} \\ 0 & -\frac{\sqrt{2}\hbar}{2i} & 0 \end{pmatrix}$ $\boxed{27}$

4.32 (a) Known the eigenspinors of S_x are

$$\chi_+^{(x)} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \text{ with eigenvalue } +\frac{\hbar}{2} \text{ and} \quad \boxed{1}$$

$$\chi_-^{(x)} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \text{ with eigenvalue } -\frac{\hbar}{2}.$$

If S_x is measured, the probability of getting $+\frac{\hbar}{2}$ is $|\chi_+^{(x)\dagger}\chi|^2$, and in the Stern-Gerlach experiment, $\chi(t) = \begin{pmatrix} \cos(\alpha/2)\, e^{i\gamma B_0 t/2} \\ \sin(\alpha/2)\, e^{-i\gamma B_0 t/2} \end{pmatrix}$, so $\boxed{2}$

$$\chi_+^{(x)\dagger}\chi = \left(\tfrac{1}{\sqrt{2}} \ \tfrac{1}{\sqrt{2}}\right) \begin{pmatrix} \cos(\alpha/2)\, e^{i\gamma B_0 t/2} \\ \sin(\alpha/2)\, e^{-i\gamma B_0 t/2} \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\left(\cos\left(\frac{\alpha}{2}\right) e^{i\gamma B_0 t/2} + \sin\left(\frac{\alpha}{2}\right) e^{-i\gamma B_0 t/2}\right), \text{ then} \quad \boxed{3}$$

$$|\chi_+^{(x)\dagger}\chi|^2 = \cancel{\frac{1}{2}\left(\cos^2\left(\frac{\alpha}{2}\right) e^{i\gamma B_0 t} + \sin^2\left(\frac{\alpha}{2}\right) e^{-i\gamma B_0 t} + 2\cos\left(\frac{\alpha}{2}\right)\sin\left(\frac{\alpha}{2}\right)\right)}$$

$$= \frac{1}{\sqrt{2}}\left(\cos\left(\frac{\alpha}{2}\right)e^{-i\gamma B_0 t/2} + \sin\left(\frac{\alpha}{2}\right)e^{i\gamma B_0 t/2}\right)\frac{1}{\sqrt{2}}\left(\cos\left(\frac{\alpha}{2}\right)e^{i\gamma B_0 t/2} + \sin\left(\frac{\alpha}{2}\right)e^{-i\gamma B_0 t/2}\right)$$

$$= \frac{1}{2}\left(\cos^2\left(\frac{\alpha}{2}\right) + \sin^2\left(\frac{\alpha}{2}\right) + \sin\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{2}\right)e^{-i\gamma B_0 t} + \sin\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{2}\right)e^{i\gamma B_0 t}\right)$$

$$= \frac{1}{2}\left(1 + \sin\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{2}\right)\left(e^{-i\gamma B_0 t} + e^{i\gamma B_0 t}\right)\right)$$

$$= \frac{1}{2} + \sin\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{2}\right)\cos(\gamma B_0 t)$$

$$= \frac{1}{2} + \frac{1}{2}\sin(\alpha)\cos(\gamma B_0 t) \quad \boxed{4}$$

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}$$

$$\sin 2A = 2\sin A\cos A$$

(b) Known the eigenspinors of S_y are

$$\chi_+^{(y)} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix} \text{ with eigenvalue of } +\frac{\hbar}{2} \text{ and} \quad \boxed{5}$$

$$\chi_-^{(y)} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} \end{pmatrix} \text{ with eigenvalue of } -\frac{\hbar}{2}$$

If S_y is measured, the probability of getting $+\frac{\hbar}{2}$ is $|\chi_+^{(y)\dagger}\chi|^2$, so

$$\chi_+^{(y)\dagger}\chi = \left(\tfrac{1}{\sqrt{2}} \ -\tfrac{i}{\sqrt{2}}\right) \begin{pmatrix} \cos(\alpha/2)\, e^{i\gamma B_0 t/2} \\ \sin(\alpha/2)\, e^{-i\gamma B_0 t/2} \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\left(\cos\left(\frac{\alpha}{2}\right)e^{i\gamma B_0 t/2} - i\sin\left(\frac{\alpha}{2}\right)e^{-i\gamma B_0 t/2}\right), \text{ then} \quad \boxed{6}$$

4.32 (b) (cont.)

$$|\chi_+^{(y)\dagger}\chi|^2 = \frac{1}{\sqrt{2}}\left(\cos\left(\frac{\alpha}{2}\right)e^{-i\gamma B_0 t/2} + i\sin\left(\frac{\alpha}{2}\right)e^{+i\gamma B_0 t/2}\right)\frac{1}{\sqrt{2}}\left(\cos\left(\frac{\alpha}{2}\right)e^{i\gamma B_0 t/2} - i\sin\left(\frac{\alpha}{2}\right)e^{-i\gamma B_0 t/2}\right)$$

$$= \frac{1}{2}\left(\cos^2\left(\frac{\alpha}{2}\right) - i\sin\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{2}\right)e^{-i\gamma B_0 t} + i\sin\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{2}\right)e^{i\gamma B_0 t} + \sin^2\left(\frac{\alpha}{2}\right)\right)$$

$$= \frac{1}{2}\left(1 + i\sin\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{2}\right)\left(-e^{-i\gamma B_0 t} + e^{i\gamma B_0 t}\right)\right)$$

$$= \frac{1}{2} + i^2\sin\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{2}\right)\sin(\gamma B_0 t)$$

$$= \frac{1}{2} - \frac{1}{2}\sin(\alpha)\sin(\gamma B_0 t) \quad \boxed{7}$$

(c) Known the eigenspinor of S_z are

$\chi_+^{(z)} = \begin{pmatrix}1\\0\end{pmatrix}$ with eigenvalue of $+\frac{\hbar}{2}$ and $\boxed{8}$

$\chi_-^{(z)} = \begin{pmatrix}0\\1\end{pmatrix}$ with eigenvalue of $\frac{\hbar}{2}$

If S_z is measured, the probability of getting $+\frac{\hbar}{2}$ is $|\chi_+^{(z)\dagger}\chi|^2$, so

$$\chi_+^{(z)\dagger}\chi = \begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}\cos(\alpha/2)\,e^{i\gamma B_0 t/2}\\ \sin(\alpha/2)\,e^{-i\gamma B_0 t/2}\end{pmatrix}$$

$$= \cos\left(\frac{\alpha}{2}\right)e^{i\gamma B_0 t/2} \quad \boxed{9}$$

$$|\chi_+^{(z)\dagger}\chi|^2 = \cos^2\left(\frac{\alpha}{2}\right)$$

8 May 2019 4.4 Spin

4.35 (a) $S_- \,|1\,0\rangle = (S_-^{(1)} + S_-^{(2)})\left(\frac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow)\right)$

$= \frac{1}{\sqrt{2}}\left((S_-^{(1)}\uparrow)\downarrow + (S_-^{(1)}\downarrow)\uparrow + \uparrow(S_-^{(2)}\downarrow) + \downarrow(S_-^{(2)}\uparrow)\right)$

$= \frac{1}{\sqrt{2}}(\hbar\downarrow\downarrow + 0 + 0 + \hbar\downarrow\downarrow)$

$= \frac{2\hbar}{\sqrt{2}}\downarrow\downarrow$

$= \sqrt{2}\hbar\,|1\;-1\rangle$

(b) $S_+ \,|0\;0\rangle = (S_+^{(1)} + S_+^{(2)})\left(\frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)\right)$

$= \frac{1}{\sqrt{2}}\left((S_+^{(1)}\uparrow)\downarrow - (S_+^{(1)}\downarrow)\uparrow + \uparrow(S_+^{(2)}\downarrow) - \downarrow(S_+^{(2)}\uparrow)\right)$

$= \frac{1}{\sqrt{2}}\left(0 - \hbar\uparrow\uparrow + \uparrow\hbar\uparrow - 0\right)$

$= 0$

$S_- \,|0\;0\rangle = (S_-^{(1)} + S_-^{(2)})\left(\frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)\right)$

$= \frac{1}{\sqrt{2}}\left((S_-^{(1)}\uparrow)\downarrow - (S_-^{(1)}\downarrow)\uparrow + \uparrow(S_-^{(2)}\downarrow) - \downarrow(S_-^{(2)}\uparrow)\right)$

$= \frac{1}{\sqrt{2}}(\hbar\downarrow\downarrow - 0 + 0 - \downarrow\hbar\downarrow)$

$= 0$

(c) Show that $|1\,1\rangle$ and $|1\;-1\rangle$ are eigenstates of S^2

Known $S^2\chi_+ = \frac{3}{4}\hbar^2\chi_+$ and $S^2\chi_- = \frac{3}{4}\hbar^2\chi_-$ $\boxed{1}$

and $S^2 = (S^{(1)} + S^{(2)})(S^{(1)} + S^{(2)}) = (S^{(1)})^2 + (S^{(2)})^2 + 2S^{(1)}S^{(2)}$ $\boxed{2}$

and $S_z = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$, $S_x = \frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$, $S_y = \frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$ $\boxed{3}$

$S^{(1)}S^{(2)}(\uparrow\downarrow) = (S_x^{(1)}\uparrow)(S_x^{(2)}\downarrow) + (S_y^{(1)}\uparrow)(S_y^{(2)}\downarrow) + (S_z^{(1)}\uparrow)(S_z^{(2)}\downarrow)$

$= \left(\frac{\hbar}{2}\downarrow\right)\left(\frac{\hbar}{2}\uparrow\right) + \left(\frac{i\hbar}{2}\downarrow\right)\left(-\frac{i\hbar}{2}\uparrow\right) + \left(\frac{\hbar}{2}\uparrow\right)\left(-\frac{\hbar}{2}\downarrow\right)$

$= \frac{\hbar^2}{4}\downarrow\uparrow + \frac{\hbar^2}{4}\downarrow\uparrow - \frac{\hbar^2}{4}\uparrow\downarrow$

$= \frac{\hbar^2}{4}(2\downarrow\uparrow - \uparrow\downarrow)$ $\boxed{4}$

$$S^{(1)}S^{(2)}(\downarrow\uparrow) = (S_x^{(1)}\downarrow)(S_x^{(2)}\uparrow) + (S_y^{(1)}\downarrow)(S_y^{(2)}\uparrow) + (S_z^{(1)}\downarrow)(S_z^{(2)}\uparrow)$$

$$= \left(\tfrac{\hbar}{2}\uparrow\right)\left(\tfrac{\hbar}{2}\downarrow\right) + \left[\begin{pmatrix} 0 & -\frac{i\hbar}{2} \\ \frac{\hbar i}{2} & 0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}\right]\left[\begin{pmatrix} 0 & -\frac{i\hbar}{2} \\ \frac{\hbar i}{2} & 0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\right] + \left[\begin{pmatrix} \frac{\hbar}{2} & 0 \\ 0 & -\frac{\hbar}{2}\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} + \begin{pmatrix} \frac{\hbar}{2} & 0 \\ 0 & -\frac{\hbar}{2}\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\right]$$

$$= \frac{\hbar^2}{4}\uparrow\downarrow + \left(-\frac{i\hbar}{2}\uparrow\right)\left(\frac{i\hbar}{2}\downarrow\right) + \left(-\frac{\hbar}{2}\downarrow\right)\left(\frac{\hbar}{2}\uparrow\right)$$

$$= \frac{\hbar^2}{4}\left(2\uparrow\downarrow - \downarrow\uparrow\right) \qquad \boxed{5}$$

For $|1\,1\rangle$,

$$S^2|1\,1\rangle = (S^{(1)})^2|1\,1\rangle + (S^{(2)})^2|1\,1\rangle + 2S^{(1)}\cdot S^{(2)}|1\,1\rangle \qquad \boxed{6}$$

$$S^{(1)}\cdot S^{(2)}|1\,1\rangle = S^{(1)}\cdot S^{(2)}(\uparrow\uparrow) = (S_x^{(1)}\uparrow)(S_x^{(2)}\uparrow) + (S_y^{(1)}\uparrow)(S_y^{(2)}\uparrow) + (S_z^{(1)}\uparrow)(S_y^{(2)}\uparrow)$$

$$= \left(\tfrac{\hbar}{2}\downarrow\right)\left(\tfrac{\hbar}{2}\downarrow\right) + \left(\tfrac{i\hbar}{2}\downarrow\right)\left(\tfrac{i\hbar}{2}\downarrow\right) + \left(\tfrac{\hbar}{2}\uparrow\right)\left(\tfrac{\hbar}{2}\uparrow\right)$$

$$= \frac{\hbar^2}{4}\downarrow\downarrow - \frac{\hbar^2}{4}\downarrow\downarrow + \frac{\hbar^2}{4}\uparrow\uparrow$$

$$= \frac{\hbar^2}{4}\uparrow\uparrow = \frac{\hbar^2}{4}|1\,1\rangle \qquad \boxed{7}$$

$$\cancel{= \frac{\hbar^2}{4}|1\,-1\rangle}$$

$$\cancel{S^{(1)}\cdot S^{(2)}|1\,1\rangle = x\,|1\,1\rangle}$$

~~since~~ $\cancel{\frac{\hbar^2}{4}\downarrow\downarrow = x\uparrow\uparrow}$ $\quad S^2|s\,m\rangle = \hbar^2 s(s+1)|s\,m\rangle$, $\qquad \boxed{8}$

$\cancel{x = 2\hbar^2}$

By $\boxed{1}$: $(S^{(1)})^2(\uparrow\uparrow) = ((S^{(1)})^2\uparrow)\uparrow = (\frac{3}{4}\hbar^2\uparrow)\uparrow = \frac{3}{4}\hbar^2\uparrow\uparrow \qquad \boxed{8}\ \boxed{9}$

$(S^{(2)})^2(\uparrow\uparrow) = \uparrow((S^{(2)})^2\uparrow) = \uparrow\frac{3}{4}\hbar^2\uparrow = \frac{3}{4}\hbar^2\uparrow\uparrow \qquad \boxed{9}\ \boxed{10}$

$\boxed{7}, \boxed{9}, \boxed{10} \to \boxed{6}$: $S^2|1\,1\rangle = \frac{3}{4}\hbar^2\uparrow\uparrow + \frac{3}{4}\hbar^2\uparrow\uparrow + 2\,\frac{\hbar^2}{4}\uparrow\uparrow$

$$= 2\hbar^2\uparrow\uparrow = 2\hbar^2|1\,1\rangle \qquad \boxed{11}$$

which follows $\boxed{8}$: $S^2|1\,1\rangle = \hbar^2(1(1+1))|1\,1\rangle = 2\hbar^2|1\,1\rangle$.

For $|1\ -1\rangle$:

$$S^2\,|1\ -1\rangle = (S^{(1)}+S^{(2)})^2\,|1\ -1\rangle = (S^{(1)})^2|1\ -1\rangle + (S^{(2)})^2|1\ -1\rangle + 2S^{(1)}\cdot S^{(2)}\,|1\ -1\rangle \quad \boxed{12}$$

$$S^{(1)}\cdot S^{(2)}\,|1\ -1\rangle = S^{(1)}\cdot S^{(2)}\,(\downarrow\downarrow) = (S_x^{(1)}\downarrow)(S_x^{(2)}\downarrow) + (S_y^{(1)}\downarrow)(S_y^{(2)}\downarrow) + (S_z^{(1)}\downarrow)(S_z^{(2)}\downarrow)$$

$$= \left(\tfrac{\hbar}{2}\uparrow\right)\left(\tfrac{\hbar}{2}\uparrow\right) + \left(\tfrac{i\hbar}{2}\uparrow\right)\left(\tfrac{i\hbar}{2}\uparrow\right) + \left(-\tfrac{\hbar}{2}\downarrow\right)\left(-\tfrac{\hbar}{2}\downarrow\right)$$

$$= \tfrac{\hbar^2}{4}\uparrow\uparrow - \tfrac{\hbar^2}{4}\uparrow\uparrow + \tfrac{\hbar^2}{4}\downarrow\downarrow$$

$$= \tfrac{\hbar^2}{4}\downarrow\downarrow \quad \boxed{13}$$

$$(S^{(1)})^2\,|1\ -1\rangle = (S^{(1)})^2\downarrow\downarrow = ((S^{(1)})^2\downarrow)\downarrow = \tfrac{3}{4}\hbar^2\downarrow\downarrow \quad \boxed{14}$$

$$(S^{(2)})^2\,|1\ -1\rangle = (S^{(2)})^2\downarrow\downarrow = \downarrow((S^{(2)})^2\downarrow) = \downarrow\tfrac{3}{4}\hbar^2\downarrow \quad \boxed{15}$$

$\boxed{13},\boxed{14},\boxed{15} \rightarrow \boxed{12}$: $S^2|1\ -1\rangle = \tfrac{3}{4}\hbar^2\downarrow\downarrow + \tfrac{3}{4}\hbar^2\downarrow\downarrow + 2\cdot\tfrac{\hbar^2}{4}\downarrow\downarrow$

$$= 2\hbar^2\downarrow\downarrow$$

which is consistant with $\boxed{8}$: $S^2\,|1\ -1\rangle = \hbar^2\,(1(1+1))\,|1\ -1\rangle = 2\hbar^2\,|1\ -1\rangle$.

5.1 (a) Let $\vec{r} \equiv \vec{r}_1 - \vec{r}_2$ $\boxed{1}$

$$\vec{R} = \frac{(m_1\vec{r}_1 + m_2\vec{r}_2)}{(m_1+m_2)} \quad \boxed{2}$$

Show that $\vec{r}_1 = \vec{R} + \frac{\mu}{m_1}\vec{r}$, $\vec{r}_2 = \vec{R} - \frac{\mu}{m_2}\vec{r}$, and

$$\nabla_1 = \frac{\mu}{m_2}\nabla_R + \nabla_r, \quad \nabla_2 = \frac{\mu}{m_1}\nabla_R - \nabla_r, \text{ where}$$

$$\mu \equiv \frac{m_1 m_2}{m_1+m_2} \quad \boxed{3}$$

(i) $\vec{r}_1 = \vec{R} + \frac{\mu}{m_1}\vec{r}$

$$= \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1+m_2} + \frac{m_1 m_2}{m_1+m_2}\frac{1}{m_1}(\vec{r}_1 - \vec{r}_2)$$

$$= \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_2\vec{r}_1 - m_2\vec{r}_2}{m_1+m_2}$$

$$= \frac{\vec{r}_1(m_1+m_2)}{m_1+m_2}$$

$$= \vec{r}_1$$

(ii) $\vec{r}_2 = \vec{R} - \frac{\mu}{m_2}\vec{r}$

$$= \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1+m_2} - \frac{m_1 m_2}{m_1+m_2}\frac{1}{m_2}(\vec{r}_1 - \vec{r}_2)$$

$$= \frac{m_1\vec{r}_1 + m_2\vec{r}_2 - m_1\vec{r}_1 + m_1\vec{r}_2}{m_1+m_2}$$

$$= \frac{\vec{r}_2(m_1+m_2)}{m_1+m_2}$$

$$= \vec{r}_2$$

(iii) $(\nabla_1)_x = \frac{\mu}{m_2}(\nabla_R)_x + (\nabla_r)_x$

$$= \frac{m_1 m_2}{m_1+m_2}\frac{1}{m_2}\frac{\partial}{\partial x_R} + \frac{\partial}{\partial x_r}$$

$$= \frac{m_1}{m_1+m_2}\frac{\partial x_1}{\partial x_R}\frac{\partial}{\partial x_1} + \frac{\partial x_1}{\partial x_r}\frac{\partial}{\partial x_1}$$

$$= \left(\frac{m_1}{m_1+m_2}\frac{\partial x_1}{\partial x_R} + \frac{\partial x_1}{\partial x_r}\right)\frac{\partial}{\partial x_1}$$

$$= \left[\frac{m_1}{m_1+m_2}\frac{\partial}{\partial x_R}\left(R_x + \frac{\mu}{m_1}r_x\right) + \frac{\partial}{\partial x_r}\left(R_x + \frac{\mu}{m_1}r_x\right)\right]\frac{\partial}{\partial x_1}$$

$$= \left[\frac{m_1}{m_1+m_2} + \frac{m_1 m_2}{m_1+m_2}\frac{1}{m_1}\right]\frac{\partial}{\partial x_1}$$

$$= \frac{\partial}{\partial x_1} = (\nabla_1)_x$$

Since it is valid for x, it's also valid for y and z.

$$(\nabla_2)_x = \frac{\mu}{m_1}(\nabla_R)_x - (\nabla_r)_x$$

(iv)

$$= \frac{m_1 m_2}{m_1+m_2}\frac{1}{m_1}\frac{\partial}{\partial R_x} - \frac{\partial}{\partial r_x}$$

$$= \frac{m_2}{m_1+m_2}\frac{\partial x_2}{\partial R_x}\frac{\partial}{\partial x_2} - \frac{\partial x_2}{\partial r_x}\frac{\partial}{\partial x_2}$$

$$= \left[\frac{m_2}{m_1+m_2}\frac{\partial}{\partial R_x}\left(R_x - \frac{\mu}{m_2}r_x\right) - \frac{\partial}{\partial r_x}\left(R_x - \frac{\mu}{m_2}r_x\right)\right]\frac{\partial}{\partial x_2}$$

$$= \left(\frac{m_2}{m_1+m_2} - \frac{m_1 m_2}{m_1+m_2}\frac{1}{m_2}\right)\frac{\partial}{\partial x_2}$$

$$= \frac{\partial}{\partial x_2} = (\nabla_2)_x \qquad (\nabla_R^2)_x = \left(\frac{\partial}{\partial R_x}\right)^2$$

(b) The time-independent Schrödinger equation is

$$-\frac{\hbar^2}{2m_1}\nabla_1^2\psi - \frac{\hbar^2}{2m_2}\nabla_2^2\psi + V\psi = E\psi \qquad \boxed{1}$$

Show that it becomes

$$-\frac{\hbar^2}{2(m_1+m_2)}\nabla_R^2\psi - \frac{\hbar^2}{2\mu}\nabla_r^2\psi + V(\vec{r})\psi = E\psi$$

~~for x: $\boxed{1}$:~~

$$\cancel{-\frac{\hbar^2}{2m_1}\left(\frac{\partial}{\partial x_1}\right)^2\psi - \frac{\hbar^2}{2m_2}\left(\frac{\partial}{\partial x_2}\right)^2\psi + V(x)\psi = E\psi}$$

$$\cancel{-\frac{\hbar^2}{2m_1}\left(\frac{\partial R_x}{\partial x_1}\right)^2\left(\frac{\partial}{\partial R_x}\right)^2\psi - \frac{\hbar^2}{2m_2}\left(\frac{\partial r_x}{\partial x_2}\right)^2\left(\frac{\partial}{\partial r_x}\right)^2\psi + V(x)\psi = E\psi}$$

$$\cancel{-\frac{\hbar^2}{2m_1}\left(\frac{\partial}{\partial x}\left[\frac{m_1x_1+m_2x_2}{m_1+m_2}\right]\right)^2(\nabla_R)_x^2\psi - \frac{\hbar^2}{2m_2}\left(\frac{\partial}{\partial x_2}\left[x_1-x_2\right]\right)^2(\nabla_r)_x^2\psi + V(x)\psi = E\psi}$$

$$\cancel{-\frac{\hbar^2}{2m_1}\left(\frac{m_1}{m_1+m_2}\right)^2(\nabla_R)_x^2\psi - \frac{\hbar^2}{2m_2}(-1)^2(\nabla_r)_x^2\psi + V(x)\psi = E\psi}$$

$$\nabla_1^2\psi = \nabla_1\cdot\nabla_1\psi = \nabla_1\cdot\left(\frac{\mu}{m_2}\nabla_R\psi + \nabla_r\psi\right) = \frac{\mu}{m_2}\nabla_R\left(\frac{\mu}{m_2}\nabla_R\psi + \nabla_r\psi\right) + \nabla_r\left(\frac{\mu}{m_2}\nabla_R\psi + \nabla_r\psi\right)$$

$$= \left(\frac{\mu}{m_2}\right)^2\nabla_R^2\psi + \frac{\mu}{m_2}\nabla_R\nabla_r\psi + \frac{\mu}{m_2}\nabla_r\nabla_R\psi + \nabla_r^2\psi = \left(\frac{\mu}{m_2}\right)^2\nabla_R^2\psi + 2\frac{\mu}{m_2}\nabla_R\nabla_r\psi + \nabla_r^2\psi \qquad \boxed{2}$$

$$\nabla_2^2\psi = \nabla_2\cdot\nabla_2\psi = \nabla_2\cdot\left(\frac{\mu}{m_1}\nabla_R\psi - \nabla_r\psi\right) = \frac{\mu}{m_1}\nabla_R\left(\frac{\mu}{m_1}\nabla_R\psi - \nabla_r\psi\right) - \nabla_r\left(\frac{\mu}{m_1}\nabla_R\psi - \nabla_r\psi\right)$$

$$= \left(\frac{\mu}{m_1}\right)^2\nabla_R^2\psi - \frac{\mu}{m_1}\nabla_R\nabla_r\psi - \frac{\mu}{m_1}\nabla_r\nabla_R\psi + \nabla_r^2\psi = \left(\frac{\mu}{m_1}\right)^2\nabla_R^2\psi - 2\frac{\mu}{m_1}\nabla_R\nabla_r\psi + \nabla_r^2\psi \qquad \boxed{3}$$

$\boxed{2}\boxed{3} \to \boxed{1}$:

$$-\frac{\hbar^2}{2m_1}\left[\left(\frac{M}{M_2}\right)^2 \nabla_R^2 \psi + 2\frac{M}{M_2}\nabla_R \nabla_r \psi + \nabla_r^2 \psi\right] - \frac{\hbar^2}{2M_2}\left[\left(\frac{M}{M_1}\right)^2 \nabla_R^2 \psi - 2\frac{M}{M_1}\nabla_R\nabla_r\psi + \nabla_r^2\psi\right] + V\psi = E\psi$$

$$\left[-\frac{\hbar^2}{2M_1}\left(\frac{M}{M_2}\right)^2 - \frac{\hbar^2}{2M_2}\left(\frac{M}{M_1}\right)^2\right]\nabla_R^2\psi + \left(-\frac{\hbar^2}{2M_1}\cdot 2\frac{M}{M_2} + \frac{\hbar^2}{2M_2}\cdot 2\frac{M}{M_1}\right)\nabla_R\nabla_r\psi - \left(+\frac{\hbar^2}{2M_1} + \frac{\hbar^2}{2M_2}\right)\nabla_r^2\psi + V\psi = E\psi$$

$$\left[-\frac{\hbar^2}{2}\frac{M_1}{(M_1+M_2)^2} - \frac{\hbar^2}{2}\frac{M_2}{(M_1+M_2)^2}\right]\nabla_R^2\psi + 0 - \left(\frac{M_1+M_2}{M_1\cdot M_2}\right)\frac{\hbar^2}{2}\nabla_r^2\psi + V\psi = E\psi$$

$$-\left(\frac{1}{M_1+M_2}\right)\frac{\hbar^2}{2}\nabla_R^2\psi - \left(\frac{M_1+M_2}{M_1 M_2}\right)\frac{\hbar^2}{2}\nabla_r^2\psi + V\psi = E\psi$$

$$-\frac{\hbar^2}{2(M_1+M_2)}\nabla_R^2\psi - \frac{\hbar^2}{2\mu}\nabla_r^2\psi + V\psi = E\psi \qquad \boxed{4}$$

(c) Separate variables, let $\psi(\vec{R},\vec{r}) = \psi_R(\vec{R})\psi_r(\vec{r})$ $\boxed{5}$

$\boxed{5} \to \boxed{4}$

$$-\frac{\hbar^2}{2(M_1+M_2)}\nabla_R^2\left[\psi_R(\vec{R})\psi_r(\vec{r})\right] - \frac{\hbar^2}{2\mu}\nabla_r^2\left[\psi_R(\vec{R})\psi_r(\vec{r})\right] + V\left[\psi_R(\vec{R})\psi_r(\vec{r})\right] = E\left[\psi_R(\vec{R})\psi_r(\vec{r})\right]$$

Divide by $\left[\psi_R(\vec{R})\psi_r(\vec{r})\right]$:

partial derivative, treat as constant

$$-\frac{\hbar^2}{2(M_1+M_2)}\frac{1}{\psi_R(\vec{R})\psi_r(\vec{r})}\nabla_R^2\left[\psi_R(\vec{R})\psi_r(\vec{r})\right] - \frac{\hbar^2}{2\mu}\frac{1}{\psi_R(\vec{R})\psi_r(\vec{r})}\nabla_r^2\left[\psi_R(\vec{R})\psi_r(\vec{r})\right] + V = E$$

$$\underbrace{-\frac{\hbar^2}{2(M_1+M_2)}\frac{1}{\psi_R(\vec{R})}\nabla_R^2\psi_R(\vec{R})}_{\text{Depend on } R} - \underbrace{\frac{\hbar^2}{2\mu}\frac{1}{\psi_r(\vec{r})}\nabla_r^2\psi_r(\vec{r})}_{\text{Depend on } r} + V = E$$

5.4 (a) Known $\psi_{\pm}(\vec{r_1},\vec{r_2}) = A[\psi_a(\vec{r_1})\psi_b(\vec{r_2}) \pm \psi_b(\vec{r_1})\psi_a(\vec{r_2})]$ ψ_a and ψ_b are orthogonal and normalized.

Find A.

Since they are orthonormal,

$$\iint \psi_+^* \psi_- \, d\vec{r_1}\, d\vec{r_2}$$

$$= \iint A^*[\psi_a^*(\vec{r_1})\psi_b^*(\vec{r_2}) + \psi_b^*(\vec{r_1})\psi_a^*(\vec{r_2})] \cdot A[\psi_a(\vec{r_1})\psi_b(\vec{r_2}) - \psi_b(\vec{r_1})\psi_a(\vec{r_2})] \, d\vec{r_1}\, d\vec{r_2}$$

$$= A^2 \iint [|\psi_a(\vec{r_1})|^2|\psi_b(\vec{r_2})|^2 - \psi_a^*(\vec{r_1})\psi_a(\vec{r_2})\psi_b^*(\vec{r_2})\psi_b(\vec{r_1}) + \psi_a^*(\vec{r_2})\psi_a(\vec{r_1})\psi_b^*(\vec{r_1})\psi_b(\vec{r_2})$$

$$- |\psi_b(\vec{r_1})|^2|\psi_a(\vec{r_2})|^2] \, d\vec{r_1}\, d\vec{r_2}$$

For ψ_+,

$$1 = \iint \psi_+^* \psi_+ \, d\vec{r_1}\, d\vec{r_2}$$

$$= A^2 \iint (\psi_a^*(\vec{r_1})\psi_b^*(\vec{r_2}) + \psi_b^*(\vec{r_1})\psi_a^*(\vec{r_2})) (\psi_a(\vec{r_1})\psi_b(\vec{r_2}) + \psi_b(\vec{r_1})\psi_a(\vec{r_2})) \, d\vec{r_1}\, d\vec{r_2}$$

$$= A^2 \iint |\psi_a(\vec{r_1})|^2|\psi_b(\vec{r_2})|^2 + \psi_a^*(\vec{r_1})\psi_a(\vec{r_2})\psi_b^*(\vec{r_2})\psi_b(\vec{r_1}) + \psi_a^*(\vec{r_2})\psi_a(\vec{r_1})\psi_b(\vec{r_2})\psi_b^*(\vec{r_1})$$

$$+ |\psi_a(\vec{r_2})|^2|\psi_b(\vec{r_1})|^2 \, d\vec{r_1}\, d\vec{r_2}$$

Since ψ_a and ψ_b are orthonormal,

$$1 = A^2 [(1\cdot 1) + (0\cdot 0) + (0\cdot 0) + (1\cdot 1)]$$

$$1 = 2A^2$$

$$A = \frac{1}{\sqrt{2}}$$

For ψ_-,

$$1 = \iint \psi_-^* \psi_- \, d\vec{r_1}\, d\vec{r_2}$$

$$= A^2 \iint (\psi_a^*(\vec{r_1})\psi_b^*(\vec{r_2}) - \psi_b^*(\vec{r_1})\psi_a^*(\vec{r_2})) \cdot (\psi_a(\vec{r_1})\psi_b(\vec{r_2}) - \psi_b(\vec{r_1})\psi_a(\vec{r_2})) \, d\vec{r_1}\, d\vec{r_2}$$

$$= A^2 \iint |\psi_a(\vec{r_1})|^2|\psi_b(\vec{r_2})|^2 - \psi_a^*(\vec{r_1})\psi_b(\vec{r_1})\psi_b^*(\vec{r_2})\psi_a(\vec{r_2}) - \psi_b^*(\vec{r_1})\psi_a(\vec{r_1})\psi_a^*(\vec{r_2})\psi_b(\vec{r_2})$$

$$+ |\psi_b(\vec{r_1})|^2|\psi_a(\vec{r_2})|^2 \, d\vec{r_1}\, d\vec{r_2}$$

$$= A^2 [(1\cdot 1) - (0\cdot 0) - (0\cdot 0) + (1\cdot 1)]$$

$$= 2A^2$$

$$A = \frac{1}{\sqrt{2}}$$

5.4 (b) Find A if $\psi_a = \psi_b$

$$\psi_\pm(\vec{r}_1,\vec{r}_2) = A(\psi_a(\vec{r}_1)\psi_a(\vec{r}_2) \pm \psi_a(\vec{r}_1)\psi_b(\vec{r}_2)) = \begin{cases} 2A\,\psi_a(\vec{r}_1)\psi_a(\vec{r}_2) & \text{if } \psi_+ \\ 0 & \text{if } \psi_- \end{cases}$$

For ψ_+: $1 = \iint \psi_+^* \psi_+ \, d\vec{r}_1 \, d\vec{r}_2$

$$= 4A^2 \iint (\psi_a^*(\vec{r}_1)\psi_a^*(\vec{r}_2)) \cdot (\psi_a(\vec{r}_1)\psi_a(\vec{r}_2)) \, d\vec{r}_1 \, d\vec{r}_2$$

$$= 4A^2 \iint |\psi_a(\vec{r}_1)|^2 |\psi_a(\vec{r}_2)|^2 \, d\vec{r}_1 \, d\vec{r}_2$$

$$= 4A^2$$

$$A = \frac{1}{2}$$

For ψ_-: $1 = \iint \psi_-^* \psi_- \, d\vec{r}_1 \, d\vec{r}_2$

$$1 = \iint 0 \, d\vec{r}_1 \, d\vec{r}_2$$

DNE

5.6 Known two particles in infinite square well, each in ψ_n and ψ_l $(l \neq n)$ state.

Find $\langle (x_1 - x_2)^2 \rangle = \langle x_1^2 \rangle^* - 2\langle x_1 x_2 \rangle + \langle x_2^2 \rangle^*$

(a) Distinguishable particle's wave function is

$$\psi(x_1, x_2) = \psi_n(x_1)\psi_l(x_2)$$

$$\langle x_1^2 \rangle^* = \iint \psi_n^*(x_1)\psi_l^*(x_2) \; x_1^2 \; \psi_n(x_1)\psi_l(x_2) \, dx_1 \, dx_2$$

$$= \iint |\psi_n(x_1)|^2 |\psi_l(x_2)|^2 x_1^2 \, dx_1 \, dx_2$$

$$= \int |\psi_n(x_1)|^2 x_1^2 \, dx_1 \int |\psi_l(x_2)|^2 dx_2$$

$$= \int \psi_n^*(x_1) \, x_1^2 \, \psi_n(x_1) \, dx_1$$

$$= \langle x^2 \rangle_n$$

$$\langle x_2^2 \rangle = \iint \psi_n^*(x_1)\psi_l^*(x_2) \; x_2^2 \; \psi_n(x_1)\psi_l(x_2) \, dx_1 \, dx_2$$

$$= \int |\psi_n(x_1)|^2 dx_1 \int |\psi_l(x_2)|^2 x_2^2 \, dx_2$$

$$= \int \psi_l^*(x_2) \, x_2^2 \, \psi_l(x_2) \, dx_2$$

$$= \langle x^2 \rangle_l$$

$$\langle x_1 x_2 \rangle = \iint \psi_n^*(x_1)\psi_l^*(x_2) \; x_1 x_2 \; \psi_n(x_1)\psi_l(x_2) \, dx_1 \, dx_2$$

$$= \int \psi_n^*(x_1) \, x_1 \, \psi_n(x_1) \, dx_1 \int \psi_l^*(x_2) \, x_2 \, \psi_l(x_2) \, dx_2$$

$$= \langle x \rangle_n \langle x \rangle_l$$

$\langle x \rangle = \frac{a}{2}$

$\langle x^2 \rangle = \frac{a^2}{3} - \frac{a^2}{2n^2\pi^2}$ in infinite square well

So for distinguishable particles in the infinite squarewell,

$$\langle (x_1 - x_2)^2 \rangle = \langle x^2 \rangle_n - 2 \langle x \rangle_n \langle x \rangle_l + \langle x^2 \rangle_l$$

$$\cancel{= \frac{a}{2} - 2\left(\frac{1}{3}a^2 - \frac{a^2}{2n^2\pi^2}\right) + \frac{a}{2}}$$

$$\cancel{= a - \frac{2a^2}{3} + \frac{8a^2}{n^2\pi^2}}$$

$$= \left(\frac{1}{3}a^2 - \frac{a^2}{2n^2\pi^2}\right) - 2 \cdot \frac{a}{2} \cdot \frac{a}{2} + \left(\frac{1}{3}a^2 - \frac{a^2}{2M^2\pi^2}\right)$$

$$= \frac{a^2}{3} - \frac{a^2}{2n^2\pi^2} - \frac{a^2}{2} + \frac{a^2}{3} - \frac{a^2}{2M^2\pi^2}$$

$$= a^2\left[\frac{1}{6} - \frac{1}{2\pi^2}\left(\frac{1}{M^2} + \frac{1}{n^2}\right)\right]$$

1b) Identical Bosons' wave functions are

$$\psi_+(x_1, x_2) = \frac{1}{\sqrt{2}}\left[\psi_a(x_1)\psi_b(x_2) + \psi_b(x_1)\psi_a(x_2)\right]$$

and identical fermions' wave functions are

$$\psi_-(x_1, x_2) = \frac{1}{\sqrt{2}}\left[\psi_a(x_1)\psi_b(x_2) + \psi_b(x_1)\psi_a(x_2)\right]$$

$$\langle x_1^2 \rangle = \iint \frac{1}{\sqrt{2}}\left[\psi_a^*(x_1)\psi_b^*(x_2) \pm \psi_b^*(x_1)\psi_a^*(x_2)\right] x_1^2 \frac{1}{\sqrt{2}}\left[\psi_a(x_1)\psi_b(x_2) \pm \psi_b(x_1)\psi_a(x_2)\right] dx_1\, dx_2$$

$$= \frac{1}{2}\iint x_1^2 \Big(\psi_a^*(x_1)\psi_a(x_1)\psi_b^*(x_2)\psi_b(x_2) \pm \psi_a^*(x_1)\psi_b(x_1)\psi_b^*(x_2)\psi_a(x_2) \pm$$
$$\psi_b^*(x_1)\psi_a(x_1)\psi_a^*(x_2)\psi_b(x_2) + \psi_b^*(x_1)\psi_b(x_1)\psi_a^*(x_2)\psi_a(x_2)\Big)\, dx_1\, dx_2$$

$$= \frac{1}{2}\Big[\int \psi_a^*(x_1)x_1^2\psi_a(x_1)\,dx_1 \int \psi_b^*(x_2)\psi_b(x_2)\,dx_2$$
$$\pm \int \psi_a^*(x_1)x_1^2\psi_b(x_1)\,dx_1 \int \psi_b^*(x_2)\psi_a(x_2)\,dx_2$$
$$\pm \int \psi_b^*(x_1)x_1^2\psi_a(x_1)\,dx_1 \int \psi_a^*(x_2)\psi_b(x_2)\,dx_2$$
$$+ \int \psi_b^*(x_1)x_1^2\psi_b(x_1)\,dx_1 \int \psi_a^*(x_2)\psi_a(x_2)\,dx_2\Big]$$

$$= \frac{1}{2}\left(\langle x_1^2 \rangle_a \cdot 1 \pm 0 \pm 0 + \langle x_1^2 \rangle_b \cdot 1\right)$$

$$= \frac{1}{2}\left(\langle x_1^2 \rangle_a + \langle x_1^2 \rangle_b\right)$$

$\langle x_2 \rangle^2 = \langle x_1 \rangle^2$ since they are identical particles.

$$\langle x_1 x_2 \rangle = \frac{1}{2}\Big[\int \psi_a^*(x_1)x_1\psi_a(x_1)\,dx_1 \int \psi_b^*(x_2)x_2\psi_b(x_2)\,dx_2$$
$$\pm \int \psi_a^*(x_1)x_1\psi_b(x_1)\,dx_1 \int \psi_b^*(x_2)x_2\psi_a(x_2)\,dx_2$$
$$\pm \int \psi_b^*(x_1)x_1\psi_a(x_1)\,dx_1 \int \psi_a^*(x_2)x_2\psi_b(x_2)\,dx_2$$
$$+ \int \psi_b^*(x_1)x_1\psi_b(x_1)\,dx_1 \int \psi_a^*(x_2)x_2\psi_a(x_2)\,dx_2\Big]$$

$$= \frac{1}{2}\Big[\langle x_1 \rangle_a \langle x_2 \rangle_b \pm \int \psi_a^*(x_1)x_1\psi_b(x_1)\,dx_1 \int \psi_b^*(x_2)x_2\psi_a(x_2)\,dx_2$$
$$+ \langle x_1 \rangle_b \langle x_2 \rangle_a \pm \int \psi_b^*(x_1)x_1\psi_a(x_1)\,dx_1 \int \psi_a^*(x_2)x_2\psi_b(x_2)\,dx_2\Big]$$

$$= \langle x \rangle_a \langle x \rangle_b \pm \left(\int \psi_a^*(x_1)x_1\psi_b(x_1)\,dx_1\right)^2$$

(b) (Cont.) The wave function of a particle in an infinite square well is

$$\psi_n = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right)$$

So $\int \psi_a^*(x_1)\, x_1\, \psi_b(x_1)\, dx_1$

$$= \frac{2}{l}\int_0^{l} \sin\left(\frac{a\pi x}{l}\right) x \sin\left(\frac{b\pi x}{l}\right) dx$$

$$= \frac{2}{l}\int_0^l x \left(\frac{\cos\left(\frac{a\pi x}{l}-\frac{b\pi x}{l}\right) - \cos\left(\frac{a\pi x}{l}+\frac{b\pi x}{l}\right)}{2}\right)dx$$

$$= \frac{1}{l}\int_0^l x\left[\cos\left((a-b)\frac{\pi x}{l}\right) - \cos\left((a+b)\frac{\pi x}{l}\right)\right]dx$$

$f(x)=x \quad g'(x)=\cos\left((a-b)\frac{\pi x}{l}\right)$

$f'(x)=1 \quad g(x)=\frac{l}{(a-b)\pi}\sin\left((a-b)\frac{\pi x}{l}\right)$

$g'(x)=\cos\left((a+b)\frac{\pi x}{l}\right)$

$g(x)=\frac{l}{(a+b)\pi}\sin\left((a+b)\frac{\pi x}{l}\right)$

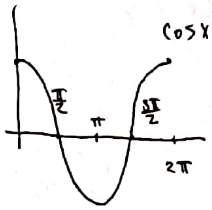
$$= \frac1l\left[\int_0^l x\cos\left((a-b)\frac{\pi x}{l}\right)dx - \int_0^l x\cos\left((a+b)\frac{\pi x}{l}\right)dx\right]$$

$$= \frac1l\left[\left[x \frac{l}{(a-b)\pi}\sin\left((a-b)\frac{\pi x}{l}\right)\right]_0^l - \int_0^l \frac{l}{(a-b)\pi}\sin\left((a-b)\frac{\pi x}{l}\right)dx - \left[x\frac{l}{(a+b)\pi}\sin\left((a+b)\frac{\pi x}{l}\right)\right]_0^l + \int_0^l \frac{l}{(a+b)\pi}\sin\left((a+b)\frac{\pi x}{l}\right)dx\right]$$

$$= \frac1l\left[\frac{l}{(a+b)\pi}\int_0^l \sin\left((a+b)\frac{\pi x}{l}\right)dx - \frac{l}{(a-b)\pi}\int_0^l\sin\left((a-b)\frac{\pi x}{l}\right)dx\right]$$

$$= \frac1l\left[\left(\frac{l}{(a+b)\pi}\right)^2\left[-\cos\left((a+b)\frac{\pi x}{l}\right)\right]_0^l - \left(\frac{l}{(a-b)\pi}\right)^2 \left[-\cos\left((a-b)\frac{\pi x}{l}\right)\right]_0^l\right]$$

$$= \frac1l\left[-\left(\frac{l}{(a+b)\pi}\right)^2\left(\cos((a+b)\pi)-1\right) + \left(\frac{l}{(a-b)\pi}\right)^2\left(\cos((a-b)\pi)-1\right)\right]$$



$$= \frac{l}{(a-b)^2\pi^2}\left((-1)^{a+b}-1\right) - \frac{l}{(a+b)^2\pi^2}\left((-1)^{a+b}-1\right)$$

$$= \frac{l}{\pi^2}\left((-1)^{a+b}-1\right)\left(\frac{1}{(a-b)^2}-\frac{1}{(a+b)^2}\right)$$

$\cos((a+b)\pi)$
$= \cos((a-b)\pi)$
$= (-1)^{a+b}$

Thus, $\langle(x_1-x_2)^2\rangle = 2\cdot\frac12\left(\cancel{\langle x\rangle_a^2+\langle x\rangle_b^2}\ \langle x^2\rangle_a + \langle x^2\rangle_b\right) + \langle x\rangle_a\langle x\rangle_b \pm \int\psi_a^*(x_1)\, x_1\, \psi_b(x_1)\, dx_1$

$$= \left(\frac{a^2}{3}-\frac{a^2}{2n^2\pi^2}\right) + \left(\frac{a^2}{3}-\frac{a^2}{2m^2\pi^2}\right) + \frac a2\cdot\frac a2 \pm \frac{1}{\pi^2}\left((-1)^{a+b}-1\right)\left(\frac{1}{(a-b)^2}-\frac{1}{(a+b)^2}\right)$$

If the particles are boson, choose +; if the particles are fermions, choose −.

5.11 (a) $\left\langle \frac{1}{|\vec{r_1}-\vec{r_2}|} \right\rangle = ?$, Known $\psi_0(\vec{r_1},\vec{r_2}) = \psi_{100}(\vec{r_1})\psi_{100}(\vec{r_2}) = \frac{8}{\pi a^3} e^{-2\left(\frac{r_1+r_2}{a}\right)}$

and let $|\vec{r_1}-\vec{r_2}| = \sqrt{r_1^2+r_2^2-2r_1r_2\cos\theta_2}$

$$\left\langle \frac{1}{|\vec{r_1}-\vec{r_2}|} \right\rangle = \iint \frac{8}{\pi a^3} e^{-2\left(\frac{r_1+r_2}{a}\right)} \frac{1}{|\vec{r_1}-\vec{r_2}|} \frac{8}{\pi a^3} e^{-2\left(\frac{r_1+r_2}{a}\right)} d^3\vec{r_1}\, d^3\vec{r_2}$$

$$= \frac{64}{\pi^2 a^6} \iint \underbrace{e^{-4\left(\frac{r_1+r_2}{a}\right)} \frac{1}{\sqrt{r_1^2+r_2^2-2r_1r_2\cos\theta_2}} d^3\vec{r_2}}_{A}\, d^3\vec{r_1}$$

$$A = \iiint e^{-4\left(\frac{r_1+r_2}{a}\right)} \frac{r^2 \sin\theta_2}{\sqrt{r_1^2+r_2^2-2r_1r_2\cos\theta_2}} dr_2\, d\theta_2\, d\phi_2$$

$$= \int_0^{2\pi} d\phi_2 \int_0^{\infty} e^{-4\left(\frac{r_1+r_2}{a}\right)} \underbrace{\int_0^{\pi} \frac{\sin\theta_2}{\sqrt{r_1^2+r_2^2-2r_1r_2\cos\theta_2}} d\theta_2}_{B}\, r_2^2\, dr_2$$

~~$u = \sin\theta_2 \cos\theta_2$~~

~~$\frac{du}{d\theta_2} = \cos\theta_2 - \sin\theta_2$~~

~~$d\theta_2 = \frac{-du}{\cos\theta_2 \sin\theta_2}$~~

~~$u(0) = 1$~~

~~$u(\pi) = -1$~~

~~$B = \int_1^{-1} \frac{\sin\theta_2}{\sqrt{r_1^2+r_2^2-2r_1r_2 u}}$~~

$u = r_1^2+r_2^2-2r_1r_2\cos\theta_2$ ~~$u(0) =$~~

$\frac{du}{d\theta_2} = +2r_1r_2 \sin\theta_2$

$d\theta_2 = \frac{du}{+2r_1r_2 \sin\theta_2}$

$$B = \int_{\theta_2=0}^{\pi} \frac{\sin\theta_2}{\sqrt{u}} \frac{du}{2r_1r_2\sin\theta_2}$$

$$= \int_{\theta_2=0}^{\pi} \frac{1}{2r_1r_2} \frac{1}{\sqrt{u}} du$$

$$= \left[\frac{1}{2r_1r_2} \cdot 2 \cdot \sqrt{u}\right]_{\theta_2=0}^{\pi}$$

$$= \left[\frac{1}{r_1r_2} \sqrt{r_1^2+r_2^2-2r_1r_2\cos\theta_2}\right]_0^{\pi}$$

$$= \frac{1}{r_1r_2}\left[\sqrt{r_1^2+r_2^2+2r_1r_2} - \sqrt{r_1^2+r_2^2-2r_1r_2}\right]$$

$$= \frac{1}{r_1r_2}\left[\sqrt{(r_1+r_2)^2} - \sqrt{(r_1-r_2)^2}\right]$$

$$= \frac{1}{r_1r_2}\left[(r_1+r_2) - |r_1-r_2|\right]$$

$$= \begin{cases} \frac{2}{r_1} & \text{if } r_1 > r_2 \\ \frac{2}{r_2} & \text{if } r_1 < r_2 \end{cases}$$

5.11 (a) (cont.) $A = 2\pi \int_0^\infty e^{-4\left(\frac{r_1+r_2}{a}\right)} r_2^2 B \, dr_2$

$$= 2\pi \left[\int_0^{r_1} e^{-4\left(\frac{r_1+r_2}{a}\right)} r_2^2 \cdot \frac{2}{r_1} \, dr_2 + \int_{r_1}^\infty e^{-4\left(\frac{r_1+r_2}{a}\right)} r_2^2 \frac{2}{r_2} \, dr_2\right]$$

$$= 2\pi \left\{\frac{2}{r_1}\int_0^{r_1} e^{-4\left(\frac{r_1+r_2}{a}\right)} r_2^2 \, dr_2 + 2\int_{r_1}^\infty e^{-4\left(\frac{r_1+r_2}{a}\right)} r_2 \, dr_2\right\}$$

$\cancel{f(x) = r_2^2 \quad g'(x) = e^{-4\left(\frac{r_1+r_2}{a}\right)} = e^{-\frac{4r_1}{a}-\frac{4r_2}{a}}}$

$\cancel{f'(x) = 2r_2 \quad g(x) = -}$

$$= 4\pi e^{-\frac{4r_1}{a}} \left[\underbrace{\frac{1}{r_1}\int_0^{r_1} e^{-\frac{4r_2}{a}} r_2^2 \, dr_2}_{C} + \underbrace{\int_{r_1}^\infty e^{-\frac{4r_2}{a}} r_2 \, dr_2}_{D}\right]$$

$f(x) = r_2^2 \qquad g'(x) = e^{-\frac{4r_2}{a}}$

$f'(x) = 2r_2 \qquad g(x) = -\frac{a}{4} e^{-\frac{4r_2}{a}}$

$$C = \left[-r_2^2 \frac{a}{4} e^{-\frac{4r_2}{a}}\right]_0^{r_1} - \int_0^{r_1} -2r_2 \frac{a}{4} e^{-\frac{4r_2}{a}} \, dr_2$$

$$= \left(-r_1^2 \frac{a}{4} e^{-\frac{4r_1}{a}}\right) + \frac{a}{2}\int_0^{r_1} r_2 e^{-\frac{4r_2}{a}} \, dr_2$$

$f(x) = r_2 \qquad g'(x) = e^{-\frac{4r_2}{a}}$

$f'(x) = 1 \qquad g(x) = -\frac{a}{4} e^{-\frac{4r_2}{a}}$

$$= \left(-r_1^2 \frac{a}{4} e^{-\frac{4r_1}{a}}\right) + \frac{a}{2}\left[\left[-\frac{ar_2}{4} e^{-\frac{4r_2}{a}}\right]_0^{r_1} - \int_0^{r_1} -\frac{a}{4} e^{-\frac{4r_2}{a}} \, dr_2\right]$$

$$= -r_1^2 \frac{a}{4} e^{-\frac{4r_1}{a}} + \frac{a}{2}\left[\left(-\frac{ar_1}{4} e^{-\frac{4r_1}{a}}\right) + \frac{a}{4}\left[-\frac{a}{4} e^{-\frac{4r_2}{a}}\right]_0^{r_1}\right]$$

$$= -r_1^2 \frac{a}{4} e^{-\frac{4r_1}{a}} - \frac{a^2 r_1}{8} e^{-\frac{4r_1}{a}} + - \frac{a^3}{32}\left(e^{-\frac{4r_1}{a}} - 1\right)$$

$$D = \left[-r_2 \frac{a}{4} e^{-\frac{4r_2}{a}}\right]_{r_1}^\infty - \int_{r_1}^\infty -\frac{a}{4} e^{-\frac{4r_2}{a}} \, dr_2$$

$$= \frac{r_1 a}{4} e^{-\frac{4r_1}{a}} + \frac{a}{4}\int_{r_1}^\infty e^{-\frac{4r_2}{a}} \, dr_2$$

$$= \frac{r_1 a}{4} e^{-\frac{4r_1}{a}} + \frac{a}{4}\left[-\frac{a}{4} e^{-\frac{4r_2}{a}}\right]_{r_1}^\infty$$

$\frac{-4r_1 a^2 \pi - a^3\pi + 2r_1 a^2 \pi}{8r_1} = \frac{a^2\pi(-2r_1 - a)}{8r_1}$

$$= \frac{r_1 a}{4} e^{-\frac{4r_1}{a}} - \frac{a^2}{4^2}\left(0 - e^{-\frac{4r_1}{a}}\right)$$

$$= \frac{r_1 a}{4} e^{-\frac{4r_1}{a}} + \left(\frac{a}{4}\right)^2 e^{-\frac{4r_1}{a}}$$

$$A = 4\pi e^{-\frac{4r_1}{a}}\left[\frac{1}{r_1}\left[-r_1^2\frac{a}{4} e^{-\frac{4r_1}{a}} - \frac{a^2 r_1}{8} e^{-\frac{4r_1}{a}} - \frac{a^3}{32}\left(e^{-\frac{4r_1}{a}} - 1\right)\right] + \left[\frac{r_1 a}{4} e^{-\frac{4r_1}{a}} + \frac{a^2}{16} e^{-\frac{4r_1}{a}}\right]\right]$$

$$= -r_1 \pi a e^{-\frac{8r_1}{a}} - \frac{a^2\pi}{2} e^{-\frac{8r_1}{a}} - \frac{a^3\pi}{8r_1} e^{-\frac{4r_1}{a}}\left(e^{-\frac{4r_1}{a}} - 1\right) + r_1 a \pi e^{-\frac{8r_1}{a}} + \frac{a^2\pi}{4} e^{-\frac{8r_1}{a}}$$

$$= \frac{a^3\pi}{8r_1} e^{-\frac{4r_1}{a}} + e^{-\frac{8r_1}{a}}\left[-\cancel{r_1\pi a} - \frac{a^2\pi}{2} - \frac{a^3\pi}{8r_1} + \cancel{r_1 a\pi} + \frac{a^2\pi}{4}\right]$$

$$= \frac{a^3\pi}{8r_1}\left[e^{-\frac{4r_1}{a}} + e^{-\frac{8r_1}{a}}\left(\frac{a^2\pi(-2r_1-a)}{8r_1}\right)\right]$$

$$= \frac{a^2\pi}{8r_1}\left[e^{-\frac{4r_1}{a}} + e^{-\frac{8r_1}{a}}(-2r_1-a)\right]$$

Thus, $\left\langle \frac{1}{|\vec{r_1}-\vec{r_2}|} \right\rangle = \frac{64}{\pi^2 a^6} \int \frac{a^2\pi}{8r_1}\left[e^{-\frac{4r_1}{a}} + e^{-\frac{8r_1}{a}}(-2r_1-a)\right] d^3\vec{r_1}$

$$= \frac{64}{\pi^2 a^6}\frac{a^2\pi}{8}\int\left(\frac{1}{r_1}e^{-\frac{4r_1}{a}} + -2e^{-\frac{8r_1}{a}} - \frac{a}{r_1}e^{-\frac{8r_1}{a}}\right) d^3\vec{r_1}$$

$$= \frac{8}{\pi a^4}$$