<<< Algebraic Equation

> relationship between unknown x and functions of x

- form: $F(x)=0$
- Solution validation: PLUG IT IN
- Solution is a number

<<< Differential Equations

> relationship between an unknown function f and its derivatives.

- Form: $F(y(x), y'(x), y''(x), \dots) = 0$
- Solution is a function
- Solution validation: PLUG IT IN

$\boxed{\text{EX1}}$ Verify $y(x) = \frac{1}{2} + ce^{-2x}$ is Solution to DE $y'+2y=1$

$$\left(\tfrac{1}{2}+ce^{-2x}\right)' + 2\left(\tfrac{1}{2}+ce^{-2x}\right) \overset{?}{=} 1$$

$$-2ce^{-2x} + 1 + 2ce^{-2x} \overset{?}{=} 1$$

$$1 = 1 \quad \checkmark$$

$\boxed{\text{EX2}}$ Verify $y = c_1\cos x + c_2\sin x$ is Solution to DE $y''+y=0$

$$(c_1\cos x + c_2\sin x)'' + (c_1\cos x + c_2\sin x) \overset{?}{=} 0$$

$$(-c_1\sin x + c_2\cos x)' + (c_1\cos x + c_2\sin x) \overset{?}{=} 0$$

$$-c_1\cos x - c_2\sin x + c_1\cos x + c_2\sin x \overset{?}{=} 0$$

$$0 = 0 \quad \checkmark$$

<<< Classifying DEs

> order - order of highest order derivative in the DE

$\boxed{\text{EX3}}$ $y^2 + y'''y' = 3$ is 3rd order DE.

> ordinary differential equation (ODE) - contain derivative w.r.t only 1 variable

> partial differential equation (PDE) - contain derivative w.r.t multiple variables

$\boxed{\text{EX4}}$ $u_t = u_{xx}$ is a PDE

> linear - unknowns are linear: they do not multiply with each other or themselves. they do not appear as arguments of nonlinear functions.

$\boxed{\text{EX5}}$ $\frac{dy}{dx} + 2y = 1$ is linear 1st order ODE

$\boxed{\text{EX6}}$ $y' = y^2$ is nonlinear 1st order ODE

$\boxed{\text{EX7}}$ $y' = \frac{1}{1+y}$ is nonlinear 1st order ODE

$\boxed{\text{EX8}}$ $y(x)\,y'(x) = x + y(x)$ is nonlinear 1st order ODE

<<< Initial Value Problems

> IVP - Given some initial conditions $y(0)=a_0,\ y'(0)=a_1, \dots, y^{(n)}(0)=a_n$, solve the ODE $F(y(t), y'(t), \dots, y^{(n)}(t))=0$

Method

1. Find the general solution to the ODE (or verify given solution)
2. Plug in any initial values to determine the values of unknown constants in the ODE general solution.
3. Check if you need to do anything else with the soln of ODE

ex1 $\begin{cases} P'(t)=2P(t) \\ P(0)=3 \end{cases}$ $P(6)=?$ $P(t)=ce^{2t}$.

Verify soln: $P(t)=ce^{2t}$ is soln to $P'(t)=2P(t)$

$$\frac{d}{dt}(ce^{2t}) \stackrel{?}{=} 2(ce^{2t})$$

$$2ce^{2t} = 2ce^{2t} \quad \checkmark$$

Plug in init value: $P(0)=3=ce^{2(0)}=c$

The solution to IVP is $P(t)=3e^{2t}$.

$$P(6)=3e^{2(6)} > 3\,2^{12} \approx 12000$$

<<< Properties of ODE Solutions

1. Existence
2. Uniqueness

$\Big\}$ often prove under strict conditions

3. Can we find it?

<<< Existence & Uniqueness Theorem

> Consider IVP $y'=f(t,y)$, $y(0)=0$.
If f and $\frac{\partial f}{\partial y}$ are continuous for $|t|\le a$, $|y|\le b$,
then there exist ($\exists$) a $|t|\le h\le a$ such that there exist a unique ($\exists!$) solution $y(t)=\phi(t)$ of the IVP.

- only true for 1^{st} order ODE
- $y(0)=0$ is not restrictive, we can do change of variables (COV)

Ex2 IVP: $\begin{cases} h'(t)=f(h,t) \\ h(0)=h_0 \end{cases} \xrightarrow{\text{COV}} \begin{cases} \text{let } g(t)=h(t)-h_0 \\ \text{so } g(0)=0 \end{cases}$

$h(t)=g(t)+h_0 \xleftarrow[\text{back}]{\text{substitute}}$ solve for $g(t)$

<<< Separable ODEs

> separable ODE (1st order) has the form $\frac{dy}{dx} = g(x)\,h(y)$

$\boxed{\text{Method}}$ Separation of variables

$\boxed{0}$ Check if separable ; write y' as $\frac{dy}{dx}$

$\boxed{1}$ Separate x and y terms

$\boxed{2}$ Integrate both sides . Add constant of integration

$\boxed{2a}$ Solve for explicit solution $y(x)$

$\boxed{3}$ IVP

$\boxed{\text{EX1}}$ $\begin{cases} \frac{dy}{dx} = 10y \\ y(0) = 3 \end{cases}$

$$\frac{dy}{dx} = 10y$$

$$\int \frac{dy}{y} = \int 10\,dx$$

$$e^{\ln|y|} = e^{(10x+c)} \quad \leftarrow \text{implicit soln}$$

$$|y| = Be^{10x}$$

$$y = Ae^{10x} \quad \leftarrow \text{explicit soln}$$

$y(0) = 3 = Ae^{10(0)} = A \Rightarrow A = 3$

The soln is $\boxed{y(x) = 3e^{10}x}$.

$\boxed{\text{Ex2}}$ $\begin{cases} \frac{dy}{dt} = \frac{1+\cos(t)}{2+3y^6} \\ y(0) = 1 \end{cases}$

$$\frac{dy}{dt} = \frac{1+\cos(t)}{2+3y^6}$$

$$\int (2+3y^6)\,dy = \int 1+\cos(t)\,dt$$

$$2y + \frac{3}{7}y^7 = t + \sin(t) + C \quad \leftarrow \text{implicit soln sufficient}$$

$y(0) = 1 \Rightarrow \quad 2(1) + \frac{3}{7}(1)^7 = 0 + \sin(0) + C$

$$\frac{17}{7} = C$$

The soln is $\boxed{2y + \frac{3}{7}y^7 = t + \sin t + \frac{17}{7}}$

$\boxed{\text{EX3}}$ $\begin{cases} \frac{dy}{dx} = \frac{e^x}{1+2y} \\ y(0) = 1 \end{cases}$

$$\frac{dy}{dx} = \frac{e^x}{1+2y}$$

$$\int (1+2y)\,dy = \int e^x\,dx$$

$$y + y^2 = e^x + C$$

$$y^2 + y - (e^x + C) = 0$$

$$y(x) = \frac{-1 \pm \sqrt{1 - 4(1)(-e^x - C)}}{2}$$

$$= \frac{-1 \pm \sqrt{1 + 4e^x + B}}{2}$$

$y(0) = 1 = \frac{-1 \pm \sqrt{1+4+B}}{2}$

$3 = \pm\sqrt{5+B} \quad (+)$

$4 = B$

The soln is $\boxed{y(x) = \frac{-1 \pm \sqrt{5+4e^x}}{2}}$

-- Interval of Validity

> valid - solution of an ODE that does not
- have complex number
- $\lim_{x \to a} = \infty$ ($\lim_{x \to \pm\infty} = \infty$ ok)
- division by 0
- undefined operation ($\ln(-2)$)

> interval of validity - largest interval where the soln is valid

$\boxed{EX4}$ $y(x) = 10e^x$

$\lim_{x \to -\infty} y(x) = 0$

$\lim_{x \to \infty} y(x) = \infty$

Interval of validity : $(-\infty, \infty)$

valid $\forall x \in \mathbb{R}$

$\boxed{EX5}$ Find max value of population of T cells, where $P(t)$ is soln of IVP

$$\begin{cases} P(0) = 2 \\ P'(t) = s - dt \end{cases}$$

$$\frac{dP}{dt} = s - dt$$

$$\int dP = \int s - dt \; dt$$

$$P(t) = st - \tfrac{1}{2}dt^2 + C$$

$P(0) = 2 = 0 - 0 + C$

$P(t) = st - \frac{1}{2}dt^2 + 2$

$P'(t) = s - dt$, $P'(t)$ exists $\forall t \in \mathbb{R}$

$P'(t^*) = 0 = s - dt^*$ $\Rightarrow$ $t^* = \frac{s}{d}$

--- Distinct Solutions

> distinct - $y_1(x) \neq y_2(x)$ for some x.

$\boxed{EX6}$ (a) Careless approach $\begin{cases} y' = y^2 \\ y(0) = 0 \end{cases}$

$$\frac{dy}{dx} = y^2$$

$$\int y^{-2}\,dy = \int dx$$

$$-y^{-1} = x + C$$

$$y = \frac{-1}{x+C}$$

$y(0) = 0 = \frac{-1}{0+C}$ not true for any $C \in \mathbb{R}$

(b) Careful approach : check if $h(y) = 0$ (* trivial soln check)

$h(y) = y^2 = 0$ $\Rightarrow$ $y(x) = 0$ is candidate soln.

Verify $y(x) = 0$ is a soln : $\frac{dy}{dx} = y^2$: $0 = 0$ ✓

The DE has solns $\begin{cases} y_1(x) = -\frac{1}{x+C} \\ y_2(x) = 0 \end{cases}$

<<< Method of Integrating Factors

→ Motivating Example

$$y' + y = 5$$

* Reduce problem to known form.

$$e^x(y'+y) = 5e^x$$

$$\underbrace{y'e^x + ye^x}_{\text{product rule}} = 5e^x$$

$$\frac{d}{dx}(ye^x) = 5e^x$$

$$\int d(ye^x) = \int 5e^x \, dx$$

$$ye^x = 5e^x + c$$

$$y(x) = \frac{5e^x + c}{e^x}$$

We can check by using separation of variable.

$$\frac{dy}{dx} = 5 - y$$

$$\int \left(\frac{1}{5-y}\right) dy = \int dx \qquad u = 5-y$$

$$\int \frac{-1}{u} \, du = x + C \qquad du = -dy$$

$$-\ln|u| = x + C$$

$$+\ln|5-y| = -(x + C)$$

$$e^{\ln|5-y|} = e^{-x-C}$$

$$|5-y| = e^{-x-c}$$

$$5 - y = Ae^{-x}$$

$$y = Ae^{-x} + 5$$

Note the solution $y(x) = \frac{5e^x + c}{e^x}$ cannot be simplified since cannot cancel out unknown x.

<<< Derivation of Integrating Factors

· Assume DE in form $y' + p(x)y = q(x)$

let $\mu(x)$ be integrating factor

$$\mu(x)\underbrace{\left(y' + p(x)y\right)}_{\text{want: product rule}} = \mu(x)q(x)$$

$$\mu(x)\left(y' + p(x)y\right) = \mu(x)y' + \mu'(x)y = (\mu(x)y)'$$

$$\mu(x)y' + \mu(x)p(x)y = \mu(x)y' + \mu'(x)y$$

$$\mu(x)p(x) = \mu'(x) \qquad \longleftarrow \text{Get separable ODE}$$

$$\frac{d\mu}{dx} = \mu p(x)$$

$$\int \frac{d\mu}{\mu} = \int p(x) \, dx$$

Plug in $\boxed{\mu(x) = e^{\int p(x)\,dx}}$ exists for integrable $p(x)$

$$\int (\mu(x)y)' = \int \mu(x)q(x)$$

$$\int d(\mu(x)y) = \int \mu(x)q(x) \, dx$$

$$\mu(x)y = \int \mu(x)q(x) \, dx + C$$

$$\boxed{y = \frac{\int \mu(x)q(x) \, dx + C}{\mu(x)}}$$

<<< Method of Integrating Factors

$\boxed{\text{Method}}$ $\boxed{0}$ Verify ODE is in form $y'(x) + p(x)\,y(x) = q(x)$

$\boxed{1}$ Find integrating factor $\mu(x) = e^{\int p(x)\,dx}$ (don't need int constant yet)

$\boxed{2}$ Find general soln $y(x) = \dfrac{\int \mu(x)\,q(x)\,dx + C}{\mu(x)}$

$\boxed{3}$ IVP

$\boxed{\text{EX1}}$ $\begin{cases} y' + 2ty = t \\ y(0) = 0 \end{cases}$

$p(t) = 2t$, $q(t) = t$

$\mu(t) = e^{\int 2t\,dt} = e^{t^2}$

$$y(t) = \frac{\int e^{t^2} t\,dt + C}{e^{t^2}} = \frac{\frac{1}{2}e^{t^2} + C}{e^{t^2}} = \frac{1}{2} + Ce^{-t^2}$$

$y(0) = 0 = \frac{1}{2} + Ce^0 = \frac{1}{2} + C \Rightarrow C = -\frac{1}{2}$

$$\boxed{y(t) = \frac{1}{2} - \frac{1}{2}e^{-t^2}}$$

$\boxed{\text{EX2}}$ $xy' + y = x$

$y' + \frac{y}{x} = 1$

$p(\cancel{t}\,x) = \frac{1}{x}$, $q(x) = 1$

$\mu(x) = e^{\int \frac{1}{x}dx} = e^{\ln|x|} = x$ (no need of int constant)

$$y(x) = \frac{\int x\,dx + C}{x} = \frac{\frac{1}{2}x^2 + C}{x} = \boxed{\frac{x^2 + B}{2x}}$$

can't simplify more
due to variable x.
(or note $x \neq 0$)

13 Jan 2021 L5 Exact ODEs AMATH 351

Derivation of verifying Exact ODEs

- Given $f(x, y(x)) = c$

$$\frac{d}{dx} f(x, y(x)) = 0$$

$$\frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0$$

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' = 0$$

let $\begin{cases} M(x,y) = \frac{\partial f}{\partial x} \\ N(x,y) = \frac{\partial f}{\partial y} \end{cases}$

$\rightarrow$ $M(x,y) + N(x,y)y' = 0$ is exact ODE with soln $f(x,y(x)) = c$.

- To check if $M(x,y) = \frac{\partial f}{\partial x}$, $N(x,y) = \frac{\partial f}{\partial y}$, use Clairaut's Theorem.
  - Clairaut's Theorem: equality of mixed derivatives

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

$$\frac{\partial}{\partial x}\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\frac{\partial f}{\partial x}$$

$$\frac{\partial}{\partial x} N = \frac{\partial}{\partial y} M$$

$$N_x = M_y$$

sufficient condition to know if they satisfy condition of Exact ODE.

Solving Exact ODEs

$\boxed{\text{Method}}$

1. Verify ODE in form $M(x,y) + N(x,y)y' = 0$
2. Verify ODE is exact by checking $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$
3. Find $f_1(x,y) \equiv \int \partial f = \int M \partial x = f_{mixed}(x,y) + f_{yterm}(y)$

   $f_2(x,y) \equiv \int \partial f = \int N \partial y = f_{mixed}(x,y) + f_{xterm}(x)$
4. Match up the terms for soln $f(x,y) = f_{mixed}(x,y) + f_{yterm}(y) + f_{xterm}(x) = C$
5. IVP

$\boxed{\text{EX1}}$ $2xyy' + 2x + y^2 = 0$ $N(x,y) = 2xy$, $M(x,y) = 2x + y^2$

$\frac{\partial M}{\partial y} = 2y$, $\frac{\partial N}{\partial x} = 2y$. $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ so the ODE is exact.

$f_1(x,y) = \int \partial f = \int M\,dx = \int 2x + y^2\ \partial x = x^2 + y^2 x$

$f_2(x,y) = \int \partial f = \int N\,\partial y = \int 2xy\ \partial y = xy^2$

$f(x,y) = xy^2 + x^2 = c$

$\boxed{EX2}$

$$0 = (\sin x + x^2 e^y - 1)y' + (y\cos x + 2xe^y)$$

$N(x,y) = \sin x + x^2 e^y - 1$ $\quad\frac{\partial N}{\partial x} = \cos x + 2e^y x$ $\quad\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$, exact ODE ✓

$M(x,y) = y\cos x + 2xe^y$ $\quad\frac{\partial M}{\partial y} = \cos x + 2e^y x$

$f_1(x,y) = \int M \, \cancel{dy}\,dx = \int y\cos x + 2xe^y \, dx = y\sin x + x^2 e^y$ #

$f_2(x,y) = \int N \, dy = \int \sin x + x^2 e^y - 1 \, dy = y\sin x + x^2 e^y - y$

$f(x,y) = y\sin x + x^2 e^y - y = C$

... Substitution

$\boxed{\text{Method}}$ [1] Write down the substitution $u(x, y(x))$ and inverse substitution $y(x, u(x))$

[2] Take derivative of u : $u' = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} y'$

[3] Replace y' with original ODE

[4] Replace y using inverse substitution (get a ODE of $u(x)$)

[5] Solve the ODE of $u(x)$

[6] Use inverse substitution to get $y(x)$.

$\boxed{\text{EX1}}$ $y' = \frac{y^2+2xy}{x^2}$ $\quad \frac{\partial M}{\partial y} = \frac{2y+2x}{x^2} \neq \frac{\partial N}{\partial x} = 0$ $\quad$ NOT $\begin{cases} \text{separable} \\ \text{linear} \\ \text{exact} \end{cases}$

① Given $u(x) = \frac{y(x)}{x}$ as substitution,

so $y(x) = u(x)\,x$ is inverse substitution.

② $u' = \frac{du}{dx} = \frac{d}{dx}(yx^{-1}) = -yx^{-2} + x^{-1}\frac{dy}{dx} = \frac{xy'-y}{x^2}$

③ Sub in y' using ODE : $u' = \frac{x\left(\frac{y^2+2xy}{x^2}\right)-y}{x^2} = \frac{y^2}{x^3} + \frac{y}{x^2}$

④ Sub in y using inverse substitution $(y=ux)$: $u' = \frac{u^2+u}{x}$ ← separable

⑤ Solve separable ODE of variable u.

① $u^2+u=0$

$u(u+1)=0$

$u=0,-1$ are soln to u'

② $u^2+u \neq 0$:

$$\int \frac{du}{u^2+u} = \int \frac{1}{x}\,dx$$

$$\int \frac{1}{u} - \frac{1}{u+1}\,du = \ln x + C$$

$$e^{\wedge}\ \ln\left(\frac{u}{u+1}\right) = \ln(Bx)\ e^{\wedge}$$

$$\frac{u}{u+1} = Bx$$

$$u = \frac{bx}{1-bx}$$

partial fraction:

$\frac{1}{u^2+u} = \frac{A}{u} + \frac{B}{u+1}$

$1 = A(u+1) + Bu$

$1 = Au + A + Bu$

$\begin{cases} A=1 \\ 0=A+B \end{cases} \Rightarrow \begin{cases} A=1 \\ B=- \end{cases}$

⑥ Sub u in inverse sub. $y=ux$:

$$\boxed{y = ux = x\left(\frac{bx}{1-bx}\right)}$$

$\boxed{Ex2}$ ① $x^2y' + 2xy = y^3$ Known $u = \frac{1}{y^2}$, $y = \pm\, u^{-\frac{1}{2}}$

$y' = \frac{y^3}{x^2} - \frac{2y}{x}$

② $u' = \frac{du}{dx} = -2y^{-3}y'$

③ $u' = -2y^{-3}y' = -2y^{-3}\left(\frac{y^3}{x^2} - \frac{2y}{x}\right) = \frac{-2}{x^2} + \frac{4}{xy^2} = \frac{-2}{x^2} + \frac{4}{x}\frac{1}{y^2}$

④ $u' = \frac{-2}{x^2} + \frac{4}{x}u \Rightarrow u' - \frac{4}{x}u = -\frac{2}{x^2}$

⑤ $p(x) = -\frac{4}{x}$ $q(x) = -\frac{2}{x^2}$ $\mu(x) = \exp\left(\int -\frac{4}{x}\,dx\right) = x^{-4}$

$$u(x) = \frac{\int q(x)\mu(x)\,dx + C}{\mu(x)} = x^4\left[\int -\frac{2}{x^2}\cdot\frac{1}{x^4}\,dx + C\right]$$

$$= x^4\left[\frac{2}{5}\,\frac{1}{x^5} + C\right]$$

$$u(x) = \frac{2}{5}x^{-1} + cx^4$$

⑥ $y = \pm\frac{1}{\sqrt{u}} = \pm\frac{1}{\sqrt{\frac{2}{5}\frac{1}{x} + cx^4}}$

<<< Mathematical Modeling

→ Formulation

- Given a description of a system, can we write models for what's happening in the system?
    - identify independent, dependent variables, parameters
    - look for "rate of change", proportionality

EX1 Radioactive decay: rate of change of mass of an isotope is proportional to current mass of isotope.

- independent var: time t
- dependent var: mass of isotope $N(t)$

In general. $$\frac{d\,\text{mass}}{dt} = \begin{pmatrix}\text{mass}\\ \text{in}\end{pmatrix} - \begin{pmatrix}\text{mass}\\ \text{out}\end{pmatrix}$$

In decay. $$\frac{dN}{dt} = 0 - K\,N(t)$$

solution: $$N(t) = N(0)\,e^{-Kt}, \quad \lim_{t\to\infty} N(t) = 0$$

EX2 Radioactive Dating: In the beginning, $\frac{U_{235}}{U_{238}} = \frac{1}{1}$, now, $\frac{U_{235}}{U_{238}} = \frac{1}{137}$.

Known half life $t_{1/2}(U_{235}) = 7\times10^{8}\ yr$, $t_{1/2}(U_{238}) = 4.5\times10^{9}\ yr$.

> half life (τ) = time it takes for half of isotopes to decay.
> $N(\tau) = \frac{N_0}{2}$, where $N(0) = N_0$

Solve for decay constant K:

$$N(t) = N_0\,e^{-Kt}$$

$$\frac{N_0}{2} = N_0\,e^{-K\tau}$$

$$\ln(\tfrac{1}{2}) = -K\tau$$

$$\boxed{K = \frac{\ln(2)}{\tau}}$$

Let now be t,

$$\begin{cases} N_{238}(t) = N_{238\,0}\,e^{-Kt} = N_{238\,0}\,e^{\frac{\ln(2)}{\tau_{238}}t} \\ N_{235}(t) = N_{235\,0}\,e^{-Kt} = N_{235\,0}\,e^{\frac{\ln(2)}{\tau_{235}}t} \end{cases}$$

$$\frac{N_{238}(t)}{N_{235}(t)} = \frac{N_{238\,0}\,e^{\frac{\ln(2)}{\tau_{238}}t}}{N_{235\,0}\,e^{\frac{\ln(2)}{\tau_{235}}t}}$$

$$137 = 1\cdot \exp\left(\left(\frac{\ln(2)}{\tau_{238}} - \frac{\ln(2)}{\tau_{235}}\right)t\right) \Rightarrow t = \frac{\ln(137)}{\ln(2)}\cdot\frac{\tau_{235}\,\tau_{238}}{\tau_{238}-\tau_{235}}$$

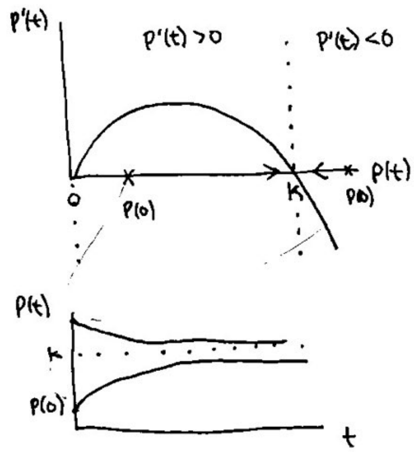
$$= 6\times10^{9}\ yr$$

22 Jan 2021 L8 Stability & Phase Plane Analysis AMATH 351

<< Motivating Example

> Logistic equation : $P'(t) = r\left(1-\frac{P}{K}\right)P \quad (r, K > 0)$

· Plot $P'(t)$ vs $P(t)$:



$P' = 0$ at $P = 0, K$

$P' > 0$, P is increasing ($\longrightarrow$ soln flow to right)

$P' = 0$, P is neither increasing nor decreasing

$P' < 0$, P is decreasing ($\longleftarrow$ soln flow to left)

<< Stability

> fixed point (equilibrium) - for ODE $y' = f(y)$, a point y^* that satisfies $f(y^*) = 0$

· a solution with the initial condition $y(0) = y^*$ is constant over time.

· The flow around a fixed point determines its stability

> stable - soln approach fixed point for some neighborhood around the fixed point.

$\longrightarrow \bullet \longleftarrow$ stable

> unstable - soln diverge from fixed point ...

$\longleftarrow \bullet \longrightarrow$ unstable

> semi-stable - soln approach from one side, diverge from another

$\longrightarrow \bullet \longrightarrow$ semi-stable $\longleftarrow \bullet \longleftarrow$

<< Phase-Plane Analysis

$\boxed{\text{Method}}$

1. Plot P' vs P
2. Find fixed point : P^* such that $P' = 0$ (x-intercept of P' vs P plot)
3. Draw flow arrows $\begin{cases} P' > 0 & \text{flow to right } \longrightarrow \\ P' < 0 & \text{flow to left } \longleftarrow \end{cases}$
4. Identify stability by flow arrows
5. Sketch soln according to ...
   - region flow to right ($\longrightarrow$) P increases
   - region flow to left ($\longleftarrow$) P decreases
   - soln approach stable fixed pt, run away from unstable fixed pt.

<< Example

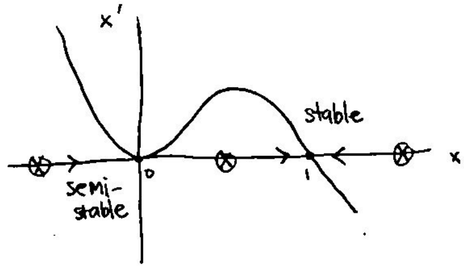
$\boxed{\text{EX1}}$ $\frac{dx}{dt} = -x^3 + x^2 = x^2(1-x)$

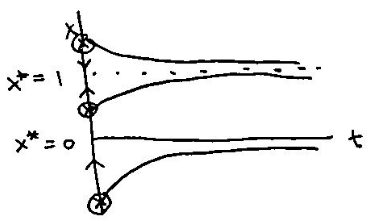
$\frac{dx}{dt} = 0$ when $x = 0, 1$

$x = 0, 1$ are fixed pt.

$x = 0$ is semi-stable

$x = 1$ is stable





<<< Second Order ODE

> second order linear ODE is of the form

$$r(x)y'' + p(x)y' + q(x)y = g(x)$$

· constant coefficient - $r(x) = a$, $p(x) = b$, $q(x) = c$

· homogeneous - $g(x) = 0$

<<< Principle of Superposition

· If y_1 and y_2 are independent soln of ODE. then the general soln of the ODE is

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

· applies to linear homogeneous second order linear ODE.

Proof :

$$0 = ay'' + b' + cy$$

$$0 = a(c_1 y_1''(x) + c_2 y_2''(x)) + b(c_1 y_1'(x) + c_2 y_2'(x)) + c(c_1 y_1(x) + c_2 y_2(x))$$

$$0 = ac_1 y_1'' + ac_2 y_2'' + bc_1 y_1' + bc_2 y_2' + cc_1 y_1 + cc_2 y_2$$

$$0 = c_1(ay_1'' + by_1' + cy_1) + c_2(ay_2'' + by_2' + cy_2)$$

$$0 = c_1(0) + c_2(0)$$

$$0 = 0 \quad \checkmark$$

· independent soln - y_1 should not be multiple of y_2

· if this were, then $y_2 = \alpha y_1$:

$$y = c_1 y_1 + c_2 y_2 = c_1 y_1 + c_2 \alpha y_1 = C y_1$$

· Second order ODE should have 2 constant of integration. Cy_1 is not general soln.

<<< Solve 2^{nd} order ODE by superposition

· To solve $ay'' + by' + c = 0$, guess that the soln have the form $y(x) = e^{\lambda x}$.

$$\left.\begin{aligned} y(x) &= e^{\lambda x} \\ y'(x) &= \lambda e^{\lambda x} \\ y''(x) &= \lambda^2 e^{\lambda x} \end{aligned}\right\}$$

$$ay'' + by' + c = 0$$

$$ae^{\lambda x}\lambda^2 + b\lambda e^{\lambda x} + ce^{\lambda x} = 0$$

$$e^{\lambda x}(a\lambda^2 + b\lambda + c) = 0$$

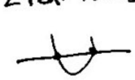
$$\boxed{a\lambda^2 + b\lambda + c = 0} \longleftarrow \text{characteristic equation of the ODE}$$

$$\begin{cases} y_1 = c_1 e^{\lambda_1 x} \\ y_2 = c_2 e^{\lambda_2 x} \end{cases}$$

$$\lambda_1, \lambda_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

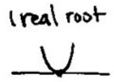
→ Possibilities of Characteristic Eq :

1. 2 real roots $\lambda_1 \neq \lambda_2$, $\lambda_1, \lambda_2 \in \mathbb{R}$



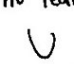
$b^2 - 4ac > 0$

2. 1 real root $\lambda_1 = \lambda_2$, $\lambda_1 \in \mathbb{R}$



$b^2 - 4ac = 0$

3. no real root $\lambda_1, \lambda_2 \in \mathbb{C}$



$b^2 - 4ac < 0$

<<< Solving 2^{nd} order const. coeff. homo. ODE

$\boxed{\text{Method}}$ $\boxed{1}$ Write ODE in form : $ay'' + by' + cy = 0$

$\boxed{2}$ Write characteristic eq : $a\lambda^2 + b\lambda + c = 0$

$\boxed{3}$ Solve characteristic eq : $\lambda_{1,2} = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$\boxed{i}$ if $\lambda_1 \neq \lambda_2$, soln is ($\lambda_1, \lambda_2 \in \mathbb{R}$)

$y(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$

$\boxed{ii}$ if $\lambda_1 = \lambda_2$, $\lambda_1, \lambda_2 \in \mathbb{R}$, soln is

$$\left.\begin{aligned} y(x) &= c_1 e^{\lambda_1 x} + c_2 x e^{\lambda_1 x} \\ &= e^{\lambda_1 x}(c_1 + c_2 x) \end{aligned}\right\}$$

Abel's Theorem
reduction of order
(L10)

$\boxed{iii}$ if $\lambda_{1,2} = \alpha + i\beta$, $\lambda_1, \lambda_2 \in \mathbb{C}$, soln is

$$\left. y(x) = c_1 e^{\alpha x} \cos(\beta x) + c_2 e^{\alpha x} \sin(\beta x) \right\}$$

Euler's Formula
(L11)

$$\text{where} \begin{cases} \alpha = -\dfrac{b}{2a} \\ \beta = \dfrac{\sqrt{4ac - b^2}}{2a} \end{cases}$$

<<< Linear Independence & Wronskian

> linear independent - functions f, g has $c_1 f(x) + c_2 f(x) = 0$ where $c_1 = c_2 = 0$ is the only soln.

$\boxed{\text{Ex1}}$ $f = e^x$, $g = 2e^x$ are linearly dependent since

$$-2f + g = -2e^x + 2e^x = 0 \qquad (c_1 = -2,\ c_2 = 1)$$

> Wronskian - $W(f,g)(x) = fg' - f'g$

Theorem Two functions f, g are linearly dependent if their Wronskian is zero.

$$W(f,g)(x) = fg' - f'g = 0 \iff \text{linearly dependent } f, g$$

$\boxed{\text{Proof}}$ If f, g are linearly dependent, then their constants c_1, c_2 satisfies

$$c_1 f + c_2 g = 0 \qquad \forall x \in \mathbb{R} \qquad (1)$$

$$c_1 f' + c_2 g' = 0 \qquad \forall x \in \mathbb{R} \qquad (2)$$

WOLOG, assume $f \neq 0$. Divide (1) by f for c_1:

$$c_1 = -c_2 \frac{g}{f} \qquad (3)$$

Sub (3) → (2)

$$-c_2 \frac{g}{f} f' + c_2 g' \frac{f}{f} = 0$$

$$\frac{c_2}{f}(fg' - f'g) = 0$$

$$\frac{c_2}{f} W = 0 \qquad (W \equiv fg' - f'g)$$

Either (i) $c_2 = 0 \Rightarrow c_1 = 0$, but that means f, g are linearly independent, violating assumption.

or (ii) $W = 0$

$\square$

$\boxed{\text{Ex2}}$ $e^x, 2e^x$

$W(e^x, 2e^x) = fg' - f'g = e^x(2e^x) - e^x(2e^x) = 0 \Rightarrow$ linearly dependent

$\boxed{\text{EX3}}$ e^x, e^{-x}

$W(e^x, e^{-x}) = e^x(-e^{-x}) - e^x e^{-x} = -2e^{x-x} = -2 \neq 0 \Rightarrow$ linearly independent

<<< SuperPosition Principle (Updated)

Theorem If y_1, y_2 are solns of $ay'' + by' + c = 0$ and $W(y_1, y_2) \neq 0$ (linearly independent), then the general soln of the ODE is

$$y = C_1 y_1 + C_2 y_2$$

Also true for variable coeff $p(x), q(x)$.

<<< Abel's Theorem

Theorem Let y_1 and y_2 be any 2 solns of $y'' + p(x)y' + q(x)y = 0$ then $\boxed{W(y_1, y_2) = ce^{-\int p(x)\,dx}}$

Proof

$$W(y_1 y_2) = y_1 y_2' - y_1' y_2$$

$$W' = (\cancel{y_1' y_2'} + y_1 y_2'') - (y_1'' y_2 + \cancel{y_1' y_2'})$$

$$= y_1 y_2'' - y_1'' y_2$$

(Since ODE gives $y'' = -p(x)y' - q(x)y$)

$$= y_1(-p(x)y_2' - q(x)y_2) - (-p(x)y_1' - q(x)y_1)y_2$$

$$= -p(x)(y_1 y_2' + y_1' y_2) - q(x)(\cancel{y_1 y_2 - y_1 y_2})$$

$$= -p(x)(y_1 y_2' + y_1 y_2)$$

$$W' = -p(x)W(y_1, y_2) \quad \leftarrow \text{separable 1st order ODE}$$

$$\Rightarrow W = ce^{-\int p(x)\,dx}$$

- Abel's Theorem allows us to solve 2nd order homogeneous ODE when $\lambda_1 = \lambda_2$ by reduction of order

<<< Reduction of Order

- We know y_1, but not y_2
- By Abel's Theorem,

$$W(y_1, y_2) = y_1 y_2' - y_1' y_2 = ce^{-\int p(x)\,dx} \quad \leftarrow \text{Known } y_1, p(x) \text{ Solve for 1st order ODE of } y_2$$

$$\Rightarrow \boxed{y_2 = y_1 \int \frac{W}{y_1^2}\,dx}$$

- We can solve for y_2 known y_1

<<< Reduction of Order

$\boxed{\text{EX4}}$ $2x^2y'' + 3xy' - y = 0 \quad (x > 0)$

$y_1 = \frac{1}{x}$ is one soln.

} Use reduction of order to find general soln

$\boxed{0}$ $y'' + \frac{3}{2x}y' - \frac{y}{2x^2} = 0$

$\boxed{1}$ $W = ce^{-\int p(x)\,dx} = ce^{-\int \frac{3}{2}\frac{1}{x}dx} = ce^{-\frac{3}{2}\ln x} = cx^{-\frac{3}{2}}$

(Abel's)

$\boxed{2}$ $y_2 = y_1 \int \frac{W}{y_1^2}dx = \frac{1}{x}\int cx^{-\frac{3}{2}}x^2\,dx = \frac{c}{x}\int x^{\frac{1}{2}}dx$

(red. of order)

$= \frac{c}{x}\frac{2}{3}x^{\frac{3}{2}} = \frac{2}{3}cx^{\frac{1}{2}}$

$\boxed{3}$ Pick coefficient c for convenience $(c \neq 0)$. There will be general coeff at the end.

- Choose $c = \frac{3}{2}$
- $y_2 = \sqrt{x}$

$\boxed{4}$ General soln $y(x) = c_1y_1 + c_2y_2$

$y = c_1\frac{1}{x} + c_2\sqrt{x}$

$\boxed{\text{Method}}$ Reduction of Order

$\boxed{0}$ Write ODE in the form $y'' + py' + qy = 0$

$\boxed{1}$ Calculate Wronskian $W = ce^{-\int p(x)dx}$

$\boxed{2}$ Find y_2 by reduction of order formula $y_2 = y_1\int\frac{W}{y_1^2}dx$

$\boxed{3}$ Pick convenient coefficient c $(c \neq 0)$

$\boxed{4}$ Write general soln $y = c_1y_1 + c_2y_2$

$\boxed{\text{EX5}}$ For $ay'' + by' + cy = 0 \Rightarrow a\lambda^2 + b\lambda + c = 0$, $\lambda_1 = \lambda_2 = -\frac{b}{2a}$

We have $y_1 = e^{\lambda_1 x}$. Find y_2 by reduction of order

$\boxed{0}$ $y'' + \frac{b}{a}y' + \frac{c}{a}y = 0$

$\boxed{1}$ $W = \tilde{c}e^{-\int p(x)dx} = \tilde{c}e^{-\frac{b}{a}x}$

$\boxed{2}$ $y_2 = y_1\int\frac{W}{y_1^2}dx = e^{\lambda_1 x}\int\frac{\tilde{c}e^{-\frac{b}{a}x}dx}{e^{2\lambda_1 x}} = \tilde{c}e^{\lambda_1 x}\int e^{-(\frac{b}{a}+2\lambda_1)x}dx$

$= \tilde{c}e^{\lambda_1 x}\int e^{-(\frac{b}{a}+2(-\frac{b}{2a}))x}dx = \tilde{c}e^{\lambda_1 x}\int dx = \tilde{c}e^{\lambda_1 x}x$

$\boxed{3}$ choose $\tilde{c} = 1$, $y_2 = xe^{\lambda_1 x}$

$\boxed{4}$ $y(x) = c_1e^{\lambda_1 x} + c_2xe^{\lambda_1 x} = e^{\lambda_1 x}(c_1 + c_2x)$

<<< Euler's Formula

$$\boxed{\begin{aligned} e^{i\beta x} &= \cos(\beta x) + i\sin(\beta x) \\ e^{-i\beta x} &= \cos(\beta x) - i\sin(\beta x)\end{aligned}}$$

$\boxed{\text{Proof 1}}$ Power Series (Taylorize)

$$\begin{aligned} e^{i\beta x} &= 1 + i\beta x + \frac{(i\beta x)^2}{2!} + \frac{(i\beta x)^3}{3!} + \frac{(i\beta x)^4}{4!} + \cdots \\ &= 1 + i\beta x - \frac{(\beta x)^2}{2!} - i\frac{(\beta x)^3}{3!} + \frac{(\beta x)^4}{4!} + \cdots \\ &= \left[1 - \frac{(\beta x)^2}{2!} + \frac{(\beta x)^4}{4!} - \cdots\right] + i\left[\beta x - \frac{(\beta x)^3}{3!} + \frac{(\beta x)^5}{5!} - \cdots\right] \\ &= \cos(\beta x) + i\sin(\beta x)\end{aligned}$$

$\boxed{\text{Proof 2}}$ Differential Equations

$$\left.\begin{aligned} u &= e^{i\beta x} \\ u' &= i\beta e^{i\beta x} \\ u'' &= -\beta^2 e^{i\beta x}\end{aligned}\right\} \quad \begin{aligned} u'' + \beta^2 u &= 0 \\ u(0) &= 1 \\ u'(0) &= i\beta\end{aligned}$$

← Euler's formula are solns of the IVP

Verify: let $v = \cos(\beta x) + i\sin(\beta x)$

$$v' = -\beta\sin(\beta x) + i\beta\cos(\beta x)$$

$$v'' = -\beta^2\cos(\beta x) - i\beta^2\sin(\beta x)$$

so

$$v'' + \beta^2 v \stackrel{?}{=} 0$$

$$(-\beta^2\cos(\beta x) - i\beta^2\sin(\beta x)) + \beta^2(\cos(\beta x) + i\sin(\beta x)) \stackrel{?}{=} 0$$

$$0 = 0 \quad \checkmark$$

<<< Complex Roots of Characteristic Equation

- For $ay'' + by' + cy = 0 \Rightarrow a\lambda^2 + b\lambda + c = 0$, $\lambda_1, \lambda_2 \in \mathbb{C}$

When $b^2 - 4ac < 0 \iff 4ac - b^2 > 0$

$$\lambda_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{(-1)(4ac - b^2)}}{2a} = \frac{-b \pm i\sqrt{4ac - b^2}}{2a} \quad \left\} \begin{aligned} \alpha &= -\frac{b}{2a} \\ \beta &= \frac{\sqrt{4ac - b^2}}{2a}\end{aligned}\right.$$

$$\left\{\begin{aligned} y_1 &= e^{\lambda_1 x} = e^{(\alpha + i\beta)x} = e^{\alpha x}e^{i\beta x} = e^{\alpha x}(\cos\beta x + i\sin\beta x) \\ y_2 &= e^{\lambda_2 x} = e^{(\alpha - i\beta)x} = e^{\alpha x}e^{-i\beta x} = e^{\alpha x}(\cos\beta x - i\sin\beta x)\end{aligned}\right.$$

← still complex

→

... Complex Roots of Characteristic Equation (cont.)

- y_1, y_2 are soln, so their linear comb. is also soln.

$$\begin{cases} y_3 = \frac{1}{2}y_1 + \frac{1}{2}y_2 = \cos(\beta x)\, e^{\alpha x} \\ y_4 = \frac{1}{2i}y_1 - \frac{1}{2i}y_2 = \sin(\beta x)\, e^{\alpha x} \end{cases}$$ now real solns !

- Check if y_3, y_4 are linearly independent :

$$\begin{aligned} W(y_3, y_4) &= W(\cos(\beta x)e^{\alpha x}, \sin(\beta x)e^{\alpha x}) \\ &= \cos(\beta x)\, e^{\alpha x}\left(\beta\cos(\beta x)e^{\alpha x} + \sin(\beta x)\,\alpha e^{\alpha x}\right) \\ &\quad - \left(-\beta\sin(\beta x)e^{\alpha x} + \cos(\beta x)\alpha e^{\alpha x}\right)\sin(\beta x)\,e^{\alpha x} \\ &= e^{2\alpha x}\left[\beta\cos^2(\beta x) + \cancel{\alpha\cos(\beta x)\sin(\beta x)}\right. \\ &\quad \left. + \beta\sin(\beta x)\sin(\beta x) - \cancel{\alpha\cos(\beta x)\sin(\beta x)}\right] \\ &= \beta e^{2\alpha x} \end{aligned}$$

$\beta \neq 0$ since $\beta = \frac{\sqrt{4ac-b^2}}{2a}$ (if $\beta = 0$, $4ac-b^2=0$ (case 2))

so $W(y_3, y_4) \neq 0 \Rightarrow y_3, y_4$ linearly independent.

- general soln :

$$\boxed{y(x) = c_1\cos(\beta x)\,e^{\alpha x} + c_2\sin(\beta x)\,e^{\alpha x}}$$ (case 3)

$\boxed{\text{Ex1}}$ $y'' + y' + y = 0 \Rightarrow \lambda^2 + \lambda + 1 = 0$

$$\lambda_{1,2} = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm i\sqrt{3}}{2}$$

$$\alpha = -\tfrac{1}{2}, \quad \beta = \tfrac{\sqrt{3}}{2}$$

general soln : $y(x) = c_1\cos\left(\frac{\sqrt{3}}{2}x\right)e^{-\frac{1}{2}x} + c_2\sin\left(\frac{\sqrt{3}}{2}x\right)e^{-\frac{1}{2}x}$.

<<< Proving Solns to Euler's Equation

> Euler Equation - $\boxed{x^2 y'' + \alpha x y' + \beta y = 0}$ $(x>0)$

· Guess soln
$$\left.\begin{aligned} y &= x^s \\ y' &= s x^{s-1} \\ y'' &= s(s-1)x^{s-2} \end{aligned}\right\} \quad \begin{aligned} &x^2\left(s(s-1)x^{s-2}\right) + \alpha x\, s x^{s-1} + \beta x^s = 0 \\ &x^s\left[s(s-1) + \alpha s + \beta\right] = 0 \quad (x>0) \end{aligned}$$

$$s(s-1) + \alpha s + \beta = 0$$

> Indicial Equation - $\boxed{s^2 + s(\alpha-1) + \beta = 0}$

$$s_{1,2} = \frac{1-\alpha \pm \sqrt{(\alpha-1)^2 - 4\beta}}{2}$$

case 1 $s_1 \neq s_2$, $s_1, s_2 \in \mathbb{R}$

$$\boxed{y(x) = c_1 x^{s_1} + c_2 x^{s_2}}$$

case 2 $s_1 = s_2 \in \mathbb{R}$ Use Abel's Theorem & Reduction of Order

$(\alpha-1)^2 - 4\beta = 0$, so $s_1 = \frac{1-\alpha-\sqrt{0}}{2} = \frac{1-\alpha}{2}$

0 $x^2 y'' + \alpha x y' + \beta y = 0$

$y'' + \frac{\alpha}{x} y' + \frac{\beta}{x^2} y = 0$ $\quad \begin{cases} p(x) = \frac{\alpha}{x} \\ q(x) = \frac{\beta}{x^2} \end{cases}$

1 $W = c e^{-\int p(x)\,dx} = c e^{-\int \frac{\alpha}{x} dx} = c x^{-\alpha}$

2 $y_2 = y_1 \int \frac{W}{y_1^2} dx = x^{s_1} \int c x^{-\alpha} x^{-2s_1} dx$

$= c x^{s_1} \int x^{-(\alpha + 2s_1)} dx = c x^{s_1} \int x^{-(\alpha + 2\frac{1-\alpha}{2})} dx$

$= c x^{s_1} \int x^{-1} dx = c x^{s_1} \ln(x)$

3 $c = 1$, $y_2 = x^{s_1} \ln(x)$

4 $\boxed{y(x) = c_1 x^{s_1} + c_2 x^{s_1} \ln(x)}$

case 3 $s_1 \neq s_2 \in \mathbb{C}$ Use Euler's formula

$s_{1,2} = \eta \pm i\mu$

$$\begin{cases} y_1 = x^{s_1} = x^{\eta + i\mu} = x^\eta x^{i\mu} = x^\eta\left(\cos(\mu \ln x) + i \sin(\mu \ln x)\right) \\ y_2 = x^{s_2} = x^{\eta - i\mu} = x^\eta x^{-i\mu} = x^\eta\left(\cos(\mu \ln x) - i \sin(\mu \ln x)\right) \end{cases}$$

$(x^{i\mu} = e^{\ln(x^{i\mu})} = e^{i\mu \ln(x)} = \cos(\mu \ln(x)) + i \sin(\mu \ln(x)))$

linear comb. of y_1, y_2 $\begin{cases} y_3 = x^\eta \cos(\mu \ln x) \\ y_4 = x^\eta \sin(\mu \ln x) \end{cases}$ linearly independent $(W \neq 0)$

General soln $\boxed{y(x) = c_1 x^\eta \cos(\mu \ln x) + c_2 x^\eta \sin(\mu \ln x)}$

<-- Method of Solving Euler Equations

$\boxed{\text{Method}}$ Euler Equations

$\boxed{0}$ Write ODE in form: $x^2y'' + \alpha x y' + \beta y = 0$

$\boxed{1}$ Write indicial equation: $s^2 + s(\alpha - 1) + \beta = 0$

$\boxed{2}$ Solve indicial equation: $s_{1,2} = \dfrac{-(\alpha-1) \pm \sqrt{(\alpha-1)^2 - 4\beta}}{2}$

$\boxed{i}$ if $s_1 \neq s_2 \in \mathbb{R}$, soln is

$$y(x) = c_1 x^{s_1} + c_2 x^{s_2}$$

$\boxed{ii}$ if $s_1 = s_2 \in \mathbb{R}$, soln is

$$\left.\begin{aligned} y(x) &= c_1 x^{s_1} + c_2 x^{s_1} \ln(x) \\ &= x^{s_1}\left(c_1 + c_2 \ln(x)\right) \end{aligned}\right\}$$

Abel's Theorem
reduction of order
(L10)

$\boxed{iii}$ if $s_1 \neq s_2 \in \mathbb{C}$, $s_{1,2} = \eta \pm i\mu$, soln is

$$\left.\begin{aligned} &y(x) = c_1 x^{\eta} \cos(\mu \ln x) + c_2 x^{\eta} \sin(\mu \ln x) \\ &\text{where } \begin{cases} \eta = \dfrac{1-\alpha}{2} \\ \mu = \dfrac{\sqrt{4\beta - (\alpha-1)^2}}{2} \end{cases} \end{aligned}\right\}$$

Euler's Formula
(L11)

$\boxed{\text{EX1}}$ Solve $x^2y'' + y = 0$

$\begin{cases} \alpha = 0 \\ \beta = 1 \end{cases}$ $\boxed{1}$ $s^2 - s + 1 = 0$

$\boxed{2}$ $s_{1,2} = \dfrac{1 \pm i\sqrt{3}}{2}$ $\left.\begin{cases} \eta = \frac{1}{2} \\ \mu = \frac{\sqrt{3}}{2} \end{cases}\right\}$ case 3

$\boxed{2iii}$ $y(x) = c_1 x^{\frac{1}{2}} \cos\left(\frac{\sqrt{3}}{2} \ln(x)\right) + c_2 x^{\frac{1}{2}} \sin\left(\frac{\sqrt{3}}{2} \ln(x)\right)$

$\boxed{\text{EX2}}$ Solve $x^2y'' - 3xy' + 4y = 0$

$\begin{cases} \alpha = -3 \\ \beta = 4 \end{cases}$ $\boxed{1}$ $s^2 + s(-4) + 4 = 0$

$\boxed{2}$ $s_{1,2} = \dfrac{4 \pm \sqrt{16-16}}{2} = 2 \Rightarrow$ case 2

$\boxed{2ii}$ $y(x) = c_1 x^2 + c_2 x^2 \ln(x)$

... Nonhomogeneous 2nd order linear ODE

> $L[y] \equiv ay'' + by' + cy$

- $L[y] = 0$ is homogeneous
- $L[y] = g(x)$ is nonhomogeneous

→ Solution of Nonhomogeneous ODE

$\boxed{\text{Theorem}}$ The general soln of $L[y]=g(x)$ is $y(x) = y_H(x) + y_p(x)$, where $y_H = C_1y_1 + C_2y_2$ is soln to homogeneous problem $L[y]=0$, and y_P is any particular soln of $L[y]=g(x)$.

$\boxed{\text{Proof}}$ $L[y] = ay'' + by' + cy$

$$L[y] = L[y_H + y_P] = L[\cancelto{0}{y_H}] + L[y_P] = g(x)$$

- Choice of particular soln to $L[y]=g(x)$ does not matter.

$\boxed{\text{Proof}}$ · Suppose we have two particular soln y_{PA}, y_{PB}, then

$$L[y_{PA} - y_{PB}] = L[y_{PA}] - L[y_{PB}]$$
$$= g(x) - g(x) = 0$$

- $L[y_{PA} - y_{PB}] = 0$ shows that $y_{PA} - y_{PB}$ is a soln to homo. prob. $L[y]=0$
- Solns of homo prob. can be written in terms of fundamental soln y_1, y_2.

$$y_{PA} - y_{PB} = c_3y_1 + c_4y_2$$
$$y_{PA} = c_3y_1 + c_4y_2 + y_{PB}$$

- For A
$$\begin{cases} y = c_1y_1 + c_2y_2 + y_{PA} \\ \quad = c_1y_1 + c_2y_2 + (c_3y_1 + c_4y_2 + y_{PB}) \\ \quad = (c_1+c_3)y_1 + (c_2+c_4)y_2 + y_{PB} \end{cases}$$

- For B : $y = c_1y_1 + c_2y_2 + y_{PB}$

these are the same!

- Summary : choice of particular soln is arbitrary.

<<< Method of Undetermined Coefficients

$\boxed{0}$ Write ODE in $L[y] \equiv ay'' + by' + cy = g(x)$

$\boxed{1}$ Calculate soln to homogeneous problem $L[y_H] = 0$

$\boxed{2}$ Guess a form for y_p

$\boxed{3}$ Substitute y_p into ODE and solve for constants in y_p

$\boxed{4}$ Write down the general soln $y(x) = y_H + y_p$

→ Guessing y_p Chart ($\boxed{2}$)

Nonhomogeniety	$g(x)$	Guess particular soln	y_p (A is constant)
constant	α	constant	A
exponent	αe^{kx}	exponent	$A e^{kx}$
polynomial	$\alpha_N x^N + \ldots + \alpha_1 x^1 + \alpha_0$	polynomial	$A_N x^N + \cdots + A_1 x^1 + A_0$
exponential-polynomial	$e^{\beta x}(\alpha_N x^N + \cdots + \alpha_0 x^0)$	exponential-polynomial	$e^{\beta x}(A_N x^N + \cdots + A_0 x^0)$
cosine-sine terms	$\alpha \cos(\omega x) + \beta \sin(\omega x)$	cosine-sine terms	$A \cos(\omega x) + B \sin(\omega x)$
polynomial, cosine, sine	$P_n(x)\cos(\omega x) + Q_m(x)\sin(\omega x)$	polynomial cosine, sine	$S_N(x)\cos(\omega x) + T_m(x)\sin(\omega x)$ (Solve for coeff. in nth & mth order polynomial)
polynomial cosine, sine exponential	$e^{\alpha x}[P_n(x)\cos(\omega x) + Q_m(x)\sin(\omega x)]$	polynomial cosine, sine exponential	$e^{\alpha x}[S_n(x)\cos(\omega x) + T_m(x)\sin(\omega x)]$

* If y_p conflicts with y_H ($y_p = y_H$), then multiply y_p by x.

<<< Examples

$\boxed{EX1}$ $y'' + 3y' - 4y = 1$

$\boxed{1}$ Homogeneous Problem $L[y] = 0$:

$$\lambda^2 + 3\lambda - 4 = 0$$
$$\lambda_1 = 1, \quad \lambda_2 = -4$$
$$y_H = c_1 e^x + c_2 e^{-4x}$$

$\boxed{2}$ Guess y_p : $y_p = A \Rightarrow y_p' = 0, \; y_p'' = 0$

$\boxed{3}$ Sub $y_p \to$ ODE : $0 + 3(0) - 4A = 1$

$$A = -\frac{1}{4}$$
$$y_p = A = -\frac{1}{4}$$

$\boxed{4}$ General soln :

$$y(x) = y_H + y_p = c_1 e^x + c_2 e^{-4x} - \frac{1}{4}$$

$\boxed{EX2}$ $y'' + 3y' - 4y = e^{-4x}$

$\boxed{1}$ $\lambda^2 + 3\lambda - 4 = 0 \Rightarrow \lambda_1 = 1, \lambda_2 = -4$

$$y_H = c_1 e^x + c_2 e^{-4x}$$

$\boxed{2}$ Guess y_p : $y_p = Ae^{-4x}$ (* This wouldnt work since it conflicts with y_H)

$$y_p' = -4Ae^{-4x}$$
$$y_p'' = 16Ae^{-4x}$$

$\boxed{3}$ $-16Ae^{-4x} - 3(-4Ae^{-4x}) - 4(Ae^{-4x}) = e^{-4x}$

$$0 \neq e^{-4x}$$

Guess $y_p x$: $y_p = Axe^{-4x}$

$$y_p' = -4Axe^{-4x} + Ae^{-4x}$$
$$y_p'' = 16Axe^{-4x} - 8Ae^{-4x}$$

$\boxed{3}$ $16A\cancel{x}e^{-4x} - 8Ae^{-4x} - 12A\cancel{x}e^{-4x} + 3Ae^{-4x} - 4A\cancel{x}e^{-4x} = e^{-4x}$

$$-5Ae^{-4x} = e^{-4x}$$
$$A = -\frac{1}{5} \quad (e^{-4x} \neq 0)$$
$$y_p = -\frac{1}{5}xe^{-4x}$$

$\boxed{4}$ $y(x) = y_H + y_p = c_1 e^x + c_2 e^{-4x} - \frac{1}{5}xe^{-4x}$

« Derivation of Variation of Parameters

- For $L[y] = y'' + p(x)y' + r(x)y = g(x)$
- Found y_H for $L[y]=0$ by reduction of order from one soln y_1.

$$y_H = c_1 y_1 + c_2 y_2$$

- Find y_p for $L[y]=g(x)$ ← hard!

→ Find y_p known y_H

- Known $y_H(x) = c_1 y_1(x) + c_2 y_2(x)$ for $L[y]=0$
- Construct y_p from y_H by allowing constants vary with x for $L[y]=g(x)$

Let:
$$y_p \equiv u_1(x)y_1(x) + u_2(x)y_2(x)$$
$$y_p' = u_1' y_1 + u_1 y_1' + u_2' y_2 + u_2 y_2'$$

- Impose arbitrary (but helpful condition on u_1, u_2: $u_1' y_1 + u_2' y_2 = 0$ $(*)$

(can do this ∵ we want a particular soln, not a unique one).

so
$$y_p' = u_1 y_1' + u_2 y_2'$$
$$y_p'' = u_1 y_1'' + u_1' y_1' + u_2 y_2'' + u_2' y_2'$$

- Plug in $L[y]=g(x)$.

$$u_1 y_1'' + u_1' y_1' + u_2 y_2'' + u_2' y_2' + p(x)\left[u_1 y_1' + u_2 y_2'\right] + r(x)\left[u_1 y_1 + u_2 y_2\right] = g(x)$$
$$u_1 \underbrace{\left[y_1'' + p(x)y_1' + r(x)y_1\right]}_{0} + u_2 \underbrace{\left[y_2'' + p(x)y_2' + r(x)y_2\right]}_{0} + u_1' y_1' + u_2' y_2' = g(x)$$

(since y_1 and y_2 satisfy $L[y_H]=0$)

$$u_1' y_1' + u_2' y_2' = g(x) \qquad (**)$$

- Solve for
$$\begin{cases} u_1' y_1 + u_2' y_2 = 0 & (*) \\ u_1' y_1' + u_2' y_2' = g(x) & (**) \end{cases}$$

$(*)$: $u_2' = \dfrac{-u_1' y_1}{y_2}$

$$g(x) = u_1' y_1' - \frac{u_1' y_1 y_2'}{y_2} = \frac{u_1' y_1' y_2 - u_1' y_1 y_2'}{y_2}$$
$$= -u_1'\left[\frac{y_1 y_2' - y_1' y_2}{y_2}\right] = -u_1'\left(\frac{W(y_1, y_2)}{y_2}\right)$$

$$\Rightarrow u_1' = -\frac{g(x)\, y_2}{W(y_1,y_2)}$$
(can divide by $W(y_1,y_2)$ since y_1 & y_2 are linearly indep $\Leftrightarrow W(y_1,y_2) \neq 0$)

$$u_2' = y_1 \frac{g(x)}{W(y_1,y_2)}$$

<<< Derivation of Variation of Parameters (cont.)

- Known u_1', u_2', solve for u_1, u_2:

$$u_1 = -\int \frac{g(x)\, y_2}{W(y_1,y_2)}\, dx$$

$$u_2 = \int \frac{g(x)\, y_1}{W(y_1,y_2)}\, dx$$

- So the particular soln is

$$y_p = u_1 y_1 + u_2 y_2$$

$$= -y_1(x)\int \frac{g(x)\, y_2}{W(y_1,y_2)}\, dx + y_2(x)\int \frac{g(x)\, y_1}{W(y_1,y_2)}\, dx$$

<<< Method of Variation of Parameters

Method Variation of Parameters (Nonhomogeneous Equations)

0. $L[y] \equiv y'' + p(x)\, y' + r(x)\, y = g(x)$
1. Find the soln of homogeneous problem $L[y] = 0$:
$$y_H = c_1 y_1 + c_2 y_2$$
2. Find the particular soln for $L[y] = g(x)$:
$$y_p = -y_1(x)\int \frac{g(x)\, y_2}{W(y_1,y_2)}\, dx + y_2(x)\int \frac{g(x)\, y_1}{W(y_1,y_2)}\, dx$$
3. Write the general soln
$$y(x) = y_H + y_p$$

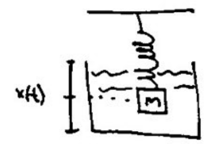
doesn't need constant ∵ can use any particular soln ; the general soln should have 2 constants

Ex $y'' - 5y' + 6y = 2e^x$, $p(x) = -5$, $r(x) = 6$, $g(x) = 2e^x$

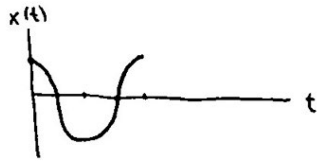
1. $\lambda^2 - 5\lambda + 6 = 0$
$\lambda_1 = 3,\ \lambda_2 = 2$ $\quad y_H = c_1 e^{3x} + c_2 e^{2x}$
2. $W(y_1,y_2) = W(e^{3x}, e^{2x}) = 2e^{3x}e^{2x} - 3e^{3x}e^{2x} = -e^{5x}$
$$y_p = -y_1\int \frac{g(x)\, y_2}{W(y_1,y_2)}\, dx + y_2\int \frac{g(x)\, y_1}{W(y_1,y_2)}\, dx = -e^{3x}\int \frac{2e^x e^{2x}}{-e^{5x}}\, dx + e^{2x}\int \frac{2e^x e^{3x}}{-e^{5x}}\, dx$$
$$= 2e^{3x}\int e^{-2x}\, dx - 2e^{2x}\int e^{-x}\, dx = -e^x + 2e^x = e^x$$
3. $y(x) = y_H + y_p = c_1 e^{3x} + c_2 e^{2x} + e^x$

Overview of Mechanical Vibration

- A mass on a spring moves vertically in a fluid bath
- Known position $x(t)$
  - velocity $v(t) = x'(t)$
  - acceleration $a(t) = x''(t)$



- If at rest, $x=0$, then

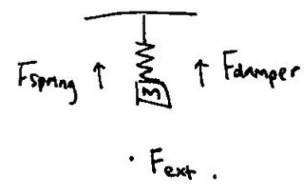


Amplitude of oscillation over time can...

1. does not change (constant amplitude)
2. decrease over time
3. increase over time
4. have other (time-dependent) behavior

Mathematical Modeling of Mech. Vibration

- Newton's second law - $\sum F = ma$
  - Damping force - $F_{damper} = -\gamma v = -\gamma x'(t)$
  - Spring force - $F_{spring} = -kx(t)$
  - External force - $F_{ext}(t) =$ (may come in diff form)



- $\sum F = F_{damper} + F_{spring} + F_{ext} = ma$

$$-\gamma x'(t) - kx(t) + F_{ext}(t) = mx''(t)$$

$$\boxed{mx''(t) + \gamma x'(t) + kx(t) = F_{ext}(t)}$$ ← 2nd order linear ODE, constant coeff

- Two situations $\begin{cases} \text{unforced oscillation} - F_{ext}(t) = 0 \rightarrow \text{homogeneous ODE} \\ \text{forced oscillation} - F_{ext}(t) \neq 0 \rightarrow \text{nonhomogeneous ODE} \end{cases}$

Unforced Oscillation

> unforced oscillation - no external force $F_{ext} \equiv 0$

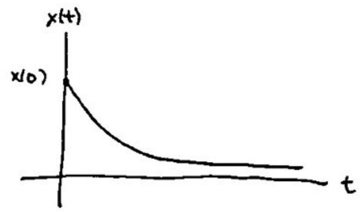
$$\boxed{mx'' + \gamma x' + kx = 0} \quad (m, \gamma, k \geq 0)$$

char. eq: $m\lambda^2 + \gamma\lambda + k = 0$

$$\lambda_{1,2} = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m} \quad \begin{cases} \lambda_1 \neq \lambda_2 \in \mathbb{R} & \text{- overdamped system} \\ \lambda_1 = \lambda_2 \in \mathbb{R} & \text{- critically damped system} \\ \lambda_1 \neq \lambda_2 \in \mathbb{C} & \text{- underdamped system} \end{cases}$$

- Unforced Oscillation

→ [1] Overdamped System



- $\lambda_1 \neq \lambda_2 \in \mathbb{R}$
- $\gamma^2 - 4mk > 0 \Rightarrow \gamma^2 > 4mk$
  - large γ, a lot of damping
- General soln $\boxed{x(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}}$ $(\lambda_1 \lambda_2 \leq 0 ;\ \gamma, m, k \geq 0)$

→ Proof of $\lambda_1, \lambda_2 \leq 0$

$$\lambda_{1,2} = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m} = -\frac{\gamma}{2m} \pm \frac{\sqrt{\gamma^2-4mk}}{2m}$$

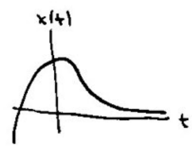
$-\frac{\gamma}{2m}$: $m \neq 0$, negative

$\frac{\sqrt{\gamma^2-4mk}}{2m}$: $m \neq 0$,
- subtract - negative
- add - $\sqrt{\gamma^2-4mk} \leq \gamma$ needed
  - $\gamma^2 - 4mk \leq \gamma^2$ is always true for $m>0,\ k \geq 0$.
  - → so is negative

→ [2] Critically Damped System

- $\lambda_1 = \lambda_2 = -\frac{\gamma}{2m} \in \mathbb{R}$
- General soln

$$\boxed{x(t) = c_1 e^{\lambda t} + c_2 t e^{\lambda t}}$$



→ [3] Underdamped System

- $\lambda_1 \neq \lambda_2 \in \mathbb{C}$
- $\lambda_{1,2} = -\frac{\gamma}{2m} \pm \frac{i\sqrt{4mk-\gamma^2}}{2m} = -\frac{\gamma}{2m} \pm i\omega$ where $\omega \equiv \frac{\sqrt{4mk-\gamma^2}}{2m}$
- General soln

$$\boxed{x(t) = c_1 e^{-\frac{\gamma}{2m}t}\cos(\omega t) + c_2 e^{-\frac{\gamma}{2m}t}\sin(\omega t)}$$

→ [3a] undamped spring

- $\gamma = 0$
- General soln reduces to

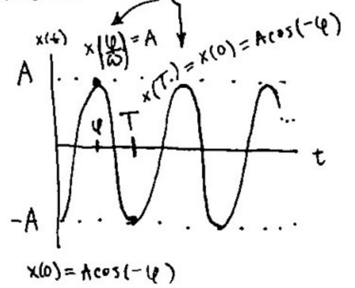
$$\boxed{x(t) = c_1\cos(\omega t) + c_2\sin(\omega t)}$$ where $\omega = \frac{\sqrt{4mk - 0^2}}{2m} = \sqrt{\frac{k}{m}}$

- natural frequency $\omega_0 = \sqrt{\frac{k}{m}}$
- phase-amplitude form
  - Let $\begin{cases} c_1 \equiv A\cos\varphi \\ c_2 \equiv A\sin\varphi \end{cases} \Rightarrow \begin{cases} A = \sqrt{c_1^2 + c_2^2} \\ \tan\varphi = \frac{c_2}{c_1} \end{cases}$
  - the soln : $x(t) = A\cos\varphi\cos(\omega t) + A\sin\varphi\sin(\omega t)$

$$\boxed{x(t) = A\cos(\omega t - \varphi)}$$

- amplitude $A = \sqrt{c_1^2 + c_2^2}$
- phase $\varphi = \arctan\left(\frac{c_2}{c_1}\right)$
- period $T = \frac{2\pi}{\omega}$
- natural freq $\omega = \sqrt{\frac{k}{m}}$

trough & peaks at $x\left(\frac{\varphi+\pi}{\omega}\right) = \pm A$



... Unforced Oscillations

→ $\boxed{3}$ Underdamped System

→ $\boxed{3b}$ Underdamped spring

- $\gamma > 0$
- Same general soln :

$$\boxed{x(t) = c_1 e^{-\frac{\gamma t}{2m}} \cos(\omega t) + c_2 e^{-\frac{\gamma t}{2m}} \sin(\omega t)}$$

- phase-amplitude form

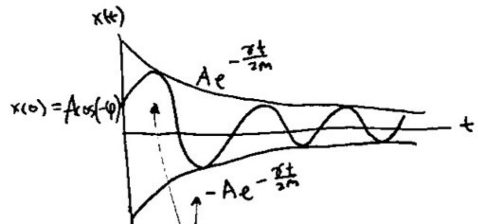
$$x(t) = e^{-\frac{\gamma t}{2m}} \left(c_1 \cos(\omega t) + c_2 \sin(\omega t)\right)$$

$$\left\{ \begin{array}{l} c_1 \equiv A\cos\varphi \\ c_2 \equiv A\sin\varphi \end{array} \right.$$

$$\boxed{x(t) = A e^{-\frac{\gamma t}{2m}} \cos(\omega t - \varphi)}$$

$$\left\{ \begin{array}{l} A = \sqrt{c_1^2 + c_2^2} \\ \varphi = \arctan\left(\frac{c_2}{c_1}\right) \\ \omega = \sqrt{\frac{k}{2m}} \\ T = \frac{2\pi}{\omega} \end{array} \right.$$

- $e^{-\frac{\gamma t}{2m}}$ is time-dependent damping



peaks & trough at $x\left(\frac{\arctan\left(-\frac{\gamma}{2m\omega} + \varphi\right)}{\omega} + n\pi\right)$

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<<< Forced Oscillations

> forced oscillation - have some external force $F_{ext} \neq 0$

· Here, consider $F_{ext} = F_0 \cos(\Omega t)$ (F_0 constant amplitude, Ω constant freq)

$$\boxed{mx'' + \gamma x' + kx = F_0 \cos(\Omega t)}$$

→ [1] No damping ($\gamma = 0$), No resonance ($\Omega \neq \omega_0$)

[0] $mx'' + kx = F_0 \cos(\Omega t)$ $\qquad \omega_0 = \sqrt{\frac{k}{m}} \Rightarrow k = \omega_0^2 m$

$$x'' + \omega_0^2 x = \frac{F_0}{m} \cos(\Omega t)$$

[1] $x_H = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t)$ from last time undamped spring

[2] Guess $x_p = A\cos(\Omega t) + B\sin(\Omega t)$ ($\Omega \neq \omega_0$ by assumption)

$$x_p' = -A\Omega \sin(\Omega t) + B\Omega \cos(\Omega t)$$

$$x_p'' = -A\Omega^2 \cos(\Omega t) - B\Omega^2 \sin(\Omega t)$$

[3] Substitute back ODE $\quad -A\Omega^2 \cos(\Omega t) - B\Omega^2 \sin(\Omega t) + \omega_0^2 [A\cos(\Omega t) + B\sin(\Omega t)] = \frac{F_0}{m} \cos(\Omega t)$

$$\cos(\Omega t)[-A\Omega^2 + A\omega_0^2] + \sin(\Omega t)[-B\Omega^2 + B\omega_0^2] = \frac{F_0}{m} \cos(\Omega t)$$

Match coeff
$$\begin{cases} -A\Omega^2 + A\omega_0^2 = \frac{F_0}{m} \\ -B\Omega^2 + B\omega_0^2 = 0 \end{cases} \Rightarrow \begin{cases} A = \frac{F_0}{m(\omega_0^2 - \Omega^2)} \\ B = 0 \end{cases}$$

So $\quad x_p = \dfrac{F_0 \cos(\Omega t)}{m(\omega_0^2 - \Omega^2)}$

[4] $$\boxed{x = x_H + x_p = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) + \frac{F_0 \cos(\Omega t)}{m(\omega_0^2 - \Omega^2)}} \quad (\gamma = 0, \Omega \neq \omega_0)$$

[5] Known IVP $x(0) = 0$, $x'(0) = 0$,

$$x'(t) = -c_1 \omega_0 \sin(\omega_0 t) + c_2 \omega_0 \cos(\omega_0 t) - \frac{F_0 \Omega \sin(\Omega t)}{m(\omega_0^2 - \Omega^2)}$$

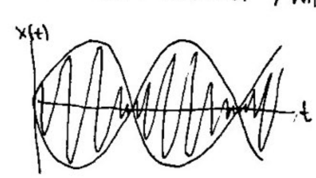
$$\begin{cases} x(0) = 0 = c_1 + \frac{F_0}{m(\omega_0^2 - \Omega^2)} \\ x'(0) = 0 = c_2 \omega_0 \end{cases} \Rightarrow \begin{cases} c_1 = -\frac{F_0}{m(\omega_0^2 - \Omega^2)} \\ c_2 = 0 \end{cases}$$

$$x(t) = \frac{F_0}{m(\omega_0^2 - \Omega^2)} [\cos(\Omega t) - \cos(\omega_0 t)]$$

Let $\omega_1 \equiv \dfrac{\omega_0 - \Omega}{2}$, $\omega_2 \equiv \dfrac{\omega_0 + \Omega}{2}$

use trig identity

$x(t) = U(t) \sin(\omega_2 t)$, where $U(t) \equiv \dfrac{2F_0 \sin(\omega_1 t)}{m(\omega_0^2 - \Omega^2)}$



Modulated wave + beats phenomenon

<<< Forced Oscillation

→ [2] No damping, Yes resonance ($\gamma = 0$, $\Omega = \omega_0$)

[0] $mx'' + kx = F_0 \cos(\omega_0 t)$ $\qquad \omega_0 = \sqrt{\frac{k}{m}} \Rightarrow k = \omega_0^2 m$

$x'' + \omega_0^2 x = \frac{F_0}{m} \cos(\omega_0 t)$

[1] $x_H = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t)$

[2] Guess $x_p = At\cos(\omega_0 t) + Bt\sin(\omega_0 t)$ (to avoid conflict with x_H)

$x_p' = A\cos(\omega_0 t) + At\omega_0(-\sin(\omega_0 t)) + B\sin(\omega_0 t) + Bt\omega_0 \cos(\omega_0 t)$

$x_p'' = -A\omega_0 \sin(\omega_0 t) - A\omega_0\left(\sin(\omega_0 t) + t\omega_0 \cos(\omega_0 t)\right)$

$+ B\cos(\omega_0 t)\omega_0 + B\omega_0\left(\cos(\omega_0 t) - t\omega_0 \sin(\omega_0 t)\right)$

$= \sin(\omega_0 t)\left[-2A\omega_0 - Bt\omega_0^2\right] + \cos(\omega_0 t)\left[-A\omega_0^2 t + 2B\omega_0\right]$

[3] Plug in ODE

$\sin(\omega_0 t)[-2A\omega_0 - Bt\omega_0^2] + \cos(\omega_0 t)[-A\omega_0^2 t + 2B\omega_0] + \omega_0^2[At\cos(\omega_0 t) + Bt\sin(\omega_0 t)] = \frac{F_0}{m}\cos(\omega_0 t)$

$\sin(\omega_0 t)[-2A\omega_0 - Bt\omega_0^2 + Bt\omega_0] + \cos(\omega_0 t)[-A\omega_0^2 t + 2B\omega_0 + A\omega_0^2 t] = \frac{F_0}{m}\cos(\omega_0 t)$

Match coeff

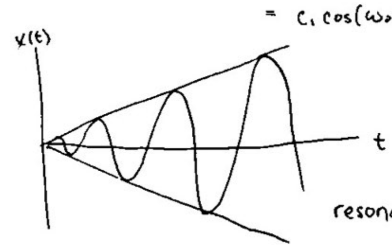
$$\begin{cases} -2A\omega_0 - \cancel{Bt\omega_0^2} + \cancel{Bt}\omega_0 = 0 \\ -A\cancel{\omega_0^2}t + 2B\omega_0 + A\cancel{\omega_0^2}t = \frac{F_0}{m} \end{cases} \rightarrow \begin{cases} A = 0 \\ B = \frac{F_0}{2\omega_0 m} \end{cases}$$

So $x_p = \frac{F_0}{2\omega_0 m} t\sin(\omega_0 t)$

[4] General soln $\boxed{x(t) = x_H + x_p = c_1\cos(\omega_0 t) + c_2 \sin(\omega_0 t) + \frac{F_0}{2\omega_0 m} t\sin(\omega_0 t)}$ $\begin{pmatrix} \gamma = 0, \\ \Omega = \omega_0 \end{pmatrix}$

$= c_1\cos(\omega_0 t) + \left(c_2 + \frac{F_0}{2\omega_0 m} t\right)\sin(\omega_0 t)$

grows linearly with t.



resonance

<<< Intro to Systems

- system of algebraic equation

$$\begin{cases} ax + by = c \\ dx + fy = g \end{cases} \Rightarrow x, y \text{ are numbers}$$

- system of differential equations

$$\begin{cases} y_1'(x) = y_2 \\ y_2'(x) = y_1 + y_2 \end{cases} \Rightarrow y_1, y_2 \text{ are functions}$$

<<< Rewriting ODEs into systems of 1st order ODEs

$\boxed{\text{Method}}$ $\boxed{1}$ Define n auxiliary variables $y_1, \dots, y_n$ for nth order ODE.

$\boxed{2}$ Write a system of ODEs with derivatives of auxiliary variables on the LHS and their expressions on the RHS in terms of the auxiliary vars.

$\boxed{\text{EX1}}$ $y'' + p(x)\, y' + r(x)\, y = g(x)$

$\boxed{1}$ $\begin{cases} y_1 = y \\ y_2 = y_1' = y' \end{cases}$ $\boxed{2}$ $\begin{cases} y_1' = y_2 \\ y_2' = -p(x)\, y_2 - r(x)\, y_1 + g(x) \end{cases}$

$$y_2' = y'' = g(x) - p(x) y' - r(x) y$$

$$= -p(x)\, y_2 - r(x) y_1 + g(x)$$

Write in terms of auxiliary variables (no original y)

$\boxed{\text{EX2}}$ $y''' + 2y' + 5y = 7$

$\boxed{1}$ $\begin{cases} y_1 = y \\ y_2 = y_1' = y' \\ y_3 = y_2' = y'' \end{cases}$ $\boxed{2}$ $\begin{cases} y_1' = y_2 \\ y_2' = y_3 \\ y_3' = 7 - 2y_2 - 5y_1 \end{cases}$

$$y_3' = y''' = 7 - 2y' - 5y = 7 - 2y_2 - 5y_1$$

$\boxed{\text{EX3}}$ $y'''' + 3y'''y + y' = 0$

$\boxed{1}$ $\begin{cases} y_1 = y \\ y_2 = y_1' = y' \\ y_3 = y_2' = y'' \\ y_4 = y_3' = y''' \end{cases}$ $\boxed{2}$ $\begin{cases} y_1' = y_2 \\ y_2' = y_3 \\ y_3' = y_4 \\ y_4' = -3y_4 y_1 - y_2 \end{cases}$

$$y_4' = y'''' = -3y'''y - y' = -3y_4 y_1 - y_2$$

Intro to Linear Algebra

- Most general linear system of ODEs :

$$\begin{cases} y_1' = a_{11} y_1 + a_{12} y_2 + \cdots + a_{1n} y_n + b_1 \\ y_2' = a_{21} y_1 + a_{22} y_2 + \cdots + a_{2n} y_n + b_2 \\ \vdots \\ y_n' = a_{n1} y_1 + a_{n2} y_2 + \cdots + y_{nn} y_n + b_n \end{cases}$$

where a_{ij} and b_i are fxn of x.

- n unknowns $y_1, \ldots, y_n$
- n ODEs

Let $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$, $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$, $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$

we have $\boxed{\frac{d}{dx} \vec{y} = A\vec{y} + \vec{b}}$

→ Elementary Properties of Matrix & Vector

1. Addition & Subtraction
   - element-wise operation
   - only defined for same dimensions

   $\vec{v} \pm \vec{w} = \vec{v} \pm \vec{w}$

   $\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \pm \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} = \begin{bmatrix} v_1 \pm w_1 \\ v_2 \pm w_2 \\ \vdots \\ v_n \pm w_n \end{bmatrix}$

2. Transposition
   - let element a_{ij} be element a_{ji}

   $\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$ $\vec{a}^T = [a_1, a_2, a_3, \ldots, a_n]$

   $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

3. Multiplication
   - if A is $n \times m$ matrix, B is $m \times p$ matrix,

   then AB is defined by

   $$AB_{ij} = i \text{ row of } A \cdot j \text{ row of } B = \sum_{r=1}^{m} A_{ir} B_{rj}$$

   $\boxed{EX4}$ $AB_{1,1} = a_{11} b_{11} + a_{12} b_{21} + a_{13} b_{31} + \cdots + a_{1m} b_{m1}$

   $\boxed{EX5}$ $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 5 & 6 & 7 \\ 8 & 9 & 0 \end{bmatrix}$

   $$AB = \begin{bmatrix} 5+16 & 6+18 & 7 \\ 15+32 & 18+36 & 21 \end{bmatrix} = \begin{bmatrix} 21 & 24 & 7 \\ 47 & 54 & 21 \end{bmatrix}$$

   - inner dimension of matrices have to agree : # column of A = # row of B

   $\boxed{EX6}$ $A : 2 \times 2$ $B : 2 \times 3$ AB : can multiply ; BA : not defined

   - dimension of the product is the outer dimension of the matrices

   $\boxed{EX7}$ $A : 2 \times 2$ $B : 2 \times 3$ $AB : 2 \times 3$; BA : not defined

   - matrix multiplication is not commutative ; $AB \neq BA$

> identity matrix $- I_n$ is $n \times n$ matrix with 1 in main diag, 0 everywhere else

- $I_n = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{bmatrix}$
- $AI_n = A = I_n A$

<<< Linear (In)dependence

> linearly dependent - vectors $\vec{x_1}, \vec{x_2}, \dots, \vec{x_n}$ satisfy equation such that the constants $c_1, c_2, \dots, c_n$ are not all zero: $c_1\vec{x_1} + c_2\vec{x_2} + \dots + c_n\vec{x_n} = 0$.

> linearly independent - vectors that are not linearly dependent

$\boxed{\text{Ex 1}}$ Are $\vec{x_1}, \vec{x_2}, \vec{x_3}$ linearly dependent?

$\vec{x_1} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$, $\vec{x_2} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, $\vec{x_3} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$

(Aka, find c_1, c_2, c_3.)

$$c_1 \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = \vec{0}$$

$$\begin{bmatrix} 2 & 0 & -1 \\ 1 & 1 & 2 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \vec{0} \quad \Rightarrow \quad A\vec{c} = \vec{0}$$

Claim: $\vec{c} = \begin{bmatrix} 1 \\ -5 \\ 2 \end{bmatrix}$

Check: $\begin{bmatrix} 2 & 0 & -1 \\ 1 & 1 & 2 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -5 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

→ Remarks

- For n-dim vectors, if we have $m > n$ of vectors, then they are linearly dependent (more variables than equations)
- If we have fewer or equal # vectors than their dimension ($m \leq n$), they they tend to be independent. (but not guaranteed!)

Matrix Inversion

→ Motivation

- In scalar algebra, we could solve system by dividing both sides by a number

$$ax = b \quad \Rightarrow \quad x = \frac{b}{a}$$

- multiply a by its multiplicative inverse, $\frac{1}{a}$, such that $a \cdot \frac{1}{a} = 1$, which is the multiplicative identity, the only number that when mult. to x gives x.
- In linear algebra, we use inverse matrix

→ Definition

> inverse of a square matrix A is A^{-1}, where $AA^{-1} = A^{-1}A = I_n$

- only defined for square matrix $(n \times n)$.
- We can solve systems using inverse.

$$A\vec{x} = \vec{b} \Rightarrow A^{-1}A\vec{x} = A^{-1}\vec{b} \Rightarrow I\vec{x} = A^{-1}\vec{b} \Rightarrow \vec{x} = A^{-1}\vec{b}$$

→ Examples

$\boxed{\text{Ex1}}$ Solve $\begin{cases} x + y = 4 \\ -x + y = 8 \end{cases} \Rightarrow \underbrace{\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 4 \\ 8 \end{bmatrix}}_{\vec{b}}$

with $A^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

Check: $A^{-1}A = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$ ✓

Solve: $\vec{x} = A^{-1}\vec{b} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 8 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -4 \\ 12 \end{bmatrix} = \begin{bmatrix} -2 \\ 6 \end{bmatrix} \Rightarrow \begin{cases} x = -2 \\ y = 6 \end{cases}$

$\boxed{\text{Ex2}}$ Solve $A\vec{x} = \vec{b}$, where

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 2 & -2 \\ 1 & 1 & 4 & 4 \\ 1 & -1 & 8 & -8 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad A^{-1} = \frac{1}{12}\begin{bmatrix} 8 & 8 & -2 & -2 \\ 8 & -2 & -2 & 2 \\ -2 & -1 & 2 & 1 \\ -2 & 1 & 2 & -1 \end{bmatrix}$$

$$\vec{x} = A^{-1}\vec{b} = \frac{1}{12}\begin{bmatrix} 8 & 8 & -2 & -2 \\ 8 & -8 & -2 & 2 \\ -2 & 1 & 2 & 1 \\ -2 & 1 & 2 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{12}\begin{bmatrix} 6 \\ 6 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \\ 0 \end{bmatrix}$$

→ Finding simple matrix inverse (2x2)

- for $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$
- check: $AA^{-1} = \frac{1}{ad-bc}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} ad-bc & 0 \\ 0 & -bc+ad \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$

Matrix Determinant

> singular – matrix with a 0 determinant

- determinant checks if a square matrix is invertable

→ 2x2 matrix

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\det(A) = ad - bc$

→ 3x3 matrix

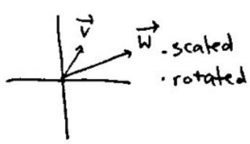
$$B = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}, \quad \det(B) = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$
$$= a(ei-hf) - b(di-gf) + c(dh-eg)$$

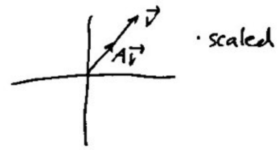
Equivalent Statements:

1. $\det(A) = 0$
2. A is singular
3. cols of A are linearly dependent
4. rows of A are linearly dependent
5. A^{-1} does not exist
6. $A\vec{x} = \vec{b}$ has either no soln or ∞ soln

<-- Linear Algebra as Transformations

- $A\vec{v} = \vec{w}$ is A rotating & scaling $\vec{v}$ to $\vec{w}$





- What if we find a vector $\vec{v}$ for a square matrix A such that $A\vec{v}$ is only scaled, but not rotated?
  - $A\vec{v} = \lambda\vec{v}$, where $\lambda \in \mathbb{C}$ is constant.

<-- Eigenvalue & Eigenvectors of Matrix

> Eigenvector - vectors $\vec{v}$ such that $A\vec{v} = \lambda\vec{v}$ for square matrix A

> Eigenvalue - constant λ corresponding to eigenvector $\vec{v}$.

- by convention, $\vec{0}$ is not an eigenvector

→ Finding Eigenvalues & Eigenvectors

$$A\vec{v} = \lambda\vec{v}$$
$$A\vec{v} - \lambda\vec{v} = \vec{0}$$
$$A\vec{v} - \lambda I\vec{v} = \vec{0}$$
$$(A-\lambda I)\vec{v} = \vec{0}$$

1. $(A-\lambda I)$ is nonsingular
   $\Rightarrow (A-\lambda I)^{-1}$ exists, and the soln of the system is
   $$\vec{v} = (A-\lambda I)^{-1}\vec{0} = \vec{0}$$
   which contradicts our convention $\Rightarrow\Leftarrow$
   (not interesting!)

2. $(A-\lambda I)$ is singular ✱
   $\Rightarrow \boxed{\det(A-\lambda I) = 0}$

→ Method : Finding Eigenvalues & Eigenvectors

$\boxed{\text{Method}}$ [1] Write $\det(A-\lambda I) = 0$ and solve for λ

- $n \times n$ matrix has n distinct λ

[2] Use $A\vec{v} = \lambda\vec{v}$ to find $\vec{v}$ for each λ

- eigenvectors are not unique (proof)
- We get relationships between eigenvector elements

If $A\vec{v} = \lambda\vec{v}$, then
$$A(\alpha\vec{v}) = \alpha(A\vec{v})$$
$$= \alpha(\lambda\vec{v})$$
$\Rightarrow$ $\alpha\vec{v}$ is also an eigvec with same λ as $\vec{v}$

→ Example

$\boxed{\text{EX1}}$ Find eigenvalue & eigenvector of $A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$

[1] $\det(A-\lambda I) = 0$

$$\det\left(\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \det\left(\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = \det\left(\begin{bmatrix} -2-\lambda & 1 \\ 1 & -2-\lambda \end{bmatrix}\right)$$

$$= (-2-\lambda)^2 - 1 = \lambda^2 + 4\lambda + 3 = (\lambda+3)(\lambda+1) = 0$$

$$\boxed{\lambda_1 = -3,\ \lambda_2 = -1} \leftarrow \text{eigenvalues}$$

[2] $A\vec{v} = \lambda\vec{v}$

① $\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = -3\begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix}$
$\Rightarrow -2v_{11} + v_{12} = -3v_{11} \Rightarrow v_{12} = -v_{11}$
$\Rightarrow v_{11} - 2v_{12} = -3v_{12} \Rightarrow v_{12} = -v_{11}$
$\}$ same $\boxed{\vec{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}}$ (not unique)

② $\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = -1\begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix}$
$\Rightarrow -2v_{21} + v_{22} = -v_{21} \Rightarrow v_{21} = v_{22}$
$\Rightarrow v_{21} - 2v_{22} = -v_{22} \Rightarrow v_{21} = v_{22}$
$\}$ same $\boxed{\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$

<<< Systems of ODEs

- $\boxed{\vec{x}' = P(t)\vec{x} + \vec{g}(t)}$ $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, $\vec{x}' = \frac{d}{dt}\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_1' \\ \vdots \\ x_n' \end{bmatrix}$
  > homogeneous - $\vec{g}(t) = \vec{0}$
  > nonhomogeneous - $\vec{g}(t) \neq \vec{0}$
- Assume $P(t)$ is constant coefficient P, we have $\vec{x}' = P\vec{x}$
  - similar to $y' = \alpha y$, $\alpha \in \mathbb{R}$, y is function
    that has soln $y(t) = y(0)e^{\alpha t}$
  - guess soln of $\vec{x} = \vec{v}e^{\lambda t}$, $\vec{v}$ is vector of constant, λ is constant
  - plug in:

$$\vec{x}' = P\vec{x}$$
$$\frac{d}{dt}\vec{v}e^{\lambda t} = P\vec{v}e^{\lambda t}$$
$$\lambda\vec{v}e^{\lambda t} = P\vec{v}e^{\lambda t}$$
$$\lambda\vec{v} = P\vec{v}$$

eigenvector eigenvalue

← eigenvalue problem

Ex1 Solve $\vec{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}\vec{x}$, $P = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$

Guess $\vec{x} = \vec{v}e^{\lambda t} \Rightarrow P\vec{v} = \lambda\vec{v}$

1. $\det(P - \lambda I) = 0$

$$\det\left(\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = \det\left(\begin{bmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{bmatrix}\right) = (1-\lambda)^2 - 4 = \lambda^2 - 2\lambda - 3 = (\lambda-3)(\lambda+1) = 0$$

$\lambda_1 = 3$, $\lambda_2 = -1$ ← eigenvalues

2. $\lambda_1 = 3$: $\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \begin{bmatrix} 3v_{11} \\ 3v_{12} \end{bmatrix} \Rightarrow v_{11} + v_{12} = 3v_{11}$, $v_{12} = 2v_{11}$ $\Rightarrow v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

   $\lambda_2 = -1$: $\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = \begin{bmatrix} -v_{21} \\ -v_{22} \end{bmatrix} \Rightarrow v_{21} + v_{22} = -v_{21}$, $v_{22} = -2v_{21}$ $\Rightarrow v_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

3. Known $\lambda_1 = 3$, $\lambda_2 = -1$, $v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $v_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, plug in to guess, and by superposition,

$$\vec{x} = c_1\vec{v_1}e^{\lambda_1 t} + c_2\vec{v_2}e^{\lambda_2 t}$$
$$\vec{x} = c_1\begin{bmatrix} 1 \\ 2 \end{bmatrix}e^{3t} + c_2\begin{bmatrix} -1 \\ 2 \end{bmatrix}e^{-t}$$

<<< Linear Independence & Combination

$\boxed{\text{Theorm}}$ If the vectors $\vec{x_1}, \vec{x_2}, \dots, \vec{x_n}$ are linearly independent soln of homogeneous problem

$$\vec{x}' = P\vec{x}$$

then the general soln $\vec{x}$ is the linear combination

$$\vec{x} = c_1\vec{x_1} + c_2\vec{x_2} + \cdots + c_n\vec{x_n}$$

→ Checking Linear Independence

> Wronskian of $\vec{x}$: $W[\vec{x_1}, \vec{x_2}, \dots, \vec{x_n}] = \det([\vec{x_1}, \vec{x_2}, \dots, \vec{x_n}]) = \det(X)$

· If $W[x_1, x_2, \dots, x_n] \neq 0$, they are linearly independent

$\boxed{\text{Ex2}}$ Check linear independence of $\vec{x_1} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t}$, $\vec{x_2} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} e^{-t}$

$$X = \begin{bmatrix} e^{3t} & -e^{-t} \\ 2e^{3t} & 2e^{-t} \end{bmatrix}$$

$$\det(X) = 2e^{2t} + 2e^{2t} = 4e^{2t} \neq 0 \Rightarrow \text{linearly independent}$$

$\boxed{\text{Ex3}}$ Solve $\vec{x}' = \begin{bmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{bmatrix} \vec{x}$ $\quad P = \begin{bmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{bmatrix}$

Guess $\vec{x}_i = \vec{v}_i e^{\lambda_i t} \Rightarrow P\vec{v} = \lambda\vec{v}$

$\boxed{1}$ $\det(P - \lambda I) = 0$

$$\det\left(\begin{bmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = \det\left(\begin{bmatrix} -3-\lambda & \sqrt{2} \\ \sqrt{2} & -2-\lambda \end{bmatrix}\right) = (-3-\lambda)(-2-\lambda) - 2$$

$$= \lambda^2 + 5\lambda + 4 = (\lambda+4)(\lambda+1) = 0$$

$$\lambda_1 = -4, \lambda_2 = -1$$

$\boxed{2}$ $\lambda_1 = -4$ $\quad \begin{bmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \begin{bmatrix} -4v_{11} \\ -4v_{12} \end{bmatrix} \Rightarrow \begin{matrix} -3v_{11} + \sqrt{2}v_{12} = -4v_{11} \\ \sqrt{2}v_{12} = -v_{11} \end{matrix} \rightarrow v_1 = \begin{bmatrix} \sqrt{2} \\ -1 \end{bmatrix}$

$\lambda_2 = -1$ $\quad \begin{bmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{bmatrix} \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = \begin{bmatrix} -v_{21} \\ -v_{22} \end{bmatrix} \Rightarrow \begin{matrix} -3v_{21} + \sqrt{2}v_{22} = -v_{21} \\ \sqrt{2}v_{22} = 2v_{21} \end{matrix} \Rightarrow v_1 = \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix}$

$\boxed{3}$

$$\vec{x} = c_1 \begin{bmatrix} \sqrt{2} \\ -1 \end{bmatrix} e^{-4t} + c_2 \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix} e^{-t}$$

<<< Dealing with Complex Eigenvalues λ (Case 2)

$\boxed{\text{Ex1}}$ Solve $\vec{x}' = \begin{bmatrix} -\frac{1}{2} & 1 \\ -1 & -\frac{1}{2} \end{bmatrix} \vec{x}$ $\quad P = \begin{bmatrix} -\frac{1}{2} & 1 \\ -1 & -\frac{1}{2} \end{bmatrix}$

Guess $\vec{x_i} = \vec{v_i} e^{\lambda_i t}$, where $\vec{v}, \lambda$ are eigvec, eigvalue of P.

$P\vec{v} = \lambda \vec{v}$

$\boxed{1}$ $\det(P - \lambda I) = \det\left(\begin{bmatrix} -\frac{1}{2} - \lambda & 1 \\ -1 & -\frac{1}{2} - \lambda \end{bmatrix}\right) = \lambda^2 + \lambda + \frac{5}{4} = 0$

$\lambda_{1,2} = -\frac{1}{2} \pm i$

$\boxed{2}$ $\lambda_1 = -\frac{1}{2} + i$ $\quad \begin{bmatrix} -\frac{1}{2} & 1 \\ -1 & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \begin{bmatrix} (-\frac{1}{2}+i)\, v_{11} \\ (-\frac{1}{2}+i)\, v_{12} \end{bmatrix}$

$\cancel{-\frac{1}{2}v_{11}} + v_{12} = \cancel{-\frac{1}{2}v_{11}} + i v_{11}$ $\quad v_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}$

$v_{12} = i v_{11}$

$\lambda_2 = -\frac{1}{2} - i$ $\quad \begin{bmatrix} -\frac{1}{2} & 1 \\ -1 & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = \begin{bmatrix} (-\frac{1}{2}-i)\, v_{21} \\ (-\frac{1}{2}-i)\, v_{22} \end{bmatrix}$

$\cancel{-\frac{1}{2}v_{21}} + v_{22} = \cancel{-\frac{1}{2}v_{21}} - i v_{21}$ $\quad v_2 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$

$v_{22} = -i v_{21}$

$\boxed{3}$ $\vec{x} = \vec{x_1} + \vec{x_2} = c_1 v_1 e^{\lambda_1 t} + c_2 v_2 e^{\lambda_2 t}$

$= c_1 \begin{bmatrix} 1 \\ i \end{bmatrix} e^{(-\frac{1}{2}+i)t} + c_2 \begin{bmatrix} 1 \\ -i \end{bmatrix} e^{(-\frac{1}{2}-i)t}$

$\boxed{4}$ Rewrite to real soln using Euler's formula $e^{it} = \cos t \pm i \sin t$

$= c_1 \begin{bmatrix} 1 \\ i \end{bmatrix} e^{-\frac{1}{2}t} e^{it} + c_2 \begin{bmatrix} 1 \\ -i \end{bmatrix} e^{-\frac{1}{2}t} e^{-it}$

$= \underbrace{c_1 \begin{bmatrix} 1 \\ i \end{bmatrix} e^{-\frac{1}{2}t} (\cos t + i \sin t)}_{\vec{x_1}} + \underbrace{c_2 \begin{bmatrix} 1 \\ -i \end{bmatrix} e^{-\frac{1}{2}t} (\cos t - i \sin t)}_{\vec{x_2}}$

complicated, we can use the "in general" statement next page

Let $\vec{x_3} \equiv \frac{\vec{x_1} + \vec{x_2}}{2} = e^{-\frac{1}{2}t} \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}$

$\vec{x_4} \equiv \frac{\vec{x_1} - \vec{x_2}}{2i} = e^{-\frac{1}{2}t} \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}$

* $\vec{x} = \vec{x_3} + \vec{x_4}$ is general soln if

✓ 1. they satisfy ODE

· if $\vec{x_1}, \vec{x_2}$ are soln of ODE, then their linear comb. is also soln. by superposition

✓ 2. they are independent

$\det(X) = \det\left(\begin{bmatrix} e^{-\frac{1}{2}t}\cos t & e^{-\frac{1}{2}t}\sin t \\ -e^{-\frac{1}{2}t}\sin t & e^{-\frac{1}{2}t}\cos t \end{bmatrix}\right) = e^{-t}\cos^2 t + e^{-t}\sin^2 t = e^{-t} > 0$

$\iff$ linearly independent

So general soln is: $\vec{x} = \vec{x_3} + \vec{x_4} = c_1 e^{-\frac{1}{2}t} \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} + c_2 e^{-\frac{1}{2}t} \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}$

<<< Dealing With Complex Eigenvalues λ (Case 2)

· In general, If $\vec{y_1} = Re(\vec{y_1}) + i\, Im(\vec{y_1})$ is a complex soln,

then a general soln can be found by $\vec{y} = c_1 Re(\vec{y_1}) + c_2 Im(\vec{y_1})$.

$\boxed{EX2}$ Use $\vec{x_1} = \begin{bmatrix} 1 \\ i \end{bmatrix} e^{-\frac{1}{2}t} (\cos t + i \sin t)$ to find general soln from EX1.

$\boxed{1}$ Rewrite to real & imaginary parts

$$\vec{x_1} = \begin{bmatrix} e^{-\frac{1}{2}t}(\cos t + i \sin t) \\ e^{-\frac{1}{2}t}(i\cos t - \sin t) \end{bmatrix} = e^{-\frac{1}{2}t} \begin{bmatrix} \cos t + i \sin t \\ -\sin t + i \cos t \end{bmatrix}$$

$$= \underbrace{e^{-\frac{1}{2}t} \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}}_{Re} + i \underbrace{e^{-\frac{1}{2}t} \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}}_{Im}$$

$\boxed{2}$ Write general soln

$$\vec{x} = c_1 Re(\vec{x_1}) + c_2 Im(\vec{x_1})$$

$$= c_1 e^{-\frac{1}{2}t} \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} + c_2 e^{-\frac{1}{2}t} \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}$$

Same as what we got in EX1

$\boxed{EX3}$ Use $\vec{x_2} = \begin{bmatrix} 1 \\ -i \end{bmatrix} e^{-\frac{1}{2}t} (\cos t - i \sin t)$ to find general soln from EX1

$\boxed{1}$ Rewrite

$$\vec{x_2} = \begin{bmatrix} e^{-\frac{1}{2}t}(\cos t - i \sin t) \\ e^{-\frac{1}{2}t}(-i\cos t - \sin t) \end{bmatrix} = e^{-\frac{1}{2}t} \begin{bmatrix} \cos t - i \sin t \\ -\sin t - i \cos t \end{bmatrix}$$

$$= \underbrace{e^{-\frac{1}{2}t} \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}}_{Re} + i \underbrace{e^{-\frac{1}{2}t} \begin{bmatrix} -\sin t \\ -\cos t \end{bmatrix}}_{Im}$$

$\boxed{2}$ General soln

$$\vec{x} = c_1 Re(\vec{x_2}) + c_2 Im(\vec{x_2})$$

$$= c_1 e^{-\frac{1}{2}t} \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} + c_2 e^{\frac{1}{2}t} \begin{bmatrix} -\sin t \\ -\cos t \end{bmatrix}$$

Same as EX1.
neg sign can be absorbed into c_2.

· Therefore, choosing $\vec{x_1}$ or $\vec{x_2}$ for the procedure is equivalent.

<<< Dealing with less distinct eigenvectors $\vec{v}$ (case 3)

$\boxed{Ex4}$ Solve $\vec{x}' = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \vec{x}$ $\qquad P = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$

Guess $\vec{x_i} = \vec{v_i} e^{\lambda_i t} \Rightarrow P\vec{v} = \lambda \vec{v}$

$\boxed{1}$ $\det(P - \lambda I) = \det\left(\begin{bmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{bmatrix}\right) = \lambda^2 - 4\lambda + 4 = (\lambda-2)^2 = 0$

$\lambda_{1,2} = 2$

$\boxed{2}$ $\lambda_1 = 2$ $\quad \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \begin{bmatrix} 2v_{11} \\ 2v_{12} \end{bmatrix}$

$v_{11} - v_{12} = 2v_{11}$

$-v_{12} = v_{11} \qquad \Rightarrow \quad v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\boxed{3}$ $\vec{x_1} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{2t}$, but this is ~~not~~ enough for general soln

$\boxed{4}$ Guess $\vec{x_2} = \vec{v_1} e^{2t} t + \vec{\eta} e^{2t}$

added multiplication to t $\qquad$ added a vector to multiply with e^{2t}.

Plug $\vec{x_2}$ into ODE $\vec{x}' = P\vec{x}$, we have

$$(\vec{v_1} e^{2t} t + \vec{\eta} e^{2t})' = P(\vec{v_1} e^{2t} t + \vec{\eta} e^{2t})$$

$$\vec{v_1} e^{2t} + 2\vec{v_1} e^{2t} t + 2\vec{\eta} e^{2t} = P\vec{v_1} e^{2t} t + P\vec{\eta} e^{2t}$$

$$\vec{v_1} + \cancel{2\vec{v_1} t} + 2\vec{\eta} = \cancel{P\vec{v_1} t} + P\vec{\eta}$$

(note that $P\vec{v_1} = \lambda \vec{v_1}$, $\lambda = 2$, so $P\vec{v_1} = 2\vec{v_1}$)

$$\vec{v_1} + 2\vec{\eta} = P\vec{\eta}$$

$$\vec{v_1} = (P - 2I_2)\vec{\eta}$$

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix}$$

$$1 = -\eta_1 - \eta_2$$

$$1 + \eta_2 = -\eta_1$$

Let $\eta_2 = k$, then $\eta_1 = -1-k$

$$\vec{\eta} = \begin{bmatrix} -1-k \\ k \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} + k\begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \text{for } k \in \mathbb{R}$$

We have $\vec{x_2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{2t} t + \begin{bmatrix} -1-k \\ k \end{bmatrix} e^{2t}$

$\boxed{5}$ General soln

$$\vec{x} = \underbrace{\vec{x_1} + \vec{x_2}}_{c_1\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{2t} +} = c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{2t} t + c_2 \begin{bmatrix} -1-k \\ k \end{bmatrix} e^{2t}$$

<<< Method of Solving Systems of ODEs

$\boxed{\text{Method}}$ First order homogeneous constant-coefficient systems of ODEs

$\boxed{1}$ Write the system in the form of $\vec{x}' = P\vec{x}$, where P is constant $n \times n$ matrix.

$\boxed{2}$ For $P\vec{v} = \lambda\vec{v}$, find eigenvalues λ by $\det(P - \lambda I_n)$

$\boxed{3}$ For $P\vec{v} = \lambda\vec{v}$, find eigenvectors $\vec{v}$ by plugging in

} eigen-problem

$\boxed{4}$ Write general soln

$\boxed{4a}$ $\leq n$ distint real λ ; n distinct real $\vec{v}$

- $\vec{x} = c_1\vec{v}_1 e^{\lambda_1 t} + \cdots + c_n\vec{v}_n e^{\lambda_n t}$

$\boxed{4b}$ $\leq n$ distinct complex λ ; n distinct complex $\vec{v}$

- Known $\vec{x}_1 = Re(\vec{x}_1) + i\, Im(\vec{x}_1)$
- $\vec{x} = c_1 Re(\vec{x}_1) + c_2 Im(\vec{x}_1)$

more generally,

λ in pairs

$$\vec{x} = \sum_{j=1}^{k}\left(c_{1j} Re(\vec{x}_{1j}) + c_{2j} Im(\vec{x}_{1j})\right) + c_{2k+1}v_{k+1}e^{\lambda_{k+1}t} + \ldots$$

λ not in pairs

$\boxed{4c}$ $\leq n$ distinct λ ; $< n$ distinct $\vec{v}$

- $\vec{x} = c_1\vec{v}e^{\lambda t} + c_2(\vec{v}te^{\lambda t} + \vec{\eta}e^{\lambda t})$

8 March 2021 L23 Laplace Transform AMATH 351

<<< Laplace Transform

> Laplace transform – an integral transform of a function $f(t)$ defined by

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

that will converge if $|f(t)| \le ke^{at}$ when $t > M$.

- Convergence argument :

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt = \underbrace{\int_0^M e^{-st} f(t)\,dt}_{\text{bounded, converge}} + \underbrace{\int_M^\infty e^{-st} f(t)\,dt}_{\text{could cause problem}}$$

$$\left| \int_M^\infty e^{-st} f(t)\,dt \right| = \int_M^\infty e^{-st} |f(t)|\,dt \le \int_M^\infty ke^{-(s-a)t}\,dt$$

$\Rightarrow$ $s > a$, this converges $\because$ decay exponentially

<<< Solving Problems with Laplace Transform

Time Domain		Laplace Domain
t, $f(t)$		s, $F(s) = \mathcal{L}\{f(t)\}$
Difficult ODE	① Laplace Transform $\longrightarrow$	Easy algebra problem
		$\downarrow$ ② Solve easy problem
Soln of original difficult ODE	$\longleftarrow$ ③ Inverse Laplace Transform	Soln to algebra prob.

* Fourier transform useful for PDEs.

$\boxed{\text{EX1}}$ Find $\mathcal{L}\{f(t)\}$ for $f(t) = 1$

$$\mathcal{L}\{f(t)=1\} = \int_0^\infty e^{-st} \cdot 1\,dt = \int_0^\infty e^{-st}\,dt = \left[-\frac{1}{s} e^{-st} \right]_0^\infty = -\frac{1}{s}(0) + \frac{1}{s} = \frac{1}{s}$$

$(s > 0)$

$\boxed{\text{EX2}}$ Find $\mathcal{L}\{f(t)\}$ for $f(t) = e^{at}$ $(s > a)$.

$$\mathcal{L}\{f(t)=e^{at}\} = \int_0^\infty e^{-st} e^{at}\,dt = \int_0^\infty e^{-t(s-a)}\,dt = \left[\frac{-1}{s-a} e^{-(s-a)t} \right]_0^\infty = \frac{1}{s-a} \quad (s > a)$$

<<< Linearity of Laplace Transform

- Laplace transform is a linear operator

Proof : $\mathcal{L}\{c_1 f(t) + c_2 g(t)\} = \int_0^\infty e^{-st} (c_1 f(t) + c_2 g(t))\,dt$

$$= \int_0^\infty c_1 e^{-st} f(t)\,dt + \int_0^\infty c_2 e^{-st} g(t)\,dt$$

$$= c_1 \int_0^\infty f(t) e^{-st}\,dt + c_2 \int_0^\infty g(t) e^{-st}\,dt$$

$$= c_1 \mathcal{L}\{f(t)\} + c_2 \mathcal{L}\{g(t)\}$$

<<< Linearity of Laplace Transform

$\boxed{EX3}$ Find $\mathcal{L}\{\frac{d}{dt}f(t)\}$

$$\mathcal{L}\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\,dt$$

$u = e^{-st}$ $\quad dv = f'(t)\,dt$

$du = -se^{-st}$ $\quad v = f(t)$

$$= \left[f(t)e^{-st}\right]_0^\infty - \int_0^\infty -se^{-st} f(t)\,dt$$

$$= -f(0) + s\int_0^\infty e^{-st} f(t)\,dt$$

$$= -f(0) + s\,\mathcal{L}\{f(t)\}$$

$$\mathcal{L}\{f'(t)\} = s\,\mathcal{L}\{f(t)\} - f(0)$$

$\boxed{EX4}$ Find $\mathcal{L}\{f''(t)\}$

$$\mathcal{L}\{f''(t)\} = s\,\mathcal{L}\{f'(t)\} - f'(0)$$

$$= s\left(s\,\mathcal{L}\{f(t)\} - f(0)\right) - f'(0)$$

$$= s^2\,\mathcal{L}\{f(t)\} - sf(0) - f'(0)$$

$\boxed{EX5}$ Find $\mathcal{L}\{f^{(n)}(t)\}$

$$\mathcal{L}\{f^{(n)}(t)\} = s^n\,\mathcal{L}\{f(t)\} - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$$

<<< Solving ODE with Laplace Transform

$\boxed{EX6}$ Solve $y'' + y = \sin(2t)$, $y(0) = 2$, $y'(0) = 1$

[1] Take Laplace transform of both sides

$$\mathcal{L}\{y'' + y\} = \mathcal{L}\{\sin(2t)\}$$

$$\mathcal{L}\{y''\} + \mathcal{L}\{y\} = \mathcal{L}\{\sin(2t)\}$$

$$s^2Y(s) - sy(0) - y'(0) + Y(s) = \frac{2}{s^2+4}$$

$/\ \mathcal{L}\{\sin(at)\} = \frac{a}{s^2+a^2}\ /$

[2] Solve the Algebra problem

$$s^2Y + Y - 2s - 1 = \frac{2}{s^2+4}$$

$$(s^2+1)Y = \frac{2}{s^2+4} + 2s + 1$$

$$Y = \frac{2}{(s^2+4)(s^2+1)} \quad \frac{2s+1}{s^2+1}$$

$$Y = \frac{2s}{s^2+1} + \frac{5}{3}\frac{1}{s^2+1} - \frac{2}{3}\frac{1}{s^2+4}$$

[3] Find inverse Laplace transform

$\mathcal{L}\{Y(s)\} = y(t)$, by table,

$$y(t) = 2\cos(t) + \frac{5}{3}\sin(t) - \frac{1}{3}\sin(2t)$$

10 March 2021 L24 The Heaviside Function AMATH 351

--- Solving ODEs with Laplace Transform

$\boxed{\text{Ex1}}$ Solve $y''-y'-2y=0$, $y(0)=1$, $y'(0)=0$

$$\mathcal{L}\{y''\}-\mathcal{L}\{y'\}-\mathcal{L}\{2y\}=\mathcal{L}\{0\}$$

$$\left[s^2Y(s)-sy(0)-y'(0)\right]-\left[sY(s)-y(0)\right]-2Y(s)=0$$

$$(s^2-s-2)Y-s+1=0$$

$$Y=\frac{s-1}{(s-2)(s+1)}=\frac{A}{s-2}+\frac{B}{s+1}$$

$$=\frac{A(s-1)+B(s-2)}{(s-2)(s+1)}$$

$$s-1=(A+B)s-(A+2B)$$

Match like terms,

$$\begin{matrix}A+B=1\\A-2B=-1\end{matrix}\Rightarrow\begin{cases}A=\frac{1}{3}\\B=\frac{2}{3}\end{cases}$$

$$Y=\frac{1}{3}\frac{1}{s-2}+\frac{2}{3}\frac{1}{s+1}$$

partial fraction decomposition

$$y(t)=\mathcal{L}^{-1}\{Y(s)\}=\frac{1}{3}e^{2t}+\frac{2}{3}e^{-t}$$

$\boxed{\text{Ex2}}$ Solve $y''-2y'+2y=e^{-t}$, $y(0)=0$, $y'(0)=1$

$$\mathcal{L}\{y''\}-\mathcal{L}\{2y'\}+2\mathcal{L}\{y\}=\mathcal{L}\{e^{-t}\}$$

$$\left[s^2Y-sy(0)-y'(0)\right]-2\left[sY-y(0)\right]+2Y=\frac{1}{s+1}$$

$$Y=\frac{1}{(s-1)^2+1}+\underbrace{\frac{1}{(s+1)((s-1)^2+1)}}\quad\frac{A}{s+1}+\frac{Bs+C}{(s-1)^2+1}=\frac{1}{(s+1)((s-1)^2+1)}$$

partial fraction decomposition

$$\frac{A((s-1)^2+1)+(s+1)(Bs+C)}{(s+1)((s-1)^2+1)}=\frac{1}{(s+1)((s-1)^2+1)}$$

$$As^2-2As+2A+Bs^2+Cs+Bs+C=1$$

$$(A+B)s^2+(-2A+B+C)s+(2A+C)=1$$

Match like terms

$$\begin{cases}A+B=0\\-2A+B+C=0\\2A+C=1\end{cases}\Rightarrow\begin{cases}A=1/5\\B=-1/5\\C=3/5\end{cases}$$

$$Y(s)=\frac{1}{(s-1)^2+1}+\frac{1}{5}\frac{1}{s+1}+\frac{-\frac{1}{5}s+\frac{3}{5}}{(s-1)^2+1}$$

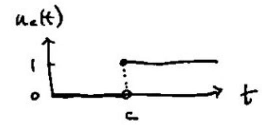
$$=\frac{7}{5}\frac{1}{(s-1)^2+1}-\frac{1}{5}\frac{s-1}{(s-1)^2+1}+\frac{1}{5}\frac{1}{s+1}$$

$$y(t)=\mathcal{L}\{Y(s)\}=\frac{7}{5}e^{t}\sin(t)-\frac{1}{5}e^{t}\cos(t)+\frac{1}{5}e^{-t}$$

- Laplace transform is easy to solve nonhomogeneous prob ∵ initial condition is in the transform
- Laplace transform useful to solve heaviside function-containing ODE.

The Heaviside Function

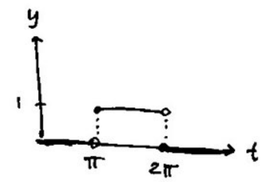
> Heaviside function : $u_c(t) = \begin{cases} 0 & t < c \\ 1 & t \geq c \end{cases}$



. think it as a switch

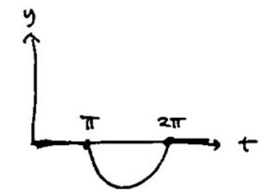
$\boxed{EX1}$ $y = u_\pi(t) - u_{2\pi}(t)$

$$= \begin{cases} 0 & t < \pi & \text{off - off} \\ 1 & \pi \leq t < 2\pi & \text{on - off} \\ 0 & t \geq 2\pi & \text{on - on} \end{cases}$$



$\boxed{EX2}$ $y(t) = \sin(t)\,(u_\pi - u_{2\pi})$

$$= \begin{cases} 0 & t < \pi \\ \sin(t) & \pi \leq t < 2\pi \\ 0 & t \geq 2\pi \end{cases}$$



→ Laplace Transform of Heaviside Fn

$$\mathcal{L}\{u_c(t)\} = \int_0^\infty e^{-st} u_c(t)\, dt$$

$(c > 0)$

$$= \int_0^c e^{-st}\, 0\, dt + \int_c^\infty e^{-st} \cdot 1\, dt$$

$$= 0 + \left[-\tfrac{1}{s} e^{-st}\right]_c^\infty \quad (s > 0)$$

$$= 0 + \left(0 + \frac{e^{-sc}}{s}\right)$$

$$= \frac{e^{-sc}}{s}$$

→ Translation Theorems

· time domain : $\mathcal{L}\{f(t-c)\, u_c(t)\} = e^{-sc} \mathcal{L}\{f(t)\}$

(inverse) : $\mathcal{L}^{-1}\{e^{-sc} \mathcal{L}\{f(t)\}\} = f(t-c)\, u_c(t)$

Proof : $\mathcal{L}\{f(t-c) u_c(t)\} = \int_0^\infty e^{-st} f(t-c) u_c(t)\, dt$

$$= \int_0^c e^{-st} f(t-\cancel{c})^{\nearrow 0} \cdot 0\, dt + \int_c^\infty e^{-st} f(t-c) \cdot 1\, dt$$

$$= \int_c^\infty e^{-st} f(t-c)\, dt$$

let $v = t - c$, $dv = dt$
$v(c) = 0$
$t = v + c$

$$= \int_0^\infty e^{-s(v+c)} f(v)\, dv$$

$$= e^{-sc} \int_0^\infty e^{-sv} f(v)\, dv$$

$$= e^{-sc} \mathcal{L}\{f(t)\}$$

· laplace domain : $\mathcal{L}\{e^{ct} f(t)\} = F(s-c)$

(inverse) : $\mathcal{L}^{-1}\{\mathcal{L}\{f(s-c)\}\} = e^{ct} f(t)$

Proof : $\mathcal{L}\{e^{ct} f(t)\} = \int_0^\infty e^{-st} e^{ct} f(t)\, dt = \int_0^\infty e^{-(s-a)t} f(t)\, dt = \mathcal{L}\{f(s-a)\}$

<<< Solving ODEs with Heaviside Fxn

$\boxed{\text{EX3}}$ $y'' + 4y = g(t)$, $y(0)=0$, $y'(0)=0$, where $g(t) = \begin{cases} 0 & 0 \le t < 5 \\ \frac{t-5}{5} & 5 \le t < 10 \\ 1 & t \ge 10 \end{cases}$ $(t > 0)$

$$g(t) = \frac{1}{5}\left[u_5(t)(t-5) - u_{10}(t)(t-10)\right]$$

$$\mathcal{L}\{y''\} + 4\mathcal{L}\{y\} = \frac{1}{5}\left[\mathcal{L}\{u_5(t)(t-5)\} - \mathcal{L}\{u_{10}(t)(t-10)\}\right]$$

$$s^2Y - sy(0) - y'(0) + 4Y = \frac{1}{5}\left[e^{-5s}\mathcal{L}\{t\} - e^{-10s}\mathcal{L}\{t\}\right]$$

$$Y = \frac{1}{5}\underbrace{\frac{1}{s^2(s^2+4)}}_{\text{fund. behavior}}\underbrace{\left(e^{-5s} - e^{-10s}\right)}_{\text{time shift}}$$

$\cancel{y = \mathcal{L}^{-1}\{Y\}}$

$$\text{fund. behavior} = \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{1}{4}\frac{1}{s^2} - \frac{1}{4}\frac{1}{s^2+4}\right\} \qquad \text{(partial fraction)}$$

$$= \frac{1}{5}\left[\frac{1}{4}t - \frac{1}{8}\sin(2t)\right]$$

$$y = \mathcal{L}^{-1}\{Y\} = \frac{1}{5}\left[u_5(t)\left(\frac{t-5}{4} - \frac{1}{8}\sin(2(t-5))\right) - u_{10}(t)\left(\frac{t-10}{4} - \frac{1}{8}\sin(2(t-10))\right)\right]$$

$\boxed{\text{EX4}}$ $y'' + y = g(t)$, $y(0)=0$, $y'(0)=0$, where $g(t) = \begin{cases} t & 0 \le t \le 1 \\ 2-t & 1 \le t < 2 \\ 0 & t \ge 2 \end{cases}$

$$g(t) = t - u_1 t + u_1(2-t) - u_2(2-t) + u_2(0)$$

$$= t + (2-2t)u_1 + (t-2)u_2$$

$$\mathcal{L}\{y''\} + \mathcal{L}\{y\} = \mathcal{L}\{t\} + \mathcal{L}\{(2-2t)u_1\} + \mathcal{L}\{(t-2)u_2\}$$

$$s^2Y - s\cancel{y(0)}^{0} - \cancel{y'(0)}^{0} + Y = \frac{1}{s^2} - 2\frac{1}{s^2}e^{-s} + \frac{1}{s^2}e^{-2s}$$

$$\mathcal{L}\{(2-2t)u_1\} = \mathcal{L}\{-2(t-1)u_1\} = -2\frac{1}{s^2}e^{-s}$$

$$\mathcal{L}\{(t-2)u_2\} = \frac{1}{s^2}e^{-2s}$$

$$(1+s^2)Y = \frac{1-2e^{-s}+e^{-2s}}{s^2}$$

$$Y = \frac{1-2e^{-s}+e^{-2s}}{s^2(1+s^2)}$$

partial fraction decomp.

$$\frac{1}{s^2(1+s^2)} = \frac{As+B}{s^2} + \frac{Cs+D}{1+s^2}$$

$$1 = (As+B)(1+s^2) + s^2(Cs+D)$$

$$1 = As + As^3 + B + Bs^2 + Cs^3 + Ds^2$$

$$\begin{cases} A+C=0 \\ B+D=0 \\ A=0 \\ B=1 \end{cases} \Rightarrow \begin{cases} A=0 \\ B=1 \\ C=0 \\ D=-1 \end{cases}$$

$$\frac{1}{s^2(1+s^2)} = \frac{1}{s^2} - \frac{1}{s^2+1}$$

$$Y = \left(\frac{1}{s^2} - \frac{1}{s^2+1}\right) - 2e^{-s}\left[\frac{1}{s^2} - \frac{1}{s^2+1}\right] + e^{-2s}\left[\frac{1}{s^2} - \frac{1}{s^2+1}\right]$$

$$Y = \mathcal{L}\{t - \sin t\} - 2e^{-s}\mathcal{L}\{t - \sin t\} + e^{-2s}\mathcal{L}\{t - \sin t\}$$

$$y(t) = \mathcal{L}^{-1}\{\mathcal{L}\{t - \sin t\}\} - \mathcal{L}^{-1}\{2e^{-s}\mathcal{L}\{t - \sin t\}\} + \mathcal{L}^{-1}\{e^{-2s}\mathcal{L}\{t - \sin t\}\}$$

$$= t - \sin t - 2\left[u_1((t-1) - \sin(t-1))\right] + u_2((t-2) - \sin(t-2))$$