MATH 324 Advanced Multivariable Calculus

2021-08-17

Double Integrals

| Double integrals in Cartesian coordinates

Description	Equations
Double integrals	$\iint\limits_R f(x,y)\,dA$ $= \lim\limits_{m,n\to\infty}\sum\limits_{i=1}^{m}\sum\limits_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A$
Fubini's Theorem $R=[a,b]\times[c,d]$	$\iint\limits_R f(x,y)\,dA$ $=\int_a^b\int_c^d f(x,y)\,dx\,dy$ $=\int_c^d\int_a^b f(x,y)\,dy\,dx$
Separation of iterative integrals $R=[a,b]\times[c,d]$	$\iint\limits_R g(x)h(y)\,dA = \int_a^b g(x)\,dx \int_c^d h(y)\,dy$
Type I region $D=x\times y=[a,b]\times[g_1(x),g_2(x)]$	$\iint\limits_D f(x,y)\,dA=\int_a^b\int_{g_1(x)}^{g_2(x)} f(x,y)\,dx\,dy$
Type II region $D=x\times y=[h_1(x),h_2(x)]\times[c,d]$	$\iint\limits_D f(x,y)\,dA=\int_c^d\int_{h_1(x)}^{h_2(x)} f(x,y)\,dy\,dx$
Addition of double integrals	$\iint\limits_D [f(x,y)+g(x,y)]\,dA$ $=\iint\limits_D f(x,y)\,dA+\iint\limits_D g(x,y)\,dA$
Constant multiple of double integrals	$\iint\limits_D cf(x,y)\,dA = c\iint\limits_D f(x,y)\,dA$
Region separation of double integrals	$\iint\limits_D f(x,y)\,dA$ $=\iint\limits_{D_1} f(x,y)\,dA+\iint\limits_{D_2} f(x,y)\,dA$
Area of a region D	$A(D)=\iint\limits_D dA$
Average value of a function	$\bar{f}=\dfrac{1}{A(R)}\iint\limits_R f(x,y)\,dA$

| Double integrals in polar coordinates

Description	Equations
Transformation to polar coordinates	$x=r\cos\theta$ $y=r\sin\theta$ $x^2+y^2=r^2$ $dA=r\,dr\,d\theta$
Double integrals in polar coordinates $R=r\times\theta=[a,b]\times[\alpha,\beta]$	$\iint\limits_R f(x,y)\,dA$ $=\int_\alpha^\beta\int_a^b f(r\cos\theta,r\sin\theta)\,r\,dr\,d\theta$
Double integrals in general polar region $R=r\times\theta=[h_1(\theta),h_2(\theta)]\times[\alpha,\beta]$	$\iint\limits_R f(x,y)\,dA$

Description	Equations
	$= \int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$

Change of variables for double integrals

Description	Equations
Transformation of two variables	$T(u,v) = (x(u,v), y(u,v))$
Jacobian of transformation of two variables	$\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}$
Change of variables for differentials	$dA = dxdy = \left\lvert \dfrac{\partial(x,y)}{\partial(u,v)} \right\rvert du\, dv$
Change of variables for double integrals	$\iint\limits_R f(x,y) dA$ $= \iint\limits_S f(x(u,v), y(u,v)) \left\lvert \dfrac{\partial(x,y)}{\partial(u,v)} \right\rvert du\, dv$

Applications of double integrals

Description	Equations
Density function	$\rho(x,y) = \dfrac{dm}{dA}$
Mass	$m = \iint\limits_D \rho(x,y)\, dA$
Moment about x-axis	$M_x = \iint\limits_D y\rho(x,y)\, dA$
Moment about y-axis	$M_x = \iint\limits_D x\rho(x,y)\, dA$
Center of mass $(\bar{x}, \bar{y})$	$\bar{x} = \dfrac{M_y}{m} = \dfrac{\iint\limits_D x\rho(x,y)\, dA}{\iint\limits_D \rho(x,y)\, dA}$ $\bar{y} = \dfrac{M_x}{m} = \dfrac{\iint\limits_D y\rho(x,y)\, dA}{\iint\limits_D \rho(x,y)\, dA}$
Moment of inertia about x-axis (second moment)	$I_x = \iint\limits_D y^2 \rho(x,y)\, dA$
Moment of inertia about y-axis (second moment)	$I_y = \iint\limits_D x^2 \rho(x,y)\, dA$
Moment of inertia about the origin (polar moment of inertia)	$I_0 = I_x + I_y = \iint\limits_D (x^2+y^2)\rho(x,y)\, dA$
Surface area	$A = \iint\limits_D \sqrt{1 + \left(\dfrac{\partial z}{\partial x}\right)^2 + \left(\dfrac{\partial z}{\partial y}\right)^2}\, dA$

Triple Integrals

Triple integrals in Cartesian coordinates

Description	Equations
Triple integrals	$\iiint\limits_B f(x,y,z)\,dV$ $= \lim\limits_{l,m,n\to\infty} \sum\limits_{i=1}^{l}\sum\limits_{j=1}^{m}\sum\limits_{k=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})\Delta V$
Fubini's Theorem \$B = x\times y\times z = \newline [a, b]\times[c, d]\times[r, s]\$	$\iiint\limits_B f(x,y,z)\,dV$ $= \int_r^s \int_c^d \int_a^b f(x,y,z)\,dx\,dy\,dz$ $= \int_c^d \int_r^s \int_a^b f(x,y,z)\,dx\,dz\,dy$ $= \ldots$
Type 1 region $D = x \times y$ $E = D \times z =$ $D \times [u_1(x,y), u_2(x,y)]$	$\iiint\limits_E f(x,y,z)\,dV = \iint\limits_D \left(\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)dz\right) dA$
Type 2 region $D = y \times z$ $E = D \times x =$ $D \times [u_1(y,z), u_2(y,z)]$	$\iiint\limits_E f(x,y,z)\,dV = \iint\limits_D \left(\int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)dx\right) dA$
Type 3 region $D = x \times z$ $E = D \times y =$ $D \times [u_1(x,z), u_2(x,z)]$	$\iiint\limits_E f(x,y,z)\,dV = \iint\limits_D \left(\int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)dy\right) dA$
Example of a general region (6 general regions) $E = [a,b] \times [g_1(x), g_2(x)] \times [u_1(x,y), u_2(x,y)]$	$\iiint\limits_E f(x,y,z)\,dV =$ $\int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\,dy\,dx$
Volume of a solid E	$V(E) = \iiint\limits_E dV$

Triple integrals in cylindrical coordinates

Description	Equations
Transformation to cylindrical coordinates	$x = r\cos\theta$ $y = r\sin\theta$ $z = z$ $x^2 + y^2 = r^2$ $dV = r\,dz\,dr\,d\theta$
Range of cylindrical coordinates	$r \in [0, \infty)$ $\theta \in [0, 2\pi]$ $z \in [0, \infty)$
Triple integrals in general cylindrical region $E = r \times \theta \times z =$ $[\alpha, \beta] \times [h_1(\theta), h_2(\theta)] \times [u_1(x,y), u_2(x,y)]$	$\iiint\limits_E f(x,y,z)\,dV =$ $\int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta, r\sin\theta)}^{u_2(r\cos\theta, r\sin\theta)} \ldots$ $\ldots f(r\cos\theta, r\sin\theta, z)\,r\,dz\,dr\,d\theta$

Triple integrals in spherical coordinates

Description	Equations
Transformation to spherical coordinates	$(r = \rho \ \sin\phi)$ $x = \rho \sin\phi \cos\theta$ $y = \rho \sin\phi \sin\theta$ $z = \rho \cos\phi$ $\rho^2 = x^2 + y^2 + z^2$ $dV = \rho^2 \ \sin\phi \ d\rho \, d\theta \, d\phi$
Range of spherical coordinates	$\rho \in [0, \infty)$ $\theta \in [0, 2\pi]$ $\phi \in [0, \pi]$
Triple integrals in general spherical region $E = \theta \times \phi \times \rho = [\alpha, \beta] \times [c, d] \times [g_1(\theta, \phi), g_2(\theta, \phi)]$	$\iiint\limits_E f(x, y, z) dV =$ $\int_c^d \int_\alpha^\beta \int_{g_1(\theta,\phi)}^{g_2(\theta,\phi)} \cdots$ $\ldots f(\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi) \ldots$ $\ldots \rho^2 \ \sin\phi \ d\rho \, d\theta \, d\phi$

Change of variables for triple integrals

Description	Equations
Transformation of three variables	$T(u, v, w) = (x(u, v, w), y(u, v, w), z(u, v, w))$
Jacobian of transformation of three variables	$\dfrac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{vmatrix}$
Change of variables for differentials	$dV = dx \, dy \, dz = \left\lvert \dfrac{\partial(x, y, z)}{\partial(u, v, w)} \right\rvert du \, dv \, dw$
Change of variables for triple integrals	$\iiint\limits_R f(x, y, z) dV$ $= \iiint\limits_S f(x(u, v, w), y(u, v, w), z(u, v, w)) \ldots$ $\ldots \left\lvert \dfrac{\partial(x, y, z)}{\partial(u, v, w)} \right\rvert du \, dv \, dw$

Applications of triple integrals

Description	Equations
Mass	$m = \iiint\limits_E \rho(x, y, z) \, dV$
Moments about coordinate planes	$M_{yz} = \iiint\limits_E x\rho(x, y, z) \, dV$ $M_{xz} = \iiint\limits_E y\rho(x, y, z) \, dV$ $M_{xy} = \iiint\limits_E z\rho(x, y, z) \, dV$
Center of mass $(\bar{x}, \bar{y}, \bar{z})$	$\bar{x} = \dfrac{M_{yz}}{m} = \dfrac{\iiint\limits_E x\rho(x, y, z) \, dV}{\iiint\limits_E \rho(x, y, z) \, dV}$

Description	Equations
	$\bar{y} = \frac{M_{xz}}{m} = \frac{\iiint\limits_E y\rho(x,y,z)\,dV}{\iiint\limits_E \rho(x,y,z)\,dV}$ $\bar{z} = \frac{M_{xy}}{m} = \frac{\iiint\limits_E z\rho(x,y,z)\,dV}{\iiint\limits_E \rho(x,y,z)\,dV}$
Moments of inertia about coordinate axes	$I_x = \iiint\limits_E (y^2+z^2)\rho(x,y,z)\,dV$ $I_y = \iiint\limits_E (x^2+z^2)\rho(x,y,z)\,dV$ $I_z = \iiint\limits_E (x^2+y^2)\rho(x,y,z)\,dV$

Partial Differentiation

| Chain rule

Description	Equations
Chain rule $z = f(x(t), y(t))$	$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$
Chain rule $z = f(x(s,t), y(s,t))$	$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$ $\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$
Chain rule (general) $z = f(x_1, \ldots, x_n)$, where $x_i = x_i(t_1, \ldots, t_m)$	$\frac{\partial z}{\partial t_i} = \frac{\partial z}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \ldots + \frac{\partial z}{\partial x_n}\frac{\partial x_n}{\partial t_i}$

| Directional derivatives and gradient vector

Description	Equations
Assumptions	Unit vector $\mathbf{u} = \langle u_1, \ldots, u_i \rangle$ Independent variables $\mathbf{x} = \langle x_1, \ldots, x_i \rangle$
General directional derivatives	$D_{\mathbf{u}} f(\mathbf{x}) = \lim\limits_{h \to 0} \frac{f(\mathbf{x} + h\mathbf{u}) - f(\mathbf{x})}{h}$
General gradient vectors	$\nabla f(\mathbf{x}) = \left\langle \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_i} \right\rangle$
General directional derivatives and gradient vectors	$D_{\mathbf{u}} f(\mathbf{x}) = \nabla f(\mathbf{x}) \cdot \mathbf{u}$
Directional derivative in 2D	$D_{\mathbf{u}} f(x,y) = f_x \cos\theta + f_y \sin\theta$
Gradient vector and maximum values	$\max(D_{\mathbf{u}} f(\mathbf{x})) = \lvert \nabla f(\mathbf{x}) \rvert$ where $\mathbf{u} = \frac{\nabla f(\mathbf{x})}{\lvert \nabla f(\mathbf{x}) \rvert}$
Gradient vector $\perp$ tangent vector for level surface $F(x,y,z) = k$	$\nabla F(x(t), y(t), z(t)) \cdot \mathbf{r}'(t) = 0$ $\nabla F(x_0, y_0, z_0) \cdot \mathbf{r}'(t_0) = 0$

Description	Equations
Tangent plane in terms of gradient vector (normal vector)	$\nabla F(x,y,z)\cdot\langle x-x_0, y-y_0, z-z_0\rangle = 0$
Symmetric equation of normal line	$\frac{x-x_0}{F_x(x_0,y_0,z_0)} = \frac{y-y_0}{F_y(x_0,y_0,z_0)} = \frac{z-z_0}{F_z(x_0,y_0,z_0)}$

Vector Calculus

Arc length and parameterization of curves

Description	Equations
Vector field	$\mathbf{F}(\mathbf{x}) = \langle P(\mathbf{x}), Q(\mathbf{x}), R(\mathbf{x})\rangle$
Conservative vector field and potential function	$\mathbf{F} = \nabla f$
Parameterization of line segments	$\mathbf{r}(t) = (1-t)\mathbf{r}_0 + t\,\mathbf{r}_1$, for $0 \le t \le 1$
Parameterization of circles	$\mathbf{r}(t) = \langle r\cos(t), r\sin(t)\rangle$
Parameterization of functions	$\mathbf{r}(t) = \langle t, f(t)\rangle$
Arc length	$L = \int_a^b \lvert\mathbf{r}'(t)\rvert\,dt$
Arc length parameter	$s(t) = \int_a^t \lvert\mathbf{r}'(u)\rvert\,du$ $s'(t) = \lvert\mathbf{r}'(t)\rvert$ $ds = \lvert\mathbf{r}'(t)\rvert\,dt$

Line integrals

Description	Equations
Line integral	$\int_C f(x,y)\,ds = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta s_i$
Line integral with respect to arc length	$\int_C f(x,y)\,ds$ $= \int_a^b f(x(t),y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$ $= \int_a^b f(\mathbf{r}(t))\,\lvert\mathbf{r}'(t)\rvert\,dt$
Line integral with respect to x and y	$\int_C f(x,y)\,dx = \int_a^b f(x(t),y(t))\,x'(t)\,dt$ $\int_C f(x,y)\,dy = \int_a^b f(x(t),y(t))\,y'(t)\,dt$
Line integrals of vector fields	$\int_C \mathbf{F}\cdot\mathbf{T}\,ds$ $= \int_C \mathbf{F}\cdot d\mathbf{r}$ $= \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt$
Line integrals of vector fields and scalar fields	$\mathbf{F} = \langle P, Q, R\rangle$ $\int_C \mathbf{F}\cdot d\mathbf{r}$ $= \int_C P\,dx + Q\,dy + R\,dz$ $= \int_C P\,x'(t) + Q\,y'(t) + R\,z'(t)\,dt$
Orientation properties of line integrals with respect to arc length, x, and y	$\int_{-C} f(x,y)\,ds = \int_C f(x,y)\,ds$ $\int_{-C} f(x,y)\,dx = -\int_C f(x,y)\,dx$

Description	Equations
	$\int_{-C} f(x,y)\,dy = -\int_C f(x,y)\,dy$
Orientation properties of line integrals of vector fields	$\int_{-C} \mathbf{F}\cdot d\mathbf{r} = -\int_C \mathbf{F}\cdot d\mathbf{r}$
Line integral of a piecewise-smooth curve	$\int_C f\,ds = \sum_{i=1}^{N} \int_{C_i} f\,ds$ $C = C_1 \cup C_2 \cup \ldots \cup C_N$

Fundamental theorem of line integrals

- path - a smooth curve with initial and terminal point
- simple curve - a curve that does not intersect itself anywhere between its endpoints
- closed curve - a curve where its terminal point coincides with its initial point
- simple region - a region that is bounded by two line segments in one direction (type-1, type-2 regions)
- open region - a region that does not contain boundary points
- closed region - a region that contains all boundary points
- connected region - two points in the region can be joined by a path that lies in the region
- simply-connected region - a region that every simple closed curve in D encloses only points that are in D
    - has no hole
    - doesn't consist of separate pieces

Description	Equations
Fundamental theorem of line integrals	$\int_C \nabla f\cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$
Line integrals of non-conservative fields are not path independent (same end points)	$\int_{C_1} \mathbf{F}\cdot d\mathbf{r} \neq \int_{C_2} \mathbf{F}\cdot d\mathbf{r}$
Line integrals of conservative fields are path independent (same end points)	$\int_{C_1} \nabla f\cdot d\mathbf{r} = \int_{C_2} \nabla f\cdot d\mathbf{r}$
Line integrals of closed path	$\int_{C_{\text{closed}}} \mathbf{F}\cdot d\mathbf{r} = 0 \Leftrightarrow$ $\int_C \mathbf{F}\cdot d\mathbf{r}$ is path independent
Path independence and conservative vector field *(open, connected region)*	$\int_C \mathbf{F}\cdot d\mathbf{r}$ is path independent $\Rightarrow$ $\mathbf{F} = \nabla f$ (conservative field)
Property of conservative vector field	$\mathbf{F} = \nabla f \Rightarrow \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$
Determine conservative vector field in 2D *(open simply-connected region)*	$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \Rightarrow \mathbf{F} = \nabla f$

Summary

- Fundamental theorem of line integral (FTL) is always true (with assumptions).
- Other statements are not true for general $\mathbf{F} = \langle P, Q\rangle$
    - They have to be verified for each given $\mathbf{F}$ or derived from theorems.

$$\begin{aligned}
\text{curl } \mathbf{F} &= \mathbf{0} && \text{(checking 3D conservative field)} \\
\frac{\partial P}{\partial y} &= \frac{\partial Q}{\partial x} && \text{(checking 2D conservative field)} \\
&\upharpoonleft\downharpoonright \text{ open, simply-connected } D \\
\mathbf{F} &= \nabla f && \text{(def. of conservative field)} \\
\int_C \nabla f \cdot d\mathbf{r} &= f(\mathbf{r}(b)) - f(\mathbf{r}(a)) && \text{(FTL)} \\
\text{open, connected } D \upharpoonleft\downharpoonright& \\
\int_{C_1} \mathbf{F} \cdot d\mathbf{r} &= \int_{C_2} \mathbf{F} \cdot d\mathbf{r} && \text{(path independence)} \\
&\upharpoonleft\downharpoonright \\
\int_C \mathbf{F} \cdot d\mathbf{r} &= 0 \text{ on a closed path} && \text{(closed path)}
\end{aligned}$$

Curl and divergence

Description	Equations
Gradient	$\text{grad } f = \nabla f$ $= \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle$
Curl	$\text{curl } \mathbf{F} = \nabla \times \mathbf{F}$ $= \langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \rangle$
Divergence	$\text{div } \mathbf{F} = \nabla \cdot \mathbf{F}$ $= \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$
Laplace operator on scalar functions	$\nabla^2 f = \nabla \cdot \nabla f = \text{div}(\nabla f)$ $= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$
Laplace operator on vector fields	$\nabla^2 \mathbf{F} = \langle \nabla^2 P, \nabla^2 Q, \nabla^2 R \rangle$
Property of conservative vector field	$\text{curl } \nabla f = \mathbf{0}$
Determine conservative vector field in 3D *(open simply-connected region)*	$\text{curl } \mathbf{F} = \mathbf{0} \Rightarrow \mathbf{F} = \nabla f$
Property of divergence and curl	$\text{div curl } \mathbf{F} = \nabla \cdot (\nabla \times \mathbf{F}) = 0$
Determine curl field	$\text{div } \mathbf{F} \neq 0 \Rightarrow \mathbf{F}$ is not curl of any field

Green's theorem

Description	Equations
Green's Theorem *(positively oriented, piecewise-smooth, simple, closed curve C enclosing D)*	$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$
Circulation-Curl form of Green's Theorem	$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D (\nabla \times \mathbf{F}) \cdot \mathbf{k} \, dA$
Flux-Divergence form of Green's Theorem	$\oint_C \mathbf{F} \cdot \mathbf{n} \, ds = \iint_D \nabla \cdot \mathbf{F} \, dA$
Area of region D enclosed by C	$A = \iint_D 1 \, dA$ $= \oint_C x \, dy$ $= -\oint_C y \, dx$ $= \frac{1}{2} \oint_C x \, dy - y \, dx$

Surface area and parameterization of surfaces

Description	Equations
General parametric surface	$\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$
Parametric equation of a plane	$\mathbf{r}(u,v) = \mathbf{r}_0 + u\mathbf{a} + v\mathbf{b}$
Parametric equation of a sphere	$\mathbf{r}(\phi,\theta) = \langle a\sin\phi\cos\theta, a\sin\phi\sin\theta, a\cos\phi\rangle$
Parametric equation of an explicit function	$\mathbf{r}(u,v) = \langle u, v, f(u,v)\rangle$
Parametric equation of a surface of revolution	$\mathbf{r}(u,\theta) = \langle u, f(u)\cos\theta, f(u)\sin\theta\rangle$
Normal vector of a tangent plane	$\mathbf{r}_u \times \mathbf{r}_v$
Surface area of a parametric surface	$\iint\limits_D \lvert\mathbf{r}_u \times \mathbf{r}_v\rvert \, dA$
Surface area of the graph of an explicit function $z = f(x,y)$	$\iint\limits_D \sqrt{1 + (\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2} \, dA$
Surface area of the graph of an implicit function $C = f(x,y,z)$	$\iint\limits_D \frac{\lvert\nabla f\rvert}{\lvert\nabla f \cdot \mathbf{k}\rvert} \, dA$

Surface integral

Description	Equations
Surface integral of a function over a parametric surface	$\iint\limits_S f(x,y,z)\, dS$ $= \iint\limits_D f(\mathbf{r}(u,v))\lvert\mathbf{r}_u \times \mathbf{r}_v\rvert \, dA$
Surface integral of an explicit function $z = f(x,y)$	$\iint\limits_S f(x,y,z)\, dS$ $= \iint\limits_D f(x,y,g(x,y))\sqrt{1 + (\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2} \, dA$
Surface integral of piecewise smooth surface	$\iint\limits_S f\, dS = \sum\limits_{i=1}^{N} \iint\limits_{S_i} f\, dS$ $S = S_1 \cup S_2 \cup \ldots \cup S_N$
Unit normal vector	$\mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{\lvert\mathbf{r}_u \times \mathbf{r}_v\rvert}$
Surface integral of a vector field over a parametric surface	$\iint\limits_S \mathbf{F} \cdot d\mathbf{S} = \iint\limits_S \mathbf{F} \cdot \mathbf{n}\, dS$ $= \iint\limits_D \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, dA$ $= \iint\limits_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\right) dA$

Stoke's theorem

Description	Equations
Stoke's Theorem *(S: oriented, piecewise-smooth surface* *C: simple, closed, piecewise-smooth curve* *F: continuous partial derivatives in open region)*	$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint\limits_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$
Alternative surface (C is a common curve of S_1 and S_2)	$\iint\limits_{S_1} (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$

Description	Equations
	$= \int_C \mathbf{F} \cdot d\mathbf{r}$ $= \iint\limits_{S_2} (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$

Divergence theorem (Gauss's theorem)

Description	Equations
Divergence Theorem *(E: simple, solid region* *S: positively oriented surface* *F: continuous partial derivatives in open region)*	$\iint\limits_S \mathbf{F} \cdot d\mathbf{S} = \iiint\limits_E \nabla \cdot \mathbf{F} \, dV$

Appendix

Types of functions

Function Type	Domain $\to$ Range	Equation	Example
Function of several variables	$\mathbb{R}^n \to \mathbb{R}$	$f(\mathbf{x})$	$f(x, y, z) = 2x^2 + e^y - 5z^3 - 7$
Vector-valued function	$\mathbb{R} \to \mathbb{R}^n$	$\mathbf{v}(t)$	$\mathbf{v}(t) = \langle t^2, -2t, e^t \rangle$
Vector field	$\mathbb{R}^n \to \mathbb{R}^n$	$\mathbf{F}(\mathbf{x})$	$\mathbf{F}(x, y, z) = \langle 3x - y, z, z^2 - x \rangle$