MATH 125 Calculus II

2021-08-17

Integrals

| Indefinite integrals

Description	Equations
Indefinite integral (antiderivative)	$F(x) = \int f(x)\ dx$ $F'(x) = f(x)$
Antiderivative as a family of functions (Plus C!)	If F is an antiderivative of f, C is a constant, then the most general antiderivative is $F(x) + C$

| Table of indefinite integrals

Function $f(x)$	Antiderivative $F(x)$	Function $f(x)$	Antiderivative $F(x)$
x^n	$\frac{x^{n+1}}{n+1} + C$	$\frac{1}{x}$	$\ln\lvert x\rvert + C$
e^x	$e^x + C$	b^x	$\frac{b^x}{\ln b} + C$
$\sin x$	$-\cos x + C$	$\cos x$	$\sin x + C$
$\sec^2 x$	$\tan x + C$	$csc^2 x$	$-\cot x + C$
$\sec x \tan x$	$\sec x + C$	$\csc x \cot x$	$-\csc x + C$
$\frac{1}{x^2 + a^2}$	$\frac{1}{a} \arctan\left(\frac{x}{a}\right) + C$	$\frac{1}{\sqrt{a^2 - x^2}}$	$\arcsin\left(\frac{x}{a}\right) + C$

| Definite integrals as Riemann sums

Description	Equations
Area	$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$
Definite integral	$\int_a^b f(x)\ dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$
Operational definition of definite integral as Riemann sum	$\int_a^b f(x)\ dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$ $\Delta x = \frac{b-a}{n}$ $x_i = a + i\Delta x$
Sums of powers of positive integers	$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ $\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$
Properties of summation	$\sum_{i=1}^{n} c = nc$

Description	Equations
	$\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$ $\sum_{i=1}^{n}(a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$

Properties of definite integrals

Description	Equations
Reversing the bounds changes the sign of definite integrals	$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
Definite integral is zero if upper and lower bounds are the same	$\int_a^a f(x)\,dx = 0$
Definite integrals of constant	$\int_a^b c\,dx = c(b-a)$
Addition and subtraction of definite integrals	$\int_a^b [f(x) \pm g(x)]\,dx$ $= \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$
Constant multiple of definite integrals	$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$
Comparison properties of definite integrals	If $f(x) \geq 0$ for $x \in [a,b]$, then $\int_a^b f(x)\,dx \geq 0$
Comparison properties of definite integrals	If $f(x) \geq g(x)$ for $x \in [a,b]$, then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$
Comparison properties of definite integrals	If $m \leq f(x) \leq M$ for $x \in [a,b]$, then $m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$

Fundamental theorem of calculus

Description	Equations
Fundamental theorem of calculus I (f is continuous on $[a,b]$)	$g(x) = \displaystyle\int_a^x f(t)\,dt$ $g'(x) = f(x)$
Fundamental theorem of calculus II (f is continuous on $[a,b]$)	$\displaystyle\int_a^b f(x)\,dx = F(b) - F(a)$ where F is any antiderivative of f
Net change theorem The integral of a rate of change is the net change	$\displaystyle\int_a^b F'(x)\,dx = F(b) - F(a)$

Substitution rule

Description	Equations
Substitution rule (u-substitution) $u \equiv g(x)$	$\displaystyle\int f(g(x))g'(x)\,dx = \int f(u)\,du$
Substitution rule for definite integrals $u \equiv g(x)$	$\displaystyle\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$
Integrals of even functions	$\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx$

Description	Equations
Integrals of odd functions	$\int_{-a}^{a} f(x)\ dx = 0$

Techniques of Integration

Integration by parts

Description	Equations
Integration by parts	$\int f(x)g'(x)\ dx$ $= f(x)g(x) - \int g(x)f'(x)\ dx$
Integration by parts	$\int u\ dv = uv - \int v\ du$
Integration by parts for definite integrals	$\int_a^b fg'\ dx = [fg]_a^b - \int_a^b f\text{'}g\ dx$

Approximating integrals

Description	Equations
Midpoint rule	$\int_a^b f(x)\ dx \approx \sum_{i=1}^{n} f(\bar{x}_i)\Delta x$ $\Delta x = \frac{b-a}{n}$ $\bar{x}_i = \frac{1}{2}(x_{i-1} + x_i)$
Error bound for midpoint rule	$\|E_M\| \leq \dfrac{K(b-a)^3}{24n^2}$
Trapezoidal rule	$\int_a^b f(x)\ dx \approx \frac{1}{2}\Delta x[f(x_0) + 2f(x_1) + \ldots +$ $2f(x_{n-1}) + f(x_n)]$ $\Delta x = \frac{b-a}{n}$ $x_i = a + i\Delta x$
Error bound for trapezoidal rule	$\|E_T\| \leq \dfrac{K(b-a)^3}{12n^2}$
Simpson's rule	$\int_a^b f(x)\ dx \approx \frac{1}{3}\Delta x[f(x_0) + 4f(x_1) + 2f(x_2) +$ $4f(x_3) + \ldots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]$ $\Delta x = \frac{b-a}{n}$, n is even
Error bound for Simpson's rule	$\|E_S\| \leq \dfrac{K(b-a)^5}{180n^4}$

Trigonometric integrals

Description	Equations
Integral of odd power of cosine $(u = \sin x)$	$\int \sin^m(x)\cos^{2k+1}(x)\ dx$ $= \int \sin^m(x)[\cos^2(x)]^k\ dx$ $= \int \sin^m(x)[1 - \sin^2(x)]^k\ dx$
Integral of odd power of sine $(u = \cos x)$	$\int \sin^{2k+1}(x)\cos^n(x)\ dx$ $= \int [\sin^2(x)]^k \cos^n(x)\sin(x)\ dx$ $= \int [1 - \cos^2(x)]^k \cos^n(x)\sin(x)\ dx$
Integral of even power of sine and cosine use trig identities	$\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$ $\cos^2(x) = \frac{1}{2}(1 + \cos(2x))$ $\sin(x)\cos(x) = \frac{1}{2}\sin(2x)$

Description	Equations
Integral of even power of secant $(u = \tan x)$	$\int \tan^m(x)\sec^{2k}(x)\,dx$ $= \int \tan^m(x)[\sec^2(x)]^{k-1}\sec^2(x)\,dx$ $= \int \tan^m(x)[1+\tan^2(x)]^{k-1}\sec^2(x)\,dx$
Integral of odd power of tangent $(u = \sec x)$	$\int tan^{2k+1}(x)\sec^n(x)\,dx$ $= \int [\tan^2(x)]^k \sec^{n-1}(x)\sec(x)\tan(x)\,dx$ $= \int [\sec^2(x)-1]^k \sec^{n-1}(x)\sec(x)\tan(x)\,dx$
Trig identity for solving $\int \sin(mx)\cos(nx)\,dx$	$\sin A\cos B = \frac{1}{2}[\sin(A-B)+\sin(A+B)]$
Trig identity for solving $\int \sin(mx)\sin(nx)\,dx$	$\sin A\sin B = \frac{1}{2}[\cos(A-B)-\cos(A+B)]$
Trig identity for solving $\int \cos(mx)\cos(nx)\,dx$	$\cos A\cos B = \frac{1}{2}[\cos(A-B)+\cos(A+B)]$

Trigonometric substitution

Expression	Substitution	Trigonometric Identity
$\sqrt{a^2-x^2}$	$x = a\sin\theta$	$1-\sin^2\theta = \cos^2\theta$
$\sqrt{a^2+x^2}$	$x = a\tan\theta$	$1+\tan^2\theta = \sec^2\theta$
$\sqrt{x^2-a^2}$	$x = a\sec\theta$	$\sec^2\theta - 1 = \tan^2\theta$

Improper integrals

Description	Equations
Improper integrals with single one-side infinite intervals	$\int_a^\infty f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx$ $\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty}\int_t^b f(x)\,dx$
Improper integrals with single two-side infinite intervals	$\int_{-\infty}^\infty f(x)\,dx$ $= \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx$
Convergence and divergence of improper integrals of power functions	$\int_1^\infty \frac{1}{x^p}\,dx$ convergent if $p > 1$ divergent if $p \le 1$
Improper integrals with discontinuous integrand on one side	$\int_a^b f(x)\,dx = \lim_{t\to b^-}\int_a^t f(x)\,dx$ $\int_a^b f(x)\,dx = \lim_{t\to a^+}\int_t^b f(x)\,dx$
Improper integrals with discontinuous integrand in the middle c	$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$
Comparison theorem $(f(x) \ge g(x) \ge 0, x \ge a)$	(a) If $\int_a^\infty f(x)\,dx$ is convergent, then $\int_a^\infty g(x)\,dx$ is convergent. (b) If $\int_a^\infty g(x)\,dx$ is divergent, then $\int_a^\infty f(x)\,dx$ is divergent.

Applications of Integration

Description	Equations
Areas between curves	$A = \int_a^b [f(x) - g(x)]\, dx$
Volume by method of disks and washers	$V = \int_a^b A(x)\, dx$
Volume by method of cylindrical shells (rotating about y-axis)	$V = \int_a^b 2\pi x f(x)\, dx$
Average value of a function	$\bar{f} = \frac{1}{b-a} \int_a^b f(x)\, dx$
The mean value theorem of integrals	If f is continuous on $[a, b]$, then there exists $c \in [a, b]$ such that $f(c) = \bar{f} = \frac{1}{b-a} \int_a^b f(x)\, dx$, $\int_a^b f(x)\, dx = f(c)(b-a)$
Arc length formula	$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$
Arc length function	$s(x) = \int_a^x \sqrt{1 + [f'(t)]^2}\, dt$
Surface area of surface of resolution about x-axis	$S = \int_a^b 2\pi f(x) \sqrt{1 + [f'(x)]^2}\, dx$