MATH 125 Calculus II Equations

Organized by Teng-Jui Lin

Warning

- **WARNING: These equations are hand-typed and for personal reference use, so it is guaranteed to have some mistakes, both innocent and unforgivable. Therefore, use with caution!**
- By using this equation sheet, you accept the risk associated with potential mistakes.
- If you find any mistakes, I welcome you to raise an issue.
- Updated: 20 March 2021

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Integrals

Indefinite integrals

Description	Equations
Indefinite integral (antiderivative)	$F(x) = \int f(x)\, dx$ $F'(x) = f(x)$
Antiderivative as a family of functions (Plus C!)	If F is an antiderivative of f, C is a constant then the most general antiderivative is $F(x) + C$

Table of indefinite integrals

Function $f(x)$	Antiderivative $F(x)$	Function $f(x)$	Antiderivative $F(x)$
x^n	$\dfrac{x^{n+1}}{n+1} + C$	$\dfrac{1}{x}$	$\ln\|x\| + C$
e^x	$e^x + C$	b^x	$\dfrac{b^x}{\ln b} + C$
$\sin x$	$-\cos x + C$	$\cos x$	$\sin x + C$
$\sec^2 x$	$\tan x + C$	$csc^2 x$	$-\cot x + C$
$\sec x \tan x$	$\sec x + C$	$\csc x \cot x$	$-\csc x + C$
$\dfrac{1}{x^2 + a^2}$	$\dfrac{1}{a} \arctan\left(\dfrac{x}{a}\right) + C$	$\dfrac{1}{\sqrt{a^2 - x^2}}$	$\arcsin\left(\dfrac{x}{a}\right) + C$

Definite integrals as Reimann sums

Description	Equations

Description	Equations
Area	$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$
Definite integral	$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$
Operational definition of definite integral as Reimann sum	$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$ $\Delta x = \frac{b-a}{n}$ $x_i = a + i\Delta x$
Sums of powers of positive integers	$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ $\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$
Properties of summation	$\sum_{i=1}^{n} c = nc$ $\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$ $\sum_{i=1}^{n}(a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$

Properties of definite integrals

Description	Equations
Reversing the bounds changes the sign of definite integrals	$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
Definite integral is zero if upper and lower bounds are the same	$\int_a^a f(x)\,dx = 0$
Definite integrals of constant	$\int_a^b c\,dx = c(b-a)$
Addition and subtraction of definite integrals	$\int_a^b [f(x) \pm g(x)]\,dx$ $= \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$
Constant multiple of definite integrals	$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$
Comparison properties of definite integrals	If $f(x) \geq 0$ for $x \in [a,b]$, then $\int_a^b f(x)\,dx \geq 0$
Comparison properties of definite integrals	If $f(x) \geq g(x)$ for $x \in [a,b]$, then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$
Comparison properties of definite integrals	If $m \leq f(x) \leq M$ for $x \in [a,b]$, then $m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$

Fundamental theorem of calculus

Description	Equations
Fundamental theorem of calculus I (f is continuous on $[a,b]$)	$g(x) = \int_a^x f(t)\,dt$ $g'(x) = f(x)$
Fundamental theorem of calculus II (f is continuous on $[a,b]$)	$\int_a^b f(x)\,dx = F(b) - F(a)$ where F is any antiderivative of f
Net change theorem The integral of a rate of change is the net change	$\int_a^b F'(x)\,dx = F(b) - F(a)$

Substitution rule

Description	Equations

Description	Equations
Substitution rule (u-substitution) $u \equiv g(x)$	$\int f(g(x))g'(x)\,dx = \int f(u)\,du$
Substitution rule for definite integrals $u \equiv g(x)$	$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$
Integrals of even functions	$\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$
Integrals of odd functions	$\int_{-a}^{a} f(x)\,dx = 0$

Techniques of Integration

Integration by parts

Description	Equations
Integration by parts	$\int f(x)g'(x)\,dx$ $= f(x)g(x) - \int g(x)f'(x)\,dx$
Integration by parts	$\int u\,dv = uv - \int v\,du$
Integration by parts for definite integrals	$\int_a^b fg'\,dx = [fg]_a^b - \int_a^b f'g\,dx$

Approximating integrals

Description	Equations
Midpoint rule	$\int_a^b f(x)\,dx \approx \sum\limits_{i=1}^{n} f(\bar{x}_i)\Delta x$ $\Delta x = \frac{b-a}{n}$ $\bar{x}_i = \frac{1}{2}(x_{i-1} + x_i)$
Error bound for midpoint rule	$\lvert E_M\rvert \leq \dfrac{K(b-a)^3}{24n^2}$
Trapezoidal rule	$\int_a^b f(x)\,dx \approx \frac{1}{2}\Delta x[f(x_0) + 2f(x_1) +$ $\ldots + 2f(x_{n-1}) + f(x_n)]$ $\Delta x = \frac{b-a}{n}$ $x_i = a + i\Delta x$
Error bound for trapezoidal rule	$\lvert E_T\rvert \leq \dfrac{K(b-a)^3}{12n^2}$
Simpson's rule	$\int_a^b f(x)\,dx \approx \frac{1}{3}\Delta x[f(x_0) + 4f(x_1) +$ $2f(x_2) + 4f(x_3) + \ldots + 2f(x_{n-2}) +$ $4f(x_{n-1}) + f(x_n)]$ $\Delta x = \frac{b-a}{n}$, n is even
Error bound for Simpson's rule	$\lvert E_S\rvert \leq \dfrac{K(b-a)^5}{180n^4}$

Trigonometric integrals

Description	Equations
Integral of odd power of cosine $(u = \sin x)$	$\int \sin^m(x)\cos^{2k+1}(x)\,dx$ $= \int \sin^m(x)[\cos^2(x)]^k\,dx$ $= \int \sin^m(x)[1 - \sin^2(x)]^k\,dx$
Integral of odd power of sine $(u = \cos x)$	$\int \sin^{2k+1}(x)\cos^n(x)\,dx$ $= \int [\sin^2(x)]^k \cos^n(x)\sin(x)\,dx$ $= \int [1 - \cos^2(x)]^k \cos^n(x)\sin(x)\,dx$
Integral of even power of sine and cosine use trig identities	$\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$ $\cos^2(x) = \frac{1}{2}(1 + \cos(2x))$ $\sin(x)\cos(x) = \frac{1}{2}\sin(2x)$
Integral of even power of secant $(u = \tan x)$	$\int \tan^m(x)\sec^{2k}(x)\,dx$ $= \int \tan^m(x)[\sec^2(x)]^{k-1}\sec^2(x)\,dx$ $= \int \tan^m(x)[1 + \tan^2(x)]^{k-1}\sec^2(x)\,dx$

Description	Equations
Integral of odd power of tangent $(u = \sec x)$	$\int tan^{2k+1}(x) \sec^n(x)\, dx$ $= \int [\tan^2(x)]^k \sec^{n-1}(x) \sec(x) \tan(x)\, dx$ $= \int [\sec^2(x) - 1]^k \sec^{n-1}(x) \sec(x) \tan(x)\, dx$
Trig identity for solving $\int \sin(mx) \cos(nx)\, dx$	$\sin A \cos B = \frac{1}{2}[\sin(A - B) + \sin(A + B)]$
Trig identity for solving $\int \sin(mx) \sin(nx)\, dx$	$\sin A \sin B = \frac{1}{2}[\cos(A - B) - \cos(A + B)]$
Trig identity for solving $\int \cos(mx) \cos(nx)\, dx$	$\cos A \cos B = \frac{1}{2}[\cos(A - B) + \cos(A + B)]$

Trigonometric substitution

Expression	Substitution	Trigonometric Identity
$\sqrt{a^2 - x^2}$	$x = a \sin \theta$	$1 - \sin^2 \theta = \cos^2 \theta$
$\sqrt{a^2 + x^2}$	$x = a \tan \theta$	$1 + \tan^2 \theta = \sec^2 \theta$
$\sqrt{x^2 - a^2}$	$x = a \sec \theta$	$\sec^2 \theta - 1 = \tan^2 \theta$

Improper integrals

Description	Equations
Improper integrals with single one-side infinite intervals	$\int_a^\infty f(x)\, dx = \lim\limits_{t \to \infty} \int_a^t f(x)\, dx$ $\int_{-\infty}^b f(x)\, dx = \lim\limits_{t \to -\infty} \int_t^b f(x)\, dx$
Improper integrals with single two-side infinite intervals	$\int_{-\infty}^\infty f(x)\, dx$ $= \int_{-\infty}^a f(x)\, dx + \int_a^\infty f(x)\, dx$
Convergence and divergence of improper integrals of power functions	$\displaystyle\int_1^\infty \frac{1}{x^p}\, dx$ convergent if $p > 1$ divergent if $p \leq 1$
Improper integrals with discontinuous integrand on one side	$\int_a^b f(x)\, dx = \lim\limits_{t \to b^-} \int_a^t f(x)\, dx$ $\int_a^b f(x)\, dx = \lim\limits_{t \to a^+} \int_t^b f(x)\, dx$
Improper integrals with discontinuous integrand in the middle c	$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx$
Comparison theorem $(f(x) \geq g(x) \geq 0, x \geq a)$	(a) If $\int_a^\infty f(x)\, dx$ is convergent, then $\int_a^\infty g(x)\, dx$ is convergent. (b) If $\int_a^\infty g(x)\, dx$ is divergent, then $\int_a^\infty f(x)\, dx$ is divergent.

Applications of Integration

Description	Equations
Areas between curves	$A = \int_a^b [f(x) - g(x)]\, dx$
Volume by method of disks and washers	$V = \int_a^b A(x)\, dx$
Volume by method of cylindrical shells (rotating about y-axis)	$V = \int_a^b 2\pi x f(x)\, dx$
Average value of a function	$\bar{f} = \frac{1}{b-a} \int_a^b f(x)\, dx$
The mean value theorem of integrals	If f is continuous on $[a, b]$, then there exists $c \in [a, b]$ such that $f(c) = \bar{f} = \frac{1}{b-a} \int_a^b f(x)\, dx$, $\int_a^b f(x)\, dx = f(c)(b - a)$
Arc length formula	$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$

Description	Equations
Arc length function	$s(x) = \int_a^x \sqrt{1 + [f'(t)]^2}\, dt$
Surface area of surface of resolution about x-axis	$S = \int_a^b 2\pi f(x) \sqrt{1 + [f'(x)]^2}\, dx$