MATH 126 Calculus III

2021-08-17

Vectors and Geometry of Space

| 3D Coordinate System

Description	Equations
Distance formula	$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$
Equation of a sphere centered at (a, b, c)	$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$
Norm/length/magnitude of vectors	$\lvert\mathbf{a}\rvert = \sqrt{a_1^2 + a_2^2 + a_3^2}$
Vector addition and subtraction	$\mathbf{a} \pm \mathbf{b} = \langle a_1 \pm b_1, a_2 \pm b_2, a_3 \pm b_3 \rangle$
Vector scalar multiplication	$c\mathbf{a} = \langle ca_1, ca_2, ca_3 \rangle$
Properties of vectors	$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$ $\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$ $\mathbf{a} + \mathbf{0} = \mathbf{a}$ $\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$ $c(\mathbf{a} + \mathbf{b}) = c\mathbf{a} + c\mathbf{b}$ $(c+d)\mathbf{a} = c\mathbf{a} + d\mathbf{a}$ $(cd)\mathbf{a} = c(d\mathbf{a})$ $1\mathbf{a} = \mathbf{a}$
Standard basis vectors	$\mathbf{i} = \langle 1, 0, 0 \rangle$ $\mathbf{j} = \langle 0, 1, 0 \rangle$ $\mathbf{k} = \langle 0, 0, 1 \rangle$

| Dot product

Description	Equations
Dot product	$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$
Properties of dot product	$\mathbf{a} \cdot \mathbf{a} = \lvert\mathbf{a}\rvert^2$ $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$ $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$ $(c\mathbf{a}) \cdot \mathbf{b} = c(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (c\mathbf{b})$ $\mathbf{0} \cdot \mathbf{a} = \mathbf{0}$
Dot product and angle between vectors	$\mathbf{a} \cdot \mathbf{b} = \lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert \cos\theta$ $\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert}$
Dot product to check orthogonal vectors	$\mathbf{a} \cdot \mathbf{b} = 0$
Direction angles and direction cosines	$\cos\alpha = \frac{\mathbf{a} \cdot \mathbf{i}}{\lvert\mathbf{a}\rvert\lvert\mathbf{i}\rvert} = \frac{a_1}{\lvert\mathbf{a}\rvert}$ $\cos\beta = \frac{a_2}{\lvert\mathbf{a}\rvert}$ $\cos\gamma = \frac{a_3}{\lvert\mathbf{a}\rvert}$

Description	Equations
Unit vector and direction cosines	$\frac{\mathbf{a}}{\lvert\mathbf{a}\rvert} = \langle\cos\alpha, \cos\beta, \cos\gamma\rangle$
Scalar projection of $\mathbf{b}$ onto $\mathbf{a}$	$\mathrm{comp}_{\mathbf{a}}\mathbf{b} = \frac{\mathbf{a}\cdot\mathbf{b}}{\lvert\mathbf{a}\rvert}$
Vector projection of $\mathbf{b}$ onto $\mathbf{a}$	$\mathrm{proj}_{\mathbf{a}}\mathbf{b} = \frac{\mathbf{a}\cdot\mathbf{b}}{\lvert\mathbf{a}\rvert}\frac{\mathbf{a}}{\lvert\mathbf{a}\rvert} = \frac{\mathbf{a}\cdot\mathbf{b}}{\lvert\mathbf{a}\rvert^2}\mathbf{a}$

Cross product

Description	Equations
Cross product	$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} =$ $\langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1\rangle$
Cross product to generate orthogonal vectors	$\mathbf{a}\times\mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$
Cross product and angle between vectors	$\lvert\mathbf{a}\times\mathbf{b}\rvert = \lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\sin\theta$
Cross product to check parallel vectors	$\mathbf{a}\times\mathbf{b} = \mathbf{0}$
Cross product as the area of parallelogram	$A = \lvert\mathbf{a}\times\mathbf{b}\rvert$
Cross products of standard basis vectors	$\mathbf{i}\times\mathbf{j} = \mathbf{k}$ $\mathbf{j}\times\mathbf{k} = \mathbf{i}$ $\mathbf{k}\times\mathbf{i} = \mathbf{j}$
Properties of cross product	$\mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a}$ $(c\mathbf{a})\times\mathbf{b} = c(\mathbf{a}\times\mathbf{b}) = \mathbf{a}\times(c\mathbf{b})$ $\mathbf{a}\times(\mathbf{b}+\mathbf{c}) = \mathbf{a}\times\mathbf{b} + \mathbf{a}\times\mathbf{c}$ $(\mathbf{a}+\mathbf{b})\times\mathbf{c} = \mathbf{a}\times\mathbf{c} + \mathbf{b}\times\mathbf{c}$ $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$ $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$
Scalar triple product as volume of parallelepiped	$V = \lvert\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})\rvert$
Scalar triple product to check three coplanar vectors	$V = \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = 0$

Equations of lines

Description	Equations
Vector equation of a line	$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$
Parametric equations of a line through (x_0, y_0, z_0), in direction of $\langle a, b, c\rangle$	$x = x_0 + at$ $y = y_0 + bt$ $z = z_0 + ct$
Symmetric equation of a line	$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$

Equations of planes

Description	Equations
Vector equation of a line segment	$\mathbf{r}(t) = (1-t)\mathbf{r}_0 + t\mathbf{r}_1$

Description	Equations
	$t \in [0, 1]$
Vector equation of a plane	$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0$ $\mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{r}_0$
Scalar equation of a plane through (x_0, y_0, z_0), normal vector $\langle a, b, c \rangle$	$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$
Linear equation of a plane	$ax + by + cz + d = 0$
Distance from a point to a plane	$D = \dfrac{\lvert ax_1 + by_1 + cz_1 + d \rvert}{\sqrt{a^2 + b^2 + c^2}}$

| Cylinders and quadratic surfaces

Description	Equations
Ellipsoid	$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$
Cone	$\dfrac{z^2}{c^2} = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}$
Elliptic paraboloid	$\dfrac{z}{c} = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}$
Hyperbolic paraboloid	$\dfrac{z}{c} = \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2}$
Hyperboloid of one sheet	$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1$
Hyperboloid of two sheets	$-\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$

Vectors Functions

| Vector functions and space curves

Description	Equations
Vector-valued function	$\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$
Limit of a vector function	$\lim\limits_{t \to a} \mathbf{r}(t) = \langle \lim\limits_{t \to a} f(t), \lim\limits_{t \to a} g(t), \lim\limits_{t \to a} h(t) \rangle$
Continuity of vector function	$\lim\limits_{t \to a} \mathbf{r}(t) = \mathbf{r}(t)$
Parametric equation of space curves	$x = f(t)$ $y = g(t)$ $z = h(t)$
Derivative of vector function	$\mathbf{r}'(t) = \lim\limits_{h \to 0} \dfrac{\mathbf{r}(t + h) - \mathbf{r}(t)}{h}$
Derivative of vector function	$\mathbf{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle$
Differentiation rules	$[\mathbf{u}(t) + \mathbf{v}(t)]' = \mathbf{u}'(t) + \mathbf{v}'(t)$ $[c\mathbf{u}(t)]' = c\mathbf{u}'(t)$ $[f(t)\mathbf{u}(t)]' = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$ $[\mathbf{u}(t) \cdot \mathbf{v}(t)]' = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$ $[\mathbf{u}(t) \times \mathbf{v}(t)]' = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$ $[\mathbf{u}(f(t))]' = f'(t)\mathbf{u}'(f(t))$

Description	Equations
Definite integral of vector function	$\int_a^b \mathbf{r}(t)\,dt$ $= \langle \int_a^b f(t)\,dt, \int_a^b g(t)\,dt, \int_a^b h(t)\,dt \rangle$
Position vector	$\mathbf{r}(t)$
Tangent (velocity) vector	$\mathbf{r}'(t)$
Unit tangent vector	$\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{\lvert \mathbf{r}'(t) \rvert}$

| Arc length and curvature

Description	Equations
Length of a curve	$L = \int_a^b \lvert \mathrm{r}'(t) \rvert\,dt$ $= \int_a^b \sqrt{[f(t)]^2 + [g(t)]^2 + [h(t)]^2}\,dt$
Arc length function	$s(t) = \int_a^t \lvert \mathrm{r}'(u) \rvert\,du$ $= \int_a^t \sqrt{[f(u)]^2 + [g(u)]^2 + [h(u)]^2}\,du$
Rate of change in arc length and the tangent vector	$\dfrac{ds}{dt} = \lvert \mathbf{r}'(t) \rvert$
Curvature	$\kappa(t) = \left\lvert \dfrac{d\mathbf{T}}{ds} \right\rvert = \dfrac{\lvert \mathbf{T}'(t) \rvert}{\lvert \mathbf{r}'(t) \rvert} = \dfrac{\lvert \mathbf{r}'(t) \times \mathbf{r}''(t) \rvert}{\lvert \mathbf{r}'(t) \rvert^3}$
Curvature in terms of function	$\kappa(x) = \dfrac{\lvert f''(x) \rvert}{[1 + (f'(x))^2]^{3/2}}$
Unit normal vector	$\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{\lvert \mathbf{T}'(t) \rvert}$
Binormal vector	$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$
Radius of osculating circle	$r = \frac{1}{\kappa}$

| Velocity and acceleration

Description	Equations
Position vector	$\mathbf{r}(t)$
Tangent (velocity) vector	$\mathbf{v}(t) = \mathbf{r}'(t)$
Acceleration vector	$\mathbf{a}(t) = \mathbf{v}'(t)$
Tangential and normal components of acceleration	$\mathbf{a}(t) = v'\mathbf{T} + \kappa v^2 \mathbf{N}$

Partial Derivatives

| Function of several variables

Description	Equations
Functions of two variables	$f(x,y) \mid (x,y) \in D$ $z = f(x,y)$
Level curves	$f(x,y) = k$

Description	Equations
Function of three variables	$f(x, y, z) \| (x, y, z) \in E$
Level surfaces	$f(x, y, z) = k$
Function of n variables	$f(\mathbf{x}) = \mathbf{c} \cdot \mathbf{x}$
Interpretation of input of functions of several variables	1. n real variables $x_1, x_2, \ldots, x_n$ 2. a single point variable $(x_1, x_2, \ldots, x_n)$ 3. a single vector variable $\mathbf{x} = \langle x_1, x_2, \ldots, x_n \rangle$
Limit of function of two variables	$\lim_{(x,y)\to(a,b)} f(x, y) = L$
Continuity of function of two variables	$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$
Limit of function of several variables	$\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = L$
Continuity of function of several variables	$\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = f(\mathbf{a})$

Partial derivatives

Description	Equations
Partial derivative with respect to x	$f_x(x, y) = \lim_{h\to 0} \frac{f(x+h, y) - f(x, y)}{h}$
Partial derivative with respect to y	$f_y(x, y) = \lim_{h\to 0} \frac{f(x, y+h) - f(x, y)}{h}$
Partial derivative rule	1. To find f_x, regard y as a constant and differentiate $f(x, y)$ with respect to x 2. To find f_y, regard x as a constant and differentiate $f(x, y)$ with respect to y
Clairaut's theorem	$f_{xy}(a, b) = f_{yx}(a, b)$

Tangent plane and linear approximations

Description	Equations
Tangent plane to a surface $z = f(x, y)$ at (x_0, y_0, z_0)	$z - z_0 = f_x(x_0, y_0) + f_y(x_0, y_0)(y - y_0)$
Linear approximation	$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$
Total differential	$dz = f_x(x, y)dx + f_y(x, y)dy$

Extreme values

Description	Equations
Critical point	a point with $f_x(a, b) = 0$ and $f_y(a, b) = 0$, ($\nabla f = \mathbf{0}$), or one of the partial derivatives does not exist
Local max/min and critical point	If f has a local max/min at (a, b), then (a, b) is a critical point

Description	Equations
Second derivative test ((a, b) is a critical point)	$D(a,b) = f_{xx}(a,b)f_{yy}(a,b) - [f_{xy}(ab)]^2$ $= \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$ (a) If $D > 0$ and $f_{xx}(a,b) > 0$, then $f(a,b)$ is a local max (b) If $D > 0$ and $f_{xx}(a,b) < 0$, then $f(a,b)$ is a local min (c) If $D < 0$, then $f(a,b)$ is a saddle point
Extreme value theorem for functions of two variables	If f is continuous on a closed, bounded set $D \in \mathbb{R}^2$, then f attains a absolute max and min at some points in D
Closed boundary method (Finding absolute max/min)	1. Find the values of f at the critical points of f in D 2. Find the extreme values of f on the boundary of D 3. The largest value is the abs max; the smallest value is the abs min

Other topics

Chain rule, directional derivative, and gradient vector are not covered in MATH 126 but covered in MATH 324.

Double Integrals

MATH 126 covers double integrals in Cartesian coordinates and polar coordinates with applications. They are reviewed in MATH 324.

Taylor Series

Linear and quadratic approximations

Description	Equations
First Taylor polynomial (Tangent line approximation)	$T_1(x) \approx f(b) + f'(b)(x-b)$
Tangent line error	$\lvert E_1 \rvert = \Big\lvert f(x) - [f(b) + f'(b)(x-b)] \Big\rvert$
Tangent line error bound	$\lvert f''(t) \rvert \leq M$ $\lvert E_1 \rvert \leq \dfrac{M}{2} \lvert x-b \rvert^2$
Second Taylor polynomial (Quadratic approximation)	$T_2(x) = f(b) + f'(b)(x-b) + \frac{1}{2}f''(b)(x-b)^2$
Quadratic approximation error	$\lvert E_2 \rvert = \lvert f(x) - T_2(x) \rvert$
Quadratic approximation error bound	$\lvert E_2 \rvert \leq \dfrac{M}{6} \lvert x-b \rvert^3$

Taylor polynomial and series

Description	Equations
nth Taylor polynomial	$T_n(x) = \displaystyle\sum_{k=0}^{n} \frac{f^{(k)}(b)}{k!}(x-b)^k$

Description	Equations
	$= f(b) + f'(b)(x-a) + \frac{f''(b)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(b)}{n!}(x-b)^n$
Taylor inequality $(\lvert f^{(n+1)}(t)\rvert \leq M)$	$\lvert f(x) - T_n(x)\rvert \leq \frac{M}{(n+1)!}\lvert x-b\rvert^{n+1}$
Taylor series	$\lim_{n\to\infty} T_n(x) = \lim_{n\to\infty}\sum_{k=0}^{n}\frac{f^{(k)}(b)}{k!}(x-b)^k = \sum_{k=0}^{\infty}\frac{f^{(k)}(b)}{k!}(x-b)^k$
Taylor series of exponential function	$e^x = \sum_{k=0}^{\infty}\frac{x^k}{k!}$
Taylor series of sine	$\cos(x) = \sum_{k=0}^{\infty}(-1)^k\frac{x^{2k}}{(2k)!}$
Taylor series of cosine	$\sin(x) = \sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{(2k+1)!}$
Geometric series as Taylor series $x \in (-1, 1)$	$\frac{1}{1-x} = \sum_{k=0}^{\infty}x^k$