MATH 124 Calculus I

2021-08-17

Limits and Continuity

Limits

Description	Equations
Limit	$\lim_{x\to a} f(x) = L$
Left-hand limit	$\lim_{x\to a^-} f(x) = L$
Right-hand limit	$\lim_{x\to a^+} f(x) = L$
Infinite limit	$\lim_{x\to a} f(x) = \pm\infty$
Limit at infinity	$\lim_{x\to \pm\infty} f(x) = L$
Limit existence	$\lim_{x\to a^-} f(x) = L = \lim_{x\to a^+} f(x)$ $\iff \lim_{x\to a} f(x) = L$
Limit non-existence	1. $\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$ 2. $\lim_{x\to a} f(x) = \pm\infty$ 3. oscillation

Limit laws

Description	Equations
Limit addition and subtraction	$\lim_{x\to a}[f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$
Limit constant multiplication	$\lim_{x\to a}[cf(x)] = c\lim_{x\to a} f(x)$
Limit multiplication	$\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$
Limit division	$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ if $\lim_{x\to a} g(x) \neq 0$
Limit power law (n is positive integer)	$\lim_{x\to a}[f(x)]^n = [\lim_{x\to a} f(x)]^n$
Limit at infinity of inverse power $(r > 0)$	$\lim_{x\to \pm\infty} \frac{1}{x^r} = 0$
Limit root law (n is positive integer; limit > 0 for even n)	$\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$
Direct substitution property of limit	If f is a polynomial or rational function, then $\lim_{x\to a} f(x) = f(a)$
Limit of functions with holes	If $f(x) = g(x)$ when $x \neq a$, then $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$ if it exists
The squeeze theorem	If $f(x) \leq g(x) \leq h(x)$ when x is near a and if $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$,

Description	Equations
	then $\lim_{x \to a} g(x) = L$

| Continuity

Description	Equations
Continuous at a number a	$\lim_{x \to a} f(x) = f(a)$
Continuous from the left at a number a	$\lim_{x \to a^-} f(x) = f(a)$
Continuous from the right at a number a	$\lim_{x \to a^+} f(x) = f(a)$
Continuous operations	addition, subtraction, multiplication, division
Continuous functions in their domain	polynomials, rational, root, trigonometric, inverse trigonometric, exponential, logarithmic functions
Types of discontinuity	1. removable discontinuity 2. jump discontinuity 3. infinite discontinuity
Continuity of function inputs and outputs	If f is continuous at b and $\lim_{x \to a} g(x) = b$, then $\lim_{x \to a} f(g(x)) = f(\lim_{x \to a} g(x))$
Continuity of composite functions	If g is continuous at a and if f is continuous at $g(a)$, then $(f \circ g)(x) = f(g(x))$ is continuous
Intermediate value theorem	If f is continuous on the closed interval $[a, b]$, and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$, then there exist a number c in (a, b) such that $f(c) = N$

Differentiation

| Derivatives

Description	Equations
Derivative of a function f at number a	$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$; $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$
Derivative as a function	$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$
Geometric interpretation of derivatives	The tangent line of $y = f(x)$ at $(a, f(a))$ has a slope of $f'(a)$ $m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = f'(a)$
Derivatives and instantaneous rate of change	The derivative $f'(a)$ is the instantaneous rate of change of $y = f(x)$ with respect to x when $x = a$: $v(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = f'(a)$
Differentiation and continuity	If f is differentiable at a, then f is continuous at a.

Description	Equations
Non-differentiable conditions	1. a corner 2. a discontinuity 3. a vertical tangent

Differentiation rules

Description	Equations
Constant multiple rule	$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}f(x)$
Addition and subtraction rule	$\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x)$
Product rule	$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$
Quotient rule (best practice: use product rule)	$\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$
Constant rule	$\frac{d}{dx}c = 0$
Power rule	$\frac{d}{dx}x^n = nx^{n-1}$
Chain rule	$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$ $\frac{d}{dx}f(g(x)) = f'(g(x)) \cdot g'(x)$
Linear approximations	$f(x) \approx f(a) + f'(a)(x-a)$
Differentials	$dy = f'(x)\,dx$

Special limits

Description	Equations
Limit associated with sine	$\lim\limits_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$
Limit associated with cosine	$\lim\limits_{\theta \to 0} \frac{\cos\theta - 1}{\theta} = 0$
Definition of e	$\lim\limits_{h \to 0} \frac{e^h - 1}{h} = 1$
e as a limit	$e = \lim\limits_{x \to 0}(1+x)^{1/x}$
e as a limit	$e = \lim\limits_{n \to \infty}(1+\frac{1}{n})^n$

Table of derivatives

Function $f(x)$	Derivative $f'(x)$	Function $f(x)$	Derivative $f'(x)$
c	0	x^n	nx^{n-1}
x	1	$\lvert x\rvert$	$\frac{x}{\lvert x\rvert}$
e^x	e^x	$\ln x$	$\frac{1}{x}$
a^x	$a^x \ln(a)$	$\log_a x$	$\frac{1}{x\ln(a)}$
$\sin x$	$\cos x$	$\sec x$	$\sec x \tan x$

Function $f(x)$	Derivative $f'(x)$	Function $f(x)$	Derivative $f'(x)$
$\cos x$	$-\sin x$	$\csc x$	$-\csc x \cot x$
$\tan x$	$\sec^2 x$	$\cot x$	$-\csc^2 x$
$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$	$\arctan x$	$\frac{1}{1+x^2}$
$\arccos x$	$\frac{-1}{\sqrt{1-x^2}}$		

Applications of Differentiation

Absolute extreme values

Description	Equations
Extreme values	Maximum and minimum values of f
Absolute maximum	$f(a) \geq f(x)$ for all $x \in D$
Absolute maximum	$f(a) \leq f(x)$ for all $x \in D$
Local maximum	$f(a) \geq f(x)$ for x near a
Local minimum	$f(a) \leq f(x)$ for x near a
Critical number c	$f'(c) = 0$ or $f'(c)$ does not exist
Extreme value theorem	If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum $f(c)$ and absolute minimum $f(d)$ at some number $c, d \in [a, b]$
Fermat's theorem	If f has a local maximum or minimum at c, then c is a critical number of f
Closed interval method (Finding the absolute max and min in $[a, b]$)	1. Find the values of f at critical numbers of f in (a, b) 2. Find the values of f at the endpoints of the interval 3. Compare the values. The largest is the abs max; the smallest is the abs min

The mean value theorem

Description	Equations
Rolle's theorem	If f is continuous on $[a, b]$, differentiable on (a, b), and endpoints $f(a) = f(b)$, then there is a number $c \in (a, b)$ such that $f'(c) = 0$
The mean value theorem	If f is continuous on $[a, b]$, differentiable on (a, b), then there is a number $c \in (a, b)$ such that $f'(c) = \frac{f(b) - f(a)}{b - a}$
Functions with derivative of zero are constants	If $f'(x) = 0$ for all $x \in (a, b)$, then f is constant on (a, b)
Functions with same derivatives are vertical translations of each other	If $f'(x) = g'(x)$ for all $x \in (a, b)$, then $f(x) = g(x) + c$ on (a, b)

Local extreme values

Description	Equations
Increasing	$f(x_1) < f(x_2)$ for $x_1 < x_2$ in I
Deceasing	$f(x_1) > f(x_2)$ for $x_1 < x_2$ in I
Increasing/decreasing test	(a) If $f'(x) > 0$ on an interval, then f is increasing on that interval. (b) If $f'(x) < 0$ on an interval, then f is decreasing on that interval.
First derivative test	If c is a critical value of continuous function f, then (a) If f' changes $+ \to -$ at c, then f has a local max at c (b) If f' changes $- \to +$ at c, then f has a local min at c (c) If f' has no sign change at c, then f has no local max/min at c
Concave upward	If the graph of f lies above all its tangents on an interval I (slope increases)
Concave downward	If the graph of f lies below all its tangents on an interval I (slope decreases)
Inflection point	The point at which the curve changes concavity
Concavity test	(a) If $f''(x) > 0$ on an interval, then f is concave upward on that interval. (b) If $f''(x) < 0$ on an interval, then f is concave downward on that interval.
Second derivative test	If f'' is continuous near c, (a) If $f'(c) = 0$ and $f''(c) > 0$, then f has a local min at c (b) If $f'(c) = 0$ and $f''(c) < 0$, then f has a local max at c

L'Hospital's rule

Description	Equations
Indeterminate forms	$\lim_{x\to a} \frac{f}{g} = \frac{0}{0}, \frac{\infty}{\infty}$
L'Hospital's rule	If f and g are differentiable and $g'(x) \neq 0$ on open interval I that contains a, and if the division has indeterminate form of $\frac{0}{0}$ or $\frac{\infty}{\infty}$, then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$
Indeterminate products	$\lim_{x\to a} fg = 0 \cdot \infty$ $\lim_{x\to a} fg = \frac{f}{1/g} = \frac{g}{1/f}$
Indeterminate differences	$\lim_{x\to a}(f-g) = \infty - \infty$

Description	Equations
Indeterminate powers	$\lim_{x \to a} [f(x)]^{g(x)} = 0^0, \infty^0, 1^\infty$ $y \equiv [f(x)]^{g(x)}$ $\ln y = g(x) \ln f(x)$

Newton's method

Description	Equations
Newton's method	$x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$