PHYS 123 Waves Equations

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Warning

- **WARNING: These equations are hand-typed and for personal reference use, so it is guaranteed to have some mistakes, both innocent and unforgivable. Therefore, use with caution!**
- By using this equation sheet, you accept the risk associated with potential mistakes.
- If you find any mistakes, I welcome you to raise an issue.
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Periodic Motion

Quantity	Unit	Definition
Period	s	$T = \dfrac{1}{f} = \dfrac{2\pi}{\omega}$
Frequency	Hz	$f = \dfrac{1}{T} = \dfrac{\omega}{2\pi}$
Angular frequency	s^{-1}	$\omega = 2\pi f = \dfrac{2\pi}{T}$

Simple harmonic motion (SHM)

Description	Equations
Angular frequency in SHM	$\omega = \sqrt{\dfrac{k}{m}}$
Spring constant	$k = m\omega^2$
Displacement in SHM	$x(t) = A\sin(\omega t + \phi)$
Velocity in SHM	$v(t) = \omega A\cos(\omega t + \phi)$
Acceleration in SHM	$a(t) = -\omega^2 A\sin(\omega t + \phi) = -\omega^2 x(t)$
Restoring force in SHM	$F = -k\Delta x$
Simple harmonic oscillator equation	$\frac{d^2x}{dt^2} = -\omega^2 x = -\frac{k}{m}x$
Conservation of energy in SHM	$E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \frac{1}{2}kA^2$
Amplitude	$A = \sqrt{x_0^2 + \dfrac{v_0^2}{\omega^2}}$
Phase angle	$\phi = \arctan\left(\dfrac{\omega x_0}{v_0}\right)$

Applications of SHM

Description	Equations
Restoring torque in angular SHM	$\tau = -\kappa\Delta\theta$
Rotational displacement in angular SHM	$\theta(t) = \theta_{\max}\sin(\omega t + \phi)$
Angular frequency in angular SHM	$\omega = \sqrt{\dfrac{\kappa}{I}}$
Angular frequency in simple pendulum	$\omega = \sqrt{\dfrac{g}{L}}$
Angular frequency in physical pendulum	$\omega = \sqrt{\dfrac{mgL}{I}}$

Damped oscillation

Description	Equations
Drag force	$F = -bv$
Time constant	$\tau = \dfrac{m}{b}$
Angular frequency of damped oscillator	$\omega_d = \sqrt{\omega^2 - \left(\frac{b}{2m}\right)^2}$
Displacement of damped oscillator	$x(t) = Ae^{-bt/2m}\sin(\omega_d t + \phi)$

1D Waves

Description	Equations
Wave speed, wavelength, and frequency	$c = \lambda f = \dfrac{\lambda}{T}$
Wave number	$k = \dfrac{2\pi}{\lambda}$
Angular frequency	$\omega = kc = 2\pi f$
Linear mass density	$\mu = \dfrac{m}{l}$
Wave speed of strings	$c = \sqrt{\dfrac{F_T}{\mu}}$

Description	Equations
Particle speed	$v = \frac{\partial f}{\partial t}$
Wave and particle speed	$c \neq v$
Energy put into a wave	$E = \frac{1}{2}\mu\lambda\omega^2 A^2$
Average power supplied to produce waves	$\overline{P} = \mu c v^2 = \frac{1}{2}\mu c A^2 \omega^2 = \frac{1}{2}\sqrt{\mu F_T}\omega^2 A^2$
Wave kinetic and potential energy	$K = U$

Wave function and boundary conditions

Description	Equations
Traveling wave functions	$f(x,t) = f(x - ct)$ to right $f(x,t) = f(x + ct)$ to left
Harmonic (sinusoidal) traveling wave functions	$f(x,t) = A\sin(kx - \omega t + \phi)$ to right $f(x,t) = A\sin(kx + \omega t + \phi)$ to left
1D wave equation	$\frac{\partial^2 f}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2}$
Principle of superposition	$f(x,t) = f_1(x,t) + f_2(x,t)$
Boundary conditions	Free end: heavy $\rightarrow$ light spring Fixed end: light $\rightarrow$ heavy spring
Shape of reflected wave	Free end: horizontal reflection only Fixed end: horizontal reflection, vertical inversion
Shape of transmitted wave	Similar to incident wave

Standing waves

Description	Equations
Standing wave function	$f(x,t) = 2A\sin(kx)\cos(\omega t)$
Standing wave of strings *with two fixed ends*	nth harmonic, n antinodes, $n+1$ nodes
Wavelength of nth harmonic	$\lambda_n = \frac{2L}{n}$
Frequency of nth harmonic	$f_n = n\frac{c}{2L} = nf_1$
Location of mth node of nth harmonic	$x_m = \frac{m\lambda_n}{2}$ $m \in [0, n+1]$

2D and 3D Waves

Quantity	Unit	Definition
2D intensity	$\mathrm{W/m}$	$I = \frac{P}{L} = \frac{P}{2\pi r}$
3D intensity	$\mathrm{W/m^2}$	$I = \frac{P}{A} = \frac{P}{4\pi r^2}$
Intensity level	dB	$\beta = (10\ \mathrm{dB})\log\left(\frac{I}{I_{\mathrm{th}}}\right)$ $I_{\mathrm{th}} = 10^{-12}\ \mathrm{W/m^2}$

Description	Equations
Path difference of constructive interference (in phase, at antinodal line)	$\delta = n\lambda$
Path difference of destructive interference (out of phase, at nodal line)	$\delta = (n - \frac{1}{2})\lambda$
Beat frequency	$f_b = \lvert f_1 - f_2 \rvert$
Average frequency	$\overline{f} = \frac{1}{2}(f_1 + f_2)$
Wave function of beats	$y(x,t) = y_1 + y_2$ $= 2A\cos(2\pi\frac{1}{2}f_b t)\sin(2\pi\overline{f}t)$
Doppler effect $(v_s < c)$ s - source; o - observer; rel to medium	$\frac{f_\text{o}}{f_\text{s}} = \frac{c \pm v_\text{o}}{c \pm v_\text{s}}$
Angle of shock wave $(v_s > c)$	$\sin\theta = \frac{c}{v_s}$
Mach number	$\text{Mach number } = \frac{v_s}{c}$

Ray Optics (Geometric Optics)

Description	Equations
Law of reflection	$\theta_1 = \theta_2$
Refraction index	$n_1 c_1 = n_2 c_2$
Wavelength of light in a new medium	$n_1 \lambda_1 = n_2 \lambda_2$
Snel's law	$n_1 \sin\theta_1 = n_2 \sin\theta_2$
Critical angle $(n_2 > n_1)$	$\arcsin\left(\frac{n_1}{n_2}\right)$
Lens equation o - object; i - image	$\frac{1}{f} = \frac{1}{d_\text{o}} + \frac{1}{d_\text{i}}$
Magnification	$M = \frac{h_\text{i}}{h_\text{o}} = -\frac{d_\text{i}}{d_\text{o}}$
Radius of curvature (distance of center) of mirror and focal length	$R = 2\lvert f \rvert$
Angular magnification	$M_\theta = \left\lvert\frac{\theta_\text{i}}{\theta_\text{o}}\right\rvert$
Small-angle (paraxial) approximation of angular magnification	$M_\theta = \frac{0.25\text{ m}}{f}$
Lens strength	$d = \frac{1\text{ m}}{f}$

Lens/mirror equation sign convention

Sign	Lens
$f > 0$ $f < 0$	converging lens diverging lens
$d_\text{o} > 0$ $d_\text{o} < 0$	object in front of lens object behind lens
$d_\text{i} > 0$ $d_\text{i} < 0$	image behind lens (in front of mirror) image in front of lens (behind mirror)

Sign	Lens
$h_\mathrm{i} > 0$	image upright
$h_\mathrm{i} < 0$	image inverted
$\lvert M \rvert > 1$	image larger than object
$\lvert M \rvert < 1$	image smaller than object

Converging lens images

Object distance d_o	Image distance $\lvert d_\mathrm{i} \rvert$	Image location	Upright Inverted	Magnification	Real/Virtual
$(0, f)$	$\lvert d_\mathrm{i} \rvert > f$	Same	Upright	Magnified	Virtual
f	∞	-	-	-	Parallel light
$(f, 2f)$	$(2f, \infty)$	Opposite	Inverted	Magnified	Real
$2f$	$2f$	Opposite	Inverted	Same	Real
$(2f, \infty)$	$(f, 2f)$	Opposite	Inverted	Demagnified	Real
∞	f	Opposite	-	-	Point

Diverging lens images

Object distance d_o	Image distance $\lvert d_\mathrm{i} \rvert$	Image Location	Upright/ Inverted	Magnification	Real/Virtual
$(0, \infty)$	$\lvert d_\mathrm{i} \rvert < d_\mathrm{o}$	Same	Upright	Demagnified	Virtual

Wave and Particle Optics

Single slit interference

Description	Equations
Variables	$n = 1, 2, 3, \ldots$ $a =$ width of the slit
Destructive interference	$a \sin\theta = \pm n\lambda$
Location of destructive interference (only for small angles)	$y_n = \pm n \dfrac{\lambda L}{a}$

Double slit interference

Description	Equations
Fringe order	$m = 0, 1, 2, \ldots$ $n = 1, 2, 3, \ldots$
Constructive interference	$d \sin\theta = \pm m\lambda$
Destructive interference	$d \sin\theta = \pm(n - \frac{1}{2})\lambda$
Distance between adjacent maxima	$D = \dfrac{L\lambda}{d}$

Multiple slit interference

Description	Equations
Variables	$m = 0, 1, 2, ...$ $N =$ number of slits $k =$ integer not multiple of N
Constructive interference (principal maxima)	$d \sin\theta = \pm m\lambda$
Destructive interference (minima)	$d \sin\theta = \pm \frac{k}{N}\lambda$
Minima adjacent to principle maxima	$d \sin\theta = \pm \frac{mN + 1}{N}\lambda$
Phasors	$\delta\varphi = 2\pi \frac{\delta s}{\lambda}$

Thin-film interference

Note: constructive and destructive interference refers to reflected light, not transmitted light.

Description	Equations
Thin-film interference t - thickness $m = 0, 1, 2, ...$ a - incident medium b - thin film medium c - transmitted medium	$\phi = \frac{4\pi n_b t \cos\theta_b}{\lambda_a} + \phi_{r2} - \phi_{r1}$ $\phi = \begin{cases} 2m\pi & \text{constructive} \\ (2m+1)\pi & \text{destructive} \end{cases}$ $\phi_r = \begin{cases} 0 & n_i > n_f \\ \pi & n_i < n_f \end{cases}$
Constructive interference if in phase (If π-shifted, use destructive condition)	$2t = m\lambda_b = m\frac{n_a}{n_b}\lambda_a$
Destructive interference if in phase (If π-shifted, use constructive condition)	$2t = (m + \frac{1}{2})\lambda_b = (m + \frac{1}{2})\frac{n_a}{n_b}\lambda_a$

Circular aperture

Description	Equations
Variables	d - diameter of the circular aperture f - focal distance of the lens
First minimum with circular aperture	$\sin\theta = 1.22\frac{\lambda}{d}$
Angular resolution Rayleigh's criterion of resolution angle	$\theta \approx \sin\theta = 1.22\frac{\lambda}{d}$
Linear resolution Radius of the first minimum by a lens	$y = 1.22\frac{\lambda f}{d}$

Wave-particle duality

Description	Equations
Bragg's condition with Bragg angle X-ray diffraction	$2d \sin\alpha = m\lambda$ $(2d\cos\theta = m\lambda, \alpha = 90^\circ - \theta)$
Energy of a photon	$E = h\nu$
Momentum of a photon	$p = \frac{h\nu}{c}$
Wavelength of a particle	$\lambda = \frac{h}{p} = \frac{h}{mv}$
Photoelectric effect	$E_k = h\nu - \Phi = eV_{\text{stop}}$

Description	Equations
Stopping potential	$V_{\text{stop}} = \frac{h\nu}{e} - \frac{\Phi}{e}$
Intensity of light	$I \propto$ rate of electron emitted from the metal

Fluid Mechanics

Quantity	Unit	Definition
Density	$\mathrm{kg/m^3}$	$\rho = \frac{m}{V}$
Pressure	Pa $\mathrm{N/m^2}$	$P = \frac{dF}{dA}$
Volumetric flow rate	$\mathrm{m^3/s}$	$Q = \dot{V} = \frac{dV}{dt}$

Description	Equations
Pressure of stationary fluid	$P = P_{\text{surf}} + \rho g h$
Pressure of stationary fluid	$P_1 + \rho g y_1 = P_2 + \rho g y_2$
Buoyant force	$F_b = F_{\text{bottom}} - F_{\text{top}}$
Archimedes' principle	$F_b = \rho_f g V_{\text{disp}}$
Volume of displaced fluid of floating object	$V_{\text{disp}} = \frac{\rho_o}{\rho_f} V_o$
Condition of object buoyancy o - object; f - fluid	$\begin{cases} \rho_o < \rho_f & \text{float} \\ \rho_o = \rho_f & \text{hang} \\ \rho_o > \rho_f & \text{sink} \end{cases}$
Absolute pressure and gauge pressure	$P_{\text{abs}} = P_{\text{atm}} + P_g$
Hydraulic system	$P = \frac{F_1}{A_1} = \frac{F_2}{A_2}$
Continuity equation Laminar flow of nonviscous fluid	$\dot{m}_1 = \dot{m}_2$ $\rho_1 Q_1 = \rho_2 Q_2$ $\rho_1 A_1 v_1 = \rho_2 A_2 v_2$
Bernoulli's equation Laminar flow of incompressible nonviscous fluid	$P_1 + \rho g y_1 + \frac{1}{2}\rho v_1^2 = P_2 + \rho g y_2 + \frac{1}{2}\rho v_2^2$

Entropy

Description	Equations
Partition	$M = \frac{V}{\delta V}$
Microstate (basic state)	$\Omega = M^N$
Entropy	$S = \ln \Omega = \ln M^N$
Linearity of entropy	$S = S_A + S_B$ $\Omega = \Omega_A \Omega_B$
Constant temperature change in entropy	$\Delta S = N \ln \left(\frac{V_f}{V_i} \right)$
Second law of thermodynamics in closed system	$\Delta S \begin{cases} > 0 & \text{toward equilibrium} \\ = 0 & \text{at equilibrium} \end{cases}$

Description	Equations
Equipartition of space	$\frac{N_A}{V_A} = \frac{N_B}{V_B}$
Root-mean-square (rms) speed	$v_{\text{rms}} = \sqrt{\overline{v^2}}$
Absolute temperature	$\frac{1}{k_B T} = \frac{dS}{dE_{\text{th}}}$
Equipartition of energy	$\frac{E_{\text{th},A}}{N_A} = \frac{E_{\text{th},B}}{N_B}$

Monoatomic ideal gas

Description	Equations
Thermal energy	$E_{\text{th}} = N\overline{K} = \frac{1}{2}Nmv_{\text{rms}}^2$
Pressure	$P = \frac{2}{3}\frac{E_{\text{th}}}{V}$
Thermal energy	$E_{\text{th}} = \frac{3}{2}Nk_B T$
Average kinetic energy	$\overline{K} = \frac{3}{2}k_B T$
rms speed	$v_{\text{rms}} = \sqrt{\frac{3k_B T}{m}}$
Ideal gas law	$PV = nRT$ $PV = Nk_B T$
Constant volume change in entropy	$\Delta S = \frac{3}{2}N\ln\left(\frac{T_f}{T_i}\right)$
Total change in entropy	$\Delta S = \frac{3}{2}N\ln\left(\frac{T_f}{T_i}\right) + N\ln\left(\frac{V_f}{V_i}\right)$

Thermodynamic Processes

Description	Equations
General energy balance	$\Delta E = W + Q$
Energy balance of ideal gas	$\Delta E_{\text{th}} = W + Q$
Change in thermal energy of ideal gas	$\Delta E_{\text{th}} = \frac{d}{2}Nk_B\Delta T$
PV Work	$W = \int_{V_i}^{V_f} P\, dV$
Entropy change	$\Delta S = N\ln\left(\frac{V_f}{V_i}\right) + \frac{d}{2}N\ln\left(\frac{T_f}{T_i}\right)$
Constant volume heat capacity	$C_V = \frac{Q}{N\Delta T} = \frac{d}{2}k_B$
Constant pressure heat capacity	$C_P = \frac{Q}{N\Delta T} = \left(\frac{d}{2} + 1\right)k_B$
Heat capacity relationship	$C_P = C_V + k_B$
Heat capacity ratio	$\gamma = \frac{C_P}{C_V} = 1 + \frac{2}{d}$

Isochoric process

Description	Equations

Description	Equations
Isochoric process	$\Delta V = 0$
Work	$W = 0$
Thermal energy and heat	$\Delta E_{\text{th}} = Q = \frac{d}{2} N k_B T = N C_V \Delta T$
Entropy	$\Delta S = \frac{d}{2} N \ln\left(\frac{T_f}{T_i}\right) = \frac{N C_V}{k_B} \ln\left(\frac{T_f}{T_i}\right)$

Isentropic process

Description	Equations
Isobaric process (quasistatic adiabatic)	$\Delta S = 0$
Heat	$Q = 0$
Thermal energy and work	$\Delta E_{\text{th}} = W = N C_V \Delta T$
PVT relationship	$P_1 V_1^{\gamma} = P_2 V_2^{\gamma}$ $T_1 V_1^{\gamma+1} = T_2 V_2^{\gamma+1}$ $P_1^{(1/\gamma)-1} T_1 = P_2^{(1/\gamma)-1} T_2$

Isobaric process

Description	Equations
Isobaric process	$\Delta P = 0$
Work	$W = -P \Delta V = -N k_B \Delta T$
Heat	$Q = N C_P \Delta T$
Thermal energy	$\Delta E_{\text{th}} = N C_V \Delta T$
Entropy	$\Delta S = \frac{N C_P}{k_B} \ln\left(\frac{T_f}{T_i}\right)$

Isothermal process

Description	Equations
Isothermal process	$\Delta T = 0$
Thermal energy	$\Delta E_{\text{th}} = 0$
Work and Heat	$Q = -W$
Work	$W = -N k_B T \ln\left(\frac{V_f}{V_i}\right)$
Heat	$Q = N k_B T \ln\left(\frac{V_f}{V_i}\right)$
Entropy	$\Delta S = N \ln\left(\frac{V_f}{V_i}\right) = \frac{Q}{k_B T}$

Degradation of Energy

Description	Equations
Complete cycle of steady device	$\Delta E = 0$ $W = Q_{\text{out}} - Q_{\text{in}}$ $\Delta S_{\text{sys}} = 0$

Description	Equations
	$\Delta S_{\text{surr}} \geq 0$
Steady device thermally transferring energy to lower temperature	$\Delta S_{\text{surr}} = \frac{Q_{\text{out}}}{k_B}\left(\frac{1}{T_{\text{out}}} - \frac{1}{T_{\text{in}}}\right)$
Steady device thermally transferring energy to higher temperature	$\Delta S_{\text{surr}} = -\frac{Q_{\text{out}}}{k_B}\left(\frac{1}{T_{\text{out}}} - \frac{1}{T_{\text{in}}}\right)$
Steady device converting mechanical energy to thermal energy	$\Delta S_{\text{surr}} = \frac{Q_{\text{out}}}{k_B T_{\text{out}}}$
Reversible heat engine	$\frac{Q_{\text{out}}}{Q_{\text{in}}} = \frac{T_{\text{out}}}{T_{\text{in}}} = \frac{T_{\text{low}}}{T_{\text{high}}}$
Energy balance	$Q_{\text{in}} + W_{\text{in}} = Q_{\text{out}} + W_{\text{out}}$
Efficiency of heat engine	$\eta = -\frac{W_{\text{out}}}{Q_{\text{in}}} = 1 - \frac{Q_{\text{out}}}{Q_{\text{in}}}$
Maximum efficiency of reversible heat engine	$\eta_{\text{max}} = 1 - \frac{T_{\text{low}}}{T_{\text{high}}}$
Coefficient of performance of heating	$\text{COP}_{\text{heating}} = \frac{Q_{\text{out}}}{W} = \frac{1}{1 - Q_{\text{in}}/Q_{\text{out}}}$
Maximum coefficient of performance of heating (reversible heat pump)	$\text{COP}_{\text{heating,max}} = \frac{1}{1 - T_{\text{in}}/T_{\text{out}}}$
Coefficient of performance of cooling	$\text{COP}_{\text{cooling}} = \frac{Q_{\text{in}}}{W} = \text{COP}_{\text{heating}} - 1$
Maximum coefficient of performance of cooling (reversible heat pump)	$\text{COP}_{\text{cooling,max}} = \frac{T_{\text{in}}}{T_{\text{out}} - T_{\text{in}}}$