Annuities and Discount Factors

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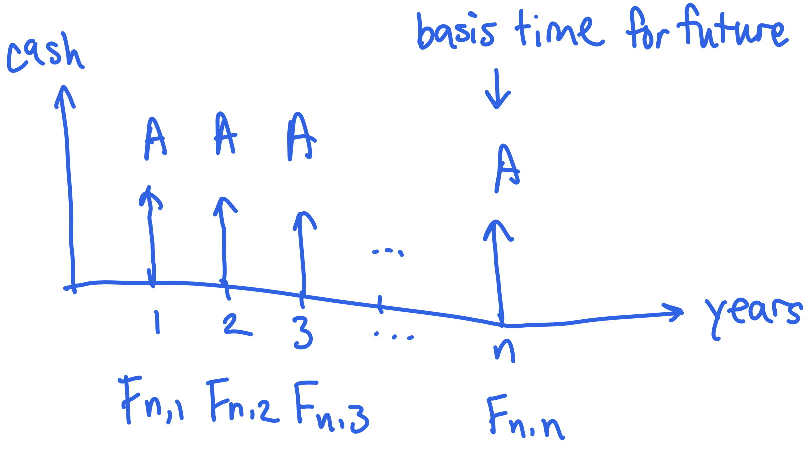
Process Design

Annuities give same amount of cash flow for each consecutive year

Ex. Verify that the future value F_n of annuities with each transaction value of A for n consecutive years is



$$\boxed{F_n = A\left[\frac{(1+i)^n - 1}{i}\right]}$$

where i is the annual interest rate.

- **Annuities** - A series of uniform value cash transactions in consecutive years
- Hint: nth partial sum of geometric series: $S_n = a_1 \dfrac{1 - r^n}{1 - r}$ $= ar^0 + ar^1 + ar^2 + \cdots + ar^n$

$a = A$

$r = 1 + i$

$n = n$

$$F_n = F_{n,1} + F_{n,2} + F_{n,3} + \cdots + F_{n,n}$$

$$= A(1+i)^n + A(1+i)^{n-1} + A(1+i)^{n-2} + \cdots + A\cancel{(1+i)}^{\cancel{n-n}}$$

$$= A\,\frac{1-(1+i)^n}{1-(1+i)} = \boxed{A\underbrace{\left[\frac{1-(1+i)^n}{i}\right]}_{f(i,n)}}$$

Discount factor converts money value between the past, present, and annuities

- Discount factor for converting A to F

$$F_n = A\left[\frac{(1+i)^n - 1}{i}\right]$$

- Notation

$$\frac{F_n}{A} = \left[\frac{(1+i)^n - 1}{i}\right] = f(i,n)$$

want known know if give i, n

$$F_n = A \left(\frac{F_n}{A}\right)$$

$$\frac{F_n}{A} = \left(\frac{F_n}{A}, i, n\right) = f(i,n) = \underbrace{\frac{F_n}{A}(i,n)}$$

Discount factors are derived from known discount factors

Ex. Derive the discount factor for converting A to P.

want given disc. f.

$$P = A\left(\frac{P}{A}\right)$$

given want

$$\frac{P}{A} = \frac{P}{1}\,\frac{1}{A} = \frac{P}{F}\,\frac{F}{A} = \frac{1}{(1+i)^n}\left[\frac{\left((1+i)^n-1\right)}{i}\right]$$

$$= \boxed{\frac{(1+i)^n-1}{i(1+i)^n}}$$

- $\frac{F}{A} = \left[\frac{(1+i)^n-1}{i}\right]$
- $\frac{P}{F} = \frac{1}{(1+i)^n}$

$$F = P\underbrace{(1+i)^n}$$

- conv from P to A
  given want

$$A = P\,\frac{A}{P}$$

$$\frac{A}{P} = \left(\frac{P}{A}\right)^{-1} = \frac{i(1+i)^n}{(1+i)^n-1}$$

Common discount factors are tabulated

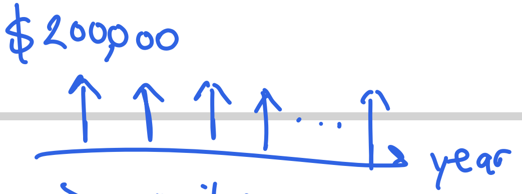
Turton Table 9.1

$F = \cancel{P}\left(\frac{F}{P}\right)$ i, n

comp. interest

Conversion (given unknown)	Symbol (unknown/given)	Common Name	Formula
P to F	F/P	Single payment compound amount factor	$(1+i)^n$
F to P	P/F	Single payment present worth factor	$\frac{1}{(1+i)^n}$
A to F	F/A	Uniform series compound amount factor; Future worth of annuity	$\frac{(1+i)^n - 1}{i}$
F to A	A/F	Sinking fund factor	$\frac{1}{(1+i)^n - 1}$
P to A	A/P	Capital recovery factor	$\frac{i(1+i)^n}{(1+i)^n - 1}$
A to P	P/A	Uniform series present worth factor; Present worth of annuity	$\frac{(1+i)^n - 1}{i(1+i)^n}$

Example of using discount factor as a conversion factor



uniform consecutive

Turton Ex. 9.14 A lottery winner will receive $200,000/year for the next 20 years. → annuity

(**a**) What is the equivalent present value of the winnings if there is a secure investment opportunity providing 7.5% p.a.?

(**b**) What rate of return in part (a) would be needed for a present value of $2.5 million?

$i = ?$

$$\frac{P}{A} = \frac{(1+i)^n - 1}{i(1+i)^n}$$

(a) $A = 200{,}000$

$P = ?$

$i = 7.5\% = 0.075$

$n = 20$

$$P = A\,\frac{P}{A} = A\,\frac{(1+i)^n - 1}{i(1+i)^n}$$

$$= (\$200{,}000)\,\frac{(1+0.075)^{20} - 1}{0.075\,(1+0.075)^{20}} = \boxed{\$2{,}038{,}900}$$

7.5%

(b) $i = ?$

✓ $P = \$2{,}500{,}000$

✓ $A = \$200{,}000$

$n = 20$

$$\frac{P}{A} = \frac{(1+i)^n - 1}{i(1+i)^n}$$

$$\frac{2{,}500{,}000}{200{,}000} = \frac{(1+i)^{20} - 1}{i(1+i)^{20}} \Rightarrow i = 0.0496 = 4.96\%$$