# Inverse Laplace Transform

Teng-Jui Lin
Department of Chemical Engineering, University of Washington
**Process Dynamics and Control**

# Definition of inverse Laplace transform

If $F(s)$ represents the Laplace transform of $f(t)$, then $f(t)$ is the **inverse Laplace transform** of $F(s)$.

- If $\mathcal{L}[f(t)] = F(s)$, then $f(t) = \mathcal{L}^{-1}[F(s)]$
- $\mathcal{L}^{-1}[\mathcal{L}[f(t)]] = f(t)$

$$\int \frac{d}{dx} \qquad \frac{d}{dx}\int$$

Inverse Laplace transform is also linear.

$$\mathcal{L}^{-1}[\alpha F(s) + \beta G(s)] = \alpha\mathcal{L}^{-1}[F(s)] + \beta\mathcal{L}^{-1}[G(s)]$$

$$\mathcal{L}^{-1}[5t + 10e^{t}] = 5\mathcal{L}^{-1}[t] + 10\,\mathcal{L}^{-1}[e^{t}]$$

# Common inverse Laplace transforms are tabulated

Use Seaborg Table 3.1... in reverse

| Inverse L.T. $f(t)$ | Laplace Transform $F(s)$ | Inverse L.T. $f(t)$ | Laplace Transform $F(s)$ |
|---|---|---|---|
| $1 = \mathcal{L}^{-1}\left[\frac{1}{s}\right]$ | $\frac{1}{s}$ | $e^{at} = \mathcal{L}^{-1}\left[\frac{1}{s-a}\right]$ | $\frac{1}{s-a}$ |
| $t^n$ | $\frac{n!}{s^{n+1}}$ | $\sqrt{t}$ | $\frac{\sqrt{\pi}}{2s^{3/2}}$ |
| $\sin(at)$ | $\frac{a}{s^2+a^2}$ | $t\sin(at)$ | $\frac{2as}{(s^2+a^2)^2}$ |
| $\cos(at)$ | $\frac{s}{s^2+a^2}$ | $t\cos(at)$ | $\frac{s^2-a^2}{(s^2+a^2)^2}$ |

# Applying inverse Laplace transform

**Zill Ex. 4.2.1a** Evaluate $\mathcal{L}^{-1}\left[\dfrac{1}{s^5}\right]$

Recall $t^n = \mathcal{L}^{-1}\left[\dfrac{n!}{s^{n+1}}\right]$

$n+1=5$

$n=4$

$$= \frac{4!}{4!}\cdot\frac{1}{s^{4+1}} = \frac{1}{4!}\left[\frac{4!}{s^{4+1}}\right]$$

$$= \boxed{\frac{1}{4!}\,t^4}$$

**Zill Ex. 4.2.1b** Evaluate $\mathcal{L}^{-1}\left[\dfrac{1}{s^2+7}\right]$

Recall $\sin(at) = \mathcal{L}^{-1}\left[\dfrac{a}{s^2+a^2}\right]$

$a^2=7$

$a=\sqrt{7}$

$$= \frac{\sqrt{7}}{\sqrt{7}}\cdot\frac{1}{s^2+(\sqrt{7})^2} = \frac{1}{\sqrt{7}}\left[\frac{\sqrt{7}}{s^2+(\sqrt{7})^2}\right]$$

$$= \boxed{\frac{1}{\sqrt{7}}\sin(\sqrt{7}\,t)}$$

# Term-wise division and linearity

**Zill Ex. 4.2.2** Evaluate $\mathcal{L}^{-1}\left[\dfrac{-2s+6}{s^2+4}\right]$

$\mathcal{L}^{-1}\left(\dfrac{a}{s^2+a^2}\right) = \sin(at)$

const

$4 = a^2$

$a = 2$

$\mathcal{L}^{-1}\left(\dfrac{s}{s^2+a^2}\right) = \cos(at)$

var in $\mathcal{L}$ space

$a^2 = 4$

$a = 2$

$\mathcal{L}^{-1}\left(\dfrac{-2s+6}{s^2+4}\right)$ ← Laplace (s) space

$= \mathcal{L}^{-1}\left[\dfrac{-2s}{s^2+4} + \dfrac{6}{s^2+4}\right]$

$= -2\mathcal{L}^{-1}\left[\dfrac{s}{s^2+4}\right] + 6\,\mathcal{L}^{-1}\left[\dfrac{1}{s^2+4}\right]$

$= \boxed{-2\cos(2t) + 6\sin(2t)}$ ← "real" (world) time + space

## Partial fractions and linearity

**Ex.** Evaluate $\mathcal{L}^{-1}\left[\dfrac{1}{(s-1)(s+4)}\right]$

$$\frac{1}{(\ )(\ )} = \frac{A}{(\ )} + \frac{B}{(\ )}$$

$$\mathcal{L}^{-1}\left(\frac{1}{s+a}\right) = e^{-at}$$

$a=-1$

$a=4$

$$\frac{1}{(s-1)(s+4)} = \frac{A}{(s-1)} + \frac{B}{(s+4)} = \frac{1}{5}\frac{1}{s-1} - \frac{1}{5}\frac{1}{s+4}$$

$$= \frac{A}{(s-1)}\cdot\frac{s+4}{s+4} + \frac{B}{(s+4)}\frac{s-1}{s-1}$$

$$\frac{1}{(s-1)(s+4)} = \frac{As+4A+Bs-B}{(s-1)(s+4)}$$

$$1 = As+4A+Bs-B$$

$$0\cdot s + 1 = s(A+B) + (4A-B)$$

$$\begin{cases} 0 = A+B \\ 1 = 4A-B \end{cases} \Rightarrow A=-B$$

$1=-4B-B$

$1=-5B$

$B=-\frac{1}{5}$, $A=\frac{1}{5}$

$$\mathcal{L}^{-1}\left[\frac{1}{(s-1)(s+4)}\right]$$

$$= \mathcal{L}^{-1}\left[\frac{1}{5}\frac{1}{s-1} - \frac{1}{5}\frac{1}{s+4}\right]$$

$$= \frac{1}{5}\mathcal{L}^{-1}\left[\frac{1}{s-1}\right] - \frac{1}{5}\mathcal{L}^{-1}\left[\frac{1}{s+4}\right]$$

$$= \boxed{\frac{1}{5}e^{t} - \frac{1}{5}e^{-4t}}$$

# Partial fraction by Heaviside expansion (cover-up method)

**Ex.** Evaluate $\mathcal{L}^{-1}\left[\dfrac{1}{(s-1)(s+4)}\right]$ with Heaviside expansion.

$a = -1$

$s-1$: $\cancel{(s-1)}\dfrac{1}{\cancel{(s-1)}(s+4)} = \left[\dfrac{A}{\cancel{s-1}} + \dfrac{B}{s+4}\right]\cancel{(s-1)}$

1. Set up partial fraction expansion
2. Multiply both sides by one denominator term $(s+a)$
3. Evaluate at $s=-a$ ←
4. Solve for coefficient

$s = 1$

$\dfrac{1}{s+4} = A + \dfrac{B\cancel{(s-1)}}{s+4} \to 0$

$\dfrac{1}{1+4} = A + 0$

$\boxed{A = \frac{1}{5}}$ ←

$\boxed{s-1}$ at $s=1$

1. Cover up denominator term $(s+a)$
2. Evaluate at $s=-a$
3. The result is the coeff for the term

$\dfrac{1}{\cancel{(s-1)}(s+4)} = \dfrac{1}{1+4} = \boxed{\frac{1}{5} = A}$

↑ ↑

Limitation: Rational function of $s$ with distinct linear factors in denominator

# Partial fraction by Heaviside expansion (cover-up method)

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1. Set up partial fraction expansion
2. Multiply both sides by one denominator term $(s+a)$
3. Evaluate at $s=-a$
4. Solve for coefficient

$$\cancel{(s+4)}\frac{1}{(s-1)\cancel{(s+4)}} = \left[\frac{A}{s-1} + \frac{B}{\cancel{s+4}}\right]\cancel{(s+4)}$$

$$\frac{1}{s-1} = \frac{A\,\cancel{(s+4)}^{\to 0}}{s-1} + B$$

$s+4=0$

$s=-4 \rightarrow$

$$\frac{1}{-5} = 0 + B \Rightarrow \boxed{B = -\frac{1}{5}}$$

1. Cover up denominator term $(s+a)$
2. Evaluate at $s=-a$
3. The result is the coeff for the term

$$\frac{1}{(s-1)\cancel{(s+4)}} = \left[\frac{1}{s-1}\right]_{s=-4} = \frac{1}{-4-1} = \boxed{\frac{1}{-5} = B}$$

$A = \frac{1}{5}$, $B = -\frac{1}{5}$ ✓

Limitation: Rational function of $s$ with distinct linear factors in denominator

# Partial fraction by Heaviside expansion (cover-up method)

**Ex.** Evaluate $\mathcal{L}^{-1}\left[\dfrac{1}{(s-1)(s+4)}\right]$ with Heaviside expansion.

$A = \frac{1}{5}$, $B = -\frac{1}{5}$ $\Rightarrow$ $\mathcal{L}^{-1}\left[\frac{1}{5}\frac{1}{s-1} - \frac{1}{5}\frac{1}{s+4}\right]$

$\mathcal{L}^{-1}\left(\frac{1}{s+a}\right) = e^{-at}$

$= \frac{1}{5}\mathcal{L}^{-1}\left[\frac{1}{s-1}\right] - \frac{1}{5}\mathcal{L}^{-1}\left[\frac{1}{s+4}\right]$

$= \boxed{\frac{1}{5}e^{t} - \frac{1}{5}e^{-4t}}$

# Cover-up method is great for more complicated expressions

**Zill Remarks 4.2.ii** Evaluate $\mathcal{L}^{-1}\left[\dfrac{s^2+6s+9}{(s-1)(s-2)(s+4)}\right]$ with Heaviside expansion.

$\boxed{1}$

$$\frac{s^2+6s+9}{(s-1)(s-2)(s+4)} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s+4} = \frac{16}{-5}\frac{1}{s-1} + \frac{25}{6}\frac{1}{s-2} + \frac{1}{30}\frac{1}{s+4}$$

$s-1=0$, $s=1$: $\dfrac{s^2+6s+9}{(s-2)(s+4)} = \dfrac{1^2+6+9}{(1-2)(1+4)} = \dfrac{16}{-5} = A$

$s-2=0$, $s=2$: $\dfrac{s^2+6s+9}{(s-1)(s+4)} = \dfrac{4+12+9}{(1)(6)} = \dfrac{25}{6} = B$

$s+4=0$, $s=-4$: $\dfrac{s^2+6s+9}{(s-1)(s-2)} = \dfrac{16-24+9}{(-5)(-6)} = \dfrac{1}{30} = C$

$\boxed{2}$ $\mathcal{L}^{-1}[\;]$

$\mathcal{L}^{-1}\left[\frac{1}{s+a}\right] = e^{-at}$

$$= \frac{16}{-5}\mathcal{L}^{-1}\left[\frac{1}{s-1}\right] + \frac{25}{6}\mathcal{L}^{-1}\left[\frac{1}{s-2}\right] + \frac{1}{30}\mathcal{L}^{-1}\left[\frac{1}{s+4}\right]$$

$$= \boxed{\frac{16}{-5}e^{t} + \frac{25}{6}e^{2t} + \frac{1}{30}e^{-4t}}$$