Definition of Laplace Transform

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Review: improper integral

The **improper integral** is defined as a limit:

$$\int_a^\infty f(t)dt = \lim_{b\to\infty} \int_a^b f(t)dt$$

- Convergent - limit exists
- Divergent - limit does not exist

Definition of Laplace transform

Laplace transform of a piece-wise continuous $f(t)$ defined for $t \geq 0$ is

$$\mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t) dt$$

if the integral converges.

- Notation
  - Lower case for original function $f(t)$
  - Upper case for Laplace-transformed function $\mathcal{L}[f(t)] \equiv F(s)$

Laplace transform simplifies solving ODE to solving algebraic equations

Time domain

$f, f(t)$

Laplace domain

$s, F(s)$

Laplace transform of a constant

Zill Ex. 4.1.1 Evaluate $\mathcal{L}[c]$, where c is a constant.

Laplace transform of a linear function

Zill Ex. 4.1.2 Evaluate $\mathcal{L}[t]$.

Laplace transform of a exponential function

Zill Ex. 4.1.3a Evaluate $\mathcal{L}[e^{-3t}]$.

Zill Ex. 4.1.3b Evaluate $\mathcal{L}[e^{6t}]$.

Laplace transform of a sine function

Zill Ex. 4.1.4 Evaluate $\mathcal{L}[\sin(2t)]$.

Laplace transform of a piecewise-continuous function

Zill Ex. 4.1.6 Evaluate $\mathcal{L}[f(t)]$ for $f(t) \begin{cases} 0 & t \in [0, 3) \\ 2 & t \in [3, \infty) \end{cases}$

Common Laplace transforms are tabulated

Use Seaborg Table 3.1

Inverse L.T. $f(t)$	**Laplace Transform** $F(s)$	**Inverse L.T.** $f(t)$	**Laplace Transform** $F(s)$
1	$\frac{1}{s}$	e^{at}	$\frac{1}{s-a}$
t^n	$\frac{n!}{s^{n+1}}$	$\sqrt{t}$	$\frac{\sqrt{\pi}}{2s^{3/2}}$
$\sin(at)$	$\frac{a}{s^2+a^2}$	$t\sin(at)$	$\frac{2as}{(s^2+a^2)^2}$
$\cos(at)$	$\frac{s}{s^2+a^2}$	$t\cos(at)$	$\frac{s^2-a^2}{(s^2+a^2)^2}$

Linearity simplifies problem

- **Linear combination** - multiplication of terms by constant and/or addition of terms

- **Linearity** - transform of a linear combination = linear combination of the transforms
  - $T[\alpha f(x) + \beta g(x)] = \alpha T[f(x)] + \beta T[g(x)]$

Laplace transform is linear

Ex. Demonstrate Laplace transform is linear.

- Laplace transform is an integral transform
  - Definition: $\mathcal{L}[f(t)] = \int_0^{\infty} e^{-st} f(t) dt$

Laplace transform is linear

$$\begin{aligned} &\mathcal{L}[\alpha f(t) + \beta g(t)] \\ =&\alpha \mathcal{L}[f(t)] + \beta \mathcal{L}[g(t)] \\ =&\alpha F(s) + \beta G(s) \end{aligned}$$

Zill Ex. 4.1.5a Use linearity of Laplace transform to evaluate $\mathcal{L}[1 + 5t]$