# Definition of Laplace Transform

Teng-Jui Lin
Department of Chemical Engineering, University of Washington

**Process Dynamics and Control**

# Review: improper integral

The **improper integral** is defined as a limit:

$$\int_a^\infty f(t)dt = \lim_{b\to\infty} \int_a^b f(t)dt$$

- Convergent - limit exists
- Divergent - limit does not exist

# Definition of Laplace transform

**Laplace transform** of a piece-wise continuous $f(t)$ defined for $t \geq 0$ is

$$\mathcal{L}[f(t)] = \int_0^\infty \underline{e^{-st}}\, \underline{f(t)}\, dt$$

if the integral converges.

- Notation
  - Lower case for original function $f(t)$ time
  - Upper case for Laplace-transformed function $\mathcal{L}[f(t)] \equiv F(s)$ Laplace

# Laplace transform simplifies solving ODE to solving algebraic equations

Time domain

$f, f(t)$

Laplace domain

$s, F(s)$

$$\begin{tikzcd}
\text{difficult ODE} \arrow[r, "\text{difficult}"] \arrow[d, "\text{Laplace transform}"'] & \text{soln of ODE} \\
\text{easy algebraic problem} \arrow[r, "\text{easy}"] & \text{soln to algebraic problem} \arrow[u, "\text{Inverse Laplace Transform}"']
\end{tikzcd}$$

# Laplace transform of a constant

$f(t) = c$

**Zill Ex. 4.1.1** Evaluate $\mathcal{L}[c]$, where $c$ is a constant.

$$\mathcal{L}[f(t)] = \int_0^\infty f(t)\, e^{-st}\, dt$$

$$\mathcal{L}[c] = \int_0^\infty c\, e^{-st}\, dt$$

$$= c \int_0^\infty e^{-st}\, dt$$

$$= c \int_0^{-\infty} e^{u}\, \frac{du}{-s}$$

$u = -st$ $\qquad u(\infty) = -\infty$

$du = -s\, dt$ $\qquad u(0) = 0$

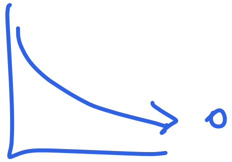
$\frac{du}{-s} = dt$

$$= \frac{c}{-s} \left[ e^{u} \right]_0^{-\infty}$$

$$= \frac{c}{-s} \left[ \cancel{e^{-\infty}}^{\,0} - e^{0} \right]$$

$$= \frac{c}{-s} (-1)$$

$$= \boxed{\frac{c}{s}}$$




$c = 1: \ \mathcal{L}[1] = \frac{1}{s}$

$c = -2: \ \mathcal{L}[-2] = -\frac{2}{s}$

# Laplace transform of a linear function

**Zill Ex. 4.1.2** Evaluate $\mathcal{L}[t]$.

$$\mathcal{L}[t] = \int_0^\infty t e^{-st}\, dt$$

$u = t$

$du = dt$

$dv = e^{-st}$

$v = \int e^{-st}\, dt = -\frac{1}{s} e^{-st}$

$$= \left[ (t)\left(-\frac{1}{s} e^{-st}\right) \right]_0^\infty - \int_0^\infty -\frac{1}{s} e^{-st}\, dt$$

$$= (0 - 0) + \frac{1}{s} \int_0^\infty e^{-st}\, dt$$

$$= \frac{1}{s} \left[ -\frac{1}{s} e^{-st} \right]_0^\infty$$

$$= -\frac{1}{s^2} \left[ \cancel{e^{-s(\infty)}} \xrightarrow{} 0 - \cancel{e^{-s(0)}} \xrightarrow{} 1 \right]$$

$$= \boxed{\frac{1}{s^2}}$$

# Laplace transform of a exponential function

**Zill Ex. 4.1.3a** Evaluate $\mathcal{L}[e^{-3t}]$.

$u = -(s+3)t$

$du = -(s+3)\,dt$

$\frac{du}{-(s+3)} = dt$

$$\mathcal{L}[e^{-3t}] = \int_0^\infty e^{-3t} e^{-st}\,dt = \int_0^\infty e^{-3t-st}\,dt = \int_0^\infty e^{-(s+3)t}\,dt$$

$$= \int_0^\infty e^{u} \frac{du}{-(s+3)} = -\frac{1}{s+3}\left[e^{u}\right]_{t=0}^{t=\infty} = -\frac{1}{s+3}\left[e^{-(s+3)t}\right]_0^\infty$$

$$= -\frac{1}{s+3}\left[\cancel{e^{-(s+3)(\infty)}}^{\,0} - \cancel{e^{-(s+3)(0)}}^{\,1}\right] = \boxed{\frac{1}{s+3}}$$

$e^{-\infty} \to 0$

exponent $< 0$

$-(s+3)t < 0$

$s+3 > 0$

$\boxed{s > -3}$ constraint

**Zill Ex. 4.1.3b** Evaluate $\mathcal{L}[e^{6t}]$.

$$\mathcal{L}[e^{6t}] = \int_0^\infty e^{6t} e^{-st}\,dt = \int_0^\infty e^{-(s-6)t}\,dt$$

$u = -(s-6)t$

$du = -(s-6)\,dt$

$\frac{du}{-(s-6)} = dt$

$$= \int_0^\infty e^{u} \frac{du}{-(s-6)} = \frac{1}{-(s-6)}\left[e^{u}\right]_{t=0}^{t=\infty}$$

$$= \frac{1}{-(s-6)}\left[e^{-(s-6)t}\right]_0^\infty$$

$$= -\frac{1}{s-6}\left[\cancel{e^{-\infty}}^{\,0} - \cancel{e^{0}}^{\,1}\right]$$

$$= \boxed{\frac{1}{s-6}}$$

exponent $< 0$

$-(s-6)t < 0$

$s-6 > 0$

$\boxed{s > 6}$ constraint

# Laplace transform of a sine function

**Zill Ex. 4.1.4** Evaluate $\mathcal{L}[\sin(2t)]$.

$u = \sin(2t) \qquad dv = e^{-st}$

$du = 2\cos(2t)\,dt \qquad v = -\frac{1}{s}e^{-st}$

$$\mathcal{L}[\sin(2t)] = \underline{\int_0^\infty \sin(2t)e^{-st}\,dt}$$

$$= \left[\sin(2t)\left(-\frac{1}{s}e^{-st}\right)\right]_0^\infty - \int_0^\infty -\frac{1}{s}e^{-st}\,2\cos(2t)\,dt$$

$$= [0 - 0] + \frac{2}{s}\int_0^\infty e^{-st}\cos(2t)\,dt$$

$u = \cos(2t) \qquad dv = e^{-st}$

$du = -2\sin(2t)\,dt \qquad v = -\frac{1}{s}e^{-st}$

$$= \frac{2}{s}\left[\left[\cos(2t)\left(-\frac{1}{s}e^{-st}\right)\right]_0^\infty - \int_0^\infty -\frac{1}{s}e^{-st}\left(-2\sin(2t)\,dt\right)\right]$$

$$= \frac{2}{s}\left[\left[0 + (1)\frac{1}{s}\right] \cancel{+} - \frac{2}{s}\int_0^\infty e^{-st}\sin(2t)\,dt\right]$$

$$= \frac{2}{s}\left[\frac{1}{s} - \frac{2}{s}\underline{\int_0^\infty e^{-st}\sin(2t)\,dt}\right] = \frac{2}{s^2} - \underline{\frac{4}{s^2}\mathcal{L}[\sin(2t)]}$$

constraint $s$:

$s^2+4 > 0 \quad \underline{s \text{ all}}$

$1/s \to \underline{s \neq 0}$

$e^{-st}\big]^\infty \to 0$

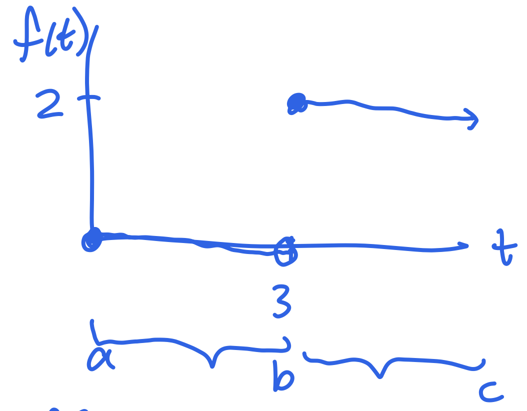
$-st < 0$

$\boxed{s < 0}$

$$\left[1 + \frac{4}{s^2}\right]\mathcal{L} = \frac{2}{s^2} \quad \Rightarrow \quad \mathcal{L}[\sin(2t)] = \frac{2/s^2}{1 + 4/s^2}\cdot\frac{s^2}{s^2} = \boxed{\frac{2}{s^2+4}}$$

# Laplace transform of a piecewise-continuous function

**Zill Ex. 4.1.6** Evaluate $\mathcal{L}[f(t)]$ for $f(t) \begin{cases} 0 & t \in [0,3) \\ 2 & t \in [3,\infty) \end{cases}$




$$\mathcal{L}[f(t)] = \int_0^\infty \begin{cases} 0 & t \in [0,3) \\ 2 & t \in [3,\infty) \end{cases} e^{-st}\,dt$$

$$= \int_0^3 \cancel{0 \cdot e^{-st}}\to 0 \; dt + \int_3^\infty 2\, e^{-st}\,dt$$

$$= 0 + 2\left[-\frac{1}{s} e^{-st}\right]_3^\infty$$

$$= -\frac{2}{s}\left[0 - e^{-3s}\right]$$

$$= \boxed{\frac{2}{s} e^{-3s}}$$

$dt \quad \int_a^c dt = \int_a^b dt + \int_b^c dt$

Constraint of s

- s anything
- $1/s \Rightarrow s \neq 0$
- $e^{-st}\Big]^\infty \to 0 \Rightarrow -st < 0$ $\boxed{s > 0}$

# Common Laplace transforms are tabulated

Use Seaborg Table 3.1

| Inverse L.T. $f(t)$ | Laplace Transform $F(s)$ | Inverse L.T. $f(t)$ | Laplace Transform $F(s)$ |
|---|---|---|---|
| $\mathcal{L}[1]$ | $\frac{1}{s}$ | $e^{at}$ | $\frac{1}{s-a}$ |
| $\mathcal{L}[t^n]$ | $\frac{n!}{s^{n+1}}$ | $\sqrt{t}$ | $\frac{\sqrt{\pi}}{2s^{3/2}}$ |
| $\sin(at)$ | $\frac{a}{s^2+a^2}$ | $t\sin(at)$ | $\frac{2as}{(s^2+a^2)^2}$ |
| $\cos(at)$ | $\frac{s}{s^2+a^2}$ | $t\cos(at)$ | $\frac{s^2-a^2}{(s^2+a^2)^2}$ |

# Linearity simplifies problem

- **Linear combination** - multiplication of terms by constant and/or addition of terms

const : $c, d$
term : $f(t), g(t)$

- $cf(t)$
- $df(t)$
- $cg(t)$
- $dg(t)$

- $cf(t) + g(t)$
- $f(t) + dg(t)$
- $\boxed{cf(t) + dg(t)}$

- $f(t) + g(t)$

$c_1 f_1(t) + c_2 f_2(t) + \dots + c_n f_n(t)$
$= \boxed{\sum_i c_i f_i(t)}$

- **Linearity** - transform of a linear combination = linear combination of the transforms
  - $T[\alpha f(x) + \beta g(x)] = \alpha T[f(x)] + \beta T[g(x)]$

- $\frac{d}{dx}[\alpha f(x) + \beta g(x)] = \frac{d}{dx}[\alpha f] + \frac{d}{dx}[\beta g] = \alpha \frac{df}{dx} + \beta \frac{dg}{dx}$ → deriv. is linear

- $\int \alpha f(x) + \beta g(x)\, dx = \alpha \int f(x)\,dx + \beta \int g(x)\,dx$ → indefinite int is linear

- $\int_a^b \alpha f(x) + \beta g(x)\, dx = \alpha \int_a^b \alpha f(x)\,dx + \beta \int_a^b g(x)\,dx$ → definite int is linear

$\mathcal{L}[f(t)] = \int_0^\infty f(t) e^{-st}\,dt$

# Laplace transform is linear

**Ex.** Demonstrate Laplace transform is linear.

- Laplace transform is an integral transform
  - Definition: $\mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t) dt$

$$\boxed{\mathcal{L}[\alpha f(t) + \beta g(t)] = \alpha \mathcal{L}[f(t)] + \beta \mathcal{L}[g(t)]}$$

$$\mathcal{L}[\alpha f(t) + \beta g(t)]$$

$$= \int_0^\infty (\alpha f(t) + \beta g(t)) e^{-st} dt$$

$$= \int_0^\infty \alpha f(t) e^{-st} + \beta g(t) e^{-st} dt$$

$$= \int_0^\infty \alpha f(t) e^{-st} dt + \int_0^\infty \beta g(t) e^{-st} dt$$

$$= \alpha \underbrace{\int_0^\infty f(t) e^{-st} dt} + \beta \underbrace{\int_0^\infty g(t) e^{-st} dt}$$

$$= \alpha \mathcal{L}[f(t)] + \beta \mathcal{L}[g(t)]$$

# Laplace transform is linear

$$\mathcal{L}[\alpha f(t) + \beta g(t)]$$
$$= \alpha \mathcal{L}[f(t)] + \beta \mathcal{L}[g(t)]$$
$$= \alpha F(s) + \beta G(s)$$

$\mathcal{L}[f(t)] = F(s)$

$\mathcal{L}[g(t)] = G(s)$

**Zill Ex. 4.1.5a** Use linearity of Laplace transform to evaluate $\mathcal{L}[1 + 5t]$

$$\mathcal{L}[1+5t] = \mathcal{L}[1] + \mathcal{L}[5t]$$
$$= \mathcal{L}[1] + 5\mathcal{L}[t]$$
$$= \frac{1}{s} + 5\frac{1}{s^2}$$