Transfer Functions of Multivariable ODEs

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Transfer functions of multivariable ODEs show linearity

total

Seborg Ex. 4.4a Given constant liquid density ρ, volume V, mass flow rates w_1, w_2, and w, the governing equation for a continuous blending process is

const

original ODE $$\rho V \frac{dx}{dt} = w_1 x_1 + w_2 x_2 - wx$$

out in const

where x are compositions. Determine the transfer function for output x and varying input x_1 while x_2 is held constant.

steady state ODE $0 = w_1 \bar{x}_1 + w_2 x_2 - w\bar{x}$

$y' = y - \bar{y}$

deviation ODE $\rho V \frac{dx}{dt} = w_1(x_1 - \bar{x}_1) + w_2\underbrace{\cancel{(x_2 - x_2)}}_{0} - w(x - \bar{x})$

$$\boxed{\rho V \frac{dx'}{dt} = w_1 x_1' - w x'}$$

$\frac{dx'}{dt} = \frac{d}{dt}(x - \bar{x}) = \frac{dx}{dt}$

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Seborg Ex. 4.4a (cont.)

$$\mathcal{L}\left[\rho V \frac{dx'}{dt} = w_1 x_1' - w x'\right]$$

init cond

$x'(0) = 0$

s.s deviation is 0

$$\rho V \left[s X'(s) - \cancel{x'(0)}^{\to 0} \right] = w_1 X_1'(s) - w X'(s)$$

$$\rho V s X'(s) + w X'(s) = w_1 X_1'(s)$$

$$G(s) = \frac{\text{out}}{\text{in}} = \frac{X'(s)}{X_1'(s)} = \boxed{\frac{w_1}{\rho V s + w}} \frac{1/w}{1/w} = \frac{K}{\tau s + 1}$$

$$\frac{w_1/w}{\frac{\rho V}{w} s + 1} = \boxed{\frac{K}{\tau s + 1} \quad \left\{ \begin{array}{l} K \equiv \frac{w_1}{w} \\ \tau \equiv \frac{\rho V}{w} \end{array} \right.} \quad \checkmark$$

Transfer functions of multivariable ODEs show linearity

Seborg Ex. 4.4a (cont.)

Transfer functions of multivariable ODEs show linearity

Seborg Ex. 4.4b Given constant liquid density ρ, volume V, mass flow rates w_1, w_2, and w, the governing equation for a continuous blending process is

original

$$\rho V \frac{dx}{dt} = w_1 x_1 + w_2 x_2 - wx$$

out

input

where x are compositions. Determine the transfer function for output x and varying inputs x_1 and x_2.

s.s. $0 = w_1 \bar{x}_1 + w_2 \bar{x}_2 - w\bar{x}$

dev. $\rho V \frac{dx}{dt} = w_1(x_1 - \bar{x}_1) + w_2(x_2 - \bar{x}_2) - w(x - \bar{x})$

$$\boxed{\rho V \frac{dx'}{dt} = w_1 x_1' + w_2 x_2' - w x'}$$

$$\frac{dx'}{dt} = \frac{d}{dt}(x(t) - \bar{x}) = \frac{dx}{dt}$$

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Seborg Ex. 4.4b (cont.)

$$\mathcal{L}\left[\rho V \frac{dx'}{dt} = w_1 x_1' + w_2 x_2' - w x'\right]$$

out: X'

in : X_1', X_2'

init cond of dev is 0 at s.s

$x'(0) = 0$

$$\rho V\left[sX'(s) - \cancel{x'(0)}^{\to 0}\right] = w_1 X_1' + w_2 X_2' - w X'$$

$$\rho V s X'(s) + w X'(s) = w_1 X_1' + w_2 X_2'$$

$$X'(s) = \frac{w_1 X_1' + w_2 X_2'}{\rho V s + w} \frac{1/w}{1/w} = \frac{\frac{w_1}{w} X_1' + \frac{w_2}{w} X_2'}{\frac{\rho V}{w} s + 1}$$

$$\begin{cases} K_1 \equiv \frac{w_1}{w} \\ K_2 \equiv \frac{w_2}{w} \\ \tau_1 = \tau_2 = \frac{\rho V}{w} \end{cases}$$

$$X'(s) = \frac{\frac{w_1}{w} X_1'}{\frac{\rho V}{w} s + 1} + \frac{\frac{w_2}{w} X_2'}{\frac{\rho V}{w} s + 1} = \frac{K_1 X_1'}{\tau_1 s + 1} + \frac{K_2 X_2'}{\tau_2 s + 1}$$

$$X'(s) = \frac{K_1 X_1'}{\tau_1 s + 1} + \frac{K_2 X_2'}{\tau_2 s + 1}$$

Transfer functions of multivariable ODEs show linearity

Seborg Ex. 4.4b (cont.)

$$X'(s) = \frac{K_1 X_1'}{\tau_1 s+1} + \frac{K_2 X_2'}{\tau_2 s+1} = G_1(s) X_1'(s) + G_2(s) X_2'(s)$$

If $x_2'(t)=0$, $X_2'(s)=0$
$x_1'(t) \neq 0$ $X_1'(s) \neq 0$: $X'(s) = \dfrac{K_1 X_1'}{\tau_1 s+1}$

$$G_1(s) = \frac{X'(s)}{X_1'(s)} = \boxed{\frac{K_1}{\tau_1 s+1}}$$

If $x_1'(t)=0$, $X_1'(s)=0$
$x_2'(t) \neq 0$, $X_2'(s) \neq 0$: $X'(s) = \dfrac{K_2 X_2'}{\tau_2 s+1}$

$$G_2(s) = \frac{X'(s)}{X_2'(s)} = \boxed{\frac{K_2}{\tau_2 s+1}}$$

Linearity and input-output relationship

Linear

$$X'(s) = \underbrace{G_1(s)X_1'(s)}_{\text{effect of } x_1(t) \text{ on } x(t)} + \underbrace{\cancel{G_2(s)X_2'(s)}}_{\text{effect of } x_2(t) \text{ on } x(t)} + \cancel{G_3(s)X_3'(s)}$$

$\cancel{G_2(s)X_2'(s)} \to 0$, $\cancel{G_3(s)X_3'(s)} \to 0$

linear!

$$\underset{\text{out}}{X'(s)} = \underset{\text{transfer}}{G_1(s)}\ \underset{\text{in}}{X_1'(s)}$$

in-out relationship

in $\mathcal{L}$ space

(s)