Motivating Transfer Functions

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Laplace-transformed output in original variables are hard to find

Ex. Find the Laplace-transformed output $Y(s)$ of the ODE

$$\mathcal{L}\left[\frac{dy}{dt} = x - y\right]$$

$X(s) = \mathcal{L}[x(t)]$

$Y(s) = \mathcal{L}[y(t)]$

where x is the input and y is the output.

$$sY(s) - y(0) = X(s) - Y(s)$$

$$sY(s) + Y(s) = X(s) - y(0)$$

$$Y(s) = \frac{X(s) - y(0)}{s+1}$$

← often nonzero

$X(s)$ not separable

– cannot write in $\frac{Y(s)}{X(s)}$

Deviation variables have vanishing initial condition at steady-state in $\mathcal{L}$ space

Ex. Find the Laplace-transformed output deviation $Y'(s)$ of the ODE

$y^{(1)} = \frac{dy}{dx}$

$$\frac{dy}{dt} = x - y$$

e.g. $+2°F = 72°F - 70°F$

deviation variable, phys, nominal steady state value

where x is the input and y is the output, by defining deviation variable of $\boxed{y' = y - \overline{y}}$.

- Derive governing ODE
- Identify variables
  - Input variable (cause) x
  - Output variable (effect) y
- Derive steady-state expression $\frac{d}{dt} = 0$ $\quad 0 = \overline{x} - \overline{y}$
- Derive ODE in deviation variables
- Get transfer function in $\mathcal{L}$ space

$\frac{dy}{dt} - 0 = (x - \overline{x}) - y + \overline{y}$

$\frac{dy}{dt} = (x - \overline{x}) - (y - \overline{y})$ $\qquad \frac{dy'}{dt} = \frac{d}{dt}(y - \overline{y}) = \frac{dy}{dt}$

$\frac{dy}{dt} = x' - y'$

⚠ $\frac{dy}{dt} = x - y$

$\frac{dy'}{dt} = x' - y'$

$\boxed{\frac{dy'}{dt} = x' - y'}$ ODE in deviation space

Deviation variables have vanishing initial condition at steady-state in $\mathcal{L}$ space

Ex. Find the Laplace-transformed output deviation $Y'(s)$ of the ODE

$$\frac{dy}{dt} = x - y$$

where x is the input and y is the output, by defining deviation variable of $y' = y - \bar{y}$.

$$\mathcal{L}\left[\frac{dy'}{dt} = x' - y'\right]$$

$$sY'(s) - \cancel{y'(0)}^{\,0} = X'(s) - Y'(s)$$

$$sY'(s) + Y'(s) = X'(s)$$

$$Y'(s) = \frac{X'(s)}{s+1}$$

no $y'(0)$

$X'(s)$ can be sep.

* $y'(0) = y(0) - \bar{y}$

normally, init would be at nominal steady state: $y(0) = \bar{y}$

$\Rightarrow y'(0) = 0$

no deviation

$$s+1 = \frac{X'(s)}{Y'(s)}$$

$\leftarrow$ scales between input & output

$$\frac{Y'(s)}{X'(s)} = \frac{1}{s+1}$$

Transfer function maps input-output relationship in the $\mathcal{L}$ space

- **Transfer function** - characterizes dynamic relationship of input and output variables in the Laplace space

transfer func

e.g. original deviation (s)

$$\boxed{G(s) = \frac{\mathcal{L}[\text{output}]}{\mathcal{L}[\text{input}]}} \cong \frac{Y(s)}{X(s)} \cong \frac{Y'(s)}{X'(s)}$$

- **Deviation variable** - deviation from a nominal steady-state
  - Have vanishing initial condition at steady-state in $\mathcal{L}$ space

$$y' = y(t) - \bar{y}$$

$$y(0) = \bar{y}$$

$$y' = \bar{y} - \bar{y} = 0$$

$$G(s) = \frac{Y'(s)}{X'(s)} = \frac{1}{s+1}$$

$$\begin{tikzcd}
\text{ODE} \arrow[r, "\text{diff.}"] \arrow[d, "\mathcal{L}"'] & \text{soln ODE} \\
\text{algeic } G(s) \arrow[r, "\text{easy}"] & \text{soln algeic} \arrow[u, "\mathcal{L}^{-1}"']
\end{tikzcd}$$

Laplace transform of multivariable ODE show linearity of transfer function

Ex. Find the Laplace-transformed output deviation $Z'(s)$ of the ODE

$$\frac{dz}{dt} = x - y - z$$

original ODE

where x and y are the inputs and z is the output.

1. steady-state $\frac{d}{dt} = 0$; bars for in & outputs

$0 = \bar{x} - \bar{y} - \bar{z}$ steady-state ODE

$\frac{dz}{dt} = (x - \bar{x}) - (y - \bar{y}) - (z - \bar{z})$

$\frac{dz'}{dt} = \frac{d}{dt}(z(t) - \bar{z}) = \frac{dz}{dt}$

$\frac{dz}{dt} = x' - y' - z'$

2. get dev. ODE

$\mathcal{L}\left[\boxed{\frac{dz'}{dt} = x' - y' - z'} \right]$ deviation ODE

3. get $Z'(s)$

$s Z'(s) - \cancelto{0}{z'(0)} = X' - Y' - Z'$ • $z'(0) = 0$

$Z'(s) = \frac{X' - Y'}{s+1} = \frac{X'}{s+1} - \frac{Y'}{s+1}$

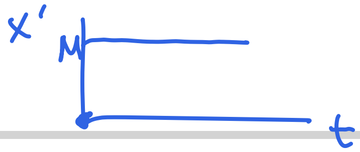
If hold $y(t)$ const.

$\Rightarrow$ dev. $= 0$, $Y'(s) = 0$

$Z'(s) = \frac{X'}{s+1}$

$\frac{Z'(s)}{X(s)} = \frac{1}{s+1}$

Transfer function can predict step change



Ex. Determine the response $y(t)$ to a step change in x of magnitude M at $t = 0$ using the transfer function

$$G(s) = \frac{1}{s+1} = \frac{Y'(s)}{X'(s)}$$

- Rearrange for $Y'(s)$ $\quad Y'(s) = G(s)\,X'(s) = \frac{1}{s+1}\,\frac{M}{s}$
- Determine $X'(s)$ from $x'(t)$ $\quad x'(t) = M \Rightarrow \mathcal{L}[x'(t)] = X'(s) = \frac{M}{s}$
- Get deviation in time space $\quad \mathcal{L}^{-1}[Y'(s)] = y'(t) = \mathcal{L}^{-1}\left[\frac{M}{(s+1)s}\right]$ $\quad b_1 = 1$, $b_2 = 0$
- Get response in time space

$$= M\frac{1}{1-0}\left(e^{0} - e^{-t}\right)$$

$$= M(1 - e^{-t})$$

$$\mathcal{L}^{-1}\left[\frac{1}{(s+b_1)(s+b_2)}\right] = \frac{1}{b_1 - b_2}\left(e^{-b_2 t} - e^{-b_1 t}\right)$$

$$y'(t) = y(t) - \bar{y}$$

$$y(t) = y'(t) + \bar{y} = \boxed{M(1-e^{-t}) + \bar{y}}$$

Obtaining transfer function from ODE

(cause) output (cause) input

Seborg Ex. 4.1a Find the transfer function model between liquid level h and inlet flow rate q_i in deviation variables with governing ODE of const. A, R_v

$$A\frac{dh}{dt} = q_i - \frac{1}{R_v}h \quad \checkmark$$

original

- Derive steady-state expression

$\frac{d}{dt} = 0$

$$0 = \bar{q}_i - \frac{1}{R_v}\bar{h}$$

S.S

- Derive ODE in deviation variables

$$A\frac{dh}{dt} = (q_i - \bar{q}_i) - \frac{1}{R_v}(h - \bar{h})$$

$$\frac{dh'}{dt} = \frac{d}{dt}(h(t) - \bar{h}) = \frac{dh}{dt}$$

$$\boxed{A\frac{dh'}{dt} = q_i' - \frac{1}{R_v}h'}$$

deviation

Obtaining transfer function from ODE

Seborg Ex. 4.1a Find the transfer function model between liquid level h and inlet flow rate q_i in deviation variables with governing ODE of

$$A\frac{dh}{dt} = q_i - \frac{1}{R_v}h$$

- Know ODE in deviation variables

$$\mathcal{L}\left[\, A\frac{dh'}{dt} = q_i' - \frac{1}{R_v}h' \,\right]$$

- Get transfer function in $\mathcal{L}$ space
  - Initial condition

$$A\left[sH'(s) - \cancelto{0}{h'(0)}\right] = Q_i'(s) - \frac{1}{R_v}H'(s)$$

$$h'(0) = h(0) - \bar{h}$$

$$h(0) = \bar{h} \;\Rightarrow\; = 0$$

$$AsH'(s) + \frac{1}{R_v}H'(s) = Q_i'(s)$$

$$H'(s) = \frac{Q_i'(s)}{As + \frac{1}{R_v}} \cdot \frac{R_v}{R_v} = \frac{Q_i'(s)\,R_v}{AR_v s + 1}$$

$$G(s) = \frac{\text{out}}{\text{in}} = \frac{H'(s)}{Q_i'(s)} = \boxed{\frac{R_v}{AR_v s + 1}} = \frac{K}{\tau s + 1} \qquad \begin{cases} K \equiv R_v \\ \tau = AR_v \end{cases}$$

Use transfer function to predict step change

Seborg Ex. 4.1b Determine the response $\overset{\text{out}}{h(t)}$ to a step change in $\overset{\text{input}}{q_i}$ of magnitude M at $t = 0$ using the transfer function

$$G(s) = \frac{H'(s)}{Q_i'(s)} = \frac{R_v}{AR_v s + 1}$$

- Rearrange for $\overset{\text{out}}{H'(s)}$ $\quad H'(s) = \overset{\checkmark}{G(s)}\,\overset{\checkmark}{Q_i'(s)} = \frac{R_v}{AR_v s+1} \cdot \frac{M}{s}$
- Determine $Q_i'(s)$ from $q_i'(t)$ $\quad q_i'(t) = M \Rightarrow Q_i'(s) = \frac{M}{s}$
- Get deviation response in time space $\quad h'(t) = \mathcal{L}^{-1}\left[H'(s)\right] = \mathcal{L}^{-1}\left[\frac{R_v M}{(AR_v s+1)s}\right] = \mathcal{L}^{-1}\left[\frac{R_v M/AR_v}{(s+\frac{1}{AR_v})s}\right]$
- Get response in time space

$$= \frac{\cancel{R_v} M}{A\cancel{R_v}} \frac{1}{\frac{1}{AR_v}-0}\left(e^{0} - e^{-t/AR_v}\right)$$

$$= \frac{M}{\cancel{A}}\cancel{A}R_v\left(1 - e^{-t/AR_v}\right)$$

$$= MR_v\left(1 - e^{-t/AR_v}\right)$$

$$h(t) = h'(t) + \bar{h} = \boxed{MR_v\left(1 - e^{-t/AR_v}\right) - \bar{h}}$$

$$\mathcal{L}^{-1}\frac{1}{(s+b_1)(s+b_2)}$$

$$= \frac{1}{b_1-b_2}\left(e^{-b_2 t} - e^{-b_1 t}\right)$$

$b_1 = \frac{1}{AR_v}$

$b_2 = 0$