Solving ODEs with Laplace Transforms

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Laplace transform is the alternative path to solve complex ODEs

"real" world
time domain

difficult ODE $\xrightarrow{\text{difficult}}$ soln of ODE

original

alternative

$\mathcal{L}$
tablulated

$\mathcal{L}^{-1}$
tabulated

Laplace domain
s domain

Easy algebraic problem $\xrightarrow{\text{easier}}$ soln of algebraic problem

- write expression in terms of tabulated $\mathcal{L}^{-1}$
- partial fraction

Initial value problem for first-order linear ODE

Zill Ex. 4.2.4 Use Laplace transform to solve the initial value problem

$$\mathcal{L}\left[\frac{dy}{dt} + 3y = 13\sin(2t)\right], \quad y(0) = 6$$

$$\begin{array}{ccc} \text{ODE} & \longrightarrow & \text{soln ODE} \\ \boxed{1}\ \mathcal{L} \downarrow & & \uparrow \mathcal{L}^{-1}\ \boxed{3} \\ \text{algebra} & \longrightarrow & \text{soln algebra} \\ \boxed{2} & & \end{array}$$

$\boxed{1}$

$$\mathcal{L}\left[\frac{dy}{dt}\right] + 3\,\mathcal{L}[y] = 13\,\mathcal{L}[\sin(2t)]$$

$$\mathcal{L}(\sin(at)) = \frac{a}{s^2+a^2}$$

$a = 2$

$$[sY(s) - y(0)] + 3Y(s) = 13\,\frac{2}{s^2+4}$$

$$(s+3)Y(s) - 6 = \frac{26}{s^2+4}$$

Algebra:

$$Y(s) = \frac{\frac{26}{s^2+4} + 6}{s+3} = \frac{26}{(s^2+4)(s+3)} + \frac{6}{s+3}$$

Initial value problem for first-order linear ODE

Zill Ex. 4.2.4 Use Laplace transform to solve the initial value problem

$$\frac{dy}{dt} + 3y = 13\sin(2t), \quad y(0) = 6$$

$$\begin{tikzcd} \text{ODE} \arrow[r] \arrow[d, "\boxed{1}\ \mathcal{L}"'] & \text{soln ODE} \\ \text{algebra} \arrow[r, "\boxed{2}"'] & \text{soln algebra} \arrow[u, "\mathcal{L}^{-1}\ \boxed{3}"'] \end{tikzcd}$$

$\boxed{2}$

$$Y(s) = \frac{26}{(s^2+4)(s+3)} + \frac{6}{s+3}$$

$$\frac{26}{(s^2+4)(s+3)} = \frac{As+B}{s^2+4} + \frac{C}{s+3} = \boxed{\frac{-2s+6}{s^2+4} + \frac{2}{s+3}}$$

$$= \frac{As+B}{s^2+4}\,\frac{s+3}{s+3} + \frac{C}{s+3}\,\frac{s^2+4}{s^2+4}$$

$$= \frac{(As^2+3As+Bs+3B) + Cs^2+4C}{(s^2+4)(s+3)}$$

$$26 = s^2\underbrace{(A+C)}_{=0} + s\underbrace{(3A+B)}_{=0} + \underbrace{(3B+4C)}_{=26}$$

$$\begin{cases} A+C=0 \\ 3A+B=0 \\ 3B+4C=26 \end{cases}$$

$A=-C$

$-3C+B=0$

$B=3C$

$9C+4C=26$

$13C=26$

$$\begin{cases} C=2 \\ B=6 \\ A=-2 \end{cases}$$

Initial value problem for first-order linear ODE

Zill Ex. 4.2.4 Use Laplace transform to solve the initial value problem

ODE $\longrightarrow$ soln ODE

$\boxed{1}$ $\mathcal{L} \downarrow$ $\quad$ $\uparrow \mathcal{L}^{-1}$ $\boxed{3}$

algebra $\longrightarrow$ soln algebra

$\boxed{2}$

$y_{general} + y_{part}$ $\quad$ $\downarrow$ non-homo term

$$\frac{dy}{dt} + 3y = 13\sin(2t), \quad y(0) = 6$$

$\boxed{3}$

$$Y(s) = \frac{26}{(s^2+4)(s+3)} + \frac{6}{s+3}$$

$$Y(s) = \boxed{\frac{-2s+6}{s^2+4} + \frac{2}{s+3}} + \frac{6}{s+3}$$

$$Y(s) = \frac{-2s+6}{s^2+4} + \frac{8}{s+3} = \frac{-2s}{s^2+4} + \frac{6}{s^2+4} + \frac{8}{s+3}$$

$$y(t) = \mathcal{L}^{-1}[Y(s)] = -2\mathcal{L}^{-1}\left(\frac{s}{s^2+4}\right) + 6\mathcal{L}^{-1}\left(\frac{1}{s^2+4}\frac{2}{2}\right) + 8\mathcal{L}^{-1}\frac{1}{s+3}$$

$$\boxed{y(t) = -2\cos(2t) + 6\sin(2t)\tfrac{1}{2} + 8e^{-3t}}$$

$$\mathcal{L}^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos(at)$$

$$\mathcal{L}^{-1}\left(\frac{a}{s^2+a^2}\right) = \sin(at)$$

$a^2 = 4$

$a = 2$

$$\mathcal{L}^{-1}\left(\frac{1}{s+a}\right) = e^{-at}$$

$a = 3$

Initial value problem for second-order linear ODE

Zill Ex. 4.2.5 Use Laplace transform to solve the initial value problem

$$\mathcal{L}\left[y'' - 3y' + 2y = e^{-4t}\right], \quad y(0) = 1, \quad y'(0) = 5$$

$$\begin{array}{ccc} \text{ODE} & \longrightarrow & \text{soln ODE} \\ \boxed{1}\ \mathcal{L} \downarrow & & \uparrow \mathcal{L}^{-1}\ \boxed{3} \\ \text{algebra} & \longrightarrow & \text{soln algebra} \\ \boxed{2} & & \end{array}$$

$\boxed{1}$ $\mathcal{L}[y''] - 3\mathcal{L}[y'] + 2\mathcal{L}[y] = \mathcal{L}[e^{-4t}]$

$\mathcal{L}[e^{-at}] = \frac{1}{s+a}$ $a = 4$

$[s^2 Y(s) - s y(0) - y'(0)] - 3[sY(s) - y(0)] + 2Y(s) = \frac{1}{s+4}$

$s^2 Y(s) - s - 5 - 3sY(s) + 3 + 2Y(s) = \frac{1}{s+4}$

$(s^2 - 3s + 2)Y(s) - s - 2 = \frac{1}{s+4}$

$Y(s) = \dfrac{\frac{1}{s+4} + s + 2}{s^2 - 3s + 2} = \dfrac{1}{(s+4)(s-1)(s-2)} + \dfrac{s+2}{(s-1)(s-2)} \cdot \dfrac{s+4}{s+4}$

$s^2 - 3s + 2$
$= (s-1)(s-2)$

$Y(s) = \dfrac{9 + s^2 + 6s}{(s+4)(s-1)(s-2)}$

Initial value problem for second-order linear ODE

Zill Ex. 4.2.5 Use Laplace transform to solve the initial value problem

$$y'' - 3y' + 2y = e^{-4t}, \quad y(0) = 1, \quad y'(0) = 5$$

$$\begin{tikzcd} \text{ODE} \arrow[r] \arrow[d, "\boxed{1}\ \mathcal{L}"'] & \text{soln ODE} \\ \text{algebra} \arrow[r, "\boxed{2}"'] & \text{soln algebra} \arrow[u, "\mathcal{L}^{-1}\ \boxed{3}"'] \end{tikzcd}$$

$\boxed{2}$

$$Y(s) = \frac{9 + s^2 + 6s}{(s+4)(s-1)(s-2)} = \frac{A}{s-4} + \frac{B}{s-1} + \frac{C}{s-2}$$

$s+4=0$, $s=-4$: $$A = \frac{9 + (-4)^2 + 6(-4)}{(-4-1)(-4-2)} = \frac{9+16-24}{(-5)(-6)} = \frac{1}{30}$$

$s-1=0$, $s=1$: $$B = \frac{9+1+6}{(1+4)(1-2)} = \frac{16}{-5}$$

$s-2=0$, $s=2$: $$C = \frac{9 + 2^2 + 6(2)}{(2+4)(2-1)} = \frac{9+4+12}{6} = \frac{25}{6}$$

$$Y(s) = \frac{1}{30}\frac{1}{s-4} - \frac{16}{5}\frac{1}{s-1} + \frac{25}{6}\frac{1}{s-2}$$

Initial value problem for second-order linear ODE

Zill Ex. 4.2.5 Use Laplace transform to solve the initial value problem

$$y'' - 3y' + 2y = e^{-4t}, \quad y(0) = 1, \quad y'(0) = 5$$

non-homo.

$y_{general} + y_{part}$

$$\text{ODE} \longrightarrow \text{soln ODE}$$

$$\boxed{1}\ \mathcal{L} \downarrow \qquad \uparrow \mathcal{L}^{-1}\ \boxed{3}$$

$$\text{algebra} \longrightarrow \text{soln algebra}$$

$$\boxed{2}$$

$\boxed{3}$

$$Y(s) = \frac{1}{30}\frac{1}{s-4} - \frac{16}{5}\frac{1}{s-1} + \frac{25}{6}\frac{1}{s-2}$$

$$\mathcal{L}^{-1}\left(\frac{1}{s+a}\right) = e^{-at}$$

$$y(t) = \mathcal{L}^{-1}[Y(s)] = \frac{1}{30}\mathcal{L}^{-1}\left[\frac{1}{s-4}\right] - \frac{16}{5}\mathcal{L}^{-1}\left[\frac{1}{s-1}\right] + \frac{25}{6}\mathcal{L}^{-1}\left[\frac{1}{s-2}\right]$$

$$\boxed{y(t) = \frac{1}{30}e^{4t} - \frac{16}{5}e^{t} + \frac{35}{6}e^{2t}}$$