Inverse Laplace Transform

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Definition of inverse Laplace transform

If $F(s)$ represents the Laplace transform of $f(t)$, then $f(t)$ is the **inverse Laplace transform** of $F(s)$.

- If $\mathcal{L}[f(t)] = F(s)$, then $f(t) = \mathcal{L}^{-1}[F(s)]$

- $\mathcal{L}^{-1}[\mathcal{L}[f(t)]] = f(t)$

Inverse Laplace transform is also linear.

$$\mathcal{L}^{-1}[\alpha F(s) + \beta G(s)] = \alpha\mathcal{L}^{-1}[F(s)] + \beta\mathcal{L}^{-1}[G(s)]$$

Common inverse Laplace transforms are tabulated

Use Seaborg Table 3.1... in reverse

Inverse L.T. $f(t)$	**Laplace Transform** $F(s)$	**Inverse L.T.** $f(t)$	**Laplace Transform** $F(s)$
1	$\frac{1}{s}$	e^{at}	$\frac{1}{s-a}$
t^n	$\frac{n!}{s^{n+1}}$	$\sqrt{t}$	$\frac{\sqrt{\pi}}{2s^{3/2}}$
$\sin(at)$	$\frac{a}{s^2+a^2}$	$t\sin(at)$	$\frac{2as}{(s^2+a^2)^2}$
$\cos(at)$	$\frac{s}{s^2+a^2}$	$t\cos(at)$	$\frac{s^2-a^2}{(s^2+a^2)^2}$

Applying inverse Laplace transform

Zill Ex. 4.2.1a Evaluate $\mathcal{L}^{-1}\left[\dfrac{1}{s^5}\right]$

Recall $t^n = \mathcal{L}^{-1}\left[\dfrac{n!}{s^{n+1}}\right]$

Zill Ex. 4.2.1b Evaluate $\mathcal{L}^{-1}\left[\dfrac{1}{s^2+7}\right]$

Recall $\sin(at) = \mathcal{L}^{-1}\left[\dfrac{a}{s^2+a^2}\right]$

Term-wise division and linearity

Zill Ex. 4.2.2 Evaluate $\mathcal{L}^{-1}\left[\dfrac{-2s+6}{s^2+4}\right]$

Partial fractions and linearity

Ex. Evaluate $\mathcal{L}^{-1}\left[\frac{1}{(s-1)(s+4)}\right]$

Partial fraction by Heaviside expansion (cover-up method)

Ex. Evaluate $\mathcal{L}^{-1}\left[\dfrac{1}{(s-1)(s+4)}\right]$ with Heaviside expansion.

1. Set up partial fraction expansion
2. Multiply both sides by one denominator term $(s+a)$
3. Evaluate at $s=-a$
4. Solve for coefficient

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3. The result is the coeff for the term

Limitation: Rational function of s with distinct linear factors in denominator

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Cover-up method is great for more complicated expressions

Zill Remarks 4.2.ii Evaluate $\mathcal{L}^{-1}\left[\dfrac{s^2+6s+9}{(s-1)(s-2)(s+4)}\right]$ with Heaviside expansion.