Derivation of the Wicking Equation for Inclined Capillary

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Surface and Colloid Science

Derivation of wicking equation for inclined capillary

- Given the wicking distance

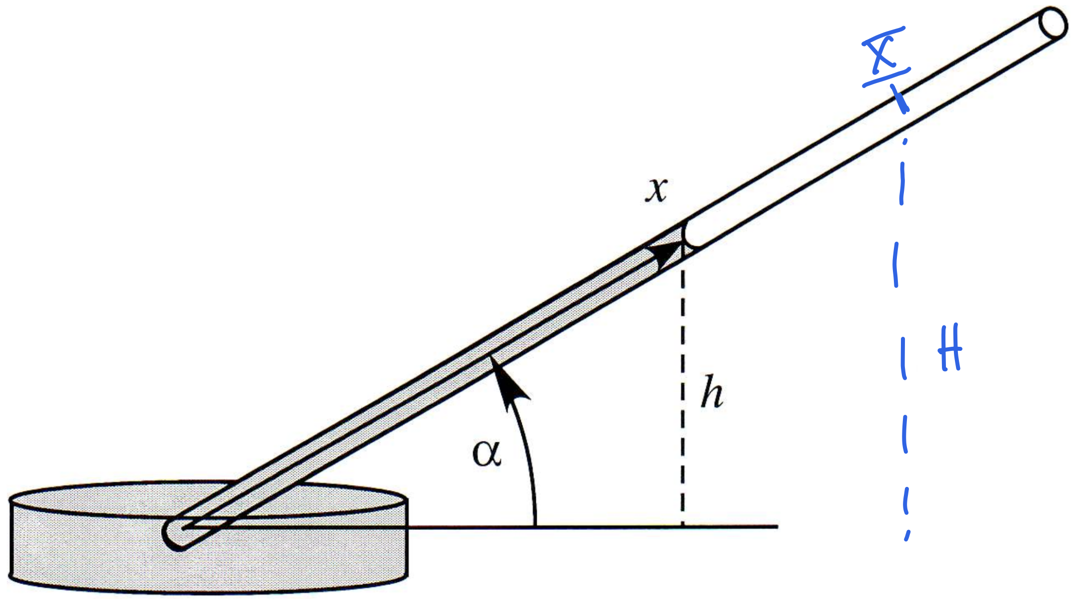
$$X = \frac{H}{\sin\alpha} = \frac{2\sigma\cos\theta}{\rho g r \sin\alpha}$$

- Use the Hagen-Poiseuille equation for inclined capillary

$$\frac{dx}{dt} = \frac{r^2}{8\mu}\left[\overbrace{\frac{2\sigma\cos\theta}{rx} - \rho g \sin\alpha}^{\Delta p = \Delta p_c - \rho g \sin\alpha}\right]$$

- Verify that the wicking equation for inclined capillary is

$$\boxed{t = \frac{8\mu X}{\rho g r^2}\frac{1}{\sin\alpha}\left[-\ln\left(1-\frac{x}{X}\right)-\frac{x}{X}\right]}$$ ✓



$$\frac{dx}{dt} = \frac{r^2}{8\mu}\left[\frac{2\sigma\cos\theta}{rx}\frac{\rho g \sin\alpha}{\rho g\sin\alpha} - \rho g \sin\alpha\right]$$

$$= \frac{r^2}{8\mu}\left[\frac{X}{x}\rho g\sin\alpha - \rho g \sin\alpha\right] = \frac{r^2}{8\mu}\rho g \sin\alpha\left[\frac{X}{x} - 1\right]$$

Derivation of wicking equation for inclined capillary

$$\frac{dx}{dt} = \frac{r^2}{8\mu}\rho g \sin\alpha\left[\frac{X}{x} - 1\right] = \frac{r^2}{8\mu}\rho g \sin\alpha\left[\frac{X - x}{x}\right]$$

$$\int dt = \int \underbrace{\frac{8\mu}{r^2}\frac{1}{\rho g \sin\alpha}}_{\text{const}} \frac{x}{X - x}\, dx$$

$$t = \frac{8\mu}{r^2}\frac{1}{\rho g \sin\alpha}\int \frac{x}{X - x}\, dx$$

$$'' \quad -X \ln(X - x) - x + c$$

$$'' \quad X\left[-\ln(X - x) - \frac{x}{X} + c\right]$$

$$'' \quad X\left[-\ln(X - x) - \frac{x}{X} + \ln(X)\right]$$

$$X\left[\ln\left(\frac{X}{X - x}\right) - \frac{x}{X}\right]$$

$$X\left[-\ln\left(1 - \frac{x}{X}\right) - \frac{x}{X}\right]$$

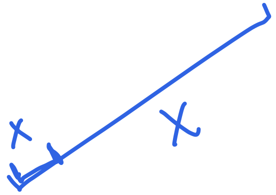
When $t = 0$, $x = 0$

$$t(0) = 0 = -\ln(X) + c$$

$$c = \ln(X)$$

$$\boxed{t = \frac{8\mu}{r^2}\frac{X}{\rho g \sin\alpha}\left[-\ln\left(1 - \frac{x}{X}\right) - \frac{x}{X}\right]}$$

Washburn equation is recovered at small x/X (far from equilibrium)



- Verify that at small $x/X < 0.3$, the wicking equation reduces to the Washburn equation

$$x = \sqrt{\frac{r\sigma\cos\theta}{2\mu}t}$$

- with Taylor series approximation

$$\ln(1-x) = \sum(-1)^n\frac{x^n}{n}$$

$$\boxed{t = \frac{8\mu}{r^2}\frac{X}{\rho g \sin\alpha}\left[-\ln\left(1-\frac{x}{X}\right)-\frac{x}{X}\right]}$$

$$t = \frac{8\mu}{r^2}\frac{X}{\rho g \sin\alpha}\left[-\left[(-1)^1\frac{\left(\frac{x}{X}\right)^1}{1} + (-1)^2\frac{\left(\frac{x}{X}\right)^2}{2} + O\left[\left(\frac{x}{X}\right)^3\right]\right]-\frac{x}{X}\right]$$

$$'' \quad \left[+\cancel{\frac{x}{X}} + \frac{1}{2}\left(\frac{x}{X}\right)^2 + O\left[\left(\frac{x}{X}\right)^3\right] - \cancel{\frac{x}{X}}\right]$$

$$'' \quad \left[\frac{1}{2}\left(\frac{x}{X}\right)^2 + O\left(\frac{x}{X}\right)^3\right]$$

Washburn equation is recovered at small x/X (far from equilibrium)

$$t = \frac{8\mu}{\rho g r^2} \frac{X}{\sin\alpha} \left[\frac{1}{2}\left(\frac{x}{X}\right)^2 + \cancel{O\left(\frac{x}{X}\right)^3}^{\,0} \right]$$

when $\frac{x}{X}$ is small, $O\left[\left(\frac{x}{X}\right)^3\right] \to 0$

$$t = \frac{4\mu}{\rho g r^2} \frac{1}{\sin\alpha} \frac{x^2}{X}$$

$$X = \frac{2\sigma\cos\theta}{\rho g r \sin\alpha}$$

$$t = \frac{2\cancel{4}\mu}{\cancel{\rho g} r^{\cancel{2}}} \frac{1}{\cancel{\sin\alpha}} x^2 \frac{\cancel{\rho g r}\, \cancel{\sin\alpha}}{\cancel{2}\sigma\cos\theta}$$

$$t = \frac{2\mu}{r} x^2 \frac{1}{\sigma\cos\theta}$$

$$x = \sqrt{\frac{t r \sigma\cos\theta}{2\mu}} = \sqrt{\frac{r\sigma\cos\theta}{2\mu} t} \Rightarrow \text{Washburn eqn}$$

At small x/X, gravitational effect is negligible: Washburn eqn for incline cap. $\Downarrow$ Washburn eqn for horizontal cap.