Laplace Transform of Derivatives

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Laplace transform of derivatives

Ex. Evaluate $\mathcal{L}[f'(t)]$

$u = e^{-st}$ $dv = f'(t)$

$du = -se^{-st}dt$ $v = f(t)$

$$\boxed{\mathcal{L}[f'(t)] = sF(s) - f(0)}$$

$\mathcal{L}$ incorporates init cond of $f(t)$

$$\mathcal{L}[f'(t)] = \int_0^\infty f'(t)e^{-st}\,dt$$

$$= \left[e^{-st}f(t)\right]_0^\infty + \int_0^\infty f(t)se^{-st}\,dt$$

$$= [0 - f(0)] + s\underbrace{\int_0^\infty f(t)e^{-st}\,dt}_{\mathcal{L}[f(t)]} = -f(0) + s\overbrace{\mathcal{L}[f(t)]}^{F(s)}$$

$$= sF(s) - f(0)$$

Ex. Evaluate $\mathcal{L}[f''(t)]$

$u = e^{-st}$ $dv = f''(t)$

$du = -se^{-st}dt$ $v = f'(t)$

$$\mathcal{L}[f''(t)] = \int_0^\infty f''(t)e^{-st}\,dt$$

$$= \left[e^{-st}f'(t)\right]_0^\infty + \int_0^\infty f'(t)se^{-st}\,dt$$

$$= [0 - f'(0)] + s\underbrace{\int_0^\infty f'(t)e^{-st}\,dt}_{\mathcal{L}[f'(t)]} = s\mathcal{L}[f'(t)] - f'(0)$$

$$= s(sF(s) - f(0)) - f'(0)$$

$$\boxed{\mathcal{L}(f''(t)) = s^2F(s) - sf(0) - f'(0)}$$

$\mathcal{L}[f^{(n)}(t)]$

Ex. Evaluate by induction $\mathcal{L}[f'''(t)]$

$$= s\mathcal{L}[f''(t)] - f''(0) = s^3F(s) - s^2f(0) - sf'(0) - f''(0)$$

Laplace transform of a derivative

$$\boxed{\mathcal{L}[f^{(n)}(t)] = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - s f^{(n-2)}(0) - f^{(n-1)}(0)}$$

$n=1$ $\quad \mathcal{L}[f^{(1)}(t)] = sF(s) - f(0)$

$n=2$ $\quad \mathcal{L}[f^{(2)}(t)] = s^2 F(s) - s f(0) - f'(0)$

$n=3$ $\quad \mathcal{L}[f^{(3)}(t)] = s^3 F(s) - s^2 f(0) - s f'(0) - f''(0)$

$n=4$ $\quad \mathcal{L}[f^{(4)}(t)] = s^4 F(s) \underbrace{- s^3 f(0) - s^2 f'(0) - s f''(0) - f'''(0)}_{\text{Initial conditions}}$

Final value theorem

Ex. Proof the final value theorem.

$$\boxed{\lim_{t\to\infty} y(t) = \lim_{s\to 0}[sY(s)]}$$

$$\mathcal{L}[f'(t)] = sF(s) - f(0)$$

$$\mathcal{L}[y'(t)] = sY(s) - y(0)$$

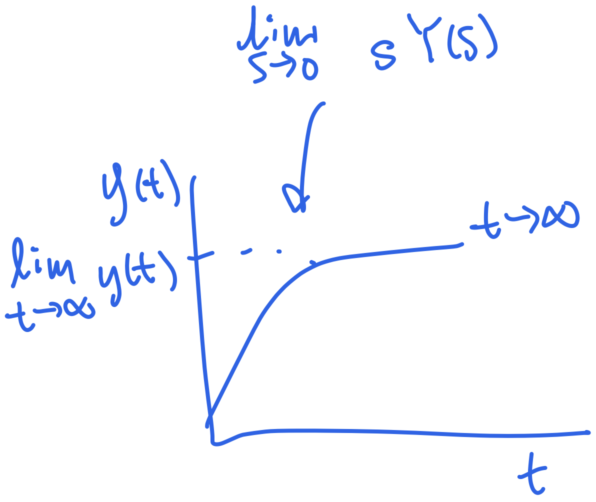
$$\lim_{s\to 0}\left[\int_0^\infty y'(t)\, e^{-st}\, dt = sY(s) - y(0)\right]$$

$$\int_0^\infty y'(t)\, dt = \lim_{s\to 0} sY(s) - y(0)$$

$$\left[y(t)\right]_0^\infty = \lim_{s\to 0} sY(s) - y(0)$$

$$\lim_{t\to\infty} y(t) - \cancel{y(0)} = \lim_{s\to 0} sY(s) - \cancel{y(0)}$$

$$\boxed{\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s)}$$



Initial value theorem

Ex. Proof the initial value theorem.

$$\boxed{\lim_{t \to 0} y(t) = \lim_{s \to \infty} [sY(s)]}$$

$$\mathcal{L}[f'(t)] = sF(s) - f(0)$$

$$\mathcal{L}[y'(t)] = sY(s) - y(0)$$

$$\lim_{s \to \infty} \left[\int_0^\infty y'(t) e^{-st} dt = sY(s) - y(0) \right]$$

$$\cancel{\int_0^\infty y'(t) e^{-\infty}} dt = \lim_{s \to \infty} sY(s) - y(0)$$

$$y(0) = \lim_{s \to \infty} sY(s)$$

$$\boxed{\lim_{t \to 0} y(t) = \lim_{s \to \infty} sY(s)}$$