

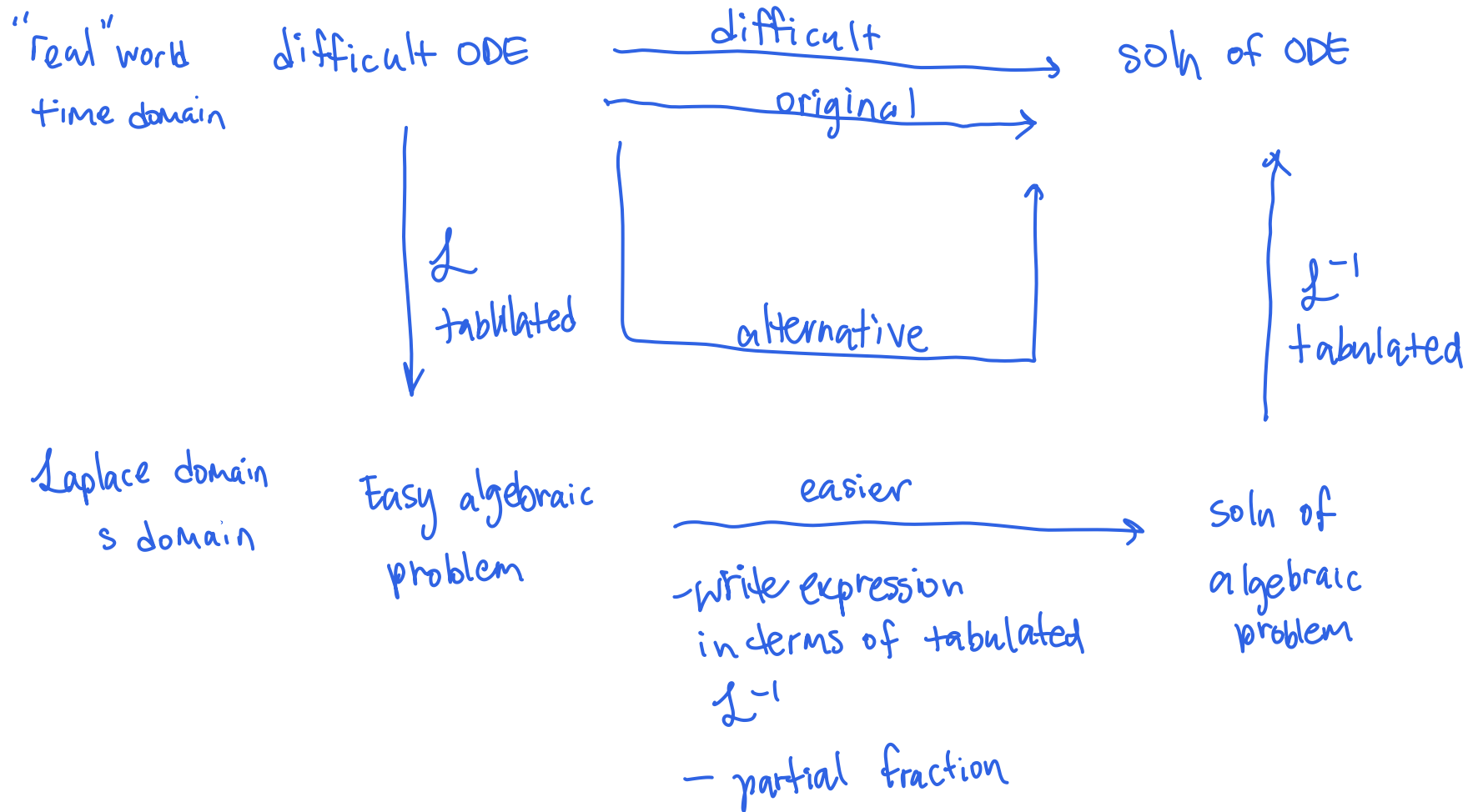
# Solving ODEs with Laplace Transforms

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**Process Dynamics and Control**

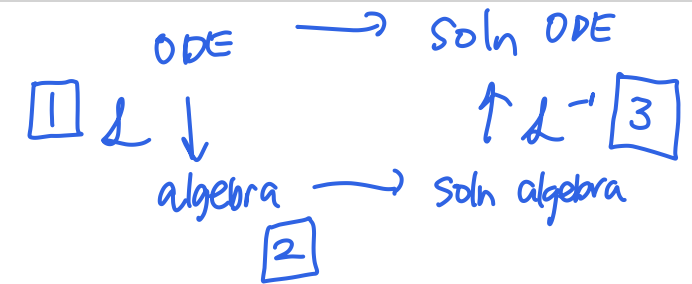
# Laplace transform is the alternative path to solve complex ODEs



# Initial value problem for first-order linear ODE

**Zill Ex. 4.2.4** Use Laplace transform to solve the initial value problem

$$\mathcal{L} \left[ \frac{dy}{dt} + 3y = 13 \sin(2t) \right] \quad \underline{y(0) = 6}$$



$$\boxed{1} \quad \mathcal{L} \left[ \frac{dy}{dt} \right] + 3 \mathcal{L}[y] = 13 \mathcal{L}[\sin(2t)]$$

$$\mathcal{L}(\sin(at)) = \frac{a}{s^2 + a^2}$$

$a=2$

$$[sY(s) - y(0)] + 3Y(s) = 13 \frac{2}{s^2 + 4}$$

$$(s+3)Y(s) - 6 = \frac{26}{s^2 + 4}$$

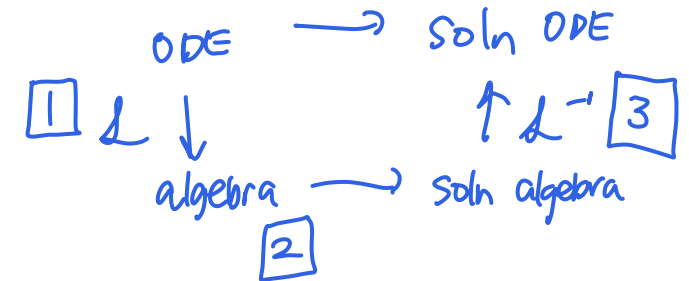
Algebra:

$$Y(s) = \frac{\frac{26}{s^2 + 4} + 6}{s+3} = \frac{26}{(s^2 + 4)(s+3)} + \frac{6}{s+3}$$

# Initial value problem for first-order linear ODE

**Zill Ex. 4.2.4** Use Laplace transform to solve the initial value problem

$$\frac{dy}{dt} + 3y = 13 \sin(2t), \quad y(0) = 6$$



$\boxed{2}$

$$Y(s) = \frac{26}{(s^2+4)(s+3)} + \frac{6}{s+3}$$

$$\begin{aligned} \frac{26}{(s^2+4)(s+3)} &= \frac{As+B}{s^2+4} + \frac{C}{s+3} = \boxed{\frac{-2s+6}{s^2+4} + \frac{2}{s+3}} \\ &= \frac{As+B}{s^2+4} \cdot \frac{s+3}{s+3} + \frac{C}{s+3} \cdot \frac{s^2+4}{s^2+4} \\ &= \frac{(As^2+3As+Bs+3B) + Cs^2+4C}{(s^2+4)(s+3)} \end{aligned}$$

$$26 = \underbrace{s^2(A+C)}_{=0} + \underbrace{s(3A+B)}_{=0} + \underbrace{(3B+4C)}_{=26}$$

$$\begin{cases} A+C=0 \\ 3A+B=0 \\ 3B+4C=26 \end{cases} \begin{cases} A=-C \\ -3C+B=0 \\ B=3C \end{cases}$$

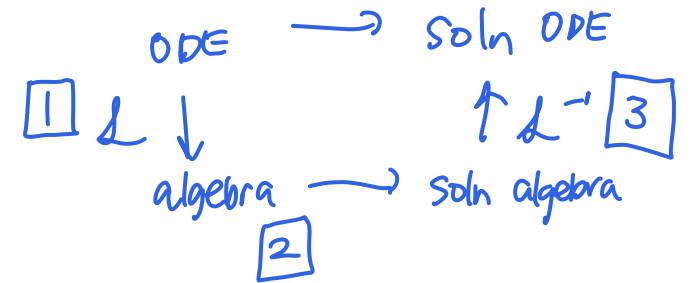
$$\begin{aligned} 9C+4C &= 26 \\ 13C &= 26 \\ C &= 2 \\ B &= 6 \\ A &= -2 \end{aligned}$$

# Initial value problem for first-order linear ODE

**Zill Ex. 4.2.4** Use Laplace transform to solve the initial value problem

$$y_{\text{general}} + y_{\text{part}} \frac{dy}{dt} + 3y = 13 \sin(2t), \quad y(0) = 6$$

↓ non-homo term



$\boxed{3}$

$$\mathcal{L}^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos(at)$$

$$\mathcal{L}^{-1}\left(\frac{a}{s^2+a^2}\right) = \sin(at)$$

$a^2=4$   
 $a=2$

$$\mathcal{L}^{-1}\left(\frac{1}{s+a}\right) = e^{-at}$$

$a=3$

$$Y(s) = \frac{26}{(s^2+4)(s+3)} + \frac{6}{s+3}$$

$$Y(s) = \boxed{\frac{-2s+6}{s^2+4} + \frac{2}{s+3}} + \frac{6}{s+3}$$

$$Y(s) = \frac{-2s+6}{s^2+4} + \frac{8}{s+3} = \frac{-2s}{s^2+4} + \frac{6}{s^2+4} + \frac{8}{s+3}$$

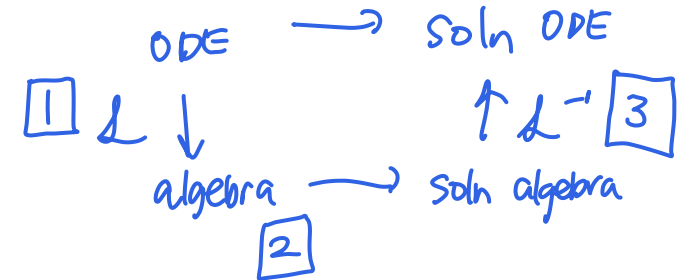
$$y(t) = \mathcal{L}^{-1}[Y(s)] = -2 \mathcal{L}^{-1}\left(\frac{s}{s^2+4}\right) + 6 \mathcal{L}^{-1}\left(\frac{1}{s^2+4}\right) + 8 \mathcal{L}^{-1}\left(\frac{1}{s+3}\right)$$

$$y(t) = -2 \cos(2t) + 6 \sin(2t) \frac{1}{2} + 8 e^{-3t}$$

# Initial value problem for second-order linear ODE

**Zill Ex. 4.2.5** Use Laplace transform to solve the initial value problem

$$\mathcal{L}[y'' - 3y' + 2y = e^{-4t}], \quad y(0) = 1, \quad y'(0) = 5$$



$$\boxed{1} \quad \mathcal{L}[y''] - 3\mathcal{L}[y'] + 2\mathcal{L}[y] = \mathcal{L}[e^{-4t}]$$

$$[s^2 Y(s) - sy(0) - y'(0)] - 3[sY(s) - y(0)] + 2Y(s) = \frac{1}{s+4}$$

$$s^2 Y(s) - s - 5 - 3sY(s) + 3 + 2Y(s) = \frac{1}{s+4}$$

$$(s^2 - 3s + 2)Y(s) - s - 2 = \frac{1}{s+4}$$

$$\begin{aligned} s^2 - 3s + 2 \\ = (s-1)(s-2) \end{aligned}$$

$$Y(s) = \frac{\frac{1}{s+4} + s + 2}{s^2 - 3s + 2} = \frac{1}{(s+4)(s-1)(s-2)} + \frac{s+2}{(s-1)(s-2)} \cdot \frac{s+4}{s+4}$$

$$Y(s) = \frac{9 + s^2 + 6s}{(s+4)(s-1)(s-2)}$$

$$\mathcal{L}[e^{-at}] = \frac{1}{s+a} \quad a=4$$

# Initial value problem for second-order linear ODE

**Zill Ex. 4.2.5** Use Laplace transform to solve the initial value problem

$$y'' - 3y' + 2y = e^{-4t}, \quad y(0) = 1, \quad y'(0) = 5$$



2

$$Y(s) = \frac{9 + s^2 + 6s}{(s+4)(s-1)(s-2)} = \frac{A}{s-4} + \frac{B}{s-1} + \frac{C}{s-2}$$

$$\begin{aligned} s+4=0 \\ s=-4 : A = \frac{9 + (-4)^2 + 6(-4)}{(-4-1)(-4-2)} = \frac{9+16-24}{(-5)(-6)} = \frac{1}{30} \end{aligned}$$

$$\begin{aligned} s-1=0 \\ s=1 : B = \frac{9+1+6}{(1+4)(1-2)} = \frac{16}{-5} \end{aligned}$$

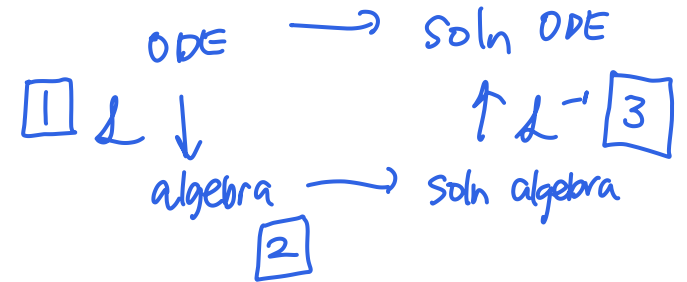
$$\begin{aligned} s-2=0 \\ s=2 : C = \frac{9+2^2+6(2)}{(2+4)(2-1)} = \frac{9+4+12}{6} = \frac{25}{6} \end{aligned}$$

$$Y(s) = \frac{1}{30} \frac{1}{s-4} - \frac{16}{5} \frac{1}{s-1} + \frac{25}{6} \frac{1}{s-2}$$

# Initial value problem for second-order linear ODE

**Zill Ex. 4.2.5** Use Laplace transform to solve the initial value problem

$y_{\text{general}} + y_{\text{part}}$   $y'' - 3y' + 2y = e^{-4t}$ ,  $y(0) = 1$ ,  $y'(0) = 5$   
*non-homo.*



[5]

$$Y(s) = \frac{1}{30} \frac{1}{s-4} - \frac{16}{5} \frac{1}{s-1} + \frac{25}{6} \frac{1}{s-2}$$

$$\mathcal{L}^{-1}\left(\frac{1}{s+a}\right) = e^{-at}$$

$$y(t) = \mathcal{L}^{-1}[Y(s)] = \frac{1}{30} \mathcal{L}^{-1}\left[\frac{1}{s-4}\right] - \frac{16}{5} \mathcal{L}^{-1}\left[\frac{1}{s-1}\right] + \frac{25}{6} \mathcal{L}^{-1}\left[\frac{1}{s-2}\right]$$

$$y(t) = \frac{1}{30} e^{4t} - \frac{16}{5} e^t + \frac{25}{6} e^{2t}$$