Fiber Optics

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1 fiber guiding from ABCD matrix formalism

In the previous lecture we learned about the solutions to the free-space paraxial wave equation, which were Hermite-Gaussian (or Laguerre-Gaussian) modes. The ABCD matrices, introduced in the context of ray optics, were useful for transforming the q-parameter of the beam as it propagates through various optical paths. This lecture will focus on the solutions to the wave equation that are confined to a cylindrical waveguide, and we will again find that Hermite-Gaussians naturally appear. We will see this in an intuitive way first by appling the ABCD matrix formalism to this problem before diving into the paraxial wave equation.

1.1 ABCD matrix of a GRIN lens

Though we haven't introduced optical fibers yet, we can approach the topic by assuming that optical fibers are strands of glass whose index of refraction is higher at the center than on the edges. This situation is remeniscent of a special type of lens called a GRaded INdex (or GRIN) lens. A GRIN lens is essentially a flat peice of glass whose index of refraction has been made spatially-dependent through some means.

Specifically, a positive GRIN lens wil result if the index of refraction falls off quadratically with distance from the center of the lens (i.e. distance from the optical axis):

$$n(x,y) = n_0\left(1 - \frac{x^2+y^2}{2\ell_{\mathrm{G}}^2}\right). \tag{1}$$

We will assume that the characteristic length scale, ℓ_{G}, is much larger than the lens diameter, such that the index of refraction is always positive and greater than 1. Our task is to derive the ABCD matrix for a GRIN lens of thickness $\Delta z \ll \ell_{\mathrm{G}}$ described by Eq. 1.

Our first step will be to model the real (thickness= Δz) GRIN lens as the combination of three ideal optical elements: (1) free propagation a distance $\Delta z/2$ though a space with unifrom refractive index n_0, (2) an ideal thin lens that produces the phase delay that is a consequence of Eq. 1, (3) and then step (1) again.

For steps (1) and (3) we can use the results of the previous lecture to write

$$\mathbf{M}(n_0, \tfrac{\Delta z}{2}) = \begin{pmatrix} 1 & \frac{\Delta z}{2n_0} \\ 0 & 1 \end{pmatrix}. \tag{2}$$

(a)

(b)

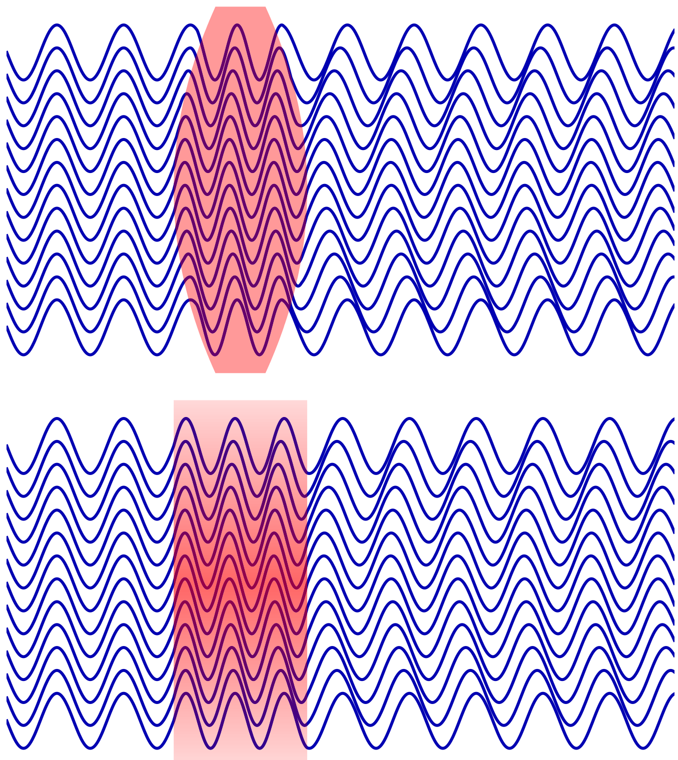

Figure 1: Effect of position-dependent phase delays introduced by (a) a bi-convex doublet lens and (b) a flat GRIN lens. The sinusoidal functions are confined to have a constant displacement from the optical axis (i.e. we are not considering ray-type refraction), and their wavelength is slightly reduced while in the lens (shown in red). The position-dependent phase delay causes the plane waves indicent from the left to develop wave-front curvature after passing through the lens, which leads to focusing. In (b), the same effect is introduced by a flat optic whose index of refraction decreases quadratically with the distance from the optical axis. Note that the ray optics picture would in this case predict (incorrectly) that incident plane waves would not focus on the other side of the lens.

For the ideal thin lens, we recall that the effect that is responsible for lens action is that a lens creates a spatially-dependent phase delay, as shown in Fig. 1. The phase delay imparted by the lens is simply $\Delta\Phi = k\Delta z$, but we need to include the spatial dependence of k:

$$\begin{aligned}\Delta\Phi(x,y) &= \tfrac{\omega}{c}\left(n_0 - n(x,y)\right)\Delta z \\ &= \frac{\omega}{c}\Delta z\, n_0 \frac{x^2+y^2}{2\ell_{\mathrm{G}}^2}.\end{aligned} \tag{3}$$

The wave fronts (some z-offset that is a function of x and y that we will call δz) that have exited the lens are defined to be iso-phase surfaces. δz is a function of x and y needed to follow a surface of constant phase; for a given position (x, y), we will either need to advance or retreat in z to find a place where the phase is equal to its value on the optical axis. We can derive an expression for δz by setting the sum of the phase shift in Eq. 3 and the $\frac{\omega}{c}\delta z$ phase from the $e^{ik\delta z}$ term equal to a constant, which might as well be zero:

$$0 = \frac{\omega}{c}\Delta z\, n_0 \frac{x^2+y^2}{2\ell_{\mathrm{G}}^2} + \frac{\omega}{z}\delta z. \tag{4}$$

Since we are moleding this as a perfect, converging lens (which would turn plane waves into spherical waves), we can parameterize δz in terms of a radius of curvature R of the wave fronts exiting the lens. Figure 2 shows a little bit of trigonometry that can be used to show that for wave front with radius of curvature R,

$$\begin{aligned}(R-\delta z)^2 &= R^2 - x^2 - y^2 \\ \delta z &\approx \frac{x^2+y^2}{2R}\end{aligned} \tag{5}$$

where in the second line we have neglected $(\delta z)^2$ compared to $R\,\delta z$. By inserting Eq. 5 into Eq. 4, we can immediately identify the magnitude of the wavefront curvature,

$$|R| = \frac{\ell_{\mathrm{G}}^2}{n_0\Delta z}. \tag{6}$$

We recall from Gaussian optics that the wave front radius of curvature is equal to the distance from the focus (minimum waist position) in the far-field limit. Eq. 6 therefore tells us the focal length of the lens (a collimated input will converge on the optical axis a distance f on the other side of a positive lens). The ABCD matrix is therefore

$$\mathbf{N}_{\mathrm{thin}} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{n_0\Delta z}{\ell_{\mathrm{G}}^2} & 1 \end{pmatrix} \tag{7}$$

We can now combine this with the ABCD matrices for the two sections of free propagatin (Eq. 2) to get the ABCD matrix of this GRIN lens of thickness Δz:

$$\mathbf{T} = \mathbf{M}(n_0, \tfrac{\Delta z}{2})\mathbf{N}_{\mathrm{thin}}\mathbf{M}(n_0, \tfrac{\Delta z}{2})$$

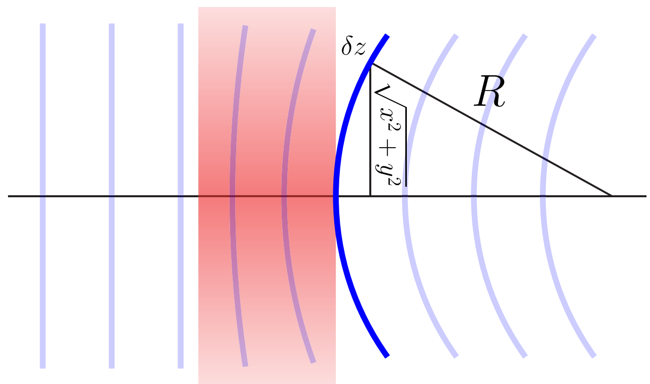

Figure 2: Wave fronts emerging from a GRIN lens illuminated by plane waves acquire a radius of curvature (-)R, which can be related to the delay distance (-)δz for a position $\sqrt{x^2+y^2}$ from the optical axis by examining the right triangle shown.

$$
\begin{aligned}
&= \begin{pmatrix} 1 & \frac{\Delta z}{2n_0} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{n_0 \Delta z}{\ell_{\mathrm{G}}^2} & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{\Delta z}{2n_0} \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 - \frac{\Delta z^2}{2\ell_{\mathrm{G}}^2} & \frac{\Delta z}{n_0} - \frac{\Delta z^3}{4n_0 \ell_{\mathrm{G}}^2} \\ -\frac{n_0 \Delta z}{\ell_{\mathrm{G}}^2} & 1 - \frac{\Delta z^2}{2\ell_{\mathrm{G}}^2} \end{pmatrix} \\
&\approx \begin{pmatrix} 1 - \frac{\Delta z^2}{2\ell_{\mathrm{G}}^2} & \frac{\Delta z}{n_0} \\ -\frac{n_0 \Delta z}{\ell_{\mathrm{G}}^2} & 1 - \frac{\Delta z^2}{2\ell_{\mathrm{G}}^2} \end{pmatrix}
\end{aligned} \tag{8}
$$

where in the last line we have neglected $(\Delta z / 2\ell_{\mathrm{G}})^2$ compared to 1.

1.2 ABCD matrix of a GRIN rod of length $p\Delta z$

Now that we have the ABCD matrix for a GRIN lens with the index profile of Eq. 1 and thickness $\Delta z \ll \ell_{\mathrm{G}}$, we are in a position to ask what happens if we cascade p of these together to made a long rod. For this, we will use two identities:

Identity 1.1 *A 2×2 matrix of the form*

$$
\mathbf{M} = \begin{pmatrix} 1 - \frac{d}{f} & 2d - \frac{d^2}{f} \\ -\frac{1}{f} & 1 - \frac{d}{f} \end{pmatrix} \tag{9}
$$

can we written as

$$
\mathbf{M} = \begin{pmatrix} \cos(\theta) & \chi \sin(\theta) \\ -\frac{1}{\chi} \sin(\theta) & \cos(\theta) \end{pmatrix} \tag{10}
$$

by defining $\cos(\theta) \equiv 1 - d/f$ *and* $\chi \equiv \sqrt{2df - d^2}$. *This can be proved simply by plugging these in.*

Identity 1.2 *If a 2×2 matrix* $\mathbf{M}$ *takes the form of Eq. 10,* $\mathbf{M}^p$ *for a positive integer* p *is given by*

$$\mathbf{M}^p = \begin{pmatrix} \cos(\theta) & \chi\sin(\theta) \\ -\frac{1}{\chi}\sin(\theta) & \cos(\theta) \end{pmatrix}^p = \begin{pmatrix} \cos(p\theta) & \chi\sin(p\theta) \\ -\frac{1}{\chi}\sin(p\theta) & \cos(p\theta) \end{pmatrix}. \tag{11}$$

This can be proved by trig identities and mathematical induction.

Note that the matrix in Eq. 9 is the ABCD matrix for free space propagation through distance d, followed by a thin lens with focal length f and another free-space propagation over distance d. So Identity 1.2 is really a nice, compact way to write the ABCD matrix of an array of p lenses of focal length f where consecutive lenses are a distance $2d$ from one another.

To find the ABCD matrix of a GRIN rod of length $p\Delta z$, we can apply identities 1.1 and 1.2 to our GRIN lens solution by making the substitution $\cos(\theta) \equiv 1 - \frac{\Delta z^2}{2\ell_{\mathrm{G}}^2}$ and $\chi \equiv \frac{1}{n_0}\sqrt{\ell_{\mathrm{G}}^2 - (\Delta z/2)^2}$, which gives us the same matrix as Eq. 11.

1.3 Guided modes

Now that we know the ABCD matrix for a GRIN rod, which is our simple model for an optical fiber, we can see how it transforms the q-parameter of a Gaussian input beam and consider whether any modes are guided. By guided, we mean that the q-parameter is identical for the input and output beams, and that this feature is independent of p. Mathematically, we set

$$q_{\mathrm{g}} = \frac{q_{\mathrm{g}}A + B}{q_{\mathrm{g}}C + D} = \frac{q_{\mathrm{g}}\cos(p\theta) + \chi\sin(p\theta)}{-\frac{1}{\chi}q_{\mathrm{g}}\sin(p\theta) + \cos(p\theta)}. \tag{12}$$

Solving for the guided q-parameter (q_{g}) gives us

$$q_{\mathrm{g}} = i\,\chi = \frac{i}{n_0}\sqrt{\ell_{\mathrm{G}}^2 - \left(\frac{\Delta z}{2}\right)^2}. \tag{13}$$

Since Δz was constructed to be thin compared to all of the other length scales in the problem (but was otherwise arbitrary), we take the limit as $\Delta z \to 0$ to ensure a spatially unform mode,

$$q_{\mathrm{guided}} = i\,\frac{\ell_{\mathrm{G}}}{n_0}. \tag{14}$$

We now know that there is a family of Hermite-Gaussian (or Laguerre-Gaussian, if you prefer) modes that will be guided in this fiber, and we can say some interesting things about them.

First, q_{guided} is purely imaginary. This means that these guided modes have perfectly flat wavefronts ($R(z=0) \to \infty$), and look a lot like the minimum waist position of a free-space beam. The confocal parameter of that free-space beam would be ℓ_{G}/n_0.

Second, the beam waist itself has been determined:

$$w_{\mathrm{o}} = \sqrt{\frac{\lambda\ell_{\mathrm{G}}}{\pi n_0}}. \tag{15}$$

This is great, because if we want to know how to focus a beam into such a fiber from free-space, this is the spot size we need to match.

Third, as we already mentioned, we have so far said that a whole family of Hermite-Gaussian modes will be guided in this fiber, with no limit. Such a fiber is called a "multi-mode fiber."

1.4 Hermite-Gaussian modes from Helmholtz equation

In this section, we will again assume a GRIN fiber of the form given in Eq. 1 and, starting from the basic E&M, solve for the guided modes. This approach will give us the same guided confocal parameter as the ABCD matrix method of the previous section, but will also give us information about mode dispersion, which is needed to understand mode cutoff.

In the previous lecture, we took advantage of the simple temporal structure of our waves ($\mathrm{e}^{-i\omega t}$) to write the wave equation as the Helmholtz equation,

$$\nabla^2\psi = -\frac{\omega^2}{v^2}\psi, \tag{16}$$

where $v = c/n$ is the phase velocity of the light. We will consider the light to be guided in our GRIN fiber from the previous section, so we seek plane-waves of the form

$$\psi(x,y,z)\mathrm{e}^{-i\omega t} = u(x,y)\mathrm{e}^{i(\beta z-\omega t)}. \tag{17}$$

Here, we have introduced the symbol β for the propagation constant instead of k because the refractive index is spatially dependent now and we do not yet know what the wavenumber of the guided modes will be.

Plugging this into 16 with $n(x,y)$ given by Eq. 1 gives us

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - \beta^2 u &= -\frac{\omega^2}{c^2}n_0^2\left(1-\frac{x^2+y^2}{2\ell_\mathrm{G}^2}\right)^2 u \\
\nabla_\perp^2 u - \beta^2 u &\approx -\frac{\omega^2}{c^2}n_0^2\left(1-\frac{x^2+y^2}{\ell_\mathrm{G}^2}\right) u \\
\nabla_\perp^2 u - \left(k_0^2\frac{x^2+y^2}{\ell_\mathrm{g}^2}\right)u &= \left(\beta^2-k_0^2\right)u
\end{aligned} \tag{18}$$

where we have introduced $k_0 \equiv n_0\omega/c$.

We now look for solutions to Eq 18 by separation of variables, which means we assume a product form for the solutions, $u(x,y) = U_x(x)U_y(y)$. We can define the constants on the right to be a sum of two eigenvalues, $\Lambda_n + \Lambda_p \equiv k_0^2 - \beta^2$, which allows us to split 18 into two ODEs:

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \left(\frac{k_0}{\ell_\mathrm{G}}\right)^2 x^2\right)U_x = -\Lambda_n U_x \tag{19}$$

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2} - \left(\frac{k_0}{\ell_\mathrm{G}}\right)^2 y^2\right)U_y = -\Lambda_p U_y. \tag{20}$$

Equations 19 and 20 *may* look a little bit familiar to you. You may recall that the quantum Hamiltonian for a simple harmonic oscillator (SHO) is $H = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2$. The time-independent Schrödinger equation can then be written (in the position representation) as

$$\begin{aligned} H\psi(x) &= E\psi(x) \\ \left(-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{1}{2}m\omega^2 x^2\right)\psi(x) &= E\psi(x) \\ \left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \left(\frac{m\omega}{\hbar}\right)^2 x^2\right)\psi(x) &= -\frac{2mE}{\hbar^2}\psi(x). \end{aligned} \tag{21}$$

Equations 19-21 are simply rescaled versions of one another. So despite the fact that we are doing completely classical physics here (i.e. Maxwell's equations), we can use the quantum SHO solutions here since they happen to be familiar to us. Specifically, we recall from quantum mechanics [2] that solutions are characterized by non-negative integers n,

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right) \tag{22}$$

and

$$\psi_n(x) = \left(\frac{m\omega}{\hbar}\right)^{1/4} \frac{1}{\pi^{1/4}\sqrt{2^n n!}} H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right) \mathrm{e}^{-\frac{m\omega x^2}{2\hbar}}, \tag{23}$$

where $H_n(z)$ is the nth Hermite polynomial. We can then identify

$$\Lambda_j = -\frac{2m}{\hbar^2}\hbar\omega\left(j + \frac{1}{2}\right) = -\frac{k_0}{\ell_\mathrm{G}}(2j+1) \tag{24}$$

and

$$U_x^{(n)}U_y^{(p)} = \sqrt{\frac{k_0}{\ell_\mathrm{G}}}\frac{1}{\sqrt{\pi}\,2^{n+p}\,n!p!} H_n\left(\sqrt{\frac{k_0}{\ell_\mathrm{G}}}x\right) H_p\left(\sqrt{\frac{k_0}{\ell_\mathrm{G}}}y\right) \mathrm{e}^{-k_0\frac{x^2+y^2}{2\ell_\mathrm{G}}}. \tag{25}$$

We can now lump that front stuff into an amplitude coefficient and write an expression for the electric field in the GRIN fiber (along the polarization direction):

$$E(x,y,z,t) = \sum_{n,p=0}^{\infty} \mathcal{E}_{n,p} H_n\left(\sqrt{\frac{k_0}{\ell_\mathrm{G}}}x\right) H_p\left(\sqrt{\frac{k_0}{\ell_\mathrm{G}}}y\right) \mathrm{e}^{-k_0\frac{x^2+y^2}{2\ell_\mathrm{G}}} \mathrm{e}^{i(\beta_{n,p}z-\omega t)}. \tag{26}$$

First, note that this solution (Eq. 26) has a (minimum) waist of $w_0 = \sqrt{\frac{2\ell_\mathrm{G}}{k_0}}$. This corresponds to a Gaussian q-parameter $q = i\frac{\ell_\mathrm{G}}{n_0}$, just as we found using the ABCD matrix formalism[1]. However, we have also gained information about the propagation constant β, which is now mode-dependent. Specifically, our definition $\Lambda_n + \Lambda_p = k_0^2 - \beta_{n,p}^2$ gives us

$$\beta_{n,p} = k_0\sqrt{1 - 2\frac{n+p+1}{k_0\ell_\mathrm{G}}}. \tag{27}$$

[1]This comes from $b = \pi w_0^2/\lambda_\mathrm{vac}$. I need to track down why that's λ_vac instead of $\lambda(n)$...

Now, $\beta_{n,p}$ tells us about the phase velocity of a particular mode,

$$v_{n,p} = \frac{\omega}{\beta_{n,p}}. \tag{28}$$

We immediately see that the phase velocity increases with increasing $n + p$. This is simply because the transverse extent (the "spot size") of the nth Hermite-Gaussian mode is approximately $w_0\sqrt{n}$ [3]. Since the higher order modes extend further into the low index part of the fiber, they have a higher phase velocity.

Next, since k_0 depends upon the optical frequency, we see that the fiber iteself causes dispersion (that is, $\mathrm{d}v/\mathrm{d}\omega \neq 0$). Most importantly, the group velocity $v_{\mathrm{group}} \equiv (\mathrm{d}\beta/\mathrm{d}\omega)^{-1}$ shows dispersion:

$$\frac{\mathrm{d}v_{\mathrm{group}}}{\mathrm{d}\omega} \neq 0. \tag{29}$$

This effect is called *waveguide dispersion*, and is very important for optical communications. Specifically, if we send a pulse train of perfect square waves through a medium with group velocity dispersion (GVD), the square waves will distort and spread out, eventually overlapping one another. However, there is a clever trick to get around this: the waveguide dispersion, which depends upon the details of the fiber geometry, can be chosen to have the same magnitude but opposite sign as the *material dispersion* from the glass used to make fibers. For (GeO_2-doped silica), the material GVD crosses zero and becomes negative at about $\lambda_{\mathrm{vac}} \geq 1.3$ μm. An appropriate fiber size (and index profile) can be designed to compensate the matrial dispersion at $\lambda_{\mathrm{vac}} \approx 1.55$ μm [4]. The low matrial dispersion at 1.3 μm and low loss at 1.55 μm (along with the easy availability of light sources near these wavelengths) is the reason these two wavelengths are popular for telecommunications.

1.5 Mode cutoff for GRIN fiber

Clearly, since the transverse extent of a fiber mode grows with n, p, at some point, our analysis that concluded guiding will fail. Specifically, though we have been considering GRIN fiber, it stands to reason that the quadratic decrease in index of refraction as the distance from the center of the fiber is increased will end at some point. The index of refraction of the (typically very think) *fiber cladding* can only be so low, and we can call this n_{clad}.

We can estimate when modes stop being guided by asking what the (combined) mode index $N \equiv n + p$ must be to ensure that $\beta_{n,p} \approx k_{\mathrm{clad}} \equiv 2\pi n_{\mathrm{clad}}/\lambda_{\mathrm{vac}}$:

$$\left(\frac{\lambda_{\mathrm{vac}}}{2\pi}\beta_{\mathrm{max}}\right)^2 \approx n_{\mathrm{clad}}^2 \approx n_0^2\left(1 - 2\frac{N_{\mathrm{max}}+1}{k_0\ell_{\mathrm{G}}}\right) \tag{30}$$

and we conclude that for $N \gg 1$,

$$N_{\mathrm{max}} \approx 2\frac{\pi\ell_{\mathrm{G}}}{n_0\lambda_{\mathrm{vac}}}\left(n_0^2 - n_{\mathrm{clad}}^2\right) \tag{31}$$

where we have in this last step inserted a factor of 2 to account for two possible polarizations.

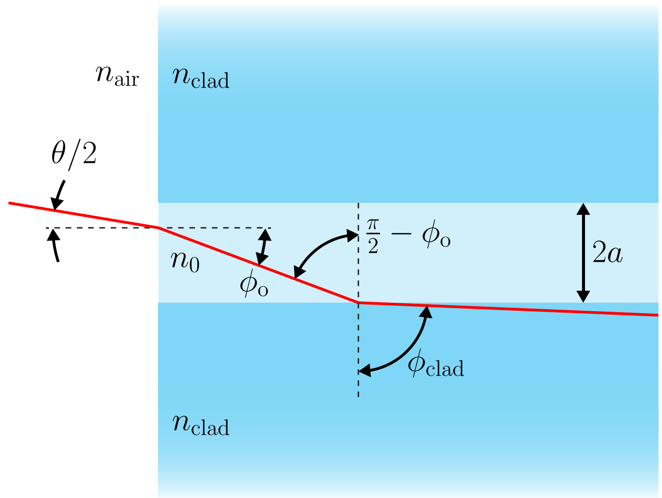

Figure 3: Step index fiber in the ray optics limit. Rays are confined to a core of index n_0 if they undergo total internal reflection at the interface between the core and cladding.

1.6 Step index fiber

Up to this point, we have been considering GRIN (or "quadratic index") fiber, mostly because it gives us the basic features of most optical fibers without special functions. However, the vast majority of optical fiber in use is actually *step index fiber*, which is cylindrically symmetric with a uniform *core* of radius a and refractive index n_0 surrounded by a *cladding* with a lower refractive index n_{clad} (Fig. 3). The guided modes in such a fiber extend into the cladding somewhat, but are essentially evanescent and eventually fall of exponentially with radius from the core. The cladding diameter is much larger than the mode and may for this discussion be considered infinite.

The guided modes of step index fiber are an exercise in special functions and are beyond the scope of this course [4]. For the moment, we will take the approach that for the situations we will encounter, they resemble the Hermite-Gaussian modes we obtained for the GRIN fiber closely enough that we can assume they are the same.

While the serious solutions of the guided modes in step index fiber are not something we will get into, the general features of step index fiber can actually be obtained straight from ray optics. First, we will ask which rays are guided when they are sent onto the face of a step index fiber, shown schematically in Fig. 3. The principle of operation in the ray optics limit for step index fiber is simply that total internal reflection from the core to cladding interface causes rays to be confined, so it is already clear that incoming rays must enter the core (of radius a). Furthermore, their angle of incidence cannot be arbitrarily steep or they will no longer satisfy the condition for total internal reflection when they encounter the core to cladding interface.

For rays incident (from air, $n_{\mathrm{air}} \approx 1$) on the core at angle $\theta/2$, their incidnce angle with

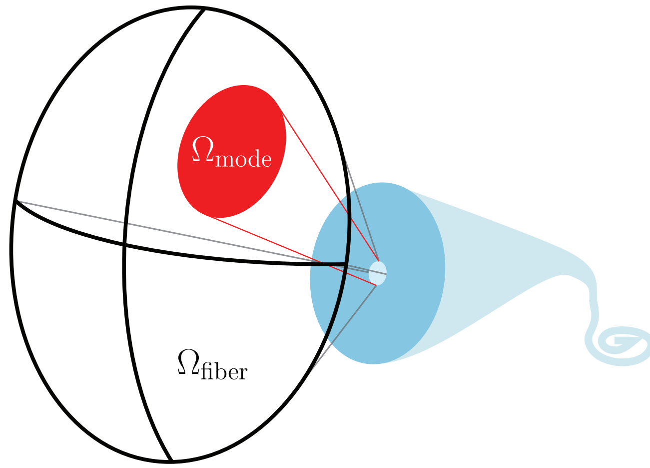

Figure 4: Step index fiber will guide light that illuminates the core from the soid angle defined by its acceptance cone. A mode whose waist matches the core radius may subtend a smaller solid angle than the fiber, which implies that the fiber will support multiple transverse modes. Such a fiber is called multimode (for a particular wavelength).

the core to cladding interface is given by Snell's law, $\sin\left(\frac{\pi}{2}-\phi_{\text{o}}\right)=\sqrt{n_0^2-\sin^2(\theta/2)}/n_0$. Applying Snell's law again and requiring that the refraction angle in the cladding is $\phi_{\text{clad}}=\frac{\pi}{2}$ gives us the critical angle for maintaining the total internal reflection, which can be written in terms of the critical full cone angle θ_{c} as

$$\sin\left(\frac{\theta_{\text{c}}}{2}\right)=\sqrt{n_0^2-n_{\text{clad}}^2}. \tag{32}$$

θ_{c} from Eq. 32 tells us the full cone angle of light that couples with the guided modes of the step index fiber. This is frequently expressed as a dimensionless number between 0 and ≈ 1 called *numerical aperture*, or NA. The NA of some optical system that supports a conical bundle of rays (in a medium with index of refraction n) whose full cone angle is θ is defined to be the sine of the *half* cone angle:

$$NA \equiv n\sin\left(\frac{\theta}{2}\right). \tag{33}$$

So the NA of a step index fiber (in $n_{\text{air}}\approx 1$) is $NA=\sqrt{n_0^2-n_{\text{clad}}^2}$. Likewise, the NA of a gaussian beam with minimum waist w_{o} is $NA=\sin\left(\frac{\lambda}{\pi w_{\text{o}}}\right)\approx\frac{\lambda}{\pi w_{\text{o}}}$.

We can use the result of Eq 32 to estimate the number of transverse modes that will be guided in a particular step index fiber [1]. Figure 4 shows the basic idea, which is to divide up the solid angle subtended by the acceptance cone (full cone angle θ_{c}) into free-space modes with minimum waist size matched to the fiber core, $w_{\text{o}}=a$. We then multiply the answer by 2 to account for two possible polarizations:

$$N_{\max} \approx 2\frac{\Omega_{\text{fiber}}}{\Omega_{\text{mode}}} = 2\frac{2\pi\left(1-\cos\left(\frac{\theta_c}{2}\right)\right)}{2\pi\left(1-\cos\left(\frac{\lambda}{\pi a}\right)\right)} \approx 2\left(\frac{\pi a}{\lambda}\right)^2\left(n_0^2 - n_{\text{clad}}^2\right). \tag{34}$$

An optical fiber with a mode cutoff low enough to ensure that only one mode is guided is called a single mode fiber. In the specifications for single mode fiber, the manufacturer will typically specify the *cutoff wavelength*, λ_c. For light that is shorter wavelength than λ_c (i.e. more blue), the fiber will be multi-mode. For longer wavelengths, the fiber will support a single mode for a given linear polarization (called LP_{01}). Single mode fibers are useful for ensuring that the spatial distribution of the light exiting the fiber is static and for eliminating modal dispersion (i.e. the dependence of GVD on N).

References

[1] D. Gloge. Weakly guiding fibers. *Applied Optics*, 10:2252, 1971.

[2] J. J. Sakurai. *Modern Quantum Mechanics*. Addison-Wesley, 1994.

[3] Anthony E. Siegman. *Lasers*. University Science Books, 1986.

[4] Amnon Yariv. *Optical Electronics in Modern Communications*. Oxford University Press, 1997.