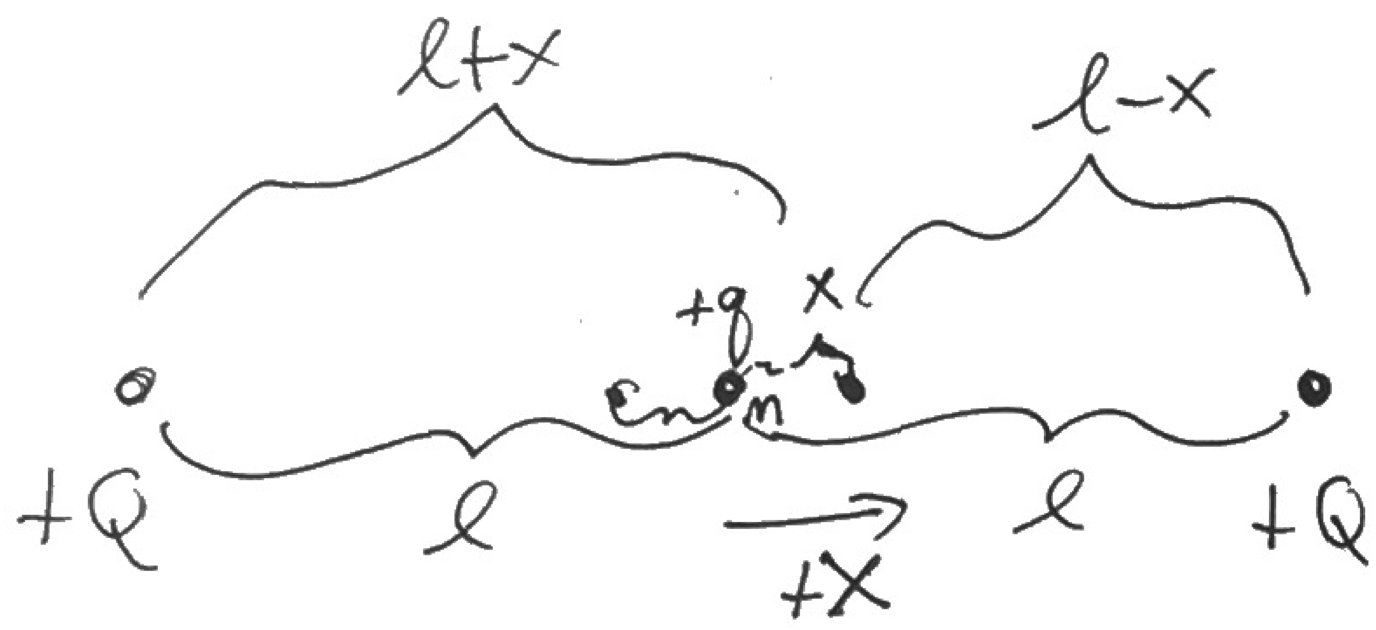
# Problem Set #1 Solutions

## #1) PM (1-38)




Let $x$ be the (small) displacement from the origin, in particular, $\frac{x}{l} \ll 1$.

~~Suppose the tie~~ The force on the particle when it is displaced by a distance $x$ is

$$F(x) = -\frac{Qq}{4\pi\varepsilon_0 (l-x)^2} + \frac{Qq}{4\pi\varepsilon_0 (l+x)^2}.$$

(Note the signs: when $x$ is positive, we know the force must be to the left (i.e., negative) because all charges are positive and $x$ being positive means that $q$ is closest to the $+Q$ on the right, and equivalently when $x$ is negative we know the force must be to the right (i.e., positive).

Thus we have

$$\boxed{\underset{\text{Newton's } 2^{\text{nd}} \text{ Law}}{m\ddot{x} =} -\frac{Qq}{4\pi\varepsilon_0}\left(\frac{1}{(l-x)^2} - \frac{1}{(l+x)^2}\right).} \quad (*)$$

This equation is exact. We now use the fact that $\frac{x}{l} \ll 1$ and Taylor expand to get the approximate motion. First, we rewrite $(*)$ (exactly) as

$$\underset{\text{still exact.}}{m\ddot{x} =} -\frac{Qq}{4\pi\varepsilon_0 l^2}\left[\left(1-\frac{x}{l}\right)^{-2} - \left(1+\frac{x}{l}\right)^{-2}\right].$$

Now a general fact, If $\alpha \ll 1$, then $(1+\alpha)^a \simeq 1 + a\alpha$.

~~Proof~~

Proof on next page →

Proof: General Taylor expansion:

For $x$ near $c$, namely, for $|x-c| \ll 1$, we have

$$f(x) = f(c) + (x-c)f'(c) + \dots \quad \text{(Taylor's theorem).}$$

Here, our "$f(x)$" is $(1+\alpha)^a$ (namely, our variable "$x$" is now $\alpha$ and our function $f(\alpha)$ is $f(\alpha) = (1+\alpha)^a$ for some $a$).

Also, here, $c$ is just $0$ since $\alpha$ is near $0$.

Thus $$f(\alpha) = (1+\alpha)^a \simeq f(0) + \alpha f'(0).$$

Now $$f(0) = (1+\alpha)^a\Big|_{\alpha=0} = 1.$$

$$f'(\alpha) = a(1+\alpha)^{a-1} \text{ so that } f'(0) = a(1+\alpha)^{a-1}\Big|_{\alpha=0} = a.$$

Thus $$f(\alpha) = (1+\alpha)^a \simeq \underset{f(0)}{\underset{\shortparallel}{1}} + \alpha \underset{f'(0)}{\underset{\shortparallel}{a}}, \text{ as desired.}$$

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Returning, we have $(*) \Rightarrow$ $$M\ddot{x} = \frac{-Qq}{4\pi\varepsilon_0 \ell^2}\left(\left(1-\frac{x}{\ell}\right)^{-2} - \left(1+\frac{x}{\ell}\right)^{-2}\right).$$

Using the above result with $\alpha = -\frac{x}{\ell} \ll 1$ and $a = -2$ gives $\left(1-\frac{x}{\ell}\right)^{-2} \simeq 1 + 2\frac{x}{\ell}$

and again with $\alpha = \frac{x}{\ell} \ll 1$ and $a = -2$ gives $\left(1+\frac{x}{\ell}\right)^{-2} \simeq 1 - 2\frac{x}{\ell}$.

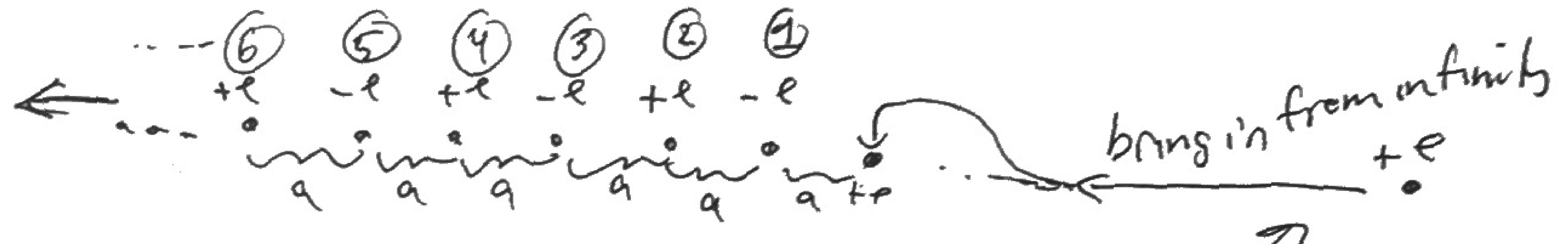
Thus $(*) \Rightarrow$ $$m\ddot{x} \simeq \frac{-Qq}{4\pi\varepsilon_0\ell^2}\left(1 + \frac{2x}{\ell} - \left(1 - \frac{2x}{\ell}\right)\right) = \frac{-Qq}{\pi\varepsilon_0\ell^3}x$$

$$\boxed{\text{Thus } m\ddot{x} = \frac{-Qq}{\pi\varepsilon_0\ell^3}x \text{ which is Hooke's law with } K = \frac{Qq}{\pi\varepsilon_0\ell^3}}$$

The frequency is thus $$\boxed{\omega = \sqrt{K/m} = \sqrt{\frac{Qq}{\pi m \varepsilon_0 \ell^3}}}$$

#2) PM #1-42).

Suppose the chain is built infinitely far off to the left up to a particular negative charge:




Now if we bring in another positive charge:

~~The potential of~~ The external work needed to do this is then

$$W_{ext} = \frac{e^2}{4\pi\varepsilon_0}\Big(\underset{\text{from particle (1)}}{-\frac{1}{a}} + \underset{\text{from particle (2)}}{\frac{1}{2a}} - \underset{\text{from particle (3)}}{\frac{1}{3a}} + \frac{1}{4a} - \frac{1}{5a} + \dots\Big)$$

$$\Rightarrow W_{ext} = \frac{-e^2}{4\pi\varepsilon_0 a}\underbrace{\Big(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots\Big)}_{(*)}.$$

Now, as the hint says, we want to know $\ln(1+x)$. After some Wikipedia-ing (or explicitly calculating it, as you're encouraged to do at least once in your life) we have $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \dots$.

Thus we see that $(*) = \ln(1+x)\big|_{x=1} = \ln(2) \simeq .693$.

Thus $\boxed{W_{ext} = \frac{-e^2}{4\pi\varepsilon_0 a}\ln(2)}$, and this is therefore the energy of this positive charge. Indeed, we'd get the same result (as you should check) if we brought in a negative charge, and since the chain is infinitely long, this is indeed the (equal) potential energy per ion.

Numerically, $\frac{-e^2}{4\pi\varepsilon_0 a}\ln(2) \simeq -\frac{(1.6\times10^{-19})^2 \cdot .693}{4\pi\,(8.85\times10^{-12})(250\times10^{-12})}\,\text{J} \simeq 6.4\times10^{-19}\,\text{J}$ per particle

$\longrightarrow$

Thus, per mole, our result is $-(6.4\times10^{-19})\times 6\times10^{23}$ J/mol $\simeq -384$ kJ/mol, which is only off by a factor of 2 from the result for NaCl of $-787$ kJ/mol, which is a pretty good approximation.

Question: what would happen if all the particles in our chain were positively charged? Is this reasonable?

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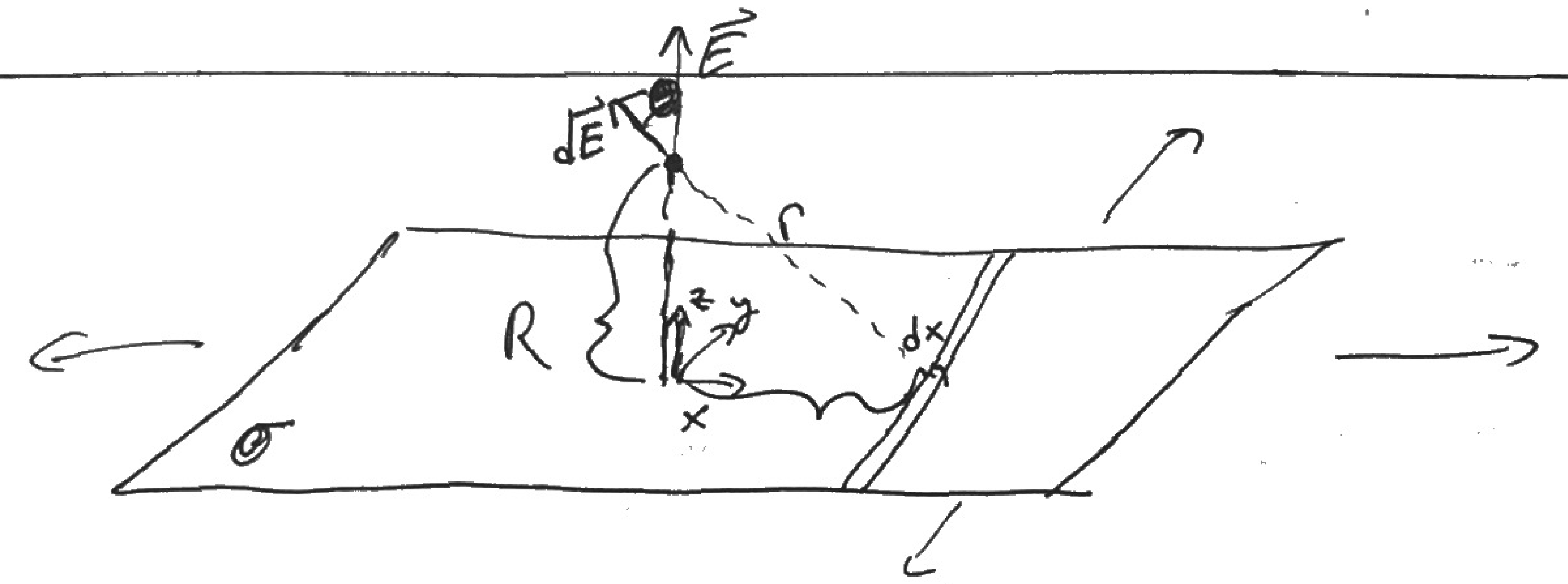
#3)




Consider a point a distance R above the infinite charged sheet with $\sigma$ (and $[\sigma]=\frac{C}{m^2}$). Due to symmetry, it doesn't matter where in the x-y plane we put the point. (Indeed, we'll soon see that the z-direction doesn't matter either!) Consider a thin, infinitely long slice in the y-direction with width dx. The E-field from this strip points as shown above, and by symmetry the only component that doesn't cancel is the z-component $dE_z$ given by $dE_z=\frac{\lambda}{2\pi\varepsilon_0 r}\cos\theta$ where $\lambda=\sigma dx$ (note that this is indeed a line charge, i.e., $[\lambda]=\frac{C}{m}$), and where the magnitude of $dE$ (of which we took the $\cos\theta$ component) is given in the problem statement as $\frac{\lambda}{2\pi\varepsilon_0 r}$. We then just need to integrate over each thin slice, i.e., $E=\int dE_z$.

We have $\quad E = \int dE_z = \int_{x=-\infty}^{\infty} \frac{\overbrace{\sigma\, dx}^{\lambda}}{2\pi \varepsilon_0 r} \cos\theta .$

Now, $\quad r = \sqrt{R^2+x^2}$ and $\cos\theta = \frac{R}{r} = \frac{R}{\sqrt{R^2+x^2}}$

Thus $\quad E = \int_{-\infty}^{\infty} \frac{\sigma\, dx}{2\pi\varepsilon_0 \sqrt{R^2+x^2}} \cdot \frac{R}{\sqrt{R^2+x^2}} = \frac{\sigma R}{2\pi\varepsilon_0} \int_{-\infty}^{\infty} \frac{dx}{R^2+x^2}$

$$\Rightarrow \quad E = \frac{\sigma}{2\pi\varepsilon_0 R} \int_{-\infty}^{\infty} \frac{dx}{1+\left(\frac{x}{R}\right)^2} .$$

Symmetry $\Rightarrow \quad E = \frac{\sigma}{2\pi\varepsilon_0 R} \cdot 2 \int_0^{\infty} \frac{dx}{1+\left(\frac{x}{R}\right)^2} .$

Now substitute $\boxed{\tan\theta = \frac{x}{R} .}$

Then $\quad \frac{dx}{d\theta} = R\sec^2\theta \Rightarrow \boxed{\frac{dx}{R} = \sec^2\theta\, d\theta .}$

Now for the limits of the integral: $\tan\theta = 0 \Rightarrow \theta = 0$
$\tan\theta = \infty \Rightarrow \theta = \pi/2 .$

Thus $\quad E = \frac{\sigma}{2\pi\varepsilon_0 R} \cdot 2 \int_0^{\infty} \frac{dx}{1+\left(\frac{x}{R}\right)^2} = \frac{\sigma}{\cancel{2}\pi\varepsilon_0} \int_0^{\pi/2} \frac{\sec^2\theta\, d\theta}{1+\tan^2\theta}$

But $\quad \sin^2\theta + \cos^2\theta = 1 \Rightarrow 1+\tan^2\theta = \sec^2\theta .$

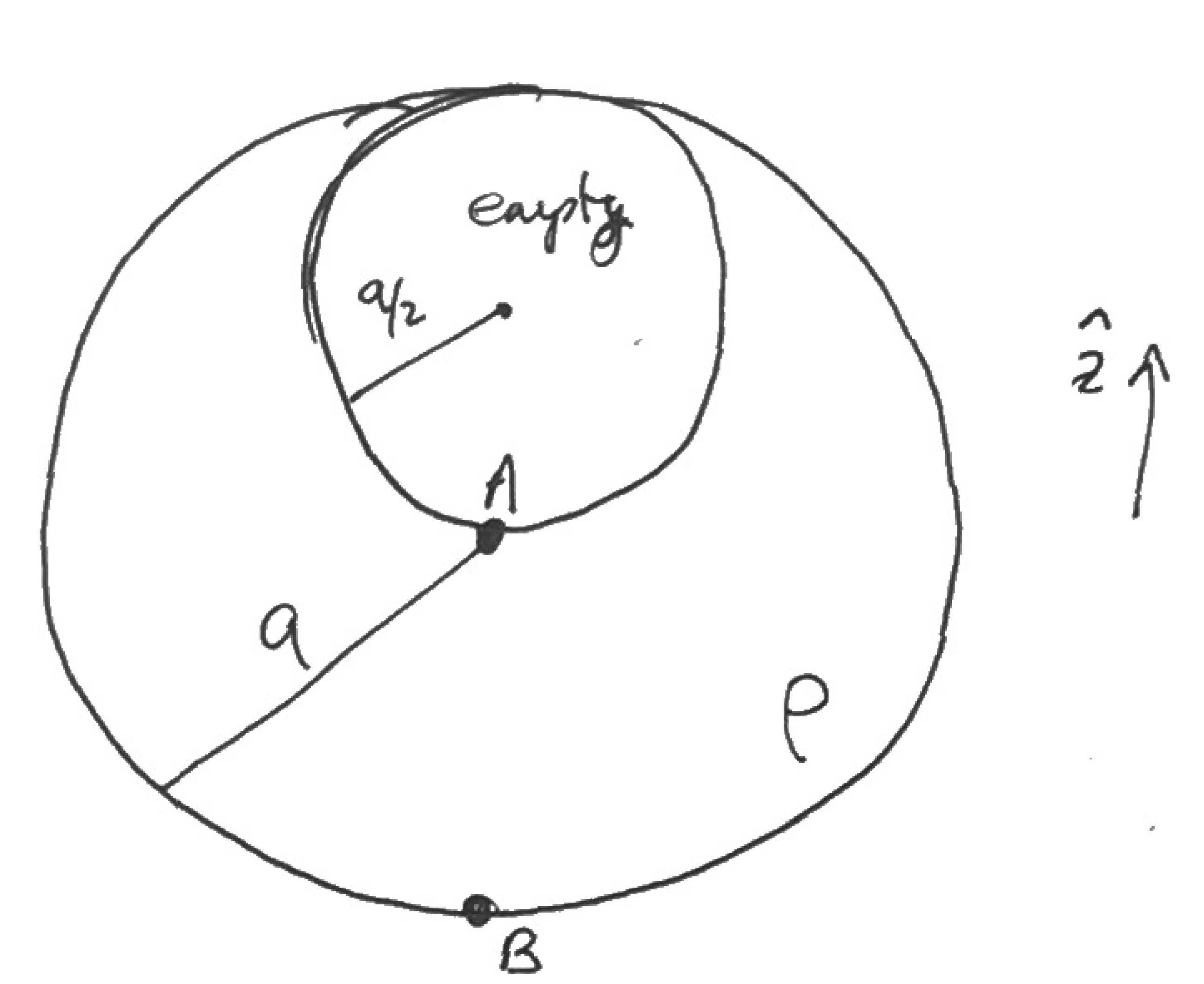
Thus, $\quad \boxed{E = \frac{\sigma}{\cancel{2}\pi\varepsilon_0} \int_0^{\pi/2} d\theta = \frac{\sigma}{2\varepsilon_0}}$ (in the positive $\hat{z}$ direction)

as desired.

Note: This is independent of $R$!

Is this surprising or expected?

#4) PM-1.69




This setup is equivalent to a uniformly charged sphere (with no cavity) of charge density ρ superimposed with an off-set sphere of radius a/2 of charge density −ρ. The E-field at A and B are the sums of these two contributions (call these respective contributions $E_\rho$ and $E_{-\rho}$).

Since A is at the center of the radius a sphere, it gets no contribution from the +ρ (radius a) sphere. ~~Using Gauss's Law~~ We can take the contribution $E_{-\rho}$ as coming from a point charge at the origin of the a/2 sphere with charge $-\rho\cdot\frac{4}{3}\pi\left(\frac{a}{2}\right)^3$. Since this sphere is negatively charged, the E-field at A will be pointing upwards.

Thus, $$\vec{E}_A = \frac{\overbrace{\rho\cdot\frac{4}{3}\pi\left(\frac{a}{2}\right)^3}^{\text{total charge}}}{4\pi\varepsilon_0\underbrace{(a/2)^2}_{\text{distance from A}}}\hat{z} = \frac{\rho}{3\varepsilon_0}\frac{a}{2}\hat{z} \Rightarrow \boxed{\vec{E}_A = \frac{a\rho}{6\varepsilon_0}\hat{z}}$$

Now $\vec{E}_B$ comes from the point charge $-\rho\frac{4}{3}\pi\left(\frac{a}{2}\right)^3$ located at the center of the −ρ, radius a/2 sphere (and the $\vec{E}$-field at B from this charge is therefore pointing up) and from the point charge $\rho\cdot\frac{4}{3}\pi a^3$ located at the center of the +ρ, radius a sphere. Thus we have

$$\vec{E}_B = \frac{1}{4\pi\varepsilon_0}\left[\frac{\rho\frac{4}{3}\pi\left(\frac{a}{2}\right)^3}{(3a/2)^2} - \frac{\rho\frac{4}{3}\pi a^3}{a^2}\right]\hat{z} = \frac{\rho a}{4\pi\varepsilon_0}\left[\frac{4\pi}{3\cdot 3^2\cdot 2} - \frac{4\pi}{3}\right] = -\frac{\rho a}{\varepsilon_0}\frac{17}{54}\hat{z}.$$

+ sign since pointing up

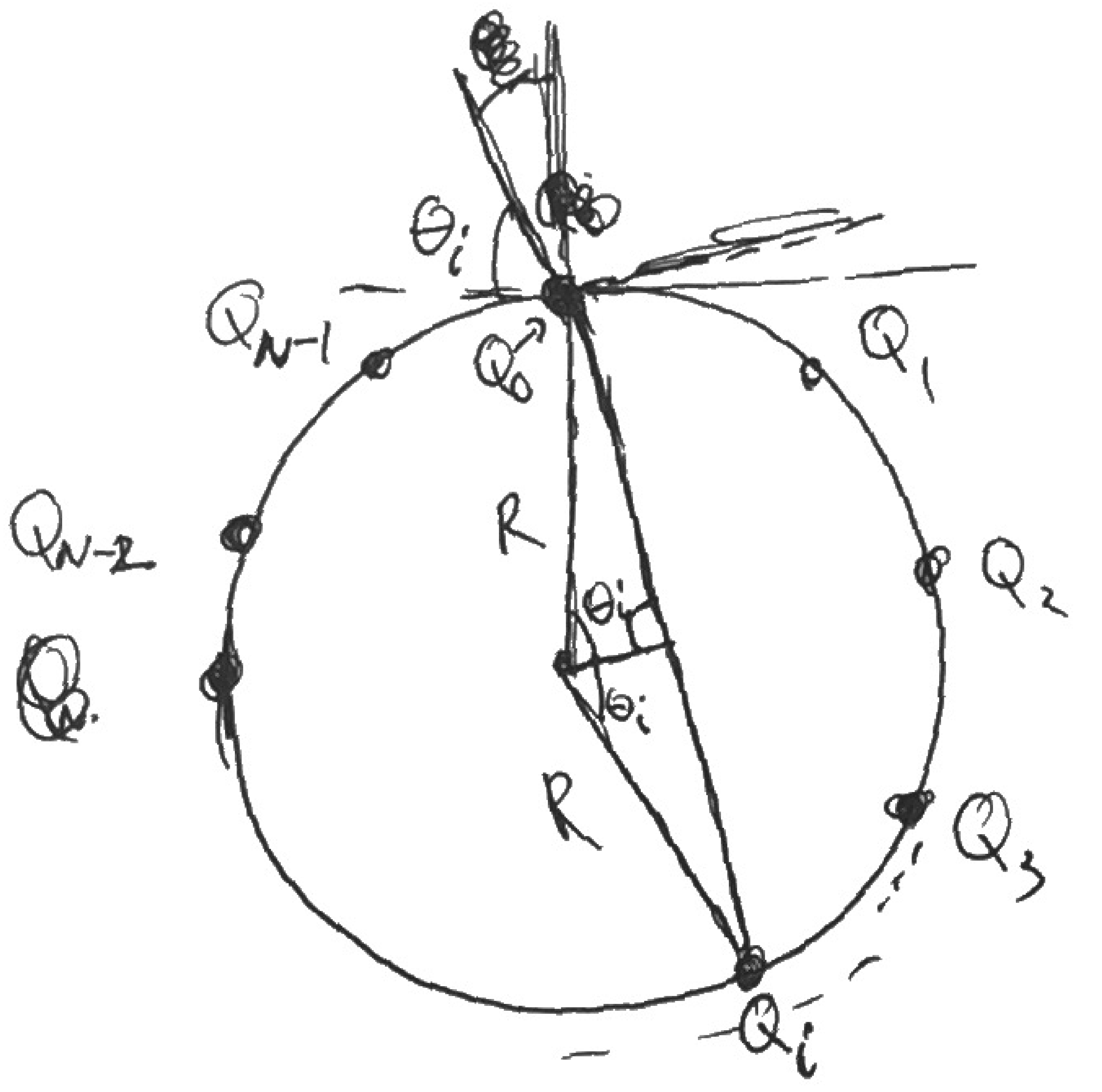
pt charge located a distance $\frac{3a}{2}$ from B

contribution from +ρ sphere points down at B

a distance a from B

Thus $\boxed{\vec{E}_B = \frac{-17}{54}\frac{\rho a}{\varepsilon_0}\hat{z}}$

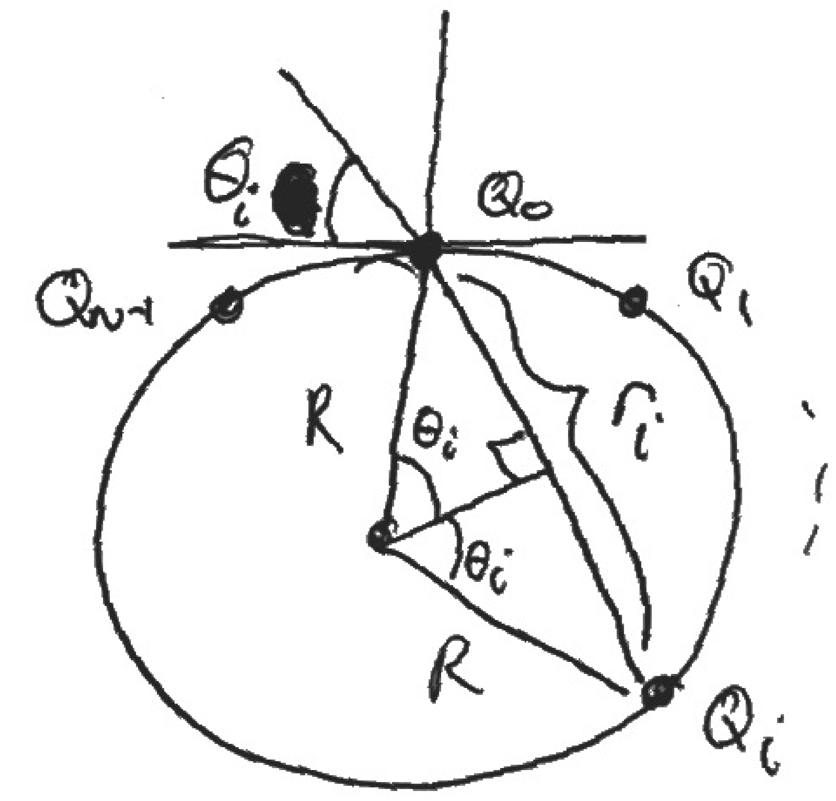
## #5) PM.1-51




N charges $Q_0, Q_1, \dots, Q_{N-1}$

each $Q_i = \frac{Q}{N}$

Let us find the field at $Q_0$. Due to symmetry, the only component that doesn't cancel is the component pointing up. Let's look at the contribution from the $i^{th}$ particle $Q_{i_0}$ (as shown above)..




Let me re-draw the above picture:

Due to the symmetric placements of the charges, we see that we have $\theta_i = \frac{i\pi}{N}$.

The distance $r_i$ is $2 \cdot R\sin\theta_i \Rightarrow r_i = 2R\sin(\frac{i\pi}{N})$.

Thus $E_i$, the contribution (in the upwards direction) from $Q_i$ at $Q_0$ is

$$E_i = \frac{Q_i}{4\pi\varepsilon_0}\frac{1}{r_i^2}\sin\theta_i = \frac{Q/N}{4\pi\varepsilon_0}\frac{\sin(\frac{i\pi}{N})}{(2R\sin(\frac{i\pi}{N}))^2}$$

$$\Rightarrow E_i = \frac{Q}{4\pi\varepsilon_0}\cdot\frac{1}{4R^2N\sin(\frac{i\pi}{N})}.$$

Thus $\boxed{E_{TOT} = \sum_{i=1}^{N-1} E_i = \frac{Q}{16\pi\varepsilon_0 R^2 N}\sum_{i=1}^{N-1}\frac{1}{\sin(\frac{i\pi}{N})}}$ in the upwards direction

Now, for small $i$ (i.e., $\frac{i}{N} \ll 1$), $\sin\left(\frac{i\pi}{N}\right) \simeq \frac{i\pi}{N}$ and our sum is

$$E_{TOT} \simeq \frac{Q}{16\pi \varepsilon_0 R^2 N} \sum_{i=1}^{N-1} \frac{1}{\frac{i\pi}{N}} \sim \sum_{i=1}^{N-1} \frac{1}{i}$$ because the N's cancel,

and this diverges like $\log(N)$

For the force, however, we ~~add~~ multiply in another factor of $Q_0 = \frac{Q}{N}$ so that the force, for large $N$, goes like $F \sim \frac{1}{N} \sum_{i=1}^{N-1} \frac{1}{i} \sim \frac{\log(N)}{N}$ and this is indeed finite as $N \to \infty$.

Note that the approximation $\sum_{i=1}^{N-1} \frac{1}{\sin\left(\frac{i\pi}{N}\right)} \overset{\text{as } N\to\infty}{\simeq} \sum_{i=1}^{N-1} \frac{1}{\frac{i\pi}{N}}$ is valid even for $i$ not necessarily small because as $N \to \infty$ the only charges that contribute to the E-field and/or force are those that are very close to $Q_0$, since all other charges die off as $\frac{Q}{N} \to 0$ so that charges must become infinitesimally close to $Q_0$ to contribute.

#6) PM 1-76) $\rho(r) = -Ce^{-2r/a_0}$

The fraction of negative charge inside ~~the~~ a sphere of radius $a_0$ is

$$\text{frac} = \frac{\int_0^{a_0} \rho\, dV}{\int_0^{\infty} \rho\, dV} = \frac{\int_0^{a_0} -Ce^{-2r/a_0} \cdot 4\pi r^2 dr}{\int_0^{\infty} -Ce^{-2r/a_0} \cdot 4\pi r^2 dr} = \frac{\int_0^{a_0} r^2 e^{-2r/a_0} dr}{\int_0^{\infty} r^2 e^{-2r/a_0} dr};$$

(Note: don't have to solve for C!)

Let $x = \frac{2r}{a_0}$ then $dr = \frac{a_0}{2}dx$ and we have (being careful with the limits of integration

$$\text{frac} = \frac{\int_0^2 x^2 e^{-x} dx}{\int_0^{\infty} x^2 e^{-x} dx}.$$

Now, using integration by parts or the cool trick that I'll show ~~below~~ in class in more detail, we have (or Google)

$$\int_a^b x^2 e^{-x} dx = -(x^2 + 2x + 2)e^{-x}\Big|_a^b.$$

Thus $\int_0^2 x^2 e^{-x} dx = -(x^2+2x+2)e^{-x}\Big|_0^2 = 2 - 10e^{-2}$

and $\int_0^{\infty} x^2 e^{-x} dx = -(x^2+2x+2)e^{-x}\Big|_0^{\infty} = 2.$

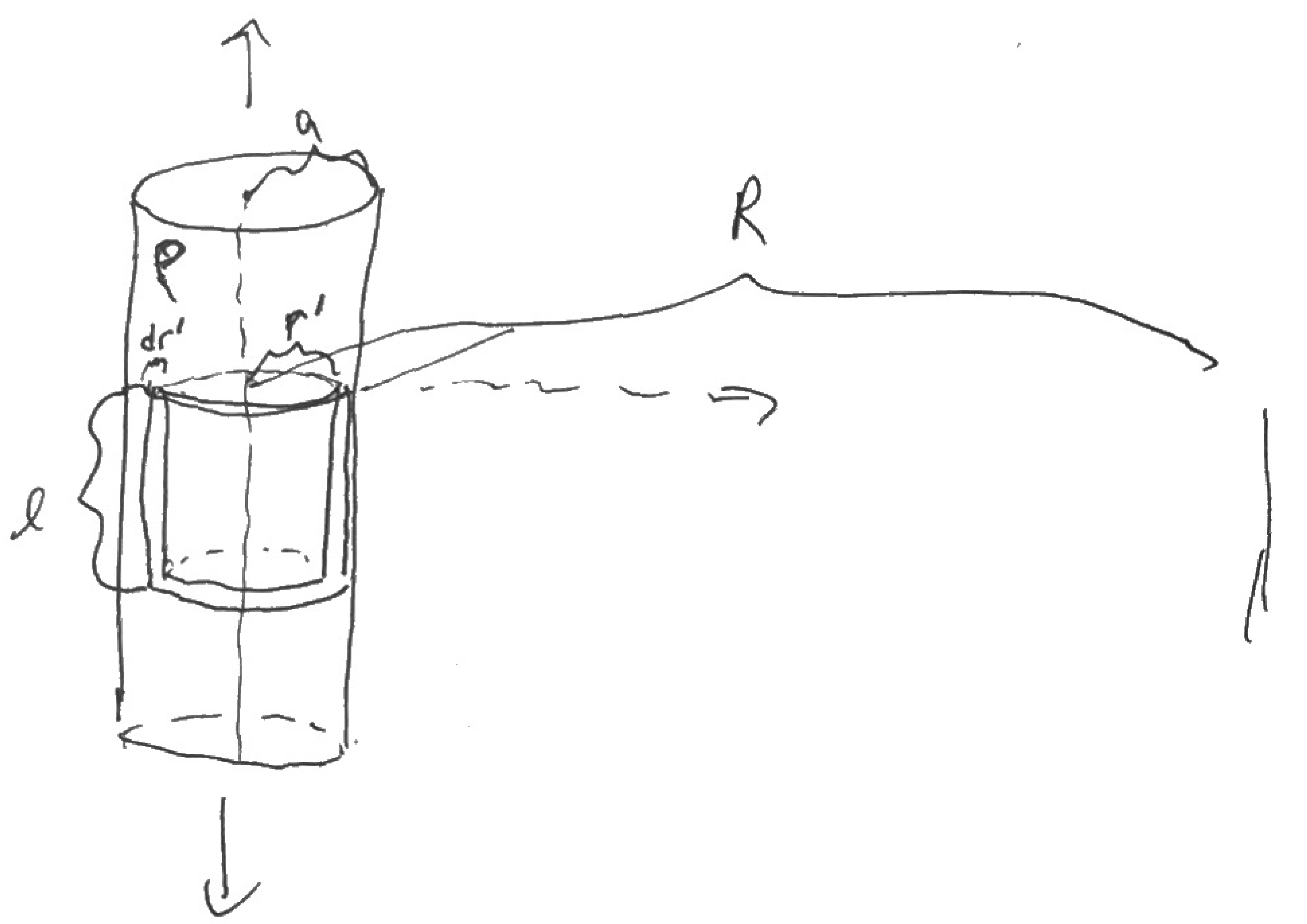
Thus $\text{frac} = \frac{2-10e^{-2}}{2} = 1 - 5/e^2 \simeq .323$ is the fraction of the full negative charge in the sphere of radius $a_0$. Thus the net charge in this sphere (including the proton at the center) is $Q_{net} = (1-.323)e = (1-.323)(1.6\times10^{-19})\,C$

$$\Rightarrow \boxed{Q_{NET} \simeq 1.08\times10^{-19} C}$$

The field at $r=a_0$ is then $\boxed{E = \frac{Q_{NET}}{4\pi\varepsilon_0 a_0^2} = \frac{1.08\times10^{-19}\,C}{(.53\times10^{-10}\,m)^2}\cdot\left(9\times10^9 \frac{kg\,m^3}{s^2C^2}\right) \simeq 3.5\times10^{11}\ N/C}$

(which is massive!)

#7) PM 1-24 and 1-83)




For the moment take the cylinder to be infinitely long, and then we'll calculate the energy per unit length.

By Gauss's law, or by problem #3, the E-field outside a uniformly charged line charge is $\vec{E} = \frac{\lambda}{2\pi\varepsilon_0 r}\hat{r}$ where $(r, \phi, z)$ are cylindrical coordinates.

Consider moving a thin shell of radius r, thickness dr, and length l (as shown above) out to some radius $R\ (<\infty)$. We want to compute the work required to do so, and then integrate over each shell. Namely, we first move the shell at $r'=a$ out to $r'=R$, then that at $r'=a-dr'$ out to $r'=R$, and so on.

For a shell at radius r', the field from the charges at radius $< r'$ is given by $\vec{E} = \frac{\lambda}{2\pi\varepsilon_0 r}\hat{r}$ with $\lambda = \frac{\overbrace{\rho\cdot(\pi r'^2 l)}^{\text{total charge inside}}}{\underbrace{l}_{\text{length}}} = \pi\rho r'^2$

Thus $\vec{E} = \frac{\rho r'^2}{2\varepsilon_0}\frac{\hat{r}}{r}$ is the $\vec{E}$-field that the cylindrical shell ~~of~~ radius r' sees.

The force on this shell is therefore the total charge of the shell times this $\vec{E}$-field, so that

$$\vec{F}_{\substack{\text{on shell}\\ \text{of } r'}} = Q_{shell}\vec{E} \qquad \text{where } Q_{shell} = \rho \cdot Vol_{shell}$$
$$= \rho \cdot l \cdot 2\pi r' dr'$$

Thus $\vec{F}_{\substack{\text{on shell}\\ \text{of } r'}} = \rho l 2\pi r' dr' \cdot \dfrac{\rho r'^2}{2\varepsilon_0}\dfrac{\hat{r}}{r}$ (Note: don't confuse $r'$ with $r$! These are different variables!)

$$= \frac{\pi \rho^2 r'^3 l\, dr'}{\varepsilon_0}\frac{\hat{r}}{r}$$

The work $dW$ done by this force to move the shell from $r=r'$ to $r=R$ is therefore (with $d\vec{r} = dr\,\hat{r}$)

$$dW = \int_{r=r'}^{r=R} \vec{F}_{\substack{\text{on shell}\\ \text{of } r'}} \cdot d\vec{r} = \frac{\pi\rho^2 r'^3 l\, dr'}{\varepsilon_0}\int_{r=r'}^{r=R}\frac{dr}{r}$$

Note: $dr'$ comes outside the integral

$$= \frac{\pi\rho^2 l}{\varepsilon_0} r'^3 dr' \ln(R/r').$$

Now for the total work done we compute $W = \int_{r'=a}^{r'=0} dW$, i.e., integrate up starting with the shell at $r'=a$ in towards the shell at $r'=0$. We have

$$W = \frac{\pi\rho^2 l}{\varepsilon_0}\int_a^0 r'^3 \ln(R/r')\,dr' = \frac{\pi\rho^2 l}{\varepsilon_0}\int_0^a r'^3 \ln\left(\frac{r'}{R}\right) dr'$$

Let $x = \frac{r'}{R}$. Then $dr' = R\,dx$, so $W = \frac{\pi\rho^2 l}{\varepsilon_0} R^4 \int_0^{a/R} x^3 \ln(x)\,dx$.

Integration by parts (or Google) gives $\int_a^b x^3 \ln(x)\,dx = x^4\left(\frac{1}{4}\ln(x) - \frac{1}{16}\right)$

Thus $W = \frac{\pi\rho^2 l}{\varepsilon_0} R^4 \cdot \left[x^4\left(\frac{1}{4}\ln(x) - \frac{1}{16}\right)\right]_0^{a/R} = \frac{\pi\rho^2 l a^4}{\varepsilon_0}\left[\frac{1}{4}\ln\left(\frac{a}{R}\right) - \frac{1}{16}\right]$

(12)

using $\ln\left(\frac{a}{R}\right) = -\ln(R/a)$ and $\lambda = \pi \rho a^2$

Thus $-\frac{W}{\ell} = \boxed{\text{potential energy per unit length} = \frac{\lambda^2}{4\pi\varepsilon_0}\left[\frac{1}{4} + \ln\left(\frac{R}{a}\right)\right] = \frac{U}{\ell}}$

Note that as $R \to \infty$, $\frac{U}{\ell} \to \infty$. Physically, why does this happen?

Now PM-183

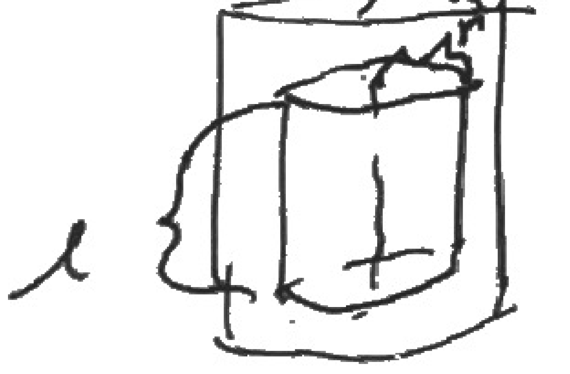
eq. (15.3) says $U = \frac{\varepsilon_0}{2}\int_{\text{entire field}} E^2 \, dV$. Instead of integrating over the entire field let us instead only integrate up to some finite R. We will first integrate the field outside the cylinder, from $r=a$ to $r=R$, then that inside the cylinder, from $r=0$ to $r=a$.

We already know that the field outside the cylinder is given by $\vec{E} = \frac{\rho a^2}{2\varepsilon_0}\frac{\hat{r}}{r}$ (just plug in $r'=a$ in the equation at the bottom of the first page of the solution to this problem).

Thus $U_{\text{outside}} = \frac{\varepsilon_0}{2}\int_{r=a}^{R}\left(\frac{\rho a^2}{2\varepsilon_0}\right)^2 \frac{1}{r^2} \cdot \underbrace{2\pi r\,dr \cdot \ell}_{dV \text{ after integrating } d\phi \to 2\pi,\ dz \to \ell.}$

$$\Rightarrow \boxed{U_{\text{outside}} = \frac{\pi\rho^2 a^4 \ell}{4\varepsilon_0}\int_a^R \frac{dr}{r} = \frac{\pi\rho^2 a^4 \ell}{4\varepsilon_0}\ln(R/a).}$$

We now use Gauss's law to find the field inside the sphere, at $0 \le r < a$.



Gauss: $\oint \vec{E}\cdot d\vec{A} = \frac{Q_{enc}}{\varepsilon_0}$. $\oint \vec{E}\cdot d\vec{A} = E \cdot 2\pi r \cdot \ell$

$Q_{enc}$ = charge inside cylinder of length $\ell$, radius $r < a$ $= \rho\pi r^2 \ell$

Thus $E 2\pi r \ell = \frac{\rho\pi r^2 \ell}{\varepsilon_0} \Rightarrow E = \frac{\rho}{2\varepsilon_0} r$

Thus $\boxed{\vec{E}_{\text{inside}} = \frac{\rho}{2\varepsilon_0} r \hat{r}}$