Homework #3 (due 10/25)
(Physics 115B, Fall 2017)

(100 points)

1-5. Griffiths 4.11, 4.13, 4.17, 4.42, 4.43

6. Find the eigenvalues and eigenfunctions of the hydrogen atom for motion in two dimensions. (Hint: Use cylindrical coordinates.)

(10') 4.11

(10') 4.13

(20) 4.17

(20) 4.42

(20) 4.43

(20) 6.

Homework 3 Solution

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4.11

(a). Normalize R_{20} :

$$1 = \int_0^{+\infty} |R_{20}|^2 r^2 dr$$

$$= \int_0^{+\infty} \frac{C_0^2}{4a^2} \left(1-\frac{r}{2a}\right)^2 e^{-\frac{r}{a}} r^2 dr$$

let $x = \frac{r}{a}$

$$= \int_0^{+\infty} \frac{C_0^2}{4a^2} a^3 \left(1-\frac{x}{2}\right)^2 e^{-x} x^2 dx$$

$$= \frac{C_0^2 a}{4} \int_0^{+\infty} x^2 \left(1-x+\frac{x^2}{4}\right) e^{-x} dx$$

$$= \frac{C_0^2 a}{4} \int_0^{+\infty} \left(x^2 - x^3 + \frac{x^4}{4}\right) e^{-x} dx$$

$$= \frac{C_0^2 a}{4} \left(\Gamma(3) - \Gamma(4) + \frac{\Gamma(5)}{4}\right)$$

$$= \frac{a}{2} C_0^2$$

$$\Rightarrow \quad C_0 = \sqrt{\frac{2}{a}}, \quad \text{then: } \psi_{200} = \frac{1}{\sqrt{4\pi}} \sqrt{\frac{2}{a}} \frac{1}{2a} \left(1-\frac{r}{2a}\right) e^{-\frac{r}{2a}}$$

$$= \frac{1}{\sqrt{2\pi a}} \frac{1}{2a} \left(1-\frac{r}{2a}\right) e^{-\frac{r}{2a}}$$

(b) similary to (a).

$$R_{21} = \frac{C_0}{4a^2} r e^{-\frac{r}{2a}}$$

$$1 = \left(\frac{C_0}{4a^2}\right)^2 a^5 \int_0^\infty x^4 e^{-x} dx = \frac{3}{2} a C_0^2$$

$$\Rightarrow \quad C_0 = \sqrt{\frac{2}{3a}}$$

Then: $R_{21} = \frac{1}{\sqrt{6a}} \frac{1}{2a^2} r e^{-\frac{r}{2a}}$

$$\psi_{2,1,\pm 1} = \frac{1}{\sqrt{6a}} \frac{1}{2a^2} r e^{-\frac{r}{2a}} \left(\mp \sqrt{\frac{3}{8\pi}} \sin\theta \, e^{\pm i\phi} \right)$$

$$= \mp \frac{1}{\sqrt{\pi a}} \frac{1}{8a^2} r e^{-\frac{r}{2a}} \sin\theta \, e^{\pm i\phi}$$

$$\psi_{210} = \frac{1}{\sqrt{6a}} \frac{1}{2a^2} r e^{-r/2a} \left(\sqrt{\frac{3}{4\pi}} \cos\theta \right)$$

$$= \frac{1}{\sqrt{2\pi a}} \frac{1}{4a^2} r e^{-\frac{r}{2a}} \cos\theta$$

4.13.

(a) Find $\langle r \rangle$, $\langle r^2 \rangle$ in ground state.

You can do in the usual way. Here I provide another way to find $\langle r \rangle$, $\langle r^2 \rangle$.

we know:

$$\psi_{100}(r,\theta,\phi) = \frac{1}{\sqrt{\pi a^3}} e^{-\frac{r}{a}}$$

$$1 = \int |\psi_{100}|^2 dv = 4\pi \int_0^{+\infty} \frac{1}{\pi a^3} e^{-\frac{2r}{a}} r^2 dr$$

$$= \frac{4}{a^3} \int_0^{+\infty} e^{-\frac{2r}{a}} r^2 dr$$

$$\rightarrow \int_0^{+\infty} e^{-\frac{2r}{a}} r^2 dr = \frac{a^3}{4}$$

Set $k = \frac{1}{a}$

$$\int_0^{+\infty} e^{-2kr} r^2 dr = \frac{1}{4} k^{-3} \quad \text{----} \quad ①$$

$\frac{d}{dk}$ acts on both sides:

$$\int_0^{+\infty} (-2r) e^{-2kr} r^2 = \frac{-3}{4} k^{-4}$$

$\Rightarrow$

$$4\kappa^3 \int_0^{+\infty} r^3 e^{-2\kappa r} dr = \frac{3}{2}\kappa^{-1} = \frac{3}{2}a$$

recognize that left hand side is $\langle r \rangle$, then:

$$\langle r \rangle = \frac{3}{2}a$$

$\frac{d^2}{d\kappa^2}$ acts on both sides of ①:

$$\int_0^{+\infty} 4r^2 e^{-2\kappa r} r^2 dr = \frac{(-3)(-4)}{4}\kappa^{-5}$$

$$\Rightarrow \quad 4\kappa^3 \int_0^{+\infty} r^4 e^{-2\kappa r} dr = 3\kappa^{-2} = 3a^2$$

which is

$$\langle r^2 \rangle = 3a^2$$

(b) We know ground state is a spherically symmetric state

$$\langle x \rangle = 0$$

$$\langle x^2 \rangle = \frac{1}{3}\langle r^2 \rangle = a^2$$

Or you can write down the integral, you'll get the result above immediately

(c)

$$\psi_{211} = -\frac{1}{\sqrt{\pi a}}\frac{1}{8a^2} r e^{-\frac{r}{2a}} \sin\theta \, e^{i\phi}$$

$$\langle x^2 \rangle = \int_0^{+\infty} dr \int_0^{\pi} d\theta \int_0^{2\pi} d\varphi \; r^2 \sin\theta \cdot (r \sin\theta \cos\phi)^2 \frac{1}{\pi a}\left(\frac{1}{8a^2}\right)^2 r^2 e^{-\frac{r}{a}} \sin^2\theta$$

$$= \frac{1}{\pi a}\frac{1}{64 a^4} \int_0^{+\infty} r^6 e^{-\frac{r}{a}} dr \int_0^{\pi} \sin^5\theta \, d\theta \int_0^{2\pi} \cos^2\varphi \, d\varphi$$

$$\int_0^{+\infty} r^6 e^{-\frac{r}{a}} dr = a^7 \, \Gamma(7)$$

according to dimension

according to r^6.

$$\int_0^{\pi} \sin^5\theta \, d\theta = -\int_0^{\pi} \sin^4\theta \, d\cos\theta = \int_{\pi}^{0} (\sin^2\theta)^2 \, d\cos\theta$$

$$= \int_{\pi}^{0} (1-\cos^2\theta)^2 \, d\cos\theta$$

$$= \int_{-1}^{1} (1-x^2)^2 \, dx$$

$$= 2 \int_0^1 (x^2-1)^2 \, dx$$

$$= 2 \int_0^1 (x^4 - 2x^2 + 1) \, dx$$

$$= 2 \left[\frac{1}{5} - \frac{2}{3} + 1 \right] = \frac{16}{15}$$

$$\int_0^{2\pi} \cos^2\varphi \, d\varphi = \int_0^{2\pi} \frac{1+\cos 2\varphi}{2} \, d\varphi = \pi$$

Then:

$$\langle x^2 \rangle = \frac{1}{\pi a} \frac{1}{64 a^4} a^7 \Gamma(7) \frac{16}{15} \pi = 12 a^2$$

4.17

(a).

Write down the ~~system~~ Hamiltonian :

hydrogen atom : $H = \frac{p^2}{2m} - \frac{e^2}{4\pi\varepsilon_0}\frac{1}{r}$

earth atom : $H = \frac{p^2}{2m} - \frac{GMm}{r}$

Then we need map : $\frac{e^2}{4\pi\varepsilon_0} \longmapsto GMm$

(b) Bohr radius :

$$a \equiv \frac{4\pi\varepsilon_0 \hbar^2}{e^2 \quad m} \longrightarrow a_g = \frac{\hbar^2}{GMm \quad m} \approx 2.34 \times 10^{-138} m$$

(c) Bohr formula:

$$E_n = -\left[\frac{m}{2\hbar^2}\left(\frac{e^2}{4\pi\varepsilon_0}\right)^2\right]\frac{1}{n^2} \longrightarrow E_n = -\frac{m}{2\hbar^2}(GMm)^2\frac{1}{n^2}$$

Classically : $E_c = \frac{1}{2}mv^2 - G\frac{Mm}{r_0}$

Note that: $\frac{GMm}{r_0^2} = \frac{mv^2}{r_0} \Rightarrow E_c = -\frac{GMm}{2r_0}$

Let $E_c = E_n$:

$$\Rightarrow n^2 = \frac{GMm^2}{\hbar^2} r_0 \quad \Rightarrow \quad n = \sqrt{\frac{r_0}{a_g}}$$

we know: r_0 = earth-sun distance $= 1.496 \times 10^{11} m$

$$\Rightarrow \quad n = \sqrt{\frac{1.496 \times 10^{11}}{2.34 \times 10^{-138}}} = 2.53 \times 10^{74}$$

(d).

$$E_n = -\frac{m}{2\hbar^2} (GMm)^2 \frac{1}{n^2}$$

$$dE_n = -\frac{m}{2\hbar^2} (GMm)^2 \frac{(-2)\, dn}{n^3}$$

$$= \frac{m}{\hbar^2} (GMm)^2 \frac{dn}{n^3}$$

$dn = -1$

then: $$dE_n = -\frac{m}{\hbar^2} (GMm)^2 \frac{1}{n^3}$$

"$-$" sign means energy release.

$$\Delta E_n = \frac{m}{\hbar^2} (GMm)^2 \frac{1}{n^3}$$

$$= 2.09 \times 10^{-41} J$$

let $\Delta E_n = \frac{hc}{\lambda}$

$$\Rightarrow \quad \lambda = 9.52 \times 10^{15} m$$

$1\,ly = 9.46 \times 10^{15}\,m$, so : $\lambda \approx 1\,ly$

Is it a coincidence ?

It's not a coincidence. The wavelength of the photon emitted in a transition from a highly excited state to next lower one is equal to the distance light wound travel in on orbital period.

From 1(c) : $n^2 = \dfrac{GMm^2 r_0}{\hbar^2}$

so :

$$\lambda = \frac{ch}{\Delta E} = c\, 2\pi\hbar \,\frac{\hbar^2 n^3}{G^2M^2m^3} = c\,\frac{2\pi\hbar^3}{G^2M^2m^3}\left(\frac{GMm^2 r_0}{\hbar^2}\right)^{\frac{3}{2}}$$

$$= c\left(2\pi\sqrt{\frac{r_0^3}{GM}}\right)$$

we know : $v = \sqrt{\dfrac{GM}{r_0}} = \dfrac{2\pi r_0}{T}$

Then :

$$\lambda = cT$$

4.42

(a)

$$\psi = \frac{1}{\sqrt{\pi a^3}} e^{-\frac{r}{a}}$$

$$\phi(\vec{p}) = \frac{1}{(2\pi\hbar)^{3/2}} \frac{1}{\sqrt{\pi a^3}} \int e^{-i\vec{p}\cdot\vec{r}/\hbar} e^{-\frac{r}{a}} r^2 \sin\theta \, dr d\theta d\phi$$

$$= \frac{1}{(2\pi\hbar)^{3/2}} \frac{2\pi}{\sqrt{\pi a^3}} \int_0^{\pi} d\theta \int_0^{+\infty} dr \; r^2 e^{-\frac{r}{a}} \sin\theta \, e^{-\frac{ipr}{\hbar}\cos\theta}$$

where :

$$\int_0^{\pi} e^{-\frac{ipr}{\hbar}\cos\theta} \sin\theta \, d\theta$$

$$= \int_{\pi}^{0} e^{-\frac{ipr}{\hbar}\cos\theta} \, d\cos\theta$$

$$= \frac{e^{-\frac{ipr}{\hbar}} - e^{\frac{ipr}{\hbar}}}{-\frac{ipr}{\hbar}}$$

$$= \frac{-2i\sin\left(\frac{pr}{\hbar}\right)}{-i\frac{pr}{\hbar}} = \frac{2\hbar\sin\left(\frac{pr}{\hbar}\right)}{pr}$$

$$\phi(\vec{p}) = \frac{1}{(2\pi\hbar)^{3/2}} \frac{2\pi}{\sqrt{\pi a^3}} \int_0^{+\infty} r^2 e^{-\frac{r}{a}} \frac{2\hbar\sin\left(\frac{pr}{\hbar}\right)}{pr} dr$$

$$= \frac{1}{(2\pi\hbar)^{3/2}} \frac{2\pi}{\sqrt{\pi a^3}} \frac{2\hbar}{p} \int_0^{+\infty} r e^{-\frac{r}{a}} \sin\left(\frac{pr}{\hbar}\right) dr$$

where :

$$\int_0^{+\infty} r\, e^{-\frac{r}{a}} \sin\left(\frac{pr}{\hbar}\right) dr$$

$$= \mathrm{Im} \int_0^{+\infty} r\, e^{-\frac{r}{a}}\, e^{\frac{ipr}{\hbar}}\, dr$$

$$= \mathrm{Im} \int_0^{+\infty} r\, e^{-\left(\frac{1}{a} - \frac{ip}{\hbar}\right) r}\, dr$$

$$= \mathrm{Im} \frac{1}{\left(\frac{1}{a} - \frac{ip}{\hbar}\right)^2}$$

$$= \mathrm{Im} \left(\frac{\frac{1}{a} + \frac{ip}{\hbar}}{\frac{1}{a^2} + \frac{p^2}{\hbar^2}} \right)^2$$

$$= \frac{\frac{2p}{a\hbar}}{\left(\frac{1}{a^2} + \frac{p^2}{\hbar^2}\right)^2} = \frac{\frac{2pa^3}{\hbar}}{\left(1 + \left(\frac{ap}{\hbar}\right)^2\right)^2}$$

Then :

$$\phi(\vec{p}) = \frac{1}{\pi} \left(\frac{2a}{\hbar}\right)^{\frac{3}{2}} \frac{1}{\left[1 + \left(\frac{ap}{\hbar}\right)^2\right]^2}$$

(b).

To prove it's normalized:

$$\int_0^{+\infty} 4\pi p^2 \cdot \frac{1}{\pi^2} \frac{(2a)^3}{(\hbar)^3} \frac{1}{\left[1+\left(\frac{ap}{\hbar}\right)^2\right]^4} dp$$

$$= \frac{4\pi}{\pi^2} \frac{(2a)^3}{\hbar^3} \int_0^{+\infty} \frac{p^2}{\left[1+\left(\frac{ap}{\hbar}\right)^2\right]^4} dp$$

set $\frac{ap}{\hbar} = \tan\theta$

$$= \frac{4\pi}{\pi^2} \frac{(2a)^3}{\hbar^3} \int_0^{\frac{\pi}{2}} \frac{\left(\frac{\hbar}{a}\right)^3 \tan^2\theta}{\left[1+\tan^2\theta\right]^4} d\tan\theta$$

$$= \frac{4\pi}{\pi^2} \times 8 \int_0^{\frac{\pi}{2}} \frac{\tan^2\theta}{\frac{1}{\cos^8\theta}} \frac{1}{\cos^2\theta} d\theta$$

$$= \frac{32}{\pi} \int_0^{\frac{\pi}{2}} \tan^2\theta \cos^6\theta \, d\theta$$

$$= \frac{32}{\pi} \int_0^{\frac{\pi}{2}} \sin^2\theta \cos^4\theta \, d\theta$$

$$= \frac{32}{\pi} \frac{1}{2} B\left(\frac{3}{2}, \frac{5}{2}\right)$$

$$= \frac{32}{\pi} \frac{1}{2} \frac{\Gamma\left(\frac{3}{2}\right) \Gamma\left(\frac{5}{2}\right)}{\Gamma(4)} = \frac{32}{\pi} \frac{1}{2} \frac{\frac{1}{2}\sqrt{\pi} \cdot \frac{3}{2} \times \frac{1}{2}\sqrt{\pi}}{3!} = 1$$

(c).

$$\langle p^2\rangle = \int p^2 |\phi|^2 d^3_p = \frac{1}{\pi^2}\left(\frac{2a}{\hbar}\right)^3 4\pi \int_0^\infty \frac{p^4}{\left[1+\left(\frac{ap}{\hbar}\right)^2\right]^4} dp$$

$$= \frac{1}{\pi^2}\left(\frac{2a}{\hbar}\right)^3 4\pi \left(\frac{\hbar}{a}\right)^5 \int_0^{\frac{\pi}{2}} \frac{\tan^4\theta}{\frac{1}{\cos^8\theta}} d\tan\theta$$

$$\int_0^{\frac{\pi}{2}} \tan^4\theta \cos^8\theta \frac{1}{\cos^2\theta} d\theta = \int_0^{\frac{\pi}{2}} \tan^4\theta \cos^6\theta \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \sin^4\theta \cos^2\theta \, d\theta = \frac{1}{2} B\left(\frac{5}{2}, \frac{3}{2}\right) = \frac{\Gamma(\frac{5}{2})\Gamma(\frac{3}{2})}{\Gamma(4)} \frac{1}{2}$$

$$= \frac{\frac{3}{2} \times \frac{1}{2}\sqrt{\pi} \times \frac{1}{2}\sqrt{\pi}}{4!} \frac{1}{2}$$

plug in :

$$\langle p^2\rangle = \frac{\hbar^2}{a^2}$$

(d)

$$\langle T\rangle = \frac{1}{2m}\langle p^2\rangle = \frac{1}{2m}\frac{\hbar^2}{a^2} = \frac{m}{2\hbar^2}\left(\frac{e^2}{4\pi\varepsilon_0}\right)^2 = -E_1$$

consistent with virial theorem.

4.43.

(a). $\psi_{321} = Y_{21} R_{32} = \frac{4}{81\sqrt{30}} \frac{1}{a^{3/2}} \left(\frac{r}{a}\right)^2 e^{-\frac{r}{3a}} \left[-\sqrt{\frac{15}{8\pi}} \sin\theta \cos\theta e^{i\phi} \right]$

$$= -\frac{1}{\sqrt{\pi}} \frac{1}{81 a^{7/2}} r^2 e^{-\frac{r}{3a}} \sin\theta \cos\theta \, e^{i\phi}$$

(b)

$$\int |\psi|^2 d^3 r = \int \frac{1}{\pi} \frac{1}{81^2 a^7} r^4 e^{-\frac{2r}{3a}} \sin^2\theta \cos^2\theta \; r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$= \frac{2\pi}{\pi \, 81^2 a^7} \int_0^{+\infty} r^6 e^{-\frac{2r}{3a}} dr \int_0^{\pi} \sin^3\theta \cos^2\theta \, d\theta$$

where :

$$\int_0^{+\infty} r^6 e^{-\frac{2r}{3a}} dr = \left(\frac{3a}{2}\right)^7 \Gamma(7)$$

$$\int_0^{\pi} \sin^3\theta \cos^2\theta \, d\theta \xlongequal{\frac{\pi}{2} - \theta = \tilde{\theta}} \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \cos^3\tilde{\theta} \sin^2\tilde{\theta} \, d\tilde{\theta}$$

$$= 2 \int_0^{\frac{\pi}{2}} \cos^3\tilde{\theta} \sin^2\tilde{\theta} \, d\tilde{\theta}$$

$$= B\left(2, \frac{3}{2}\right)$$

Then:

$$\int |\psi|^2 \, d^3_r = \frac{2\pi}{\pi\, 81^2 a^7} \left(\frac{3}{2}\right)^7 a^7 \, \Gamma(7) \, B(2, \tfrac{3}{2})$$

$$= \frac{2}{81^2} \left(\frac{3}{2}\right)^7 \Gamma(7) \, \frac{\Gamma(2)\,\Gamma(\frac{3}{2})}{\Gamma(\frac{7}{2})}$$

$$= 1$$

(c).

$$\langle r^s \rangle = \int_0^{+\infty} r^s \, |R_{32}|^2 r^2 \, dr$$

$$= \left(\frac{4}{81}\right)^2 \frac{1}{30} \frac{1}{a^7} \int_0^{+\infty} r^{s+6} e^{-\frac{2r}{3a}} \, dr$$

$$= \left(\frac{4}{81}\right)^2 \frac{1}{30} \frac{1}{a^7} \left(\frac{3a}{2}\right)^{s+7} \Gamma(s+7)$$

$$= \left(\frac{4}{81}\right)^2 \frac{1}{30} \left(\frac{3}{2}\right)^7 \left(\frac{3a}{2}\right)^s (s+6)!$$

$$= \frac{(s+6)!}{720} \left(\frac{3a}{2}\right)^s$$

Finite for $s > -7$

6.

We want to find eigenvalues & eigenfunctions of hydrogen atom for motion in two dimension. Before start,

① we only confine our attention to discrete bound state.

② We say motion in two dimension which means we confine our atom in two dimension instead of saying the space time is (2+1) dimension. So, our $V(r) = -\frac{e^2}{4\pi\varepsilon_0}\frac{1}{r}$ instead of $V(r) \propto \ln r$.

$\rightarrow$ Write down S.E.:

$$\left[\frac{\hat{p}^2}{2m} + V(\vec{r})\right]\psi(\vec{r}) = E\,\psi(\vec{r})$$

$$\rightarrow \left[-\frac{\hbar^2\nabla^2}{2m} + V(\vec{r})\right]\psi(\vec{r}) = E\,\psi(\vec{r})$$

$$\rightarrow \left[\nabla^2 + \left(-\frac{2m}{\hbar^2}V(\vec{r})\right)\right]\psi(\vec{r}) = -\frac{2mE}{\hbar^2}\psi(\vec{r})$$

Bound state $E<0$, set: $\kappa^2 = \frac{-2mE}{\hbar^2}$

$$\left[\nabla^2 - \frac{2m}{\hbar^2} V(r)\right] \psi(\vec{r}) = \kappa^2 \psi(\vec{r})$$

$$\rightarrow \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right] \psi(\vec{r}) = \left[\kappa^2 + \frac{2m}{\hbar^2} V(r)\right] \psi(\vec{r})$$

Seperation of variables:

$$\psi(\vec{r}) = \psi(r, \theta) = \Theta(\theta) R(r)$$

Then we have:

$$\left(\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) - \kappa^2 - \frac{2m}{\hbar^2} V(r)\right] R(r)\right) \Theta(\theta) + \frac{1}{r^2} \Theta'' R(r) = 0$$

we know the constrain: $\Theta(\theta) = \Theta(\theta + 2\pi)$

$$\int_0^{2\pi} \Theta(\theta)\, \Theta^*(\theta)\, d\theta = 1$$

$$\Rightarrow \quad \Theta(\theta) = \frac{1}{\sqrt{2\pi}} e^{i l \theta}, \quad l = 0, \pm 1, \pm 2 \ldots$$

Then:

$$\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\right) - \kappa^2 - \frac{2m}{\hbar^2} V(r)\right] R(r) + \frac{-l^2}{r^2} R(r) = 0$$

$$\boxed{\frac{1}{r}\frac{d}{dr}\left(r\frac{dR}{dr}\right) = \left(\kappa^2 + \frac{2m}{\hbar^2} V(r) + \frac{l^2}{r^2}\right) R(r)}$$

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dR}{dr}\right) = \left(\kappa^2 - \frac{2m}{\hbar^2}\frac{e^2}{4\pi\varepsilon_0}\frac{1}{r} + \frac{l^2}{r^2}\right)R$$

set $\quad R(r) = \dfrac{u(r)}{\sqrt{r}}$

$$r^{-\frac{1}{2}}u'' + \frac{1}{4}r^{-\frac{5}{2}}u = \left(\kappa^2 - \frac{2m}{\hbar^2}\frac{e^2}{4\pi\varepsilon_0}\frac{1}{r} + \frac{l^2}{r^2}\right)r^{-\frac{1}{2}}u$$

$$u'' + \frac{1}{4}\frac{1}{r^2}u = \left(\kappa^2 - \frac{2m}{\hbar^2}\frac{e^2}{4\pi\varepsilon_0}\frac{1}{r} + \frac{l^2}{r^2}\right)u$$

set $\quad \rho = \kappa r, \quad \rho_0 = \dfrac{2m}{\hbar^2}\dfrac{e^2}{4\pi\varepsilon_0 \kappa}$

$$\frac{d^2u}{d\rho^2} + \frac{1}{4}\frac{1}{\rho^2}u = \left(1 - \frac{\rho_0}{\rho} + \frac{l^2}{\rho^2}\right)u$$

$$\boxed{\frac{d^2u}{d\rho^2} = \left[1 - \frac{\rho_0}{\rho} + \frac{(l+\frac{1}{2})(l-\frac{1}{2})}{\rho^2}\right]u} \quad \cdots\cdots ①$$

Consider the limit:

1° $\quad \rho \to 0$:

$$\rho^2\frac{d^2u}{d\rho^2} = (l+\tfrac{1}{2})(l-\tfrac{1}{2})u$$

$\Rightarrow$ assume $\quad u = \rho^n$

$$\Rightarrow n(n-1) = (l+\tfrac{1}{2})(l-\tfrac{1}{2})$$
$$= (-l+\tfrac{1}{2})(-l-\tfrac{1}{2})$$

Then: $n = l+\frac{1}{2}$, or $n = -l+\frac{1}{2}$. then:

$$u = A\,\rho^{l+\frac{1}{2}} + B\,\rho^{-l+\frac{1}{2}} \qquad \rho \to 0.$$

we need 'u' finite : $u = \rho^{|l|+\frac{1}{2}}$

2° $\rho \to \infty$:

$$\frac{d^2u}{d\rho^2} = u \Rightarrow u = e^{-\rho}$$

→ Follow the idea in textbook: introduce the new function: $v(\rho)$:

$$u(\rho) = \rho^{|l|+\frac{1}{2}} e^{-\rho}\, v(\rho)$$

plug into ① and get the differential equation for $v(\rho)$:

$$u' = (|l|+\tfrac{1}{2})\,\rho^{|l|-\frac{1}{2}} e^{-\rho} v(\rho) + \rho^{|l|+\frac{1}{2}} (-1)\, e^{-\rho} v(\rho) + \rho^{|l|+\frac{1}{2}} e^{-\rho}\, v'(\rho)$$

$$u'' = (l^2-\tfrac{1}{4})\,\rho^{|l|-\frac{3}{2}} e^{-\rho} v(\rho) + (|l|+\tfrac{1}{2})\,\rho^{|l|-\frac{1}{2}} (-1)\, e^{-\rho} v(\rho)$$
$$+ (|l|+\tfrac{1}{2})\,\rho^{|l|-\frac{1}{2}} e^{-\rho} v' + \text{other page.}$$

$$(-1)\left(|l|+\tfrac{1}{2}\right)\rho^{|l|-\frac{1}{2}}e^{-\rho}v(\rho)+\rho^{|l|+\frac{1}{2}}e^{-\rho}v(\rho)$$

$$+(-1)\rho^{|l|+\frac{1}{2}}e^{-\rho}v'+\left(|l|+\tfrac{1}{2}\right)\rho^{|l|-\frac{1}{2}}e^{-\rho}v'$$

$$-\rho^{|l|+\frac{1}{2}}e^{-\rho}v'+\rho^{|l|+\frac{1}{2}}e^{-\rho}v''$$

$$=\rho^{|l|-\frac{1}{2}}e^{-\rho}\Big[\frac{(l^2-\frac{1}{4})}{\rho}v-\left(|l|+\tfrac{1}{2}\right)v+\left(|l|+\tfrac{1}{2}\right)v'$$

$$-\left(|l|+\tfrac{1}{2}\right)v+\rho v-\rho v'+\left(|l|+\tfrac{1}{2}\right)v'$$

$$-\rho v'+\rho v''\Big]$$

$$=\rho^{|l|-\frac{1}{2}}e^{-\rho}\left[\left(-2|l|-1+\rho+\frac{l^2-\frac{1}{4}}{\rho}\right)v+2\left(|l|+\tfrac{1}{2}-\rho\right)v'+\rho v''\right]$$

Then, we have:

$$\boxed{\rho v''+2\left(|l|+\tfrac{1}{2}-\rho\right)v'+\left(-2|l|-1+\rho_0\right)v=0} \cdots ②$$

we assume the solution:

$$v(\rho)=\sum_{j=0}^{\infty}c_j\rho^j$$

$$v'(\rho) = \sum_{j=0}^{\infty} j\, c_j\, \rho^{j-1} = \sum_{j=0}^{\infty} (j+1)\, c_{j+1}\, \rho^j$$

$$v''(\rho) = \sum_{j=0}^{\infty} j(j+1)\, c_{j+1}\, \rho^{j-1}$$

plug into ②:

$$\sum_j j(j+1)\, c_{j+1}\, \rho^j + 2\left(|l|+\tfrac{1}{2}\right) \sum_j (j+1)\, c_{j+1}\, \rho^j$$

$$- 2\sum_j j\, c_j\, \rho^j + \left(-2|l|-1+\rho_0\right) \sum_j c_j\, \rho^j = 0$$

we have:

$$j(j+1)\, c_{j+1} + (2|l|+1)(j+1)\, c_{j+1} = 2j\, c_j + \left(2|l|+1-\rho_0\right) c_j$$

$$(2|l|+1+j)(j+1)\, c_{j+1} = \left(2j+2|l|+1-\rho_0\right) c_j$$

$$\boxed{c_{j+1} = \frac{2j+2|l|+1-\rho_0}{(2|l|+1+j)(j+1)}\, c_j}$$

we start with c_0 (this becomes an overall constant, to be fixed eventually by normalization) and the equation above gives us c_1, putting this back in, we obtain c_2 and so on.

look at large j:

$$C_{j+1} \sim \frac{2}{j} c_j$$

$$\Rightarrow \quad c_{j+1} = \frac{2^j}{j!} c_0$$

Then: $$v(\rho) = c_0 \sum_{j=0}^{\infty} \frac{2^j}{j!} \rho^j = c_0 e^{2\rho}$$

hence:

$$u(\rho) = c_0 e^{2\rho} \cdot e^{-\rho} \rho^{|l|+\frac{1}{2}}$$

$$= c_0 e^{\rho} \rho^{|l|+\frac{1}{2}}$$

which blows up at large ρ. Then the series must terminate. There must occur some maximal integer, j_{max}, such that:

$$C_{j_{max}+1} = 0$$

$$\Updownarrow$$

$$2\left(j_{max} + |l| + \tfrac{1}{2}\right) = \rho_0$$

we set $n = j_{max} + |l| + 1$

then: $2\left(n - \frac{1}{2}\right) = \rho_0$

$\Rightarrow$

$$2\left(n - \frac{1}{2}\right) = \frac{2m}{\hbar^2} \frac{e^2}{4\pi\varepsilon_0} \frac{1}{\kappa}$$

$\Rightarrow$

$$E_n = -\frac{m}{2\hbar^2}\left(\frac{e^2}{4\pi\varepsilon_0}\right)^2 \frac{1}{\left(n-\frac{1}{2}\right)^2}$$

where: $n = 1, 2, \cdots$

$n > |l|$

$l = 0, \pm 1, \cdots$

For eigenstates:

$$\Psi_{nl}(r,\theta) = R_{nl}(r)\, \Theta_l(\theta)$$

where:

$$R_{nl}(r) = \frac{1}{\sqrt{r}} \rho^{|l|+\frac{1}{2}} e^{-\rho} v(\rho)$$

$$= \rho^{|l|} e^{-\rho} v(\rho)$$

there is constant. we can absorb it into C_0.

$$\Theta_l = \frac{1}{\sqrt{2\pi}} e^{il\theta}$$

introduce Bohr radius:

$$a = \frac{4\pi \epsilon_0 \hbar^2}{m e^2}$$

Then:

$$\kappa = \frac{e^2 m}{4\pi \epsilon_0 \hbar^2 (n-\frac{1}{2})} = \frac{1}{a(n-\frac{1}{2})}$$

then:

$$\rho = \kappa r = \frac{r}{a(n-\frac{1}{2})}$$

Let's work out the ground state, where $n=1$, $l=0$.

$$R_{10} = e^{-\rho} v(\rho) = c_0 e^{-\rho}$$

$$\Theta_0 = \frac{1}{\sqrt{2\pi}}$$

Then:

$$\psi_{10}(\vec{r}) = \frac{c_0}{\sqrt{2\pi}} e^{-\rho} = \frac{c_0}{\sqrt{2\pi}} e^{-\frac{2r}{a}}$$

Find the normalization factor:

$$1 = \int_0^{+\infty} \frac{c_0^2}{2\pi} e^{-\frac{4r}{a}} 2\pi r\, dr$$

$$= c_0^2 \int_0^{+\infty} e^{-\frac{4}{a}r} r\, dr = \left(\frac{a}{4}\right)^2 c_0^2 \Rightarrow c_0 = \frac{4}{a}$$

$$\psi_{10}(\vec{r}) = \frac{4}{a\sqrt{2\pi}} e^{-\frac{2r}{a}}$$