## 8.7 Planetary Motion—Kepler's Problem

The equation for the path of a particle moving under the influence of a central force whose magnitude is inversely proportional to the square of the distance between the particle and the force center can be obtained (see Equation 8.17) from

$$\theta(r) = \int \frac{(l/r^2)\,dr}{\sqrt{2\mu\left(E + \dfrac{k}{r} - \dfrac{l^2}{2\mu r^2}\right)}} + \text{constant} \tag{8.38}$$

The integral can be evaluated if the variable is changed to $u \equiv l/r$ (see Problem 8-2). If we define the origin of $\theta$ so that the minimum value of $r$ is at $\theta = 0$, we find

$$\cos\theta = \frac{\dfrac{l^2}{\mu k}\cdot\dfrac{1}{r} - 1}{\sqrt{1 + \dfrac{2El^2}{\mu k^2}}} \tag{8.39}$$

Let us now define the following constants:

$$\left.\begin{aligned} \alpha &\equiv \frac{l^2}{\mu k} \\ \varepsilon &\equiv \sqrt{1 + \frac{2El^2}{\mu k^2}} \end{aligned}\right\} \tag{8.40}$$

Equation 8.39 can thus be written as

$$\boxed{\frac{\alpha}{r} = 1 + \varepsilon\cos\theta} \tag{8.41}$$

This is the equation of a conic section with one focus at the origin.* The quantity $\varepsilon$ is called the **eccentricity,** and $2\alpha$ is termed the **latus rectum** of the orbit. Conic sections are formed by the intersection of a plane and a cone. A conic section is formed by the loci of points (formed in a plane), where the ratio of the distance from a fixed point (the focus) to a fixed line (called the directrix) is a constant. The directrix for the parabola is shown in Figure 8-8 by the vertical dashed line, drawn so that $r/r' = 1$.

The minimum value for $r$ in Equation 8.41 occurs when $\theta = 0$, or when $\cos\theta$ is a maximum. Thus the choice of the integration constant in Equation 8.38 corresponds to measuring $\theta$ from $r_{\min}$, which position is called the **pericenter;** $r_{\max}$ corresponds to the **apocenter.** The general term for turning points is **apsides.** The corresponding terms for motion about the Sun are *perihelion* and *aphelion,* and for motion about Earth, *perigee* and *apogee.*

*Johann Bernoulli (1667–1748) appears to have been the first to prove that *all* possible orbits of a body moving in a potential proportional to $1/r$ are conic sections (1710).



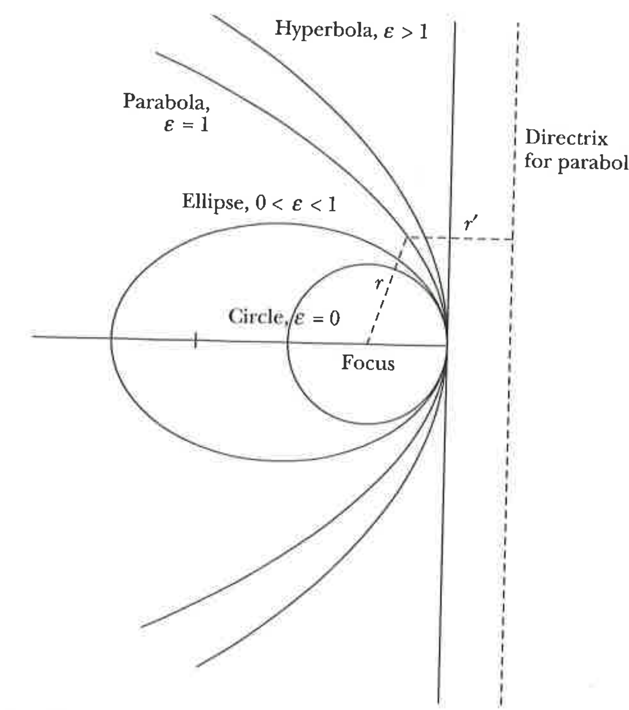


**FIGURE 8-8** The orbits of the various conic sections are shown together with their eccentricities $\varepsilon$.

Various values of the eccentricity (and hence of the energy $E$) classify the orbits according to different conic sections (see Figure 8-8):

| | | |
|---|---|---|
| $\varepsilon > 1,$ | $E > 0$ | Hyperbola |
| $\varepsilon = 1,$ | $E = 0$ | Parabola |
| $0 < \varepsilon < 1,$ | $V_{\min} < E < 0$ | Ellipse |
| $\varepsilon = 0,$ | $E = V_{\min}$ | Circle |

For planetary motion, the orbits are ellipses with major and minor axes (equal to $2a$ and $2b$, respectively) given by

$$a = \frac{\alpha}{1 - \varepsilon^2} = \frac{k}{2|E|} \tag{8.42}$$

$$b = \frac{\alpha}{\sqrt{1 - \varepsilon^2}} = \frac{l}{\sqrt{2\mu|E|}} \tag{8.43}$$

Thus, the major axis depends only on the energy of the particle, whereas the minor axis is a function of both first integrals of the motion, $E$ and $l$. The geometry of elliptic orbits in terms of the parameters $\alpha$, $\varepsilon$, $a$, and $b$ is shown in Figure 8-9; $P$

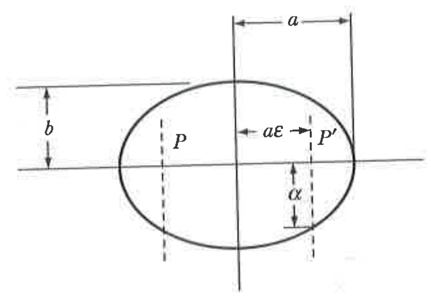


**FIGURE 8-9** The geometry of elliptic orbits is shown in terms of parameters $\alpha$, $\varepsilon$, $a$, and $b$. $P$ and $P'$ are the foci.

and $P'$ are the foci. From this diagram, we see that the apsidal distances ($r_{\min}$ and $r_{\max}$ as measured from the foci to the orbit) are given by

$$\left.\begin{aligned} r_{\min} &= a(1-\varepsilon) = \frac{\alpha}{1+\varepsilon} \\ r_{\max} &= a(1+\varepsilon) = \frac{\alpha}{1-\varepsilon} \end{aligned}\right\} \tag{8.44}$$

To find the period for elliptic motion, we rewrite Equation 8.12 for the areal velocity as

$$dt = \frac{2\mu}{l}\,dA$$

Because the entire area $A$ of the ellipse is swept out in one complete period $\tau$,

$$\int_0^\tau dt = \frac{2\mu}{l}\int_0^A dA$$

$$\tau = \frac{2\mu}{l}A \tag{8.45}$$

The area of an ellipse is given by $A = \pi ab$, and using $a$ and $b$ from Equations 8.42 and 8.43, we find

$$\tau = \frac{2\mu}{l}\cdot\pi ab = \frac{2\mu}{l}\cdot\pi\cdot\frac{k}{2|E|}\cdot\frac{l}{\sqrt{2\mu|E|}}$$

$$= \pi k\sqrt{\frac{\mu}{2}}\cdot|E|^{-3/2} \tag{8.46}$$

We also note from Equations 8.42 and 8.43 that the semiminor axis* can be written as

$$b = \sqrt{\alpha a} \tag{8.47}$$

*The quantities $a$ and $b$ are called *semimajor* and *semiminor axes*, respectively.



Therefore, because $\alpha = l^2/\mu k$, the period $\tau$ can also be expressed as

$$\boxed{\tau^2 = \frac{4\pi^2\mu}{k}a^3} \tag{8.48}$$

This result, that the square of the period is proportional to the cube of the semimajor axis of the elliptic orbit, is known as **Kepler's Third Law.*** Note that this result is concerned with the equivalent one-body problem, so account must be taken of the fact that it is the *reduced* mass $\mu$ that occurs in Equation 8.48. Kepler actually concluded that the squares of the periods of the planets were proportional to the cubes of the major axes of their orbits—with the same proportionality constant for all planets. In this sense, the statement is only approximately correct, because the reduced mass is different for each planet. In particular, because the gravitational force is given by

$$F(r) = -\frac{Gm_1m_2}{r^2} = -\frac{k}{r^2}$$

we identify $k = Gm_1m_2$. The expression for the square of the period therefore becomes

$$\tau^2 = \frac{4\pi^2a^3}{G(m_1+m_2)} \cong \frac{4\pi^2a^3}{Gm_2}, \qquad m_1 \ll m_2 \tag{8.49}$$

and Kepler's statement is correct only if the mass $m_1$ of a planet can be neglected with respect to the mass $m_2$ of the Sun. (But note, for example, that the mass of Jupiter is about 1/1000 of the mass of the Sun, so the departure from the approximate law is not difficult to observe in this case.)

Kepler's laws can now be summarized:

**I.** *Planets move in elliptical orbits about the Sun with the Sun at one focus.*

**II.** *The area per unit time swept out by a radius vector from the Sun to a planet is constant.*

**III.** *The square of a planet's period is proportional to the cube of the major axis of the planet's orbit.*

See Table 8-1 for some properties of the principal objects in the solar system.

*Published by Kepler in 1619. Kepler's Second Law is stated in Section 8.3. The First Law (1609) dictates that the planets move in elliptical orbits with the Sun at one focus. Kepler's work preceded by almost 80 years Newton's enunciation of his general laws of motion. Indeed, Newton's conclusions were based to a great extent on Kepler's pioneering studies (and on those of Galileo and Huygens).