Moment of Inertia

Moment of inertia is the name given to rotational inertia, the rotational analog of mass for linear motion. The moment of inertia must be specified with respect to a chosen axis of rotation. For a point mass the moment of inertia is just the mass times the square of perpendicular distance to the rotation axis, I = mr^2. That point mass relationship becomes the basis for all other moments of inertia since any object can be built up from a collection of point masses. The most general form is:

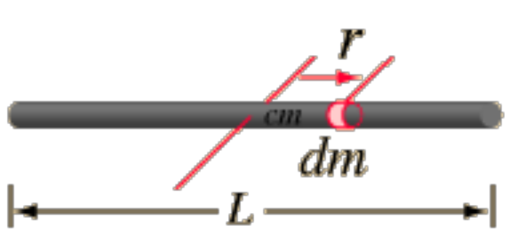
$$I = \int_0^M r^2 dm$$

<u>Some useful moment of inertia examples:</u>

1. For a point mass the moment of inertia is just the mass times the radios from the axis squared.

$$I = mr^2$$

2. For a uniform rod about the center:



$$I = \int_0^M r^2 dm$$

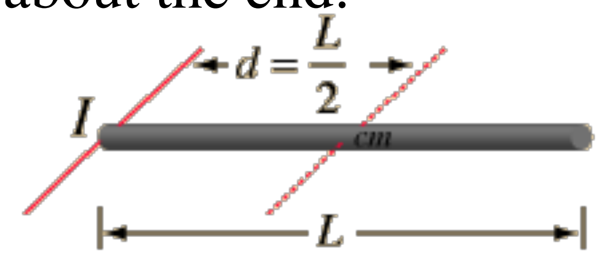
$$I = \int_{-L/2}^{L/2} r^2 \frac{M}{L} dr = \frac{M}{L}\frac{r^3}{3}\Big|_{-L/2}^{L/2} = \frac{M}{3L}\left[\frac{L^3}{8} - \frac{-L^3}{8}\right]$$

Mass of intinitesmal length dr : $dm = \frac{M}{L} dr$

$$I_{cm} = \frac{1}{12} ML^2$$

3. For a uniform rod about the end:

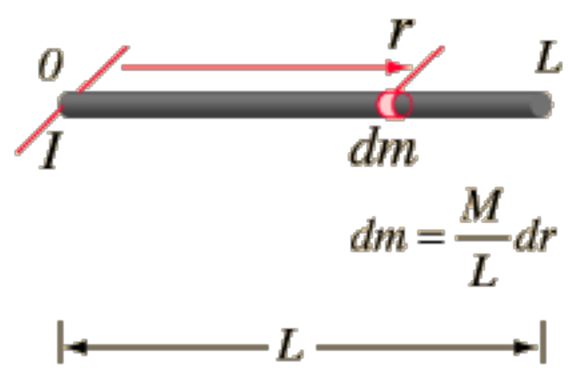
$$I = I_{cm} + Md^2$$



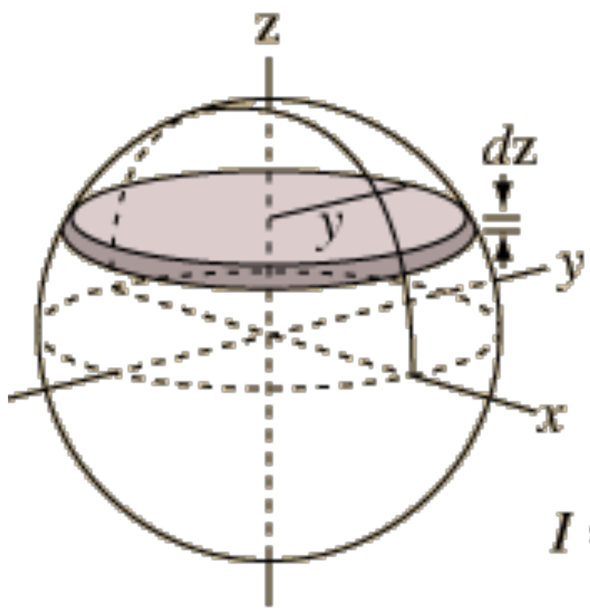
$$I_{end} = \frac{1}{12}ML^2 + M\frac{L^2}{4} = \frac{1}{3}ML^2$$

This can be confirmed by direct integration

$$I = \int_0^L r^2 \frac{M}{L} dr = \frac{M}{L}\frac{r^3}{3}\Bigg|_0^L = \frac{1}{3}ML^2$$



4. The expression for the moment of inertia of a sphere can be developed by summing the moments of infintesmally thin disks about the z axis. The moment of inertia of a thin disk is



$$dI = \frac{1}{2}y^2 dm = \frac{1}{2}y^2 \rho dV = \frac{1}{2}y^2 \rho \pi y^2 dz$$

and the integral becomes

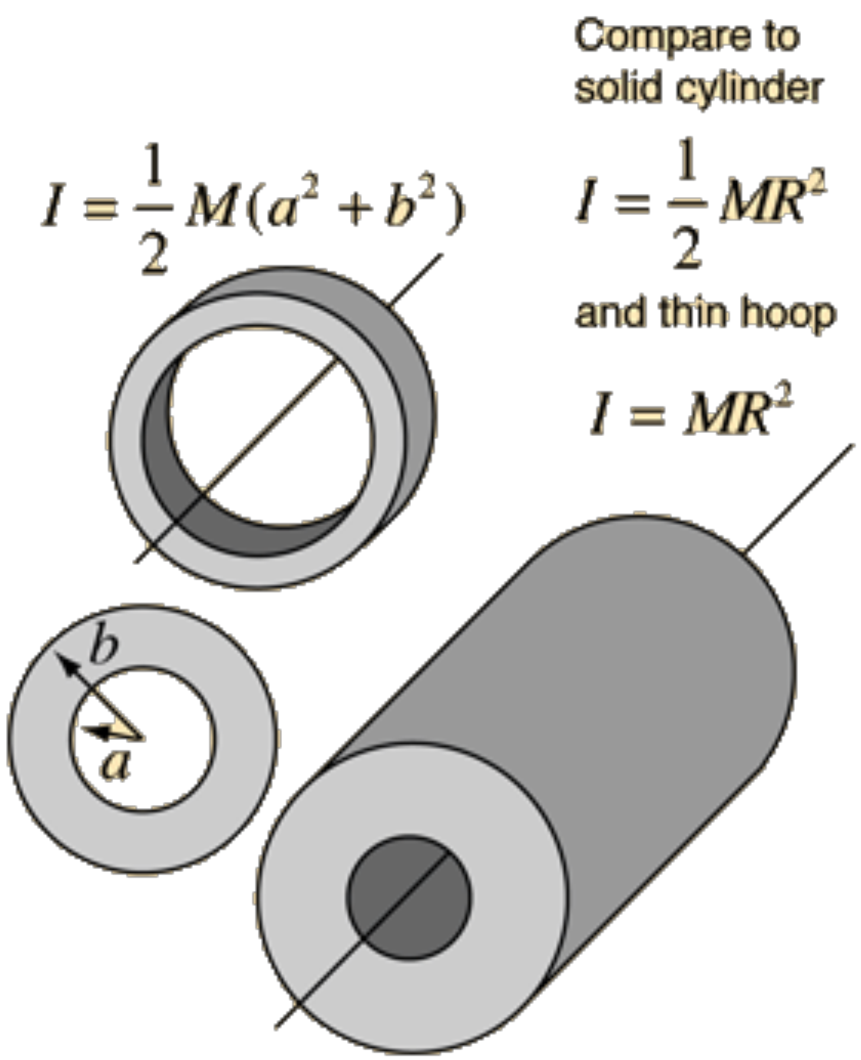
$$I = \frac{1}{2}\rho\pi \int_{-R}^{R} y^4 dz = \frac{1}{2}\rho\pi \int_{-R}^{R} (R^2 - z^2)^2 dz = \frac{8}{15}\rho\pi R^5$$

Radius $= R$
Mass $= M$
Density $= \rho = \frac{M}{V} = \frac{M}{\frac{4}{3}\pi R^3}$

Substituting the density expression gives

$$I = \frac{8}{15}\left[\frac{M}{\frac{4}{3}\pi R^3}\right]\pi R^5 = \frac{2}{5}MR^2$$

Thick Hoops and Hollow Cylinders



Hoop