Optical Cavities

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1 Steady-state response of linear optical systems

We are interested in studying the behavior of light incident on an optical cavity, *e.g.* two mirrors with some distance between their reflecting surfaces. The naïve expectation is that something interesting will happen when the spacing between the mirrors gets close to being an integer number of wavelengths. We will start by examining the 1D (infinite plane waves and flat, parallel mirrors) case first, which will utilize the concept of transfer matrices (T-matrices).

1.1 Scattering and transfer matrices

Consider the case where infinite, monochromatic plane waves are incident on an optical system from the left with amplitude $\mathcal{E}_{\text{incident}} = a^{(1)}$, as shown in Fig. 1. We'll assume they are linearly polarized in some transverse direction and work only with that component of the electric field. Some light will be reflected from the optical system so that in steady state, the total electric field (in the direction of the optical polarization) in the region left of the optical system includes amplitudes for the incident ($a^{(1)}$) and reflected ($b^{(1)}$) light:

$$E_{\text{left}} \equiv E^{(1)} = \left(a^{(1)}\mathrm{e}^{\mathrm{i}kz^{(1)}} + b^{(1)}\mathrm{e}^{-\mathrm{i}kz^{(1)}}\right)\mathrm{e}^{-\mathrm{i}\omega t} \tag{1}$$

Likewise, to keep our formalism completely general, we will also allow for incident ($a^{(2)}$) and reflected ($b^{(2)}$) fields on the right side of the optical system, given by

$$E_{\text{right}} \equiv E^{(2)} = \left(a^{(2)}\mathrm{e}^{-\mathrm{i}kz^{(2)}} + b^{(2)}\mathrm{e}^{\mathrm{i}kz^{(2)}}\right)\mathrm{e}^{-\mathrm{i}\omega t}. \tag{2}$$

A general situation of this type is shown in Fig. 1.

With these definitions, we describe a system with light incident from either side (or both) by choosing $a^{(1)}$ and $a^{(2)}$ and the optical system, which we will assume is linear (as well as reciprocal, lossless, and time-reversal invariant, though we won't go into details about these assumptions). There is a matrix that transforms the inputs into the outputs called the S-matrix:

$$\mathbf{b} = \mathbf{S}\mathbf{a} \tag{3}$$

$$\begin{pmatrix} b^{(1)} \\ b^{(2)} \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}\begin{pmatrix} a^{(1)} \\ a^{(2)} \end{pmatrix} \tag{4}$$

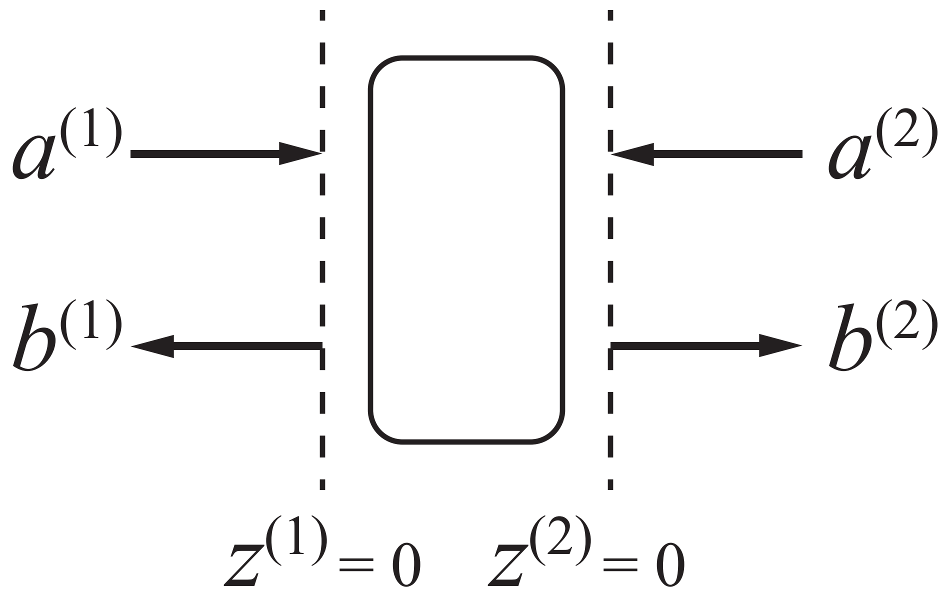

Figure 1: A linear system is illuminated from the left by a field $E^{(1)}$ and the right by a field $E^{(2)}$. Each side has its own $z = 0$ reference plane, which we often choose to be somewhere that keeps the algebra from getting cluttered.

The S-matrix is most useful if it is known for an entire optical system, but is less useful if it is only known for individual optical components within an optical system. The reason is that cascaded elements will be encountered in a different order by the light that is incident from the left than by the light that is incident from the right. So we clearly cannot simply multiply S-matrices together to get the composite S-matrix of a whole optical system (in what order would we perform the multiplication?).

To remedy this, we can develop a complimentary formalism that is useful for relating the left side amplitudes ($a^{(1)}$ and $b^{(1)}$) to the right side amplitudes ($a^{(2)}$ and $b^{(2)}$) of the system, which is called the transfer matrix (or T-matrix):

$$\begin{pmatrix} b^{(2)} \\ a^{(2)} \end{pmatrix} = \begin{pmatrix} T_{ba} & T_{bb} \\ T_{aa} & T_{ab} \end{pmatrix} \begin{pmatrix} a^{(1)} \\ b^{(1)} \end{pmatrix} \tag{5}$$

Note that the vectors of the T-matrix definition flip a and b on each side of this equation, and that the elements of the T-matrix with double, identical subscripts are the off-diagonal elements. All of this is to make sure that if a series of 2-port linear systems are cascaded, the T-matrix of the composite system is just the matrix product of the individual T-matrices.

The elements of the S-matrix and T-matrix are related to each other via

$$
\begin{aligned}
S_{11} &= -\frac{T_{aa}}{T_{ab}} \\
S_{12} &= \frac{1}{T_{ab}} \\
S_{21} &= T_{ba} - \frac{T_{aa}T_{bb}}{T_{ab}} \\
S_{22} &= \frac{T_{bb}}{T_{ab}}
\end{aligned}
\tag{6}
$$

$$
\begin{aligned}
T_{ba} &= S_{21} - \frac{S_{11}S_{22}}{S_{12}} \\
T_{bb} &= \frac{S_{22}}{S_{12}} \\
T_{aa} &= -\frac{S_{11}}{S_{12}} \\
T_{ab} &= \frac{1}{S_{12}}
\end{aligned}
\tag{7}
$$

Last, in order to derive the correct S- and T-matrices for a certain optical system, it is important to clearly define reference planes where these fields are being described, as shown in Fig. 1.

So, the typical way of using this formalism is that we will be interested in what happens when we send light at an optical system made of a few optical elements. Often, we will be interested in the relationship between $a^{(1)}$ and $b^{(2)}$ (i.e. with $a^{(2)} = 0$). Under these conditions, if we solve Eq. 5 for $b^{(2)}$ we get an expression for the amplitude transfer function,

$$
\frac{b^{(2)}}{a^{(1)}} = T_{ba} - \frac{T_{aa}T_{bb}}{T_{ab}}, \tag{8}
$$

which is also S_{21}. For a more complicated set of inputs, it is probably worth it to calculate the T-matrix of the composite system and then convert it into S-matrix form. That said, the T-matrix is really the useful thing for us when thinking about how to build optical cavities, and we can almost forget about S-matrices entirely.

1.2 Example S- and T-matrices

1. Free propagation over a distance $\Delta z = L$:
$$
\mathbf{S}_\mathrm{f} = \begin{pmatrix} 0 & \mathrm{e}^{\mathrm{i}kL} \\ \mathrm{e}^{\mathrm{i}kL} & 0 \end{pmatrix} \tag{9}
\qquad
\mathbf{T}_\mathrm{f} = \begin{pmatrix} \mathrm{e}^{\mathrm{i}kL} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}kL} \end{pmatrix} \tag{10}
$$

2. Free propagation over L through a medium with amplitude absorption coefficient α:
$$
\mathbf{S}_{\mathrm{f},\alpha} = \begin{pmatrix} 0 & \mathrm{e}^{\mathrm{i}kL}\mathrm{e}^{-\alpha L} \\ \mathrm{e}^{\mathrm{i}kL}\mathrm{e}^{-\alpha L} & 0 \end{pmatrix} \tag{11}
\qquad
\mathbf{T}_{\mathrm{f},\alpha} = \begin{pmatrix} \mathrm{e}^{\mathrm{i}kL}\mathrm{e}^{-\alpha L} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}kL}\mathrm{e}^{-\alpha L} \end{pmatrix} \tag{12}
$$

3. A partially reflecting, lossless mirror with amplitude reflectivity $r = \sqrt{1-t^2}$:
$$
\mathbf{S}_\mathrm{m} = -\begin{pmatrix} r & \mathrm{i}t \\ \mathrm{i}t & r \end{pmatrix} \tag{13}
\qquad
\mathbf{T}_\mathrm{m} = -\frac{\mathrm{i}}{t}\begin{pmatrix} 1 & r \\ -r & -1 \end{pmatrix} \tag{14}
$$

4. Nonmagnetic dielectric interface with left side constants $\epsilon_1 = \epsilon_\mathrm{o} n_1^2$ and right side $\epsilon_2 = \epsilon_\mathrm{o} n_2^2$, reference planes are both right at the interface:

$$\mathbf{S}_{n_1,n_2} = \frac{1}{n_1+n_2}\begin{pmatrix} n_1-n_2 & 2\sqrt{n_1n_2} \\ 2\sqrt{n_1n_2} & n_2-n_1 \end{pmatrix} \quad (15) \qquad \mathbf{T}_{n_1,n_2} = \frac{1}{2\sqrt{n_1n_2}}\begin{pmatrix} n_1+n_2 & n_2-n_1 \\ n_2-n_1 & n_1+n_2 \end{pmatrix} \quad (16)$$

2 Planar Fabry-Perot cavity

Armed with Eq. 10 and Eq. 14, we can find the transfer matrix of two, identical, lossless mirrors spaced apart by a distance L:

$$\begin{aligned} T_{\mathrm{FP}} &= \left(-\frac{\mathrm{i}}{t}\right)^2 \begin{pmatrix} 1 & r \\ -r & -1 \end{pmatrix}\begin{pmatrix} \mathrm{e}^{\mathrm{i}kL} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}kL} \end{pmatrix}\begin{pmatrix} 1 & r \\ -r & -1 \end{pmatrix} \\ &= -\frac{1}{t^2}\begin{pmatrix} \mathrm{e}^{\mathrm{i}kL} & r\mathrm{e}^{-\mathrm{i}kL} \\ -r\mathrm{e}^{\mathrm{i}kL} & -\mathrm{e}^{-\mathrm{i}kL} \end{pmatrix}\begin{pmatrix} 1 & r \\ -r & -1 \end{pmatrix} \\ &= -\frac{1}{t^2}\begin{pmatrix} \mathrm{e}^{\mathrm{i}kL} - r^2\mathrm{e}^{-\mathrm{i}kL} & \mathrm{i}2r\sin(kL) \\ -\mathrm{i}2r\sin(kL) & \mathrm{e}^{-\mathrm{i}kL} - r^2\mathrm{e}^{\mathrm{i}kL} \end{pmatrix} \\ &= -\frac{\mathrm{e}^{\mathrm{i}kL}}{t^2}\begin{pmatrix} 1 - r^2\mathrm{e}^{-\mathrm{i}2kL} & r - r\mathrm{e}^{-\mathrm{i}2kL} \\ -r + r\mathrm{e}^{-\mathrm{i}2kL} & \mathrm{e}^{-\mathrm{i}2kL} - r^2 \end{pmatrix} \end{aligned} \tag{17}$$

The amplitude transfer function for input coming from the left can be found using Eq. 8:

$$\frac{b^{(2)}}{a^{(1)}} = -\mathrm{e}^{\mathrm{i}kL}\frac{(1-r^2)}{1-r^2\mathrm{e}^{\mathrm{i}2kL}}, \tag{18}$$

where we have used $t^2 = 1 - r^2$ to get rid of references to t.

While Eq. 18 is correct, it is easier to tell what is happening if we instead look at the *power* transfer function,

$$\left|\frac{b^{(2)}}{a^{(1)}}\right|^2 = \frac{(1-r^2)^2}{(1-r^2)^2 + 4r^2\sin^2(kL)}. \tag{19}$$

Equation 19 is the main result of this section – it shows the steady-state behavior of an optical cavity illuminated by a laser, and contains lots of physics. Eq. 19 is plotted in Fig. 2 for a fixed length cavity and three different reflectivities as a function of optical frequency shift. Resonances occur whenever the argument of the sine function is an integer multiple of π, which happens whenever $\omega L/c = m\pi$ for some integer m. For light with a frequency that is resonant in the cavity, a full round-trip in the cavity results in a total phase accumulation of an integer times 2π, and we see that resonant modes are those that repeat their behavior every round trip.

The properties of Fabry-Perot cavities like this are typically characterized by some of the following definitions:

- The m^{th} (longitudinal) resonance has a frequency given by

$$f_m = m\frac{c}{2L}. \tag{20}$$

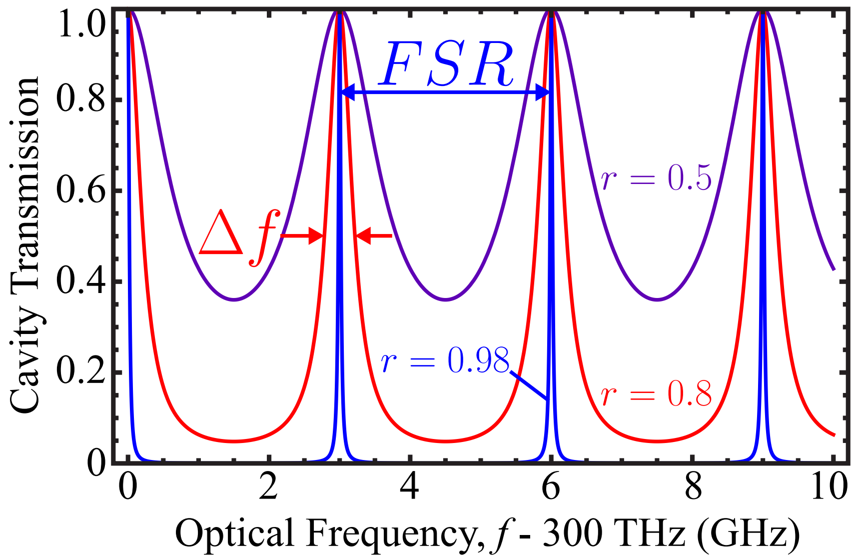

Figure 2: Fractional transmission of a planar, 5 cm Fabry-Perot cavity. Curves are shown for cases with both (lossless) mirrors having *amplitude* reflectivities of $r = 0.5$ (purple curve), $r = 0.8$ (red curve), and $r = 0.98$ (blue curve). The highest reflectivity case has a finesse of $\mathcal{F} \approx 80$.

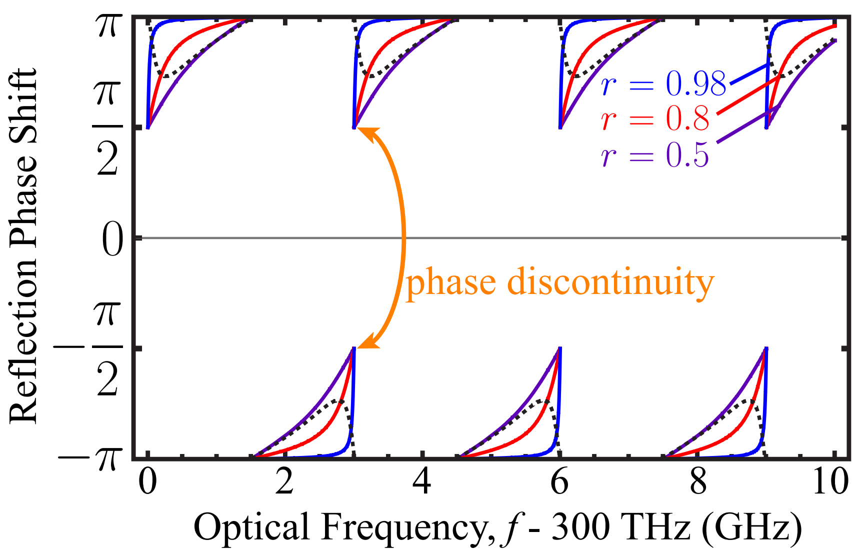

Figure 3: Phase of the reflection coefficient $(b^{(1)}/a^{(1)})$ for light reflected from a planar, 5 cm Fabry-Perot cavity. The apparent phase discontinuity on resonance is due to the complete disappearance of reflected power on resonance for a lossless cavity. If cavity losses are included, the phase shift remains continuous through resonance, as shown by the dashed, black curve.

- The *linewidth* of a resonance (often simply called the linewidth of the cavity itself) is the frequency full width at half maximum (FWHM) of the power transfer function, given by setting Eq. 19 equal to 1/2:

$$\Delta f = \frac{c}{\pi L} \arcsin\left(\frac{1-r^2}{2r}\right) \approx \frac{(1-r^2)c}{2\pi L r} \tag{21}$$

  where the approximation above is valid when the reflectivity is high ($1 - r^2 \ll 1$).

- The *free spectral range* (FSR) is the frequency difference between adjacent resonances, given in cyclic frequency units by

$$FSR = \frac{c}{2L}. \tag{22}$$

  This is sometimes called the longitudinal mode spacing or (for mode-locked lasers) the *repetition rate* (f_r). Note that this is also the inverse of the *round trip time* in the cavity.

- The *finesse* of a cavity is the free spectral range divided by the linewidth,

$$\mathcal{F} \equiv \frac{FSR}{\Delta f} \approx \frac{\pi r}{1-r^2}. \tag{23}$$

- The *quality factor*, or Q of a resonance is the ratio of the resonant frequency to the linewidth,

$$Q_m \equiv \frac{f_m}{\Delta f}. \tag{24}$$

- The number of reflections required to reduce the probability that a photon is still in the cavity by a factor of e is a quantity we will call the *bounce number* n_o. We can find n_o by noting that each round trip reduces the retention probability by $(r^2)^2$,

$$\left(r^2\right)^{2n_\mathrm{o}} = \frac{1}{\mathrm{e}}. \tag{25}$$

  Solving this for n_o gives us

$$n_\mathrm{o} = -\frac{1}{2\ln(r^2)} \approx \frac{1}{2\,(1-r^2)}. \tag{26}$$

- The *ring-down time* is the amount of time required for a photon to make n_o bounces, which sets the timescale for the onset of steady-state behavior,

$$\tau = n_\mathrm{o}\frac{2L}{c} = \frac{L}{c\,(1-r^2)} \approx \frac{1}{2\pi\Delta f}. \tag{27}$$

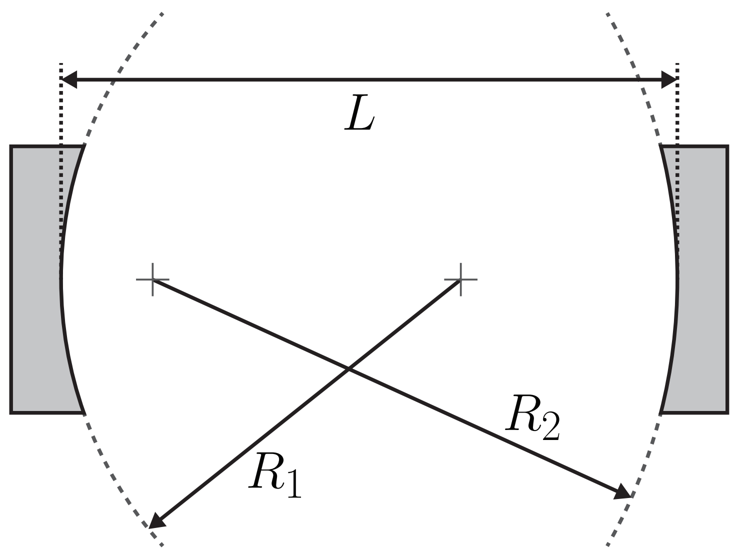

Figure 4: A Fabry-Perot cavity made of two curved mirrors with radii of curvature R_1 and R_2 spaced apart by some distance L.

3 Paraxial wave cavities

The infinite plane waves we have been using to describe Fabry-Perot cavities are in most practical situations not a good approximation. We showed previously that in cases where the light is confined to a finite region close to the optical axis, the paraxial wave equation is applicable and its solutions (in x, y, z coordinates) are Hermite-Gaussians. The lowest order solution was the TEM_{00} mode, which is the simple Gaussian beam one often sees coming out of a laser. This beam is entirely characterized by its confocal parameter b and the position of its (minimum) waist, which we often take to be $z_0 = 0$.

However, what chooses the value for b? In most cases, the answer is that there are boundary conditions that choose b and z_0. We will begin by looking at how the boundary conditions select for particular solutions and then, for a given b and z_0, what the impact of the higher-order transverse modes is on the details of these solutions.

3.1 Cavity stability

Consider an optical cavity that is made from two curved mirrors, such as shown in Fig. 4. In analogy to the planar Fabry-Perot cavity, where the resonant (longitudinal) modes were those for which a full round trip in the cavity left the phase (mod 2π) unchanged, the value of $q = z_0 + ib$ that repeats itself after one round trip will be resonant in this cavity. Resonant paraxial cavity modes will not only have to be resonant *longitudinally*, but also in terms of their *transverse* properties. The transformation of q as the beam traverses the cavity for a full round trip can be described by an ABCD matrix, so we seek q such that

$$q = \frac{Aq+B}{Cq+D}. \tag{28}$$

Even before we calculate the ABCD matrix for a particular cavity, we can make some important observations. First, since a full round trip starts and ends in a material with the same index of refraction, we see that $AD - CB = 1$ (recall that the determinant of the ABCD matrix is the ratio $\frac{n_1}{n_2}$). Solving Eq. 28 for q yields the Gaussian beam properties for the resonant transverse modes of the cavity,

$$q_{\text{res}} = \frac{A - D \pm \sqrt{(A+D)^2 - 4}}{2C}. \tag{29}$$

Since A, B, C, and D are real, we know that $0 \leq (A+D)^2$. Further, in order for the confocal parameter to be nonzero (a requirement for paraxial waves), we require a finite imaginary part for q_{res}, which gives an upper bound as well, and we can write all of this as

$$0 \leq \frac{(A+D)^2}{4} \leq 1. \tag{30}$$

For reasons that will be clear in a moment, let us suppose that there is some real number η such that $(A+D)^2/4 = (2\eta - 1)^2$. The condition above then reduces to $0 \leq \eta \leq 1$.

We can now examine the general case of a two-mirror paraxial optical cavity (Fig. 4). The ABCD matrix for a round-trip in the cavity is given by

$$\begin{aligned}
&\begin{pmatrix} 1 & 0 \\ -\frac{2}{R_1} & 1 \end{pmatrix}\begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -\frac{2}{R_2} & 1 \end{pmatrix}\begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \\
=& \begin{pmatrix} 1 & L \\ -\frac{2}{R_1} & -\frac{2L}{R_1}+1 \end{pmatrix}\begin{pmatrix} 1 & L \\ -\frac{2}{R_2} & -\frac{2L}{R_2}+1 \end{pmatrix} \\
=& \begin{pmatrix} 1 - \frac{2L}{R_2} & 2L - \frac{2L^2}{R_2} \\ \frac{4L}{R_1R_2} - \frac{2}{R_2} - \frac{2}{R_1} & 1 - \frac{2L}{R_2} - \frac{4L}{R_1} + \frac{4L^2}{R_1R_2} \end{pmatrix}
\end{aligned} \tag{31}$$

where we have been careful to remember that we build these matrix products up from right to left, using free propagation, reflection off the back mirror, free propagation, and then reflection off the front mirror. Using the observation made in the previous paragraph, we identify $\eta = (1 - L/R_1)(1 - L/R_2)$ and we have the condition for stability of a two-mirror paraxial cavity,

$$0 \leq \left(1 - \frac{L}{R_1}\right)\left(1 - \frac{L}{R_2}\right) \leq 1. \tag{32}$$

The stable region for a two-mirror cavity is shown in Fig. 5. A *concentric* cavity is (part of) a full spherical shell with a reflective internal surface. For a (symmetric) *confocal* cavity, the radius of curvature for each mirror is centered where the opposite mirror intersects the optical axis, so the foci of each mirror coincide at $z = L/2$. While it is also possible to make

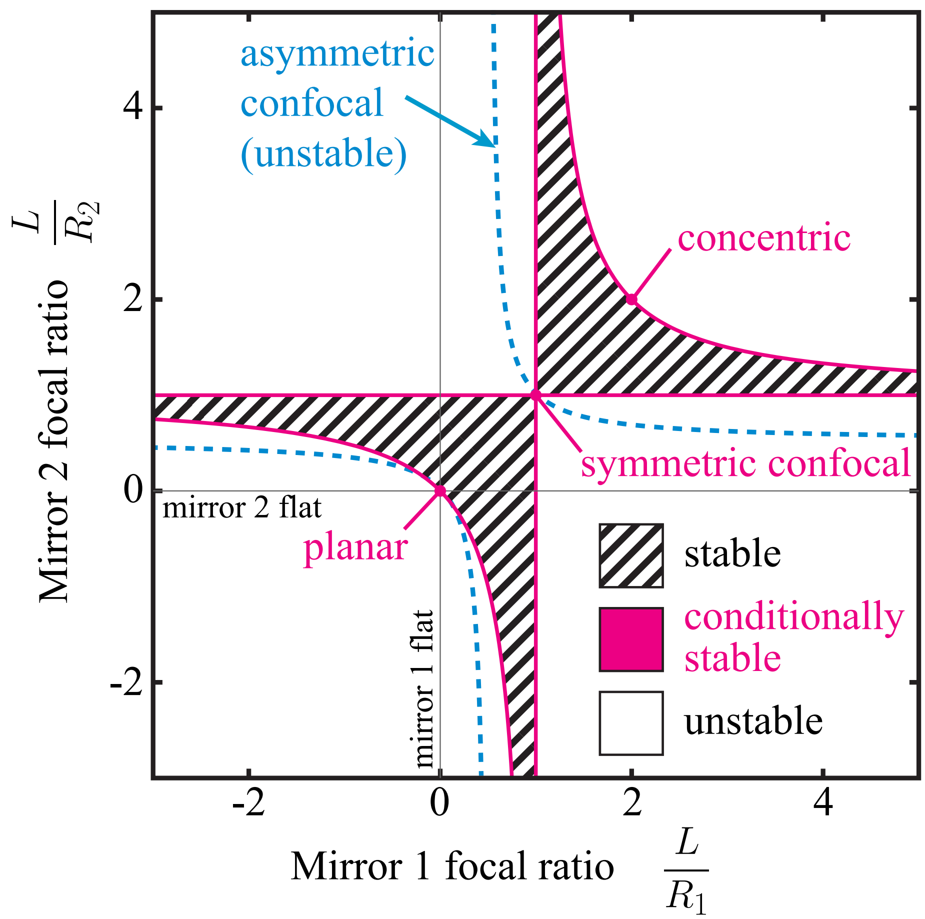

Figure 5: Transverse mode stability region for a Fabry-Perot cavity made of two curved mirrors with radii of curvature R_1 and R_2 spaced apart by some distance L.

an asymmetric "confocal" cavity (one where the foci coincide, but the radii of curvature are unequal), only the *symmetric* confocal cavity is stable, which is what people tend to mean when they say a cavity is "confocal."

The boundary between stable and unstable cavity geometries is often called *conditionally stable*. This is a little bit of nonsense, but the practical implication is that passive cavities can be made very close to these parameters as long as the finesse is finite. For instance, if rays in a barely unstable cavity take an average of N bounces before they diverge so far from the optical axis that they miss the mirror, if the bounce number enforced by the mirror reflectivity (n_o) is less than N, the cavity could look fairly resonant. The cavity linewidth therefore sort of smears out the boundary of the unstable region a little, and it is possible to construct "nearly confocal" and "nearly concentric" cavities in the lab.

3.2 The resonant Gaussian

Using Eq. 29 and Eq. 31, we can write the resonant q parameter for a two-mirror cavity as

$$q_{\mathrm{res}} = z_0 + \mathrm{i}b = -\frac{L(R_2 - L)}{R_1 + R_2 - 2L} + \mathrm{i}\frac{\sqrt{L(R_1 - L)(R_2 - L)(R_1 + R_2 - L)}}{R_1 + R_2 - 2L}. \tag{33}$$

Eq. 33 tells us everything we need to know about the position of the minimum waist and the spot size at the waist. In order to couple light into this cavity, therefore, one needs to send in a beam that has the same waist size and position at the cavity mode; this is called (transverse) *mode matching*.

There are a couple of observations that are worth making about Eq. 33. First, for a confocal cavity, the confocal parameter of the resonant mode has $b = L/2$. This is essentially why b is called the "confocal parameter." It tells you how long a confocal cavity needs to be to mode-match this beam. At a distance $z = b$ from the focus of a Gaussian beam, the wave-fronts have radius of curvature $R = 2b$.

Second, we can examine the wave front curvature at the surface of a mirror. This is done most easily by rewriting Eq. 28 for q^{-1} instead of q,

$$\frac{1}{q} = \frac{C + D\frac{1}{q}}{A + B\frac{1}{q}} \tag{34}$$

which, along with Eq. 30, gives us

$$\frac{1}{q} = \frac{D - A}{2B} - \mathrm{i}\frac{\sqrt{4 - (A + D)^2}}{2B}. \tag{35}$$

We can now write an expression for $1/q$:

$$\frac{1}{q_{\mathrm{res}}} = -\frac{1}{R_1} - \mathrm{i}\frac{1}{R_1}\sqrt{\frac{(R_1 - L)(R_1 + R_2 - L)}{L(R_2 - L)}} \tag{36}$$

We see that the wave-front radius of curvature *at the surface of the mirror* is equal to the mirror's radius of curvature. The phase fronts match their shape to the mirrors at the mirror surface in a Fabry-Perot cavity.

3.3 Higher-order resonant transverse modes

When we derived the solutions to the paraxial wave equation, we found that for a particular waist position and confocal parameter, there was a whole family of higher-order transverse modes associated with them. In Cartesian coordinates, these were the Hermite-Gaussian modes, which I reproduce here for your reference:

$$E = \sum_{n,p} \mathcal{E}_{n,p} \frac{w_0}{w(z)} H_n\left(\frac{\sqrt{2}\,x}{w(z)}\right) H_p\left(\frac{\sqrt{2}\,y}{w(z)}\right) \mathrm{e}^{-\mathrm{i}(n+p+1)\varphi(z)} \mathrm{e}^{-\frac{x^2+y^2}{w^2(z)}} \mathrm{e}^{\mathrm{i}k\frac{x^2+y^2}{2R(z)}} \mathrm{e}^{\mathrm{i}(kz-\omega t)}. \tag{37}$$

You may recall that the Gouy phase of the Gaussian was defined by $\varphi(z) \equiv \arctan(z/b)$, and we therefore expect that the total phase accumulated by the $\mathrm{TEM}_{n,p}$ mode during a full round trip in the cavity is

$$\Phi = 2kL - 2(n+p+1)\left(\varphi(z_2) - \varphi(z_1)\right) \tag{38}$$

where z_1 and z_2 are the positions of mirrors 1 and 2. We can then apply our conclusions from the infinite, planar cavity to conclude that the resonant frequencies of such a cavity will be those for which $\Phi = m2\pi$ for some integer longitudinal mode number m. We therefore have an expression for the resonant frequencies of the (m, n, p) mode of a paraxial cavity,

$$f_{np}^{(m)} = \frac{c}{2L}\left(m + \frac{1}{\pi}(n+p+1)\Delta\varphi\right) \tag{39}$$

where

$$\Delta\varphi \equiv \arctan\left[\frac{z_2}{b}\right] - \arctan\left[\frac{z_1}{b}\right]. \tag{40}$$

For a symmetric cavity,

$$\Delta\varphi = 2\arctan\left[\frac{L}{2b}\right] = 2\arctan\left[\frac{L}{\sqrt{2LR - L^2}}\right] = \arccos\left[1 - \frac{L}{R}\right] \tag{41}$$

where the reader is warned that the last step there requires some minor trigonometric sorcery.

For the general case,

$$\Delta\varphi = \arccos\left[\sqrt{\left(1 - \frac{L}{R_1}\right)\left(1 - \frac{L}{R_2}\right)}\right] \tag{42}$$

and we have

$$f_{np}^{(m)} = \frac{c}{2L}\left(m + \frac{1}{\pi}(n+p+1)\arccos\left[\sqrt{\left(1 - \frac{L}{R_1}\right)\left(1 - \frac{L}{R_2}\right)}\right]\right) \tag{43}$$

3.4 The symmetric confocal cavity

A symmetric cavity (both mirrors have the same radius of curvature) with $L = R$ has mode frequencies (directly from Eq. 41)

$$f_{np}^{(m)} = \frac{c}{2L}\left(m + \frac{1}{2}(n+p+1)\right). \tag{44}$$

All of the "even" transverse modes ($n+p$ is even) overlap with each other since their spacing is equal to the FSR. The $\mathrm{TEM}_{1,1}$ transverse mode for longitudinal mode number m has the exact same optical frequency as the TEM_{00} transverse mode of longitudinal mode $m+1$. Likewise, all of the "odd" transverse modes overlap with each other since their spacing is also equal to the FSR. However, the odd modes are offset in frequency by $FSR/2$ from the even modes.

As a practical matter, this mode overlap is what makes a confocal cavity so useful. Even poorly mode-matched input light will couple to half of the available modes so long as it is has the right frequency. In practice, scanning either the optical frequency or the cavity length while monitoring cavity transmission will produce peaks that are separated by $FSR/2 = c/4L$. This effect, combined with the fact that ray optics methods suggest that rays in a symmetric confocal cavity will repeat themselves every *two* round trips leads to people believing that the FSR of a confocal cavity is actually $c/4L$. However, if the input light is well mode-matched into a single transverse mode (or at least *only even* or *only odd* modes), those other peaks do not appear and the spectrum clearly repeats itself every $c/2L$. The free spectral range of the symmetric confocal cavity is therefore $c/2L$, just like every other cavity.