HW #8 (due Friday, 12/8)
(Physics 115B, Fall 2017)

1. A system of two-dimensional electron gas has area density of n,

(a) Show that the Fermi energy ε_F of the system is $\pi n\hbar^2/m$?

(b) What is its density of states?

(c) Show that the chemical potential is given by:

$$\mu(T) = k_B T \ln\{(\exp(\varepsilon_f / k_B T) - 1)]$$

(d) Show that the average kinetic energy is $k_B T$ at the high temperature limit ($T \to \infty$) and is 1/2 ε_F at the low temperature limit ($T \to 0$).

2. Show that the internal energy of an ideal Bose gas is given by

$$E = \frac{3}{2} k_B T V \left(\frac{2\pi m k_B T}{h^2}\right)^{3/2} \sum_{i=1}^{\infty} \frac{e^{i\mu / k_B T}}{i^{5/2}}$$

when the degeneracy is weak.

3-5. Griffiths 5.16, 5.20, 5.23.

N 1.

Repeat the method from the lecture, but for 2D.

Unit cell now is $\frac{2\pi}{L_x} \frac{2\pi}{L_y}$

Energy $E = \frac{\hbar^2}{2m}\left(k_x^2 + k_y^2\right)$

Number of electrons with $k < k_F$

$$N = 2 \cdot \frac{(\pi k_F^2)}{\left(\frac{2\pi}{L_x}\right)\left(\frac{2\pi}{L_y}\right)} = \frac{V k_F^2}{2\pi}$$

$$\Rightarrow k_F^2 = \frac{2\pi N}{V}$$

$$E = \frac{\hbar^2}{m} \frac{\pi N}{V} = \frac{\pi \hbar^2 n}{m} \quad QED$$

b) $n = \frac{m E_F}{\pi \hbar^2}$

$$n = \frac{m E}{\pi \hbar^2}$$

$$\rho = \frac{dn}{dE} = \frac{m}{\pi \hbar^2}$$ (for $E < E_F$ and 0 for $E > E_F$)

c) From Griffiths, eq. 5.91

$$N_k = \frac{d_k}{e^{\alpha + \beta E_k} + 1}$$

for fermions

$$d_k = \frac{V k dk}{\pi}$$ (we take into account that spin up and spin down configurations give factor 2)

$$N_k = \frac{V k dk}{\pi \left(e^{\alpha + \beta \frac{\hbar^2 k^2}{2m}} + 1\right)}$$

$$N = \sum N_k =$$

$$= \int_0^{+\infty} \frac{V k\, dk}{\pi\left(e^{\alpha + \beta \hbar^2 k^2/2m} + 1\right)} =$$

$$= \int_0^{+\infty} \frac{V\, dk^2}{2\pi\left(e^{\alpha + \beta \hbar^2 k^2/2m} + 1\right)} =$$

$$= \int_0^{+\infty} \frac{\frac{2mV}{\beta \hbar^2} dx}{2\pi\left(e^{x+\alpha} + 1\right)} = \frac{mV}{\beta \hbar^2 \pi} \ln\left(1 + e^{-\alpha}\right)$$

$$\Rightarrow \quad N = \frac{mV}{\beta \hbar^2 \pi} \ln\left(1 + e^{-\alpha}\right)$$

But we know

$$E_F = \frac{\hbar^2}{m} \frac{\pi N}{V}$$

Hence

$$E_F = \frac{1}{\beta} \ln\left(1 + e^{-\alpha}\right)$$

$$1 + e^{-\alpha} = e^{\beta E_F}$$

$$e^{-\alpha} = e^{\beta E_F} - 1$$

$$-\alpha = \ln(e^{\beta E_F} - 1)$$

$$\mu = \beta \ln(e^{\beta E_F} - 1) \quad QED.$$

d) We are interested in the cases $T \to 0$ and $T \to \infty$.

Let us look at $n(\varepsilon)$ in both regimes.

$$n(\varepsilon) = \frac{1}{e^{\alpha + \beta\varepsilon} + 1}$$

$$\alpha = -\ln(e^{\beta\varepsilon_F} - 1)$$

$$e^{\alpha + \beta\varepsilon} = e^{-\ln(e^{\beta\varepsilon_F} - 1) + \beta\varepsilon} =$$

$$= \frac{e^{\beta\varepsilon}}{e^{\beta\varepsilon_F} - 1}$$

$$e^{\alpha+\beta\varepsilon}+1=\frac{e^{\beta\varepsilon}+e^{\beta\varepsilon_F}-1}{e^{\beta\varepsilon_F}}$$

$$n(\varepsilon)=\frac{e^{\beta\varepsilon_F}}{e^{\beta\varepsilon}+e^{\beta\varepsilon_F}-1}$$

For $\beta\to\infty$ $(T\to 0)$

$n(\varepsilon)=1$ for $\varepsilon<\varepsilon_F$

$n(\varepsilon)=0$ for $\varepsilon>\varepsilon_F$

For $\beta\to 0$ $(T\to\infty)$

$$n(\varepsilon)=\frac{1+\beta\varepsilon_F}{1+\beta(\varepsilon+\varepsilon_F)}\simeq 1-\beta\varepsilon\simeq e^{-\beta\varepsilon}$$

For total number of particles we have

$$N = \int_0^{+\infty} n(\varepsilon)\, d_k = \int_0^{+\infty} n\left(\frac{\hbar^2 k^2}{2m}\right) \frac{Vk}{\pi} dk$$

For $T \to 0$

$$N = \int_0^{k_F} \frac{Vk}{\pi} dk = \frac{V}{2\pi} k_F^2$$

$$E = \int_0^{+\infty} n(\varepsilon)\, \varepsilon\, d_k = \int_0^{k_F} \frac{\hbar^2 k^2}{2m} \frac{Vk}{\pi} dk =$$

$$= \frac{\hbar^2}{2m} V \frac{k_F^4}{4\pi}$$

Energy per particle

$$\varepsilon = E/N = \frac{1}{2} \frac{\hbar^2}{2m} k_F^2 = \frac{1}{2} \varepsilon_F \quad QED$$

For $T \to \infty$

$$N = \int_0^{+\infty} e^{-\beta \frac{\hbar^2 k^2}{2m}} \frac{Vk}{\pi} dk =$$

$$= \frac{V}{2\pi} \int_0^{+\infty} e^{-\beta \frac{\hbar^2 k^2}{2m}} dk^2 =$$

$$= \frac{V}{2\pi} \frac{2m}{\beta \hbar^2}$$

While

$$E = \int_0^{+\infty} n(\varepsilon)\, \varepsilon\, d_k =$$

$$= \frac{V}{2\pi} \int_0^{+\infty} e^{-\beta \hbar^2 k^2 / 2m} \frac{\hbar^2 k^2}{2m} k\, dk =$$

$$= -\frac{\partial N}{\partial \beta} = \frac{V}{2\pi} \frac{2m}{\beta^2 \hbar^2}$$

$$\Downarrow$$

$$E/N = \frac{1}{\beta} = k_B T \qquad QED$$

N2.

We know that (formula 5.109 Griffits)

$$E = \frac{V}{2\pi^2} \frac{\hbar^2}{2m} \int_0^{\infty} \frac{k^4}{e^{[\hbar^2 k^2/2m - \mu]\beta} - 1}$$

Weak degeneracy means μ is large and negative.

Hence, $\frac{\hbar^2 k^2}{2m} - \mu$ is large and positive (because $\frac{\hbar^2 k^2}{2m}$ is always positive).

Denote $A(k) = \left(\frac{\hbar^2 k^2}{2m} - \mu\right)\beta$

$A(k) \gg 1$ for all k

Let us rewrite our integral

$$E = \frac{V}{2\pi^2}\frac{\hbar^2}{2m}\int_0^\infty \frac{k^4}{e^A - 1}\,dk$$

$$\frac{1}{e^A - 1} = \frac{e^{-A}}{1 - e^{-A}}$$

$e^{-A} \ll 1 \Rightarrow$ we can use

a Taylor expansion with small parameter $x = e^{-A}$

$$\frac{x}{1-x} = x \sum_{i=0}^{\infty} x^i = \sum_{i=0}^{\infty} x^{i+1} =$$

$$= \sum_{i=1}^{\infty} x^i = \sum_{i=1}^{\infty} e^{-iA}$$

So our integral is then

$$\int_0^{\infty} k^4 \sum_{i=1}^{\infty} e^{-i\left(\frac{\hbar^2 k^2}{2m} - \mu\right)\beta} dk =$$

$$= \sum_{i=1}^{\infty} \int_0^{\infty} k^4 e^{-i\left(\frac{\hbar^2 k^2}{2m} - \mu\right)\beta} dk$$

The integrals under summation can be taken explicitly, because they are gaussian. We get after taking the integrals

$$\int_0^\infty k^4 e^{-i\left(\frac{\hbar^2}{2m}k^2-\mu\right)\beta} dk =$$

$$= \frac{3}{2} \frac{e^{i\beta\mu}\sqrt{2\pi}}{\left(\frac{i\beta\hbar^2}{m}\right)^{5/2}}$$

After we substitute it back to the expression for the energy we get

$$E = \frac{3}{2} \frac{V}{2\pi^2} \frac{\hbar^2}{2m} \frac{\sqrt{2\pi}}{\left(\frac{\beta \hbar^2}{m}\right)^{5/2}} \sum_{i=1}^{\infty} \frac{e^{i\beta\mu}}{i^{5/2}}$$

QED.

Griffits 5.16.

a) $E_F = \frac{\hbar^2}{2m}(3\rho\pi^2)^{2/3}$ = after substitution of the values =

$= 7.04\ eV$

b) $E_F = \frac{1}{2} m v_F^2 \Rightarrow v_F = \sqrt{2mE_F}$

$v_F = 1.57 \cdot 10^6\ m/s$

speed of light is $3 \cdot 10^8\ m/s$

$\Rightarrow v_F \ll c$, electrons in a metal can be treated as non-relativistic.

c) $T = \frac{E_F}{k_B} = 8.17 \cdot 10^4\ K$

d) $P = \frac{(3\pi^2)^{2/3} \hbar^2}{5m} \rho^{5/3} =$

$= 3.84 \cdot 10^{10} \frac{N}{m^2}$

5.20

Positive energy solutions –
– the same as before with the exception that α and β now negative.

Negative-energy solutions. On $0 < x < a$

$$\frac{d^2\psi}{dx^2} = \kappa^2 \psi \quad \text{where}$$

$$\kappa \equiv \frac{\sqrt{-2mE}}{\hbar} \Rightarrow$$

$$\Rightarrow \psi(x) = A \sinh \kappa x + B \cosh \kappa x$$

According to Bloch's theorem the solution on $-a<x<0$ is

$$\psi(x)=e^{-iKa}\left[A\sinh \kappa(x+a)+B\cosh \kappa(x+a)\right]$$

Continuity at $x=0 \Rightarrow$

$$B=e^{-iKa}\left[A\sinh \kappa a+\right.$$

$$\left.+B\cosh \kappa a\right],\quad \text{or}$$

$$A\sinh \kappa a=B\left[e^{iKa}-\cosh \kappa a\right]$$

The discontinuity in ψ' (eq. 2.125)

$$\kappa A-e^{-iKa}\kappa\left[A\cosh \kappa a+B\sinh \kappa a\right]=$$

$$=\frac{2m\alpha}{\hbar^2}B$$

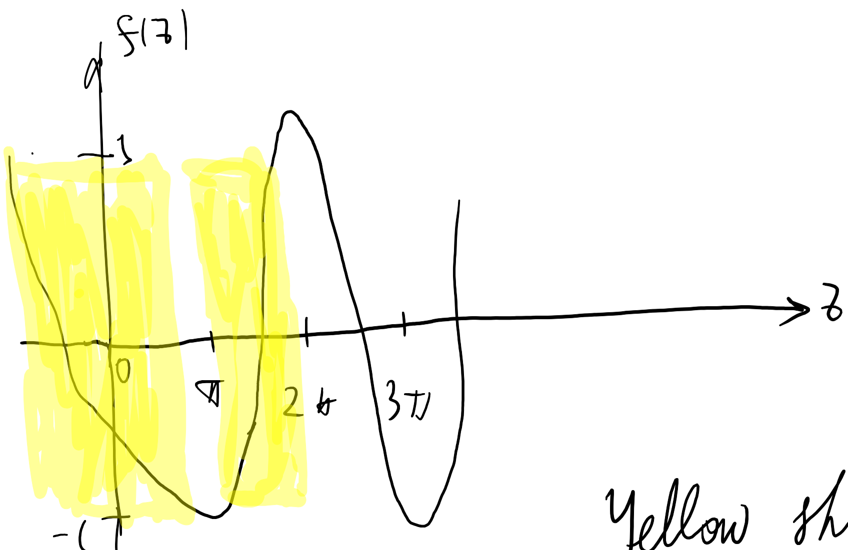
Solving this two we get

$$\cos Ka = \cosh \kappa a + \frac{m\alpha}{\hbar^2 \kappa} \sinh \kappa a$$

Define $z \equiv -\kappa a$ and

$$f(z) = \cosh z + \beta \frac{\sinh z}{z}$$

As before we plot it and see



Yellow shows the band structure

N 5.23.

a) $E_{n_1 n_2 n_3} = \left(n_1 + n_2 + n_3 + \frac{3}{2}\right)\hbar\omega =$

$= \frac{9}{2}\hbar\omega \Rightarrow n_1 + n_2 + n_3 = 3$

$(n_1, n_2, n_3 = 0, 1, 2, 3 \ldots)$

State n_1	n_2	n_3	Configuration $(N_0, N_1, N_2 \ldots)$	# of States
0	0	3	(2, 0, 0, 1, 0, 0 ..	3
0	3	0		
3	0	0		
0	1	2	(1, 1, 1, 0, 0, 0 ...)	6
0	2	1		
1	0	2		
1	2	0		
2	0	1		
2	1	0		
1	1	1	(0, 3, 0 ...)	1

Most probable configuration

$(1,1,1,0,0,\dots)$

Most probable single-particle energy

$$E_0 = \frac{1}{2}\hbar\omega$$

b) For identical fermions the only configuration is $(1,1,1,0\dots)$ so this is also the most probable configuration. The possible one-particle energies are

$$E_0 \left(P_0 = \frac{1}{3}\right)$$

$$E_1 \left(P_1 = \frac{1}{3}\right)$$

$$E_2 \left(P_2 = \frac{1}{3}\right)$$

and they are equally likely.

c) For identical bosons all three configurations are possible, and there is one state for each.

Possible one-particle energies

$E_0 \left(P_0 = \frac{1}{3}\right)$

$E_1 \left(P_1 = \frac{4}{9}\right)$

$E_2 \left(P_2 = \frac{1}{9}\right)$

$E_3 \left(P_3 = \frac{1}{9}\right)$

Most probable energy is E_1