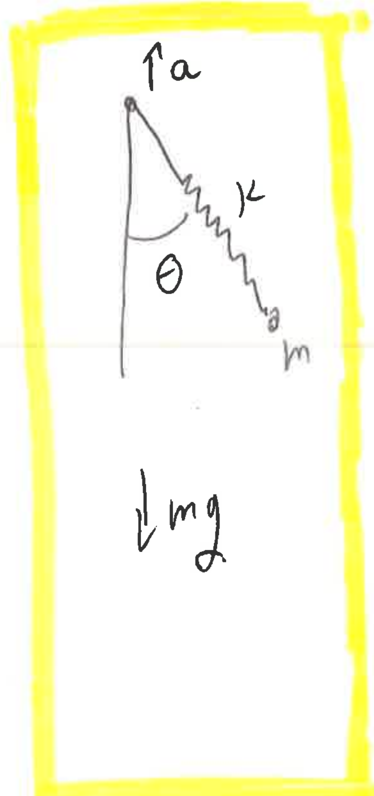
AP.1

Example

Consider a pendulum of a mass m suspended with a massless spring with unextended length b and spring const. k

The pendulum point of attachment rises vertically with a const acceleration a



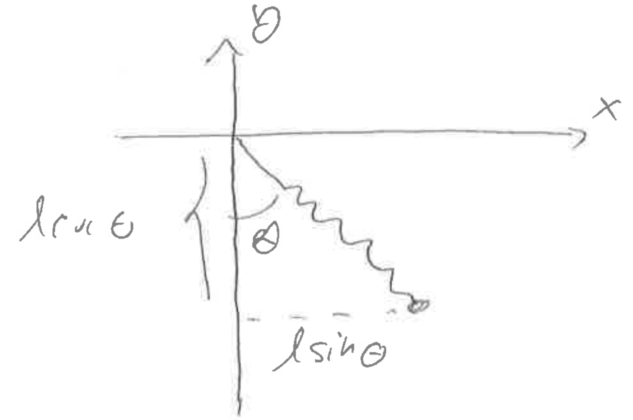
1. find $\mathcal{L}$
2. find E.O.M
3. find the momenta and conserved quantities
4. find $\mathcal{H}$
5. assume small angles $\theta \ll 1$ and find the period of oscillations
6. find the solution of the E.O.M.

AP.2

first we define the coordinates:

b - unextended length

l - variable length



(1) $x = l\sin\theta$

(2) $y = \frac{1}{2}at^2 - l\cos\theta$

This factor comes because we have const. acceleration.

So

(3) $\dot{x} = \dot{l}\sin\theta + l\dot{\theta}\cos\theta$

(4) $\dot{y} = at - \dot{l}\cos\theta + l\dot{\theta}\sin\theta$

(5) $T = \frac{1}{2}m(\dot{x}^2+\dot{y}^2) = \frac{1}{2}m\left[\dot{l}^2 + l^2\dot{\theta}^2 + a^2t^2 + 2at(l\dot{\theta}\sin\theta - \dot{l}\cos\theta)\right]$

(6) $U = mgy + \frac{1}{2}k(l-b)^2$

(7) $$\boxed{\mathcal{L} = T - U = \frac{1}{2}m\left[\dot{l}^2 + l^2\dot{\theta}^2 + a^2t^2 + 2at(l\dot{\theta}\sin\theta - \dot{l}\cos\theta)\right] + mg\left[l\cos\theta - \frac{at^2}{2}\right] - \frac{k}{2}(l-b)^2}$$

AP.3 To find the E.O.Ms we have:

$$(8) \quad \frac{\partial \mathcal{L}}{\partial l} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{l}}\right) = 0 \ ; \quad \frac{\partial \mathcal{L}}{\partial \theta} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) = 0$$

Let's start with l:

$$(9) \quad \frac{\partial \mathcal{L}}{\partial l} = m l \dot{\theta}^2 + mat\dot{\theta}\sin\theta + mg\cos\theta - k(l-b)$$

$$(10) \quad \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{l}}\right) = \frac{d}{dt}\left[m\dot{l} - amt\cos\theta\right] = m\ddot{l} - am\cos\theta + amt\dot{\theta}\sin\theta$$

(10) = (9)

$$(11) \quad m\ddot{l} - am\cos\theta + \cancel{amt\dot{\theta}\sin\theta} = ml\dot{\theta}^2 + \cancel{mat\dot{\theta}\sin\theta} + mg\cos\theta - k(l-b)$$

arranging we have:

$$(12) \quad \boxed{\ddot{l} - l\dot{\theta}^2 - (a+g)\cos\theta + \frac{k}{m}(l-b) = 0}$$

Now for θ:

$$(13) \quad \frac{\partial \mathcal{L}}{\partial \theta} = mat\dot{l}\sin\theta - mgl\sin\theta + matl\dot{\theta}\cos\theta$$

$$(14) \quad \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) = \frac{d}{dt}\left[ml^2\dot{\theta} + matl\sin\theta\right] =$$

$$= ml^2\ddot{\theta} + 2ml\dot{\theta}\dot{l} + mal\sin\theta + mat\dot{l}\sin\theta + matl\dot{\theta}\cos\theta$$

Since (13) = (14) we have

$$\cancel{mat\dot{l}\sin\theta} - mgl\sin\theta + \cancel{matl\dot{\theta}\cos\theta} = ml^2\ddot{\theta} + 2ml\dot{\theta}\dot{l}$$

(15) $$+ ml\dot{\theta}\dot{l} + mal\sin\theta + \cancel{mat\dot{l}\sin\theta} + \cancel{matl\dot{\theta}\cos\theta}$$

(16) $$\boxed{\ddot{\theta} + \frac{2}{l}\dot{l}\dot{\theta} + \frac{a+g}{l}\sin\theta = 0}$$

Eqs. (12) and (16) give the EMos

The momenta.

(17) $$p_l = \frac{\partial L}{\partial \dot{l}} = m\dot{l} - mat\cos\theta$$

(18) $$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = ml^2\ddot{\theta} + matl\sin\theta$$

Note that both momenta are Not constant as $\frac{\partial L}{\partial l}; \frac{\partial L}{\partial \theta} \neq 0$

The Energy, E, is also not conserved since $L = L(t)$

There are no intergral of motion here.

AP.5 To find the Hamiltonian we'll use (17) and (18) to express the coordinates via the momenta

$$\dot{l} = \frac{p_l}{m} + at\cos\theta \tag{19}$$

$$\dot{\theta} = \frac{p_\theta}{ml^2} - \frac{at\sin\theta}{l} \tag{20}$$

So then

$$H = p_l\dot{l} + p_\theta\dot{\theta} - \mathcal{L} \tag{21}$$

$$\boxed{H = \frac{p_l^2}{2m} + \frac{p_\theta^2}{2ml^2} - \frac{at}{l}p_\theta\sin\theta + atp_l\cos\theta + \frac{1}{2}k(l-b)^2 + \frac{1}{2}mgat^2 - mgl\cos\theta} \tag{22}$$

Assuming $\theta << 1$ we have

$\sin\theta \approx \theta$ $\quad\cos\theta \approx 1$, keeping only linear terms we have for l:

$$\ddot{l} - (a+g) + \frac{k}{m}(l-b) = 0 \tag{23}$$

which we can write

$$\boxed{\ddot{l} + \frac{k}{m}l = a + g + \frac{k}{m}b} \tag{24}$$

AP. 6

For Θ we have

(25) $$\boxed{\ddot{\Theta} + \frac{a+g}{l}\Theta = 0}$$

So the pendulum has small angles oscillations of

(26) $$\omega = \sqrt{\frac{a+g}{l}}$$ with period of

(27) $$\boxed{T_\Theta = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{l}{a+g}}}$$

To find the sol we know that

(28) $$\boxed{\Theta(t) = \Theta_0 \cos(\omega t + \ell_0)}$$ where $\omega = \sqrt{\frac{a+g}{l}}$

and for l we know that

(29) $$l = l_0 + l_p$$

(30) $$l_0 = A\cos(\omega_l t + \phi_l)$$

(31) where $\boxed{\omega_l = \sqrt{\frac{k}{m}}}$

(32) $$l_p = \frac{m}{k}(a+g) + b$$ So

(33) $$\boxed{l(t) = A\cos\left(\sqrt{\frac{k}{m}}\,t + \phi_l\right) + \frac{m}{k}(a+g) + b}$$