105A - Midterm 2 - solutions

(Grades are out of 150)

A damped linear oscillator, with a restoring acceleration of $-\omega_0^2 x$, and friction acceleration of $-2\beta v$, where v is the velocity and $\beta > 0$ is set originally in its equilibrium position and given a velocity $v_0 = \frac{a}{\tau\omega_0^2}$. The oscillator is subjected to a forcing function given by:

$$\frac{f(t)}{m} = a(t/\tau) \tag{1}$$

The equation of motion of this oscillator is:

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = a(t/\tau) \tag{2}$$

1. Find the particular solution in terms of a, β, ω_0 and τ.
   Answer: For the particular solution we do an extension to the complex regime so

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = a\frac{t}{\tau} \tag{3}$$

We guess the following solution $x_p = C + Dt$. Plugging this in we have

$$2\beta D + \omega_0^2(C + Dt) = a\frac{t}{\tau} \tag{4}$$

which gives:

$$\omega_0^2 D = \frac{a}{\tau} \tag{5}$$

$$2\beta D + \omega_0^2 C = 0 \tag{6}$$

So

$$D = \frac{a}{\tau\omega_0^2} \tag{7}$$

$$C = -\frac{2\beta a}{\omega_0^4\tau} \tag{8}$$

So

$$x_p(t) = -\frac{2\beta a}{\omega_0^4\tau} + \frac{a}{\tau\omega_0^2}t \tag{9}$$

2. Find the homogenous solution for $x_0(t)$ (i.e., solve: $\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$). And from that find the general solution $x(t) = x_p(t) + x_0(t)$. Find all the parameters in terms of β, ω_0, a and τ.
   Answer: The homogenous solution, $x_0(t)$ is easy, we did it many times.

$$x_0(t) = Re[Ae^{i(\omega t+\phi)}] \tag{10}$$

Plugging this into the homogenous equation: $\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$ we are left with the simple solution:

$$\omega^2 - i2\beta\omega - \omega_0^2 = 0 \tag{11}$$

Finding the frequencies we have:

$$\omega_{1,2} = \frac{i2\beta \pm \sqrt{-4\beta^2 + 4\omega_0^2}}{2} = \beta i \pm \sqrt{\omega_0^2 - \beta^2} \tag{12}$$

identifying

$$\omega_r = \sqrt{\omega_0^2 - \beta^2} \quad \text{and} \quad \omega_i = \beta \tag{13}$$

we have $\omega = i\omega_i \pm \omega_r$ and the solution is $x(t) = Re[e^{i(\omega t+\phi)}] = Re[e^{i(\{i\omega_i \pm \omega_r\}t+\phi)}]$ so

$$\begin{aligned} x_0(t) &= ARe[e^{-\omega_i t \pm i\omega_r t + i\phi}] = Ae^{-\omega_i t} Re[\cos(\omega_r t + \phi) \pm i \sin(\omega_r t + \phi)] \\ &= Ae^{-\omega_i t}\cos(\omega_r t + \phi) = Ae^{-\beta t}\cos(\sqrt{\omega_0^2 - \beta^2} t + \phi) \end{aligned} \tag{14}$$

So we have

$$x(t) = x_0 + x_p = Ae^{-\beta t}\cos(\sqrt{\omega_0^2 - \beta^2} t + \phi) - \frac{2\beta a}{\omega_0^4 \tau} + \frac{a}{\tau\omega_0^2} t \tag{15}$$

Plugging in the initial conditions ($x(t=0) = 0$ and $\dot{x}(t=0) = v_0 = a\tau$) we have

$$x(t=0) = A\cos(\phi) - \frac{2\beta a}{\omega_0^4 \tau} = 0 \tag{16}$$

which gives:

$$A = \frac{2\beta a}{\omega_0^4 \tau \cos(\phi)} \tag{17}$$

For the first derivative note that

$$\dot{x}(t) = -\omega_r A \sin(\omega_r t + \phi) e^{-\omega_i t} - A\omega_i e^{-\omega_i t}\cos(\omega_r t + \phi) + \frac{a}{\tau\omega_0^2} \tag{18}$$

which is then

$$\dot{x}(t=0) = v_0 = \frac{a}{\tau\omega_0^2} = -\omega_r A \sin(\phi) - A\omega_i \cos(\phi) + \frac{a}{\tau\omega_0^2} \tag{19}$$

So this equation gives

$$-\omega_r \sin(\phi) = \omega_i \cos(\phi) \tag{20}$$

or

$$\tan\phi = \frac{-\omega_i}{\omega_r} = \frac{-\beta}{\sqrt{\omega_0^2 - \beta^2}} \tag{21}$$

3. Studying for the midterm two students solved a similar problem. Student A said that if $\beta = \omega_0$ then the system will be in resonance However, student B disagreed with student A. Who is right? Student A or student B? If student A is correct, show how the system can enter a resonance, if student B is correct, explain what will happen when $\beta = \omega_0$. Explain your answer.
Answer: Student A is wrong. When $\beta = \omega_0$ we have critical damping, the argument under the square root is zero and the system simply damps, with no oscillations. Resonance will take place in this system if the forcing will be in the same frequency as one of the natural frequencies of the damped system.