# Problem Sets 1 solutions

**Physics 262**
*Winter 2018*

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1. An arbitrary state can be written as

$$|\psi\rangle = a\,|\uparrow\rangle + b\,|\downarrow\rangle = |a|\, e^{i\arg a}\,|\uparrow\rangle + |b|\, e^{i\arg b}\,|\downarrow\rangle$$

where $|a|^2 + |b|^2 = 1$. Now let $\cos\frac{\theta}{2} = |a|$. Then $|b| = \sqrt{1-|a|^2} = \left|\sin\frac{\theta}{2}\right|$. Finally we are free to multiply by an overall phase factor $e^{-i\arg a}$. Choosing $\phi = \arg b - \arg a$ then gives (5 pts)

$$|\psi\rangle = |a|\,|\uparrow\rangle + |b|\, e^{i(\arg b - \arg a)}\,|\downarrow\rangle = \cos\left(\frac{\theta}{2}\right)|\uparrow\rangle + \sin\left(\frac{\theta}{2}\right)e^{i\phi}\,|\downarrow\rangle\,.$$

$$\begin{aligned}
\langle S_x\rangle &= \begin{pmatrix}\cos\frac{\theta}{2} & e^{-i\phi}\sin\frac{\theta}{2}\end{pmatrix}\begin{pmatrix}0 & \hbar/2\\ \hbar/2 & 0\end{pmatrix}\begin{pmatrix}\cos\frac{\theta}{2}\\ e^{i\phi}\sin\frac{\theta}{2}\end{pmatrix}\\
&= \cos\frac{\theta}{2}\sin\frac{\theta}{2}(e^{i\phi}+e^{-i\phi})\frac{\hbar}{2}\\
&= \frac{\hbar}{2}\sin\theta\cos\phi.
\end{aligned}$$

Similarly one finds

$$\begin{aligned}
\langle S_y\rangle &= \frac{\hbar}{2}\sin\theta\sin\phi\\
\langle S_z\rangle &= \frac{\hbar}{2}\cos\theta\\
\langle\vec{S}\rangle &= \frac{\hbar}{2}\begin{pmatrix}\sin\theta\cos\phi\\ \sin\theta\sin\phi\\ \cos\theta\end{pmatrix}
\end{aligned}$$

which is a vector of length $\hbar/2$ pointing in the $(\theta,\phi)$ direction. (5 pts)

2. The rotation operator can be expressed as

$$\begin{aligned}
\mathscr{D}(\hat{n},\phi) = e^{-\frac{i\vec{\sigma}\cdot\hat{n}\phi}{2}} &= \left[1 - \frac{(\vec{\sigma}\cdot\hat{n})^2}{2!}\left(\frac{\phi}{2}\right)^2 + \cdots\right] - i\left[(\vec{\sigma}\cdot\hat{n})\frac{\phi}{2} - \frac{(\vec{\sigma}\cdot\hat{n})^3}{3!}\left(\frac{\phi}{2}\right)^3 + \cdots\right]\\
&= \mathbb{1}\cos\left(\frac{\phi}{2}\right) - i(\vec{\sigma}\cdot\hat{n})\sin\left(\frac{\phi}{2}\right)
\end{aligned}$$

where we have used

$$(\vec{\sigma}\cdot\hat{n})^n = \begin{cases} 1 & n \text{ even} \\ \vec{\sigma}\cdot\hat{n} & n \text{ odd} \end{cases}$$

Therefore (4 pts)

$$\mathscr{D}(\hat{z}, -\omega_0 t)\,|\psi(0)\rangle = \begin{pmatrix} e^{i\omega_0 t/2} & 0 \\ 0 & e^{-i\omega_0 t/2} \end{pmatrix}\begin{pmatrix} a_\uparrow \\ a_\downarrow \end{pmatrix} = a_\uparrow e^{i\omega_0 t/2}\,|\uparrow\rangle + a_\downarrow e^{-i\omega_0 t/2}\,|\downarrow\rangle = |\psi(t)\rangle\,.$$

The expectation value is given by

$$\begin{aligned} \langle \vec{S(t)}\rangle &= \langle\psi(t)|\vec{S}|\psi(t)\rangle \\ &= \langle\psi(0)|\mathscr{D}^\dagger(\hat{z},-\omega_0 t)\vec{S}\mathscr{D}(\hat{z},-\omega_0 t)|\psi(0)\rangle \\ &= \mathscr{R}(\hat{z},-\omega_0 t)\,\langle\psi(0)|\vec{S}|\psi(0)\rangle \end{aligned}$$

where $\mathscr{R}(\hat{z},-\omega_0 t)$ is the rotation matrix for a classical vector by an angle $-\omega_0 t$ around $\hat{z}$. From problem 1 we can write

$$\langle\vec{S}(0)\rangle = \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix}$$

where $\theta = 2\cos^{-1}\left|a_\uparrow\right|, \phi = \arg a_\downarrow - \arg a_\uparrow$. Therefore (3 pts)

$$\langle\vec{S}(t)\rangle = \begin{pmatrix} \sin\theta\cos(\phi-\omega_0 t) \\ \sin\theta\sin(\phi-\omega_0 t) \\ \cos\theta \end{pmatrix}$$

Taking the time derivative gives (3 pts)

$$\frac{\mathrm{d}\,\langle\vec{S}(t)\rangle}{\mathrm{d}t} = \begin{pmatrix} \omega_0\sin\theta\sin(\phi-\omega_0 t) \\ -\omega\sin\theta\cos(\phi-\omega_0 t) \\ 0 \end{pmatrix} = \omega_0\,\langle\vec{S}(t)\rangle\times\hat{z} = \gamma\,\langle\vec{S}(t)\rangle\times\vec{B}$$

since $\omega_0 = 2\vec{\mu}\cdot\vec{B} = \gamma B$.

3. One way to show the selection rules is to first express the interaction operator in terms of the rank-1 spherical tensor, $\boldsymbol{r} = (\hat{x},\hat{y},\hat{z})$, where

$$\hat{x} = \frac{\hat{r}_{-1}-\hat{r}_{+1}}{\sqrt{2}} \qquad \hat{y} = \frac{i(\hat{r}_{+1}+\hat{r}_{-1})}{\sqrt{2}} \qquad \hat{z} = \hat{r}_0$$

Then the interaction operator can be written as

$$H'(t) = e\vec{r}\cdot\vec{E} = eE\left[\cos(\omega t)\hat{x} - \sin(\omega t)\hat{y}\right] = \frac{eE}{\sqrt{2}}\left[e^{-i\omega t}\hat{r}_{-1} - e^{i\omega t}\hat{r}_{+1}\right].$$

Intuitively we can see that the field is rotating counterclockwise giving a negative projection of angular momentum along the $\hat{z}$ axis. Absorbing a photon from this field will give $\Delta m_l = -1$ (i.e. the $\hat{r}_{-1}$ term) while stimulated emission gives $\Delta m_l = +1$ (the $\hat{r}_{+1}$ term). Formally one can use the Wigner-Eckert theorem to show that $\langle \psi_{n'l'm'} | H' | \psi_{nlm} \rangle \neq 0$ only if $m' = m \pm 1$ which rules out $1s \to 2p, m_l = 0$. (2 pts) To find the matrix element to the $2p$,$|m| = 1$ states we need the wavefunctions

$$\begin{aligned}
\psi^*_{21\pm1} &= \pm\sqrt{\frac{3}{8\pi}} \sin\theta e^{\mp i\phi} \left(\frac{1}{2a_0}\right)^{3/2} \frac{r}{\sqrt{3}a_0} e^{-r/2a_0} \\
\psi_{100} &= \frac{1}{\sqrt{4\pi}} \left(\frac{1}{a_0}\right)^{3/2} 2e^{-r/a_0}
\end{aligned}$$

In spherical coordinates $x = r\cos\phi\sin\theta, y = r\sin\phi\sin\theta$, and the interaction matrix element is (7 pts)

$$\begin{aligned}
\langle \psi^*_{21\pm1} | H' | \psi_{100} \rangle = &\pm \frac{eE}{8\pi a_0^4} \int_0^\infty r^4 e^{-3r/2a_0}\, dr \\
&\times \int_0^\pi \int_0^{2\pi} \sin^2\theta e^{\mp i\phi} \left[\cos(\omega t)\cos\phi\sin\theta - \sin(\omega t)\sin\phi\sin\theta\right] \, d\phi\, d\theta \\
&= \pm \frac{eE}{8\pi a_0^4} \times \frac{256 a_0^5}{81} \times \pi \int_0^\pi \sin^3\theta \left[\cos(\omega t) \pm i\sin(\omega t)\right] \, d\theta \\
&= \pm \frac{eE}{8\pi a_0^4} \times \frac{256 a_0^5}{81} \times \pi \times \frac{4}{3} e^{\pm i\omega t} = \pm \frac{128}{243} eEa_0 e^{\pm i\omega t}
\end{aligned}$$

$$V = \frac{256}{243} ea_0.$$

In our derivation for Rabi flopping (keeping track of signs) one can show that the detuning for the transition $1s \to 2p, m = \pm1$, which depends of the sign of the phase, is $\Delta = \mp\omega - \omega_0$. As we anticipated for $\omega \approx \omega_0$ only the $m = -1$ final state is near resonance for absorption. One can think of this as coming from the fact that, for angular momentum to be conserved, there would have to be stimulated emission of the photon to reach $m = +1$ which is far off resonance by $2\omega_0$. (1 pt)