Question 1

A damped harmonic oscillator, with mass m, and damping coefficient β, that is driven sinusoidally with angular frequency ω and acceleration for a finite time interval

$$\frac{F(t)}{m} = \begin{cases} 0, & t < 0 \\ a \sin \omega t, & 0 < t < \pi/\omega \\ 0, & t > \pi/\omega \end{cases}$$

Find $x(t)$ express all the constants using the parameter given in the problem.

Solution to question 1

We would like to solve for the motion of a damped harmonic oscillator that is driven sinusoidally with angular frequency ω and acceleration a for a finite time interval $0 < t < \pi/\omega$. For $t < 0$ the oscillator is at rest: $x(t) = 0$. This trivially solves the homogeneous differential equation

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0. \tag{1}$$

The general solution of this equation is the complementary function

$$x_c(t) = e^{-\beta t}\left[A\cos(\omega_1 t) + B\sin(\omega_1 t)\right], \tag{2}$$

where $\omega_1 \equiv \sqrt{\omega_0^2 - \beta^2}$ and the two integration constants A and B are to be determined. For $0 < t < \pi/\omega$ we have to solve the inhomogeneous equation

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = a\sin\omega t, \tag{3}$$

whose the solutions are the sum of the complementary function $x_c(t)$ given by (2) and the particular integral

$$x_p(t) = \frac{a}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\omega^2\beta^2}}\sin(\omega t - \delta), \tag{4}$$

where

$$\delta = \tan^{-1}\left(\frac{2\omega\beta}{\omega_0^2 - \omega^2}\right). \tag{5}$$

This can easily be derived by substituting the ansatz $x_p(t) \propto \sin(\omega t - \delta)$ into (3). Matching boundary conditions at $t = 0$ we find that the equations $x_c(0) + x_p(0) = 0$ and $\dot{x}_c(0) + \dot{x}_p(0) = 0$ are satisfied if

$$\begin{aligned} A &= \frac{2a\omega\beta}{(\omega_0^2 - \omega^2)^2 + 4\omega^2\beta^2}, && (6) \\ B &= -\frac{a\omega(\omega_0^2 - \omega^2 - 2\beta^2)}{\omega_1[(\omega_0^2 - \omega^2)^2 + 4\omega^2\beta^2]}. && (7) \end{aligned}$$

This specifies the solution for as long as the system is driven. For $t > \pi/\omega$ the system is undriven and obeys (1), thus

$$x(t) = x_c(t) = e^{-\beta t}\left[C\cos(\omega_1 t) + D\sin(\omega_1 t)\right]. \tag{8}$$

Again matching x and $\dot{x}$ and $t = \pi/\omega$ we find after a little bit of algebra

$$\begin{aligned} C &= A + e^{\beta\pi/\omega}\left[A\cos\left(\frac{\omega_1}{\omega}\pi\right) - B\sin\left(\frac{\omega_1}{\omega}\pi\right)\right], && (9) \\ D &= B + e^{\beta\pi/\omega}\left[A\sin\left(\frac{\omega_1}{\omega}\pi\right) + B\cos\left(\frac{\omega_1}{\omega}\pi\right)\right]. && (10) \end{aligned}$$

With these constants equation (8) describes the behaviour for all times $t > \pi/\omega$.

Question 2

An automobile with a mass of 1000 kg, including passengers, settles 1.0 cm closer to the road for every additional 100 kg of passengers. It is driven with a constant horizontal component of speed 20 km/h over a washboard road with sinusoidal bumps. The amplitude and wavelength of the sine curve are 5.0 cm and 20 cm, respectively. The distance between the front and back wheels is 2.4 m. Find the amplitude of oscillation of the automobile, assuming it moves vertically as an undamped driven harmonic oscillator. Neglect the mass of the wheels and springs and assume that the wheels are always in contact with the road.

Solution to question 1

This is a simple application of the forced undamped harmonic oscillator. The car drives at speed $v = 20$ km/h over a road with sinusoidal profile, so that its springs are compressed by an amount $y(t) = A\sin(\omega t)$, where $A = 0.05$ m and $\omega = 2\pi v/\lambda = 174.5\,\mathrm{s}^{-1}$. You are told that the car simply oscillates vertically, which is consistent with the separation between front and rear axle being an integer multiple of the wavelength of the bumps λ.

We treat the car as a mass m attached to a spring characterized by k, which leads to the resonance frequency $\omega_0^2 = k/m = 98.1\,\mathrm{s}^{-2}$. The system satisfies the equation

$$\ddot{x} + \omega_0^2 x = A\omega_0^2 \sin(\omega t). \tag{11}$$

We can then directly read off the amplitude of oscillations from the particular solution (4) we found in the previous problem:

$$\max[x_p(t)] = \frac{A\omega_0^2}{|\omega_0^2 - \omega^2|} = 1.6\ 10^{-4}\,\mathrm{m}. \tag{12}$$