Problem Set #9 Solutions (Last two problems)
(The rest have answers in the back)

#1) PM #8.17) After Q cycles, the angle ωt is equal to $2\pi \cdot Q$ radians (just the usual cycles $\Leftrightarrow$ radians conversion). Using equation (8.13) we have $Q = \frac{\omega}{2\alpha}$, thus

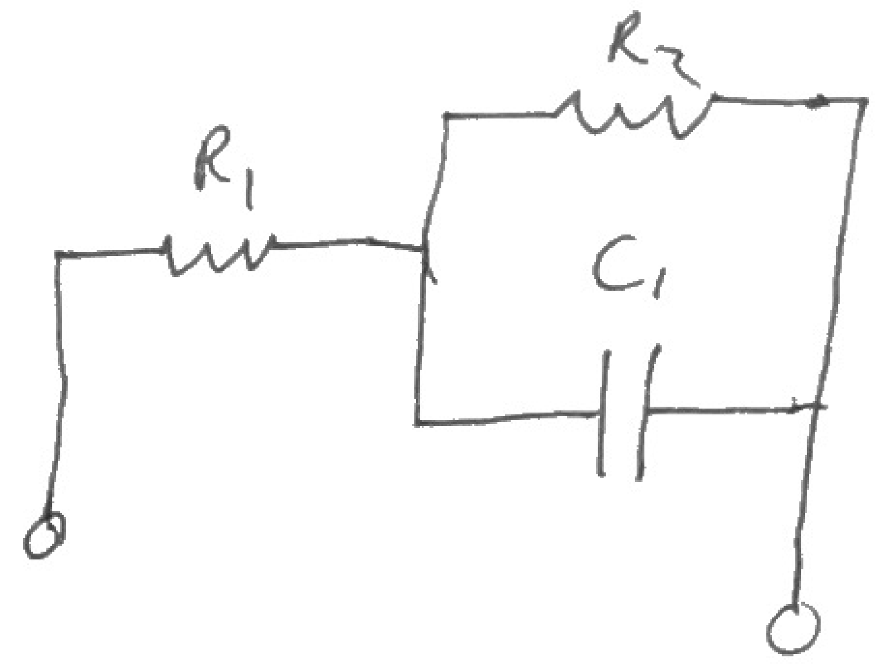
$$2\pi Q = 2\pi \cdot \frac{\omega}{2\alpha} = \frac{\pi\omega}{\alpha} = \omega t.$$

Thus, it takes $t = \frac{\pi}{\alpha}$ seconds to go Q cycles.

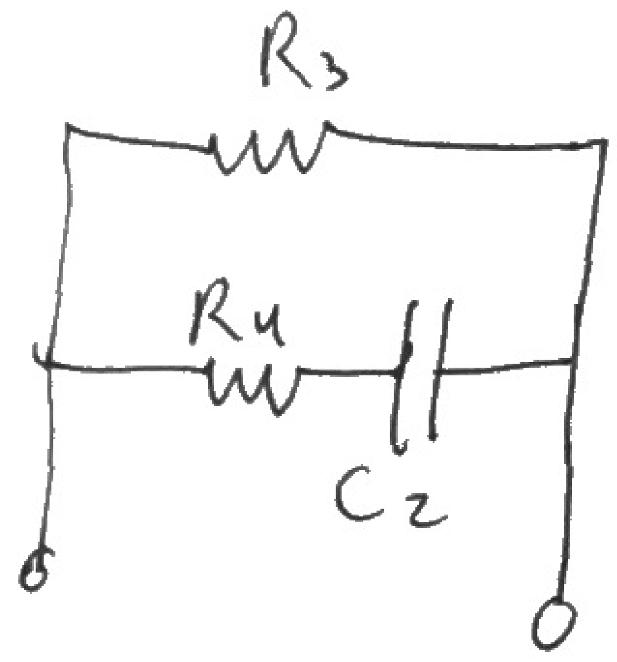
Since ~~both~~ $I(t)$ and $V(t)$ go as $\sim e^{-\alpha t}$, we have after Q cycles (or $t = \frac{\pi}{\alpha}$ seconds) that $I(t)$ and $V(t)$ have decreased by $e^{-\alpha\left(\frac{\pi}{\alpha}\right)} = e^{-\pi}$, as desired.

#2) PM #8.33

Box 1:



Box 2:



$R_1 = 1000\,\Omega$, $R_2 = 4000\,\Omega$, $C_1 = 1\mu F$

$R_3 = 5000\,\Omega$, $R_4 = 1250\,\Omega$, $C_2 = .64\mu F$

For box 1: R_1 in series with $R_2 \| C_1$.

Recall: $Z_C = \frac{1}{i\omega C}$ $\quad Z_R = R.$

impedance for capacitor

for resistor

just like normal resistors

$$\text{so} \quad \frac{1}{Z_{R_2\|C_1}} = \frac{1}{Z_{R_2}} + \frac{1}{Z_{C_1}} \Rightarrow Z_{R_2\|C_1} = \frac{Z_{R_2} Z_{C_1}}{Z_{R_2} + Z_{C_1}} = \frac{R_2}{i\omega C} \frac{1}{R_2 + \frac{1}{i\omega C_1}} = \frac{R_2}{1 + i\omega C_1 R_2}$$

$$\text{So} \quad Z_{eff\,1} = Z_{R_1} + Z_{R_2\|C_1} = R_1 + \frac{R_2}{1 + i\omega C_1 R_2} = R_1 + \frac{R_2(1 - i\omega C_1 R_2)}{1 + \omega^2 C_1^2 R_2^2}$$

$$= \frac{(R_1 + R_2) + C_1^2 R_2^2 R_1 \omega^2 - i C_1 R_2^2 \omega}{1 + \omega^2 C_1^2 R_2^2}.$$

Plugging in numbers

$$\boxed{Z_{eff\,1} = \frac{5000 + 16\times10^{-3}\,\omega^2 - 16i\omega}{1 + 16\times10^{-6}\,\omega^2}}$$

as desired.

Similary $Z_{eff\,2} = \underbrace{Z_{R_3} \parallel \underbrace{\left(Z_{R_4} + Z_{C_2}\right)}_{\text{series}}}_{\text{parallel}}$

For box 2

using $Z_1 \parallel Z_2 = \frac{Z_1 Z_2}{Z_1 + Z_2}$

$$= \frac{Z_{R_3}\left(Z_{R_4} + Z_{C_2}\right)}{Z_{R_3} + Z_{R_4} + Z_{C_2}} = \frac{R_3\left(R_4 + \frac{1}{i\omega C_2}\right)}{R_3 + R_4 + \frac{1}{i\omega C_2}}$$

$\cancel{R_3R_4+R_3}$

multiply by $i\omega C_2$ on top and bottom

$$= \frac{R_3 + i\omega C_2 R_3 R_4}{1 + i\omega C_2 (R_3+R_4)} = \frac{\left(R_3 + i\omega C_2 R_3 R_4\right)\left(1 - i\omega C_2 (R_3+R_4)\right)}{1 + \omega^2 C_2^2 (R_3+R_4)^2}$$

$$\overset{(*)}{=} \frac{R_3 + C_2^2 R_3 R_4 (R_3+R_4)\omega^2 - C_2 R_3^2 \omega i}{1 + C_2^2 (R_3+R_4)^2 \omega^2}$$

plugging in numbers

$$\Rightarrow \boxed{Z_{eff\,2} = \frac{5000 + 16\times10^{-3}\,\omega^2 - 16 i\omega}{1 + 16\times10^{-6}\,\omega^2}}$$ as desired

~~Now, to get R_3, R_4, and C_2 from a given R_1, R_2 and C_1 with the constraint that $Z_{eff\,1} = Z_{eff\,2}$ for all ω, we explore (from page ② and (*) on this page)~~

~~$Z_{eff\,1} = Z_{eff\,2} \Rightarrow \frac{(R_1+R_2) + C_1^2 R_2^2 R_1 \omega^2 - i C_1 R_2^2 \omega}{1 + \omega^2 C_1^2 R_2^2} = \frac{R_3 + C_2^2 R_3 R_4 (R_3+R_4)\omega^2 - C_2 R_3^2 \omega i}{1 + C_2^2 (R_3+R_4)^2 \omega^2}$~~

~~By clearing denominators and bringing all the terms together we get 5 different powers of ω: the 0^{th} power, ω^3, and ω^4~~

Now, to get R_3, R_4, and C_2 from R_1, R_2 and C_1, subject to the constraint that $Z_{eff\,1} = Z_{eff\,2}$, ~~find~~ we need to explore

$$Z_{eff\,1} = \underbrace{R_1 + \frac{1}{\frac{1}{R_2} + i\omega C_1}}_{Z_{R_1} + \underbrace{\frac{1}{\frac{1}{Z_{R_2}} + \frac{1}{Z_{C_1}}}}_{Z_{R_2 \| C_1}}} = \frac{1}{\frac{1}{R_3} + \frac{1}{R_4 + \frac{1}{i\omega C_2}}}$$

$$\Rightarrow \left(\frac{1}{R_3} + \frac{1}{R_4 + \frac{1}{i\omega C_2}}\right)\left(R_1 + \frac{1}{\frac{1}{R_2} + i\omega C_1}\right) = 1$$

$$\Rightarrow \left(\frac{1}{R_3} + \frac{1}{R_4 + \frac{1}{i\omega C_2}}\right)\left(\frac{R_1/R_2 + i\omega C_1 R_1 + 1}{\frac{1}{R_2} + i\omega C_1}\right) = 1$$

$$\Rightarrow \left(\frac{R_4 + \frac{1}{i\omega C_2} + R_3}{R_3\left(R_4 + \frac{1}{i\omega C_2}\right)}\right)\left(\frac{R_1}{R_2} + i\omega C_1 R_1 + 1\right) = \frac{1}{R_2} + i\omega C_1$$

$$\Rightarrow \left(1 + \frac{R_4}{R_3} + \frac{1}{i\omega C_2 R_3}\right)\left(\frac{R_1}{R_2} + i\omega C_1 R_1 + 1\right) = \left(\frac{1}{R_2} + i\omega C_1\right)\left(R_4 + \frac{1}{i\omega C_2}\right)$$

multiply by $i\omega C_2 R_3$

$$\Rightarrow \left(1 + i\omega C_2 R_3 + i\omega C_2 R_4\right)\left(\frac{R_1}{R_2} + i\omega C_1 R_1 + 1\right) = \left(\frac{1}{R_2} + i\omega C_1\right)\left(i\omega C_2 R_3 R_4 + R_3\right)$$

So we get terms linear in ω, quadratic in ω, and constants. For these to be equal for all ω, we need each of these coefficient to be equal.

The constant terms are:

$$\frac{R_1}{R_2} + 1 = \frac{R_3}{R_2} \Rightarrow \boxed{R_1 + R_2 = R_3}$$

see next page

The linear terms are: $\left(1+\frac{R_1}{R_2}\right)\left(C_2R_3 + C_2R_4\right) i\omega + i\omega C_1 R_1 = i\omega\left(C_2\frac{R_3R_4}{R_2} + C_1R_3\right)$

$$\Rightarrow C_2\left(1+\frac{R_1}{R_2}\right)\left(R_3+R_4\right) + C_1R_1 = C_1R_3 + C_2\frac{R_3R_4}{R_2}$$

$$\Rightarrow C_2(R_1+R_2)(R_3+R_4) + C_1R_1R_2 = (C_1R_2 + C_2R_4)R_3$$

$R_3 = R_1+R_2$

$$\Rightarrow C_2R_3(R_3+R_4) + C_1R_1R_2 = (C_1R_2 + C_2R_4)R_3$$

$$\text{(i).}\quad \Rightarrow \boxed{C_2R_3^2 + C_1R_1R_2 = C_1R_2R_3}$$

Equating quadratic terms gives: $(i\omega)^2\left(C_2R_3 + C_2R_4\right)C_1R_1 = (i\omega)^2 C_1C_2R_3R_4$

$$\rightarrow C_1C_2R_1R_3 + C_1C_2R_1R_4 = C_1C_2R_3R_4$$

$$\Rightarrow R_1(R_3+R_4) = R_3R_4$$

$R_3 = R_1+R_2$

$$\Rightarrow R_1(R_1+R_2) = R_4\left((R_1+R_2)-R_1\right)$$

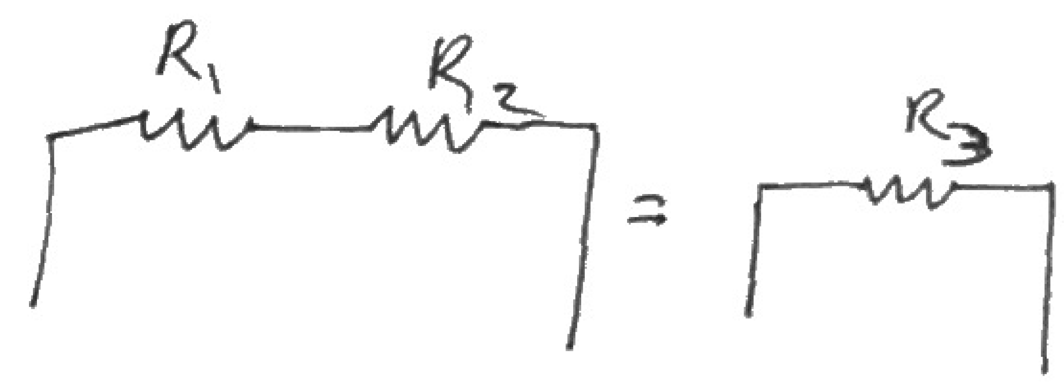
$$\Rightarrow \boxed{R_4 = \frac{R_1(R_1+R_2)}{R_2}}$$

Also, (i) $\Rightarrow C_2 = C_1R_2\frac{(R_3-R_1)}{R_3^2} = C_1R_2\frac{(R_1+R_2-R_1)}{(R_1+R_2)^2} = \frac{R_2^2}{(R_1+R_2)^2}C_1 \Rightarrow \boxed{C_2 = \frac{R_2^2}{(R_1+R_2)^2}C_1}$

$R_3 = R_1+R_2$

$\longrightarrow$

Note that ☺ and ☺ ☺ can be obtained directly by taking the $\omega \to 0$ limit (for ☺), where the capacitors don't (i.e., DC) let any current through so that we have



and $\omega \to \infty$ where the impedance of C_1, C_2 vanishes so that we have