Physics 262

Homework Set 1

Due: Beginning of class 2/2

Two-level systems

1. Show that for a spin-1/2 system any arbitrary state can be written in the form

$$|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|\uparrow\rangle + \sin\left(\frac{\theta}{2}\right)e^{i\phi}|\downarrow\rangle.$$

   Show that the expectation value of the vector spin operator, $\langle \boldsymbol{S} \rangle$, is a vector with polar coordinates (θ, ϕ). This is called the "Bloch sphere representation" and validates our claim in class that the behavior of a spin-1/2 state has the same dynamics as a classical spin vector.

2. In class we compared the behavior of a classical spin precessing around a magnetic field, $\mathbf{B} = B_0\hat{z}$, with a spin-1/2 system. Show explicitly that an arbitrary spin ½ state

$$|\Psi\rangle(t) = a_\uparrow e^{i\omega_0 t/2}|\uparrow\rangle + a_\downarrow e^{-i\omega_0 t/2}|\downarrow\rangle,$$

   can be written as a rotation of the initial state about the *z*-axis

$$|\Psi\rangle(t) = \mathcal{D}(\hat{z}, \phi = -\omega_0 t)|\Psi\rangle(0),$$

   where $\mathcal{D}(\hat{n}, \phi) = \exp\left(-i\vec{\sigma}\cdot\hat{n}\frac{\phi}{2}\right)$ is the operator which rotates the system by angle ϕ about the direction $\hat{n}$. (see, e.g. Sakurai 3.1-3.2). Find $\langle \boldsymbol{S} \rangle$ for this state and show that $\frac{d\langle \boldsymbol{S} \rangle}{dt} = \gamma\langle \boldsymbol{S} \rangle \times \boldsymbol{B_0}$, where $\boldsymbol{\mu} = \gamma\boldsymbol{S}$.

3. For the two-level system in hydrogen we discussed in class with states $|\mathrm{a}\rangle = |1\mathrm{s}\rangle, |\mathrm{b}\rangle = |2p, m_l = -1\rangle$, and an electric field after $t > 0$ of $E(t) = E(\cos(\omega t)\,\hat{x} - \sin(\omega t)\,\hat{y})$ show that the interaction $H'(t) = -\boldsymbol{d}\cdot\boldsymbol{E}$ takes the form:

$$H'(t) = \begin{pmatrix} 0 & \left(\frac{V}{2}\right)e^{i\omega t} \\ \left(\frac{V}{2}\right)e^{-i\omega t} & 0 \end{pmatrix}.$$

   Express *V* in terms of *E* and argue why it is valid to ignore the other m_l projections of the *2p* state.