Additionally, one type of generalization of mechanics is due to a subtler form of equivalence. We have seen that the Lagrangian and Hamilton's principle together form a compact invariant way of obtaining the mechanical equations of motion. This possibility is not reserved for mechanics only; in almost every field of physics variational principles can be used to express the "equations of motion," whether they be Newton's equations, Maxwell's equations, or the Schrödinger equation. Consequently, when a variational principle is used as the basis of the formulation, all such fields will exhibit, at least to some degree, a *structural analogy*. When the results of experiments show the need for alterating the physical content in the theory of one field, this degree of analogy has often indicated how similar alterations may be carried out in other fields. Thus, the experiments performed early in this century showed the need for quantization of both electromagnetic radiation and elementary particles. The methods of quantization, however, were first developed for particle mechanics, starting essentially from the Lagrangian formulation of classical mechanics. By describing the electromagnetic field by a Lagrangian and corresponding Hamilton's variational principle, it is possible to carry over the methods of particle quantization to construct a quantum electrodynamics (cf. Sections 13.5 and 13.6).

2.6 ■ CONSERVATION THEOREMS AND SYMMETRY PROPERTIES

Thus far, we have been concerned primarily with obtaining the equations of motion, but little has been said about how to solve them for a particular problem once they are obtained. In general, this is a question of mathematics. A system of n degrees of freedom will have n differential equations that are second order in time. The solution of each equation will require two integrations resulting, all told, in $2n$ constants of integration. In a specific problem these constants will be determined by the initial conditions, i.e., the initial values of the nq_j's and the $n\dot{q}_j$'s. Sometimes the equations of motion will be integrable in terms of known functions, but not always. In fact, the majority of problems are not completely integrable. However, even when complete solutions cannot be obtained, it is often possible to extract a large amount of information about the physical nature of the system motion. Indeed, such information may be of greater interest to the physicist than the complete solution for the generalized coordinates as a function of time. It is important, therefore, to see how much can be stated about the motion of a given system without requiring a complete integration of the problem.*

In many problems a number of first integrals of the equations of motion can be obtained immediately; by this we mean relations of the type

$$f(q_1, q_2, \ldots, \dot{q}_1, \dot{q}_2, \ldots, t) = \text{constant}, \tag{2.43}$$

*In this and succeeding sections it will be assumed, unless otherwise specified, the system is such that its motion is completely described by a Hamilton's principle of the form (2.2).

which are first-order differential equations. These first integrals are of interest because they tell us something physically about the system. They include, in fact, the conservation laws obtained in Chapter 1.

Let us consider as an example a system of mass points under the influence of forces derived from potentials dependent on position only. Then

$$\begin{aligned}\frac{\partial L}{\partial \dot{x}_i} &\equiv \frac{\partial T}{\partial \dot{x}_i} - \frac{\partial V}{\partial \dot{x}_i} = \frac{\partial T}{\partial \dot{x}_i} = \frac{\partial}{\partial \dot{x}_i} \sum \frac{1}{2} m_i \left(\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2 \right) \\ &= m_i \dot{x}_i = p_{ix},\end{aligned}$$

which is the x component of the linear momentum associated with the ith particle. This result suggests an obvious extension to the concept of momentum. The generalized momentum associated with the coordinate q_j shall be defined as

$$p_j = \frac{\partial L}{\partial \dot{q}_j}. \tag{2.44}$$

The terms *canonical momentum* and *conjugate momentum* are often also used for p_j. Notice that if q_j is not a Cartesian coordinate, p_j does not necessarily have the dimensions of a linear momentum. Further, if there is a velocity-dependent potential, then even with a Cartesian coordinate q_j the associated *generalized* momentum will not be identical with the usual *mechanical* momentum. Thus, in the case of a group of particles in an electromagnetic field, the Lagrangian is (cf. 1.63)

$$L = \sum_i \frac{1}{2} m_i \dot{r}_i^2 - \sum_i q_i \phi(x_i) + \sum_i q_i \mathbf{A}(x_i) \cdot \dot{\mathbf{r}}_i$$

(q_i here denotes charge) and the generalized momentum conjugate to x_i is

$$p_{ix} = \frac{\partial L}{\partial \dot{x}_i} = m_i \dot{x}_i + q_i A_x, \tag{2.45}$$

i.e., mechanical momentum plus an additional term.

If the Lagrangian of a system does not contain a given coordinate q_j (although it may contain the corresponding velocity $\dot{q}_j$), then the coordinate is said to be *cyclic* or *ignorable*. This definition is not universal, but it is the customary one and will be used here. The Lagrange equation of motion,

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0,$$

reduces, for a cyclic coordinate, to

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_j} = 0$$

or

$$\frac{dp_j}{dt} = 0,$$

which mean that

$$p_j = \text{constant}. \tag{2.46}$$

Hence, we can state as a general conservation theorem that *the generalized momentum conjugate to a cyclic coordinate is conserved.*

Note that the derivation of Eq. (2.46) assumes that q_j is a generalized coordinate; one that is linearly independent of all the other coordinates. When equations of constraint exist, all the coordinates are not linearly independent. For example, the angular coordinate θ is not present in the Lagrangian of a hoop rolling without slipping in a horizontal plane that was previously discussed, but the angle appears in the constraint equations $r d\theta = dx$. As a result, the angular momentum, $p_\theta = mr^2\dot{\theta}$, is not a constant of the motion.

Equation (2.46) constitutes a first integral of the form (2.43) for the equations of motion. It can be used formally to eliminate the cyclic coordinate from the problem, which can then be solved entirely in terms of the remaining generalized coordinates. Briefly, the procedure, originated by Routh, consists in modifying the Lagrangian so that it is no longer a function of the generalized velocity corresponding to the cyclic coordinate, but instead involves only its conjugate momentum. The advantage in so doing is that p_j can then be considered one of the constants of integration, and the remaining integrations involve only the noncyclic coordinates. We shall defer a detailed discussion of Routh's method until the Hamiltonian formulation (to which it is closely related) is treated.

Note that the conditions for the conservation of generalized momenta are more general than the two momentum conservation theorems previously derived. For example, they furnish a conservation theorem for a case in which the law of action and reaction is violated, namely, when electromagnetic forces are present. Suppose we have a single particle in a field in which neither ϕ nor $\mathbf{A}$ depends on x. Then x nowhere appears in L and is therefore cyclic. The corresponding canonical momentum p_x must therefore be conserved. From (1.63) this momentum now has the form

$$p_x = m\dot{x} + qA_x = \text{constant}. \tag{2.47}$$

In this case, it is not the mechanical linear momentum $m\dot{x}$ that is conserved but rather its sum with qA_x.* Nevertheless, it should still be true that the conservation theorems of Chapter 1 are contained within the general rule for cyclic coordinates; with proper restrictions (2.46) should reduce to the theorems of Section 1.2.

*It can be shown from classical electrodynamics that under these conditions, i.e., neither $\mathbf{A}$ nor ϕ depending on x, that qA_x is exactly the x-component of the electromagnetic linear momentum of the field associated with the charge q.

We first consider a generalized coordinate q_j, for which a change dq_j represents a translation of the system as a whole in some given direction. An example would be one of the Cartesian coordinates of the center of mass of the system. Then clearly q_j cannot appear in T, for velocities are not affected by a shift in the origin, and therefore the partial derivative of T with respect to q_j must be zero. Further, we will assume conservative systems for which V is not a function of the velocities, so as to eliminate such complications as electromagnetic forces. The Lagrange equation of motion for a coordinate so defined then reduces to

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} \equiv \dot{p}_j = -\frac{\partial V}{\partial q_j} \equiv Q_j. \tag{2.48}$$

We will now show that (2.48) is the equation of motion for the total linear momentum, i.e., that Q_j represents the component of the total force along the direction of translation of q_j, and p_j is the component of the total linear momentum along this direction. In general, the generalized force Q_j is given by Eq. (1.49):

$$Q_j = \sum_i \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j}.$$

Since dq_j corresponds to a translation of the system along some axis, the vectors $\mathbf{r}_i(q_j)$ and $\mathbf{r}_i(q_j + dq_j)$ are related as shown in Fig. 2.7. By the definition of a derivative, we have

$$\frac{\partial \mathbf{r}_i}{\partial q_j} = \lim_{dq_j \to 0} \frac{\mathbf{r}_i\left(q_j + dq_j\right) - \mathbf{r}_i\left(q_j\right)}{dq_j} = \frac{dq_j}{dq_j}\mathbf{n} = \mathbf{n}, \tag{2.49}$$

where $\mathbf{n}$ is the unit vector along the direction of the translation. Hence,

$$Q_j = \sum \mathbf{F}_i \cdot \mathbf{n} = \mathbf{n} \cdot \mathbf{F},$$

which (as was stated) is the component of the total force in the direction of $\mathbf{n}$. To prove the other half of the statement, note that with the kinetic energy in the form

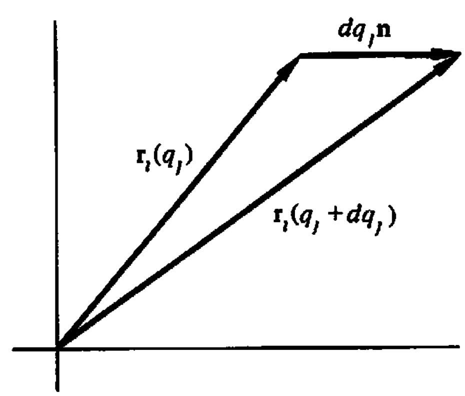

FIGURE 2.7 Change in a position vector under translation of the system.

$$T = \frac{1}{2}\sum m_i \dot{\mathbf{r}}_i^2,$$

the conjugate momentum is

$$p_j = \frac{\partial T}{\partial \dot{q}_j} = \sum_i m_i \dot{\mathbf{r}}_i \cdot \frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_j}$$

$$= \sum_i m_i \mathbf{v}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j},$$

using Eq. (1.51). Then from Eq. (2.49)

$$p_j = \mathbf{n} \cdot \sum_i m_i \mathbf{v}_i,$$

which again, as predicted, is the component of the total system linear momentum along **n**.

Suppose now that the translation coordinate q_j that we have been discussing is cyclic. Then q_j cannot appear in V and therefore

$$-\frac{\partial V}{\partial q_j} \equiv Q_j = 0.$$

But this is simply the familiar conservation theorem for linear momentum—that if a given component of the total applied force vanishes, the corresponding component of the linear momentum is conserved.

In a similar fashion, it can be shown that if a cyclic coordinate q_j is such that dq_j corresponds to a rotation of the system of particles around some axis, then the conservation of its conjugate momentum corresponds to conservation of an angular momentum. By the same argument used above, T cannot contain q_j, for a rotation of the coordinate system cannot affect the magnitude of the velocities. Hence, the partial derivative of T with respect to q_j must again be zero, and since V is independent of $\dot{q}_j$, we once more get Eq. (2.48). But now we wish to show that with q_j a rotation coordinate the generalized force is the component of the total applied torque about the axis of rotation, and p_j is the component of the total angular momentum along the same axis.

The generalized force Q_j is again given by

$$Q_j = \sum_i \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j},$$

only the derivative now has a different meaning. Here the change in q_j must correspond to an infinitesimal rotation of the vector $\mathbf{r}_i$, keeping the magnitude of the vector constant. From Fig. 2.8, the magnitude of the derivative can easily be obtained:

$$|d\mathbf{r}_i| = r_i \sin\theta \, dq_j$$

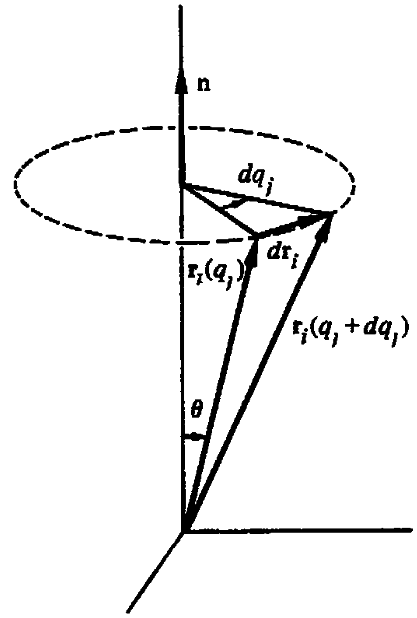

FIGURE 2.8 Change of a position vector under rotation of the system.

and

$$\left|\frac{\partial \mathbf{r}_i}{\partial q_j}\right| = r_i \sin\theta,$$

and its direction is perpendicular to both $\mathbf{r}_i$ and $\mathbf{n}$. Clearly, the derivative can be written in vector form as

$$\frac{\partial \mathbf{r}_i}{\partial q_j} = \mathbf{n} \times \mathbf{r}_i. \tag{2.50}$$

With this result, the generalized force becomes

$$\begin{aligned} Q_j &= \sum_i \mathbf{F}_i \cdot \mathbf{n} \times \mathbf{r}_i \\ &= \sum_i \mathbf{n} \cdot \mathbf{r}_i \times \mathbf{F}_i, \end{aligned}$$

reducing to

$$Q_j = \mathbf{n} \cdot \sum_i \mathbf{N}_i = \mathbf{n} \cdot \mathbf{N},$$

which proves the first part. A similar manipulation of p_j with the aid of Eq. (2.50) provides proof of the second part of the statement:

$$p_j = \frac{\partial T}{\partial \dot{q}_j} = \sum_i m_i \mathbf{v}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} = \sum_i \mathbf{n} \cdot \mathbf{r}_i \times m_i \mathbf{v}_i = \mathbf{n} \cdot \sum \mathbf{L}_i = \mathbf{n} \cdot \mathbf{L}.$$

Summarizing these results, we see that if the rotation coordinate q_j is cyclic, then Q_j, which is the component of the applied torque along $\mathbf{n}$, vanishes, and the component of $\mathbf{L}$ along $\mathbf{n}$ is constant. Here we have recovered the angular momentum conservation theorem out of the general conservation theorem relating to cyclic coordinates.

The significance of cyclic translation or rotation coordinates in relation to the properties of the system deserves some comment at this point. If a generalized coordinate corresponding to a displacement is cyclic, it means that a translation of the system, as if rigid, has no effect on the problem. In other words, if the system is *invariant* under translation along a given direction, the corresponding linear momentum is conserved. Similarly, the fact that a generalized rotation coordinate is cyclic (and therefore the conjugate angular momentum conserved) indicates that the system is invariant under rotation about the given axis. Thus, the momentum conservation theorems are closely connected with the *symmetry properties* of the system. If the system is spherically symmetric, we can say without further ado that all components of angular momentum are conserved. Or, if the system is symmetric only about the z axis, then only L_z will be conserved, and so on for the other axes. These symmetry considerations can often be used with relatively complicated problems to determine by inspection whether certain constants of the motion exist. (cf. Noether's theorem—Sec. 13.7.)

Suppose, for example, the system consists of a set of mass points moving in a potential field generated by fixed sources uniformly distributed on an infinite plane, say, the $z = 0$ plane. (The sources might be a mass distribution if the forces were gravitational, or a charge distribution for electrostatic forces.) Then the symmetry of the problem is such that the Lagrangian is invariant under a translation of the system of particles in the x- or y-directions (but not in the z-direction) and also under a rotation about the z axis. It immediately follows that the x- and y-components of the total linear momentum, P_x and P_y, are constants of the motion along with L_z, the z-component of the total angular momentum. However, if the sources were restricted only to the half plane, $x \geq 0$, then the symmetry for translation along the x axis and for rotation about the z axis would be destroyed. In that case, P_x and L_z could not be conserved, but P_y would remain a constant of the motion. We will encounter the connections between the constants of motion and the symmetry properties of the system several times in the following chapters.

2.7 ■ ENERGY FUNCTION AND THE CONSERVATION OF ENERGY

Another conservation theorem we should expect to obtain in the Lagrangian formulation is the conservation of total energy for systems where the forces are derivable from potentials dependent only upon position. Indeed, it is possible to demonstrate a conservation theorem for which conservation of total energy represents only a special case. Consider a general Lagrangian, which will be a function of the coordinates q_j and the velocities $\dot{q}_j$ and may also depend explicitly on the time. (The explicit time dependence may arise from the time variation of external potentials, or from time-dependent constraints.) Then the total time derivative of L is

$$\frac{dL}{dt} = \sum_j \frac{\partial L}{\partial q_j}\frac{dq_j}{dt} + \sum_j \frac{\partial L}{\partial \dot{q}_j}\frac{d\dot{q}_j}{dt} + \frac{\partial L}{\partial t}. \tag{2.51}$$

From Lagrange's equations,

$$\frac{\partial L}{\partial q_j} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right),$$

and (2.51) can be rewritten as

$$\frac{dL}{dt} = \sum_j \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)\dot{q}_j + \sum_j \frac{\partial L}{\partial \dot{q}_j}\frac{d\dot{q}_j}{dt} + \frac{\partial L}{\partial t}$$

or

$$\frac{dL}{dt} = \sum_j \frac{d}{dt}\left(\dot{q}_j\frac{\partial L}{\partial \dot{q}_j}\right) + \frac{\partial L}{\partial t}.$$

It therefore follows that

$$\frac{d}{dt}\left(\sum_j \dot{q}_j\frac{\partial L}{\partial \dot{q}_j} - L\right) + \frac{\partial L}{\partial t} = 0. \tag{2.52}$$

The quantity in parentheses is oftentimes called the *energy function** and will be denoted by h:

$$h(q_1, \ldots, q_n;\quad \dot{q}_1, \ldots, \dot{q}_n;\quad t) = \sum_j \dot{q}_j\frac{\partial L}{\partial \dot{q}_j} - L, \tag{2.53}$$

and Eq. (2.52) can be looked on as giving the total time derivative of h:

$$\frac{dh}{dt} = -\frac{\partial L}{\partial t}. \tag{2.54}$$

If the Lagrangian is not an explicit function of time, i.e., if t does not appear in L explicitly but only implicitly through the time variation of q and $\dot{q}$, then Eq. (2.54) says that h is conserved. It is one of the first integrals of the motion and is sometimes referred to as Jacobi's integral.†

*The energy function h is identical in value with the Hamiltonian H (See Chapter 8) It is given a different name and symbol here to emphasize that h is considered a function of n independent variables q_j and their time derivatives $\dot{q}_j$ (along with the time), whereas the Hamiltonian will be treated as a function of $2n$ independent variables, q_j, p_j (and possibly the time)

†This designation is most often confined to a first integral in the restricted three-body problem. However, the integral there is merely a special case of the energy function h, and there is some historical precedent to apply the name Jacobi integral to the more general situation

Under certain circumstances, the function h is the total energy of the system. To determine what these circumstances are, we recall that the total kinetic energy of a system can always be written as

$$T = T_0 + T_1 + T_2, \tag{1.73}$$

where T_0 is a function of the generalized coordinates only, $T_1(q, \dot{q})$ is linear in the generalized velocities, and $T_2(q, \dot{q})$ is a quadratic function of the $\dot{q}$'s. For a very wide range of systems and sets of generalized coordinates, the Lagrangian can be similarly decomposed as regards its functional behavior in the $\dot{q}$ variables:

$$L(q, \dot{q}, t) = L_0(q, t) + L_1(q, \dot{q}, t) + L_2(q, \dot{q}, t). \tag{2.55}$$

Here L_2 is a homogeneous function of the second degree (not merely quadratic) in $\dot{q}$, while L_1 is homogeneous of the first degree in $\dot{q}$. There is no reason intrinsic to mechanics that requires the Lagrangian to conform to Eq. (2.55), but in fact it does for most problems of interest. The Lagrangian clearly has this form when the forces are derivable from a potential not involving the velocities. Even with the velocity-dependent potentials, we note that the Lagrangian for a charged particle in an electromagnetic field, Eq. (1.63), satisfies Eq. (2.55). Now, recall that Euler's theorem states that if f is a homogeneous function of degree n in the variables x_i, then

$$\sum_i x_i \frac{\partial f}{\partial x_i} = nf. \tag{2.56}$$

Applied to the function h, Eq. (2.53), for the Lagrangians of the form (2.55), this theorem implies that

$$h = 2L_2 + L_1 - L = L_2 - L_0. \tag{2.57}$$

If the transformation equations defining the generalized coordinates, Eqs. (1.38), do not involve the time explicitly, then by Eqs. (1.73) $T = T_2$. If, further, the potential does not depend on the generalized velocities, then $L_2 = T$ and $L_0 = -V$, so that

$$h = T + V = E, \tag{2.58}$$

and the energy function is indeed the total energy. Under these circumstances, if V does not involve the time explicitly, neither will L. Thus, by Eq. (2.54), h (which is here the total energy), will be conserved.

Note that the conditions for conservation of h are in principle quite distinct from those that identify h as the total energy. We can have a set of generalized coordinates such that in a particular problem h is conserved but is not the total energy. On the other hand, h can be the total energy, in the form $T + V$, but not be conserved. Also note that whereas the Lagrangian is uniquely fixed for each system by the prescription

$$L = T - U$$

independent of the choice of generalized coordinates, the energy function h depends in magnitude and functional form on the specific set of generalized coordinates. For one and the same system, various energy functions h of different physical content can be generated depending on how the generalized coordinates are chosen.

The most common case that occurs in classical mechanics is one in which the kinetic energy terms are all of the form $m\dot{q}_i^2/2$ or $p_i^2/2m$ and the potential energy depends only upon the coordinates. For these conditions, the energy function is both conserved and is also the total energy.

Finally, note that where the system is not conservative, but there are frictional forces derivable from a dissipation function $\mathcal{F}$, it can be easily shown that $\mathcal{F}$ is related to the decay rate of h. When the equations of motion are given by Eq. (1.70), including dissipation, then Eq. (2.52) has the form

$$\frac{dh}{dt} + \frac{\partial L}{\partial t} = \sum_j \frac{\partial \mathcal{F}}{\partial \dot{q}_j}\dot{q}_j.$$

By the definition of $\mathcal{F}$, Eq. (1.67), it is a homogeneous function of the $\dot{q}$'s of degree 2. Hence, applying Euler's theorem again, we have

$$\frac{dh}{dt} = -2\mathcal{F} - \frac{\partial L}{\partial t}. \tag{2.59}$$

If L is not an explicit function of time, *and* the system is such that h is the same as the energy, then Eq. (2.59) says that $2\mathcal{F}$ is the rate of energy dissipation,

$$\frac{dE}{dt} = -2\mathcal{F}, \tag{2.60}$$

a statement proved above (cf. Sec. 1.5) in less general circumstances.

DERIVATIONS

1. Complete the solution of the brachistochrone problem begun in Section 2.2 and show that the desired curve is a cycloid with a cusp at the initial point at which the particle is released. Show also that if the particle is projected with an initial kinetic energy $\frac{1}{2}mv_0^2$ that the brachistochrone is still a cycloid passing through the two points with a cusp at a height z above the initial point given by $v_0^2 = 2gz$.

2. Show that if the potential in the Lagrangian contains velocity-dependent terms, the canonical momentum corresponding to a coordinate of rotation θ of the entire system