105A - Examples

1. Consider a damped (with a damping force of $-\beta v$, where v is the velocity) harmonic oscillator ($\omega^2 = k/m$, where m) which at $t = 0$ is at rest and with a position $x = 0$. The oscillator is subject t an external impulse at time $t = 0$ for a complete cycle of the unforced oscillation.

$$f(x) = \begin{cases} mv_0/t_0 & \text{for} \quad 0 \leq t < t_0 \\ mlt/(\Delta t)^3 & \text{for} \quad t_0 \leq t \end{cases}$$

   (a) Determine $x(t)$ using the parameters given in the problem for the different regimes. Assume that t_0 represent a complete cycle of the oscillation. *Hint, this is the real part of the frequency.*

   Answer: We need to solve the equation:

$$m\ddot{x} + \beta\dot{x} + kx = \begin{cases} mv_0/t_0 & \text{for} \quad 0 \leq t < t_0 \\ mlt/(\Delta t)^3 & \text{for} \quad t_0 \leq t \end{cases} \tag{1}$$

   Note that here I already moved the damping and retiring force to the left hand side of the equation.

   First lets solve it in the regime: $0 \leq t < t_0$

   To solve this for $0 \leq t < t_0$ we need the solution of the homogeneous equation $x_0(t)$ plus the particular solution $x_p(t)$, so $x_1(t) = x_0(t) + x_p(t)$. (subscript "1" denote the first regime). The homogeneous solution is simply:

$$x_0(t) = \tilde{A}e^{i(\omega t + \phi)} \tag{2}$$

   The particular solution here is very simple and its only $x_p(t) = C_1$. Plugging this in equation (1) we find:

$$kC_1 = \frac{mv_0}{t_0} \tag{3}$$

   So we find:

$$C_1 = \frac{mv_0}{t_0 k} \tag{4}$$

   Now for the homogenous solution we have:

$$\omega^2 - \frac{\beta i}{m}\omega - \frac{k}{m} = 0 \tag{5}$$

   of which the solution

$$\omega_{1,2} = \frac{i\beta/m \pm \sqrt{-\beta^2/m^2 + 4\omega_0^2}}{2} \tag{6}$$

   where we defined $\omega_0 = \sqrt{k/m}$. To help with carrying big terms 'll define:

$$\omega_r = \frac{\pm\sqrt{-\beta^2/m^2 + 4\omega_0^2}}{2} \tag{7}$$

So then

$$x_0(t) = \tilde{A}e^{i(i\beta/mt \pm \omega_r t + \phi)} = \tilde{A}e^{-\beta/mt}e^{i(\pm\omega_r t+\phi)} \tag{8}$$

So the solution is:

$$\begin{aligned} x_1(t) &= Re[x_0(t) + x_p(t)] = Re[\tilde{A}e^{-\beta/mt}e^{i(\pm\omega_r t+\phi)} + \frac{mv_0}{t_0 k}] \\ &= e^{-\beta/mt}\left(A_1\cos(\omega_r t) + B_1\sin(\omega_r t)\right) + \frac{mv_0}{t_0 k} \end{aligned} \tag{9}$$

Now we need to find the constants using the initial conditions. We know that at $t = 0$ we had $x = 0$ and $\dot{x} = 0$. So we get:

$$x_1(t=0) = 0 = A_1 + \frac{mv_0}{t_0 k} \tag{10}$$

So

$$A_1 = -\frac{mv_0}{t_0 k} \tag{11}$$

And

$$\dot{x}_1(t=0) = 0 = \frac{-\beta}{m}A_1 + B_1\omega_r \tag{12}$$

Or

$$B_1 = \frac{\beta}{\omega_r m}A_1 = -\frac{\beta}{\omega_r m}\frac{mv_0}{t_0 k} = -\frac{\beta v_0}{\omega_r t_0 k} \tag{13}$$

So we may write:

$$x_1(0 \le t < t_0) = \frac{mv_0}{t_0 k} - e^{-\beta/mt}\left(\frac{mv_0}{t_0 k}\cos(\omega_r t) + \frac{\beta v_0}{\omega_r t_0 k}\sin(\omega_r t)\right) \tag{14}$$

Now lets solve it in the regime: $t_0 \le t$
Again we need to have a homogenous solution in addition to a particular solution $x_2(t) = x_0 + x_p$. For it the homogenous solution is exactly the same as equation (8) and the particular solution is: and $x_p(t) = C_2 + D_2 t$. Plugging the particular solution to equation 1 we get:

$$\beta D_2 + k(C_2 + D + 2t) = \frac{mlt}{(\Delta t)^3} \tag{15}$$

which I can write:

$$\beta D_2 + kC_2 = \frac{mlt}{(\Delta t)^3} - kD_2 t \tag{16}$$

Both sides need to be equal to zero for equation (1) to be true. So we get:

$$D_2 = \frac{ml}{k(\Delta t)^3} \tag{17}$$

and

$$C_2 = \frac{ml\beta}{k^2(\Delta t)^3} \tag{18}$$

[Check units, — all is good!] So we may write:

$$\begin{aligned} x_2(t) &= Re[x_0(t)+x_p(t)] = Re[\tilde{A}e^{-\beta/mt}e^{i(\pm\omega_r t+\phi)} + C_2 + D_2 t] \\ &= e^{-\beta/mt}\left(A_2\cos(\omega_r t)+B_2\sin(\omega_r t)\right) + \frac{ml\beta}{k^2(\Delta t)^3} + \frac{ml}{k(\Delta t)^3}t \end{aligned} \tag{19}$$

Now we need to implement the conditions at the transition at t_0 which represent a complete cycle of ω_r (as given in the question). So we can write: $t_0 = 2\pi/\omega_r$. So then plugging in Eq. (19) we have

$$\begin{aligned} x(t=t_0=2\pi/\omega_r) &= e^{-\beta/mt_0}\left(A_2\cos(\omega_r 2\pi/\omega_r)+B_2\sin(\omega_r 2\pi/\omega_r)\right) \\ &+ \frac{ml\beta}{k^2(\Delta t)^3} + \frac{ml}{k(\Delta t)^3}2\pi/\omega_r \\ &= e^{-\beta/mt_0}A_2 + \frac{ml\beta}{k^2(\Delta t)^3} + \frac{ml}{k(\Delta t)^3}2\pi/\omega_r \end{aligned} \tag{20}$$

This should be equal to the $x_1(t_0)$ at the $0 \le t < t_0$ regime. Plugging in $t_0 = 2\pi/\omega_r$ we get $x_1(t_0) = 0$ which make sense since we completed a full cycle. In other words:

$$e^{-\beta/mt_0}A_2 + \frac{ml\beta}{k^2(\Delta t)^3} + \frac{ml}{k(\Delta t)^3}2\pi/\omega_r = x_1(t_0) = 0 \tag{21}$$

From which we can extract A and find:

$$A_2 = -\frac{ml\beta}{k^2(\Delta t)^3}e^{\beta/mt_0} - \frac{ml}{k(\Delta t)^3}2\pi/\omega_r e^{\beta/mt_0} \tag{22}$$

To find B we need to take the derivative of Eq. (19), which is:

$$\begin{aligned} \dot{x}_2(t) &= \frac{-\beta}{m}e^{-\beta/mt}\left(A_2\cos(\omega_r t)+B\sin(\omega_r t)\right) + e^{-\beta/mt}(-A\omega_r\sin(\omega_r t)+B_2\omega_r\cos(\omega_r t)) \\ &+ \frac{ml}{k(\Delta t)^3} \end{aligned} \tag{23}$$

Plugging $t_0 = 2\pi/\omega_r$ we get:

$$\dot{x}_2(t=t_0) = \frac{-\beta}{m}e^{-\beta/mt_0}A_2 + e^{-\beta/mt_0}B_2\omega_r + \frac{ml}{k(\Delta t)^3} = \dot{x}_1(t_0) \tag{24}$$

which needs to be equal to the velocity at the other regime. Where

$$\dot{x}_1(t_0) = e^{-\beta/mt_0}\left(\frac{-\beta}{m}\frac{mv_0}{t_0 k} - \omega_r\frac{\beta v_0}{\omega_r t_0 k}\right) = 0 \tag{25}$$

(which make sense since we completed a whole cycle. In other words:

$$0 = \frac{-\beta}{m}e^{-\beta/mt_0}A_2 + e^{-\beta/mt_0}B_2\omega_r + \frac{ml}{k(\Delta t)^3} \tag{26}$$

which can be written as:

$$B_2 = \frac{\beta}{m\omega_r}A_2 - \frac{ml}{k(\Delta t)^3\omega_r}e^{\beta/mt_0} \tag{27}$$

Plugging in A_2 we get:

$$B_2 = e^{\beta/mt_0}\left(\frac{\beta}{m\omega_r}\left[-\frac{ml\beta}{k^2(\Delta t)^3} - \frac{ml}{k(\Delta t)^3}2\pi/\omega_r\right] - \frac{ml}{k(\Delta t)^3\omega_r}\right) \tag{28}$$

So we found all the parameters we needed to find.

2. A damped harmonic oscillator obeys the dimensionless equation of motion

$$\ddot{x} + 8\dot{x} + 36x = 0 \tag{29}$$

(a) Write down the solution $x(t)$ under the conditions $x(0) = 1$ and $\dot{x}(0) = 3$.
Answer: This is straight forward: this is an equation of the form $\ddot{x} + b/m\dot{x} + k/mx = 0$, so for this we have $b/m = 4$ and $k/m = \omega_0^2 = 36$, or $\omega_0 = 6$ so since $\omega_0^2 > (b/m)^2$ we have an underdamped solution for which

$$\begin{aligned} x(t) &= Re[\tilde{A}e^{-0.5b/mt}e^{i(\pm\omega_r t+\phi)}] \\ &= e^{-0.5b/mt}\left(A_1\cos(\omega_r t) + B_1\sin(\omega_r t)\right) \end{aligned} \tag{30}$$

Plugging this into the equation of motion and finding the solution of ω (we already did it in the pervious question, so I'll not repeat it here).

$$\omega_r = \frac{\pm\sqrt{-64 + 4 * 36}}{2} = \frac{\pm\sqrt{80}}{2} = 2\sqrt{5} \tag{31}$$

so $x(t)$ can be written as:

$$x(t) = e^{-4t}\left(A_1\cos(2\sqrt{5}t) + B_1\sin(2\sqrt{5}t)\right) \tag{32}$$

and the time derivative is:

$$\dot{x}(t) = -4e^{-8t}\left(A_1\cos(2\sqrt{5}t) + B_1\sin(2\sqrt{5}t)\right) + e^{-4t}\left(-2\sqrt{5}A_1\sin(2\sqrt{5}t) + 2\sqrt{5}B_1\cos(2\sqrt{5}t)\right) \tag{33}$$

Plugging $x(0) = 0$ we get $A = 1$ and $B = 1/(2\sqrt{5})$ so we can write:

$$x(t) = e^{-4t}\left(\cos(2\sqrt{5}t) + \frac{1}{2\sqrt{5}}\sin(2\sqrt{5}t)\right) \tag{34}$$

(b) Let the oscillator now be forced at (dimensionless) angular frequency ω

$$\ddot{x} + 8\dot{x} + 36x = \cos(\omega t) \tag{35}$$

Find the frequency ω_4 at which the amplitude of the oscillations is maximized. You may assume that t is great enough that the homogeneous component has decayed to zero (i.e. ignore the homogeneous component). Find the amplitude of oscillation (i.e. the maximum value of the displacement) at this frequency. Also find the phase angle by which the response lags the forcing.
Answer: Here we need to find out particular solution, and it will look like

$$x(t) = Re[D(\omega_{1,2})e^{\omega t-\phi}] \tag{36}$$

We already solved this in class where we showed that

$$De^{i\phi} = \frac{f_0/m}{(\omega-\omega_1)(\omega-\omega_2)} \tag{37}$$

and in our case $\omega_1 = 4i + 2\sqrt{5}$ and $\omega_2 = 4i - 2\sqrt{5}$, also $f_0/m = 1$ in our case, so we can write:

$$De^{i\phi} = \frac{1}{(\omega - 4i - 2\sqrt{5})(\omega - 4i + 2\sqrt{5})} \tag{38}$$

From this you can show that $\phi = \tan^{-1}(1/2)$ and $D = 1/(16\sqrt{5})$. So

$$x(t) = \frac{1}{16\sqrt{5}} \cos\left(\omega t - \tan^{-1}(1/2)\right) \tag{39}$$