CHAPTER

9 Canonical Transformations

When applied in a straightforward manner, the Hamiltonian formulation usually does not materially decrease the difficulty of solving any given problem in mechanics. We wind up with practically the same differential equations to be solved as are provided by the Lagrangian procedure. The advantages of the Hamiltonian formulation lie not in its use as a calculational tool, but rather in the deeper insight it affords into the formal structure of mechanics. The equal status accorded to coordinates and momenta as independent variables encourages a greater freedom in selecting the physical quantities to be designated as "coordinates" and "momenta." As a result we are led to newer, more abstract ways of presenting the physical content of mechanics. While often of considerable help in practical applications to mechanical problems, these more abstract formulations are primarily of interest to us today because of their essential role in constructing the more modern theories of matter. Thus, one or another of these formulations of classical mechanics serves as a point of departure for both statistical mechanics and quantum theory. It is to such formulations, arising as outgrowths of the Hamiltonian procedure, that this and the next chapter are devoted.

9.1 ■ THE EQUATIONS OF CANONICAL TRANSFORMATION

There is one type of problem for which the solution of the Hamilton's equations is trivial. Consider a situation in which the Hamiltonian is a constant of the motion, and where *all* coordinates q_i are cyclic. Under these conditions, the conjugate momenta p_i are all constant:

$$p_i = \alpha_i,$$

and since the Hamiltonian cannot be an explicit function of either the time or the cyclic coordinates, it may be written as

$$H = H(\alpha_1, \dots, \alpha_n).$$

Consequently, the Hamilton's equations for $\dot{q}_i$ are simply

$$\dot{q}_i = \frac{\partial H}{\partial \alpha_i} = \omega_i, \tag{9.1}$$

where the ω_i's are functions of the α_i's only and therefore are also constant in time. Equations (9.1) have the immediate solutions

$$q_i = \omega_i t + \beta_i, \tag{9.2}$$

where the β_i's are constants of integration, determined by the initial conditions.

It would seem that the solution to this type of problem, easy as it is, can only be of academic interest, for it rarely happens that all the generalized coordinates are cyclic. But a given system can be described by more than one set of generalized coordinates. Thus, to discuss motion of a particle in a plane, we may use as generalized coordinates either the Cartesian coordinates

$$q_1 = x, \qquad q_2 = y,$$

or the plane polar coordinates

$$q_1 = r, \qquad q_2 = \theta.$$

Both choices are equally valid, but one of the other set may be more convenient for the problem under consideration. Note that for central forces neither x nor y is cyclic, while the second set does contain a cyclic coordinate in the angle θ. The number of cyclic coordinates can thus depend upon the choice of generalized coordinates, and for each problem there may be one particular choice for which all coordinates are cyclic. If we can find this set, the remainder of the job is trivial. Since the obvious generalized coordinates suggested by the problem will not normally be cyclic, we must first derive a specific procedure for *transforming* from one set of variables to some other set that may be more suitable.

The transformations considered in the previous chapters have involved going from one set of coordinates q_i to a new set Q_i by transformation equations of the form

$$Q_i = Q_i(q, t). \tag{9.3}$$

For example, the equations of an orthogonal transformation, or of the change from Cartesian to plane polar coordinates, have the general form of Eqs. (9.3). As has been previously noted in Derivation 10 of Chapter 1, such transformations are known as *point transformations*. But in the Hamiltonian formulation the momenta are also independent variables on the same level as the generalized coordinates. The concept of transformation of coordinates must therefore be widened to include the simultaneous transformation of the independent *coordinates* and *momenta*, q_i, p_i, to a new set Q_i, P_i, with (invertible) equations of transformation:

$$\begin{aligned} Q_i &= Q_i(q, p, t), \\ P_i &= P_i(q, p, t). \end{aligned} \tag{9.4}$$

Thus, the new coordinates will be defined not only in terms of the old coordinates but also in terms of the old momenta. Equations (9.3) may be said to define a *point transformation of configuration space*; correspondingly Eqs. (9.4) define a *point transformation of phase space*.

In developing Hamiltonian mechanics, only those transformations can be of interest for which the new Q, P are canonical coordinates. This requirement will be satisfied provided there exists some function $K(Q, P, t)$ such that the equations of motion in the new set are in the Hamiltonian form

$$\dot{Q}_i = \frac{\partial K}{\partial P_i}, \qquad \dot{P}_i = -\frac{\partial K}{\partial Q_i}. \tag{9.5}$$

The function K plays the role of the Hamiltonian in the new coordinate set.* It is important for future considerations that the transformations considered be problem-independent. That is to say, (Q, P) must be canonical coordinates not only for some specific mechanical systems, but for all systems of the same number of degrees of freedom. Equations (9.5) must be the form of the equations of motion in the new coordinates and momenta no matter what the particular initial form of H. We may indeed be incited to develop a particular transformation from (q, p) to (Q, P) to handle, say, a plane harmonic oscillator. But the same transformation must then also lead to Hamilton's equations of motion when applied, for example, to the two-dimensional Kepler problem.

As was seen in Section 8.5, if Q_i and P_i are to be canonical coordinates, they must satisfy a modified Hamilton's principle that can be put in the form

$$\delta \int_{t_1}^{t_2} (P_i \dot{Q}_i - K(Q, P, t))\, dt = 0, \tag{9.6}$$

(where summation over the repeated index i is implied). At the same time the old canonical coordinates of course satisfy a similar principle:

$$\delta \int_{t_1}^{t_2} (p_i \dot{q}_i - H(q, p, t))\, dt = 0. \tag{9.7}$$

The simultaneous validity of Eqs. (9.6) and (9.7) does not mean of course that the integrands in both expressions are equal. Since the general form of the modified Hamilton's principle has zero variation at the end points, both statements will be satisfied if the integrands are connected by a relation of the form

$$\lambda(p_i \dot{q}_i - H) = P_i \dot{Q}_i - K + \frac{dF}{dt}. \tag{9.8}$$

Here F is any function of the phase space coordinates with continuous second derivatives, and λ is a constant independent of the canonical coordinates and the time. The multiplicative constant λ is related to a particularly simple type of transformation of canonical coordinates known as a *scale transformation*.

*It has been remarked in a jocular vein that if H stands for the Hamiltonian, K must stand for the Kamiltonian! Of course, K is every bit as much a Hamiltonian as H, but the designation is occasionally a convenient substitute for the longer term "transformed Hamiltonian."

Suppose we change the size of the units used to measure the coordinates and momenta so that in effect we transform them to a set (Q', P') defined by

$$Q'_i = \mu q_i, \qquad P'_i = \nu p_i. \tag{9.9}$$

Then it is clear Hamilton's equations in the form of Eqs. (9.5) will be satisfied for a transformed Hamiltonian $K'(Q', P') = \mu\nu H(q, p)$. The integrands of the corresponding modified Hamilton's principles are, also obviously, related as

$$\mu\nu(p_i\dot{q}_i - H) = P'_i\dot{Q}'_i - K', \tag{9.10}$$

which is of the form of Eq. (9.8) with $\lambda = \mu\nu$. With the aid of suitable scale transformation, it will always be possible to confine our attention to transformations of canonical coordinates for which $\lambda = 1$. Thus, if we have a transformation of canonical coordinates $(q, p) \to (Q', P')$ for some $\lambda \neq 1$, then we can always find an intermediate set of canonical coordinates (Q, P) related to (Q', P') by a simple scale transformation of the form (9.9) such that $\mu\nu$ also has the same value λ. The transformation between the two sets of canonical coordinates (q, p) and (Q, P) will satisfy Eq. (9.8), but now with $\lambda = 1$:

$$p_i\dot{q}_i - H = P_i\dot{Q}_i - K + \frac{dF}{dt}. \tag{9.11}$$

Since the scale transformation is basically trivial, the significant transformations to be examined are those for which Eq. (9.11) holds.

A transformation of canonical coordinates for which $\lambda \neq 1$ will be called an *extended canonical transformation*. Where $\lambda = 1$, and Eq. (9.11) holds, we will speak simply of a *canonical transformation*. The conclusion of the previous paragraph may then be stated as saying that any extended canonical transformation can be made up of a canonical transformation followed by a scale transformation. Except where otherwise stated, all future considerations of transformations between canonical coordinates will involve only canonical transformations. It is also convenient to give a specific name to canonical transformations for which the equations of transformation Eqs. (9.4) do not contain the time explicitly; they will be called *restricted canonical transformations*.

The last term on the right in Eq. (9.11) contributes to the variation of the action integral only at the end points and will therefore vanish if F is a function of (q, p, t) or (Q, P, t) or any mixture of the phase space coordinates since these have zero variation at the end points. Further, through the equations of transformation, Eqs. (9.4) and their inverses F can be expressed partly in terms of the old set of variables and partly of the new. Indeed, F is useful for specifying the exact form of the canonical transformation only when half of the variables (beside the time) are from the old set and half are from the new. It then acts, as it were, as a bridge between the two sets of canonical variables and is called the *generating function* of the transformation.

To show how the generating function specifies the equations of transformation, suppose F were given as a function of the old and new generalized space coordinates:

$$F = F_1(q, Q, t). \tag{9.12}$$

Equation (9.11) then takes the form

$$\begin{aligned} p_i\dot{q}_i - H &= P_i\dot{Q}_i - K + \frac{dF_1}{dt} \\ &= P_i\dot{Q}_i - K + \frac{\partial F_1}{\partial t} + \frac{\partial F_1}{\partial q_i}\dot{q}_i + \frac{\partial F_1}{\partial Q_i}\dot{Q}_i. \end{aligned} \tag{9.13}$$

Since the old and the new coordinates, q_i and Q_i, are separately independent, Eq. (9.13) can hold identically only if the coefficients of $\dot{q}_i$ and $\dot{Q}_i$ each vanish:

$$p_i = \frac{\partial F_1}{\partial q_i}, \tag{9.14a}$$

$$P_i = -\frac{\partial F_1}{\partial Q_i}, \tag{9.14b}$$

leaving finally

$$K = H + \frac{\partial F_1}{\partial t}. \tag{9.14c}$$

Equations (9.14a) are n relations defining the p_i as functions of q_j, Q_j, and t. Assuming they can be inverted, they could then be solved for the n Q_i's in terms of q_j, p_j, and t, thus yielding the first half of the transformation equations (9.4). Once the relations between the Q_i's and the old canonical variables (q, p) have been established, they can be substituted into Eqs. (9.14b) so that they give the n P_i's as functions of q_j, p_j, and t, that is, the second half of the transformation equations (9.4). To complete the story, Eq. (9.14c) provides the connection between the new Hamiltonian, K, and the old one, H. We must be careful to read Eq. (9.14c) properly. First q and p in H are expressed as functions of Q and P through the inverses of Eqs. (9.4). Then the q_i in $\partial F_1/\partial t$ are expressed in terms of Q, P in a similar manner and the two functions are added to yield $K(Q, P, t)$.

The procedure described shows how, starting from a given generating function F_1, the equations of the canonical transformation can be obtained. We can usually reverse the process: Given the equations of transformation (9.4), an appropriate generating function F_1 may be derived. Equations (9.4) are first inverted to express p_i and P_i as functions of q, Q, and t. Equations (9.14a, b) then constitute a coupled set of partial differential equations than can be integrated, in principle, to find F_1 providing the transformation is indeed canonical. Thus, F_1 is always uncertain to within an additive arbitrary function of t alone (which doesn't affect the equations of transformation), and there may at times be other ambiguities.

It sometimes happens that it is not suitable to describe the canonical transformation by a generating function of the type $F_1(q, Q, t)$. For example, the transformation may be such that p_i cannot be written as functions of q, Q, and t, but rather will be functions of q, P, and t. We would then seek a generating function that is a function of the old coordinates q and the new momenta P. Clearly Eq. (9.13) must then be replaced by an equivalent relation involving $\dot{P}_i$ rather than $\dot{Q}_i$. This can be accomplished by writing F in Eq. (9.11) as

$$F = F_2(q, P, t) - Q_i P_i. \tag{9.15}$$

Substituting this F in Eq. (9.11) leads to

$$p_i \dot{q}_i - H = -Q_i \dot{P}_i - K + \frac{d}{dt} F_2(q, P, t). \tag{9.16}$$

Again, the total derivative of F_2 is expanded and the coefficients of $\dot{q}_i$ and $\dot{P}_i$ collected, leading to the equations

$$p_i = \frac{\partial F_2}{\partial q_i}, \tag{9.17a}$$

$$Q_i = \frac{\partial F_2}{\partial P_i}, \tag{9.17b}$$

with

$$K = H + \frac{\partial F_2}{\partial t}. \tag{9.17c}$$

As before, Eqs. (9.17a) are to be solved for P_i as functions of q_j, p_j, and t to correspond to the second half of the transformation equations (9.4). The remaining half of the transformation equations is then provided by Eqs. (9.17b).

The corresponding procedures for the remaining two basic types of generating functions are obvious, and the general results are displayed in Table 9.1.

It is tempting to look upon the four basic types of generating functions as being related to each other through Legendre transformations. For example, the

TABLE 9.1 Properties of the Four Basic Canonical Transformations

Generating Function	Generating Function Derivatives		Trivial Special Case		
$F = F_1(q, Q, t)$	$p_i = \frac{\partial F_1}{\partial q_i}$	$P_i = -\frac{\partial F_1}{\partial Q_i}$	$F_1 = q_i Q_i$,	$Q_i = p_i$,	$P_i = -q_i$
$F = F_2(q, P, t) - Q_i P_i$	$p_i = \frac{\partial F_2}{\partial q_i}$	$Q_i = \frac{\partial F_2}{\partial P_i}$	$F_2 = q_i P_i$,	$Q_i = q_i$,	$P_i = p_i$
$F = F_3(p, Q, t) + q_i p_i$	$q_i = -\frac{\partial F_3}{\partial p_i}$	$P_i = -\frac{\partial F_3}{\partial Q_i}$	$F_3 = p_i Q_i$,	$Q_i = -q_i$,	$P_i = -p_i$
$F = F_4(p, P, t) + q_i p_i - Q_i P_i$	$q_i = -\frac{\partial F_4}{\partial p_i}$	$Q_i = \frac{\partial F_4}{\partial P_i}$	$F_4 = p_i P_i$,	$Q_i = p_i$,	$P_i = -q_i$

transition from F_1 to F_2 is equivalent to going from the variables q, Q to q, P with the relation

$$-P_i = \frac{\partial F_1}{\partial Q_i}. \tag{9.18}$$

This is just the form required for a Legendre transformation of the basis variables, as described in Section 8.1, and in analogy to Eq. (8.5) we would set

$$F_2(q, P, t) = F_1(q, Q, t) + P_i Q_i, \tag{9.19}$$

which is equivalent to Eq. (9.15) combined with Eq. (9.12). All the other defining equations for the generating functions can similarly be looked on, in combination with Eq. (9.12) as Legendre transformations from F_1, with the last entry in Table 9.1 describing a double Legendre transformation. The only drawback to this picture is that it might erroneously lead us to believe that any given canonical transformation can be expressed in terms of the four basic types of Legendre transformations listed in Table 9.1. This is not always possible. Some transformations are just not suitable for description in terms of these or other elementary forms of generating functions, as has been noted above and as will be illustrated in the next section with specific examples. If we try to apply the Legendre transformation process, we are then led to generating functions that are identically zero or are indeterminate. For this reason, we have preferred to define each type of generating function relative to F, which is some unspecified function of $2n$ independent coordinates and momenta.

Finally, note that a suitable generating function doesn't have to conform to one of the four basic types for *all* the degrees of freedom of the system. It is possible, and for some canonical transformations necessary, to use a generating function that is a mixture of the four types. To take a simple example, it may be desirable for a particular canonical transformation with two degrees of freedom to be defined by a generating function of the form

$$F'(q_1, p_2, P_1, Q_2, t). \tag{9.20}$$

This generating function would be related to F in Eq. (9.11) by the equation

$$F = F'(q_1, p_2, P_1, Q_2, t) - Q_1 P_1 + q_2 p_2, \tag{9.21}$$

and the equations of transformation would be obtained from the relations

$$\begin{aligned} p_1 &= \frac{\partial F'}{\partial q_1}, \qquad & Q_1 &= \frac{\partial F'}{\partial P_1}, \\ q_2 &= -\frac{\partial F'}{\partial p_2}, \qquad & P_2 &= -\frac{\partial F'}{\partial Q_2}, \end{aligned} \tag{9.22}$$

with

$$K = H + \frac{\partial F'}{\partial t}. \tag{9.23}$$

Specific illustrations are given in the next section and in the exercises.

9.2 ■ EXAMPLES OF CANONICAL TRANSFORMATIONS

The nature of canonical transformations and the role played by the generating function can best be illustrated by some simple yet important examples. Let us consider, first, a generating function of the second type with the particular form

$$F_2 = q_i P_i \tag{9.24}$$

found in column 3 of Table 9.1. From Eqs. (9.17), the transformation equations are

$$\begin{aligned} p_i &= \frac{\partial F_2}{\partial q_i} = P_i, \\ Q_i &= \frac{\partial F_2}{\partial P_i} = q_i, \\ K &= H. \end{aligned} \tag{9.25}$$

The new and old coordinates are the same; hence F_2 merely generates the *identity transformation* (cf. Table 9.1). We also note, referring to Table 9.1, that the particular generating function $F_3 = p_i Q_i$ generates an identity transformation with negative signs; that is, $Q_i = -q_i$, $P_i = -p_i$.

A more general type of transformation is described by the generating function

$$F_2 = f_i(q_1, \ldots, q_n;\ t) P_i, \tag{9.26}$$

where the f_i may be any desired set of independent functions. By Eqs. (9.17b), the new coordinates Q_i are given by

$$Q_i = \frac{\partial F_2}{\partial P_i} = f_i(q_1, \ldots, q_n;\ t) \tag{9.27}$$

Thus, with this generating function the new coordinates depend only upon the old coordinates and the time and do not involve the old momenta. Such a transformation is therefore an example of the class of point transformations defined by Eqs. (9.3). In order to define a point transformation, the functions f_i must be independent and invertible, so that the q_j can be expressed in terms of the Q_i. Since the f_i are otherwise completely arbitrary, we may conclude that *all point transformations are canonical.* Equation (9.17c) furnishes the new Hamiltonian in terms of the old and of the time derivatives of the f_i functions.

Note that F_2 as given by Eq. (9.26) is not the only generating function leading to the point transformation specified by the f_i. Clearly the same point transformation is implicit in the more general form

$$F_2 = f_i(q_1, \ldots, q_n;\ t)P_i + g(q_1, \ldots, q_n;\ t), \tag{9.28}$$

where $g(q, t)$ is any (differentiable) function of the old coordinates and the time. Equations (9.27), the transformation equations for the coordinates, remain unaltered for this generating function. But the transformation equations of the momenta differ for the two forms. From Eqs. (9.17a), we have

$$p_j = \frac{\partial F_2}{\partial q_j} = \frac{\partial f_i}{\partial q_j}P_i + \frac{\partial g}{\partial q_j}, \tag{9.29}$$

using the form of F_2 given by Eq. (9.28). These equations may be inverted to give P as a function of (q, p), most easily by writing them in matrix notation:

$$\mathbf{p} = \frac{\partial \mathbf{f}}{\partial \mathbf{q}}\mathbf{P} + \frac{\partial g}{\partial \mathbf{q}}. \tag{9.29$'$}$$

Here $\mathbf{p}$, $\mathbf{P}$, and $\partial g/\partial \mathbf{q}$ are n-elements of single-column matrices, and $\partial \mathbf{f}/\partial \mathbf{q}$ is a square matrix whose ijth element is $\partial f_i/\partial q_j$. In two dimensions, Eq. (9.29$'$) can be written as

$$\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} \dfrac{\partial f_1}{\partial q_1} & \dfrac{\partial f_1}{\partial q_2} \\ \dfrac{\partial f_2}{\partial q_1} & \dfrac{\partial f_2}{\partial q_2} \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} + \begin{bmatrix} \dfrac{\partial g}{\partial q_1} \\ \dfrac{\partial g}{\partial q_2} \end{bmatrix}.$$

It follows that $\mathbf{P}$ is a linear function of $\mathbf{p}$ given by

$$\mathbf{P} = \left[\frac{\partial \mathbf{f}}{\partial \mathbf{q}}\right]^{-1} \left[\mathbf{p} - \frac{\partial g}{\partial \mathbf{q}}\right]. \tag{9.30}$$

In two dimensions, (9.30) becomes

$$\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} \dfrac{\partial f_1}{\partial q_1} & \dfrac{\partial f_1}{\partial q_2} \\ \dfrac{\partial f_2}{\partial q_1} & \dfrac{\partial f_2}{\partial q_2} \end{bmatrix}^{-1} \left[\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} - \begin{bmatrix} \dfrac{\partial g}{\partial q_1} \\ \dfrac{\partial g}{\partial q_2} \end{bmatrix} \right]. \tag{9.31}$$

Thus, the transformation equations (9.27) for Q are independent of g and depend only upon the $f_i(q, t)$, but the transformation equations (9.29) for P do depend upon the form of g and are in general functions of both the old coordinates and momenta. The generating function given by Eq. (9.26) is only a special case of Eq. (9.28) for which $g = 0$, with correspondingly specialized transformation equations for P.

An instructive transformation is provided by the generating function of the first kind, $F_1(q, Q, t)$, of the form

$$F_1 = q_k Q_k.$$

The corresponding transformation equations, from (9.14a, b) are

$$p_i = \frac{\partial F_i}{\partial q_i} = Q_i, \tag{9.32a}$$

$$P_i = -\frac{\partial F_1}{\partial Q_i} = -q_i. \tag{9.32b}$$

In effect, the transformation interchanges the momenta and the coordinates; the new coordinates are the old momenta and the new momenta are essentially the old coordinates. Table 9.1 shows that the particular generating function of type $F_4 = p_i P_i$ produces the same transformation. These simple examples should emphasize the independent status of generalized coordinates and momenta. They are both needed to describe the motion of the system in the Hamiltonian formulation. The distinction between them is basically one of nomenclature. We can shift the names around with at most no more than a change in sign. There is no longer present in the theory any lingering remnant of the concept of q_i as a spatial coordinate and p_i as a mass times a velocity. Incidentally, we may see directly from Hamilton's equations,

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}, \qquad \dot{q}_i = \frac{\partial H}{\partial p_i},$$

that this exchange transformation is canonical. If q_i is substituted for p_i, the equations remain in the canonical form only if $-p_i$ is substituted for q_i.

A transformation that leaves some of the (q, p) pairs unchanged, and interchanges the rest (with a sign change), is obviously a canonical transformation of a "mixed" form. Thus, in a system of two degrees of freedom, the transformation

$$Q_1 = q_1, \qquad P_1 = p_1,$$
$$Q_2 = p_2, \qquad P_2 = -q_2,$$

is generated by the function

$$F = q_1 P_1 + q_2 Q_2, \tag{9.33}$$

which is a mixture of the F_1 and F_2 types.

9.3 ■ THE HARMONIC OSCILLATOR

As a final example, let us consider a canonical transformation that can be used to solve the problem of the simple harmonic oscillator in one dimension. If the force constant is k, the Hamiltonian for this problem in terms of the usual coordinates is

$$H = \frac{p^2}{2m} + \frac{kq^2}{2}. \tag{9.34a}$$

Designating the ratio k/m by ω^2, H can also be written as

$$H = \frac{1}{2m}(p^2 + m^2\omega^2 q^2). \tag{9.34b}$$

This form of the Hamiltonian, as the sum of two squares, suggests a transformation in which H is cyclic in the new coordinate. If we could find a canonical transformation of the form

$$p = f(P)\cos Q, \tag{9.35a}$$

$$q = \frac{f(P)}{m\omega}\sin Q, \tag{9.35b}$$

then the Hamiltonian as a function of Q and P would be simply

$$K = H = \frac{f^2(P)}{2m}(\cos^2 Q + \sin^2 Q) = \frac{f^2(P)}{2m}, \tag{9.36}$$

so that Q is cyclic. The problem is to find the form of the yet unspecified function $f(P)$ that makes the transformation canonical. If we use a generating function of the first kind given by

$$F_1 = \frac{m\omega q^2}{2}\cot Q, \tag{9.37}$$

Eqs. (9.14) then provide the equations of transformation,

$$p = \frac{\partial F_1}{\partial q} = m\omega q \cot Q, \tag{9.38a}$$

$$P = -\frac{\partial F_1}{\partial Q} = \frac{m\omega q^2}{2\sin^2 Q}. \tag{9.38b}$$

Solving for q and p, we have*

$$q = \sqrt{\frac{2P}{m\omega}}\sin Q, \tag{9.39a}$$

*It can be argued that F_1 does not unambiguously specify the canonical transformation, because in solving Eq (9 38b) for q we could have taken the negative square root instead of the positive root as (implied) in Eqs. (9-39). However, the two canonical transformations thus derived from F_1 differ only trivially; a shift in α by π corresponds to going from one transformation to the other. Nonetheless, it should be kept in mind that the transformations derived from a generating function may at times be double-valued or even have local singularities.

$$p = \sqrt{2pm\omega} \cos Q, \tag{9.39b}$$

and comparison with Eq. (9.35a) evaluates $f(P)$:

$$f(P) = \sqrt{2m\omega P}. \tag{9.40}$$

It follows then that the Hamiltonian in the transformed variables is

$$H = \omega P. \tag{9.41}$$

Since the Hamiltonian is cyclic in Q, the conjugate momentum P is a constant. It is seen from Eq. (9.41) that P is in fact equal to the constant energy divided by ω:

$$P = \frac{E}{\omega}.$$

The equation of motion for Q reduces to the simple form

$$\dot{Q} = \frac{\partial H}{\partial P} = \omega,$$

with the immediate solution

$$Q = \omega t + \alpha, \tag{9.42}$$

where α is a constant of integration fixed by the initial conditions. From Eqs. (9.39), the solutions for q and p are

$$q = \sqrt{\frac{2E}{m\omega^2}} \sin(\omega t + \alpha), \tag{9.43a}$$

$$p = \sqrt{2mE} \cos(\omega t + \alpha). \tag{9.43b}$$

It is instructive to plot the time dependence of the old and new variables as is shown in Fig. 9.1. We see that q and p oscillate (Fig. 9.1a, b) whereas Q and P are linear plots (Fig. 9.1d, e). The figure also shows the phase space plots for p versus q (Fig. 9.1c) and for P versus Q (Fig. 9.1f). Fig. 9.1c is an ellipse with the following semimajor axes (for the q and p directions, respectively):

$$a = \sqrt{\frac{2E}{m\omega^2}} \qquad \text{and} \qquad b = \sqrt{2mE},$$

where m is the mass of the oscillator, ω its frequency, and E the oscillator's energy. The area, A, of this ellipse in phase space is

$$A = \pi ab = \frac{2\pi E}{\omega}.$$