$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B} \tag{D.23}$$

$$\tan^2 \frac{A}{2} = \frac{1 - \cos A}{1 + \cos A} \tag{D.24}$$

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i} \tag{D.25}$$

$$\cos x = \frac{e^{ix} + e^{-ix}}{2} \tag{D.26}$$

$$e^{ix} = \cos x + i \sin x \tag{D.27}$$

D.3 Trigonometric Series

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \tag{D.28}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \tag{D.29}$$

$$\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \cdots, \qquad |x| < \pi/2 \tag{D.30}$$

$$\sin^{-1} x = x + \frac{x^3}{6} + \frac{3}{40}x^5 + \cdots, \qquad \begin{cases} |x| < 1 \\ |\sin^{-1} x| < \pi/2 \end{cases} \tag{D.31}$$

$$\cos^{-1} x = \frac{\pi}{2} - x - \frac{x^3}{6} - \frac{3}{40}x^5 - \cdots, \qquad \begin{cases} |x| < 1 \\ 0 < \cos^{-1} x < \pi \end{cases} \tag{D.32}$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots, \qquad |x| < 1 \tag{D.33}$$

D.4 Exponential and Logarithmic Series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \tag{D.34}$$

$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots, \qquad |x| < 1, \quad x = 1 \tag{D.35}$$

$$\ln\left[\sqrt{(x^2/a^2) + 1} + (x/a)\right] = \sinh^{-1} x/a \tag{D.36}$$

$$= -\ln\left[\sqrt{(x^2/a^2) + 1} - (x/a)\right] \tag{D.37}$$

D.5 Complex Quantities

Cartesian form: $z = x + iy$, complex conjugate $z^* = x - iy$, $i = \sqrt{-1}$ **(D.38)**

Polar form:
$$z = |z|\, e^{i\theta} \tag{D.39}$$

$$z^* = |z|\, e^{-i\theta} \tag{D.40}$$

$$zz^* = |z|^2 = x^2 + y^2 \tag{D.41}$$

Real part of z:
$$\text{Re}\, z = \frac{1}{2}(z + z^*) = x \tag{D.42}$$

Imaginary part of z:
$$\text{Im}\, z = -\frac{1}{2}(z - z^*) = y \tag{D.43}$$

Euler's formula:
$$e^{i\theta} = \cos\theta + i\sin\theta \tag{D.44}$$

D.6 Hyperbolic Functions

$$\sinh x = \frac{e^x - e^{-x}}{2} \tag{D.45}$$

$$\cosh x = \frac{e^x + e^{-x}}{2} \tag{D.46}$$

$$\tanh x = \frac{e^{2x} - 1}{e^{2x} + 1} \tag{D.47}$$

$$\sin ix = i \sinh x \tag{D.48}$$

$$\cos ix = \cosh x \tag{D.49}$$

$$\sinh ix = i \sin x \tag{D.50}$$

$$\cosh ix = \cos x \tag{D.51}$$

$$\sinh^{-1} x = \tanh^{-1}\left(\frac{x}{\sqrt{x^2 + 1}}\right) \tag{D.52}$$

$$= \ln(x + \sqrt{x^2 + 1}) \tag{D.53}$$

$$= \cosh^{-1}(\sqrt{x^2 + 1}), \quad \begin{cases} > 0, & x > 0 \\ < 0, & x < 0 \end{cases} \tag{D.54}$$

$$\cosh^{-1} x = \pm \tanh^{-1}\left(\frac{\sqrt{x^2 - 1}}{x}\right), \quad x > 1 \tag{D.55}$$

$$= \pm \ln(x + \sqrt{x^2 - 1}), \quad x > 1 \tag{D.56}$$

$$\cosh^{-1} x = \pm \sinh^{-1}(\sqrt{x^2 - 1}), \quad x > 1 \tag{D.57}$$

$$\frac{d}{dy} \sinh y = \cosh y \tag{D.58}$$

$$\frac{d}{dy} \cosh y = \sinh y \tag{D.59}$$

$$\sinh(x_1 + x_2) = \sinh x_1 \cosh x_2 + \cosh x_1 \sinh x_2 \tag{D.60}$$

$$\cosh(x_1 + x_2) = \cosh x_1 \cosh x_2 + \sinh x_1 \sinh x_2 \tag{D.61}$$

$$\cosh^2 x - \sinh^2 x = 1 \tag{D.62}$$

PROBLEMS

D-1. Is it possible to ascribe a meaning to the inequality $z_1 < z_2$? Explain. Does the inequality $|z_1| < |z_2|$ have a different meaning?

D-2. Solve the following equations:
(a) $z^2 + 2z + 2 = 0$ **(b)** $2z^2 + z + 2 = 0$

D-3. Express the following in polar form:
(a) $z_1 = i$ **(b)** $z_2 = -1$
(c) $z_3 = 1 + i\sqrt{3}$ **(d)** $z_4 = 1 + 2i$
(e) Find the product $z_1 z_2$ **(f)** Find the product $z_1 z_3$
(g) Find the product $z_3 z_4$

D-4. Express $(z^2 - 1)^{-1/2}$ in polar form.

D-5. If the function $w = \sin^{-1} z$ is defined as the inverse of $z = \sin w$, then use the Euler relation for $\sin w$ to find an equation for $\exp(iw)$. Solve this equation and obtain the result

$$w = \sin^{-1} z = -i \ln\left(iz + \sqrt{1 - z^2}\right)$$

D-6. Show that

$$y = Ae^{ix} + Be^{-ix}$$

can be written as

$$y = C \cos(x - \delta)$$

where A and B are *complex* but where C and δ are *real.*

D-7. Show that
(a) $\sinh(x_1 + x_2) = \sinh x_1 \cosh x_2 + \cosh x_1 \sinh x_2$
(b) $\cosh(x_1 + x_2) = \cosh x_1 \cosh x_2 + \sinh x_1 \sinh x_2$

APPENDIX E

Useful Integrals*

E.1 Algebraic Functions

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right), \qquad \left|\tan^{-1}\left(\frac{x}{a}\right)\right| < \frac{\pi}{2} \tag{E.1}$$

$$\int \frac{x\,dx}{a^2 + x^2} = \frac{1}{2}\ln(a^2 + x^2) \tag{E.2}$$

$$\int \frac{dx}{x(a^2 + x^2)} = \frac{1}{2a^2}\ln\left(\frac{x^2}{a^2 + x^2}\right) \tag{E.3}$$

$$\int \frac{dx}{a^2x^2 - b^2} = \frac{1}{2ab}\ln\left(\frac{ax - b}{ax + b}\right) \tag{E.4a}$$

$$= -\frac{1}{ab}\coth^{-1}\left(\frac{ax}{b}\right), \qquad a^2x^2 > b^2 \tag{E.4b}$$

$$= -\frac{1}{ab}\tanh^{-1}\left(\frac{ax}{b}\right), \qquad a^2x^2 < b^2 \tag{E.4c}$$

$$\int \frac{dx}{\sqrt{a + bx}} = \frac{2}{b}\sqrt{a + bx} \tag{E.5}$$

$$\int \frac{dx}{\sqrt{x^2 + a^2}} = \ln(x + \sqrt{x^2 + a^2}) \tag{E.6}$$

*This list is confined to those (nontrivial) integrals that arise in the text and in the problems. Extremely useful compilations are, for example, Pierce and Foster (Pi57) and Dwight (Dw61).

$$\int \frac{x^2\,dx}{\sqrt{a^2 - x^2}} = -\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} \tag{E.7}$$

$$\int \frac{dx}{\sqrt{ax^2 + bx + c}} = \frac{1}{\sqrt{a}}\ln(2\sqrt{a}\sqrt{ax^2 + bx + c} + 2ax + b), \quad a > 0 \tag{E.8a}$$

$$= \frac{1}{\sqrt{a}}\sinh^{-1}\left(\frac{2ax + b}{\sqrt{4ac - b^2}}\right), \quad \begin{cases} a > 0 \\ 4ac > b^2 \end{cases} \tag{E.8b}$$

$$= -\frac{1}{\sqrt{-a}}\sin^{-1}\left(\frac{2ax + b}{\sqrt{b^2 - 4ac}}\right), \quad \begin{cases} a < 0 \\ b^2 > 4ac \\ |2ax + b| < \sqrt{b^2 - 4ac} \end{cases} \tag{E.8c}$$

$$\int \frac{x\,dx}{\sqrt{ax^2 + bx + c}} = \frac{1}{a}\sqrt{ax^2 + bx + c} - \frac{b}{2a}\int \frac{dx}{\sqrt{ax^2 + bx + c}} \tag{E.9}$$

$$\int \frac{dx}{x\sqrt{ax^2 + bx + c}} = -\frac{1}{\sqrt{c}}\sinh^{-1}\left(\frac{bx + 2c}{|x|\sqrt{4ac - b^2}}\right), \quad \begin{cases} c > 0 \\ 4ac > b^2 \end{cases} \tag{E.10a}$$

$$= \frac{1}{\sqrt{-c}}\sin^{-1}\left(\frac{bx + 2c}{|x|\sqrt{b^2 - 4ac}}\right), \quad \begin{cases} c < 0 \\ b^2 > 4ac \end{cases} \tag{E.10b}$$

$$= -\frac{1}{\sqrt{c}}\ln\left(\frac{2\sqrt{c}}{x}\sqrt{ax^2 + bx + c} + \frac{2c}{x} + b\right), \quad c > 0 \tag{E.10c}$$

$$\int \sqrt{ax^2 + bx + c}\,dx = \frac{2ax + b}{4a}\sqrt{ax^2 + bx + c} + \frac{4ac - b^2}{8a}\int \frac{dx}{\sqrt{ax^2 + bx + c}} \tag{E.11}$$

E.2 Trigonometric Functions

$$\int \sin^2 x\,dx = \frac{x}{2} - \frac{1}{4}\sin 2x \tag{E.12}$$

$$\int \cos^2 x\,dx = \frac{x}{2} + \frac{1}{4}\sin 2x \tag{E.13}$$

$$\int \frac{dx}{a + b\sin x} = \frac{2}{\sqrt{a^2 - b^2}}\tan^{-1}\left[\frac{a\tan(x/2) + b}{\sqrt{a^2 - b^2}}\right], \quad a^2 > b^2 \tag{E.14}$$

$$\int \frac{dx}{a + b\cos x} = \frac{2}{\sqrt{a^2 - b^2}} \tan^{-1}\left[\frac{(a-b)\tan(x/2)}{\sqrt{a^2 - b^2}}\right], \quad a^2 > b^2 \tag{E.15}$$

$$\int \frac{dx}{(a + b\cos x)^2} = \frac{b\sin x}{(b^2 - a^2)(a + b\cos x)} - \frac{a}{b^2 - a^2}\int \frac{dx}{a + b\cos x} \tag{E.16}$$

$$\int \tan x\, dx = -\ln|\cos x| \tag{E.17a}$$

$$\int \tanh x\, dx = \ln\cosh x \tag{E.17b}$$

$$\int e^{ax}\sin x\, dx = \frac{e^{ax}}{a^2 + 1}(a\sin x - \cos x) \tag{E.18a}$$

$$\int e^{ax}\sin^2 x\, dx = \frac{e^{ax}}{a^2 + 4}\left(a\sin^2 x - 2\sin x\cos x + \frac{2}{a}\right) \tag{E.18b}$$

$$\int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\pi/a} \tag{E.18c}$$

E.3 Gamma Functions

$$\Gamma(n) = \int_0^{\infty} x^{n-1}e^{-x}\, dx \tag{E.19a}$$

$$= \int_0^1 [\ln(1/x)]^{n-1} dx \tag{E.19b}$$

$$\Gamma(n) = (n-1)!, \quad \text{for } n = \text{positive integer} \tag{E.19c}$$

$$n\Gamma(n) = \Gamma(n+1) \tag{E.20}$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \tag{E.21}$$

$$\Gamma(1) = 1 \tag{E.22}$$

$$\Gamma\left(1\frac{1}{4}\right) = 0.906 \tag{E.23}$$

$$\Gamma\left(1\frac{3}{4}\right) = 0.919 \tag{E.24}$$

$$\Gamma(2) = 1 \tag{E.25}$$

$$\int_0^1 \frac{dx}{\sqrt{1 - x^n}} = \frac{\sqrt{\pi}}{n} \frac{\Gamma\left(\frac{1}{n}\right)}{\Gamma\left(\frac{1}{n} + \frac{1}{2}\right)} \tag{E.26}$$

$$\int_0^1 x^m (1 - x^2)^n \, dx = \frac{\Gamma(n + 1)\Gamma\left(\frac{m + 1}{2}\right)}{2\Gamma\left(n + \frac{m + 3}{2}\right)} \tag{E.27a}$$

$$\int_0^{\pi/2} \cos^n x \, dx = \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{n + 1}{2}\right)}{\Gamma\left(\frac{n}{2} + 1\right)}, \quad n > -1 \tag{E.27b}$$

APPENDIX **F**

Differential Relations in Different Coordinate Systems

F.1 Rectangular Coordinates

$$\textbf{grad}\, U = \nabla U = \sum_i \mathbf{e}_i \frac{\partial U}{\partial x_i} \tag{F.1}$$

$$\text{div}\, \mathbf{A} = \nabla \cdot \mathbf{A} = \sum_i \frac{\partial A_i}{\partial x_i} \tag{F.2}$$

$$\textbf{curl}\, \mathbf{A} = \nabla \times \mathbf{A} = \sum_{i,j,k} \varepsilon_{ijk} \frac{\partial A_k}{\partial x_j} \mathbf{e}_i \tag{F.3}$$

$$\nabla^2 U = \nabla \cdot \nabla U = \sum_i \frac{\partial^2 U}{\partial x_i^2} \tag{F.4}$$

F.2 Cylindrical Coordinates

Refer to Figures F-1 and F-2.

$$x_1 = r\cos\phi, \quad x_2 = r\sin\phi, \quad x_3 = z \tag{F.5}$$

$$r = \sqrt{x_1^2 + x_2^2}, \quad \phi = \tan^{-1}\frac{x_2}{x_1}, \quad z = x_3 \tag{F.6}$$

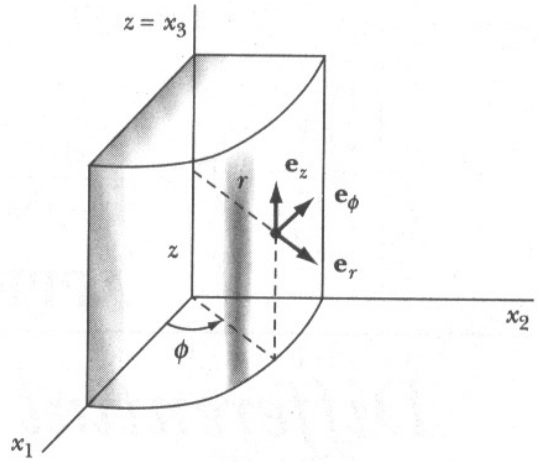

FIGURE F-1

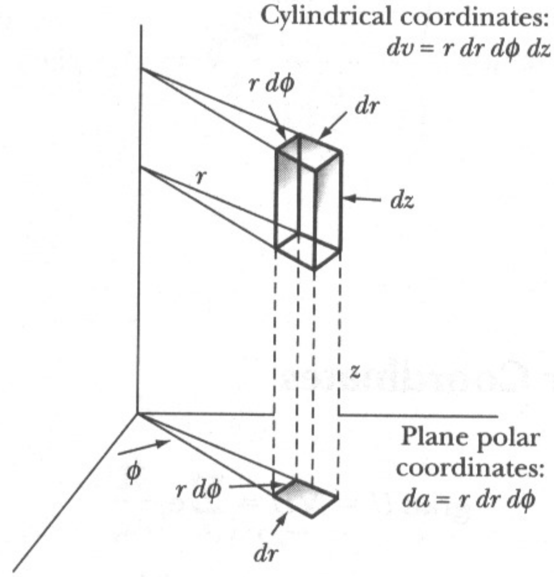

FIGURE F-2

$$ds^2 = dr^2 + r^2\,d\phi^2 + dz^2 \tag{F.7}$$

$$dv = r\,dr\,d\phi\,dz \tag{F.8}$$

$$\textbf{grad}\ \psi = \nabla\psi = \mathbf{e}_r\frac{\partial\psi}{\partial r} + \mathbf{e}_\phi\frac{1}{r}\frac{\partial\psi}{\partial\phi} + \mathbf{e}_z\frac{\partial\psi}{\partial z} \tag{F.9}$$

$$\text{div}\ \mathbf{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z} \tag{F.10}$$

$$\textbf{curl A} = \mathbf{e}_r\left(\frac{1}{r}\frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z}\right) + \mathbf{e}_\phi\left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right) + \mathbf{e}_z\left(\frac{1}{r}\frac{\partial}{\partial r}(rA_\phi) - \frac{1}{r}\frac{\partial A_r}{\partial\phi}\right) \tag{F.11}$$

$$\nabla^2\psi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2} \tag{F.12}$$

F.3 Spherical Coordinates

Refer to Figures F-3 and F-4

$$x_1 = r \sin\theta \cos\phi, \quad x_2 = r \sin\theta \sin\phi, \quad x_3 = r \cos\theta \tag{F.13}$$

$$r = \sqrt{x_1^2 + x_2^2 + x_3^2}, \quad \theta = \cos^{-1}\frac{x_3}{r}, \quad \phi = \tan^{-1}\frac{x_2}{x_1} \tag{F.14}$$

$$ds^2 = dr^2 + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\phi^2 \tag{F.15}$$

$$dv = r^2 \sin\theta\, dr\, d\theta\, d\phi \tag{F.16}$$

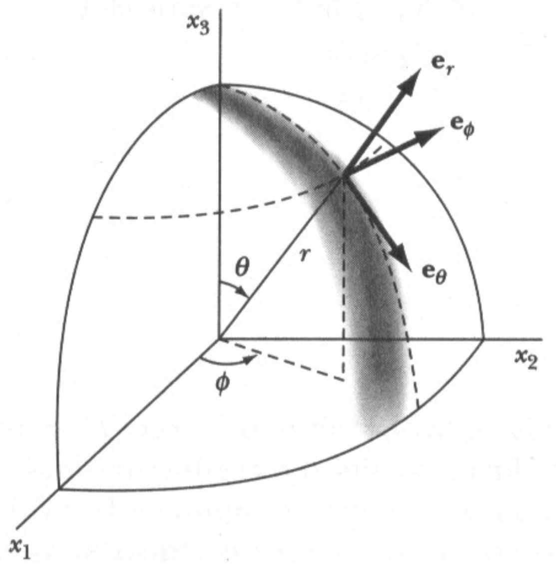

FIGURE F-3

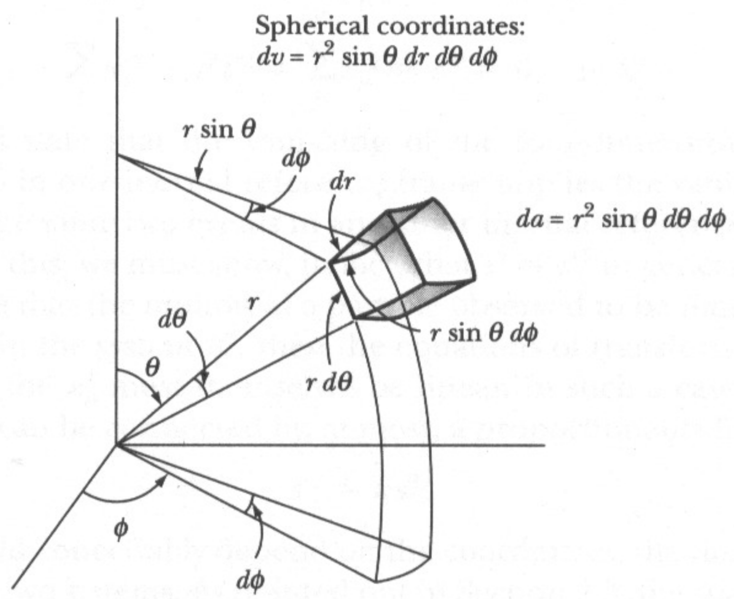

FIGURE F-4

$$\mathbf{grad}\,\psi = \nabla\psi = \mathbf{e}_r\frac{\partial\psi}{\partial r} + \mathbf{e}_\theta\frac{1}{r}\frac{\partial\psi}{\partial\theta} + \mathbf{e}_\phi\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi} \tag{F.17}$$

$$\text{div}\mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial\mathbf{r}}(r^2A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\theta\sin\theta) + \frac{1}{r\sin\theta} + \frac{\partial A_\phi}{\partial\phi} \tag{F.18}$$

$$\begin{aligned}\mathbf{curl}\,\mathbf{A} = {} & \mathbf{e}_r\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(A_\phi\sin\theta) - \frac{\partial A_\theta}{\partial\phi}\right] \\ & + \mathbf{e}_\theta\frac{1}{r\sin\theta}\left[\frac{\partial A_r}{\partial\phi} - \sin\theta\frac{\partial}{\partial r}(rA_\phi)\right] + \mathbf{e}_\phi\frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right]\end{aligned} \tag{F.19}$$

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2} \tag{F.20}$$