Equations (3.18) and (3.20) are the two remaining integrations, and formally the problem has been reduced to quadratures, with four constants of integration $E$, $l$, $r_0$, $\theta_0$. These constants are not the only ones that can be considered. We might equally as well have taken $r_0$, $\theta_0$, $\dot{r}_0$, $\dot{\theta}_0$, but of course $E$ and $l$ can always be determined in terms of this set. For many applications, however, the set containing the energy and angular momentum is the natural one. In quantum mechanics, such constants as the initial values of $r$ and $\theta$, or of $\dot{r}$ and $\dot{\theta}$, become meaningless, but we can still talk in terms of the system energy or of the system angular momentum. Indeed, two salient differences between classical and quantum mechanics appear in the properties of $E$ and $l$ in the two theories. In order to discuss the transition to quantum theories, it is therefore important that the classical description of the system be in terms of its energy and angular momentum.

## 3.3 ■ THE EQUIVALENT ONE-DIMENSIONAL PROBLEM, AND CLASSIFICATION OF ORBITS

Although we have solved the one-dimensional problem formally, practically speaking the integrals (3.18) and (3.20) are usually quite unmanageable, and in any specific case it is often more convenient to perform the integration in some other fashion. But before obtaining the solution for any specific force laws, let us see what can be learned about the motion in the general case, using only the equations of motion and the conservation theorems, without requiring explicit solutions.

For example, with a system of known energy and angular momentum, the magnitude and direction of the velocity of the particle can be immediately determined in terms of the distance $r$. The magnitude $v$ follows at once from the conservation of energy in the form

$$E = \tfrac{1}{2}mv^2 + V(r)$$

or

$$v = \sqrt{\frac{2}{m}\left(E - V(r)\right)}. \tag{3.21}$$

The radial velocity—the component of $\dot{\mathbf{r}}$ along the radius vector—has been given in Eq. (3.16). Combined with the magnitude $v$, this is sufficient information to furnish the direction of the velocity.* These results, and much more, can also be obtained from consideration of an equivalent one-dimensional problem.

The equation of motion in $r$, with $\dot{\theta}$ expressed in terms of $l$, Eq. (3.12), involves only $r$ and its derivatives. It is the same equation as would be obtained for a

*Alternatively, the conservation of angular momentum furnishes $\dot{\theta}$, the angular velocity, and this together with $\dot{r}$ gives both the magnitude and direction of $\dot{\mathbf{r}}$.

fictitious one-dimensional problem in which a particle of mass $m$ is subject to a force

$$f' = f + \frac{l^2}{mr^3}. \tag{3.22}$$

The significance of the additional term is clear if it is written as $mr\dot{\theta}^2 = mv_\theta^2/r$, which is the familiar centrifugal force. An equivalent statement can be obtained from the conservation theorem for energy. By Eq. (3.15) the motion of the particle in $r$ is that of a one-dimensional problem with a fictitious potential energy:

$$V' = V + \frac{1}{2}\frac{l^2}{mr^2}. \tag{3.22$'$}$$

As a check, note that

$$f' = -\frac{\partial V'}{\partial r} = f(r) + \frac{l^2}{mr^3},$$

which agrees with Eq. (3.22). The energy conservation theorem (3.15) can thus also be written as

$$E = V' + \tfrac{1}{2}m\dot{r}^2. \tag{3.15$'$}$$

As an illustration of this method of examining the motion, consider a plot of $V'$ against $r$ for the specific case of an attractive inverse-square law of force:

$$f = -\frac{k}{r^2}.$$

(For positive $k$, the minus sign ensures that the force is *toward* the center of force.) The potential energy for this force is

$$V = -\frac{k}{r},$$

and the corresponding fictitious potential is

$$V' = -\frac{k}{r} + \frac{l^2}{2mr^2}.$$

Such a plot is shown in Fig. 3.3; the two dashed lines represent the separate components

$$-\frac{k}{r} \quad \text{and} \quad \frac{l^2}{2mr^2},$$

and the solid line is the sum $V'$.

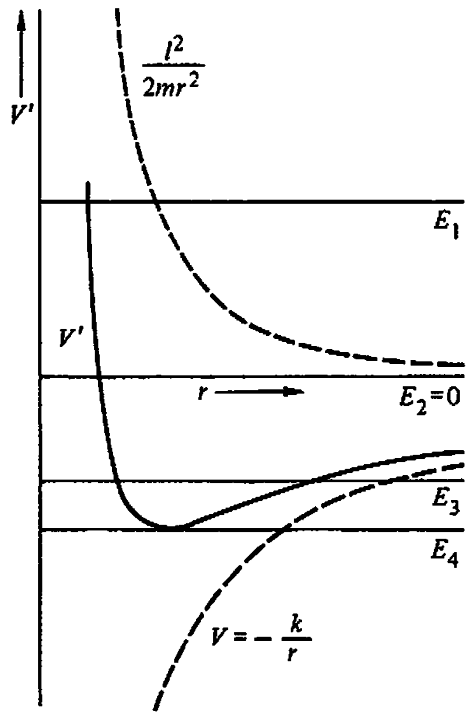


**FIGURE 3.3** The equivalent one-dimensional potential for attractive inverse-square law of force.

Let us consider now the motion of a particle having the energy $E_1$, as shown in Figs. 3.3 and 3.4. Clearly this particle can never come closer than $r_1$ (cf. Fig. 3.4). Otherwise with $r < r_1$, $V'$ exceeds $E_1$ and by Eq. (3.15′) the kinetic energy would have to be negative, corresponding to an imaginary velocity! On the other hand, there is no upper limit to the possible value of $r$, so the orbit is not bounded. A particle will come in from infinity, strike the "repulsive centrifugal barrier," be repelled, and travel back out to infinity (cf. Fig. 3.5). The distance between $E$ and $V'$ is $\frac{1}{2}m\dot{r}^2$, i.e., proportional to the square of the radial velocity, and becomes zero, naturally, at the *turning point* $r_1$. At the same time, the distance between $E$ and $V$ on the plot is the kinetic energy $\frac{1}{2}mv^2$ at the given value of $r$. Hence, the distance between the $V$ and $V'$ curves is $\frac{1}{2}mr^2\dot{\theta}^2$. These curves therefore supply the magnitude of the particle velocity and its components for any distance $r$, at the given energy and angular momentum. This information is sufficient to produce an approximate picture of the form of the orbit.

For the energy $E_2 = 0$ (cf. Fig. 3.3), a roughly similar picture of the orbit behavior is obtained. But for any lower energy, such as $E_3$ indicated in Fig. 3.6, we have a different story. In addition to a lower bound $r_1$, there is also a maximum value $r_2$ that cannot be exceeded by $r$ with positive kinetic energy. The motion is then "bounded," and there are two turning points, $r_1$ and $r_2$, also known as *apsidal distances*. This does not necessarily mean that the orbits are closed. All that can be said is that they are bounded, contained between two circles of radius $r_1$ and $r_2$ with turning points always lying on the circles (cf. Fig. 3.7).

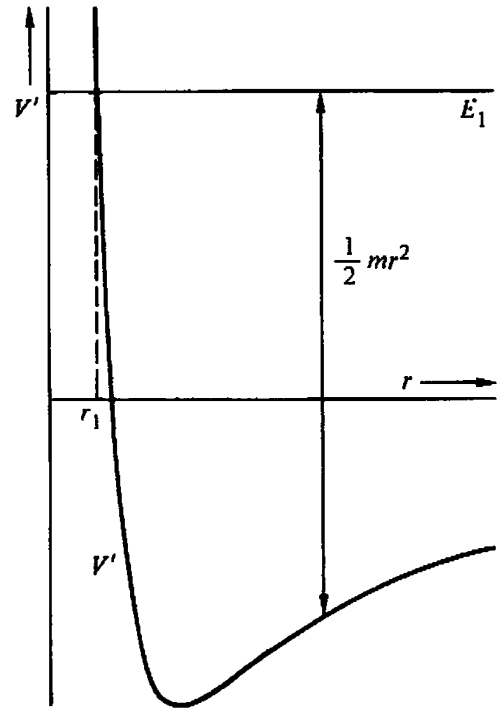


**FIGURE 3.4** Unbounded motion at positive energies for inverse-square law of force.

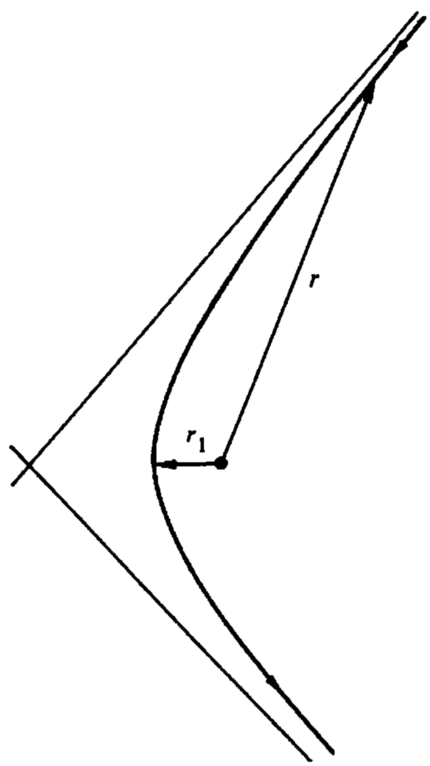


**FIGURE 3.5** The orbit for $E_1$ corresponding to unbounded motion.

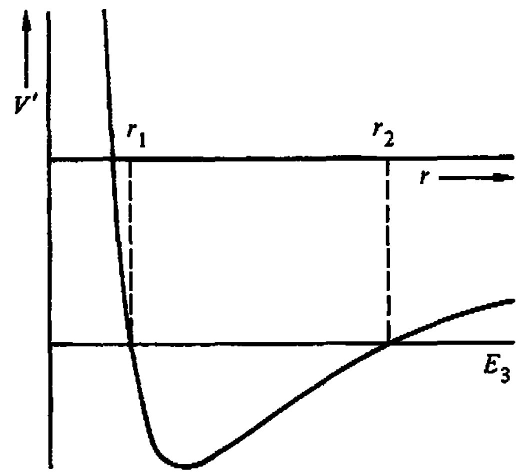


**FIGURE 3.6** The equivalent one-dimensional potential for inverse-square law of force, illustrating bounded motion at negative energies.

If the energy is $E_4$ at the minimum of the fictitious potential as shown in Fig. 3.8, then the two bounds coincide. In such case, motion is possible at only one radius; $\dot{r} = 0$, and the orbit is a circle. Remembering that the effective "force" is the negative of the slope of the $V'$ curve, the requirement for circular orbits is simply that $f'$ be zero, or

$$f(r) = -\frac{l^2}{mr^3} = -mr\dot{\theta}^2.$$

We have here the familiar elementary condition for a circular orbit, that the applied force be equal and opposite to the "reversed effective force" of centripetal

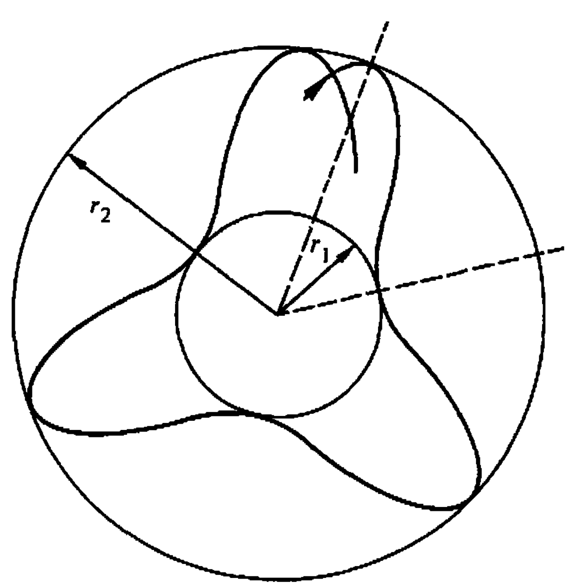


**FIGURE 3.7** The nature of the orbits for bounded motion.

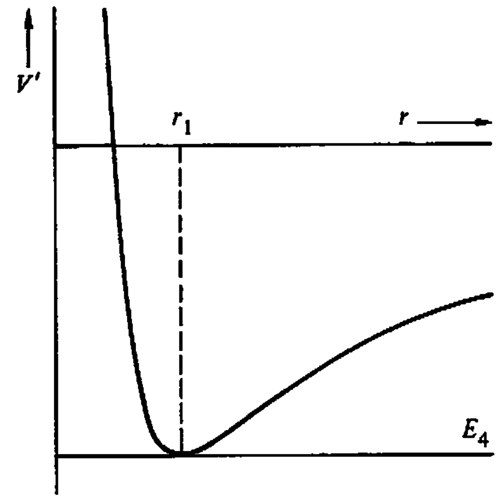


**FIGURE 3.8** The equivalent one-dimensional potential of inverse-square law of force, illustrating the condition for circular orbits.

acceleration.* The properties of circular orbits and the conditions for them will be studied in greater detail in Section 3.6.

Note that all of this discussion of the orbits for various energies has been at one value of the angular momentum. Changing $l$ changes the quantitative details of the $V'$ curve, but it does not affect the general classification of the types of orbits.

For the attractive inverse-square law of force discussed above, we shall see that the orbit for $E_1$ is a hyperbola, for $E_2$ a parabola, and for $E_3$ an ellipse. With other forces the orbits may not have such simple forms. However, the same general qualitative division into open, bounded, and circular orbits will be true for any attractive potential that (1) falls off slower than $1/r^2$ as $r \to \infty$, and (2) becomes infinite slower than $1/r^2$ as $r \to 0$. The first condition ensures that the potential predominates over the centrifugal term for large $r$, while the second condition is such that for small $r$ it is the centrifugal term that is important.

The qualitative nature of the motion will be altered if the potential does not satisfy these requirements, but we may still use the method of the equivalent potential to examine features of the orbits. As an example, let us consider the attractive potential

$$V(r) = -\frac{a}{r^3}, \qquad \text{with} \quad f = -\frac{3a}{r^4}.$$

The energy diagram is then as shown in Fig. 3.9. For an energy $E$, there are two possible types of motion, depending upon the initial value of $r$. If $r_0$ is less than $r_1$ the motion will be bounded, $r$ will always remain less than $r_1$, and the particle will pass through the center of force. If $r$ is initially greater than $r_2$, then it will

*The case $E < E_4$ does not correspond to physically possible motion, for then $\dot{r}^2$ would have to be negative, or $\dot{r}$ imaginary.

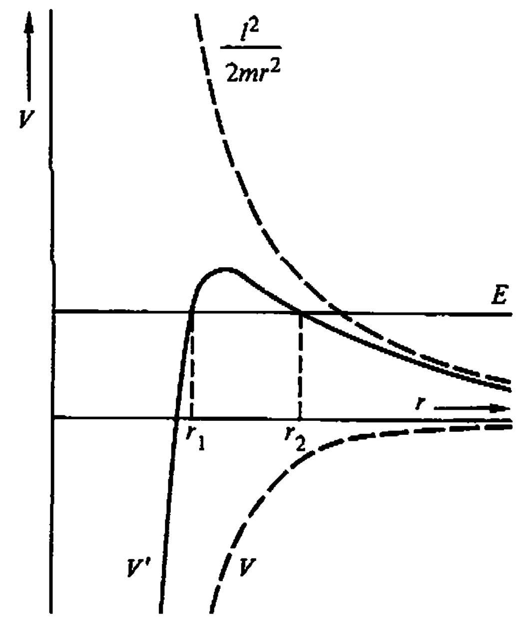


**FIGURE 3.9** The equivalent one-dimensional potential for an attractive inverse-fourth law of force

always remain so; the motion is unbounded, and the particle can never get inside the "potential" hole. The initial condition $r_1 < r_0 < r_2$ is again not physically possible.

Another interesting example of the method occurs for a linear restoring force (isotropic harmonic oscillator):

$$f = -kr, \qquad V = \tfrac{1}{2}kr^2.$$

For zero angular momentum, corresponding to motion along a straight line, $V' = V$ and the situation is as shown in Fig. 3.10. For any positive energy the motion is bounded and, as we know. simple harmonic. If $l \neq 0$, we have the state of affairs shown in Fig. 3.11. The motion then is always bounded for all physically possible

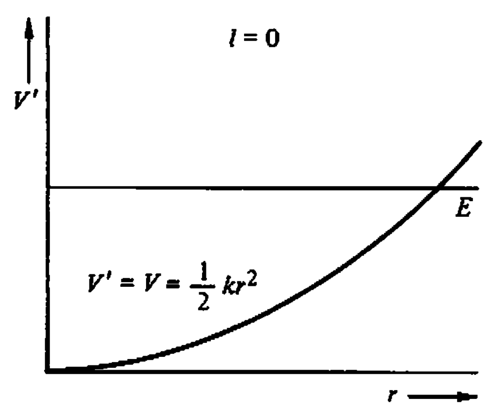


**FIGURE 3.10** Effective potential for zero angular momentum.

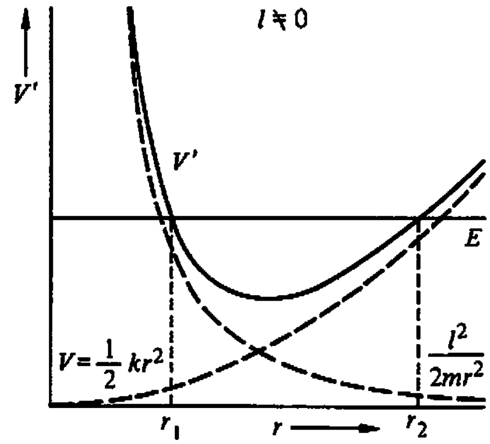


**FIGURE 3.11** The equivalent one-dimensional potential for a linear restoring force.

energies and does not pass through the center of force. In this particular case, it is easily seen that the orbit is elliptic, for if $\mathbf{f} = -k\mathbf{r}$, the $x$- and $y$-components of the force are

$$f_x = -kx, \qquad f_y = -ky.$$

The total motion is thus the resultant of two simple harmonic oscillations at right angles, and of the same frequency, which in general leads to an elliptic orbit.

A well-known example is the spherical pendulum for small amplitudes. The familiar Lissajous figures are obtained as the composition of two sinusoidal oscillations at right angles where the ratio of the frequencies is a rational number. For two oscillations at the same frequency, the figure is a straight line when the oscillations are in phase, a circle when they are 90° out of phase, and an elliptic shape otherwise. Thus, central force motion under a linear restoring force therefore provides the simplest of the Lissajous figures.

## 3.4 ■ THE VIRIAL THEOREM

Another property of central force motion can be derived as a special case of a general theorem valid for a large variety of systems—the *virial theorem*. It differs in character from the theorems previously discussed in being *statistical* in nature; i.e., it is concerned with the time averages of various mechanical quantities.

Consider a general system of mass points with position vectors $\mathbf{r}_i$ and applied forces $\mathbf{F}_i$ (including any forces of constraint). The fundamental equations of motion are then

$$\dot{\mathbf{p}}_i = \mathbf{F}_i. \tag{1.3}$$

We are interested in the quantity