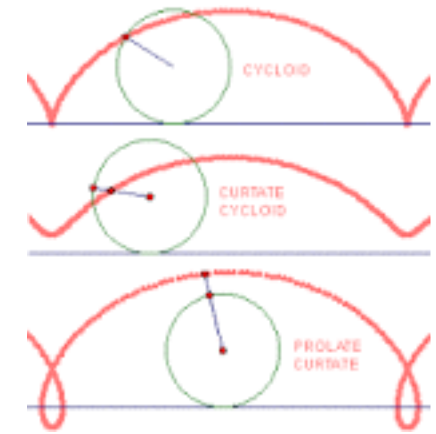
A **cycloid** is the curve traced by a point on the rim of a circular wheel as the wheel rolls along a straight line without slippage. It is an example of a roulette, a curve generated by a curve rolling on another curve.



A pendulum is suspended from the cusp of a cycloid cut in rigid support (see figure below). The path described by the pendulum bob is cycloidal and is given by:

$$x = a(\phi - \sin\phi)$$

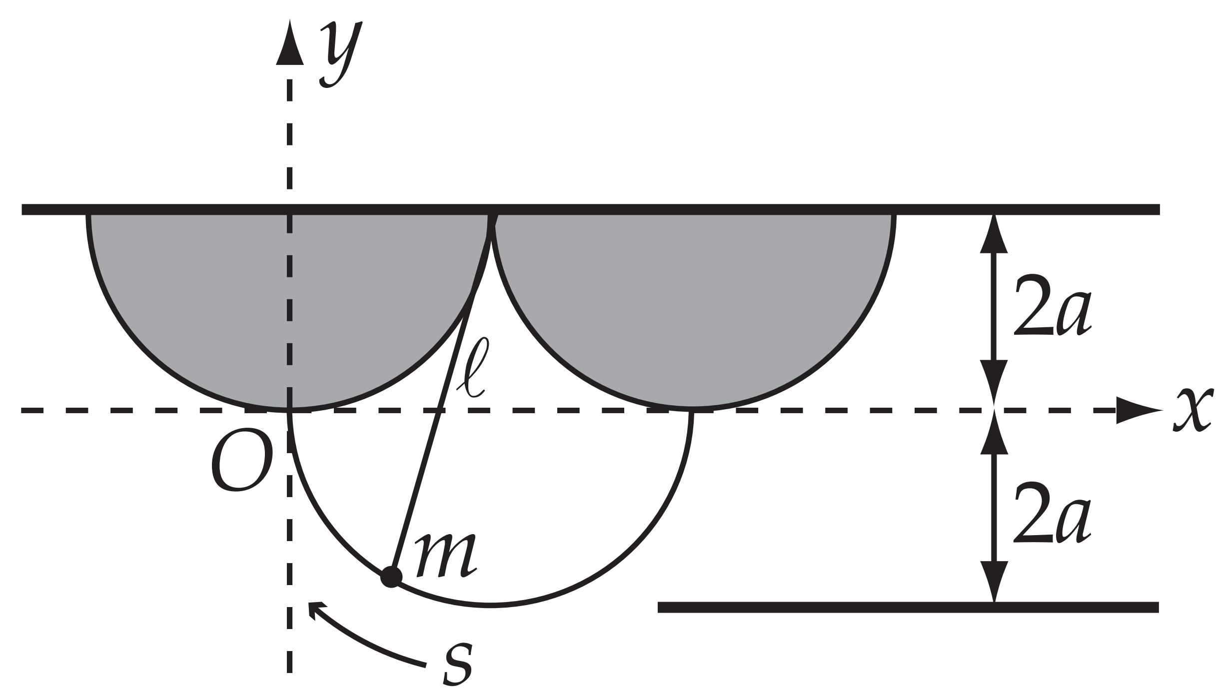
$$y = a(\cos\phi - 1)$$

where the length of the pendulum is $l = 4a$
and where ϕ is the angle of rotation of the circle generating the cycloid.

Show that the oscillations are exactly isochronous (i.e., occur regularly, or at equal time intervals) with a frequency

$$\omega_0 = \sqrt{\frac{g}{l}}$$

independent of the amplitude.



The force responsible for the motion of the pendulum bob is the component of the gravitational force on m that acts perpendicular to the straight portion of the suspension string. This component is seen, from the figure (a) below, to be

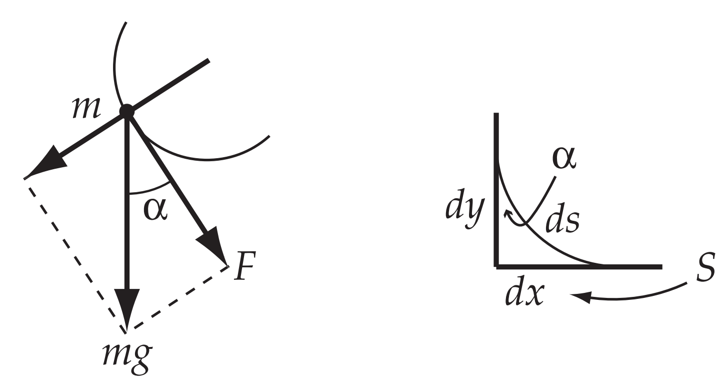
$$F = ma = m\dot{v} = -mg\cos\alpha \tag{1}$$

where α is the angle between the vertical and the tangent to the cycloidal path at the position of m. The cosine of α is expressed in terms of the differentials shown in the figure (b) as

$$\cos\alpha = \frac{dy}{ds} \tag{2}$$

where

$$ds = \sqrt{dx^2 + dy^2} \tag{3}$$



(a) **(b)**

The differentials, dx and dy, can be computed from the defining equations for $x(\phi)$ and $y(\phi)$ above:

$$\left.\begin{aligned} dx &= a(1-\cos\phi)\,d\phi \\ dy &= -a\sin\phi\,d\phi \end{aligned}\right] \tag{4}$$

Therefore,

$$\begin{aligned} ds^2 &= dx^2 + dy^2 \\ &= a^2\left[\left(1-\cos\phi\right)^2 + \sin^2\phi\right]d\phi^2 = 2a^2\left(1-\cos\phi\right)d\phi^2 \\ &= 4a^2\sin^2\frac{\phi}{2}\,d\phi^2 \end{aligned} \tag{5}$$

so that

$$ds = 2a\sin\frac{\phi}{2}\,d\phi \tag{6}$$

Thus,

$$\begin{aligned} \frac{dy}{ds} &= \frac{-a\sin\phi\,d\phi}{2a\sin\frac{\phi}{2}\,d\phi} \\ &= -\cos\frac{\phi}{2} = \cos\alpha \end{aligned} \tag{7}$$

The velocity of the pendulum bob is

$$\begin{aligned} v = \frac{ds}{dt} &= 2a\sin\frac{\phi}{2}\frac{d\phi}{dt} \\ &= -4a\frac{d}{dt}\left[\cos\frac{\phi}{2}\right] \end{aligned} \tag{8}$$

from which

$$\dot{v} = -4a\,\frac{d^2}{dt^2}\left[\cos\frac{\phi}{2}\right] \tag{9}$$

Letting $z \equiv \cos\frac{\phi}{2}$ be the new variable, and substituting (7) and (9) into (1), we have

$$-4ma\ddot{z} = mgz \tag{10}$$

or,

$$\ddot{z} + \frac{g}{4a}\,z = 0 \tag{11}$$

which is the standard equation for simple harmonic motion,

$$\ddot{z} + \omega_0^2 z = 0 \tag{12}$$

If we identify

$$\boxed{\omega_0 = \sqrt{\frac{g}{\ell}}} \tag{13}$$

where we have used the fact that $\ell = 4a$.

Thus, the motion is exactly isochronous, independent of the amplitude of the oscillations. This fact was discovered by Christian Huygene (1673).

If the amplitude of a damped oscillator decreases to 1/e of its initial value after n periods, show that the frequency of the oscillator is approximately

$$1-\frac{1}{8\pi^2 n^2}$$

Times the frequency of the corresponding undamped oscillator.

The amplitude of a damped oscillator is expressed by

$$x(t)=Ae^{-\beta t}\cos\left(\omega_1 t+\delta\right) \tag{1}$$

Since the amplitude decreases to $1/e$ after n periods, we have

$$\beta nT=\beta n\frac{2\pi}{\omega_1}=1 \tag{2}$$

Substituting this relation into the equation connecting ω_1 and ω_0 (the frequency of undamped oscillations), $\omega_1^2=\omega_0^2-\beta^2$, we have

$$\omega_0^2=\omega_1^2+\left[\frac{\omega_1}{2\pi n}\right]^2=\omega_1^2\left[1+\frac{1}{4\pi^2 n^2}\right] \tag{3}$$

Therefore,

$$\frac{\omega_1}{\omega_0}=\left[1+\frac{1}{4\pi^2 n^2}\right]^{-1/2} \tag{4}$$

so that

$$\boxed{\frac{\omega_1}{\omega_2}\cong 1-\frac{1}{8\pi^2 n^2}}$$

Two mass points of mass m_1 and m_2 are connected by a string passing through a hole in a smooth table so that m_1 rests on the table and m_2 hangs suspended. Assuming m_2 moves only in a vertical line, what are the generalized coordinates for the system? Write down the Lagrange equations for the system and, if possible, discuss the physical significance any of them might have. Reduce the problem to a single second-order differential equation and obtain a first integral of the equation. What is its physical significance? (Consider the motion only so long as neither m_1 nor m_2 passes through the hole).