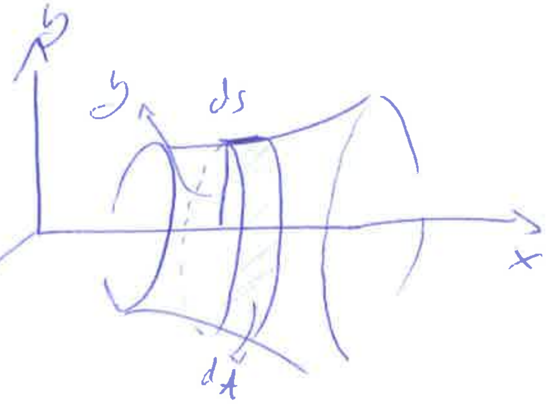
5.1

Soap Bubble

find the surface that minimaies the area.



(1) $dA = 2\pi y\, ds = 2\pi y\sqrt{dx^2+dy^2} = 2\pi y\sqrt{1+y'^2}\,dx$

(2) $$A = \int_{x_1}^{x_2} 2\pi y\sqrt{1+y'^2}\,dx$$

So we get:

(3) $f(y, y') = y\sqrt{1+y'^2}$

Euler Eq: $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$

So (4) $\frac{\partial f}{\partial y} = \sqrt{1+y'^2}$

and

(5) $$\frac{\partial f}{\partial y'} = \frac{yy'}{\sqrt{1+y'^2}}$$

and

(6) $$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = \frac{y'^2 + yy'' + y'^4}{(1+y'^2)^{\frac{3}{2}}}$$

plugging this to Euler Eq we get:

$$(7) \quad \sqrt{1+y'^2} = \frac{y'^2 + yy'' + y'^4}{(1+y'^2)^{\frac{3}{2}}}$$

$$(8) \quad (1+y'^2)^2 = y'^2 + yy'' + y'^4$$

$$1 + 2y'^2 + y'^4 = y'^2 + yy'' + y'^4$$

$$(9) \quad 1 + y'^2 - yy'' = 0$$

How to solve?

set:

$$(10) \quad p = y' \quad ; \quad p' = \frac{dp}{dy}\frac{dy}{dx} = p\frac{dp}{dy}$$

$$(11) \quad 1 + p^2 - yp\frac{dp}{dy} = 0$$

So

Thus,

$$(12) \quad \int \frac{p\,dp}{1+p^2} = \int \frac{dy}{y}$$

right hand side

Smart choice of constant.

$$(13) \quad \int \frac{dy}{y} = \ln y + \ln c^{-1} = \ln\left(\frac{y}{c}\right)$$

5.2

(14) $\int \frac{p}{1+p^2}\,dp \underset{\uparrow}{=} \int \frac{p\,\frac{du}{2p}}{u} = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln u = \frac{1}{2}\ln(1+p^2)$

$u = 1+p^2$; $du = 2p\,dp$

So we have

(15) $\frac{1}{2}\ln(1+p^2) = \ln\left(\frac{y}{c}\right)$

the 2:

$2\ln x = \ln x^2$

(16) $1+p^2 = \frac{y^2}{c^2}$

So

(17) $p^2 = \frac{y^2-c^2}{c^2}$

(18) $\frac{dy}{dx} = \frac{\sqrt{y^2-c^2}}{c}$

(19) $\int \frac{dy}{\sqrt{y^2-c^2}} = \int c\,dx = cx + K$

we are left with:

(20) $\int \frac{dy}{\sqrt{y^2-c^2}} \underset{\uparrow}{=} \int \frac{c\,du}{c\sqrt{u^2-1}}$

$u = \frac{y}{c}$;

$du = \frac{1}{c}\,dy$

Now I'll make a different def for a variable:

(21) $V = \sqrt{u^2-1} + u$; $dV = \dfrac{u\,du}{\sqrt{u^2-1}} + du$

$$dV = du\left[\frac{u}{\sqrt{\ }} + 1\right] = du\left[\frac{\overbrace{u+\sqrt{\ }}^{V}}{\sqrt{\ }}\right]$$

So

(22) $$\int \frac{du}{\sqrt{u^2-1}} = \int \frac{\sqrt{u^2-1}\;dV}{\sqrt{u^2-1}\;V} = \int \frac{dV}{V} =$$

$$= \ln V = \ln\left(\sqrt{u^2-1}+u\right) \equiv \cosh^{-1}\left(\frac{y}{c}\right)$$

Def of $\cosh^{-1}(u)$

So finally we get

(23) $$\cosh^{-1}\left(\frac{y}{c}\right) = \frac{x+k}{c}$$

(24) $$\boxed{y = c\cdot\cosh\left(\frac{x+k}{c}\right)}$$

$\square$ //