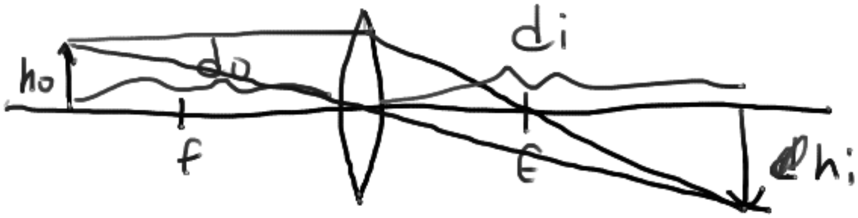
Classical Optics: Gaussian beams, cavities, lasers, acousto-optical modulators, ABCD and Jones matrices.

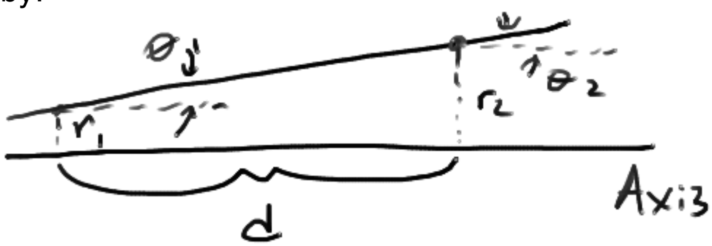
Ray tracing with ABCD matrices (astigmatism, chroma, etc.) (Based on Chp 2 of Verdeyen)

We're all familiar with classical ray tracing for an imaging system:



This is fine, but a bit cumbersome for more complicated imaging systems. We now develop a simple matrix method to treat optical systems.

First, because light travels in a straight path, it is described (assuming axial symmetry) by:



$\theta_1 = \theta_2$ for homogenous media

So, you only need to know:
1. Where it is wrt to some arbitrarily chosen axis (r).
2. In which direction it's heading (theta or r').

First, we make the paraxial assumption: all rays are close enough to the axis that tanθ = sinθ = θ. Thus, θ = r'.

So, for the case of homogenous media we have:

$$r_2 = 1 \cdot r_1 + d\, r_1'$$

$$r_2' = 0 \cdot r_1 + 1 \cdot r_1'$$

OR

$$\begin{pmatrix} r_2 \\ r_2' \end{pmatrix} = \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} r_1 \\ r_1' \end{pmatrix}$$

In general:

$$\begin{pmatrix} r_2 \\ r_2' \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} r_1 \\ r_1' \end{pmatrix}$$

Also, you can show that the determinant of an ABCD matrix is unity:

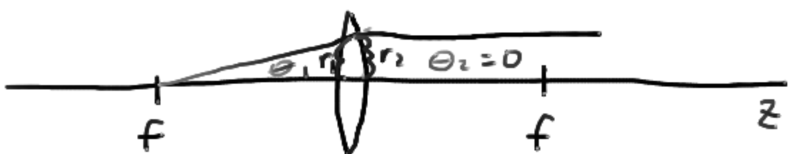
$AD - BC = 1$, as long n doesn't change entrance & exit planes

So, we have the "T" matrix for free space as:

$$T = \begin{pmatrix} 1 & \overset{\leftarrow \text{optical path length}}{d} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & L/n \\ 0 & 1 \end{pmatrix} \qquad L \equiv \text{real space separation}$$

Now, we just need to work these out for a few more cases and we can treat most any optical system.

1. Thin lens of focal length f:
For a thin lens r1 = r2 and the derivative is modified as:

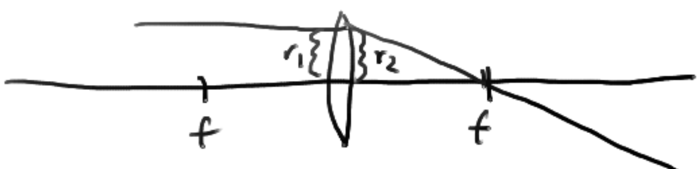


$$r_1 = r_2$$

$$r_2' = C r_1 + D r_1'$$

$$r_2' = 0 = C r_1 + D\left(\frac{r_1}{f}\right)$$

$$r_1' = \frac{dr}{dz} = \tan\theta = \frac{r_1}{\sqrt{r_1^2 + f^2}} \simeq \frac{r_1}{f}$$



$$r_1 = r_2$$

$$r_2' = C r_1 + D r_1' \qquad r_1' = 0$$

$$r_2' = C r_1$$

Now,

$$r_2' = -\frac{r_1}{f} \quad \therefore \quad C = -\frac{1}{f}$$

And $D = 1$.

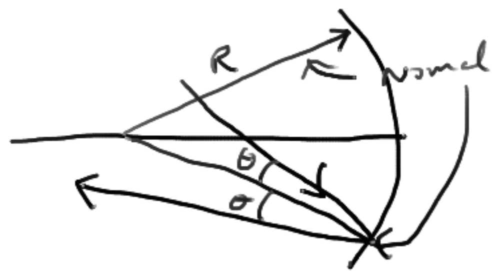
Thus, for a thin lens we have the transmission matrix as:

$$T = \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix}$$

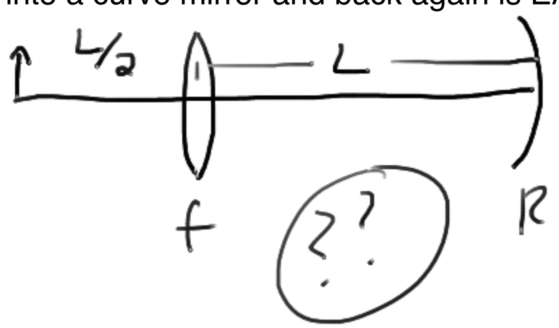
Via similar arguments you can easily derive the following T matrices:

Curved Mirror:

$$T = \begin{pmatrix} 1 & 0 \\ -\frac{2}{R} & 1 \end{pmatrix}$$



So, now a complicated system where we shoot light into a lens then through free space into a curve mirror and back again is EASILY treated as:



$$T = \begin{pmatrix} 1 & L/2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix}\begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -\frac{2}{R} & 1 \end{pmatrix}\begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix}\begin{pmatrix} 1 & L/2 \\ 0 & 1 \end{pmatrix}$$

last element $\longleftarrow$ 1st element

$T =$ easy

Problems:

1. Work out a thick lens like in Verdeyen's Chapter 2 problems.

~~ABCD matrices and the stability of an optical cavity.~~

Gaussian Beams (and ABCD matrices for Gaussian beams)

Ray tracing is great. Simple and intuitive, however, real laser beams for example don't make perfect rays. They make beams of electromagnetic radiation that can have complex phase fronts. To completely describe them requires more information than can be carried in a ray (e.g. {r,r'} --> {r,r', waist, divergence}). We now develop the theory of propagating electromagnetic waves.

Maxwell's equations allow for traveling wave solutions that transport energy from one place to another. For a field propagating in free space the most important/simplest solutions are what we call transverse waves (that is polarization is perpendicular to the direction of propagation). This fact is pretty easily seen from Maxwell's equations in a source free region:

$$\vec{\nabla}\cdot\vec{E} = 0 = \vec{\nabla}_t\cdot\vec{E}_t + \frac{\partial E_z}{\partial z},$$

where z is the propagation direction. Wave is basically a traveling wave with speed c, thus the major thing that is varying in Ez is a term exp(-i*k*z) , where k = ωn/c = 2πn/c. Thus,

$$E_z \approx -\frac{2\pi n}{\lambda} i E_z$$

And a beam of finite transverse diameter D has transverse divergence of ~ E_t/D, thus:

$$|E_z| \approx \frac{\lambda_0}{2\pi n D}|E_t|$$

now usually Lambda/D is a very small number so the transverse component dominates. Similar argument can be made for Del*H = 0.

Though at a focus or maybe in a CO2 laser the ratio doesn't have to be so small, the assumption of a transverse electromagnetic wave (TEM wave) is usually great.

Now, from the above ~~it's~~ we posit a solution for a traveling wave of:

$$\vec{E}(x,y,z) = \vec{E}_0\,\psi(x,y,z)\,e^{-i(kz-\omega t)}$$

The Maxwell equations are:

$$\vec{\nabla}\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = 0 \qquad \vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{H} - \frac{\partial\vec{D}}{\partial t} = 0 \qquad \vec{\nabla}\cdot\vec{D} = 0$$

And for isotropic linear media we have after the time derivative:

$$\vec{D} = \epsilon \vec{E} \quad , \quad \vec{B} = \mu \vec{H}$$

$$\vec{\nabla} \times \vec{E} - i\omega \vec{B} = 0 \quad , \quad \vec{\nabla} \times \vec{B} + i\omega\mu\epsilon \vec{E} = 0$$

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) - i\omega \vec{\nabla} \times \vec{B} = 0$$

$$\vec{\nabla}(\cancelto{0}{\vec{\nabla} \cdot \vec{E}}) - \nabla^2 \vec{E} - i\omega(-i\omega\mu\epsilon \vec{E}) = 0$$

$$\nabla^2 E + \omega^2 \mu\epsilon \vec{E} = 0$$

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) - i\omega \vec{\nabla} \times \vec{B} = 0$$

$$\vec{\nabla}(\cancelto{0}{\vec{\nabla} \cdot \vec{E}}) - \nabla^2 \vec{E} - i\omega(-i\omega\mu\epsilon \vec{E}) = 0$$

$$\nabla^2 \vec{E} + \omega^2 \mu\epsilon \vec{E} = 0$$

$$\boxed{\nabla^2 \vec{E} + \frac{\omega^2}{c^2} n^2 \vec{E} = 0}$$

The traveling wave equation.

Now we insert our ansatz for the spatial part.

$$\nabla_T^2 \vec{E} + \frac{\partial^2 \vec{E}}{\partial z^2} + \left(\frac{\omega}{c}\right)^2 n^2 \vec{E} = 0$$

n = constant.

$$\nabla_T^2 \vec{E} = \vec{E}_0 (\nabla_T^2 \psi) e^{-ikz}$$

$$\frac{\partial \vec{E}}{\partial z} = \vec{E}_0 \left(-ik\psi + \frac{\partial \psi}{\partial z}\right) e^{-ikz}$$

$$\frac{\partial \vec{E}}{\partial z} = \vec{E}_0 \left(-k^2 \psi - 2ik \frac{\partial \psi}{\partial z} + \frac{\partial^2 \psi}{\partial z^2}\right) e^{-ikz}$$

Finally,

$$\nabla_t^2 \psi - 2ik \frac{\partial \psi}{\partial z} + \frac{\partial^2 \psi}{\partial z^2} = 0$$

Since k is huge you typically neglect the last term to arrive at the PARAXIAL WAVE EQUATION:

$$\boxed{\nabla_t^2 \psi - 2ik \frac{\partial \psi}{\partial z} = 0}$$

Now, assuming cylindrical symmetry we have:

$$\frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial \psi}{\partial r} \right) - 2ik \frac{\partial \psi}{\partial z} = 0$$

Now this equation is solved by a class of solutions, which we call the TEM modes. Solving for Psi and plugging back into the E-field ansantz leads to:

$$\vec{E}(x,y,z) = \vec{E}_{m,p}\, H_m\left(\frac{\sqrt{2}\,x}{w(z)}\right) H_p\left(\frac{\sqrt{2}\,y}{w(z)}\right)$$

$$\times \frac{w_0}{w(z)} \exp\left(-\frac{x^2+y^2}{w^2(z)}\right)$$

$$\times \exp\left(-i\left(kz - (1+m+p)\tan^{-1}\left(\frac{z}{z_0}\right)\right)\right)$$

$$\times \exp\left(\frac{-ikr^2}{2R(z)}\right),$$

where:

$$H_m(u) = (-1)^m e^{u^2} \frac{d^m e^{-u^2}}{du^m}$$

, Hermite polynomial.

$$w^2(z) = w_0^2\left(1 + \left(\frac{\lambda z}{\pi n w_0^2}\right)^2\right)$$

$$= w_0^2\left(1 + \left(\frac{z}{z_0}\right)^2\right)$$

$$R(z) = z\left(1 + \left(\frac{\pi n w_0^2}{\lambda z}\right)^2\right) = z\left(1 + \left(\frac{z_0}{z}\right)^2\right)$$

w(z) is the z-dependent beam waist and R(z) is the z-dependent radius of curvature. wo is the minimum waist of the beam which exist at z = 0. zo is the Rayleigh range of the focus. Note that a guassian beam always diverges. A laser beam is not a plane wave. Only a beam of infinite extent has zero divergence.
These are called the Hermite Gaussian TEM modes.

The $TEM_{0,0}$ mode

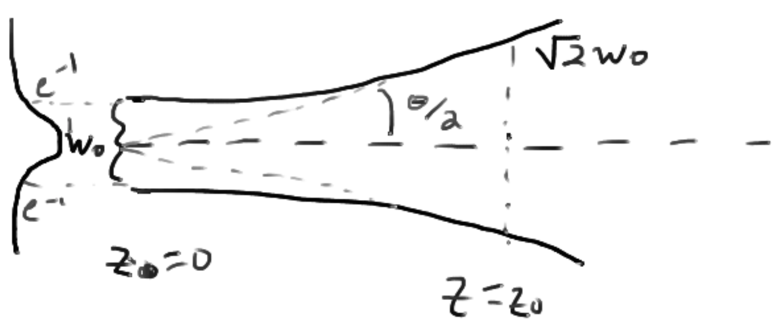
To get some intuition for this let's look at the lowest order solution -- the $TEM_{0,0}$ mode.

$$\vec{E}(x,y,z) = \vec{E}_{0,0}\left\{\frac{w_0}{w(z)}\exp\left(-\frac{r^2}{w^2(z)}\right)\right\} \leftarrow \text{Amplitude part}$$

$$\times\left\{\exp\left[-i\left(Kz - \tan^{-1}\left(\frac{z}{z_0}\right)\right)\right]\right\} \leftarrow \text{longitudinal phase}$$

$$\times\left\{\exp\left(-i\frac{Kr^2}{2R(z)}\right)\right\} \quad \text{radial phase}$$

<u>Amplitude Part:</u>



$$w^2(z) = w_0^2\left(1 + \left(\frac{z}{z_0}\right)^2\right)$$

For, z>> zo (far-field) the beam looks like it is diverging linearly:

$$w(z \gg z_0) = \frac{w_0 z}{z_0} = \frac{\lambda_0 z}{\pi n w_0}$$

$$\frac{\theta}{2} = \frac{dw}{dz} \Rightarrow \boxed{\theta = \frac{2\lambda_0}{\pi n w_0}}$$

Divergence of 00 mode

We see:
1. The shorter the wavelength the less divergent a beam can be.
2. The smaller the beam waist is the faster it diverges (slow/fast focus).

Note that these are FIELDS, usually in the lab you measure intensity or power.

$$I = \frac{1}{2}\epsilon_0 c |E|^2$$

$$I = \left(\frac{w_0}{w(z)}\right)^2 e^{-2r^2/w(z)^2}$$

(so $w(z)$ is the e^{-2} intensity waist.) Everybody screws this up. Don't.

<u>Longitudinal phase factor:</u>
The longitudinal phase of a gaussian beam slightly differs from a plane wave. We'll see this is important for resonators because it changes the resonant condition. The phase is:

$$\phi_{00} = \underset{\substack{\uparrow \\ \text{plane wave}}}{Kz} - \tan^{-1}\left(\frac{z}{z_0}\right)$$

Phase velocity: $$\underset{\substack{\uparrow \\ \text{slightly > plane wave } v_p}}{v_p} = \left(\frac{\phi}{\omega z}\right)^{-1} = \frac{c/n}{1 - \left(\frac{\lambda_0}{2\pi n z}\right)\tan^{-1}\left(\frac{z}{z_0}\right)}$$

<u>Radial phase factor:</u>
Because the total phase includes a part that depends on kr^2/(2R(z)) any z plane is not an equiphase surface (again, it's not a plane wave). So, what does the equiphase surface look like? A curved surface with radius of curvature R(z). The total phase is:

$$\phi \cong kz \cancel{- \tan^{-1}\left(\frac{z}{z_0}\right)} + \frac{kr^2}{2R(z)}$$

$\leq \pi/2$

$-\frac{\pi}{2}$

$$\phi = k\left(z + \frac{r^2}{2R(z)}\right)$$

The quantity in parentheses is constant on a surface of curvature R(z).

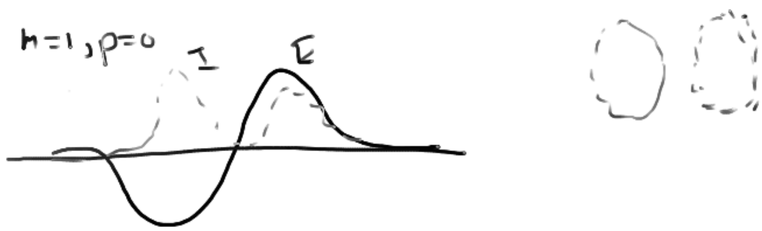
$z = \text{const} - \frac{r^2}{2R(z)}$

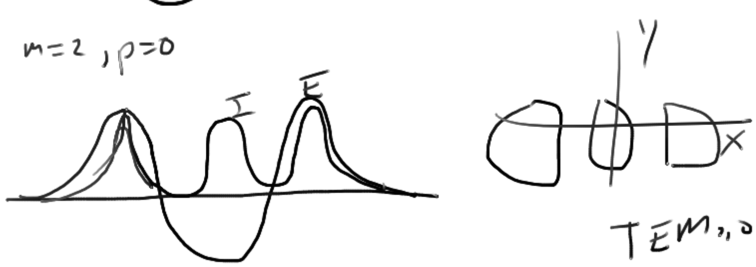
R(z) is radius of curvature!

Note @ $z=0$ $R(0) = \infty$

↳ wave front is planar

So, now we have a feel for TEM$_{OO}$ let's look at the intensity pattern of it and others:





If you ever look carefully at the output of a cavity you'll see exactly this. (A real laser beam can be described as a combination of these Hermite-Gaussian basis functions.)

Problem: Align a cavity and see this?

ABCD Law for Gaussian Beams

If we had done the standard derivation of the Hermite gaussian by solving that awful differential equation then at one point we would have found in convenient to define a complex function:

$$\frac{1}{q(z)} = \frac{1}{R(z)} - i\frac{\lambda_0}{\pi n W^2(z)}$$

In this parameter is everything you care about for a Gaussian beam. And, as such, people use it to describe their beams -- it's called the "q" parameter.

It turns out there is an extremely useful relationship for the q parameter at one z location and the q parameter at another z location if you know the ABCD matrix that connects the two spots (obviously for free space you can just change z in q). It is:

q2 = (Aq1+ B)/(Cq1 + D) -- no formal proof has been shown as far as Verdeyen knows.

Since it's usually easier to work with 1/q, we rearrange:

1/q2 = (C + D(1/q1))/(A + B(1/q1)).
An example?

~~**Cavities**~~

~~So, we now have seen how to find out whether a cavity is stable or not. And we also know how to describe a real laser beam being affected by curved mirrors, lenses, and so on; but, how do we put them together? I mean with ray-tracing we can show a cavity should be stable, but what does the light inside of it really look like? Answer: Use our gaussian beam analysis... but first let's look at a simple case where we can intuit the answer. Later we'll do it generally/formally.~~

~~Plane Mirror (R1 = infinity) and a curved mirror (R2) for a cavity of length L~~.