2.3.8 Damped linear oscillator under the influence of an external force

Because of the unavoidable friction every oscillating process is exponentially damped unless an additional external force acts. We will now include the latter in our considerations. The equation of motion (2.169) is then to be replaced by

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = \frac{1}{m}F(t) \; . \tag{2.186}$$

We choose the same denotations as in the last section and restrict ourselves to the important special case of a periodic force:

$$F(t) = f\cos\overline{\omega}t \; . \tag{2.187}$$

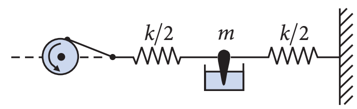

Figure 2.31: *Mechanical realization of the damped harmonic oscillator under the influence of a periodic external force*

One can realize the periodic force by, for instance, a wheel spinning with constant angular velocity and being connected via a drive rod to the oscillating body (Fig. 2.31).

Here again we have an exact non-mechanical realization (Fig. 2.32) by the electrical oscillator circuit if one applies to it a periodic alternating voltage $U_0 \sin \overline{\omega} t$:

$$L\ddot{I} + R\dot{I} + \frac{1}{C} I = U_0 \overline{\omega} \cos \overline{\omega} t \; . \tag{2.188}$$

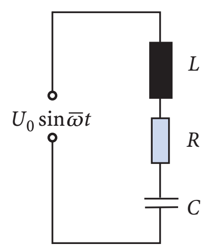

Figure 2.32: *Electrical realization of the damperd harmonic oscillator under the influence of a periodic external force*

The eigen frequency of the oscillator circuit is obviously:

$$\omega_0^2 = \frac{1}{LC} \; ,$$

while the damping constant is given by:

$$\beta = \frac{R}{2L}$$

We look for the general solution of the inhomogeneous differential equation of 2^{nd} order (2.186). We know already the general solution of the associated homogeneous equation from the last section. Therefore we first try to find a special solution of the inhomogeneous differential equation. The easiest way to do this is probably if we first rewrite the differential equation (2.186) by use of complex quantities:

$$\ddot{z} + 2\beta \dot{z} + \omega_0^2 z = \frac{f}{m} e^{i \overline{\omega} t} \; . \tag{2.189}$$

Naturally, physical forces are always real. However, to calculate with the exponential function is especially comfortable. That is the reason why one uses such complex ansatz-functions. One therewith comes to a complex solution from which one eventually takes the real part as the physically relevant result. This works because of the linearity of the differential equation which prevents the real and imaginary parts from mixing.

After a certain **settling time** the oscillator will essentially follow the driving force $F(t)$. A self-evident solution ansatz therefore should be

$$z(t) = A\, e^{i\overline{\omega}t}$$

Insertion into (2.189) yields in this case a conditional equation for the *amplitude* A:

$$\left[A\left(-\overline{\omega}^2 + 2i\beta\,\overline{\omega} + \omega_0^2\right) - \frac{f}{m}\right] e^{i\overline{\omega}t} = 0\,.$$

Thus for A must hold:

$$A = -\frac{f}{m}\,\frac{1}{\left(\overline{\omega}^2 - \omega_0^2\right) - 2i\beta\,\overline{\omega}} = |A|\, e^{i\overline{\varphi}}\,. \tag{2.190}$$

A is of course complex:

$$A = -\frac{f}{m}\,\frac{\left(\overline{\omega}^2 - \omega_0^2\right) + 2i\beta\,\overline{\omega}}{\left(\overline{\omega}^2 - \omega_0^2\right)^2 + 4\beta^2\overline{\omega}^2}$$

whose magnitude is

$$|A| = \frac{f/m}{\sqrt{\left(\overline{\omega}^2 - \omega_0^2\right)^2 + 4\beta^2\overline{\omega}^2}}\,. \tag{2.191}$$

Real and imaginary parts can then be written as follows:

$$\begin{aligned} \mathrm{Re}A &= -\frac{m}{f}|A|^2\left(\overline{\omega}^2 - \omega_0^2\right)\,, \\ \mathrm{Im}A &= -2\frac{m}{f}\beta|A|^2\overline{\omega}\,. \end{aligned} \tag{2.192}$$

For $\overline{\varphi} = \arg(A)$ it therefore holds:

$$\tan\overline{\varphi} = \frac{\mathrm{Im}A}{\mathrm{Re}A} = \frac{2\beta\,\overline{\omega}}{\overline{\omega}^2 - \omega_0^2}\,. \tag{2.193}$$

Since for positive $\overline{\omega}$ the numerator $\mathrm{Im}A$ is always less than zero, $\overline{\varphi}$ will always lie in between $-\pi$ and 0.

We have now found a special solution for (2.189), namely:

$$z(t) = |A|\, e^{i(\overline{\omega}t + \overline{\varphi})}\,.$$

Only the real part is physically relevant which represents a special solution of (2.186):

$$x_0(t) = |A|\cos\left(\overline{\omega}t + \overline{\varphi}\right)\,. \tag{2.194}$$

Therewith the problem is in principle solved because we know the general solution of the associated homogeneous equation:

$$x_{\mathrm{inh}}(t) = x_{\mathrm{hom}}(t) + x_0(t) \ . \tag{2.195}$$

Independently of which of the three cases discussed in the last section (oscillatory case, aperiodic limiting case, creeping case) does appear, the homogeneous solution exhibits in any case an exponentially damped motion which after a sufficiently long time ($t > 1/\beta$) will hardly carry any significant weight. It plays a role only during the so-called **'settling process'**. One can use it to fulfill the given preconditions. After a certain time the mass point m oscillates with the frequency $\overline{\omega}$ of the driving force. The motion then becomes independent of the initial conditions. Therefore, we can concentrate the following discussions on the special solution $x_0(t)$.

The amplitude $|A|$ of the enforced oscillation is proportional to the amplitude f of the driving force and otherwise is essentially dependent on system properties such as (m, ω_0, β) as well as the frequency $\overline{\omega}$. Furthermore, $|A|$ is a symmetric function of $\overline{\omega}$. The limiting cases

$$\begin{aligned} |A|_{\overline{\omega}=0} &= \frac{f}{m\,\omega_0^2} = \frac{f}{k} \ , \\ |A|_{\overline{\omega}\to\infty} &\sim \frac{1}{\overline{\omega}^2} \longrightarrow 0 \end{aligned} \tag{2.196}$$

one can read off directly from (2.191).

If one sets the derivative of $|A|$ with respect to $\overline{\omega}$ equal to zero one finds a conditional equation for the extreme values of $|A|$:

$$\overline{\omega}_1 = 0 \ ; \quad \overline{\omega}_\pm = \pm\sqrt{\omega_0^2 - 2\beta^2} \ . \tag{2.197}$$

The values $\overline{\omega}_\pm$ have a certain formal similarity to the eigen frequency ω of the damped harmonic oscillator (2.172) being, however, because of the factor 2 in front of β, **not** identical to it. The $\overline{\omega}_\pm$ are of course frequencies for A extreme values only as long as they are real, i.e. for $2\beta^2 < \omega_0^2$. In case $\overline{\omega}_\pm$ are real, then one finds at $\overline{\omega}_1$ a minimum and at $\overline{\omega}_\pm$ maxima. If however the $\overline{\omega}_\pm$ turn out to be imaginary numbers, then $|A|$ has a single maximum at $\overline{\omega}_1 = 0$ (Fig. 2.33).

The appearance of a pronounced maximum of the amplitude is called

'resonance'

The **resonance frequency** $\sqrt{\omega_0^2 - 2\beta^2}$ shifts with increasing friction to lower values. In the special case of the undamped oscillator it coincides with the eigen frequency ω_0 of the oscillator. The amplitude then becomes infinitely large and one speaks of a **resonance catastrophy**. For real systems, however, one has to take into consideration that near the resonance the amplitude can become so big that the preconditions of the harmonic oscillator are no longer fulfilled. We think, as an example, of the assumed *small* deflections of the simple thread pendulum.

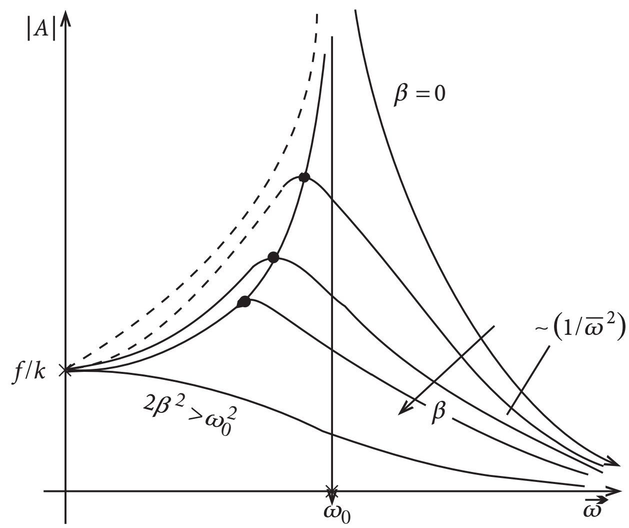

Figure 2.33: *Resonance behaviour of the amplitude of the harmonic oscillator under the influence of a periodic external force for different damping strengths* β

Let us finally still consider the phase shift $\overline{\varphi}$ of the oscillation amplitude $|A|$ relatively to the driving force for which we have already found out in (2.192) and (2.193) that always

$$-\pi \leq \overline{\varphi} \leq 0$$

holds. The amplitude thus drags behind the force (Fig. 2.34). The displacement maximum is reached only **after** the force reaches maximum. For $\overline{\omega} = \omega_0$ the phase shift $\overline{\varphi}$ independently of β is always equal to $-\pi/2$. For the undamped oscillator $\overline{\varphi}$ jumps at $\overline{\omega} = \omega_0$ discontinuously from 0 to $-\pi$. With $\beta \neq 0$ the phase shift $\overline{\varphi}$ becomes a continuous function of $\overline{\omega}$.