# 105A - Set 2 - Solutions

1. A particle is under the influence of a force $F = -kx + kx^3/\alpha^2$ where $k$ and $\alpha$ are constant and $k > 0$. Determine the potential energy $U(x)$ and find the possible particle motions. What happens when $E = k\alpha^2/4$? **there was a typo in the HW, it was suppose both solutions are OK.**
**Answer:**

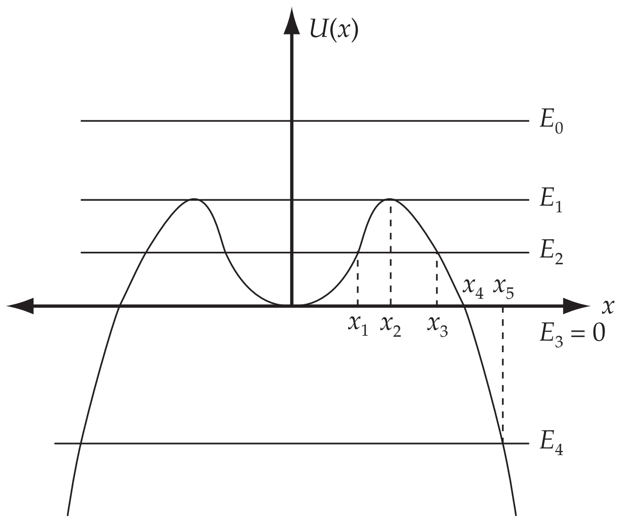


*Figure 1: The potential energy*

$$U(x) = -\int F dx = \frac{1}{2}kx^2 - \frac{1}{4}k\frac{x^4}{\alpha^2} \tag{1}$$

We setch $U(x)$, in Figure 1, not that for small $x$, $U(x)$ behaves like a parabola $kx^2/2$. For large x, the behavior is determanined by $-kx^4/(4\alpha^2)$. The energy marked in Figure 1 is $E = mv^2/2 + U(x)$. We explore the possible motions:
For $E = E_0$, the motion is unbounded; the particle may be anywhere.
For $E = E_1$ (the maxima of $U(x)$) the particle is at a point of unstable equilibrium. It may remain at rest where it is, but if perturbed slightly, it will move away from the equilibrium. Finding the value of the unstable equilibrium we set $dU/dx = 0$, so

$$0 = kx - kx^3/\alpha^2 \tag{2}$$

the solutions are: $x = 0, \pm\alpha$ and $x = \pm\alpha$ are the unstable equilibrium points. So

$$U(\pm\alpha) = X_1 = \frac{1}{2}k\alpha^2 - \frac{1}{4}k\alpha^2 = \frac{1}{4}k\alpha^2 \tag{3}$$

So this is the energy at the unstable equilibrium points.
For $E = E_2$ the particle is either bounded and oscillates between $-x_2$ and $x_2$ or the particle comes from $\pm\infty$ to $\pm x_3$ and returns to $\pm\infty$.
For $E_3 = 0$ the particle is either at the stable point at $x = 0$ or beyond $x = \pm x_4$.
For $E_4$ the particle comes in form $\pm\infty$ to $\pm x_5$ and returns.

2. Two masses $m_1 = 100$ g and $m_2 = 200$ g slide freely in a horizontal frictionless track and are connected by a sling whose force constant is $k = 0.5$ N m$^{-1}$. Find the frequency of oscillatory motion for this system.

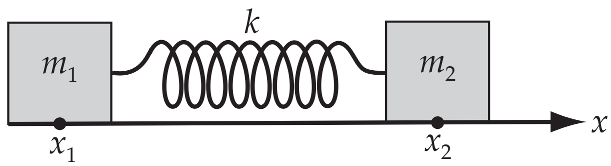


*Figure 2: Two masses connected with a spring*

**Answer:** Suppose the coordinates of $m_1$ and $m_2$ are $x_1$ and $x_2$ respectively, and the spring length is $l$. Then the equations of motion of $m_1$ and $m_2$ are:

$$\begin{aligned} m_1\ddot{x}_1 &= -k(x_1 - x_2 + l) \quad (4)\\ m_2\ddot{x}_2 &= -k(x_2 - x_1 + l) \quad (5)\end{aligned}$$

From the latter equation we can write: $x_1 = (m_2\ddot{x}_2 + kx_2 - kl)/k$. Substituting this expression into (4) we find:

$$\frac{d^2}{dt^2}\left(m_1 m_2 \ddot{x}_2 + (m_1+m_2)kx_2\right) = 0 \quad (6)$$

from which

$$\ddot{x}_2 = -\frac{m_1+m_2}{m_1 m_2}kx_2 \quad (7)$$

So $x_2$ oscillates with the frequency

$$\omega = \sqrt{\frac{m_1+m_2}{m_1 m_2}k} \quad (8)$$

We obtain the same result for $x_1$. If we notice that the reduced mass of the system is

$$\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2} \quad (9)$$

we can write the frequency as

$$\omega = \sqrt{\frac{k}{\mu}} \quad (10)$$

This means the system oscillates in the same way as a system consisting of a single mass $\mu$. Inserting the given value we find: $\mu = 66.7$ g and $\omega = 2.74$ rad s$^{-1}$.

3. Given the equation of motion of a damped oscillations:

$$m\ddot{x} + b\dot{x} + kx = 0 \quad (11)$$

and $b < 0$, i.e, the damping resistance is negative. Find the solution for $x(t)$ and discuss the three oscillatory solutions.
**Answer:** Rearranging the equation

$$m\ddot{x} + b\dot{x} + kx = 0 \tag{12}$$

we can write it as

$$\ddot{x} - 2\beta\dot{x} + \omega^2 x = 0 \tag{13}$$

where $\beta = -b/(2m) > 0$, because $b < 0$. The general solution is then:

$$x(t) = e^{\beta t}\left[A_q e^{\sqrt{\beta^2-\omega^2}t} + A_2 e^{-\sqrt{\beta^2-\omega^2}t}\right] \tag{14}$$

From this equation, we see that the motion is not bounded, irrespective of the relative values of $\beta^2$ and $\omega^2$. The three cases are:

(a) For $\omega^2 > \beta^2$ the motion consists of an oscillatory solution of frequency $\omega_1 = \sqrt{\omega^2 - \beta^2}$, multiplied by ever increasing exponential:

$$x(t) = e^{\beta t}\left[A_1 e^{i\omega_1 t} + A_2 e^{-i\omega_1 t}\right] \tag{15}$$

(b) For $\omega^2 = \beta^2$ the solution is

$$x(t) = e^{\beta t}(A + Bt) \tag{16}$$

which again is increasing.

(c) For $\omega^2 < \beta^2$ the solution is

$$x(t) = e^{\beta t}\left[A_1 e^{\omega_2 t} + A_2 e^{-\omega_2 t}\right] \tag{17}$$

where $\omega_2 = \sqrt{\beta^2 - \omega^2} \leq \beta$. This solution also increases continuously with time.

The three cases describe motions in which the particle is either always moving away from its initial position, as in cases b) or c), or it is oscillating around its initial position, but with an amplitude that grows with the time, as in a). Because $b < 0$ the medium in which the particle moves continually gives energy to the particle and the motion grows without bound.

4. A simple pendulum is made from a wooden block of mass $M$ suspended from a string of length $l$. Initially the pendulum is at rest. A bullet of mass $m$ is fired horizontally with velocity $v_0$ and gets imbedded inside the block at $t = 0$. Write down an equation for subsequent motion of the block $\theta(t)$ assuming the amplitude of oscillations is small. You can also assume $m << M$. Then solve for $\theta(t)$.
**Answer:** The differential equation that governs the motion of a simple pendulum of length $l$ and with mass $M$ is

$$Ml^2\frac{d^2\theta}{dt^2} = -gMl\sin\theta \sim -gMl\theta \tag{18}$$

where for the last transition we have used the small angle approximation. The general solution can be written as:

$$\theta(t) = A\cos(\omega t) + B\sin(\omega t) \tag{19}$$

where $\omega = \sqrt{g/l}$. From conservation of momentum we can write at time $t = 0$ : $Mld\theta/dt_{t=0} = mv_0$. So the boundary conditions at the instant the bullet hits are

$$\theta(t=0) = 0 \qquad \text{and} \qquad \left.\frac{d\theta}{dt}\right|_{t=0} = \frac{mv_0}{Ml} \tag{20}$$

Using this we can solve for $A$ and $B$ and get:

$$\theta(t) = \frac{mv_0}{Ml\omega}\sin(\omega t) \tag{21}$$

5. **Bonus Question + 20pt**
In my colloquium I have discussed the dynamical evolution of hierarchical triple systems and showed that the orbits undergoes eccentricity and inclination oscillations. Adding tidal friction between the two inner objects (just like the tides between the earth and the moon - they convert orbital energy to heat), and general relativity (not important for the question) I have calculated the evolution of triple stars. In Figure 3 I am showing two different examples of the evolution of hierarchical triple systems. The left column shows a system that resulted in merger while the right column is of an inner binary that shirnked its separation to a stable tight configuration. Both had different initial conditions that resulted in these different fates.
Look at the figure and the oscillations of the eccentricity of each system [eccentricity is a value between 1 and 0 that measures how elliptical an orbit is. If its $e = 0$ the orbit is circular and if $e = 1$ its a radial orbit - we will learn more about eccentricity later in the course, but its easy to understand it already now]. Can you relate these two examples to the subject learned in class? How would you classify those two examples in relation to what we discussed in class?
**Answer:** From looking at the Figure we can identify that we have damped oscillations here. The tides are indeed a friction mechanism that converts orbital energy to heat deposited on the stars. The right panel which describes a binary that its separation shirnked, is very similar to **underdamped** oscillation as the amplitude of oscillation in the eccentricity were damped. The panel to the right is very similar to over-damped or critically damped (we cannot know for sure without solving the dispersion relation, which is too complicated anyhow here). The oscillations didn't have time to slowly damp and as soon as they reached a high eccentricity value the system merged.

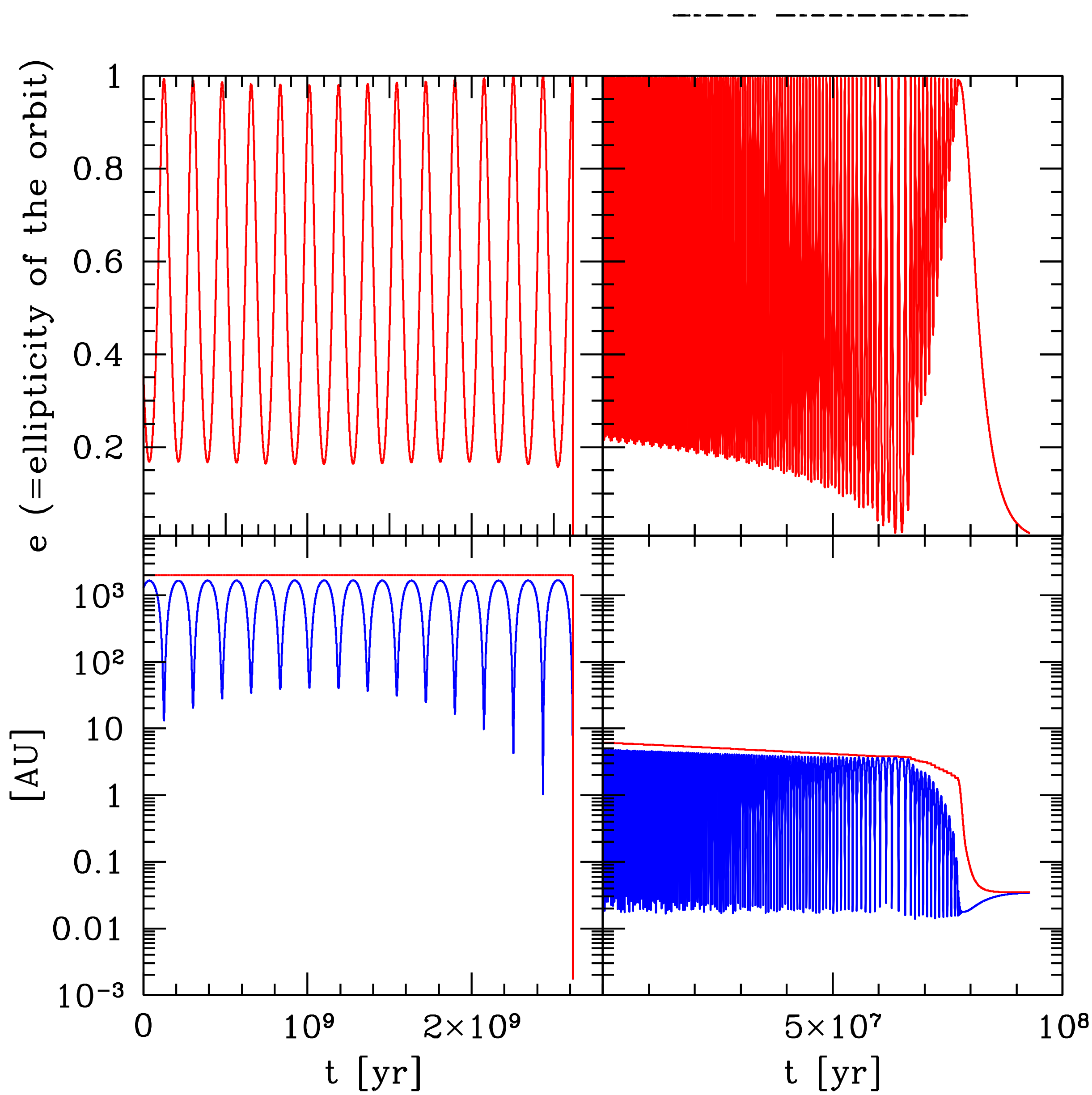


*Figure 3:* **The evolution of hierarchical stellar systems**. The left column shows a system that resulted in merger while the right column is of an inner binary that shirnked its separation to a stable tight configuration. In the top panel we show the eccentricity (a value between 1 and 0 which measures how elliptical an orbit is. if its 0 its circular and if $e = 1$ its a radial orbit) for the inner binary (red lines). Bottom panel shown the inner binary separation (red lines) and the pericenter distance, closest approach in an orbit (blue lines). Figure adopted (and somewhat reconfigured) from Naoz and Fabrycky (2014).