also degenerate. This is indeed the case, except that the natural way to formulate the constant of the motion leads not to a vector but to a tensor of the second rank (cf. Section 7.5). Thus, the existence of an additional constant or integral of the motion, beyond E and $\mathbf{L}$, that is a simple algebraic function of the motion is sufficient to indicate that the motion is degenerate and the bounded orbits are closed.

3.10 ■ SCATTERING IN A CENTRAL FORCE FIELD

Historically, the interest in central forces arose out of the astronomical problems of planetary motion. There is no reason, however, why central force motion must be thought of only in terms of such problems; mention has already been made of the orbits in the Bohr atom. Another field that can be investigated in terms of classical mechanics is the *scattering* of particles by central force fields. Of course, if the particles are on the atomic scale, it must be expected that the specific results of a classical treatment will often be incorrect physically, for quantum effects are usually large in such regions. Nevertheless, many classical predictions remain valid to a good approximation. More important, the procedures for *describing* scattering phenomena are the same whether the mechanics is classical or quantum; we can learn to speak the language equally as well on the basis of classical physics.

In its one-body formulation, the scattering problem is concerned with the scattering of particles by a *center of force*. We consider a uniform beam of particles—whether electrons, or α-particles, or planets is irrelevant—all of the same mass and energy incident upon a center of force. It will be assumed that the force falls off to zero for very large distances. The incident beam is characterized by specifying its *intensity* I (also called flux density), which gives the number of particles crossing unit area normal to the beam in unit time. As a particle approaches the center of force, it will be either attracted or repelled, and its orbit will deviate from the incident straight-line trajectory. After passing the center of force, the force acting on the particle will eventually diminish so that the orbit once again approaches a straight line. In general, the final direction of motion is not the same as the incident direction, and the particle is said to be scattered. The *cross section for scattering in a given direction*, $\sigma(\mathbf{\Omega})$, is defined by

$$\sigma(\mathbf{\Omega})\, d\Omega = \frac{\text{number of particles scattered into solid angle } d\Omega \text{ per unit time}}{\text{incident intensity}}, \tag{3.88}$$

where $d\Omega$ is an element of solid angle in the direction $\mathbf{\Omega}$. Often $\sigma(\mathbf{\Omega})$ is also designated as the *differential scattering cross section*. With central forces there must be complete symmetry around the axis of the incident beam; hence the element of solid angle can be written

$$d\Omega = 2\pi \sin\Theta\, d\Theta, \tag{3.89}$$

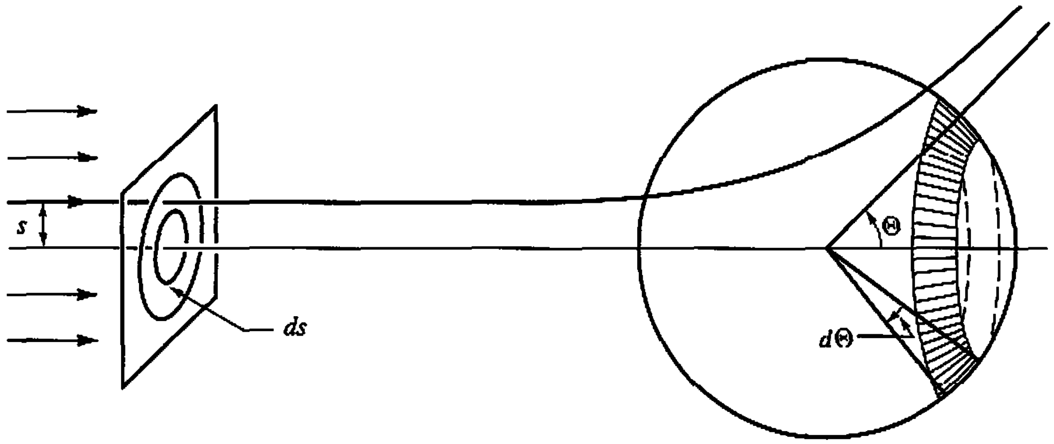

FIGURE 3.19 Scattering of an incident beam of particles by a center of force.

where Θ is the angle between the scattered and incident directions, known as the *scattering angle* (cf. Fig. 3.19, where repulsive scattering is illustrated). Note that the name "cross section" is deserved in that $\sigma(\mathbf{\Omega})$ has the dimensions of an area.

For any given particle the constants of the orbit, and hence the amount of scattering, are determined by its energy and angular momentum. It is convenient to express the angular momentum in terms of the energy and a quantity known as the *impact parameter*, s, defined as the perpendicular distance between the center of force and the incident velocity. If v_0 is the incident speed of the particle, then

$$l = mv_0 s = s\sqrt{2mE}. \tag{3.90}$$

Once E and s are fixed, the angle of scattering Θ is then determined uniquely.* For the moment, it will be assumed that different values of s cannot lead to the same scattering angle. Therefore, the number of particles scattered into a solid angle $d\Omega$ lying between Θ and $\Theta + d\Theta$ must be equal to the number of the incident particles with impact parameter lying between the corresponding s and $s + ds$:

$$2\pi I s|ds| = 2\pi\sigma(\Theta) I \sin\Theta\,|d\Theta|. \tag{3.91}$$

Absolute value signs are introduced in Eq. (3.91) because numbers of particles must of course always be positive, while s and Θ often vary in opposite directions. If s is considered as a function of the energy and the corresponding scattering angle,

$$s = s(\Theta, E), \tag{3.92}$$

*It is at this point in the formulation that classical and quantum mechanics part company. Indeed, it is fundamentally characteristic of quantum mechanics that we cannot unequivocally predict the trajectory of any particular particle. We can only give probabilities for scattering in various directions.

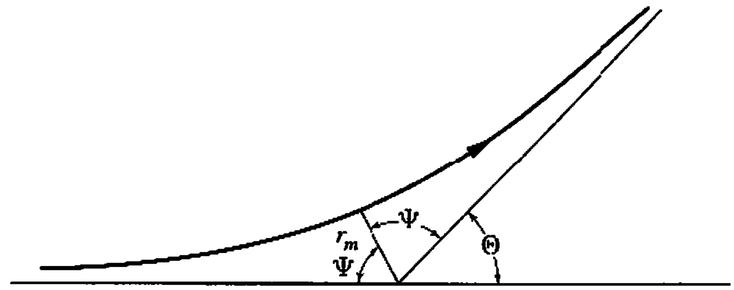

FIGURE 3.20 Relation of orbit parameters and scattering angle in an example of repulsive scattering.

then the dependence of the differential cross section on Θ is given by

$$\sigma(\Theta) = \frac{s}{\sin\Theta}\left|\frac{ds}{d\Theta}\right|. \tag{3.93}$$

A formal expression for the scattering angle Θ as a function of s can be directly obtained from the orbit equation, Eq. (3.36). Again, for simplicity, we will consider the case of purely repulsive scattering (cf. Fig. 3.20). As the orbit must be symmetric about the direction of the periapsis, the scattering angle is given by

$$\Theta = \pi - 2\Psi, \tag{3.94}$$

where Ψ is the angle between the direction of the incoming asymptote and the periapsis (closest approach) direction. In turn, Ψ can be obtained from Eq. (3.36) by setting $r_0 = \infty$ when $\theta_0 = \pi$ (the incoming direction), whence $\theta = \pi - \Psi$ when $r = r_m$, the distance of closest approach. A trivial rearrangement then leads to

$$\Psi = \int_{r_m}^{\infty} \frac{dr}{r^2\sqrt{\frac{2mE}{l^2} - \frac{2mV}{l^2} - \frac{1}{r^2}}}. \tag{3.95}$$

Expressing l in terms of the impact parameter s (Eq. (3.90)), the resultant expression for $\Theta(s)$ is

$$\Theta(s) = \pi - 2\int_{r_m}^{\infty} \frac{s\,dr}{r\sqrt{r^2\left(1 - \frac{V(r)}{E}\right) - s^2}}, \tag{3.96}$$

or, changing r to $1/u$

$$\Theta(s) = \pi - 2\int_{0}^{u_m} \frac{s\,du}{\sqrt{1 - \frac{V(u)}{E} - s^2u^2}}. \tag{3.97}$$

Equations (3.96) and (3.97) are rarely used except for direct numerical computation of the scattering angle. However, when an analytic expression is available for the orbits, the relation between Θ and s can often be obtained almost by inspection. An historically important illustration of such a procedure is the repulsive scattering of charged particles by a Coulomb field. The scattering force field is that produced by a fixed charge $-Ze$ acting on the incident particles having a charge $-Z'e$ so that the force can be written as

$$f = \frac{ZZ'e^2}{r^2},$$

i.e., a repulsive inverse-square law. The results of Section 3.7 can be taken over here with no more change that writing the force constant as

$$k = -ZZ'e^2. \tag{3.98}$$

The energy E is greater than zero, and the orbit is a hyperbola with the eccentricity given by*

$$\epsilon = \sqrt{1 + \frac{2El^2}{m(ZZ'e^2)^2}} = \sqrt{1 + \left(\frac{2Es}{ZZ'e}\right)^2}, \tag{3.99}$$

where use has been made of Eq. (3.90). If θ' in Eq. (3.55) is chosen to be π, periapsis corresponds to $\theta = 0$ and the orbit equation becomes

$$\frac{1}{r} = \frac{mZZ'e}{l^2}(\epsilon \cos\theta - 1). \tag{3.100}$$

This hyperbolic orbit equation has the same form as the elliptic orbit equation (3.56) except for a change in sign. The direction of the incoming asymptote, Ψ, is then determined by the condition $r \to \infty$:

$$\cos \Psi = \frac{1}{\epsilon}$$

or, by Eq. (3.94),

$$\sin \frac{\Theta}{2} = \frac{1}{\epsilon}.$$

Hence,

$$\cot^2 \frac{\Theta}{2} = \epsilon^2 - 1,$$

and using Eq. (3.99)

*To avoid confusion with the electron charge e, the eccentricity will temporarily be denoted by ϵ.

$$\cot\frac{\Theta}{2} = \frac{2Es}{ZZ'e}.$$

The desired functional relationship between the impact parameter and the scattering angle is therefore

$$s = \frac{ZZ'e^2}{2E}\cot\frac{\Theta}{2}, \tag{3.101}$$

so that on carrying through the manipulation required by Eq. (3.93), we find that $\sigma(\Theta)$ is given by

$$\sigma(\Theta) = \frac{1}{4}\left(\frac{ZZ'e^2}{2E}\right)^2 \csc^4\frac{\Theta}{2}. \tag{3.102}$$

Equation (3.102) gives the famous Rutherford scattering cross section, originally derived by Rutherford for the scattering of α particles by atomic nuclei. Quantum mechanics in the nonrelativistic limit yields a cross section identical with this classical result.

In atomic physics, the concept of a *total scattering cross section* σ_T, defined as

$$\sigma_T = \int_{4\pi} \sigma(\boldsymbol{\Omega})\, d\Omega = 2\pi \int_0^\pi \sigma(\Theta)\sin\Theta\, d\Theta.$$

is of considerable importance. However, if we attempt to calculate the total cross section for Coulomb scattering by substituting Eq. (3.102) in this definition, we obtain an infinite result! The physical reason behind this behavior is not difficult to discern. From its definition the total cross section is the number of particles scattered in all directions per unit time for unit incident intensity. Now, the Coulomb field is an example of a "long-range" force; its effects extend to infinity. The very small deflections occur only for particles with very large impact parameters. Hence, all particles in an incident beam of infinite lateral extent will be scattered to some extent and must be included in the total scattering cross section. It is therefore clear that the infinite value for σ_T is not peculiar to the Coulomb field; it occurs in classical mechanics whenever the scattering field is different from zero at all distances, no matter how large.* Only if the force field "cuts off," i.e., is zero beyond a certain distance, will the scattering cross section be finite. Physically, such a cut-off occurs for the Coulomb field of a nucleus as a result of the presence of the atomic electrons, which "screen" the nucleus and effectively cancel its charge outside the atom.

*σ_T is also infinite for the Coulomb field in quantum mechanics, since it has been stated that Eq. (3.102) remains valid there. However, not all "long-range" forces give rise to infinite total cross sections in quantum mechanics. It turns out that all potentials that fall off faster at larger distances than $1/r^2$ produce a finite quantum-mechanical total scattering cross section.

In Rutherford scattering, the scattering angle Θ is a smooth monotonic function of the impact parameter s. From Eq. (3.101) we see that as s decreases from infinity, Θ increases monotonically from zero, reaching the value π as s goes to zero. However, other types of behavior are possible in classical systems, requiring some modification in the prescription, Eq. (3.93), for the classical cross section. For example, with a repulsive potential and particle energy qualitatively of the nature shown in Fig. 3.21(a), it is easy to see physically that the curve of Θ versus s may behave as indicated in Fig. 3.21(b). Thus, with very large values of the impact parameter, as noted above, the particle always remains at large radial distances from the center of force and suffers only minor deflection. At the other extreme, for $s = 0$, the particle travels in a straight line into the center of force, and if the energy is greater than the maximum of the potential, it will continue on through the center without being scattered at all. Hence, for both limits in s, the scattering angle goes to zero. For some intermediate value of s, the scattering angle must pass through a maximum Θ_m. When $\Theta < \Theta_m$, there will be *two* values of s that can give rise to the same scattering angle. Each will contribute to the scattering cross section at that angle, and Eq. (3.93) should accordingly be modified to the form

$$\sigma(\Theta) = \sum_i \frac{s_i}{\sin\Theta} \left|\frac{ds}{d\Theta}\right|_i , \tag{3.103}$$

where for $\Theta \neq \Theta_m$ the index i takes on the values 1 and 2. Here the subscript i distinguishes the various values of s giving rise to the same value of Θ.

Of particular interest is the cross section at the maximum angle of scattering Θ_m. As the derivative of Θ with respect to s vanishes at this angle, it follows from Eq. (3.93) or (3.103) that the cross section must become infinite at $\Theta \to \Theta_m$. But for all larger angles the cross section is zero, since the scattering angle cannot exceed Θ_m. The phenomenon of the infinite rise of the cross section followed by abrupt disappearance is very similar to what occurs in the geometrical optics of the scattering of sunlight by raindrops. On the basis of this similarity, the phenomenon is called *rainbow scattering*.

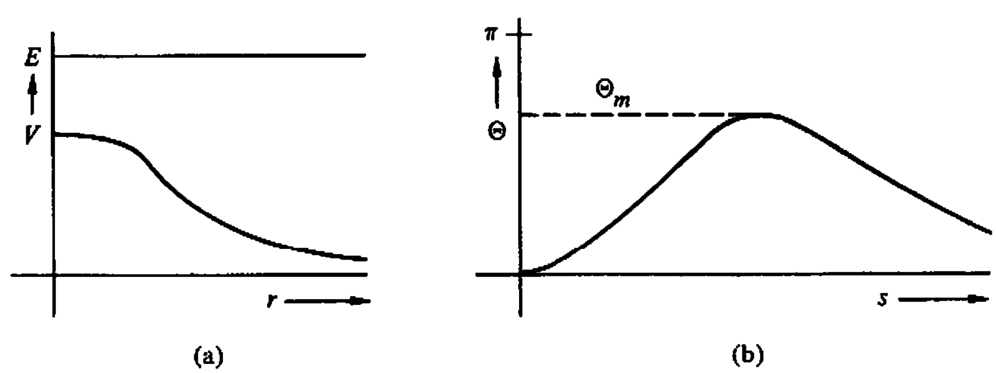

FIGURE 3.21 Repulsive nonsingular scattering potential and double-valued curve of scattering angle Θ versus impact parameter s_0 for sufficiently high energy.

So far, the examples have been for purely repulsive scattering. If the scattering involves attractive forces, further complications may arise. The effect of attraction will be to pull the particle in toward the center instead of the repulsive deflection outward shown in Fig. 3.20. In consequence, the angle Ψ between the incoming direction and the periapsis direction may be greater than $\pi/2$, and the scattering angle as given by Eq. (3.94) is then negative. This in itself is no great difficulty as clearly it is the magnitude of Θ that is involved in finding the cross section. But, under circumstances Θ as calculated by Eq. (3.96) may be greater than 2π. That is, the particle undergoing scattering may circle the center of force for one or more revolutions before going off finally in the scattered direction.

To see how this may occur physically, consider a scattering potential shown as the $s = 0$ curve in Fig. 3.22. It is typical of the intermolecular potentials assumed in many kinetic theory problems—an attractive potential at large distances falling off more rapidly than $1/r^2$, with a rapidly rising repulsive potential at small distances. The other curves in Fig. 3.22 show the effective one-dimensional potential $V'(r)$, Eq. (3.22′), for various values of the impact parameter s (equivalently various values of l). Since the repulsive centrifugal barrier dominates at large r for all values of $s > 0$, the equivalent potential for small s will exhibit a hump.

Now let us consider an incoming particle with impact parameter s_1 and at the energy E_1 corresponding to the maximum of the hump. As noted in Section 3.3, the difference between E_1 and $V'(r)$ is proportional to the square of the radial velocity at that distance. When the incoming particle reaches r_1, the location of the maximum in V', the radial velocity is zero. Indeed, recall from the discussion

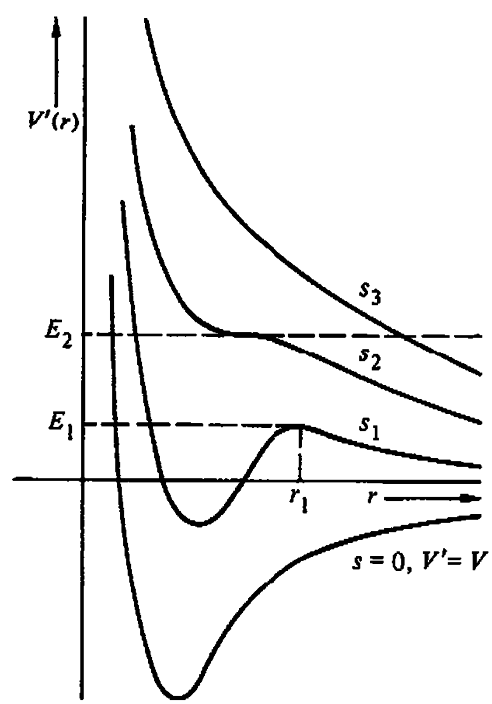

FIGURE 3.22 A combined attractive and repulsive scattering potential, and the corresponding equivalent one-dimensional potential at several values of the impact parameter s.

in Section 3.6 that we have here the conditions for an unstable circular orbit at the distance r_1. In the absence of any perturbation, the incoming particle with parameters E_1 and s_1, once having reached r, would circle around the center of force indefinitely at that distance without ever emerging! For the same impact parameter but at an energy E slightly higher than E_1, no true circular orbit would be established. However, when the particle is in the immediate vicinity of r_1 the radial speed would be very small, and the particle would spend a disproportionately large time in the neighbourhood of the hump The angular velocity, $\dot{\theta}$, meanwhile would not be affected by the existence of a maximum, being given at r, by (3.90)

$$\dot{\theta} = \frac{l}{mr_1^2} = \frac{s_1}{r_1^2}\sqrt{\frac{2E}{m}}.$$

Thus, in the time it takes the particle to get through the region of the hump, the angular velocity may have carried the particle through angles larger than 2π or even multiples thereof. In such instances, the classical scattering is said to exhibit *orbiting* or *spiraling*.

As the impact parameter is increased, the well and hump in the equivalent potential V' tend to flatten out, until at some parameter s_2 there is only a point of inflection in V' at an energy E_2 (cf. Fig. 3.22). For particle energies above E_2, there will no longer be orbiting. But the combined effects of the attractive and repulsive components of the effective potential can lead even in such cases to zero deflection for some finite value of the impact parameter. At large energies and small impact parameters, the major scattering effects are caused by the strongly repulsive potentials at small distances, and the scattering qualitatively resembles the behavior of Rutherford scattering.

We have seen that the scattered particle may be deflected by more than π when orbiting takes place. On the other hand, the observed scattering angle in the laboratory lies between 0 and π. It is therefore helpful in such ambiguous cases to distinguish between a *deflection angle* Φ, as calculated by the right-hand sides of Eqs. (3.96) or (3.97), and the observed scattering angle Θ. For given Φ, the angle Θ is to be determined from the relation

$$\Theta = \pm\Phi - 2m\pi, \qquad m \text{ a positive integer.}$$

The sign and the value of m are to be chosen so that Θ lies between 0 and π. The sum in Eq. (3.103) then covers all values of Φ leading to the same Θ. Figure 3.23 shows curves of Θ versus s for the potential of Fig. 3.22 at two different energies The orbiting that takes place for $E = E_1$ shows up as a singularity in the curve at $s = s_1$. When $E > E_2$, orbiting no longer takes place, but there is a rainbow effect at $\Theta = -\Phi'$ (although there is a nonvanishing cross section at higher scattering angles). Note that Θ vanishes at $s = s_3$, which means from Eq. (3.93) that the cross section becomes infinite in the forward direction through the vanishing of $\sin\Theta$. The cross section can similarly become infinite in the backward direction

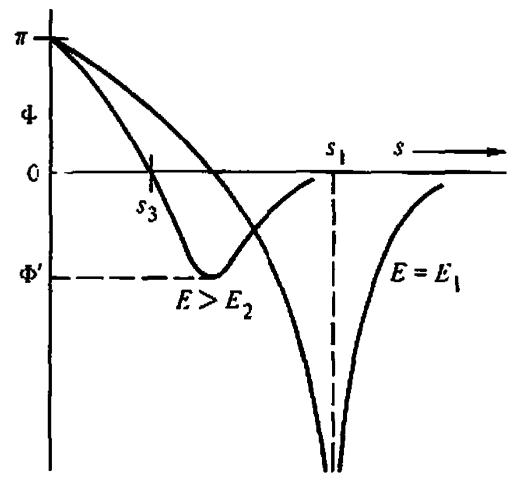

FIGURE 3.23 Curves of deflection angle Φ versus s, for the potential of Fig. 3.22 at two different energies.

providing

$$s\left|\frac{ds}{d\Theta}\right|$$

remains finite at $\Theta = \pi$. These infinities in the forward or backward scattering angles are referred to as *glory scattering*. again in analogy to the corresponding phenomenon in meteorological optics.*

A more general treatment would involve quantum corrections, but in some instances quantum effects are small, as in the scattering of low-energy ions in crystal lattices, and the classical calculations are directly useful. Even when quantum-mechanical corrections are important, it often suffices to use an approximation method (the "semiclassical" approximation) for which a knowledge of the classical trajectory is required. For almost all potentials of practical interest, it is impossible to find an analytic form for the orbit, and Eq. (3.96) (or variant forms) is either approximated for particular regions of s or integrated numerically.

3.11 ■ TRANSFORMATION OF THE SCATTERING PROBLEM TO LABORATORY COORDINATES

In the previous section we were concerned with the one-body problem of the scattering of a particle by a fixed center of force. In practice, the scattering always involved two bodies; e.g., in Rutherford scattering we have the α particle and the atomic nucleus. The second particle, m_2, is not fixed but recoils from its initial position as a result of the scattering. Since it has been shown that any two-body

*The backward glory is familiar to airplane travelers as the ring of light observed to encircle the shadow of the plane projected on clouds underneath