Math 115A: Sample final exam

Sections 1 and 3. Instructor: James Freitag

For the exam, you may use one 8 inch by 11 inch (normal sized paper) piece of paper with anything at all written on **one side** - theorems, example problems, inspirational sayings - anything goes. There will be 8 problems on the final. The difficulty will be on the level of the exams.

Keep in mind this sample review is not comprhensive. I will post more problems throughout the week.

Problem 1 Eigenvalues

Let $\theta \in (0, \pi/2)$. Let

$$A = \begin{pmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{pmatrix}.$$

The entries of A are in $\mathbb{R}$, so we can regard A as either a matrix over the reals *or* the complex numbers. Are there any eigenvectors over $\mathbb{R}$? Explain why not intuitively.

Calculate the eigenvalues and eigenvectors over $\mathbb{C}$.

Problem 2 Some basics

Let

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{pmatrix}.$$

Find a basis of $N(A)$. Find a basis of $R(A)$. Diagonalize A.

Problem 3 A subspace

Prove that the set of all functions that can be written in the form $a \cdot sin(x + b)$ for $a \in \mathbb{C}$ and $b \in \mathbb{R}$ is a vector space. Is it finite dimensional?

Problem 4 From class Friday

Let $S : U \to V$, $T : V \to W$ be linear maps of finite dimensional vector spaces. Suppose that TS is bijective. Prove that S is surjective if and only T is injective.

Problem 5 Use elementary matrices?

Let $A \in M_{n\times n}(\mathbb{F})$ be an invertible matrix, and let B be any other matrix of $M_{n\times n}(\mathbb{F})$. Prove that $det(AB) = det(A) \cdot det(B)$.

Problem 6 A map with specified kernel

Let V be a finite dimensional innter product space. Let W be a subspace. Constuct a linear operator S on V with $N(S) = W^{\perp}$ and $R(S) = W$.

Problem 7 Using a map with specified kernel

Let V be a finite dimensional innter product space. Let W be a subspace. Use the previous problem to prove that $dim(W) + dim(W^{\perp}) = dim(V)$.

Problem 8 Examples or lack thereof

Give an example of a 2×2 matrix M over $\mathbb{R}$ such that M has no eigenvalues in $\mathbb{R}$. Can you give an example of a 3×3 matrix M over $\mathbb{R}$ such that M has no eigenvalues in $\mathbb{R}$?

Problem 9 Representation is better than working by hand!

Find a polynomial $q \in P_3(\mathbb{R})$ such that

$$p\left(\frac{1}{4}\right) = \int_0^1 p(x)q(x)dx$$

for all $p \in P_3(\mathbb{R})$. You can write down your polynomial in terms of an inner product.

#9: First, lets find an orthogonal basis: $\langle 1,1\rangle = \int_0^1 1\cdot 1\,dx = x\Big|_0^1 = \boxed{1} = v_1$

$x - \langle x,1\rangle\cdot 1 = x - \int_0^1 x\,dx = x - \left(\frac{x^2}{2}\right)\Big|_0^1 = \boxed{x-\frac{1}{2}} = v_2 \quad \left\| x-\frac{1}{2}\right\|^2 = \frac{1}{48}$

$$x^2 - \frac{\langle x^2,1\rangle\cdot 1}{\|1\|^2} - \frac{\langle x^2, x-\frac{1}{2}\rangle}{\|x-\frac{1}{2}\|^2}\left(x-\frac{1}{2}\right) = x^2 - \int_0^1 x^2\,dx - \frac{\int_0^1 x^3 - \frac{x^2}{2}\,dx}{\frac{1}{48}}\cdot\left(x-\frac{1}{2}\right)$$

$$= x^2 - \frac{1}{3} - \frac{\left(\frac{1}{4}-\frac{1}{6}\right)}{\frac{1}{48}}\left(x-\frac{1}{2}\right)$$

$$= x^2 - \frac{1}{3} - 4\left(x-\frac{1}{2}\right)$$

$$\boxed{= x^2 - 4x + \frac{1}{6}} = v_3$$

$$\boxed{x^3 - \langle x^3, 1\rangle - \frac{\langle x^3, x-\frac{1}{2}\rangle}{\|x-\frac{1}{2}\|^2}\cdot\left(x-\frac{1}{2}\right) - \frac{\langle x^3, x^2-4x+\frac{1}{6}\rangle}{\|x^2-4x+\frac{1}{6}\|^2}\cdot\left(x^2-4x+\frac{1}{6}\right)}$$

$$\|$$

$$v_4$$

Let u_1, u_2, u_3, u_4 be given by $\frac{v_i}{\|v_i\|}$ for $i=1\ldots 4$.

Then $(u_1,\ldots,u_4)$ is an orthonormal basis of which each element is a polynomial.

Let $y := \sum_{i=1}^{4} u_i\left(\frac{1}{4}\right)\cdot u_i$. Then by Rep. Thm $\langle p, y\rangle = p\left(\frac{1}{4}\right)$ for all $p\in \mathcal{P}_3(\mathbb{R})$.

$\langle p, y\rangle = \int_0^1 p(x)\cdot y(x)\,dx$