Practice Problems For Final

Problem 1 Let $|a\rangle$ and $|b\rangle$ be eigenstates of a Hermitian operator A with eigenvalues a and b, respectively ($a \neq b$). The Hamiltonian operator is given by

$$H = |a\rangle \, \delta \, \langle b| + |b\rangle \, \delta \, \langle a|$$

where δ is just a real number.

(1) Clearly, $|a\rangle$ and $|b\rangle$ are not eigenstates of the Hamiltonian. Write down the eigenstates of the Hamiltonian. What are their energy eigenvalues?

(2) Suppose the system is known to be in state $|a\rangle$ at $t = 0$. Write down the state vector for $t > 0$.

Problem 2 Suppose an electron is in a state described by the wave function

$$\psi = \frac{1}{\sqrt{4\pi}}(e^{i\phi} sin\theta + cos\theta) g(r)$$

where

$$\int_0^\infty |g(r)|^2 r^2 dr = 1$$

and ϕ, θ are the azimuth and polar angles respectively.

(1) What are the possible results of a measurement of the z-component L_z of the angular momentum of the electron in this state?

(2) What is the probability of obtaining each of the possible results in part (1)?

(3) What is the expectation value of L_z?

Problem 3 A particle of is subject to the Hamiltonian

$$H = A\hat{L}^2 + B\hat{L}_z$$

where $\vec{L} = \vec{x} \times \vec{p}$.

(1) What are the eigenvalues and eigenfunctions?

(2) At time zero, we measured $\hat{L}^2$ and got $6\hbar^2$ with probability 100%. When we measure L_z, there are 50% to get 0 and 50% to get $\hbar$. What is the state at time $t > 0$?

Problem 4 (1)Consider a system of spin 1/2. What are the eigenvalues and normalized eigenvector of the operator $A\hat{s}_y + B\hat{s}_z$, where $\hat{s}_y, \hat{s}_z$ are the angular momentum operators, and A, B are real constants.

(2) Assume that the system is in a state corresponding to the upper eigenvalue. What is the probability that a measurement of $\hat{s}_y$ will yield the value $\hbar/2$?

Problem 5 A particle of spin one is subject to the Hamiltonian $H = As_z + Bs_x^2$ where A and B are constants.

(1) Construct matrix representation for s_x, s_y, s_z.

(2) Calculate the energy levels of this systems. If at time zero the spin is in an eigenstate of s with $s_z = +\hbar$, calculate the expectation value of s_x, s_y, s_z at time t.

Problem 1:

means $|a\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $|b\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

If we choose $|a\rangle$, $|b\rangle$ as our basis, we have:

$$H = \begin{pmatrix} 0 & \delta \\ \delta & 0 \end{pmatrix}$$

(1) The eigenvalues:

$$\det \begin{pmatrix} -\lambda & \delta \\ \delta & -\lambda \end{pmatrix} = 0 \iff \begin{array}{l} \lambda_1 = \delta \\ \lambda_2 = -\delta \end{array}$$

when $\lambda = \delta$

$$\begin{pmatrix} 0 & \delta \\ \delta & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \delta \begin{pmatrix} a \\ b \end{pmatrix} \iff b\delta = a\delta$$

$$|\lambda = \delta\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$|\lambda = -\delta\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

eigenvalues: δ $\quad |\lambda = \delta\rangle = \frac{1}{\sqrt{2}} |a\rangle + \frac{1}{\sqrt{2}} |b\rangle$

$-\delta$ $\quad |\lambda = -\delta\rangle = \frac{1}{\sqrt{2}} |a\rangle - \frac{1}{\sqrt{2}} |b\rangle$

(2)

Method One : initial state $|\psi_0\rangle = |a\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

~~expers~~ initial state in terms of eigenstates

$$|\psi_0\rangle = \left(|\delta\rangle\langle\delta| + |-\delta\rangle\langle-\delta| \right) |\psi_0\rangle$$

$$= \langle\delta|\psi_0\rangle\, |\delta\rangle + \langle-\delta|\psi_0\rangle\, |-\delta\rangle$$

$$= \left(\tfrac{1}{\sqrt{2}} \ \ \tfrac{1}{\sqrt{2}}\right) \begin{pmatrix} 1 \\ 0 \end{pmatrix} |\delta\rangle + \left(\tfrac{1}{\sqrt{2}} \ \ -\tfrac{1}{\sqrt{2}}\right) \begin{pmatrix} 1 \\ 0 \end{pmatrix} |-\delta\rangle$$

$$= \tfrac{1}{\sqrt{2}} |\delta\rangle + \tfrac{1}{\sqrt{2}} |-\delta\rangle$$

$$|\psi(t)\rangle = e^{-\frac{iHt}{\hbar}} |\psi_0\rangle = \frac{1}{\sqrt{2}} \left(e^{-\frac{i\delta t}{\hbar}} |\delta\rangle + e^{\frac{i\delta t}{\hbar}} |-\delta\rangle \right)$$

$$= \frac{1}{\sqrt{2}} \left\{ e^{-\frac{it\delta}{\hbar}} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} + e^{\frac{i\delta t}{\hbar}} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \right\}$$

$$= \begin{pmatrix} \dfrac{e^{-it\delta/\hbar} + e^{it\delta/\hbar}}{2} \\ \dfrac{e^{-it\delta/\hbar} - e^{it\delta/\hbar}}{2} \end{pmatrix} = \begin{pmatrix} \cos\left(\frac{t\delta}{\hbar}\right) \\ -i\sin\left(\frac{t\delta}{\hbar}\right) \end{pmatrix}$$

$$= \cos\left(\frac{t\delta}{\hbar}\right) |a\rangle - i\sin\left(\frac{t\delta}{\hbar}\right) |b\rangle$$

Method two: use the algebra of Pauli matrix:

$$H = \begin{pmatrix} 0 & \delta \\ \delta & 0 \end{pmatrix} = \delta\, \sigma_x$$

$$U = e^{\frac{-iHt}{\hbar}} = e^{-\frac{i\delta t}{\hbar}\sigma_x} = \sum_{n=0}^{+\infty} \frac{\left(-\frac{i\delta t}{\hbar}\sigma_x\right)^n}{n!}$$

$$= \sum_{n=0}^{+\infty} \frac{\left(-\frac{i\delta t}{\hbar}\right)^{2n} \sigma_x^{2n}}{(2n)!} + \sum_{n=0}^{+\infty} \frac{\left(-\frac{i\delta t}{\hbar}\right)^{2n+1} \sigma_x^{2n+1}}{(2n+1)!}$$

$$= ch\left(\frac{-it\delta}{\hbar}\right) + \sigma_x\, sh\left(-\frac{it\delta}{\hbar}\right)$$

$$= \cos\left(\frac{t\delta}{\hbar}\right) + \sigma_x (-i) \sin\left(\frac{t\delta}{\hbar}\right)$$

$$= \begin{pmatrix} \cos\left(\frac{t\delta}{\hbar}\right) & -i\sin\left(\frac{t\delta}{\hbar}\right) \\ -i\sin\left(\frac{t\delta}{\hbar}\right) & \cos\left(\frac{t\delta}{\hbar}\right) \end{pmatrix}$$

$$|\psi(t)\rangle = U\,|a\rangle = \begin{pmatrix} \cos\left(\frac{t\delta}{\hbar}\right) & -i\sin\left(\frac{t\delta}{\hbar}\right) \\ -i\sin\left(\frac{t\delta}{\hbar}\right) & \cos\left(\frac{t\delta}{\hbar}\right) \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\left(\frac{t\delta}{\hbar}\right) \\ -i\sin\left(\frac{t\delta}{\hbar}\right) \end{pmatrix}$$

$$= \cos\left(\frac{t\delta}{\hbar}\right) |a\rangle - i\sin\left(\frac{t\delta}{\hbar}\right) |b\rangle$$

Problem 2 :

(1) As

$$Y_{10} = \sqrt{\frac{3}{4\pi}} \cos\theta$$

$$Y_{1,\pm1} = \mp \sqrt{\frac{3}{8\pi}} \sin\theta \; e^{\pm i\phi}$$

the wave function can be written as:

$$\psi = \sqrt{\frac{1}{3}} \left(-\sqrt{2}\, Y_{11} + Y_{10}\right) g(r)$$

Hence the possible value of L_z are $\hbar$, 0

(2)

the probability of $L_z = +\hbar$ is $\left(\sqrt{\frac{2}{3}}\right)^2 = \frac{2}{3}$

$L_z = 0$ is $\left(\frac{1}{\sqrt{3}}\right)^2 = \frac{1}{3}$

(3)

$$\langle L_z \rangle = \frac{2}{3} \times \hbar + \frac{1}{3} \times 0 = \frac{2\hbar}{3}$$

Problem 3 :

$$H = A \hat{L}^2 + B \hat{L}_z$$

(1) the eigenvalues , eigenstates :

we know:

$$\hat{L}^2 |lm\rangle = l(l+1)\hbar^2 |lm\rangle$$

$$L_z |lm\rangle = m\hbar |lm\rangle$$

so : consider:

$$H |lm\rangle = \left[A\, l(l+1)\hbar^2 + B m \hbar \right] |lm\rangle$$

$|lm\rangle$ is the eigenstate , the corresponding eigenvalues :

$$E_{lm} = A l(l+1)\hbar^2 + B m \hbar$$

labelled by l, m

$l = 0, 1, 2, \ldots$

For given l, $m = l, l-1, \cdots, -l$

(2) At time $t=0$:

$$|\psi_0\rangle = \frac{1}{\sqrt{2}} |2,1\rangle + \frac{e^{i\delta}}{\sqrt{2}} |2,0\rangle$$

where δ is a undetermined phase fator.

$$|\psi(t)\rangle = e^{-\frac{i\hat{H}t}{\hbar}} |\psi_0\rangle = \frac{1}{\sqrt{2}} e^{-\frac{it}{\hbar}E_{21}} |2,1\rangle + \frac{e^{i\delta}}{\sqrt{2}} e^{-\frac{it}{\hbar}E_{2,0}} |2,0\rangle$$

Problem 4.

(1). denote: $\frac{\hbar A}{2} = a$

$\frac{\hbar B}{2} = b$

$$A\hat{S}_y + B\hat{S}_z = \frac{\hbar A}{2}\sigma_y + \frac{\hbar B}{2}\sigma_z$$

$$= a\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + b\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

$$= \begin{pmatrix} b & -ia \\ ia & -b \end{pmatrix}$$

$$\det\begin{pmatrix} b-\lambda & -ia \\ ia & -b-\lambda \end{pmatrix} = (\lambda-b)(\lambda+b) - (-ia)(ia)$$

$$= \lambda^2 - b^2 - a^2$$

$$\Rightarrow \lambda = \pm\sqrt{b^2+a^2}$$

$\lambda_1 = \sqrt{b^2+a^2}$

$$\begin{pmatrix} b & -ia \\ ia & -b \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \sqrt{b^2+a^2}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$b\cdot x - iay = \sqrt{b^2+a^2}\,x$$

$$x\left(b - \sqrt{b^2+a^2}\right) = iay$$

$$\Rightarrow \quad x = ia$$

$$y = b - \sqrt{b^2+a^2}$$

$$\lambda_2 = -\sqrt{b^2 + a^2}$$

$$\begin{pmatrix} b & -ia \\ ia & -b \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = -\sqrt{b^2+a^2} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$b \cdot x - ia\, y = -\sqrt{b^2+a^2}\, x$$

$$\left(b + \sqrt{b^2+a^2}\right) x = ia\, y$$

$$x = ia$$

$$y = b + \sqrt{b^2+a^2}$$

So: the eigenvalues and eigenstates:

$$\lambda_1 = \sqrt{a^2+b^2}$$

$$|\lambda_1\rangle = \frac{1}{\left[a^2 + (b - \sqrt{b^2+a^2})^2\right]^{\frac{1}{2}}} \begin{pmatrix} ia \\ b - \sqrt{a^2+b^2} \end{pmatrix}$$

$$\lambda_2 = -\sqrt{a^2+b^2}$$

$$|\lambda_2\rangle = \frac{1}{\left[a^2 + (b + \sqrt{b^2+a^2})^2\right]^{\frac{1}{2}}} \begin{pmatrix} ia \\ b + \sqrt{a^2+b^2} \end{pmatrix}$$

where

$$a = \frac{\hbar A}{2}$$

$$b = \frac{\hbar B}{2}$$

(2) $$|S_y = \tfrac{\hbar}{2}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} -i \\ 1 \end{pmatrix}$$

The probability :

$$P = \left| \langle S_y = \tfrac{\hbar}{2} | \lambda_1 \rangle \right|^2$$

$$= \frac{\left(B - \sqrt{A^2 + B^2} - A\right)^2}{2\left[\left(B - \sqrt{A^2+B^2}\right)^2 + A^2\right]}$$

Problem 5.

(1). we choose $|1,1\rangle$ $|1,0\rangle$ $|1-1\rangle$

$|1,1\rangle \to$ first basis, $|1,0\rangle \to$ second basis, $|1,-1\rangle \to$ third basis

Then:

$$S_z = \hbar \begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix}$$

$$S_+ = S_x + i S_y \quad , \quad S_- = S_x - i S_y$$

work out S_+:

$$S_+ |1,1\rangle = 0$$

$$S_+ |1,0\rangle = \hbar \sqrt{1 \times 2 - 0} \; |1,1\rangle = \sqrt{2}\hbar |1,1\rangle$$

$$S_+ |1,-1\rangle = \hbar \sqrt{2 \times 1 - 0} \; |1,0\rangle = \sqrt{2}\hbar |1,0\rangle$$

$$S_+ = \hbar \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}$$

$$S_- = (S_+)^\dagger = \hbar \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix}$$

$$S_x = \frac{1}{2}(S_+ + S_-)$$

$$= \frac{\hbar}{2}\begin{pmatrix} 0 & \sqrt{2} & 0 \\ \sqrt{2} & 0 & \sqrt{2} \\ 0 & \sqrt{2} & 0 \end{pmatrix}$$

$$S_y = \frac{1}{2i}(S_+ - S_-)$$

$$= \frac{\hbar}{2i}\begin{pmatrix} 0 & \sqrt{2} & 0 \\ -\sqrt{2} & 0 & \sqrt{2} \\ 0 & -\sqrt{2} & 0 \end{pmatrix}$$

(2) $$S_x^2 = \hbar^2 \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$$

$$H = A\hbar \begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix} + B\hbar^2 \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$$

$$= \begin{pmatrix} A\hbar & & \\ & 0 & \\ & & -A\hbar \end{pmatrix} + \begin{pmatrix} \frac{B\hbar^2}{2} & 0 & \frac{B\hbar^2}{2} \\ 0 & B\hbar^2 & 0 \\ \frac{B\hbar^2}{2} & 0 & \frac{B\hbar^2}{2} \end{pmatrix}$$

$$= \begin{pmatrix} A\hbar + \frac{B\hbar^2}{2} & 0 & \frac{B\hbar^2}{2} \\ 0 & B\hbar^2 & 0 \\ \frac{B\hbar^2}{2} & 0 & -A\hbar + \frac{B\hbar^2}{2} \end{pmatrix}$$

$$\det\begin{pmatrix} a+b-\lambda & 0 & b \\ 0 & 2b-\lambda & 0 \\ b & 0 & -a+b-\lambda \end{pmatrix} = 0$$

$$\Rightarrow \quad \det\begin{pmatrix} a+b-\lambda & b \\ b & -a+b-\lambda \end{pmatrix} \cdot (2b-\lambda) = 0$$

$$(2b-\lambda)\left[(a+b-\lambda)(-a+b-\lambda)-b^2\right] = 0$$

$$\Rightarrow \quad (\lambda-2b)(\lambda^2-2b\lambda-a^2)=0$$

$$\boxed{\begin{aligned} \lambda_1 &= b+\sqrt{b^2+a^2} \\ \lambda_2 &= 2b \\ \lambda_3 &= b-\sqrt{b^2+a^2} \end{aligned}}$$

where $a = A\hbar$

$b = \dfrac{B\hbar^2}{2}$

The corresponding eigenvectors:

$$|\lambda_1\rangle = \frac{1}{\sqrt{\Delta}}\begin{pmatrix} b \\ 0 \\ \sqrt{b^2+a^2}-a \end{pmatrix}$$

$$\boxed{\Delta = b^2 + \left(\sqrt{b^2+a^2}-a\right)^2}$$

$$|\lambda_2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$|\lambda_3\rangle = \frac{1}{\sqrt{\Delta}}\begin{pmatrix} \sqrt{b^2+a^2}-a \\ 0 \\ 1 \end{pmatrix}$$

$$|\psi_0\rangle = |S_z = +\hbar\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$|\psi_0\rangle = |\lambda_1\rangle\langle\lambda_1|\psi_0\rangle + |\lambda_2\rangle\langle\lambda_2|\psi_0\rangle + |\lambda_3\rangle\langle\lambda_3|\psi_0\rangle$$

$$= \frac{b}{\sqrt{\Delta}}|\lambda_1\rangle + \frac{\sqrt{b^2+a^2}-a}{\sqrt{\Delta}}|\lambda_3\rangle$$

$$|\psi(t)\rangle = e^{-\frac{iHt}{\hbar}}|\psi_0\rangle$$

$$= \frac{b}{\sqrt{\Delta}} e^{-\frac{i\lambda_1 t}{\hbar}} |\lambda_1\rangle + \frac{\sqrt{b^2+a^2}-a}{\sqrt{\Delta}} e^{-\frac{i\lambda_3 t}{\hbar}} |\lambda_3\rangle$$

$$= \frac{1}{\Delta}\begin{pmatrix} b^2 e^{-\frac{i\lambda_1 t}{\hbar}} + (\sqrt{b^2+a^2}-a)^2 e^{-i\lambda_3 t/\hbar} \\ 0 \\ b(\sqrt{b^2+a^2}-a)\left(e^{-i\lambda_1 t/\hbar} - e^{-i\lambda_3 t/\hbar}\right) \end{pmatrix}$$

$$= \frac{1}{\Delta}\begin{pmatrix} b^2 e^{-i\lambda_1 t/\hbar} + (\sqrt{b^2+a^2}-a)^2 e^{-i\lambda_3 t/\hbar} \\ 0 \\ b(\sqrt{b^2+a^2}-a)\, e^{-\frac{itb}{\hbar}}\left[-2i\sin\left(\frac{\sqrt{b^2+a^2}\,t}{\hbar}\right)\right] \end{pmatrix} = \begin{pmatrix} \frac{\Delta_1}{\Delta} \\ 0 \\ \frac{\Delta_2}{\Delta} \end{pmatrix}$$

$$\langle S_z \rangle = \langle \psi(t) | \, \hat{S}_z \, | \psi(t) \rangle = \begin{pmatrix} \frac{\Delta_1^*}{\Delta} & 0 & \frac{\Delta_2^*}{\Delta} \end{pmatrix} \begin{pmatrix} \hbar & & \\ & 0 & \\ & & -\hbar \end{pmatrix} \begin{pmatrix} \frac{\Delta_1}{\Delta} \\ 0 \\ \frac{\Delta_2}{\Delta} \end{pmatrix}$$

$$= \frac{|\Delta_1|^2 - |\Delta_2|^2}{\Delta^2} \hbar$$

$$= \hbar \left[1 - \frac{8 b^2 \left(\sqrt{b^2+a^2} - a \right)^2 \sin^2\left(\frac{\sqrt{a^2+b^2}}{\hbar} t \right)}{\left(b^2 + \left(\sqrt{b^2+a^2} - a \right)^2 \right)^2} \right]$$

$$= \hbar \left(1 - \frac{2 b^2}{b^2+a^2} \sin^2\left(\frac{\sqrt{a^2+b^2}}{\hbar} t \right) \right)$$

$$\boxed{\begin{aligned} \langle S_x \rangle &= \langle \psi(t) | \, \hat{S}_x \, | \psi(t) \rangle = 0 \\ \langle S_y \rangle &= \langle \psi(t) | \, \hat{S}_y \, | \psi(t) \rangle = 0 \\ \langle S_z \rangle &= \hbar - \frac{2 b^2 \hbar}{b^2+a^2} \sin^2\left(\frac{\sqrt{a^2+b^2}}{\hbar} t \right) \end{aligned}}$$