Physics 262

Homework Set 2

Due Friday 2/23/18

Time-dependent perturbation theory – adiabatic series

1. **Ramsey's method of separated oscillating fields.** This problem outlines the deceptively simple but powerful technique which won Norman Ramsey a share of the Nobel prize. A 2-level system (states $|a\rangle, |b\rangle$) starts out in state $|a\rangle$ and is subjected to near-resonant radiation (perturbing Hamiltonian $H'(t) = V\cos(\omega t)$) over the time $0 < t < \tau$. Next the system evolves for a time $T \gg \tau$ and is finally subjected to a second pulse of radiation, coherent with the first pulse, but possibly shifted in phase, $H''(t) = V\cos(\omega t + \phi)$, for the same pulse pulse length, τ (i.e. in the interval $T + \tau < t < T + 2\tau$). Suppose that the "area" of each pulse is adjusted to be "$\pi/2$," i.e. $\Omega_R \tau = V\tau = \pi/2$.
   a. Use the rotating-wave approximation and the exact solutions to the Schrödinger equation to find the probability P_b to find the system in state $|b\rangle$ at time $T + 2\tau$, as a function of detuning Δ and the free evolution time *T*, in the limit $\tau \to 0$. Plot $P_b(\Delta)$ for $\phi = 0, \frac{\pi}{2}, \pi$.
   b. Suppose a small perturbation is applied during the "free evolution" time *T*, which shifts the resonant frequency from ω_0 to $\omega_0 + \delta$. How would you design the experiment to be maximally sensitive to δ?
   c. Now suppose that the atoms are in a beam with a velocity distribution given by a Maxwell-Boltzmann distribution with average velocity *v*. The two pulses come from the atoms passing through two regions with resonant radiation separated by a distance L. What effect does the velocity distribution have on the probability $P_b(\Delta)$? Consider limits near resonance (how do you define near?) and where $\frac{\Delta L}{\sigma_v} \gg 1$, where σ_v is the width of the velocity distribution.

2. In TDPT we chose a basis of unperturbed eigenfunctions of H_0. Alternatively we could use a time-dependent basis of eigenfunctions, $\phi_n(t)$, of the instantaneous Hamiltonian $H(t)$, where $\phi_n(0) = \psi_n$, are the unperturbed *n*th eigenfunctions. Now we can express the wavefunction as

$$\psi(t) = \sum_n a_n(t) e^{-i\theta_n(t)} \phi_n(t)$$

   Where the phase has been generalized to a "dynamic phase," $\theta_n(t) = \int_0^t \omega_n(t')dt'$, that takes into account the time dependence of the energies ω_n. Expanding through perturbation theory gives the **adiabatic series** where the first term represents the adiabatic approximation itself (see e.g. Griffiths) and higher order terms represent departures from adiabaticity.
   a. Using this expansion in the Schrödinger equation derive the differential equation for the coefficients:

$$\dot{a}_m = -\sum_n \left\langle \phi_m \middle| \frac{\partial \phi_n}{\partial t} \right\rangle a_n e^{-i(\theta_n - \theta_m)}$$

   b. Derive the 0th and 1st order approximations if the system starts in the *N*th state:

$$a_n^{(0)}(t) = \delta_{nN} e^{i\Gamma_N(t)}$$
$$a_m^{(1)} = a_m(0) - \int_0^t \left\langle \phi_m \middle| \frac{\partial \phi_N}{\partial t'} \right\rangle e^{i\Gamma_N(t')} e^{-i[\theta_n(t') - \theta_m(t')]} dt'$$

where $\Gamma_N(t)$ is the time-dependent phase picked up by adiabatic evolution.

c. Apply this to the case of "adiabatic passage" we discussed in class. Use the Hamiltonian in the rotating frame

$$H = \begin{pmatrix} -\frac{\Delta}{2} & \frac{\Omega_R}{2} \\ \frac{\Omega_R}{2} & \frac{\Delta}{2} \end{pmatrix},$$

where $\Delta = -\Delta_0\ (t < 0); \Delta = +\Delta_0 (t > 0);$ and $\Delta = -\Delta_0 + 2\Delta_0 t/T\ (0 < t < T)$. To ensure adiabatic evolution we require that $\partial\Delta/\partial t \ll \Omega_R^2$ and for simplicity $\Delta_0 \gg \Omega_R$. If the system is initially spin-up find the amplitude for spin-down for $t > T$, with a first order correction to adiabaticity (i.e. first order in $\partial\Delta/\partial t$).

3. **Optical Bloch Equations.** Solve the optical Bloch equations numerically in the programming language of your choice. Play around with values of the detuning, Rabi frequency, and natural lifetime of the upper state to get a feel for the dynamics. Plot the upper state population as a function of time for the following cases:
   a. Atoms start in the lower state and $\gamma = 0, \Delta = 0, \Omega, 5\Omega$.
   b. Atoms start in the lower state and $\Delta = 0, \gamma = 0.2\Omega, \Omega, 5\Omega$.
   c. $\Delta = 0, \gamma = 0$. Atoms start in 50%/50% mixture of lower and upper states versus atoms starting in the state $|\psi\rangle = \frac{1}{\sqrt{2}}(|a\rangle + |b\rangle)$.
4. **Density operator.**
   a. Show that $tr(\rho^2) \leq 1$. When does the equal sign hold?
   b. Quantum mechanically one can define entropy as
   $$S = -k_B tr(\rho \ln \rho)\ .$$
   Show that *S* vanishes for a pure state. Extra credit: Show that the populations of the energy eigenstates follow the Boltzmann distribution by maximizing the entropy subject to the constraints, $tr(\rho H) = const.$, $tr(\rho) = 1$, using the method of Lagrange multipliers.
   c. Consider an ensemble of spin ½ particles in an identically prepared pure state. Given $\langle S_x \rangle, \langle S_z \rangle$, and the sign of $\langle S_y \rangle$ determine the state wavefunction. Now consider an ensemble in a mixed state with ensemble averages $[S_x]$, $[S_y]$, $[S_z]$. What is the density matrix?