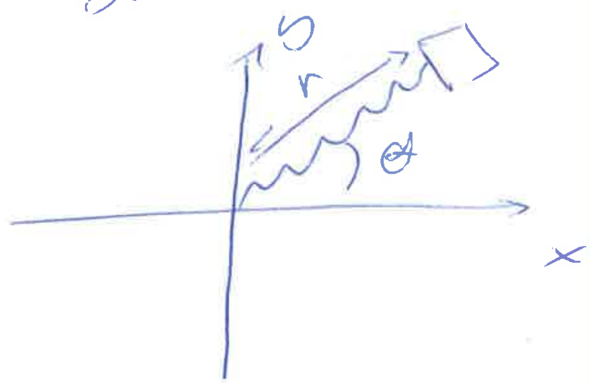
Example - phase diagram

Harmonic Oscillations in 2-D

have an Isotropic spring free to move in the 2D plane



(1) $\bar{F} = -k\bar{r}$

Coordinates:

(2) $x = r\cos\theta$

(3) $y = r\sin\theta$

So the forces:

(4) $\hat{x}$: $F_x = -kx = -kr\cos\theta$

(5) $\hat{y}$: $F_y = -ky = -kr\sin\theta$

So E.O.M:

(6) $\ddot{x} = -\omega^2 x$ where $\omega = \sqrt{\frac{k}{m}}$

(7) $\ddot{y} = -\omega^2 y$

(8) $x(t) = A_x\cos(\omega t + \delta_x)$

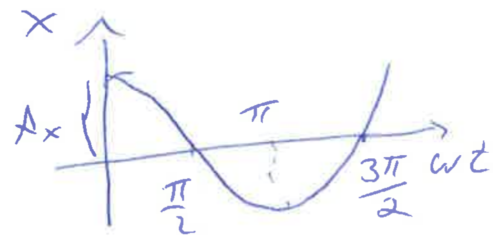
(9) $y(t) = A_y\cos(\omega t - \delta_y)$

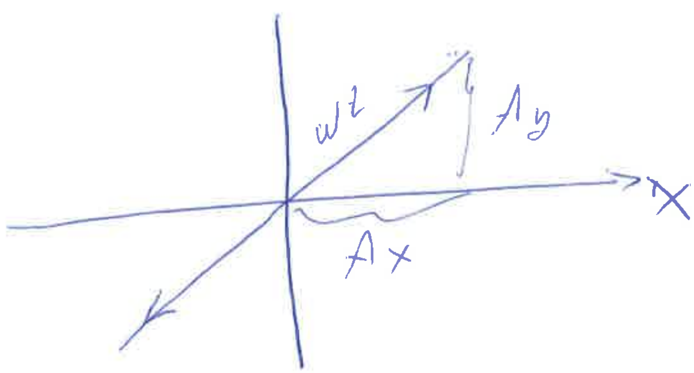
where A_x, A_y δ_x and δ_y are being determine from initial conditions.
We can write Eqs (8) and (9) as

(10) $x(t) = A_x \cos(\omega t)$ where $\delta = \delta_y - \delta_x$

(11) $y(t) = A_y \cos(\omega t - \delta)$

(A) for $\delta = 0$ and $A_x > A_y$





← The motion

← $y(t) = \frac{A_y}{A_x} x(t)$

(B) if $\delta = \frac{\pi}{2}$, $A_x > A_y$

(12) $x(t) = A_x \cos \omega t$

(13) $y(t) = A_y \cos(\omega t - \frac{\pi}{2}) = A_y \sin \omega t$

(14) $$\frac{y(t)}{x(t)} = \frac{A_y}{A_x} \tan \omega t = \frac{A_y}{A_x}\sqrt{\frac{1}{\cos^2 \omega t} - 1} = \frac{A_y}{A_x}\sqrt{\frac{A_x^2}{x^2} - 1}$$

$y(t)^2 = A_y^2 \sin^2 \omega t$
$x(t)^2 = A_x^2 \cos^2 \omega t$
$$\boxed{\frac{y(t)^2}{A_y^2} + \frac{x(t)^2}{A_x^2} = 1}$$

quicker ←

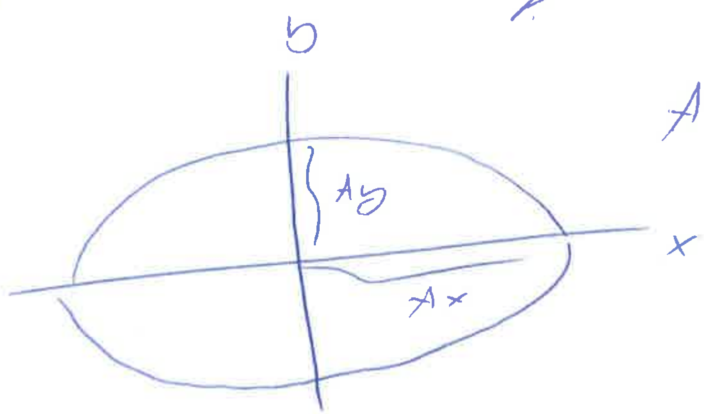
which I can write as

$$(15)\quad \frac{y^2}{x^2} = \frac{A_y^2}{A_x^2}\left(\frac{A_x^2}{x^2} - 1\right) = \frac{A_y^2}{x^2} - \frac{A_y^2}{A_x^2}$$

arranging

$$(16)\quad \frac{y^2}{A_y^2} + \frac{x^2}{A_x^2} = 1$$

which is the general eq for an ellipse



$A_x > A_y$

---

if $\delta = \frac{\pi}{4}$ we again get an ellipse

note that

@ $t=0$ $\hat{x}$: $wt=0$, $\cancel{x} = A_x$, $y=0$

@ $t=\frac{\pi}{2w}$ $\hat{y}$: $wt=\frac{\pi}{2}$ $x=0$ $y=A_y$

so the ellipse is tilted.