105A - Set 5

(Grades are out of 150)

1. Two blocks connected by a spring of spring constant k are free to slide frictionlessly along a horizontal surface, as shown in Fig. 1. The unstretched length of the spring is a.

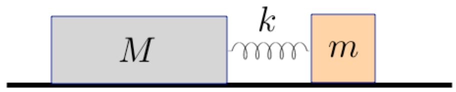

Figure 1: Two masses connected by a spring sliding horizontally along a frictionless surface.

(a) Identify a set of generalized coordinates and write the Lagrangian.
Answer As generalized coordinates I choose $x_1 = X$ and u, where X is the position of the right edge of the block of mass M, and $x_2 = X + u + a$ is the position of the left edge of the block of mass m, where a is the unstretched length of the spring. Thus, the extension of the spring is u. The kinetic energy is then: $M\dot{x}_1^2/2 + m\dot{x}_2^2/2 = M\dot{X}^2/2 + m(\dot{X} + \dot{u})^2/2$ and the potential energy is $U = k(x_2 - x_1 - a)^2/2 = ku^2/2$. The Lagrangian is then

$$L = T - U = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}m(\dot{X} + \dot{u})^2 - \frac{1}{2}ku^2 \tag{1}$$

which we can write:

$$L = \frac{1}{2}(M + m)\dot{X}^2 + \frac{1}{2}m\dot{u}^2 + m\dot{X}\dot{u} - \frac{1}{2}ku^2 \tag{2}$$

Another possible choice is -and a much better choice as you'll see below! to use the fact that $x_1 M + x_2 m = (m + M)x_{cm}$ and then to have $\eta = x_2 - x_1$, so then

$$x_1 = x_{cm} - \frac{m}{m + M}\eta \tag{3}$$

and

$$x_2 = x_{cm} + \frac{M}{m + M}\eta \tag{4}$$

So

$$L = \frac{1}{2}(M + m)\dot{x}_{cm}^2 + \frac{1}{2}\frac{mM}{m + M}\dot{\eta}^2 - \frac{1}{2}k\eta^2 \tag{5}$$

Note that the first choice is not so elegant since we have a $\dot{u}$ and $\dot{X}$ part, the second choice is better.

(b) Find the equations of motion.

Answer: For the first choice we find:

$$\frac{\partial L}{\partial X} = 0 \tag{6}$$
$$\frac{\partial L}{\partial u} = -ku \tag{7}$$
$$\frac{\partial L}{\partial \dot{X}} = (M+m)\dot{X} + m\dot{u} \tag{8}$$
$$\frac{\partial L}{\partial \dot{u}} = m(\dot{X} + \dot{u}) \tag{9}$$

So the equation of motions are:

$$X : (M+m)\ddot{X} + m\ddot{u} = 0 \tag{10}$$
$$u : m(\ddot{X} + \ddot{u}) = -ku \tag{11}$$

For the second choice we find:

$$\frac{\partial L}{\partial x_{cm}} = 0 \tag{12}$$
$$\frac{\partial L}{\partial \eta} = -k\eta \tag{13}$$
$$\frac{\partial L}{\partial \dot{x}_{cm}} = (M+m)\dot{x}_{cm} \tag{14}$$
$$\frac{\partial L}{\partial \dot{\eta}} = \frac{mM}{m+M}\dot{\eta} \tag{15}$$

(c) Find a complete solution to the equations of motion.

Answer: From the equation of motions we have $\ddot{X} = -m\ddot{u}/(M+m)$, plugging this into the equation of motion for u we can eliminate X from the equation and get:

$$\left(1 - \frac{m}{m+M}\right)\ddot{u} = -\frac{k}{m}u \tag{16}$$

rearranging:

$$\frac{M}{m+M}\ddot{u} = -\frac{k}{m}u \tag{17}$$

and rearranging more we have

$$\ddot{u} = -\frac{k(m+M)}{Mm}u = -\Omega^2 u \tag{18}$$

where we defined:

$$\Omega^2 = \frac{k(m+M)}{Mm} \tag{19}$$

The solution is simply:

$$u(t) = A\cos(\Omega t) + B\sin(\Omega t) \tag{20}$$

Now for X; using this equation $\ddot{X} = -m\ddot{u}/(M+m)$ we can write:

$$\ddot{X} = \frac{m\Omega^2}{M+m}u = \frac{k}{M}u = \frac{k}{M}\left(A\cos(\Omega t) + B\sin(\Omega t)\right) \tag{21}$$

Integrating this once we get:

$$\dot{X} = C + \frac{k}{M\Omega}\left(A\sin(\Omega t) - B\cos(\Omega t)\right) \tag{22}$$

And again,

$$X(t) = D + Ct - \frac{k}{M\Omega^2}\left(A\cos(\Omega t) + B\sin(\Omega t)\right) = D + Ct - \frac{m}{M+m}\left(A\cos(\Omega t) + B\sin(\Omega t)\right) \tag{23}$$

For the second choice:

$$\ddot{x}_{cm} = 0 \tag{24}$$

and the other equation is:

$$\frac{mM}{m+M}\ddot{\eta} = -k\eta \tag{25}$$

so we again get the same frequency, i.e,

$$\Omega^2 = \frac{k(m+M)}{Mm} \tag{26}$$

The solution is simply:

$$\eta(t) = A\cos(\Omega t) + B\sin(\Omega t) \tag{27}$$

and then $x_{cm} = D + Ct$.

Note that in the end the two choices (the better one and the not so good one), give the same functional behavior as a function of time, just different constants.

2. In the previous problem you had to find the Lagrangian and equation of motion for two blocks connected by a spring of spring constant k are free to slide frictionlessly along a horizontal surface, as shown in Fig. 1. After we've learned about conserved quantities two students decided to reexamine this problem. Student A assigned x_1 with mass M and x_2 with mass m and wrote the Lagrangian as

$$L = \frac{1}{2}M\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2 - \frac{k}{2}(x_2 - x_1)^2 \ . \tag{28}$$

He then decided to move to the center of mass, where the definition of the center of mass (basically moving the origin of the coordinate system to the center of mass) $Mx_1 + mx_2 = 0$ so $x_1 = -mx_2/M$. Plugging this into the Lagrangian he wrote:

$$L = \frac{1}{2}M\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2 - \frac{k}{2}\left(1 + \frac{m}{M}\right)^2 x_2^2 \ . \tag{29}$$

He then said that since

$$\frac{\partial L}{\partial x_1} = 0 \ . \tag{30}$$

the conjugate momentum $p_{x_1} = M\dot{x}_1$ is constant. Student B said she thinks he is wrong. Is she correct? If so what were her arguments? If not, what was her mistake?
Answer: Of course she is correct. There are few ways to show that she is correct. Student A process was incomplete, he should have also plug in $\dot{x}_1 = -m\dot{x}_2/M$, which means that since the Lagrangian is explicitly depended x_2 then $p_{x_2} = m\dot{x}_2$ is not constant, and since $\dot{x}_1 = -m\dot{x}_2/M$ then of course that $p_{x,1}$ is not constant. The problem is that x_1 is not a canonical (generalized) coordinate once we reduced the problem to x_2 (we could have eliminate x_2 and left with just x_1 in the Lagrangian, which then x_1 would have been the canonical coordinate and x_2 not).

3. A particle of mass m is attracted to a force centered with a forced magnitude k/r^2. Use plane polar coordinates.

   (a) Write the Lagrangian of the system.
   Answer: For a force $F = k/r^2$ the potential is $U = -\int \mathbf{F}\cdot\mathbf{dr} = -k/r$. So the Lagrangian is:

   $$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{k}{r} \ . \tag{31}$$

   (b) Find the momenta. What are the conserved quantities ?
   Answer: The momenta are:

   $$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r} \ . \tag{32}$$

   and

   $$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \ . \tag{33}$$

   To find if the momenta are conserved we need to do:

   $$\dot{p}_r = \frac{\partial L}{\partial r} = mr\dot{\theta}^2 - \frac{k}{r^2} \ . \tag{34}$$

   $$\dot{p}_\theta = \frac{\partial L}{\partial \theta} = 0 \ . \tag{35}$$

   so $p_\theta = \text{Const}$. This is the angular momentum. In addition

   $$\frac{\partial L}{\partial t} = 0 \ . \tag{36}$$

   so the energy is also conserved.

4. A particle of mass m moves under the influence of gravity along a helix $z = k\theta, r = \text{Const}$, where k is constant and z is vertical.
   (i) Choose a coordinate system
   (ii) How many degrees of freedom the system has?

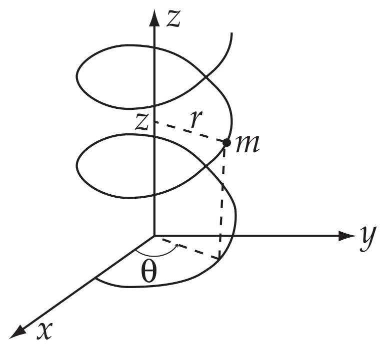

Figure 2: mass on a helix

(iii) Write the Lagrangian
(iv) Find all the conserved quantities.
(v) Find the Lagrangian equation of motion.
Answer: We choose to work in cylindrical coordinates $x = r\cos\theta$, $y = r\sin\theta$ and z. In cylindrical coordinates the kinetic energy and the potential energy of the spiraling particle are expressed by

$$T = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2\right) \tag{37}$$

and

$$U = mgz \tag{38}$$

Since $z = k\theta$ then $\dot{z} = k\dot{\theta}$, and also $r = \text{const}$. Thus, we will eliminate θ as a coordinate and will have **one degree of freedom**. Its also possible of course to choose θ as the preferred coordinate. So combining this we get (if we choose z as our coordinate):

$$L = \frac{1}{2}m\left(\frac{r^2}{k^2}\dot{z}^2 + \dot{z}^2\right) - mgz \tag{39}$$

If we choose θ the Lagrangian would have been (one of the two is an acceptable answer)

$$L = \frac{1}{2}m\left(r^2\dot{\theta}^2 + k^2\dot{\theta}^2\right) - mgk\theta \tag{40}$$

To find the equation of motion (for z) we first have

$$\frac{\partial L}{\partial z} = -mg \tag{41}$$

and

$$p_z = \frac{\partial L}{\partial \dot{z}} = m\left(\frac{r^2}{k^2} + 1\right)\dot{z} \tag{42}$$

Thus, the equation of motion is:

$$m\left(\frac{r^2}{k^2}+1\right)\ddot{z}=-mg \tag{43}$$

or

$$\ddot{z}=-g\left(\frac{r^2}{k^2}+1\right)^{-1} \tag{44}$$

(not exactly free fall)
The conserved quantities are E the energy and P_θ because the Lagrangian doesn't depends on θ.
If you choose θ as your coordinate than your solution is:

$$\frac{\partial L}{\partial \theta}=-mgk \tag{45}$$

and

$$p_\theta=\frac{\partial L}{\partial \dot{\theta}}=m\left(r^2+k^2\right)\dot{\theta} \tag{46}$$

and the equation of motion is then

$$\ddot{\theta}=-gk\left(r^2+k^2\ \right)^{-1} \tag{47}$$

The conserved quantities are E the energy and P_z because the Lagrangian doesn't depends on z.