symmetric in time, this angular change is twice that which would result from the passage from r_{min} to r_{max}; thus

$$\Delta\theta = 2\int_{r_{\text{min}}}^{r_{\text{max}}} \frac{(l/r^2)\,dr}{\sqrt{2\mu\left(E - U - \dfrac{l^2}{2\mu r^2}\right)}} \tag{8.31}$$

The path is closed only if $\Delta\theta$ is a rational fraction of 2π—that is, if $\Delta\theta = 2\pi \cdot (a/b)$, where a and b are integers. Under these conditions, after b periods the radius vector of the particle will have made a complete revolutions and will have returned to its original position. We can show (see Problem 8-35) that if the potential varies with some integer power of the radial distance, $U(r) \propto r^{n+1}$, then a closed noncircular path can result *only** if $n = -2$ or $+1$. The case $n = -2$ corresponds to an inverse-square-law force—for example, the gravitational or electrostatic force. The $n = +1$ case corresponds to the harmonic oscillator potential. For the two-dimensional case discussed in Section 3.4, we found that a closed path for the motion resulted if the ratio of the angular frequencies for the x and y motions were rational.

8.6 Centrifugal Energy and the Effective Potential

In the preceding expressions for $\dot{r}$, $\Delta\theta$, and so forth, a common term is the radical

$$\sqrt{E - U - \frac{l^2}{2\mu r^2}}$$

The last term in the radical has the dimensions of energy and, according to Equation 8.10, can also be written as

$$\frac{l^2}{2\mu r^2} = \frac{1}{2}\mu r^2\dot{\theta}^2$$

If we interpret this quantity as a "potential energy,"

$$U_c \equiv \frac{l^2}{2\mu r^2} \tag{8.32}$$

then the "force" that must be associated with U_c is

$$F_c = -\frac{\partial U_c}{\partial r} = \frac{l^2}{\mu r^3} = \mu r\dot{\theta}^2 \tag{8.33}$$

*Certain fractional values of n also lead to closed orbits, but in general these cases are uninteresting from a physical standpoint.

This quantity is traditionally called the **centrifugal force,*** although it is not a force in the ordinary sense of the word.† We shall, however, continue to use this unfortunate terminology, because it is customary and convenient.

We see that the term $l^2/2\mu r^2$ can be interpreted as the *centrifugal potential energy* of the particle and, as such, can be included with $U(r)$ in an *effective potential energy* defined by

$$\boxed{V(r) \equiv U(r) + \frac{l^2}{2\mu r^2}} \tag{8.34}$$

$V(r)$ is therefore a *fictitious* potential that combines the real potential function $U(r)$ with the energy term associated with the angular motion about the center of force. For the case of inverse-square-law central-force motion, the force is given by

$$F(r) = -\frac{k}{r^2} \tag{8.35}$$

from which

$$U(r) = -\int F(r)\,dr = -\frac{k}{r} \tag{8.36}$$

The effective potential function for gravitational attraction is therefore

$$V(r) = -\frac{k}{r} + \frac{l^2}{2\mu r^2} \tag{8.37}$$

This effective potential and its components are shown in Figure 8-5. The value of the potential is arbitrarily taken to be zero at $r = \infty$. (This is implicit in Equation 8.36, where we omitted the constant of integration.)

We may now draw conclusions similar to those in Section 2.6 on the motion of a particle in an arbitrary potential well. If we plot the total energy E of the particle on a diagram similar to Figure 8-5, we may identify three regions of interest (see Figure 8-6). If the total energy is positive or zero (e.g., $E_1 \geq 0$), then the motion is unbounded; the particle moves toward the force center (located at $r = 0$) from infinitely far away until it "strikes" the potential barrier at the *turning point* $r = r_1$ and is reflected back toward infinitely large r. Note that the height of the constant total energy line above $V(r)$ at any r, such as r_5 in Figure 8-6, is equal to $\frac{1}{2}\mu\dot{r}^2$. Thus the radial velocity $\dot{r}$ vanishes and changes sign at the turning point (or points).

*The expression is more readily recognized in the form $F_c = mr\omega^2$. The first real appreciation of centrifugal force was by Huygens, who made a detailed examination in his study of the conical pendulum in 1659.

†See Section 10.3 for a more critical discussion of centrifugal force.

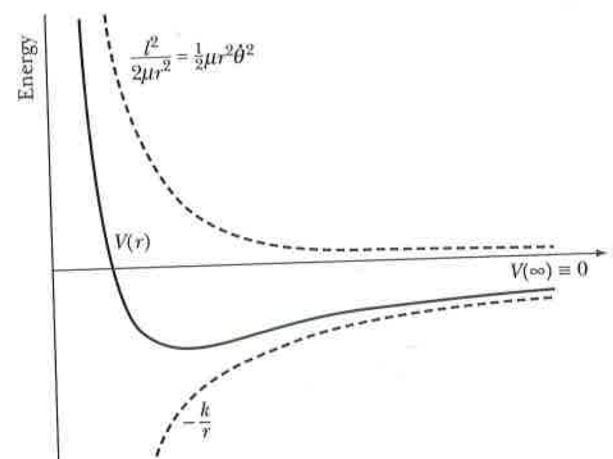

FIGURE 8-5 The effective potential for gravitational attraction $V(r)$ is composed of the real potential $-k/r$ term and the centrifugal potential energy $\ell^2/2\mu r^2$.

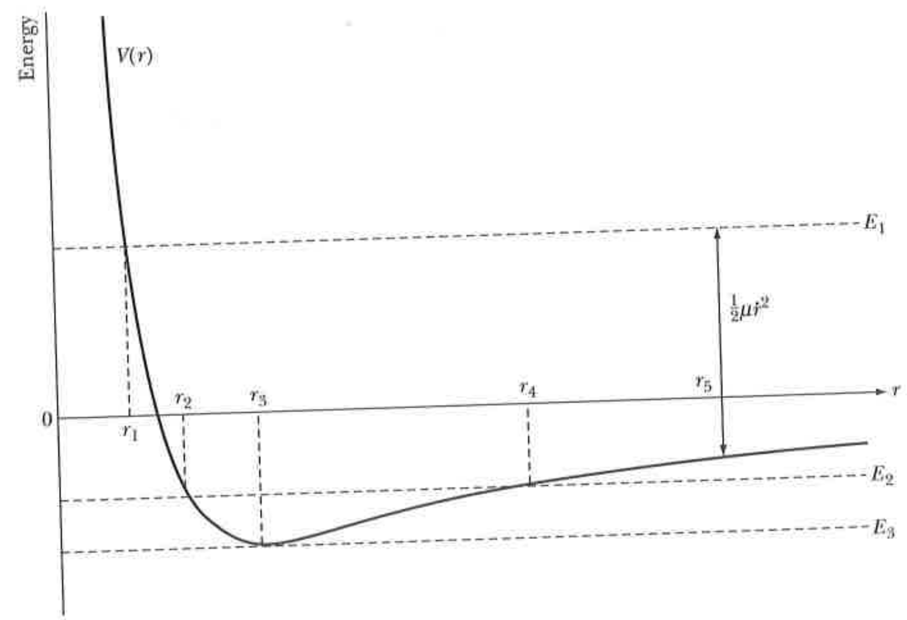

FIGURE 8-6 We can tell much about motion by looking at the total energy E on a potential energy plot. For example, for energy E_1 the particle's motion is unbounded. For energy E_2 the particle is bounded with $r_2 \leq r \leq r_4$. For energy E_3 the motion has $r = r_3$ and is circular.

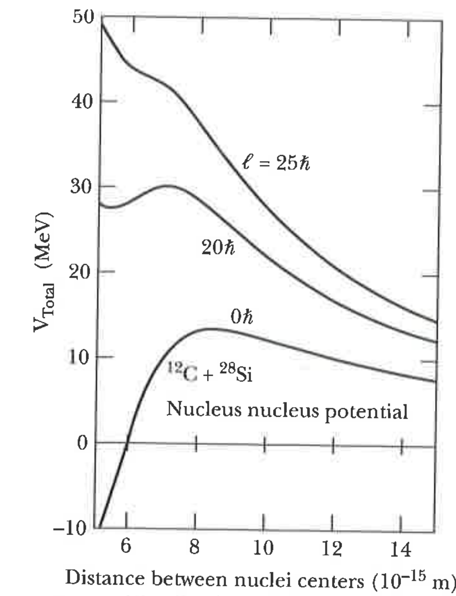

FIGURE 8-7 The total potential (coulomb, nuclear, and centrifugal) for scattering ^{28}Si nuclei from ^{12}C for various angular momentum l values as a function of distance between nuclei. For $l = 20\hbar$ a shallow pocket exists where the two nuclei may be bound together for a short time. For $l = 25\hbar$ the nuclei are not bound together.

If the total energy is negative* and lies between zero and the minimum value of $V(r)$, as does E_2, then the motion is bounded, with $r_2 \leq r \leq r_4$. The values r_2 and r_4 are the turning points, or the **apsidal distances,** of the orbit. If E equals the minimum value of the effective potential energy (see E_3 in Figure 8-6), then the radius of the particle's path is limited to the single value r_3, and then $\dot{r} = 0$ for all values of the time; hence the motion is circular.

Values of E less than $V_{\min} = -(\mu k^2/2l^2)$ do not result in physically real motion; for such cases $\dot{r}^2 < 0$ and the velocity is imaginary.

The methods discussed in this section are often used in present-day research in general fields, especially atomic, molecular, and nuclear physics. For example, Figure 8-7 shows effective total nucleus-nucleus potentials for the scattering of ^{28}Si and ^{12}C. The total potential includes the coulomb, nuclear, and the centrifugal contributions. The potential for $l = 0\hbar$ indicates the potential with no centrifugal term. For a relative angular momentum value of $l = 20\hbar$, a "pocket" exists where the two scattering nuclei may be bound together (even if only for a short time). For $l = 25\hbar$, the centrifugal "barrier" dominates, and the nuclei cannot form a bound state at all.

*Note that negative values of the total energy arise only because of the arbitrary choice of $V(r) = 0$ at $r = \infty$.