

made to roll around in a circle tangent to the path and of arbitrary radius. At the end of the process, x , y , and θ have been returned to their original values, but θ has changed by an amount depending on the radius of the circle.

Nonintegrable differential constraints of the form of Eqs. (1.39) are of course not the only type of nonholonomic constraints. The constraint conditions may involve higher-order derivatives, or may appear in the form of inequalities, as we have seen.

Partly because the dependent coordinates can be eliminated, problems involving holonomic constraints are always amenable to a formal solution. But there is no general way to attack nonholonomic examples. True, if the constraint is nonintegrable, the differential equations of constraint can be introduced into the problem along with the differential equations of motion, and the dependent equations eliminated, in effect, by the method of Lagrange multipliers.

We shall return to this method at a later point. However, the more vicious cases of nonholonomic constraint must be tackled individually, and consequently in the development of the more formal aspects of classical mechanics, it is almost invariably assumed that any constraint, if present, is holonomic. This restriction does not greatly limit the applicability of the theory, despite the fact that many of the constraints encountered in everyday life are nonholonomic. The reason is that the entire concept of constraints imposed in the system through the medium of wires or surfaces or walls is particularly appropriate only in macroscopic or large-scale problems. But today physicists are more interested in atomic and nuclear problems. On this scale all objects, both in and out of the system, consist alike of molecules, atoms, or smaller particles, exerting definite forces, and the notion of constraint becomes artificial and rarely appears. Constraints are then used only as mathematical idealizations to the actual physical case or as classical approximations to a quantum-mechanical property, e.g., rigid body rotations for "spin." Such constraints are always holonomic and fit smoothly into the framework of the theory.

To surmount the second difficulty, namely, that the forces of constraint are unknown *a priori*, we should like to so formulate the mechanics that the forces of constraint disappear. We need then deal only with the known applied forces. A hint as to the procedure to be followed is provided by the fact that in a particular system with constraints, i.e., a rigid body, the work done by internal forces (which are here the forces of constraint) vanishes. We shall follow up this clue in the ensuing sections and generalize the ideas contained in it.

D'ALEMBERT'S PRINCIPLE AND LAGRANGE'S EQUATIONS

A virtual (infinitesimal) displacement of a system refers to a change in the configuration of the system as the result of any arbitrary infinitesimal change of the coordinates $\delta\mathbf{r}_i$, *consistent with the forces and constraints imposed on the system at the given instant t*. The displacement is called virtual to distinguish it from an actual displacement of the system occurring in a time interval dt , during which

the forces and constraints may be changing. Suppose the system is in equilibrium; i.e., the total force on each particle vanishes, $\mathbf{F}_i = 0$. Then clearly the dot product $\mathbf{F}_i \cdot \delta\mathbf{r}_i$, which is the virtual work of the force \mathbf{F}_i in the displacement $\delta\mathbf{r}_i$, also vanishes. The sum of these vanishing products over all particles must likewise be zero:

$$\sum_i \mathbf{F}_i \cdot \delta\mathbf{r}_i = 0. \quad (1.40)$$

As yet nothing has been said that has any new physical content. Decompose \mathbf{F}_i into the applied force, $\mathbf{F}_i^{(a)}$, and the force of constraint, \mathbf{f}_i ,

$$\mathbf{F}_i = \mathbf{F}_i^{(a)} + \mathbf{f}_i, \quad (1.41)$$

so that Eq. (1.40) becomes

$$\sum_i \mathbf{F}_i^{(a)} \cdot \delta\mathbf{r}_i + \sum_i \mathbf{f}_i \cdot \delta\mathbf{r}_i = 0. \quad (1.42)$$

We now restrict ourselves to systems for which *the net virtual work of the forces of constraint is zero*. We have seen that this condition holds true for rigid bodies and it is valid for a large number of other constraints. Thus, if a particle is constrained to move on a surface, the force of constraint is perpendicular to the surface, while the virtual displacement must be tangent to it, and hence the virtual work vanishes. This is no longer true if sliding friction forces are present, and we must exclude such systems from our formulation. The restriction is not unduly hampering, since the friction is essentially a macroscopic phenomenon. On the other hand, the forces of rolling friction do not violate this condition, since the forces act on a point that is momentarily at rest and can do no work in an infinitesimal displacement consistent with the rolling constraint. Note that if a particle is constrained to a surface that is itself moving in time, the force of constraint is instantaneously perpendicular to the surface and the work during a virtual displacement is still zero even though the work during an actual displacement in the time dt does not necessarily vanish.

We therefore have as the condition for equilibrium of a system that the virtual work of the *applied forces* vanishes:

$$\sum_i \mathbf{F}_i^{(a)} \cdot \delta\mathbf{r}_i = 0. \quad (1.43)$$

Equation (1.43) is often called the *principle of virtual work*. Note that the coefficients of $\delta\mathbf{r}_i$ can no longer be set equal to zero; i.e., in general $\mathbf{F}_i^{(a)} \neq 0$, since the $\delta\mathbf{r}_i$ are not completely independent but are connected by the constraints. In order to equate the coefficients to zero, we must transform the principle into a form involving the virtual displacements of the q_i , which are independent. Equation (1.43) satisfies our needs in that it does not contain the \mathbf{f}_i , but it deals only with statics; we want a condition involving the general motion of the system.

To obtain such a principle, we use a device first thought of by James Bernoulli and developed by D'Alembert. The equation of motion,

$$\mathbf{F}_i = \dot{\mathbf{p}}_i,$$

can be written as

$$\mathbf{F}_i - \dot{\mathbf{p}}_i = 0,$$

which states that the particles in the system will be in equilibrium under a force equal to the actual force plus a “reversed effective force” $-\dot{\mathbf{p}}_i$. Instead of (1.40) we can immediately write

$$\sum_i (\mathbf{F}_i - \dot{\mathbf{p}}_i) \cdot \delta \mathbf{r}_i = 0, \quad (1.44)$$

and, making the same resolution into applied forces and forces of constraint, the results

$$\sum_i (\mathbf{F}_i^{(a)} - \dot{\mathbf{p}}_i) \cdot \delta \mathbf{r}_i + \sum_i \mathbf{f}_i \cdot \delta \mathbf{r}_i = 0.$$

We again restrict ourselves to systems for which the virtual work of the forces of constraint vanishes and therefore obtain

$$\sum_i (\mathbf{F}_i^{(a)} - \dot{\mathbf{p}}_i) \cdot \delta \mathbf{r}_i = 0, \quad (1.45)$$

which is often called *D'Alembert's principle*. We have achieved our aim, in that the forces of constraint no longer appear, and the superscript ^(a) can now be dropped without ambiguity. It is still not in a useful form to furnish equations of motion for the system. We must now transform the principle into an expression involving virtual displacements of the generalized coordinates, which are then independent of each other (for holonomic constraints), so that the coefficients of the δq_i can be set separately equal to zero.

The translation from \mathbf{r}_i to q_j language starts from the transformation equation (1.38),

$$\mathbf{r}_i = \mathbf{r}_i(q_1, q_2, \dots, q_n, t) \quad (1.46)$$

(assuming n independent coordinates), and is carried out by means of the usual “chain rules” of the calculus of partial differentiation. Thus, \mathbf{v}_i is expressed in terms of the \dot{q}_k by the formula

$$\mathbf{v}_i \equiv \frac{d\mathbf{r}_i}{dt} = \sum_k \frac{\partial \mathbf{r}_i}{\partial q_k} \dot{q}_k + \frac{\partial \mathbf{r}_i}{\partial t}. \quad (1.47)$$

1.4 D'Alembert's Principle and Lagrange's Equations

Similarly, the arbitrary virtual displacement $\delta \mathbf{r}_i$ can be connected with the virtual displacements δq_i by

$$\delta \mathbf{r}_i = \sum_j \frac{\partial \mathbf{r}_i}{\partial q_j} \delta q_j. \quad (1.47)$$

Note that no variation of time, δt , is involved here, since a virtual displacement by definition considers only displacements of the coordinates. (Only then is the virtual displacement perpendicular to the force of constraint if the constraint itself is changing in time.)

In terms of the generalized coordinates, the virtual work of the \mathbf{F}_i becomes

$$\begin{aligned} \sum_i \mathbf{F}_i \cdot \delta \mathbf{r}_i &= \sum_{i,j} \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} \delta q_j \\ &= \sum_j Q_j \delta q_j, \end{aligned} \quad (1.48)$$

where the Q_j are called the components of the *generalized force*, defined as

$$Q_j = \sum_i \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j}. \quad (1.49)$$

Note that just as the q 's need not have the dimensions of length, so the Q 's do not necessarily have the dimensions of force, but $Q_j \delta q_j$ must always have the dimensions of work. For example, Q_j might be a torque N_j and dq_j a differential angle $d\theta_j$, which makes $N_j d\theta_j$ a differential of work.

We turn now to the other term involved in Eq. (1.45), which may be written as

$$\sum_i \dot{\mathbf{p}}_i \cdot \delta \mathbf{r}_i = \sum_i m_i \ddot{\mathbf{r}}_i \cdot \delta \mathbf{r}_i.$$

Expressing $\delta \mathbf{r}_i$ by (1.47), this becomes

$$\sum_{i,j} m_i \ddot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} \delta q_j.$$

Consider now the relation

$$\sum_i m_i \ddot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} = \sum_i \left[\frac{d}{dt} \left(m_i \dot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} \right) - m_i \dot{\mathbf{r}}_i \cdot \frac{d}{dt} \left(\frac{\partial \mathbf{r}_i}{\partial q_j} \right) \right]. \quad (1.50)$$

In the last term of Eq. (1.50) we can interchange the differentiation with respect to t and q_j , for, in analogy to (1.46),

$$\begin{aligned} \frac{d}{dt} \left(\frac{\partial \mathbf{r}_i}{\partial q_j} \right) &= \frac{\partial \dot{\mathbf{r}}_i}{\partial q_j} = \sum_k \frac{\partial^2 \mathbf{r}_i}{\partial q_j \partial q_k} \dot{q}_k + \frac{\partial^2 \mathbf{r}_i}{\partial q_j \partial t}, \\ &= \frac{\partial \mathbf{v}_i}{\partial q_j}, \end{aligned}$$

by Eq. (1.46). Further, we also see from Eq. (1.46) that

$$\frac{\partial \mathbf{v}_i}{\partial \dot{q}_j} = \frac{\partial \mathbf{r}_i}{\partial q_j}. \quad (1.51)$$

Substitution of these changes in (1.50) leads to the result that

$$\sum_i m_i \ddot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} = \sum_i \left[\frac{d}{dt} \left(m_i \mathbf{v}_i \cdot \frac{\partial \mathbf{v}_i}{\partial \dot{q}_j} \right) - m_i \mathbf{v}_i \cdot \frac{\partial \mathbf{v}_i}{\partial q_j} \right],$$

and the second term on the left-hand side of Eq. (1.45) can be expanded into

$$\sum_j \left\{ \frac{d}{dt} \left[\frac{\partial}{\partial \dot{q}_j} \left(\sum_i \frac{1}{2} m_i v_i^2 \right) \right] - \frac{\partial}{\partial q_j} \left(\sum_i \frac{1}{2} m_i v_i^2 \right) - Q_j \right\} \delta q_j.$$

Identifying $\sum_i \frac{1}{2} m_i v_i^2$ with the system kinetic energy T , D'Alembert's principle (cf. Eq. (1.45)) becomes

$$\sum_j \left\{ \left[\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} \right] - Q_j \right\} \delta q_j = 0. \quad (1.52)$$

Note that in a system of Cartesian coordinates the partial derivative of T with respect to q_j vanishes. Thus, speaking in the language of differential geometry, this term arises from the curvature of the coordinates q_j . In polar coordinates, e.g., it is in the partial derivative of T with respect to the angle coordinate that the centripetal acceleration term appears.

Thus far, no restriction has been made on the nature of the constraints other than that they be workless in a virtual displacement. The variables q_j can be any set of coordinates used to describe the motion of the system. If, however, the constraints are holonomic, then it is possible to find sets of independent coordinates q_j that contain the constraint conditions implicitly in the transformation equations (1.38). Any virtual displacement δq_j is then independent of δq_k , and therefore the only way for (1.52) to hold is for the individual coefficients to vanish:

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} = Q_j. \quad (1.53)$$

There are n such equations in all.

When the forces are derivable from a scalar potential function V ,

$$\mathbf{F}_i = -\nabla_i V.$$

1.4 D'Alembert's Principle and Lagrange's Equations

Then the generalized forces can be written as

$$Q_j = \sum_i \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} = - \sum_i \nabla_i V \cdot \frac{\partial \mathbf{r}_i}{\partial q_j},$$

which is exactly the same expression for the partial derivative of a function $-V(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N, t)$ with respect to q_j :

$$Q_j \equiv - \frac{\partial V}{\partial q_j}. \quad (1.54)$$

See Eq. (1.47). Equations (1.53) can then be rewritten as

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial (T - V)}{\partial q_j} = 0. \quad (1.55)$$

The equations of motion in the form (1.55) are not necessarily restricted to conservative systems; only if V is not an explicit function of time is the system conservative (cf. p. 4). As here defined, the potential V does not depend on the generalized velocities. Hence, we can include a term in V in the partial derivative with respect to \dot{q}_j :

$$\frac{d}{dt} \left(\frac{\partial (T - V)}{\partial \dot{q}_j} \right) - \frac{\partial (T - V)}{\partial q_j} = 0.$$

Or, defining a new function, the *Lagrangian* L , as

$$L = T - V, \quad (1.56)$$

the Eqs. (1.53) become

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_j} \right) - \frac{\partial L}{\partial q_j} = 0, \quad (1.57)$$

expressions referred to as "Lagrange's equations."

Note that for a particular set of equations of motion there is no unique choice of Lagrangian such that Eqs. (1.57) lead to the equations of motion in the given generalized coordinates. Thus, in Derivations 8 and 10 it is shown that if $L(q, \dot{q}, t)$ is an approximate Lagrangian and $F(q, t)$ is *any* differentiable function of the generalized coordinates and time, then

$$L'(q, \dot{q}, t) = L(q, \dot{q}, t) + \frac{dF}{dt} \quad (1.57')$$

is a Lagrangian also resulting in the same equations of motion. It is also often possible to find alternative Lagrangians beside those constructed by this prescription (see Exercise 20). While Eq. (1.56) is always a suitable way to construct a Lagrangian for a conservative system, it does not provide the *only* Lagrangian suitable for the given system.

VELOCITY-DEPENDENT POTENTIALS AND THE DISSIPATION FUNCTION

Lagrange's equations can be put in the form (1.57) even if there is no potential function, V , in the usual sense, providing the generalized forces are obtained from a function $U(q_j, \dot{q}_j)$ by the prescription

$$Q_j = -\frac{\partial U}{\partial q_j} + \frac{d}{dt} \left(\frac{\partial U}{\partial \dot{q}_j} \right). \quad (1.58)$$

In such case, Eqs. (1.57) still follow from Eqs. (1.53) with the Lagrangian given by

$$L = T - U. \quad (1.59)$$

Here U may be called a "generalized potential," or "velocity-dependent potential." The possibility of using such a "potential" is not academic; it applies to one very important type of force field, namely, the electromagnetic forces on moving charges. Considering its importance, a digression on this subject is well worthwhile.

Consider an electric charge, q , of mass m moving at a velocity, \mathbf{v} , in an otherwise charge-free region containing both an electric field, \mathbf{E} , and a magnetic field, \mathbf{B} , which may depend upon time and position. The charge experiences a force, called the Lorentz force, given by

$$\mathbf{F} = q[\mathbf{E} + (\mathbf{v} \times \mathbf{B})]. \quad (1.60)$$

Both $\mathbf{E}(x, y, z, t)$ and $\mathbf{B}(x, y, z, t)$ are continuous functions of time and position derivable from a scalar potential $\phi(x, y, z, t)$ and a vector potential $\mathbf{A}(x, y, z, t)$ by

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} \quad (1.61a)$$

$$\mathbf{B} = \nabla \times \mathbf{A}. \quad (1.61b)$$

The force on the charge can be derived from the following velocity-dependent potential energy

$$U = q\phi - q\mathbf{A} \cdot \mathbf{v}, \quad (1.62)$$

so the Lagrangian, $L = T - U$, is

$$L = \frac{1}{2}m\mathbf{v}^2 - q\phi + q\mathbf{A} \cdot \mathbf{v}. \quad (1.63)$$

1.5 Velocity-Dependent Potentials and the Dissipation Function

Considering just the x -component of Lagrange's equations gives

$$m\ddot{x} = q \left(v_x \frac{\partial A_x}{\partial x} + v_y \frac{\partial A_y}{\partial x} + v_z \frac{\partial A_z}{\partial x} \right) - q \left(\frac{\partial \phi}{\partial x} + \frac{dA_x}{dt} \right). \quad (1.64)$$

The total time derivative of A_x is related to the partial time derivative through

$$\begin{aligned} \frac{dA_x}{dt} &= \frac{\partial A_x}{\partial t} + \mathbf{v} \cdot \nabla A_x \\ &= \frac{\partial A_x}{\partial t} + v_x \frac{\partial A_x}{\partial x} + v_y \frac{\partial A_x}{\partial y} + v_z \frac{\partial A_x}{\partial z}. \end{aligned} \quad (1.65)$$

Equation (1.61b) gives

$$(\mathbf{v} \times \mathbf{B})_x = v_y \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) + v_z \left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z} \right).$$

Combining these expressions gives the equation of motion in the x -direction

$$m\ddot{x} = q [E_x + (\mathbf{v} \times \mathbf{B})_x]. \quad (1.66)$$

On a component-by-component comparison, Eqs. (1.66) and (1.60) are identical, showing that the Lorentz force equation is derivable from Eqs. (1.61) and (1.62).

Note that if not all the forces acting on the system are derivable from a potential, then Lagrange's equations can always be written in the form

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_j} \right) - \frac{\partial L}{\partial q_j} = Q_j,$$

where L contains the potential of the conservative forces as before, and Q_j represents the forces *not* arising from a potential. Such a situation often occurs when frictional forces are present. It frequently happens that the frictional force is proportional to the velocity of the particle, so that its x -component has the form

$$F_{fx} = -k_x v_x.$$

Frictional forces of this type may be derived in terms of a function \mathcal{F} , known as *Rayleigh's dissipation function*, and defined as

$$\mathcal{F} = \frac{1}{2} \sum_i (k_x v_{ix}^2 + k_y v_{iy}^2 + k_z v_{iz}^2), \quad (1.67)$$

where the summation is over the particles of the system. From this definition it is clear that

$$F_{fx_i} = -\frac{\partial \mathcal{F}}{\partial v_{xi}},$$

or, symbolically,

$$\mathbf{F}_f = -\nabla_v \mathcal{F}. \quad (1.68)$$

We can also give a physical interpretation to the dissipation function. The work done by the system *against* friction is

$$dW_f = -\mathbf{F}_f \cdot d\mathbf{r} = -\mathbf{F}_f \cdot \mathbf{v} dt = \left(k_x v_x^2 + k_y v_y^2 + k_z v_z^2 \right) dt.$$

Hence, $2\mathcal{F}$ is the rate of energy dissipation due to friction. The component of the generalized force resulting from the force of friction is then given by

$$\begin{aligned} Q_j &= \sum_i \mathbf{F}_{f_i} \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} = - \sum \nabla_v \mathcal{F} \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} \\ &= - \sum \nabla_v \mathcal{F} \cdot \frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_j}, \quad \text{by (1.51),} \\ &= - \frac{\partial \mathcal{F}}{\partial \dot{q}_j}. \end{aligned} \quad (1.69)$$

An example is Stokes' law, whereby a sphere of radius a moving at a speed v , in a medium of viscosity η experiences the frictional drag force $\mathbf{F}_f = -6\pi\eta av$. The Lagrange equations with dissipation become

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_j} \right) - \frac{\partial L}{\partial q_j} + \frac{\partial \mathcal{F}}{\partial \dot{q}_j} = 0, \quad (1.70)$$

so that two scalar functions, L and \mathcal{F} , must be specified to obtain the equations of motion.