CHAPTER 3

The Central Force Problem

In this chapter we shall discuss the problem of two bodies moving under the influence of a mutual central force as an application of the Lagrangian formulation. Not all the problems of central force motion are integrable in terms of well-known functions. However, we shall attempt to explore the problem as thoroughly as is possible with the tools already developed. In the last section of this chapter we consider some of the complications that follow by the presence of a third body.

3.1 ■ REDUCTION TO THE EQUIVALENT ONE-BODY PROBLEM

Consider a monogenic system of two mass points, m_1 and m_2 (cf. Fig. 3.1), where the only forces are those due to an interaction potential U. We will assume at first that U is any function of the vector between the two particles, $\mathbf{r}_2 - \mathbf{r}_1$, or of their relative velocity, $\dot{\mathbf{r}}_2 - \dot{\mathbf{r}}_1$, or of any higher derivatives of $\mathbf{r}_2 - \mathbf{r}_1$. Such a system has six degrees of freedom and hence six independent generalized coordinates. We choose these to be the three components of the radius vector to the center of mass, $\mathbf{R}$, plus the three components of the difference vector $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$. The Lagrangian will then have the form

$$L = T(\dot{\mathbf{R}}, \dot{\mathbf{r}}) - U(\mathbf{r}, \dot{\mathbf{r}}, \ldots). \tag{3.1}$$

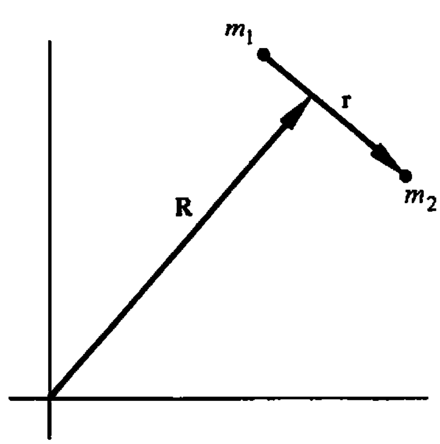

FIGURE 3.1 Coordinates for the two-body problem.

The kinetic energy T can be written as the sum of the kinetic energy of the motion of the center of mass, plus the kinetic energy of motion about the center of mass, T':

$$T = \tfrac{1}{2}(m_1 + m_2)\dot{\mathbf{R}}^2 + T'$$

with

$$T' = \tfrac{1}{2}m_1\dot{\mathbf{r}}_1'^2 + \tfrac{1}{2}m_2\dot{\mathbf{r}}_2'^2.$$

Here $\mathbf{r}_1'$ and $\mathbf{r}_2'$ are the radii vectors of the two particles relative to the center of mass and are related to $\mathbf{r}$ by

$$\begin{aligned} \mathbf{r}_1' &= -\frac{m_2}{m_1 + m_2}\mathbf{r}, \\ \mathbf{r}_2' &= \frac{m_1}{m_1 + m_2}\mathbf{r} \end{aligned} \tag{3.2}$$

Expressed in terms of $\mathbf{r}$ by means of Eq. (3.2), T' takes on the form

$$T' = \frac{1}{2}\frac{m_1 m_2}{m_1 + m_2}\dot{\mathbf{r}}^2$$

and the total Lagrangian (3.1) is

$$L = \frac{m_1 + m_2}{2}\dot{\mathbf{R}}^2 + \frac{1}{2}\frac{m_1 m_2}{m_1 + m_2}\dot{\mathbf{r}}^2 - U(\mathbf{r}, \dot{\mathbf{r}}, \ldots). \tag{3.3}$$

It is seen that the three coordinates $\mathbf{R}$ are cyclic, so that the center of mass is either at rest or moving uniformly. None of the equations of motion for $\mathbf{r}$ will contain terms involving $\mathbf{R}$ or $\dot{\mathbf{R}}$. Consequently, the process of integration is particularly simple here. We merely drop the first term from the Lagrangian in all subsequent discussion.

The rest of the Lagrangian is exactly what would be expected if we had a fixed center of force with a single particle at a distance $\mathbf{r}$ from it, having a mass

$$\mu = \frac{m_1 m_2}{m_1 + m_2}, \tag{3.4}$$

where μ is known as the *reduced mass*. Frequently, Eq. (3.4) is written in the form

$$\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}. \tag{3.5}$$

Thus, the central force motion of two bodies about their center of mass can always be reduced to an equivalent one-body problem.

3.2 ■ THE EQUATIONS OF MOTION AND FIRST INTEGRALS

We now restrict ourselves to conservative central forces, where the potential is $V(r)$, a function of r only, so that the force is always along $\mathbf{r}$. By the results of the preceding section, we need only consider the problem of a single particle of reduced mass m moving about a fixed center of force, which will be taken as the origin of the coordinate system. Since potential energy involves only the radial distance, the problem has spherical symmetry; i.e., any rotation, about any fixed axis, can have no effect on the solution. Hence, an angle coordinate representing rotation about a fixed axis must be cyclic. These symmetry properties result in a considerable simplification in the problem.

Since the problem is spherically symmetric, the total angular momentum vector,

$$\mathbf{L} = \mathbf{r} \times \mathbf{p},$$

is conserved. It therefore follows that $\mathbf{r}$ is always perpendicular to the fixed direction of $\mathbf{L}$ in space. This can be true only if $\mathbf{r}$ always lies in a plane whose normal is parallel to $\mathbf{L}$. While this reasoning breaks down if $\mathbf{L}$ is zero, the motion in that case must be along a straight line going through the center of force, for $\mathbf{L} = 0$ requires $\mathbf{r}$ to be parallel to $\dot{\mathbf{r}}$, which can be satisfied only in straight-line motion.* Thus, central force motion is always motion in a plane.

Now, the motion of a single particle in space is described by three coordinates; in spherical polar coordinates these are the azimuth angle θ, the zenith angle (or colatitude) ψ, and the radial distance r. By choosing the polar axis to be in the direction of $\mathbf{L}$, the motion is always in the plane perpendicular to the polar axis. The coordinate ψ then has only the constant value $\pi/2$ and can be dropped from the subsequent discussion. The conservation of the angular momentum vector furnishes three independent constants of motion (corresponding to the three Cartesian components). In effect, two of these, expressing the constant *direction* of the angular momentum, have been used to reduce the problem from three to two degrees of freedom. The third of these constants, corresponding to the conservation of the magnitude of $\mathbf{L}$, remains still at our disposal in completing the solution.

Expressed now in plane polar coordinates, the Lagrangian is

$$\begin{aligned} L &= T - V \\ &= \tfrac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - V(r). \end{aligned} \tag{3.6}$$

As was forseen, θ is a cyclic coordinate, whose corresponding canonical momentum is the angular momentum of the system:

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}.$$

*Formally $\dot{\mathbf{r}} = \dot{r}\mathbf{n}_r + r\dot{\theta}\mathbf{n}_\theta$, hence $\mathbf{r} \times \dot{\mathbf{r}} = 0$ requires $\dot{\theta} = 0$.

One of the two equations of motion is then simply

$$\dot{p}_\theta = \frac{d}{dt}\left(mr^2\dot{\theta}\right) = 0. \tag{3.7}$$

with the immediate integral

$$mr^2\dot{\theta} = l. \tag{3.8}$$

where l is the constant magnitude of the angular momentum. From (3.7) is also follows that

$$\frac{d}{dt}\left(\frac{1}{2}r^2\dot{\theta}\right) = 0. \tag{3.9}$$

The factor $\frac{1}{2}$ is inserted because $\frac{1}{2}r^2\dot{\theta}$ is just the *areal velocity*—the area swept out by the radius vector per unit time. This interpretation follows from Fig. 3.2, the differential area swept out in time dt being

$$dA = \tfrac{1}{2}r(r\,d\theta),$$

and hence

$$\frac{dA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt}.$$

The conservation of angular momentum is thus equivalent to saying the areal velocity is constant. Here we have the proof of the well-known Kepler's second law of planetary motion: The radius vector sweeps out equal areas in equal times. It should be emphasized however that the conservation of the areal velocity is a general property of central force motion and is not restricted to an inverse-square law of force.

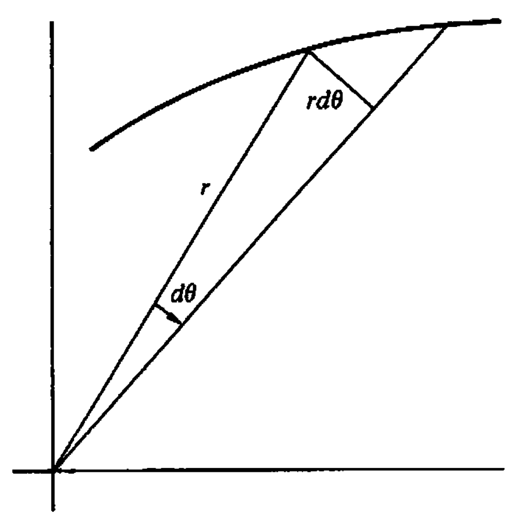

FIGURE 3.2 The area swept out by the radius vector in a time dt.

The remaining Lagrange equation, for the coordinate r, is

$$\frac{d}{dt}(m\dot{r}) - mr\dot{\theta}^2 + \frac{\partial V}{\partial r} = 0. \tag{3.10}$$

Designating the value of the force along $\mathbf{r}$, $-\partial V/\partial r$, by $f(r)$ the equation can be rewritten as

$$m\ddot{r} - mr\dot{\theta}^2 = f(r). \tag{3.11}$$

By making use of the first integral, Eq. (3.8), $\dot{\theta}$ can be eliminated from the equation of motion, yielding a second-order differential equation involving r only:

$$m\ddot{r} - \frac{l^2}{mr^3} = f(r). \tag{3.12}$$

There is another first integral of motion available, namely the total energy, since the forces are conservative. On the basis of the general energy conservation theorem, we can immediately state that a constant of the motion is

$$E = \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + V(r), \tag{3.13}$$

where E is the energy of the system. Alternatively, this first integral could be derived again directly from the equations of motion (3.7) and (3.12). The latter can be written as

$$m\ddot{r} = -\frac{d}{dr}\left(V + \frac{1}{2}\frac{l^2}{mr^2}\right). \tag{3.14}$$

If both sides of Eq. (3.14) are multiplied by $\dot{r}$ the left side becomes

$$m\ddot{r}\dot{r} = \frac{d}{dt}\left(\frac{1}{2}m\dot{r}^2\right).$$

The right side similarly can be written as a total time derivative, for if $g(r)$ is any function of r, then the total time derivative of g has the form

$$\frac{d}{dt}g(r) = \frac{dg}{dr}\frac{dr}{dt}.$$

Hence, Eq. (3.14) is equivalent to

$$\frac{d}{dt}\left(\frac{1}{2}m\dot{r}^2\right) = -\frac{d}{dt}\left(V + \frac{1}{2}\frac{l^2}{mr^2}\right)$$

or

$$\frac{d}{dt}\left(\frac{1}{2}m\dot{r}^2 + \frac{1}{2}\frac{l^2}{mr^2} + V\right) = 0,$$

and therefore

$$\frac{1}{2}m\dot{r}^2 + \frac{1}{2}\frac{l^2}{mr^2} + V = \text{constant}. \tag{3.15}$$

Equation (3.15) is the statement of the conservation of total energy, for by using (3.8) for l, the middle term can be written

$$\frac{1}{2}\frac{l^2}{mr^2} = \frac{1}{2mr^2}m^2r^4\dot{\theta}^2 = \frac{mr^2\dot{\theta}^2}{2}.$$

and (3.15) reduces to (3.13).

These first two integrals give us in effect two of the quadratures necessary to complete the problem. As there are two variables, r and θ, a total of four integrations are needed to solve the equations of motion. The first two integrations have left the Lagrange equations as two first-order equations (3.8) and (3.15); the two remaining integrations can be accomplished (formally) in a variety of ways. Perhaps the simplest procedure starts from Eq. (3.15). Solving for $\dot{r}$, we have

$$\dot{r} = \sqrt{\frac{2}{m}\left(E - V - \frac{l^2}{2mr^2}\right)}, \tag{3.16}$$

or

$$dt = \frac{dr}{\sqrt{\frac{2}{m}\left(E - V - \frac{l^2}{2mr^2}\right)}}. \tag{3.17}$$

At time $t = 0$, let r have the initial value r_0. Then the integral of both sides of the equation from the initial state to the state at time t takes the form

$$t = \int_{r_0}^{r} \frac{dr}{\sqrt{\frac{2}{m}\left(E - V - \frac{l^2}{2mr^2}\right)}}. \tag{3.18}$$

As it stands, Eq. (3.18) gives t as a function of r and the constants of integration E, l, and r_0. However, it may be inverted, at least formally, to give r as a function of t and the constants. Once the solution for r is found, the solution θ follows immediately from Eq. (3.8), which can be written as

$$d\theta = \frac{l\,dt}{mr^2}. \tag{3.19}$$

If the initial value of θ is θ_0, then the integral of (3.19) is simply

$$\theta = l\int_0^t \frac{dt}{mr^2(t)} + \theta_0. \tag{3.20}$$

Equations (3.18) and (3.20) are the two remaining integrations, and formally the problem has been reduced to quadratures, with four constants of integration E, l, r_0, θ_0. These constants are not the only ones that can be considered. We might equally as well have taken r_0, θ_0, $\dot{r}_0$, $\dot{\theta}_0$, but of course E and l can always be determined in terms of this set. For many applications, however, the set containing the energy and angular momentum is the natural one. In quantum mechanics, such constants as the initial values of r and θ, or of $\dot{r}$ and $\dot{\theta}$, become meaningless, but we can still talk in terms of the system energy or of the system angular momentum. Indeed, two salient differences between classical and quantum mechanics appear in the properties of E and l in the two theories. In order to discuss the transition to quantum theories, it is therefore important that the classical description of the system be in terms of its energy and angular momentum.

3.3 ■ THE EQUIVALENT ONE-DIMENSIONAL PROBLEM, AND CLASSIFICATION OF ORBITS

Although we have solved the one-dimensional problem formally, practically speaking the integrals (3.18) and (3.20) are usually quite unmanageable, and in any specific case it is often more convenient to perform the integration in some other fashion. But before obtaining the solution for any specific force laws, let us see what can be learned about the motion in the general case, using only the equations of motion and the conservation theorems, without requiring explicit solutions.

For example, with a system of known energy and angular momentum, the magnitude and direction of the velocity of the particle can be immediately determined in terms of the distance r. The magnitude v follows at once from the conservation of energy in the form

$$E = \tfrac{1}{2}mv^2 + V(r)$$

or

$$v = \sqrt{\frac{2}{m}\,(E - V(r))}. \tag{3.21}$$

The radial velocity—the component of $\dot{\mathbf{r}}$ along the radius vector—has been given in Eq. (3.16). Combined with the magnitude v, this is sufficient information to furnish the direction of the velocity.* These results, and much more, can also be obtained from consideration of an equivalent one-dimensional problem.

The equation of motion in r, with $\dot{\theta}$ expressed in terms of l, Eq. (3.12), involves only r and its derivatives. It is the same equation as would be obtained for a

*Alternatively, the conservation of angular momentum furnishes $\dot{\theta}$, the angular velocity, and this together with $\dot{r}$ gives both the magnitude and direction of $\dot{\mathbf{r}}$.

fictitious one-dimensional problem in which a particle of mass m is subject to a force

$$f' = f + \frac{l^2}{mr^3}. \tag{3.22}$$

The significance of the additional term is clear if it is written as $mr\dot{\theta}^2 = mv_\theta^2/r$, which is the familiar centrifugal force. An equivalent statement can be obtained from the conservation theorem for energy. By Eq. (3.15) the motion of the particle in r is that of a one-dimensional problem with a fictitious potential energy:

$$V' = V + \frac{1}{2}\frac{l^2}{mr^2}. \tag{3.22'}$$

As a check, note that

$$f' = -\frac{\partial V'}{\partial r} = f(r) + \frac{l^2}{mr^3},$$

which agrees with Eq. (3.22). The energy conservation theorem (3.15) can thus also be written as

$$E = V' + \tfrac{1}{2}m\dot{r}^2. \tag{3.15'}$$

As an illustration of this method of examining the motion, consider a plot of V' against r for the specific case of an attractive inverse-square law of force:

$$f = -\frac{k}{r^2}.$$

(For positive k, the minus sign ensures that the force is *toward* the center of force.) The potential energy for this force is

$$V = -\frac{k}{r},$$

and the corresponding fictitious potential is

$$V' = -\frac{k}{r} + \frac{l^2}{2mr^2}.$$

Such a plot is shown in Fig. 3.3; the two dashed lines represent the separate components

$$-\frac{k}{r} \quad \text{and} \quad \frac{l^2}{2mr^2},$$

and the solid line is the sum V'.

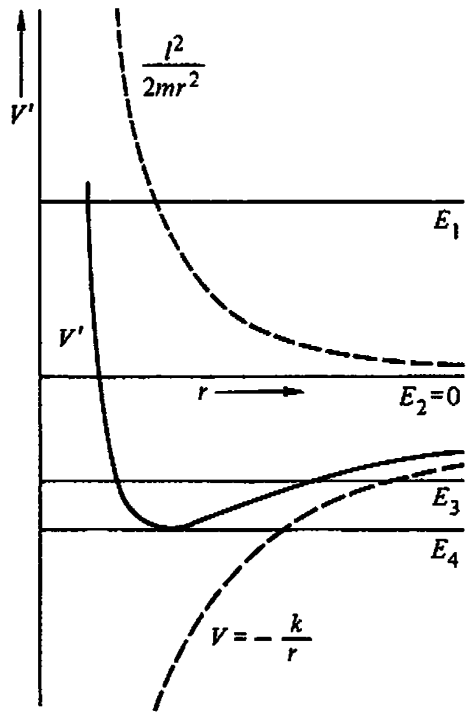

FIGURE 3.3 The equivalent one-dimensional potential for attractive inverse-square law of force.

Let us consider now the motion of a particle having the energy E_1, as shown in Figs. 3.3 and 3.4. Clearly this particle can never come closer than r_1 (cf. Fig. 3.4). Otherwise with $r < r_1$, V' exceeds E_1 and by Eq. (3.15′) the kinetic energy would have to be negative, corresponding to an imaginary velocity! On the other hand, there is no upper limit to the possible value of r, so the orbit is not bounded. A particle will come in from infinity, strike the "repulsive centrifugal barrier," be repelled, and travel back out to infinity (cf. Fig. 3.5). The distance between E and V' is $\frac{1}{2}m\dot{r}^2$, i.e., proportional to the square of the radial velocity, and becomes zero, naturally, at the *turning point* r_1. At the same time, the distance between E and V on the plot is the kinetic energy $\frac{1}{2}mv^2$ at the given value of r. Hence, the distance between the V and V' curves is $\frac{1}{2}mr^2\dot{\theta}^2$. These curves therefore supply the magnitude of the particle velocity and its components for any distance r, at the given energy and angular momentum. This information is sufficient to produce an approximate picture of the form of the orbit.

For the energy $E_2 = 0$ (cf. Fig. 3.3), a roughly similar picture of the orbit behavior is obtained. But for any lower energy, such as E_3 indicated in Fig. 3.6, we have a different story. In addition to a lower bound r_1, there is also a maximum value r_2 that cannot be exceeded by r with positive kinetic energy. The motion is then "bounded," and there are two turning points, r_1 and r_2, also known as *apsidal distances*. This does not necessarily mean that the orbits are closed. All that can be said is that they are bounded, contained between two circles of radius r_1 and r_2 with turning points always lying on the circles (cf. Fig. 3.7).

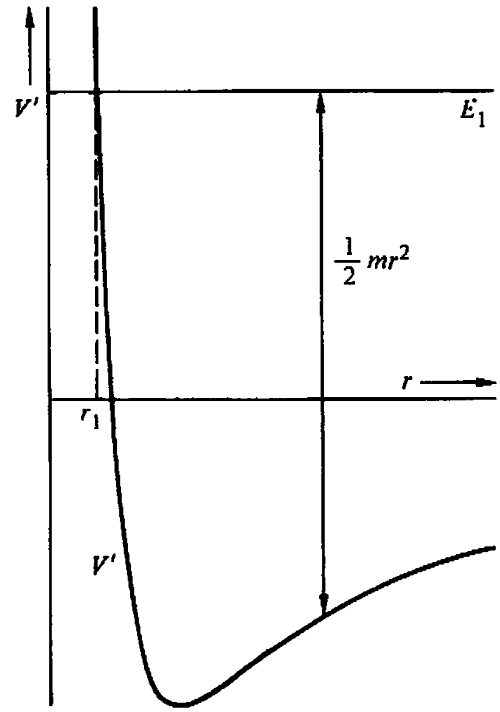

FIGURE 3.4 Unbounded motion at positive energies for inverse-square law of force.

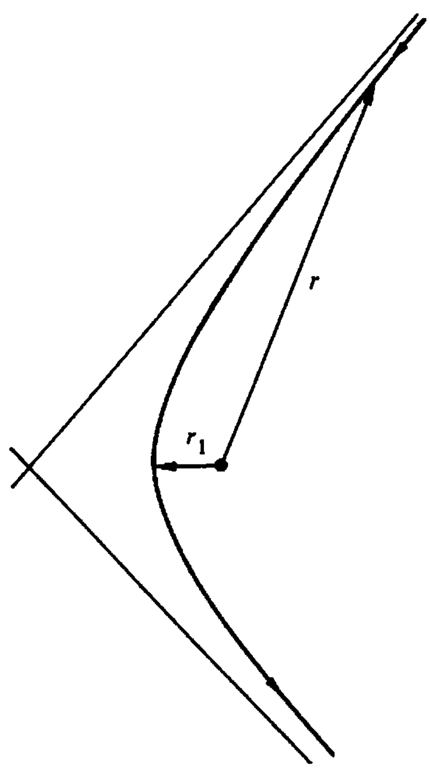

FIGURE 3.5 The orbit for E_1 corresponding to unbounded motion.

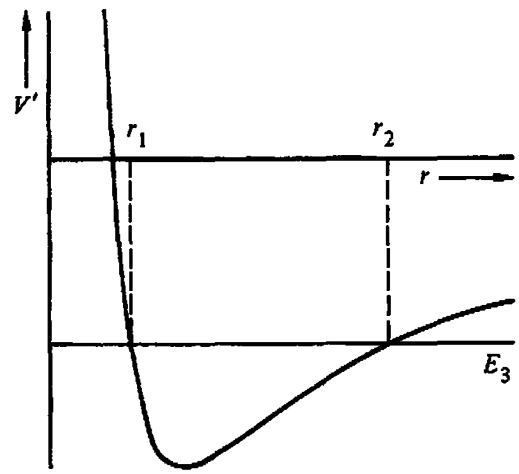

FIGURE 3.6 The equivalent one-dimensional potential for inverse-square law of force, illustrating bounded motion at negative energies.

If the energy is E_4 at the minimum of the fictitious potential as shown in Fig. 3.8, then the two bounds coincide. In such case, motion is possible at only one radius; $\dot{r} = 0$, and the orbit is a circle. Remembering that the effective "force" is the negative of the slope of the V' curve, the requirement for circular orbits is simply that f' be zero, or

$$f(r) = -\frac{l^2}{mr^3} = -mr\dot{\theta}^2.$$

We have here the familiar elementary condition for a circular orbit, that the applied force be equal and opposite to the "reversed effective force" of centripetal

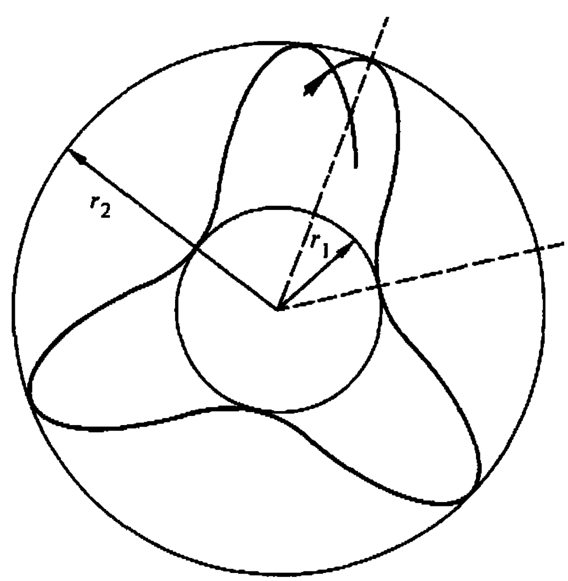

FIGURE 3.7 The nature of the orbits for bounded motion.

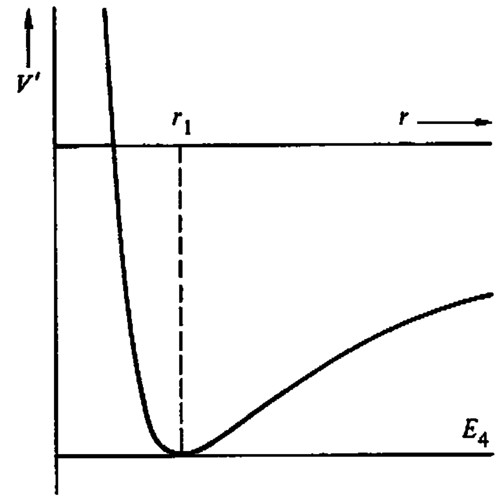

FIGURE 3.8 The equivalent one-dimensional potential of inverse-square law of force, illustrating the condition for circular orbits.

acceleration.* The properties of circular orbits and the conditions for them will be studied in greater detail in Section 3.6.

Note that all of this discussion of the orbits for various energies has been at one value of the angular momentum. Changing l changes the quantitative details of the V' curve, but it does not affect the general classification of the types of orbits.

For the attractive inverse-square law of force discussed above, we shall see that the orbit for E_1 is a hyperbola, for E_2 a parabola, and for E_3 an ellipse. With other forces the orbits may not have such simple forms. However, the same general qualitative division into open, bounded, and circular orbits will be true for any attractive potential that (1) falls off slower than $1/r^2$ as $r \to \infty$, and (2) becomes infinite slower than $1/r^2$ as $r \to 0$. The first condition ensures that the potential predominates over the centrifugal term for large r, while the second condition is such that for small r it is the centrifugal term that is important.

The qualitative nature of the motion will be altered if the potential does not satisfy these requirements, but we may still use the method of the equivalent potential to examine features of the orbits. As an example, let us consider the attractive potential

$$V(r) = -\frac{a}{r^3}, \qquad \text{with} \quad f = -\frac{3a}{r^4}.$$

The energy diagram is then as shown in Fig. 3.9. For an energy E, there are two possible types of motion, depending upon the initial value of r. If r_0 is less than r_1 the motion will be bounded, r will always remain less than r_1, and the particle will pass through the center of force. If r is initially greater than r_2, then it will

*The case $E < E_4$ does not correspond to physically possible motion, for then $\dot{r}^2$ would have to be negative, or $\dot{r}$ imaginary.

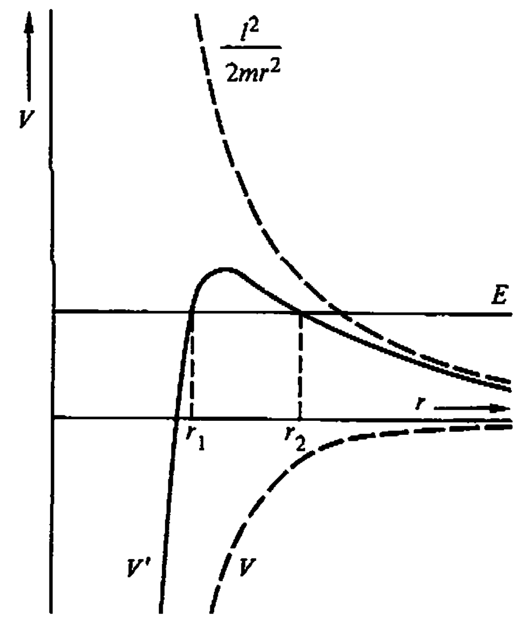

FIGURE 3.9 The equivalent one-dimensional potential for an attractive inverse-fourth law of force

always remain so; the motion is unbounded, and the particle can never get inside the "potential" hole. The initial condition $r_1 < r_0 < r_2$ is again not physically possible.

Another interesting example of the method occurs for a linear restoring force (isotropic harmonic oscillator):

$$f = -kr, \qquad V = \tfrac{1}{2}kr^2.$$

For zero angular momentum, corresponding to motion along a straight line, $V' = V$ and the situation is as shown in Fig. 3.10. For any positive energy the motion is bounded and, as we know. simple harmonic. If $l \neq 0$, we have the state of affairs shown in Fig. 3.11. The motion then is always bounded for all physically possible

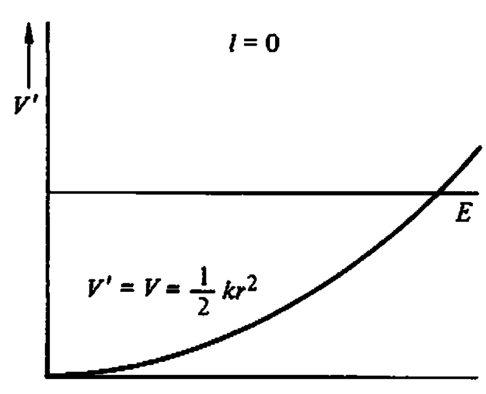

FIGURE 3.10 Effective potential for zero angular momentum.

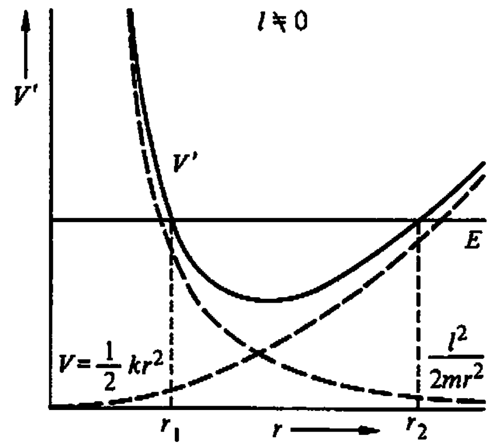

FIGURE 3.11 The equivalent one-dimensional potential for a linear restoring force.

energies and does not pass through the center of force. In this particular case, it is easily seen that the orbit is elliptic, for if $\mathbf{f} = -k\mathbf{r}$, the x- and y-components of the force are

$$f_x = -kx, \qquad f_y = -ky.$$

The total motion is thus the resultant of two simple harmonic oscillations at right angles, and of the same frequency, which in general leads to an elliptic orbit.

A well-known example is the spherical pendulum for small amplitudes. The familiar Lissajous figures are obtained as the composition of two sinusoidal oscillations at right angles where the ratio of the frequencies is a rational number. For two oscillations at the same frequency, the figure is a straight line when the oscillations are in phase, a circle when they are 90° out of phase, and an elliptic shape otherwise. Thus, central force motion under a linear restoring force therefore provides the simplest of the Lissajous figures.

3.4 ■ THE VIRIAL THEOREM

Another property of central force motion can be derived as a special case of a general theorem valid for a large variety of systems—the *virial theorem*. It differs in character from the theorems previously discussed in being *statistical* in nature; i.e., it is concerned with the time averages of various mechanical quantities.

Consider a general system of mass points with position vectors $\mathbf{r}_i$ and applied forces $\mathbf{F}_i$ (including any forces of constraint). The fundamental equations of motion are then

$$\dot{\mathbf{p}}_i = \mathbf{F}_i. \tag{1.3}$$

We are interested in the quantity

$$G = \sum_i \mathbf{p}_i \cdot \mathbf{r}_i,$$

where the summation is over all particles in the system. The total time derivative of this quantity is

$$\frac{dG}{dt} = \sum_i \dot{\mathbf{r}}_i \cdot \mathbf{p}_i + \sum_i \dot{\mathbf{p}}_i \cdot \mathbf{r}_i. \tag{3.23}$$

The first term can be transformed to

$$\sum_i \dot{\mathbf{r}}_i \cdot \mathbf{p}_i = \sum_i m_i \dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i = \sum_i m_i v_i^2 = 2T,$$

while the second term by (1.3) is

$$\sum_i \dot{\mathbf{p}}_i \cdot \mathbf{r}_i = \sum_i \mathbf{F}_i \cdot \mathbf{r}_i.$$

Equation (3.23) therefore reduces to

$$\frac{d}{dt} \sum_i \mathbf{p}_i \cdot \mathbf{r}_i = 2T + \sum_i \mathbf{F}_i \cdot \mathbf{r}_i. \tag{3.24}$$

The time average of Eq. (3.24) over a time interval τ is obtained by integrating both sides with respect to t from 0 to τ, and dividing by τ:

$$\frac{1}{\tau} \int_0^\tau \frac{dG}{dt} dt \equiv \overline{\frac{dG}{dt}} = \overline{2T} + \overline{\sum_i \mathbf{F}_i \cdot \mathbf{r}_i}$$

or

$$\overline{2T} + \overline{\sum_i \mathbf{F}_i \cdot \mathbf{r}_i} = \frac{1}{\tau}\left[G(\tau) - G(0)\right]. \tag{3.25}$$

If the motion is periodic, i.e., all coordinates repeat after a certain time, and if τ is chosen to be the period, then the right-hand side of (3.25) vanishes. A similar conclusion can be reached even if the motion is not periodic, provided that the coordinates and velocities for all particles remain finite so that there is an upper bound to G. By choosing τ sufficiently long, the right-hand side of Eq. (3.25) can be made as small as desired. In both cases, it then follows that

$$\overline{T} = -\frac{1}{2} \overline{\sum_i \mathbf{F}_i \cdot \mathbf{r}_i}. \tag{3.26}$$

Equation (3.26) is known as the *virial theorem*, and the right-hand side is called the *virial of Clausius*. In this form the theorem is imporant in the kinetic theory of gases since it can be used to derive ideal gas law for perfect gases by means of the following brief argument.

We consider a gas consisting of N atoms confined within a container of volume V. The gas is further assumed to be at a Kelvin temperature T (not to be confused with the symbol for kinetic energy). Then by the equipartition theorem of kinetic theory, the average kinetic energy of each atom is given by $\frac{3}{2}k_B T$, k_B being the Boltzmann constant, a relation that in effect is the definition of temperature. The left-hand side of Eq. (3.26) is therefore

$$\tfrac{3}{2}Nk_B T.$$

On the right-hand side of Eq. (3.26), the forces $\mathbf{F}_i$ include both the forces of interaction between atoms and the forces of constraint on the system. A perfect gas is defined as one for which the forces of interaction contribute negligibly to the virial. This occurs, e.g., if the gas is so tenuous that collisions between atoms occur rarely, compared to collisions with the walls of the container. It is these walls that constitute the constraint on the system, and the forces of constraint, $\mathbf{F}_c$, are localized at the wall and come into existence whenever a gas atom collides with the wall. The sum on the right-hand side of Eq. (3.26) can therefore be replaced in the average by an integral over the surface of the container. The force of constraint represents the reaction of the wall to the collision forces exerted by the atoms on the wall, i.e., to the pressure P. With the usual outward convention for the unit vector $\mathbf{n}$ in the direction of the normal to the surface, we can therefore write

$$d\mathbf{F}_i = -P\mathbf{n}\,dA,$$

or

$$\frac{1}{2}\sum_i \mathbf{F}_i \cdot \mathbf{r}_i = -\frac{P}{2}\int \mathbf{n}\cdot\mathbf{r}\,dA.$$

But, by Gauss's theorem,

$$\int \mathbf{n}\cdot\mathbf{r}\,dA = \int \nabla\cdot\mathbf{r}\,dV = 3V.$$

The virial theorem, Eq. (3.26), for the system representing a perfect gas can therefore be written

$$\tfrac{3}{2}Nk_B T = \tfrac{3}{2}PV,$$

which, cancelling the common factor of $\frac{3}{2}$ on both sides, is the familiar ideal gas law. Where the interparticle forces contribute to the virial, the perfect gas law of course no longer holds. The virial theorem is then the principal tool, in classical kinetic theory, for calculating the equation of state corresponding to such imperfect gases.

We can further show that if the forces $\mathbf{F}_i$ are the sum of nonfrictional forces $\mathbf{F}'_i$ and frictional forces $\mathbf{f}_i$ proportional to the velocity, then the virial depends only on the $\mathbf{F}'_i$; there is no contribution from the $\mathbf{f}_i$. Of course, the motion of the system must not be allowed to die down as a result of the frictional forces. Energy must constantly be pumped into the system to maintain the motion; otherwise *all* time averages would vanish as τ increases indefinitely (cf. Derivation 1.)

If the forces are derivable from a potential, then the theorem becomes

$$\overline{T} = \frac{1}{2}\overline{\sum_i \nabla V \cdot \mathbf{r}_i}, \tag{3.27}$$

and for a single particle moving under a central force it reduces to

$$\overline{T} = \frac{1}{2}\overline{\frac{\partial V}{\partial r}r}. \tag{3.28}$$

If V is a power-law function of r,

$$V = ar^{n+1},$$

where the exponent is chosen so that the force law goes as r^n, then

$$\frac{\partial V}{\partial r}r = (n+1)V,$$

and Eq. (3.28) becomes

$$\overline{T} = \frac{n+1}{2}\overline{V}. \tag{3.29}$$

By an application of Euler's theorem for homogeneous functions (cf. p. 62), it is clear that Eq. (3.29) also holds whenever V is a homogeneous function in r of degree $n+1$. For the further special case of inverse-square law forces, n is -2, and the virial theorem takes on a well-known form:

$$\overline{T} = -\tfrac{1}{2}\overline{V}. \tag{3.30}$$

3.5 ■ THE DIFFERENTIAL EQUATION FOR THE ORBIT, AND INTEGRABLE POWER-LAW POTENTIALS

In treating specific details of actual central force problems, a change in the orientation of our discussion is desirable. Hitherto solving a problem has meant finding r and θ as functions of time with E, l, etc., as constants of integration. But most often what we really seek is the equation of the orbit, i.e., the dependence of r upon θ, eliminating the parameter t. For central force problems, the elimination is particularly simple, since t occurs in the equations of motion only as a variable of differentiation. Indeed, one equation of motion, (3.8), simply provides a definite