Some Algebra about Pauli Matrices[1]

1) Try to prove $\{\sigma_i, \sigma_j\} = 2\delta_{ij}$ and $[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k$ using the expression of Pauli matrices. Then try to get $\sigma_i\sigma_j = \delta_{ij} + i\epsilon_{jkl}\sigma_l$

2) Prove $(\vec{\sigma} \cdot \vec{a})(\vec{\sigma} \cdot \vec{b}) = \vec{a} \cdot \vec{b} + i\vec{\sigma} \cdot (\vec{a} \times \vec{b})$, where $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$.

3) Prove $e^{ia(\hat{n}\cdot\vec{\sigma})} = Icosa + i(\hat{n} \cdot \vec{\sigma})sina$.

4) Assume our spin-1/2 particle moves in a magnetic field $\vec{B}$, then Hamiltonian is $H = -\gamma\vec{B} \cdot \vec{S}$. Assume magnetic field is in x-y plane $\vec{B} = B(cos\theta, sin\theta, 0)$. Initial state is $\psi(t = 0) = \begin{bmatrix} a \\ b \end{bmatrix}$. What's the state at time t? There are two ways to solve it. First, you can solve it by writing down the Schrodinger equation and solve it. Second, you can use the formula we have proved above to work out the unitary evolution operator $e^{-\frac{iHt}{\hbar}}$ explicitly. Then we can get our final result immediately by doing a matrix multiplication $\psi(t) = e^{-\frac{iHt}{\hbar}}\psi(t = 0)$. Try to do it using the second way.

[1]Pauli Matrices play an important role in modern physics. For example, identity matrix and three Pauli matrices form a complete basis for 2×2 hermitian matrices (Can you prove it? Try it!). Also, these four matrices are building blocks of representation of Clifford algebra or spinor representation of $so(N)$ which is the key when we formulate Dirac equation. Another example is that we need σ_y to construct time reversal operator in spin-1/2 system. Besides, they can also related with quaternion. So, make friends with them by doing some problems!

Solution:

1) $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $\quad \sigma_z = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$

$$\sigma_x \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\{\sigma_x, \sigma_x\} = 2\sigma_x^2 = 2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 2\hat{I}$$

You can prove similarly:

$$\{\sigma_y, \sigma_y\} = 2\hat{I}$$

$$\{\sigma_z, \sigma_z\} = 2\hat{I}$$

$$\sigma_x \sigma_y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

$$\sigma_y \sigma_x = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

So: $\{\sigma_x, \sigma_y\} = 0$ in general: $\{\sigma_i, \sigma_j\} = 2\delta_{ij}$

$$\sigma_x \sigma_y - \sigma_y \sigma_x = 2i \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} = 2i\,\sigma_z$$

in general: $[\sigma_i, \sigma_j] = 2i\,\epsilon_{ijk}\,\sigma_k$

$$\sigma_i\sigma_j + \sigma_j\sigma_i = 2\delta_{ij}$$

$$\sigma_i\sigma_j - \sigma_j\sigma_i = 2i\epsilon_{ijk}\sigma_k$$

$$2\sigma_i\sigma_j = 2\delta_{ij} + 2i\epsilon_{ijk}\sigma_k$$

$$\sigma_i\sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k$$

2)

$$(\vec{\sigma}\cdot\vec{a})(\vec{\sigma}\cdot\vec{b})$$

$$= \sigma_i a_i \sigma_j b_j$$

$$= a_i b_j \sigma_i \sigma_j$$

Note that a_i, b_j ~~must~~ just numbers we can change their positions.

But we can't do it to $\sigma_i\sigma_j$.

$\sigma_i\sigma_j \neq \sigma_j\sigma_i$

Using the result of (1).

$$= a_i b_j(\delta_{ij} + i\epsilon_{ijk}\sigma_k)$$

$$= a_i b_i + i\epsilon_{ijk}a_i b_j\sigma_k = \vec{a}\cdot\vec{b} + i\vec{\sigma}\cdot(\vec{a}\times\vec{b})$$

3)

$$e^{ia(\hat{n}\cdot\vec{\sigma})} = \sum_{n=0}^{+\infty} \frac{(ia(\hat{n}\cdot\vec{\sigma}))^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{(ia)^{2n}(\hat{n}\cdot\vec{\sigma})^{2n}}{(2n)!}$$

$$+ \sum_{n=0}^{+\infty} \frac{(ia)^{2n+1}(\hat{n}\cdot\vec{\sigma})^{2n+1}}{(2n+1)!}$$

set $\vec{a} = \vec{b} = \hat{n}$:

$$(\vec{\sigma}\cdot\hat{n})^2 = 1 + i\,\vec{\sigma}\cdot(\hat{n}\times\hat{n}) = 1$$

$$= \sum_{n=0}^{\infty} \frac{(ia)^{2n}}{(2n)!} + \hat{n}\cdot\vec{\sigma} \sum_{n=0}^{+\infty} \frac{(ia)^{2n+1}}{(2n+1)!}$$

$$= ch\,(ia) + sh\,(ia)\ \hat{n}\cdot\vec{\sigma}$$

$$= \cos(a) + i\sin(a)\ \hat{n}\cdot\vec{\sigma}$$

4)

$$H = -\gamma B \;\hat{n}\cdot\frac{\hbar}{2}\vec{\sigma}$$

$$= -\frac{\gamma B\hbar}{2}\,\hat{n}\cdot\vec{\sigma} \qquad \text{where } \hat{n} = (\cos\theta, \sin\theta, 0)$$

$$U(t) = e^{-i\frac{\hat{H}t}{\hbar}}$$

$$= e^{-\frac{it}{\hbar}\left(-\frac{\gamma B\hbar}{2}\hat{n}\cdot\vec{\sigma}\right)}$$

$$= e^{\frac{it}{\hbar}\cdot\frac{\gamma B\hbar}{2}\,\hat{n}\cdot\vec{\sigma}}$$

$$= e^{\frac{i\gamma Bt}{2}\,\hat{n}\cdot\vec{\sigma}}$$

$$\boxed{a = \frac{\gamma Bt}{2}}$$

$$= e^{i a\cdot(\hat{n}\cdot\vec{\sigma})}$$

$$= \hat{I}\cos(a) + i(\hat{n}\cdot\vec{\sigma})\sin a$$

$$= \begin{pmatrix} \cos(a) & \\ & \cos a \end{pmatrix} + i\sin a\;\hat{n}\cdot\vec{\sigma}$$

$$\hat{n}\cdot\vec{\sigma} = \cos\theta\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \sin\theta\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \cos\theta - i\sin\theta \\ \cos\theta + i\sin\theta & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & e^{-i\theta} \\ e^{i\theta} & 0 \end{pmatrix}$$

$$\Rightarrow \quad U(t) = \begin{pmatrix} \cos a & \\ & \cos a \end{pmatrix} + i\sin a \begin{pmatrix} 0 & e^{-i\theta} \\ e^{i\theta} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \cos a & 0 \\ 0 & \cos a \end{pmatrix} + \begin{pmatrix} 0 & i\sin a\, e^{-i\theta} \\ i\sin a\, e^{i\theta} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \cos a & i\sin a\, e^{-i\theta} \\ i\sin a\, e^{i\theta} & \cos a \end{pmatrix}$$

$$= \begin{pmatrix} \cos \frac{\gamma B t}{2} & i\sin \frac{\gamma B t}{2}\, e^{-i\theta} \\ i\sin\left(\frac{\gamma B t}{2}\right) e^{i\theta} & \cos \frac{\gamma B t}{2} \end{pmatrix}$$

$$|\psi(t)\rangle = \begin{pmatrix} \cos \frac{\gamma B t}{2} & i\sin \frac{\gamma B t}{2}\, e^{-i\theta} \\ i\sin\left(\frac{\gamma B t}{2}\right) e^{i\theta} & \cos \frac{\gamma B t}{2} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

$$= \begin{pmatrix} a \cos \frac{\gamma B t}{2} + i\, b \sin \frac{\gamma B t}{2}\, e^{-i\theta} \\ i a \sin\left(\frac{\gamma B t}{2}\right) e^{i\theta} + b \cos \frac{\gamma B t}{2} \end{pmatrix}$$

Consider the electron in a constant magnetic field $\vec{B} = (B_x, B_y, B_z)$. For this problem, stay in the given cooordinate system (i.e., do not rotate your axis Z in the same direction as magnetic field as we did before).

1. Write a hamiltonian of the system. $H = -\vec{\mu}\vec{B}$

2. Find an eigenvalues. You should see that the eigenvalues are the same as for the case when you specify the direction of Z axis along the magnetic field.

3. Go to another eigenbasis by rotation of the spinor by matrix $e^{i\phi\sigma_z}$. Ho w magnetic field changes?

4. Check that rotation of the coordinate system by angle ϕ around Z axis is equivalent to the rotation of the spinor by matrix $e^{i\phi\sigma_z/2}$

5. What happens with a spinor, when you rotate your coordinate system by the angle 2π ?

1. $\hat{H} = -\bar{\mu}\cdot\bar{B} = -\gamma\bar{S}\cdot\bar{B} =$

$$= \frac{-\gamma\hbar}{2}\begin{pmatrix} B_z & B_x+iB_y \\ B_x-iB_y & -B_z \end{pmatrix}$$

2. Obviously eigenvalues of $\hat{H}$ are equal to eigenvalues of matrix $\begin{pmatrix} B_z & B_x+iB_y \\ B_x-iB_y & -B_z \end{pmatrix}$ times $\frac{-\gamma\hbar}{2}$

$$\det\begin{pmatrix} B_z-\lambda & B_x+iB_y \\ B_x-iB_y & -B_z-\lambda \end{pmatrix} = 0$$

$$(B_z-\lambda)(-B_z-\lambda) - (B_x+iB_y)(B_x-iB_y) = 0$$

$$\lambda^2 - B_z^2 - B_x^2 - B_y^2 = 0$$

$\lambda = \pm|B|$ $\Rightarrow$ coincide with the case where we rotate $\bar{B} = B_z\cdot\hat{z}$

3.

$e^{i\varphi\sigma_z/2} = \exp \begin{pmatrix} i\phi/2 & 0 \\ 0 & -i\phi/2 \end{pmatrix} = \begin{pmatrix} e^{i\varphi/2} & 0 \\ 0 & e^{-i\varphi/2} \end{pmatrix} = \hat{u}$

$$\hat{u} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} e^{i\varphi/2}\alpha \\ e^{-i\varphi/2}\beta \end{pmatrix}$$

To see how magnetic field changes, let's say $B_x + iB_y = B_\perp e^{i\psi}$, $B_x - iB_y = B_\perp e^{-i\psi}$

Then

$H_{\text{new coord}} = u^+ H_{old}\, u =$

$$= -\frac{\gamma\hbar}{2} \begin{pmatrix} e^{-i\varphi/2} & 0 \\ 0 & e^{i\varphi/2} \end{pmatrix} \begin{pmatrix} B_z & B_\perp e^{i\psi} \\ B_\perp e^{-i\psi} & B_z \end{pmatrix} \begin{pmatrix} e^{i\varphi/2} & 0 \\ 0 & e^{-i\varphi/2} \end{pmatrix} =$$

$$= \begin{pmatrix} B_z & B_\perp e^{i(\psi-\varphi)} \\ B_\perp e^{-i(\psi-\varphi)} & B_z \end{pmatrix}$$

$\Rightarrow \psi \to \psi - \varphi$

4. ~~When~~ When we rotate coordinate system by angle α around z axis

$B_z \to B_z$

$B_\perp \to B_\perp$

$\psi \to \psi + \alpha$

$\Rightarrow$ ~~the~~ The same ~~effect~~ effect as after rotation by matrices

$$\begin{pmatrix} e^{i\varphi/2} & 0 \\ 0 & e^{-i\varphi/2} \end{pmatrix}$$

5. If $\alpha = 2\pi$, $\varphi = \pi \Rightarrow$

$$\Rightarrow \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \Rightarrow \begin{pmatrix} e^{i\pi}\alpha \\ e^{-i\pi}\beta \end{pmatrix} = \begin{pmatrix} -\alpha \\ -\beta \end{pmatrix}$$

When you rotate your system by 2π, wave function of the fermion changes its sign!