105A - Midterm 1 - solutions

(Grades are out of 150)

A surfer on a surfboard with mass M floats near the edge of the pier. The surfer is given a push so its initial speed at $t = 0$ is v_0. The drag by the water on the surfboard depends on the speed v of the surfboard as $-bv$, where $b =$ Const..

1. Find an expression for the speed $v(t)$ of the surfer as a function of time.

   Answer: The equation of motion is $M\ddot{x} = -bv$ which can be written as: $M\dot{v} = -bv$. To solve this we simple have

$$\frac{M}{b}\int_{v_0}^{v(t)} \frac{dv'}{v'} = -\int_0^t dt \tag{1}$$

   Which is:

$$-t = \frac{M}{b}\ln(v)\Big|_{v_0}^{v(t)} = \frac{M}{b}\left(\ln(v) - \ln v_0\right) \tag{2}$$

   After organizing we get:

$$v(t) = e^{-tb/M + \ln v_0} = v_0 e^{-tb/M} \tag{3}$$

2. How far can the surfer travel given the initial velocity ?
   Answer: From the above equation we simply have:

$$\frac{dx}{dt} = v_0 e^{-tb/M} \tag{4}$$

   so

$$L = \int dx = \int_0^t v_0 e^{-tb/M} dt = \frac{-v_0 M}{b} e^{-tb/M}\Big|_0^t = \frac{-v_0 M}{b} e^{-tb/M} - \frac{-v_0 M}{b} \tag{5}$$

   Which is simply:

$$L = \frac{v_0 M}{b}\left(1 - e^{-tb/M}\right) \tag{6}$$

   Since when $t \to \infty$, $x \to v_0 M/b$, this is how far the surfer can travel.

3. Examining the situation in reality you find that the surfboard is in fact bobbing back and front with a restoring force that scales as $-17b^2x/(4M)$.
   (i) Write the equation of motion.
   (ii) Find the frequency of oscillations.
   (iii) Find the expression for the distance of the surfboard as a function of time $x(t)$. Express your answer as a function of b, M and the initial conditions (note that at $t = 0$ the surfboard is at $x = 0$).
   Answer: The equation of motion is simply $M\ddot{x} = -17b^2x/(4M) - b\dot{x}$ which can be written as:

$$\ddot{x} + \frac{b}{M}\dot{x} + \frac{17b^2}{4M^2}x = 0 \tag{7}$$

we guess a solution of the sort $x(t) = Ae^{i(\omega t+\phi)}$ which results in the dispersion relation of the form:

$$-\omega^2 + \frac{b}{M}i\omega + \frac{17b^2}{(2M)^2} = 0 \tag{8}$$

or

$$\omega^2 - \frac{b}{M}i\omega - \frac{17b^2}{(2M)^2} = 0 \tag{9}$$

The solution is

$$\omega_{1,2} = \frac{1}{2}\left(\frac{ib}{M} \pm \sqrt{\left(\frac{ib}{M}\right)^2 + 4\frac{17b^2}{(2M)^2}}\right) = \frac{1}{2}\left(\frac{ib}{M} \pm \sqrt{16\frac{b^2}{M^2}}\right) = \frac{1}{2}\left(\frac{ib}{M} \pm 4\frac{b}{M}\right) \tag{10}$$

identifying

$$\omega_r = 2\frac{b}{M} \quad \text{and} \quad \omega_i = \frac{b}{2M} \tag{11}$$

we have $\omega = i\omega_i \pm \omega_r$ and the solution is $x(t) = Re[e^{i(\omega t+\phi)}] = Re[e^{i(\{i\omega_i \pm \omega_r\}t+\phi)}]$ so

$$x(t) = ARe[e^{-\omega_i t \pm i\omega_r t + i\phi}] = Ae^{-\omega_i t}Re[\cos(\omega_r t+\phi) \pm i\sin(\omega_r t+\phi)] = Ae^{-\omega_i t}\cos(\omega_r t+\phi) \tag{12}$$

Plugging in the initial conditions ($x(t=0) = 0$ and $\dot{x}(t=0) = v_0$) we have

$$x(t=0) = A\cos(\phi) = 0 \tag{13}$$

From here of course so $A \neq 0$ it means that $\cos(\phi) = 0$ or $\phi = \pm\pi/2$. for the first derivative note that

$$\dot{x}(t) = -\omega_r A\sin(\omega_r t+\phi)e^{-\omega_i t} - A\omega_i e^{-\omega_i t}\cos(\omega_r t+\phi) \tag{14}$$

which is then

$$\dot{x}(t=0) = v_0 = -\omega_r A\sin(\phi) - A\omega_i\cos(\phi) \tag{15}$$

Since $\phi = \pm\pi/2$ we get that $v_0 = -\omega_r A\sin(\pm pi/2)$ so since $A > 0$ we know that $\phi = -pi/2$ and $A = v_0/\omega_r$. So finally we can write:

$$x(t) = 2\frac{v_0 M}{b}e^{-bt/(2M)}\sin\left(2\frac{bt}{M}\right) \tag{16}$$