CHSH "explanation"

$$A(0) \equiv |H_1\rangle\langle H_1| - |V_1\rangle\langle V_1|$$

$$A(\tfrac{\pi}{4}) \equiv |V_1\rangle\langle H_1| + |H_1\rangle\langle V_1|$$

$$B(-\tfrac{\pi}{8}) \equiv \tfrac{1}{\sqrt{2}}\left(|H_2\rangle\langle H_2| - |H_2\rangle\langle V_2| - |V_2\rangle\langle H_2| - |V_2\rangle\langle V_2|\right)$$

$$B(\tfrac{5\pi}{8}) \equiv \tfrac{1}{\sqrt{2}}\left(-|H_2\rangle\langle H_2| - |H_2\rangle\langle V_2| - |V_2\rangle\langle H_2| + |V_2\rangle\langle V_2|\right)$$

Let's check these in matrix form.

$$C(0) \equiv |H\rangle\langle H| - |V\rangle\langle V| = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{where} \quad |H\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } |V\rangle \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

In 2x2 matrix form, an operator

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Will correspond to

$$M = a|H\rangle\langle H| + b|V\rangle\langle H| + c|H\rangle\langle V| + d|V\rangle\langle V|.$$

$$C(\alpha) = R(\alpha)\, C(0)\, R^{-1}(\alpha) \quad \text{with}$$

$$R(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \qquad \therefore\ C(\alpha) = \begin{pmatrix} \cos(2\alpha) & \sin(2\alpha) \\ \sin(2\alpha) & -\cos(2\alpha) \end{pmatrix}$$

So

$$A(\tfrac{\pi}{4}) = R(\tfrac{\pi}{4})\, A(0)\, R(-\tfrac{\pi}{4})$$

$$= \begin{pmatrix} \cos(\frac{\pi}{2}) & \sin(\frac{\pi}{2}) \\ \sin(\frac{\pi}{2}) & -\cos(\frac{\pi}{2}) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = |V\rangle\langle H| + |H\rangle\langle V|.$$

$$B\left(-\frac{\pi}{8}\right) = \begin{pmatrix} \cos\left(-\frac{\pi}{4}\right) & \sin\left(-\frac{\pi}{4}\right) \\ \sin\left(-\frac{\pi}{4}\right) & -\cos\left(-\frac{\pi}{4}\right) \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix} = \frac{1}{\sqrt{2}}\left(|H\rangle\langle H| - |V\rangle\langle H| - |H\rangle\langle V| - |V\rangle\langle V| \right)$$

$$B\left(\frac{5\pi}{8}\right) = \begin{pmatrix} \cos\left(\frac{5\pi}{4}\right) & \sin\left(\frac{5\pi}{4}\right) \\ \sin\left(\frac{5\pi}{4}\right) & -\cos\left(\frac{5\pi}{4}\right) \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\left(-|H\rangle\langle H| - |V\rangle\langle H| - |H\rangle\langle V| + |V\rangle\langle V| \right)$$

Let's find the +1 eigenstate of each operator:

$$A(0)|H\rangle = +1\,|H\rangle$$

$$A(0)|D\rangle = \left(|V\rangle\langle H| + |H\rangle\langle V|\right)\frac{1}{\sqrt{2}}\left(|H\rangle + |V\rangle\right)$$

$$= +1\,|D\rangle$$

So we try the general case

$$\begin{pmatrix} \cos(2\alpha) & \sin(2\alpha) \\ \sin(2\alpha) & -\cos(2\alpha) \end{pmatrix}\begin{pmatrix} \cos(\alpha) \\ \sin(\alpha) \end{pmatrix}$$

$$= R(\alpha)\, C(0)\, R^{-1}(\alpha)\, R(\alpha)\, |H\rangle$$

$$= R(\alpha)\, C(0)|H\rangle$$

$$= R(\alpha)|H\rangle$$

and we have $\begin{pmatrix} \cos\left(-\frac{\pi}{8}\right) \\ \sin\left(-\frac{\pi}{8}\right) \end{pmatrix}$ and $\begin{pmatrix} \cos\left(\frac{5\pi}{8}\right) \\ \sin\left(\frac{5\pi}{8}\right) \end{pmatrix}$ for

$B\left(-\frac{\pi}{8}\right)$ and $B\left(\frac{5\pi}{8}\right)$.

Bell's observable is

$$S = A(\tfrac{\pi}{4})B(-\tfrac{\pi}{8}) + A(0)\,B(-\tfrac{\pi}{8}) + A(\tfrac{\pi}{4})B(\tfrac{5\pi}{8}) - A(0)\,B(\tfrac{5\pi}{8})$$

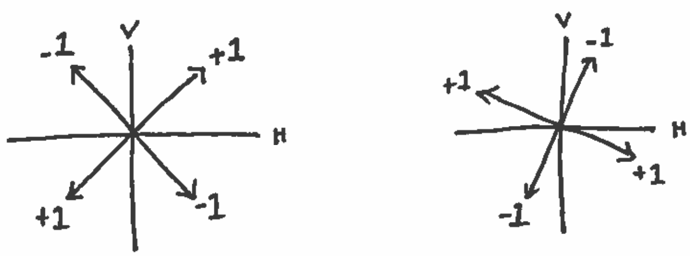
Let's try to make sense of the expectation values of each term for the singlet state

$$|\psi_s\rangle = \tfrac{1}{\sqrt{2}}\left(|H_1H_2\rangle - |V_1V_2\rangle\right)$$

The first term,

$$S_1 \equiv A(\tfrac{\pi}{4})B(-\tfrac{\pi}{8})$$

corresponds to measurements in the bases that look like



Alice Bob

Here's the algebra:

$\langle\psi_s|S_1|\psi_s\rangle$

$$= \tfrac{1}{\sqrt{2}}\left(\langle H_1H_2| - \langle V_1V_2|\right)\left(|V_1\rangle\langle H_1| + |H_1\rangle\langle V_1|\right)\otimes B(-\tfrac{\pi}{8})|\psi_s\rangle$$

$$= \tfrac{1}{\sqrt{2}}\left(\langle V_1H_2| - \langle H_1V_2|\right)\left(\tfrac{1}{\sqrt{2}}\right)\left(|H_2\rangle\langle H_2| - |H_2\rangle\langle V_2| - |V_2\rangle\langle H_2| - |V_2\rangle\langle V_2|\right)|\psi_s\rangle$$

$$= \tfrac{1}{2}\left(\langle V_1H_2| - \langle V_1V_2| + \langle H_1H_2| + \langle H_1V_2|\right)\tfrac{1}{\sqrt{2}}\left(|H_1H_2\rangle - |V_1V_2\rangle\right)$$

$$= \tfrac{1}{2\sqrt{2}}(1+1) = \tfrac{1}{\sqrt{2}}.$$

If Alice measures +1 for her photon in the $A(\frac{\pi}{4})$ basis, we can write the state of photon #2 as

$$|\phi_2\rangle \equiv c\langle D_1|\Psi_S\rangle$$

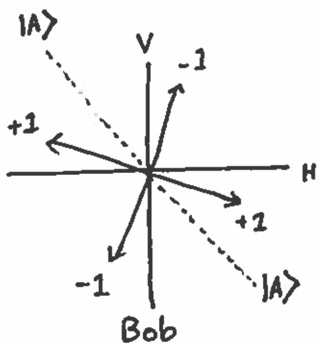
where c is a normalization constant. We have

$$|\phi_2\rangle = \frac{c}{\sqrt{2}}\left(\langle H_1| + \langle V_1|\right)\frac{1}{\sqrt{2}}\left(|H_1H_2\rangle - |V_1V_2\rangle\right)$$

$$= \frac{c}{2}\left(|H_2\rangle - |V_2\rangle\right)$$

$$= |A_2\rangle$$

If we look at what an Anti-diagonal photon will do if measured in the $B(-\frac{\pi}{8})$ basis,

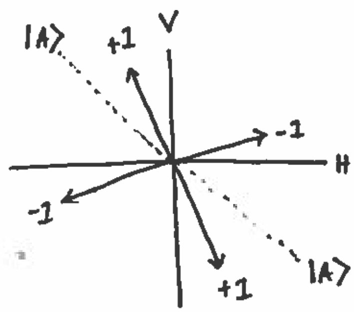


it will be more-likely to return +1 than −1, so we expect a positive number smaller than 1 for

$$\langle\Psi_S|S_1|\Psi_S\rangle$$

which is consistent with our algebra.

The same is true for $S_3 \equiv A(\frac{\pi}{4})B(\frac{5\pi}{8})$, where the $B(\frac{5\pi}{8})$ measurement basis looks like

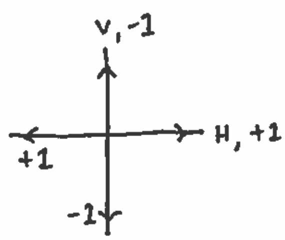


Bob

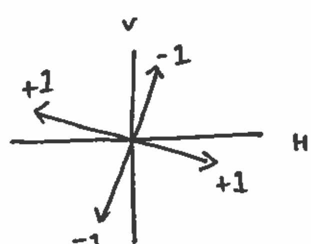
$\therefore |A_2\rangle$ is more likely to give $+1$ than -1 if measured in $B(\frac{5\pi}{8})$.

So one way to think about how to come up with the four measurement bases is that you want three pairs that are positively-correlated and one pair that are negatively-correlated.

Clearly, $A(0)$ will project the second photon onto the polarization found in the measurement of the first photon (to within a minus sign), so we expect positive correlations (i.e. a positive expectation value) for $A(0)B(-\frac{\pi}{8}) = S_2$:



Alice



Bob

But negative correlations for $A(0)\,B(\frac{5\pi}{8})$: