Phys 180Q
Week 4 Lecture – Fiber optics
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In this week's lab we will learn to handle and utilize optical fibers. The optical fiber is qualitatively very intuitive. To get a completely quantitative description is straightforward, but tiresome – lots of Bessel functions. Our approach will be to look at what is called the "slab model" – turn the cylindrical symmetry of the fiber into a 2D problem. From this model, we can see the main features of a fiber. Namely, that there are guided optical modes and to couple light into these modes requires some care.

From there we will go to a graded index fiber, which is also used in the real world, but is more easily treated. Here we will again see the main features of the fiber arise from the mathematics.

The treatment we are about to go through is not terribly challenging, but should give you excellent intuition for using fibers. As you'll find in the lab, the challenge of fibers is typically not understanding them, but aligning the optics carefully enough.

The development of the optical fiber

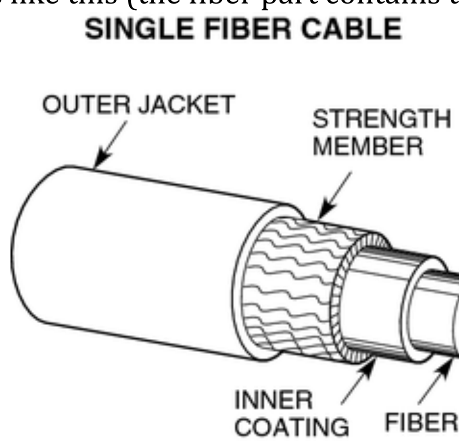
Though the principle behind fibers, total internal reflection, was understood hundreds of years earlier – see for example John Tyndall's light pipe experiment (which UCLA has a cool demo of) – the modern optical fiber has its roots in the fiberscope. The fiberscope was developed in the 1950s. (The fiber camera things hitmen use.)

The fiberscope was all glass and as we shall see very lossy. To combat this, a cladding was developed around the glass to lessen loss and modern fiber optics was born. (Brian O'Brien was the first to suggest this; University of Rochester and American Optical Company).

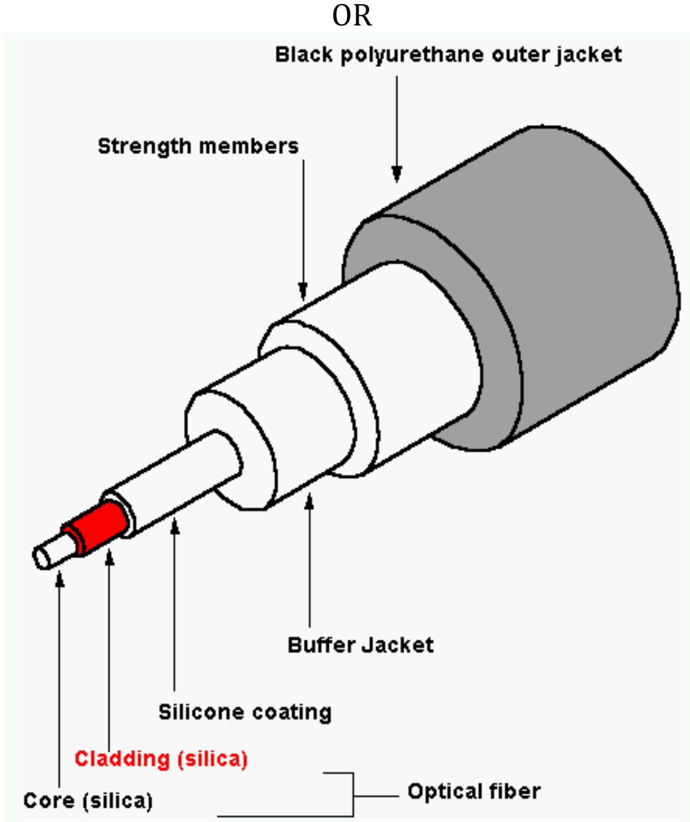
From here companies (Corning and Bell Labs) develop the technology into the field we know today. (Nowadays, there are all kinds of fibers and waveguides being made and new companies popping up everyday.)

The basics of the optical fiber and The Slab Model

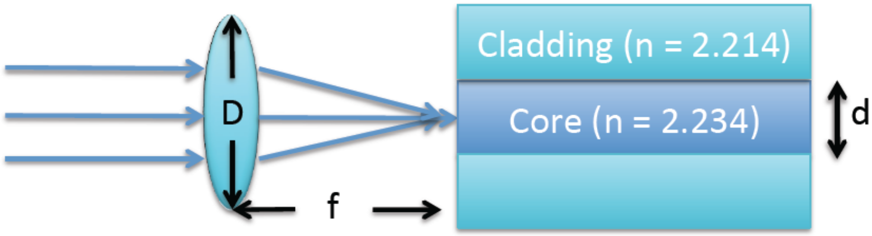
As you may know, the optical fiber works through total internal reflection. The fiber typically looks like this (the fiber part contains the core and cladding):



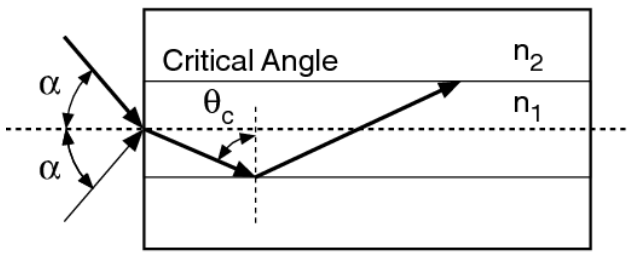
OR



And in the simplest approximation (zig-zag analysis) work like this:



Numerical Aperture



$$NA = \sin\alpha = \sqrt{n_1^2 - n_2^2}$$

Full Acceptance Angle = 2α

In the ray-tracing picture, light entering the fiber and resulting in an angle less than θ_c is totally internally reflected. So, for light to be transmitted through the fiber it has to be incident on the core and have the right angle. The core diameters can be as small as a few microns so you will typically place a lens before the fiber tip to aid in coupling. However, not just any lens or input beam size will work because if the angle α is too large the light will leak out. We can quickly see this from Snell's law:

1. Recall Snell's law: $n_1 sin\,\theta_1 = n_2 sin\theta_2$
2. At the core-cladding interface total internal reflection happens when the refracted angle in the cladding becomes imaginary. Thus, the critical angle is given as: $n_1 \sin\theta_c = n_2$
3. At the input we have: $1\sin\alpha = n_1 \sin\frac{\pi}{2} - \theta_c = n_1\cos\theta_c$
4. Thus, $\sin^2\alpha = n_1^2\cos^2\theta_c = n_1^2(1-\sin^2\theta_c) = n_1^2 - n_2^2$
5. And the numerical aperture of the fiber is defined as: $NA = \sin\alpha = \sqrt{n_1^2 - n_2^2}$.
6. The full acceptance angle is defined as 2α.

So, if you want to couple light efficiently into a fiber you must satisfy two things:

1. Focus onto tip, such that all of your light hits the core (In fiber specifications this is basically the mode field diameter or MFD).

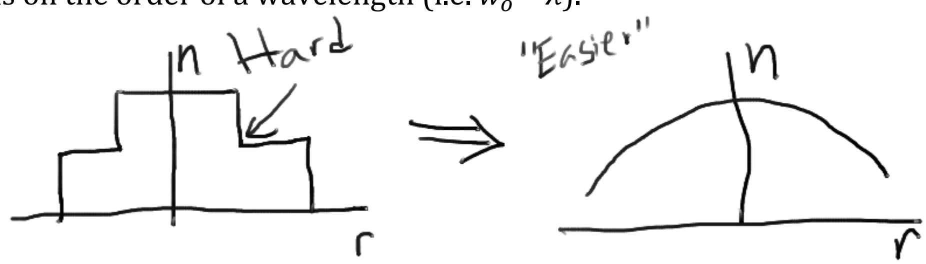
2. Match the NA ($NA = \sin\alpha = \frac{\frac{D}{2}}{\sqrt{f^2+\left(\frac{D}{2}\right)^2}} \approx \frac{D}{2f}$ needs to equal or be smaller than $\sqrt{n_1^2 - n_2^2}$.

Now that you know how they work, here's how they are made:
http://www.youtube.com/watch?v=llI8Mf_faVo

The graded-index fiber

We can of course do better than this. Previously we learned that propagating laser beams aren't rays, but can be better described by Gaussian beams. We learned this by solving the wave equation (Helmholtz equation) for free space. Here we simply have to solve the wave equation for the fiber and apply boundary conditions at the different interfaces. (Think about it: somehow a fiber confines light to a diameter of just a few wavelengths, if this was a free-space propagating beam we know it would quickly diverge – $\theta \propto \frac{\lambda}{w}$).

The problem with the required analysis though is that it is mathematically challenging. In the full-blown treatment, the fiber is cylindrically symmetric and you get a bunch of Bessel functions. This is fine, but the math tends to obscure the physics. What you will find is that the fiber is a dielectric waveguide. Inside the core the wave propagates with something similar to (but not quite) the normal $e^{i(kz-\omega t)}$ phase factor. Outside the core it decays away exponentially with a length scale on the order of the wavelength. Inside the core the field amplitude has a variation transversely like the Hermite Gaussians we saw last week – and of course the oscillations depend on which mode is propagating. Here though, the variation happens on the order of a wavelength (i.e. $w_o \sim \lambda$).



To make the math a little easier we will not analyze the step-indexed fiber, but a different kind of fiber: the graded-index fiber (aka gradient-index fiber). These fibers really exist[1], so this isn't just a mathematical exercise. The results we find are qualitatively similar to the much harder to achieve results of the step-index fiber.

Recall the travelling wave equation (we derived it last week):

$$\nabla^2 \vec{E} + \frac{\omega^2}{c^2} n^2 \vec{E} = 0$$

[1] From my personal sampling, multimode graded index fibers are more common that single mode graded index fiber.

Suppose the core of the fiber has a graded index of refraction:

$$n^2 = n_0^2 (1 - C r^2)$$

Now, because the beam is not travelling in free space our usual ansatz for the spatial part ($\vec{E} = \vec{E}_o \psi(x, y) e^{ikz}$) has to be modified to $\vec{E} = \vec{E}_o \psi(x, y, z) e^{i\beta z}$. Here β is the phase constant, which is no longer simply equal to k. (Note: Since we anticipate guiding, we look for a solution that keeps it shape as it propagates along z, so $\psi(x, y, z) = \psi(x, y)$. This is different than the Gaussian beams we just derived since we allowed $\psi = f(z)$. Again, here it's guided so the transverse profile does not evolve like a Gaussian Beam.) With this form, the travelling wave equation becomes:

$$\nabla_t^2 \vec{E} + \frac{\partial^2 \vec{E}}{\partial z^2} + \left(\frac{\omega}{c}\right)^2 n_0^2 (1 - C r^2) \vec{E} = 0$$

$$\nabla_t^2 \vec{E} = \vec{E}_0 (\nabla_t^2 \psi) e^{i\beta z}$$

$$\frac{\partial^2 \vec{E}}{\partial z} = \vec{E}_0 (-\beta^2) \psi e^{i\beta z}$$

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial z^2} + \Big(\underbrace{\frac{n_0^2 \omega^2}{c^2}}_{k^2} (1 - C r^2) - \beta^2 \Big) \psi = 0$$

From here it's just maths… Assume a separable solution and see where you get.

$$\psi = X(x)\, Y(y)$$

$$Y \frac{\partial^2 X}{\partial x^2} + X \frac{\partial^2 Y}{\partial y^2} + (k^2 - \beta^2 - C k^2 (x^2 + y^2)) X Y = 0$$

$$-\frac{1}{X} \frac{\partial^2 X}{\partial x^2} + C K x^2 = \frac{1}{Y} \frac{\partial^2 Y}{\partial y^2} + K^2 - \beta^2 - C K^2 y^2$$

$$= \underbrace{T}_{\text{seperation constant}}$$

Leading to two, uncoupled ODEs.

$$\frac{\partial^2 X}{\partial x^2} + (T - Ck^2x^2)X = 0$$

$$\frac{\partial^2 Y}{\partial y^2} + (k^2 - \beta^2 - T - Ck^2y^2)Y = 0$$

This has the same form as a quantum harmonic oscillator and so is solved by the same Hermite-Gaussian functions. Like the QMHO we see the solution by recasting the x dimension as:

Let: $u = (Ck^2)^{1/4}x$

$$du = (Ck^2)^{1/4}dx$$

$$\frac{\partial^2}{\partial u^2} = \frac{1}{\sqrt{Ck^2}}\frac{\partial^2}{\partial x^2}$$

$$\sqrt{Ck^2}\frac{\partial^2 X}{\partial x^2} + (T - \sqrt{Ck^2}u^2)X = 0$$

$$\frac{\partial^2 X}{\partial x^2} + \left(\frac{T}{\sqrt{C}k} - u^2\right)X = 0$$

Which the Hermite-Gaussian of order m, $H_m(u)e^{-u^2/2}$, solves as long as $\frac{T}{\sqrt{C}k} = 2m + 1$. (Just go check your favorite math book.)

So, require that our separation constant: $T = (2m + 1)\sqrt{C}k$

And for Y we have the same business leading to:

$$\frac{\partial^2 Y}{\partial u^2} + \left(\frac{k^2 - \beta^2 - T}{\sqrt{C}k} - u^2\right)Y = 0$$

And the defining equation for the Hermite-Gaussian requires: $\frac{k^2-\beta^2-T}{\sqrt{C}k} = 2p + 1$. And from these two equations for T, we can get the propagation constant in terms of the mode numbers…

$$\beta^2 = K^2 - T - \sqrt{C}K(2p+1)$$

$$\beta^2 = K^2 - \sqrt{C}K(2p+2m+2)$$

$$\beta = K\sqrt{1 - \frac{2\sqrt{C}}{K}(p+m+1)}$$

And we have the electric field in the fiber as:

$$\psi = H_m\left((CK^2)^{1/4}x\right) H_p\left((CK^2)^{1/4}y\right)$$

$$\times \exp\left(-(CK^2)^{1/2} r^2\right)$$

$$\times \exp\left(iK\left(1 - \frac{2\sqrt{C}}{K}(p+m+1)\right)^{1/2} z\right)$$

And if we define a characteristic spot size:

$$w^2 = \frac{2}{\sqrt{C}K} = \frac{\lambda_0}{\sqrt{C}\pi n_0}$$

$\sqrt{C} \sim 1/\ell \leftarrow$ length scale of n variation

We have a result quite similar to the Gaussian beam result:

$$\psi = H_m\left(\frac{\sqrt{2}x}{w}\right) H_p\left(\frac{\sqrt{2}y}{w}\right) \exp\left(\frac{-r^2}{w^2}\right)$$

$$\times \exp\left(iK\sqrt{1 - (p+m+1)\frac{\sqrt{C}2}{K}}\, z\right)$$

Put in some numbers:

Suppose, we have a typical multimode fiber diameter of ~50 microns and $n^2 = no^2\left(1 - \left(\frac{r}{200\mu m}\right)^2\right)$. Thus, $C = \left(\frac{1}{200\mu m}\right)^2 = 2.5(10^6)m^{-2}$ and we're at a wavelength of 635 nm. Then the waist spot size is: $w = \sqrt{\frac{\lambda_o}{\sqrt{C}\pi n_o}} = \sqrt{\frac{635\ nm}{5x10^3}} = \sqrt{100\ (10^{-12})m^2} = 10\ \mu m$. Thus, the mode is very tightly confined to the axis and our ignoring of the core-cladding boundary conditions is somewhat justified.

Since β depends on the mode number, the different modes propagate with different speeds (i.e. different m's and p's have different phase velocities: $v_p = \omega/\beta$). This is called modal dispersion, since the different modes will separate.

Now, is this fiber multimode or single mode? Well, it depends. You can see that not ever p and m will work. Once $\frac{(p+m+1)2\sqrt{C}}{k} > 1$, the phase factor becomes a decaying exponential and therefore the mode will not propagate. So, for a given wavelength and C you can see how many p's and m's are allowed. For the example above: $\frac{2\sqrt{C}}{k} \approx \frac{3(10^3)}{6} 635(10^{-9}) = 300(10^{-6})$ and thus any mode such that $p + m + 1 < \frac{1}{3} 10^4 \approx 300$ will propagate. Clearly this is a multimode fiber.

There are two ways to achieve a single mode fiber. Crank up the index variation a lot, so that $\frac{2\sqrt{C}}{k} \approx 1$, then only the (0,0) mode could propagate – i.e. $C \approx 2\pi^2/\lambda^2$, which for our example is $C \approx 5(10^{13})m^{-2}$. Or, make the fiber smaller so that the core-cladding boundary conditions matter. The first way is obvious now (at least in principle). The second one may not be as clear, but is what is used in a stepped index single mode fiber (i.e. most single mode fiber). To really see how single mode operation is achieved in these fibers requires including the boundary conditions (i.e. the waveguide nature of things), which is really not worth it. If you do this though, you can find that the modes look similar to what we calculated and that the number of transverse modes that can propagate in the step-index fiber is given by:

$$n_t = 2\frac{\pi a^2}{\lambda^2}\pi(n_{core}^2 - n_{cladding}^2).$$

This formula is important enough to remember or at least write down somewhere safe, as you will often encounter a fiber made for some purpose other than your own and you can use this to decide if the fiber will work for you.

(e.g. Typical multimode fiber has $a = 62.5/2\ \mu m$ and $n_{core} = 1.53$ and $n_{cladding} = 1.51, so\ n_t = 1662$, while a SMF has $a = \frac{3}{2}\mu m$, resulting in $n_t \sim 1$. (Typically, you find that you can get a few modes to propagate down SMF – i.e. not truly SMF.)