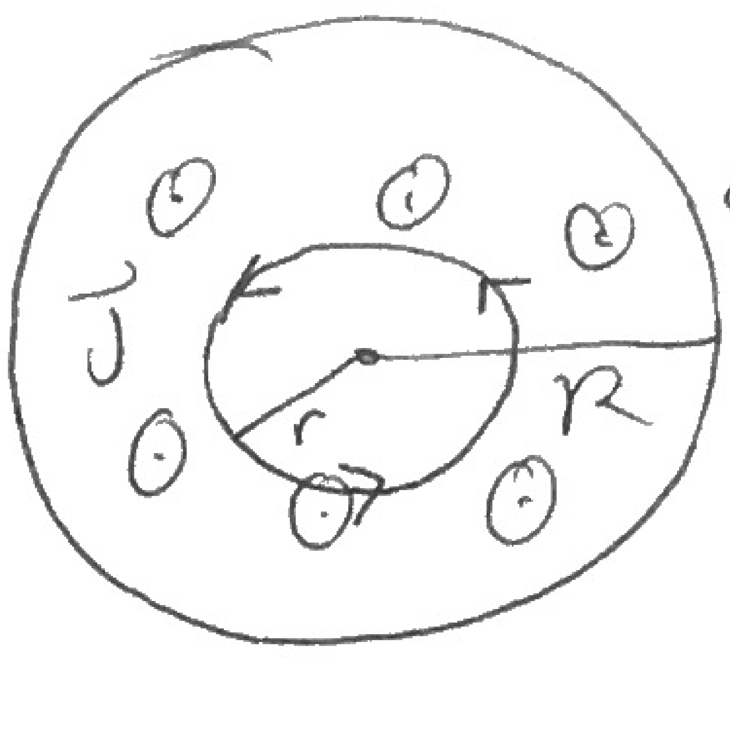
Problem Set #6

#1) PM 6.39



Cylindrical wire, coming out of the page, with current also coming out of the page.

Inside the wire, Ampere's law states that $\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc}$

where, assuming $\vec{J} = \vec{J}(r)$ (i.e., the current density is independent of the angle ϕ)

we have $I_{enc} = \int_0^r |\vec{J}| da = \int_0^r j(r') r' dr' d\phi = 2\pi \int j(r') r' dr'$

and, by symmetry, $\oint \vec{B} \cdot d\vec{l} = \underset{\text{magnitude}}{B} \cdot 2\pi r$,

Thus, we have $2\pi B r = 2\pi \mu_0 \int_0^r j(r') r' dr'$

$$\Rightarrow B = \frac{\mu_0}{r} \int_0^r j(r') r' dr'.$$

Thus we want $\int_0^r j(r') r' dr'$ to be proportional to r in order for B to be independent of r. This means we need $j(r') r'$ to be constant, so we need $j(r') = \frac{j_0 R}{r}$. (j_0 has units of current/area, and R has units of distance) constants

Then $B = \frac{\mu_0}{r} \int_0^r \frac{j_0 R}{r'} \cdot r' dr' = \frac{\mu_0}{r} j_0 R \int_0^r dr' = \mu_0 j_0 R = \text{constant}$, as desired.

$\boxed{\text{Thus, we need } j(r) \sim \frac{1}{r}.}$

#2 PM 6.41

We consider $(I) = \oint_C \vec{A} \cdot d\vec{l}$ where $d\vec{l}$ is the infinitesimal line element around the curve C.

We then have $(I) = \oint_C \vec{A} \cdot d\vec{l} \overset{\text{Stokes' Theorem}}{=} \int_S (\nabla \times \vec{A}) \cdot d\vec{a}$

where S is ~~the~~ a surface (any surface) bounded by the curve C.

Using $\nabla \times \vec{A} = \vec{B}$, we have $(I) = \int_S (\vec{\nabla} \times \vec{A}) \cdot d\vec{a} = \int_S \vec{B} \cdot d\vec{a} = \Phi$ where the last equality comes from the fact that $\int_S \vec{B} \cdot d\vec{a}$ is the definition of the (magnetic) flux through S.

#3) PM 6.42

We want to find $\vec{A}$ such that $B_x = 0, B_y = 0, B_z = B_0$,

i.e., such that $\nabla \times \vec{A} = (0, 0, B_0)$.

Well, $\nabla \times \vec{A} = (\partial_y A_z - \partial_z A_y, \partial_z A_x - \partial_x A_z, \partial_x A_y - \partial_y A_x)$

There are several possibilities, and we'll list ~~the~~ only a few.

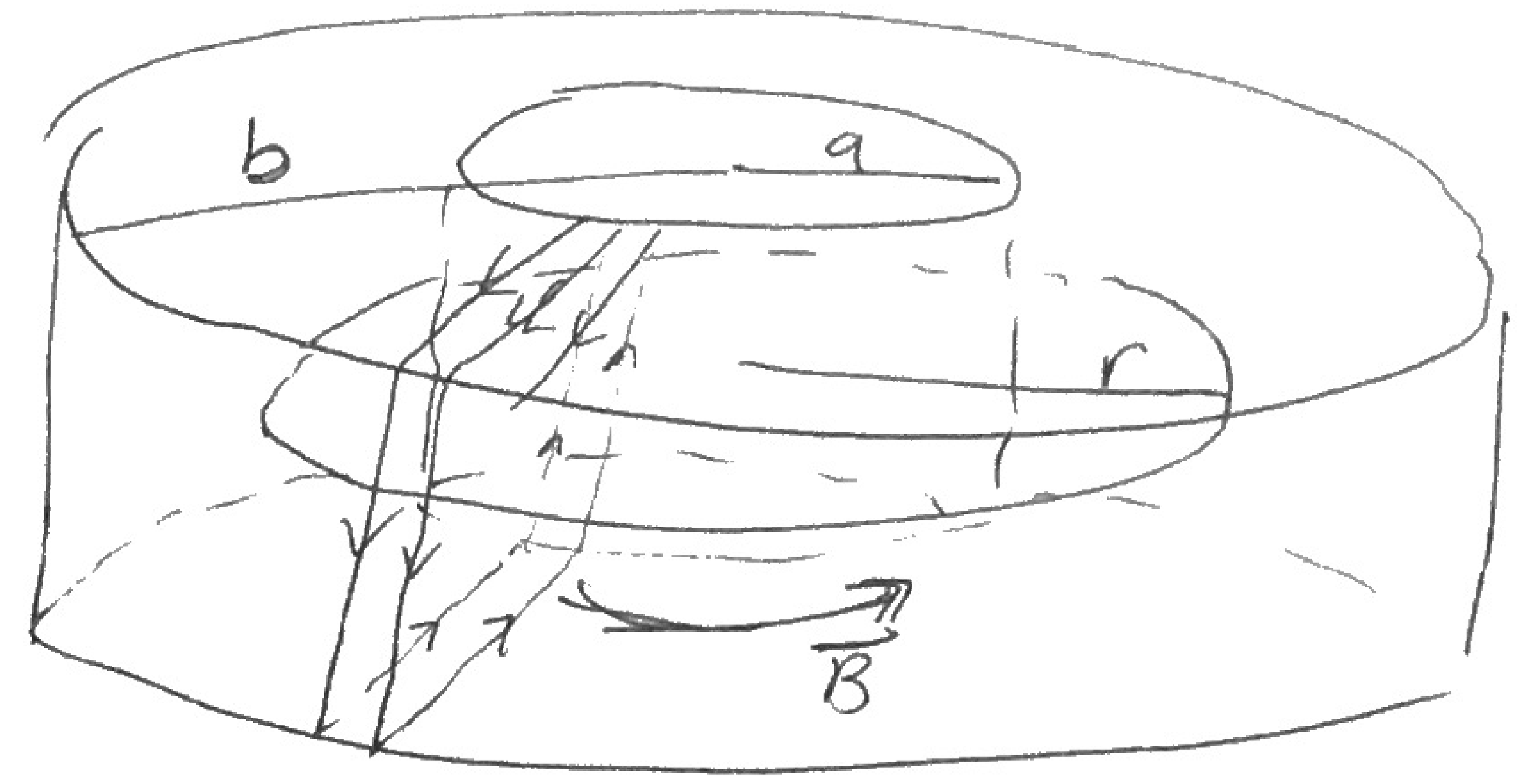
The simplest is $A_y = B_0 x$ (and $A_x = A_z = 0$)

or $A_x = -B_0 y$ (and $A_y = A_z = 0$)

or $A_y = \frac{B_0}{2} x$, $A_x = \frac{-B_0 y}{2}$, $A_z = 0$.

It is straightforward to check that these all work, and also to read off many other possibilities. For example, we could have $A_y = B_0 x + f(y)$ where $f(y)$ is any function of y. What others can you find?

#4) PM 6.61)



For the "convincing yourself" parts, I'll leave you to do that on your own (feel free to ask if you have any questions).

Now, assuming that $\vec{B}$ is circular, we again use Ampere's law for a circle of radius r, $a<r<b$. We have

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enc}$$ where $\oint \vec{B}\cdot d\vec{l} = 2\pi r B$ and where

I_{enc} is the total current piercing the surface whose boundary (as always) is the circle of radius r. We can choose any surface with this circle as its boundary, but for simplicity we will take it to be just the flat disk of radius r. Then, if the current in the wire is I, since there are N turns total we have that $I_{enc} = NI$ (the number of times the wire "pierces" the surface). Thus

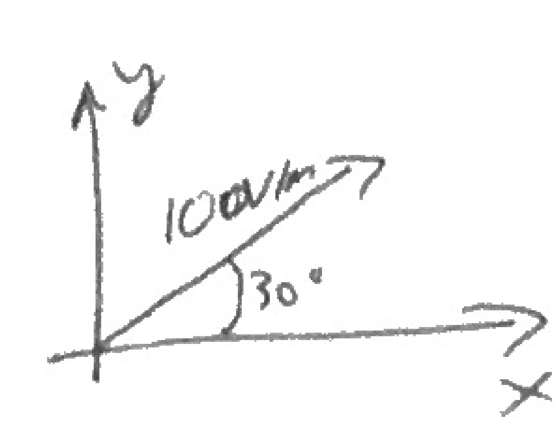
$$2\pi r B = \mu_0 I_{enc} = \mu_0 N I \Rightarrow \boxed{\vec{B} = \frac{\mu_0}{2\pi}\frac{N}{r} I \hat{\phi}}$$ angular direction.

In the $b-a \ll a$ limit the curvature is negligible and we expect to get $|B| = \mu_0 n I$ where n = # of turns per unit length. This is indeed exactly what we get since $\frac{N}{2\pi r}$ is exactly the # of turns per unit length on the above torus.

(5)

#5 PM 6.66

In frame F, ~~$|\vec{E}| = 100$ V/m~~ frame F



$$\vec{E} = 100\,\text{V/m}\left(\cos 30^\circ\,\hat{x} + \sin 30^\circ\,\hat{y}\right)$$
$$\vec{B} = 0$$

~~In frame F~~ Frame F' moves with speed $v = .6c$ in the positive $\hat{y}$ direction

Using equation (6.76), we have $\vec{E}_{\parallel}' = \vec{E}_{\parallel}$, $\vec{E}_{\perp}' = \gamma\vec{E}_{\perp}$, $\vec{B}_{\parallel}' = 0$, $\vec{B}_{\perp}' = -\frac{\gamma}{c^2}\vec{v}\times\vec{E}_{\perp}$ where the primed quantities are the observed quantities in F', and the unprimed quantities those in F, and where components are either parallel (denoted by $\parallel$) or perpendicular (denoted by $\perp$) to $\vec{v} = .6c\,\hat{y}$. (and $\vec{E}' = \vec{E}_{\parallel}' + \vec{E}_{\perp}'$, etc.)

Thus, $\vec{E}_{\parallel} = E_y\hat{y} = 100\sin 30^\circ\,\text{V/m}\,\hat{y} = 50\,\text{V/m}\,\hat{y}$ so that $\boxed{\vec{E}_{\parallel}' = 50\,\text{V/m}\,\hat{y}}$

Now, $\gamma = \frac{1}{\sqrt{1-(.6)^2}} = \frac{1}{\sqrt{1-.36}} = \frac{1}{\sqrt{.64}} = \frac{1}{.8} = \frac{5}{4}$.

Thus $\vec{E}_{\perp}' = \gamma\vec{E}_{\perp} = \frac{5}{4}(100\,\text{V/m})\underbrace{\cos 30^\circ}_{\frac{\sqrt{3}}{2}}\,\hat{x} = \frac{500\sqrt{3}}{8}\,\text{V/m}\,\hat{x}$.

Thus $\boxed{\begin{aligned}\vec{E}' &= \vec{E}_{\parallel}' + \vec{E}_{\perp}' \\ &= 50\left(\tfrac{5}{4}\sqrt{3}\,\hat{x} + \hat{y}\right)\text{V/m}\end{aligned}}$ (Direction and magnitude can now be read off in the usual way).

Similarly, $\vec{B}' \left(= \left(-\frac{\gamma}{c}\vec{v}\times\vec{E}_{\perp}\right)\right) = -\frac{5}{4}\frac{1}{c^2}\underbrace{(.6c)}_{\frac{3}{5}}(100\,\text{V/m})\underbrace{\cos 30^\circ}_{\frac{\sqrt{3}}{2}}\underbrace{\hat{y}\times\hat{x}}_{-\hat{z}} = \frac{3\sqrt{3}}{8c}\cdot 100\,\text{V/m}\,\hat{z}$

$$\Rightarrow \vec{B}' = \frac{3\sqrt{3}\cdot 100}{8\cdot 3\times 10^8}\,\frac{\text{V}\cdot\text{s}}{\text{m}^2}\,\hat{z} \Rightarrow \boxed{\vec{B}' \approx 2.165\times 10^{-7}\,\text{T}\,\hat{z}}$$