ton's second law of motion can be written vectorially as

$$\dot{\mathbf{p}} = f(r)\frac{\mathbf{r}}{r}. \tag{3.79}$$

The cross product of $\dot{\mathbf{p}}$ with the constant angular momentum vector $\mathbf{L}$ therefore can be expanded as

$$\begin{aligned}\dot{\mathbf{p}} \times \mathbf{L} &= \frac{mf(r)}{r}\left[\mathbf{r} \times (\mathbf{r} \times \dot{\mathbf{r}})\right] \\ &= \frac{mf(r)}{r}\left[\mathbf{r}(\mathbf{r} \cdot \dot{\mathbf{r}}) - r^2\dot{\mathbf{r}}\right].\end{aligned} \tag{3.80}$$

Equation (3.80) can be further simplified by noting that

$$\mathbf{r} \cdot \dot{\mathbf{r}} = \frac{1}{2}\frac{d}{dt}(\mathbf{r} \cdot \mathbf{r}) = r\dot{r}$$

(or, in less formal terms, the component of the velocity in the radial direction is $\dot{r}$). As $\mathbf{L}$ is constant, Eq. (3.80) can then be rewritten, after a little manipulation, as

$$\frac{d}{dt}(\mathbf{p} \times \mathbf{L}) = -mf(r)r^2\left(\frac{\dot{\mathbf{r}}}{r} - \frac{\mathbf{r}\dot{r}}{r^2}\right).$$

or

$$\frac{d}{dt}(\mathbf{p} \times \mathbf{L}) = -mf(r)r^2\frac{d}{dt}\left(\frac{\mathbf{r}}{r}\right). \tag{3.81}$$

Without specifying the form of $f(r)$, we can go no further. But Eq. (3.81) can be immediately integrated if $f(r)$ is inversely proportional to r^2—the Kepler problem. Writing $f(r)$ in the form prescribed by Eq. (3.49), Eq. (3.81) then becomes

$$\frac{d}{dt}(\mathbf{p} \times \mathbf{L}) = \frac{d}{dt}\left(\frac{mk\mathbf{r}}{r}\right),$$

which says that for the Kepler problem there exists a *conserved vector* $\mathbf{A}$ defined by

$$\mathbf{A} = \mathbf{p} \times \mathbf{L} - mk\frac{\mathbf{r}}{r}. \tag{3.82}$$

The relationships between the three vectors in Eq. (3.82) and the conservation of $\mathbf{A}$ are illustrated in Fig. 3.18, which shows the three vectors at different positions in the orbit. In recent times, the vector $\mathbf{A}$ has become known amongst physicists as the Runge–Lenz vector, but priority belongs to Laplace.

From the definition of $\mathbf{A}$, we can easily see that

$$\mathbf{A} \cdot \mathbf{L} = 0, \tag{3.83}$$

since $\mathbf{L}$ is perpendicular to $\mathbf{p} \times \mathbf{L}$ and $\mathbf{r}$ is perpendicular to $\mathbf{L} = \mathbf{r} \times \mathbf{p}$. It follows from this orthogonality of $\mathbf{A}$ to $\mathbf{L}$ that $\mathbf{A}$ must be some fixed vector in the plane of

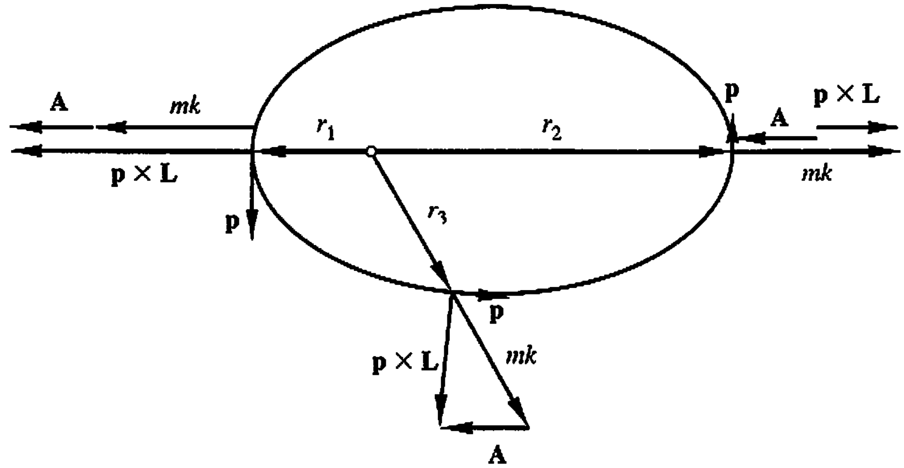

FIGURE 3.18 The vectors **p**, **L**, and **A** at three positions in a Keplerian orbit. At perihelion (extreme left) $|\mathbf{p}\times\mathbf{L}| = mk(1+e)$ and at aphelion (extreme right) $|\mathbf{p}\times\mathbf{L}| = mk(1-e)$. The vector **A** always points in the same direction with a magnitude mke.

the orbit. If θ is used to denote the angle between **r** and the fixed direction of **A**, then the dot product of **r** and **A** is given by

$$\mathbf{A}\cdot\mathbf{r} = Ar\cos\theta = \mathbf{r}\cdot(\mathbf{p}\times\mathbf{L}) - mkr. \tag{3.84}$$

Now, by permutation of the terms in the triple dot product, we have

$$\mathbf{r}\cdot(\mathbf{p}\times\mathbf{L}) = \mathbf{L}\cdot(\mathbf{r}\times\mathbf{p}) = l^2,$$

so that Eq. (3.84) becomes

$$Ar\cos\theta = l^2 - mkr,$$

or

$$\frac{1}{r} = \frac{mk}{l^2}\left(1 + \frac{A}{mk}\cos\theta\right). \tag{3.85}$$

The Laplace–Runge–Lenz vector thus provides still another way of deriving the orbit equation for the Kepler problem! Comparing Eq. (3.85) with the orbit equation in the form of Eq. (3.55) shows that **A** is in the direction of the radius vector to the perihelion point on the orbit, and has a magnitude

$$A = mke. \tag{3.86}$$

For the Kepler problem we have thus identified two vector constants of the motion **L** and **A**, and a scalar E. Since a vector must have all three independent components, this corresponds to seven conserved quantities in all. Now, a system such as this with three degrees of freedom has six independent constants of the motion, corresponding, say to the three components of both the initial position and the initial velocity of the particle. Further, the constants of the motion we have found are all algebraic functions of **r** and **p** that describe the orbit as a whole (orientation in space, eccentricity, etc.); none of these seven conserved quantities relate to where the particle is located in the orbit at the initial time. Since one constant of the motion must relate to this information, say in the form of T, the time of the perihelion passage, there can be only five independent constants of the motion describing the size, shape, and orientation of the orbit. We can therefore conclude that not all of the quantities making up **L**, **A**, and E can be independent; there must in fact be two relations connecting these quantities. One such relation has already been obtained as the orthogonality of **A** and **L**, Eq. (3.83). The other follows from Eq. (3.86) when the eccentricity is expressed in terms of E and l from Eq. (3.57), leading to

$$A^2 = m^2k^2 + 2mEl^2, \tag{3.87}$$

thus confirming that there are only five *independent* constants out of the seven.

The angular momentum vector and the energy alone contain only four independent constants of the motion: The Laplace–Runge–Lenz vector thus adds one more. It is natural to ask why there should not exist for any general central force law some conserved quantity that together with **L** and E serves to define the orbit in a manner similar to the Laplace–Runge–Lenz vector for the special case of the Kepler problem. The answer seems to be that such conserved quantities can in fact be constructed, but that they are in general rather peculiar functions of the motion. The constants of the motion relating to the orbit between them define the orbit, i.e., lead to the orbit equation giving r as a function of θ. We have seen that in general orbits for central force motion are not closed; the arguments of Section 3.6 show that closed orbits imply rather stringent conditions on the form of the force law. It is a property of nonclosed orbits that the curve will eventually pass through any arbitrary (r, θ) point that lies between the bounds of the turning points of r. Intuitively this can be seen from the nonclosed nature of the orbit; as θ goes around a full cycle, the particle must never retrace its footsteps on any previous orbit. Thus, the orbit equation is such that r is a multivalued function of θ (modulo 2π); in fact, it is an *infinite-valued function* of θ. The corresponding conserved quantity additional to **L** and E defining the orbit must similarly involve an infinite-valued function of the particle motion. Suppose the **r** variable is periodic with angular frequency ω_r and the angular coordinate θ is periodic with angular frequency ω_θ. If these two frequencies have a ratio (ω_r/ω_θ) that is an integer or integer fraction, periods are said to be *commensurate*. Commensurate orbits are closed with the orbiting mass continually retracing its path. When $\omega_\theta > \omega_r$ the orbit will spiral about the origin as the distance varies between the apsidal (maximum and minimum) values, closing only if the frequencies are commensurate. If, as in the Kepler problem, $\omega_r = \omega_e$, the periods are said to be degenerate. If the orbits are degenerate there exists an additional conserved quantity that is an algebraic function of **r** and **p**, such as the Runge–Lenz vector.

From these arguments we would expect a simple analog of such a vector to exist for the case of a Hooke's law force, where, as we have seen, the orbits are