105A - Set 3.5 - Bonus Solutions

(Grades are out of 150)

1. Consider the motion of a skier of mass m down a track with a constant slope θ. The friction of the skies is negligible, but the motion is subject to air resistance. The resistance force is given by $f = -kv^2$. The initial velocity of the skier is equal to zero.

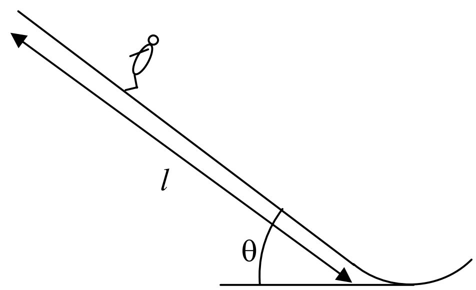

Figure 1: A skier.

(a) What is the velocity of the skier as a function of time?
Answer: Newton second law gives:

$$m\frac{dv}{dt} = mg\sin\theta - kv^2 \tag{1}$$

which we can write as:

$$\frac{dv}{g\sin\theta - kv^2/m} = dt \tag{2}$$

and integrating both sides we have:

$$t = \frac{m}{k}\int_0^v \frac{dv}{mg\sin\theta/k - v^2} \tag{3}$$

for which the solution is:

$$t = \frac{m}{k}\frac{1}{\sqrt{mg\sin\theta/k}}\tanh^{-1}\frac{v}{\sqrt{mg\sin\theta/k}} \tag{4}$$

From which we find:

$$v = \sqrt{vg\sin\theta/k}\,\tanh(t\sqrt{kg\sin\theta/m}) \tag{5}$$

(b) What is the distance l traveled by the skier as a function of time?
Answer: We simply integrate over the expression for the velocity we found above:

$$l = \int_0^t vdt = \sqrt{vg\sin\theta/k}\int_0^t \tanh(t\sqrt{kg\sin\theta/m})dt = \frac{m}{k}\ln\cosh(t\sqrt{kg\sin\theta/m}) \tag{6}$$

(c) How far does he travel by the time his velocity reaches approximately 90% of its maximum possible value?
Answer The maximum velocity will be reached if the argument inside the $\tanh^{-1}$ will be equal to 1, see Equations (4) and (5). So all we need to do is to plug in 0.9. Note that $\tanh^{-1}(0.9) = 1.47$ so

$$t_{90}\sqrt{kg\sin\theta/m} = 1.47 \tag{7}$$

so we can plug it now in the expression for l and find:

$$l_{90} = \frac{m}{k}\ln\cosh(1.47) = 0.83\frac{m}{k} \tag{8}$$

(d) The coefficient of air resistance is approximately given by $k = \rho A/2$, where ρ is the density of air and A is the cross-sectional area of the skier. Taking $\rho = 1$ kg m^{-3}, $A = 1$ m^2, $m = 100$ kg and $\theta = 45°$,estimate the length l needed to achieve 90% of the maximum velocity. How does it compare with the length of a typical ski jump track of about 100 m?
Answer: plugging in we have

$$l_{90} = \frac{100kg}{1kg/m^2 1m^2/2}0.833 = 166m \tag{9}$$

So, the track is nearly as long as it takes to reach the maximum velocity. The coefficient of air resistance is probably actually smaller because of the aerodynamic suits.

2. Find the shortest distance between two points that are located on a spherical surface with radius R.
Hint1: use spherical coordinates.
Hint2: Guess the solution by knowing that the shortest distance between points in the sphere is a great circle, which I can write as:

$$\frac{\cos\psi}{\sin\psi} = A\cos\theta + B\sin\theta \quad \textit{with} \quad A^2 + B^2 < 1 \tag{10}$$

Answer: Points on a sphere of radius R are determined by two angular coordinates, an azimuthal angle and a polar angle θ, in other words:

$$\mathbf{r} = x\hat{i} + y\hat{j} + z\hat{k} = R(\sin\phi\cos\theta\hat{i} + \sin\psi\sin\theta\hat{j} + \cos\psi\hat{k}) \tag{11}$$

When moving on the sphere, the differential arc length ds is

$$\begin{aligned} ds^2 &= dx^2 + dy^2 + dz^2 \\ &= R^2([\cos\psi\cos\theta d\psi - \sin\psi\sin\theta d\theta]^2 + [\cos\psi\sin\theta d\psi + \sin\psi\cos\theta d\theta]^2 +) - \sin\psi d\psi]^2) \\ &= R^2(d\psi^2 + \sin^2\psi d\theta^2) \end{aligned} \tag{12}$$

The distance on the sphere between two points is then

$$
\begin{aligned}
l &= \int ds = R\int\sqrt{d\psi^2+\sin^2\psi d\theta^2} \\
&= R\int d\theta\sqrt{\left(\frac{d\psi}{d\theta}\right)^2+\sin^2\psi} = R\int d\theta f(\psi,\psi') \qquad (13)
\end{aligned}
$$

We can use a variational principle for finding the path with minimum length between two point. The path is described by a function $\psi(\theta)$ and the (differential) equation for ψ can be obtained from the Euler-Lagrange equation using $f(\psi,\psi')=\sqrt{\sin^2\psi+\psi'^2}$. Thus the equation is:

$$
\begin{aligned}
0 &= \frac{d}{d\theta}\frac{\partial f}{\partial\psi'}-\frac{\partial f}{\partial\psi} && (14)\\
&= \frac{d}{d\theta}\left(\frac{\psi'}{\sqrt{\sin^2\psi+\psi'^2}}\right)-\frac{\sin\psi\cos\psi}{\sqrt{\sin^2\psi+\psi'^2}} && (15)\\
&= \frac{\psi''}{\sqrt{\sin^2\psi+\psi'^2}}-\frac{\psi'}{\sin^2\psi+\psi'^2}\frac{d}{d\theta}\sqrt{\sin^2\psi+\psi'^2}-\frac{\sin\psi\cos\psi}{\sqrt{\sin^2\psi+\psi'^2}} && (16)\\
&= \frac{\psi''}{\sqrt{\sin^2\psi+\psi'^2}}-\frac{\psi'\sqrt{\sin^2\psi+\psi'^2}}{\sin^2\psi+\psi'^2}\frac{\psi'\sin\psi\cos\psi+\psi'\psi''}{\sqrt{\sin^2\psi+\psi'^2}}-\frac{\sin\psi\cos\psi}{\sqrt{\sin^2\psi+\psi'^2}} && (17)
\end{aligned}
$$

Multiplying both sides by $(\sin^2\psi+\psi'^2)^{3/2}$ we can write:

$$
\begin{aligned}
0 &= (\psi''-\sin\psi\cos\psi)(\sin^2\psi+\psi'^2)-\psi'^2(\psi''+\sin\psi\cos\psi) && (18)\\
0 &= \psi''\sin^2\psi-2\psi'^2\sin\psi\cos\psi-\sin^3\psi\cos\psi && (19)
\end{aligned}
$$

This looks like a complicated equation to solve! Its always useful if we know the solution before we obtain it, admittedly not the most common case, but true in this case. We know that the shortest path between points in the sphere are great circles. Great circles are the intersection between the sphere and a plane. If the unit vector normal to the plane as $\hat{n}=a\hat{i}+b\hat{j}+c\hat{k}$, the points in the great circle are those points in the sphere that satisfy $\hat{n}\cdot\mathbf{r}=0$ in other words we guess the solution:

$$q(\theta)=\frac{\cos\psi}{\sin\psi}=A\cos\theta+B\sin\theta \quad \text{with} \quad A^2+B^2<1 \qquad (20)$$

which describes a great circle. We are now looking for an equation of the form

$$\frac{dq^2}{d^2\theta}=-q \qquad (21)$$

If $q=1/\tan\theta$ than

$$\frac{dq}{d\theta}=q'=-\frac{\psi'}{\sin^2\psi} \qquad (22)$$

and

$$\frac{d^2q}{d\theta^2}=q''=-\frac{\psi''}{\sin^2\psi}+2\psi'^2\frac{\cos\psi}{\sin^3\psi} \qquad (23)$$

In other words, $\psi'' \sin^2 \psi - 2\psi'^2 \sin\psi \cos\psi = -q'' \sin^4 \psi$ so we can write Lagranges equation in the following way:

$$\psi'' \sin^2 \psi - 2\psi'^2 \sin\psi\cos\psi - \sin^3\psi\cos\psi = -q''\sin^4\psi - \sin^3\psi\cos\psi = 0 \tag{24}$$

which I can rearrange and write:

$$q'' = -\frac{\cos\psi}{\sin\psi} = -q \tag{25}$$

which is the equation we were looking for, with a general solution

$$q(\theta) = \frac{\cos\psi}{\sin\psi} = A\cos\theta + B\sin\theta \quad \text{with} \quad A^2 + B^2 < 1 \tag{26}$$

which describes a great circle. So indeed a great circle is the shortest path between two points.

3. A small mass m can slide without friction on the inside of a conical surface with opening angle α. Use cylindrical coordinates ρ, θ and z to describe the position of the mass with z =0 at the apex of the cone. Gravity points down.

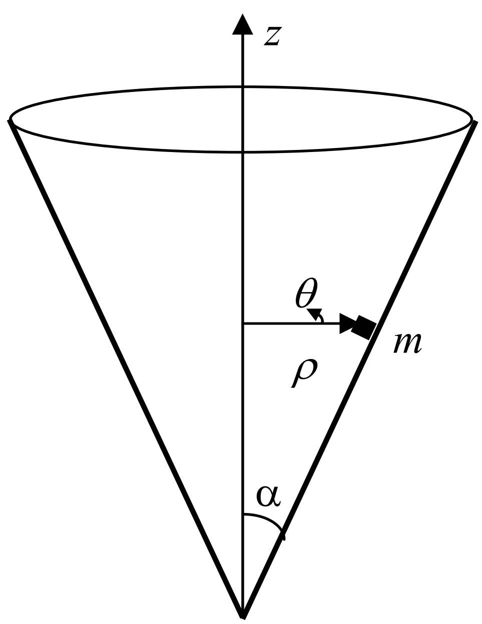

Figure 2: Conical surface

(a) Find the Lagrangian for the mass m in terms of coordinates ρ and θ.
Answer: The relation between z and ρ can be easily expresses as $z = \rho\cot\alpha$. So the Lagrangian is:

$$L = \frac{1}{2}m(\dot\rho^2 + \dot z^2 + \rho^2\dot\theta^2) - mgz = \frac{1}{2}m(\dot\rho^2[1+\cot^2\alpha] + \rho^2\dot\theta^2) - mg\rho\cot\alpha \tag{27}$$

(b) Find the equation of motions using the Euler Lagrange differential equations for ρ and θ.
Answer: So first we need to find the following:

$$\begin{aligned}
\frac{\partial L}{\partial \rho} &= m\rho\dot{\theta}^2 - mg\cot\alpha && (28)\\
\frac{\partial L}{\partial \theta} &= 0 && (29)\\
\frac{\partial L}{\partial \dot{\rho}} &= m\dot{\rho}(1+\cot^2\alpha) && (30)\\
\frac{\partial L}{\partial \dot{\theta}} &= m\rho^2\dot{\theta} && (31)
\end{aligned}$$

so:

$$\begin{aligned}
\rho : \quad \frac{\partial L}{\partial \rho} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\rho}} &= m\rho\dot{\theta}^2 - mg\cot\alpha - m\ddot{\rho}(1+\cot^2\alpha) = 0 && (32)\\
\theta : \quad \frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} &= m\rho^2\ddot{\theta} = 0 && (33)
\end{aligned}$$

(c) Find a general solution for the problem.
Hint: remember that α is some constant, and simplify your equation so it will look familiar
Answer: We'll start from the equation for $\ddot{\theta}$ which is true for every ρ so we have basically $\ddot{\theta} = 0$ integrating once we have $\dot{\theta} = C$ where C is constant. Plygging this into the equation for ρ we have:

$$m\rho C^2 - mg\cot\alpha - m\ddot{\rho}(1+\cot^2\alpha) = 0 \quad (34)$$

which I can write as:

$$\ddot{\rho} - \frac{C^2}{1+\cot^2\alpha}\rho = -\frac{g\cot\alpha}{1+\cot^2\alpha} \quad (35)$$

Denoting $\lambda^2 = C^2/(1+\cot^2\alpha)$ I'll guess a solution of the form $\rho = \rho_0 + \rho_p = Ae^{(\lambda t+\phi)} + B$.
Plugging this into the homogeneous equation we get:

$$\lambda^2 - \lambda^2 = 0 \quad (36)$$

So indeed this is the result of the homogeneous equation. To find the particular solution we have:

$$\lambda^2 B = -\frac{g\cot\alpha}{1+\cot^2\alpha} \quad (37)$$

so

$$B = -\frac{g\cot\alpha}{\lambda^2(1+\cot^2\alpha)} \quad (38)$$

So the general solution is:

$$\rho = Ae^{(\lambda t+\phi)} - \frac{g\cot\alpha}{\lambda^2(1+\cot^2\alpha)} \quad (39)$$