Name: ______________________

Student ID number: ______________________

105A -Practice Final

A similar format to the actual final

Please read the following very carefully

- This is a closed book exam. You may use a calculator. All other electronic devices should **not** be around!
- You have **3 hours** to complete the exam.
- Grades are out of 150
- Answer all **three** subquestions, i.e., (a), (b), and (c).
- Make sure to write your name at the top of each page of this exam. Use the space provided on the exam pages to do your work. You may use the back of the pages also, but please mark clearly which problem you are working on (and also state underneath that problem that you have done work on the back of the page).
- Partial credit will be given. Show as much work/justification as possible (diagrams where appropriate). If you can not figure out how to complete a particular computation, a written statement of the concepts involved and qualitative comments on what you think the answer should be may be assigned partial credit.
- Mistakes in grading: If you find a mistake in the grading of your exam, alert the instructor within one week of the exams being returned DO NOT write on the returned graded exam you may make a note of the problems you thought were misgraded on a separate page.
- In class we showed that the equation of motion can be expressed using the force $F = -dU/dr$ in the form (i.e., the Binnet's equation):

$$\frac{l^2u^2}{\mu}\left(\frac{d^2u}{d\theta^2}+u\right) = -F(1/u)\ , \tag{1}$$

where $l = mr^2\dot{\theta}$ is the angular momentum and $u = 1/r$.

1. A pendulum consists of a mass m suspended by a massless spring with unextended length b and spring constant k (see Figure 1). Gravity is pointed downwards.

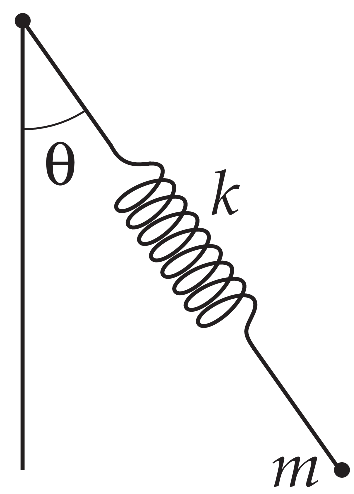

Figure 1: A spring pendulum .

(a) Write the Lagrangian of the system.
Answer: Setting l as the variable length of spring we can write the kinetic energy as:

$$T = \frac{m}{2}\left(\dot{l}^2 + l^2\dot{\theta}^2\right) \tag{2}$$

and the potential is:

$$U = \frac{1}{2}k\left(l-b\right)^2 + mgy = \frac{1}{2}k\left(l-b\right)^2 - mgl\cos\theta \tag{3}$$

And the Lagrangian is:

$$L = T - U = \frac{m}{2}\left(\dot{l}^2 + l^2\dot{\theta}^2\right) - \frac{1}{2}k\left(l-b\right)^2 + mgl\cos\theta \tag{4}$$

(b) Find the equations of motion.

$$l \quad : \quad \frac{d}{dt}(m\dot{l}) = ml\dot{\theta}^2 - k(l-b) + mg\cos\theta \tag{5}$$

$$\theta \quad : \quad \frac{d}{dt}(ml^2\dot{\theta}) = -mgl\sin\theta \tag{6}$$

So

$$l \quad : \quad \ddot{l} - l\dot{\theta}^2 + \frac{k}{m}(l-b) - g\cos\theta = 0 \tag{7}$$

$$\theta \quad : \quad \ddot{\theta} + \frac{2}{l}\dot{l}\dot{\theta} + \frac{g}{l}\sin\theta = 0 \tag{8}$$

(c) Assume small oscillation $\theta << 1$ rad. Which means that you can approximate $\sin\theta \sim \theta$ and $\cos\theta \sim 1$. Keep only first order terms (i.e., $\theta^2 \to 0$ and also $\dot{\theta} \to 0$). Find the solution (coordinate as a function of time) for **only** the **radial** component of the pendulum (i.e., no need to work on the solution of $\theta(t)$).
Assume that the pendulum started at rest and that the spring is initially unextended (i.e., initial length of the spring is b).
Express your answer **only** it terms of k, b, m, g and of course t.
Hint: Remember: A solution of a second order differential equation of the form $y'' + py' + qy = C = \text{Const}$ *is* $y = Y_0 + y_p$ *where* Y_0 *is the solution of the homogenous equation, i.e.,* $y'' + py' + qy = 0$ *and* $y_p = A$ *where* A *is a constant to be found by plugging in the solution to the equation*
Answer: To solve we will start with

$$\ddot{l} - l\dot{\theta}^2 + \frac{k}{m}(l - b) - g\cos\theta = 0 \tag{9}$$

Which for small oscillations is reduced to

$$\ddot{l} + \frac{k}{m}(l - b) - g = 0 \tag{10}$$

Changing $u = l - b$ and $\ddot{u} = \ddot{l}$ we write:

$$\ddot{u} + \frac{k}{m}u = g \tag{11}$$

The corresponding homogeneous equation is

$$\ddot{u} + \frac{k}{m}u = \ddot{u} + \omega_0^2 u = 0 \tag{12}$$

where $\omega_0 = \sqrt{k/m}$. For which the solution is

$$u = A\cos(\omega_0 t + \delta) \tag{13}$$

The peculiar solution is: $u = C$, for which $C = g/\omega_0^2$. So the final solution is:

$$l = b + A\cos(\omega_0 t + \delta) + \frac{g}{\omega_0^2} \tag{14}$$

To find the constants we will set the initial conditions, i.e.,:

$$l(t = 0) = b = b + A\cos(\delta) + \frac{g}{\omega_0^2} \tag{15}$$

and find the first time derivative:

$$\dot{l} = -A\omega_0 \sin(\omega_0 t + \delta) \tag{16}$$

for which the initial condition is:

$$\dot{l}(t = 0) = 0 = -A\omega_0 \sin(\delta) \tag{17}$$

For which we find that $\delta = 0$. Plugging this to equation (18), we find:

$$A = -\frac{g}{\omega_0^2} \tag{18}$$

So finally we have:

$$l = b - \frac{gm}{k}\cos\left(\frac{k}{m}t\right) + \frac{gm}{k} \tag{19}$$

Or

$$l = b - \frac{g}{\omega_0^2}\cos(\omega_0 t) + \frac{g}{\omega_0^2} \tag{20}$$

2. A hollow cylinder of mass M and radius R rolls without slipping on a table. The end of the cylinder hangs off the edge of the table and is constructed in such a way that a pendulum is suspended from the axis of the cylinder. The pendulum rod, assumed rigid and massless, has length l, and the pendulum bob has mass m. Assume that the pendulum can rotate without friction in its support.

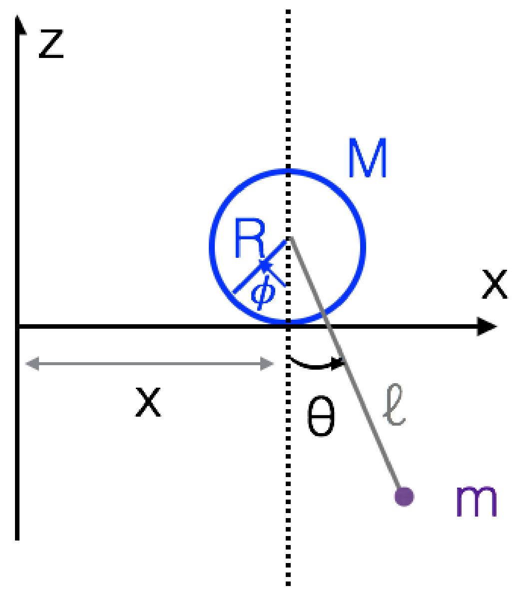

Figure 2: A hollow cylinder and a pendulum

(a) Define carefully a set of independent generalized coordinates for this system. Think about the direction of your angles!
Answer The coordinates are drawn on the figure. They are X and θ, where X is the displacement of the center of mass of the cylinder. The rolling of the cylinder implies that $X = R\phi$. θ is the angle of the pendulum relative to the vertical. The Cartesian coordinates can be expressed in the following way:

$$x = X + l\sin\theta = R\phi + l\sin\theta \tag{21}$$
$$z = R - l\cos\theta \tag{22}$$

A choice of θ and ϕ is also good.

(b) Write down the Lagrangian in terms of the generalized coordinates you defined in (a).

Answer: The kinetic energy is $T = T_{pendulum} + T_{M,Center of Mass} + T_{M,Ror}$

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{z}^2) + \frac{1}{2}M\dot{X}^2 + \frac{1}{2}MR^2\dot{\phi}^2 = \frac{1}{2}m(\dot{x}^2 + \dot{z}^2) + \frac{1}{2}M\dot{X}^2 + \frac{1}{2}M\dot{X}^2 \quad (23)$$

The time derivatives of the coordinates are

$$\dot{x} = \dot{X} + l\dot{\theta}\cos\theta = R\dot{\phi} + l\dot{\theta}\cos\theta \quad (24)$$

$$\dot{z} = l\dot{\theta}\sin\theta \quad (25)$$

So in terms of X and θ the kinetic energy is:

$$\begin{aligned} T &= \frac{1}{2}m(\dot{X}^2 + 2\dot{X}l\dot{\theta}\cos\theta + (l\dot{\theta}\cos\theta)^2 + (l\dot{\theta}\sin\theta)^2) + M\dot{X}^2 \\ &= \left(M + \frac{1}{2}m\right)\dot{X}^2 + \frac{1}{2}ml^2\dot{\theta}^2 + m\dot{X}l\dot{\theta}\cos\theta \end{aligned} \quad (26)$$

and in terms of θ and ϕ:

$$T = \left(M + \frac{1}{2}m\right)R^2\dot{\phi}^2 + \frac{1}{2}ml^2\dot{\theta}^2 + m\dot{X}l\dot{\theta}\cos\theta + mR\dot{\phi}l \quad (27)$$

The potential energy is $U = -mgl\cos\theta$. And the Lagrangian is:
In terms of θ and X:

$$L = \left(M + \frac{1}{2}m\right)\dot{X}^2 + \frac{1}{2}ml^2\dot{\theta}^2 + m\dot{X}l\dot{\theta}\cos\theta + m\dot{X}l + mgl\cos\theta \quad (28)$$

In terms of θ and ϕ:

$$L = \left(M + \frac{1}{2}m\right)R^2\dot{\phi}^2 + \frac{1}{2}ml^2\dot{\theta}^2 + mR\dot{\phi}l\dot{\theta}\cos\theta + mR\dot{\phi}l + mgl\cos\theta \quad (29)$$

(c) What are the conserved quantities? Explain your answer. Express the conserved quantities in terms of your coordinates.
Answer: P_X is conserved because the Lagrangian does not depend on X.

$$\frac{\partial L}{\partial \dot{X}} = P_X = (2M + m)\,\dot{X} + ml\dot{\theta}\cos\theta = \text{Const} \tag{30}$$

In terms of ϕ, which is cyclic thus P_ϕ is conserved

$$\frac{\partial L}{\partial \dot{\phi}} = P_\phi = (2M + m)\,R\dot{\phi} + mlR\dot{\theta}\cos\theta = \text{Const} \tag{31}$$

The second quantity is the energy which is conserved because the Lagrangian does not depend explicitly on time In terms of θ and X:

$$E = \left(M + \frac{1}{2}m\right)\dot{X}^2 + \frac{1}{2}ml^2\dot{\theta}^2 + m\dot{X}l\dot{\theta}\cos\theta + m\dot{X}l - mgl\cos\theta = \text{Const} \tag{32}$$

In terms of θ and ϕ:

$$E = \left(M + \frac{1}{2}m\right)R^2\dot{\phi}^2 + \frac{1}{2}ml^2\dot{\theta}^2 + mR\dot{\phi}l\dot{\theta}\cos\theta + mR\dot{\phi}l - mgl\cos\theta = \text{Const} \tag{33}$$

3. A particle is moving in a central inverse-square-law force field for a superimposed force which magnitude is inversely proportional to the cube of the distance from the particle to the force center. in other words:

$$F = -\frac{k}{r^2} - \frac{\lambda}{r^3} \quad k, \lambda > 0 \tag{34}$$

(a) Write down the energy and the effective potential.
Answer: The potential is:

$$U = -\int F dr = -\frac{k}{r} - \frac{\lambda}{2r^2} \tag{35}$$

The energy is:

$$E = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + U(r) = \frac{1}{2}m\dot{r}^2 + \frac{l^2}{2mr^2} - \frac{k}{r} - \frac{\lambda}{2r^2} \tag{36}$$

and the effective potential is

$$U_{eff} = \frac{l^2}{2mr^2} - \frac{k}{r} - \frac{\lambda}{2r^2} = -\frac{k}{r} + \left(\frac{l^2}{2m} - \frac{\lambda}{2}\right)\frac{1}{r^2} \tag{37}$$

(b) Find the extremum point of this potential. Express your answer in l, m, λ and k.
Answer:

$$\frac{dU_{eff}}{dr} = \frac{k}{r^2} - \left(\frac{l^2}{m} - \lambda\right)\frac{1}{r^3} = 0 \tag{38}$$

so

$$kr = \left(\frac{l^2}{m} - \lambda\right) \tag{39}$$

$$r_0 = \left(\frac{l^2}{m} - \lambda\right)\frac{1}{k} \tag{40}$$

(c) What should be the condition for a stable point? Meaning should it be $\lambda < l^2/\mu$ or $\lambda > l^2/\mu$?

Hint: It will help if you'll define:

$$\frac{l^2}{m} - \lambda = \alpha^2 \tag{41}$$

and check if α^2 is larger or smaller than zero.

Answer: To find if the point is stable or unstable we need to take the second derivative of the effective potential, i.e,

$$\begin{aligned} \left.\frac{d^2U_{eff}}{dr^2}\right|_{r_0} &= -2\frac{k}{r_0^3} + 3\left(\frac{l^2}{m} - \lambda\right)\frac{1}{r_0^4} \\ &= -2\frac{k}{(\alpha^2/k)^3} + \frac{3\alpha^2}{(\alpha^2/k)^4} = \frac{k^4}{\alpha^6} > 0 \end{aligned} \tag{42}$$

where the > 0 means that the point is stable. Thus we need that $\alpha^6 > 0$ which means that for

$$\alpha^2 = \frac{l^2}{m} - \lambda > 0 \tag{43}$$

we have a stable point. So $\lambda < l^2/\mu$.

(d) What type of orbit this stable point represent?

Answer: Circle.

(e) Assume that $\lambda = l^2/\mu$, and find the most general solution of $r(\theta)$.

Answer: Using Binnet's equation we can write:

$$\frac{l^2u^2}{\mu}\left(\frac{d^2u}{d\theta^2} + u\right) = -F(1/u) \ , \tag{44}$$

where in our case

$$F = -\frac{k}{r^2} - \frac{\lambda}{r^3} = -ku^2 - \lambda u^3 \tag{45}$$

So Binnet's equation is then

$$\frac{l^2u^2}{\mu}\left(\frac{d^2u}{d\theta^2} + u\right) = ku^2 + \lambda u^3 \ , \tag{46}$$

arranging we can write:

$$\frac{d^2u}{d\theta^2} = \frac{\mu}{l^2}k \ , \tag{47}$$

so

$$\frac{1}{r} = u = \frac{\mu k}{2l^2}\theta^2 + A\theta + B \tag{48}$$

from which we see that r continuously decreases as θ increases; that is, the particle spirals in toward the force center.