Week 6 QM Discussion

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Office Hours: Tuesday 10am-12pm, Tutoring Center.

Add Spin-1/2 and Spin-1

In class we add two spin-1/2 particles together and we get spin-1 and spin-0 which correspond to triplet states and single state. Now, can you add a spin-1/2 particle and spin-1 particle together? Follow the steps below.

1) We denote spin-1/2 particle as $\vec{S}^{(1)}$, spin-1 particle as $\vec{S}^{(2)}$. When we add them together, we have $\vec{S} = \vec{S}^{(1)} + \vec{S}^{(2)}$. What's the possible values of S?

2) Before we add them together, we use the decoupled basis which are labeled by $\left|S^1, S_z^1; S^2, S_z^2\right\rangle$. After we add them together, we use coupled basis which are labeled by $|S, S_z\rangle$ where $S_z = S_z^1 + S_z^2$. Express the coupled basis in terms the coupled basis.

Solution:

add spin $\frac{1}{2}$ and spin 1

$$\vec{S} = \vec{S}^{(1)} + \vec{S}^{(2)}$$

in old basis: $|S^{(1)}, S_z^{(1)}; S^{(2)}, S_z^{(2)}\rangle$ to label the basis.

1° What's the possible ~~value~~ value of S ?

S: $|S^{(1)} + S^{(2)}|$, $|S^{(1)} + S^{(2)} - 1|$, $\cdots$, $|S^{(1)} - S^{(2)}|$

here: $\frac{3}{2}$, $\frac{1}{2}$

so: $S = \frac{3}{2}$ or $S = \frac{1}{2}$

Then:

in coupled basis, we have: $|S, S_z\rangle$

$$\left.\begin{array}{l} |\frac{3}{2}, \frac{3}{2}\rangle \\ |\frac{3}{2}, \frac{1}{2}\rangle \\ |\frac{3}{2}, -\frac{1}{2}\rangle \\ |\frac{3}{2}, -\frac{3}{2}\rangle \end{array}\right\} \text{spin } \frac{3}{2}$$

$$\left.\begin{array}{l} |\frac{1}{2}, \frac{1}{2}\rangle \\ |\frac{1}{2}, -\frac{1}{2}\rangle \end{array}\right\} \text{spin } \frac{1}{2}$$

we need express

$|s, s_z\rangle$ in terms of $|s_z^{(1)}, s_z^{(2)}\rangle$

can be:

$|\frac{1}{2}, 1\rangle$

$|\frac{1}{2}, 0\rangle$

$|\frac{1}{2}, -1\rangle$

$|-\frac{1}{2}, 1\rangle$

$|-\frac{1}{2}, 0\rangle$

$|-\frac{1}{2}, -1\rangle$

act S_z on states $|s_z^{(1)}\ s_z^{(2)}\rangle$

$$S_z |\tfrac{1}{2}, 1\rangle = (S_z^1 + S_z^2)\ |\tfrac{1}{2}, 1\rangle = \tfrac{3}{2}\ |\tfrac{1}{2}, 1\rangle$$

$$S_z |\tfrac{1}{2}, 0\rangle = \tfrac{1}{2} |\tfrac{1}{2}, 0\rangle$$

$$S_z |\tfrac{1}{2}, -1\rangle = -\tfrac{1}{2} |\tfrac{1}{2}, 0\rangle$$

$$S_z |-\tfrac{1}{2}, 1\rangle = \tfrac{1}{2} |-\tfrac{1}{2}, 1\rangle$$

$$S_z |-\tfrac{1}{2}, 0\rangle = -\tfrac{1}{2} |-\tfrac{1}{2}, 0\rangle$$

$$S_z |-\tfrac{1}{2}, -1\rangle = -\tfrac{3}{2} |-\tfrac{1}{2}, -1\rangle$$

Then:

$\boxed{|\tfrac{3}{2}, \tfrac{3}{2}\rangle = |\tfrac{1}{2}, 1\rangle}$ (because there is no other choice)

$\boxed{|\tfrac{3}{2}, -\tfrac{3}{2}\rangle = |-\tfrac{1}{2}, -1\rangle}$ (the same reason above)

For $|\frac{3}{2},\frac{1}{2}\rangle$ or $|\frac{1}{2},\frac{1}{2}\rangle$, they can be linear superposition of $|\frac{1}{2},0\rangle$ and $|-\frac{1}{2},1\rangle$

Then: we need work it out explicitly:

starting from $|\frac{3}{2},\frac{3}{2}\rangle = |\frac{1}{2},1\rangle$

we have: $S_-|\frac{3}{2},\frac{3}{2}\rangle = (S_-^1 + S_-^2)|\frac{1}{2},1\rangle$

$$L.H.S = \sqrt{\frac{3}{2}\times\frac{5}{2} - \frac{3}{2}\times\frac{1}{2}}\ |\frac{3}{2},\frac{1}{2}\rangle = \sqrt{3}\ |\frac{3}{2},\frac{1}{2}\rangle$$

$$R.H.S = S_{1-}|\frac{1}{2},1\rangle + S_-^2|\frac{1}{2},1\rangle$$

$$= \sqrt{\frac{1}{2}\times\frac{3}{2} - \frac{1}{2}\times(-\frac{1}{2})}\ |-\frac{1}{2},1\rangle + \sqrt{1\times2 - 1\times0}\ |\frac{1}{2},0\rangle$$

$$= |-\frac{1}{2},1\rangle + \sqrt{2}\,|\frac{1}{2},0\rangle$$

$$\Rightarrow \boxed{|\frac{3}{2},\frac{1}{2}\rangle = \frac{1}{\sqrt{3}}|-\frac{1}{2},1\rangle + \sqrt{\frac{2}{3}}\,|\frac{1}{2},0\rangle}$$

due to: $\langle\frac{1}{2},\frac{1}{2}|\frac{3}{2},\frac{1}{2}\rangle = 0.$

So: we can choose:

$$\boxed{|\frac{1}{2}\frac{1}{2}\rangle = \sqrt{\frac{2}{3}}\,|-\frac{1}{2},1\rangle - \sqrt{\frac{1}{3}}\,|\frac{1}{2},0\rangle}$$

For $|\frac{3}{2},-\frac{1}{2}\rangle$ or $|\frac{1}{2},-\frac{1}{2}\rangle$ they can be superposition of $|\frac{1}{2},-1\rangle$ and $|-\frac{1}{2},0\rangle$

Start from: $\left|\frac{3}{2},-\frac{3}{2}\right\rangle = \left|-\frac{1}{2},-1\right\rangle$

$$S_+\left|\frac{3}{2},-\frac{3}{2}\right\rangle = \left(S_+^1 + S_+^2\right)\left|-\frac{1}{2},-1\right\rangle$$

$$L.H.S = \sqrt{\frac{3}{2}\times\frac{5}{2}-\left(-\frac{3}{2}\right)\times\left(-\frac{1}{2}\right)}\ \left|\frac{3}{2},-\frac{1}{2}\right\rangle = \sqrt{3}\left|\frac{3}{2},-\frac{1}{2}\right\rangle$$

$$R.H.S = S_+^1\left|-\frac{1}{2},-1\right\rangle + S_{2+}\left|-\frac{1}{2},-1\right\rangle$$

$$= \sqrt{\frac{1}{2}\times\frac{3}{2}-\left(-\frac{1}{2}\right)\times\frac{1}{2}}\ \left|\frac{1}{2},-1\right\rangle + \sqrt{1\times 2-(-1)\times 0}\ \left|-\frac{1}{2},0\right\rangle$$

$$= \left|\frac{1}{2},-1\right\rangle + \sqrt{2}\left|-\frac{1}{2},0\right\rangle$$

Then:

$$\boxed{\left|\frac{3}{2},-\frac{1}{2}\right\rangle = \frac{1}{\sqrt{3}}\left|\frac{1}{2},-1\right\rangle + \frac{\sqrt{2}}{\sqrt{3}}\left|-\frac{1}{2},0\right\rangle}$$

due to $\left\langle\frac{1}{2},-\frac{1}{2}\middle|\frac{3}{2},-\frac{1}{2}\right\rangle = 0$ we have:

$$\boxed{\left|\frac{1}{2},-\frac{1}{2}\right\rangle = \sqrt{\frac{2}{3}}\left|\frac{1}{2},-1\right\rangle - \frac{1}{\sqrt{3}}\left|-\frac{1}{2},0\right\rangle}$$