Week 4 QM Discussion

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Office Hours: Tuesday 10am-12pm, Tutoring Center.

Problem 1

Construct the matrix representation for operators L_x, L_y, L_z for the case $l = 1$. Using the fact that $L_+ |l, m\rangle = \hbar\sqrt{l(l+1) - m(m+1)}\, |l, m+1\rangle$, $L_- |l, m\rangle = \hbar\sqrt{l(l+1) - m(m-1)}\, |l, m-1\rangle$ and $L_z |l, m\rangle = m\hbar\, |l, m\rangle$. What if $l = 1/2$?[1]

[1] when l represents angular momentum, l can be only integers. If we start from the basic commutation relation $[L_i, L_j] = i\hbar\epsilon_{ijk}L_k$ and forget that L is angular momentum, the three equations given in the problem are still true and l can be integer or half integer in this case. It turns out we can use l to represent the spin of a particle. Also, this is known as representations of $su(2)$ Lie algebra in group theory.

Spinless Charged Particle on a Sphere with Weak Magnetic Field Applied

Consider a particle of charge e, mass m_0, constrained to move on the surface of a sphere of radius R (we do not consider spin in this problem). There is a weak uniform magnetic field $\vec{B} = B\hat{z}$. Find the energy levels of the system by following the steps below:

a) You may have no idea how to start. OK, remember that the starting point is always Hamiltonian. So, write down the Hamilton of the system. (Hint: the Hamiltonian with magnetic field applied is $H = \frac{(\vec{p}-e\vec{A}/c)^2}{2m_0}$, we use symmetric gauge here: $\vec{A} = \frac{1}{2}(-By, Bx, 0)$)

b) Congratulations! You have finished the first step. Now, you need simplify the Hamiltonian. You should get $H = \frac{\vec{p}^2}{2m_0} - \frac{eB}{2m_0c}L_z$. (Hint: remember x, p are operators. So, for example,you should write $(p_x - eA_x/c)^2 = (p_x - eA_x/c)(p_x - eA_x/c)$ when you expand the square and do term by term carefully. Also, you can drop all B^2 terms because B is small.)

c) Use the condition that the particle is on the sphere to simplify Hamiltonian further. You should get $H = \frac{\vec{L}^2}{2m_0R^2} - \frac{eB}{2m_0c}L_z$. (You may find the formula below useful.)

d) Now, write down the eigenfunctions and energy levels from what you have learned.

Useful Formula: $\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r}) - \frac{\vec{L}^2}{\hbar^2 r^2}$. To get this result, write down the expression ∇^2 and $\vec{L}^2$ explicitly and compare them.

Problem 1:

$$L_+ = L_x + iL_y \qquad \Rightarrow \qquad L_x = \frac{1}{2}(L_+ + L_-)$$

$$L_- = L_x - iL_y \qquad\qquad L_y = \frac{1}{2i}(L_+ - L_-)$$

three basis:

$|1,1\rangle$, $|1,0\rangle$, $|1,-1\rangle$

$$L_+|1,1\rangle = 0$$

$$L_+|1,0\rangle = \hbar\sqrt{1\times 2}\;|1,1\rangle = \hbar\sqrt{2}\,|1,1\rangle$$

$$L_+|1,-1\rangle = \hbar\sqrt{2}\;|1,0\rangle$$

$$L_+ = \begin{array}{c} \begin{array}{ccc} |1,1\rangle & |1,0\rangle & |1,-1\rangle \end{array} \\ \begin{pmatrix} 0 & \sqrt{2}\hbar & 0 \\ 0 & 0 & \sqrt{2}\hbar \\ 0 & 0 & 0 \end{pmatrix} \end{array}$$

$$L_- = (L_+)^\dagger = \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2}\hbar & 0 & 0 \\ 0 & \sqrt{2}\hbar & 0 \end{pmatrix}$$

Then:

$$L_x = \frac{1}{2}\begin{pmatrix} 0 & \sqrt{2}\hbar & 0 \\ \sqrt{2}\hbar & 0 & \sqrt{2}\hbar \\ 0 & \sqrt{2}\hbar & 0 \end{pmatrix}$$

$$L_y = \frac{1}{2i}\begin{pmatrix} 0 & \sqrt{2}\,\hbar & 0 \\ -\sqrt{2}\hbar & 0 & \sqrt{2}\hbar \\ 0 & -\sqrt{2}\hbar & 0 \end{pmatrix}$$

$$L_z = \begin{pmatrix} \hbar & & \\ & 0 & \\ & & -\hbar \end{pmatrix}$$

For $l = \frac{1}{2}$

basis :

$$|\tfrac{1}{2}, \tfrac{1}{2}\rangle \;, \quad |\tfrac{1}{2}, -\tfrac{1}{2}\rangle$$

$$L_+ |\tfrac{1}{2}\,\tfrac{1}{2}\rangle = 0$$

$$L_+ |\tfrac{1}{2}, -\tfrac{1}{2}\rangle = \hbar\sqrt{\tfrac{1}{2}\times\tfrac{3}{2} - (-\tfrac{1}{2})(\tfrac{1}{2})}\; |\tfrac{1}{2}\,\tfrac{1}{2}\rangle$$

$$= \hbar\, |\tfrac{1}{2}\,\tfrac{1}{2}\rangle$$

$$L_+ = \begin{pmatrix} 0 & \hbar \\ 0 & 0 \end{pmatrix} \;, \quad L_- = \begin{pmatrix} 0 & 0 \\ \hbar & 0 \end{pmatrix}$$

$$L_x = \frac{1}{2}\begin{pmatrix} 0 & \hbar \\ \hbar & 0 \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{\hbar}{2}\sigma_x$$

$$L_y = \frac{1}{2i}\begin{pmatrix} 0 & \hbar \\ -\hbar & 0 \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \frac{\hbar}{2}\sigma_y$$

$$L_z = \begin{pmatrix} \frac{\hbar}{2} & \\ & -\frac{\hbar}{2} \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 1 & \\ & -1 \end{pmatrix} = \frac{\hbar}{2}\sigma_z$$

where $\sigma_{x,y,z}$ are known as

Pauli matrix.

Solution : Problem 2.

a)

$$\hat{H} = \frac{(\vec{p} - \frac{e}{c}\vec{A})^2}{2m_0} = \frac{(P_x - \frac{e}{c}A_x)^2}{2m_0} + \frac{(P_y - \frac{e}{c}A_y)^2}{2m_0} + \frac{(P_z - \frac{e}{c}A_z)^2}{2m_0}$$

$$= \frac{(P_x + \frac{eB}{2c}y)^2}{2m_0} + \frac{(P_y - \frac{eB}{2c}x)^2}{2m_0} + \frac{P_z^2}{2m_0}$$

b)

$$\left(P_x + \frac{eB}{2c}y\right)\left(P_x + \frac{eB}{2c}y\right)$$

$$= P_x^2 + \frac{eB}{2c}P_x y + \frac{eB}{2c}y P_x$$

$$= P_x^2 + \frac{eB}{c}y P_x$$ due to $[y, P_x] = 0$.

$$\left(P_y - \frac{eB}{2c}x\right)\left(P_y - \frac{eB}{2c}x\right)$$

$$= P_y^2 - \frac{eB}{2c}P_y x - \frac{eB}{2c}x P_y$$

$$= P_y^2 - \frac{eB}{c}x P_y$$

Then : $$\hat{H} = \frac{P_x^2 + \frac{eB}{c}y P_x}{2m_0} + \frac{P_y^2 - \frac{eB}{c}x P_y}{2m_0} + \frac{P_z^2}{2m_0}$$

$$= \frac{\vec{P}^2}{2m_0} + \frac{eB}{2m_0 c}\left(-x P_y + y P_x\right)$$

$$= \frac{\vec{P}^2}{2m_0} - \frac{eB}{2m_0 c}\hat{L}_z$$

c)

$$\hat{H} = \frac{-\hbar^2 \vec{\nabla}^2}{2m_0} - \frac{eB}{2m_0 c}\hat{L}_z$$

where
$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) - \frac{\vec{L}^2}{\hbar^2 r^2}$$

$$= -\frac{\vec{L}^2}{\hbar^2 R^2}$$

due to the particle is constrained on a sphere.

$\psi(\vec{r}) = \psi(r=R, \theta, \varphi)$

$$\hat{H} = \frac{\vec{L}^2}{2m_0 R^2} - \frac{eB}{2m_0 c}\hat{L}_z$$

d) $\vec{L}^2$, L_z commute and they have the same eigenfunction

$$Y_{lm}(\theta, \varphi)$$

Then:

$$\hat{H}\, Y_{lm}(\theta,\varphi) = \left(\frac{\hbar^2 l(l-1)}{2m_0} - \frac{eB\, m\hbar}{2m_0 c}\right) Y_{lm}(\theta,\varphi)$$

$$E_{l,m} = \frac{l(l-1)\hbar^2}{2m_0} - \frac{eB\hbar}{2m_0 c}m.$$