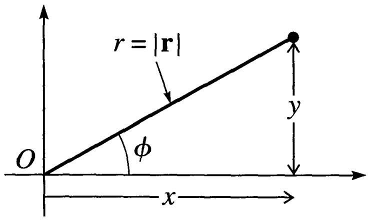

Figure 1.10 The definition of the polar coordinates r and ϕ.

1.7 Two-Dimensional Polar Coordinates

While Cartesian coordinates have the merit of simplicity, we are going to find that it is almost impossible to solve certain problems without the use of various non-Cartesian coordinate systems. To illustrate the complexities of non-Cartesian coordinates, let us consider the form of Newton's second law in a two-dimensional problem using polar coordinates. These coordinates are defined in Figure 1.10. Instead of using the two rectangular coordinates x, y, we label the position of a particle with its distance r from O and the angle ϕ measured up from the x axis. Given the rectangular coordinates x and y, you can calculate the polar coordinates r and ϕ, or vice versa, using the following relations. (Make sure you understand all four equations.[12])

$$\left.\begin{aligned} x &= r\cos\phi \\ y &= r\sin\phi \end{aligned}\right\} \longleftrightarrow \left\{\begin{aligned} r &= \sqrt{x^2+y^2} \\ \phi &= \arctan(y/x) \end{aligned}\right. \tag{1.37}$$

Just as with rectangular coordinates, it is convenient to introduce two unit vectors, which I shall denote by $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\phi}}$. To understand their definitions, notice that we can define the unit vector $\hat{\mathbf{x}}$ as the unit vector that points in the direction of increasing x when y is fixed, as shown in Figure 1.11(a). In the same way we shall define $\hat{\mathbf{r}}$ as the unit vector that points in the direction we move when r increases with ϕ fixed; likewise, $\hat{\boldsymbol{\phi}}$ is the unit vector that points in the direction we move when ϕ increases with r fixed. Figure 1.11 makes clear a most important difference between the unit vectors $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ of rectangular coordinates and our new unit vectors $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\phi}}$. The vectors $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ are the same at all points in the plane, whereas the new vectors $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\phi}}$ change their directions as the position vector $\mathbf{r}$ moves around. We shall see that this complicates the use of Newton's second law in polar coordinates.

Figure 1.11 suggests another way to write the unit vector $\hat{\mathbf{r}}$. Since $\hat{\mathbf{r}}$ is in the same direction as $\mathbf{r}$, but has magnitude 1, you can see that

$$\hat{\mathbf{r}} = \frac{\mathbf{r}}{|\mathbf{r}|}. \tag{1.38}$$

This result suggests a second role for the "hat" notation. For *any* vector $\mathbf{a}$, we can define $\hat{\mathbf{a}}$ as the unit vector in the direction of $\mathbf{a}$, namely $\hat{\mathbf{a}} = \mathbf{a}/|\mathbf{a}|$.

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[12] There is a small subtlety concerning the equation for ϕ: You need to make sure ϕ lands in the proper quadrant, since the first and third quadrants give the same values for y/x (and likewise the second and fourth). See Problem 1.42.

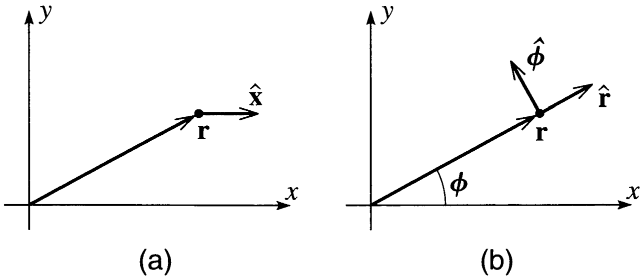

Figure 1.11 **(a)** The unit vector $\hat{\mathbf{x}}$ points in the direction of increasing x with y fixed. **(b)** The unit vector $\hat{\mathbf{r}}$ points in the direction of increasing r with ϕ fixed; $\hat{\boldsymbol{\phi}}$ points in the direction of increasing ϕ with r fixed. Unlike $\hat{\mathbf{x}}$, the vectors $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\phi}}$ change as the position vector $\mathbf{r}$ moves.

Since the two unit vectors $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\phi}}$ are perpendicular vectors in our two-dimensional space, any vector can be expanded in terms of them. For instance, the net force $\mathbf{F}$ on an object can be written

$$\mathbf{F} = F_r\hat{\mathbf{r}} + F_\phi\hat{\boldsymbol{\phi}}. \tag{1.39}$$

If, for example, the object in question is a stone that I am twirling in a circle on the end of a string (with my hand at the origin), then F_r would be the tension in the string and F_ϕ the force of air resistance retarding the stone in the tangential direction. The expansion of the position vector itself is especially simple in polar coordinates. From Figure 1.11(b) it is clear that

$$\mathbf{r} = r\hat{\mathbf{r}}. \tag{1.40}$$

We are now ready to ask about the form of Newton's second law, $\mathbf{F} = m\ddot{\mathbf{r}}$, in polar coordinates. In rectangular coordinates, we saw that the x component of $\ddot{\mathbf{r}}$ is just $\ddot{x}$, and this is what led to the very simple result (1.35). We must now find the components of $\ddot{\mathbf{r}}$ in polar coordinates; that is, we must differentiate (1.40) with respect to t. Although (1.40) is very simple, the vector $\hat{\mathbf{r}}$ changes as $\mathbf{r}$ moves. Thus when we differentiate (1.40), we shall pick up a term involving the derivative of $\hat{\mathbf{r}}$. Our first task is to find this derivative of $\hat{\mathbf{r}}$.

Figure 1.12(a) shows the position of the particle of interest at two successive times, t_1 and $t_2 = t_1 + \Delta t$. If the corresponding angles $\phi(t_1)$ and $\phi(t_2)$ are different, then the two unit vectors $\hat{\mathbf{r}}(t_1)$ and $\hat{\mathbf{r}}(t_2)$ point in different directions. The change in $\hat{\mathbf{r}}$ is shown in Figure 1.12(b), and (provided Δt is small) is approximately

$$\begin{aligned}\Delta\hat{\mathbf{r}} &\approx \Delta\phi\,\hat{\boldsymbol{\phi}}\\ &\approx \dot{\phi}\,\Delta t\,\hat{\boldsymbol{\phi}}.\end{aligned} \tag{1.41}$$

(Notice that the direction of $\Delta\hat{\mathbf{r}}$ is perpendicular to $\hat{\mathbf{r}}$, namely the direction of $\hat{\boldsymbol{\phi}}$.) If we divide both sides by Δt and take the limit as $\Delta t \to 0$, then $\Delta\hat{\mathbf{r}}/\Delta t \to d\hat{\mathbf{r}}/dt$ and we find that

$$\frac{d\hat{\mathbf{r}}}{dt} = \dot{\phi}\,\hat{\boldsymbol{\phi}}. \tag{1.42}$$

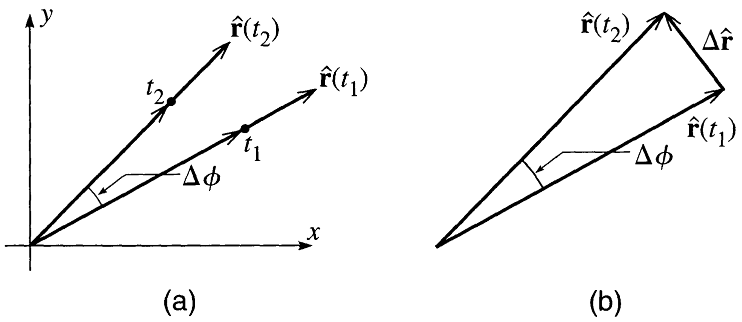

Figure 1.12 **(a)** The positions of a particle at two successive times, t_1 and t_2. Unless the particle is moving exactly radially, the corresponding unit vectors $\hat{\mathbf{r}}(t_1)$ and $\hat{\mathbf{r}}(t_2)$ point in different directions. **(b)** The change $\Delta\hat{\mathbf{r}}$ in $\hat{\mathbf{r}}$ is given by the triangle shown.

(For an alternative proof of this important result, see Problem 1.43.) Notice that $d\hat{\mathbf{r}}/dt$ is in the direction of $\hat{\boldsymbol{\phi}}$ and is proportional to the rate of change of the angle ϕ — both of which properties we would expect based on Figure 1.12.

Now that we know the derivative of $\hat{\mathbf{r}}$, we are ready to differentiate Equation (1.40). Using the product rule, we get two terms:

$$\dot{\mathbf{r}} = \dot{r}\hat{\mathbf{r}} + r\frac{d\hat{\mathbf{r}}}{dt},$$

and, substituting (1.42), we find for the velocity $\dot{\mathbf{r}}$, or $\mathbf{v}$,

$$\mathbf{v} \equiv \dot{\mathbf{r}} = \dot{r}\hat{\mathbf{r}} + r\dot{\phi}\,\hat{\boldsymbol{\phi}}. \tag{1.43}$$

From this we can read off the polar components of the velocity:

$$v_r = \dot{r} \quad \text{and} \quad v_\phi = r\dot{\phi} = r\omega \tag{1.44}$$

where in the second equation I have introduced the traditional notation ω for the angular velocity $\dot{\phi}$. While the results in (1.44) should be familiar from your introductory physics course, they are undeniably more complicated than the corresponding results in Cartesian coordinates ($v_x = \dot{x}$ and $v_y = \dot{y}$).

Before we can write down Newton's second law, we have to differentiate a second time to find the acceleration:

$$\mathbf{a} \equiv \ddot{\mathbf{r}} = \frac{d}{dt}\dot{\mathbf{r}} = \frac{d}{dt}(\dot{r}\hat{\mathbf{r}} + r\dot{\phi}\,\hat{\boldsymbol{\phi}}), \tag{1.45}$$

where the final expression comes from substituting (1.43) for $\dot{\mathbf{r}}$. To complete the differentiation in (1.45), we must calculate the derivative of $\hat{\boldsymbol{\phi}}$. This calculation is completely analogous to the argument leading to (1.42) and is illustrated in Figure 1.13. By inspecting this figure, you should be able to convince yourself that

$$\frac{d\hat{\boldsymbol{\phi}}}{dt} = -\dot{\phi}\hat{\mathbf{r}}. \tag{1.46}$$

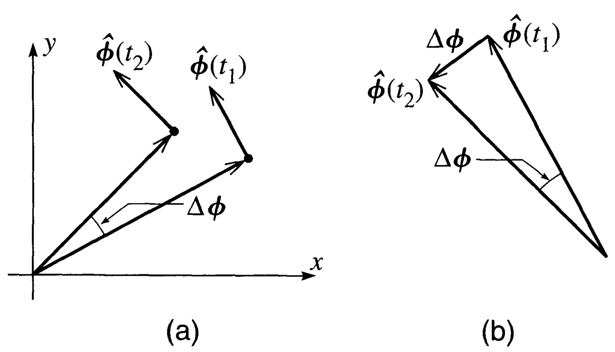

Figure 1.13 **(a)** The unit vector $\hat{\boldsymbol{\phi}}$ at two successive times t_1 and t_2. **(b)** The change $\Delta\hat{\boldsymbol{\phi}}$.

Returning to Equation (1.45), we can now carry out the differentiation to give the following five terms:

$$\mathbf{a} = \left(\ddot{r}\hat{\mathbf{r}} + \dot{r}\frac{d\hat{\mathbf{r}}}{dt}\right) + \left((\dot{r}\dot{\phi} + r\ddot{\phi})\hat{\boldsymbol{\phi}} + r\dot{\phi}\frac{d\hat{\boldsymbol{\phi}}}{dt}\right)$$

or, if we use (1.42) and (1.46) to replace the derivatives of the two unit vectors,

$$\mathbf{a} = \left(\ddot{r} - r\dot{\phi}^2\right)\hat{\mathbf{r}} + \left(r\ddot{\phi} + 2\dot{r}\dot{\phi}\right)\hat{\boldsymbol{\phi}}. \tag{1.47}$$

This horrible result is a little easier to understand if we consider the special case that r is constant, as is the case for a stone that I twirl on the end of a string of fixed length. With r constant, both derivatives of r are zero, and (1.47) has just two terms:

$$\mathbf{a} = -r\dot{\phi}^2\hat{\mathbf{r}} + r\ddot{\phi}\hat{\boldsymbol{\phi}}$$

or

$$\mathbf{a} = -r\omega^2\hat{\mathbf{r}} + r\alpha\hat{\boldsymbol{\phi}},$$

where $\omega = \dot{\phi}$ denotes the angular velocity and $\alpha = \ddot{\phi}$ is the angular acceleration. This is the familiar result from elementary physics that when a particle moves around a fixed circle, it has an inward "centripetal" acceleration $r\omega^2$ (or v^2/r) and a tangential acceleration, $r\alpha$. Nevertheless, when r is not constant, the acceleration includes all four of the terms in (1.47). The first term, $\ddot{r}$ in the radial direction is what you would probably expect when r varies, but the final term, $2\dot{r}\dot{\phi}$ in the ϕ direction, is harder to understand. It is called the Coriolis acceleration, and I shall discuss it in detail in Chapter 9.

Having calculated the acceleration as in (1.47), we can finally write down Newton's second law in terms of polar coordinates:

$$\mathbf{F} = m\mathbf{a} \qquad \Longleftrightarrow \qquad \begin{cases} F_r = m(\ddot{r} - r\dot{\phi}^2) \\ F_\phi = m(r\ddot{\phi} + 2\dot{r}\dot{\phi}). \end{cases} \tag{1.48}$$

These equations in polar coordinates are a far cry from the beautifully simple equations (1.35) for rectangular coordinates. In fact, one of the main reasons for taking the trouble to recast Newtonian mechanics in the Lagrangian formulation (Chapter 7) is that the latter is able to handle nonrectangular coordinates just as easily as rectangular.

You may justifiably be feeling that the second law in polar coordinates is so complicated that there could be no occasion to use it. In fact, however, there are many problems which are most easily solved using polar coordinates, and I conclude this section with an elementary example.

EXAMPLE 1.2 An Oscillating Skateboard

A "half-pipe" at a skateboard park consists of a concrete trough with a semicircular cross section of radius $R = 5$ m, as shown in Figure 1.14. I hold a frictionless skateboard on the side of the trough pointing down toward the bottom and release it. Discuss the subsequent motion using Newton's second law. In particular, if I release the board just a short way from the bottom, how long will it take to come back to the point of release?

Because the skateboard is constrained to move on a circular path, this problem is most easily solved using polar coordinates with origin O at the center of the pipe as shown. (At some point in the following calculation, try writing the second law in rectangular coordinates and observe what a tangle you get.) With this choice of polar coordinates, the coordinate r of the skateboard is constant, $r = R$, and the position of the skateboard is completely specified by the angle ϕ. With r constant, the second law (1.48) takes the relatively simple form

$$F_r = -mR\dot{\phi}^2 \tag{1.49}$$

and

$$F_\phi = mR\ddot{\phi}. \tag{1.50}$$

The two forces on the skateboard are its weight $\mathbf{w} = m\mathbf{g}$ and the normal force $\mathbf{N}$ of the wall, as shown in Figure 1.14. The components of the net force $\mathbf{F} = \mathbf{w} + \mathbf{N}$ are easily seen to be

$$F_r = mg\cos\phi - N \quad \text{and} \quad F_\phi = -mg\sin\phi.$$

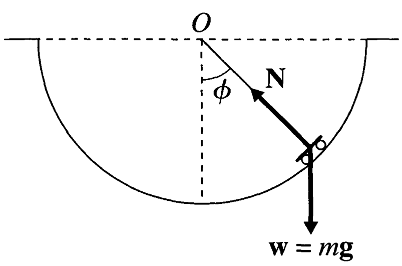

Figure 1.14 A skateboard in a semicircular trough of radius R. The board's position is specified by the angle ϕ measured up from the bottom. The two forces on the skateboard are its weight $\mathbf{w} = m\mathbf{g}$ and the normal force $\mathbf{N}$.

Substituting for F_r into (1.49) we get an equation involving N, ϕ, and $\dot{\phi}$. Fortunately, we are not really interested in N, and — even more fortunately — when we substitute for F_ϕ into (1.50), we get an equation that does not involve N at all:

$$-mg\sin\phi = mR\ddot{\phi}$$

or, canceling the m's and rearranging,

$$\ddot{\phi} = -\frac{g}{R}\sin\phi. \tag{1.51}$$

Equation (1.51) is the differential equation for $\phi(t)$ that determines the motion of the skateboard. Qualitatively, we can easily see the kind of motion that it implies. First, if $\phi = 0$, (1.51) says that $\ddot{\phi} = 0$. Therefore, if we place the board at rest ($\dot{\phi} = 0$) at the point $\phi = 0$, the board will never move (unless someone pushes it); that is, $\phi = 0$ is an equilibrium position, as you would certainly have guessed. Next, suppose that at some time, ϕ is not zero and, to be definite, suppose that $\phi > 0$; that is, the skateboard is on the right-hand side of the half-pipe. In this case, (1.51) implies that $\ddot{\phi} < 0$, so the acceleration is directed to the left. If the board is moving to the right it must slow down and eventually start moving to the left.[13] Once it is moving toward the left, it speeds up and returns to the bottom, where it moves over to the left. As soon as the board is on the left, the argument reverses ($\phi < 0$, so $\ddot{\phi} > 0$) and the board must eventually return to the bottom and move over to the right again. In other words, the differential equation (1.51) implies that the skateboard oscillates back and forth, from right to left and back to the right.

The equation of motion (1.51) cannot be solved in terms of elementary functions, such as polynomials, trigonometric functions, or logs and exponentials.[14] Thus, if we want more quantitative information about the motion, the simplest course is to use a computer to solve it numerically (see Problem 1.50). However, if the initial angle ϕ_o is *small*, we can use the small angle approximation

$$\sin\phi \approx \phi \tag{1.52}$$

and, within this approximation, (1.51) becomes

$$\ddot{\phi} = -\frac{g}{R}\phi \tag{1.53}$$

which *can* be solved using elementary functions. [By this stage, you have almost certainly recognized that our discussion of the skateboard problem closely parallels the analysis of the simple pendulum. In particular, the small-angle

[13] I am taking for granted that it doesn't reach the top and jump out of the trough. Since it was released from rest inside the trough, this is correct. Much the easiest way to prove this claim is to invoke conservation of energy, which we shan't be discussing for a while. Perhaps, for now, you could agree to accept it as a matter of common sense.

[14] Actually the solution of (1.51) is a Jacobi elliptic function. However, I shall take the point of view that for most of us the Jacobi function is not "elementary."

approximation (1.52) is what let you solve the simple pendulum in your introductory physics course. This parallel is, of course, no accident. Mathematically the two problems are exactly equivalent.] If we define the parameter

$$\omega = \sqrt{\frac{g}{R}}, \tag{1.54}$$

then (1.53) becomes

$$\ddot{\phi} = -\omega^2 \phi. \tag{1.55}$$

This is the equation of motion for our skateboard in the small-angle approximation. I would like to discuss its solution in some detail to introduce several ideas that we'll be using again and again in what follows. (If you've studied differential equations before, just see the next three paragraphs as a quick review.)

We first observe that it is easy to find two solutions of the equation (1.55) by inspection (that is, by inspired guessing). The function $\phi(t) = A\sin(\omega t)$ is clearly a solution for any value of the constant A. [Differentiating $\sin(\omega t)$ brings out a factor of ω and changes the sin to a cos; differentiating it again brings out another ω and changes the cos back to $-$sin. Thus the proposed solution does satisfy $\ddot{\phi} = -\omega^2\phi$.] Similarly, the function $\phi(t) = B\cos(\omega t)$ is another solution for any constant B. Furthermore, as you can easily check, the sum of these two solutions is itself a solution. Thus we have now found a whole family of solutions:

$$\phi(t) = A\sin(\omega t) + B\cos(\omega t) \tag{1.56}$$

is a solution for any values of the two constants A and B.

I now want to argue that *every* solution of the equation of motion (1.55) has the form (1.56). In other words, (1.56) is the *general solution* — we have found *all* solutions, and we need seek no further. To get some idea of why this is, note that the differential equation (1.55) is a statement about the second derivative $\ddot{\phi}$ of the unknown ϕ. Now, if we had actually been told what $\ddot{\phi}$ is, then we know from elementary calculus that we could find ϕ by two integrations, and the result would contain two unknown constants — the two constants of integration — that would have to be determined by looking (for example) at the initial values of ϕ and $\dot{\phi}$. In other words, knowledge of $\ddot{\phi}$ would tell us that ϕ itself is one of a family of functions containing precisely two undetermined constants. Of course, the differential equation (1.55) does not actually tell us $\ddot{\phi}$ — it is an equation for $\ddot{\phi}$ in terms of ϕ. Nevetheless, it is plausible that such an equation would imply that ϕ is one of a family of functions that contain precisely two undetermined constants. If you have studied differential equations, you know that this is the case; if you have not, then I must ask you to accept it as a plausible fact: For any given second-order differential equation [in a large class of "reasonable" equations, including (1.55) and all of the equations we shall encounter in this book], the solutions all belong to a family of functions containing precisely two independent constants — like the constants A and B in (1.56). (More generally, the solutions of an nth-order equation contain precisely n independent constants.)

This theorem sheds a new light on our solution (1.56). We already knew that any function of the form (1.56) is a solution of the equation of motion. Our theorem now guarantees that *every* solution of the equation of motion is of this form. This same argument applies to all the second-order differential equations we shall encounter. If, by hook or by crook, we can find a solution like (1.56) involving two arbitrary constants, then we are guaranteed that we have found the general solution of our equation.

All that remains is to pin down the two constants A and B for our skateboard. To do so, we must look at the initial conditions. At $t = 0$, Equation (1.56) implies that $\phi = B$. Therefore B is just the initial value of ϕ, which we are calling ϕ_o, so $B = \phi_o$. At $t = 0$, Equation (1.56) implies that $\dot{\phi} = \omega A$. Since I released the board from rest, this means that $A = 0$, and our solution is

$$\phi(t) = \phi_o \cos(\omega t). \tag{1.57}$$

The first thing to note about this solution is that, as we anticipated on general grounds, $\phi(t)$ oscillates, moving from positive to negative and back to positive periodically and indefinitely. In particular, the board first returns to its initial position ϕ_o when $\omega t = 2\pi$. The time that this takes is called the period of the motion and is denoted τ. Thus our conclusion is that the period of the skateboard's oscillations is

$$\tau = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{R}{g}}. \tag{1.58}$$

We were given that $R = 5\,\text{m}$, and $g = 9.8\,\text{m/s}^2$. Substituting these numbers, we conclude that the skateboard returns to its starting point in a time $\tau = 4.5$ seconds.

Principal Definitions and Equations of Chapter 1

Dot and Cross Products

$$\mathbf{r}\cdot\mathbf{s} = rs\cos\theta = r_x s_x + r_y s_y + r_z s_z \qquad \text{[Eqs. (1.6) \& (1.7)]}$$

$$\mathbf{r}\times\mathbf{s} = (r_y s_z - r_z s_y,\ r_z s_x - r_x s_z,\ r_x s_y - r_y s_x) = \det\begin{bmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ r_x & r_y & r_z \\ s_x & s_y & s_z \end{bmatrix} \qquad \text{[Eq. (1.9)]}$$