105A - Set 6 - Solution

(Grades are out of 150)

1. Its late at night, early 17th century, Johannes Kepler is working hard on his notes, the candle light is flickering. He just found that planets orbit the sun in an ellipse and he expressed their radius as a function of the angle. He is trying to give meaning to the constants he found, can he relate these constants to the semi-major axis of the ellipse? All of sudden a figure materialized in front of him. Its a person, that says that she came from the future to help him out with his First law of planetary motion. She draws an ellipse, (see Figure) and says that the relation between the radius and the angle can be worked out from the geometry and one can simply start from the Ellipse equation
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{1}$$
Unfrouvntently the time traveler had to come back quickly to her own time to preserve the time continuum. Help Johannes Kepler out and show that starting from the above equation you can find:
$$r = \frac{a(1-e^2)}{1+e\cos\theta} \ , \tag{2}$$
where $e = \sqrt{1 - b^2/a^2}$ is the eccentricity.

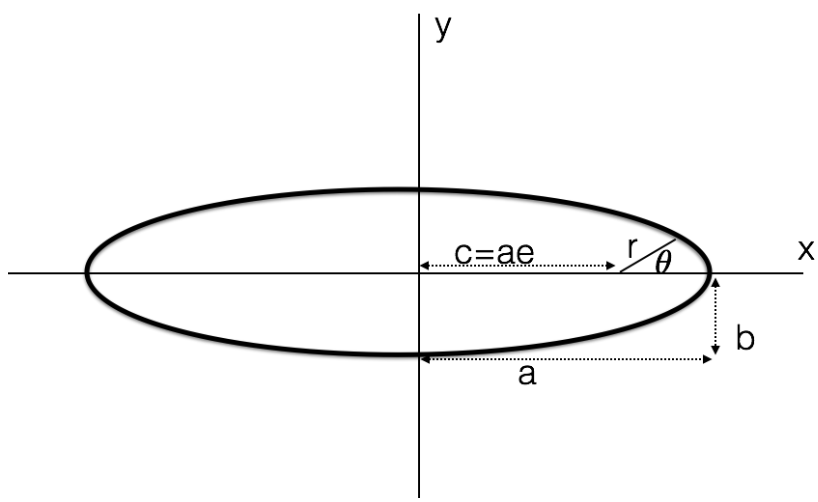

Figure 1: Ellipse

Answer Starting from
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{3}$$
we can define the x and y coordinates as $x = c + r\cos\theta$ and $y = r\sin\theta$ plugging this in to the ellipse equation we have
$$\frac{(c + r\cos\theta)^2}{a^2} + \frac{(r\sin\theta)^2}{b^2} = 1 \tag{4}$$

which we can write as:

$$\frac{c^2 + 2cr\cos\theta + r^2\cos^2\theta}{a^2} + \frac{r^2\sin^2\theta}{b^2} = 1 \tag{5}$$

From the figure we have $c = ae$ so

$$(ae)^2b^2 + 2aerb^2\cos\theta + r^2b^2\cos^2\theta + r^2\sin^2\theta a^2 = b^2a^2 \tag{6}$$

replacing $b^2 = a^2(1-e^2)$ we find:

$$(ae)^2a^2(1-e^2) + 2aera^2(1-e^2)\cos\theta + r^2a^2(1-e^2)\cos^2\theta + r^2\sin^2\theta a^2 = a^2(1-e^2)a^2 \tag{7}$$

dividing by a^2 we have

$$(ae)^2(1-e^2) + 2aer(1-e^2)\cos\theta + r^2(1-e^2)\cos^2\theta + r^2\sin^2\theta = a^2(1-e^2) \tag{8}$$

$$(ae)^2(1-e^2) + 2aer(1-e^2)\cos\theta - r^2e^2\cos^2\theta + r^2 = a^2(1-e^2) \tag{9}$$

arranging we have

$$r^2 = (er\cos\theta - a(1-e^2))^2 \tag{10}$$

so

$$r = \pm(er\cos\theta - a(1-e^2)) \tag{11}$$

but when $e = 0$ r has to be positive so

$$r = er\cos\theta - a(1-e^2) \tag{12}$$

Solving for r we can write:

$$r = \frac{a(1-e^2)}{1+e\cos\theta}\ , \tag{13}$$

2. Assumes Earth's orbit to be circular and that the Sun's mass suddenly decreases by half. What orbit does the Earth then have? Will the Earth escape the solar system? **Answer:** The potential energy is

$$U = -\frac{GM_\odot m_\oplus}{r_\oplus} \tag{14}$$

The kinetic every is $m_\oplus v^2/2$ where $v = \omega r$ which we can find from by equating the gravitational force to the centripetal force

$$m_\oplus \omega^2 r_\oplus = \frac{GM_\odot m_\oplus}{r_\oplus^2} \tag{15}$$

so

$$\omega^2 = \frac{GM_\odot}{r_\oplus^3} \tag{16}$$

and then before the Sun's mass decreased we have:

$$\begin{aligned} E &= T+U = \frac{1}{2}m_\oplus\omega^2 r_\oplus^2 - \frac{GM_\odot m_\oplus}{r_\oplus} = \frac{1}{2}m_\oplus\frac{GM_\odot}{r_\oplus^3}r_\oplus^2 - \frac{GM_\odot m_\oplus}{r_\oplus} \\ &= -\frac{1}{2}\frac{GM_\odot m_\oplus}{r_\oplus} = \frac{1}{2}U \end{aligned} \tag{17}$$

So $E = T+U = U/2$ so $T = -U/2$.
If the suns mass suddenly goes to half its original value, T remains unchanged but U is halved. So the Energy after E_A:

$$E_A = T_A + U_A = T + \frac{1}{2}U = -\frac{1}{2}U + \frac{1}{2}U = 0 \tag{18}$$

So The energy is 0, so the orbit is a parabola. For a parabolic orbit, the earth will escape the solar system.

3. Two particles move about each other in circular orbits under the influence of gravitational forces, with a period τ. Their motion is suddenly stopped, and they are then released and allowed to fall into each other. Prove that they collide after a time $\tau/(4\sqrt{2})$. *Hint 1: find τ for circular motion.*
Hint 2: when you get to $\ddot{r}$ equal to some thing and its hard to solve it, I suggest to multiply by $\dot{r}$ - an alternative way is to think about the energy.
Answer: Since we are dealing with gravitational forces, the potential energy between the particles is

$$U(r) = -\frac{k}{r} \tag{19}$$

and, after reduction to the equivalent one-body problem, the Lagrangian is

$$L = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + \frac{k}{r} \tag{20}$$

where μ is the reduced mass. The equation of motion for r is

$$\mu\ddot{r} = \mu r\dot{\theta}^2 - \frac{k}{r^2} = \frac{l^2}{\mu r^2} - \frac{k}{r^2} \tag{21}$$

We can also use the fact that the motion is circular $r = r_0$, i.e.,

$$\mu\omega^2 r_0 = \mu\dot{\theta}^2 r_0 = \frac{k}{r_0^2} \tag{22}$$

Which means that the

$$\omega^2 = \frac{k}{r_0^3\mu} \tag{23}$$

So the period τ is

$$\tau = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{r_0^3\mu}{k}} \tag{24}$$

Now if the particles stopped, the angular momentum goes to zero. So plugging this in eq. (21) we get

$$\mu\ddot{r} = -\frac{k}{r^2} \tag{25}$$

Then we use the hist and multiply by $\dot{r}$

$$\mu\ddot{r}\dot{r} = -\frac{k}{r^2}\dot{r} \tag{26}$$

which is equivalent to writing

$$\frac{d}{dt}(\dot{r}^2) = +\frac{2k}{\mu r} \tag{27}$$

So then:

$$\dot{r}^2 = +\frac{2k}{\mu r} + C, \tag{28}$$

where C is constant and is determined from the boundary condition on $\dot{r}$, which is when $\dot{r} = 0$ then $r = r_0$ since initially the particles are not moving at all. So we find that $C = -2k/r_0$ so

$$\dot{r} = \frac{dr}{dt} = \sqrt{\frac{2k}{\mu}}\sqrt{\frac{1}{r} - \frac{1}{r_0}}, \tag{29}$$

We could now proceed to solve this differential equation for $r(t)$, but since in fact were interested in solving for the time difference corresponding to given boundary values of r, its easier to invert this equation and solve for $t(r)$:

$$t = \int_{r_0}^{0} \frac{dr}{dt}\, dr = \sqrt{\frac{\mu}{2k}} \int_{r_0}^{0} \sqrt{\frac{r r_0}{r_0 - r}} dr \tag{30}$$

We change variables to $u = r/r_0$ and $du = dr/r_0$ so

$$t = \sqrt{\frac{\mu}{2k}} r_0^{3/2} \int_{1}^{0} \sqrt{\frac{u}{1-u}} du \tag{31}$$

And then another change so $u = \sin^2 x$ and $du = 2\sin x \cos x dx$ so

$$t = \sqrt{\frac{\mu r_0^3}{2k}} \int_{\pi/2}^{0} \sin^2 x dx = \sqrt{\frac{\mu r_0^3}{2k}} \frac{\pi}{2} \tag{32}$$

Now we found that the period is

$$\tau = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{r_0^3 \mu}{k}} \tag{33}$$

so

$$t = \sqrt{\frac{1}{2}} \frac{\tau}{2\pi} \frac{\pi}{2} = \frac{\tau}{4\sqrt{2}} \tag{34}$$

Alternatively we could have used the energy equation

$$E = -\frac{k}{r_0} = \frac{1}{2}\mu\dot{r}^2 - \frac{k}{r} \tag{35}$$

and solve from this, this gives the same equation as equation (29) and from there the solution is identical.

4. Consider a comet moving in a parabolic orbit in the plane of Earth's orbit. If the distance of the closest approach of the comment to the sun is $\beta \times r_E$, where r_E is the radius of the Earth's orbit (assumed circular) around the Sun and $\beta < 1$. Show that the time of the comet spends within the orbit of the Earth is given by

$$\sqrt{2(1-\beta)}\frac{1+2\beta}{3\pi} \times 1\text{yr} \tag{36}$$

If the comet approaches the Sun to a distance of the perihelion (closest approach) of Mercury, how many days is it within Earth's orbit?
Answer: The orbit of the comet is a parabola ($e = 1$), so that the equation of the

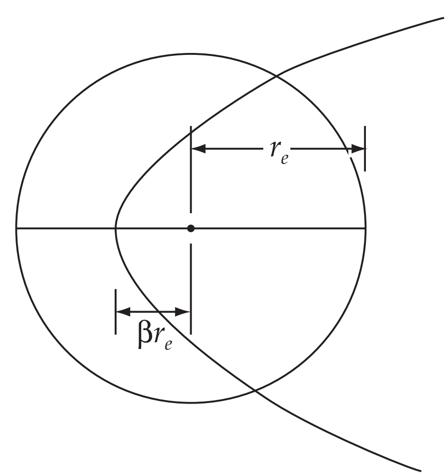

Figure 2: Comet

orbit is

$$\frac{1}{r} = \frac{\mu k}{l^2}\left(1 + \cos\theta\right) \tag{37}$$

We choose to measure θ from perihelion; hence $r(\theta = 0) = \beta r_E$. Therefore setting $\theta = 0$ in the orbit equation

$$\frac{1}{r(\theta = 0)} = \frac{1}{\beta r_E} = \frac{\mu k}{l^2}2 \tag{38}$$

or

$$\frac{l^2}{\mu k} = 2\beta r_E \tag{39}$$

Since the total energy is zero (the orbit is parabolic) and the potential energy is $U = -k/r$, we can write:

$$E = 0 = \frac{\mu}{2}\dot{r}^2 + U_{eff} = \frac{\mu}{2}\dot{r}^2 - \frac{k}{r} + \frac{l^2}{2\mu r^2} \tag{40}$$

So solving for $\dot{r}$

$$\frac{dr}{dt} = \dot{r} = \sqrt{\frac{2}{\mu}}\sqrt{\frac{k}{r} - \frac{l^2}{2\mu r^2}} \tag{41}$$

so

$$T = \int dt = \sqrt{\frac{\mu}{2}}2\int_{\beta r_E}^{r_E} \frac{dr}{\sqrt{\frac{k}{r} - \frac{l^2}{2\mu r^2}}} \ , \tag{42}$$

where the factor 2 comes because there is a symmetry to integrating from r_E to βr_E and from βr_E to r_E. replacing the angular momentum with the perihelion and arranging we can write the integral as:

$$T = \int dt = \sqrt{\frac{2\mu}{k}}2\int_{\beta r_E}^{r_E} \frac{r dr}{\sqrt{r - \beta r_E}} = \sqrt{\frac{2\mu}{k}}\left[-\frac{2(-2\beta r_E - r)}{2}\sqrt{r - \beta r_E}\right]_{\beta r_E}^{r_E} \ , \tag{43}$$

so finally we have

$$T = \sqrt{\frac{2\mu}{k}}\left[\frac{2}{3}r_E^{3/2}(2\beta + 1)\sqrt{1-\beta}\right] \ , \tag{44}$$

Now, the period and the radius of the Earth are related by

$$P_E^2 = \frac{4\pi^2 \mu_E r_E^3}{k_E} = 1 \text{ yr} \ , \tag{45}$$

where $k_E = GM_\odot \mu_E$. so

$$r_E^{3/2} = \sqrt{\frac{GM_\odot \mu_E}{\mu_E}}\frac{P_E}{2\pi} \ , \tag{46}$$

so plugging it to the comet period we get

$$\begin{aligned} T &= \sqrt{\frac{2\mu}{GM_\odot \mu}}\left[\frac{2}{3}\sqrt{\frac{GM_\odot \mu_E}{\mu_E}}\frac{P_E}{2\pi}(2\beta+1)\sqrt{1-\beta}\right] \\ &= \sqrt{2(1-\beta)}\frac{1+2\beta}{3\pi}P_E = \sqrt{2(1-\beta)}\frac{1+2\beta}{3\pi} \times 1\text{yr} \ , \end{aligned} \tag{47}$$

For mercury $\beta = r_{mercury}/r_E = 0.387$ so

$$T = \sqrt{2(1-0.387)}\frac{1+2\times 0.387}{3\pi} \times 1\text{yr} = 76 \text{ days} \ , \tag{48}$$

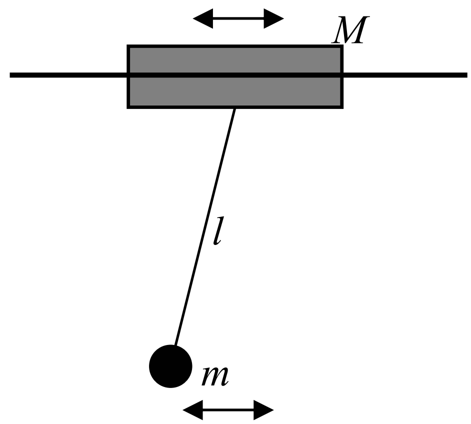

Figure 3: Sliding pendulum

5. (35pt) **Practice power expansion and initial condition**
A block of mass M is free to slide on a horizontal bar without any friction. A mass m is attached to the bottom of the block with a massless rod of length l and can oscillate freely in the same plane as the horizontal bar.

   (a) (5pt) Write down the Lagrangian of the system
   Answer: Let X be the horizontal displacement of the black, the position of the pendulum bob is given by $(x, y) = (X + l\sin\theta, -l\cos\theta)$. So the Lagrangian of the system is:

$$\begin{aligned} L &= \frac{1}{2}M\dot{X}^2 + \frac{1}{2}m([\dot{X} + l\dot{\theta}\cos\theta]^2 + [l\dot{\theta}\sin\theta]^2) + mgl\cos\theta \\ &= \frac{1}{2}M\dot{X}^2 + \frac{1}{2}m(\dot{X}^2 + 2\dot{X}l\dot{\theta}\cos\theta + l^2\dot{\theta}^2) + mgl\cos\theta \end{aligned} \tag{49}$$

   (b) (5pt) Find the equations of motion
   Answer: The equations of motion are:

$$X : \quad (M+m)\ddot{X} + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta = 0 \tag{50}$$

$$\theta : \quad ml\ddot{X}\cos\theta + ml^2\ddot{\theta} + mlg\sin\theta = 0 \tag{51}$$

   Note that P_X, the linear momentum of the X coordinate is conserved. (X is not in the Lagrangian). And also

$$\frac{\partial L}{\partial \dot{X}} = M\dot{X} + m\dot{X} + ml\dot{\theta}\cos\theta \tag{52}$$

   and

$$\frac{d}{dt}P_X = \frac{d}{dt}\frac{\partial L}{\partial \dot{X}} = \frac{\partial L}{\partial X} = 0 \tag{53}$$

so

$$\dot{X} = -\frac{m}{M+m} l\dot{\theta}\cos\theta + \text{Const} \tag{54}$$

Without loss of generality I am setting Const $= 0$. This is equivalent for having to work in a farm of reference that moves with P_X. we can then solve for it.

An answer that has Const $\neq 0$ **is also valid!**

So

$$\dot{X} = -\frac{M}{M+m} l\dot{\theta}\cos\theta \tag{55}$$

So the problem is reduced to

$$\ddot{\theta}\left(1 - \frac{m}{M+m}\cos^2\theta\right) + \frac{m}{M+m}\dot{\theta}^2\sin\theta\cos\theta + \frac{g}{l}\sin\theta = 0 \tag{56}$$

(c) (10pt) Assume that the pendulum oscillation is constrained to small angles around zero. Write the equation of motion in that case.

Answer: In small angle approximation, keeping only linear terms) we get that $\cos^2\theta \sim 1$, $\sin\theta \sim \theta$ and $\cos\theta \sin 1$ so the equation reduces to

$$\ddot{\theta}\left(1 - \frac{m}{M+m}\right) + \frac{g}{l}\theta = 0\ , \tag{57}$$

where the second term was dropped because it is second order ($\dot{\theta} \times \theta$ is second order). We can write this equation as:

$$\ddot{\theta} = -\frac{g(M+m)}{lM}\theta \tag{58}$$

Which is the equation for simple pendulum.

(d) (5pt) Find the frequency of the small oscillations of the system.

Answer: Then from the above equation the oscillation of the system is:

$$\omega^2 = \frac{g(M+m)}{lM} \tag{59}$$

(e) (10pt) Given the initial conditions of $\theta(t=0) = \theta_0$ and $\dot{\theta}(t=0) = \theta_0\sqrt{g(M+m)/(lM)}$, find $\theta(t)$ as a function of the g, M, m, l and θ_0.

Answer: the most general solution here is $\sim e^{\pm i\omega t}$ which we can write as:

$$\theta(t) = A\cos(\omega t) + B\sin(\omega t) \tag{60}$$

Using the initial conditions we can find:

$$\theta(t=0) = \theta_0 = A \tag{61}$$

and

$$\dot{\theta}(t=0) = \theta_0\omega = B\omega \tag{62}$$

So we get that

$$\theta(t) = \theta_0(\cos(\omega t) + \sin(\omega t)) \tag{63}$$

(f) Given that $X(t=0) = x_0$ find the equation that describes the movement of the bob in the horizontal direction.

Answer For small amplitude oscillations we have that the horizontal direction is $x = X + l\sin\theta \sim X + l\theta$. We have from the conservation law that

$$\dot{X} = -\frac{M}{M+m} l\dot{\theta}\cos\theta \sim -\frac{M}{M+m} l\dot{\theta} \tag{64}$$

where the last transition was for small angle approximation. So

$$X = -\frac{M}{M+m} l\theta(t) + C \ , \tag{65}$$

where C is constant. Given the initial conditions that $X(t=0) = x_0$ we can write

$$X(t=0) = x_0 = -\frac{M}{M+m} l\theta_0 + C \ , \tag{66}$$

so

$$C = x_0 + \frac{M}{M+m} l\theta_0 \ , \tag{67}$$

So finally the horizontal movement can be written as:

$$\begin{aligned} x &= X + l\theta = x_0 + \frac{M\theta_0}{M+m} l + \left(l - \frac{M}{M+m} l\right)\theta(t) \\ &= x_0 + \frac{M\theta_0}{M+m} l + \frac{ml}{m+M}\theta_0(\cos(\omega t) + \sin(\omega t)) \end{aligned} \tag{68}$$