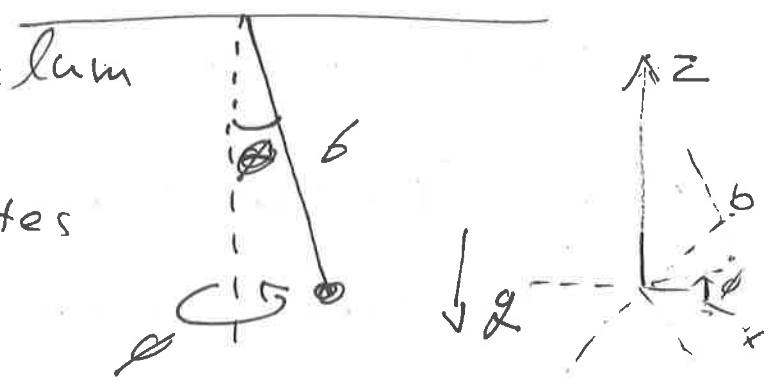
find the conserved quntaties in the following system:

Given a spherical pendulum



The generalized coordinates are ϕ and θ;

there are 2 D.O.F.

first we should find T and U

(1) $T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$

The coordinates:

(2) $x = b \cos\phi \sin\theta$ $\qquad \dot{x} = b(-\dot{\phi} \sin\phi \sin\theta + \dot{\theta} \cos\phi \cos\theta)$

(3) $y = b \sin\phi \sin\theta$ $\qquad \dot{y} = b(\dot{\phi} \cos\phi \sin\theta + \dot{\theta} \sin\phi \cos\theta)$

(4) $z = b \cos\theta$ $\qquad \dot{z} = -b \dot{\theta} \sin\theta$

So

(5)
$$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = b^2 \{ \dot{\phi}^2 \sin^2\phi \sin^2\theta - 2\dot{\phi}\dot{\theta} \sin\phi \cos\phi \sin\theta \cos\theta + \dot{\theta}^2 \cos^2\phi \cos^2\theta + \dot{\phi}^2 \cos^2\phi \sin^2\theta + 2\dot{\phi}\dot{\theta} \sin\phi \cos\phi \sin\theta \cos\theta + \dot{\theta}^2 \sin^2\phi \cos^2\theta + \dot{\theta}^2 \sin^2\theta \} = b^2 \{ \dot{\phi}^2 \sin^2\theta + \dot{\theta}^2 (\cos^2\theta + \sin^2\theta) \}$$

So

(6) $T = \frac{1}{2} m b^2 [\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta]$

Now U, we define the potential to be zero at the pendulum's point of attachment. so

(7) $U = -mgb\cos\theta$

Thus the Lagrangian is:

(8) $$\boxed{\mathcal{L} = T - U = \frac{1}{2}mb^2\dot{\theta}^2 + \frac{1}{2}mb^2\sin^2\theta\,\dot{\phi}^2 + mgb\cos\theta}$$

The generalized momenta are

(9) $p_\theta = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = mb^2\dot{\theta}$

(10) $p_\phi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = mb^2\sin^2\theta\,\dot{\phi}$

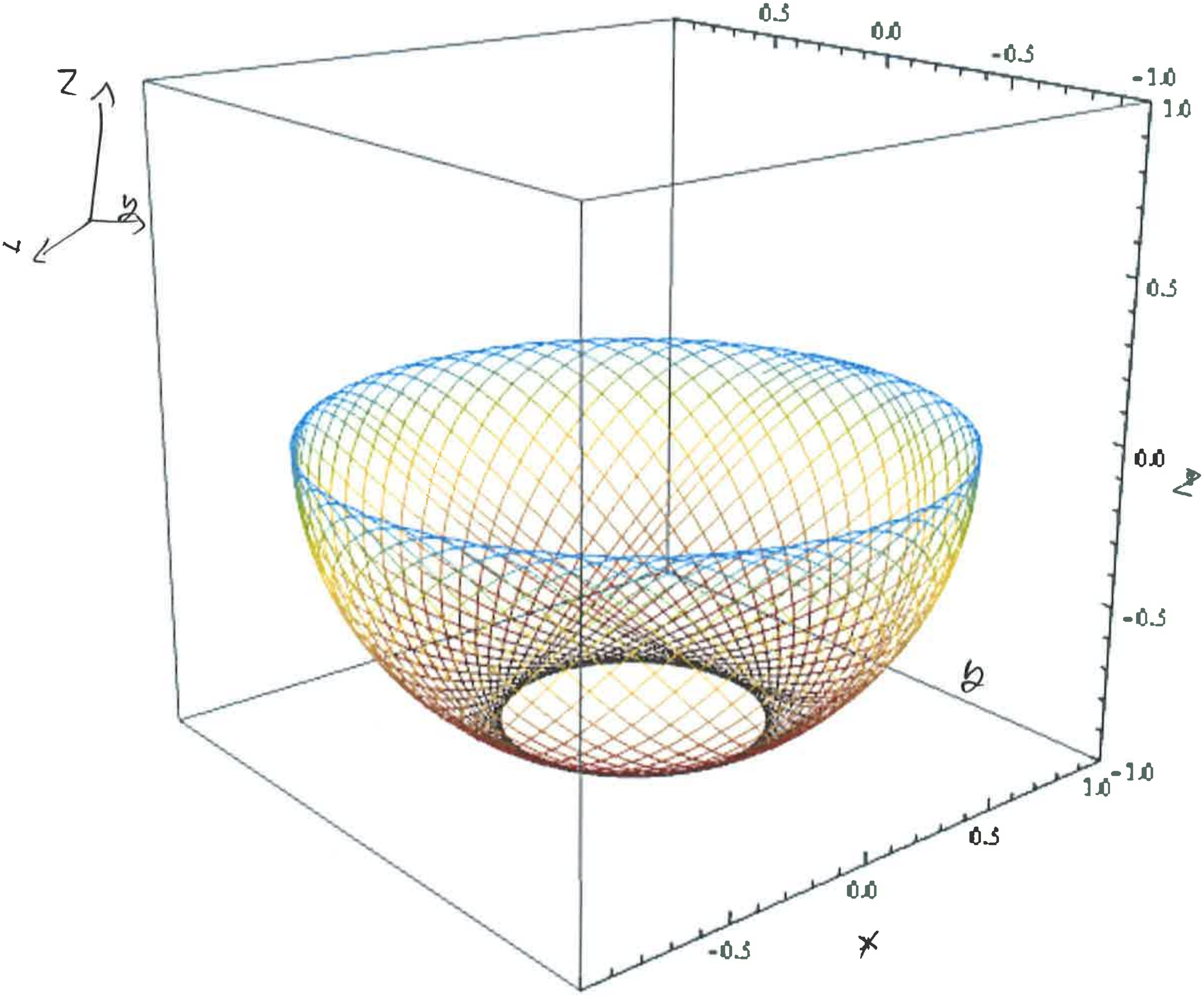
(11) $\dot{p}_\theta = \frac{\partial \mathcal{L}}{\partial \theta} = \dot{\phi}^2 mb^2\sin\theta\cos\theta - mgb\sin\theta$

(12) $\dot{p}_\phi = \frac{\partial \mathcal{L}}{\partial \phi} = 0$ Because ϕ is cyclic the momentum p_ϕ about the symmetry is constant

$\Downarrow$

(13) $p_\phi = \text{Const.}$

(14) $\underline{E}$ is also Const since $\mathcal{L} \neq \mathcal{L}(t)$



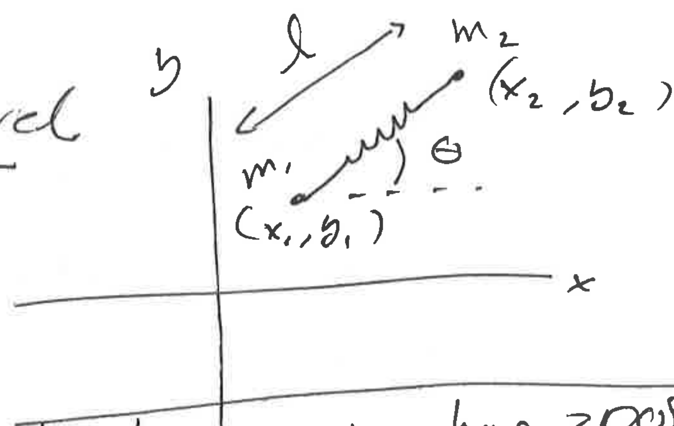
on the x-y plane it looks like a rotating ellipse

2D.1 Example 2D spring

A massless spring of length b and spring constant k connects two particles of mass m_1 and m_2. The system rests on a smooth table and may oscillate and rotate

(1) $L = ?$
(2) E.O.M
(3) Conserved

using polar coordinates we can write.



$$x_2 = x_1 + l\cos\theta$$

$$y_2 = y_1 + l\sin\theta$$

This system has 2DOF we will find that P_{x_1} and P_{y_1} are conserved

So

$$\dot{x}_2 = \dot{x}_1 + \dot{l}\cos\theta - l\dot{\theta}\sin\theta$$

$$\dot{y}_2 = \dot{y}_1 + \dot{l}\sin\theta + l\dot{\theta}\cos\theta$$

$$\mathcal{L} = T - U = \tfrac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2) + \tfrac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2) - \tfrac{1}{2}k(l-b)^2$$

Which gives:

$$\mathcal{L} = \tfrac{1}{2}(m_1+m_2)(\dot{x}_1^2 + \dot{y}_1^2) + m_2\dot{l}(\dot{x}_1\cos\theta + \dot{y}_1\sin\theta)$$
$$+ m_2 l\dot{\theta}(\dot{y}_1\cos\theta - \dot{x}_1\sin\theta) - \tfrac{1}{2}k(l-b)^2.$$

↓

E.O.M.

x_1: $\frac{\partial \mathcal{L}}{\partial x_1} = 0$; $\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_1}\right) \frac{d}{dt}\left[(m_1+m_2)\dot{x}_1 + m_2\dot{l}\cos\theta - m_2 l\dot{\theta}\sin\theta\right] = 0$

Note that what is written here is simply:

$$\frac{d}{dt}(m_1\dot{x}_1 + m_2\dot{x}_2) = 0$$

So the momentum $P_x = m_1\dot{x}_1 + m_2\dot{x}_2 = \text{Const}$

Similarly:

y_1: $\frac{\partial \mathcal{L}}{\partial y_1} = 0$; $\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{y}_1}\right) = \frac{d}{dt}\left[(m_1+m_2)\dot{y}_1 + m_2\dot{l}\sin\theta + m_2 l\dot{\theta}\cos\theta\right) = 0$

Which again can be written as

$$\frac{d}{dt}(m_1\dot{y}_1 + m_2\dot{y}_2) = 0$$

So $P_y = m_1\dot{y}_1 + m_2\dot{y}_2 = \text{Const}$.

l: $\frac{d}{dt}\left(m_2\dot{l} + m_2(\dot{x}_1\cos\theta + \dot{y}_1\sin\theta)\right] =$

$= m_2 l\dot{\theta}^2 - k(l-b) + m_2\dot{\theta}(\dot{y}_1\cos\theta - \dot{x}_1\sin\theta)$

which is

$$\ddot{l} - l\dot{\theta}^2 + \ddot{x}_1\cos\theta + \ddot{y}_1\sin\theta + \frac{k}{m_2}(l-b) = 0$$

2D.2

θ: $\frac{d}{dt}\left(m_2 l^2 \dot\theta + m_2 l(\dot y_1 \cos\theta - \dot x_1 \sin\theta)\right] =$

$= -m_2 \dot l(\dot x_1 \sin\theta - \dot y_1 \cos\theta) + m_2 l \dot\theta(-\dot x_1 \cos\theta - \dot y_1 \sin\theta)$

which reduces to

$$\boxed{\ddot\theta + \frac{2}{l}\dot l\dot\theta + \frac{\cos\theta}{l}\ddot y_1 - \frac{\sin\theta}{l}\ddot x_1 = 0}$$

P_x and P_y are conserved (total linear momentum)

are also E since $\mathcal{L} \neq f(t)$