CHAPTER 2

Newtonian Mechanics—Single Particle

2.1 Introduction

The science of mechanics seeks to provide a precise and consistent description of the dynamics of particles and systems of particles, that is, a set of physical laws mathematically describing the motions of bodies and aggregates of bodies. For this, we need certain fundamental concepts such as distance and time. The combination of the concepts of distance and time allows us to define the **velocity** and **acceleration** of a particle. The third fundamental concept, **mass,** requires some elaboration, which we give when we discuss Newton's laws.

Physical laws must be based on experimental fact. We cannot expect *a priori* that the gravitational attraction between two bodies must vary exactly as the inverse square of the distance between them. But experiment indicates that this is so. Once a set of experimental data has been correlated and a postulate has been formulated regarding the phenomena to which the data refer, then various implications can be worked out. If these implications are all verified by experiment, we may believe that the postulate is generally true. The postulate then assumes the status of a **physical law**. If some experiments disagree with the predictions of the law, the theory must be modified to be consistent with the facts.

Newton provided us with the fundamental laws of mechanics. We state these laws here in modern terms, discuss their meaning, and then derive the implications of the laws in various situations.* But the logical structure of the science of mechanics is not straightforward. Our line of reasoning in interpreting Newton's laws is not the only one possible.† We do not pursue in any detail the philosophy of mechanics but rather give only sufficient elaboration of Newton's laws to allow us to continue with the discussion of classical dynamics. We devote our attention in this chapter to the motion of a single particle, leaving systems of particles to be discussed in Chapters 9 and 11–13.

2.2 Newton's Laws

We begin by simply stating in conventional form Newton's laws of mechanics‡:

I. *A body remains at rest or in uniform motion unless acted upon by a force.*

II. *A body acted upon by a force moves in such a manner that the time rate of change of momentum equals the force.*

III. *If two bodies exert forces on each other, these forces are equal in magnitude and opposite in direction.*

These laws are so familiar that we sometimes tend to lose sight of their true significance (or lack of it) as physical laws. The First Law, for example, is meaningless without the concept of "force," a word Newton used in all three laws. In fact, standing alone, the First Law conveys a precise meaning only for *zero force*; that is, a body remaining at rest or in uniform (i.e., unaccelerated, rectilinear) motion is subject to no force whatsoever. A body moving in this manner is termed a **free body** (or **free particle**). The question of the frame of reference with respect to which the "uniform motion" is to be measured is discussed in the following section.

In pointing out the lack of content in Newton's First Law, Sir Arthur Eddington§ observed, somewhat facetiously, that all the law actually says is that "every particle continues in its state of rest or uniform motion in a straight line

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*Truesdell (Tr68) points out that Leonhard Euler (1707–1783) clarified and developed the Newtonian concepts. Euler "put most of mechanics into its modern form" and "made mechanics simple and easy" (p. 106).

†Ernst Mach (1838–1916) expressed his view in his famous book first published in 1883; E. Mach, *Die Mechanic in ihrer Entwicklung historisch-kritisch dargestellt* [The science of mechanics] (Prague, 1883). A translation of a later edition is available (Ma60). Interesting discussions are also given by R. B. Lindsay and H. Margeneau (Li36) and N. Feather (Fe59).

‡Enunciated in 1687 by Sir Isaac Newton (1642–1727) in his *Philosophiae naturalis principia mathematica* [*Mathematical principles of natural philosophy*, normally called *Principia*] (London, 1687). Previously, Galileo (1564–1642) generalized the results of his own mathematical experiments with statements equivalent to Newton's First and Second Laws. But Galileo was unable to complete the description of dynamics because he did not appreciate the significance of what would become Newton's Third Law—and therefore lacked a precise meaning of force.

§Sir Arthur Eddington (Ed30, p. 124).

except insofar as it doesn't." This is hardly fair to Newton, who meant something very definite by his statement. But it does emphasize that the First Law by itself provides us with only a qualitative notion regarding "force."

The Second Law provides an explicit statement: Force is related to the time rate of change of *momentum.* Newton appropriately defined **momentum** (although he used the term *quantity of motion*) to be the product of mass and velocity, such that

$$\mathbf{p} \equiv m\mathbf{v} \tag{2.1}$$

Therefore, Newton's Second Law can be expressed as

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = \frac{d}{dt}(m\mathbf{v}) \tag{2.2}$$

The definition of force becomes complete and precise only when "mass" is defined. Thus the First and Second Laws are not really "laws" in the usual sense; rather, they may be considered *definitions.* Because length, time, and mass are concepts normally already understood, we use Newton's First and Second Laws as the operational definition of force. Newton's Third Law, however, is indeed a *law.* It is a statement concerning the real physical world and contains all of the physics in Newton's laws of motion.*

We must hasten to add, however, that the Third Law is not a *general* law of nature. The law does apply when the force exerted by one (point) object on another (point) object is directed along the line connecting the objects. Such forces are called **central forces;** the Third Law applies whether a central force is attractive or repulsive. Gravitational and electrostatic forces are central forces, so Newton's laws can be used in problems involving these types of forces. Sometimes, elastic forces (which are actually macroscopic manifestations of microscopic electrostatic forces) are central. For example, two point objects connected by a straight spring or elastic string are subject to forces that obey the Third Law. Any force that depends on the velocities of the interacting bodies is noncentral, and the Third Law may not apply. Velocity-dependent forces are characteristic of interactions that propagate with finite velocity. Thus the force between *moving* electric charges does not obey the Third Law, because the force propagates with the velocity of light. Even the gravitational force between *moving* bodies is velocity dependent, but the effect is small and difficult to detect. The only observable effect is the precession of the perihelia of the inner planets (see Section 8.9). We will return to a discussion of Newton's Third Law in Chapter 9.

To demonstrate the significance of Newton's Third Law, let us paraphrase it in the following way, which incorporates the appropriate definition of mass:

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*The reasoning presented here, viz., that the First and Second Laws are actually definitions and that the Third Law contains the physics, is not the only possible interpretation. Lindsay and Margenau (Li36), for example, present the first two Laws as physical laws and then derive the Third Law as a consequence.

III′. *If two bodies constitute an ideal, isolated system, then the accelerations of these bodies are always in opposite directions, and the ratio of the magnitudes of the accelerations is constant. This constant ratio is the inverse ratio of the masses of the bodies.*

With this statement, we can give a practical definition of mass and therefore give precise meaning to the equations summarizing Newtonian dynamics. For two isolated bodies, 1 and 2, the Third Law states that

$$\mathbf{F}_1 = -\mathbf{F}_2 \tag{2.3}$$

Using the definition of force as given by the Second Law, we have

$$\frac{d\mathbf{p}_1}{dt} = -\frac{d\mathbf{p}_2}{dt} \tag{2.4a}$$

or, with constant masses,

$$m_1\left(\frac{d\mathbf{v}_1}{dt}\right) = m_2\left(-\frac{d\mathbf{v}_2}{dt}\right) \tag{2.4b}$$

and, because acceleration is the time derivative of velocity,

$$m_1(\mathbf{a}_1) = m_2(-\mathbf{a}_2) \tag{2.4c}$$

Hence,

$$\frac{m_2}{m_1} = -\frac{a_1}{a_2} \tag{2.5}$$

where the negative sign indicates only that the two acceleration vectors are oppositely directed. Mass is taken to be a positive quantity.

We can always select, say, m_1 as the *unit mass*. Then, by comparing the ratio of accelerations when m_1 is allowed to interact with any other body, we can determine the mass of the other body. To measure the accelerations, we must have appropriate clocks and measuring rods; also, we must choose a suitable coordinate system or reference frame. The question of a "suitable reference frame" is discussed in the next section.

One of the more common methods of determining the mass of an object is by *weighing*—for example, by comparing its weight to that of a standard by means of a beam balance. This procedure makes use of the fact that in a gravitational field the weight of a body is just the gravitational force acting on the body; that is, Newton's equation $\mathbf{F} = m\mathbf{a}$ becomes $\mathbf{W} = m\mathbf{g}$, where $\mathbf{g}$ is the acceleration due to gravity. The validity of using this procedure rests on a fundamental assumption: that the mass m appearing in Newton's equation and defined according to Statement III′ is equal to the mass m that appears in the gravitational force equation. These two masses are called the **inertial mass** and **gravitational mass,** respectively. The definitions may be stated as follows:

Inertial Mass: *That mass determining the acceleration of a body under the action of a given force.*

Gravitational Mass: *That mass determining the gravitational forces between a body and other bodies.*

Galileo was the first to test the equivalence of inertial and gravitational mass in his (perhaps apocryphal) experiment with falling weights at the Tower of Pisa. Newton also considered the problem and measured the periods of pendula of equal lengths but with bobs of different materials. Neither Newton nor Galileo found any difference, but the methods were quite crude.* In 1890 Eötvös† devised an ingenious method to test the equivalence of inertial and gravitational masses. Using two objects made of different materials, he compared the effect of the Earth's gravitational force (i.e., the weight) with the effect of the inertial force caused by the Earth's rotation. The experiment involved a *null* method using a sensitive torsion balance and was therefore highly accurate. More recent experiments (notably those of Dicke‡), using essentially the same method, have improved the accuracy, and we know now that inertial and gravitational mass are identical to within a few parts in 10^{12}. This result is considerably important in the general theory of relativity.§ The assertion of the *exact* equality of inertial and gravitational mass is termed the **principle of equivalence.**

Newton's Third Law is stated in terms of two bodies that constitute an isolated system. It is impossible to achieve such an ideal condition; every body in the universe interacts with every other body, although the force of interaction may be far too weak to be of any practical importance if great distances are involved. Newton avoided the question of how to disentangle the desired effects from all the extraneous effects. But this practical difficulty only emphasizes the enormity of Newton's assertion made in the Third Law. It is a tribute to the depth of his perception and physical insight that the conclusion, based on limited observations, has successfully borne the test of experiment for 300 years. Only within the 20th century did measurements of sufficient detail reveal certain discrepancies with the predictions of Newtonian theory. The pursuit of these details led to the development of relativity theory and quantum mechanics.‖

Another interpretation of Newton's Third Law is based on the concept of momentum. Rearranging Equation 2.4a gives

$$\frac{d}{dt}(\mathbf{p}_1 + \mathbf{p}_2) = 0$$

or

$$\mathbf{p}_1 + \mathbf{p}_2 = constant \tag{2.6}$$

The statement that momentum is conserved in the isolated interaction of two particles is a special case of the more general **conservation of linear momentum.** Physicists cherish general conservation laws, and the conservation of linear momentum is believed always to be obeyed. Later we shall modify our defi-

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*In Newton's experiment, he could have detected a difference of only one part in 10^3.

†Roland von Eötvös (1848-1919), a Hungarian baron; his research in gravitational problems led to the development of a gravimeter, which was used in geological studies.

‡P. G. Roll, R. Krotkov, and R. H. Dicke, Ann. Phys. (N.Y.) **26**, 442 (1964). See also Braginsky and Pavov, Sov. Phys.-JETP **34**, 463 (1972).

§See, for example, the discussions by P. G. Bergmann (Be46) and J. Weber (We61). Weber's book also provides an analysis of the Eötvös experiment.

‖See also Section 2.8.

nition of momentum from Equation 2.1 for high velocities approaching the speed of light.

2.3 Frames of Reference

Newton realized that, for the laws of motion to have meaning, the motion of bodies must be measured relative to some reference frame. A reference frame is called an **inertial frame** if Newton's laws are indeed valid in that frame; that is, if a body subject to no external force moves in a straight line with constant velocity (or remains at rest), then the coordinate system establishing this fact is an inertial reference frame. This is a clear-cut operational definition and one that also follows from the general theory of relativity.

If Newton's laws are valid in one reference frame, then they are also valid in any reference frame in uniform motion (i.e., not accelerated) with respect to the first system.* This is a result of the fact that the equation $\mathbf{F} = m\ddot{\mathbf{r}}$ involves the second time derivative of $\mathbf{r}$: A change of coordinates involving a constant velocity does not influence the equation. This result is called **Galilean invariance** or the **principle of Newtonian relativity.**

Relativity theory has shown us that the concepts of *absolute rest* and an *absolute* inertial reference frame are meaningless. Therefore, even though we conventionally adopt a reference frame described with respect to the "fixed" stars—and, indeed, in such a frame the Newtonian equations are valid to a high degree of accuracy—such a frame is, in fact, not an absolute inertial frame. We may, however, consider the "fixed" stars to define a reference frame that approximates an "absolute" inertial frame to an extent quite sufficient for our present purposes.

Although the fixed-star reference frame is a conveniently definable system and one suitable for many purposes, we must emphasize that the fundamental definition of an inertial frame makes no mention of stars, fixed or otherwise. If a body subject to no force moves with constant velocity in a certain coordinate system, that system is, by definition, an inertial frame. Because precisely describing the motion of a real physical object in the real physical world is normally difficult, we usually resort to idealizations and approximations of varying degree; that is, we ordinarily neglect the lesser forces on a body if these forces do not significantly affect the body's motion.

If we wish to describe the motion of, say, a free particle and if we choose for this purpose some coordinate system in an inertial frame, then we require that the (vector) equation of motion of the particle be independent of the *position* of the origin of the coordinate system and independent of its *orientation* in **space.** We further require that **time** be homogeneous; that is, a free particle moving with a certain constant velocity in the coordinate system during a certain time

*In Chapter 10, we discuss the modification of Newton's equations that must be made if it is desired to describe the motion of a body with respect to a *noninertial* frame of reference, that is, a frame that is accelerated with respect to an inertial frame.

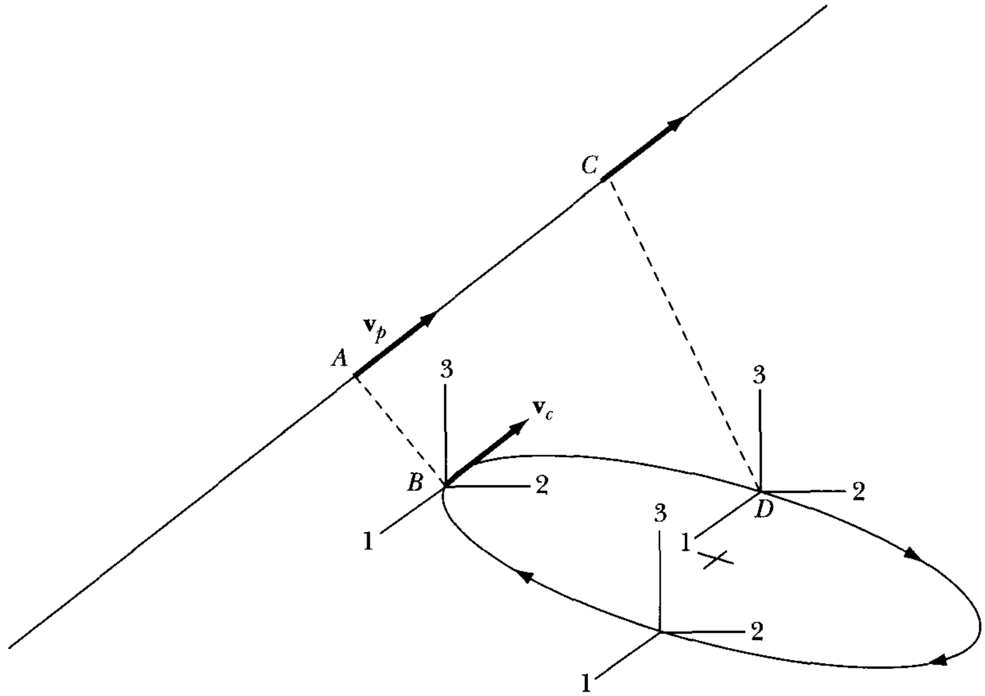

FIGURE 2-1 We choose to describe the path of a free particle moving along the path *AC* in a rectangular coordinate system whose origin moves in a circle. Such a system is not an inertial reference frame.

interval must not, during a later time interval, be found to move with a different velocity.

We can illustrate the importance of these properties by the following example. Consider, as in Figure 2-1, a free particle moving along a certain path *AC*. To describe the particle's motion, let us choose a rectangular coordinate system whose origin moves in a circle, as shown. For simplicity, we let the orientation of the axes be fixed in space. The particle moves with a velocity $\mathbf{v}_p$ relative to an inertial reference frame. If the coordinate system moves with a linear velocity $\mathbf{v}_c$ when at the point *B*, and if $\mathbf{v}_c = \mathbf{v}_p$, then to an observer in the moving coordinate system the particle (at *A*) will appear to be *at rest*. At some later time, however, when the particle is at *C* and the coordinate system is at *D*, the particle will appear to accelerate with respect to the observer. We must, therefore, conclude that the rotating coordinate system does not qualify as an inertial reference frame.

These observations are not sufficient to decide whether time is homogeneous. To reach such a conclusion, repeated measurements must be made in identical situations at various times; identical results would indicate the homogeneity of time.

Newton's equations do not describe the motion of bodies in noninertial systems. We can devise a method to describe the motion of a particle by a rotating coordinate system, but, as we shall see in Chapter 10, the resulting equations contain several terms that do not appear in the simple Newtonian equation $\mathbf{F} = m\mathbf{a}$. For the moment, then, we restrict our attention to inertial reference frames to describe the dynamics of particles.

2.4 The Equation of Motion for a Particle

Newton's equation $\mathbf{F} = d\mathbf{p}/dt$ can be expressed alternatively as

$$\mathbf{F} = \frac{d}{dt}(m\mathbf{v}) = m\frac{d\mathbf{v}}{dt} = m\ddot{\mathbf{r}} \tag{2.7}$$

if we assume that the mass m does not vary with time. This is a second-order differential equation that may be integrated to find $\mathbf{r} = \mathbf{r}(t)$ if the function $\mathbf{F}$ is known. Specifying the initial values of $\mathbf{r}$ and $\dot{\mathbf{r}} = \mathbf{v}$ then allows us to evaluate the two arbitrary constants of integration. We then determine the motion of a particle by the force function $\mathbf{F}$ and the initial values of position $\mathbf{r}$ and velocity $\mathbf{v}$.

The force $\mathbf{F}$ may be a function of any combination of position, velocity, and time and is generally denoted as $\mathbf{F}(\mathbf{r}, \mathbf{v}, t)$. For a given dynamic system, we normally want to know $\mathbf{r}$ and $\mathbf{v}$ as a function of time. Solving Equation 2.7 will help us do this by solving for $\ddot{\mathbf{r}}$. Applying Equation 2.7 to physical situations is an important part of mechanics.

In this chapter, we examine several examples in which the force function is known. We begin by looking at simple force functions (either constant or dependent on only one of $\mathbf{r}$, $\mathbf{v}$, and t) in only one spatial dimension as a refresher of earlier physics courses. It is important to form good habits in problem solving. Here are some useful problem-solving techniques.

1. Make a sketch of the problem, indicating forces, velocities, and so forth.
2. Write down the given quantities.
3. Write down useful equations and what is to be determined.
4. Strategy and the principles of physics must be used to manipulate the equations to find the quantity sought. Algebraic manipulations as well as differentiation or integration is usually required. Sometimes numerical calculations using a computer are the easiest, if not the only, method of solution.
5. Finally, put in the actual values for the assumed variable names to determine the quantity sought.

Let us first consider the problem of a block sliding on an inclined plane. Let the angle of the inclined plane be θ and the mass of the block be 100 g. The sketch of the problem is shown in Figure 2-2a.

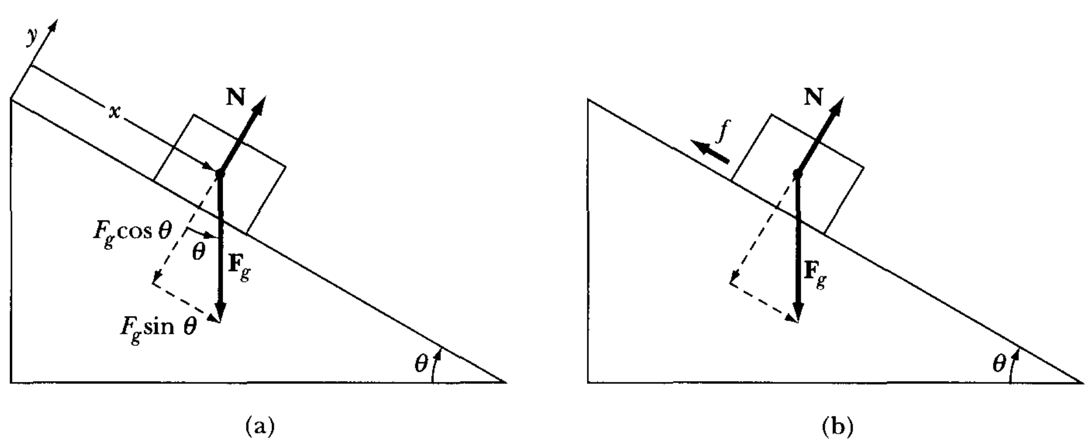

FIGURE 2-2 Examples 2.1 and 2.2.