105A Double pendulum example

A double pendulum is consists of two simple pendular, with one pendulum suspended from a the bob of the other (see Figure reffig:1). If he two pendular have equal lengths and have bobs of equal mass and if both pendular are confined to move in the same plane, fins the Lagrange's equations of motion for the system. Do **not** assume small angles.

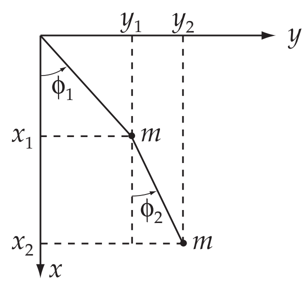

Figure 1: Double pendulum

Answer: Assigning coordinates of the double pendulum in the usual way we have

$$
\begin{aligned}
x_1 &= l\cos\phi_1 &(1)\\
y_1 &= l\sin\phi_1 &(2)\\
x_2 &= l(\cos\phi_1+\cos\phi_2) &(3)\\
y_2 &= l(\sin\phi_1+\sin\phi_2) &(4)
\end{aligned}
$$

we find the kinetic energy of the system to be

$$
\begin{aligned}
T &= \frac{m}{2}(\dot{x}_1^2+\dot{y}_1^2)+\frac{m}{2}(\dot{x}_2^2+\dot{y}_2^2)\\
&= \frac{m}{2}l^2\left(\dot{\phi}_1^2+\dot{\phi}_1^2+\dot{\phi}_2^2+2\dot{\phi}_1\dot{\phi}_2(\sin\phi_1\sin\phi_2+\cos\phi_1\cos\phi_2)\right)\\
&= \frac{m}{2}l^2\left(2\dot{\phi}_1^2+\dot{\phi}_2^2+2\dot{\phi}_1\dot{\phi}_2\cos(\phi_1-\phi_2)\right) \qquad (5)
\end{aligned}
$$

The potential energy is

$$U=-mgx_1-mgx_2=-mgl(2\cos\phi_1+\cos\phi_2) \qquad (6)$$

Therefore, the Lagrangian is

$$L=T-U=\frac{m}{2}l^2\left(2\dot{\phi}_1^2+\dot{\phi}_2^2+2\dot{\phi}_1\dot{\phi}_2\cos(\phi_1-\phi_2)\right)+mgl(2\cos\phi_1+\cos\phi_2) \qquad (7)$$

Taking the relevant derivatives we have:

$$\begin{aligned}
\frac{\partial L}{\partial \phi_1} &= ml^2\dot{\phi}_1\dot{\phi}_2\sin(\phi_1-\phi_2) - 2mgl\sin\phi_1 && (8)\\
\frac{\partial L}{\partial \dot{\phi}_1} &= 2ml^2\dot{\phi}_1 + ml^2\dot{\phi}_2\cos(\phi_1-\phi_2) && (9)\\
\frac{\partial L}{\partial \phi_2} &= -ml^2\dot{\phi}_1\dot{\phi}_2\sin(\phi_1-\phi_2) - mgl\sin\phi_2 && (10)\\
\frac{\partial L}{\partial \dot{\phi}_2} &= ml^2\dot{\phi}_2 + ml^2\dot{\phi}_1\cos(\phi_1-\phi_2) && (11)
\end{aligned}$$

The Lagrange equations for ϕ_1 and ϕ_2 are

$$\begin{aligned}
\phi_1 &: \; 2\phi_1 + \ddot{\phi}_2\cos(\phi_1-\phi_2) + \dot{\phi}_2^2\sin(\phi_1-\phi_2) + 2\frac{g}{l}\sin\phi_1 = 0 && (12)\\
\phi_2 &: \; \ddot{\phi}_2 + \ddot{\phi}_1\cos(\phi_1-\phi_2) - \dot{\phi}_1^2\sin(\phi_1-\phi_2) + 2\frac{g}{l}\sin\phi_2 = 0 && (13)
\end{aligned}$$