Two mass points of mass m_1 and m_2 are connected by a string passing through a hole in a smooth table so that m_1 rests on the table and m_2 hangs suspended. Assuming m_2 moves only in a vertical line, what are the generalized coordinates for the system? Write down the Lagrange equations for the system and, if possible, discuss the physical significance any of them might have. Reduce the problem to a single second-order differential equation and obtain a first integral of the equation. What is its physical significance? (Consider the motion only so long as neither m_1 nor m_2 passes through the hole).

Let d be the height of m_2 above its lowest possible position, so that $d = 0$ when the string is fully extended beneath the table and m_1 is just about to fall through the hole. Also, let θ be the angular coordinate of m_1 on the table. Then the kinetic energy of m_2 is just $m_2\dot{d}^2/2$, while the kinetic energy of m_1 is $m_1\dot{d}^2/2 + m_1 d^2\dot{\theta}^2/2$, and the potential energy of the system is just the gravitational potential energy of m_2, $U = m_2 g d$. Then the Lagrangian is

$$L = \frac{1}{2}(m_1 + m_2)\dot{d}^2 + \frac{1}{2}m_1 d^2\dot{\theta}^2 - m_2 g d$$

and the Euler-Lagrange equations are

$$\frac{d}{dt}(m_1 d^2\dot{\theta}) = 0$$
$$(m_1 + m_2)\ddot{d} = -m_2 g + m_1 d\dot{\theta}^2$$

From the first equation we can identify a first integral, $m_1 d^2\dot{\theta} = l$ where l is a constant. With this we can substitute for $\dot{\theta}$ in the second equation:

$$(m_1 + m_2)\ddot{d} = -m_2 g + \frac{l^2}{m_1 d^3}$$

Because the sign of the two terms on the RHS is different, this is saying that, if l is big enough (if m_1 is spinning fast enough), the centrifugal force of m_1 can balance the downward pull of m_2, and the system can be in equilibrium.

A particle of mass m is constrained to move under gravity without friction on the inside of a paraboloid of revolution whose axis is vertical. Find the one-dimensional problem equivalent to its motion. What is the condition on the particle's initial velocity to produce circular motion? Find the period of small oscillations about this circular motion.

We'll take the paraboloid to be defined by the equation $z = \alpha r^2$. The kinetic and potential energies of the particle are

$$\begin{aligned} T &= \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2) \\ &= \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2 + 4\alpha^2 r^2 \dot{r}^2) \\ V &= mgz = mg\alpha r^2. \end{aligned}$$

Hence the Lagrangian is

$$L = \frac{m}{2}\left[(1 + 4\alpha^2 r^2)\dot{r}^2 + r^2\dot{\theta}^2\right] - mg\alpha r^2.$$

This is cyclic in θ, so the angular momentum is conserved:

$$l = mr^2\dot{\theta} = \text{constant}.$$

For r we have the derivatives

$$\begin{aligned}
\frac{\partial L}{\partial r} &= 4\alpha^2 mr\dot{r}^2 + mr\dot{\theta}^2 - 2mg\alpha r \\
\frac{\partial L}{\partial \dot{r}} &= m(1+4\alpha^2 r^2)\dot{r} \\
\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} &= 8m\alpha^2 r\dot{r}^2 + m(1+4\alpha^2 r^2)\ddot{r}.
\end{aligned}$$

Hence the equation of motion for r is

$$8m\alpha^2 r\dot{r}^2 + m(1+4\alpha^2 r^2)\ddot{r} = 4\alpha^2 mr\dot{r}^2 + mr\dot{\theta}^2 - 2mg\alpha r$$

or

$$m(1+4\alpha^2 r^2)\ddot{r} + 4m\alpha^2 r\dot{r}^2 - mr\dot{\theta}^2 + 2mg\alpha r = 0.$$

In terms of the constant angular momentum, we may rewrite this as

$$m(1+4\alpha^2 r^2)\ddot{r} + 4m\alpha^2 r\dot{r}^2 - \frac{l^2}{mr^3} + 2mg\alpha r = 0.$$

So this is the differential equation that determines the time evolution of r.

If initially $\dot{r} = 0$, then we have

$$m(1+4\alpha^2 r^2)\ddot{r} + -\frac{l^2}{mr^3} + 2mg\alpha r = 0.$$

Evidently, $\ddot{r}$ will then vanish—and hence $\dot{r}$ will remain 0, giving circular motion—if

$$\frac{l^2}{mr^3} = 2mg\alpha r$$

or

$$\dot{\theta} = \sqrt{2g\alpha}.$$

So if this condition is satisfied, the particle will execute circular motion (assuming its initial r velocity was zero). It's interesting to note that the condition on $\dot{\theta}$ for circular motion is independent of r.