Week 1 QM Discussion 10/5/2017

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We can always represent a physical quantity using a hermitian operator. In certain basis, operators can be matrices or differential operators. The eigenvalues give us the possible results when we do measurements. The eigenkets are always orthogonal and complete when there is no degeneracy.

Example

Consider a physical quantity, let's say energy, corresponding to an operator

$$\hat{H} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

in the basis: $|1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $|2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Our system is in state $|\phi(t = 0)\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

a) What are possible results when we measure the energy in the state $|\phi(t = 0)\rangle$? What's the probability for each result?

b) Verify that the eigenkets are orthogonal and complete.

c) What's the average energy in state $|\phi(t = 0)\rangle$?

d) Can you work out the state $|\phi(t)\rangle$ of our system at time t? What's the average energy at time t? (You will see the average energy is independent of time which means the average energy is conserved. Can you prove this statement using Heisenberg equation of motion? Can you give a physical argument for this statement?)

e) Now suppose we have another hermitian operator $\hat{A} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, compute $[\hat{H}, \hat{A}]$ and verify uncertainty relation holds for our system (use $|\phi(t = 0)\rangle$).

a). only need eigenvalues of $\hat{H}$:

$$\det\begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix} = \lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1$$

To get the prob., we need the eigenkets:

$$|\lambda=1\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow a = b \Rightarrow |\lambda=1\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$P_{\lambda=1} = \left| \langle \lambda=1 | \phi(t=0)\rangle \right|^2$$

$$= \left| \left(\tfrac{1}{\sqrt{2}} \;\; \tfrac{1}{\sqrt{2}}\right) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right|^2$$

$$= \left| \frac{1}{\sqrt{2}} \right|^2$$

$$= \frac{1}{2}$$

$$|\lambda=-1\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = -\begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow a = -b \Rightarrow |\lambda=-1\rangle = \begin{pmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$P_{\lambda=-1} = \left| \langle \lambda=-1 | \phi(t=0) \rangle \right|^2$$

$$= \left| \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right|^2$$

$$= \frac{1}{2}$$

b). $\langle \lambda=1 | \lambda=-1 \rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = 0$

$$\hat{I} \stackrel{?}{=} |\lambda=1\rangle\langle\lambda=1| + |\lambda=-1\rangle\langle\lambda=-1|$$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} + \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} + \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

c). one way:

$$\begin{array}{cc} \lambda=1 & \lambda=-1 \\ \uparrow & \uparrow \\ \frac{1}{2} & \frac{1}{2} \end{array}$$

$$\Leftrightarrow \langle H \rangle = 1 \times \frac{1}{2} + (-1)\frac{1}{2} = 0$$

another way:

$$\langle H \rangle = \langle \phi(t=0) | \hat{H} | \phi(t=0) \rangle$$

$$= (1 \quad 0) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= (1 \quad 0) \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= 0.$$

d).

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} - \frac{1}{\sqrt{2}} \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}} |+\rangle - \frac{1}{\sqrt{2}} |-\rangle$$

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}} e^{-\frac{iE_+ t}{\hbar}} |+\rangle - \frac{1}{\sqrt{2}} e^{-\frac{iE_- t}{\hbar}} |-\rangle$$

$$= \frac{1}{\sqrt{2}} e^{-\frac{it}{\hbar}} |+\rangle - \frac{1}{\sqrt{2}} e^{+\frac{it}{\hbar}} |-\rangle$$

$\langle H \rangle = 0$:

From Hengerten equation of motion, we know:

$$\frac{d\langle \hat{A} \rangle}{dt} = \left\langle \frac{\partial \hat{A}}{\partial t} \right\rangle + \frac{1}{i\hbar} \langle [\hat{A}, \hat{H}] \rangle$$

set $\hat{A} = \hat{H}$:

$$\frac{d\langle \hat{H} \rangle}{dt} = 0.$$

system independent of t: $\rightarrow$ Energy is conserved.

e).

$$[\hat{H}, \hat{A}] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} - \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

$$= 2i \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Delta A^2 \Delta B^2 \geq \frac{1}{4} \left| \langle [\hat{A}, \hat{B}] \rangle \right|^2$$

$$R.H.S = \frac{1}{4} \cdot 4 \left| (1 \;\; 0) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right|^2$$

$$= \left| (1 \;\; 0) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right|^2 = 1$$

$$\Delta \hat{H}^2 = \langle \hat{H}^2 \rangle - (\langle \hat{H} \rangle)^2 = \langle \hat{H}^2 \rangle$$

$$\hat{H}^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\langle \hat{H}^2 \rangle = (1 \quad 0)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = (1 \quad 0)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1$$

$$\Delta\hat{A}^2 = \langle \hat{A}^2 \rangle - \langle \hat{A} \rangle^2$$

$$\langle \hat{A} \rangle = (1 \quad 0)\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = (1 \quad 0)\begin{pmatrix} 0 \\ i \end{pmatrix} = 0$$

$$\hat{A}^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$1 \geq 1.$ ✓