105A - Set 7

(Grades are out of 150)

1. (39pt) A particle is moving in a central inverse-square-law force field for a superimposed force which magnitude is inversely proportional to the cube of the distance from the particle to the force center. in other words:

$$F = -\frac{k}{r^2} - \frac{\lambda}{r^3} \quad k, \lambda > 0 \tag{1}$$

describe the motion (i.e., r as a function of θ) and show that the motion can be described as a precessing ellipse. Consider the following cases:

   (a) (13pt) $\lambda < l^2/\mu$

   (b) (13pt) $\lambda = l^2/\mu$

   (c) (13pt) $\lambda > l^2/\mu$

   where l is the angular momentum and μ is the mass of the particle.
   Hint 1: Use Binnet's equation
   Hint 2: Note that

$$\frac{\mu k}{l^2}\left(1 - \frac{\mu\lambda}{l^2}\right)^{-1} \tag{2}$$

   Is constant.

2. (34pt) A particle moves in a central force field given by the potential

$$V = -k\frac{e^{-ar}}{r} \tag{3}$$

where k and a are positive constants.

   (a) (5pt) Write down the Lagrangian

   (b) (8pt) Find the equations of motions

   (c) (10pt) When is circular orbit is possible?

   (d) (5pt) What is the effective potential?

   (e) (6pt) Which point does a circular orbit represent on the effective potential?

3. (38pt) A satellite of mass m in a Kepler central potential $U(r) = -k/r$ has orbits described by

$$\frac{1}{r} = \frac{\mu k}{l^2}(1 + e\cos\theta) \tag{4}$$

where

$$e = \sqrt{1 + \frac{2l^2 E}{\mu k^2}} \tag{5}$$

(a) (19pt) Suppose the particle is initially in a parabolic orbit. An impulse is applied at periastron (closest approached) to place the particle in a circular orbit. Give the energy and angular momentum of the circular orbit in terms of the energy and angular momentum of the initial parabolic orbit.

(b) (19pt) Suppose the particle is initially in an arbitrary elliptical orbit. An impulse is applied at $\theta = \pi/2$ to place the particle in a circular orbit. Give the energy and angular momentum of the circular orbit in terms of the energy and angular momentum of the initial orbit.

4. (39pt) Two point particles of masses m_1 and m_2 interact via the central potential

$$U(r) = U_0 \ln\left(\frac{r^2}{r^2 + b^2}\right) \tag{6}$$

where b is a constant with dimensions of length

(a) (5pt) Write the effective potential.

(b) (17pt) For what values of the angular momentum l does a circular orbit exist? Find the radius r_0 of the circular orbit. Is it stable or unstable?

(c) (17pt) Suppose the orbit is nearly circular, with $r = r_0 + \eta$, where $\eta << r_0$. Find the equation for the shape $\eta(\theta)$ of the perturbation. - a general function as an answer is good enough.
Hint 1: Use the conservation of angular momentum to find a relation between $\dot{r}$ and $\dot{\theta}$, just as we did in class and plug this into the expression for energy. Then expand to the **second** *order in η.*
Hint 2: Remember that the Energy can be expanded as $E(r_0 + \eta) = E_0 + E(\eta)$, where E_0 is the energy of the circular orbit and $E(\eta)$ is constant.
Hint 3: Keep in the equation $(d\eta/d\theta)^2$, you'll need it.