Math 115A: midterm 1

Sections 1. Instructor: James Freitag

Problem 1 Bases and linear transformations.

Let $\beta = ((1,0),(0,1))$ be the standard ordered basis for $\mathbb{R}^2$. Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation such that $T(1,2) = (3,5)$ and $T(1,1) = (12,-3)$. Calculate $[T]_\beta^\beta$.

Recall from class that $[T]_\beta^\beta = \left(\left[T\begin{pmatrix}1\\0\end{pmatrix}\right]^\beta \left[T\begin{pmatrix}0\\1\end{pmatrix}\right]^\beta \right)$, so lets calculate these coordinate vectors.

2pts

$$\boxed{T\left(\begin{pmatrix}0\\1\end{pmatrix}\right) = T\left(\begin{pmatrix}1\\2\end{pmatrix} - \begin{pmatrix}1\\1\end{pmatrix}\right) = T\left(\begin{pmatrix}1\\2\end{pmatrix}\right) - T\left(\begin{pmatrix}1\\1\end{pmatrix}\right) = \begin{pmatrix}3\\5\end{pmatrix} - \begin{pmatrix}12\\-3\end{pmatrix} = \begin{pmatrix}-9\\8\end{pmatrix}}$$

and

2pts

$$\boxed{T\left(\begin{pmatrix}1\\0\end{pmatrix}\right) = T\left(\begin{pmatrix}1\\2\end{pmatrix} - \begin{pmatrix}0\\2\end{pmatrix}\right) = T\begin{pmatrix}1\\2\end{pmatrix} - T\begin{pmatrix}0\\2\end{pmatrix} = T\begin{pmatrix}1\\2\end{pmatrix} - 2T\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}3\\5\end{pmatrix} - 2\begin{pmatrix}-9\\8\end{pmatrix} = \begin{pmatrix}21\\-11\end{pmatrix}}$$

Now, in β, the coordinate vector of any element is simply the element, so

1pt

$$\boxed{[T]_\beta^\beta = \begin{pmatrix}21 & -9\\-11 & 8\end{pmatrix}}$$

* Not all of the problems have such a clear way to break down the grading

e.g. most other problems can be solved in many ways.

Problem 2 How to span a space

Let $T : V \to W$ be a linear transformation. Suppose that $v_1, \ldots, v_n$ are vectors in V such that $Span(\{v_1, \ldots, v_n\})$ contains $N(T)$ and that $Span(\{T(v_1), \ldots, T(v_n)\}) = R(T)$. Show that $Span(\{v_1, \ldots, v_n\}) = V$.

As noted during the exam, you may assume that V, W are fin.-dim.

Let $U = span(\{v_1 \ldots v_n\})$. Now we have two possible maps:

$T : V \longrightarrow W$

$T\restriction_U : U \longrightarrow W$, the restriction of T to U.

Let $k = nullity(T)$.

Then since $N(T) \leq U$,

we see $k = nullity(T\restriction_U)$.

But now $R(T) = R(T\restriction_U)$, since $span(\{Tv_1 \ldots Tv_n\}) = R(T)$.

Thus $rank(T) = rank(T\restriction_U)$.

$$dim(V) \underset{\text{by dimension Thm.}}{=} rank(T) + nullity(T) = rank(T\restriction_U) + nullity(T\restriction_U) \underset{\text{by dimension Thm.}}{=} dim(U)$$

We proved in class that $U \leq V$ and $dim(U) = dim(V) \Rightarrow U = V$.

Problem 3 Rank and nullity

Let U, V, W be vector spaces such that $dim(U) = 4, dim(V) = 3, dim(W) = 5$ and let $T : U \to V$ and $S : V \to W$ be linear. Let $R = S \circ T$ be the composition. Prove that R is not surjective. Prove that R is not injective.

$$\begin{tikzcd} \overset{4}{U} \arrow[r, "T"'] \arrow[rr, bend right, "R"'] & \overset{3}{V} \arrow[r, "S"'] & \overset{5}{W} \end{tikzcd}$$

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$\dim(U) = \operatorname{rank}(R) + \operatorname{nullity}(R)$, so in particular, $\dim(U) \geq \operatorname{rank}(R)$

by dimension Thm.

Thus $\operatorname{rank}(R) < 5$ and so R is not surjective

$N(T) \subseteq N(R)$ since if $T(v) = \bar{0}$ then $R(v) = S(T(v)) = S(\bar{0}) = \bar{0}$

Now $\dim(U) = \operatorname{nullity}(T) + \operatorname{rank}(T)$

by dimension Thm.

$4 = \operatorname{nullity}(T) + \operatorname{rank}(T)$. But $R(T) \subseteq V$, so $\operatorname{rank}(T) \leq 3$, so

$\operatorname{nullity}(T) \geq 1$. Thus $N(T) \neq \{\bar{0}\}$ and so $N(R) \neq \bar{0}$ and so

R is not injective.

Problem 4 Functions and vector spaces

Let $\mathfrak{F}(\mathbb{R})$ denote the vector space of all functions $f : \mathbb{R} \to \mathbb{R}$. Let W be the subspace of $\mathcal{F}(V)$ consisting of all functions f such that

$$f(x) = a \cdot \sin(x + b) + c \cdot \cos(x + d).$$

You don't need to show that W is a subspace. Just assume this. Is W finite-dimensional? If so, what is the dimension of W.

Hint: Try to show that $\{\sin(x), \cos(x)\}$ is a basis of W. Note the formulas

$$\sin(x + y) = \sin(x)\cos(y) + \cos(x)\sin(y)$$

$$\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y)$$

Let $a, b, c, d \in \mathbb{R}$.
Then:

$$a \cdot \sin(x+b) + c \cdot \cos(x+d) = a \cdot \sin(x) \cdot \cos(b) + a\cos(x) \cdot \sin(b) + c \cdot \cos(x)\cos(d) - c \cdot \sin(x)\sin(d)$$

$$= \underline{(a \cdot \cos(b) - c \cdot \sin(d))}(\sin(x)) + \underline{(a\sin(b) + c \cdot \cos(d))}(\cos(x))$$

These are real numbers, so $W \subseteq \mathrm{Span}(\{\sin(x), \cos(x)\})$

* Thus $\dim(W) \leq 2$.

*Actually, it is easy to show also that $W = \mathrm{span}(\{\sin(x), \cos(x)\})$ and that $\sin(x)$ and $\cos(x)$ are Lin. Ind., so $\dim(W) = 2$.