Math 115A: midterm 1

Section 3. Instructor: James Freitag

Problem 1 Bases and linear transformations.

Let $\beta = ((1,0),(0,1))$ be the standard ordered basis for $\mathbb{R}^2$. Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation such that $T(1,3) = (3,4)$ and $T(1,1) = (1,-3)$. Calculate $[T]_\beta^\beta$.

Recall from class that $[T]_\beta^\beta = \left([T\begin{pmatrix}1\\0\end{pmatrix}]^\beta \; [T\begin{pmatrix}0\\1\end{pmatrix}]^\beta \right)$, so lets calculate these coordinate vectors.

2 pts.

$$T\left(\begin{pmatrix}0\\1\end{pmatrix}\right) = T\left(\tfrac{1}{2}\left(\begin{pmatrix}1\\3\end{pmatrix} - \begin{pmatrix}1\\1\end{pmatrix}\right)\right) = \tfrac{1}{2}\left(T\begin{pmatrix}1\\3\end{pmatrix} - T\begin{pmatrix}1\\1\end{pmatrix}\right) = \tfrac{1}{2}\left(\begin{pmatrix}3\\4\end{pmatrix} - \begin{pmatrix}1\\-3\end{pmatrix}\right) =$$

$$= \tfrac{1}{2}\begin{pmatrix}2\\7\end{pmatrix} = \begin{pmatrix}1\\7/2\end{pmatrix}$$

2 pts

$$T\left(\begin{pmatrix}1\\0\end{pmatrix}\right) = T\left(\begin{pmatrix}1\\1\end{pmatrix} - \begin{pmatrix}0\\1\end{pmatrix}\right) = T\begin{pmatrix}1\\1\end{pmatrix} - T\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\-3\end{pmatrix} - \begin{pmatrix}1\\7/2\end{pmatrix} = \begin{pmatrix}0\\-13/2\end{pmatrix}$$

Now, in β, the coordinate vector of any element is simply the element, so

$$[T]_\beta^\beta = \begin{pmatrix}0 & 1\\ -13/2 & 7/2\end{pmatrix}.$$

1 pt.

* Not all of the problems have such a clear way to break down the grading

e.g. most other problems can be solved in many ways.

Problem 2 How to span a space

Let $T : V \to W$ be a linear transformation. Suppose that $v_1, \dots, v_n$ are vectors in V such that $Span(\{v_1, \dots, v_n\})$ contains $N(T)$ and that $Span(\{T(v_1), \dots, T(v_n)\}) = R(T)$. Show that $Span(\{v_1, \dots, v_n\}) = V$.

As noted during the exam, you may assume that V, W are fin.-dim.
Let $U = span(\{v_1, \dots v_n\})$. Now we have two possible maps:

$T : V \longrightarrow W$

$T\restriction_U : U \longrightarrow W$, the restriction of T to U.

Let $k = nullity(T)$.
Then since $N(T) \leq U$,
we see $k = nullity(T\restriction_U)$.

But now $R(T) = R(T\restriction_U)$, since $span(\{Tv_1, \dots Tv_n\}) = R(T)$.

Thus $rank(T) = rank(T\restriction_U)$.

$$\dim(V) = rank(T) + nullity(T) = rank(T\restriction_U) + nullity(T\restriction_U) = \dim(U)$$

by dimension Thm. by dimension Thm.

We proved in class that $U \leq V$ and $\dim(U) = \dim(V) \Rightarrow U = V$.

Problem 3 Rank and nullity

Let U, V, W be vector spaces such that $dim(U) = 4, dim(V) = 5, dim(W) = 4$ and let $T : U \to V$ and $S : V \to W$ be linear. Let $R = S \circ T$ be the composition. Prove that S is not injective. Give an example in which R is injective. Prove that T is not surjective.

$\dim(W) = \operatorname{rank}(S) + \operatorname{nullity}(S)$ and $\operatorname{rank}(S) = \dim(R(S))$, so since $R(S) \leq W$, $\dim(R(S)) \leq 4$, so $\operatorname{nullity}(S) \geq 1$, so $N(S) \neq \{\bar{0}\}$, and S is not injective.

by dimension Thm.

Let $U = \mathbb{R}^4$, $V = \mathbb{R}^5$, $W = \mathbb{R}^4$.

Let $T(x_1, x_2, x_3, x_4) = (x_1, x_2, x_3, x_4, 0)$

$S(y_1, y_2, y_3, y_4, y_5) = (y_1, y_2, y_3, y_4)$.

$\dim(V) = \operatorname{rank}(T) + \operatorname{nullity}(T)$ so, $\operatorname{rank}(T) = \dim(R(T)) \leq 4$.

by dimension Thm.

Thus $R(T) \neq W$ and so T is not surjective.

Problem 4 Functions and vector spaces

Let $\mathfrak{F}(\mathbb{R})$ denote the vector space of all functions $f : \mathbb{R} \to \mathbb{R}$. Let W be the subspace of $\mathcal{F}(V)$ consisting of all functions f such that

$$f(x) = a \cdot \sin(x + b) + c \cdot \cos(x + d).$$

You don't need to show that W is a subspace. Just assume this. Is W finite-dimensional? If so, what is the dimension of W.

Hint: Try to show that $\{\sin(x), \cos(x)\}$ is a basis of W. Note the formulas

$$\sin(x + y) = \sin(x)\cos(y) + \cos(x)\sin(y)$$

$$\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y)$$

Let $a, b, c, d \in \mathbb{R}$.
Then:

$$a \cdot \sin(x+b) + c \cdot \cos(x+d) = a \cdot \sin(x) \cdot \cos(b) + a\cos(x) \cdot \sin(b) + c \cdot \cos(x)\cos(d) - c \cdot \sin(x)\sin(d)$$

$$= \underline{(a \cdot \cos(b) - c \cdot \sin(d))}(\sin(x)) + \underline{(a\sin(b) + c \cdot \cos(d))}(\cos(x))$$

These are real numbers, so $W \subseteq \operatorname{Span}(\{\sin(x), \cos(x)\})$

* Thus $\dim(W) \leq 2$.

*Actually, it is easy to show also that $W = \operatorname{span}(\{\sin(x), \cos(x)\})$ and that $\sin(x)$ and $\cos(x)$ are Lin. Ind., so $\dim(W) = 2$.