cf. 16

Laplace Runge-Lenz vector.

apart from l and E there is another conserved vector in the system.

$$\bar{F}f(|\bar{r}|) = f(|\bar{r}|)\,\hat{r}$$

so Newton's second law is

$$\dot{\bar{p}} = f(|\bar{r}|)\frac{\bar{r}}{|\bar{r}|} \quad ; \quad \bar{L} = m\,\bar{r}\times\dot{\bar{r}}$$

$$\dot{\bar{p}}\times\bar{L} = \frac{m f(|\bar{r}|)}{|\bar{r}|}\left(\bar{r}\times(\bar{r}\times\dot{\bar{r}})\right) =$$

$$= \frac{m f(r)}{|\bar{r}|}\left[\bar{r}(\bar{r}\cdot\dot{\bar{r}}) - r^2\dot{\bar{r}}\right]$$

$$\bar{r}\cdot\dot{\bar{r}} = \frac{1}{2}\frac{d}{dt}(\bar{r}\cdot\bar{r}) = r\dot{r}$$

$$\dot{\bar{p}}\times\bar{L} = \frac{m f(r)}{r}\left[\bar{r}r\dot{r} - r^2\dot{\bar{r}}\right] = m f(r) r^2\left(\frac{\bar{r}\dot{r}}{r^2} - \frac{\dot{\bar{r}}}{r}\right)$$

l.h.s: $\dot{\bar{p}}\times\bar{L} = \frac{d}{dt}(\vec{\bar{p}}\times\bar{L})$ Q. is this ok to write?

A - yes since L is const

r.h.s: $-\left(\frac{\dot{\bar{r}}}{r} - \frac{\bar{r}\dot{r}}{r^2}\right) = \frac{d}{dt}\left(\frac{\bar{r}}{r}\right)$

combining them both we get: $\frac{d}{dt}(\bar{p}\times\bar{L}) = -m f(r) r^2\frac{d}{dt}\left(\frac{\bar{r}}{r}\right)$

Now lets plug in kepler orbit

$$f(r) = -\frac{k}{r^2}$$

So

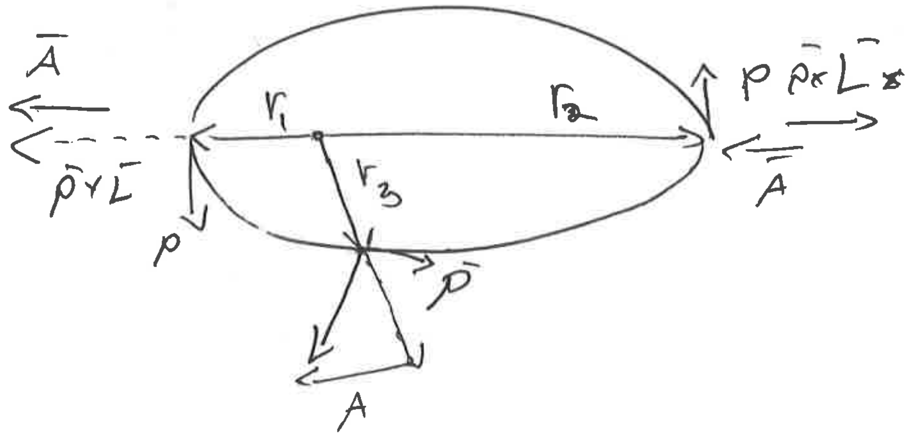
$$\frac{d}{dt}(\bar{p} \times \bar{L}) = \frac{d}{dt}\left(\frac{mk\bar{r}}{r}\right)$$

So:

$$\frac{d}{dt}\left[\bar{p} \times \bar{L} - \frac{mk\bar{r}}{r}\right] = 0$$

so the vector

$$\bar{A} = \bar{p} \times \bar{L} - \frac{\mu k \bar{r}}{r} = \text{cnst}$$



$\bar{A}$ always points toward the pericenter.

$$\left(\bar{A} = \mu \dot{\bar{r}} \times (\mu \bar{r} \times \dot{\bar{r}}) - \frac{\mu k \bar{r}}{r}\right)$$ skip!

$\bar{A} \cdot \bar{L} = 0$ ← perpendicular to $\bar{L} = \bar{r} \times \bar{p}$

$\bar{r} \cdot \bar{L} = 0 \qquad \bar{p} \cdot \bar{L} = 0$

Ex.19 what is the magnitud of A?

$$|\bar{A}| = |\bar{p} \times \bar{L} - \mu k \hat{r}|$$

$$\bar{A}^2 = (\bar{p} \times \bar{L})(\bar{p} \times \bar{L}) - 2\mu k \hat{r} \cdot (\bar{p} \times \bar{L}) + (\mu k)^2$$

using
$$\bar{a} \cdot (b \times c) = b(c \times a) = c(a \times b)$$
$$a \times (b \times c) = b(ac) - c(ab)$$

$$(\bar{p} \times \bar{L})(\bar{p} \times \bar{L}) = \bar{p}(\bar{L} \times (\bar{p} \times \bar{L})) = \bar{p}\left[\bar{p} L^2 - \bar{L}(\underbrace{\bar{L} \cdot \bar{p}}_{0})\right]$$
$$= \bar{p}^2 \bar{L}^2$$

also

$$\hat{r}(\bar{p} \times \bar{L}) = \bar{L}(\hat{r} \times \bar{p}) = \frac{L^2}{r}$$

So finally

$$A^2 = p^2 L^2 - 2\mu k \frac{L^2}{r} + (\mu k)^2 = 2\mu L^2 \underbrace{\left(\frac{p^2}{2\mu} - \frac{k}{r}\right)}_{} + (\mu k)^2$$

$$E = \frac{1}{2}\mu \dot{r}^2 - \frac{k}{r} =$$
$$= \frac{1}{2}\frac{\bar{p}^2}{\mu} - \frac{k}{r} = -\frac{k}{2a}$$

$$A^2 = 2\mu L^2 E + (\mu k)^2 = 2\mu a k(1-e^2)\left(-\frac{k}{2a}\right) + (\mu k)^2$$

$$= (\mu k)^2\left[-1 + e^2 + 1\right] = (\mu k)^2 e^2$$

So $$e^2 = \frac{A^2}{(\mu k)^2}$$

$$e = \frac{A}{\mu k}$$

So not only A is almost the eccentricity it also points toward the pericenter all the time. so we define

$$\bar{e} = \frac{1}{\mu k}(\bar{p} \times \bar{L} - \mu k \hat{r}) =$$

$$= \frac{\bar{p} \times \bar{L}}{\mu k} - \hat{r}$$