Math 115A: Sample midterm 2

Sections 1 and 3. Instructor: James Freitag

You need to know how to calculate the matrix representing a given transformation with respect to given ordered bases. Make sure you know this.

Problem 1 Coordinates of a transformation

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ and $S : \mathbb{R}^3 \to \mathbb{R}^3$ be linear transformations. Let α, β be ordered bases of $\mathbb{R}^3$. Suppose that $[T]_\alpha^\beta [S]_\beta^\alpha = Id_{3\times 3}$. Prove that $S = T^{-1}$. (Recall that inverse in general means *two sided* inverse. That is $S \circ T = T \circ S = id$, so you need to prove both of these identities.)

Problem 2 Determining the determinant

Suppose that A, B, C are square matrices. Prove that

$$det \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} = det(A) \cdot det(C).$$

Here as usual, 0 denotes the appropriate (square) matrix of all zeros.

Problem 3 Eigenvalues

Let $\theta \in (0, \pi/2)$. Let

$$A = \begin{pmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{pmatrix}.$$

The entries of A are in $\mathbb{R}$, so we can regard A as either a matrix over the reals *or* the complex numbers. Are there any eigenvectors over $\mathbb{R}$? Explain why not intuitively.

Calculate the eigenvalues and eigenvectors over $\mathbb{C}$.

Problem 4 You should consider diagonalizing

Let

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.$$

Give an expression (it can involve *numbers* raised to exponents and it can involve finitely many matrix multiplications) for A^n for $n \in \mathbb{N}$.

$\boxed{1}$ $[T\circ S]^{\beta}_{\beta} = Id_{3\times3}$ because $[T\circ S]^{\beta}_{\beta} = [T]^{\beta}_{\alpha}[S]^{\alpha}_{\beta}$. $[T\circ S]^{\beta}_{\beta} = Id_{3\times3}$, so $T\circ S = Id$ (note for L a lin. trans, if $L|_{\beta} = id$ for some basis, β, then $L = id$).

Now we need to show that $S\circ T = id$.

$T\circ S$ is a bijection, so T must be a surjection from $\mathbb{R}^3 \to \mathbb{R}^3$. By rank nullity Thm, T is bijective. Since $T\circ S = Id$, $T\circ S\circ T = Id\circ T = T$. Also by rank-nullity, $S\circ T$ is a bijection. For a composition of bijections, $f\circ g = f\circ h \Leftrightarrow g = h$. So, we se $S\circ T = id$.

$\boxed{2}$ $\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$. Assume $A_{1,1} \neq 0$. Doing row reduction, we can convert A to a matrix A', each of whose entries $A_{m,1} = 0$ for $m > 1$.

Further $\det A = \det A'$ and $\det\begin{pmatrix} A & B \\ 0 & C \end{pmatrix} = \det\begin{pmatrix} A' & B \\ 0 & C \end{pmatrix}$.

Now $\det\begin{pmatrix} A' & B \\ 0 & C \end{pmatrix} = A_{11} \cdot \det\begin{pmatrix} \tilde{A}'_{11} & B \\ 0 & C \end{pmatrix}$ and by induction

$$= A_{11} \cdot \det(\tilde{A}'_{11}) \cdot \det(C)$$

$$= \det(A') \cdot \det(C)$$

$$= \det(A) \cdot \det(C).$$

$\boxed{3}$ Eigenvalues: $\cos(\theta) + i\sin(\theta)$ $\quad \cos(\theta) - i\sin(\theta)$.

eigenvectors: $\begin{pmatrix} 1 \\ -i \end{pmatrix}$ $\quad \begin{pmatrix} 1 \\ i \end{pmatrix}$

$\boxed{4}$ http://www.wolframalpha.com/input/?i=diagonalize+%7B%7B1%2C2%7D%2C%7B3%2C4%7D%7D