5

LINEAR SYSTEMS

5.0 Introduction

As we've seen, in one-dimensional phase spaces the flow is extremely confined—all trajectories are forced to move monotonically or remain constant. In higher-dimensional phase spaces, trajectories have much more room to maneuver, and so a wider range of dynamical behavior becomes possible. Rather than attack all this complexity at once, we begin with the simplest class of higher-dimensional systems, namely *linear systems in two dimensions*. These systems are interesting in their own right, and, as we'll see later, they also play an important role in the classification of fixed points of *nonlinear* systems. We begin with some definitions and examples.

5.1 Definitions and Examples

A ***two-dimensional linear system*** is a system of the form

$$\dot{x} = ax + by$$
$$\dot{y} = cx + dy$$

where a, b, c, d are parameters. If we use boldface to denote vectors, this system can be written more compactly in matrix form as

$$\dot{\mathbf{x}} = A\mathbf{x},$$

where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

Such a system is ***linear*** in the sense that if $\mathbf{x}_1$ and $\mathbf{x}_2$ are solutions, then so is any linear combination $c_1\mathbf{x}_1 + c_2\mathbf{x}_2$. Notice that $\dot{\mathbf{x}} = \mathbf{0}$ when $\mathbf{x} = \mathbf{0}$, so $\mathbf{x}^* = \mathbf{0}$ is always a fixed point for any choice of A.

The solutions of $\dot{\mathbf{x}} = A\mathbf{x}$ can be visualized as trajectories moving on the (x, y) plane, in this context called the ***phase plane***. Our first example presents the phase plane analysis of a familiar system.

EXAMPLE 5.1.1:

As discussed in elementary physics courses, the vibrations of a mass hanging from a linear spring are governed by the linear differential equation

$$m\ddot{x} + kx = 0 \qquad (1)$$

where m is the mass, k is the spring constant, and x is the displacement of the mass from equilibrium (Figure 5.1.1). Give a phase plane analysis of this ***simple harmonic oscillator***.

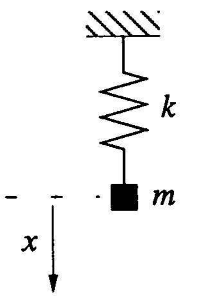


Figure 5.1.1

Solution: As you probably recall, it's easy to solve (1) analytically in terms of sines and cosines. But that's precisely what makes linear equations so special! For the *nonlinear* equations of ultimate interest to us, it's usually impossible to find an analytical solution. We want to develop methods for deducing the behavior of equations like (1) *without actually solving them.*

The motion in the phase plane is determined by a vector field that comes from the differential equation (1). To find this vector field, we note that the ***state*** of the system is characterized by its current position x and velocity v; if we know the values of *both* x and v, then (1) uniquely determines the future states of the system. Therefore we rewrite (1) in terms of x and v, as follows:

$$\dot{x} = v \qquad (2a)$$
$$\dot{v} = -\tfrac{k}{m}x. \qquad (2b)$$

Equation (2a) is just the definition of velocity, and (2b) is the differential equation (1) rewritten in terms of v. To simplify the notation, let $\omega^2 = k/m$. Then (2) becomes

$$\dot{x} = v \qquad (3a)$$
$$\dot{v} = -\omega^2 x. \qquad (3b)$$

The system (3) assigns a vector $(\dot{x}, \dot{v}) = (v, -\omega^2 x)$ at each point (x, v), and therefore represents a ***vector field*** on the phase plane.

For example, let's see what the vector field looks like when we're on the x-axis. Then $v = 0$ and so $(\dot{x}, \dot{v}) = (0, -\omega^2 x)$. Hence the vectors point vertically downward for positive x and vertically upward for negative x (Figure 5.1.2). As x gets larger in magnitude, the vectors $(0, -\omega^2 x)$ get longer. Similarly, on the v-axis, the vector field is $(\dot{x}, \dot{v}) = (v, 0)$, which points to the right when $v > 0$ and to the left when $v < 0$. As we move around in phase space, the vectors change direction as shown in Figure 5.1.2.

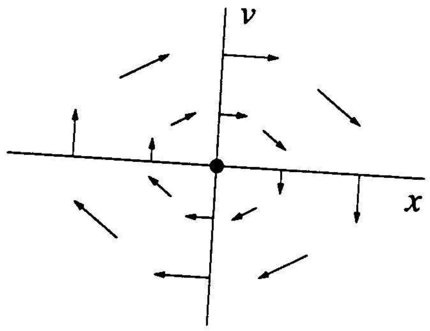


Figure 5.1.2

Just as in Chapter 2, it is helpful to visualize the vector field in terms of the motion of an imaginary fluid. In the present case, we imagine that a fluid is flowing steadily on the phase plane with a local velocity given by $(\dot{x}, \dot{v}) = (v, -\omega^2 x)$. Then, to find the trajectory starting at (x_0, v_0), we place an imaginary particle or ***phase point*** at (x_0, v_0) and watch how it is carried around by the flow.

The flow in Figure 5.1.2 swirls about the origin. The origin is special, like the eye of a hurricane: a phase point placed there would remain motionless, because $(\dot{x}, \dot{v}) = (0, 0)$ when $(x, v) = (0, 0)$; hence the origin is a ***fixed point***. But a phase point starting anywhere else would circulate around the origin and eventually return to its starting point. Such trajectories form ***closed orbits***, as shown in Figure 5.1.3. Figure 5.1.3 is called the ***phase portrait*** of the system—it shows the overall picture of trajectories in phase space.

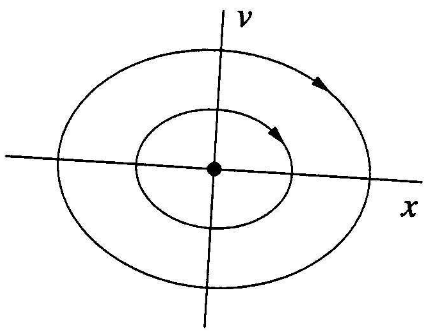


Figure 5.1.3

What do fixed points and closed orbits have to do with the original problem of a mass on a spring? The answers are beautifully simple. The fixed point $(x, v) = (0, 0)$ corresponds to static equilibrium of the system: the mass is at rest at its equilibrium position and will remain there forever, since the spring is relaxed. The closed orbits have a more interesting interpretation: they correspond to periodic motions, i.e., oscillations of the mass. To see this, just look at some points on a closed orbit (Figure 5.1.4). When the displacement x is most negative, the velocity v is zero; this corresponds to one extreme of the oscillation, where the spring is most compressed (Figure 5.1.4).

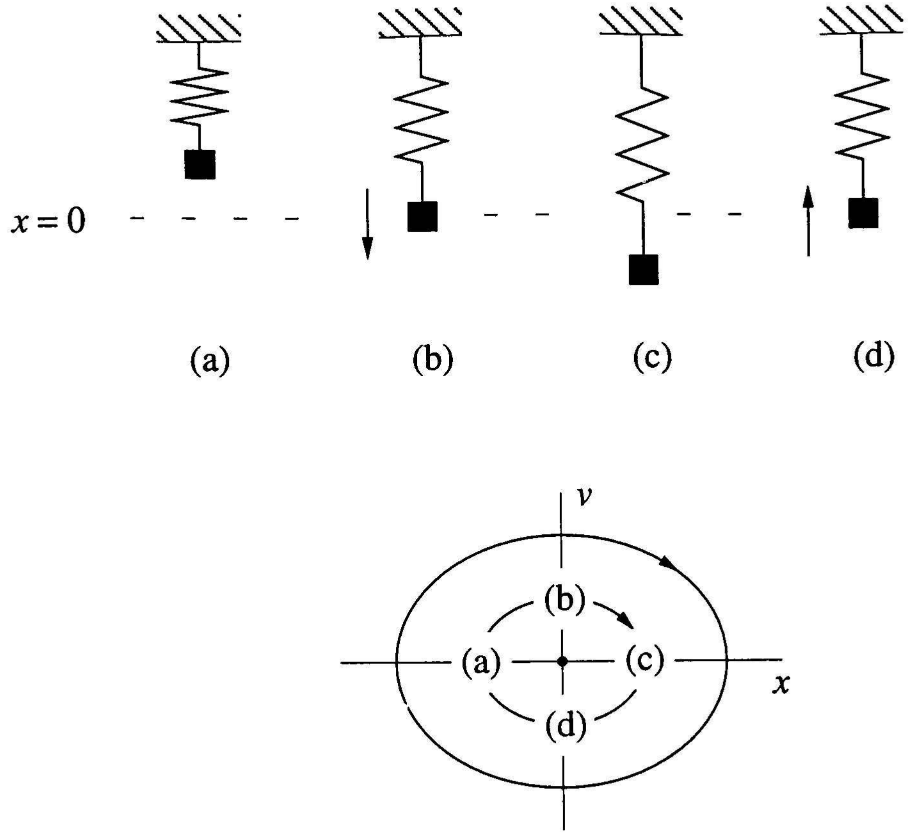


Figure 5.1.4

In the next instant as the phase point flows along the orbit, it is carried to points where x has increased and v is now positive; the mass is being pushed back toward its equilibrium position. But by the time the mass has reached $x=0$, it has a large positive velocity (Figure 5.1.4b) and so it overshoots $x=0$. The mass eventually comes to rest at the other end of its swing, where x is most positive and v is zero again (Figure 5.1.4c). Then the mass gets pulled up again and eventually completes the cycle (Figure 5.1.4d).

The shape of the closed orbits also has an interesting physical interpretation. The orbits in Figures 5.1.3 and 5.1.4 are actually *ellipses* given by the equation $\omega^2 x^2 + v^2 = C$, where $C \geq 0$ is a constant. In Exercise 5.1.1, you are asked to derive this geometric result, and to show that it is equivalent to conservation of energy. ■

EXAMPLE 5.1.2:

Solve the linear system $\dot{\mathbf{x}} = A\mathbf{x}$, where $A = \begin{pmatrix} a & 0 \\ 0 & -1 \end{pmatrix}$. Graph the phase portrait as a varies from $-\infty$ to $+\infty$, showing the qualitatively different cases.

Solution: The system is

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

Matrix multiplication yields

$$\dot{x} = ax$$
$$\dot{y} = -y$$

which shows that the two equations are ***uncoupled***; there's no x in the y-equation and vice versa. In this simple case, each equation may be solved separately. The solution is

$$x(t) = x_0 e^{at} \qquad (1a)$$
$$y(t) = y_0 e^{-t}. \qquad (1b)$$

The phase portraits for different values of a are shown in Figure 5.1.5. In each case, $y(t)$ decays exponentially. When $a<0$, $x(t)$ also decays exponentially and so all trajectories approach the origin as $t \to \infty$. However, the direction of approach depends on the size of a compared to -1.

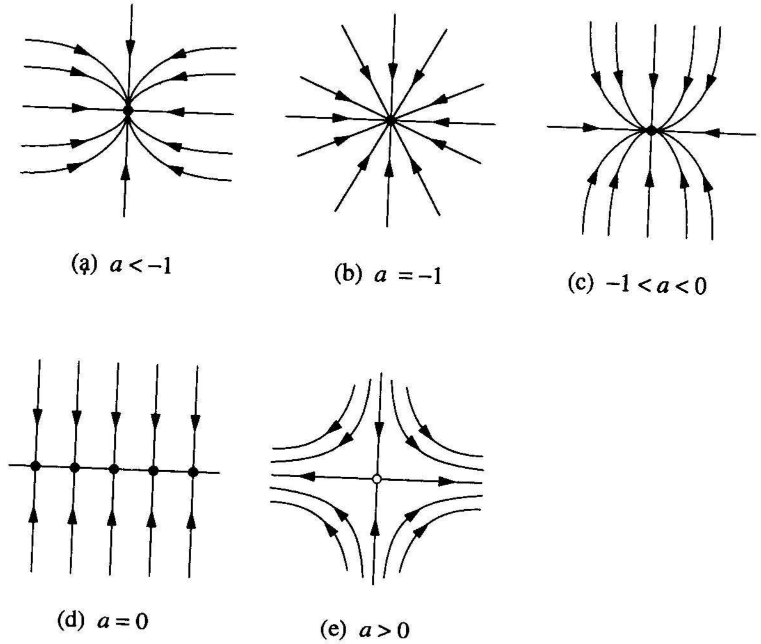


Figure 5.1.5

than $y(t)$. The trajectories approach the origin tangent to the *slower* direction (here, the y-direction). The intuitive explanation is that when a is very negative, the trajectory slams horizontally onto the y-axis, because the decay of $x(t)$ is almost instantaneous. Then the trajectory dawdles along the y-axis toward the origin, and so the approach is tangent to the y-axis. On the other hand, if we look *backwards* along a trajectory ($t \to -\infty$), then the trajectories all become parallel to the faster decaying direction (here, the x-direction). These conclusions are easily proved by looking at the slope $dy/dx = \dot{y}/\dot{x}$ along the trajectories; see Exercise 5.1.2. In Figure 5.1.5a, the fixed point $\mathbf{x}^* = \mathbf{0}$ is called a ***stable node***.

Figure 5.1.5b shows the case $a = -1$. Equation (1) shows that $y(t)/x(t) = y_0/x_0 =$ constant, and so all trajectories are straight lines through the origin. This is a very special case—it occurs because the decay rates in the two directions are precisely equal. In this case, $\mathbf{x}^*$ is called a symmetrical node or ***star***.

When $-1 < a < 0$, we again have a node, but now the trajectories approach $\mathbf{x}^*$ along the x-direction, which is the more slowly decaying direction for this range of a (Figure 5.1.5c).

Something dramatic happens when $a = 0$ (Figure 5.1.5d). Now (1a) becomes $x(t) \equiv x_0$ and so there's an entire ***line of fixed points*** along the x-axis. All trajectories approach these fixed points along vertical lines.

Finally when $a > 0$ (Figure 5.1.5e), $\mathbf{x}^*$ becomes unstable, due to the exponential growth in the x-direction. Most trajectories veer away from $\mathbf{x}^*$ and head out to infinity. An exception occurs if the trajectory starts on the y-axis; then it walks a tightrope to the origin. In forward time, the trajectories are asymptotic to the x-axis; in backward time, to the y-axis. Here $\mathbf{x}^* = \mathbf{0}$ is called a ***saddle point***. The y-axis is called the ***stable manifold*** of the saddle point $\mathbf{x}^*$, defined as the set of initial conditions $\mathbf{x}_0$ such that $\mathbf{x}(t) \to \mathbf{x}^*$ as $t \to \infty$. Likewise, the ***unstable manifold*** of $\mathbf{x}^*$ is the set of initial conditions such that $\mathbf{x}(t) \to \mathbf{x}^*$ as $t \to -\infty$. Here the unstable manifold is the x-axis. Note that a typical trajectory asymptotically approaches the unstable manifold as $t \to \infty$, and approaches the stable manifold as $t \to -\infty$. This sounds backwards, but it's right! ■

Stability Language

It's useful to introduce some language that allows us to discuss the stability of different types of fixed points. This language will be especially useful when we analyze fixed points of *nonlinear* systems. For now we'll be informal; precise definitions of the different types of stability will be given in Exercise 5.1.10.

We say that $\mathbf{x}^* = \mathbf{0}$ is an ***attracting*** fixed point in Figures 5.1.5a–c; all trajectories that start near $\mathbf{x}^*$ approach it as $t \to \infty$. That is, $\mathbf{x}(t) \to \mathbf{x}^*$ as $t \to \infty$. In fact $\mathbf{x}^*$ attracts *all* trajectories in the phase plane, so it could be called ***globally attracting***.

There's a completely different notion of stability which relates to the behavior

punov stable if all trajectories that start sufficiently close to $\mathbf{x}^*$ remain close to it for all time. In Figures 5.1.5a–d, the origin is Liapunov stable.

Figure 5.1.5d shows that a fixed point can be Liapunov stable but not attracting. This situation comes up often enough that there is a special name for it. When a fixed point is Liapunov stable but not attracting, it is called ***neutrally stable***. Nearby trajectories are neither attracted to nor repelled from a neutrally stable point. As a second example, the equilibrium point of the simple harmonic oscillator (Figure 5.1.3) is neutrally stable. Neutral stability is commonly encountered in mechanical systems in the absence of friction. Conversely, it's possible for a fixed point to be attracting but not Liapunov stable; thus, neither notion of stability implies the other. An example is given by the following vector field on the circle: $\dot{\theta} = 1 - \cos\theta$ (Figure 5.1.6). Here $\theta^* = 0$ attracts all trajectories as $t \to \infty$, but it is not Liapunov stable; there are trajectories that start infinitesimally close to θ^* but go on a very large excursion before returning to θ^*.

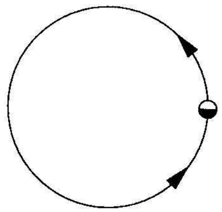


Figure 5.1.6

However, in practice the two types of stability often occur together. If a fixed point is *both* Liapunov stable and attracting, we'll call it ***stable***, or sometimes ***asymptotically stable***.

Finally, $\mathbf{x}^*$ is ***unstable*** in Figure 5.1.5e, because it is neither attracting nor Liapunov stable.

A graphical convention: we'll use open dots to denote unstable fixed points, and solid black dots to denote Liapunov stable fixed points. This convention is consistent with that used in previous chapters.

5.2 Classification of Linear Systems

The examples in the last section had the special feature that two of the entries in the matrix A were zero. Now we want to study the general case of an arbitrary 2×2 matrix, with the aim of classifying all the possible phase portraits that can occur.

Example 5.1.2 provides a clue about how to proceed. Recall that the x and y axes played a crucial geometric role. They determined the direction of the trajectories as $t \to \pm\infty$. They also contained special ***straight-line trajectories***: a trajectory starting on one of the coordinate axes stayed on that axis forever, and exhibited simple exponential growth or decay along it.

For the general case, we would like to find the analog of these straight-line trajectories. That is, we seek trajectories of the form

$$\mathbf{x}(t) = e^{\lambda t}\mathbf{v}, \tag{2}$$

where $\mathbf{v} \neq \mathbf{0}$ is some fixed vector to be determined, and [illegible] rate, also to be determined. If such solutions exist, they correspond to exponential motion along the line spanned by the vector $\mathbf{v}$.

To find the conditions on $\mathbf{v}$ and λ, we substitute $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$ into $\dot{\mathbf{x}} = A\mathbf{x}$, and obtain $\lambda e^{\lambda t}\mathbf{v} = e^{\lambda t}A\mathbf{v}$. Canceling the nonzero scalar factor $e^{\lambda t}$ yields

$$A\mathbf{v} = \lambda\mathbf{v}, \tag{3}$$

which says that the desired straight line solutions exist if $\mathbf{v}$ is an ***eigenvector*** of A with corresponding ***eigenvalue*** λ. In this case we call the solution (2) an ***eigensolution***.

Let's recall how to find eigenvalues and eigenvectors. (If your memory needs more refreshing, see any text on linear algebra.) In general, the eigenvalues of a matrix A are given by the ***characteristic equation*** $\det(A - \lambda I) = 0$, where I is the identity matrix. For a 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

the characteristic equation becomes

$$\det\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} = 0.$$

Expanding the determinant yields

$$\lambda^2 - \tau\lambda + \Delta = 0 \tag{4}$$

where

$$\tau = \text{trace}(A) = a + d,$$
$$\Delta = \det(A) = ad - bc.$$

Then

$$\lambda_1 = \frac{\tau + \sqrt{\tau^2 - 4\Delta}}{2}, \qquad \lambda_2 = \frac{\tau - \sqrt{\tau^2 - 4\Delta}}{2} \tag{5}$$

are the solutions of the quadratic equation (4). In other words, the eigenvalues depend only on the trace and determinant of the matrix A.

The typical situation is for the eigenvalues to be distinct: $\lambda_1 \neq \lambda_2$. In this case, a theorem of linear algebra states that the corresponding eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent, and hence span the entire plane (Figure 5.2.1). In particular, any initial condition $\mathbf{x}_0$ can be written as a linear combination of eigenvectors, say $\mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$.

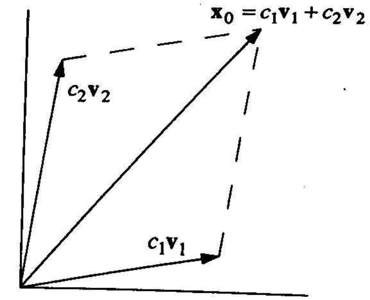


Figure 5.2.1

This observation allows us to write down the general solution for $\mathbf{x}(t)$—it is simply

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2. \tag{6}$$

Why is this the general solution? First of all, it is a linear combination of solutions to $\dot{\mathbf{x}} = A\mathbf{x}$, and hence is itself a solution. Second, it satisfies the initial condition $\mathbf{x}(0) = \mathbf{x}_0$, and so by the existence and uniqueness theorem, it is the *only* solution. (See Section 6.2 for a general statement of the existence and uniqueness theorem.)

EXAMPLE 5.2.1:

Solve the initial value problem $\dot{x} = x + y$, $\dot{y} = 4x - 2y$, subject to the initial condition $(x_0, y_0) = (2, -3)$.

Solution: The corresponding matrix equation is

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

First we find the eigenvalues of the matrix A. The matrix has $\tau = -1$ and $\Delta = -6$, so the characteristic equation is $\lambda^2 + \lambda - 6 = 0$. Hence

$$\lambda_1 = 2, \quad \lambda_2 = -3.$$

Next we find the eigenvectors. Given an eigenvalue λ, the corresponding eigenvector $\mathbf{v} = (v_1, v_2)$ satisfies

$$\begin{pmatrix} 1-\lambda & 1 \\ 4 & -2-\lambda \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For $\lambda_1 = 2$, this yields $\begin{pmatrix} -1 & 1 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, which has a nontrivial solution

$(v_1, v_2) = (1,1)$, or any scalar multiple thereof. (Of course, any multiple of an eigenvector is always an eigenvector; we try to pick the simplest multiple, but any one will do.) Similarly, for $\lambda_2 = -3$, the eigenvector equation becomes $\begin{pmatrix} 4 & 1 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, which has a nontrivial solution $(v_1, v_2) = (1,-4)$. In summary,

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \mathbf{v}_2 = \begin{pmatrix} 1 \\ -4 \end{pmatrix}.$$

Next we write the general solution as a linear combination of eigensolutions. From (6), the general solution is

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-3t}. \qquad (7)$$

Finally, we compute c_1 and c_2 to satisfy the initial condition $(x_0, y_0) = (2,-3)$. At $t = 0$, (7) becomes

$$\begin{pmatrix} 2 \\ -3 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -4 \end{pmatrix},$$

which is equivalent to the algebraic system

$$2 = c_1 + c_2,$$
$$-3 = c_1 - 4c_2.$$

The solution is $c_1 = 1$, $c_2 = 1$. Substituting back into (7) yields

$$x(t) = e^{2t} + e^{-3t},$$
$$y(t) = e^{2t} - 4e^{-3t}$$

for the solution to the initial value problem. ■

Whew! Fortunately we don't need to go through all this to draw the phase portrait of a linear system. All we need to know are the eigenvectors and eigenvalues.

EXAMPLE 5.2.2:

Draw the phase portrait for the system of Example 5.2.1.

Solution: The system has eigenvalues $\lambda_1 = 2$, $\lambda_2 = -3$. Hence the first eigensolution grows exponentially, and the second eigensolution decays. This means the origin is a *saddle point*. Its stable manifold is the line spanned by the eigenvector $\mathbf{v}_2 = (1,-4)$, corresponding to the decaying eigensolution. Similarly, the unstable manifold is the line spanned by $\mathbf{v}_1 = (1,1)$. As with all saddle points, a typical trajectory approaches the unstable manifold as $t \to \infty$, and the stable manifold as $t \to -\infty$. Figure 5.2.2 shows the phase portrait. ■

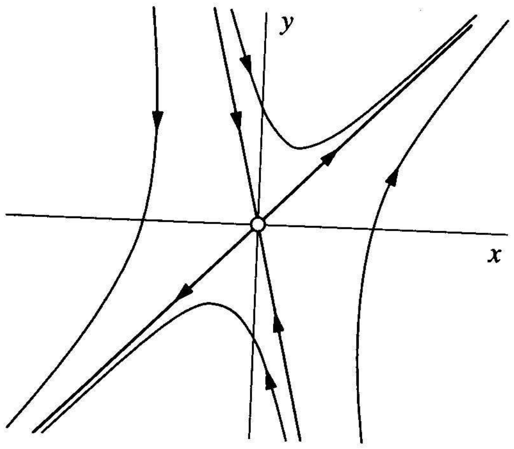


Figure 5.2.2

EXAMPLE 5.2.3:

Sketch a typical phase portrait for the case $\lambda_2 < \lambda_1 < 0$.

Solution: First suppose $\lambda_2 < \lambda_1 < 0$. Then both eigensolutions decay exponentially. The fixed point is a stable node, as in Figures 5.1.5a and 5.1.5c, except now the eigenvectors are not mutually perpendicular, in general. Trajectories typically approach the origin tangent to the ***slow eigendirection***, defined as the direction spanned by the eigenvector with the smaller $|\lambda|$. In backwards time ($t \to -\infty$), the trajectories become parallel to the fast eigendirection. Figure 5.2.3 shows the phase portrait. (If we reverse all the arrows in Figure 5.2.3, we obtain a typical phase portrait for an ***unstable node***.) ■

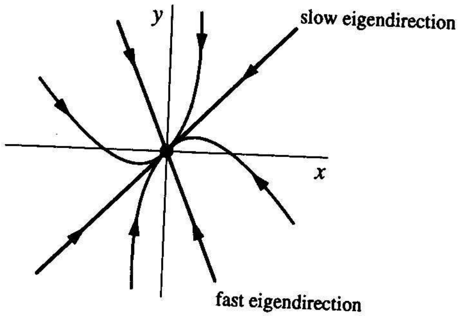
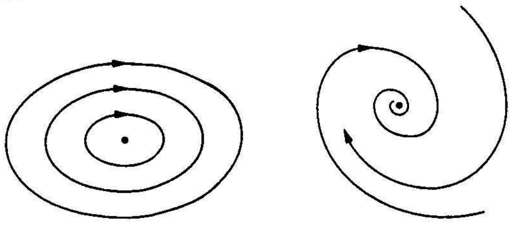


Figure 5.2.3

EXAMPLE 5.2.4:

What happens if the eigenvalues are *complex* numbers?

Solution: If the eigenvalues are complex, the fixed point is either a ***center*** (Figure 5.2.4a) or a ***spiral*** (Figure 5.2.4b). We've already seen an example of a center in the simple harmonic oscillator of Section 5.1; the origin is surrounded by a family of closed orbits. Note that centers are *neutrally stable,* since nearby trajectories are neither attracted to nor repelled from the fixed point. A spiral would occur if the harmonic oscillator were lightly damped. Then the trajectory would just fail to close, because the oscillator loses a bit of energy on each cycle.



(a) center (b) spiral

Figure 5.2.4

To justify these statements, recall that the eigenvalues are $\lambda_{1,2} = \frac{1}{2}\left(\tau \pm \sqrt{\tau^2 - 4\Delta}\right)$. Thus complex eigenvalues occur when

$$\tau^2 - 4\Delta < 0.$$

To simplify the notation, let's write the eigenvalues as

$$\lambda_{1,2} = \alpha \pm i\omega$$

where

$$\alpha = \tau/2, \qquad \omega = \tfrac{1}{2}\sqrt{4\Delta - \tau^2}.$$

By assumption, $\omega \neq 0$. Then the eigenvalues are distinct and so the general solution is still given by

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2.$$

But now the c's and $\mathbf{v}$'s are *complex*, since the λ's are. This means that $\mathbf{x}(t)$ involves linear combinations of $e^{(\alpha \pm i\omega)t}$. By Euler's formula, $e^{i\omega t} = \cos\omega t + i\sin\omega t$. Hence $\mathbf{x}(t)$ is a combination of terms involving $e^{\alpha t}\cos\omega t$ and $e^{\alpha t}\sin\omega t$. Such terms represent exponentially *decaying oscillations* if $\alpha = \mathrm{Re}(\lambda) < 0$ and *growing oscillations* if $\alpha > 0$. The corresponding fixed points are ***stable*** and ***unstable spirals***, respectively. Figure 5.2.4b shows the stable case.

If the eigenvalues are pure imaginary ($\alpha = 0$), then all the solutions are periodic with period $T = 2\pi/\omega$. The oscillations have fixed amplitude and the fixed point is a center.

For both centers and spirals, it's easy to determine whether the rotation is clockwise or counterclockwise; just compute a few vectors in the vector field and the sense of rotation should be obvious. ■

EXAMPLE 5.2.5:

In our analysis of the general case, we have been assuming that the eigenvalues are distinct. What happens if the eigenvalues are *equal*?

Solution: Suppose $\lambda_1 = \lambda_2 = \lambda$. There are two possibilities: either there are two independent eigenvectors corresponding to λ, or there's only one.

If there are two independent eigenvectors, then they span the plane and so *every vector is an eigenvector with this same eigenvalue* λ. To see this, write an arbitrary vector $\mathbf{x}_0$ as a linear combination of the two eigenvectors: $\mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$. Then

$$A\mathbf{x}_0 = A(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) = c_1\lambda\mathbf{v}_1 + c_2\lambda\mathbf{v}_2 = \lambda\mathbf{x}_0$$

so $\mathbf{x}_0$ is also an eigenvector with eigenvalue λ. Since multiplication by A simply stretches every vector by a factor λ, the matrix must be a multiple of the identity:

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}.$$

Then if $\lambda \neq 0$, all trajectories are straight lines through the origin ($\mathbf{x}(t) = e^{\lambda t}\mathbf{x}_0$) and the fixed point is a ***star node*** (Figure 5.2.5).

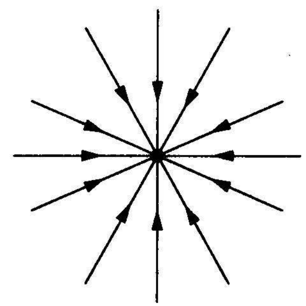


Figure 5.2.5

On the other hand, if $\lambda = 0$, the whole plane is filled with fixed points! (No surprise—the system is $\dot{\mathbf{x}} = \mathbf{0}$.)

The other possibility is that there's only one eigenvector (more accurately, the eigenspace corresponding to λ is one-dimensional.) For example, any matrix of the form $A = \begin{pmatrix} \lambda & b \\ 0 & \lambda \end{pmatrix}$, with $b \neq 0$ has only a one-dimensional eigenspace (Exercise 5.2.11).

When there's only one eigendirection, the fixed point is a ***degenerate node***. A

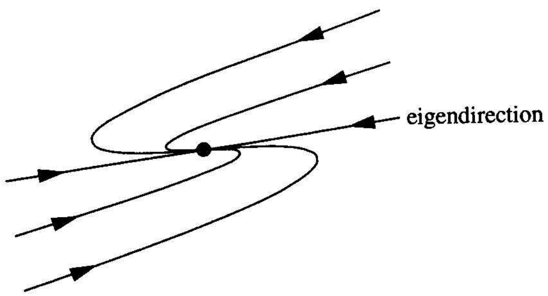
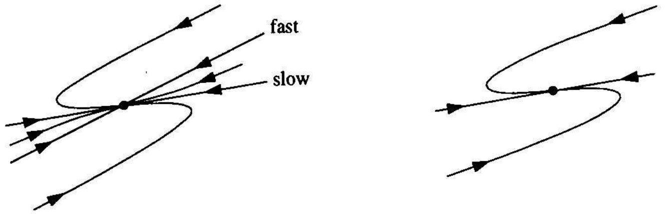


Figure 5.2.6

typical phase portrait is shown in Figure 5.2.6. As $t \to +\infty$ and also as $t \to -\infty$, all trajectories become parallel to the one available eigendirection.

A good way to think about the degenerate node is to imagine that it has been created by deforming an ordinary node. The ordinary node has two independent eigendirections; all trajectories are parallel to the slow eigendirection as $t \to \infty$, and to the fast eigendirection as $t \to -\infty$ (Figure 5.2.7a).



(a) node

(b) degenerate node

Figure 5.2.7

Now suppose we start changing the parameters of the system in such a way that the two eigendirections are scissored together. Then some of the trajectories will get squashed in the collapsing region between the two eigendirections, while the surviving trajectories get pulled around to form the degenerate node (Figure 5.2.7b).

Another way to get intuition about this case is to realize that the degenerate node is on the *borderline between a spiral and a node*. The trajectories are trying to wind around in a spiral, but they don't quite make it. ■

Classification of Fixed Points

By now you're probably tired of all the examples and ready for a simple classification scheme. Happily, there is one. We can show the type and stability of all the different fixed points on a single diagram (Figure 5.2.8).

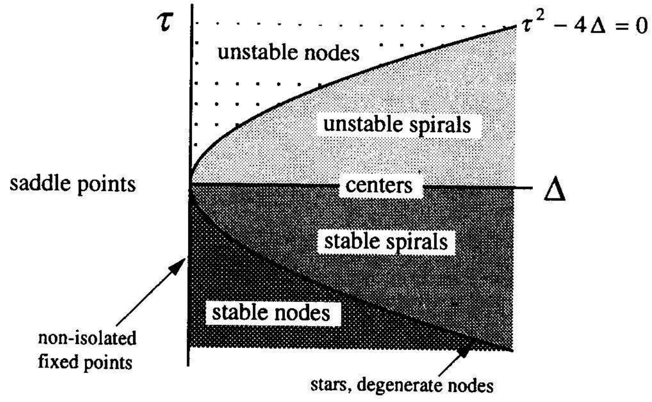


Figure 5.2.8

The axes are the trace τ and the determinant Δ of the matrix A. All of the information in the diagram is implied by the following formulas:

$$\lambda_{1,2} = \tfrac{1}{2}\left(\tau \pm \sqrt{\tau^2 - 4\Delta}\right), \qquad \Delta = \lambda_1\lambda_2, \qquad \tau = \lambda_1 + \lambda_2.$$

The first equation is just (5). The second and third can be obtained by writing the characteristic equation in the form $(\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 - \tau\lambda + \Delta = 0$.

To arrive at Figure 5.2.8, we make the following observations:

If $\Delta < 0$, the eigenvalues are real and have opposite signs; hence the fixed point is a *saddle point*.

If $\Delta > 0$, the eigenvalues are either real with the same sign (*nodes*), or complex conjugate (*spirals* and *centers*). Nodes satisfy $\tau^2 - 4\Delta > 0$ and spirals satisfy $\tau^2 - 4\Delta < 0$. The parabola $\tau^2 - 4\Delta = 0$ is the borderline between nodes and spirals; star nodes and degenerate nodes live on this parabola. The stability of the nodes and spirals is determined by τ. When $\tau < 0$, both eigenvalues have negative real parts, so the fixed point is stable. Unstable spirals and nodes have $\tau > 0$. Neutrally stable centers live on the borderline $\tau = 0$, where the eigenvalues are purely imaginary.

If $\Delta = 0$, at least one of the eigenvalues is zero. Then the origin is not an isolated fixed point. There is either a whole line of fixed points, as in Figure 5.1.5d, or a plane of fixed points, if $A = 0$.

Figure 5.2.8 shows that saddle points, nodes, and spirals are the major types of fixed points; they occur in large open regions of the (Δ, τ) plane. Centers, stars, degenerate nodes, and non-isolated fixed points are ***borderline cases*** that occur along curves in the (Δ, τ) plane. Of these borderline cases, centers are by far the most important. They occur very commonly in frictionless mechanical systems where energy is conserved.

EXAMPLE 5.2.6:

Classify the fixed point $\mathbf{x}^* = \mathbf{0}$ for the system $\dot{\mathbf{x}} = A\mathbf{x}$, where $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.

Solution: The matrix has $\Delta = -2$; hence the fixed point is a saddle point. ■

EXAMPLE 5.2.7:

Redo Example 5.2.6 for $A = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}$.

Solution: Now $\Delta = 5$ and $\tau = 6$. Since $\Delta > 0$ and $\tau^2 - 4\Delta = 16 > 0$, the fixed point is a node. It is unstable, since $\tau > 0$. ■

5.3 Love Affairs

To arouse your interest in the classification of linear systems, we now discuss a simple model for the dynamics of love affairs (Strogatz 1988). The following story illustrates the idea.

Romeo is in love with Juliet, but in our version of this story, Juliet is a fickle lover. The more Romeo loves her, the more Juliet wants to run away and hide. But when Romeo gets discouraged and backs off, Juliet begins to find him strangely attractive. Romeo, on the other hand, tends to echo her: he warms up when she loves him, and grows cold when she hates him.

Let

$R(t)$ = Romeo's love/hate for Juliet at time t

$J(t)$ = Juliet's love/hate for Romeo at time t.

Positive values of R, J signify love, negative values signify hate. Then a model for their star-crossed romance is

$$\dot{R} = aJ$$
$$\dot{J} = -bR$$

where the parameters a and b are positive, to be consistent with the story.

The sad outcome of their affair is, of course, a neverending cycle of love and hate; the governing system has a center at $(R, J) = (0,0)$. At least they manage to achieve simultaneous love one-quarter of the time (Figure 5.3.1).

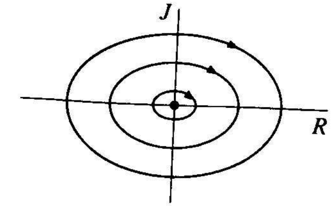


Figure 5.3.1

Now consider the forecast for lovers governed by the general linear system

$$\dot{R} = aR + bJ$$
$$\dot{J} = cR + dJ$$

where the parameters a, b, c, d may have either sign. A choice of signs specifies the romantic styles. As named by one of my students, the choice $a > 0$, $b > 0$ means that Romeo is an "eager beaver"—he gets excited by Juliet's love for him, and is further spurred on by his own affectionate feelings for her. It's entertaining to name the other three romantic styles, and to predict the outcomes for the various pairings. For example, can a "cautious lover" ($a < 0$, $b > 0$) find true love with an eager beaver? These and other pressing questions will be considered in the exercises.

EXAMPLE 5.3.1:

What happens when two identically cautious lovers get together?

Solution: The system is

$$\dot{R} = aR + bJ$$
$$\dot{J} = bR + aJ$$

with $a < 0$, $b > 0$. Here a is a measure of cautiousness (they each try to avoid throwing themselves at the other) and b is a measure of responsiveness (they both get excited by the other's advances). We might suspect that the outcome depends on the relative size of a and b. Let's see what happens.

The corresponding matrix is

$$A = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$$

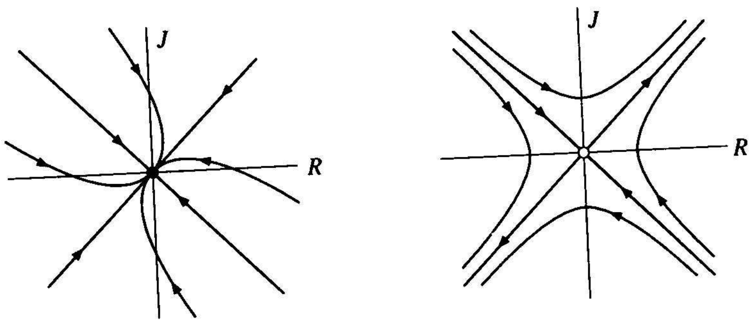
which has

$$\tau = 2a < 0, \qquad \Delta = a^2 - b^2, \qquad \tau^2 - 4\Delta = 4b^2 > 0.$$

Hence the fixed point $(R,J)=(0,0)$ is a saddle point if $a^2<b^2$ and a stable node if $a^2>b^2$. The eigenvalues and corresponding eigenvectors are

$$\lambda_1=a+b, \qquad \mathbf{v}_1=(1,1), \qquad \lambda_2=a-b, \qquad \mathbf{v}_2=(1,-1).$$

Since $a+b>a-b$, the eigenvector $(1,1)$ spans the unstable manifold when the origin is a saddle point, and it spans the slow eigendirection when the origin is a stable node. Figure 5.3.2 shows the phase portrait for the two cases.



$a^2>b^2$ $\qquad\qquad$ $a^2<b^2$

Figure 5.3.2

If $a^2>b^2$, the relationship always fizzles out to mutual indifference. The lesson seems to be that excessive caution can lead to apathy.

If $a^2<b^2$, the lovers are more daring, or perhaps more sensitive to each other. Now the relationship is explosive. Depending on their feelings initially, their relationship either becomes a love fest or a war. In either case, all trajectories approach the line $R=J$, so their feelings are eventually mutual. ■

EXERCISES FOR CHAPTER 5

5.1 Definitions and Examples

5.1.1 (Ellipses and energy conservation for the harmonic oscillator) Consider the harmonic oscillator $\dot{x}=v$, $\dot{v}=-\omega^2 x$.

a) Show that the orbits are given by ellipses $\omega^2x^2+v^2=C$, where C is any non-negative constant. (Hint: Divide the $\dot{x}$ equation by the $\dot{v}$ equation, separate the v's from the x's, and integrate the resulting separable equation.)

b) Show that this condition is equivalent to conservation of energy.

5.1.2 Consider the system $\dot{x}=ax$, $\dot{y}=-y$, where $a<-1$. Show that all trajectories become parallel to the y-direction as $t\to\infty$, and parallel to the x-direction as $t\to-\infty$.

(Hint: Examine the slope $dy/dx=\dot{y}/\dot{x}$.)

Write the following systems in matrix form.

5.1.3 $\dot{x}=-y,\ \dot{y}=-x$

5.1.4 $\dot{x}=3x-2y,\ \dot{y}=2y-x$

5.1.5 $\dot{x}=0,\ \dot{y}=x+y$

5.1.6 $\dot{x}=x,\ \dot{y}=5x+y$

Sketch the vector field for the following systems. Indicate the length and direction of the vectors with reasonable accuracy. Sketch some typical trajectories.

5.1.7 $\dot{x}=x,\ \dot{y}=x+y$

5.1.8 $\dot{x}=-2y,\ \dot{y}=x$

5.1.9 Consider the system $\dot{x}=-y,\ \dot{y}=-x$.

a) Sketch the vector field.

b) Show that the trajectories of the system are hyperbolas of the form $x^2-y^2=C$. (Hint: Show that the governing equations imply $x\dot{x}-y\dot{y}=0$ and then integrate both sides.)

c) The origin is a saddle point; find equations for its stable and unstable manifolds.

d) The system can be decoupled and solved as follows. Introduce new variables u and v, where $u=x+y,\ v=x-y$. Then rewrite the system in terms of u and v. Solve for $u(t)$ and $v(t)$, starting from an arbitrary initial condition (u_0,v_0).

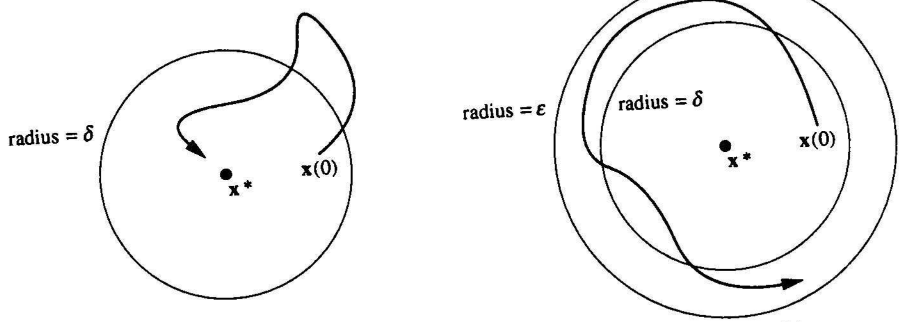
e) What are the equations for the stable and unstable manifolds in terms of u and v?

f) Finally, using the answer to (d), write the general solution for $x(t)$ and $y(t)$, starting from an initial condition (x_0,y_0).

5.1.10 (Attracting and Liapunov stable) Here are the official definitions of the various types of stability. Consider a fixed point $\mathbf{x}^*$ of a system $\dot{\mathbf{x}}=\mathbf{f}(\mathbf{x})$.

We say that $\mathbf{x}^*$ is ***attracting*** if there is a $\delta>0$ such that $\lim_{t\to\infty}\mathbf{x}(t)=\mathbf{x}^*$ whenever $\|\mathbf{x}(0)-\mathbf{x}^*\|<\delta$. In other words, any trajectory that starts within a distance δ of $\mathbf{x}^*$ is guaranteed to converge to $\mathbf{x}^*$ *eventually*. As shown schematically in Figure 1, trajectories that start nearby are allowed to stray from $\mathbf{x}^*$ in the short run, but they must approach $\mathbf{x}^*$ in the long run.

In contrast, Liapunov stability requires that nearby trajectories remain close for *all* time. We say that $\mathbf{x}^*$ is ***Liapunov stable*** if for each $\varepsilon>0$, there is a $\delta>0$ such that $\|\mathbf{x}(t)-\mathbf{x}^*\|<\varepsilon$ whenever $t\geq 0$ and $\|\mathbf{x}(0)-\mathbf{x}^*\|<\delta$. Thus, trajectories that start within δ of $\mathbf{x}^*$ remain within ε of $\mathbf{x}^*$ for all positive time (Figure 1).



Attracting

Liapunov stable

Figure 1

Finally, $\mathbf{x}^*$ is ***asymptotically stable*** if it is both attracting and Liapunov stable.

For each of the following systems, decide whether the origin is attracting, Liapunov stable, asymptotically stable, or none of the above.

a) $\dot{x} = y$, $\dot{y} = -4x$.

b) $\dot{x} = 2y$, $\dot{y} = x$

c) $\dot{x} = 0$, $\dot{y} = x$

d) $\dot{x} = 0$, $\dot{y} = -y$

e) $\dot{x} = -x$, $\dot{y} = -5y$

f) $\dot{x} = x$, $\dot{y} = y$

5.1.11 (Stability proofs) Prove that your answers to 5.1.10 are correct, using the definitions of the different types of stability. (You must produce a suitable δ to prove that the origin is attracting, or a suitable $\delta(\varepsilon)$ to prove Liapunov stability.)

5.1.12 (Closed orbits from symmetry arguments) Give a simple proof that orbits are closed for the simple harmonic oscillator $\dot{x} = v$, $\dot{v} = -x$, using *only* the symmetry properties of the vector field. (Hint: Consider a trajectory that starts on the v-axis at $(0, -v_0)$, and suppose that the trajectory intersects the x-axis at $(x, 0)$. Then use symmetry arguments to find the subsequent intersections with the v-axis and x-axis.)

5.1.13 Why do you think a "saddle point" is called by that name? What's the connection to real saddles (the kind used on horses)?

5.2 Classification of Linear Systems

5.2.1 Consider the system $\dot{x} = 4x - y$, $\dot{y} = 2x + y$.

a) Write the system as $\dot{\mathbf{x}} = A\mathbf{x}$. Show that the characteristic polynomial is $\lambda^2 - 5\lambda + 6$, and find the eigenvalues and eigenvectors of A.

b) Find the general solution of the system.

c) Classify the fixed point at the origin.

d) Solve the system subject to the initial condition $(x_0, y_0) = (3, 4)$.

5.2.2 (Complex eigenvalues) This exercise leads you through the solution of a linear system where the eigenvalues are complex. The system is $\dot{x} = x - y$, $\dot{y} = x + y$.

a) Find A and show that it has eigenvalues $\lambda_1 = 1 + i$, $\lambda_2 = 1 - i$, with eigenvectors $\mathbf{v}_1 = (i, 1)$, $\mathbf{v}_2 = (-i, 1)$. (Note that the eigenvalues are complex conjugates, and so are the eigenvectors—this is always the case for real A with complex eigenvalues.)

b) The general solution is $\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2$. So in one sense we're done! But this way of writing $\mathbf{x}(t)$ involves complex coefficients and looks unfamiliar. Express $\mathbf{x}(t)$ purely in terms of real-valued functions. (Hint: Use $e^{i\omega t} = \cos\omega t + i\sin\omega t$ to rewrite $\mathbf{x}(t)$ in terms of sines and cosines, and then separate the terms that have a prefactor of i from those that don't.)

Plot the phase portrait and classify the fixed point of the following linear systems. If the eigenvectors are real, indicate them in your sketch.

5.2.3 $\dot{x} = y$, $\dot{y} = -2x - 3y$

5.2.4 $\dot{x} = 5x + 10y$, $\dot{y} = -x - y$

5.2.5 $\dot{x} = 3x - 4y$, $\dot{y} = x - y$

5.2.6 $\dot{x} = -3x + 2y$, $\dot{y} = x - 2y$

5.2.7 $\dot{x} = 5x + 2y$, $\dot{y} = -17x - 5y$

5.2.8 $\dot{x} = -3x + 4y$, $\dot{y} = -2x + 3y$

5.2.9 $\dot{x} = 4x - 3y$, $\dot{y} = 8x - 6y$

5.2.10 $\dot{x} = y$, $\dot{y} = -x - 2y$.

5.2.11 Show that any matrix of the form $A = \begin{pmatrix} \lambda & b \\ 0 & \lambda \end{pmatrix}$, with $b \neq 0$, has only a one-dimensional eigenspace corresponding to the eigenvalue λ. Then solve the system $\dot{\mathbf{x}} = A\mathbf{x}$ and sketch the phase portrait.

5.2.12 (*LRC* circuit) Consider the circuit equation $L\ddot{I} + R\dot{I} + I/C = 0$, where $L, C > 0$ and $R \geq 0$.

a) Rewrite the equation as a two-dimensional linear system.

b) Show that the origin is asymptotically stable if $R > 0$ and neutrally stable if $R = 0$.

c) Classify the fixed point at the origin, depending on whether $R^2C - 4L$ is positive, negative, or zero, and sketch the phase portrait in all three cases.

5.2.13 (Damped harmonic oscillator) The motion of a damped harmonic oscillator is described by $m\ddot{x} + b\dot{x} + kx = 0$, where $b > 0$ is the damping constant.

a) Rewrite the equation as a two-dimensional linear system.

b) Classify the fixed point at the origin and sketch the phase portrait. Be sure to show all the different cases that can occur, depending on the relative sizes of the parameters.

c) How do your results relate to the standard notions of overdamped, critically damped, and underdamped vibrations?

5.2.14 (A project about random systems) Suppose we pick a linear system at random, what is the probability that the origin is, say, an unstable spiral? To be more specific, consider the system $\dot{\mathbf{x}} = A\mathbf{x}$, where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Suppose we pick the entries a, b, c, d independently and at random from a uniform distribution on the interval $[-1, 1]$. Find the probabilities of all the different kinds of fixed points.

To check your answers (or if you hit an analytical roadblock), try the *Monte Carlo method*. Generate millions of random matrices on the computer and have the machine count the relative frequency of saddles, unstable spirals, etc.

Are the answers the same if you use a normal distribution instead of a uniform distribution?

5.3 Love Affairs

5.3.1 (Name-calling) Suggest names for the four romantic styles, determined by the signs of a and b in $\dot{R} = aR + bJ$.

5.3.2 Consider the affair described by $\dot{R} = J$, $\dot{J} = -R + J$.
a) Characterize the romantic styles of Romeo and Juliet.
b) Classify the fixed point at the origin. What does this imply for the affair?
c) Sketch $R(t)$ and $J(t)$ as functions of t, assuming $R(0) = 1$, $J(0) = 0$.

In each of the following problems, predict the course of the love affair, depending on the signs and relative sizes of a and b.

5.3.3 (Out of touch with their own feelings) Suppose Romeo and Juliet react to each other, but not to themselves: $\dot{R} = aJ$, $\dot{J} = bR$. What happens?

5.3.4 (Fire and water) Do opposites attract? Analyze $\dot{R} = aR + bJ$, $\dot{J} = -bR - aJ$.

5.3.5 (Peas in a pod) If Romeo and Juliet are romantic clones ($\dot{R} = aR + bJ$, $\dot{J} = bR + aJ$), should they expect boredom or bliss?

5.3.6 (Romeo the robot) Nothing could ever change the way Romeo feels about Juliet: $\dot{R} = 0$, $\dot{J} = aR + bJ$. Does Juliet end up loving him or hating him?

6

PHASE PLANE

6.0 Introduction

This chapter begins our study of two-dimensional *nonlinear* systems. First we consider some of their general properties. Then we classify the kinds of fixed points that can arise, building on our knowledge of linear systems (Chapter 5). The theory is further developed through a series of examples from biology (competition between two species) and physics (conservative systems, reversible systems, and the pendulum). The chapter concludes with a discussion of index theory, a topological method that provides global information about the phase portrait.

This chapter is mainly about fixed points. The next two chapters will discuss closed orbits and bifurcations in two-dimensional systems.

6.1 Phase Portraits

The general form of a vector field on the phase plane is

$$\dot{x}_1 = f_1(x_1, x_2)$$
$$\dot{x}_2 = f_2(x_1, x_2)$$

where f_1 and f_2 are given functions. This system can be written more compactly in vector notation as

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$$

where $\mathbf{x} = (x_1, x_2)$ and $\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}))$. Here $\mathbf{x}$ represents a point in the phase plane, and $\dot{\mathbf{x}}$ is the velocity vector at that point. By flowing along the vector field, a phase point traces out a solution $\mathbf{x}(t)$, corresponding to a trajectory winding through the phase plane (Figure 6.1.1).

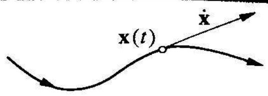


Figure 6.1.1

Furthermore, the entire phase plane is filled with trajectories, since each point can play the role of an initial condition.

For nonlinear systems, there's typically no hope of finding the trajectories analytically. Even when explicit formulas are available, they are often too complicated to provide much insight. Instead we will try to determine the *qualitative* behavior of the solutions. Our goal is to find the system's phase portrait directly from the properties of $\mathbf{f}(\mathbf{x})$. An enormous variety of phase portraits is possible; one example is shown in Figure 6.1.2.

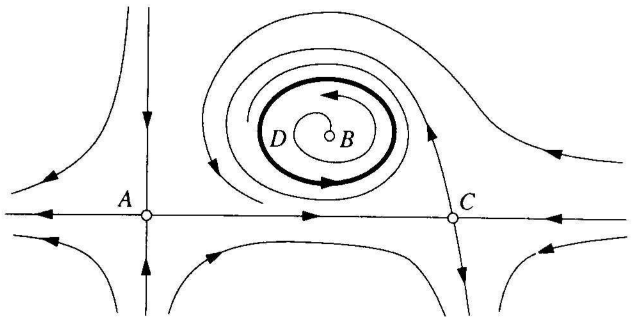


Figure 6.1.2

Some of the most salient features of any phase portrait are:

1. The ***fixed points***, like A, B, and C in Figure 6.1.2. Fixed points satisfy $\mathbf{f}(\mathbf{x}^*) = \mathbf{0}$, and correspond to steady states or equilibria of the system.
2. The ***closed orbits***, like D in Figure 6.1.2. These correspond to periodic solutions, i.e., solutions for which $\mathbf{x}(t+T) = \mathbf{x}(t)$ for all t, for some $T > 0$.
3. The arrangement of trajectories near the fixed points and closed orbits. For example, the flow pattern near A and C is similar, and different from that near B.
4. The stability or instability of the fixed points and closed orbits. Here, the fixed points A, B, and C are unstable, because nearby trajectories tend to move away from them, whereas the closed orbit D is stable.

Numerical Computation of Phase Portraits

Sometimes we are also interested in *quantitative* aspects of the phase portrait. Fortunately, numerical integration of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is not much harder than that of $\dot{x} = f(x)$. The numerical methods of Section 2.8 still work, as long as we replace the numbers x and $f(x)$ by the vectors $\mathbf{x}$ and $\mathbf{f}(\mathbf{x})$. We will always use the Runge-Kutta method, which in vector form is

$$\mathbf{x}_{n+1} = \mathbf{x}_n + \tfrac{1}{6}(\mathbf{k}_1 + 2\mathbf{k}_2 + 2\mathbf{k}_3 + \mathbf{k}_4)$$

where

$$\begin{aligned}
\mathbf{k}_1 &= \mathbf{f}(\mathbf{x}_n)\Delta t \\
\mathbf{k}_2 &= \mathbf{f}(\mathbf{x}_n + \tfrac{1}{2}\mathbf{k}_1)\Delta t \\
\mathbf{k}_3 &= \mathbf{f}(\mathbf{x}_n + \tfrac{1}{2}\mathbf{k}_2)\Delta t \\
\mathbf{k}_4 &= \mathbf{f}(\mathbf{x}_n + \mathbf{k}_3)\Delta t.
\end{aligned}$$

A stepsize $\Delta t = 0.1$ usually provides sufficient accuracy for our purposes.

When plotting the phase portrait, it often helps to see a grid of representative vectors in the vector field. Unfortunately, the arrowheads and different lengths of the vectors tend to clutter such pictures. A plot of the ***direction field*** is clearer: short line segments are used to indicate the local direction of flow.

EXAMPLE 6.1.1:

Consider the system $\dot{x} = x + e^{-y}$, $\dot{y} = -y$. First use qualitative arguments to obtain information about the phase portrait. Then, using a computer, plot the direction field. Finally, use the Runge–Kutta method to compute several trajectories, and plot them on the phase plane.

Solution: First we find the fixed points by solving $\dot{x} = 0$, $\dot{y} = 0$ simultaneously. The only solution is $(x^*, y^*) = (-1, 0)$. To determine its stability, note that $y(t) \to 0$ as $t \to \infty$, since the solution to $\dot{y} = -y$ is $y(t) = y_0 e^{-t}$. Hence $e^{-y} \to 1$ and so in the long run, the equation for x becomes $\dot{x} \approx x + 1$; this has exponentially growing solutions, which suggests that the fixed point is unstable. In fact, if we restrict our attention to initial conditions on the x-axis, then $y_0 = 0$ and so $y(t) = 0$ for all time. Hence the flow on the x-axis is governed *strictly* by $\dot{x} = x + 1$. Therefore the fixed point is unstable.

To sketch the phase portrait, it is helpful to plot the ***nullclines***, defined as the curves where either $\dot{x} = 0$ or $\dot{y} = 0$. The nullclines indicate where the flow is purely horizontal or vertical (Figure 6.1.3). For example, the flow is horizontal where $\dot{y} = 0$, and since $\dot{y} = -y$, this occurs on the line $y = 0$. Along this line, the flow is to the right where $\dot{x} = x + 1 > 0$, that is, where $x > -1$.

Similarly, the flow is vertical where $\dot{x} = x + e^{-y} = 0$, which occurs on the curve shown in Figure 6.1.3. On the upper part of the curve where $y > 0$, the flow is downward, since $\dot{y} < 0$.

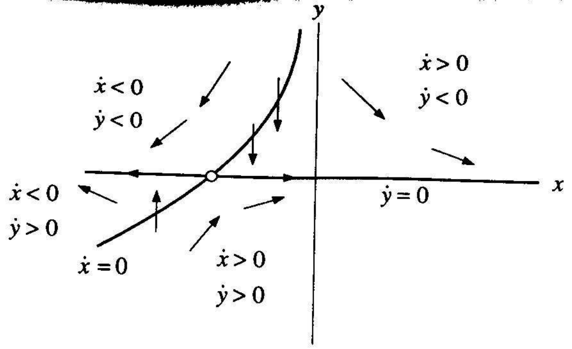


Figure 6.1.3

The nullclines also partition the plane into regions where $\dot{x}$ and $\dot{y}$ have various signs. Some of the typical vectors are sketched above in Figure 6.1.3. Even with the limited information obtained so far, Figure 6.1.3 gives a good sense of the overall flow pattern.

Now we use the computer to finish the problem. The direction field is indicated by the line segments in Figure 6.1.4, and several trajectories are shown. Note how the trajectories always follow the local slope.

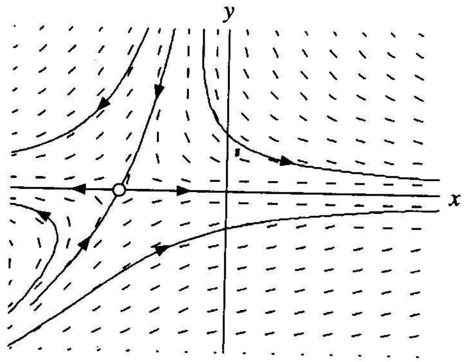


Figure 6.1.4

The fixed point is now seen to be a nonlinear version of a saddle point. ■

6.2 Existence, Uniqueness, and Topological Consequences

We have been a bit optimistic so far—at this stage, we have no guarantee that the general nonlinear system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ even *has* solutions! Fortunately the existence and uniqueness theorem given in Section 2.5 can be generalized to two-dimen-

sional systems. We state the result for n-dimensional systems, since no extra effort is involved:

Existence and Uniqueness Theorem: Consider the initial value problem $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, $\mathbf{x}(0) = \mathbf{x}_0$. Suppose that $\mathbf{f}$ is continuous and that all its partial derivatives $\partial f_i/\partial x_j$, $i, j = 1, \ldots, n$, are continuous for $\mathbf{x}$ in some open connected set $D \subset \mathbf{R}^n$. Then for $\mathbf{x}_0 \in D$, the initial value problem has a solution $\mathbf{x}(t)$ on some time interval $(-\tau, \tau)$ about $t = 0$, and the solution is unique.

In other words, existence and uniqueness of solutions are guaranteed if $\mathbf{f}$ is continuously differentiable. The proof of the theorem is similar to that for the case $n = 1$, and can be found in most texts on differential equations. Stronger versions of the theorem are available, but this one suffices for most applications.

From now on, we'll assume that all our vector fields are smooth enough to ensure the existence and uniqueness of solutions, starting from any point in phase space.

The existence and uniqueness theorem has an important corollary: *different trajectories never intersect.* If two trajectories *did* intersect, then there would be two solutions starting from the same point (the crossing point), and this would violate the uniqueness part of the theorem. In more intuitive language, a trajectory can't move in two directions at once.

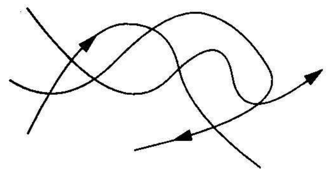


Figure 6.2.1

Because trajectories can't intersect, phase portraits always have a well-groomed look to them. Otherwise they might degenerate into a snarl of criss-crossed curves (Figure 6.2.1). The existence and uniqueness theorem prevents this from happening.

In two-dimensional phase spaces (as opposed to higher-dimensional phase spaces), these results have especially strong topological consequences. For example, suppose there is a closed orbit C in the phase plane. Then any trajectory starting inside C is trapped in there forever (Figure 6.2.2).

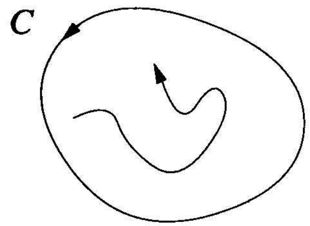


Figure 6.2.2

What is the fate of such a bounded trajectory? If there are fixed points inside C, then of course the trajectory might eventually approach one of them. But what if there *aren't* any fixed points? Your intuition may tell you that the trajectory can't meander around forever—if so, you're right. For vector fields on the plane, the ***Poincaré–Bendixson theorem*** states that if a trajectory is confined to a closed, bounded region and there are no fixed points in the region, then the trajectory must electrostatics, namely that the electric flux through a surface is proportional to the total charge enclosed. See Exercise 6.8.12 for a further exploration of this analogy between index and charge.

Theorem 6.8.2: Any closed orbit in the phase plane must enclose fixed points whose indices sum to +1.

Proof: Let C denote the closed orbit. From property (4) above, $I_C = +1$. Then Theorem 6.8.1 implies $\sum_{k=1}^{n} I_k = +1$. ■

Theorem 6.8.2 has many practical consequences. For instance, it implies that there is always at least one fixed point inside any closed orbit in the phase plane (as you may have noticed on your own). If there is *only* one fixed point inside, it cannot be a saddle point. Furthermore, Theorem 6.8.2 can sometimes be used to rule out the possible occurrence of closed trajectories, as seen in the following examples.

EXAMPLE 6.8.5:

Show that closed orbits are impossible for the "rabbit vs. sheep" system

$$\dot{x} = x(3 - x - 2y)$$
$$\dot{y} = y(2 - x - y)$$

studied in Section 6.4. Here $x, y \geq 0$.

Solution: As shown previously, the system has four fixed points: $(0,0)$ = unstable node; $(0,2)$ and $(3,0)$ = stable nodes; and $(1,1)$ = saddle point. The index at each of these points is shown in Figure 6.8.9. Now suppose that the system had a closed trajectory. Where could it lie? There are three qualitatively different locations, indicated by the dotted curves C_1, C_2, C_3. They can be ruled out as follows: orbits like C_1 are impossible because they don't enclose any fixed points, and orbits like C_2 violate the requirement that the indices inside must sum to +1. But what is wrong with orbits like C_3, which satisfy the index requirement? The trouble is that such orbits always cross the x-axis or the y-axis, and these axes contain straight-line trajectories. Hence C_3 violates the rule that trajectories can't cross (recall Section 6.2). ■

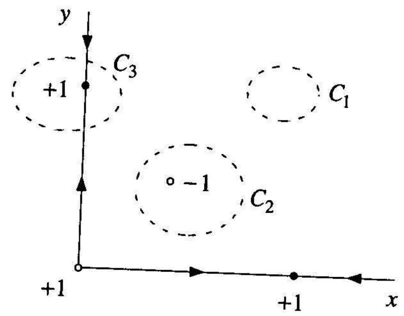


Figure 6.8.9

EXAMPLE 6.8.6:

Show that the system $\dot{x} = xe^{-x}$, $\dot{y} = 1 + x + y^2$ has no closed orbits.

Solution: This system has no fixed points: if $\dot{x} = 0$, then $x = 0$ and so $\dot{y} = 1 + y^2 \neq 0$. By Theorem 6.8.2, closed orbits cannot exist. ■

EXERCISES FOR CHAPTER 6

6.1 Phase Portraits

For each of the following systems, find the fixed points. Then sketch the nullclines, the vector field, and a plausible phase portrait.

6.1.1 $\dot{x} = x - y$, $\dot{y} = 1 - e^x$

6.1.2 $\dot{x} = x - x^3$, $\dot{y} = -y$

6.1.3 $\dot{x} = x(x - y)$, $\dot{y} = y(2x - y)$

6.1.4 $\dot{x} = y$, $\dot{y} = x(1 + y) - 1$

6.1.5 $\dot{x} = x(2 - x - y)$, $\dot{y} = x - y$

6.1.6 $\dot{x} = x^2 - y$, $\dot{y} = x - y$

6.1.7 (Nullcline vs. stable manifold) There's a confusing aspect of Example 6.1.1. The nullcline $\dot{x} = 0$ in Figure 6.1.3 has a similar shape and location as the stable manifold of the saddle, shown in Figure 6.1.4. But they're not the same curve! To clarify the relation between the two curves, sketch both of them on the same phase portrait.

(Computer work) Plot computer-generated phase portraits of the following systems. As always, you may write your own computer programs or use any ready-made software, e.g., *MacMath* (Hubbard and West 1992).

6.1.8 (van der Pol oscillator) $\dot{x} = y$, $\dot{y} = -x + y(1 - x^2)$

6.1.9 (Dipole fixed point) $\dot{x} = 2xy$, $\dot{y} = y^2 - x^2$

6.1.10 (Two-eyed monster) $\dot{x} = y + y^2$, $\dot{y} = -\frac{1}{2}x + \frac{1}{5}y - xy + \frac{6}{5}y^2$ (from Borrelli and Coleman 1987, p. 385.)

6.1.11 (Parrot) $\dot{x} = y + y^2$, $\dot{y} = -x + \frac{1}{5}y - xy + \frac{6}{5}y^2$ (from Borrelli and Coleman 1987, p. 384.)

6.1.12 (Saddle connections) A certain system is known to have exactly two fixed points, both of which are saddles. Sketch phase portraits in which
a) there is a single trajectory that connects the saddles;
b) there is no trajectory that connects the saddles.

6.1.13 Draw a phase portrait that has exactly three closed orbits and one fixed point.

6.1.14 (Series approximation for the stable manifold of a saddle point) Recall the system $\dot{x} = x + e^{-y}$, $\dot{y} = -y$ from Example 6.1.1. We showed that this system