2.3 ■ DERIVATION OF LAGRANGE'S EQUATIONS FROM HAMILTON'S PRINCIPLE

The fundamental problem of the calculus of variations is easily generalized to the case where f is a function of many independent variables y_i, and their derivatives $\dot{y}_i$. (Of course, all these quantities are considered as functions of the parametric variable x.) Then a variation of the integral J,

$$\delta J = \delta \int_1^2 f(y_1(x); y_2(x), \ldots, \dot{y}_1(x); \dot{y}_2(x), \ldots, x)\, dx, \tag{2.14}$$

is obtained, as before, by considering J as a function of parameter α that labels a possible set of curves $y_1(x, \alpha)$. Thus, we may introduce α by setting

$$\begin{aligned} y_1(x, \alpha) &= y_1(x, 0) + \alpha\eta_1(x), \\ y_2(x, \alpha) &= y_2(x, 0) + \alpha\eta_2(x), \\ \vdots \quad & \quad\quad \vdots \quad\quad\quad \vdots \end{aligned} \tag{2.15}$$

where $y_1(x, 0)$, $y_2(x, 0)$, etc., are the solutions of the extremum problem (to be obtained) and η_1, η_2, etc., are independent functions of x that vanish at the end points and that are continuous through the second derivative, but otherwise are completely arbitrary.

The calculation proceeds as before. The variation of J is given in terms of

$$\frac{\partial J}{\partial \alpha} d\alpha = \int_1^2 \sum_i \left(\frac{\partial f}{\partial y_i} \frac{\partial y_i}{\partial \alpha} d\alpha + \frac{\partial f}{\partial \dot{y}_i} \frac{\partial \dot{y}_i}{\partial \alpha} d\alpha \right) dx. \tag{2.16}$$

Again we integrate by parts the integral involved in the second sum of Eq. (2.16):

$$\int_1^2 \frac{\delta f}{\partial \dot{y}_i} \frac{\partial^2 y_i}{\partial \alpha\, \partial x} dx = \left. \frac{\partial f}{\partial \dot{y}_i} \frac{\partial y_i}{\partial \alpha} \right|_1^2 - \int_1^2 \frac{\partial y_i}{\partial \alpha} \frac{d}{dx} \left(\frac{\partial f}{\partial \dot{y}_i} \right) dx,$$

where the first term vanishes because all curves pass through the fixed end points. Substituting in (2.16), δJ becomes

$$\delta J = \int_1^2 \sum_i \left(\frac{\partial f}{\partial y_i} - \frac{d}{dx} \frac{\partial f}{\partial \dot{y}_i} \right) \delta y_i \, dx, \tag{2.17}$$

where, in analogy with (2.12), the variation δy_i is

$$\delta y_i = \left(\frac{\partial y_i}{\partial \alpha} \right)_0 d\alpha.$$

Since the y variables are independent, the variations δy_i are independent (e.g., the functions $\eta_i(x)$ will be independent of each other). Hence, by an obvious extension of the fundamental lemma (cf. Eq. (2.10)), the condition that δJ is zero requires that the coefficients of the δy_i separately vanish:

$$\frac{\partial f}{\partial y_i} - \frac{d}{dx}\frac{\partial f}{\partial \dot{y}_i} = 0, \qquad i = 1, 2, \ldots, n. \tag{2.18}$$

Equations (2.18) represent the appropriate generalization of (2.11) to several variables and are known as the *Euler-Lagrange differential equations*. Their solutions represent curves for which the variation of an integral of the form given in (2.14) vanishes. Further generalizations of the fundamental variational problem are easily possible. Thus, we can take f as a function of higher derivatives $\ddot{y}$, $\dddot{y}$, etc., leading to equations different from (2.18). Or we can extend it to cases where there are several parameters x_j and the integral is then multiple, with f also involving as variables derivatives of y_i with respect to each of the parameters x_j. Finally, it is possible to consider variations in which the end points are *not* held fixed.

For present purposes, what we have derived here suffices, for the integral in Hamilton's principle,

$$I = \int_1^2 L(q_i, \dot{q}_i, t)\, dt, \tag{2.19}$$

has just the form stipulated in (2.14) with the transformation

$$x \to t$$

$$y_i \to q_i$$

$$f(y_i, \dot{y}_i, x) \to L(q_i, \dot{q}_i, t).$$

In deriving Eqs. (2.18), we assumed that the y_i variables are independent. The corresponding condition in connection with Hamilton's principle is that the generalized coordinates q_i be independent, which requires that the constraints be holonomic. The Euler-Lagrange equations corresponding to the integral I then become the Lagrange equations of motion,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0, \qquad i = 1, 2, \ldots, n,$$

and we have accomplished our original aim, to show that Lagrange's equations follow from Hamilton's principle—for monogenic systems with holonomic constraints.

2.4 ■ EXTENSION OF HAMILTON'S PRINCIPLE TO NONHOLONOMIC SYSTEMS

It is possible to extend Hamilton's principle, at least in a formal sense, to cover certain types of nonholonomic systems. In deriving Lagrange's equations from either Hamilton's or D'Alembert's principle, the requirement of holonomic constraints does not appear until the last step, when the variations q_i are considered as independent of each other. With nonholonomic systems the generalized coordinates are not independent of each other, and it is not possible to reduce them further by means of equations of constraint of the form $f(q_1, q_2, \ldots, q_n, t) = 0$. Hence, it is no longer true that the q_i's are all independent.

Another difference that must be considered in treating the variational principle is the manner in which the varied paths are constructed. In the discussion of Section 2.2, we pointed out that δy (or δq) represents a virtual displacement from a point on the actual path to some point on the neighboring varied path. But, with independent coordinates it is the final varied path that is significant, not how it is constructed. When the coordinates are not independent, but subject to constraint relations, it becomes important whether the varied path is or is not constructed by displacements consistent with the constraints. Virtual displacements, in particular, may or may not satisfy the constraints.

It appears that a reasonably straightforward treatment of nonholonomic systems by a variational principle is possible only when the equations of constraint can be put in the form

$$f_\alpha(q_1, \ldots, q_n; \dot{q}_1 \ldots, \dot{q}_n) = 0, \tag{2.20}$$

when this can be done the constraints are called semi-holonomic. The index α indicates that there may be more than one such equation. We will assume there are m equations in all, i.e., $\alpha = 1, 2, \ldots, m$. Equation (2.20) commonly appears in the restricted form

$$\sum_k a_{ik}\, dq_k + a_{it}\, dt = 0. \tag{2.20$'$}$$

We might expect that the varied paths, or equivalently, the displacements constructing the varied path, should satisfy the constraints of Eq. (2.20). However, it has been proven that no such varied path can be constructed unless Eqs. (2.20) are integrable, in which case the constraints are actually holonomic. A variational principle leading to the correct equations of motion can nonetheless be obtained when the varied paths are constructed from the actual motion by virtual displacements.

The procedure for eliminating these extra virtual displacements is the method of *Lagrange undetermined multipliers*. If Eqs. (2.20) hold, then it is also true that

$$\sum_{\alpha=1}^{m} \lambda_\alpha f_\alpha = 0, \tag{2.21}$$

where the λ_α, $\alpha = 1, 2, \ldots, m$, are some undetermined quantities, functions in general of the coordinates and of the time t. In addition, Hamilton's principle,

$$\delta \int_{t_1}^{t_2} L\, dt = 0, \tag{2.2}$$

is assumed to hold for this semiholonomic system. Following the development of Section 2.3, Hamilton's principle then implies that

$$\int_1^2 dt \sum_k \left(\frac{\partial L}{\partial q_k} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} \right) \delta q_k = 0. \tag{2.22}$$

The variation cannot be taken as before since the q_k are not independent; however, combining (2.21) with (2.2) gives

$$\delta \int_{t_1}^{t_2} \left(L + \sum_{\alpha=1}^{m} \lambda_\alpha f_\alpha \right) dt = 0 \tag{2.23}$$

The variation can now be performed with the n δq_i and m λ_α for $m+n$ independent variables. For the simplifying assumption that $\lambda_\alpha = \lambda_\alpha(t)$, the resulting equations from δq_i become*

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k} \right) - \frac{\partial L}{\partial q_k} = Q_k, \tag{2.24}$$

where

$$Q_k = \sum_{\alpha=1}^{m} \left\{ \lambda_\alpha \left[\frac{\partial f_\alpha}{\partial q_k} - \frac{d}{dt}\left(\frac{\partial f_\alpha}{\partial \dot{q}_k} \right) \right] - \frac{d\lambda_\alpha}{dt}\frac{\partial f_\alpha}{\partial \dot{q}_k} \right\}, \tag{2.25}$$

while the $\delta\lambda_\alpha$ give the equations of constraint (2.20). Equations (2.24) and (2.20) together constitute $n + m$ equations for $n + m$ unknowns. The system can now be interpreted as an $m + n$ holonomic system with generalized forces Q_k. The generalization to $\lambda_\alpha = \lambda_\alpha(q_1, \ldots, q_n; \dot{q}_1, \ldots, \dot{q}_n; t)$ is straightforward.

As an example, let us consider a particle whose Lagrangian is

$$L = \tfrac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - V(x, y, z) \tag{2.26}$$

subject to the constraint

$$f(\dot{x}, \dot{y}, y) = \dot{x}\dot{y} + ky = 0 \tag{2.27}$$

with k a constant. The resulting equations of motion are

$$m\ddot{x} + \lambda\ddot{y} + \dot{\lambda}\dot{y} + \frac{\partial V}{\partial x} = 0, \tag{2.28}$$

$$m\ddot{y} + \lambda\ddot{x} - k\lambda + \dot{\lambda}\dot{x} + \frac{\partial V}{\partial y} = 0, \tag{2.29}$$

$$m\ddot{z} + \frac{\partial V}{\partial z} = 0, \tag{2.30}$$

*J. Ray, *Amer. J. Phys.* **34** (406–8), 1996.

and the equation of constraint, (2.20), becomes

$$\dot{y}\dot{x} + ky = 0.$$

In this process we have obtained more information than was originally sought. Not only do we get the q_k's we set out to find, but we also get $m\lambda_l$'s. What is the physical significance of the λ_l's? Suppose we remove the constraints on the system, but instead apply external forces Q'_k in such a manner as to keep the motion of the system unchanged. The equations of motion likewise remain the same. Clearly these extra applied forces must be equal to the forces of constraint, for they are the forces applied to the system so as to satisfy the condition of constraint. Under the influence of these forces Q'_k, the equations of motion are

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = Q'_k. \tag{2.31}$$

But these must be identical with Eqs. (2.24). Hence, we can identify (2.25) with Q'_k, the generalized forces of constraint. In this type of problem we really do not eliminate the forces of constraint from the formulation. They are supplied as part of the answer.

Although it is not obvious, the version of Hamilton's principle adopted here for semiholonomic systems also requires that the constraints do no work in virtual displacements. This can be most easily seen by rewriting Hamilton's principle in the form

$$\delta \int_{t_1}^{t_2} L\, dt = \delta \int_{t_1}^{t_2} T\, dt - \delta \int_{t_1}^{t_2} U\, dt = 0. \tag{2.32}$$

If the variation of the integral over the generalized potential is carried out by the procedures of Section 2.3, the principle takes the form

$$\delta \int_{t_1}^{t_2} T\, dt = \int_{t_1}^{t_2} \sum_k \left[\frac{\partial U}{\partial q_k} - \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_k}\right)\right] \delta q_k dt; \tag{2.33}$$

or, by Eq. (1.58),

$$\delta \int_{t_1}^{t_2} T\, dt = -\int_{t_1}^{t_2} \sum_k Q_k \delta q_k dt. \tag{2.34}$$

In this dress, Hamilton's principle says that the difference in the time integral of the kinetic energy between two neighboring paths is equal to the negative of the time integral of the work done in the virtual displacements between the paths. The work involved is that done only by the forces derivable from the generalized potential. The same Hamilton's principle holds for both holonomic and semiholonomic systems, it must be required that the additional forces of semiholonomic constraints do no work in the displacements δq_k. This restriction parallels the earlier condition that the virtual work of the forces of holonomic constraint also be zero (cf. Section 1.4). In practice, the restriction presents little handicap to the applications, as many problems in which the semiholonomic formalism is used relate to rolling without slipping, where the constraints are obviously workless.

Note that Eq. (2.20) is not the most general type of nonholonomic constraint; e.g., it does not include equations of constraint in the form of inequalities. On the other hand, it does include holonomic constraints. A holonomic equation of constraint,

$$f(q_1, q_2, q_3, \ldots, q_n, t) = 0, \tag{2.35}$$

is equivalent to (2.20) with no dependence on $\dot{q}_k$. Thus, the Lagrange multiplier method can be used also for holonomic constraints when (1) it is inconvenient to reduce all the q's to independent coordinates or (2) we might wish to obtain the forces of constraint.

As another example of the method, let us consider the following somewhat trivial illustration—a hoop rolling, without slipping, down an inclined plane. In this example, the constraint of "rolling" is actually holonomic, but this fact will be immaterial to our discussion. On the other hand, the holonomic constraint that the hoop be on the inclined plane will be contained implicitly in our choice of generalized coordinates.

The two generalized coordinates are x, θ, as in Fig. 2.5, and the equation of rolling constraint is

$$r\,d\theta = dx.$$

The kinetic energy can be resolved into kinetic energy of motion of the center of mass plus the kinetic energy of motion about the center of mass:

$$T = \tfrac{1}{2}M\dot{x}^2 + \tfrac{1}{2}Mr^2\dot{\theta}^2.$$

The potential energy is

$$V = Mg(l - x)\sin\phi,$$

where l is the length of the inclined plane and the Lagrangian is

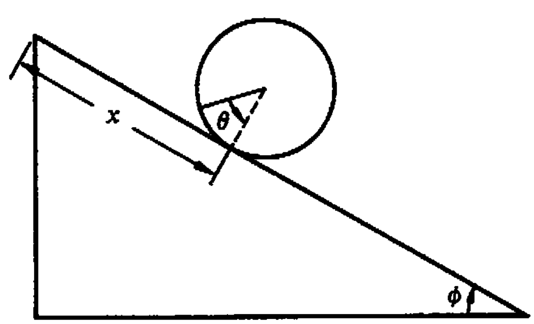

FIGURE 2.5 A hoop rolling down an inclined plane.