Week 2 QM Discussion

Ji Zou, jzeeucla@physics.ucla.edu
Office Hours: Tuesday 10am-12pm, Tutoring Center.

3-D Harmonic Potential

A particle with mass m is in the potential $V(x,y,z) = \frac{mw^2}{2}(x^2 + 2x + y^2 + z^2)$.

Find the energy.
(What if there is a cross term in the potential: $V(x,y,z) = \frac{mw^2}{2}(x^2 + xy + y^2 + z^2)$)

Write down 3-D S.E.

↓

Try to solve this equation using separation variables

↓

For x part:

$\tilde{x} = x + 1$

↓

3 decoupled harmonic ~~oscia~~ oscillation

Write down 3D S.E

↓

Try to decouple, x, y. by diagonizing:

$x^2 + xy + y^2$

$= (x \;\; y)\begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

↓

Then solve it using sepation of variat

Particle in 3-D Delta Function Potential

A particle of mass m interacts in three dimension with a spherically symmetric potential of the form:

$$V(r) = -c\delta(|\vec{r}| - a)$$

In other words, the potential is a delta function that vanishes unless the particle is precisely a distance a from the center of the potential. Here c is a positive constant.

What's the minimum value of c for which there is a bound state?

Write down 3-D equation in Spherical coordinate.

$\downarrow$

Seperation of variables, get the equation for r component.

$\downarrow$

match boundary condition

$\gamma = r\,\frac{R(r)}{\cancel{\quad}}$ usful.

The first step is to write down S.E:

$$\left[\frac{\hat{p}^2}{2m} + V(x,y,z)\right]\psi(x,y,z) = E\,\psi(x,y,z)$$

$$\cancel{V(x,y,z) = \frac{m\omega^2}{2}\left[x^2+2x+y^2+z^2\right]}$$

$$= \cancel{\frac{m\omega^2}{2}\left[(x+1)^2 - 1 + y^2 + z^2\right]}$$

$$\psi(x,y,z) = X(x)\,Y(y)\,Z(z)$$

$$\frac{(p_x^2 + 2mV_x)X(x)}{2m}\,Y(y)\,Z(z) + \frac{(p_y^2 + 2mV_y)Y(y)}{2m}\,Z(z)\,X(x) + \frac{(p_z^2 + 2mV_z)Z(z)}{2m}\,XY$$

$$= EXYZ$$

$$\left(\frac{p_x^2}{2m} + V_x\right)X = E_x X$$

$$\left(\frac{p_y^2}{2m} + V_y\right)Y = E_y Y \qquad E = E_x + E_y + E_z$$

$$\left(\frac{p_z^2}{2m} + V_z\right)Z = E_z Z$$

$$\left[\frac{p_x^2}{2m} + \frac{m\omega^2}{2}(x^2+2x)\right]X = E_x X$$

$$\hat{H}_x = \frac{p_x^2}{2m} + \frac{m\omega^2}{2}(x^2+2x)$$

$$= \frac{p_x^2}{2m} + \frac{m\omega^2}{2}\left[(x+1)^2 - 1\right]$$

$$= \frac{p_x^2}{2m} + \frac{m\omega^2}{2}\left[x+1\right]^2 - \frac{m\omega^2}{2}$$

$$\tilde{x} = x+1$$

Then: $p_x = p_{\tilde{x}}$

$$\hat{H} = \frac{p_{\tilde{x}}^2}{2m} + \frac{m\omega^2}{2}\tilde{x}^2 - \frac{m\omega^2}{2}$$

$$= \frac{\hbar\omega}{2}\left(n+\frac{1}{2}\right) - \frac{m\omega^2}{2}$$

$$H_y = \frac{\hbar\omega}{2}\left(l+\frac{1}{2}\right)$$

$n, l, k = 0, 1, 2 \ldots$

$$H_z = \frac{\hbar\omega}{2}\left(k+\frac{1}{2}\right)$$

$$E = \frac{\hbar\omega}{2}\left(n+l+k+\frac{3}{2}\right) - \frac{m\omega^2}{2}$$

$$V(x,y,z) = \frac{m\omega^2}{2}\left(x^2 + xy + y^2 + z^2\right)$$

$$(x \quad y)\begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\det\begin{pmatrix} 1-\lambda & \frac{1}{2} \\ \frac{1}{2} & 1-\lambda \end{pmatrix} = (\lambda-1)^2 - \frac{1}{4} = 0$$

$$\lambda - 1 = \pm\frac{1}{2} \Rightarrow \lambda = \frac{1}{2}, \frac{3}{2}$$

$$\begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{2}\begin{pmatrix} a \\ b \end{pmatrix}$$

$$a + \frac{b}{2} = \frac{a}{2} \Rightarrow \frac{b}{2} = -\frac{a}{2}$$

$$|\tfrac{1}{2}\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$

$$|\tfrac{3}{2}\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$U = \begin{pmatrix} |\tfrac{3}{2}\rangle & |\tfrac{1}{2}\rangle \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\tilde{x} = \frac{1}{\sqrt{2}} x + \frac{1}{\sqrt{2}} y$$

$$\tilde{y} = \frac{1}{\sqrt{2}} x - \frac{1}{\sqrt{2}} y$$

$$\frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{m\omega^2}{2}\left(x^2 + xy + y^2\right)$$

$$= \frac{p_x^2 + p_y^2}{2m} + \frac{m\omega^2}{2}\left[\frac{3}{2}\tilde{x}^2 + \frac{\tilde{y}^2}{2}\right]$$

$$\tilde{p}_x = \left(\frac{p_x}{\sqrt{2}} + \frac{p_y}{\sqrt{2}}\right)$$

$$\tilde{p}_y = \left(\frac{p_x}{\sqrt{2}} - \frac{p_y}{\sqrt{2}}\right)$$

$$[\tilde{p}_x, \tilde{x}] = -i\hbar$$

$$[\tilde{p}_x, \tilde{y}] = 0$$

$$[\tilde{p}_y, \tilde{y}] = -i\hbar$$

$$[\tilde{p}_y, \tilde{x}] = 0$$

$$= \frac{\tilde{p}_x^2 + \tilde{p}_y^2}{2m} + \frac{m\omega^2}{2}\left[\frac{3}{2}\tilde{x}^2 + \frac{\tilde{y}^2}{2}\right]$$

$$E_{n,l,k} = \underbrace{\frac{\hbar}{2}\sqrt{\frac{3}{2}}\omega\left(n + \frac{1}{2}\right) + \frac{\hbar}{2}\sqrt{\frac{1}{2}}\omega\left(l + \frac{1}{2}\right)}_{x,\,y} + \underbrace{\frac{\hbar}{2}\omega\left(k + \frac{1}{2}\right)}_{z}$$

Solution:

First step: Write down schrödinger equation:

$$\left(\frac{\hat{\vec{p}}^2}{2m} + V(r)\right)\psi = E\,\psi(\vec{r})$$

$$\left(-\frac{\nabla^2}{2m} + V(r)\right)\psi = E\,\psi(\vec{r})$$

$$\left(\nabla^2 - 2m\,V(r)\right)\psi = -2mE\,\psi(\vec{r})$$

using: $\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{\hat{L}^2}{r^2}$

$$\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{\hat{L}^2}{r^2} + 2m\left(E - V(r)\right)\right]\psi(\vec{r}) = 0$$

we choose $\psi(\vec{r}) = R(r)\,Y_{lm}(\theta,\varphi)$ due to it's a spherical symmetrical potential

For R component:

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}R(r)\right) - \frac{l(l+1)}{r^2}R(r) + 2m\left(E - V(r)\right)R = 0.$$

set: $R = \frac{\chi(r)}{r}$

$$\chi'' + 2m\left[E - V(r) - \frac{l(l+1)}{2mr^2}\right]\chi = 0$$

The condition for the min. value of e will arise when there is only one bound state, which will obviously be the ground state, with $l=0$.

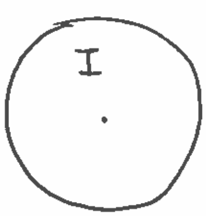
$$\chi'' + 2m\left(E - V(r)\right)\chi = 0$$

The boundary conditions we have:

$$\chi(0) = 0 \quad , \quad \chi(\infty) = 0$$

$\delta(|\vec{r}| - a) \rightarrow$ two conditions.

II.



Bound state $E < 0$.

1° I: $\chi'' + 2mE\chi = 0$ $\quad \boxed{-k^2 = 2mE}$

$$\chi_{I}(r) = A\, sh(kr)$$

2° II:

$$\chi_{II}(r) = B e^{-kr}$$

continuous at a: $\boxed{A\ sh(\kappa a) = B e^{-\kappa a}}$

$$\chi'' + 2mE\,\chi = 2m\,V(r)\,\chi$$

$$\int_{a-\varepsilon}^{a+\varepsilon} \chi''\,dr + 2mE \int_{a-\varepsilon}^{a+\varepsilon} \chi\,dr = 2m \int_{a-\varepsilon}^{a+\varepsilon} V(r)\,\chi(r)\,dr$$

$$\chi'_{II}(a) - \chi'_{I}(a) = 2m\,(-c)\,V(a)$$

$$\chi'_{II}(r) = -\kappa B e^{-\kappa r}$$

$$\chi'_{I}(r) = \kappa A\, ch(\kappa r)$$

$$-\kappa B e^{-\kappa a} - \kappa A\, ch(\kappa a) = -2mc B e^{-\kappa a}$$

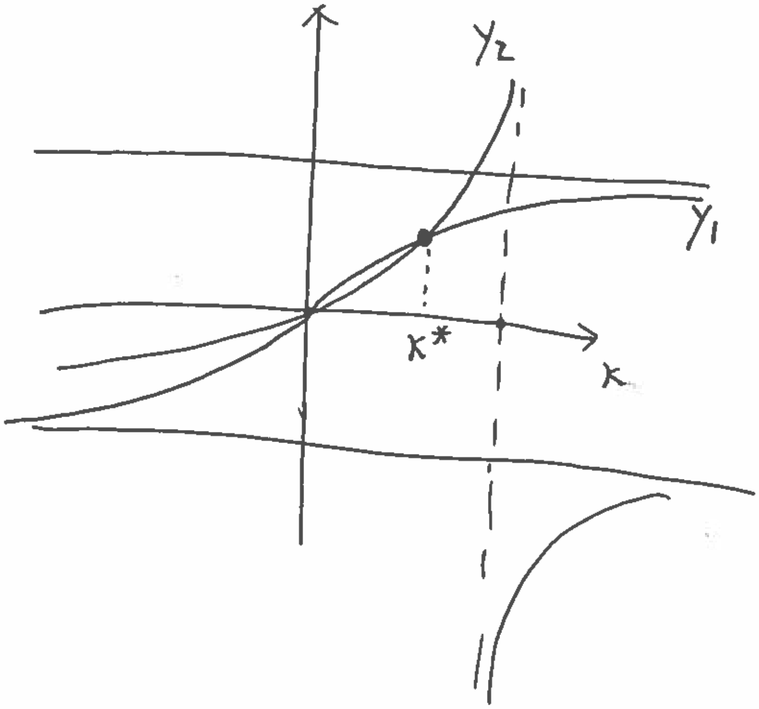
$$(2mcB - \kappa B)\,e^{-\kappa a} = \kappa A\, ch(\kappa a)$$

$$\Rightarrow \quad (2mc - \kappa)\,B e^{-\kappa a} = \kappa A\, ch(\kappa a)$$

$$(2mc - \kappa)\,\cancel{A}\,sh(\kappa a) = \kappa\,\cancel{A}\,ch(\kappa a)$$

$$th(\kappa a) = \frac{\kappa}{2mc - \kappa}$$ — ½

v.



$$y_2 = -1 - \frac{2mc}{x - 2mc}$$

the condition :
$$\left. y_2' \right|_{x=0} < \left. y_1' \right|_{x=0}$$

$$\left. \frac{2mc}{(x-2mc)^2} \right|_{x=0} < \left. \frac{a}{ch(xa)} \right|_{x=0}$$

$$\boxed{c > \frac{1}{2ma}}$$

restore $\hbar$, $c > \dfrac{\hbar^2}{2ma}$