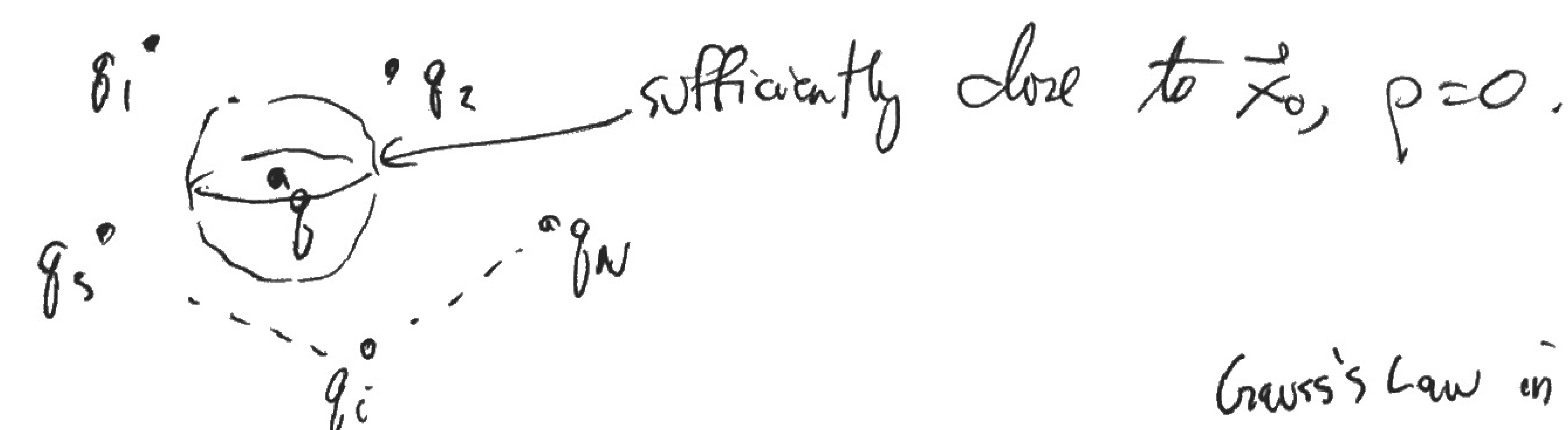
Problem Set #3

#1) Consider a collection of point charges, and let q be one of them, where q is located at the position $\vec{x}_0$.

For this particle to be in stable equilibrium, the force lines near this point ($\vec{x}_0$) acting on q from all the other charges (i.e., not including q) must point towards $\vec{x}_0$. Said mathematically, we need $\nabla\cdot\vec{F} < 0$ near $\vec{x}_0$.

Now, $\nabla\cdot\vec{F} = q\nabla\cdot\vec{E}$ where $\vec{E}$ is the $\vec{E}$-field created by all of the other point charges. But, since there is a finite distance between point charges, we can assume that close enough to $\vec{x}_0$ we have $\rho = 0$ (see picture:)

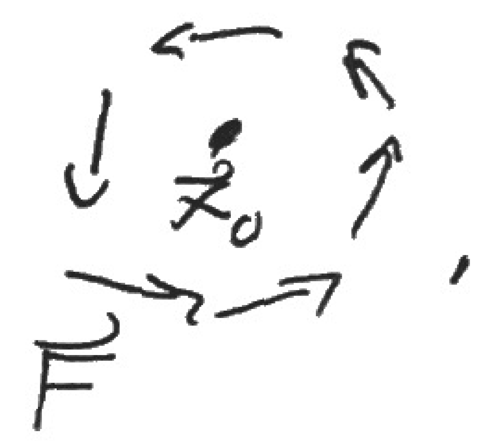


Gauss's Law in differential form.

Thus, near enough to $\vec{x}_0$, we have $\nabla\cdot\vec{E} = \frac{\rho}{\varepsilon_0} = 0$.

Thus, near $\vec{x}_0$, $\nabla\cdot\vec{F} = q\nabla\cdot\vec{E} = 0$, so that $\nabla\cdot\vec{F}$ can't be negative, as it must be. Therefore no stable equilibrium can be obtained.

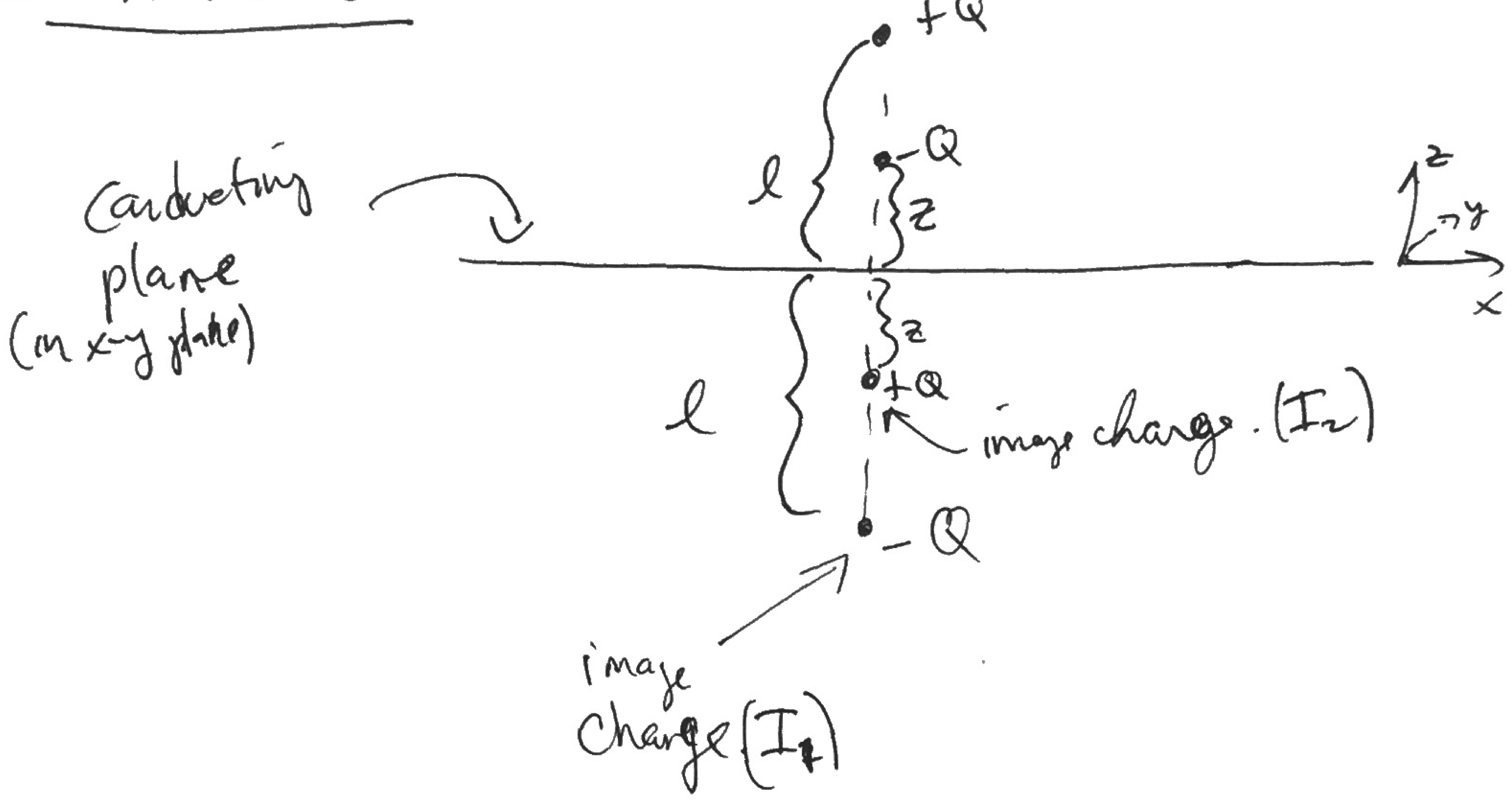
More rigorously, if $\nabla \cdot \vec{F} = 0$ (as we just proved), we might also call a force field that spirals around $\vec{x}_0$ as a stable force:



For this, we would need $\nabla \times \vec{F} \neq 0$.

However, we know $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ so that when there are no magnetic fields (as is the case when we only have stationary point charges) then $\nabla \times \vec{E} = 0$. Thus $\nabla \times \vec{F} = q \nabla \times \vec{E} = 0$ so that this possible loophole is also taken care of.

#2) PM 3.38



If we place a $-Q$ pt. charge a distance z above the plane, then our system is equivalent to that of 4 pt charges: The initial $+Q$, its image charge $-Q$ a distance l below the plane, the $-Q$ that we are trying to place, and its image $+Q$ charge. We note that such a z should exist: for $z \simeq 0$, the attraction between $-Q$ and its $+Q$ image will dominate, pulling the $-Q$ down, and for $z \lesssim l$ the attraction between the initial $+Q$ and our $-Q$ will dominate, pulling the $-Q$ upwards. Thus, somewhere in between should be our point of zero force.

We have:

$$F_{on -Q} = \frac{Q^2}{4\pi\varepsilon_0}\left(\underbrace{\frac{1}{(l-z)^2}}_{\text{from initial } +Q} + \underbrace{\frac{1}{(l+z)^2}}_{\text{from image } I_1} - \underbrace{\frac{1}{(2z)^2}}_{\text{from image } I_2}\right).$$

We want $F_{on-Q} = 0$ so

$$\frac{1}{(2z)^2} = \frac{1}{(l-z)^2} + \frac{1}{(l+z)^2} \iff 4z^2 = \frac{(l-z)^2(l+z)^2}{l^2+2lz+z^2+l^2-2lz+z^2}$$

$$\iff 8z^2 = \frac{(l^2-z^2)^2}{l^2+z^2} \iff 8l^2z^2+8z^4 = l^4+z^4-2l^2z^2$$

$$\iff 7z^4+10l^2z^2-l^4=0.$$

→

This is a quadratic equation for z^2:

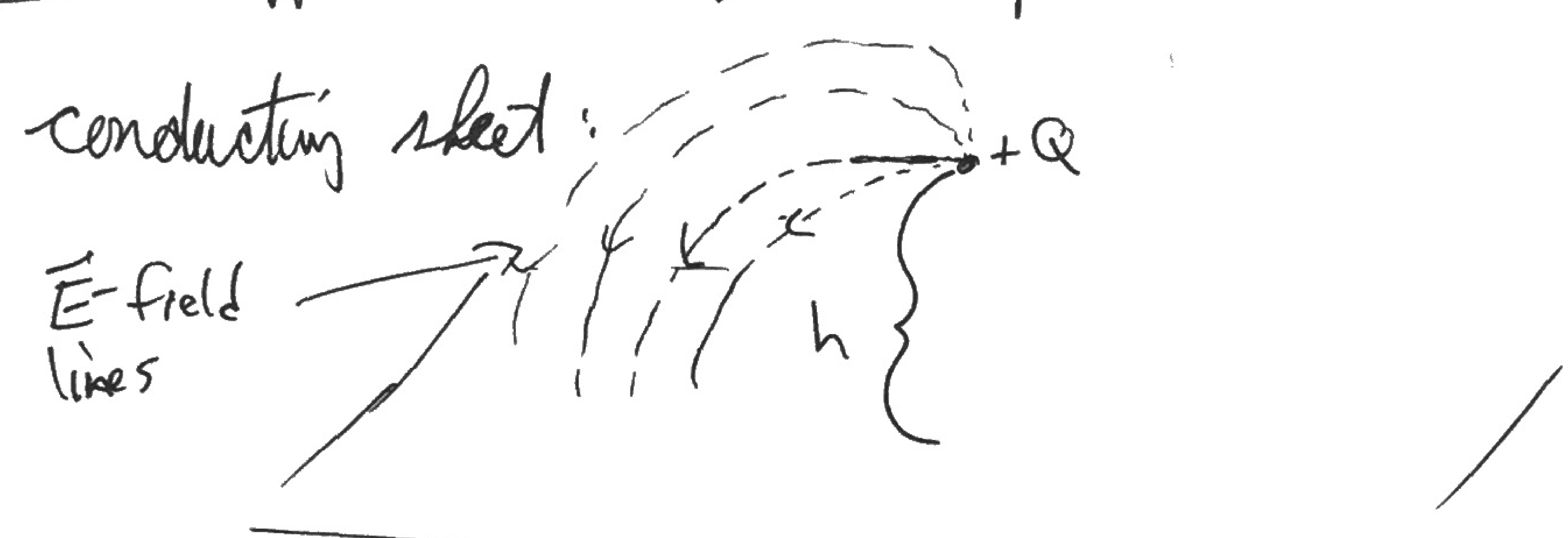
$$z^2 = \frac{1}{14}\left(-10\ell^2 \pm \sqrt{100\ell^4 + 28\ell^4}\right) = \frac{\ell^2}{14}\left(-10 + \sqrt{64\cdot 2}\right) = \frac{\ell^2}{14}\left(-10 + 8\sqrt{2}\right)$$

take positive answer because $z^2 \geq 0$.

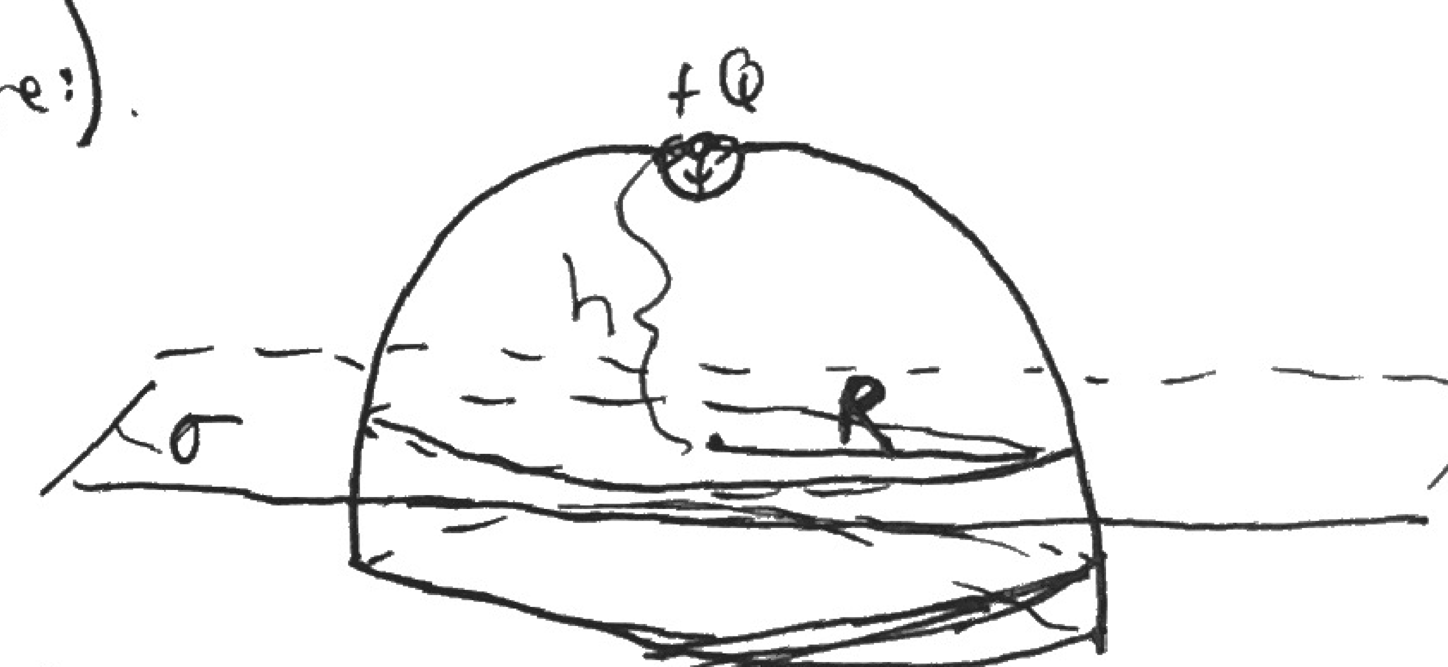
$$\Rightarrow \quad z^2 = \left(\frac{4}{7}\sqrt{2} - \frac{5}{7}\right)\ell^2 \Rightarrow \boxed{z \approx .31\,\ell}$$

(Note: There are also the (trivial) solutions of $z = 0, \ell$)
(very trivial though)

#3) PM 3.41) Suppose a charge Q is placed a distance h above an infinite conducting sheet:



Choose the Gaussian surface that follows the horizontal $\vec{E}$-field lines, closer below the conducting plane (where $\vec{E}=0$), but which forms a hemisphere around the charge (see picture).



Then, the only place on the Gaussian surface that ~~gets~~ electric flux is from the small hemisphere near the charge, and, since we know it is half of the full flux from Q, we have

$$\int_{\text{full surface}} \vec{E}\cdot d\vec{A} = -\frac{Q}{2\varepsilon_0}$$

where the negative sign comes from the fact that the flux goes into the surface. But we also know $\int \vec{E}\cdot d\vec{A} = \frac{Q_{enc}}{\varepsilon_0}$ and $Q_{enc} = 2\pi\int_0^R \sigma\, r\,dr$ where σ is the charge density on the plane, given by eq (3.4) in PM: $\sigma(r) = \frac{-Qh}{2\pi(r^2+h^2)^{3/2}}$.

Thus

$$\frac{-Q}{2\varepsilon_0} = \frac{Q_{enc}}{\varepsilon_0} = -\frac{2\pi}{\varepsilon_0}\int_0^R \frac{Qh\,r\,dr}{2\pi(r^2+h^2)^{3/2}} \Rightarrow \frac{Q}{2} = Qh\int_0^R \frac{r\,dr}{(r^2+h^2)^{3/2}}$$

$\longrightarrow$

$$\Rightarrow \quad \frac{1}{2} = h \int_0^R \frac{r\,dr}{(r^2+h^2)^{3/2}}$$

Let $u = r^2 + h^2$

$du = 2r\,dr$

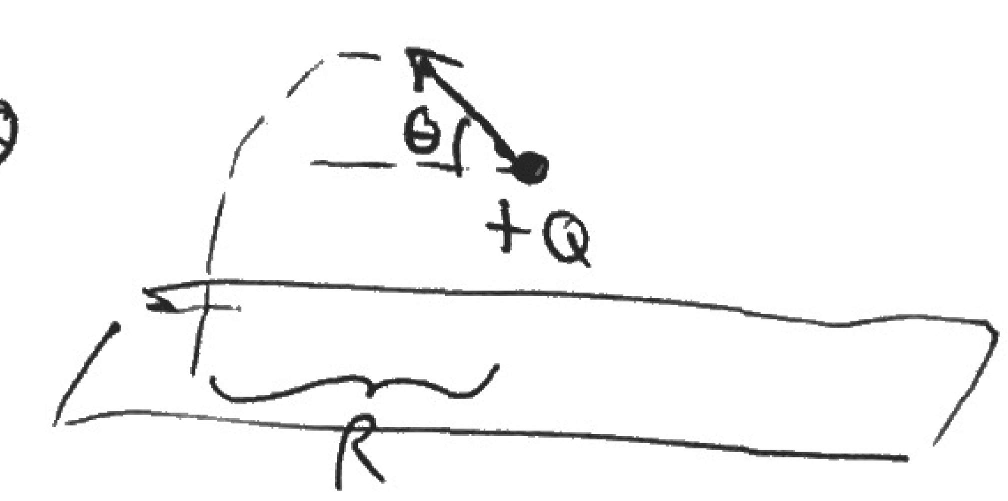
$$\Rightarrow \quad \frac{1}{2} = \frac{h}{2} \int_{h^2}^{R^2+h^2} \frac{du}{u^{3/2}} \Rightarrow 1 = -h \cdot 2\, u^{-1/2} \Big|_{h^2}^{R^2+h^2}$$

$$\Rightarrow \quad 1 = 2h\left(\frac{1}{h} - \frac{1}{\sqrt{R^2+h^2}}\right) \Rightarrow 1 = 2 - \frac{2h}{\sqrt{R^2+h^2}}$$

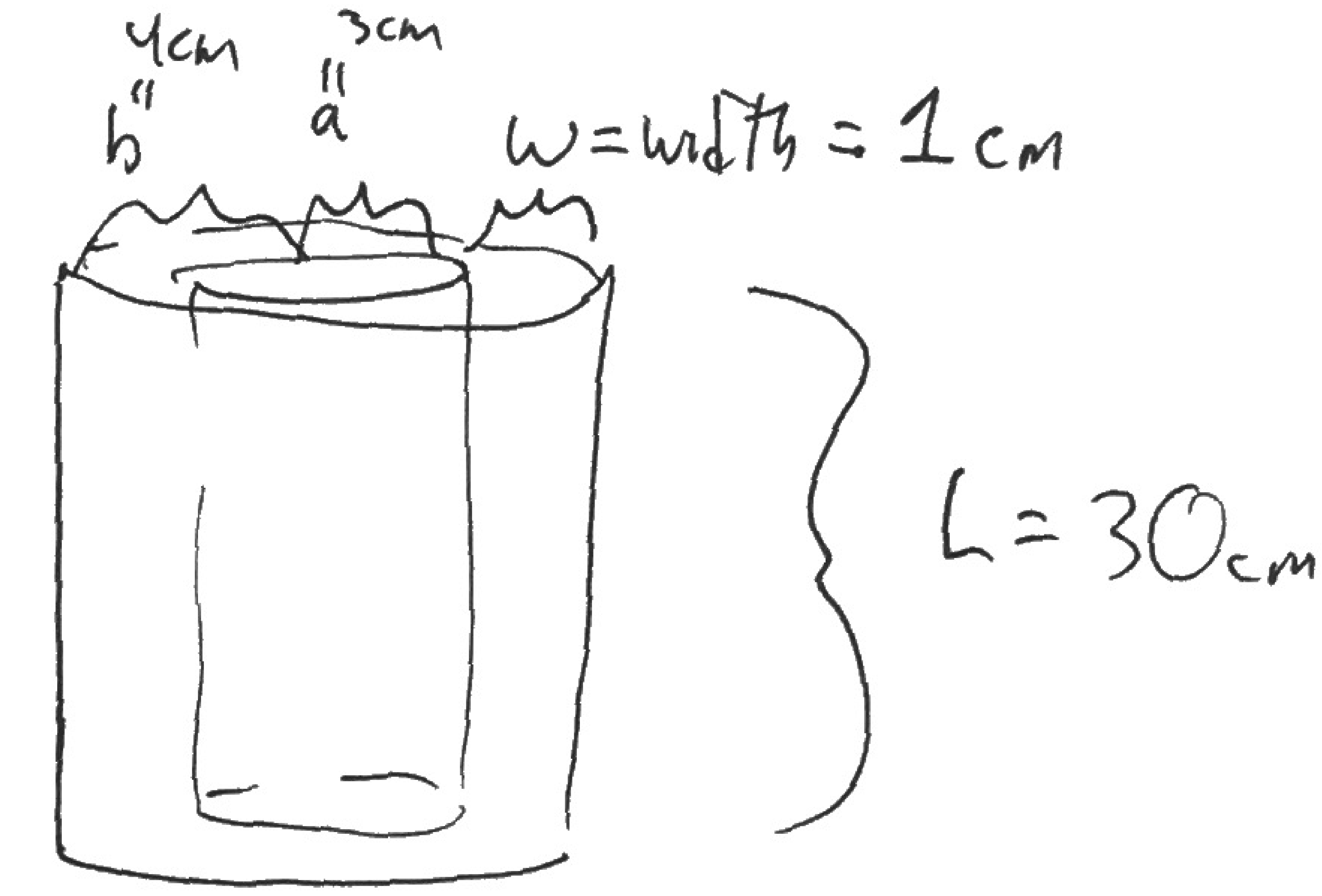
$$\Rightarrow \quad \frac{2h}{\sqrt{R^2+h^2}} = 1 \Rightarrow h = \frac{1}{2}\sqrt{R^2+h^2} \Rightarrow h^2 = \frac{1}{4}R^2 + \frac{1}{4}h^2$$

$$\Rightarrow \quad \frac{3}{4}h^2 = \frac{1}{4}R^2 \Rightarrow \boxed{R = \sqrt{3}\,h}$$

Question: how can we solve this if we care about where the field lines that leave $+Q$ at an angle θ hit the conducting plane?



#4 PM 3.67



V = Voltage between cables = 45 volts

Since $L \gg w$, we can treat the field inside as that of an infinite cable, i.e, there are no fringing effects at the ends.

~~Let~~ There are a couple ways to solve this, and we'll go the route of computing the capacitance C of the system, for then $\text{Energy} = \frac{1}{2}CV^2$.

We compute the capacitance by placing $+Q$ on the inner ~~plate~~ cable, $-Q$ on the outer cable, ~~and~~ computing the potential difference ~~of~~ between them, ($= V$) to then give ~~and we have~~ $C = \frac{Q}{V}$.

As usual, choosing a Gaussian ~~surface~~ cylinder of length l and radius $a<r<b$, we have $\oint \vec{E}\cdot d\vec{A} = E\cdot 2\pi r l = \frac{Q_{enc}}{\varepsilon_0} = \frac{\lambda \cdot l}{\varepsilon_0}$ so that $\vec{E} = \frac{\lambda}{2\pi\varepsilon_0 r}\hat{r}$

where $\lambda = \frac{Q}{L}$. So $\vec{E} = \frac{Q}{2\pi\varepsilon_0 L r}\hat{r}$ so that the (magnitude of the) potential difference V between $r=a$ and $r=b$ is $V = |\phi| = \left|-\int_a^b \vec{E}\cdot d\vec{l}\right| = \frac{\lambda}{2\pi\varepsilon_0}\left|-\int_a^b \frac{dr}{r}\right|$

$\Rightarrow V = \frac{\lambda}{2\pi\varepsilon_0}\ln\left(\frac{b}{a}\right) = \frac{Q}{2\pi\varepsilon_0 L}\ln\left(\frac{b}{a}\right)$.

Thus $\boxed{C = \frac{Q}{V} = \frac{2\pi\varepsilon_0 L}{\ln(b/a)}}$ So $\text{Energy} = \frac{1}{2}CV^2 = \frac{\pi\varepsilon_0 L}{\ln(b/a)}(45\,\text{Volts})^2$

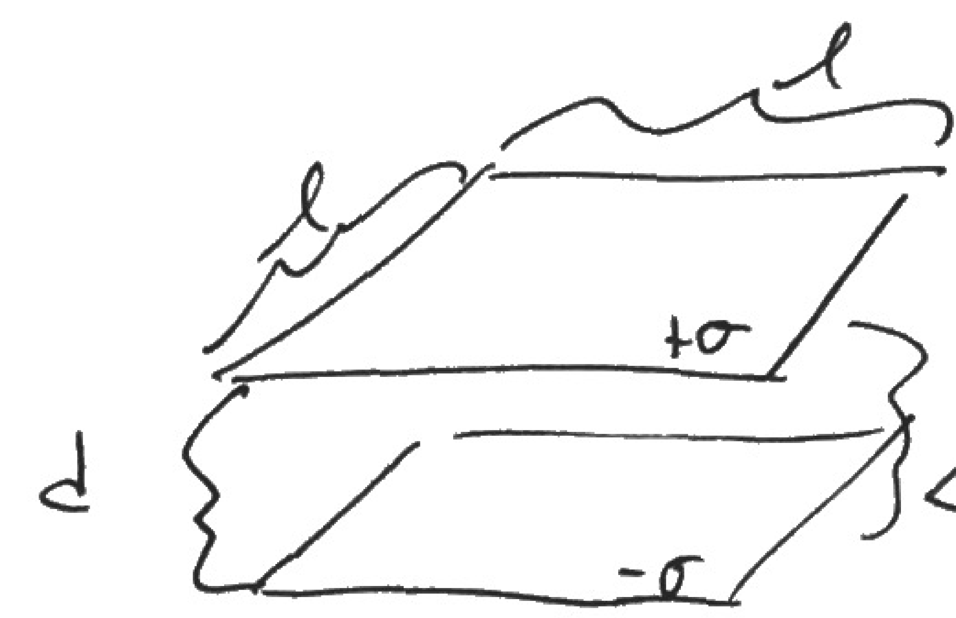
plugging in numbers $\Longrightarrow$ $\boxed{\text{Energy} = 5.9\times10^{-8}\ \text{J}}$ $\longrightarrow$

Note: This can also be done by computing the energy directly: $U = \frac{1}{2}\varepsilon_0 \int |\vec{E}|^2 dV$

and expressing your answer in terms of $\phi = V = \frac{Q}{2\pi\varepsilon_0 L}\ln(b/a) = 45\,V$.

You are encouraged to check that this gives the same result.

#5) PM 3.70



$\ell = 20\,cm$

$d = 3\,cm$

$\Delta V = 10$ Volts.

The force on the top plate is $F = QE$ ~~and~~ where $Q = \sigma A$

and where $E = \frac{\sigma}{2\varepsilon_0}$ ← The field from the other plate. ~~which |E| is constant over whole plate~~ Additionally,

we know for parallel plates, $\Delta V = E_c d$. Thus $\Delta V = \frac{\sigma d}{\varepsilon_0} \Rightarrow \sigma = \frac{(\Delta V)\varepsilon_0}{d}$.

↑ The E-field in the capacitor. $(E_c = \frac{\sigma}{\varepsilon_0})$

Thus $F = \sigma A E = \frac{(\Delta V)\varepsilon_0}{d} \cdot A \cdot \frac{\Delta V}{2d} = \frac{\varepsilon_0 \ell^2}{2d^2}(\Delta V)^2 = \frac{\varepsilon_0 \ell^2}{2}\left(\frac{\Delta V}{d}\right)^2$

$= \frac{\varepsilon_0 \ell^2}{2d^2} E_c^2$

$\boxed{\Rightarrow F \simeq 2\times10^{-8}\,N}$

↑ plugging in numbers

b) If the charge on the plates is constant, then the $\vec{E}$-field between the plates E_c is constant and following the above derivation through shows that the force F is also constant. ~~Namely, everything: E, ΔV only depends on σ = constant~~

Namely, $F = (\text{const})\cdot E_c^2 = \text{constant}$.

Thus, the work done in bringing the plates together is $\boxed{W = F\cdot d = \frac{\varepsilon_0 \ell^2}{2d} E_c^2 = \frac{\varepsilon_0 \ell^2}{2d}(\Delta V)^2}$

→

But we know that the Energy stored in the capacitor $= \frac{1}{2} C(\Delta V)^2$

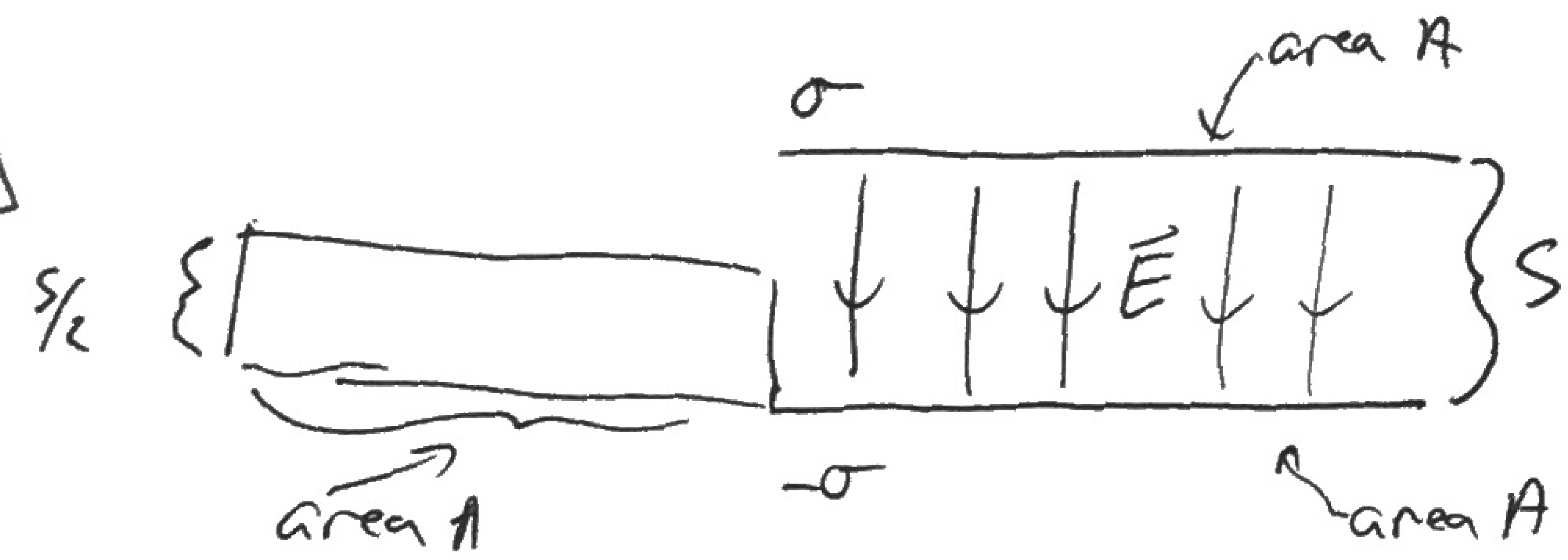
where, for a parallel plate capacitor, $C = \frac{\varepsilon_0 A}{d} = \frac{\varepsilon_0 \ell^2}{d}$.

from previous page

Thus Energy stored in capacitor $= \frac{1}{2} C(\Delta V)^2 = \frac{1}{2} \frac{\varepsilon_0}{d} \ell^2 (\Delta V)^2 = W$ = work to bring the plates together.

#6)

(a)



Initial energy in capacitor: $U_i = \frac{\varepsilon_0}{2}\int E^2\, dV$

(Note: There are several ways to compute this)

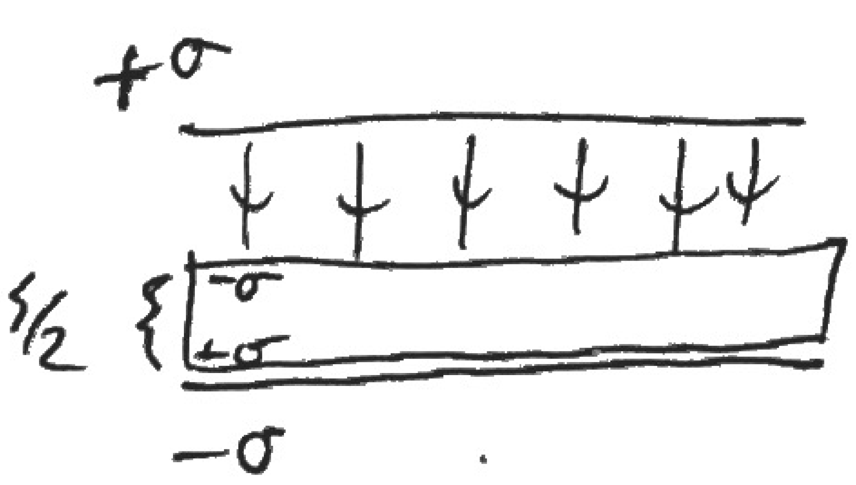
The only place $E \neq 0$ is in the capacitor, where it has the constant value $E = \frac{\sigma}{\varepsilon_0}$. So

$$U_i = \frac{\varepsilon_0}{2}\left(\frac{\sigma}{\varepsilon_0}\right)^2 \underbrace{\int dV}_{\substack{\text{Volume} \\ \text{(of Capacitor)}}}$$

$As \leftarrow$ = Volume of capacitor.

$$\Rightarrow \quad U_i = \frac{\varepsilon_0}{2}\frac{\sigma^2}{\varepsilon_0^2} A\cdot s \Rightarrow \boxed{U_i = \frac{\sigma^2}{2\varepsilon_0} A\cdot s}$$

At the moment the slab is fully inside the capacitor, we have:



recall: The conductor perfectly cancels the $\vec{E}$-field, and therefore must develop a $+\sigma$ on its bottom and a $-\sigma$ on its top.

Now the only place $E \neq 0$ is in the top half of the capacitor, where $E = \frac{\sigma}{\varepsilon_0}$.

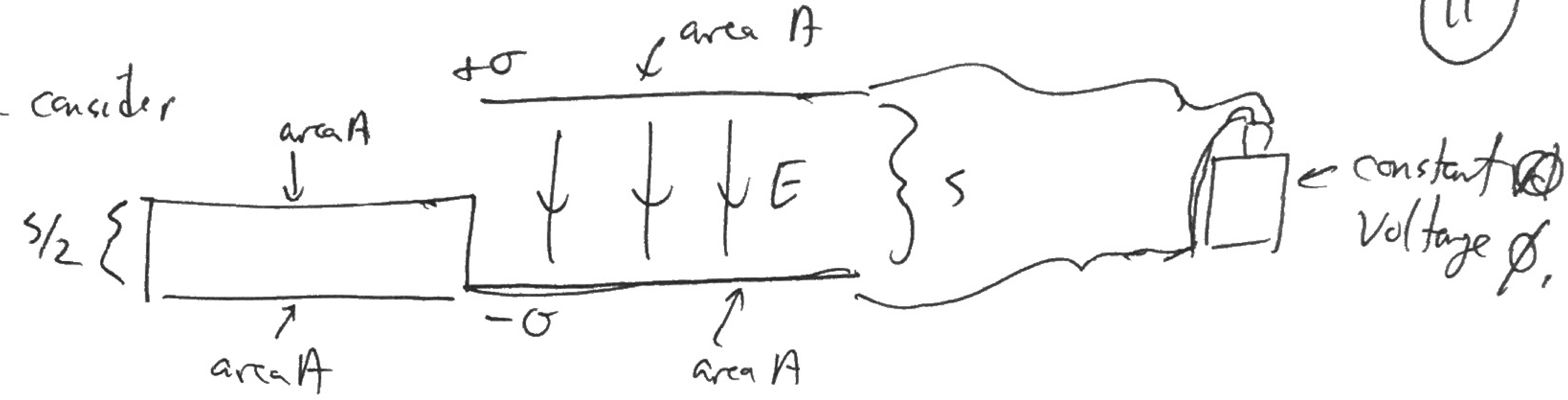
Thus $U_f = \frac{\varepsilon_0}{2}\int E^2\, dV = \frac{\varepsilon_0}{2}\left(\frac{\sigma}{\varepsilon_0}\right)^2 \underbrace{\int_{\text{top half}} dV}_{A\frac{s}{2}} = \frac{\sigma^2}{4\varepsilon_0} As.$

Thus, since $\underbrace{KE_i}_{=0} + U_i = KE_f + U_f \Rightarrow KE_f = U_i - U_f = \frac{\sigma^2 As}{\varepsilon_0}\left(\frac{1}{2} - \frac{1}{4}\right)$

KE_f ← of the slab (the only moving part)

$$\Rightarrow \boxed{KE_f = \frac{\sigma^2 As}{4\varepsilon_0}} \rightarrow$$

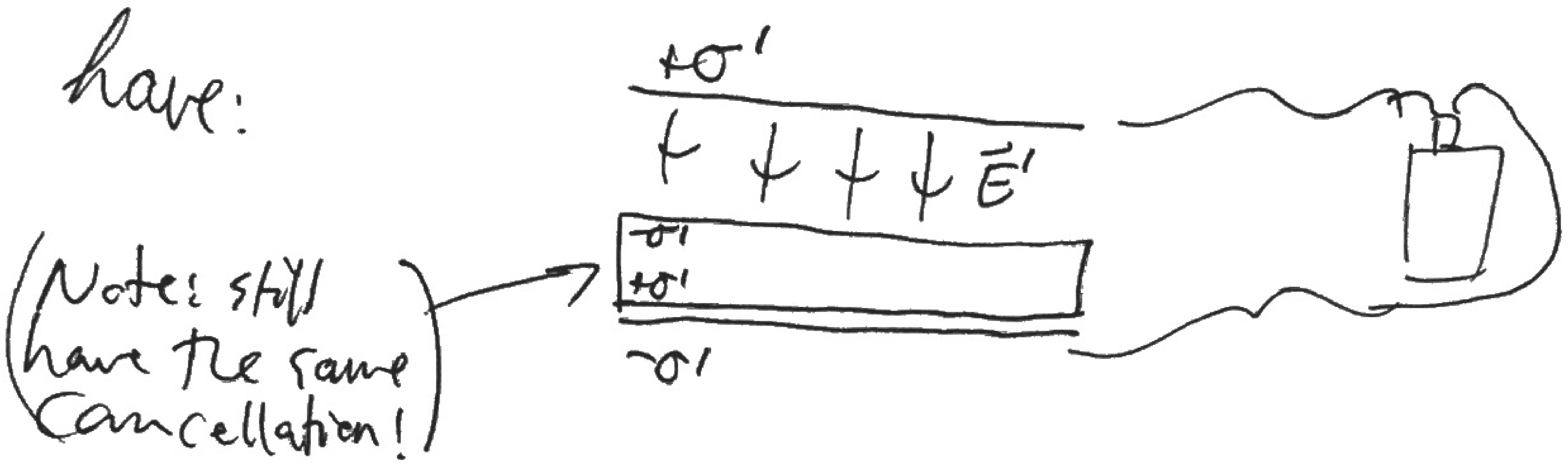
(b) Now, we consider



As before, $\boxed{U_i = \frac{\varepsilon_0}{2}\int E^2 dV = \frac{\varepsilon_0}{2}\left(\frac{\sigma}{\varepsilon_0}\right)^2 A\cdot s = \frac{\sigma^2}{2\varepsilon_0}A\cdot s}$.

After shoving the slab in, the charge densities may have changed (since now it's voltage that we're holding fixed).

Thus we have:



Thus $\underset{\text{of capacitor}}{U_f} = \frac{\varepsilon_0}{2}\underset{\substack{\text{top half}\\ \text{of capacitor}}}{\int}(E')^2 dV = \frac{\varepsilon_0}{2}\left(\frac{\sigma'}{\varepsilon_0}\right)^2 A\cdot\frac{s}{2} = \frac{As}{4\varepsilon_0}(\sigma')^2$

still need to solve for σ'.

Indeed, since $\phi = \text{const}$ (before and after) we have

$$\phi_{before} = E\cdot s = \frac{\sigma}{\varepsilon_0}s = \phi_{after} = E'\cdot\left(\frac{s}{2}\right) = \frac{\sigma'}{\varepsilon_0}\cdot\frac{s}{2} \Longrightarrow \sigma' = 2\sigma.$$

Thus $\boxed{U_f = \frac{As}{4\varepsilon_0}(2\sigma)^2 = \frac{As}{\varepsilon_0}\sigma^2}$

Note: $U_f > U_i$, but energy conservation is not violated because the battery has ~~done~~ done work to move the extra charge ($\sigma' = 2\sigma$) onto the plates. Namely, it has done work to move more positive charge from the bottom (negative) plate to the top plate.

$\longrightarrow$

The total amount of charge it has moved was $Q_{tot} = (\sigma' - \sigma)A$

so $Q_{TOT} = (2\sigma - \sigma)A = \sigma A$ and it has moved this charge across the (constant) voltage ϕ, so that $W_{battery} = Q_{tot}\,\phi = \sigma A \phi$

$= \sigma A \left(\frac{\sigma s}{\varepsilon_0}\right).$

Work done by the battery

Thus $W_{battery} = \sigma^2 \frac{As}{\varepsilon_0}$.

Thus, since $\underbrace{KE_i}_{=0} + U_i + W_{battery} = KE_f + U_f$

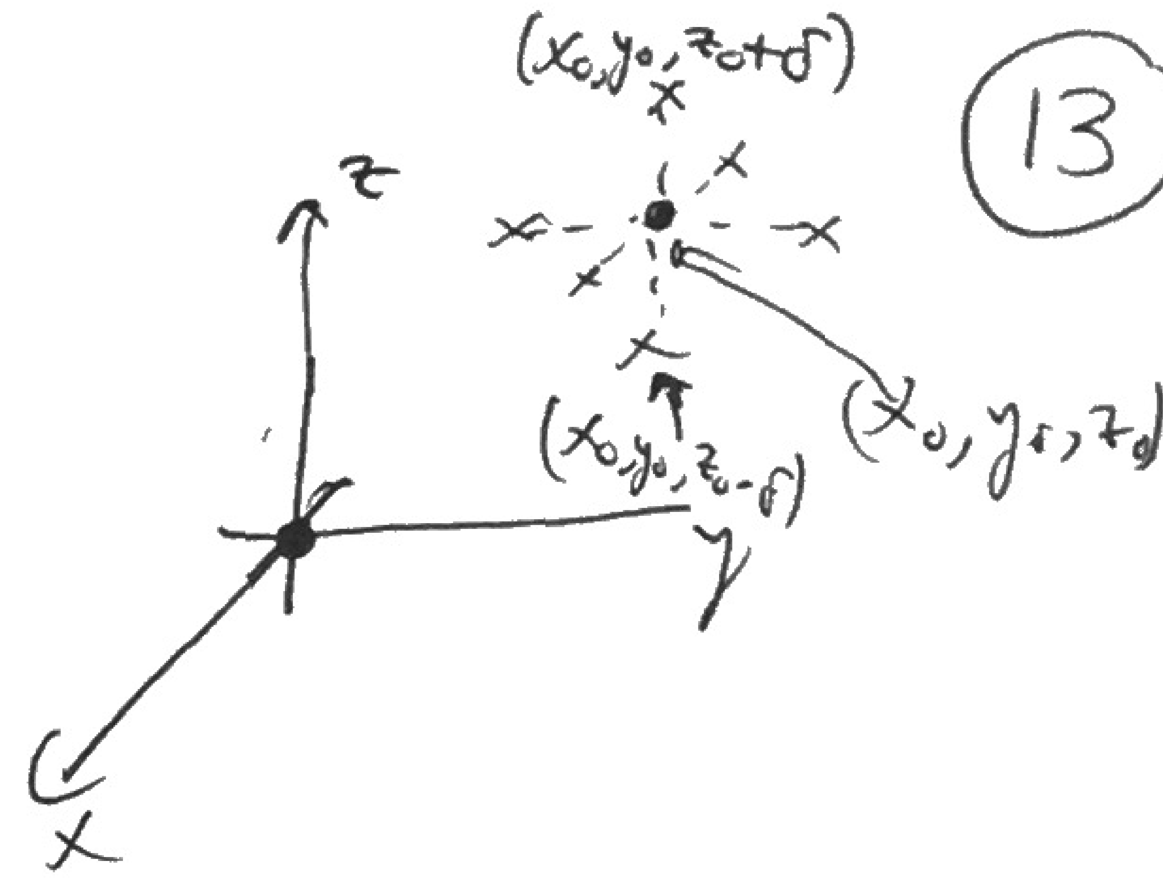
we have $\frac{\sigma^2}{2\varepsilon_0} As + \frac{\sigma^2}{\varepsilon_0} As = KE_f + \frac{\sigma^2}{\varepsilon_0} As \implies \boxed{KE_f = \frac{\sigma^2}{2\varepsilon_0} As}$

Note: This is greater than what we found in part (a), can you make sense of this?

Bonus: Why/how does the slab get pushed into the capacitor in the first place?

(Hint: Look at ~~fringing~~ fringing fields).

#7 PM 3.75



We have

$$\phi(x_0+\delta, y_0, z_0) \simeq \phi(x_0, y_0, z_0) + \delta\, \partial_x \phi(x_0, y_0, z_0) + \frac{1}{2}\delta^2 \partial_x^2 \phi(x_0, y_0, z_0)$$

$$\left(3! = \right) + \frac{1}{6}\delta^3 \partial_x^3 \phi(x_0, y_0, z_0) + \mathcal{O}(\delta^4)$$

$$\phi(x_0-\delta, y_0, z_0) \simeq \phi(x_0, y_0, z_0) - \delta\, \partial_x \phi(x_0, y_0, z_0) + \frac{1}{2}\delta^2 \partial_x^2 \phi(x_0, y_0, z_0)$$

$$- \frac{1}{6}\delta^3 \partial_x^3 \phi(x_0, y_0, z_0) + \mathcal{O}(\delta^4)$$

$$\phi(x_0, y_0+\delta, z_0) \simeq \phi(x_0, y_0, z_0) + \delta\, \partial_y \phi(x_0, y_0, z_0) + \frac{1}{2}\delta^2 \partial_y^2 \phi(x_0, y_0, z_0) + \frac{1}{6}\delta^3 \partial_y^3 \phi(x_0, y_0, z_0) + \mathcal{O}(\delta^4)$$

$$\phi(x_0, y_0-\delta, z_0) \simeq \phi(x_0, y_0, z_0) - \delta\, \partial_y \phi(x_0, y_0, z_0) + \frac{1}{2}\delta^2 \partial_y^2 \phi(x_0, y_0, z_0) - \frac{1}{6}\delta^3 \partial_y^3 \phi(x_0, y_0, z_0) + \mathcal{O}(\delta^4)$$

$$\phi(x_0, y_0, z_0+\delta) \simeq \phi(x_0, y_0, z_0) + \delta\, \partial_z \phi(x_0, y_0, z_0) + \frac{1}{2}\delta^2 \partial_z^2 \phi(x_0, y_0, z_0) + \frac{1}{6}\delta^3 \partial_z^3 \phi(x_0, y_0, z_0) + \mathcal{O}(\delta^4)$$

$$\phi(x_0, y_0, z_0-\delta) \simeq \phi(x_0, y_0, z_0) - \delta\, \partial_z \phi(x_0, y_0, z_0) + \frac{1}{2}\delta^2 \partial_z^2 \phi(x_0, y_0, z_0) - \frac{1}{6}\delta^3 \partial_z^3 \phi(x_0, y_0, z_0) + \mathcal{O}(\delta^4)$$

So, the average of all the points comes from adding all of these terms of. We see that terms with odd powers of δ will cancel pairwise and we have

$$\phi_{average} = \frac{1}{6}\left\{ 6\phi(x_0, y_0, z_0) + \delta^2\left(\partial_x^2 + \partial_y^2 + \partial_z^2\right)\phi(x_0, y_0, z_0) + \mathcal{O}(\delta^4)\right\}$$

Thus, if $\nabla^2 \phi = 0$ (i.e., ϕ satisfies Laplace's equation) we have $\boxed{\phi_{average} = \phi(x_0, y_0, z_0) + \mathcal{O}(\delta^4)}$, as desired.