105A - Set 6

(Grades are out of 150)

1. (30pt) Its late at night, early 17th century, Johannes Kepler is working hard on his notes, the candle light is flickering. He just found that planets orbit the sun in an ellipse and he expressed their radius as a function of the angle. He is trying to give meaning to the constants he found, can he relate these constants to the semi-major axis of the ellipse? All of sudden a figure materialized in front of him. Its a person, that says that she came from the future to help him out with his First law of planetary motion. She draws an ellipse, (see Figure) and says that the relation between the radius and the angle can be worked out from the geometry and one can simply start from the Ellipse equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{1}$$

Unfrouvntently the time traveler had to come back quickly to her own time to preserve the time continuum. Help Johannes Kepler out and show that starting from the above equation you can find:

$$r = \frac{a(1-e^2)}{1+e\cos\theta}\ , \tag{2}$$

where $e = \sqrt{1 - b^2/a^2}$ is the eccentricity.

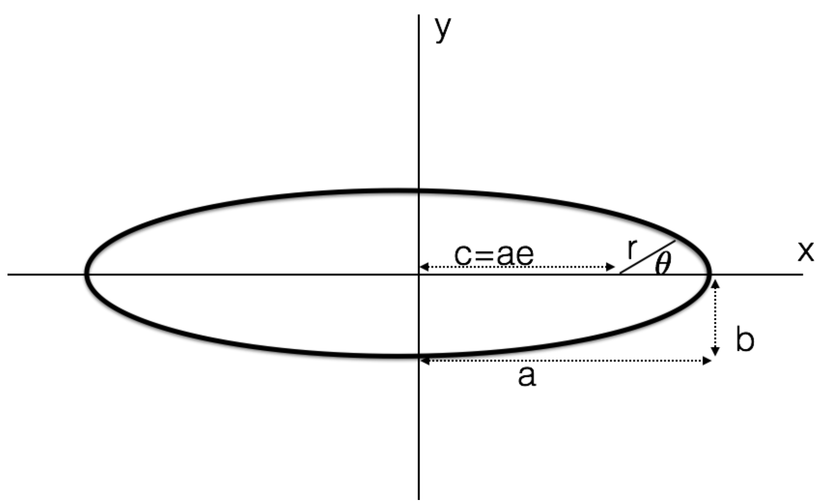

Figure 1: Ellipse

2. (25pt) Assumes Earth's orbit to be circular and that the Sun's mass suddenly decreases by half. What orbit does the Earth then have? Will the Earth escape the solar system?

3. (30pt) Two particles move about each other in circular orbits under the influence of gravitational forces, with a period τ. Their motion is suddenly stopped, and they are then released and allowed to fall into each other. Prove that they collide after a time

$\tau/(4\sqrt{2})$. *Hint 1: find τ for circular motion.*
Hint 2: when you get to $\ddot{r}$ equal to some thing and its hard to solve it, I suggest to multiply by $\dot{r}$ - an alternative way is to think about the energy.

4. (30pt) Consider a comet moving in a parabolic orbit in the plane of Earth's orbit. If the distance of the closest approach of the comment to the sun is $\beta \times r_E$, where r_E is the radius of the Earth's orbit (assumed circular) around the Sun and $\beta < 1$. Show that the time of the comet spends within the orbit of the Earth is given by

$$\sqrt{2(1-\beta)}\frac{1+2\beta}{3\pi} \times 1\text{yr} \tag{3}$$

If the comet approaches the Sun to a distance of the perihelion (closest approach) of Mercury, how many days is it within Earth's orbit?

5. (35pt) **Practice power expansion and initial condition**
A block of mass M is free to slide on a horizontal bar without any friction. A mass m is attached to the bottom of the block with a massless rod of length l and can oscillate freely in the same plane as the horizontal bar.

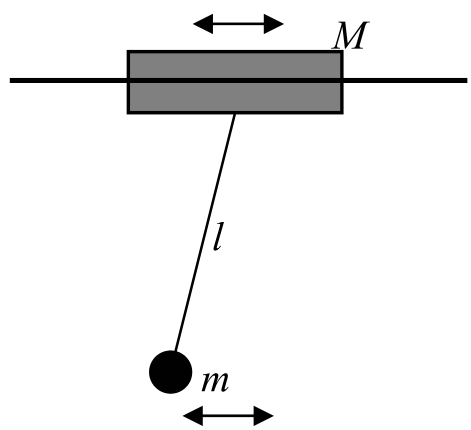

Figure 2: Sliding pendulum

(a) (3pt) Write down the Lagrangian of the system
(b) (3pt) Find the equations of motion.
(c) (8pt) Assume that the pendulum oscillation is constrained to small angles around zero. Write the equation of motion in that case.
(d) (4pt) Find the frequency of the small oscillations of the system.
(e) (10pt) Given the initial conditions of $\theta(t=0) = \theta_0$ and $\dot{\theta}(t=0) = \theta_0\sqrt{g(M+m)/(lM)}$, find $\theta(t)$ as a function of the g, M, m, l and θ_0.
(f) (7pt) Given that $X(t=0) = x_0$ find the equation that describes the movement of the bob in the horizontal direction.