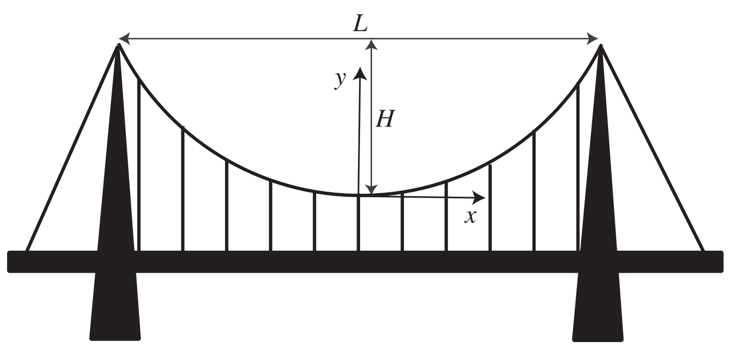
Determine the equilibrium shape $y(x)$ of the suspension cable in a bridge shown in the figure. Assume that the mass of the roadway is much larger than the mass of the cable. The length of the vertical ropes is adjusted so there is no shear stress in the roadway. You can also assume that vertical ropes are spaced by a very small distance.



There are many approaches to this problem. We choose to solve it by writing Newton's equations for a piece of rope with projection dx over the bridge. We define a function $T(x)$ which is the tension along the rope and another $y(x)$ which is just the shape of the rope. Newton's equation in the $\hat{x}$ direction is:

$$-T(x-\frac{dx}{2})\cos\left(\theta(x-\frac{dx}{2})\right)+T(x+\frac{dx}{2})\cos\left(\theta(x+\frac{dx}{2})\right)=0,$$

where $\theta(x)$ is just the angle of the tangent to the rope with respect to the $\hat{x}$ axis at position x. We now expand this in a Taylor series around x to first order and get:

$$\frac{d}{dx}\left(T(x)\cos\theta(x)\right)=0.$$

From this equation we see that $T(x)\cos\theta(x)$ is just a constant we call A. We now need to use Newton's equation for the $\hat{y}$ direction.

$$-\frac{gM}{L}dx-T(x-\frac{dx}{2})\sin\left(\theta(x-\frac{dx}{2})\right)+T(x+\frac{dx}{2})\sin\left(\theta(x+\frac{dx}{2})\right)=0,$$

where $\frac{gM}{L}dx$ is just the weight of the piece of bridge hanging from the piece of rope. M is the total mass of the bridge and L its length. Again, we expand around x, and obtain to first order:

$$\frac{d}{dx}\left(T(x)\sin\theta(x)\right)=\frac{gM}{L}.$$

We can now divide this equation by A. Since it is a constant we can put it inside the derivative in the l.h.s. This yields:

$$\frac{d}{dx}\left(\frac{T(x)\sin\theta(x)}{T(x)\cos\theta(x)}\right)=\frac{gM}{LA}.$$

Now we use that $\frac{dy}{dx}=\tan\theta(x)$. We obtain:

$$\frac{d^2y}{dx^2}=\frac{gM}{LA}.$$

This means that $y(x)=\frac{gM}{2LA}x^2+Bx+C$, where B and C are just constants. B is determined to be zero by fixing the extremes of the rope at, say, $x=-\frac{L}{2}$ and $x=\frac{L}{2}$. C is just related to the origin of coordinates for y, so we can take it to be zero (this amounts to taking the origin at the point where rope lies in the middle of the bridge). Finally, A is determined by knowing the height of the bridge towers with respect to the origin of coordinates. If we take this height to be H,

$$H=\frac{gML}{8A}.$$

The final solution is then

$$y(x)=\frac{4H}{L^2}x^2.$$