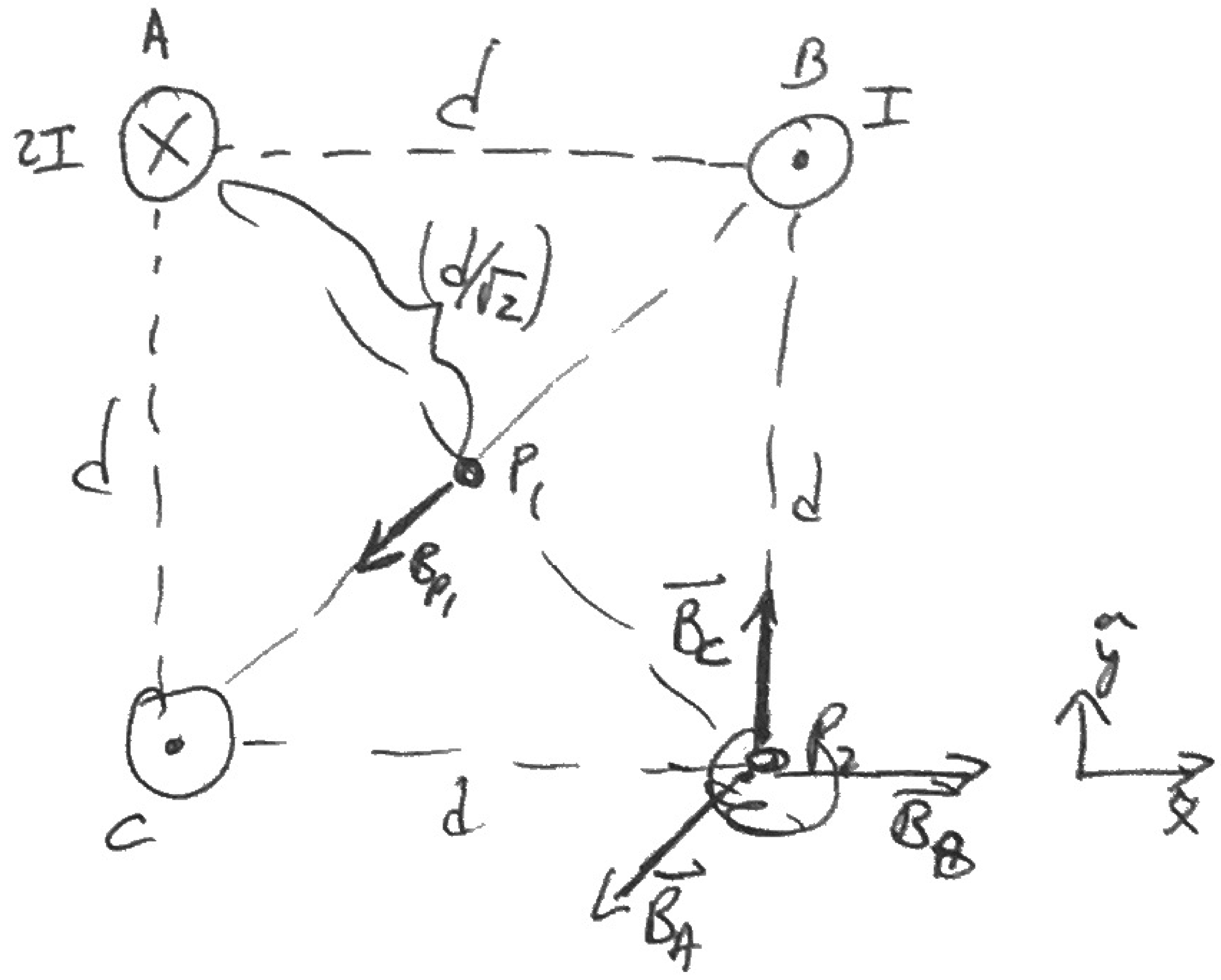
Problem Set #5 Solutions

#1) PM 6.31 We know that the $(\vec{B})$ field a distance r from a wire with current I is given by $\vec{B} = \frac{\mu_0}{2\pi r} I \hat{\phi}$ where $\hat{\phi}$ signifies curling around the wire according to the right hand rule.



At P_1, therefore, the contribution from wire B and C cancel each other, so that only wire A contributes and it does so with

$$B_{P_1} = \frac{\mu_0}{2\pi (d/\sqrt{2})} \cdot (2I)$$

$$\Rightarrow \boxed{B_{P_1} = \frac{\sqrt{2}\,\mu_0 I}{\pi d} \text{ and it is pointing towards wire C}}$$

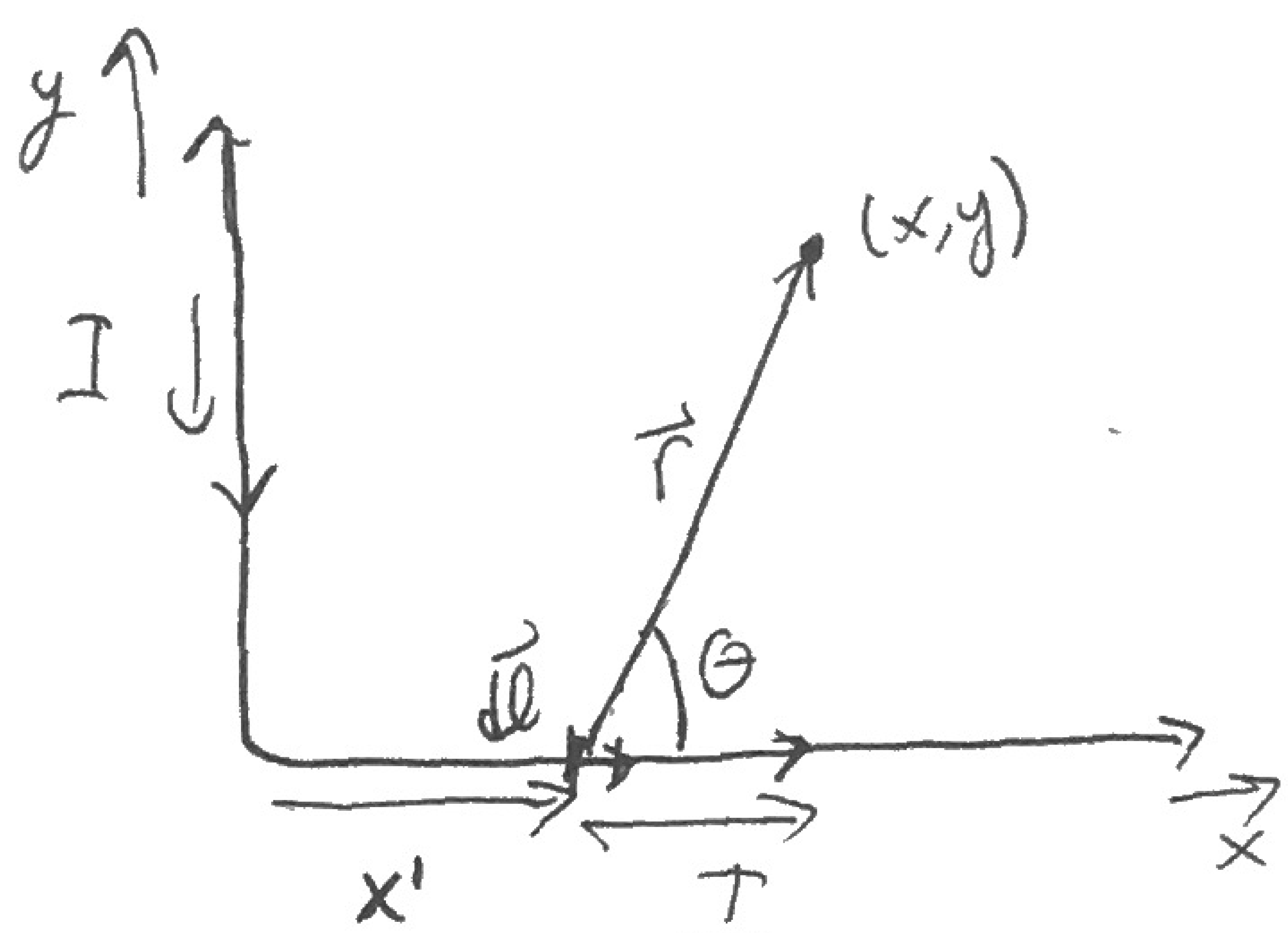
Call the fields produced by wires A, B, and C at point P_2 $\vec{B}_A, \vec{B}_B, \vec{B}_C$, respectively (as shown).

$\vec{B}_C = \frac{\mu_0 I}{2\pi d}\hat{y}$ (upwards), $\vec{B}_B = \frac{\mu_0 I}{2\pi d}\hat{x}$ (to the right)

$$\vec{B}_A = \frac{\mu_0 \cdot 2I}{2\pi \cdot (\sqrt{2}d)} \underbrace{\left(\frac{-\hat{x}-\hat{y}}{\sqrt{2}}\right)}_{\text{unit vector diagonally down and to the left.}} = \frac{-\mu_0 I}{2\pi d}(\hat{x}+\hat{y})$$

Thus $\vec{B}_{P_2} = \vec{B}_A + \vec{B}_B + \vec{B}_C = \frac{\mu_0 I}{2\pi d}(\hat{x}+\hat{y}-(\hat{x}+\hat{y})) = 0. \Rightarrow \boxed{\vec{B}_{P_2} = 0}$

#2) PM 6.52



Choose a point (x,y) in the plane (not on a positive axis).

Consider first the contribution from the part of the wire lying on the x-axis. Call this contribution $\vec{B}_1$ and that from the y-axis $\vec{B}_2$. Then

$\vec{B}_{TOT}(x,y) = \vec{B}_1 + \vec{B}_2$. Note that no matter which of the 4 quadrants (x,y) is in, $\vec{B}_1$ and $\vec{B}_2$ will point in the same direction. For $\vec{B}_1$, from Biot-Savart, we have

$$\vec{B}_1 = \frac{\mu_0 I}{4\pi}\int \frac{d\vec{l}\times\hat{r}}{r^2} \quad \text{with } d\vec{l} = dx'\,\hat{x}'$$

$$\vec{r} = (x,y,0)-(x',0,0)$$

$$\Rightarrow \vec{r} = (x-x', y, 0)$$

so that $r = \sqrt{(x-x')^2+y^2}$.

unit vectors

Now, $d\vec{l}\times\hat{r} = |d\vec{l}||\hat{r}|\sin\theta\,\hat{z} = \sin\theta\,\hat{z}\,dx'$

so $$\vec{B}_1 = \frac{\mu_0 I}{4\pi}\hat{z}\int_{x'=0}^{x'=\infty}\frac{\sin\theta\,dx'}{(x-x')^2+y^2}.$$ Now, $\sin\theta = \frac{y}{r} = \frac{y}{\sqrt{(x-x')^2+y^2}}$

$$\Rightarrow \vec{B}_1(x,y) = \frac{\mu_0 I}{4\pi}\hat{z}\cdot y\int_0^\infty \frac{dx'}{\left((x-x')^2+y^2\right)^{3/2}}.$$

Let $u = x'-x \rightarrow$ $$\vec{B}_1(x,y) = \frac{\mu_0 I}{4\pi}y\,\hat{z}\int_{-x}^{\infty}\frac{du}{(u^2+y^2)^{3/2}}$$

Now $\vec{B}_1(x,y) = \frac{\mu_0 I}{4\pi}\hat{z}\,\frac{1}{y^2}\int_{-x}^{\infty}\frac{du}{\left(1+\left(\frac{u}{y}\right)^2\right)^{3/2}}$. Let $\frac{u}{y} = \tan\theta$

$\Rightarrow du = y\sec^2\theta\, d\theta$

$$\Rightarrow \vec{B}_1(x,y) = \frac{\mu_0 I}{4\pi}\hat{z}\,\frac{1}{y}\underbrace{\int_{\tan^{-1}\left(\frac{-x}{y}\right)}^{\pi/2}\frac{\sec^2\theta\, d\theta}{(\underbrace{1+\tan^2\theta}_{\sec^2\theta})^{3/2}}}$$

$$\int_{\tan^{-1}\left(\frac{-x}{y}\right)}^{\pi/2}\cos\theta\, d\theta = \sin\theta\Big|_{\tan^{-1}\left(\frac{-x}{y}\right)}^{\pi/2} = 1-\sin\left(\tan^{-1}\left(-\frac{x}{y}\right)\right).$$

Cool trick: $\sin(\tan^{-1}(\alpha)) = ?$ Let $\phi = \tan^{-1}(\alpha)$. Then

$$\sin(\tan^{-1}(\alpha)) = \sin\phi = \underbrace{(\tan\phi)(\cos\phi)}_{=\frac{\sin\phi}{\cos\phi}\cos\phi} = \tan(\tan^{-1}(\alpha))\cos\phi = \alpha\cos\phi = \alpha\sqrt{1-\sin^2\phi}$$

$$\Rightarrow \sin\phi = \alpha\sqrt{1-\sin^2\phi} \Rightarrow \sin^2\phi = \alpha^2(1-\sin^2\phi)$$

$$\Rightarrow \sin^2\phi(1+\alpha^2) = \alpha^2 \Rightarrow \sin\phi = \pm\frac{\alpha}{\sqrt{1+\alpha^2}}.$$

Thus, $\sin\left(\tan^{-1}\left(\frac{-x}{y}\right)\right) = \frac{-x/y}{\sqrt{1+\frac{x^2}{y^2}}}$

so $\vec{B}_1(x,y) = \frac{\mu_0 I}{4\pi}\left(1+\frac{x}{y\sqrt{1+\frac{x^2}{y^2}}}\right)\cdot\frac{1}{y}\hat{z} = \frac{\mu_0 I}{4\pi}\left(\frac{1}{y}+\frac{x}{y\sqrt{x^2+y^2}}\right)\hat{z}$,

Now, from symmetry (as you can check explicitly if you'd like), we have that $\vec{B}_2(x,y) = \vec{B}_1(y,x)$, namely, we get $\vec{B}_2$ simply by switching the role of x and y in $\vec{B}_1$ (as should be clear from the picture).

Thus $\boxed{\vec{B}_{TOT}(x,y) = \frac{\mu_0 I}{4\pi}\left(\frac{1}{x}+\frac{1}{y}+\frac{x}{y\sqrt{x^2+y^2}}+\frac{y}{x\sqrt{x^2+y^2}}\right)\hat{z},}$ as desired.

Now for $x \ll y$, we have $\frac{1}{x} \gg \frac{1}{y}$ so we drop $\frac{1}{y}$

and similarly, we have $\frac{y}{x\sqrt{x^2+y^2}} + \frac{x}{y\sqrt{x^2+y^2}} = \frac{1}{\sqrt{x^2+y^2}}\left(\frac{x}{y}+\frac{y}{x}\right)$

and $\frac{y}{x} \gg \frac{x}{y}$ so we drop the $\frac{x}{y}$ term.

Moreover, $\frac{1}{\sqrt{x^2+y^2}} = \frac{1}{y}\left(1+\left(\frac{x}{y}\right)^2\right)^{-1/2} \simeq \frac{1}{y}\left(1-\frac{1}{2}\frac{x^2}{y^2}\right) \simeq \frac{1}{y}$

Thus $\frac{y}{x\sqrt{x^2+y^2}} \simeq \frac{1}{x}$

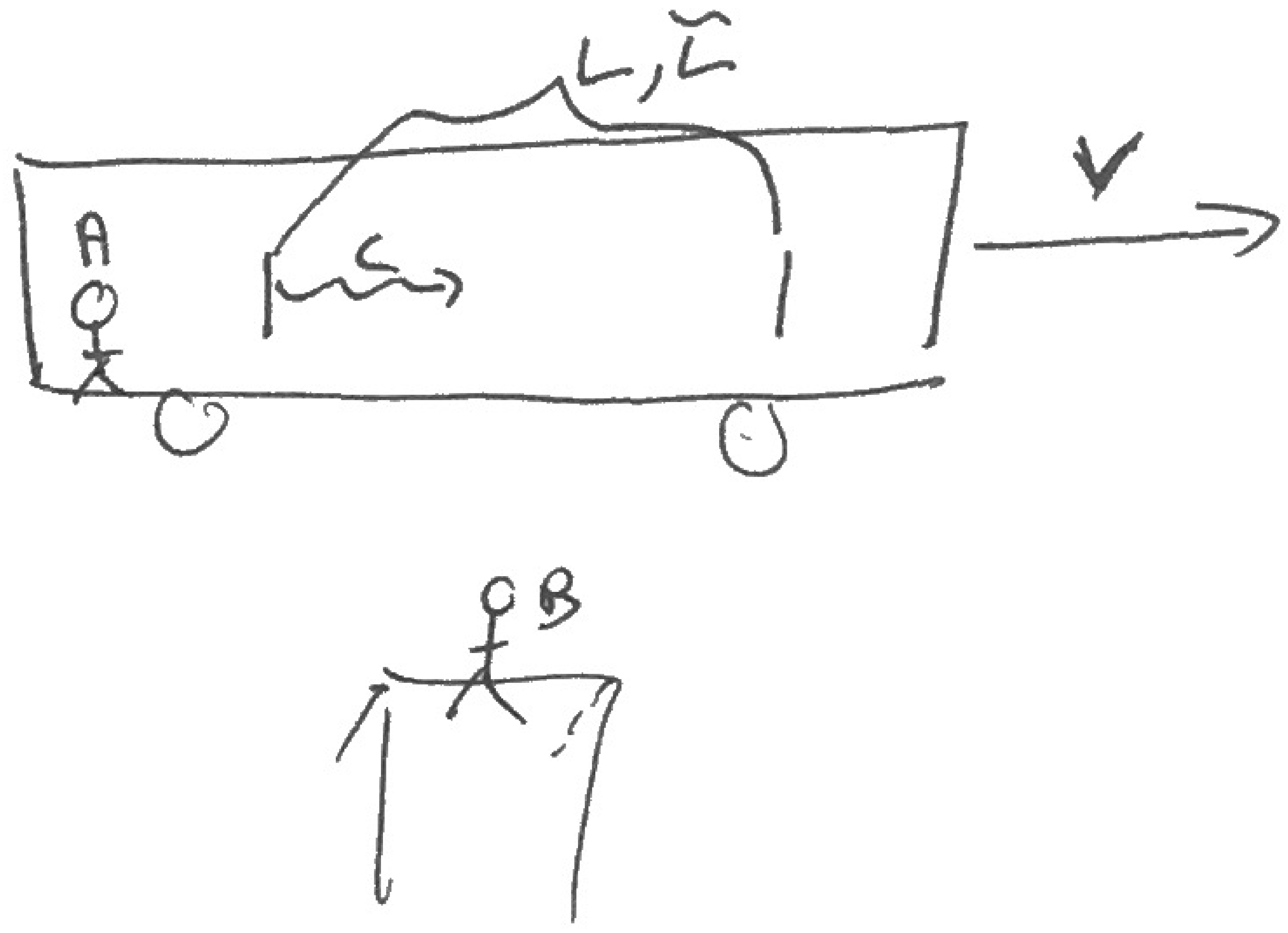
and so we have $B_{TOT}(x \ll y) = \frac{\mu_0 I}{4\pi}\left(\frac{1}{x} + \cancel{\frac{1}{y}} + \cancel{\frac{x}{y\sqrt{x^2+y^2}}} + \underbrace{\frac{y}{x\sqrt{x^2+y^2}}}_{\simeq \frac{1}{x}}\right)$

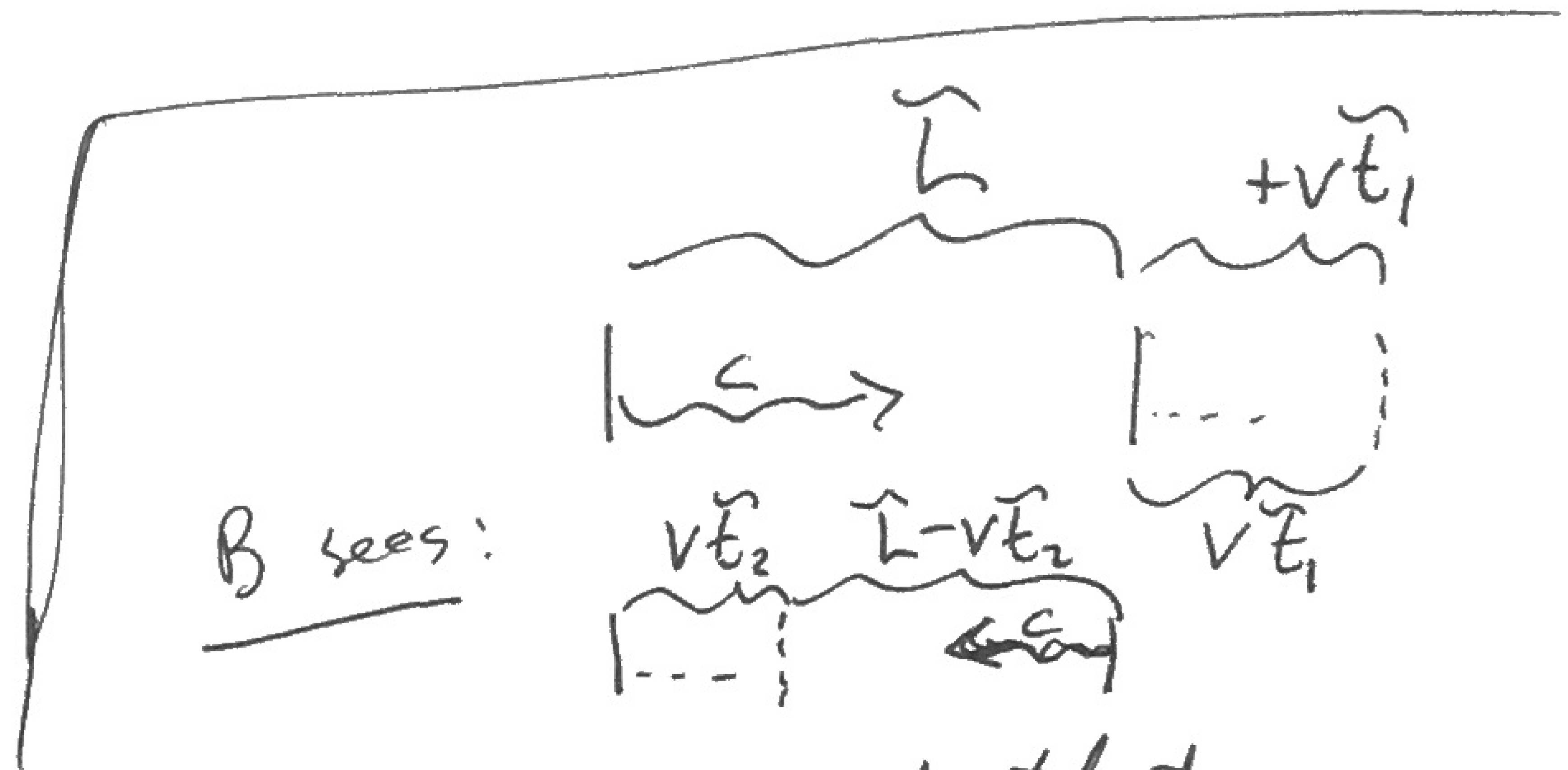
$$\Rightarrow \boxed{B_{TOT}(x \ll y) \simeq \frac{\mu_0 I}{2\pi x}}$$

which is the result for the infinite wire only in the y direction, which makes sense because for $x \ll y$ we are close enough to the y-axis wire for the other wire to be negligible.

Since B_{TOT} is symmetric in x and y, the exact same reasoning will give the analogous result in the $y \ll x$ limit (as you can (should) check).

#3) Consider a train car with the following light clock on it:





Let us not assume that the observer in the train and that on the platform measure the same length. Instead let us assume the observer A measures the clock to have length L and B measures it to have length $\tilde{L}$. If we find that $L=\tilde{L}$ (as our intuition tells us), then fine. The punchline, though, is that we won't. Let t_1 be the time A measures for the photon to move from the left plate to the right plate, and t_2 be the time (for A) for the photon to come back. I.e., $\boxed{t_1 = L/c = t_2}$. Let $\tilde{t}_1, \tilde{t}_2$ be the analogous times for B. Then $\boxed{c\tilde{t}_1 = \tilde{L} + v\tilde{t}_1 \text{ and } c\tilde{t}_2 = \tilde{L} - v\tilde{t}_2}$

(see above-right diagram)

Then $\tilde{t}_1 = \dfrac{\tilde{L}}{c-v}$ and $\tilde{t}_2 = \dfrac{\tilde{L}}{c+v}$. Now, from time dilation, we know $(\tilde{t}_1+\tilde{t}_2) = \gamma(t_1+t_2)$. Thus

$$\tilde{t}_1+\tilde{t}_2 = \tilde{L}\left(\frac{1}{c-v}+\frac{1}{c+v}\right) = \gamma(t_1+t_2) = \gamma\cdot\frac{2L}{c} \Rightarrow \tilde{L}\left(\frac{2c}{c^2-v^2}\right) = \frac{2L\gamma}{c}$$

$$\Rightarrow \tilde{L}\left(\frac{c^2}{c^2-v^2}\right) = L\gamma \Rightarrow \tilde{L}\underbrace{\left(\frac{1}{1-\frac{v^2}{c^2}}\right)}_{\gamma^2} = L\gamma \Rightarrow \boxed{\tilde{L} = L/\gamma}$$

which is the length contraction that we derived before in class

#4 (at rest)
Half life $\Gamma = 1.5 \times 10^{-6}$ seconds.

$V = .995c$. Height $H = 10km = 10^4 m$.

wrt Earth's surfac

Let T = travel time

w/o relativity: $H = VT \Rightarrow T = \frac{H}{V}$

plug in numbers

The fraction f left over: $\boxed{f = \left(\frac{1}{2}\right)^{\frac{T}{\Gamma}} = \left(\frac{1}{2}\right)^{\frac{H}{V\Gamma}} = 1.9 \times 10^{-7} \quad \left(\simeq \frac{1}{2^{22}}\right)}$

With relativity, from the detector's frame, the height is still H and the muon's speed is still $V = .995c$, but now the muon's clock runs slow, so that its observed half-life $\hat{\Gamma} = \gamma\Gamma$ where $\gamma = \frac{1}{\sqrt{1-(.995)^2}} \simeq 10$. Thus $\boxed{f_D = \left(\frac{1}{2}\right)^{\frac{T}{\hat{\Gamma}}} = \left(\frac{1}{2}\right)^{\frac{H}{10V\Gamma}} \simeq .2}$

Detector's frame

plug in numbers

(an impressive increase!)

In the muon's rest frame, its half-life is still Γ, and the detector is still moving towards it with speed $V = .995c$, but now its perceived height is contracted to $\hat{H} = H/\gamma$. With γ the same as above, we see that $\boxed{f_M = f_D} = \left(\frac{1}{2}\right)^{\frac{H}{\gamma V\Gamma}} = \left(\frac{1}{2}\right)^{\frac{H}{10V\Gamma}}$, as it should be since f is an observable quantity and hence shouldn't depend on the observer.

Muon's frame

#5 a) Under a Lorentz transformation, we have $c\Delta t' = (c\Delta t - \frac{v}{c}\Delta x)\gamma$

and $\Delta x' = (\Delta x - v\Delta t)\gamma$.

Then $c^2\Delta t'^2 - (\Delta x')^2 = (c\Delta t - \frac{v}{c}\Delta x)^2\gamma^2 - (\Delta x - v\Delta t)^2\gamma^2$

$$= c^2\Delta t^2\gamma^2 + \frac{v^2}{c^2}\Delta x^2\gamma^2 - \cancel{(2v\Delta x\Delta t)\gamma^2}$$

$$- (\Delta x)^2\gamma^2 - v^2(\Delta t)^2\gamma^2 + \cancel{2v\Delta x\Delta t\gamma^2}$$

$$= c^2\Delta t^2\underbrace{\left(\gamma^2 - \gamma^2\frac{v^2}{c^2}\right)}_{1} - (\Delta x)^2\gamma^2\underbrace{\left(1 - \frac{v^2}{c^2}\right)}_{1}$$

$$= c^2\Delta t^2 - \Delta x^2, \text{ as desired.}$$

I wrote those before the problem was changed. To get $c^2(\Delta t)^2 - \Delta x^2 - \Delta y^2 - \Delta z^2 = c^2\Delta t'^2 - \Delta x'^2 - \Delta y'^2 - \Delta z'^2$ we just note that under an x-boost, $\Delta y' = \Delta y$ and $\Delta z' = \Delta z$.

~~b) Imagine a flash of light being emitted at the origin of two observers moving with a relative velocity such that at $t = t' = 0$ the flash is sent. Then we have, in the unprimed frame $(c\Delta t)^2 = (\Delta x)^2$ and in the primed frame $(c\Delta t')^2 = (\Delta x')^2$ where we have assumed c is the same for both (a la Einstein).~~

~~Then $(c\Delta t)^2 - (\Delta x)^2 = 0 = (c\Delta t')^2 - (\Delta x')^2$~~

~~$\Rightarrow \boxed{c^2\Delta t^2 - \Delta x^2 = c\Delta t'^2 - \Delta x'^2}$, as desired.~~

~~(The same reasoning can be used in 3, 4, 5, 6, ... N spacetime dimensions)~~

#6) PM 5.15

Equation 5.15 says: $\vec{E}' = \frac{Q}{4\pi\epsilon_0 (r')^2} \frac{1-\beta^2}{(1-\beta^2\sin^2\theta')^{3/2}} \hat{r}'$ where $\beta = \frac{v}{c}$ and v is the velocity of the charge.

Now let us integrate $\vec{E}' \cdot d\vec{a}$ over a sphere of radius R.

We have
$$\int \vec{E} \cdot d\vec{a} = \frac{Q}{4\pi\epsilon_0} \frac{(1-\beta^2)}{\underset{(r'=R)}{R^2}} \int_0^{\pi} d\theta' \int_0^{2\pi} d\phi' \frac{1}{(1-\beta^2\sin^2\theta')^{3/2}} \underbrace{R^2 \sin\theta'}_{da}$$

R's cancel, $\int_0^{2\pi} d\phi = 2\pi$
$$= \frac{Q}{4\pi\epsilon_0}(1-\beta^2)\cdot 2\pi \int_0^{\pi} \frac{\sin\theta}{(1-\beta^2\sin^2\theta)^{3/2}} d\theta$$

$$= \frac{Q}{2\epsilon_0}(1-\beta^2) \underbrace{\int_0^{\pi} \frac{\sin\theta \, d\theta}{(1-\beta^2\sin^2\theta)^{3/2}}}$$

$\parallel \leftarrow$ using (K.15) in appendix

$$\underbrace{\frac{-\cos\theta}{(1-\beta^2)\sqrt{1-\beta^2\sin^2\theta}}\Bigg|_0^{\pi}}$$

$$\frac{1}{(1-\beta^2)}\left[1-(-1)\right] = \frac{2}{(1-\beta^2)}$$

$$\Rightarrow \boxed{\int \vec{E}\cdot d\vec{a} = \frac{Q}{\epsilon_0}}$$, as desired

#7 PM 5.27 We refer to Fig 5.22 in PM.

If $v = v_0$, then $\beta = \beta_0$ and so $\gamma = \frac{1}{\sqrt{1-\beta^2}} = \gamma_0 \left(= \frac{1}{\sqrt{1-\beta_0^2}} \right)$.
$\left(\beta = \frac{v}{c}\right)$

Thus, the positive charge density in the test charge's frame is $\lambda_+ = \gamma_0 \lambda_0$.

Using equation (5.29), we have that the negative charge density λ_- is given by

$$\lambda_- = -\frac{\gamma_0'}{\gamma_0}\lambda_0 = -\frac{\gamma\gamma_0}{\gamma_0}(1-\beta\beta_0)\lambda_0 \underset{\substack{\beta=\beta_0 \\ \gamma=\gamma_0}}{=} -\gamma_0\underbrace{(1-\beta_0^2)}_{1/\gamma_0^2}\lambda_0 = \frac{-\lambda_0}{\gamma_0}$$

Thus $\boxed{\lambda_+ = \gamma_0\lambda_0}$ and $\boxed{\lambda_- = \frac{-\lambda_0}{\gamma_0}}$

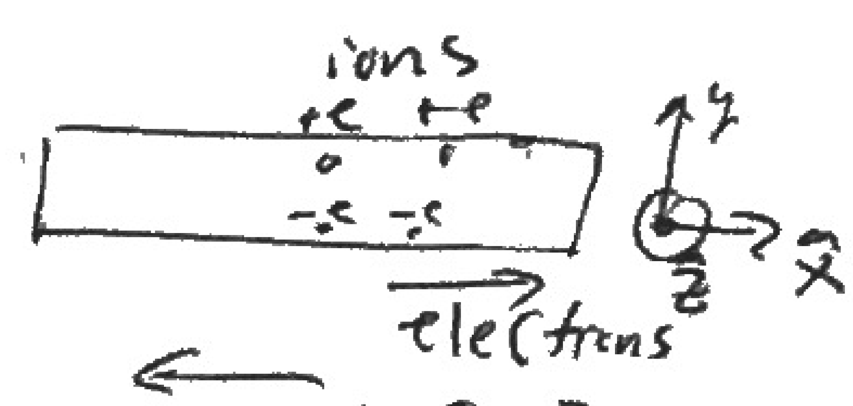
Now, the net $\lambda = \lambda_+ + \lambda_- = \lambda_0\left(\gamma_0 - \frac{1}{\gamma_0}\right) = \lambda_0\gamma_0\left(1 - \frac{1}{\gamma_0^2}\right)$

$$= \lambda_0\gamma_0\big(\underbrace{1-(1-\beta_0^2)}_{\beta_0^2}\big) \Rightarrow \lambda = \lambda_0\gamma_0\beta_0^2.$$

We know that for an infinite charged wire of line charge λ,

$\vec{E} = \frac{\lambda}{2\pi\varepsilon_0 r}\hat{r}$, Thus, the force on this test charge in its own rest frame is given by
cylindrical

$$\boxed{\vec{F}_{\text{rest frame}} = q_0\vec{E} = \frac{q_0\lambda_0\gamma_0\beta_0^2}{2\pi\varepsilon_0 r}\hat{r}} \quad \left(\beta_0 = \frac{v_0}{c}\right)$$



In the wire's rest frame, we have a current of $\vec{I} = -\lambda_0 v_0\hat{x}$ $\quad$ current $\overleftarrow{\ }$ $\quad$ $\overrightarrow{q_0 \; v_0}$
$(\vec{v} = v_0\hat{x})$

so $B(r) = \frac{\mu_0}{2\pi}\frac{|I|}{r}\hat{z} = \frac{\mu_0\lambda_0 v_0}{2\pi r}\hat{z}$.

Thus, $F_{\text{magnetic}} = q_0\vec{v}\times\vec{B} = \frac{\mu_0\lambda_0 v_0^2 q_0}{2\pi r}\underbrace{\hat{x}\times\hat{z}}_{-\hat{y}} = \frac{\mu_0\lambda_0 v_0^2 q_0}{2\pi r}(-\hat{y})$.

$\longrightarrow$

Note: at the test charge's location (namely, under the wire), $-\hat{y} = \hat{r}$. Thus $\vec{F}_{magnetic} = \frac{q_0 \lambda_0 \mu_0 v_0^2}{2\pi r}\hat{r} \underset{\uparrow}{=} \frac{q_0 \lambda_0}{2\pi \varepsilon_0 r}\left(\frac{v_0}{c}\right)^2 \hat{r}$

using $\mu_0 = \frac{1}{\varepsilon_0 c^2}$

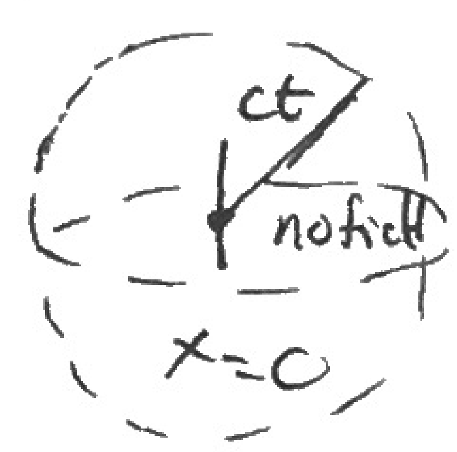
$$\Rightarrow \boxed{\vec{F}_{magnetic} = \frac{q_0 \lambda_0}{2\pi \varepsilon_0 r}\beta^2 \hat{r} = \frac{1}{\gamma_0}\vec{F}_{\substack{rest\\frame}}}$$

#8) PM 5.19

Initially:

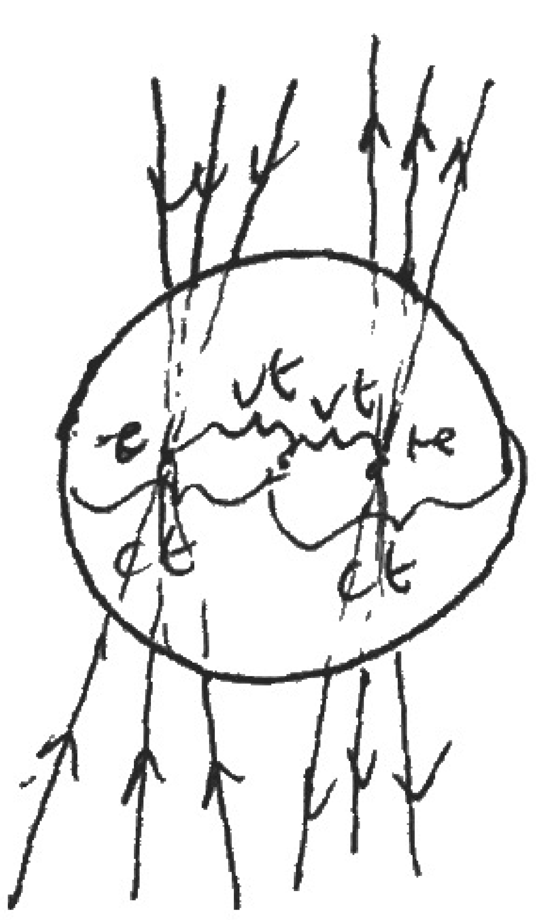
(Fig 5.29)

After the collision (at $t=0$), no charge remains so there can be no field inside a sphere of radius ct:

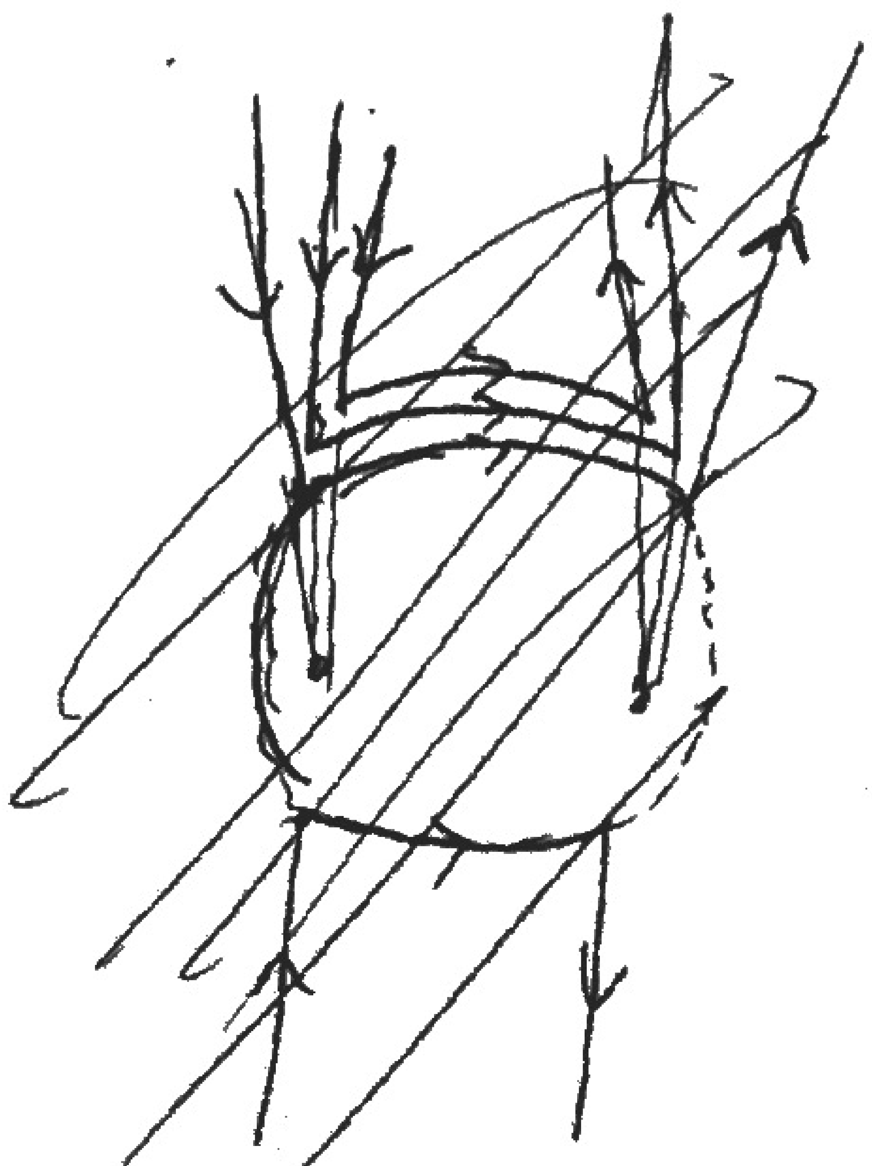


Outside of this sphere, however, the field lines still don't "know" that the collision ever took place, so the field lines look as if the particles are still moving. At time t, the $+e$ would have been at $x=vt$ and the $-e$ at $x=-vt$. Thus we'd have

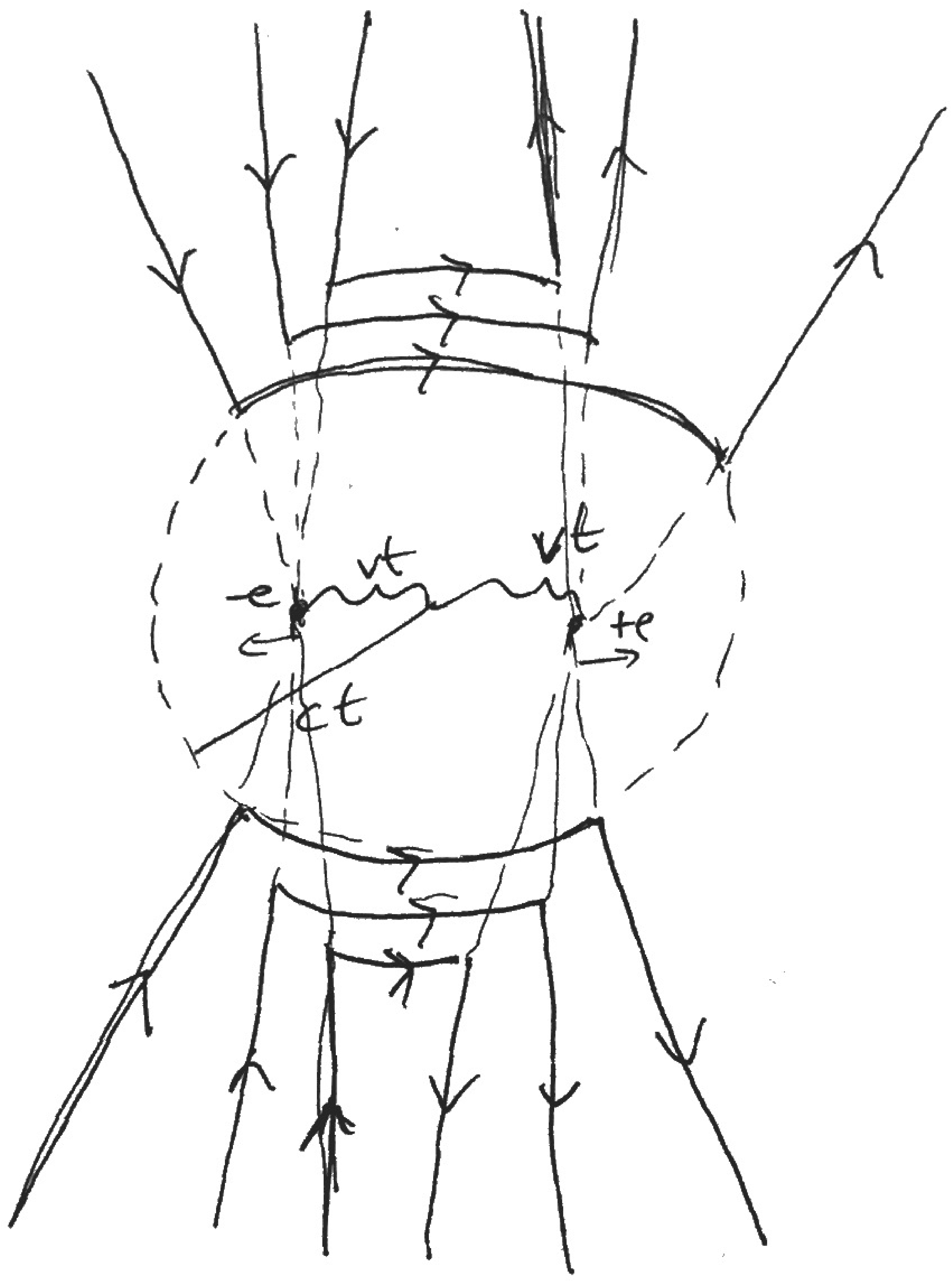
something like



~~And~~ since $\bar{E}$-field lines can't end, we have



(better drawing on back)



(okay, not ~~one~~ a <u>much</u> better drawing)