Name: ______________________

Student ID number: ______________________

105A - Final

Please read the following very carefully

- This is a closed book exam. You may **not** use a calculator. All electronic devices should **not** be around!
- You have **3 hours** to complete the exam.
- Grades are out of 150
- Answer all **three** questions, and their subsections.
- Make sure to write your name at the top of each page of this exam. Use the space provided on the exam pages to do your work. You may use the back of the pages also, but please mark clearly which problem you are working on (and also state underneath that problem that you have done work on the back of the page).
- Partial credit will be given. Show as much work/justification as possible (diagrams where appropriate). If you can not figure out how to complete a particular computation, a written statement of the concepts involved and qualitative comments on what you think the answer should be may be assigned partial credit.
- Make sure you check your units. A unit problem will cost 20% of the points.
- Mistakes in grading: If you find a mistake in the grading of your exam, alert the instructor within one week of the exams being returned DO NOT write on the returned graded exam you may make a note of the problems you thought were misgraded on a separate page.

- In class we showed that the equation of motion can be expressed using the force $F = -dU/dr$ in the form (i.e., the Binnet's equation):

$$\frac{l^2u^2}{\mu}\left(\frac{d^2u}{d\theta^2}+u\right) = -F(1/u)\ , \tag{1}$$

where $l = mr^2\dot{\theta}$ is the angular momentum and $u = 1/r$.

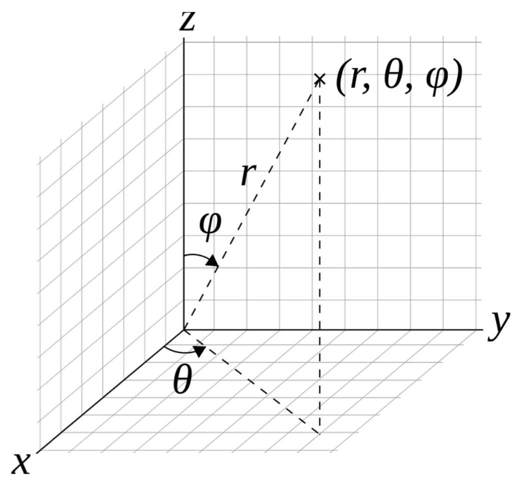

Figure 1: Spherical coordinates

- Remember that spherical coordinates are defines as shown in Figure 1:

$$\begin{aligned} x &= r\cos\varphi\sin\theta & (2)\\ y &= r\sin\varphi\sin\theta & (3)\\ z &= r\cos\theta & (4) \end{aligned}$$

Questions

1. (50pt) A spherical pendulum has a massless wire, at length b, attached to a mass m. Assume that there is no friction in the attachment point and that gravity is pointing down.

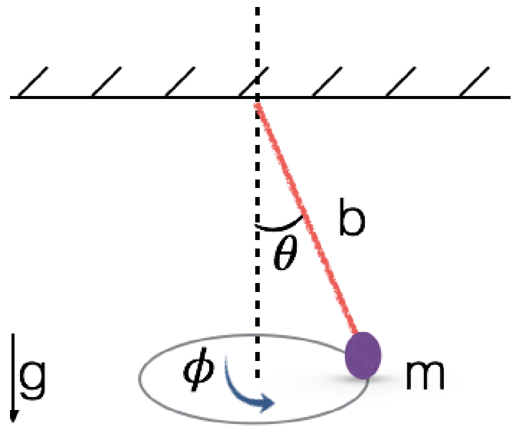

Figure 2: A Spherical pendulum

   (a) (8pt) Define carefully a set of independent generalized coordinates for this system. Define your farm of reference. Write down the Lagrangian in terms of the generalized coordinates.

you may continue writing down your answer here...

(b) (12pt) Find the generalized momenta and the conserved quantities.

(c) (10pt) Write down the equation of motions. Using one of the conserved quantities you found in (b) to express your answer, and write down only one equation of motion, using that quantity.

(d) (10pt) What does an assumption of $p_\phi = 0$ tells you about the initial conditions?

(e) (10pt) Plug in $p_\phi = 0$ and assume small oscillation $\theta << 1$ rad. Which means that you can approximate $\sin\theta \sim \theta$ and $\cos\theta \sim 1$. Find the solution in that case, assuming that the system is set initially at rest with a displacement $\theta_0 \neq 0$.

2. (50pt) A block of mass M is attached to a spring with a spring constant k and is able to slide on a horizontal bar (the x axis). A pendulum consists of a mass m suspended by a massless string with unextended length l and can move in the $x - z$ plane is attached to the block. (see Figure 3). Gravity is pointed downwards.

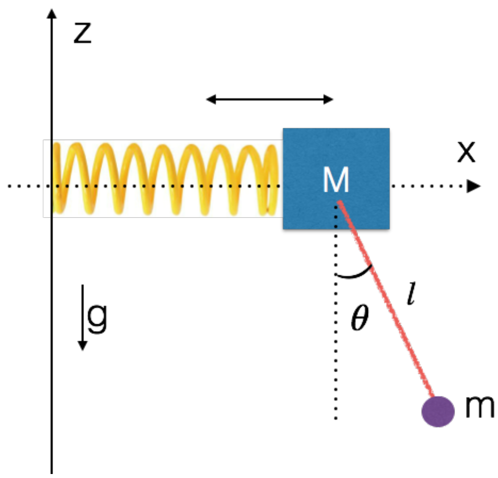

Figure 3: A pendulum attached to horizontal spring.

(a) (15pt) Chose carefully your generalized coordinates and write the Lagrangian of the system.

you may continue writing down your answer here...

(b) (10pt) Find the equations of motion.

(c) (13pt) Assume a very weak spring, compared to the pendulum, so the motion of the mass M is dominated by the pendulum. In other words:

$$\frac{k}{m+M} << \frac{m\ddot{\theta}}{m+M} \ , \tag{5}$$

or you can simply take $k \to 0$ corresponding to a free moving mass. Also assume small oscillation $\theta << 1$ rad. Which means that you can approximate $\sin\theta \sim \theta$ and $\cos\theta \sim 1$. Keep only first order terms (i.e., $\theta^2 \to 0$ and also $\dot{\theta}^2 \to 0$). Write the equation of motions in this case.

(d) (12pt) Given the initial conditions of $\theta(t=0)=\theta_0$, $\dot{\theta}(t=0)=\theta_0\sqrt{g(M+m)/(lM)}$ find the time solution of the angle that describes the oscillations of the pendulum using the equations you found in (c) as a function of the g, M, m, l and θ_0.

3. A particle is moving in a a force field describes by

$$U = -\frac{k}{r}e^{-r/a} \quad k, a > 0 \tag{6}$$

(a) (7pt) Write down the energy and the effective potential in therms of r, l, k, m, a and the velocity $\dot{r}$.

(b) (15pt) Find the angular momentum which describes the a circular motion in this system. Express your answer in terms of r, l, k, m and a. Explain why this point represents a circular motion.

(c) (15pt) Is this a stable point, if not, when will that represent a stable point? In other words, what should be the condition for a stable point?

(d) (13pt) Assume that a represent some characteristic scale in the problem and we are interested in smaller scales, i.e., $r << a$. In that regime you find that

$$F \sim -\left(\frac{k}{r^2} + \frac{k}{ar}\right) \tag{7}$$

And

$$a = \frac{km}{l^2} r^2 \tag{8}$$

Find the most general solution of $r(\theta)$ in this case. Express your answer in terms of m, k, l, θ and the relevant constant if you have them.
Hint: use the Binnet's equation given above.

you may continue writing down your answer here...