Problem Set #8 Solutions

#1) PM 9.19 We want a maximum at $x=0$ and $t=0$. The wave travels in the $\vec{E}\times\vec{B}$ direction, which we want to be the $-\hat{x}$ direction. Thus, since $\hat{y}\times\hat{z}=\hat{x}$, we have that $\vec{E}$ is in the $\pm\hat{y}$ direction, so that $\vec{B}$ is then in the $\mp\hat{z}$ direction (see Right hand rule). (Note, $+\hat{y}$ for $\vec{E} \Rightarrow -\hat{z}$ for $\vec{B}$ in order for $\vec{E}\times\vec{B}$ to be in $-\hat{x}$ direction, and similarly for $-\hat{y}$ for $\vec{E}$).

Choosing (arbitrarily) $+\hat{y}$ for $\vec{E}$, with the requirements at $x=0, t=0$, gives $\boxed{\vec{E} = E_0\cos(kx+\omega t)\hat{y}}$ where the "+" sign is chosen inside the parentheses because the wave is traveling in the negative $\hat{x}$ direction. Namely, if we follow a peak of the wave for increasing t, we see that x is decreasing, as desired. Then, using the fact that $\vec{E}\times\vec{B}$ points in the direction of motion, and that $B_0 = |\vec{B}|_{max} = E_0/c$, we have $\boxed{\vec{B} = -(E_0/c)\cos(kx+\omega t)\hat{z}}$

Using x and vt, we have $\boxed{\vec{E} = E_0\cos\left(k\left(x+\frac{\omega}{k}t\right)\right)\hat{y} = E_0\cos(k(x+vt))\hat{y}}$

and $\boxed{\vec{B} = -\frac{E_0}{c}\cos\left(k\left(x+\frac{\omega}{k}t\right)\right)\hat{z} = -\frac{E_0}{c}\cos(k(x+vt))\hat{z}}$

(with $v=\frac{\omega}{k}$.)

#2) PM 9.20) (9.28): $\vec{E} = \dfrac{E_0\hat{y}}{1+\frac{(x+ct)^2}{\ell^2}}$ $\vec{B} = \dfrac{-(E_0/c)\hat{z}}{1+\frac{(x+ct)^2}{\ell^2}}$

We have $E_0 = 100$ kV/m and $\ell = 1$ ft $\simeq$ .305 m.

The proton is at the origin so we care about $\vec{E}$ at $x=0$.

We can ignore the magnetic force because $|\vec{F}_{magnetic}| \sim qvB \sim qvE_0/c \sim \frac{v}{c}(qE_0)$

whereas $|\vec{F}_{electric}| \sim qE_0$

so that as long as $v \ll c$,

$$\frac{|\vec{F}_{magnetic}|}{|F_{Electric}|} \sim \frac{v}{c} \ll 1.$$

Since $\frac{d\vec{p}}{dt} = \vec{F}$, $\underbrace{p_y}_{\text{momentum aquired during entire pulse}} = \int_{-\infty}^{\infty} F_y\,dt = \int_{-\infty}^{\infty} eE_y\,dt$

$$= eE_0\int_{-\infty}^{\infty}\frac{dt}{1+\frac{c^2}{\ell^2}t^2} \qquad \text{Let } \tan\theta = \frac{c}{\ell}t$$

$$dt = \frac{\ell}{c}\sec^2\theta\,d\theta$$

$$= \frac{eE_0\ell}{c}\int_{-\pi/2}^{\pi/2}\frac{\sec^2\theta\,d\theta}{\underbrace{1+\tan^2\theta}_{\sec^2\theta}} = \frac{eE_0\ell}{c}\underbrace{\int_{-\pi/2}^{\pi/2}d\theta}_{\pi}$$

$$\Rightarrow \boxed{p_y = \frac{\pi eE_0\ell}{c}}$$

The proton's final speed is then $v_y = \frac{p_y}{m} = \frac{\pi eE_0\ell}{mc} \simeq 3.1\times10^4$ m/s ($\ll 3\times10^8$ m/s $= c$) (plugging in numbers)

so that $\frac{v}{c} \ll 1$ indeed.

The displacement during the few nanoseconds of the pulse passing is therefore negligible and after 1 microsecond, the displacement d will be about

$$d = v_y\cdot(1\,\text{microsecond}) = 3.1\times10^4\,\text{m/s}\cdot10^{-6}\,\text{s} = 3.1\times10^{-2}\,\text{m/s} \Rightarrow \boxed{d = 3.1\,\text{cm}}$$

#3) PM 9.25 As found directly after equation (9.33) in the text, the average energy density u in a sinusoidal E+M wave is given by $u = \frac{\varepsilon_0}{2} E_0^2 = \varepsilon_0 E_{rms}^2$ (recall: $E_{rms} = \sqrt{\frac{1}{T}\int_0^T E_0^2 \sin^2(\omega t)\,dt}$ where $T = \frac{2\pi}{\omega}$ (for a sinusoidal wave))

(Good exercise to check that this means $E_{rms}^2 = \frac{1}{2} E_0^2$)

Here, we're given u and want E_{rms},

so $E_{rms}^2 = \frac{u}{\varepsilon_0} = \frac{4\times10^{-14}\ J/m^3}{\left(8.85\times10^{-12}\right)\frac{s^2 C^2}{kg\, m^3}} = 4.5\times10^{-3}\ V^2/m^2$

$\Rightarrow \boxed{E_{rms} = 6.7\times10^{-2}\ V/m}$

Note: Recall that in the above expression $u = \frac{\varepsilon_0}{2} E_0^2$, even though this looks like the energy density of a static $\vec{E}$-field, this is just a nice coincidence. Namely, to get $u = \frac{\varepsilon_0}{2} E_0^2$ here, we start with $u = \frac{\varepsilon_0}{2} E^2(t) + \frac{1}{2\mu_0} B^2(t)$ where $E(t)$, $B(t)$ are sinusoidal waves. $B(t)$ has amplitude E_0/c, and use $c^2 = \frac{1}{\varepsilon_0 \mu_0}$. This effectively gives $u = \varepsilon_0 E^2(t)$, but then we must time average to get the factor of 2: $u = \frac{\varepsilon_0}{2} E_0^2$. So, even though this looks like the old formula, it's important to know where it really comes from.

#3) b) If we want to see a comparable intensity at a distance R from a 1 kW radio transmitter, then, assuming the transmissions are sent out in spherical waves, we want

$$\underbrace{\frac{1}{c}}_{\text{speed of wave}} \frac{P_{transmitted}}{\underbrace{4\pi R^2}_{\text{surface area of sphere of radius } R}} \cong \text{Energy density} = 4\times10^{-14}\ J/m^3$$

$$\Rightarrow R^2 = \frac{\overbrace{P_{transmitted}}^{= 1kW}}{4\pi\, c\, (4\times10^{-14}\ J/m^3)}$$

plug in numbers $\Rightarrow$ $\boxed{R \cong 2600\ m}$

#4) Pm 9.26 $E_x = 0$ $E_y = E_0 \sin(kx+\omega t)$ $E_z = 0$ $(\Rightarrow \vec{E} = E_0 \sin(kx+\omega t)\hat{y})$

$B_x = 0$, $B_y = 0$, $B_z = -E_0/c \sin(kx+\omega t) \Rightarrow \vec{B} = -\frac{E_0}{c}\sin(kx+\omega t)\hat{z})$

a) Need to check: $\nabla\cdot\vec{E} = 0$, $\nabla\cdot\vec{B} = 0$, $\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$, and $\nabla\times\vec{B} = \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}$

because we're in free space, i.e., $\rho = 0$

because $\vec{J} = 0$ in free space.

$\nabla\cdot\vec{E} = \underbrace{\partial_x E_x}_{=0} + \partial_y E_y + \underbrace{\partial_z E_z}_{=0} = 0$ because E_y has no y dependence.

$\nabla\cdot\vec{B} = \underbrace{\partial_x B_x}_{=0} + \underbrace{\partial_y B_y}_{=0} + \partial_z B_z = 0$ because B_z has no z dependence.

$\nabla\times\vec{E} = (\underbrace{\partial_y E_z}_{=0} - \underbrace{\partial_z E_y}_{=0}, \underbrace{\partial_z E_x}_{0} - \underbrace{\partial_x E_z}_{0}, \partial_x E_y - \underbrace{\partial_y E_z}_{=0}) = (0, 0, kE_0\cos(kx+\omega t))$

$\frac{\partial\vec{B}}{\partial t} = (0, 0, \partial_t B_z) = (0, 0, -\frac{\omega}{c}E_0\cos(kx+\omega t))$.

Thus $\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \Leftrightarrow \boxed{k = \frac{\omega}{c}}$

Now, $\nabla\times\vec{B} = (\underbrace{\partial_y B_z}_{=0} - \underbrace{\partial_z B_y}_{=0}, \underbrace{\partial_z B_x}_{=0} - \partial_x B_z, \underbrace{\partial_x B_y}_{=0} - \underbrace{\partial_y B_x}_{=0}) = (0, E_0\frac{k}{c}\cos(kx+\omega t), 0)$

and $\frac{\partial\vec{E}}{\partial t} = (0, E_0\omega\cos(kx+\omega t), 0)$. Thus $\nabla\times\vec{B} = \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t} \Leftrightarrow \frac{k}{c} = \frac{\omega}{c^2}$

but again $k = \frac{\omega}{c}$ works for this ✓

$\boxed{\text{Thus the above E+M wave is a solution to Maxwell's equations in free space} \Leftrightarrow k = \frac{\omega}{c}.}$

#4) (cont'd). With $\omega = 10^{10}\,s^{-1}$ and $E_0 = 1\,kV/m$, we have that the wavelength

$$\lambda = \frac{2\pi}{k} = \frac{2\pi c}{\omega} \approx \frac{2\pi \cdot 3\times 10^8\,m/s}{10^{10}\,s^{-1}} \approx \boxed{.19\,m}$$

$(k = \frac{\omega}{c})$

As discussed in the previous problem, the energy density U is

$$U = \frac{1}{2}\varepsilon_0 E_0^2 = \frac{1}{2}\left(8.85\times 10^{-12}\,\frac{s^2C^2}{kg\,m^3}\right)\left(10^3\,V/m\right) \approx \boxed{4.4\times 10^{-6}\,J/m^3}$$

Since the power density S, i.e., the energy flow per area per time is $S = Uc$ (see picture:

speed of wave

(also see units: $Uc = \left[{energy}/{volume}\right]\cdot\left[\frac{length}{second}\right] = \left[\frac{Energy}{Area}/second\right]$.

We have $S = Uc = \frac{1}{2}\varepsilon_0 E_0^2 c \approx (4.4\times 10^{-6}\,J/m^3)\cdot(3\times 10^8\,m/s)$

$$\Rightarrow \boxed{S \approx 1300\,J/m^2 s}$$

#5) PM 9.27 Let E_i, E_r be the magnitudes of the incident ~~wave~~ and reflected waves, respectively. Since half of the incident energy is absorbed, and since Energy $\sim |E|^2$, we have that $E_r = \frac{1}{\sqrt{2}} E_i$.

Let us, for concreteness, take the incident wave to be moving in the $+y$ direction, and the reflected wave in the $-y$ direction, so we can use equations (9.29) and (9.30). ~~Namely~~

Namely, $\vec{E}_i = E_i \sin\left(\frac{2\pi}{\lambda}(y-ct)\right)\hat{z}$ and $\vec{E}_r = \underbrace{E_r}_{= \frac{1}{\sqrt{2}}E_i} \sin\left(\frac{2\pi}{\lambda}(y+ct)\right)\hat{z}$.

Thus
$$\vec{E}_{TOT} = \vec{E}_i + \vec{E}_r$$
$$= E_i \hat{z}\left[\sin\left(\frac{2\pi}{\lambda}(y-ct)\right) + \frac{1}{\sqrt{2}}\sin\left(\frac{2\pi}{\lambda}(y+ct)\right)\right].$$

Using $\sin(a \pm b) = \sin(a)\cos(b) \pm \cos(a)\sin(b)$ gives
$$\vec{E}_{TOT} = E_i \hat{z}\left[\sin\left(\frac{2\pi}{\lambda}y\right)\cos\left(\frac{2\pi c t}{\lambda}\right) - \cos\left(\frac{2\pi}{\lambda}y\right)\sin\left(\frac{2\pi c t}{\lambda}\right) + \frac{1}{\sqrt{2}}\sin\left(\frac{2\pi}{\lambda}y\right)\cos\left(\frac{2\pi c t}{\lambda}\right) + \frac{1}{\sqrt{2}}\cos\left(\frac{2\pi y}{\lambda}\right)\sin\left(\frac{2\pi c t}{\lambda}\right)\right]$$

$$\Rightarrow \vec{E}_{TOT} = E_i \hat{z}\left[\left(1+\frac{1}{\sqrt{2}}\right)\sin\left(\frac{2\pi y}{\lambda}\right)\cos\left(\frac{2\pi c t}{\lambda}\right) - \left(1-\frac{1}{\sqrt{2}}\right)\cos\left(\frac{2\pi y}{\lambda}\right)\sin\left(\frac{2\pi c t}{\lambda}\right)\right].$$

When $\frac{2\pi y}{\lambda} = \frac{\pi}{2}\underbrace{(2n-1)}_{\text{some odd number}}$, i.e., when $y = \frac{\lambda}{4}(2n-1)$ for $n = 1, 2, 3, \ldots$, the second term above vanishes and the $\sin\left(\frac{2\pi y}{\lambda}\right)$ term is ± 1, so the total E_{TOT} oscillates with magnitude $\underbrace{\left(1+\frac{1}{\sqrt{2}}\right)E_i}_{(= E_i + E_r)}$. Similarly, when $\frac{2\pi y}{\lambda} = \pi n$, i.e., when $y = \frac{\lambda}{2}n$ for $n = 1, 2, 3, \ldots$, the first term vanishes and $\cos\left(\frac{2\pi y}{\lambda}\right) = \pm 1$ so that the total $\vec{E}_{TOT}$ oscillates with magnitude $\left(1-\frac{1}{\sqrt{2}}\right)E_i$ $(= E_i - E_r)$. Thus, the ratio r that we want is
$$\boxed{r = \frac{E_i + E_r}{E_i - E_r} = \frac{1 + 1/\sqrt{2}}{1 - 1/\sqrt{2}} \simeq 5.83}$$

#6) PM 9.28

Inside the wire, $\vec{J} = \sigma \vec{E} \Rightarrow \vec{E} = \vec{J}/\sigma$ and indeed, right outside of the wire, because the curl of $\vec{E}$ is zero, this still holds (as the problem statement hints). Thus at some r (~~x~~ greater than the radius of the wire) $\vec{E} = \vec{J}/\sigma$. Using cylindrical coordinates with $\hat{z}$ as shown above, we have $E_z \equiv E = J/\sigma$.

We also know ~~B~~ $\vec{B}$ at this r. Namely, if the wire has ~~area~~ cross-sectional area A, then $\oint \vec{B} \cdot d\vec{l} = 2\pi r B = \mu_0 J A$. In particular, if we take r arbitrarily close to the radius of the wire, then we have

$$2\pi r B = \mu_0 J \pi r^2 \Rightarrow B = \frac{\mu_0 J}{2} r. \quad \left(\text{so that } \vec{B} = \frac{\mu_0 J}{2} r \hat{\phi} \text{ at the surface of the wire}\right)$$

Now, at the surface of the wire, we have $\vec{E} = \frac{J}{\sigma}\hat{z}$ and $\vec{B} = \frac{\mu_0 J}{2} r \hat{\phi}$

and so
$$\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B} = \frac{1}{\mu_0} \frac{J}{\sigma} \frac{\mu_0 J}{2} r \underbrace{\hat{z} \times \hat{\phi}}_{\substack{= -\hat{r} \\ \text{in cylindrical coords}}} = \frac{-J^2}{2\sigma} r \hat{r},$$

i.e., the power per unit area $\vec{S}$ has magnitude $\frac{J^2}{2\sigma} r$ (radius of wire) and points radially inwards towards the wire. Thus, the amount of power P entering a cylinder of length l

of wire is $P = 2\pi r l \cdot |\vec{S}| = 2\pi r l \frac{J^2}{2\sigma} r = \frac{J^2}{\sigma} l \pi r^2$.

But $J\pi r^2 = I$ so $\boxed{P = I \frac{J l}{\sigma} = I \underbrace{E l}_{V} = IV}$ as desired.

#7) PM 9.32

(6.76) says that $\vec{E}'_{\parallel} = \vec{E}_{\parallel}$, $\vec{E}'_{\perp} = \gamma(\vec{E}_{\perp} + \vec{v}\times\vec{B}_{\perp})$

$\vec{B}'_{\parallel} = \vec{B}_{\parallel}$, $\vec{B}'_{\perp} = \gamma(\vec{B}_{\perp} - (\vec{v}/c^2)\times\vec{E}_{\perp})$.

Now $|\vec{E}'|^2 - c^2|\vec{B}'|^2 = (\vec{E}'_{\perp} + \vec{E}'_{\parallel})\cdot(\vec{E}'_{\perp} + \vec{E}'_{\parallel}) - c^2(\vec{B}'_{\perp} + \vec{B}'_{\parallel})\cdot(\vec{B}'_{\perp} + \vec{B}'_{\parallel})$

since $\vec{E}_{\perp}\cdot\vec{E}_{\parallel} = \vec{B}_{\perp}\cdot\vec{B}_{\parallel} = 0$

$$\stackrel{\downarrow}{=} |\vec{E}'_{\perp}|^2 + |\vec{E}'_{\parallel}|^2 - c^2(|\vec{B}'_{\perp}|^2 + |\vec{B}'_{\parallel}|^2)$$

since $\vec{E}'_{\parallel} = \vec{E}_{\parallel}$ and $\vec{B}'_{\parallel} = \vec{B}_{\parallel}$

$$\stackrel{\downarrow}{=} E_{\parallel}^2 + |\vec{E}'_{\perp}|^2 - c^2(|\vec{B}_{\parallel}|^2 + |\vec{B}'_{\perp}|^2)$$

$$= E_{\parallel}^2 + \gamma^2(\vec{E}_{\perp} + \vec{v}\times\vec{B}_{\perp})\cdot(\vec{E}_{\perp} + \vec{v}\times\vec{B}_{\perp})$$

$$- c^2|\vec{B}_{\parallel}|^2 - c^2\gamma^2\left(\vec{B}_{\perp} - \frac{\vec{v}}{c^2}\times\vec{E}_{\perp}\right)\cdot\left(\vec{B}_{\perp} - \frac{\vec{v}}{c^2}\times\vec{E}_{\perp}\right)$$

$$= E_{\parallel}^2 - c^2B_{\parallel}^2 + \gamma^2\left(E_{\perp}^2 + |\vec{v}\times\vec{B}_{\perp}|^2 + 2\vec{E}_{\perp}\cdot(\vec{v}\times\vec{B}_{\perp})\right)$$

$$- c^2\gamma^2B_{\perp}^2 - c^2\gamma^2\frac{1}{c^4}|\vec{v}\times\vec{E}_{\perp}|^2$$

$$+ 2c^2\gamma^2\cdot\frac{1}{c^2}\vec{B}_{\perp}\cdot(\vec{v}\times\vec{E}_{\perp}).$$

Since $\vec{v}$ is perpendicular to both $\vec{E}_{\perp}$ and $\vec{B}_{\perp}$ (by definition of $\vec{E}_{\perp}$ and $\vec{B}_{\perp}$), we have $|\vec{v}\times\vec{E}_{\perp}|^2 = v^2E_{\perp}^2$ and $|\vec{v}\times\vec{B}_{\perp}|^2 = v^2B_{\perp}^2$, and since $\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{B}\cdot(\vec{C}\times\vec{A}) = \vec{C}\cdot(\vec{A}\times\vec{B})$, we have $\vec{B}_{\perp}\cdot(\vec{v}\times\vec{E}_{\perp}) = -\vec{E}_{\perp}\cdot(\vec{v}\times\vec{B})$

"$\vec{E}_{\perp}\cdot(\vec{B}_{\perp}\times\vec{v})$"

so the terms cancel and we have

$$E'^2 - c^2B'^2 = E_{\parallel}^2 - c^2B_{\parallel}^2 + \gamma^2E_{\perp}^2 + \gamma^2v^2B_{\perp}^2 - c^2\gamma^2B_{\perp}^2 - \gamma^2\frac{v^2}{c^2}E_{\perp}^2$$

$$= E_{\parallel}^2 + \underbrace{\gamma^2\left(1-\frac{v^2}{c^2}\right)}_{1}E_{\perp}^2 - c^2\left(B_{\parallel}^2 + \underbrace{\gamma^2\left(1-\frac{v^2}{c^2}\right)}_{1}B_{\perp}^2\right) = (E_{\parallel}^2 + E_{\perp}^2) - c^2(B_{\parallel}^2 + B_{\perp}^2)$$

$$\Rightarrow \boxed{E'^2 - c^2B'^2 = E^2 - c^2B^2,}$$ as desired.

$\longrightarrow$

For an electromagnetic wave, $|E| = c|B|$ always, so $E^2 - c^2B^2 = 0$ in any frame. Simple as that.