Gaussian Beams

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1 Paraxial Wave Equation

Just like the in previous discussion of polarization, we will start by considering the case of electromagnetic waves propagating in the $+z$ direction in free space (with a little wiggle room). However, this week we will be abandoning the "infinite plane wave" assumption we had invoked, and will be investigating a more realistic *beam* of light. By beam, we mean that the field amplitude will go to zero as x and y go to $\pm\infty$. The electric field will be given by

$$|\mathbf{E}| = \mathcal{E}_0 \psi(x, y, z)\ \mathrm{e}^{-\mathrm{i}\omega t}. \tag{1}$$

Note that we have introduced a new parameter called $\psi(x, y, z)$ that has nothing to do with the polarization unit vector $\hat{\boldsymbol{\psi}}$ we used last week (I am insisting upon calling both of them by the same Greek letter since they both bear striking resemblance to the quantum-mechanical state vector $|\psi\rangle$, albeit in altogether different ways). Maxwell's equations can be used to show that $\psi(x, y, z)$ satisfies the following scalar wave equation:

$$\nabla^2 \psi(x, y, z) = -k^2\ \psi(x, y, z) \tag{2}$$

with $k^2 = \mu_o \epsilon_o \omega^2$. To tie this back in to last week, infinite plane waves propagating toward $+z$ are valid solutions:

$$\psi(x, y, z) = \mathrm{e}^{\mathrm{i}k_z z}, \tag{3}$$

but we will be starting with the slightly more general solution

$$\psi(x, y, z) = \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}} = \mathrm{e}^{\mathrm{i}(k_x x + k_y y + k_z z)}. \tag{4}$$

If $\mathbf{k}$ is inclined by a very small angle w.r.t. the z-axis, we can make the following approximation, which is known as the *paraxial approximation*:

$$k_z = \sqrt{k^2 - k_x^2 - k_y^2} \approx k - \frac{k_x^2 + k_y^2}{2k}. \tag{5}$$

Our plane-wave solution then takes the form

$$\psi(x, y, z) = \mathrm{e}^{\mathrm{i}(k_x x + k_y y)} \mathrm{e}^{\mathrm{i}(k - \frac{k_x^2 + k_y^2}{2k})z}. \tag{6}$$

The $\mathrm{e}^{\mathrm{i}kz}$ term oscillates quickly, so we can separate out the slowly-varying part of this by defining

$$u(x,y,z) \equiv \psi(x,y,z)\mathrm{e}^{-\mathrm{i}kz} = \mathrm{e}^{\mathrm{i}(k_x x + k_y y)}\,\mathrm{e}^{-\mathrm{i}(\frac{k_x^2+k_y^2}{2k})z}. \tag{7}$$

If we plug $\psi(x,y,z)$ back into the scalar wave equation (Eq. 2), we find that $u(x,y,z)$ obeys the following equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + \mathrm{i}2k\frac{\partial u}{\partial z} = 0 \tag{8}$$

The first two terms can be gathered into a definition of the transverse Laplacian

$$\nabla_{\mathrm{T}}^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial x^2}. \tag{9}$$

The last two terms can be written as

$$\begin{aligned}
\frac{\partial^2 u}{\partial z^2} + \mathrm{i}2k\frac{\partial u}{\partial z} &= \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z} + \mathrm{i}2k\;u\right) \\
&= \frac{\partial}{\partial z}\left(-\mathrm{i}\frac{k_x^2+k_y^2}{2k}u + \mathrm{i}2k\;u\right) \\
&\approx \frac{\partial}{\partial z}\left(\mathrm{i}2k\;u\right).
\end{aligned} \tag{10}$$

The last approximation is (once again) the paraxial approximation and we have the equation that $u(x,y,z)$ must obey, which is called the *paraxial wave equation*:

$$\nabla_{\mathrm{T}}^2 u = -\mathrm{i}\,2k\,\frac{\partial u}{\partial z}. \tag{11}$$

To construct solutions to this, we can start with the point source solution to the regular wave equation (Eq. 2):

$$\psi(r) = \frac{r_{\mathrm{o}}}{r}\mathrm{e}^{\mathrm{i}kr} \tag{12}$$

for some constant r_{o} and note that for locations closer to the z-axis than the location of the point source (here $r=0$),

$$r = \sqrt{x^2+y^2+z^2} = z\sqrt{1+\frac{x^2+y^2}{z^2}} \approx z + \frac{x^2+y^2}{2z} \tag{13}$$

to write

$$\psi(x,y,z) = \frac{r_{\mathrm{o}}}{r}\mathrm{e}^{\mathrm{i}kr} \approx \frac{r_{\mathrm{o}}}{z}\,\mathrm{e}^{\mathrm{i}k(x^2+y^2)/2z}\,\mathrm{e}^{\mathrm{i}kz}. \tag{14}$$

We can again take just the slowly-varying part, $u = (r_{\mathrm{o}}/z)\exp(\mathrm{i}k(x^2+y^2)/2z)$, which is also typically multiplied by an overall normalization factor of $-\mathrm{i}/\lambda r_{\mathrm{o}}$ to give the following form for a function proportional to $u(x,y,z)$ for a point source in the paraxial approximation:

$$h(x,y,z) = -\frac{\mathrm{i}}{\lambda z}\mathrm{e}^{\mathrm{i}k(x^2+y^2)/2z}. \tag{15}$$

You will notice that I have given this a special symbol, h, instead of u. That's because this particular solution, which describes the field at some point (x, y, z) that is close to the z-axis due to a point-source at the origin is the Green's function for the paraxial wave equation. $h(x, y, z)$ is called the *impulse response*, the *Green's function*, the *point spread function*, or the *Fresnel kernel*.

To use this as a Green's function, consider the case where the field distribution is known everywhere on some plane $z = z_\mathrm{o}$ (which is to say we know $u(x, y, z_\mathrm{o})$). In order to find the field at some other location in space $u(x, y, z)$, we just convolve the source distribution with the Green's function for sources in that plane creating fields at the new location:

$$u(x,y,z) = -\frac{\mathrm{i}}{\lambda(z-z_\mathrm{o})}\int \mathrm{d}x'\mathrm{d}y' u(x',y',z_\mathrm{o})h(x-x',y-y',z-z_\mathrm{o}). \tag{16}$$

Eq. 16 is called the *Fresnel integral* or *Huygens' integral in the Fresnel approximation.*

2 Gaussian beams

The Fresnel kernel (Eq. 15) is very instructive, but there are some aspects of it that make it difficult to use in realistic situations. For instance, the amplitude of the field extends out infinitely in the x and y directions, far past the point where the paraxial approximation is valid. Second, it diverges at the source, whereas real sources on an optics table do not behave this way.

There is a clever solution to these issues that relies on analytic continuation of the z-coordinate. Since a solution of the paraxial wave equation at $z = z_a$ is also a solution at $z = z_a + \mathrm{i}b$, we can replace the real coordinate z with a complex coordinate $q(z) = z + \mathrm{i}b$:

$$u(x,y,q(z)) = -\frac{\mathrm{i}}{\lambda q(z)}\mathrm{e}^{\mathrm{i}k(x^2+y^2)/2q(z)} = -\frac{\mathrm{i}}{\lambda(z+\mathrm{i}b)}\mathrm{e}^{\mathrm{i}k(x^2+y^2)/2(z+\mathrm{i}b)}. \tag{17}$$

The real length b is called the *confocal parameter*[1] and we will show that this is equal to the *Rayleigh range* $z_\mathrm{R} = b$. Verdeyen [2] uses the symbol z_0 for b.

If we separate out the real and imaginary parts, rearrange terms with some definitions, and then normalize our solution so that $\int \mathrm{d}x\mathrm{d}y|u(x,y,z)|^2 = 1$ for all z, we have the following power-normalized solution to the paraxial wave equation

$$U_{00}(x,y,z) = \sqrt{\frac{2}{\pi}}\frac{1}{w(z)}\mathrm{e}^{-\mathrm{i}\varphi(z)}\mathrm{e}^{-\frac{x^2+y^2}{w^2(z)}}\mathrm{e}^{\mathrm{i}k\frac{x^2+y^2}{2R(z)}}, \tag{18}$$

which gives us a complete expression for the electric field magnitude

$$|\mathbf{E}| = \mathcal{E}_0\frac{w_\mathrm{o}}{w(z)}\mathrm{e}^{-\mathrm{i}\varphi(z)}\mathrm{e}^{-\frac{x^2+y^2}{w^2(z)}}\mathrm{e}^{\mathrm{i}k\frac{x^2+y^2}{2R(z)}}\mathrm{e}^{\mathrm{i}(kz-\omega t)}. \tag{19}$$

These expressions utilize the following properties and definitions:

[1] Older texts have a factor of 2 in there, so beware.

1. We have expanded $1/q(z)$ in terms of its real and imaginary parts:

$$\frac{1}{q(z)} = \frac{1}{R(z)} - \mathrm{i}\frac{\lambda}{\pi w^2(z)}. \tag{20}$$

2. $R(z)$ is the *radius of curvature of the wave fronts*:

$$R(z) = z + \frac{b^2}{z}. \tag{21}$$

3. $w(z)$ is a sort of effective radius of the Gaussian beam (often called the *spot size*) at position z, which is the $1/e$ point of the electric field (a.k.a the $1/e$ field radius and also the $1/e^2$ intensity radius):

$$w^2(z) = \frac{2b}{k}\left(1 + \frac{z^2}{b^2}\right) \tag{22}$$

4. This parameter $q(z)$, which may be regarded as a sort of "complex radius of curvature", contains all of the information we need to figure out what is happening with the field of a Gaussian mode of given frequency! If we know q at some plane $z = z_{\mathrm{o}}$, we can find the entire field distribution at plane $z = z'$ by just plugging in

$$q(z') = q(z_{\mathrm{o}}) + z' - z_{\mathrm{o}}. \tag{23}$$

   This is far easier than using Eq. 16, and allows us to trivially propagate a Gaussian mode forward and backward if q is known somewhere else in space.

5. $\varphi(z)$ is called the *Gouy phase* and comes from splitting up the front factor into phase and magnitude:

$$-\frac{\mathrm{i}}{q(z)} = \sqrt{\frac{2}{kb}}\,\frac{1}{w(z)}\,\mathrm{e}^{-\mathrm{i}\varphi(z)} \tag{24}$$

   where

$$\varphi(z) \equiv \arctan\left[\frac{z}{b}\right]. \tag{25}$$

6. Looking at Eq. 22, we can see that the *minimum beam waist* (or simply the *beam waist*) w_{o} is determined entirely by the wavelength and the confocal parameter:

$$w_{\mathrm{o}} = \sqrt{\frac{\lambda b}{\pi}} \tag{26}$$

   and we can write our earlier definitions in terms of w_{o}:

   - $$b = z_{\mathrm{R}} = \frac{\pi w_{\mathrm{o}}^2}{\lambda} \tag{27}$$

- $$w^2(z) = w_o^2 \left[1 + \left(\frac{\lambda z}{\pi w_o^2} \right)^2 \right] \tag{28}$$

- $$\frac{1}{R(z)} = \frac{z}{z^2 + \left(\frac{\pi w_o^2}{\lambda} \right)^2} \tag{29}$$

- $$\varphi(z) = \arctan \left[\frac{\lambda z}{\pi w_o^2} \right] \tag{30}$$

7. The *divergence* of the mode is a function of its confocal parameter, which is typically stated as the far-field (full-cone) divergence angle θ in terms of the (minimum) beam waist,
$$\theta = \frac{2\lambda}{\pi w_o}. \tag{31}$$

3 ABCD matrices

The complex q-parameter of a beam at a certain location tells us everything we need to know about that beam. $\mathbb{R}\{q\}$ is the distance to the plane where the minimum waist occurs, and the imaginary part, $\mathbb{I}\{q\} \equiv b$, gives the size of that minimum waist (via Eq. 26), and so on. This gives us enough information to propagate a mode through free space forward and backward to any plane we want, because we only need to plug in the new z-location to $\mathbb{R}\{q\}$. The confocal parameter b remains fixed, and is a property of the particular mode.

However, what do we do if there is an optical element that *changes* the confocal parameter, such as a lens? We need to be able to find the transformation laws that relate q at some known plane in the system to q' at some other plane, possibly with a bunch of optics in between.

For a free-space distance L, this is easy:

$$q' = q + L. \tag{32}$$

For something like a thin lens, however, it is more complicated. The transformation that relates q right before the lens to q' right after it is

$$q' = \frac{q}{-\frac{q}{f} + 1}. \tag{33}$$

It turns out that for all of the linear optics examples we are going to discuss, the transformation can always be written as

$$q' = \frac{Aq + B}{Cq + D}. \tag{34}$$

The beauty of this discovery is that if we cascade two optical elements,

$$\begin{aligned} q' &= \frac{A_1 q + B_1}{C_1 q + D_1} \\ q'' &= \frac{A_2 q' + B_2}{C_2 q' + D_2} \end{aligned} \tag{35}$$

the resulting transformation between the input and the output of the composite system (which can be written in the same form, of course) is

$$q'' = \frac{(A_2 A_1 + B_2 C_1) q + (A_2 B_1 + B_2 D_1)}{(C_2 A_1 + D_2 C_1) q + (C_2 B_1 + D_2 D_1)}. \tag{36}$$

The terms in parentheses are the effective ABCD coefficients for the composite system, and the point is that they can be calculated directly through matrix multiplication:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix} \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}. \tag{37}$$

These matrices are eloquently named "ABCD matrices," and are *identical* to the ABCD matrices of ray optics. We can calculate the ABCD matrix for a whole series of optics through simple matrix multiplication, and a classic review article that talks about how to use these for cavities is given by Kogelnik and Li [1]. Note that these matrices don't really operate on an obvious vector involving q, but rather that they give you a way to figure out what A, B, C, and D are for an optical system so that you can use Eq. 34 to figure out what this does to q.

3.1 Connection to ray optics

The ABCD matrices may actually be familiar to you if you have spent some time with ray optics. In ray optics, if we wish to find the relationship between a ray in a certain plane to where it appears in another plane, we need to know two things about it. First, we need to know how far the ray is from the optical axis at the plane of interest, which we can denote by r. Second, we need to know which way it points, which in the paraxial approximation we can say is given by its slope where is crosses the plane of interest, denoted by r'. The ABCD matrices connect two such pairs together via

$$\begin{pmatrix} r_2 \\ r_2' \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} r_1 \\ r_1' \end{pmatrix}. \tag{38}$$

So what is the connection here? We seem to be describing very different concepts (ray optics and Gaussian waves), and the way we use these ABCD matrices is even very different (compare Eq. 34 to Eq. 38).

If we look at the *ratio* of these two ray parameters r/r', we find that this ratio transforms as

$$\frac{r_2}{r_2'} = \frac{A\left(\frac{r_1}{r_1'}\right) + B}{C\left(\frac{r_1}{r_1'}\right) + D}. \tag{39}$$

which is exactly the same form as our definition of the ABCD parameters for Gaussian beams, Eq. 34.

So what is this ratio, physically? A little geometry can convince you that this ratio is the distance between the plane of interest and the plane where the ray crosses the optical axis. So it is the "distance to the focus" in some sense, and it was by allowing this distance z to be complex (at which point we started calling it q) that we were able to find convenient ways to describe Gaussian beams.

3.2 Examples of ABCD matrices

1. Free propagation over a distance $\Delta z = L$:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \tag{40}$$

2. Planar dielectric interface:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{n_1}{n_2} \end{pmatrix} \tag{41}$$

3. Thin lens with reference planes right at the lens

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \tag{42}$$

4. Thin lens with front reference plane ℓ_1 in front of the lens and back reference plane ℓ_2 behind the lens

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 - \frac{\ell_2}{f} & \ell_1 + \ell_2 - \frac{\ell_1 \ell_2}{f} \\ -\frac{1}{f} & 1 - \frac{\ell_1}{f} \end{pmatrix} \tag{43}$$

5. Nearly-normal incidence reflection from a concave spherical mirror with radius of curvature R_{oc}. (For a *convex* mirror, R_{oc} is negative):

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{2}{R_{oc}} & 1 \end{pmatrix} \tag{44}$$

6. Convex spherical dielectric interface with radius of curvature R_{oc}. (For concave interface, R_{oc} is negative.)

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{n_1 - n_2}{n_2 R_{oc}} & \frac{n_1}{n_2} \end{pmatrix} \tag{45}$$

3.3 properties of ABCD matrices

- Using our definitions, it turns out that if the input plane is in a medium with refractive index n_1 and the output plane is in refractive index n_2, the determinant of any ABCD matrix is given by
$$\det \begin{vmatrix} A & B \\ C & D \end{vmatrix} = \frac{n_1}{n_2}. \tag{46}$$
- For imaging, there is of course the concept of an image being "in focus", meaning that an object at the input plane will create an image at the output plane. In this case, thinking about the ray optics interpretation of the ABCD matrices is helpful. If the input and output planes really are the object and image planes of the system, r_2 cannot depend upon r_1', since *all* rays that originate at r_1 will end up at r_2, regardless of where they point. This condition is satisfied if $B = 0$.
- For an imaging system that is in focus, the transverse magnification M is given by
$$M = \frac{r_2}{r_1} = A. \tag{47}$$

4 Higher Order Beams

The electric field of a Gaussian mode (Eq. 19) is only one possible solution to the paraxial wave equation. The general solution can be written a couple of ways, and we will simply state the solutions here.

The most common basis to use for writing the solutions to Eq. 11 is to write them in terms of the Cartesian coordinates x and y, which yields the Hermite-Gaussian modes, given here in their fully-general form (just to get this written down in one place):

$$\mathbf{E} = \hat{\boldsymbol{\psi}} \sum_{n,p} \mathcal{E}_{n,p} \frac{w_0}{w(z)} H_n\left(\frac{\sqrt{2}\,x}{w(z)}\right) H_p\left(\frac{\sqrt{2}\,y}{w(z)}\right) \mathrm{e}^{-\mathrm{i}(n+p+1)\varphi(z)}\, \mathrm{e}^{-\frac{x^2+y^2}{w^2(z)}}\, \mathrm{e}^{\mathrm{i}k\frac{x^2+y^2}{2R(z)}}\, \mathrm{e}^{\mathrm{i}(kz-\omega t)}. \tag{48}$$

H_j is the j^{th} Hermite polynomial and n and p are non-negative integers. The Hermite-Gaussian transverse electromagnetic (TEM) modes are often referred to with the notation $\text{TEM}_{n,p}$. The Gaussian beam is $\text{TEM}_{0,0}$, and the $\text{TEM}_{n,p}$ Hermite-Gaussian mode roughly resembles a rectangular array of spots with $(n+1)$ "rows" along x and $(p+1)$ "columns" along y. We also note that the Gouy phase is now $(n+p+1)$ times larger than the $\text{TEM}_{0,0}$ mode, an observation that will become important when we talk about optical cavities.

Last, there is another basis for writing the full solutions to the paraxial wave equation. (I am only bringing this up because I want you to be aware of it, so don't worry too much about catching all the subtle details.) You will notice that the Hermite-Gaussian modes above are not rotationally symmetric; x and y are somehow specified, and it is probably not hard to imagine that a little bit of generalization of Eq. 48 would allow you to write modes with different confocal parameters in the two directions. In many situations, this type of symmetry breaking happens naturally due to tiny imperfections that lift the rotation symmetry about the optical axis. However, strictly speaking, the more natural solutions to the paraxial wave equation should be rotationally symmetric about the optical axis, so we should be able to replace x and y with cylindrical coordinates ρ and ϕ (note the difference between this symbol ϕ and the Gouy phase $\varphi(z)$). These solutions are the Laguerre-Gaussian modes:

$$\mathbf{E} = \hat{\boldsymbol{\psi}} \sum_{\ell,m} \mathcal{E}_{\ell,m} \frac{w_0}{w(z)} \left(\frac{\sqrt{2}\,\rho}{w(z)}\right)^m L_m^\ell\!\left(\frac{\sqrt{2}\,\rho}{w(z)}\right) \mathrm{e}^{-\mathrm{i}(2\ell+m+1)\varphi(z)}\, \mathrm{e}^{-\frac{\rho^2}{w^2(z)}}\, \mathrm{e}^{\mathrm{i}k\frac{\rho^2}{2R(z)}}\, \mathrm{e}^{\mathrm{i}m\phi}\, \mathrm{e}^{\mathrm{i}(kz-\omega t)} \tag{49}$$

where the L_m^ℓ are the generalized Laguerre polynomials and ℓ and m follow the same rules as they do for atomic angular momentum. Note that the lowest-order Laguerre-Gaussian mode is the same as the lowest-order Hermite-Gaussian mode, Eq. 19.

5 Preview of what's to come

After all of this discussion of how to solve for the paraxial modes of free space, we never figured out how to constrain the confocal parameter, b. How do we figure out what b is? The answer is that the confocal parameter is typically chosen by *boundary conditions* of each situation of interest. For instance, a single transverse mode laser beam will come out of the laser with certain confocal parameter b. Propagation through free-space will not change b, so what chose this value in the first place? The answer is that the boundary conditions of the laser cavity itself only support resonant modes with a certain confocal parameter. We will work with this in detail in later labs, and you will use the formalism here to be able to solve for b in a given cavity.

Last, if the beam exiting a laser has a certain confocal parameter, can we do anything to it to change b for that beam? Well, let's find out by looking at what happens if we send a beam with a certain $q(z) = z + \mathrm{i}b$ through a simple thin lens with focal length f. Using Eq. 42 and Eq. 34, we get

$$q' = \frac{z + \mathrm{i}b}{-\frac{z+\mathrm{i}b}{f} + 1} = \frac{z - \frac{z^2+b^2}{f}}{\left(1 - \frac{z}{f}\right)^2 + \left(\frac{b}{f}\right)^2} + \mathrm{i}\frac{b}{\left(1 - \frac{z}{f}\right)^2 + \left(\frac{b}{f}\right)^2}, \tag{50}$$

which clearly has a different imaginary part than q. For instance, if $z = 0$ in the equation above, the lens is right at the focus of the input beam and the lens transforms the input mode (with confocal parameter b) into an output mode with a confocal parameter of $\mathbb{I}\{q\} = f^2 b/(b^2 + f^2)$.

References

[1] H. Kogelnik and T. Li. Laser beams and resonators. *Applied Optics*, 5:1550, 1966.

[2] Joseph T. Verdeyen. *Laser Electronics*. Prentice Hall, 3rd edition, 1995.