A gun can fire shells in any direction with a fixed speed v_0. Show that by adjusting the angle of fire, it can hit any object inside a surface given by

$$z = \frac{v_0^2}{2g} - \frac{g}{2v_0^2}(x^2 + y^2) \tag{1}$$

with the origin of the coordinate system at the location of the gun.

1 Firing Shells at Constant Speed

Here we would like to find the set of all points that lie on some trajectory of a shell fired with fixed speed v_0, but at an arbitrary angle α to the horizontal. In particular we would like to find an equation for the surface that can just be reached by such an trajectory (such every point inside the volume bounded by this surface lies on some trajectory, but no point outside this surface does).

First we note that there is rotational symmetry about the z axis. Therefore we have essentially a two-dimensional problem and will choose x as our horizontal coordinate from here on.

The equations of motion are:

$$x(t) = v_0 t \cos\alpha, \tag{1}$$

$$z(t) = v_0 t \sin\alpha - \frac{1}{2} g t^2, \tag{2}$$

Eliminating t we have the trajectory in parametric form:

$$z = x\tan\alpha - \frac{gx^2}{2v_0^2\cos^2\alpha}. \tag{3}$$

The trick now is not to look at one particular trajectory (since we don't a priori know which point on it might touch the required surface), but instead to look at all possible trajectories at once and maximize some appropriate distance over all possible values of α. E.g. we may choose to look at fixed x coordinate and maximize the height z reached at that x:

$$\left.\frac{\partial z}{\partial \alpha}\right|_x = \frac{x}{\cos^2\alpha} - \frac{gx^2}{v_0^2}\frac{\sin\alpha}{\cos^3\alpha} = 0 \qquad \Rightarrow \qquad \tan\alpha = \frac{v_0^2}{gx}. \tag{4}$$

Substituting back this expressions for $\tan\alpha$ we obtain the equation for the "boundary" surface we were looking for:

$$z = \frac{v_0^2}{2g} - \frac{gx^2}{2v_0^2}, \tag{5}$$

which becomes the required expression once we reinstate the y coordinate by replacing $x^2 \to x^2 + y^2$.

Alternatively one might choose to write the surface in polar coordinates as $r(\theta)$ and then maximize r over all α at fixed θ. This gives

$$\alpha = \frac{\pi}{4} + \frac{\theta}{2}, \tag{6}$$

and converting back to Cartesian coordinates leads to the same result.