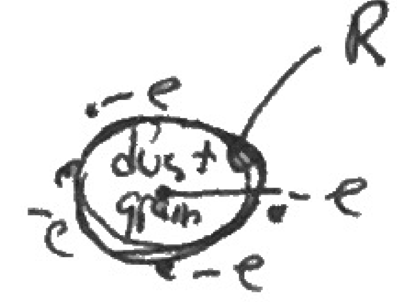
#1) PM 2.38

$R = 3\times10^{-7}\,m$



Let Q be the total (negative) charge on the grain, and write $Q=-Ne$ where $N=$ # of electrons and $e=1.6\times10^{-19}\,C$. We want to find N.

We know $\dfrac{Q}{4\pi\varepsilon_0 R} = -.15\,V$ (since the grain is a sphere at $-.15V$ potential, as given in the problem)

Thus $Q=-Ne = (-.15V)\cdot 4\pi\cdot\varepsilon_0\cdot R$ $\quad(\varepsilon_0 = 8.85\times10^{-12}\,m^{-3}kg^{-1}s^2C^2)$

$$\Rightarrow N = \frac{(-.15V)\cdot 4\pi\cdot\left(8.85\times10^{-12}\,\frac{C^2 s^2}{m^3 kg}\right)\cdot 3\times10^{-7}\,m}{1.6\times10^{-19}\,C}$$

$$\Rightarrow \boxed{N \simeq 31 \text{ electrons}}$$

At the surface, $\vec{E} = \dfrac{Q}{4\pi\varepsilon_0 R^2}\hat{r} = -\dfrac{(31\cdot 1.6\times10^{-19}\,C)}{4\pi\cdot 8.85\times10^{-12}\cdot(3\times10^{-7})^2}\hat{r}\ N/C$

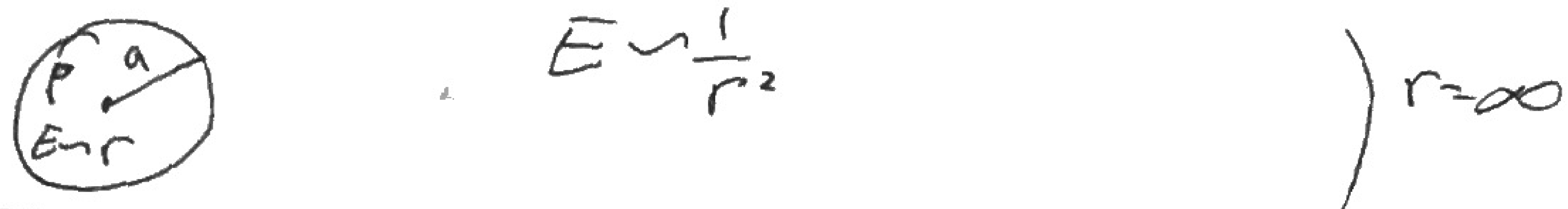
$$\Rightarrow \boxed{\vec{E} \simeq -5\times10^{5}\ N/C}$$

(Note, could also compute $\vec{E} = -\dfrac{\overset{\text{potential}}{.15V}}{R}$ to get the same answer).

#2) PM 2.4Q) To get the potential at the center of the sphere, we need to integrate up the potential from $r=\infty$ to $r=0$. However, the form of the E-field is different in the $a \le r < \infty$ region than it is in the $0 = r < a$ region, ~~Also~~ so we need to be careful.

Namely $(Q = 79e)$

$(a = 6\times10^{-15}\,m)$



for $r \geq a$, $\boxed{\vec{E}_{\text{outside}} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{r} = \frac{79e}{4\pi\varepsilon_0 r^2}\hat{r}.}$

charges enclosed in a sphere of radius r.

For $0 \le r < a$, Gauss says $\oint_{S(r)} \vec{E}\cdot d\vec{A} = \frac{Q_{enc}}{\varepsilon_0}$.

$S(r)$: sphere of radius r

$Q_{enc} = \frac{4}{3}\pi r^3 \rho$

$\Rightarrow Q_{enc} = 79e\frac{r^3}{a^3}$

$\rho = \frac{Q_{TOT}}{\frac{4}{3}\pi a^3} = \frac{79\cdot 3e}{4\pi a^3}$

$\Rightarrow \rho = \frac{237e}{4\pi a^3}$

$$\Rightarrow \quad E\cdot 4\pi r^2 = \frac{79e}{\varepsilon_0}\frac{r^3}{a^3}$$

$$\Rightarrow \boxed{\vec{E}_{\text{inside}} = \frac{79e}{4\pi\varepsilon_0}\frac{r}{a^3}\hat{r}.}$$

Thus, (Take $d\vec{l} = dr\,\hat{r}$

$$\phi(r=0) = -\int_\infty^0 \vec{E}\cdot d\vec{l} = -\int_\infty^a \vec{E}_{\text{outside}}\cdot d\vec{l} - \int_a^0 \vec{E}_{\text{inside}}\cdot d\vec{l}$$

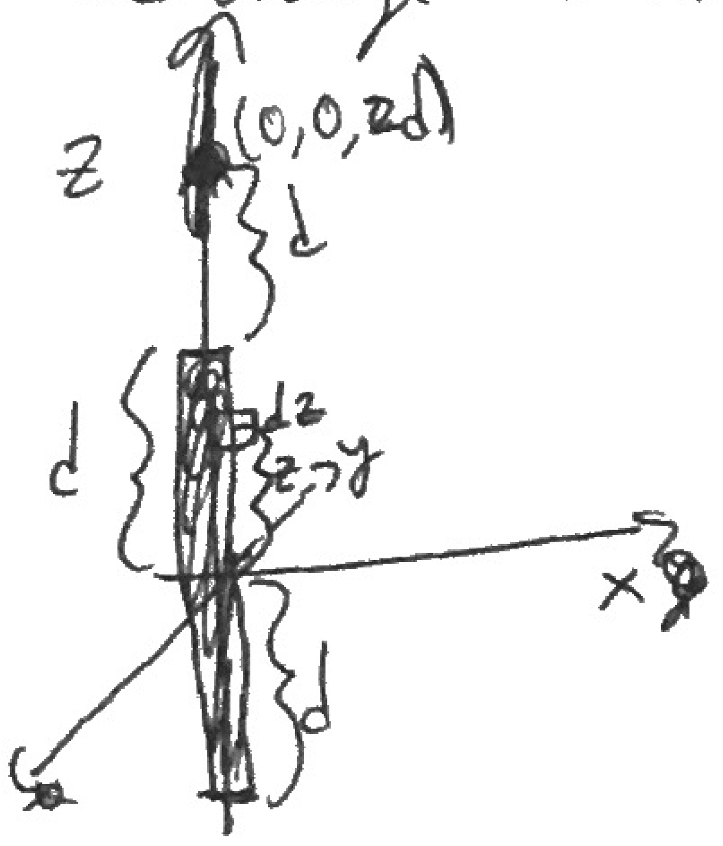
$$\Rightarrow \phi(r=0) = -\int_\infty^a \frac{79e}{4\pi\varepsilon_0}\frac{dr}{r^2} - \int_a^0 \frac{79e}{4\pi\varepsilon_0 a^3}\,r\,dr$$

$$\Rightarrow \phi(r=0) = \frac{-79e}{4\pi\varepsilon_0}\Big[\underbrace{\int_\infty^a \frac{dr}{r^2}}_{-\frac{1}{r}\big|_\infty^a} + \frac{1}{a^3}\underbrace{\int_a^0 r\,dr}_{\frac{1}{2}r^2\big|_a^0 = -\frac{a^2}{2}}\Big] = \frac{79e}{4\pi\varepsilon_0}\Big[\frac{1}{a} + \frac{1}{2a}\Big]$$

Thus $\boxed{\phi(0) = \frac{3}{2}\cdot\frac{79e}{4\pi\varepsilon_0 a} \simeq 28 \text{ megavolts}}$

#3) PM 2.43) As we know, the potential ϕ from a point charge q at a distance r from the origin when $\phi=0$ is defined at $r=\infty$ is given by $\phi(r) = \frac{q}{4\pi\varepsilon_0 r}$. (Note: all of these requirements are important).

As is usually the case for problems like these, we want to break up the distribution of charge (here, the rod) into lot of point charges dq ($=\lambda dz$ in this case) ~~then integrate~~ compute the $d\phi$ from this dq at our observation point, then integrate over all the dq's, i.e., over the whole charge distribution. Here, we have:

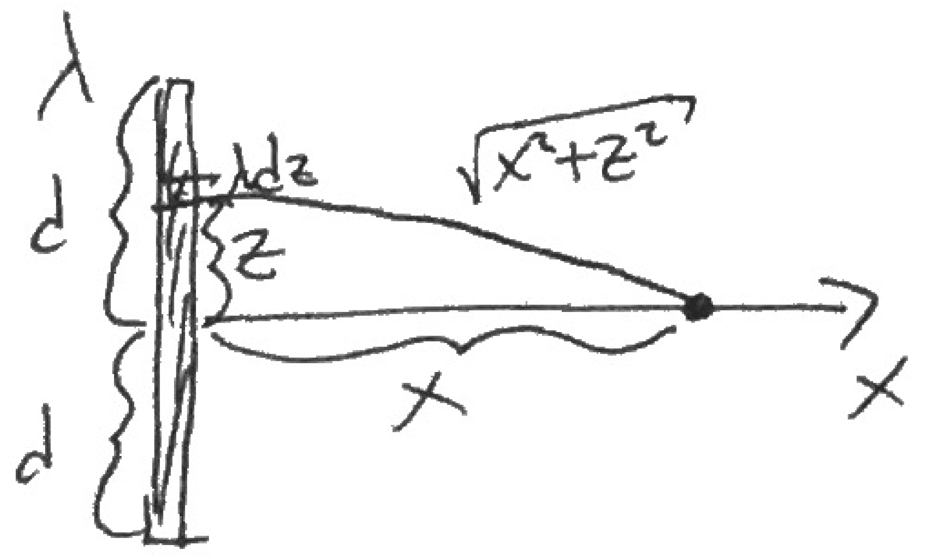


Consider the point charge λdz located on the rod a distance z from the origin. It is then $2d-z$ from our observation point $(0,0,2d)$. Thus, the contribution $d\phi$ from this λdz at $(0,0,2d)$ is $d\phi = \frac{\lambda dz}{4\pi\varepsilon_0 (2d-z)}$. We now just need to integrate this from $z=-d$ to $z=d$ to get ϕ at $(0,0,2d)$. We therefore have

$$\phi = \int d\phi = \frac{\lambda}{4\pi\varepsilon_0}\int_{-d}^{d}\frac{dz}{2d-z} \underset{\substack{\text{let } u=2d-z \\ du=-dz}}{=} \frac{-\lambda}{4\pi\varepsilon_0}\int_{3d}^{d}\frac{du}{u} = \frac{-\lambda}{4\pi\varepsilon_0}\ln\left(\frac{d}{3d}\right)$$

Thus $\boxed{\phi(0,0,2d) = \frac{\lambda}{4\pi\varepsilon_0}\ln(3)}$ (Note: independent of d! Is this surprising? It should not be surprising, can you see why it's not surprising?)

We now have our observation point at $(x, 0, 0)$:



Now, the contribution from each λdz at $(x,0,0)$ is given by

$$d\phi = \frac{\lambda\, dz}{4\pi\varepsilon_0 \sqrt{x^2+z^2}}.$$

Thus, $\phi(x,0,0) = \dfrac{\lambda}{4\pi\varepsilon_0}\displaystyle\int_{-d}^{d} \frac{dz}{\sqrt{x^2+z^2}}$ Look up: $\displaystyle\int_{-d}^{d} \frac{dz}{\sqrt{x^2+z^2}} = \ln\left(\sqrt{x^2+z^2}+z\right)\Big|_{-d}^{d}$.

integrating over ~~the charge~~ distribution

$$\cancel{= \frac{\lambda}{4\pi\varepsilon_0 x}\int_{-d}^{d} \frac{dz}{\sqrt{1+\frac{z^2}{x^2}}}}$$

$$\cancel{= \frac{\lambda}{2\pi\varepsilon_0 x}\int_{0}^{d} \frac{dz}{\sqrt{1+\frac{z^2}{x^2}}}}$$

$$\Rightarrow \boxed{\phi(x,0,0) = \frac{\lambda}{4\pi\varepsilon_0}\ln\left(\sqrt{x^2+z^2}+z\right)\Big|_{-d}^{d} = \frac{\lambda}{4\pi\varepsilon_0}\ln\left(\frac{\sqrt{x^2+d^2}+d}{\sqrt{x^2+d^2}-d}\right)}$$

For $\phi(x,0,0) = \phi(0,0,2d)$, we need $\dfrac{\sqrt{x^2+d^2}+d}{\sqrt{x^2+d^2}-d} = 3$

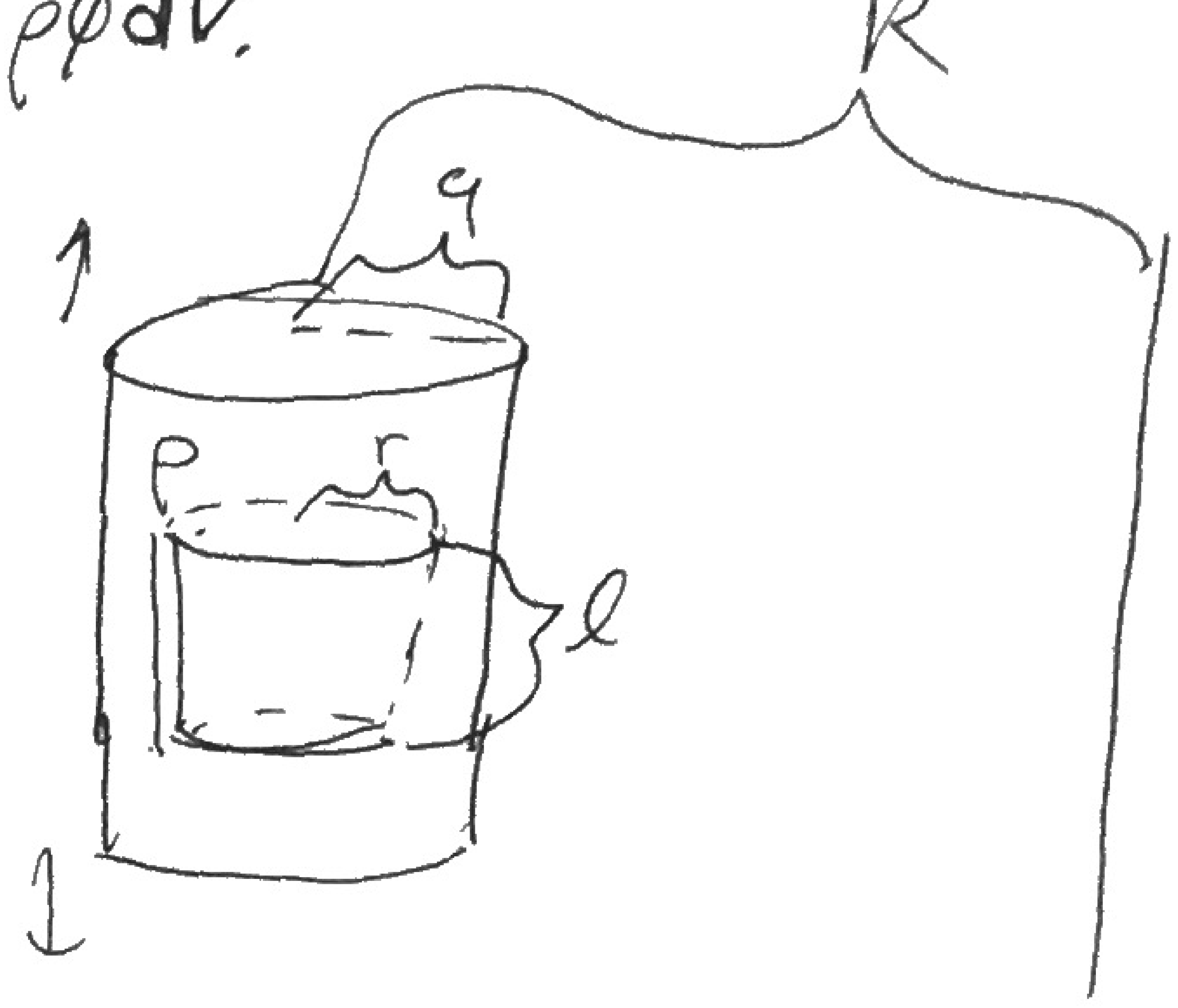
$$\Rightarrow \sqrt{x^2+d^2}+d = 3\sqrt{x^2+d^2} - 3d$$

$$\Rightarrow 4d = 2\sqrt{x^2+d^2}$$

$$\Rightarrow 4d^2 = x^2+d^2$$

$$\Rightarrow \boxed{x = \sqrt{3}\,d}$$

#4) PM 2.59) Plan: First compute the $\vec{E}$-field outside of the cylinder, then integrate that up to get the potential, then use Eq. (2.32) as suggested, which says $U = \frac{1}{2}\int \rho \phi \, dV$.



We know that for $r > a$ we can view the cylinder as a line charge with $\lambda = \pi a^2 \rho$, since the total charge in the cylinder of length L is $Q_{TOT} = \pi a^2 L \rho$ so that the charge per length $\lambda = \frac{Q_{TOT}}{L} = \pi a^2 \rho$.

Thus, from Gauss's Law, and/or problem set 1 we have

$$\vec{E}_{out}(r > a) = \frac{\lambda}{2\pi\varepsilon_0 r}\underset{\text{cylindrical } \hat{r}}{\hat{r}} = \frac{\rho a^2}{2\varepsilon_0 r}\hat{r}. \Rightarrow \boxed{\vec{E}_{out} = \frac{\rho a^2}{2\varepsilon_0 r}\hat{r}}$$

Again using Gauss's law for $r < a$ we have

$$\oint \vec{E} \cdot d\vec{A} = E \cdot \underbrace{2\pi r l}_{\substack{\text{surface area of} \\ \text{cylinder w/ radius } r \\ \text{length } l \text{ (see picture)}}} = \frac{Q_{enc}}{\varepsilon_0} = \frac{\rho \cdot \pi r^2 l}{\varepsilon_0} \Rightarrow \boxed{\vec{E}_{inside}(r < a) = \frac{\rho}{2\varepsilon_0} r \hat{r}.}$$

Now, for $U = \frac{1}{2}\int \rho \phi \, dV$, we note that $\rho = 0$ outside of the cylinder, so we only need to know ϕ inside the cylinder. We again need to be careful and break up ϕ into the integral from R to a and then from a to some $r < a$.

$$\phi(r<a) = -\int_R^{r<a} \vec{E}\cdot d\vec{l} = -\int_R^a \vec{E}\cdot d\vec{l} - \int_a^r \vec{E}\cdot d\vec{l} \quad \text{where } d\vec{l} = dr\,\hat{r}$$

is chosen to be

Thus $$\phi(r<a) = -\int_R^a \frac{\rho a^2}{2\varepsilon_0 r}\,dr - \int_a^r \frac{\rho}{2\varepsilon_0} r\,dr$$

$$= -\frac{\rho}{2\varepsilon_0}\left[a^2\int_R^a \frac{dr}{r} + \int_a^r r\,dr\right]$$

$$= -\frac{\rho}{2\varepsilon_0}\left[a^2 \ln(a/R) + \tfrac{1}{2}(r^2 - a^2)\right].$$

integrates to 2π

$$dV = r\,dr\,d\phi\,dz$$

integrates to L

Now ρ is constant so $$U = \frac{1}{2}\int \rho\phi\,dV = -\frac{\rho^2}{4\varepsilon_0}\int_{\text{cylinder}} \left[a^2\ln(a/R) + \tfrac{1}{2}(r^2-a^2)\right] dV$$

volume of cylinder of length L

only integrate over cylinder of length L to get U per length.

$$= -\frac{\rho^2}{4\varepsilon_0}\cdot 2\pi \cdot L \int_0^a \left(a^2\ln(a/R) + \tfrac{1}{2}(r^2-a^2)\right) r\,dr$$

$$\Rightarrow U = -\frac{\rho^2 \pi L}{2\varepsilon_0}\left[a^2\ln(a/R)\underbrace{\int_0^a r\,dr}_{\frac{1}{2}a^2} + \frac{1}{2}\underbrace{\int_0^a r^3\,dr}_{\frac{1}{4}a^4} - \frac{1}{2}a^2\underbrace{\int_0^a r\,dr}_{\frac{1}{2}a^2}\right]$$

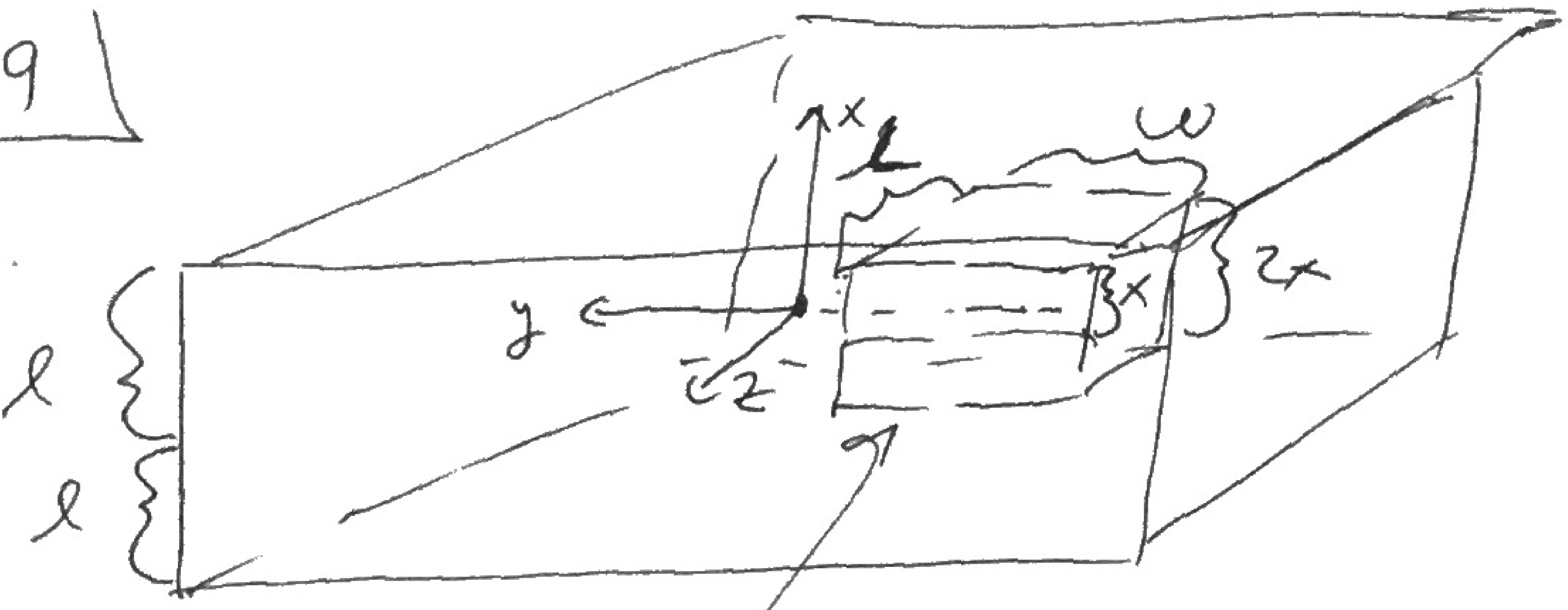
$$\Rightarrow \frac{U}{L} = \frac{-\pi\rho^2}{2\varepsilon_0}\left[\tfrac{1}{2}a^4\ln(a/R) + \tfrac{1}{8}a^4 - \tfrac{1}{4}a^4\right]$$

$$= \frac{-\pi\rho^2}{2\varepsilon_0}\left[-\tfrac{1}{2}a^4\ln(R/a) - \tfrac{1}{8}a^4\right]$$

$$\Rightarrow \boxed{\frac{U}{L} = \frac{1}{4}\frac{\pi a^4\rho^2}{\varepsilon_0}\left[\frac{1}{4} + \ln(R/a)\right] = \frac{\lambda^2}{4\pi\varepsilon_0}\left[\frac{1}{4} + \ln(R/a)\right]}$$

as desired, where $\underline{\lambda = \pi a^2 \rho}$

#5 PM 2.69



Gaussian surface centered on $x=0$ plane, i.e. has height $2x$, width w, length (into and out of page) L.

Due to symmetry the E-field must point perpendicularly out through the top ($x=$ constant) and bottom faces. (outward if $\rho>0$, inward if $\rho<0$)

Thus $\oint_{\text{rectangular prism}} \vec{E}\cdot d\vec{A} = 2\cdot w \cdot L E = \frac{Q_{enc}}{\varepsilon_0} = \frac{\rho\cdot \overbrace{2x\cdot w\cdot L}^{\text{volume of rectangular prism}}}{\varepsilon_0}$

$\Rightarrow \boxed{\vec{E}_{in} = \frac{\rho x}{\varepsilon_0}\hat{x} \text{ for } |x|<l}$ (note: this points in the correct direction both for $\rho>0$ and $\rho<0$, as well as for $x>0$, $x<0$).

For $|x|>l$, this sheet acts as a surface charge with surface charge density $\sigma = \rho\cdot 2l$. To see this, note that in a rectangular patch of width Δy and length Δz (and height $2l$), the total charge enclosed is $Q_{TOT} = 2l\Delta y\rho$

Thus, $\sigma = \frac{Q_{TOT}}{\Delta y \Delta z} = 2l\rho$. We found in problem set #1 that the $\vec{E}$ from an infinite sheet of charge is $\boxed{\vec{E} = \frac{\sigma}{2\varepsilon_0}\hat{x} = \frac{\rho l}{\varepsilon_0}\hat{x} \text{ for } x>l}$

and $\boxed{\vec{E} = -\frac{\sigma}{\varepsilon_0}\hat{x} = -\frac{\rho l}{\varepsilon_0}\hat{x} \text{ for } x<l}$

(b) With $\phi = 0$ at $x = 0$, we have (with $d\vec{l} = dx\,\hat{x}$)

for $x < l$: $\phi(x) = -\int_0^x \vec{E}\cdot d\vec{l} = -\frac{\rho}{\epsilon_0}\int_0^x x'\,dx' = -\frac{\rho}{2\epsilon_0}x^2$

$$\Rightarrow \boxed{\phi_{in}(x<l) = -\frac{\rho}{2\epsilon_0}x^2}$$

Now, for $x > l$, we have $\phi(x) = -\int_0^l \vec{E}\cdot d\vec{l} - \int_l^x \vec{E}\cdot d\vec{l}$

need to consider $x<-l$ separately because $\vec{E}$ is different for $x>l$ and $x<-l$

$$= -\int_0^l \frac{\rho}{\epsilon_0}x'\,dx' - \int_l^x \frac{\rho l}{\epsilon_0}\,dx'$$

$$\Rightarrow \phi(x) = -\frac{1}{2}\frac{\rho}{\epsilon_0}l^2 - \frac{\rho l}{\epsilon_0}(x-l)$$

$$\Rightarrow \boxed{\phi(x) = \frac{\rho}{2\epsilon_0}l^2 - \frac{\rho l x}{\epsilon_0}} \quad (\text{for } x>l.$$

Similarly, for $x < -l$, we have

$$\phi(x) = -\int_0^{-l}\vec{E}\cdot d\vec{l} - \int_{-l}^{x}\vec{E}\cdot d\vec{l} = -\frac{\rho}{\epsilon_0}\int_0^{-l}x'\,dx' - \int_{-l}^{x}\frac{-\rho l}{\epsilon_0}\,dx'$$

important minus sign!

$$= -\frac{\rho l^2}{2\epsilon_0} + \frac{\rho l}{\epsilon_0}(x+l)$$

$$\Rightarrow \boxed{\phi(x) = \frac{\rho l^2}{2\epsilon_0} + \frac{\rho l x}{\epsilon_0} \quad \text{for } x<-l.}$$

More succinctly, for $|x| > l$, we have $\boxed{\phi_{out}(|x|>l) = \frac{\rho l^2}{2\epsilon_0} - \frac{\rho l |x|}{\epsilon_0}}$

$\vec{E}_{in} = \frac{\rho x}{\varepsilon_0}\hat{x}$ from pg 7

(c) For $\vec{E}_{in}$, we have $\nabla\cdot\vec{E}_{in} = \partial_x E_x = \partial_x\left(\frac{\rho x}{\varepsilon_0}\right) = \frac{\rho}{\varepsilon_0}$, as desired. ✓

For $\vec{E}_{out}$, we have $\nabla\cdot\vec{E}_{out} = \partial_x E_x = \partial_x 0 = 0$ and $\rho = 0$ for $|x| > l$ as well, thus indeed $\varepsilon_0\nabla\cdot\vec{E}_{out} = \rho_{out}\,(=0)$ ✓

For ϕ_{in}, we have $\nabla^2\phi_{in}(x) = \partial_x^2\phi = \partial_x^2\left(-\frac{\rho}{2\varepsilon_0}x^2\right) = -\frac{\rho}{\varepsilon_0}$, as desired ✓.

For ϕ_{out}, either for $x > l$ or $x < -l$, we see that ϕ_{out} is <u>linear</u> in x, thus $\nabla^2\phi_{out} = \partial_x^2\phi_{out} = 0$, in line with the fact that $\rho = 0$ out here, thus $-\varepsilon_0\nabla^2\phi_{out} = \rho\,(=0)$, as desired.

~~#~~6) PM 2.72

We have $\phi = \begin{cases} \frac{\rho_0}{4\pi\varepsilon_0}(x^2+y^2+z^2) & \text{for } x^2+y^2+z^2<a^2 \\ \frac{\rho_0}{4\pi\varepsilon_0}\left(-a^2+\frac{2a^3}{(x^2+y^2+z^2)^{1/2}}\right) & \text{for } x^2+y^2+z^2>a^2 \end{cases}$

This is more conveniently written in spherical coordinates as:

$$\phi(r) = \begin{cases} \frac{\rho_0}{4\pi\varepsilon_0} r^2 & \text{for } r<a \\ \frac{\rho_0}{4\pi\varepsilon_0}\left(-a^2+\frac{2a^3}{r}\right) & \text{for } r>a. \end{cases}$$

We have $\vec{E} = -\nabla\phi = -\frac{d}{dr}\phi\,\hat{r}$

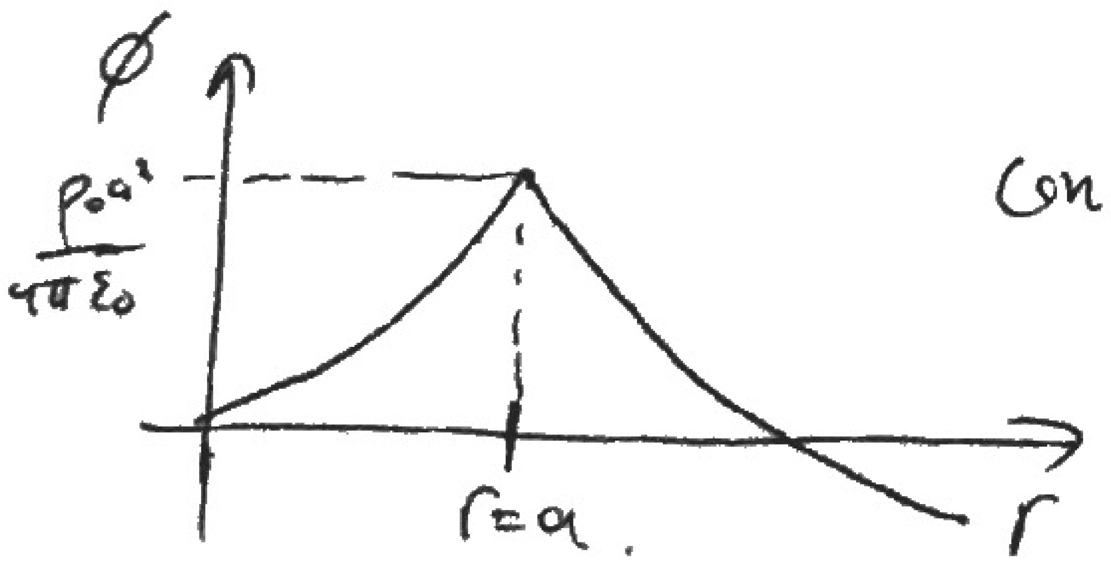
in spherical coordinates for a function only of r

Thus for $r<a$, $\vec{E} = -\frac{\rho_0}{2\pi\varepsilon_0} r\hat{r}$

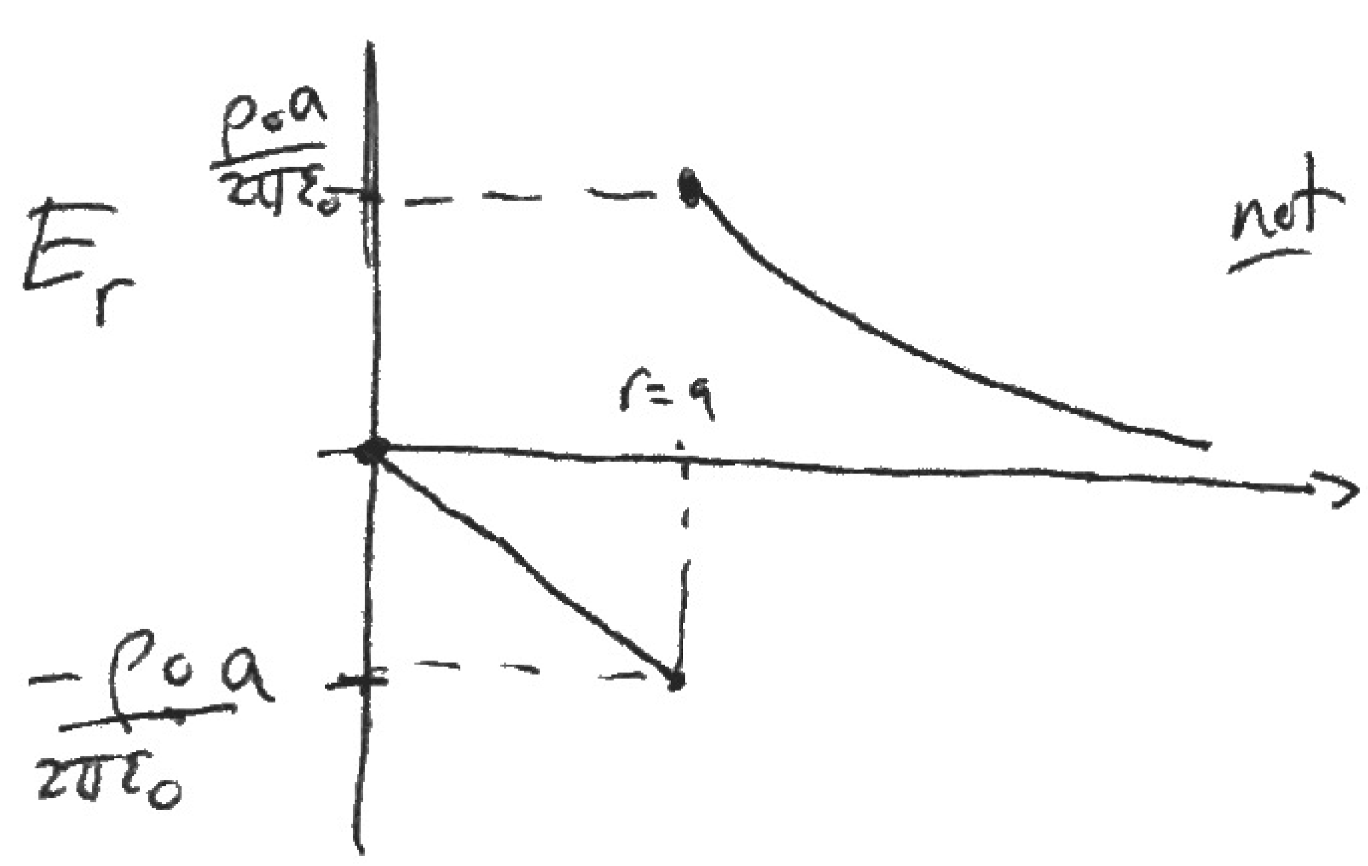
and for $r>a$, $\vec{E} = \frac{\rho_0 a^3}{2\pi\varepsilon_0 r^2}\hat{r}$. i.e.,

$$\boxed{\vec{E} = \begin{cases} -\frac{\rho_0}{2\pi\varepsilon_0} r\hat{r} & \text{for } r<a \\ \frac{\rho_0 a^3}{2\pi\varepsilon_0 r^2}\hat{r} & \text{for } r>a. \end{cases}}$$

Thus, we have



continuous but not smooth



not continuous.

Now, for ρ, we use the fact that $\rho = -\varepsilon_0 \nabla^2 \phi$.

In spherical coordinates for a function only of r, we have

$$\nabla^2 \phi = \frac{1}{r^2}\partial_r \left(r^2 \partial_r \phi\right)$$

$$\left(\phi_{in} = \frac{\rho_0}{4\pi\varepsilon_0} r^2\right) \text{ (for } r<a)$$

Thus, $\nabla^2 \phi_{in} = \frac{1}{r^2}\partial_r\left(r^2 \cdot \frac{\rho}{4\pi\varepsilon}\cdot 2r\right) = \frac{\rho}{2\pi\varepsilon_0}\frac{1}{r^2}\partial_r\left(r^3\right) = \frac{3}{2}\frac{\rho_0}{\pi\varepsilon_0}$.

$$\boxed{\text{Thus, for } r<a, \quad \rho(r) = -\frac{3}{2}\frac{\rho_0}{\pi}}$$

For $r>a$,

$$\nabla^2\phi = \frac{1}{r^2}\partial_r\left(r^2\partial_r\left(\frac{\rho_0}{4\pi\varepsilon}\left(-a^2+\frac{2a^3}{r}\right)\right)\right)$$

$$= \frac{1}{r^2}\partial_r\left(r^2\cdot\left(-\frac{\rho_0 a^3}{2\pi\varepsilon_0 r^2}\right)\right)$$

$$= \frac{1}{r^2}\partial_r\left(\underbrace{\frac{-\rho_0 a^3}{2\pi\varepsilon}}_{\text{constant!}}\right) = 0.$$

$$\boxed{\text{Thus for } r>a, \quad \rho(r)=0}$$

However, $\rho(r)$ is not continuous at $r=a$, because $\nabla^2\phi$ is not continuous there. Indeed, there is a surface charge at $r=a$, which can be computed using an infinitesimally small Gaussian pillbox. Namely, let us consider an infinitesimally small box around a portion of the sphere:

In the limit that this box is infinitely small, the sphere looks like an infinite plane, in which case we know $\vec{E}$ on either side is $|\vec{E}| = \frac{\sigma}{2\varepsilon_0}$

Now, $\int \vec{E}\cdot d\vec{A}$ over this small box is $E_{top}\cdot A - E_{bottom}\cdot A$,

where $E_{top} = \cancel{E(r>a)} \lim_{r\to a} E(r>a)$ and $E_{bottom} = \lim_{r\to a} E(r<a)$

so that $E_{top} = \frac{\rho_0 a}{2\pi\varepsilon_0}$ and $E_{bottom} = -\frac{\rho_0 a}{2\pi\varepsilon_0}$.

Thus $\int \vec{E}\cdot d\vec{A} = \frac{\rho_0 a}{2\pi\varepsilon_0} A - \left(-\frac{\rho_0 a}{2\pi\varepsilon_0}\right) A = \frac{\rho_0 a A}{\pi\varepsilon_0}$. $(*)$

We also know (from Gauss's Law) $\int \vec{E}\cdot d\vec{A} = \frac{Q_{enc}}{\varepsilon_0}$ $(**)$ where now $Q_{enc} = \sigma A$ where σ is the surface charge density that we're lookin for.

Thus $(*)$ and $(**)$ $\Rightarrow$ $\frac{\rho_0 a A}{\pi\varepsilon_0} = \frac{\sigma A}{\varepsilon_0} \Rightarrow \sigma = \frac{\rho_0 a}{\pi}$.

Thus,

$$\boxed{\rho = \begin{cases} -\frac{3}{2}\frac{\rho_0}{\pi} & \text{for } r<a \\ 0 & \text{for } r>a \end{cases} \quad \text{and there is a surface charge of } \sigma = \frac{\rho_0 a}{\pi} \text{ at } r=a.}$$

#7) PM 2.75

(a) $\vec{F} = (x+y, -x+y, -2z) \Rightarrow F_x = x+y,\ F_y = -x+y,\ F_z = -2z$

$$\Rightarrow \nabla\cdot\vec{F} = \partial_x F_x + \partial_y F_y + \partial_z F_z = \partial_x(x+y) + \partial_y(-x+y) + \partial_z(-2z)$$

$$= 1 + 1 - 2 = 0.$$

Thus $\boxed{\nabla\cdot\vec{F} = 0}$

$$\nabla\times\vec{F} = (\partial_y F_z - \partial_z F_y,\ \partial_z F_x - \partial_x F_z,\ \partial_x F_y - \partial_y F_x)$$
$$= (0-0,\ 0-0,\ -1-1) = (0,0,-2). \Rightarrow \boxed{\nabla\times\vec{F} = (0,0,-2)}$$

(b) $\vec{G} = (\underbrace{2y}_{G_x}, \underbrace{2x+3z}_{G_y}, \underbrace{3y}_{G_z}) \Rightarrow \nabla\cdot\vec{G} = \underbrace{\partial_x G_x}_{0} + \underbrace{\partial_y G_y}_{0} + \underbrace{\partial_z G_z}_{0} = 0 \to \boxed{\nabla\cdot\vec{G} = 0}$

$$\nabla\times\vec{G} = (\partial_y G_z - \partial_z G_y,\ \partial_z G_x - \partial_x G_z,\ \partial_x G_y - \partial_y G_x)$$
$$= (3-3,\ 0-0,\ 2-2) = (0,0,0) \Rightarrow \boxed{\nabla\times\vec{G} = (0,0,0)}$$

Thus there exists a g such that $\vec{G} = \nabla g$. One possible g is found by integrating up $\vec{G}$ from $(0,0,0)$ to some (a,b,c) along a path that first goes in the x-direction, then the y, then the z. We have

$$g(a,b,c) = \int_{(0,0,0)}^{(a,b,c)} \vec{G}\cdot d\vec{s} = \int_0^a G_x(x,0,0)\,dx + \int_0^b G_y(a,y,0)\,dy + \int_0^c G_z(a,b,z)\,dz$$
$$= \int_0^a 0\cdot dx + \underbrace{\int_0^b (2a+0)\,dy}_{2ab} + \underbrace{\int_0^c 3b\,dz}_{3bc}$$
$$= 2ab + 3bc.$$

Thus $\boxed{g(x,y,z) = 2xy + 3yz}$. $\longrightarrow$

One can straightforwardly check that indeed $\vec{G} = \nabla(2xy+3yz)$
$= \nabla g$.

(c) $\vec{H} = (\overbrace{x^2-z^2}^{H_x}, \overbrace{z}^{H_y}, \overbrace{2xz}^{H_z})$

$$\Rightarrow \nabla\cdot\vec{H} = 2x + 0 + 2x = 4x. \Rightarrow \boxed{\nabla\cdot\vec{H} = 4x}$$

$$\nabla\times\vec{H} = (\partial_y H_z - \partial_z H_y, \partial_z H_x - \partial_x H_z, \partial_x H_y - \partial_y H_x)$$
$$= (0-0, -2z-2z, 0-0) = (0,-4z,0).$$

Thus $\boxed{\nabla\times\vec{H} = (0,-4z,0)}$