105A - Set 4 - Solutions

(Grades are out of 150)

1. A disk of mass M and radius R rolls without slipping down a plane inclined from the horizontal by an angle α, in the presence of gravity. The disk has a short weightless axle of negligible radius. From this axis is suspended a simple pendulum of length $l < R$ and whose bob has a mass m. Consider that the motion of the pendulum takes place in the plane of the disk, and find the Lagrange's equation of the system.
   Hint: make a sketch of the system.
   Answer The first thing to do, after you sketch the problem is to choose your coordinate system, see Fig. 4 Naively we have two set of coordinates, one for the pendulum (x_p, y_p),

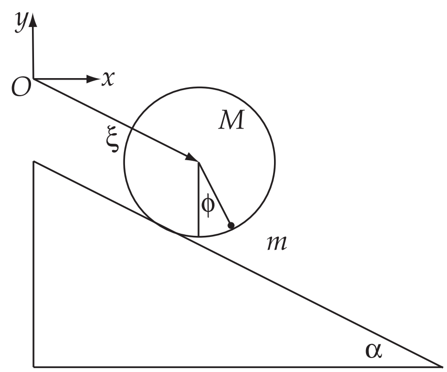

Figure 1: A disk and a pendulum on an inclined plane.

which the describes the position of the mass m in the pendulum and one for the disk x_d, y_d, for the center of mass of the disk. These coordinates can be connected by the following transformations:

$$\begin{aligned} x_d &= \xi\cos\alpha & (1)\\ y_d &= -\xi\sin\alpha & (2)\\ x_p &= l\sin\phi + \xi\cos\alpha & (3)\\ y_p &= -l\cos\phi - \xi\sin\alpha & (4) \end{aligned}$$

The time derivatives are:

$$\begin{aligned} \dot{x}_d &= \dot{\xi}\cos\alpha & (5)\\ \dot{y}_d &= -\dot{\xi}\sin\alpha & (6)\\ \dot{x}_p &= \dot{\phi}l\cos\phi + \dot{\xi}\cos\alpha & (7)\\ \dot{y}_p &= \dot{\phi}l\sin\phi - \dot{\xi}\sin\alpha & (8) \end{aligned}$$

This system has two degrees of freedom. The kinetic energy is given by

$$T = T_{disk} + T_{pendulum} = \frac{1}{2}M\left(\dot{x}_d^2 + \dot{y}_d^2\right) + \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}m\left(\dot{x}_p^2 + \dot{y}_p^2\right)\ , \qquad (9)$$

where $I = MR^2/2$ is the moment of inertia around the axis of rotation and $\theta = \xi/R$. Substituting the coordinates we defined for the pendulum and the disk we can write:

$$\begin{aligned}
T &= \frac{1}{2}M\left(\dot{\xi}^2\cos^2\alpha + \dot{\xi}^2\sin^2\alpha\right) + \frac{1}{4}MR^2\frac{\dot{\xi}^2}{R^2} & (10)\\
&+ \frac{1}{2}m\left(\dot{\phi}^2l^2\cos^2\phi + \dot{\xi}^2\cos^2\alpha + 2\dot{\phi}l\cos\phi\dot{\xi}\cos\alpha + \dot{\phi}^2l^2\sin^2\phi + \dot{\xi}^2\sin^2\alpha - 2\dot{\phi}l\sin\phi\dot{\xi}\sin\alpha\right)\\
&\quad \text{arranging :}\\
&= \frac{1}{2}M\dot{\xi}^2 + \frac{1}{4}M\dot{\xi} + \frac{1}{2}m\left(\dot{\phi}^2l^2 + \dot{\xi}^2 + 2\dot{\phi}l\dot{\xi}\cos(\phi+\alpha)\right) & (11)\\
& & (12)
\end{aligned}$$

which we can write as:

$$T = \frac{1}{2}(M+m)\dot{\xi}^2 + \frac{1}{4}M\dot{\xi} + \frac{1}{2}m\dot{\phi}^2l^2 + m\dot{\phi}l\dot{\xi}\cos(\phi+\alpha)\ , \tag{13}$$

The potential energy is given by

$$U = U_{disk} + U_{pendulum} = Mgy_d + mgy_p = -Mg\xi\sin\alpha - mg(l\cos\phi + \xi\sin\alpha)\ , \tag{14}$$

after arranging we can write:

$$U = -(M+m)g\xi\sin\alpha - mgl\cos\phi\ . \tag{15}$$

The Lagrangian is:

$$L = T - U = \left(\frac{3}{4}M + \frac{1}{2}m\right)\dot{\xi}^2 + \frac{1}{2}ml^2\dot{\phi}^2 + m\dot{\phi}l\dot{\xi}\cos(\phi+\alpha) + (M+m)g\xi\sin\alpha + mgl\cos\phi\ . \tag{16}$$

To calculate the equation of motion we first consider the following derivatives:

$$\begin{aligned}
\frac{\partial L}{\partial \xi} &= (M+m)g\sin\alpha & (17)\\
\frac{\partial L}{\partial \phi} &= -m\dot{\phi}l\dot{\xi}\sin(\phi+\alpha) - mgl\sin\phi & (18)\\
\frac{\partial L}{\partial \dot{\xi}} &= \left(\frac{3}{2}M + m\right)\dot{\xi} + m\dot{\phi}l\cos(\phi+\alpha) & (19)\\
\frac{\partial L}{\partial \dot{\phi}} &= ml^2\dot{\phi} + ml\dot{\xi}\cos(\phi+\alpha) & (20)
\end{aligned}$$

And then:

$$\begin{aligned}
\frac{d}{dt}\frac{\partial L}{\partial \dot{\xi}} &= \left(\frac{3}{2}M + m\right)\ddot{\xi} + m\ddot{\phi}l\cos(\phi+\alpha) - m\dot{\phi}^2l\sin(\phi+\alpha) & (21)\\
\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} &= ml^2\ddot{\phi} + ml\ddot{\xi}\cos(\phi+\alpha) - ml\dot{\xi}\dot{\phi}\sin(\phi+\alpha) & (22)
\end{aligned}$$

The resulting equations of motion for our two generalized coordinates are

$$\xi \;:\; (M+m)g\sin\alpha = \left(\frac{3}{2}M+m\right)\ddot{\xi} + m\ddot{\phi}l\cos(\phi+\alpha) - m\dot{\phi}^2 l\sin(\phi+\alpha) \tag{23}$$

$$\phi \;:\; -m\dot{\phi}l\dot{\xi}\sin(\phi+\alpha) - mgl\sin\phi = ml^2\ddot{\phi} + ml\ddot{\xi}\cos(\phi+\alpha) - ml\dot{\xi}\dot{\phi}\sin(\phi+\alpha) \tag{24}$$

After arranging we get:

$$\xi \;:\; \left(\frac{3}{2}M+m\right)\ddot{\xi} - (M+m)g\sin\alpha + ml[\ddot{\phi}\cos(\phi+\alpha) - \dot{\phi}^2\sin(\phi+\alpha)] = 0 \tag{25}$$

$$\phi \;:\; \ddot{\phi} + \frac{g}{l}\sin\phi + \frac{1}{l}\ddot{\xi}\cos(\phi+\alpha) = 0 \tag{26}$$

2. Two blocks, each of mass M, are connected by an extension less, uniform string of length l. One bloc is places on a smooth horizontal surface and the other block hangs over the side. The string passing over a frictionless pulley (see Figure 3). For the following, (i) write the Lagrangian of the system, (ii) find the equation of motion (iii) and find the solution.

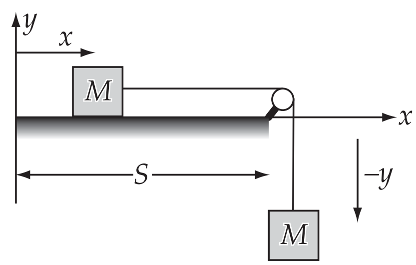

Figure 2: Two masses connected by a string with length l.

(a) when the mass of the string is negligible.
Answer: Let the length of the string be l so that

$$(S-x) - y = l\ , \tag{27}$$

where S is defined in the figure. So that:

$$\dot{x} = -\dot{y}\ . \tag{28}$$

(i) The Lagrangian of the system is

$$L = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}M\dot{y}^2 - Mgy = M\dot{y}^2 - Mgy\ , \tag{29}$$

where in the last transition we used equation (55). Therefore Lagranges equation for y is (since there is one degree of freedom y is the general coordinate):

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} = 2M\ddot{y} + Mg = 0\ , \tag{30}$$

which we can write as:
(ii)

$$\ddot{y} = -\frac{g}{2} \ , \tag{31}$$

(iii) The solution is:

$$y(t) = \frac{-g}{4}t^2 + C_1 t + C_2 \ , \tag{32}$$

using the initial conditions of $y(t=0) = 0$ and $\dot{y}(t=0) = 0$ we have:

$$y(t) = \frac{-g}{4}t^2 \ , \tag{33}$$

(b) when the string has a mass m.
Answer: If the string has a mass m, we must consider its kinetic energy and potential energy. These are

$$T_{string} = \frac{1}{2}m\dot{y}^2 \ , \tag{34}$$

and to calculate the potential energy on the string we first need to calculate the tension (the string is uniform): $T = gy(m/l)$ and than

$$U_{string} = \int_y^0 \frac{m}{l} gy = -\frac{mg}{2l}y^2 \ , \tag{35}$$

(i) So the Lagrangian of the system is

$$\begin{aligned} L &= M\dot{y}^2 - Mgy + T_{string} - U_{string} = M\dot{y}^2 - Mgy + \frac{1}{2}m\dot{y}^2 + \frac{mg}{2l}y^2 \qquad (36) \\ &= \left(M + \frac{1}{2}m\right)\dot{y}^2 - Mgy + \frac{mg}{2l}y^2 \ , \qquad (37) \end{aligned}$$

Therefore, Lagranges equation for y now becomes

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} = (2M + m)\ddot{y} + Mg - \frac{mg}{l}y = 0 \ , \tag{38}$$

(ii) Arranging this we can write the equation of motion as:

$$(2M + m)\ddot{y} = \frac{mg}{l}\left(y - \frac{Ml}{m}\right) \ , \tag{39}$$

Note that:

$$\frac{d^2}{dt^2}\left(y - \frac{Ml}{m}\right) = \frac{d^2y}{dt^2} \ , \tag{40}$$

So we can define a new variable

$$u = \left(y - \frac{Ml}{m}\right) \ , \tag{41}$$

so the E.O.M is simply

$$\ddot{u} = \frac{mg}{l(2M+m)} u \ , \tag{42}$$

and the solution is:

$$y - \frac{Ml}{m} = u(t) = Ae^{\alpha t} + Be^{-\alpha t} \ , \tag{43}$$

where

$$\alpha = \sqrt{\frac{mg}{l(2M+m)}} \tag{44}$$

given the initial conditions $y(t=0) = 0$ and $\dot{y}(t=0) = 0$ we get: $A = B = -Ml/(2m)$ so

(iii)

$$y(t) = \frac{Ml}{m}(1 - \cosh(\alpha t)) \ , \tag{45}$$

3. A ladder of mass m and length $2l$ is standing up against a vertical wall with initial angleα relative to the horizontal. There is no friction between the ladder and the wall or the floor. The ladder begins to slide down with zero initial velocity. Denote by$\theta(t)$ the angle the ladder makes with the horizontal after it starts to slide.

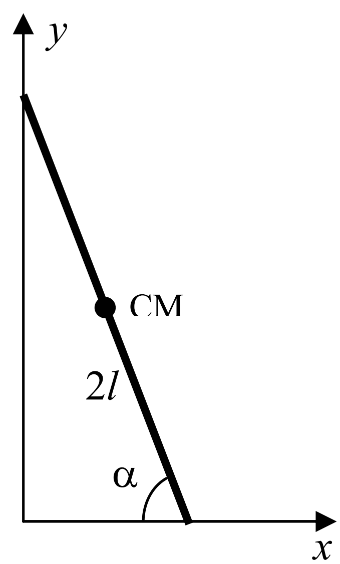

Figure 3: A sliding ladder

(a) Write down Lagrangian equations of motion with two constraints describing the contact of the ladder with the vertical wall and the floor. Use the coordinates x, y and θ.

Answer: We'll start with the kinetic energy:

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I\dot{\theta}^2 \ , \tag{46}$$

We need to find the moment of inertia $I = \int r^2 dm$ so this is for a uniform rod about the center of mass, see online material if you don't remember how to do this, in other words:

$$I_{cm} = \int_{-l}^{l} r^2 dm = \int_{-l}^{l} r^2 \frac{m}{2l} dr = \frac{r^3}{3}\frac{m}{2l}\Big|_{-l}^{l} = m\frac{l^2}{3} \tag{47}$$

so

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}m\frac{l^2}{3}\dot{\theta}^2 \ , \tag{48}$$

the potential energy is:

$$U = mgy \tag{49}$$

and the Lagrangian is

$$L = T - U = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}m\frac{l^2}{3}\dot{\theta}^2 - mgy \tag{50}$$

The constrains are:

$$g_1 = l\cos\theta - x = 0 \tag{51}$$

$$g_2 = l\sin\theta - y = 0 \tag{52}$$

So:

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} + \lambda_1 \frac{\partial g_1}{\partial x} = 0 \tag{53}$$

$$\frac{\partial L}{\partial y} - \frac{d}{dt}\frac{\partial L}{\partial \dot{y}} + \lambda_2 \frac{\partial g_2}{\partial y} = 0 \tag{54}$$

and

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} + \lambda_1 \frac{\partial g_1}{\partial \theta} + \lambda \frac{\partial g_2}{\partial \theta} = 0 \tag{55}$$

Plugging in the derivatives we get:

$$-m\ddot{x} - \lambda_1 \quad = \quad 0 \tag{56}$$

$$-mg - m\ddot{y} - \lambda_2 \quad = 0 \tag{57}$$

$$-\frac{1}{3}ml^2\ddot{\theta} - \lambda_1 l\sin\theta + \lambda_2\cos\theta \quad = \quad 0 \tag{58}$$

(b) Find the expression for $\ddot{\theta}$ as a function of θ.
Hint: think, what is the generalized coordinate here?
Answer Differentiating the constraint equations we get:

$$\dot{x} \quad = \quad -l\sin\theta\dot{\theta} \tag{59}$$

$$\dot{y} \quad = \quad l\cos\theta\dot{\theta} \tag{60}$$

plugging them into the equation for kinetic energy we get:

$$T = \frac{1}{2}ml^2(\sin^2\theta + \cos^2\theta)\dot{\theta} + \frac{1}{6}ml^2\dot{\theta}^2 = \frac{2}{3}ml^2\dot{\theta}^2 \tag{61}$$

and the potential energy is

$$U = mgl\sin\theta \tag{62}$$

The lagrangian is:

$$L = T - U = U = \frac{2}{3}ml^2\dot{\theta}^2 - mgl\sin\theta \tag{63}$$

We can now just write the Euler Lagrange equation for θ, **where now we don't need the constrains any more!!**

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = 0 \tag{64}$$

So

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = -mgl\cos\theta - \frac{4}{3}ml^2\ddot{\theta} = 0 \tag{65}$$

In other words:

$$\ddot{\theta} = \frac{-3g}{4l}\cos\theta \tag{66}$$

(c) Find the expression for $\dot{\theta}^2$.
Hint: use energy conservation
Answer: Using energy conservation we have

$$mlg\sin\alpha = mlg\sin\theta + \frac{2}{3}ml^2\dot{\theta}^2 \tag{67}$$

so from that we get:

$$\dot{\theta}^2 = \frac{3g}{2l}(\sin\alpha - \sin\theta) \tag{68}$$

(d) **Bonus + 15pt** Find the angle θ_c when the ladder losses contact with the vertical wall.
Answer The constraint force given by λ_1 is equal to the normal force from the vertical wall. We'll find when it goes to zero.

$$\lambda_1 = -m\ddot{x} = ml\sin\theta\ddot{\theta} + ml\cos\theta\dot{\theta}^2 = 0 \tag{69}$$

which we can write:

$$\sin\theta\ddot{\theta} + \cos\theta\dot{\theta}^2 = \frac{-3g}{4l}\cos\theta\sin\theta + \frac{3g}{2l}(\sin\alpha - \sin\theta)\cos\theta = 0 \tag{70}$$

where in the last transition we plugged in the expressions we found. Solving this we find:

$$\sin\theta_c = \frac{2}{3}\sin\alpha \tag{71}$$

4. We consider the gravitational forces created on particles by different mass distributions. If the mass distribution has a particular symmetry, so will the potential associated with the force, and so will the Lagrangian. Since symmetries are associated with conserved quantities through Noethers theorem, we can find the conserved quantities: translational symmetries are associated with components of the linear momentum; rotational symmetries with components of the angular momentum, and time independence with conservation of energy. Since the potential is fixed in all cases (i.e., independent of time), the energy is conserved in all systems.

   (a) The mass is uniformly distributed in the plane $z = 0$ (an infinite, flat, Earth): the forces do not depend on the coordinates x, y, and thus the components of the linear momentum p_x, p_y will be conserved. Also, the force is invariant under a rotation about the z axis, so L_z is conserved.

   (b) The mass is uniformly distributed in the half plane $z = 0, y > 0$ (a finite, flat, Earth, like Columbus feared): theres only translational symmetry with respect to x, and no rotational symmetries: only p_x will be conserved.

   (c) The mass is uniformly distributed in a circular cylinder of infinite length, with axis along the z-axis: the configuration has translational symmetry along z, and rotational symmetry about z, so p_z and L_z are conserved.

   (d) The mass is uniformly distributed in a circular cylinder of finite length, with axis along the z-axis: there is now no translational symmetry, but there is still rotational symmetry about z: only L_z is conserved.

   (e) The mass is uniformly distributed in a right cylinder of elliptical cross section and infinite length, with axis along the z axis: there is now no rotational symmetry, but because the cylinder is infinite along z, there is translational symmetry along z: only p_z is conserved.

   (f) The mass is uniformly distributed in a dumbbell whose axis is oriented along the z axis: no translational symmetries, but there is rotational symmetry about the z axis, so L_z is conserved.

   (g) The mass is the form of a uniform wire wound in the geometry of an infinite helical solenoid, with axis along the z axis. There are no pure translational or rotational symmetries, but there is a symmetry combining a z-translation of distance h (the distance between coils), and a rotation about z of 2π. Thus, although p_z or L_z are not individually conserved, $hp_z + L_z$ will be conserved.