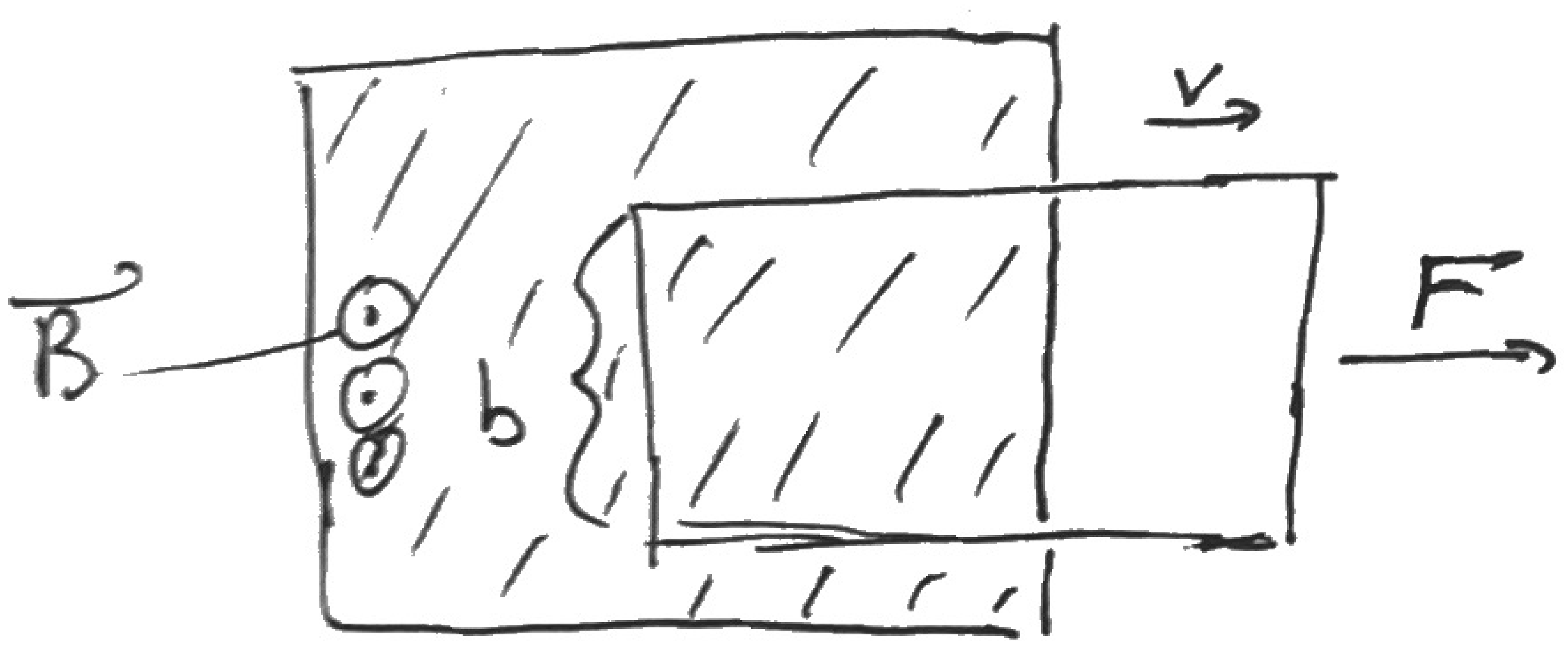
Problem Set #7

#1) PM 7.24 Suppose the wire has total perimeter l and let b the length of the side that is fully above the magnet:



(Note: we ignore signs in general, using Lenz's law whenever we need to worry about the directionality).

The current in the wire is given by $I = \mathcal{E}/R$ where $\mathcal{E} = \text{Emf} = \frac{d\Phi}{dt}$ where Φ = magnetic flux through the frame at any given time.

We readily see that $\frac{d\Phi}{dt} = B \cdot b \cdot v$ ($|B|$ times (constant) vertical distance b times $\frac{d}{dt}$ of the horizontal distance, which is v).

Thus $I = \frac{Bbv}{R}$. Now, with a diameter d of the wire, and $R = \rho\frac{l}{A}$, we have $R = \rho\frac{l}{\pi(d/2)^2} = 4\rho\frac{l}{\pi d^2}$. Thus $I = \frac{\pi B b v d^2}{4\rho l}$.

Now, since there is current flowing through the wire, which is in the ~~magnet~~ presence of a $\vec{B}$-field, there is a force on the wire, but the force on the top and bottom of the wire will cancel (as can be seen by the right hand rule), thus the only force that isn't cancelled is the force along the side of length b.

Thus, $F = \left|\int I\, d\vec{l} \times \vec{B}\right| = IbB$ (due to the constancy of B and I.)

we have $\boxed{F = \frac{\pi B^2 b^2 v d^2}{4\rho l}}$. (Note: due to constant force, we have constant v).

~~We therefore see that if the force~~

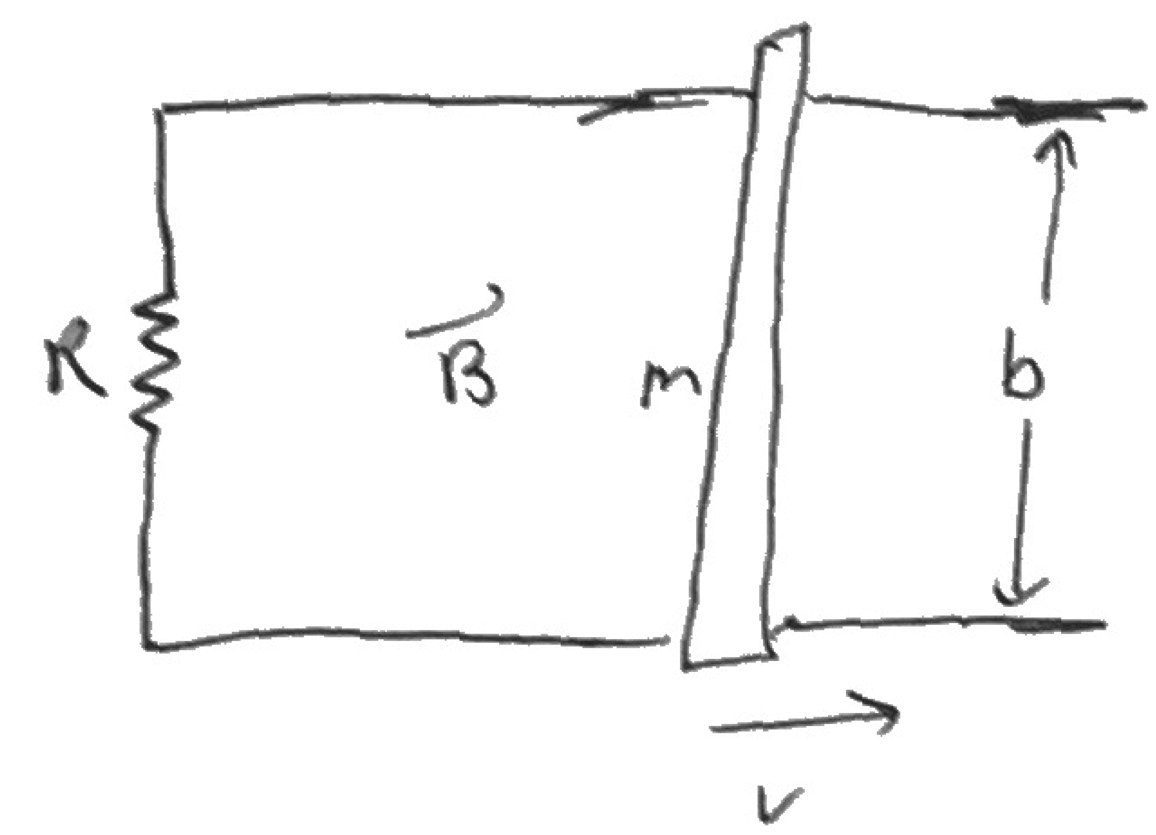
From the expression $F = \frac{\pi B^2 b^2 v d^2}{4 \rho l}$, we see that

if the force is doubled, so is the velocity and thus the time is cut in half, so if for a 1N force the time required is 1 sec, then for a 2N force the time required is .5 sec. (1st blank in question)

If ρ is doubled and everything else stays the same (namely, the thickness of the wire and the time required), we see that the required force is halved, to ½ N. (2nd blank)

If the diameter is doubled and all else stays the same, the required force ~~is~~ increases by a factor of $2^2 = 4$ to 4N. (3rd blank)

#2) PM 7.26



a) Let $v(t)$ be the velocity of the bar at time $t \geq 0$. $(v(0) = v_0)$

magnetic flux through rectangular part of circuit

We then have that $\frac{d\Phi}{dt} = Bbv$ (for the same reasons as #1).

$$\mathcal{E} = \frac{d\Phi}{dt} \Rightarrow \mathcal{E} = Bbv \quad \text{and} \quad I = \mathcal{E}/R \Rightarrow I = \frac{Bbv}{R}.$$

Here, the metal bar is acting as a wire with current I through it in the presence of $\vec{B}$, and thus the induced force on it is $F = IbB = \frac{b^2B^2v}{R}$.

Due to Lenz's law, this force opposes the motion, so we have

$$\boxed{m\frac{dv}{dt} = -F = -\frac{b^2B^2}{R}v} \Rightarrow \frac{dv}{v} = -\frac{b^2B^2}{mR}dt$$

$$\Rightarrow \int_{v_0}^{v(t)} \frac{dv'}{v'} = -\frac{b^2B^2}{mR}\int_0^t dt'$$

$$\Rightarrow \ln\left(\frac{v(t)}{v_0}\right) = -\frac{b^2B^2}{mR}t \Rightarrow \boxed{v(t) = v_0 e^{-\frac{b^2B^2}{mR}t}} = v_0 e^{-t/T} \text{ with } T = \frac{mR}{b^2B^2}$$

So in an ideal world (i.e., no friction or thermal fluctuations) the bar never truly stops but rather slows down exponentially forever.

b) Even though the bar never truly stops, it does only travel a finite distance and approaches the total displacement Δx given by

$$\Delta x = \int_{t=0}^{t=\infty} v(t)\,dt = v_0 \int_0^{\infty} e^{-\frac{b^2B^2 t}{mR}}\,dt = -\frac{mRv_0}{b^2B^2}\,\underbrace{e^{-\frac{b^2B^2 t}{mR}}\Big|_0^{\infty}}_{\underbrace{e^{-\infty}-e^{-0}}_{-1}}$$

$$\Rightarrow \boxed{\Delta x = \frac{mRv_0}{b^2B^2}}$$

(Sanity checks: does this make sense in the limits of $B \to 0, \infty$

$v_0 \to \infty$

$R \to \infty$ ← This one is interesting

$b \to 0, \infty$?)

c) The initial KE (and thus total energy) is $\frac{1}{2}mv_0^2$. After an infinite time, nothing is moving, so we need to check that all this energy was dissipated in the heating of the resistor. We have

$$E_{dissipated} = -\int_0^{\infty} \overset{\text{power dissipated}}{P}\,dt = -\int_0^{\infty} I^2 R\,dt = -\int_0^{\infty}\left(\frac{Bbv}{R}\right)^2 R\,dt = -\frac{B^2b^2}{R}\int_0^{\infty} v^2\,dt$$

$$= -\frac{B^2b^2}{R}v_0^2 \int_0^{\infty} e^{-\frac{2B^2b^2 t}{mR}}\,dt = -\frac{B^2b^2}{R}v_0^2 \cdot \left(\frac{-mR}{2B^2b^2}\right)\underbrace{e^{-\frac{2B^2b^2}{mR}}\Big|_0^{\infty}}_{-1}$$

$$\Rightarrow \boxed{E_{dissipated} = \frac{1}{2}mv_0^2}$$, as desired.

#3

PM 7.40) Here we are only meant to do the part involving assuming the B field is constant throughout the solenoid.

We recall the definition of inductance: $\mathcal{E} = L\frac{dI}{dt}$ where $\mathcal{E}$ is the emf and I is the current (with L the inductance).

Suppose that at some moment, the current through the solenoid is I, so that $B_{\text{inside solenoid}} = \mu_0 n I = \mu_0\left(\frac{N}{L}\right)I$ where N is the total number of turns and L is the total length of the solenoid.

The magnetic flux through the solenoid, (being sure to account for each turn!) is given by $\Phi = \underbrace{\pi r^2}_{\text{cross sectional area of solenoid}} \cdot \underbrace{N}_{\text{total \# of turns}} \cdot B$

so that $\Phi = \mu_0 \pi r^2 \frac{N^2}{L} I$.

Thus $\mathcal{E} = \frac{d\Phi}{dt} = \mu_0 \pi r^2 \frac{N^2}{L}\frac{dI}{dt}$ (everything else — μ_0, π, r, N, L — are constant)

Thus we read off that $\boxed{L = \frac{\pi \mu_0 N^2 r^2}{L} \simeq 7.1\times10^{-3}\ H}$

"Henry"s

#4) PM 7.42 $\qquad R = .01\,\Omega$, $L = .5\times10^{-3}\,H$

Battery has $\mathcal{E} = 12V$.

From equation (7.69) in PM we have $I(t) = I_0\left(1 - e^{-(R/L)t}\right)$

where $I_0 = \frac{\mathcal{E}}{R} = \frac{12V}{.01\,\Omega} = 1200\,A$ and $R/L = \frac{.01\,\Omega}{.5\times10^{-3}\,H} = 20\,s^{-1}$

The t (call it t^*) we are interested in is when $I(t^*) = .9\,I_0$,

i.e., when $1 - e^{-(R/L)t^*} = .9 \iff e^{-\frac{R}{L}t^*} = .1$.

Thus, $t^* = -\frac{L}{R}\ln(.1) = -\frac{L}{R}\ln\left(\frac{1}{10}\right) = \frac{L}{R}\ln(10) = \frac{1}{20}\ln(10)$ seconds

$\Rightarrow \boxed{t^* \approx .115 \text{ seconds}}$

The energy stored in the B-field is $\frac{1}{2}LI^2$, and at time t^* we have $I(t^*) = .9I_0$

$\rightarrow E_{\substack{\text{in B}\\\text{field}}} = \frac{1}{2}LI^2 = \frac{1}{2}L(.9)^2 I_0^2 = \frac{1}{2}(.5\times10^{-3})\times(.81)(1200)^2$

$\Rightarrow \boxed{E_{\text{in B field}} = 291.6\text{ J}}$

#4) (cont'd).

The instantaneous power P delivered by the battery is $P = \mathcal{E} I$
$= (12V) I$

so we must integrate this up from $t=0$ to $t = t^* = .115$ seconds since I changes with time.

Thus,
$$E_{\substack{\text{delivered}\\\text{by battery}}} = \mathcal{E}\int_0^{t^*} I\,dt = \mathcal{E} I_0 \int_0^{t^*}\left(1 - e^{-(R/L)t}\right)dt$$

$$= \mathcal{E} I_0 \left[t + \frac{L}{R} e^{-\frac{R}{L}t}\right]_0^{t^*}$$

$$= \mathcal{E} I_0 \left[t^* + \frac{L}{R}\left[e^{-\frac{R}{L}t^*} - 1\right]\right]$$

$$= (12V)(1200A)\left[.115s + \frac{1}{20}\left[e^{-20 \cdot .115} - 1\right]\right]$$

$$\Rightarrow \boxed{E_{\substack{\text{delivered}\\\text{by battery}}} \simeq 1008\ J}$$

Now for the energy dissipated by the resistor, we have

$$E_{\text{dissipated}} = \int_0^{t^*} I^2 R\,dt = I_0^2 R \int_0^{t^*}\left(1 - e^{-\frac{R}{L}t}\right)^2 dt$$

$$= I_0^2 R \int_0^{t^*}\left[1 - 2e^{-\frac{R}{L}t} + e^{-\frac{2R}{L}t}\right]dt$$

$$= I_0^2 R\left[t^* + 2\frac{L}{R}\left(e^{-\frac{R}{L}t^*} - 1\right) - \frac{L}{2R}\left(e^{-\frac{2R}{L}t^*} - 1\right)\right]$$

$$\Rightarrow \boxed{E_{\text{dissipated}} \simeq 716.\ J.}$$

(Note: The "$\simeq$" is only because we approximated decimals along the way.)

Indeed, $\boxed{E_{\text{m B field}} + E_{\text{dissipated}} \simeq 1008 J = E_{\substack{\text{delivered by}\\\text{battery}}}}$, as it should

#5) $\boxed{\text{PM } 7.44}$ (Assume the ~~universe~~ galaxy is a cylinder of base area πr^2 and height h, $r = \frac{10^{21}}{2}$ m, $h = 10^{19}$ m)

We take $B = 3 \times 10^{-6}$ Gauss $= 3 \times 10^{-10}$ Tesla

and the magnetic energy density u is $u = \dfrac{B^2}{2\mu_0}$.

Thus the total magnetic energy $U_{magnetic}$ is

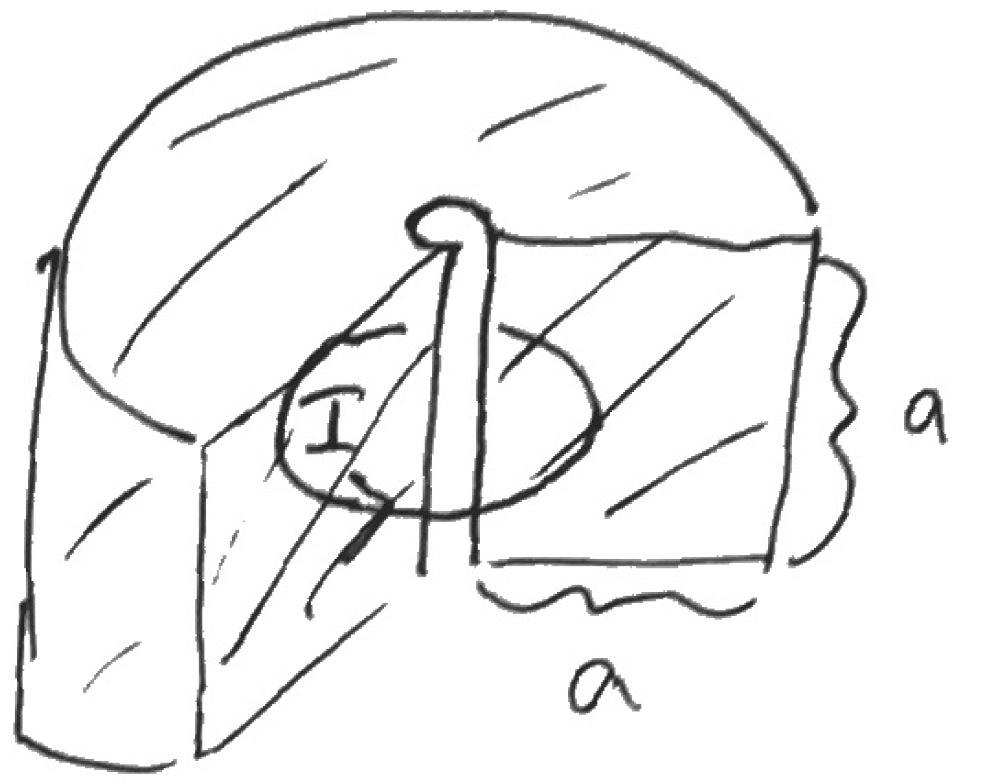
$$U_{magnetic} = u \cdot Vol = u \cdot \pi r^2 \cdot h$$

$$= \frac{B^2}{2\mu_0} \cdot \pi r^2 h$$

$$\Rightarrow \boxed{U_{magnetic} \simeq 2.8 \times 10^{47} \text{ J}}$$

Stars in the galaxy radiate a total of 10^{37} J/s, so $U_{magnetic}$ is about $\boxed{2.8 \times 10^{10} \text{ second's worth of radiation,}}$ or about 890 years

#6



From Eq (6.54) in PM, the B field at the center of a ring of radius r is $\frac{\mu_0 I}{2r}$,

so that here $B = \frac{\mu_0 I}{2(a/2)} = \frac{\mu_0 I}{a}$. The stored energy is then

$$U_{stored} = \underbrace{\frac{B^2}{2\mu_0}}_{\text{energy density}} \cdot \underbrace{(\pi a^2 \cdot a)}_{\text{volume}} = \frac{\pi}{2\mu_0} a^3 \cdot \frac{\mu_0^2 I^2}{a^2} = \frac{\pi}{2}\mu_0 a I^2 .$$

With $R = \frac{\pi}{a\sigma}$, we have for the Ohmic dissipation (i.e. dissipated power) $P_{dissipated} = I^2 R = \frac{\pi}{a\sigma} I^2$

Using $\tau = \frac{U}{I^2 R}$, we have $\tau = \frac{\frac{\pi}{2}\mu_0 a I^2}{\frac{\pi}{a\sigma} I^2} = \frac{\mu_0 a^2 \sigma}{2}$.

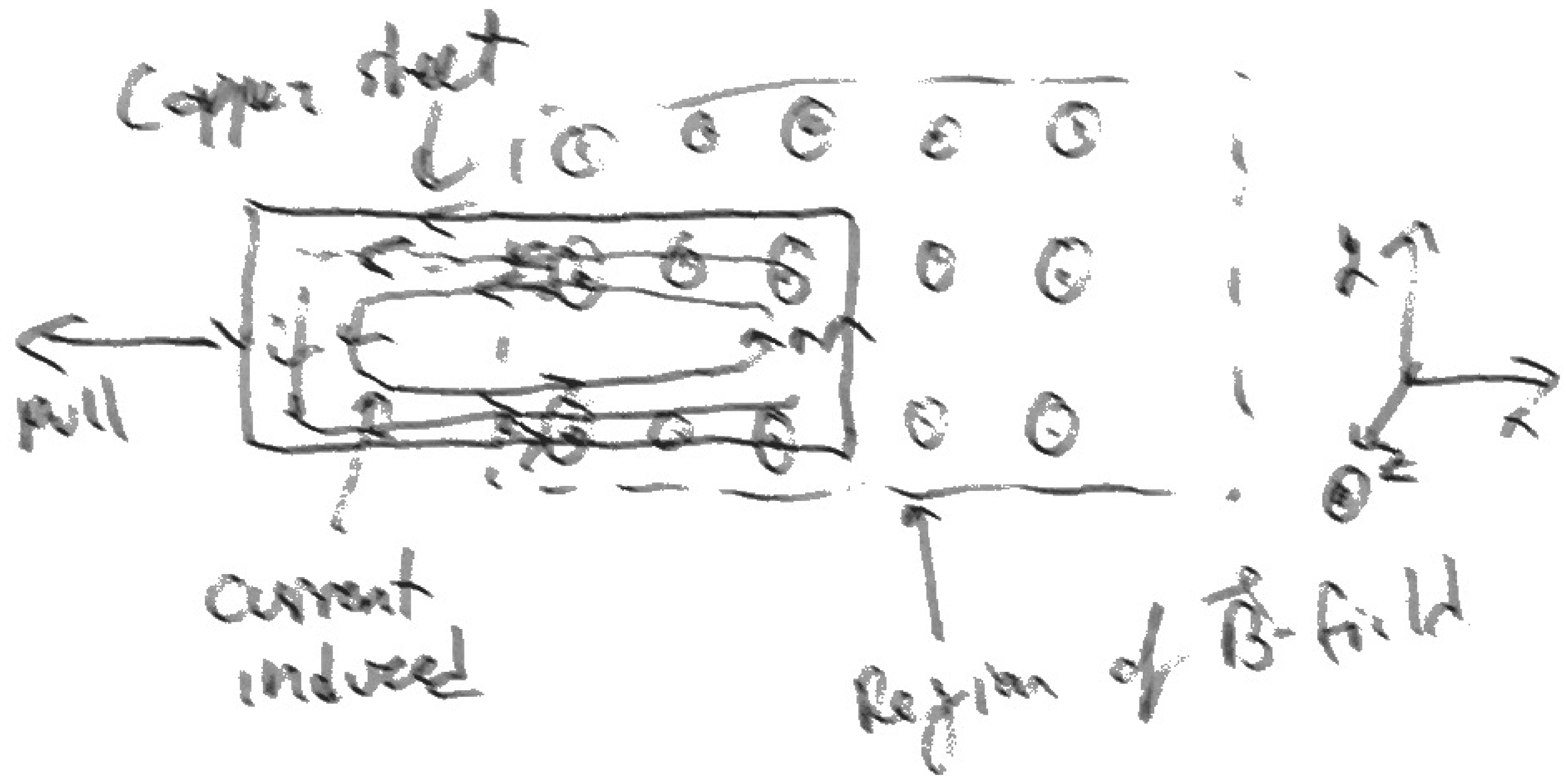
With $a = 3\times10^6$ m, $\sigma = 10^6\ (\Omega\text{-m})^{-1}$, we have

$$\boxed{\tau \simeq 5.5\times10^{12}\ (\text{or } 10^{13}) \text{ seconds} \quad \text{or} \approx 1700 \text{ centuries}}$$

#7) PM 29.70)

Top down view: (B field coming out at us)

Case 1: pulling copper out:



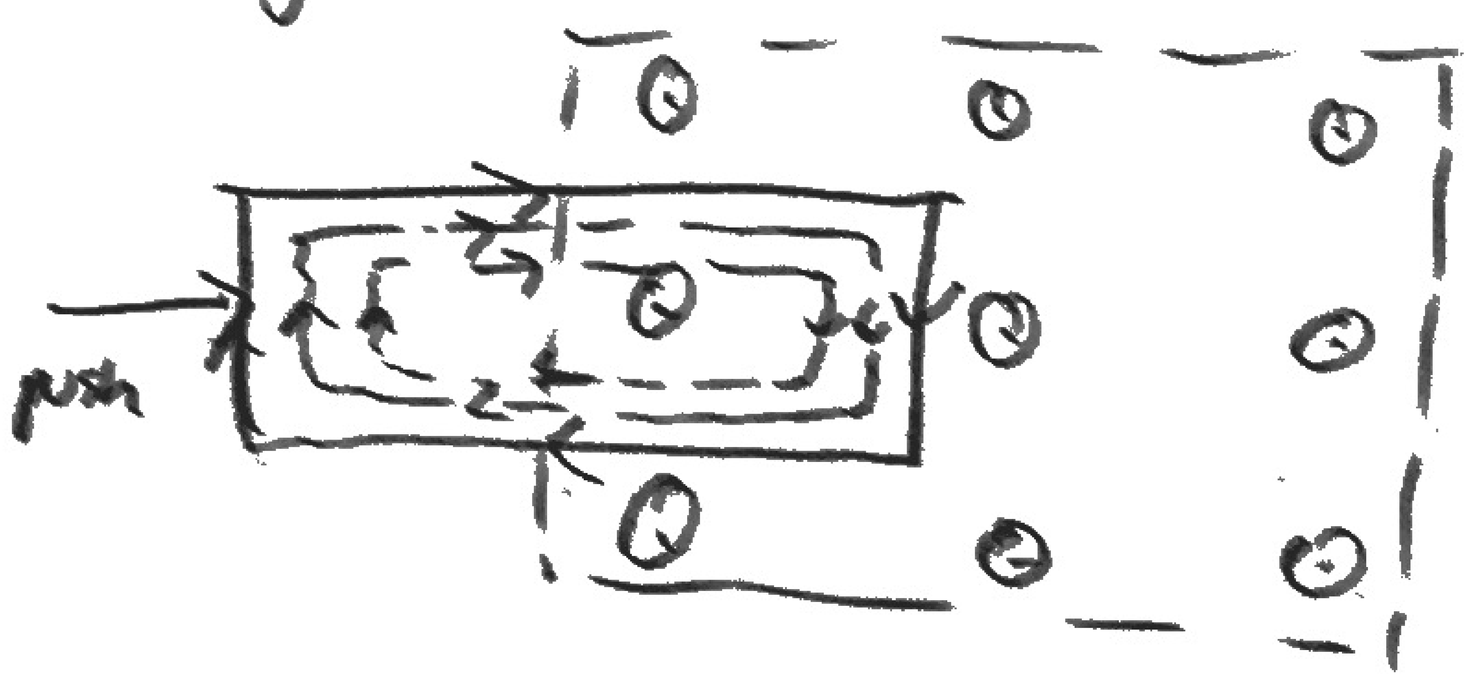
pull $\Rightarrow$ $\frac{d\Phi}{dt} < 0$ (i.e., magnetic flux through copper is decreasing)

$\Rightarrow$ current induced is <u>counter</u>clockwise to counteract (partially) the loss of flux. — Faraday's and Lenz's law

$\Rightarrow$ only right hand side has ~~non-cancelling~~ non cancelling $\vec{I} \times \vec{B}$ force, which ~~points~~ points to the right: $\vec{I} \sim I\hat{y}$, $\vec{B} = B\hat{z}$

$I_{induced}$ $\Rightarrow \vec{I} \times \vec{B} = IB\,\hat{y}\times\hat{z}$
$= IB\hat{x}$ ✓.

Case 2: Pushing in:



push $\Rightarrow$ $\frac{d\Phi}{dt} > 0$

$\Rightarrow$ $I_{induced}$ is now <u>clockwise</u> (to combat <u>increase</u> in flux)

Now $\vec{I} \times \vec{B} \sim -\hat{x}$ on the non-cancelling side.

So in either case, the force is resistive, both coming from a changing magnetic flux giving rise to an induced current which itself (the current, that is) interacts with the $\vec{B}$-field to produce the force.