Name: ______________________

Student ID number: ______________________

105A - Final Solutions

Please read the following very carefully

- This is a closed book exam. You may **not** use a calculator. All electronic devices should **not** be around!
- You have **3 hours** to complete the exam.
- Grades are out of 150
- Answer all **three** questions, and their subsections.
- Make sure to write your name at the top of each page of this exam. Use the space provided on the exam pages to do your work. You may use the back of the pages also, but please mark clearly which problem you are working on (and also state underneath that problem that you have done work on the back of the page).
- Partial credit will be given. Show as much work/justification as possible (diagrams where appropriate). If you can not figure out how to complete a particular computation, a written statement of the concepts involved and qualitative comments on what you think the answer should be may be assigned partial credit.
- Make sure you check your units. A unit problem will cost 20% of the points.
- Mistakes in grading: If you find a mistake in the grading of your exam, alert the instructor within one week of the exams being returned DO NOT write on the returned graded exam you may make a note of the problems you thought were misgraded on a separate page.

- In class we showed that the equation of motion can be expressed using the force $F = -dU/dr$ in the form (i.e., the Binnet's equation):

$$\frac{l^2 u^2}{\mu}\left(\frac{d^2 u}{d\theta^2} + u\right) = -F(1/u) \ , \tag{1}$$

where $l = mr^2\dot{\theta}$ is the angular momentum and $u = 1/r$.

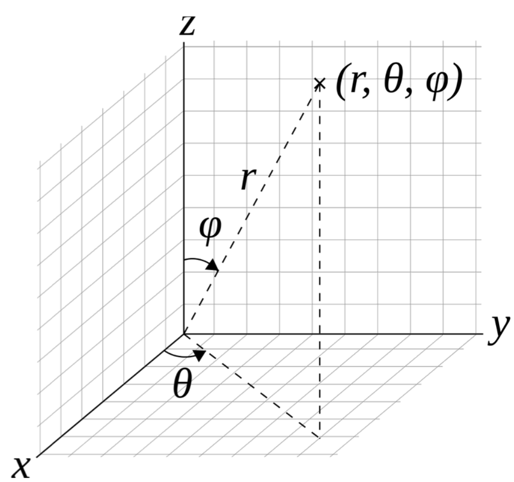

Figure 1: Spherical coordinates

- Remember that spherical coordinates are defines as shown in Figure 1:

$$\begin{aligned} x &= r\cos\varphi\sin\theta & (2)\\ y &= r\sin\varphi\sin\theta & (3)\\ z &= r\cos\theta & (4) \end{aligned}$$

Questions

1. (50pt) A spherical pendulum has a massless wire, at length b, attached to a mass m. Assume that there is no friction in the attachment point and that gravity is pointing down.

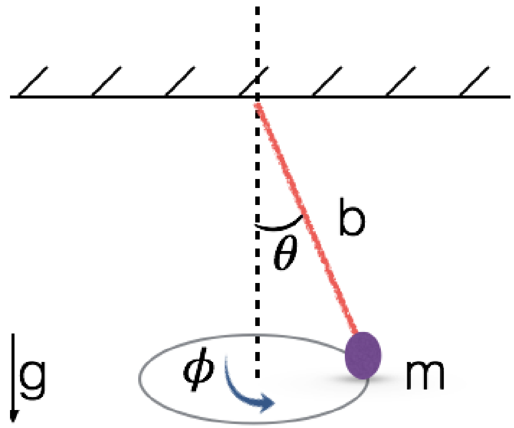

Figure 2: A Spherical pendulum

(a) (8pt) Define carefully a set of independent generalized coordinates for this system. Define your farm of reference. Write down the Lagrangian in terms of the generalized coordinates.
Answer We'll choose z pointing up. θ is defined already, we'll define ϕ on the 2D $x-y$ plane with ϕ between the projected line on this plane (i.e., $b\sin\theta$) and the x axis. A sketch on the drawing is enough.

$$x = b\cos\phi\sin\theta \quad (5)$$
$$y = b\sin\phi\sin\theta \quad (6)$$
$$z = b\cos\theta \quad (7)$$

The independent generalized coordinates are θ and ϕ. To find the kinetic energy we'll have:

$$\dot{x} = b(-\dot{\phi}\sin\phi\sin\theta + \dot{\theta}\cos\phi\cos\theta) \quad (8)$$
$$\dot{y} = b(\dot{\phi}\cos\phi\sin\theta + \dot{\theta}\sin\phi\cos\theta) \quad (9)$$
$$\dot{z} = -b\sin\theta \quad (10)$$

After some arrangement the kinetic energy is:

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \frac{1}{2}mb^2(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) \quad (11)$$

We define the potential to be zero at the pendulum's point of attachment. The potential energy is $U = -mgb\cos\theta$, where we set it to be zero at the point of the

attachment. **Note that I have specified where the potential is zero.** And the Lagrangian is:
In terms of θ and X:

$$L = T - U = \frac{1}{2}mb^2(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta) + mgb\cos\theta \tag{12}$$

(b) (12pt) Find the generalized momenta and the conserved quantities.

Answer: The generalized momenta are:

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mb^2\dot{\theta} \tag{13}$$

and

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = mb^2 \sin^2\theta\dot{\phi} \tag{14}$$

To find the conserve quantities we'll take the derivatives:

$$\dot{p}_\theta = \frac{\partial L}{\partial \theta} = \dot{\phi}^2 mb^2 \sin\theta\cos\theta - mgb\sin\theta \tag{15}$$

and

$$\dot{p}_\phi = 0 \tag{16}$$

Because ϕ is cyclic the momentum p_ϕ is constant (saying this in words is also OK). so $p_\phi = \text{const.}$ E is also conserved because the Lagrangian does not depends on the time explicitly. [Each momentum worth 3pt and each conserved quantity worth 3pt]

(c) (10pt) Write down the equation of motions. Using one of the conserved quantities you found in (b) to express your answer, and write down only one equation of motion, using that quantity.

Answer: We already have everything that we need:

$$\theta \quad : \quad \frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = 0 \tag{17}$$

$$\phi \quad : \quad \frac{\partial L}{\partial \phi} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} = 0 \tag{18}$$

so:

$$\theta \quad : \quad \dot{\phi}^2 mb^2 \sin\theta\cos\theta - mgb\sin\theta - mb^2\ddot{\theta} = 0 \tag{19}$$

$$\phi \quad : \quad \frac{d}{dt}(mb^2\sin^2\theta\dot{\phi}) = \frac{d}{dt}p_\phi = mb^2(2\sin\theta\cos\theta\dot{\phi}\dot{\theta} + mb^2\sin^2\theta\ddot{\phi}) = 0 \tag{20}$$

So we'll can write:

$$p_\phi = \text{Const} = mb^2\sin^2\theta\dot{\phi} \tag{21}$$

so

$$\dot{\phi} = \frac{p_\phi}{mb^2\sin^2\theta} \tag{22}$$

We'll plug it into Eq. (19) and we'll get

$$\frac{p_\phi^2}{mb^2\sin^3\theta}\cos\theta - mgb\sin\theta - mb^2\ddot{\theta} = 0 \tag{23}$$

which we can write as:

$$\ddot{\theta} = \frac{p_\phi^2}{m^2b^4\sin^3\theta}\cos\theta - \frac{g}{b}\sin\theta \tag{24}$$

(d) (10pt) What does an assumption of $p_\phi = 0$ tells you about the initial conditions?
Answer: Since p_ϕ = Const we can plug in the initial conditions, i.e., $p_\phi = mb^2 \sin^2 \theta_0 \dot{\phi}_0$, so plugging in $p_\phi = 0$ means that either $\dot{\phi}_0 = 0$, or $\theta_0 = 0$. [each one worth 5pt]

(e) (10pt) Plug in $p_\phi = 0$ and assume small oscillation $\theta << 1$ rad. Which means that you can approximate $\sin\theta \sim \theta$ and $\cos\theta \sim 1$. Find the solution in that case, assuming that the system is set initially at rest with a displacement $\theta_0 \neq 0$.
Answer: In that case the equation is very simple:

$$\ddot{\theta} = -\frac{g}{b}\theta \tag{25}$$

The solution is also stightforward:

$$\theta(t) = A\cos(\omega t + \delta) \tag{26}$$

Setting the initial conditions $\theta(t = 0) = \theta_0$ and $\dot{\theta}(t = 0) = 0$ we have

$$\theta(t) = \theta_0 \cos(\omega t) \tag{27}$$

2. (50pt) A block of mass M is attached to a spring with a spring constant k and is able to slide on a horizontal bar (the x axis). A pendulum consists of a mass m suspended by a massless string with unextended length l and can move in the $x - z$ plane is attached to the block. (see Figure 3). Gravity is pointed downwards.

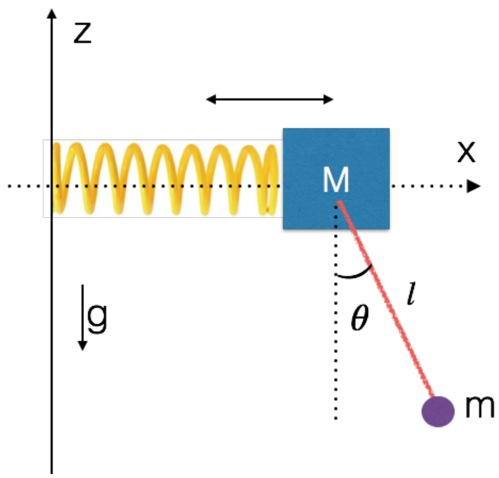

Figure 3: A pendulum attached to horizontal spring.

(a) (15pt) Chose carefully your generalized coordinates and write the Lagrangian of the system.
Answer: Let X be the horizontal displacement of the black, the position of the pendulum bob is given by $(x, z) = (X + l\sin\theta, -l\cos\theta)$. So the kinetic energy is

$$\begin{aligned} T &= \frac{m}{2}\left(\dot{X}^2 + \dot{z}^2\right) = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}m([\dot{X} + l\dot{\theta}\cos\theta]^2 + [l\dot{\theta}\sin\theta]^2) \\ &= \frac{1}{2}M\dot{X}^2 + \frac{1}{2}m(\dot{X}^2 + 2\dot{X}l\dot{\theta}\cos\theta + l^2\dot{\theta}^2) \qquad (28) \end{aligned}$$

and the potential is:

$$U = \frac{1}{2}kX^2 + mgz = \frac{1}{2}kX^2 - mgl\cos\theta \qquad (29)$$

[A potential defined up to a constant is also OK] So the Lagrangian of the system is:

$$L = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}m(\dot{X}^2 + 2\dot{X}l\dot{\theta}\cos\theta + l^2\dot{\theta}^2) - \frac{1}{2}kX^2 + mgl\cos\theta \qquad (30)$$

missing the $kX^2/2$ part is minus 7pt.

(b) (10pt) Find the equations of motion.
Answer: The equations of motion (after some arranging) are:

$$\begin{aligned} X: &\quad \frac{\partial L}{\partial X} - \frac{d}{dt}\frac{\partial L}{\partial \dot{X}} = 0 && (31) \\ \text{so} &\quad (M+m)\ddot{X} + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta + kX = 0 && (32) \\ \theta: &\quad \frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = 0 && (33) \\ \text{so}: &\quad ml\ddot{X}\cos\theta + ml^2\ddot{\theta} + mlg\sin\theta = 0 && (34) \end{aligned}$$

(c) (13pt) Assume a very weak spring, compared to the pendulum, so the motion of the mass M is dominated by the pendulum. In other words:

$$\frac{k}{m+M} << \frac{m\ddot{\theta}}{m+M} \tag{35}$$

or you can simply take $k \to 0$ corresponding to a free moving mass. Also assume small oscillation $\theta << 1$ rad. Which means that you can approximate $\sin\theta \sim \theta$ and $\cos\theta \sim 1$. Keep only first order terms (i.e., $\theta^2 \to 0$ and also $\dot{\theta}^2 \to 0$). Write the equation of motions in this case.

Answer: The equation of motions for small angles are:

$$X: \quad (M+m)\ddot{X} + ml\ddot{\theta} - ml\dot{\theta}^2\theta + kX = 0 \tag{36}$$

$$\theta: \quad ml\ddot{X} + ml^2\ddot{\theta} + mlg\theta = 0 \tag{37}$$

Which we can write: (after eliminating $\dot{\theta}^2\theta \to 0$ and $\frac{k}{M+m}X$)

$$X: \quad \ddot{X} + \frac{ml}{M+m}\ddot{\theta} = 0 \tag{38}$$

$$\theta: \quad \ddot{X} + l\ddot{\theta} + g\theta = 0 \tag{39}$$

(d) (12pt) Given the initial conditions of $\theta(t=0)=\theta_0$, $\dot{\theta}(t=0)=\theta_0\sqrt{g(M+m)/(lM)}$ find the time solution of the angle that describes the oscillations of the pendulum using the equations you found in (c) as a function of the g, M, m, l and θ_0.
Answer: The first equation gives:

$$\ddot{X} = -\frac{ml}{M+m}\ddot{\theta} \tag{40}$$

Plugging this we can write the equation for θ as:

$$\ddot{\theta} = -\frac{g(M+m)}{lM}\theta \tag{41}$$

Which is the equation for simple pendulum. We'll denote the oscillation frequency as:

$$\omega^2 = \frac{g(M+m)}{lM} \tag{42}$$

The most general solution here is $\sim e^{\pm i\omega t}$ which we can write as:

$$\theta(t) = A\cos(\omega t) + B\sin(\omega t) \tag{43}$$

Using the initial conditions we can find:

$$\theta(t=0) = \theta_0 = A \tag{44}$$

and

$$\dot{\theta}(t=0) = \theta_0\omega = B\omega \tag{45}$$

So we get that

$$\theta(t) = \theta_0(\cos(\omega t) + \sin(\omega t)) \tag{46}$$

Alternatively, this can be written as:

$$\theta(t) = \sqrt{2}\theta_0\cos(\omega t - \pi/4) \tag{47}$$

3. A particle is moving in a a force field describes by

$$U = -\frac{k}{r}e^{-r/a} \quad k, a > 0 \tag{48}$$

(a) (7pt) Write down the energy and the effective potential.
Answer: The energy is:

$$E = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + U(r) = \frac{1}{2}m\dot{r}^2 + \frac{l^2}{2mr^2} - \frac{k}{r}e^{-r/a} \tag{49}$$

and the effective potential is

$$U_{eff} = \frac{l^2}{2mr^2} - \frac{k}{r}e^{-r/a} \tag{50}$$

[The energy worth 4pt and effective potential 3pt]

(b) (15pt) Find the angular momentum which describes the a circular motion. Explain why this point is a circular motion.
Answer: A circular point is defined at the extremum of the effective potential, there $E = U_{eft}$ since $\dot{r} = 0$. [an explanation such as this worth 5pt].
We take the derivative of the effective potential as a function to r and find the extremum point.

$$\frac{dU_{eff}}{dr} = -\frac{l^2}{mr^3} + \left(\frac{k}{r^2} + \frac{k}{ar}\right)e^{-r/a} = 0 \tag{51}$$

So

$$\left(k + \frac{kr}{a}\right)\frac{e^{-r/a}}{r^2} = \frac{l^2}{mr^3} \tag{52}$$

So the angular momentum of the circular orbit is:

$$l^2 = kmr\left(1 + \frac{r}{a}\right)e^{-r/a} \tag{53}$$

which can be written as:

$$l = \sqrt{kmr\left(1 + \frac{r}{a}\right)}e^{-r/2a} \tag{54}$$

(c) (15pt) Is this a stable point, if not, when will that represent a stable point? In other words, what should be the condition for a stable point?
Answer: To find if the point is stable or unstable we need to take the second derivative of the effective potential, i.e,

$$\frac{d^2U_{eff}}{dr^2} = \frac{3l^2}{mr^4} + \left[\left(\frac{k}{r^2} + \frac{k}{ar}\right)\frac{-1}{a} - \frac{2k}{r^3} - \frac{k}{ar^2}\right]e^{-r/a} \tag{55}$$

where the > 0 means that the point is stable. So we'll find what is the condition for an effective potential which is larger than zero. Arranging and setting to zero this we have:

$$\frac{d^2U_{eff}}{dr^2} = \frac{3l^2}{mr^4} - \left(\frac{2k}{ar^2} + \frac{k}{a^2r} + \frac{2k}{r^3}\right) e^{-r/a} = 0 \tag{56}$$

Plugging in l^2 that we found before we have

$$\frac{3}{mr^4} kmr \left(1 + \frac{r}{a}\right) e^{-r/a} - \left(\frac{2k}{ar^2} + \frac{k}{a^2r} + \frac{2k}{r^3}\right) e^{-r/a} = 0 \tag{57}$$

Arranging this we have

$$\frac{1}{r^3} + \frac{1}{ar^2} - \frac{1}{a^2r} = 0 \tag{58}$$

or

$$1 + \frac{r}{a} - \frac{r^2}{a^2} = 0 \tag{59}$$

The solutions here are:

$$r = \frac{a}{2}\left(1 \pm \sqrt{5}\right) \tag{60}$$

And the stable solution will be when

$$0 < r \leq \frac{a}{2}\left(1 + \sqrt{5}\right) \tag{61}$$

(d) (13pt) Assume that

$$a = \frac{km}{l^2} r^2 \tag{62}$$

(in other words once $a > r$ you can easily set the exponent to be one) and find the most general solution of $r(\theta)$ in this case.

Hint: Set only the exponent to unity and don't do any other approximations.

Answer: Using Binnet's equation we can write:

$$\frac{l^2u^2}{m}\left(\frac{d^2u}{d\theta^2} + u\right) = -F(1/u)\ , \tag{63}$$

where in our case

$$F = -\frac{dU}{dr} = -\left(\frac{k}{r^2} + \frac{k}{ar}\right) e^{-r/a} \sim -\frac{k}{r^2} - \frac{l^2}{mr^3} \tag{64}$$

So Binnet's equation is then

$$\frac{l^2u^2}{m}\left(\frac{d^2u}{d\theta^2} + u\right) = ku^2 + \frac{l^2}{m}u^3\ , \tag{65}$$

arranging we can write:

$$\frac{d^2u}{d\theta^2} = \frac{m}{l^2}k\ , \tag{66}$$

so

$$\frac{1}{r} = u = \frac{mk}{2l^2}\theta^2 + A\theta + B \tag{67}$$

from which we see that r continuously decreases as θ increases; that is, the particle spirals in toward the force center.