3.5 Damped Oscillations

The motion represented by the simple harmonic oscillator is termed a **free oscillation;** once set into oscillation, the motion would never cease. This oversimplifies the actual physical case, in which dissipative or frictional forces would eventually damp the motion to the point that the oscillations would no longer occur. We can analyze the motion in such a case by incorporating into the differential equation a term representing the damping force. It does not seem reasonable that the damping force should, in general, depend on the displacement, but it could be a function of the velocity or perhaps of some higher time derivative of the displacement. It is frequently assumed that the damping force is a linear function of the velocity,* $\mathbf{F}_d = \alpha\mathbf{v}$. We consider here only one-dimensional damped oscillations so that we can represent the damping term by $-b\dot{x}$. The parameter b must be *positive* in order that the force indeed be *resisting*. (A force $-b\dot{x}$ with $b < 0$ would act to *increase* the speed instead of decreasing it as any resisting force must.) Thus, if a particle of mass m moves under the combined influence of a linear restoring force $-kx$ and a resisting force $-b\dot{x}$, the differential equation describing the motion is

$$m\ddot{x} + b\dot{x} + kx = 0 \tag{3.34}$$

which we can write as

$$\boxed{\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0} \tag{3.35}$$

Here $\beta \equiv b/2m$ is the **damping parameter** and $\omega_0 = \sqrt{k/m}$ is the characteristic angular frequency in the absence of damping. The roots of the auxiliary equation are (cf. Equation C.8, Appendix C)

$$\left.\begin{aligned} r_1 &= -\beta + \sqrt{\beta^2 - \omega_0^2} \\ r_2 &= -\beta - \sqrt{\beta^2 - \omega_0^2} \end{aligned}\right\} \tag{3.36}$$

The general solution of Equation 3.35 is therefore

$$\boxed{x(t) = e^{-\beta t}\left[A_1 \exp\left(\sqrt{\beta^2 - \omega_0^2}\,t\right) + A_2 \exp\left(-\sqrt{\beta^2 - \omega_0^2}\,t\right)\right]} \tag{3.37}$$

There are three general cases of interest:

Underdamping: $\omega_0^2 > \beta^2$
Critical damping: $\omega_0^2 = \beta^2$
Overdamping: $\omega_0^2 < \beta^2$

The motion of the three cases is shown schematically in Figure 3-6 for specific initial conditions. We shall see that only the case of underdamping results in oscillatory motion. These three cases are discussed separately.

Underdamped Motion

For the case of underdamped motion, it is convenient to define

$$\omega_1^2 \equiv \omega_0^2 - \beta^2 \tag{3.38}$$

*See Section 2.4 for a discussion of the dependence of resisting forces on velocity.

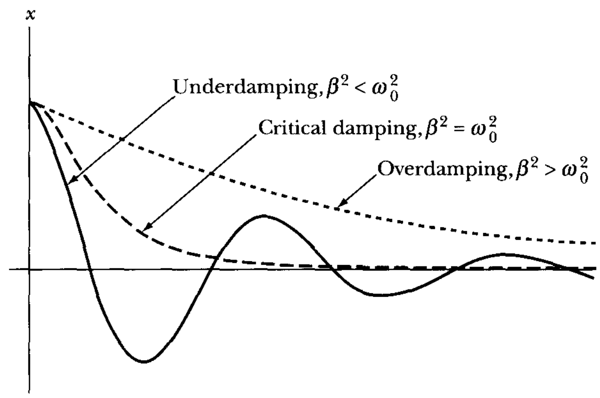

FIGURE 3-6 Damped oscillator motion for three cases of damping.

where $\omega_1^2 > 0$; then the exponents in the brackets of Equation 3.37 are imaginary, and the solution becomes

$$x(t) = e^{-\beta t}\left[A_1 e^{i\omega_1 t} + A_2 e^{-i\omega_1 t}\right] \tag{3.39}$$

Equation 3.39 can be rewritten as*

$$x(t) = Ae^{-\beta t}\cos(\omega_1 t - \delta) \tag{3.40}$$

We call the quantity ω_1 the *angular frequency* of the damped oscillator. Strictly speaking, we cannot define a frequency when damping is present, because the motion is not periodic—that is, the oscillator never passes twice through a given point with the same velocity. However, because $\omega_1 = 2\pi/(2T_1)$, where T_1 is the time between adjacent zero x-axis crossings, the angular frequency ω_1 has meaning for a given time period. Note that $2T_1$ would be the "period" in this case, not T_1. For simplicity, we refer to ω_1 as the "angular frequency" of the damped oscillator, and we note that this quantity is less than the frequency of the oscillator in the absence of damping (i.e., $\omega_1 < \omega_0$). If the damping is small, then

$$\omega_1 = \sqrt{\omega_0^2 - \beta^2} \cong \omega_0$$

so the term angular *frequency* may be used. But the meaning is not precise unless $\beta = 0$.

The maximum amplitude of the motion of the damped oscillator decreases with time because of the factor $\exp(-\beta t)$, where $\beta > 0$, and the envelope of the displacement *versus* time curve is given by

$$x_{\text{en}} = \pm Ae^{-\beta t} \tag{3.41}$$

This envelope and the displacement curve are shown in Figure 3-7 for the case $\delta = 0$. The sinusoidal curve for undamped motion ($\beta = 0$) is also shown in this figure. A close comparison of the two curves indicates that the frequency for the damped case is *less* (i.e., that the period is *longer*) than that for the undamped case.

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*See Exercise D-6, Appendix D.

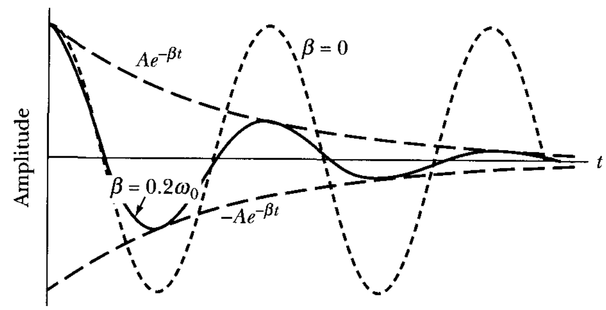

FIGURE 3-7 The underdamped motion (solid line) is an oscillatory motion (short dashes) that decreases within the exponential envelope (long dashes).

The ratio of the amplitudes of the oscillation at two successive maxima is

$$\frac{Ae^{-\beta T}}{Ae^{-\beta(T+\tau_1)}} = e^{\beta\tau_1} \qquad \textbf{(3.42)}$$

where the first of any pair of maxima occurs at $t = T$ and where $\tau_1 = 2\pi/\omega_1$. The quantity $\exp(\beta\tau_1)$ is called the **decrement** of the motion; the logarithm of $\exp(\beta\tau_1)$—that is, $\beta\tau_1$—is known as the **logarithmic decrement** of the motion.

Unlike the simple harmonic oscillator discussed previously, the energy of the damped oscillator is not constant in time; rather, energy is continually given up to the damping medium and dissipated as heat (or, perhaps, as radiation in the form of fluid waves). The rate of energy loss is proportional to the square of the velocity (see Problem 3-11), so the decrease of energy does not take place uniformly. The loss rate will be a maximum when the particle attains its maximum velocity near (but not exactly at) the equilibrium position, and it will instantaneously vanish when the particle is at maximum amplitude and has zero velocity. Figure 3-8 shows the total energy and the rate of energy loss for the damped oscillator.