TABLE 7-1

Characteristic of inertial frame	Property of Lagrangian	Conserved quantity
Time homogeneous	Not explicit function of time	Total energy
Space homogeneous	Invariant to translation	Linear momentum
Space isotropic	Invariant to rotation	Angular momentum

7.10 Canonical Equations of Motion—Hamiltonian Dynamics

In the previous section, we found that if the potential energy of a system is velocity independent, then the linear momentum components in rectangular coordinates are given by

$$p_i = \frac{\partial L}{\partial \dot{x}_i} \tag{7.150}$$

By analogy, we extend this result to the case in which the Lagrangian is expressed in generalized coordinates and define the **generalized momenta*** according to

$$\boxed{p_j \equiv \frac{\partial L}{\partial \dot{q}_j}} \tag{7.151}$$

(Unfortunately, the customary notations for ordinary momentum and generalized momentum are the same, even though the two quantities may be quite different.) The Lagrange equations of motion are then expressed by

$$\boxed{\dot{p}_j = \frac{\partial L}{\partial q_j}} \tag{7.152}$$

Using the definition of the generalized momenta, Equation 7.128 for the Hamiltonian may be written as

$$H = \sum_j p_j \dot{q}_j - L \tag{7.153}$$

The Lagrangian is considered to be a function of the generalized coordinates, the generalized velocities, and possibly the time. The dependence of L on the time may arise either if the constraints are time dependent or if the transformation equations connecting the rectangular and generalized coordinates explicitly contain the time. (Recall that we do not consider time-dependent potentials.) We may solve Equation 7.151 for the generalized velocities and express them as

$$\dot{q}_j = \dot{q}_j(q_k, p_k, t) \tag{7.154}$$

*The terms *generalized coordinates, generalized velocities,* and *generalized momenta* were introduced in 1867 by Sir William Thomson (later, Lord Kelvin) and P. G. Tait in their famous treatise *Natural Philosophy.*

Thus, in Equation 7.153, we may make a change of variables from the $(q_j, \dot{q}_j, t)$ set to the (q_j, p_j, t) set* and express the Hamiltonian as

$$\boxed{H(q_k, p_k, t) = \sum_j p_j\dot{q}_j - L(q_k, \dot{q}_k, t)} \tag{7.155}$$

This equation is written in a manner that stresses the fact that *the Hamiltonian is always considered as a function of the* (q_k, p_k, t) *set, whereas the Lagrangian is a function of the* $(q_k, \dot{q}_k, t)$ *set*:

$$\boxed{H = H(q_k, p_k, t), \qquad L = L(q_k, \dot{q}_k, t)} \tag{7.156}$$

The total differential of H is therefore

$$dH = \sum_k \left(\frac{\partial H}{\partial q_k}\, dq_k + \frac{\partial H}{\partial p_k}\, dp_k\right) + \frac{\partial H}{\partial t}\, dt \tag{7.157}$$

According to Equation 7.155, we can also write

$$dH = \sum_k \left(\dot{q}_k\, dp_k + p_k\, d\dot{q}_k - \frac{\partial L}{\partial q_k}\, dq_k - \frac{\partial L}{\partial \dot{q}_k}\, d\dot{q}_k\right) - \frac{\partial L}{\partial t}\, dt \tag{7.158}$$

Using Equations 7.151 and 7.152 to substitute for $\partial L/\partial q_k$ and $\partial L/\partial \dot{q}_k$, the second and fourth terms in the parentheses in Equation 7.158 cancel, and there remains

$$dH = \sum_k (\dot{q}_k dp_k - \dot{p}_k dq_k) - \frac{\partial L}{\partial t}\, dt \tag{7.159}$$

If we identify the coefficients† of dq_k, dp_k, and dt between Equations 7.157 and 7.159, we find

$$\boxed{\dot{q}_k = \frac{\partial H}{\partial p_k}} \tag{7.160}$$

Hamilton's equations of motion

$$\boxed{-\dot{p}_k = \frac{\partial H}{\partial q_k}} \tag{7.161}$$

and

$$-\frac{\partial L}{\partial t} = \frac{\partial H}{\partial t} \tag{7.162}$$

Furthermore, using Equations 7.160 and 7.161 in Equation 7.157, the term in the parentheses vanishes, and it follows that

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} \tag{7.163}$$

*This change of variables is similar to that frequently encountered in thermodynamics and falls in the general class of the so-called Legendre transformations (used first by Euler and perhaps even by Leibniz). A general discussion of Legendre transformations with emphasis on their importance in mechanics is given by Lanczos (La49, Chapter 6).

†The assumptions implicitly contained in this procedure are examined in the following section.

Equations 7.160 and 7.161 are **Hamilton's equations of motion.*** Because of their symmetric appearance, they are also known as the **canonical equations of motion.** The description of motion by these equations is termed **Hamiltonian dynamics.**

Equation 7.163 expresses the fact that if H does not explicitly contain the time, then the Hamiltonian is a conserved quantity. We have seen previously (Section 7.9) that the Hamiltonian equals the total energy $T + U$ if the potential energy is velocity independent and the transformation equations between $x_{\alpha,i}$ and q_j do not explicitly contain the time. Under these conditions, and if $\partial H/\partial t = 0$, then $H = E =$ constant.

There are $2s$ canonical equations and they replace the s Lagrange equations. (Recall that $s = 3n - m$ is the number of degrees of freedom of the system.) But the canonical equations are *first-order* differential equations, whereas the Lagrange equations are of *second order.*† To use the canonical equations in solving a problem, we must first construct the Hamiltonian as a function of the generalized coordinates and momenta. It may be possible in some instances to do this directly. In more complicated cases, it may be necessary first to set up the Lagrangian and then to calculate the generalized momenta according to Equation 7.151. The equations of motion are then given by the canonical equations.

EXAMPLE 7.11

Use the Hamiltonian method to find the equations of motion of a particle of mass m constrained to move on the surface of a cylinder defined by $x^2 + y^2 = R^2$. The particle is subject to a force directed toward the origin and proportional to the distance of the particle from the origin: $\mathbf{F} = -k\mathbf{r}$.

Solution. The situation is illustrated in Figure 7-9. The potential corresponding to the force $\mathbf{F}$ is

$$\begin{aligned} U &= \frac{1}{2}kr^2 = \frac{1}{2}k(x^2 + y^2 + z^2) \\ &= \frac{1}{2}k(R^2 + z^2) \end{aligned} \tag{7.164}$$

We can write the square of the velocity in cylindrical coordinates (see Equation 1.101) as

$$v^2 = \dot{R}^2 + R^2\dot{\theta}^2 + \dot{z}^2 \tag{7.165}$$

But in this case, R is a constant, so the kinetic energy is

$$T = \frac{1}{2}m(R^2\dot{\theta}^2 + \dot{z}^2) \tag{7.166}$$

*This set of equations was first obtained by Lagrange in 1809, and Poisson also derived similar equations in the same year. But neither recognized the equations as a basic set of equations of motion; this point was first realized by Cauchy in 1831. Hamilton first derived the equations in 1834 from a fundamental variational principle and made them the basis for a far-reaching theory of dynamics. Thus the designation "Hamilton's" equations is fully deserved.

†This is not a special result; any set of s second-order equations can always be replaced by a set of $2s$ first-order equations.

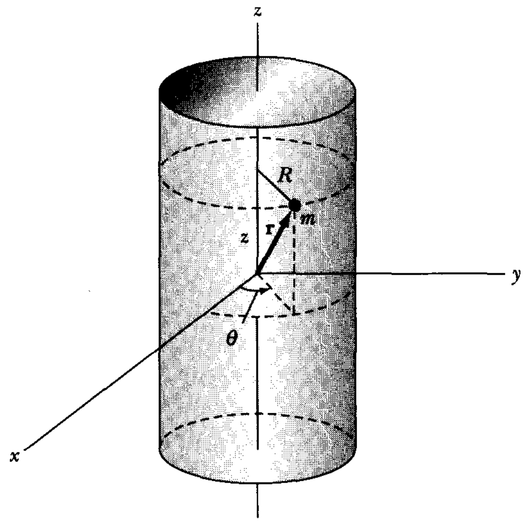

FIGURE 7-9 Example 7.11. A particle is constrained to move on the surface of a cylinder.

We may now write the Lagrangian as

$$L = T - U = \frac{1}{2} m(R^2\dot{\theta}^2 + \dot{z}^2) - \frac{1}{2} k(R^2 + z^2) \tag{7.167}$$

The generalized coordinates are θ and z, and the generalized momenta are

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mR^2\dot{\theta} \tag{7.168}$$

$$p_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z} \tag{7.169}$$

Because the system is conservative and because the equations of transformation between rectangular and cylindrical coordinates do not explicitly involve the time, the Hamiltonian H is just the total energy expressed in terms of the variables θ, p_θ, z, and p_z. But θ does not occur explicitly, so

$$\begin{aligned} H(z, p_\theta, p_z) &= T + U \\ &= \frac{p_\theta^2}{2mR^2} + \frac{p_z^2}{2m} + \frac{1}{2} kz^2 \end{aligned} \tag{7.170}$$

where the constant term $\frac{1}{2} kR^2$ has been suppressed. The equations of motion are therefore found from the canonical equations:

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = 0 \tag{7.171}$$

$$\dot{p}_z = -\frac{\partial H}{\partial z} = -kz \tag{7.172}$$

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mR^2} \tag{7.173}$$

$$\dot{z} = \frac{\partial H}{\partial p_z} = \frac{p_z}{m} \tag{7.174}$$

Equations 7.173 and 1.174 just duplicate Equations 7.168 and 7.169. Equations 7.168 and 7.171 give

$$p_\theta = mR^2\dot{\theta} = \text{constant} \tag{7.175}$$

The angular momentum about the z-axis is thus a constant of the motion. This result is ensured, because the z-axis is the symmetry axis of the problem. Combining Equations 7.169 and 7.172, we find

$$\ddot{z} + \omega_0^2 z = 0 \tag{7.176}$$

where

$$\omega_0^2 \equiv k/m \tag{7.177}$$

The motion in the z direction is therefore simple harmonic.

The equations of motion for the preceding problem can also be found by the Lagrangian method using the function L defined by Equation 7.167. In this case, the Lagrange equations of motion are easier to obtain than are the canonical equations. In fact, it is quite often true that the Lagrangian method leads more readily to the equations of motion that does the Hamiltonian method. But because we have greater freedom in choosing the variable in the Hamiltonian formulation of a problem (the q_k and the p_k are independent, whereas the q_k and the $\dot{q}_k$ are not), we often gain a certain practical advantage by using the Hamiltonian method. For example, in celestial mechanics—particularly in the event that the motions are subject to perturbations caused by the influence of other bodies—it proves convenient to formulate the problem in terms of Hamiltonian dynamics. Generally speaking, however, the great power of the Hamiltonian approach to dynamics does not manifest itself in simplifying the solutions to mechanics problems; rather, it provides a base we can extend to other fields.

The generalized coordinate q_k and the generalized momentum p_k are **canonically conjugate** quantities. According to Equations 7.160 and 7.161, if q_k does not appear in the Hamiltonian, then $\dot{p}_k = 0$, and the conjugate momentum p_k is a constant of the motion. Coordinates not appearing explicitly in the expressions for T and U are said to be *cyclic*. A coordinate cyclic in H is also cyclic in L. But, even if q_k does not appear in L, the generalized velocity $\dot{q}_k$ related to this coordinate is in general still present. Thus

$$L = L(q_1, \ldots, q_{k-1}, q_{k+1}, \ldots, q_s, \dot{q}_1 \ldots, \dot{q}_s, t)$$

and we accomplish no reduction in the number of degrees of freedom of the system, even though one coordinate is cyclic; there are still s second-order equations to be solved. However, in the canonical formulation, if q_k is cyclic, p_k is constant, $p_k = \alpha_k$, and

$$H = H(q_1, \ldots, q_{k-1}, q_{k+1}, \ldots, q_s, p_1, \ldots, p_{k-1}, \alpha_k, p_{k+1}, \ldots, p_s, t)$$

Thus, there are $2s - 2$ first-order equations to be solved, and the problem has, in fact, been reduced in complexity; there are in effect only $s - 1$ degrees of freedom remaining. The coordinate q_k is completely separated, and it is *ignorable* as far as the remainder of the problem is concerned. We calculate the constant α_k by applying the initial conditions, and the equation of motion for the cyclic coordinate is

$$\dot{q}_k = \frac{\partial H}{\partial \alpha_k} \equiv \omega_k \tag{7.178}$$

which can be immediately integrated to yield

$$q_k(t) = \int \omega_k \, dt \tag{7.179}$$

The solution for a cyclic coordinate is therefore trivial to reduce to quadrature. Consequently, the canonical formulation of Hamilton is particularly well suited for dealing with problems in which one or more of the coordinates are cyclic. The simplest possible solution to a problem would result if the problem could be formulated in such a way that *all* the coordinates were cyclic. Then, each coordinate would be described in a trivial manner as in Equation 7.179. It is, in fact, possible to find transformations that render all the coordinates cyclic,* and these procedures lead naturally to a formulation of dynamics particularly useful in constructing modern theories of matter. The general discussion of these topics, however, is beyond the scope of this book.†

EXAMPLE 7.12

Use the Hamiltonian method to find the equations of motion for a spherical pendulum of mass m and length b (see Figure 7-10).

Solution. The generalized coordinates are θ and ϕ. The kinetic energy is

$$T = \frac{1}{2} mb^2\dot{\theta}^2 + \frac{1}{2} mb^2 \sin^2 \theta \dot{\phi}^2$$

The only force acting on the pendulum (other than at the point of support) is gravity, and we define the potential zero to be at the pendulum's point of attachment.

$$U = -mgb \cos \theta$$

*Transformations of this type were derived by Carl Gustav Jacob Jacobi (1804–1851). Jacobi's investigations greatly extended the usefulness of Hamilton's methods, and these developments are known as *Hamilton-Jacobi theory*.

†See, for example, Goldstein (Go80, Chapter 10).

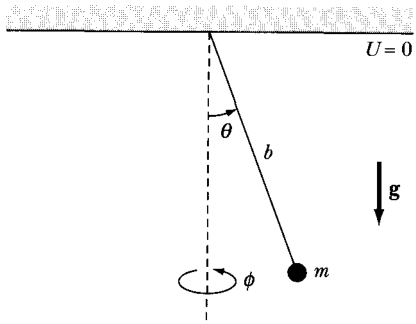

FIGURE 7-10 Example 7.12. A spherical pendulum with generalized coordinates θ and ϕ.

The generalized momenta are then

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mb^2\dot{\theta} \tag{7.180}$$

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = mb^2 \sin^2 \theta \dot{\phi} \tag{7.181}$$

We can solve Equations 7.180 and 7.181 for $\dot{\theta}$ and $\dot{\phi}$ in terms of p_θ and p_ϕ.

We determine the Hamiltonian from Equation 7.155 or from $H = T + U$ (because the conditions for Equation 7.130 apply).

$$\begin{aligned} H &= T + U \\ &= \frac{1}{2} mb^2 \frac{p_\theta^2}{(mb^2)^2} + \frac{1}{2} \frac{mb^2 \sin^2 \theta p_\phi^2}{(mb^2 \sin^2 \theta)^2} - mgb \cos \theta \\ &= \frac{p_\theta^2}{2mb^2} + \frac{p_\phi^2}{2mb^2 \sin^2 \theta} - mgb \cos \theta \end{aligned}$$

The equations of motion are

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mb^2}$$

$$\dot{\phi} = \frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{mb^2 \sin^2 \theta}$$

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = \frac{p_\phi^2 \cos \theta}{mb^2 \sin^3 \theta} - mgb \sin \theta$$

$$\dot{p}_\phi = -\frac{\partial H}{\partial \phi} = 0$$

Because ϕ is cyclic, the momentum p_ϕ about the symmetry axis is constant.