

What entanglement is and how to use parametric down conversion to realize it

ERH

Last week in lecture we finished off the last of the preliminary modern optics topics of this class. At this point you should have a good idea of most of the tools of modern optics: lasers, cavities, fibers, polarization and beam shaping optics, etc. This week we begin our journey into true quantum optics. At the center of everything we do in the remainder of this class is a non-linear optics technique, called spontaneous parametric down conversion, which we will use to generate entangled photons. This week we will review entanglement and the parametric down conversion process that we will use. In the lab you will prepare entangled states for the first time and begin our march towards Bell's inequality and two-photon interference. The purpose of studying entanglement here is that classically we like to ascribe a value to observables – like position – even before we measure it, but entanglement proves to us (quite strikingly) that you cannot do this. Only ψ is physical and it contains all knowable information.

Entanglement

Entanglement is a fancy quantum mechanics word, which implies that two or more particles are **correlated** in a way that cannot be realized with classical mechanics. The wavefunction of entangled particles cannot be separated into a simple product of one particle wavefunctions. This means that the state of one particle cannot be described without reference to the other particle. Therefore the particles are said to be "entangled". The road to fully coming to grips with entanglement is fraught with precise definitions, so let us begin as simply as possible.

First, what do we mean by a **correlation** between two particles? What we mean is that when we measure a certain property of particle 1 and a certain property of particle 2, the outcomes of the two measurements show some sort of relationship – that is, they are correlated. As a specific example of a classical (non-quantum mechanical) correlation, imagine the following situation:

Two Frisbees:

Imagine that David Griffiths is standing between you and a friend, but you can't see him. Now you don't know this, but what he is doing is throwing two Frisbees, one with each arm, and you and your friend have to catch them and record the direction they are spinning as CCW or CW – as viewed from above.



Since you can't see Griffiths, you don't know which way he is facing. And to make it more interesting, he turns around randomly before each throw of the Frisbees. Thus, if his left arm is nearest you, you will receive a CCW rotating Frisbee and if his right hand is nearest you, you will receive a CW rotating Frisbee. Your friend will receive the other type.

Now, if Griffiths is really turning around randomly, your record of the outcomes would look something like:

CCW
CW
CCW
CCW
CW
CCW
CW
CW

So, the probability of getting CCW is: $P(CCW) = 4/8 = \frac{1}{2}$ and the $P(CW) = \frac{1}{2}$ as well. (Of course, I have "cooked the books" here so that it works out perfectly. We would not really expect to see exactly $\frac{1}{2}$ after only 8 trials.) Clearly this is just a random process when viewed by one "experimenter" alone.

However, when you and your friends compare notes after the fact you see something pretty astounding:

You	Your friend
CCW	CW
CW	CCW
CCW	CW
CCW	CW
CW	CCW
CCW	CW
CW	CCW
CW	CCW

That is, whenever you measured CCW, your friend measured CW and vice versa. These measurements are said to be perfectly anti-correlated, and even though you can't see Griffiths, you know that he is up to something.

Now the example we just looked at is quite similar to the problem we will treat today, but we will see that when you look at correlations of quantum mechanical things, they are allowed to be extremely different than anything allowed by classical mechanics. And it is in this case, when you see these "crazy or impossible correlations" that the particles are said to be "entangled".

An entangled state:

Now, I've been making entanglement sound spooky and esoteric, but the truth is you have already encountered at least two entangled states! Recall in your Griffiths class the total spinor part of the wavefunction for two spin-1/2 particles. As you learned there, the total wavefunction could carry either a total spin of 1 or a total spin of 0. Denoting the states as $|S m_s\rangle$, where S is the total spin and m_s its projection, you found that there were four allowed quantum states:

Three, $S = 1$ states:

$$|11\rangle = \uparrow\uparrow$$

$$|10\rangle = \frac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow)$$

$$|1 - 1\rangle = \downarrow\downarrow$$

And one, $S = 0$ state:

$$|00\rangle = \frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow),$$

Of these states, the $|10\rangle$ and $|00\rangle$ are entangled states.

Now you might be thinking that really all of them should be entangled, because clearly the outcome of a measurement of particle 1 when the whole system is in e.g. the $|11\rangle$, is perfectly correlated with the result of measurements of particle 2's spin. However, the correlation of this state and the $|1 - 1\rangle$ turn out to be no different than the classical correlations of our Frisbee example. Entanglement is more than just being correlated, it is a certain type of correlation that only exist in quantum mechanics. As we will soon see, the $|10\rangle$ and $|00\rangle$ exhibit this "quantum mechanics only" type of correlation and therefore they are said to be an entangled state. (To use the word in a sentence: If the system is in either of the $|10\rangle$ or $|00\rangle$ states, we would say the two particles are entangled.)

As you may be able to guess at this point, to really get entanglement requires that there are two or more outcomes for the particle and those outcomes are correlated

with different outcomes of another particle. So for the singlet state ($S=0$), if you measure up for particle one, you know particle two is down and vice versa. Having the ability to get different correlated outcomes is entanglement.

The mystery of the quantum cakes

The clearest way to see the hallmarks of entanglement are to perform a measurement of particle 1 for one observable and a measurement of a non-commuting observable on particle 2. (In the original EPR paper, they considered a measurement of the position of one particle and a measurement of the momentum of the other particle.) In the above example the two measurements could be the spin along two different axes.

The result can be seen with relatively simple math, but in my opinion even the spin- $1/2$ system is peculiar enough to obscure the astonishing result. So, before we delve into the real problem, let's look at a beautiful example problem put forward by Kwiat and Hardy, called "the mystery of the quantum cakes".



Fig. 1. Lucy and Ricardo explore nonlocal correlations through quantum mechanically (non-maximally) entangled cakes. Because Ricardo's first cake (far right) rose early, Lucy's cake tastes good.

Let us suppose there is a cake factory that makes cakes that are really described by quantum mechanics. There are two opposing doors of the cake factory that are connected to conveyor belts. The doors open at the same time and an oven with a cake inside of it comes out on each side and moves away from the factory along the respective conveyor belt.

This cake factory only makes soufflés. In case you don't remember, soufflés are those cakes that due to the use of beaten egg whites "rise" considerably as they are baked. If you open the oven once the batter starts to rise, the vibration can cause it fall, ruining the cake.

Further down the conveyor belt there are two experimenters, Lucy on the left and Ricardo on the right. Lucy and Ricardo make two different types of measurements on the ovens. After making many measurements, they will come together and compare their results. The types of measurements they make are:

Type 1: The experimenter waits until the oven gets to the end of the conveyor belt. Then s/he opens the oven and measures if the cake inside tastes Good or Bad. (This is one observable: whether or not the cake tastes Good or Bad).

Type 2: Instead of waiting until the oven gets to the end of conveyor belt, the experimenter can open it midway on its journey to the end of the belt to see whether or not the batter has Risen or Not Risen. (This is the second observable: whether or not the batter has Risen or Not Risen.)

Now because these cakes are soufflés we have something like two non-commuting measurements. If we try to measure the second observable (Risen/Not Risen), when we close the oven door after the measurement we will cause the batter to fall, ruining the cake. Therefore only one of the observables can be measured on a given cake.

Now, when an oven is released, each experimenter randomly decides which measurement they will make. They write down the result for that run of the experiment. If the cakes are entangled (i.e. quantum mechanically correlated), later when Lucy and Ricardo compare their list the strangeness of quantum mechanics will be in full effect.

Since we obviously want to see this strangeness, let us suppose the two ovens are described by the entangled wavefunction:

$|\psi\rangle = \frac{1}{2} |B_L\rangle|B_R\rangle - \sqrt{\frac{3}{8}} (|B_L\rangle|G_R\rangle + |G_L\rangle|B_R\rangle)$, where B and G are the Bad- and Good-tasting eigenstates, which are related to the R (Risen) and N (Not Risen) eigenstates by:

$$|B\rangle = \sqrt{\frac{2}{5}}|N\rangle + \sqrt{\frac{3}{5}}|R\rangle,$$

$$|G\rangle = -\sqrt{\frac{3}{5}}|N\rangle + \sqrt{\frac{2}{5}}|R\rangle.$$

Note, we can really pick whatever we want here as long as we are using an orthonormal and complete basis and our wavefunctions are properly normalized. (We are.) This is the quantum mechanical analog of saying the classical statement that of 3/8 of the time, the cake factory spits a good cake out of the left side and a bad cake out of the right side. Further, regarding the decomposition into N and R states, we are also free to pick whatever we want because in principle any batter is realizable. Here the one we have picked leads to bad cakes that rise 3/5 of the time and don't rise 2/5 of the time. If this all feels very arbitrary to you that is okay. Remember we are doing a thought experiment here and can pick any physically realizable state – whether or not we could actually make such a state doesn't matter. And finally, if you would prefer a more quantum mechanical description, you could replace cakes with spin-1/2 particles and then the B and G states are spin-up and spin-down along e.g. the z-axis, while the N and R states are spin-up and spin-down along an axis rotated by 50.8° CCW to the z-axis, which leads to the decomposition shown. (We'll prove this in a minute.)

With that out of the way, let's think about the measurements that Lucy and Ricardo can perform. Clearly, there are three (technically four) main cases to consider.

- Case 1: Both Lucy and Ricardo check their ovens midway to see if they have risen.
- Case 2a: Lucy checks her oven midway to see if the batter has Risen/Not Risen and Ricardo tastes his cake at the end to see if it tastes Good/Bad.
- Case 2b: Ricardo checks his oven midway to see if the batter has Risen/Not Risen and Lucy tastes her cake at the end to see if it tastes Good/Bad.
- Case 3: Both Lucy and Ricardo check their ovens at the end to see if the cake tastes Good/Bad.

Now, since we know the wavefunction of our system we can trivially calculate the probabilities of different outcomes in these cases. We'll focus on a few.

First, it will make our lives easier to know the projections of R onto B/G:

$$\langle R|B \rangle = \sqrt{\frac{3}{5}} \text{ and } \langle R|G \rangle = \sqrt{\frac{2}{5}}$$

Case 1: Let's calculate the probability that both cakes rose early:

$$|\langle R_L | \langle R_R | \psi \rangle|^2 = \left| \frac{1}{2} \sqrt{\frac{3}{5}} \sqrt{\frac{3}{5}} - 2 \sqrt{\frac{3}{8}} \sqrt{\frac{3}{5}} \sqrt{\frac{2}{5}} \right|^2 = \left| \frac{3}{10} - 2 \frac{3}{5} \frac{1}{2} \right|^2 = \frac{9}{100} = 9\%. \text{ The rest of the time only one cake or neither rose early.}$$

Case 2: Whenever Lucy measures that her cake has Risen, we see that the wavefunction collapses to:

$$\langle R_L | \psi \rangle = \frac{1}{2} \sqrt{\frac{3}{5}} |B_R\rangle - \sqrt{\frac{3}{8}} \left(\sqrt{\frac{3}{5}} |G_R\rangle + \sqrt{\frac{2}{5}} |B_R\rangle \right) = \frac{3}{2\sqrt{10}} |G_R\rangle \rightarrow |G_R\rangle.$$

Therefore, Ricardo always measures that his cake taste Good whenever Lucy measures that her cake has Risen.

Case 2b: ψ is symmetric with respect to $L \leftrightarrow R$, thus the result is the same. Whenever Ricardo measures that his cake rose early, we see that the wavefunction collapses to:

$$\langle R_R | \psi \rangle = \frac{1}{2} \sqrt{\frac{3}{5}} |B_L\rangle - \sqrt{\frac{3}{8}} \left(\sqrt{\frac{3}{5}} |G_L\rangle + \sqrt{\frac{2}{5}} |B_L\rangle \right) = \frac{3}{2\sqrt{10}} |G_L\rangle \rightarrow |G_L\rangle.$$

Therefore, Lucy always measures that her cake taste Good whenever Ricardo measures that his cake has Risen.

Case 3: Now, this is where things get interesting. Since 9% of the time Ricardo and Lucy both measure that their cakes rise early and every single time Ricardo or Lucy measure that their cake rose the other one measured that their cake tastes Good. Therefore by classical reasoning we conclude when Ricardo and Lucy both measure the taste of their cake **at least** 9% of the time they both measure that their cakes taste Good. Classically speaking this is the right answer without question.

Simple enough, right? Well, what does quantum mechanics say:

$$|\langle G_L | \langle G_R | \psi \rangle|^2 = 0.$$

It says that both cakes never taste Good at the same time! (i.e. at least one of the cakes always taste Bad.)

How can we explain this!? The classical argument that got us here is obviously solid, assuming the cakes are left alone inside of the oven once produced. But it turns out we can push this discrepancy even further. Let's allow the chef to have two apprentices that travel along secretly with the oven. Can these three cooks reproduce the quantum result using any means necessary?

So, first off the chef could prepare a batter that rises early and use it 9% of the time for both cakes. The other 91% of the time he will not use this batter for both cakes. This satisfies the results of Case 1. Now, if the batter that rises early also leads to cakes that taste Good, then he has satisfied Case 1, 2a, and 2b. But, what about Case 3? Clearly not.

Alternatively, the batter that rises early could always lead to Bad tasting cakes, thus satisfying 1 and 3, but not 2a and 2b.

The problem for the classical cakes is that in quantum mechanics the outcome of one measurement is tied to the outcome of the other measurement. For example, suppose Ricardo decides to taste his cake¹. Then we know that he can measure Good or Bad. However, if Lucy decides to check if the batter has risen, his cake has to taste Good, but if she decides to taste her cake then he can get Good or Bad – and even worse, if her cake taste Good then his must taste Bad.

Clearly, we will never be able to mimic the quantum correlations by the preparation of the batter alone. But, remember, we allowed the chef to have two apprentices, so perhaps they can use radios to talk with one another and try to make the results

¹ To put more succinctly, in the classical world events at one end of the conveyor belt are uninfluenced by random choices and events at the other end – i.e. Lucy's cake taste the same whether or not Ricardo measures his. (This can be made rigorous by making the places where the cakes are measured extremely far apart so that no-slower-than-light influence could propagate to the other cake before it is measured.)

match the quantum mechanical prediction. Well, if the conveyor belts are close by, the apprentice chefs could watch what is being measured to the cake on the other conveyor belt and “adjust” their cake accordingly. However, we are allowed to make these conveyor belts very long and fast, so that the places where the cakes are measured are so far apart that the time it takes light to propagate between them is longer than the time between the random measurements. In this case quantum mechanics still predicts the same result, but the chef’s apprentices are no longer able to know what happens on the other conveyor belt before their own cake is measured. Thus, unless they can somehow communicate faster than the speed of light, even with two apprentice chefs, they cannot fake the quantum result!

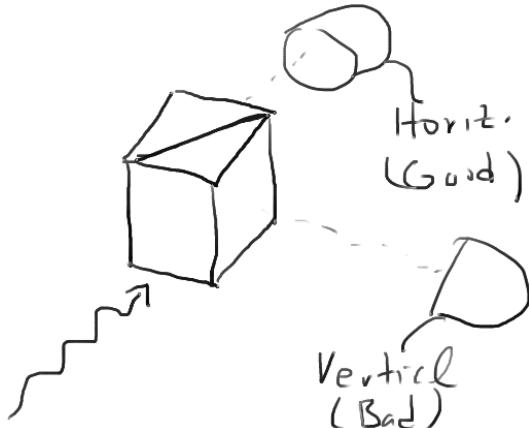
So, what does this all mean? Well first off it means the wavefunction collapse is non-local. If two particles are entangled and allowed to propagate to the edge of the universe, a measurement on one of them collapses the wavefunction of the other!!!!!! (As an FYI, Einstein hated this. In fact, he used this result to say that QM must be wrong. It turns out that as near as experiments can tell, wavefunction collapse is instantaneous!)

It also rules out what are known as local hidden variable theories. We’ll talk more about this next week, but for now these theories postulate that things like the position and momentum of a particle are in principle knowable exactly – i.e. the wavefunction is not the ultimate theory. Thus, they would ascribe a “classical” reality to Good/Bad and Risen/Not Risen before the measurement. Clearly, this can’t be the case. QM says: The cakes are not in either state until one is measured.

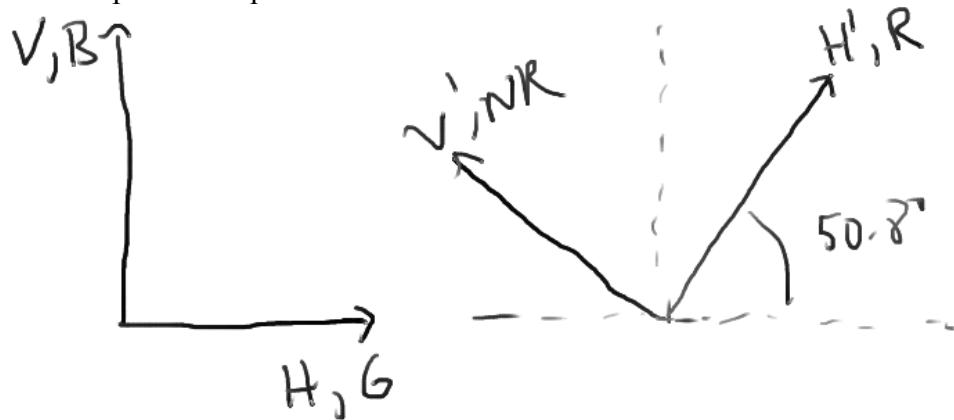
Parametric down conversion

At this point, we’re starting to get a feel for QMical entanglement, but something as weird as this needs to be verified experimentally. Clearly, the quantum cake is an idealization. Cakes are big classical things and though people still debate the quantum-to-classical transition, I think we all agree that Cakes reside soundly in the classical world. So, we need a different system. Luckily, there is a fairly straightforward way of generating entangled photons, called spontaneous parametric down conversion. We’ll talk about that in just a second, but let me first prove to you that entangled photons are a perfect stand-in for our quantum cakes.

As you know, the photon has two distinct polarization states. Therefore, we could use a polarizing beam splitter to measure whether its polarization is Horizontal (0°) or Vertical (90°), which we can let be the analog of the cake tasting Good or Bad. (Good/Bad = Horizontal/Vertical). So, if we put detectors on each output of the polarizer beam splitter, if we get a ‘click’ on the horizontal port, we know the photon was Horizontal (Good) and if we get a click on the vertical detector we know the photon was Vertical (Bad).



Now to mimic the Risen/Not Risen basis we need to measure the polarization along a different set of non-commuting axes. We can do this by rotating our PBS. Based on our decomposition of Good and Bad into Risen and Not Risen, we rotate the PBS so that horizontal port passes polarization at $+50.8^\circ$ and the vertical passes light polarized at 140.8° . Then a click in the rotated horizontal port corresponds to Rise and a click on the rotated vertical port corresponds to Not Risen.



Specifically,

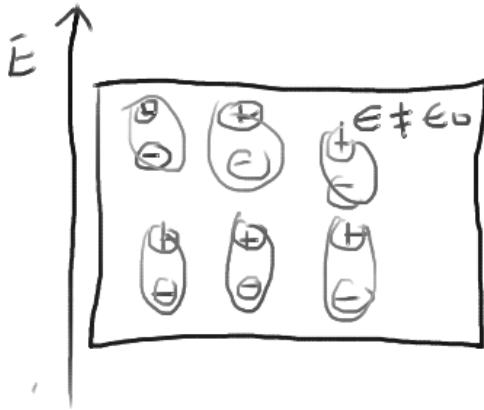
$$|B\rangle = \cos 50.8^\circ |N\rangle + \cos 39.2^\circ |R\rangle = \sqrt{\frac{2}{5}} |N\rangle + \sqrt{\frac{3}{5}} |R\rangle$$

$$|G\rangle = -\sin 50.8^\circ |N\rangle + \cos 50.8^\circ |R\rangle = -\sqrt{\frac{3}{5}} |N\rangle + \sqrt{\frac{2}{5}} |R\rangle$$

With that out of the way, how can we make entangled photons? That is, how do we make states like the cake $|\psi\rangle$? One answer is spontaneous parametric down conversion (SPDC).

SPDC is a non-linear optical effect, which means that it requires a material that responds to an electric field in a non-linear way.

Specifically, when a dielectric material is in an electric field it responds to the field creating a polarization, which gives rise to another electric field in the medium:



$$\begin{aligned} \vec{D} &= \epsilon \vec{E} \\ &= \epsilon_0 \vec{E} + \vec{P} \\ &= \epsilon_0 (1 + \chi) \vec{E} \end{aligned}$$

Within Maxwell's equations this is handled through the use of \vec{D} the electric displacement field (or sometimes called the electric flux density) – units are C/m² or N/(V/m). Here, P is the polarization density and is defined as the average dipole moment per unit volume (units Cm/m³ = C/m²). Thus, our earlier derivation of the traveling wave equation is now modified as:

Recall the Maxwell-Faraday equation and Ampere's circuit law, in the absence of currents, we have:

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \text{ and } \nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t}$$

Now, in the usual way we use $\vec{B} = \mu \vec{H}$ to write:

$$\nabla \times \nabla \times \vec{E} = -\frac{\partial (\nabla \times \vec{B})}{\partial t} = -\frac{\partial^2 \vec{D}}{\partial t^2} = -\epsilon_0 \mu \frac{\partial^2 \vec{E}}{\partial t^2} - \mu \frac{\partial^2 \vec{P}}{\partial t^2}$$

And the curl of the curl relation gives:

$$\nabla \times \nabla \times \vec{E} = \nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E}$$

So, we have the normal traveling wave equation, but now with an extra source term.

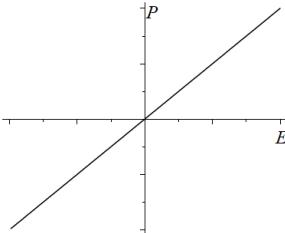
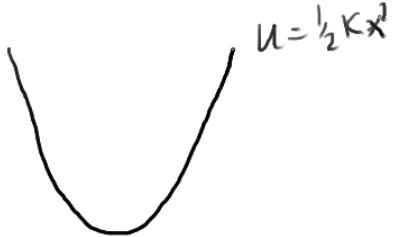
$$\nabla^2 \vec{E} - \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} = \mu \frac{\partial^2 \vec{P}}{\partial t^2}$$

Now, in most materials the polarization is linear in E as we have written above. Intuitively, this makes sense as the polarization is arising from the electric field pulling the electrons and nucleus a bit. In most materials, the binding force of the charges is spring-like ($F = -kx$), so for a given E -field the charge feels a force of qE , which displaces it from equilibrium by: $\Delta x = qE/k$. And thus, leading to a polarization of order $q\Delta x = \frac{q^2}{k} E$. Thus, the polarization density P is linear in E .

Linear

$$P \propto E$$

$$P = \chi^{(1)} E$$



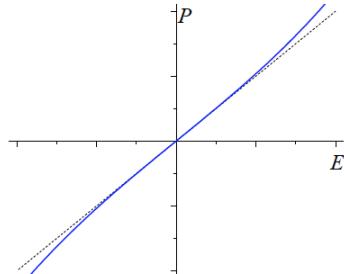
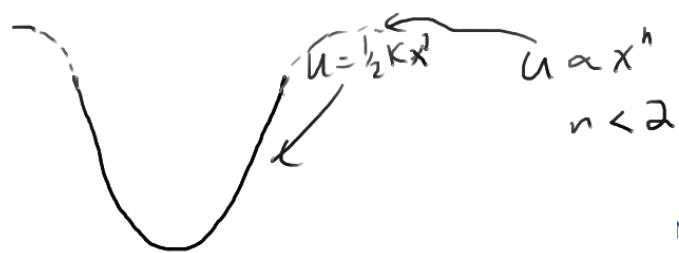
[linear]

Linear optical properties:
index of refraction
absorption

However, at higher intensities the force on the electrons becomes un-spring-like and the potential is anharmonic. It can be shown that for all materials at large enough intensity, the polarization increases faster than E^1 , at high intensities. Effectively, the force becomes less strong than $-kx$ at large x , so the electric field can displace it more, leading to a larger polarization. This leads to a third order correction to the polarization:

3rd order

$$P = \chi^{(1)} E + \chi^{(3)} E^3$$



[nonlinear]

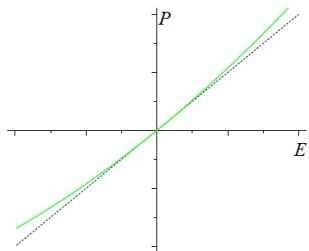
Nonlinearities:
third harmonic generation
nonlinear index of refraction (Kerr)

Occurs in all materials

This effect is used in quite a bit of technology, but is not the subject of what we'll talk about today. Instead, we're going to look at the 2nd order correction to the polarization:

2nd order

$$P = \chi^{(1)}E + \chi^{(2)}E^2$$



For there to be a second order nonlinearity, the force the electrons feel must do something pretty odd. For one direction of the polarization, let's say +, the polarization will increase faster than E^1 . Thus, like the 3rd order situation above, the force must get weaker than $-kx$ for large E . However, when the electric field reverses direction to the - direction, the polarization grows at a rate slower than linear. Thus, the force must be getting stronger as the electron is pushed in this direction! Clearly, the potential must look something like:

Nonlinearities:
second harmonic generation
optical rectification
difference frequency generation

Requires inversion asymmetry



Now, unlike the third order effect, this effect only occurs in certain materials. You can show that to have a $\chi^{(2)}$ -nonlinearity requires a dielectric crystal structure that does not have inversion symmetry – i.e. if you invert the coordinates of the atoms in the crystal you get a different crystal structure. However, that doesn't really concern what we are doing today. We just need to know that materials exist with $\chi^{(2)} \neq 0$.

Now, let us return to the wave equation and see what happens with this type of source. We assume that $\chi^{(3)} = 0$ and will ignore the linear term (Really, we just group it with the $1/c^2$ term: $\frac{1}{c^2} \frac{\partial^2 E}{\partial t^2} + \chi^{(1)} \frac{\partial^2 E}{\partial t^2} = \mu\epsilon_0 (1 + \chi^{(1)}) \frac{\partial^2 E}{\partial t^2} \approx \frac{1}{c^2} \frac{\partial^2 E}{\partial t^2}$); we just want to see what the effect of the $\chi^{(2)}$ term is. Now, technically, $\chi^{(2)}$ is a tensor and the E^2 is really a dyadic product of two copies of \vec{E} . This happens because if there are electric fields with more than one polarization present they can both influence the polarization in complicated ways. Generally speaking it is a nightmare, so let's suppose we are just dealing with electric fields polarization along one direction and we can write $\chi^{(2)} = 2d$ for that direction, where d is a constant.

$$\nabla^2 E - \frac{1}{c^2} \frac{\partial^2 E}{\partial t^2} = 2\mu d \frac{\partial^2 E^2}{\partial t^2}$$

Now, as usual we need to guess a solution and see what happens. I'm going to guess a solution that is a superposition of 3-waves. Why would I do this? Well, I know the answer we're looking for and want to take the most efficient route to it. Don't worry if you think this is not an intuitive step. It isn't.

Let's first guess a trivial time dependence for a three wave solution:

$$E = \sum_{q=\pm 1, \pm 2, \pm 3} \frac{1}{2} E_q e^{i\omega_q t}$$

Here the minus sign in the summation just means flip the sign in the phase factor and use the complex conjugate of E_q -- i.e. the sum includes the c.c.'s.

Then, the source term takes the form:

$$2\mu d \frac{\partial^2 E^2}{\partial t^2} = -\frac{\mu d}{2} \sum_{q,r=\pm 1, \pm 2, \pm 3} (\omega_q + \omega_r)^2 E_q E_r e^{i(\omega_q + \omega_r)t}$$

Now, the resulting differential equation is fairly complicated until we realize that we can equate terms that oscillate at the same frequency. That's easy for the left-hand side, but the right hand side is going to oscillate at all sorts of sum and difference frequencies.

Thus, most likely the effects will average out. However, if the frequencies are commensurate, e.g. $\omega_3 = \omega_1 + \omega_2$, then we can get terms that oscillate at frequencies that are the same as terms on the left hand-side. Thus we have:

$$\begin{aligned} \left(\nabla^2 + \left(\frac{\omega_1}{c} \right)^2 \right) E_1 &= 2e^{-i\omega_1 t} S_1 \\ \left(\nabla^2 + \left(\frac{\omega_2}{c} \right)^2 \right) E_2 &= 2e^{-i\omega_2 t} S_2 \\ \left(\nabla^2 + \left(\frac{\omega_3}{c} \right)^2 \right) E_3 &= 2e^{-i\omega_3 t} S_3 \end{aligned}$$

where S_q will be given by the terms of the polarization source that oscillate at frequency ω_q . So, for example in the first equation we need terms that oscillate at ω_1 . That will be given by the term that goes as $\omega_3 - \omega_2$, so:

$$S_1 = -\frac{\mu d}{2} (\omega_3 - \omega_2)^2 E_3 E_2^* e^{i(\omega_3 - \omega_2)t} = -\frac{\mu d}{2} \omega_1^2 E_3 E_2^* e^{i\omega_1 t}$$

And similarly for the other S 's, so that we have:

$$\begin{aligned} (\nabla^2 + k_1^2) E_1 &= -\mu d \omega_1^2 E_3 E_2^* \\ (\nabla^2 + k_2^2) E_2 &= -\mu d \omega_2^2 E_3 E_1^* \quad + \text{c.c.} \\ (\nabla^2 + k_3^2) E_3 &= -\mu d \omega_3^2 E_1 E_2 \end{aligned}$$

Since clearly the amplitudes will evolve as the light propagates, let's try plane wave solutions with amplitudes that are functions of propagation distance: $E_q = A_q(z) e^{-ik_q z}$

$$\begin{aligned} \nabla E_q &= \nabla A_q e^{-ik_q z} - ik_q A_q e^{-ik_q z} \text{ and} \\ \nabla^2 E_q &= \nabla^2 A_q e^{-ik_q z} - 2ik_q \nabla A_q e^{-ik_q z} - k^2 A_q e^{-ik_q z} \end{aligned}$$

With the normal paraxial approximation we neglect the $\nabla^2 A_q$ term because k is huge and so the second term dominates. Also, $\nabla A_q = \frac{dA_q}{dz}$. Thus,

$$\frac{dA_1}{dz} = -\frac{i}{2} \mu d \omega_1 c A_3 A_2^* e^{-i(k_3 - k_2 - k_1)z}$$

$$\begin{aligned}\frac{dA_2}{dz} &= -\frac{i}{2} \mu d\omega_2 c A_3 A_1^* e^{-i(k_3 - k_2 - k_1)z} \\ \frac{dA_3}{dz} &= -\frac{i}{2} \mu d\omega_3 c A_2 A_1 e^{i(k_3 - k_2 - k_1)z}\end{aligned}$$

Now, we see that unless $\Delta k = k_3 - k_2 - k_1 = 0$, not much will happen. The source term oscillates rapidly and the A 's don't really change as a result of it. However, if $\Delta k = 0$ then something neat can happen. To see this, let us assume that $\omega_1 = \omega_2 = \omega$ and therefore $\omega_3 = 2\omega$, then we just have two coupled equations (since the first two are the same equation):

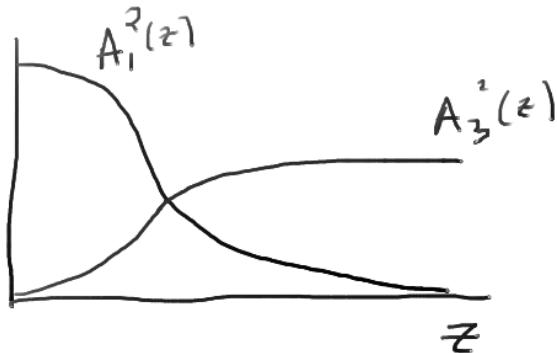
$$\begin{aligned}\frac{dA_1}{dz} &= -\frac{i}{2} \mu d\omega_1 c A_3 A_1^* = -\frac{i}{2} \mu d\omega c A_3 A_1^* \\ \frac{dA_3}{dz} &= -\frac{i}{2} \mu d\omega_3 c A_1 A_1 = -i \mu d\omega c A_1^2\end{aligned}$$

A quick aside: Second harmonic generation

Since we've done all of this work, let's stop and take a quick detour. Suppose that $A_3(0) = 0$ -- i.e. there is no light at frequency 2ω -- but that there is some light at frequency ω -- i.e. $A_1(0) \neq 0$. Then with some maths you can show, the solutions of the differential equations above are:

$$\begin{aligned}A_1(z) &= A_1(0) \operatorname{sech} \frac{\mu d\omega A_1(0) z}{2\sqrt{2}} \\ A_3(z) &= -\frac{i}{\sqrt{2}} A_1(0) \tanh \frac{\mu d\omega A_1(0) z}{2\sqrt{2}}\end{aligned}$$

Thus, the evolution looks like:

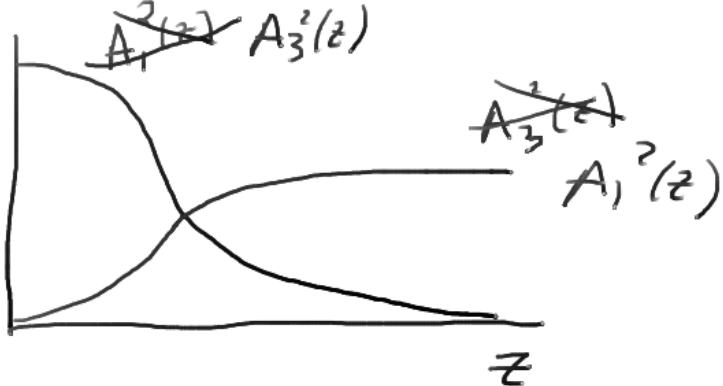


The light at frequency ω disappears and light at frequency 2ω appears! Evidently two photons at ω combine to make one photon at 2ω ! This is called second harmonic generation and it's how green laser pointers work.

Back to our story

Now let's look at the opposite situation of what we just looked at. Let's let $A_1(0) = 0$. Because $A_1(z)$ appears on the RHS of both of the equations above, nothing happens.

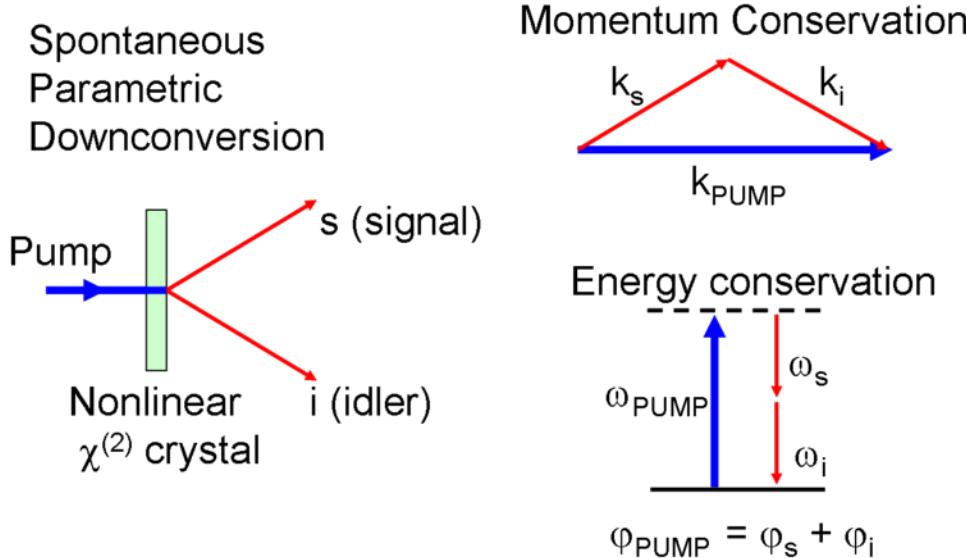
The field amplitudes do not change with z . (NOTE: We got a different answer for the Second Harmonic case because the equation for $\frac{dA_3}{dz} \neq 0$. Thus evolution of the amplitudes with position could happen.) That's okay. We know that even in vacuum there are a few photons around, so let's suppose $A_1 = 0 + \epsilon$, what happens? Well, you guessed it: the opposite of above. Light at 2ω disappears and more light at ω appears!



(I'm recycling the graph here, because the general behavior is the same. Note that it does not have the exact same functional form b/c of the initial conditions.)

Thus, a photon at frequency 2ω , can break up into 2-photons of frequency ω ! This is called parametric down conversion. Now, to get this solution we had to make there already be a little bit of light at ω , but if you treat the problem in QED you'll find you don't need to have any light at ω to get the same behavior – the vacuum is enough. (This is just like spontaneous and stimulated emission. You can see stimulated emission in regular ol' QM, but spontaneous requires QED.) When this process happens without any light at frequency ω to start with, it is called spontaneous parametric down conversion.

And if we have a crystal with $\chi^{(2)} \neq 0$, we will get this as long as we can ensure that $\omega_1 + \omega_1 = \omega_3$ and $\Delta k = k_3 - k_2 - k_1 = 0$. The first equation is energy conservation and the second one is just conservation of momentum!



Satisfying $\Delta k = k_3 - k_2 - k_1 = 0$, in a crystal is beyond the scope of what we want to do in this class. You might think that it is obviously satisfied since $\lambda_3 = \lambda_1/2$. However, you cannot forget about the index of refraction:

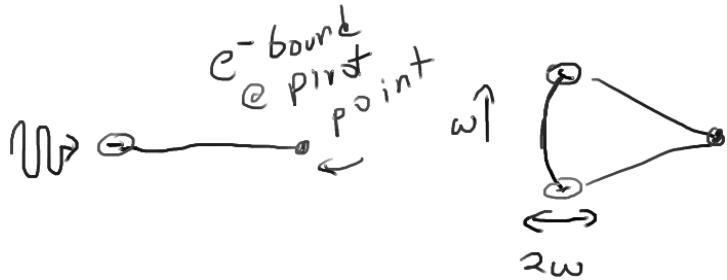
$$\Delta k = k_3 - k_2 - k_1 = k_3 - 2k_1 = 2\pi \left(\frac{n(\lambda_3)}{\lambda_3} - \frac{n(\lambda_1)}{\lambda_1} \right) = 0$$

Since most materials have dispersion ($n = f(\lambda)$) it is usually non-trivial to satisfy this equation and often requires making use of birefringence to engineer an effective n . Making this equation be true is called “phase matching” since it also ensures that the waves stay in phase as they propagate through the crystal.

What's really going on here?

The physically phenomenon responsible for SHG and (S)PDC tend to get obscured by the math. What is really going on here? Well, it's actually pretty simply. The electrons are bound in materials by a force that looks very much like a spring (around the equilibrium position), so they are basic harmonic oscillators. When you drive them at a frequency ω they oscillate at their natural frequency and at ω . However, at high intensities (and depending on the material as per the above), the electron restoring force looks anharmonic and you can get electron motion at harmonics and subharmonics, which can radiate the SHG or (S)PDC light!

In some nonlinear processes the polarization of the radiated photons differs from the input polarization. In these cases, the process depends on the off-diagonal components of $\chi^{(2)}$, which couples electron motion in one direction to motion in the other direction. In this case, you can imagine the process as the electron is bound at a pivot point. When you drive at ω in one direction, it oscillates at 2ω in the transverse direction.

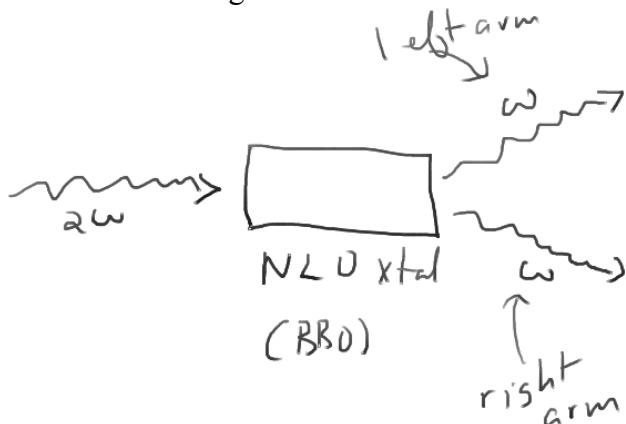


(With more time we could just do the CM problem here.)

Proving your SPDC photons are entangled

Alright, now we know how to turn a single photon into two photons, but how does that help us with generating entangled photons. Well, it turns out that the two photons that are generated in SPDC are entangled.

In the type of SPDC we will be using in class, we will satisfy the phase matching conditions by orienting the crystal so that the two down-converted photons have perpendicular polarization. Since the crystal is birefringent this means the phase matching condition becomes: $\left(\frac{n(\lambda_3)}{\lambda_3} - \frac{n(\lambda_1)}{\lambda_1} - \frac{n'(\lambda_1)}{\lambda_1}\right) = 0$, where the primed denotes that the polarization is along a different axis and thus the index of refraction is different. (For the expert, this is called Type II phase matching.) This gives you a bit more flexibility and when you couple it with the fact that by rotating the crystal you can change the effective n for a given polarization, it allows you to find a situation where $\Delta k = 0$. Therefore, we have the following situation:

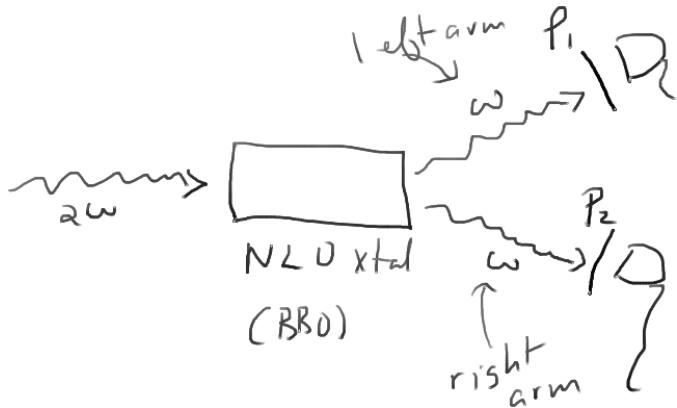


So, we have two photons going off in different directions and we know they are oppositely polarized, but we cannot know which one is H and which one is V, so the state of the system has to be:

$|\psi\rangle = \frac{1}{\sqrt{2}}(|H_L\rangle|V_R\rangle + e^{i\phi}|V_L\rangle|H_R\rangle)$, which is an entangled state since the outcome of the measurements in each arm depends on the measurements in the other arm!

Quantifying the degree of entanglement (preview)

In this weeks lab we are going to play around with SPDC and the preparation of entangled photons. The lab will walk you through some preliminary verifications that the photons are in fact entangled, but we can already see a very simple (crude) way of seeing some **correlations**:



If we place polarizers and detectors in each arm, we've now made real versions of Lucy and Ricardo. If we set $P_1 = H$ and $P_2 = V$ or vice versa we expect to see a maximum in coincidence counts – that is both detectors click at the same time. If we set them to both H or both V we expect to see a minimum in V. Because we get maxima when set $\{P_1, P_2\} = \{H, V\}$ or $\{P_1, P_2\} = \{V, H\}$, we have recovered the correlations of the original Griffiths Frisbee situation. By itself this is not significant enough proof to claim entanglement since you could get this same result with the Frisbees. But it is very suggestive and consistent with our entangled wavefunction. Next week we will discuss how to prove entanglement unequivocally.