Math 115A: Sample final exam

Sections 1 and 3. Instructor: James Freitag

For the exam, you may use one 8 inch by 11 inch (normal sized paper) piece of paper with anything at all written on **one side** - theorems, example problems, inspirational sayings - anything goes. There will be 8 problems on the final. The difficulty will be on the level of the exams.

Keep in mind this sample review is not comprhensive. I will post more problems throughout the week.

Problem 1 Eigenvalues

Let $\theta \in (0, \pi/2)$. Let

$$A = \begin{pmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{pmatrix}.$$

The entries of A are in $\mathbb{R}$, so we can regard A as either a matrix over the reals *or* the complex numbers. Are there any eigenvectors over $\mathbb{R}$? Explain why not intuitively.

Calculate the eigenvalues and eigenvectors over $\mathbb{C}$.

First, calculate the characteristic polynomial: this is given by

$$p(\lambda) = \left| \begin{pmatrix} cos(\theta) - \lambda & -sin(\theta) \\ sin(\theta) & cos(\theta) - \lambda \end{pmatrix} \right| = cos^2(\theta) - 2\lambda cos(\theta) + \lambda^2 + sin^2(\theta) = 1 - 2\lambda cos(\theta) + \lambda^2$$

The roots are given by

$$\frac{2cos(\theta) \pm \sqrt{4cos^2(\theta) - 4}}{2}$$

and these two values are imaginary whenever $cos(\theta) < 1$, which holds for all $\theta \in (0, \pi/2)$. Now, why does this make sense?

Well, draw the real plane. Now think about a vector and rotate it 30 degrees. It changes direction - it doesn't just scale. This means it is not an eigenvector. There are complex eigenvectors - in this case it might be helpful to think about hte complex numbers as pointing out of the page at you and the rotation as happening on the page.