105A - Set 7 - Solution

(Grades are out of 150)

1. A particle is moving in a central inverse-square-law force field for a superimposed force which magnitude is inversely proportional to the cube of the distance from the particle to the force center. in other words:

$$F = -\frac{k}{r^2} - \frac{\lambda}{r^3} \quad k, \lambda > 0 \tag{1}$$

describe the motion (i.e., r as a function of θ) and show that the motion can be described as a precessing ellipse. Consider the following cases:

(a) $\lambda < l^2/\mu$

(b) $\lambda = l^2/\mu$

(c) $\lambda > l^2/\mu$

where l is the angular momentum and μ is the mass of the particle.
Hint 1: Use Binnet's equation
Hint 2: Note that

$$\frac{\mu k}{l^2}\left(1 - \frac{\mu\lambda}{l^2}\right)^{-1} \tag{2}$$

Is constant.
Answer: Using Binnet's equation we can write:

$$\frac{l^2 u^2}{\mu}\left(\frac{d^2 u}{d\theta^2} + u\right) = -F(1/u) \ , \tag{3}$$

where in our case

$$F = -\frac{k}{r^2} - \frac{\lambda}{r^3} = -ku^2 - \lambda u^3 \tag{4}$$

So Binnet's equation is then

$$\frac{l^2 u^2}{\mu}\left(\frac{d^2 u}{d\theta^2} + u\right) = ku^2 + \lambda u^3 \ , \tag{5}$$

arranging we can write:

$$\frac{d^2 u}{d\theta^2} + u = \frac{\mu}{l^2}\left(k + \lambda u\right) \ , \tag{6}$$

And arranging yet again:

$$\frac{d^2 u}{d\theta^2} + u\left(1 - \frac{\mu\lambda}{l^2}\right) = \frac{\mu}{l^2}k \ , \tag{7}$$

defining

$$\alpha^2 = \left(1 - \frac{\mu\lambda}{l^2}\right) \tag{8}$$

we have

$$\frac{d^2u}{d\theta^2} + u\alpha^2 - \frac{\mu}{l^2}k = 0 \ , \tag{9}$$

Or

$$\frac{d^2u}{d\theta^2} + \alpha^2\left(u - \frac{\mu}{l^2\alpha^2}k\right) = 0 \ , \tag{10}$$

We define:

$$x = u - \frac{\mu}{l^2\alpha^2}k \tag{11}$$

So the equation is simply:

$$\frac{d^2x}{d\theta^2} + \alpha^2 x = 0 \ , \tag{12}$$

We can consider now the different cases:

(a) $\lambda < l^2/\mu$
so $\alpha^2 > 0$ and the solution is:

$$x = A\cos(\alpha\theta - \delta) \tag{13}$$

or

$$u = A\cos(\alpha\theta - \delta) + \frac{\mu}{l^2\alpha^2}k \tag{14}$$

which we can write:

$$\frac{1}{r} = A\cos(\alpha\theta - \delta) + \frac{\mu}{l^2 - \mu\lambda}k \tag{15}$$

This is a good enough answer, but we can do better. When $\alpha = 1, \lambda = 0$, this equation describes a conic section. Since we do not know the value of the constant A, we need to use what we have learned from Keplers problem to describe the motion. We know that for $\lambda = 0$

$$\frac{1}{r} = \frac{\mu k}{l^2}(1 + e\cos\theta) \tag{16}$$

and that we have an ellipse or circle $(0 \leq e \leq 1)$ when $E < 1$, a parabola $(e = 1)$ when$E = 0$, and a hyperbola otherwise. It is clear that for this problem, if $E \geq 0$, we will have some sort of parabolic or hyperbolic orbit. An ellipse should result when$E < 0$, this being the only bound orbit. When $\alpha \neq 0$, the orbit, whatever it is, precesses. This is most easily seen in the case of the ellipse, where the two turning points do not have an angular separation of π.

(b) $\lambda = l^2/\mu$
For this case $\alpha = 0$ and then

$$\frac{d^2u}{d\theta^2} = \frac{\mu}{l^2}k \ , \tag{17}$$

so

$$\frac{1}{r} = u = \frac{\mu k}{2l^2}\theta^2 + A\theta + B \tag{18}$$

from which we see that r continuously decreases as θ increases; that is, the particle spirals in toward the force center.

(c) $\lambda > l^2/\mu$ so $\alpha^2 < 0$ and the solution is:

$$x = A\cosh(\sqrt{-\alpha^2}\theta - \delta) \tag{19}$$

or

$$\frac{1}{r} = A\cosh(\sqrt{-\alpha^2}\theta - \delta) + \frac{\mu}{l^2 - \mu\lambda}k \tag{20}$$

Again, the particle spirals in toward the force center.

2. A particle moves in a central force field given by the potential

$$V = -k\frac{e^{-ar}}{r} \tag{21}$$

where k and a are positive constants.

(a) Write down the Lagrangian
Answer: The Lagrangian is

$$L = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + k\frac{e^{-ar}}{r} \tag{22}$$

(b) Find the equations of motions
Answer: We need:

$$\begin{aligned}
\frac{\partial L}{\partial r} &= mr\dot{\theta}^2 - k\frac{e^{-ar}}{r^2} - ka\frac{e^{-ar}}{r} = mr\dot{\theta}^2 - k(1+ar)\frac{e^{-ar}}{r^2} && (23)\\
\frac{\partial L}{\partial \theta} &= 0 && (24)
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial L}{\partial \dot{r}} &= m\dot{r} && (25)\\
\frac{\partial L}{\partial \dot{\theta}} &= mr^2\dot{\theta} = 0 \quad \text{i.e.,} \quad l = mr^2\dot{\theta} = \text{const} && (26)
\end{aligned}$$

so finally we write:

$$\ddot{r} = r\dot{\theta}^2 - \frac{k(1+ar)}{m}\frac{e^{-ar}}{r^2} = \frac{l^2}{m^2r^3} - \frac{k(1+ar)}{m}\frac{e^{-ar}}{r^2} \tag{27}$$

i.e., the angular momentum is conserved.

(c) When is circular orbit is possible?
Answer: At circular orbit: $\ddot{r} = 0$ so

$$\frac{l^2}{mr^3} = k(1+ar)\frac{e^{-ar}}{r^2} \tag{28}$$

(d) What is the effective potential?
Answer:

$$V_{eff} = -k\frac{e^{-ar}}{r} + \frac{l^2}{2mr^2} \tag{29}$$

(e) Which point does a circular orbit represent on the effective potential?
Answer: Its the minimum point since

$$\frac{dV_{eff}}{dr} = (k + ar)\frac{e^{-ar}}{r^2} - \frac{l^2}{mr^3} \tag{30}$$

Equating this to zero we find that

$$\frac{l^2}{mr^3} = k(1 + ar)\frac{e^{-ar}}{r^2} \tag{31}$$

which represent the minimum point.

3. A particle of mass m in a Kepler central potential $U(r) = -k/r$ has orbits described by

$$\frac{1}{r} = \frac{\mu k}{l^2}(1 + e\cos\theta) \tag{32}$$

where

$$e = \sqrt{1 + \frac{2l^2 E}{\mu k^2}} \tag{33}$$

(a) Suppose the particle is initially in a parabolic orbit. An impulse is applied at periastron (closest approached) to place the particle in a circular orbit. Give the energy and angular momentum of the circular orbit in terms of the energy and angular momentum of the initial parabolic orbit.
Answer: The initial orbit is parabolic so $E = 0$ and $e = 1$ and

$$\frac{1}{r} = \frac{\mu k}{l^2}(1 + \cos\theta) \tag{34}$$

Closest approach takes place when $\theta = 0$ and then

$$r_p = \frac{l_p^2}{2\mu k}\ , \tag{35}$$

where we denote l_p as the angular momentum of the parabolic orbit. The final orbit is circular so $e = 0$ and

$$r_c = \frac{l_c^2}{\mu k} = r_p = \frac{l_p^2}{2\mu k} \tag{36}$$

So:

$$l_c = \frac{l_p}{\sqrt{2}} \tag{37}$$

The energy is of course not conserved. The energy of the parabolic orbit ($E_p = 0$) while the one for the circular orbit (E_c) is derived from:

$$e = 0 = \sqrt{1 + \frac{2l_c^2 E_c}{\mu k^2}} \tag{38}$$

So

$$E_c = -\frac{\mu k^2}{2l_c^2} = -\frac{\mu k^2}{l_p^2} \tag{39}$$

Note that since at periastron $\dot{r} = 0$ there is no radial imp use.

(b) Suppose the particle is initially in an arbitrary elliptical orbit. An impulse is applied at $\theta = \pi/2$ to place the particle in a circular orbit. Give the energy and angular momentum of the circular orbit in terms of the energy and angular momentum of the initial orbit.
Answer: Elliptical orbit is described by

$$\frac{1}{r_e} = \frac{\mu k}{l_e^2}(1 + e\cos\theta) \quad 0 < e < 1 \ , \tag{40}$$

where r_e is the for the elliptical orbit. Impulse at $\theta = \pi/2$ means that

$$r_e = \frac{l_e^2}{\mu k} \ , \tag{41}$$

The final orbit is circular so $e = 0$ and

$$r_c = \frac{l_c^2}{\mu k} = r_e = \frac{l_e^2}{\mu k} \tag{42}$$

So $l_e = l_c$ no change in the angular momentum. For the energy of the circular orbit:

$$E_c = -\frac{\mu k^2}{2l_c^2} = -\frac{\mu k^2}{2l_e^2} \tag{43}$$

4. Two point particles of masses m_1 and m_2 interact via the central potential

$$U(r) = U_0 \ln\left(\frac{r^2}{r^2 + b^2}\right) \tag{44}$$

where b is a constant with dimensions of length

(a) Write the effective potential.
Answer: The effective potential is

$$U_{eff}(r) = U_0 \ln\left(\frac{r^2}{r^2 + b^2}\right) + \frac{l^2}{2\mu r^2} \tag{45}$$

(b) For what values of the angular momentum l does a circular orbit exist? Find the radius r_0 of the circular orbit. Is it stable or unstable?
Answer: For circular orbit $dU_{eff}/dr = 0$ so

$$\frac{dU_{eff}(r)}{dr} = U_0 \frac{2rb^2}{r^2(r^2 + b^2)} - \frac{l^2}{\mu r_0^3} = 0 \tag{46}$$

so the circular orbit is (solving for r_0):

$$r_0 = \sqrt{\frac{b^2 l^2}{2\mu b^2 U_0 - l^2}} \tag{47}$$

The condition is that r_0 is real, which means that $2\mu b^2 U_0 - l^2 > 0$ or $2\mu b^2 U_0 > l^2$ or in other words $l < \sqrt{2\mu b^2 U_0}$. We can define $\sqrt{2\mu b^2 U_0} = l_c$ which is the critical angular momentum for circular orbit, and then we get that $l < l_c$ is our condition.

(c) Suppose the orbit is nearly circular, with $r = r_0 + \eta$, where $\eta << r_0$. Find the equation for the shape $\eta(\theta)$ of the perturbation. - a general function as an answer is good enough.
Hint 1: Use the conservation of angular momentum to find a relation between $\dot{r}$ and $\dot{\theta}$, just as we did in class and plug this into the expression for energy. Then expand to the **second** *order in η.*
Hint 2: Remember that the Energy can be expanded as $E(r_0 + \eta) = E_0 + E(\eta)$, where E_0 is the energy of the circular orbit and $E(\eta)$ is constant.
Hint 3: Keep in the equation $(d\eta/d\theta)^2$, you'll need it.
Answer: Using the conservation of angular momentum we have $l = \mu r^2 \dot{\theta} = \text{Const}$ so

$$\frac{dr}{d\theta} = \frac{dr}{dt}\frac{dt}{d\theta} = \frac{\dot{r}}{\dot{\theta}} \tag{48}$$

The energy is:

$$E = \frac{1}{2}\mu \dot{r}^2 + U_{eff}(r) = \frac{1}{2}\mu\left(\frac{dr}{d\theta}\right)^2 \dot{\theta}^2 + U_{eff}(r) = \frac{l^2}{2\mu r^4}\left(\frac{dr}{d\theta}\right)^2 + U_{eff}(r) \tag{49}$$

Then we set $r = r_0 + \eta$ and thus

$$\begin{aligned} U_{eff}(r_0+\eta) &= U_0 \ln\left(\frac{(r_0+\eta)^2}{(r_0+\eta)^2 + b^2}\right) + \frac{l^2}{2\mu(r_0+\eta)^2} \\ &\sim U_0 \ln\left(\frac{r_0^2}{b^2 + r_0^2}\right) + U_0 \frac{2b^2\eta}{r_0(r_0^2 + b^2)} + U_0 \frac{(-b^4 - 3b^2 r_0^2)\eta^2}{r_0^2(b^2 + r_0^2)^2} \\ &+ \frac{l^2}{2\mu r_0^2} - \frac{l^2}{\mu r_0^3}\eta + \frac{l^2}{2\mu}\frac{3\eta^2}{r_0^4} \end{aligned} \tag{50}$$

And we identify the effective potential of the circular orbit $U_{eft}(r_0)$ as

$$U_{eft}(r_0) = U_0 \ln\left(\frac{r_0^2}{b^2 + r_0^2}\right) + \frac{l^2}{2\mu r_0^2} = E_0 \tag{51}$$

Also Expanding the first term in the left hand side of the energy equation is:

$$\frac{l^2}{2\mu(r_0+\eta)^4}\left(\frac{d(r_0+\eta)}{d\theta}\right)^2 = \frac{l^2}{2\mu}\left(\frac{1}{r_0^4} - 4\frac{\eta}{r_0^5}\right)\left(\frac{d\eta}{d\theta}\right)^2 \sim \frac{l^2}{2\mu r_0^4}\left(\frac{d\eta}{d\theta}\right)^2 \tag{52}$$

where in the last transition we kept only second order effects. So the equation of the energy is:

$$E_0+E(\eta) = E_0+\frac{l^2}{2\mu r_0^4}\left(\frac{d\eta}{d\theta}\right)^2+U_0\frac{2b^2\eta}{r_0(r_0^2+b^2)}-\frac{l^2}{\mu r_0^3}\eta+U_0\frac{(-b^4-3b^2r_0^2)\eta^2}{r_0^2(b^2+r_0^2)^2}+\frac{l^2}{2\mu}\frac{3\eta^2}{r_0^4} \tag{53}$$

So eliminating the E_0 from both sides and recognize that $\sqrt{2\mu b^2 U_0} = l_c$, the same critical angular momentum fro before, so we have

$$U_0\frac{2b^2}{r_0(r_0^2+b^2)} - \frac{l^2}{\mu r_0^3} = \frac{(l_c^2-l^2)r_0^2-l^2b^2}{\mu r_0^3(r_0^2+b^2)} = A = \text{Const} \tag{54}$$

And defining

$$\beta = \frac{l^2}{2\mu r_0^4} \tag{55}$$

And also

$$\begin{aligned} B &= U_0\frac{-b^4-3b^2r_0^2}{r_0^2(b^2+r_0^2)^2}+\frac{3}{r_0^4}\frac{l^2}{2\mu} = \frac{1}{r_0^2}\frac{2\mu U_0 b^2(-b^2-3r_0^2)r_0^2+3l^2(b_0^2+r_0^2)^2}{2\mu r_0^2(b^2+r_0^2)^2} \\ &= \frac{1}{r_0^2}\frac{3l^2(b^2+r_0^2)^2-l_c^2r_0^2(b^2+3r_0^2)}{2\mu r_0^2(b^2+r^2)^2} \end{aligned} \tag{56}$$

This is because we don't have a circular orbit so in fact $l^2 > 2\mu b^2 U_0$.
We can write the energy equation as

$$E(\eta) = \beta\left(\frac{d\eta}{d\theta}\right)^2 + A\eta + B\eta^2 \tag{57}$$

Or

$$\frac{d\eta}{d\theta} = \sqrt{\frac{E(\eta)-A\eta-B\eta^2}{\beta}} \tag{58}$$

So the equation we need to solve is

$$\frac{d\eta}{\sqrt{E(\eta)-A\eta-B\eta^2}} = \frac{1}{\sqrt{\beta}}d\theta \tag{59}$$

since $B>0$ and we can assume that $A^2 > 4BE$ the solution for the left hand side (E.8c) in the tables from the first week is

$$-\frac{1}{\sqrt{B}}\sin^{-1}\left(\frac{-2B\eta-A}{\sqrt{-4BE-A^2}}\right) = \frac{1}{\sqrt{\beta}}(\theta+\theta_0) \tag{60}$$

So $\eta(\theta) = C_1\sin(\theta C_2+C_3)$ where C_1, C_2 and C_3 are constant of the system, that can be expressed from A , B and β.