Math 115A: Sample final exam

Sections 1 and 3. Instructor: James Freitag

For the exam, you may use one 8 inch by 11 inch (normal sized paper) piece of paper with anything at all written on **one side** - theorems, example problems, inspirational sayings - anything goes. There will be 8 problems on the final. The difficulty will be on the level of the exams.

Keep in mind this sample review is not comprhensive. I will post more problems throughout the week.

Problem 1 Eigenvalues

Let $\theta \in (0, \pi/2)$. Let

$$A = \begin{pmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{pmatrix}.$$

The entries of A are in $\mathbb{R}$, so we can regard A as either a matrix over the reals *or* the complex numbers. Are there any eigenvectors over $\mathbb{R}$? Explain why not intuitively.

Calculate the eigenvalues and eigenvectors over $\mathbb{C}$.

Problem 2 Some basics

Let

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{pmatrix}.$$

Find a basis of $N(A)$. Find a basis of $R(A)$. Diagonalize A.

Problem 3 A subspace

Prove that the set of all functions that can be written in the form $a \cdot sin(x + b)$ for $a \in \mathbb{C}$ and $b \in \mathbb{R}$ is a vector space. Is it finite dimensional?

Problem 4 From class Friday

Let $S : U \to V$, $T : V \to W$ be linear maps of finite dimensional vector spaces. Suppose that TS is bijective. Prove that S is surjective if and only T is injective.

Problem 5 Use elementary matrices?

Let $A \in M_{n\times n}(\mathbb{F})$ be an invertible matrix, and let B be any other matrix of $M_{n\times n}(\mathbb{F})$. Prove that $det(AB) = det(A) \cdot det(B)$. So, assuming that A is invertible, we proved that A is a product of elementary matrices. Elementary matrices E have the property $det(EB) = det(E)det(B)$, as we proved in class (recall there were only three cases to check). Now, finish the problem with an induction on the number of elementary matrices in the product representing A.

Problem 6 A map with specified kernel

Let V be a finite dimensional innter product space. Let W be a subspace. Constuct a linear operator S on V with $N(S) = W^{\perp}$ and $R(S) = W$.

Let $w_1, \ldots, w_d$ be an orthonormal basis of W. Then define $T_W : V \to V$ via $T_W(v) = \sum_{i=1}^{d} \langle v, w_i \rangle w_i$ - note that the coordinates of a point $v \in W$ are given by the inner products in the sum. So, the map is the identity for vectors in W. For vectors in $W^{\perp}$, it is easy to see the map is zero. Further, it is not hard to show that the map is zero only for vectors in $W^{\perp}$.

Problem 7 Using a map with specified kernel

Let V be a finite dimensional innter product space. Let W be a subspace. Use the previous problem to prove that $dim(W) + dim(W^{\perp}) = dim(V)$.

Problem 8 Examples or lack thereof

Give an example of a 2×2 matrix M over $\mathbb{R}$ such that M has no eigenvalues in $\mathbb{R}$. Can you give an example of a 3×3 matrix M over $\mathbb{R}$ such that M has no eigenvalues in $\mathbb{R}$?

Problem 9 Representation is better than working by hand!

Find a polynomial $q \in P_3(\mathbb{R})$ such that

$$p\left(\frac{1}{4}\right) = \int_0^1 p(x)q(x)dx$$

for all $p \in P_3(\mathbb{R})$. You can write down your polynomial in terms of an inner product.