Spontaneous Parametric Down Conversion: Quantum Treatment

Having seen that classical physics prohibits SPDC, I want to give you a brief glance at the full Quantum treatment just so you see it at least once. Don't worry about following all of the details since this is graduate-level material; I just want to provide a plausibility argument for the fact that vacuum fluctuations (i.e. zero-point fluctuations of the quantum-mechanical electromagnetic vacuum) give rise to SPDC.

The interaction Hamiltonian describing degenerate parametric amplification is given by

$$V = \hbar\kappa\left(a^{\dagger 2} b + a^2 b^{\dagger}\right).$$

What on Earth does this mean? Well, κ is some constant that describes the strength of the optical parametric effect, so we expect that it's something like $\chi^{(2)}$.

The $a^{\dagger}, a, b^{\dagger}$, and b are the harmonic oscillator raising and lowering operators you know from Griffiths.

What are harmonic oscillator ladder operators doing here? Well, it turns out that the electromagnetic field (which can oscillate in time) is a harmonic oscillator. The math is exactly the same. In fact, if you think back to week 1 when we wrote

$$\vec{E} = \hat{\psi}\, \mathcal{E}_0 \tfrac{1}{2}\left(e^{i(kz-\omega t)} + e^{-i(kz-\omega t)}\right)$$

We can treat the E&M field quantum-mechanically by simply adding a and $a^\dagger$ as follows:

$$\vec{E} = \hat{\psi}\, \frac{\mathcal{E}_0}{2}\left(a\, e^{i(kz-\omega t)} + a^\dagger e^{-i(kz-\omega t)}\right)$$

The electric field becomes an operator! An the thing it operates <u>on</u> is the QM state vector of the field, which we already identified as a harmonic oscillator.

So all this talk about <u>modes</u> we've done in this class is really referring to distinguishable harmonic oscillators. Each independent h.o. (harmonic oscillator) will have it's own set of quantum numbers (typically things like polarization, confocal parameter, transverse mode index, optical frequency) that determine which one you're talking about. And each mode will have its own a and $a^\dagger$ raising and lowering operators.

So what does it mean to "raise" or "lower" the harmonic oscillator corresponding to a particular mode? Well, those are <u>photons</u>!

$a_i^\dagger$ photon creation operator for mode i

a photon annihilation operator for mode i

So if we look back at the Hamiltonian describing our $\chi^{(2)}$ medium, we see

$$V = \hbar\kappa \left(a^{\dagger 2} b + a^2 b^\dagger \right)$$

and the interpretation of the physics this describes presents itself:

$$a^{\dagger 2} b \qquad\qquad a^2 b^\dagger$$

two red photons are created — one blue photon is annihilated

two red photons die — one blue photon is born

So the first term is our SPDC term, and the second one is ~~the~~ responsible for second harmonic generation (SHG).

Now, all I want to do with this is to give you a feel for how QM lets you do SPDC when classical physics clearly does not.

Let's see what the transition matrix element looks like if we put in a blue laser beam (and no red photons) and look for the appearance of two red photons

$$V_{0_r \to 2_r} \equiv \langle 2_r, \beta_b | V | \beta_b, 0_r \rangle$$

here, the state of the electromagnetic field is being split up into just the two modes we care about. The initial state of the red mode ($|0_r\rangle$) is the h.o. ground state, also known as the <u>vacuum state</u>. The final state of the red mode is $|2_r\rangle$, which is the $n=2$ state of the red h.o., a.k.a. "two photons."

The blue mode is in what is called a "coherent state." This is fancy talk for essentially a very classical-looking state, and all you really need to know is that $b|\beta\rangle = \beta|\beta\rangle$.

Now, recalling from undergraduate QM that

$$a|n\rangle = \sqrt{n}\,|n-1\rangle$$

$$a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle \quad , \quad \text{we see that}$$

$$V_{02} = \langle 2_r, \beta_b |\, \hbar\kappa \left(a^{\dagger 2} b + a^2 b^\dagger\right) | \beta_b, 0_r \rangle$$

$$= \hbar\kappa \langle 2_r | a^{\dagger 2} | 0_r \rangle \langle \beta_b | b | \beta_b \rangle + \hbar\kappa \langle 2_r | a^2 | 0_r \rangle \langle \beta_b | b^\dagger | \beta_b \rangle$$

$\sqrt{0} = 0$

$$= \hbar\kappa \langle 2_r | a^\dagger a^\dagger | 0_r \rangle \, \beta$$

$$= \hbar\kappa\beta \langle 2_r | a^\dagger \sqrt{1} | 1_r \rangle$$

$$= \hbar\kappa\beta \langle 2_r | \sqrt{2} | 2_r \rangle$$

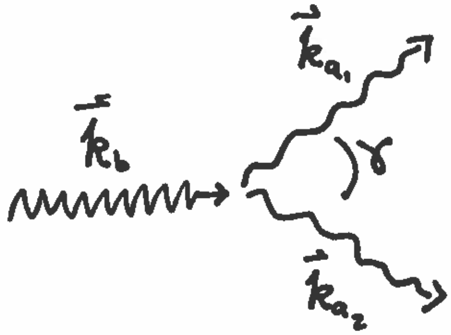
$$= \sqrt{2}\,\hbar\kappa\beta$$

and we see that the transition matrix element is nonzero, even when we send in zero red photons! This is the S in SPDC — the vacuum state is able to support the creation of red photons from an input blue photon even in the absence of any red photons.

Type I SPDC

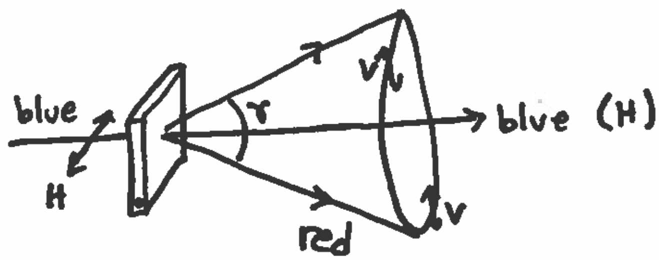
For 180Q, we will be using an apparatus that creates photons using what is called "Type I phase matching."

Basically, in order to satisfy conservation of momentum during the SPDC process, the dispersion of the $\chi^{(2)}$ crystal has to be overcome somehow. The way this works for Type I phase matching is that blue photons polarized along the xtal optical axis get converted into orthogonally-polarized red photons. And the two red photons (which have the same polarization) even make an angle with each other:



$$\underbrace{-\vec{k}_b + \vec{k}_{a_1} + \vec{k}_{a_2} = 0}_{\text{phase-matching}}$$

really, this results in a cone of red light emerging from the $\chi^{(2)}$ crystal:



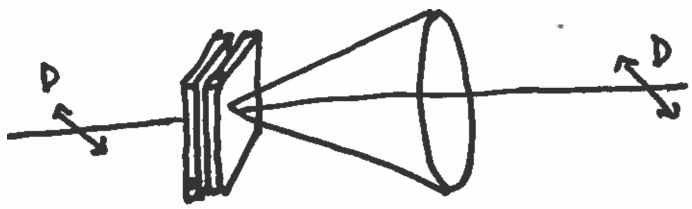
Photons that are gathered from opposite sides of the cone can be pairs created by single blue photons.

Now, because we're using type I phase matching, if the blue beam is H-polarized and the optical axis of the crystal is along H, the red photons will all be V-polarized.

If, for the same crystal orientation, the pump is V-polarized, no red light will come out since it can't phase-match.

So ~~A~~ Diagonally-polarized pump (blue) light will give us V-polarized red photons since the V-polarized part of the pump beam bascically does nothing.

Now lets sandwitch two, orthogonal SPDC crystals together and send in diagonal pump light:



If we collect pairs of red photons from opposite sides of the cone in a way that does not allow distinguishability in terms of <u>which</u> Xtal did the SPDC, the pairs are in a superposition of both having been created in the first crystal ($|V_1 V_2\rangle$) and both having been created in the second crystal ($|H_1 H_2\rangle$):

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|V_1 V_2\rangle + e^{i\phi}|H_1 H_2\rangle\right)$$

This is an entangled state!