6.4 The "Second Form" of the Euler Equation

A second equation may be derived from Euler's equation that is convenient for functions that do not explicitly depend on x: $\partial f/\partial x = 0$. We first note that for any function $f(y, y'; x)$ the derivative is a sum of terms

$$\begin{aligned}\frac{df}{dx} = \frac{d}{dx} f\{y, y'; x\} &= \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} + \frac{\partial f}{\partial x} \\ &= y'\frac{\partial f}{\partial y} + y''\frac{\partial f}{\partial y'} + \frac{\partial f}{\partial x}\end{aligned} \tag{6.37}$$

Also

$$\frac{d}{dx}\left(y'\frac{\partial f}{\partial y'}\right) = y''\frac{\partial f}{\partial y'} + y'\frac{d}{dx}\frac{\partial f}{\partial y'}$$

or, substituting from Equation 6.37 for $y''(\partial f/\partial y')$,

$$\frac{d}{dx}\left(y'\frac{\partial f}{\partial y'}\right) = \frac{df}{dx} - \frac{\partial f}{\partial x} - y'\frac{\partial f}{\partial y} + y'\frac{d}{dx}\frac{\partial f}{\partial y'} \tag{6.38}$$

The last two terms in Equation 6.38 may be written as

$$y'\left(\frac{d}{dx}\frac{\partial f}{\partial y'} - \frac{\partial f}{\partial y}\right)$$

which vanishes in view of the Euler equation (Equation 6.18). Therefore,

$$\boxed{\frac{\partial f}{\partial x} - \frac{d}{dx}\left(f - y'\frac{\partial f}{\partial y'}\right) = 0} \tag{6.39}$$

We can use this so-called "second form" of the Euler equation in cases in which f does not depend explicitly on x, and $\partial f/\partial x = 0$. Then,

$$f - y'\frac{\partial f}{\partial y'} = \text{constant} \qquad \left(\text{for } \frac{\partial f}{\partial x} = 0\right) \tag{6.40}$$