105A - Set 3 - Solutions

(Grades are out of 150)

1. **Laser Beam in Refractive Medium**
   A well known example of the use of the theory of variations is the **Fermat's principle**: light travels by the path that takes the least amount of time. Consider a medium with an index of refraction given by $n(x, y) = n_0(1 + ky)$. Recall that the speed of light in a medium with index n is given by $v = c/n$, where c is the speed of light. Find the function that describes the path of light in this medium. Determine a specific equation for the path of a laser beam that initially starts at the origin propagating in the x direction, as shown in figure 1. *Hint: Use the second form of the Euler equation (see*

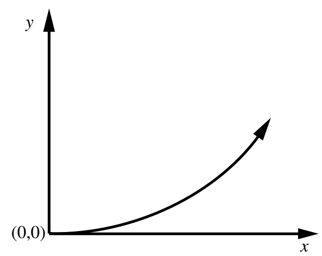

Figure 1: Fermat's principle

   reading material online!) and use the plot to find the constant.
   Answer: Here we find the path of a light ray using Fermats principle. The travel time is:

$$T = \int \frac{ds}{v} = \int \frac{n}{c} ds = \int \frac{n_0(1 + ky)}{c} ds \tag{1}$$

   Since the system is two dimension we know that $ds = \sqrt{dx^2 + dy^2}$ so the above equation can be written as:

$$T = \int \frac{n_0(1 + ky)}{c} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \tag{2}$$

   So we can identify our f as:

$$f(y, y'; x) = \frac{n_0(1 + ky)}{c} \sqrt{1 + (y')^2} \tag{3}$$

   So using the second form of the Eulers equation we write:

$$f - y' \frac{\partial f}{\partial y'} = \text{Const} \tag{4}$$

   The second term is simply:

$$\frac{\partial f}{\partial y'} = \frac{n_0(1 + ky)y'}{c\sqrt{1 + (y')^2}} \tag{5}$$

So the second form of the Eulers equation is simply

$$\frac{n_0(1+ky)}{c}\sqrt{1+(y')^2} - y'\frac{n_0(1+ky)y'}{c\sqrt{1+(y')^2}} = \text{Const} \tag{6}$$

using the plot for the boundary condition, i.e., at the beginning $x = 0, y = 0, y' = 0$ so we get that Const $= n_0/c$ So than equation (6) is simply (after dividing in n_0/c):

$$(1+ky)\sqrt{1+(y')^2} - y'\frac{(1+ky)y'}{\sqrt{1+(y')^2}} = 1 \tag{7}$$

which after little algebra we can write:

$$1+ky = \sqrt{1+(y')^2} \tag{8}$$

which is simply:

$$\frac{dy}{dx} = \sqrt{(1+ky)^2 - 1} \tag{9}$$

Separating variables we can write:

$$\frac{dy}{\sqrt{(1+ky)^2-1}} = dx \tag{10}$$

The solution of the left hand side is simply

$$\frac{dy}{\sqrt{(1+ky)^2-1}} = \frac{2}{k}\sinh^{-1}(\sqrt{ky/2}) = x \tag{11}$$

so

$$\sqrt{ky/2} = \sinh\left(\frac{kx}{2}\right) \tag{12}$$

or finally:

$$y = \frac{2}{k}\sinh^2\left(\frac{kx}{2}\right) \tag{13}$$

2. Show that the shortest distance between two points in **three dimensional** space is a straight line.
   Hint 1: Define a family of solution (similar to what we did in class), where your coordinates will depend on the parameter that defines this family of solutions, i.e., $x(\alpha), y(\alpha)$ *and* $z(\alpha)$.
   Hint 2: You may find this definition useful:

$$dl = d\alpha\sqrt{\left(\frac{dx}{d\alpha}\right)^2 + \left(\frac{dy}{d\alpha}\right)^2 + \left(\frac{dz}{d\alpha}\right)^2} \tag{14}$$

Answer: Our goal is to show that the shortest distance between two points in three dimensional space is a straight line. The distance between two points that are infinitesimally close is given by $ds = \sqrt{dx^2 + dy^2 + dz^2}$. So the distance between two points is given by

$$S = \int ds = \int \sqrt{dx^2 + dy^2 + dz^2} \tag{15}$$

We can consider a family of solution that are differ from one another by α, just like you did in class so the three demential curve as being defined by the equation $\mathbf{r}(\alpha) = (\mathbf{x}(\alpha), \mathbf{y}(\alpha), \mathbf{z}(\alpha))$. So if we move along this curve the distance traveled will be:

$$S = \int ds = \int \sqrt{\left(\frac{dx}{d\alpha}\right)^2 + \left(\frac{dy}{d\alpha}\right)^2 + \left(\frac{dz}{d\alpha}\right)^2}\, d\alpha \tag{16}$$

so we can define:

$$f = \sqrt{\left(\frac{dx}{d\alpha}\right)^2 + \left(\frac{dy}{d\alpha}\right)^2 + \left(\frac{dz}{d\alpha}\right)^2} \tag{17}$$

Eulers equation for the three components (i.e., x, y and z) are simply:

$$\frac{\partial f}{\partial x} - \frac{d}{d\alpha}\frac{\partial f}{\partial (dx/d\alpha)} = 0 \tag{18}$$

replacing x with y and z we get similar equations for these components as well. Since

$$\frac{\partial f}{\partial x} = 0 \tag{19}$$

(f does not depends explicitly on x) we get:

$$\frac{d}{d\alpha}\frac{\partial f}{\partial (dx/d\alpha)} = 0 \tag{20}$$

Or

$$\frac{\partial f}{\partial (dx/d\alpha)} = \text{Const} \tag{21}$$

On the other hand from the definition of f we can find:

$$\frac{\partial f}{\partial (dx/d\alpha)} = \frac{dx}{d\alpha}\left\{\left(\frac{dx}{d\alpha}\right)^2 + \left(\frac{dy}{d\alpha}\right)^2 + \left(\frac{dz}{d\alpha}\right)^2\right\}^{-1/2} = \text{Const} \tag{22}$$

Which is constant as we got. We note that we can define a new variable

$$dl = d\alpha\sqrt{\left(\frac{dx}{d\alpha}\right)^2 + \left(\frac{dy}{d\alpha}\right)^2 + \left(\frac{dz}{d\alpha}\right)^2} \tag{23}$$

So equation (22) is then can be written as:

$$\frac{\partial f}{\partial (dx/d\alpha)} = \frac{dx}{d\alpha}\left\{\left(\frac{dx}{d\alpha}\right)^2 + \left(\frac{dy}{d\alpha}\right)^2 + \left(\frac{dz}{d\alpha}\right)^2\right\}^{-1/2} = \frac{dx}{dl} = A = \text{Const} \tag{24}$$

So $x = Al + B$, where B is also constant. Similarly we can have $y = Cl + D$ and $z = El + F$. All are equations of line in three dimensional space.

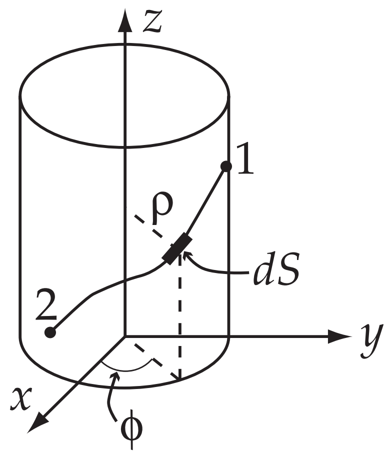

Figure 2: Surface of a cylinder

3. Show that the geodesic on the surface of a circular cylinder is a segment of a helix (see Figure 2 behind the page).
Answer: The element of distance along the surface is

$$dS = \sqrt{dx^2 + dy^2 + dz^2} \tag{25}$$

In cylindrical coordinates (x, y, z) are related to (ρ, ϕ, z) by

$$x = \rho \cos \phi \tag{26}$$
$$y = \rho \sin \phi \tag{27}$$
$$z = z \tag{28}$$

from which

$$dx = -\rho \sin \phi d\phi \tag{29}$$
$$dy = \rho \cos \phi d\phi \tag{30}$$
$$dz = dz \tag{31}$$

Substituting (29) into (25) and integrating along the entire path, we find

$$dS = \int_1^2 \sqrt{\rho^2 d\phi^2 + dz^2} = \int_{\phi_1}^{\phi_2} \sqrt{\rho^2 + (dz/d\phi)^2} d\phi \tag{32}$$

we identify $f = \sqrt{\rho^2 + (dz/d\phi)^2}$ and it must satisfy the Euler equation:

$$\frac{\partial f}{\partial z} - \frac{\partial}{\partial \phi} \frac{\partial f}{\partial (dz/d\phi)} = 0 \tag{33}$$

Since $\partial f/\partial z = 0$ the Euler equation becomes

$$\frac{\partial}{\partial \phi} \frac{dz/d\phi}{\sqrt{\rho^2 + (dz/d\phi)^2}} = 0 \tag{34}$$

or

$$\frac{dz/d\phi}{\sqrt{\rho^2 + (dz/d\phi)^2}} = C = \text{Const.} \tag{35}$$

Rearranging we can write:

$$\frac{dz}{d\phi} = \sqrt{\frac{C^2}{1 - C^2}}\phi = \text{Const.} \tag{36}$$

wince ρ is also constant. So for any point along the path z and ϕ change at the same rate. The curve described by this condition is a helix.