105A - Set 1 - Solutions

1. A particle moves in the x-y plan (center $\mathcal{O}$) with constant angular velocity ω counter-clockwise. The particle's position is given by

$$\mathbf{r}(t) = 2b\cos(\omega t)\hat{\mathbf{x}} + b\sin(\omega t)\hat{\mathbf{y}} \ , \tag{1}$$

where the convention is that bold face represents a vector, i.e., instead of writing $\vec{r}$ we write $\mathbf{r}$.

   (a) Find the particle's velocity.
   Answer: taking the time derivative we write:

$$\dot{\mathbf{r}}(t) = -2b\omega\sin(\omega t)\hat{\mathbf{x}} + b\omega\cos(\omega t)\hat{\mathbf{y}} \ , \tag{2}$$

   (b) Find the particle's velocity magnitude (speed).
   Answer:

$$|v| = \sqrt{\dot{\mathbf{r}}(t)\cdot\dot{\mathbf{r}}(t)} = \sqrt{(2b\omega\sin(\omega t))^2 + (b\omega\cos(\omega t))^2} = b\omega\sqrt{4\sin^2\omega t + \cos^2\omega t} \ . \tag{3}$$

   Since $\cos^2\omega t + \sin^2\omega t = 1$ we can write:

$$|v| = b\omega\sqrt{3\sin^2\omega t + 1} \ . \tag{4}$$

   (c) Find the particle's acceleration. Express your answer in terms of ω and $\mathbf{r}$. What are the magnitude and direction of the acceleration?
   Answer:

$$\mathbf{a} = \ddot{\mathbf{r}}(t) = \frac{d}{dt}\left(-2b\omega\sin(\omega t)\hat{\mathbf{x}} + b\omega\cos(\omega t)\hat{\mathbf{y}}\right) = -2b\omega^2\cos(\omega t)\hat{\mathbf{x}} - b\omega^2\sin(\omega t)\hat{\mathbf{y}} = \omega^2\mathbf{r} \ , \tag{5}$$

   where in the last transition we have used the definition of $\mathbf{r}$. So its direction is at the opposite direction to $\mathbf{r}$. The magnitude is simply $|a| = \omega^2 r$.

   (d) What is the angle between $\mathbf{v}$ and $\mathbf{a}$ at time $t = \pi/(2\omega)$?
   Answer: At that time $\mathbf{a} = \omega^2 b\hat{\mathbf{y}}$ and $\mathbf{v} = -2b\omega\hat{\mathbf{x}}$ so the velocity and acceleration are parallel, so the angle is 90°.

2. Find the components of the acceleration vector $\mathbf{a}$ in spherical coordinates. Be as detailed as possible.
Answer:
The unit vectors in spherical coordinates are expressed in terms of rectangular coordinates by Thus, Similarly ,

$$\begin{aligned}
\mathbf{e}_r &= (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) && (6)\\
\mathbf{e}_\theta &= (\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta) && (7)\\
\mathbf{e}_\phi &= (-\sin\phi, \cos\phi, 0) && (8)
\end{aligned}$$

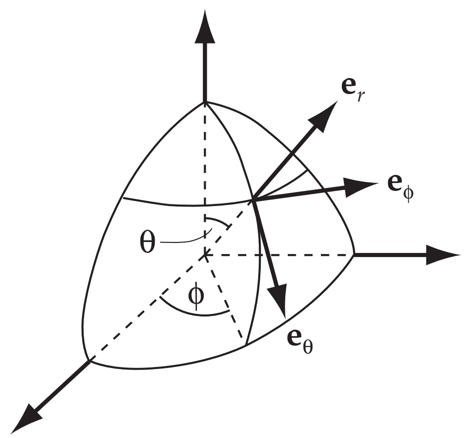

Figure 1: Spherical coordinates

The derivatives of the the unit vectors:

$$\frac{\partial \mathbf{e}_r}{\partial r} = 0 \tag{9}$$

$$\frac{\partial \mathbf{e}_r}{\partial \phi} = \sin\theta \mathbf{e}_\phi \tag{10}$$

$$\frac{\partial \mathbf{e}_r}{\partial \theta} = \mathbf{e}_\theta \tag{11}$$

$$\frac{\partial \mathbf{e}_\phi}{\partial r} = 0 \tag{12}$$

$$\frac{\partial \mathbf{e}_\phi}{\partial \phi} = \cos\theta \mathbf{e}_\theta - \sin\theta \mathbf{e}_r \tag{13}$$

$$\frac{\partial \mathbf{e}_\phi}{\partial \theta} = 0 \tag{14}$$

$$\frac{\partial \mathbf{e}_\theta}{\partial r} = 0 \tag{15}$$

$$\frac{\partial \mathbf{e}_\theta}{\partial \phi} = \cos\theta \mathbf{e}_\phi \tag{16}$$

$$\frac{\partial \mathbf{e}_\theta}{\partial \theta} = -\mathbf{e}_r \tag{17}$$

So from these we can find the unit vector time derivative:

$$\dot{\mathbf{e}}_r = \frac{\partial \mathbf{e}_r}{\partial r}\frac{\partial r}{\partial t} + \frac{\partial \mathbf{e}_r}{\partial \phi}\frac{\partial \phi}{\partial t} + \frac{\partial \mathbf{e}_r}{\partial \theta}\frac{\partial \theta}{\partial t} \tag{18}$$

using Eqs (9)-(17) we can write:

$$\dot{\mathbf{e}}_r = 0 + \sin\theta \mathbf{e}_\phi \dot{\phi} + \mathbf{e}_\theta \dot{\theta} = \sin\theta \dot{\phi} \mathbf{e}_\phi + \dot{\theta} \mathbf{e}_\theta \tag{19}$$

Similarly we find:

$$\begin{aligned} \dot{\mathbf{e}}_\theta &= -\dot{\theta}\mathbf{e}_r + \dot{\phi}\cos\theta\mathbf{e}_\phi && (20) \\ \dot{\mathbf{e}}_\phi &= -\dot{\phi}\cos\theta\mathbf{e}_\theta - \dot{\phi}\sin\theta\mathbf{e}_r\ . && (21) \end{aligned}$$

So, the velocity: $\dot{\mathbf{r}} = r\dot{\mathbf{e}}_r + \dot{r}\mathbf{e}_r$ we find that

$$\dot{\mathbf{r}} = r\sin\theta\dot{\phi}\mathbf{e}_\phi + r\dot{\theta}\mathbf{e}_\theta + \dot{r}\mathbf{e}_r \tag{22}$$

Now for the acceleration,

$$\begin{aligned} \ddot{\mathbf{r}} &= \dot{r}\sin\theta\dot{\phi}\mathbf{e}_\phi + r\cos\theta\dot{\theta}\dot{\phi}\mathbf{e}_\phi + r\sin\theta\ddot{\phi}\mathbf{e}_\phi + r\sin\theta\dot{\phi}\dot{\mathbf{e}}_\phi \\ &+ \dot{r}\dot{\theta}\mathbf{e}_\theta + r\ddot{\theta}\mathbf{e}_\theta + r\dot{\theta}\dot{\mathbf{e}}_\theta \\ &+ \ddot{r}\mathbf{e}_r + \dot{r}\dot{\mathbf{e}}_r \end{aligned} \tag{23}$$

Plugging in the relevant derivatives we find:

$$\begin{aligned} \ddot{\mathbf{r}} &= \dot{r}\sin\theta\dot{\phi}\mathbf{e}_\phi + r\cos\theta\dot{\theta}\dot{\phi}\mathbf{e}_\phi + r\sin\theta\ddot{\phi}\mathbf{e}_\phi + r\sin\theta\dot{\phi}(-\dot{\phi}\cos\theta\mathbf{e}_\theta - \dot{\phi}\sin\theta\mathbf{e}_r) \\ &+ \dot{r}\dot{\theta}\mathbf{e}_\theta + r\ddot{\theta}\mathbf{e}_\theta + r\dot{\theta}(-\dot{\theta}\mathbf{e}_r + \dot{\phi}\cos\theta\mathbf{e}_\phi) \\ &+ \ddot{r}\mathbf{e}_r + \dot{r}(\sin\theta\dot{\phi}\mathbf{e}_\phi + \dot{\theta}\mathbf{e}_\theta) \end{aligned} \tag{24}$$

collecting the relevant terms for each component we can write:

$$\begin{aligned} \ddot{\mathbf{r}} &= (\ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2\sin^2\theta)\mathbf{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta} - r\dot{\phi}^2\sin\theta\cos\theta)\mathbf{e}_\theta \\ &+ (2\dot{r}\dot{\phi}\sin\theta 2r + \dot{\theta}\dot{\phi}\cos\theta + r\ddot{\phi}\sin\theta)\mathbf{e}_\phi\ , \end{aligned} \tag{25}$$

which we can write:

$$\begin{aligned} \ddot{\mathbf{r}} &= (\ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2\sin^2\theta)\mathbf{e}_r + \left(\frac{1}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right) - r\dot{\phi}^2\sin\theta\cos\theta\right)\mathbf{e}_\theta \\ &+ \frac{1}{r\sin\theta}\frac{d}{dt}\left(r^2\dot{\phi}\sin^2\theta\right)\mathbf{e}_\phi\ . \end{aligned} \tag{26}$$

3. **Work and kinetic energy**
Consider the definition of Work:

$$W_{12} = \int_1^2 \mathbf{F}\cdot d\mathbf{s} \tag{27}$$

Show that for a constant mass the

$$W_{12} = \frac{m}{2}(v_2^2 - v_1^2) = T_2 - T_1\ , \tag{28}$$

where T_1 (T_2) is the kinetic energy of the system at state "1" ("2").
Answer: Simply:

$$W_{12} = \int_1^2 \mathbf{F}\cdot d\mathbf{s} = \int_1^2 \frac{dm\mathbf{v}}{dt}\cdot d\mathbf{s} \tag{29}$$

using $d\mathbf{s} = \mathbf{v}dt$ we can write:

$$W_{12} = m \int_1^2 \frac{d\mathbf{v}}{dt} \cdot \mathbf{v} dt \ , \tag{30}$$

where in the last transition we assumed that m is constant. Then, not that $v^2 = \mathbf{v} \cdot \mathbf{v}$ and then note that $\dot{v^2} = 2\mathbf{v} \cdot \dot{\mathbf{v}}$ so

$$W_{12} = m \int_1^2 \frac{d\mathbf{v}}{dt} \cdot \mathbf{v} dt = \frac{m}{2} \int_1^2 \frac{d(v^2)}{dt} dt = \frac{m}{2} v^2 \Big|_1^2 \ , \tag{31}$$

so

$$W_{12} = \frac{m}{2}(v_2^2 - v_1^2) = T_2 - T_1 \ , \tag{32}$$

4. A particle with mass m moves in a medium under the influence of a retarding force equal to $mk(v^3 + a^2 v)$, where k and a are constant. Show that for any value of the initial speed the particle will never move a grater distance than $\pi/(2ka)$ and that the particle comes to rest only at $t \to \infty$.
Answer: The equation of motion is:

$$m\ddot{x} = -mk(v^3 + a^2 v) = -mk(\dot{x}^3 + a^2 \dot{x}) \tag{33}$$

or

$$m\dot{v} = -mk(v^3 + a^2 v) \tag{34}$$

integrating we can write:

$$\int \frac{dv}{v^3 + a^2 v} = -k \int dt \tag{35}$$

so

$$\frac{1}{2a^2} \ln \left(\frac{v^2}{v^2 + a^2} \right) = -kt + C \tag{36}$$

So

$$\frac{v^2}{v^2 + a^2} = \tilde{C} e^{-2a^2 kt} \tag{37}$$

where $\tilde{C}$ is determined from initial conditions, i.e., at $t = 0$ we set $v = v_0$, so

$$\tilde{C} = \frac{v_0^2}{v_0^2 + a^2} \tag{38}$$

Now to find $x(t)$ we simply rearrange equation (37):

$$\frac{dx}{dt} = v = \sqrt{\frac{a^2 \tilde{C} e^{-2a^2 kt}}{1 - \tilde{C} e^{-2a^2 kt}}} \tag{39}$$

Note that when $t \to \infty$, this equation yields $v \to 0$.
So we integrate on the following equation:

$$\int dx = \int \sqrt{\frac{a^2 \tilde{C} e^{-2a^2 kt}}{1 - \tilde{C} e^{-2a^2 kt}}} dt \tag{40}$$

setting $u = e^{-2a^2kt}$ so $du = -2a^2kudt$ we can have

$$\int dx = \int \sqrt{\frac{a^2\tilde{C}u}{1-\tilde{C}u}} \frac{-du}{2a^2ku} = -\frac{\tilde{C}}{2ak}\int \frac{du}{\sqrt{u-\tilde{C}u^2}} = \frac{1}{2ak}\sin^{-1}(1-2\tilde{C}u) + C_1 \quad (41)$$

to find C_1, we set again the initial conditions $x = 0$, $t = 0$, which gives $u = e^{-2a^2kt=0} = 1$ so

$$C_1 = -\frac{1}{2ak}\sin^{-1}(1-2\tilde{C}) \quad (42)$$

So finally we can write:

$$x = \frac{1}{2ak}\left\{ \sin^{-1}\left(\frac{-v^2+a^2}{v^2+a^2}\right) - \sin^{-1}\left(\frac{-v_0^2+a^2}{v_0^2+a^2}\right) \right\} \quad (43)$$

As we found from the velocity equation, when $t \to \infty$, $v \to 0$. so at $t \to \infty$ the above equation yields

$$x = \frac{1}{2ak}\left\{ \sin^{-1}\left(\frac{a^2}{a^2}\right) - \sin^{-1}\left(\frac{-v_0^2+a^2}{v_0^2+a^2}\right) \right\} = \frac{1}{2ak}\left\{ \pi - \sin^{-1}\left(\frac{-v_0^2+a^2}{v_0^2+a^2}\right) \right\} \quad (44)$$

Now the maximum value that a particle can travel is indeed $\pi/(2ak)$ where the last term can only make it smaller. Large initial velocities can only help the particle reach its maximal distance. This can be seen by once plugging zero for v_0 , and than getting $\pi - \pi$. And once plugging in $v_0 \to \infty$ which yields $\sin^{-1}\pi/2 = 0$.

5. A particle of mass m is sliding down an inclined plane under the influence of gravity. If the motion is resisted by a force $f = kmv^2$, show that the time required to move a distance d after starting from rest is

$$t = \frac{\cosh^{-1}(e^{kd})}{\sqrt{kg\sin\theta}}\ , \quad (45)$$

where θ is the angle of the inclination of the plane.
Answer See figure 2. The equation of motion along the plane is:

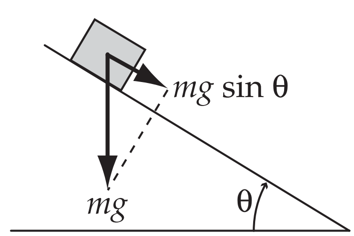

Figure 2: Inclined plane and mas m

$$m\frac{dv}{dt} = mg\sin\theta - kmv^2 \ , \tag{46}$$

which we can write, after rearranging:

$$\frac{dv/k}{(g/k)\sin\theta - v^2} = dt \ , \tag{47}$$

the solution of the integral is:

$$\frac{1}{k}\frac{k}{\sqrt{(g/k)\sin\theta}}\tanh^{-1}\left(\frac{v}{\sqrt{(g/k)\sin\theta}}\right) = t + C \ , \tag{48}$$

The initial condition is that $v(t=0) = 0$ so $C = 0$. So we can easily extract v from this equation and get:

$$v = \frac{dx}{dt} = \sqrt{\frac{g}{k}\sin\theta}\tanh(\sqrt{gk\sin\theta}t) \ , \tag{49}$$

so to get x we should integrate:

$$x = \int dx = \int \sqrt{\frac{g}{k}\sin\theta}\tanh(\sqrt{gk\sin\theta}t)dt = \sqrt{\frac{g}{k}\sin\theta}\frac{\ln\cosh(\sqrt{gk\sin\theta}t)}{\sqrt{gk\sin\theta}} + C \ , \tag{50}$$

setting $X(t=0) = 0$ we find $C = 0$. So:

$$x = \frac{1}{k}\ln\cosh(\sqrt{gk\sin\theta}t) \ , \tag{51}$$

Extracting t and setting $x = d$ we have

$$t = \frac{\cosh^{-1}(e^{kd})}{\sqrt{kg\sin\theta}} \ , \tag{52}$$