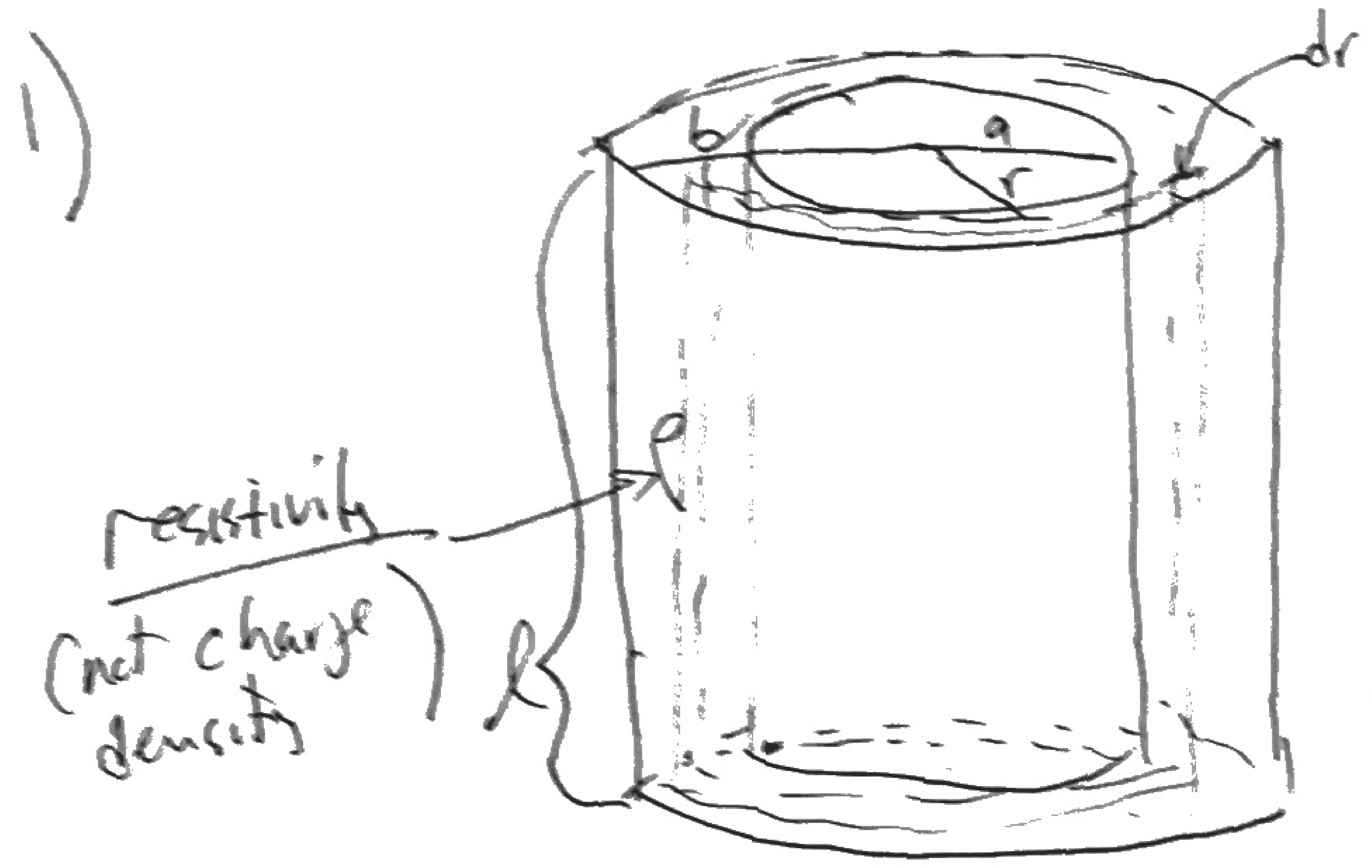
Problem Set #4 Solutions

1)



Note: we want resistance between inner and outer cylinders, not along the length of the cylinder.

In general: $R = \rho \frac{L}{A}$ ← length, ← cross sectional area [eq (4.17) in PM]

, A = cross-sectional area of a thin cylindrical shell at radius r $= \ell \cdot 2\pi r$.

Now, we can view this resistor as a collection of series resistors, one from each cylindrical shell of radius r. In other words,

$$R_{TOT} = \int_{r=a}^{r=b} dR$$ where dR is the small amount of resistance

coming from the shell at radius r. The length of this small resistor (shell) is dr, so that the resistance dR of one shell is given by

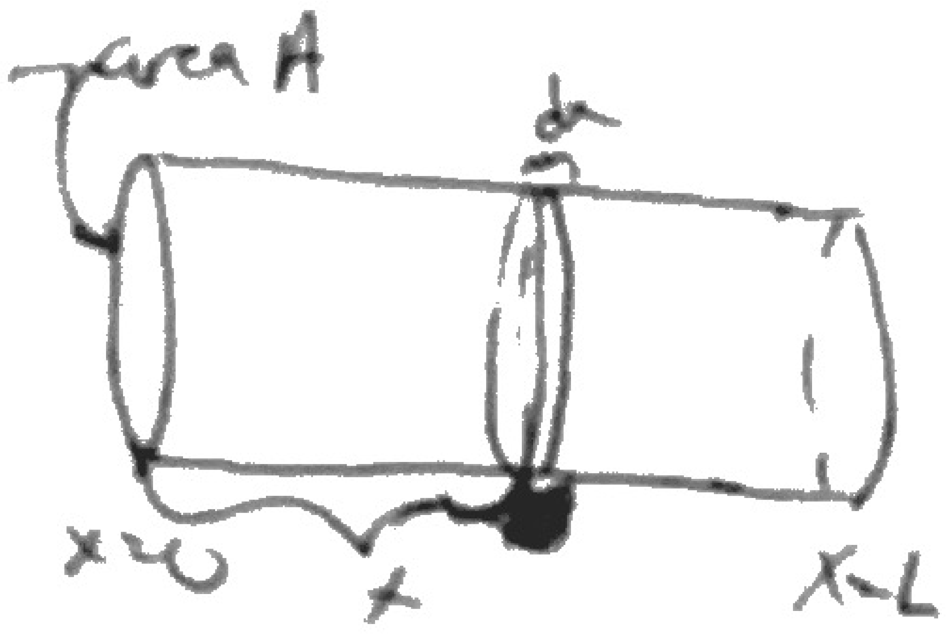
$$dR = \rho \frac{L}{A} = \rho \frac{dr}{\ell \cdot 2\pi r}.$$ Thus we have, summing over all the little shells,

$$\boxed{R_{TOT} = \int_{r=a}^{r=b} \frac{\rho\, dr}{2\pi \ell r} = \frac{\rho}{2\pi \ell} \ln(b/a).}$$

units ✓

Note: makes sense in limits $\ell \to \infty$ ($R \to 0$) and $b \to a$ ($R \to 0$).

#2 Again we can (and should) think of this rod as a bunch of thin resistors, each located at $0 \le x \le L$ with thickness dx and resistance dR given by $dR = \rho(x)\frac{dx}{A}$ with $\rho(x) = \rho_0 \exp(-\frac{x}{L})$.



Then we have
$$R_{TOT} = \int_{x=0}^{x=L} dR = \frac{1}{A}\int_0^L \rho_0 \exp(-\tfrac{x}{L})\,dx$$
$$= \frac{\rho_0}{A}(-L)\exp(-\tfrac{x}{L})\Big|_0^L = \frac{\rho_0 L}{A}\left(1 - \frac{1}{e}\right)$$

so $\boxed{R_{TOT} = \frac{\rho_0 L}{A}(1 - e^{-1})}$ (units ✓)

Now suppose a potential difference of V is set up, so that $\phi(0) = 0$ $(\phi(x=0))$ and $\phi(x=L) = V$.

By symmetry, ϕ can only depend on x. Ohm's Law says
(eq 4.16 in PM) $\vec{J} = \frac{1}{\rho}\vec{E} = -\frac{1}{\rho}\nabla\phi$. ϕ only depends on $x \Rightarrow \nabla\phi = (\partial_x \phi)\hat{x}$

and so, since the current must be constant in this resistor (resistors in series) we have
$$\frac{1}{\rho}\partial_x \phi = G = \text{constant}.$$

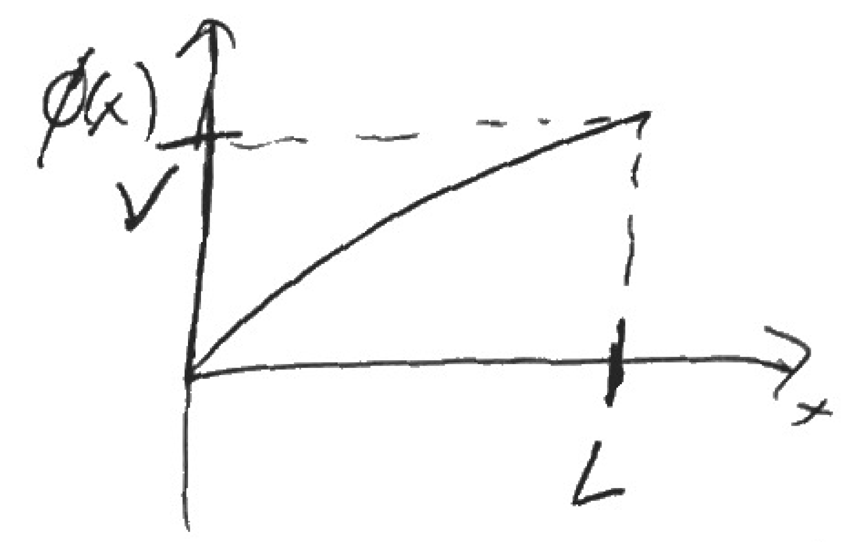
~~Then~~ $\cancel{\partial_x\phi = A\exp(-\tfrac{x}{L})}$ Thus $\partial_x\phi = G\rho_0 \exp(-\frac{x}{L})$

so that $\phi(x) = A\exp(-\frac{x}{L}) + B$ where A, B are constants.

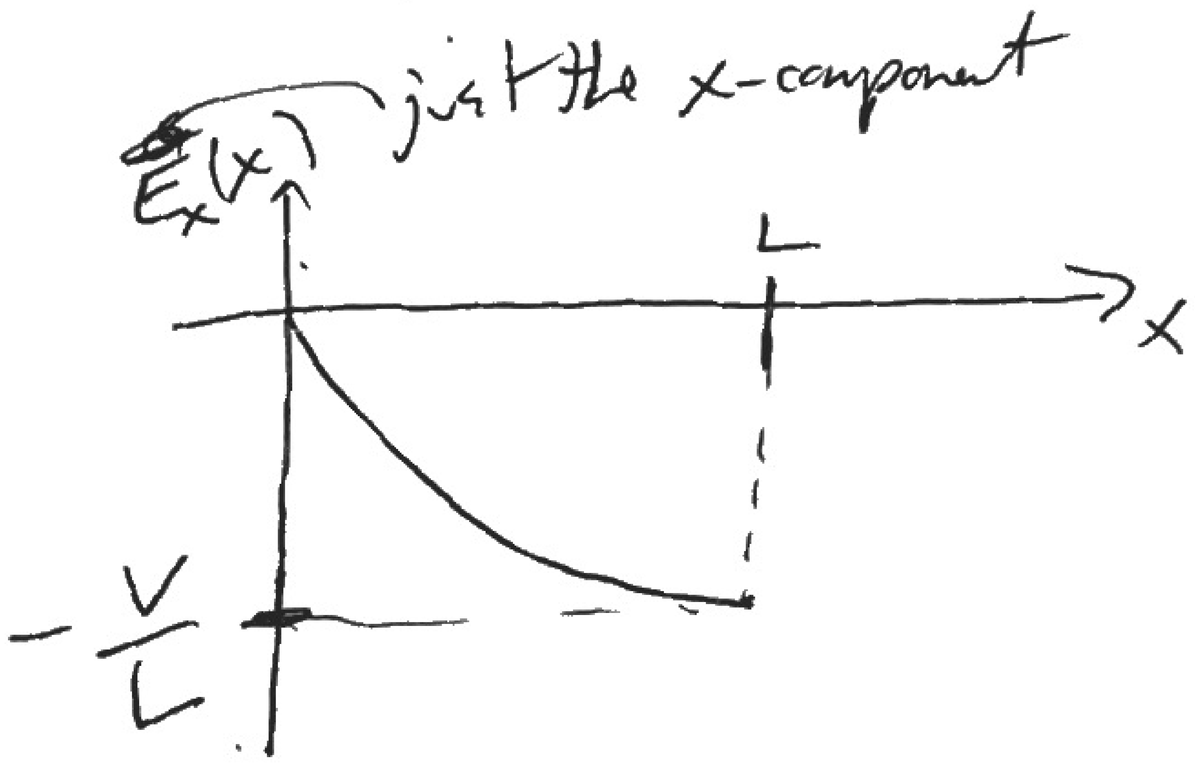
→

$$\phi(0)=0 \Rightarrow A+B=0$$
$$\phi(L)=V \Rightarrow Ae^{-1}+B=V$$
$$\Rightarrow (-B)e^{-1}+B=V$$
$$\Rightarrow B(1-e^{-1})=V$$
$$\Rightarrow B=\frac{V}{1-e^{-1}}$$
$$(A=-B)$$

Thus $\boxed{\phi(x)=\frac{V}{1-e^{-1}}\left(1-\exp\left(\frac{-x}{L}\right)\right)}$



$$\boxed{\vec{E}=-(\partial_x\phi)\hat{x}=-\frac{V}{L(1-e^{-1})}\left(1-\exp\left(\frac{-x}{L}\right)\right)}$$



#3) PM 4.22 a) We have $\rho=3\times10^{-8}$ ohm-meter, each wire has diameter $d=.73$ mm so that each wire's crossesectional area is $A=\pi r^2=\frac{\pi}{4}d^2$ and so the crossesectional area of the whole cable (with 7 wires) is $7A$. ($L=3000$ km)

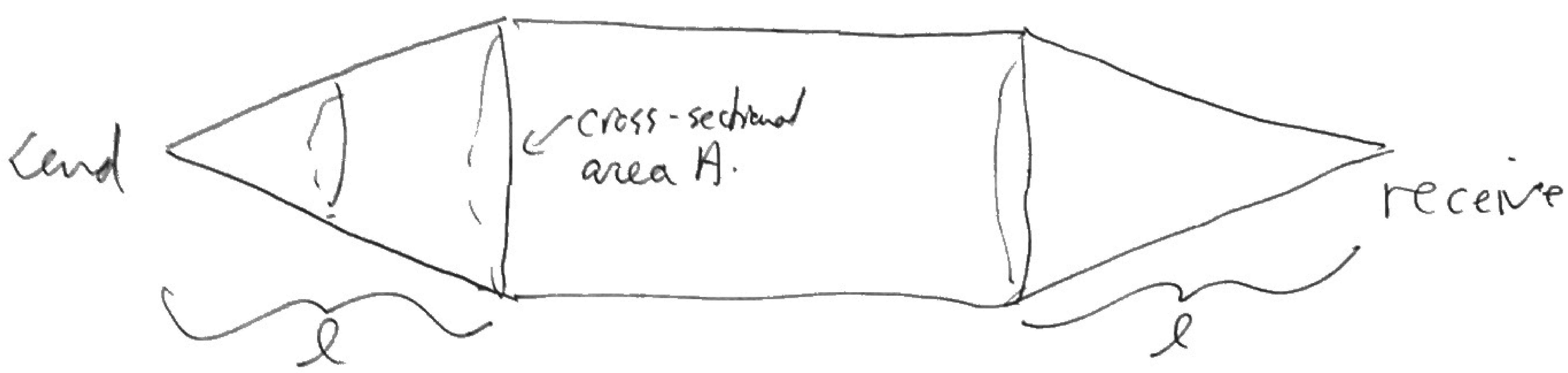
Thus $$R_{TOT}=\rho\frac{L}{7A}=\frac{3\times10^6\cdot3\times10^{-8}}{7\cdot\frac{\pi}{4}(.73\times10^{-3})^2}\approx3.1\times10^4\,\Omega.$$

So $\boxed{R_{TOT}\simeq3.1\times10^4\,\Omega}$

→

#3) (b) Let us assume the return path in the water looks something like

$\rho_{water} = .25$ ohm-meter.



We want an upper bound on the resistance of the return, so we can imagine the cross-sectional area A of the middle part is rather small (on oceanic scales), perhaps a radius of .5 km.

Then the resistance R_{tube} of the tubular part (even if we took its length to be the full 3,000 km) would be

$$R_{tube} = \frac{\rho L}{A} = \frac{.25 \cdot 3 \times 10^6}{\pi (.5 \times 10^3)^2} \simeq 1 \, \Omega$$

, which is tiny compared to what we got in part a).

~~Similarly, for the cone edges, we can break them up into cross sectional circles, from radius 0 to radius .5 km, and then (again taking each cone to have the full 3,000 km length (since after all, we just want an upper bound)) we get~~

$\cancel{R_{cone} = \rho \int}$

→

#3) (b) (cont'd)

(assume the cones are "right" cones, i.e. )

Similarly, for the conic ends, we break R_{cone} up into circular cross-sections of length dr and cross-sectional area πr^2, and let us make the (generous) assumption that the initial radius is, say, .1 meter and the length of each cone is 1,000 km. (i.e. 1/3 of the way across). After all, we just want an upper bound on R_{TOT}.

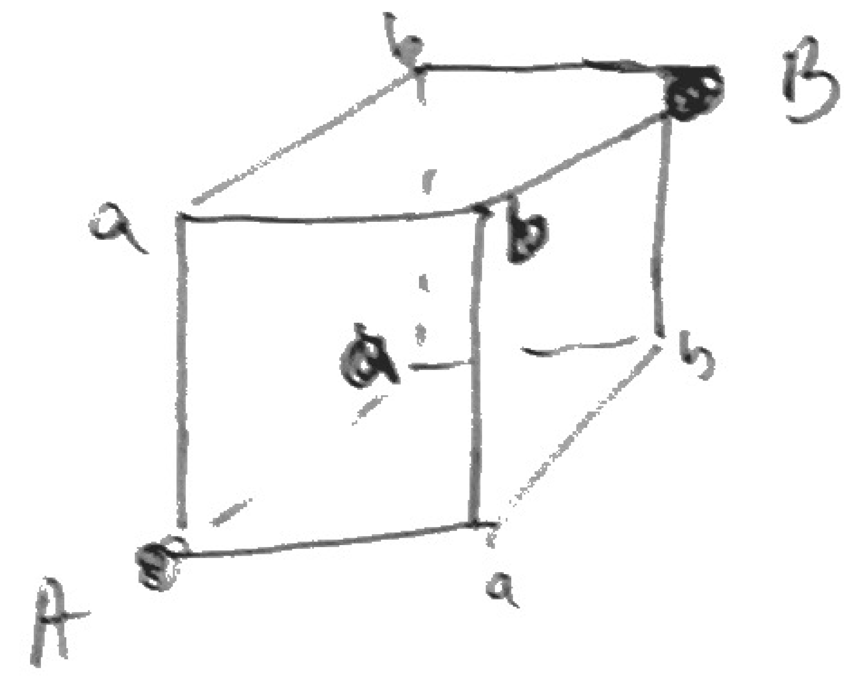
We then have
$$R_{cone} = \rho \cdot \underbrace{\int_{r=.1m}^{r=1{,}000km}}_{\text{summing over all resistors}} \frac{\overset{\text{length of each resister}}{dr}}{\underbrace{\pi r^2}_{\text{cross sectional area } A}} = \frac{\rho}{\pi}\left(\frac{1}{.1m} - \underbrace{\frac{1}{10^6 m}}_{\text{much smaller than first term, so can drop it.}}\right)$$

so that $R_{cone} \simeq \frac{10}{\pi}(.25)\,\text{Ohm} \simeq 1\,\Omega$.

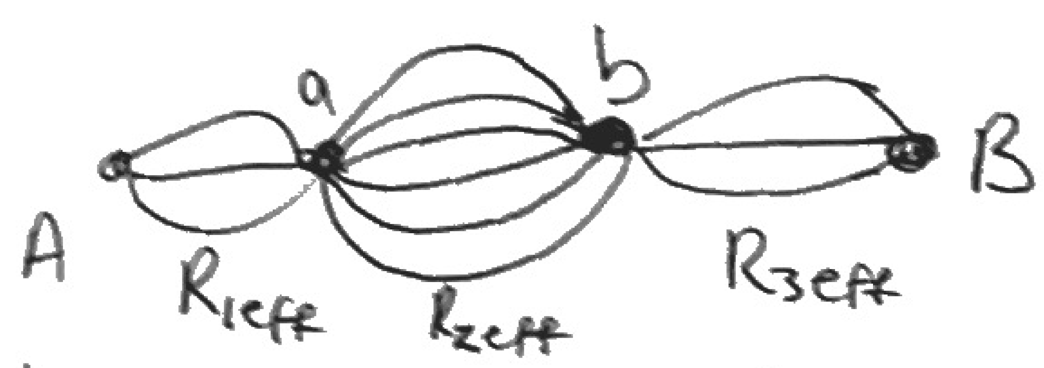
Thus the total resistance $R_{TOT} = R_{tube} + 2R_{cone}$ $(\simeq 3\Omega)$ of the return is <u>much</u> smaller than the ~~transmission~~ sending resistance.

#4) PM 4.35)

a)



Due to ~~rotational symmetry~~ (rotational symmetry), all the "a" vertices are at an equal potential, as are ~~these~~ the "b" vertices. Thus, shrinking these down to points, we get

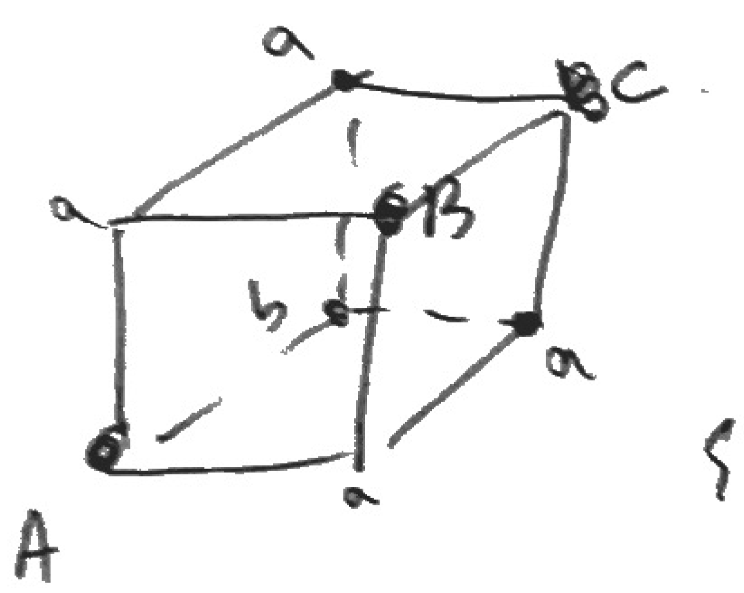


Since each line carries resistance R, we have $R_{1eff} = R_{3eff}$

and $\frac{1}{R_{1eff}} = \frac{1}{R} + \frac{1}{R} + \frac{1}{R} \Rightarrow R_{eff} = \frac{R}{3} = R_{3eff}$ and similarly $R_{2eff} = \frac{R}{6}$.

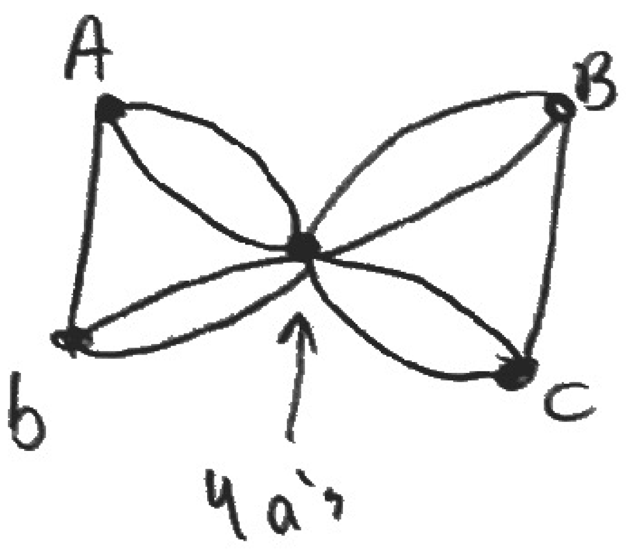
Thus $\boxed{R_{\substack{TOT\\AB}} = \frac{R}{3} + \frac{R}{6} + \frac{R}{3} = \frac{5}{6}R}$

b)



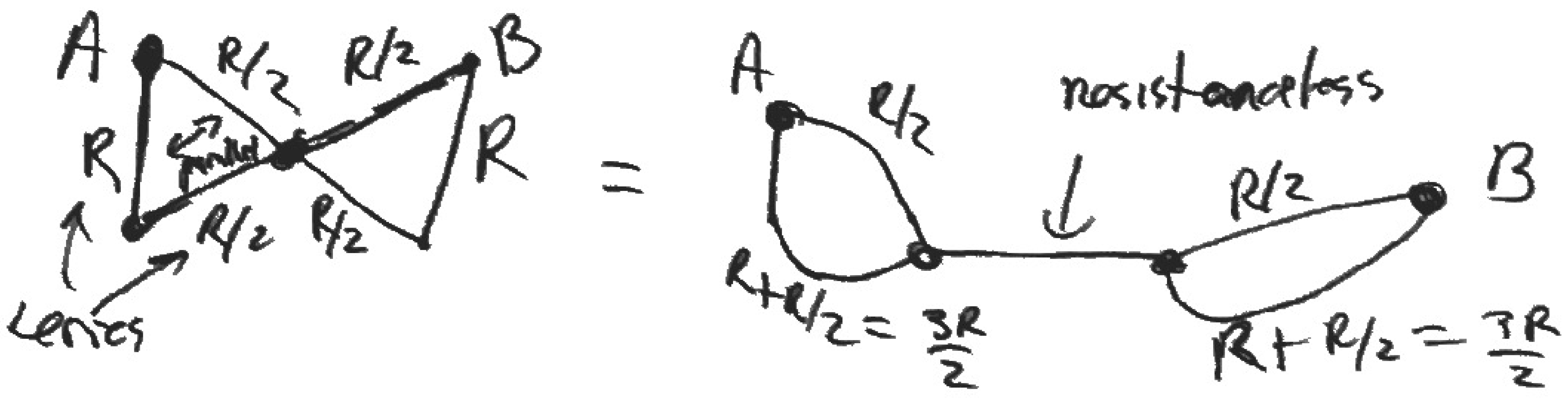
All "a" vertices are at equal potential since they lie on a plane equidistant from A and B

Shrinking these down gives

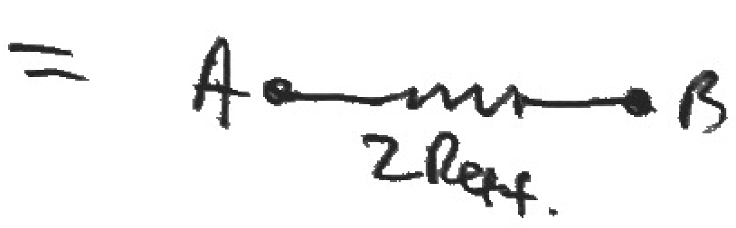


Note: we have lost two lines that connect the "a"s to eachother.

This circuit is equivalent to



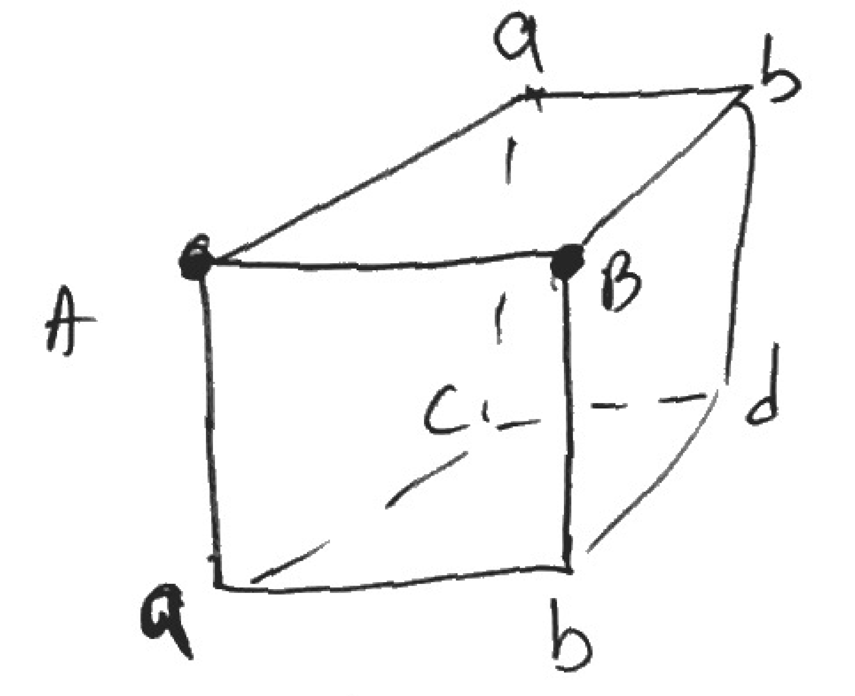
where $\frac{1}{R_{eff}} = \frac{2}{3R} + \frac{2}{R} = \frac{8}{3R} \Rightarrow R_{eff} = \frac{3R}{8}$



Thus $\boxed{R_{\substack{TOT\\AB}} = \frac{3}{4}R}$

#4

c)



Similarly labelled vertices are at the same potential. For example, the 2 "a" vertices can be turned into each other via a rotation that keeps A and B fixed.

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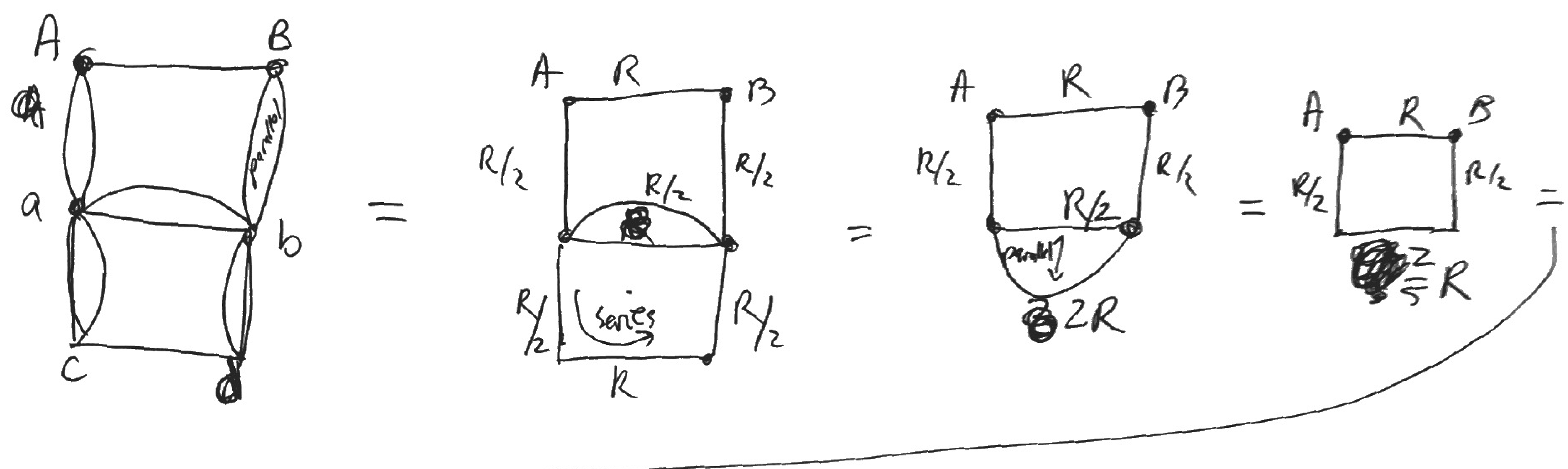

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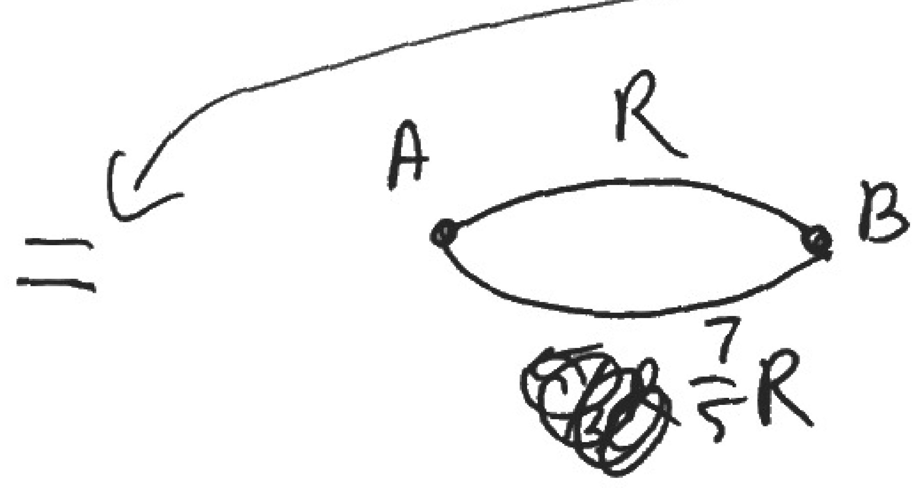
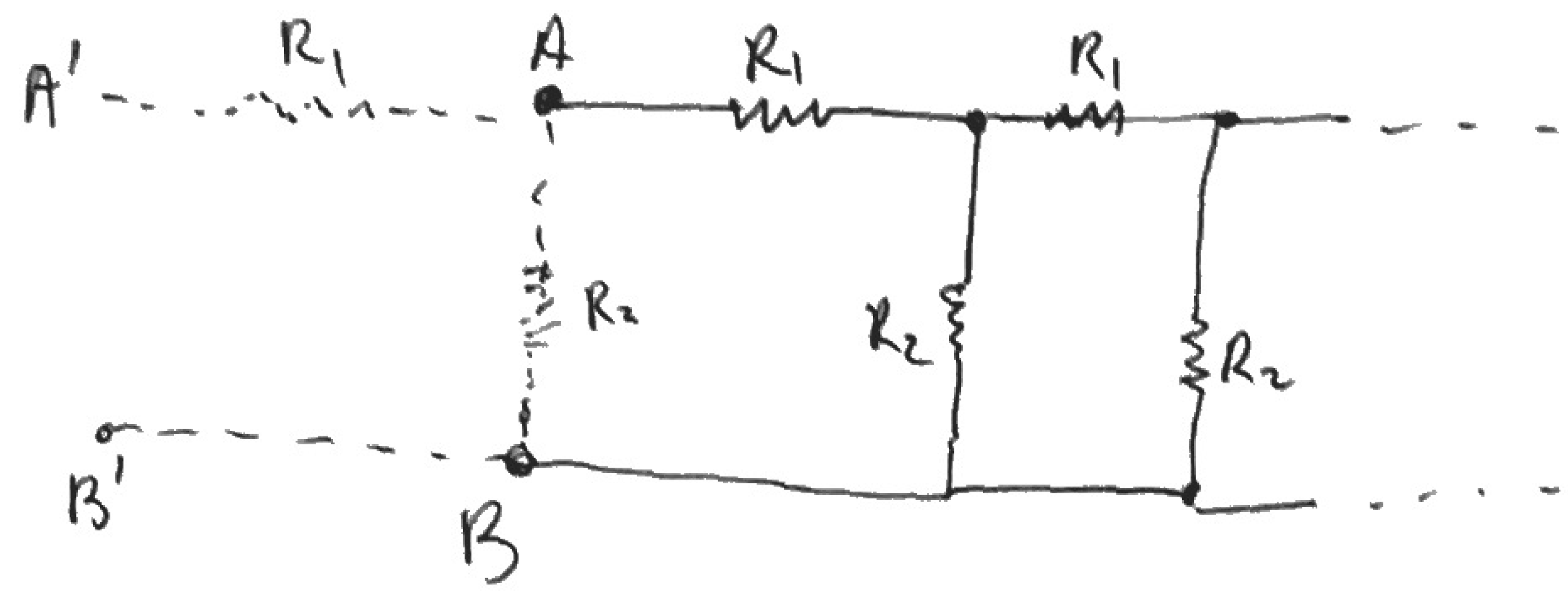


$\cancel{\frac{1}{R_{TOT,AB}} = \frac{1}{R} + \frac{3}{5R} =}$

$$\Rightarrow \frac{1}{R_{TOT,AB}} = \frac{1}{R} + \frac{5}{7R} = \frac{12}{7R} \Rightarrow \boxed{R_{TOT,AB} = \frac{7R}{12}}$$

#5)



Let R be the total resistance of the full unit. By adding the suggested R_1 and R_2 (in dotted lines) we know we don't change the result.

But this addition is just

and the effective resistance R_{eff} (which we know is R) is

$$R_{eff} = R = R_1 + \left(\frac{1}{\frac{1}{R} + \frac{1}{R_2}}\right) = R_1 + \frac{RR_2}{R+R_2}$$

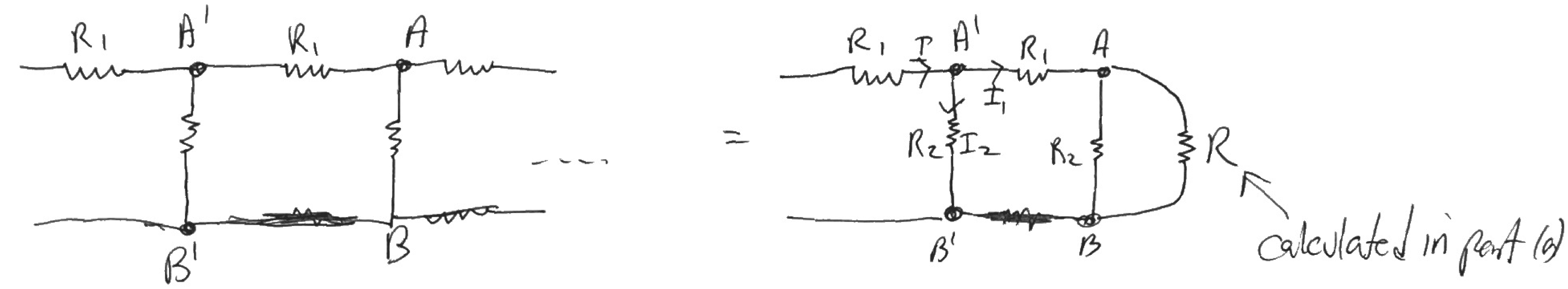
Thus $R = R_1 + \frac{RR_2}{R+R_2} \Rightarrow R(R+R_2) = R_1(R+R_2) + RR_2$

$\Rightarrow R^2 + \cancel{RR_2} = RR_1 + R_1R_2 + \cancel{RR_2} \Rightarrow R^2 - R_1R - R_1R_2 = 0$

$\Rightarrow R = \frac{1}{2}\left(R_1 \pm \sqrt{R_1^2 + 4R_1R_2}\right)$ (only take positive value).

so $\boxed{R = \frac{1}{2}\left(R_1 + \sqrt{R_1^2 + 4R_1R_2}\right)}$

#5) (cont'd) Now consider a square somewhere in the circuit. (with corners A, A', B, B'):



with current I going into A' and I_1, I_2 as shown, so that in particular $I_1 + I_2 = I$. Now, the effective resistance between A and B is $\frac{1}{R_{eff}} = \frac{1}{R_2} + \frac{1}{R} \Rightarrow R_{eff} = \frac{RR_2}{R+R_2}$, so that the voltage drop V_{AB} between A and B is $V_{AB} = I_1\left(\frac{RR_2}{R+R_2}\right)$. However, the circuit to the right of A' and B' is also just an effective resistor with resistance R (even though it has an extra $R_1 + R_2$ attached (recall part (a))). Thus the effective resistance between A' and B' is also $\frac{RR_2}{R+R_2}$ so that the voltage difference $V_{A'B'}$ between A' and B' is $V_{A'B'} = I\frac{RR_2}{R+R_2}$. Thus,

$$\frac{V_{AB}}{V_{A'B'}} = \frac{I_1\left(\frac{RR_2}{R+R_2}\right)}{I\left(\frac{RR_2}{R+R_2}\right)} = \frac{I_1}{I}.$$

Now, since the circuit to the right of A' and B' is just a resistor R, we have that $I_1 R = I_2 R_2$. This with $I_1 + I_2 = I \Rightarrow I_1 + I_1\frac{R}{R_2} = I \Rightarrow I_1 = \frac{I}{1+\frac{R}{R_2}}$.

Thus $\boxed{\frac{V_{AB}}{V_{A'B'}} = \frac{R_2}{R_2+R}}$ If we want $\frac{V_{AB}}{V_{A'B'}} = \frac{1}{2}$ then we take $R = R_2$,

so that, from (a), we have $R_2 = R = \frac{1}{2}\left(R_1 + \sqrt{R_1^2 + 4R_1R_2}\right)$

This gives $2R_2 = R_1 + \sqrt{R_1^2+4R_1R_2}$

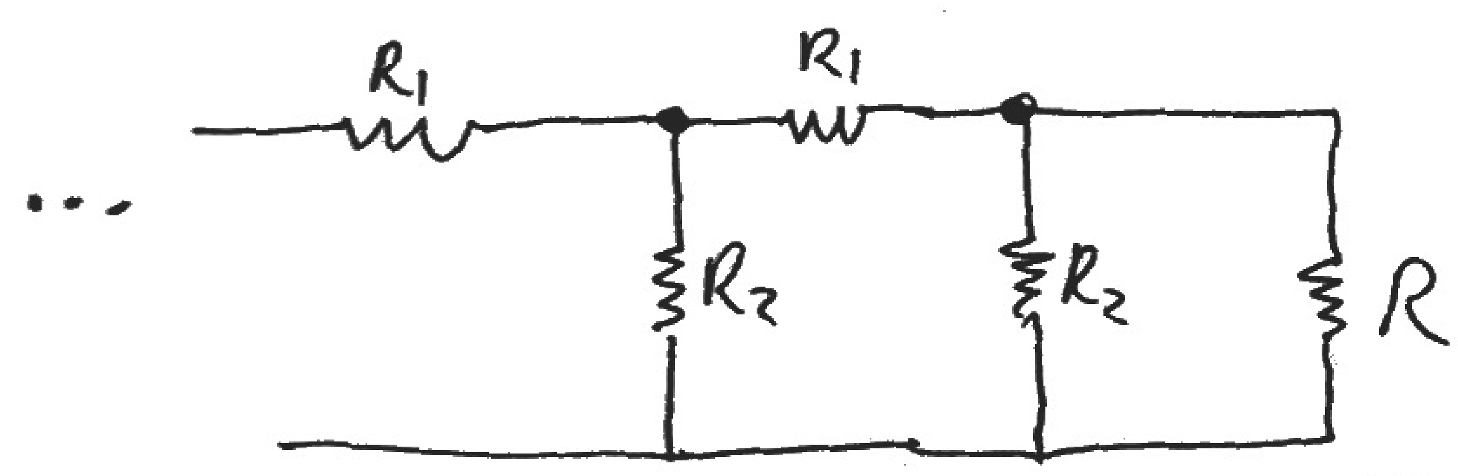
Don't cancel (typo)

$$\Rightarrow (2R_2-R_1)^2 = R_1^2+4R_1R_2 \Rightarrow 4R_2^2 + \cancel{R_1^2} - \underline{4R_1R_2} = \cancel{R_1^2} + \underline{4R_1R_2}$$

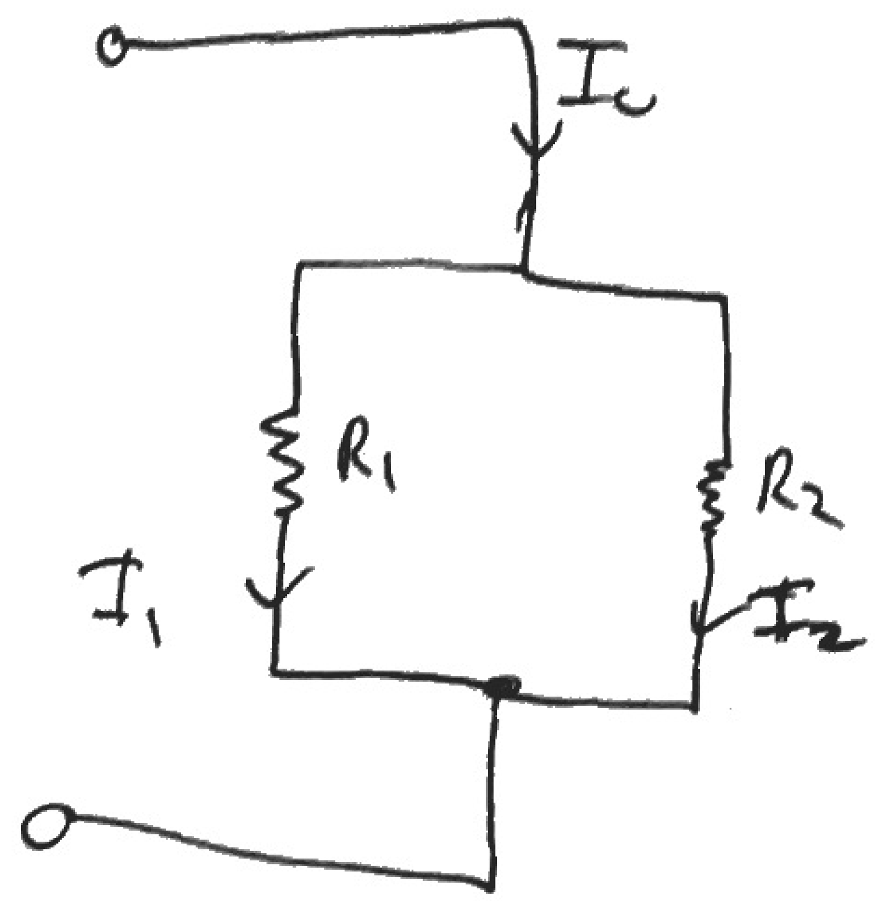
$$\Rightarrow R_2^2 = 2R_1R_2 \Rightarrow \boxed{R_2 = 2R_1}$$

To emulate an infinite ladder with finitely many resistors, we simply need to cut the ladder off and attach a resistor with value $R = \frac{1}{2}\left(R_1 + \sqrt{R_1^2+4R_1R_2}\right)$ in parallel with an R_2.

Namely, as shown on the right of the top of the previous page, we can use



#6) PM 4.40



The total power dissipated is $P_{TOT} = I_1^2 R_1 + I_2^2 R_2$. Using $I_2 = I_0 - I_1$ gives

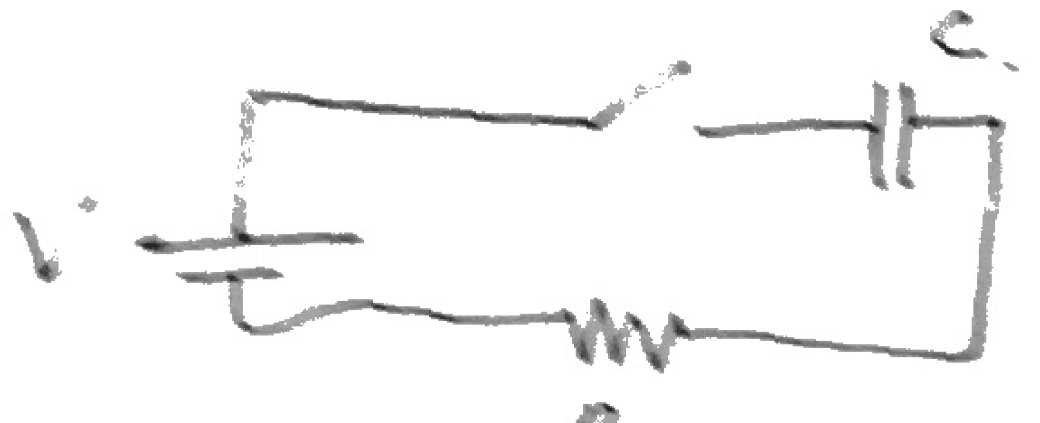
$$
\begin{aligned}
P_{TOT} &= I_1^2 R_1 + (I_0 - I_1)^2 R_2 \\
&= I_1^2 R_1 + I_0^2 R_2 - 2 I_0 I_1 R_2 + I_1^2 R_2 \\
&= I_0^2 R_2 + I_1^2 (R_1 + R_2) - 2 I_0 I_1 R_2 .
\end{aligned}
$$

Minimizing this with respect to I_1 gives

$$
0 = \frac{dP_{TOT}}{dI_1} = 2 I_1 (R_1 + R_2) - 2 I_0 R_2 \Rightarrow \boxed{I_1 = \frac{R_2}{R_1 + R_2} I_0}
$$

just as we expect.

PM 4.48 We first do problem 4.17:



Kirchoff's rule, when the switch is closed, tells us that $V - \underbrace{\overset{\text{charge on capacitor}}{\frac{Q}{C}}}_{\text{voltage drop across capacitor}} - IR = 0$.

$$\Rightarrow IR = \frac{Q}{C} - V.$$

$I = \frac{dQ}{dt} \Rightarrow \frac{dQ}{dt} = \frac{Q}{RC} - V.$

This equation has the standard solution (as can/should be checked) $Q(t) = CV(1 - e^{-t/RC})$

(to "derive" this, try $Q(t) = A(1 - e^{-t/\tau})$ and solve for τ, A by differentiating and also requiring $Q(t \to \infty) = CV$, as we know it should). (Note: we guess this because we want $Q(0) = 0$)

Then we see that $I(t) = \frac{dQ}{dt} = \frac{V}{R} e^{-t/RC}$.

Now, the total work done by the battery in charging up the capacitor is $QV \overset{(Q=CV)}{=} CV^2$ since it must move a total charge of $Q = CV$ against the constant potential V. So $W_{battery} = CV^2$.

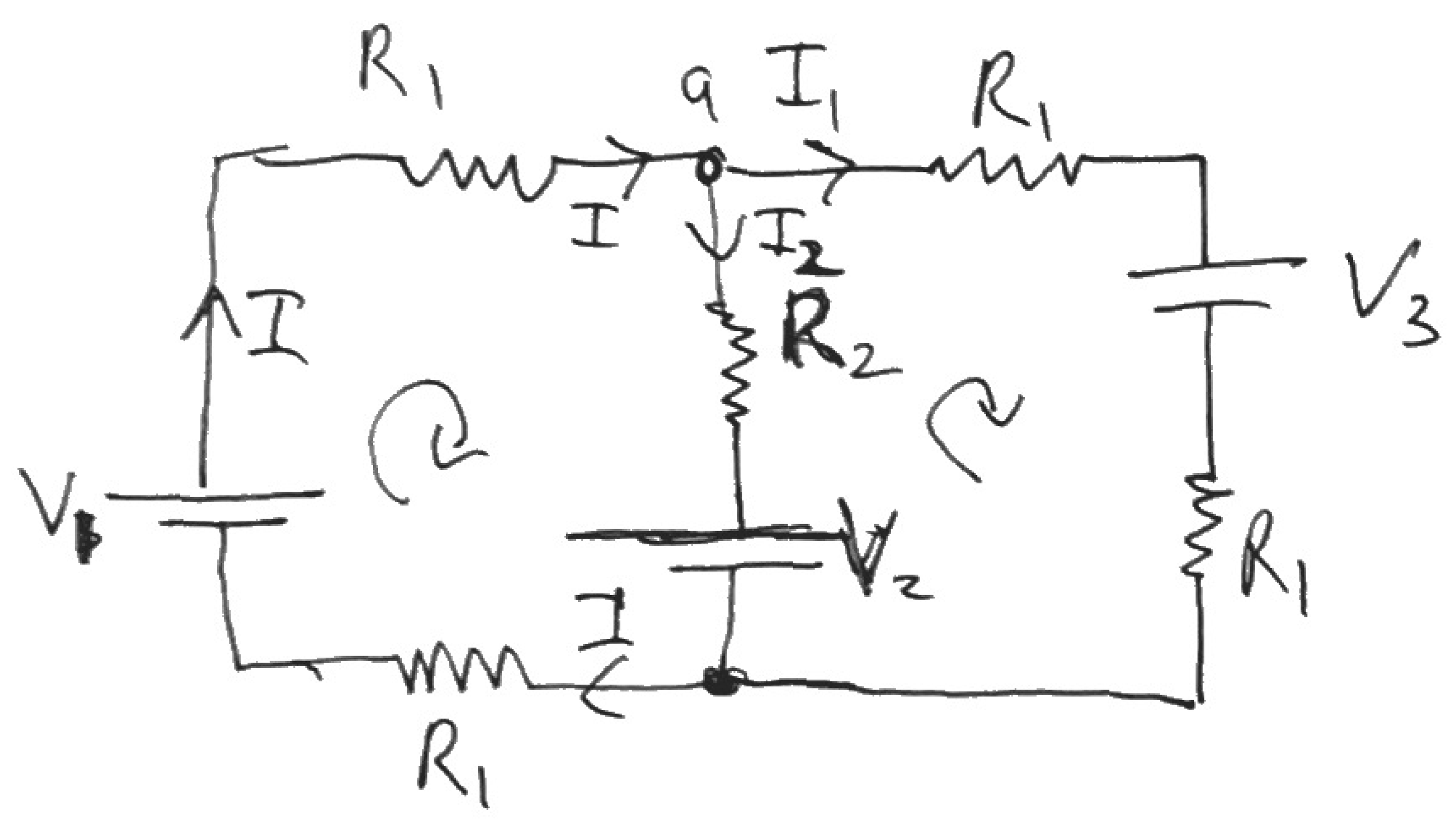
The final energy stored in the ~~battery~~ capacitor $E_{capacitor}$ is of course $E_{capacitor} = \frac{1}{2}CV^2$.

And the total power dissipated $U_{dissipated} = \int_{t=0}^{\infty} I^2 R\, dt = \frac{V^2}{R}\int_0^{\infty} e^{\frac{-2t}{RC}} dt = -\frac{RC}{2R} V^2 e^{\frac{-2t}{RC}}\Big|_0^{\infty}$

so that $U_{dissipated} = \frac{1}{2}CV^2$.

Indeed, $\boxed{W_{battery} = CV^2 = U_{dissipated} + E_{capacitor} = \frac{1}{2}CV^2 + \frac{1}{2}CV^2,}$ as desired.

#8



$R_1 = 1\,\Omega$

$R_2 = 2\,\Omega$ $\quad V_2 = V_3 = 4$ Volts

$V_1 = 2$ Volts

Equations: $\underline{I_1 + I_2 = I}$ $(*)$

Kirkhoff on RHS $\Rightarrow$ $-I_2R_2 - V_2 = -I_1R_1 - V_3 - I_1R_1$

$\Rightarrow$ $I_2R_2 + V_2 = 2I_1R_1 + V_3$

$\Rightarrow$ $I_2R_2 = 2I_1R_1$ (since $V_2 = V_3$)

$\Rightarrow$ $2I_2 = 2I_1 \Rightarrow \underline{I_1 = I_2}$.

since $R_2 = 2R_1$

$\Rightarrow I_1 = I_2 = \frac{I}{2}$

Kirkhoff on LHS: $V_1 - IR_1 - I_2R_2 - V_2 - IR_1 = 0$

$\Rightarrow$ $-2IR_1 + \underbrace{V_1 - V_2}_{-2} = I_2R_2$

$\Rightarrow$ $-2IR_1 = I_2R_2 + 2 \Rightarrow \underline{-2I = 2I_2 + 2}$

$\Rightarrow$ $-I = \frac{I}{2} + 1 \Rightarrow -\frac{3}{2}I = 1 \Rightarrow \underline{I = -\frac{2}{3}\text{ Amps.}}$

Thus I actually goes down with $\frac{2}{3}$ Amps.
so I_1 goes up with $\frac{1}{3}$ Amps, as does I_2 (up with $\frac{1}{3}$ Amps)

$V_{ab} = -I_2R_2 + V_2 = \left(\left(-\frac{1}{3}\right)\cdot 2 + 4\right)$ Volts $= \frac{10}{3}$ Volts. $\quad \boxed{V_{ab} = \frac{10}{3}\text{ Volts}}$