PHYSICS 115B, Fall 2017
Final Exam (100 points in total)

- You are allowed to bring a formula sheet (both sides).
- Please write down the necessary intermediate steps.
- Write your answers in the space provided. Use additional paper if necessary.

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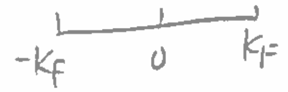
Total	77

1. Derive the density of states for 1D free electron gas at $T = 0$. (12 points)

Accessible space for 1D



$$N = \frac{2\cdot(2k_F)}{\left(\frac{2\pi}{L}\right)} = \frac{2k_F L}{\pi} \qquad k^2 = \frac{2mE}{\hbar^2}$$

Since $E_F = \frac{\hbar^2 k_F^2}{2m}$

$$\Rightarrow k_F = \frac{\sqrt{2mE_F}}{\hbar}$$

$$\Rightarrow N = \frac{2L}{\pi\hbar}(2mE_F)^{\frac{1}{2}}, \quad n = \frac{N}{V} = \frac{2L}{\pi\hbar V}(2mE_F)^{\frac{1}{2}}$$

$L = V$

$$\Rightarrow D(E) = \frac{dn}{dE} = \frac{2L}{\pi\hbar V}\left(\frac{1}{2}\right)(2mE)^{-\frac{1}{2}}$$

$$= \frac{L\sqrt{2m}}{\pi\hbar V}(E)^{-\frac{1}{2}}$$

12

15

2. Suppose there are three noninteracting particles (all of mass m) in the 1D infinite square well of width L.
   (a) Construct the completely antisymmetric wave function $\psi(x_A, x_B, x_C)$ for three identical fermions, one in the state ψ_5, one in the state ψ_7, and one in the state ψ_{17}. (6 points)
   (b) Construct the completely symmetric wave function $\psi(x_A, x_B, x_C)$ for three identical bosons, (i) if all three are in state ψ_{11}, (ii) if two are in state ψ_1 and one is in state ψ_{19}, and (iii) if one is in the state ψ_5, one in the state ψ_7 and one in the state ψ_{17}. (9 points)

Note: I'm using $\psi(x_1, x_2, x_3)$ instead of $\psi(x_A, x_B, x_C)$

(a)
$$\psi = \frac{1}{\sqrt{3!}}\begin{vmatrix} \psi_5(x_1) & \psi_5(x_2) & \psi_5(x_3) \\ \psi_7(x_1) & \psi_7(x_2) & \psi_7(x_3) \\ \psi_{17}(x_1) & \psi_{17}(x_2) & \psi_{17}(x_3) \end{vmatrix}$$

$$= \frac{1}{\sqrt{6}}\Big\{\psi_5(x_1)\left[\psi_7(x_2)\psi_{17}(x_3) - \psi_7(x_3)\psi_{17}(x_2)\right] - \psi_5(x_2)\left[\psi_7(x_1)\psi_{17}(x_3) - \psi_7(x_3)\psi_{17}(x_1)\right] + \psi_5(x_3)\left[\psi_7(x_1)\psi_{17}(x_2) - \psi_7(x_2)\psi_{17}(x_1)\right]\Big\}$$

$\psi(x_1, x_2, x_3)$
$$= \frac{1}{\sqrt{6}}\Big[\psi_5(x_1)\psi_7(x_2)\psi_{17}(x_3) - \psi_5(x_1)\psi_7(x_3)\psi_{17}(x_2) - \psi_5(x_2)\psi_7(x_1)\psi_{17}(x_3) + \psi_5(x_2)\psi_7(x_3)\psi_{17}(x_1) + \psi_5(x_3)\psi_7(x_1)\psi_{17}(x_2) - \psi_5(x_3)\psi_7(x_2)\psi_{17}(x_1)\Big]$$

(b) (i)
$$\psi = \frac{1}{\sqrt{6}}\begin{vmatrix} \psi_{11}(x_1) & \psi_{11}(x_2) & \psi_{11}(x_3) \\ \psi_{11}(x_1) & \psi_{11}(x_2) & \psi_{11}(x_3) \\ \psi_{11}(x_1) & \psi_{11}(x_2) & \psi_{11}(x_3) \end{vmatrix}$$

$$= \frac{1}{\sqrt{6}}\Big[\psi_{11}(x_1)\left(\psi_{11}(x_2)\psi_{11}(x_3) + \psi_{11}(x_2)\psi_{11}(x_3)\right) + \psi_{11}(x_2)\left[\psi_{11}(x_1)\psi_{11}(x_3) + \psi_{11}(x_3)\psi_{11}(x_1)\right] + \psi_{11}(x_3)\left[\psi_{11}(x_1)\psi_{11}(x_2) + \psi_{11}(x_2)\psi_{11}(x_1)\right]\Big]$$

$$= \frac{2}{\sqrt{6}}\left[3\,\psi_{11}(x_1)\psi_{11}(x_2)\psi_{11}(x_3)\right] = \frac{6}{\sqrt{6}}\psi_{11}(x_1)\psi_{11}(x_2)\psi_{11}(x_3)$$

(ii)
$$\frac{1}{\sqrt{6}}\begin{vmatrix} \psi_1(x_1) & \psi_1(x_2) & \psi_1(x_3) \\ \psi_1(x_1) & \psi_1(x_2) & \psi_1(x_3) \\ \psi_{19}(x_1) & \psi_{19}(x_2) & \psi_{19}(x_3) \end{vmatrix} = \frac{1}{\sqrt{6}}\Big[\psi_1(x_1)\left[\psi_1(x_2)\psi_{19}(x_3) + \psi_1(x_3)\psi_{19}(x_2)\right] + \psi_1(x_2)\left[\psi_1(x_1)\psi_{19}(x_3) + \psi_1(x_3)\psi_{19}(x_1)\right] + \psi_1(x_3)\left[\psi_1(x_1)\psi_{19}(x_2) + \psi_1(x_2)\psi_{19}(x_1)\right]\Big]$$

(iii)
$$\frac{1}{\sqrt{6}}\begin{vmatrix} \psi_5(x_1) & \psi_5(x_2) & \psi_5(x_3) \\ \psi_7(x_1) & \psi_7(x_2) & \psi_7(x_3) \\ \psi_{17}(x_1) & \psi_{17}(x_2) & \psi_{17}(x_3) \end{vmatrix} = \frac{1}{\sqrt{6}}\Big[\psi_5(x_1)\left[\psi_7(x_2)\psi_{17}(x_3) + \psi_7(x_3)\psi_{17}(x_2)\right] + \psi_5(x_2)\left[\psi_7(x_1)\psi_{17}(x_3) + \psi_{17}(x_1)\psi_7(x_3)\right] + \psi_5(x_3)\left[\psi_7(x_1)\psi_{17}(x_2) + \psi_7(x_2)\psi_{17}(x_1)\right]\Big]$$

$$= \frac{1}{\sqrt{6}}\Big[\psi_5(x_1)\psi_7(x_2)\psi_{17}(x_3) + \psi_5(x_1)\psi_7(x_3)\psi_{17}(x_2) + \psi_5(x_2)\psi_7(x_1)\psi_{17}(x_3) + \psi_5(x_2)\psi_{17}(x_1)\psi_7(x_3) + \psi_5(x_3)\psi_7(x_1)\psi_{17}(x_2) + \psi_5(x_3)\psi_7(x_2)\psi_{17}(x_1)\Big]$$

$1J = 6.242\times10^{12}$ MeV

3. **A quark (mass = $m_p/3$) is confined in a cubical box with sides of length 2 fermis = 2×10^{-15} m. Find the excitation energy from the ground state to the first excited state in MeV. [proton mass: $m_p = 1.67\times10^{-27}$ *kg* and $\hbar = 1.05\times10^{-34}$ *J·s*] (10 points)**

$E = h\nu$

$L = 2\times10^{-15}$ m.

Particle in a cubical box: $-\frac{\hbar^2}{2m}\nabla^2 = E\psi$. , $k^2 = \frac{2mE}{\hbar^2}$

$\psi(x,y,z) = \left(\frac{L}{2}\right)^{\frac{3}{2}} \sin\left(\frac{n_x\pi}{L}x\right)\sin\left(\frac{n_y\pi}{L}y\right)\sin\left(\frac{n_z\pi}{L}z\right)$; $kL = n\pi$

$k = \frac{n\pi}{L}$

$\Rightarrow E_n = \frac{\hbar^2 k^2}{2m} = \frac{\hbar^2}{2m}\left(\frac{\pi}{L}\right)^2(n_x^2+n_y^2+n_z^2)$

ground state: $E_{111} = \frac{\hbar^2}{2m}\left(\frac{\pi}{L}\right)^2\cdot 3 = \frac{3\hbar^2}{2m}\left(\frac{\pi}{L}\right)^2$

1st excited $E_{112} = E_{121} = E_{211} = \frac{\hbar^2}{2m}\left(\frac{\pi}{L}\right)^2(2^2+1^2+1^2)$

$= 3\frac{\hbar^2}{m}\left(\frac{\pi}{L}\right)^2$

$\Rightarrow \Delta E = E_{112} - E_{111} = \frac{3}{2}\frac{\hbar^2}{m}\left(\frac{\pi}{L}\right)^2$

$= \frac{3}{2}\frac{(1.05\times10^{-34}Js)^2}{\left(\frac{1.67\times10^{-27}kg}{3}\right)}\left(\frac{\pi}{2\times10^{-15}m}\right)^2$

$(1.05\times10^{-34})^2 = 1.10\times10^{-68}$

$\frac{1.67\times10^{-27}}{3} = 0.557\times10^{-27}$

$\left(\frac{\pi}{2\times10^{-15}}\right)^2 = 2.47\times10^{30}$

10

$\simeq 7.4\times10^{-11}$ J

$= (7.4\times10^{-11})\cdot 6.242\times10^{12}$ MeV

$= 462$ MeV

note: Could vary from 6.70 - 7.41 depending on decimal of $\frac{(1.05\times10^{-34})}{\left(\frac{1.67\times10^{-27}}{3}\right)}$

1.81

13

$j=1$

4. Consider an angular momentun 1 system, represented by the state vector $\psi = \frac{1}{\sqrt{26}}\begin{pmatrix}1\\4\\3\end{pmatrix}$. What is the probability that a mesaurement of L_x yields the value 0? (15 points)

For $L=1$, since $L^2|lm\rangle = l(l+1)\hbar^2|lm\rangle$
and $L_\pm|lm\rangle = \hbar\sqrt{l(l+1)-m(m\pm1)}\,|l\,m\pm\rangle$

$L=1 \Rightarrow m_l = 0, +1, -1$
$\begin{pmatrix}1\\0\\0\end{pmatrix} \begin{pmatrix}0\\1\\0\end{pmatrix} \begin{pmatrix}0\\0\\1\end{pmatrix}$ basis

we can see that the matrices for $L+, L-$ should be: $L_+ = \hbar\begin{pmatrix}0 & \sqrt2 & 0\\ 0 & 0 & \sqrt2\\ 0 & 0 & 0\end{pmatrix}$

$L_- = \hbar\begin{pmatrix}0&0&0\\ \sqrt2 & 0 & 0\\ 0 & \sqrt2 & 0\end{pmatrix}$

So $L_x = \frac12(L_+ + L_-) = \frac12\hbar\begin{pmatrix}0&\sqrt2&0\\ \sqrt2&0&\sqrt2\\ 0&\sqrt2&0\end{pmatrix}$

$\Rightarrow \langle\psi|L_x|\psi\rangle = \frac{1}{26}\cdot\frac{\hbar}{2}\left[(1\;4\;3)\cdot\begin{pmatrix}0&\sqrt2&0\\ \sqrt2&0&\sqrt2\\ 0&\sqrt2&0\end{pmatrix}\begin{pmatrix}1\\4\\3\end{pmatrix}\right]$

$= \frac{\hbar}{48}\left[(1\;4\;3)\begin{pmatrix}4\sqrt2\\ 4\sqrt2\\ 4\sqrt2\end{pmatrix}\right]$ 3×3 3×1

$= \frac{\sqrt2\hbar}{12}\left[1+4+3\right] = \frac{\sqrt2\hbar}{12}\cdot 8 = \frac{2\sqrt2}{3}\hbar$

8

So $P(L_x=0)=0$

Another way to look at it:

eigenvals of L_x: $\det\begin{pmatrix}-\lambda & \sqrt2\hbar & 0\\ \sqrt2\hbar & -\lambda & \sqrt2\hbar\\ 0 & \sqrt2\hbar & -\lambda\end{pmatrix}=0$

$-\lambda(\lambda^2-2\hbar^2)+\sqrt2\hbar(-\sqrt2\hbar\lambda)=0$

$\Rightarrow -\lambda(\lambda^2-2\hbar^2)+2\hbar^2\lambda=0$

$\lambda=0$ or $-\lambda^2+2\hbar^2+2\hbar^2=0$

$\lambda=\pm2\hbar$

$|\lambda=0\rangle$

$\Rightarrow \hbar\begin{pmatrix}0&\sqrt2&0\\ \sqrt2&0&\sqrt2\\ 0&\sqrt2&0\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix}=0$

$\Rightarrow \sqrt2 b=0 \Rightarrow b=0$
$a+c=0 \Rightarrow a=-c$ $\Rightarrow \begin{pmatrix}1\\0\\-1\end{pmatrix} = \frac{1}{\sqrt2}\begin{pmatrix}1\\0\\-1\end{pmatrix}$

$|\lambda=2\hbar\rangle$: $\sqrt2\hbar b = 2\hbar a \Rightarrow \sqrt2 b = 2a$
$\sqrt2\hbar(a+c) = 2\hbar b \Rightarrow \sqrt2(a+c)=2b$
$\sqrt2\hbar b = 2\hbar c \Rightarrow \sqrt2 b = 2c$
$b=\frac{2}{\sqrt2}a$ $\Rightarrow \sqrt{\frac{2}{4}}\begin{pmatrix}1\\ \sqrt2/2\\ 1\end{pmatrix}$

$|\lambda=-2\rangle$

$\Rightarrow \Rightarrow \sqrt{\frac{5}{2}}\begin{pmatrix}1\\ \sqrt{2}/2\\ -1\end{pmatrix}$

$|\lambda=0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\end{pmatrix}$, $|\lambda=2\rangle = \frac{1}{\sqrt{5}}\begin{pmatrix}1\\ 2/\sqrt{2}\\ 1\end{pmatrix}$, $|\lambda=-2\rangle = \frac{1}{\sqrt{5}}\begin{pmatrix}1\\ 2/\sqrt{2}\\ -1\end{pmatrix}$

$\Rightarrow \psi = \begin{pmatrix}\frac{1}{\sqrt{26}}\\ \frac{4}{\sqrt{26}}\\ \frac{3}{\sqrt{26}}\end{pmatrix} = a\,\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\-1\end{pmatrix} + b\,\sqrt{\frac{5}{2}}\begin{pmatrix}1\\ 2/\sqrt{2}\\ -1\end{pmatrix} + \left(\frac{-1}{\sqrt{65}}\right)\sqrt{\frac{5}{2}}\begin{pmatrix}1\\ 2/\sqrt{2}\\ -1\end{pmatrix}$

$P(\lambda=0) = |\langle\lambda=0|\psi\rangle|^2$

$= a^2$

-2

$\frac{a}{\sqrt{2}} + b\sqrt{\frac{5}{2}} + c\cdot\sqrt{\frac{5}{2}} = \frac{1}{\sqrt{26}}$

$\sqrt{13}\,a + b\sqrt{5\cdot13} + c\sqrt{5\cdot13} = 1$

$b\,2\sqrt{5} + c\sqrt{5} = \frac{4}{\sqrt{26}}$

$b+c = \frac{4}{\sqrt{26}}\cdot\frac{1}{\sqrt{5}} = \frac{4}{\sqrt{130}}$

$\frac{a}{\sqrt{2}} + b\sqrt{\frac{5}{2}} - c\sqrt{\frac{5}{2}} = \frac{3}{\sqrt{26}}$

$\sqrt{13}\,a + \sqrt{5\cdot13}\,b - \sqrt{5\cdot13}\,c = 3$

$2\sqrt{5\cdot13}\,c = -2$

$c = -\frac{1}{\sqrt{65}}$

$b = \frac{4}{\sqrt{130}} + \frac{1}{\sqrt{65}}$

$\sqrt{2}\,c = -1$

5. A system of two particles each with spin 1/2 is described by an effective Hamiltonian $\hat{H} = A(\hat{S}_{1z} + \hat{S}_{2z}) + B\hat{S}_1 \cdot \hat{S}_2$, where $\hat{S}_1$ and $\hat{S}_2$ are the two spins, $\hat{S}_{1z}$ and $\hat{S}_{2z}$ are their z-components, and A and B are constants. Find all the energy levels of this Hamiltonian. (16 points)

$$S_{1z} = S_{2z} = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Rightarrow S_{1z} + S_{2z} = \hbar\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$S_1 = (S_x^1, S_y^1, S_z^1), \quad S_2 = (S_x^2, S_y^2, S_y^2)$$

$$\hat{S}_1 \cdot \hat{S}_2 = (S_x^1, S_y^1, S_z^1) \cdot (S_x^2, S_y^2, S_z^2)$$

$$= S_x^1 \cdot S_x^2 + S_y^1 \cdot S_y^2 + S_z^1 \cdot S_z^2$$

Tensor
Pauli
product

$$S_x^1 \cdot S_x^2 = \left(\frac{\hbar}{2}\right)^2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{\hbar^2}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$S_y^1 \cdot S_y^2 = \left(\frac{\hbar}{2}\right)^2 \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \frac{\hbar^2}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$S_z^1 \cdot S_z^2 = \left(\frac{\hbar}{2}\right)^2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{\hbar^2}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\text{So } H = A\hbar\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + B\frac{\hbar^2}{4}\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]$$

0

$$= A\hbar\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{3}{4}B\hbar^2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$H = \begin{pmatrix} A\hbar + \frac{3}{4}B\hbar^2 & 0 \\ 0 & -A\hbar + \frac{3}{4}B\hbar^2 \end{pmatrix}$$

Eigenvals of $\hat{H}$: Since H must be Hermitian.

eigenvals are diagonal terms: $E_+ = A\hbar + \frac{3}{4}B\hbar^2$,
$E_- = -A\hbar + \frac{3}{4}B\hbar^2$

B_x 13

6. A preparatory Stern-Gerlach experiment has established that the z-component of the spin of an electron is $-\hbar/2$. A uniform magnetic field in the x-direction of magnitude B is then switched on at time $t = 0$.
   (a) Predict the result of a single mesaurement of the z-component of the spin after elapse of time T. (8 points)
   (b) If, istead of mesauring the z-component of the spin, the x-component is mesasured, predict the result of such a single mesaurement after elapse of time T. (8 points)

$-\frac{\hbar}{2}.$

$\langle S_z \rangle = -\frac{\hbar}{2}$

(a) $\vec{B} = B_x \hat{x} \Rightarrow H = \gamma S_x B_x = \gamma B_x \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ +4

$e^{\frac{i\gamma B_x \frac{\hbar}{2} t}{\hbar}}$

Initially $\langle S_z \rangle = -\frac{\hbar}{2} \Rightarrow |\Psi(0)\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |\chi^{(z)}_-\rangle$

$H\Psi = i\hbar \frac{\partial}{\partial t}\Psi$, H is time indep $\Rightarrow |\Psi(T)\rangle = e^{\frac{iHT}{\hbar}}|\Psi(0)\rangle$

eigenvect of H:

$\det\begin{pmatrix} -\lambda & \gamma B_x \frac{\hbar}{2} \\ \gamma B_x \frac{\hbar}{2} & -\lambda \end{pmatrix} = 0$

$\Rightarrow \lambda = \pm \frac{\gamma \hbar B_x}{2}$

$|\lambda_+\rangle : \begin{pmatrix} 0 & \gamma B_x \frac{\hbar}{2} \\ \gamma B_x \frac{\hbar}{2} & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \frac{\gamma \hbar B_x}{2}\begin{pmatrix} a \\ b \end{pmatrix}$

$\Rightarrow b = a \Rightarrow \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$|\lambda_-\rangle = b = -a = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\Rightarrow |\Psi(0)\rangle = \frac{1}{\sqrt{2}}\left(|\lambda_+\rangle - |\lambda_-\rangle\right)$ algebra wrong

$\Rightarrow |\Psi(T)\rangle = \frac{1}{\sqrt{2}}\left(e^{\frac{-i\gamma B_x T}{2}}|\lambda_+\rangle + e^{\frac{+i\gamma B_x T}{2}}|\lambda_-\rangle\right)$ +3

$= \frac{1}{2}\begin{pmatrix} e^{\frac{-i\gamma B_x T}{2}} + e^{\frac{i\gamma B_x T}{2}} \\ e^{\frac{i\gamma B_x T}{2}} - e^{\frac{-i\gamma B_x T}{2}} \end{pmatrix} = \begin{pmatrix} \cos\left(\frac{\gamma B_x T}{2}\right) \\ \sin\left(\frac{\gamma B_x T}{2}\right) \end{pmatrix}$

So $\langle \Psi(T)|S_z|\Psi(T)\rangle = \left[\cos\left(\frac{\gamma B_x T}{2}\right), \sin\left(\frac{\gamma B_x T}{2}\right)\right]\frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \cos\left(\frac{\gamma B_x T}{2}\right) \\ \sin\left(\frac{\gamma B_x T}{2}\right) \end{pmatrix}$

$\begin{pmatrix} \cos\left(\frac{\gamma B_x T}{2}\right) \\ -\sin\left(\frac{\gamma B_x T}{2}\right) \end{pmatrix}$

$\Rightarrow \boxed{\frac{\hbar}{2}\left[\cos^2\left(\frac{\gamma B_x T}{2}\right) - \sin^2\left(\frac{\gamma B_x T}{2}\right)\right]}$ when B_x is x-comp of $\vec{B}$

(b) $\langle \Psi(T)|S_x|\Psi(T)\rangle$

$= \left[\cos\left(\frac{\gamma B_x T}{2}\right), \sin\left(\frac{\gamma B_x T}{2}\right)\right]\frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \cos\left(\frac{\gamma B_x T}{2}\right) \\ \sin\left(\frac{\gamma B_x T}{2}\right) \end{pmatrix}$

$\begin{pmatrix} \sin\left(\frac{\gamma B_x T}{2}\right) \\ \cos\left(\frac{\gamma B_x T}{2}\right) \end{pmatrix}$

$\Rightarrow \boxed{\hbar\left[\cos\left(\frac{\gamma B_x}{2}T\right)\sin\left(\frac{\gamma B_x}{2}T\right)\right]}$

+6

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{\hat{L}^2}{\hbar^2 r^2}$$

7. A particle of mass m is constrained to move between two concentric impermeable spheres of radii $r = a$ and $r = b$. There is no other potential. Find the ground state energy and normalized wave function. (16 points)

$$V(r) = \begin{cases} 0 & a<r<b \\ \infty & \text{otherwise} \end{cases}$$

In between 2 spheres:

$V(r)=0 \Rightarrow$ free particle: $\frac{-\hbar^2}{2m}\nabla^2\Psi(r) = E\Psi(r)$, $E<0$. bound state

$$\frac{-\hbar^2}{2m}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{L^2}{\hbar^2 r^2}\right]\Psi(r) = E\Psi(r)$$

we know Radial equation for free particle: $\frac{-\hbar^2}{2m}\left(\frac{d^2}{dr^2} + \frac{l(l+1)}{r^2}\right)u = Eu$, when $u = r R(r)$

4-6

For $l=0 \Rightarrow \frac{d^2u}{dr^2} = k^2u \Rightarrow u = Ae^{kr} + Be^{-kr}$, $k = \frac{\sqrt{-2mE}}{\hbar}$

$$\Rightarrow R(r) = \frac{A}{r}e^{kr} + \frac{B}{r}e^{-kr}$$

$kr = \frac{1}{\hbar}n\pi$

Impose Boundary condition:

① $R(r=a) = R(r=b) = 0 \Rightarrow \frac{A}{a}e^{ka} + \frac{B}{a}e^{-ka} = 0$

so $Ae^{ka} + Be^{-ka} = Ae^{kb} + Be^{-kb}$

$A(e^{ka} + e^{kb}) = B(e^{-kb} - e^{-ka})$

$2A\sinh(ka) = 2B\cosh(kb)$

$A = \cot(kb)B$

② $R(r=a)=0$

$$\left.\frac{\partial R(r)}{\partial r}\right|_{r=a} = \left.\frac{\partial R(r)}{\partial r}\right|_{r<a} = 0$$

$$\Rightarrow \left[-\frac{A}{r^2}ke^{kr} \pm \frac{B}{r^2}ke^{-kr}\right]_{r=a} \Rightarrow -\frac{A}{a^2}ke^{ka} - \frac{B}{a^2}ke^{-ka} = 0$$

$Ae^{ka} = -Be^{-ka}$

$A = -Be^{-2ka}$

$$\Rightarrow R(r) = -\frac{B}{r}e^{-2ka}e^{kr} + \frac{B}{r}e^{-kr} = \frac{-B}{r}\left(e^{k(r-2a)} + e^{-kr}\right)$$

$$R(r) = \frac{B}{r}\left[e^{-2ka}e^{kr} + e^{-kr}\right]$$

P7 cont.

Normalization $\int |R(r)|^2 d^3r = 1$

$$B^2 \int \frac{1}{r^2}\left(e^{-2ka}e^{kr} + e^{-kr}\right)^2 d^3r$$

$$\Rightarrow B^2 \int_a^b \frac{1}{r^2}\left(e^{-4ka}e^{2kr} + 2e^{-2ka}e^{kr} + e^{-2kr}\right) r^2 dr \int_0^{\pi} \sin\theta \, d\theta \int_0^{2\pi} d\phi$$

$$\Rightarrow 4\pi B^2 \left[\int_a^b e^{-4ka} e^{2kr} dr + 2e^{-2ka} \int_a^b e^{kr} dr + \int_a^b e^{-2kr} dr\right]$$

$$\Rightarrow 4\pi B^2 \left[e^{-4ka} \frac{1}{2k} e^{2k(b-a)} + 2e^{-2ka} e^{k(b-a)} + \frac{1}{2k} e^{-2k(b-a)}\right] = 1$$

$$\Rightarrow B = \sqrt{\frac{1}{4\pi}} \left(\frac{1}{2k} e^{-4ka} e^{2k(b-a)} + 2e^{-2ka} e^{k(b-a)} + \frac{1}{2k} e^{-2k(b-a)}\right)^{-\frac{1}{2}}$$

Algebra

Energy: Since by B.C.

$$k(b-a) = i n\pi \qquad kb = i n\pi$$

$$\Rightarrow k = \frac{i n\pi}{(b-a)}$$

$$\Rightarrow E = \frac{-\hbar^2 k^2}{2m} \Rightarrow \frac{\hbar^2}{2m}\left(\frac{\pi}{b-a}\right)^2 n^2$$

ground state $n=1 \Rightarrow E = \frac{\hbar^2}{2m}\left(\frac{\pi}{b-a}\right)^2$

angular part $Y(\theta,\phi) = A\, P_l^m(\cos\theta)\, e^{im\phi}$

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