Tensors of minimal border rank II

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A problem in geometry

Let $V = \mathbb{C}^N$, let $G \subset GL(V)$ be reductive, and let $[v] \in \mathbb{P}V$.

(GL_k is reductive, finite groups are reductive, group of invertible diagonal matrices is reductive, product of reductive groups is reductive)

Consider the orbit closure $\overline{G \cdot [v]} \subset \mathbb{P}V$.

Question: What is in the boundary $\partial \overline{G \cdot [v]} := \overline{G \cdot [v]} \backslash G \cdot [v]$?

Example: $V = \mathbb{C}^{\mathbf{b}} \otimes \mathbb{C}^{\mathbf{c}} = B \otimes C$ space of $\mathbf{b} \times \mathbf{c}$ matrices,
$G = GL(B) \times GL(C)$, $v = \mathrm{Id}_r$ some $r < \min \mathbf{b}, \mathbf{c}$.

Answer: space of matrices of rank less than r.

Million dollar example

Let $\mathbf{b} = \mathbf{c} = n > m$, let $V = S^n(B \otimes C)$, homogeneous polynomials of deg n on $n \times n$ matrices,

$G = GL_{n^2} = GL(B \otimes C)$,

$v = \det_n$, the determinant.

Let $\mathrm{perm}_m \in S^m(\mathbb{C}^{m^2})$ permanent polynomial, and let $\ell \in S^1(\mathbb{C}^1)$ be a linear form. Linearly include $\mathbb{C}^{m^2+1} \subset \mathbb{C}^{n^2}$.

Question: Is $\ell^{n-m}\mathrm{perm}_m \in \partial \overline{G \cdot [v]}$ when $n = poly(m)$?

Valiant's algebraic variant of P v. NP (Mulmuley-Sohoni variant)

Example of interest to us

Set $A = B = C = \mathbb{C}^m$, and let $V = A \otimes B \otimes C$, let
$G = GL(A) \times GL(B) \times GL(C) \ltimes \mathfrak{S}_3$,

and let $v = M_{\langle 1 \rangle}^{\oplus m} = \sum_{j=1}^m a_j \otimes b_j \otimes c_j$,

Then $\overline{G \cdot [v]} = \sigma_m$ tensors of border rank at most m.

Last time: saw motivation from the complexity of matrix multiplication. We were looking for good tensors for the laser method to prove new upper bounds on ω, the exponent of matrix multiplication.

Above: fits into context of classical algebraic geometry. Below: additional motivation from quantum information theory.

Comment on our case

Classical theorem (Matsushima) If subgroup of G stabilizing $[v]$ is also reductive then $\partial\overline{G\cdot[v]}$ is of pure codimension one in $\overline{G\cdot[v]}$

Applies to our case as group preserving $M_{\langle 1\rangle}^{\oplus m}$ is isomorphic to product of diagonal matrices in two spaces

In general, may have many components, each is a union of a family of orbit closures. How many components? What is the geometry of a general element of a component? When is a component defined by a single orbit closure?

Quantum information theory

Classical information: “bits” as resource.

Quantum: “qubits” $\rightsquigarrow$ tensors

not all tensors are equivalent as resources. Two aspects: cost (e.g., to build in lab) and value. (e.g., how much classical information can it store?)

Want low cost high value.

$\rightsquigarrow$

asymptotic rank $\underset{\sim}{\mathbf{R}}(T)$ and asymptotic subrank $\underset{\sim}{\mathbf{Q}}(T)$.

Back to classification problem

$G \cdot [M_{\langle 1 \rangle}^{\oplus m}] \sim$ tensors isomorphic to $M_{\langle 1 \rangle}^{\oplus m}$

So question becomes what is $\partial \sigma_m$?

Irrelevant part: nonconcise locus.

Recall $T \in A \otimes B \otimes C$ is *concise* if the maps $T_A : A^* \to B \otimes C$, $T_B : B^* \to A \otimes C$, $T_C : C^* \to A \otimes B$ are injective. (if not concise should study tensors in $\mathbb{C}^{m-1} \otimes \mathbb{C}^m \otimes \mathbb{C}^m$)

Let $\partial \sigma_m^{con} := \overline{\partial \sigma_m \cap \{concise\}}$

Classification of $\partial\sigma_m^{con}$

$m = 1$: empty

$m = 2$: up to isomorphism, single tensor
$W = a_1 \otimes b_1 \otimes c_2 + a_1 \otimes b_2 \otimes c_1 + a_2 \otimes b_1 \otimes c_1$: general tangent vector to Segre

$m = 3$: up to isomorphism, three points:

$M_{\langle 1 \rangle} \oplus W$: point plus tangent vector to Segre

$a_1 \otimes b_2 \otimes c_2 + a_2 \otimes b_1 \otimes c_2 + a_2 \otimes b_2 \otimes c_1 + a_1 \otimes b_1 \otimes c_3 + a_1 \otimes b_3 \otimes c_1 + a_3 \otimes b_1 \otimes c_1$

and

$a_2 \otimes b_1 \otimes c_2 + a_2 \otimes b_2 \otimes c_1 + a_1 \otimes b_1 \otimes c_3 + a_1 \otimes b_3 \otimes c_1 + a_3 \otimes b_1 \otimes c_1$

Exercise: Second is sum $x' + x''$ of tangent vector and second derivative of a curve at a point. Third is sum of two tangent vectors at two distinct points that lie on a line on Segre

$m = 4$ there are 10 tensors up to isomorphism $m = 5$ there are 36 tensors up to isomorphism (Jagiella-Jelisiejew 2024)

JLP Classification $m \leq 5$

Two consequences

- $m \leq 5$: $\partial\sigma_m^{con}$ consists of a single component that is an orbit closure, namely

  $$\overline{G \cdot (M_{\langle 1 \rangle}^{\oplus m-2} \oplus W)}$$

  where $W = a_1 \otimes b_1 \otimes c_2 + a_1 \otimes b_2 \otimes c_1 + a_2 \otimes b_1 \otimes c_1$ is general tangent vector of the Segre. (a component of $\partial\sigma_m$ for all m).
- normal forms, i.e., no moduli

More modest classification goal: classify components of $\partial\sigma_m^{con}$

Q: Do $m \leq 5$ results persist? Answer: NO!

Thm. (GJLM 2025) There exists a component of $\partial\sigma_6$ that is *not* the closure of a single orbit, so in particular $\partial\sigma_6^{con}$ has moduli.

It is described as follows: Let $Z \subset Mat_{3\times 3}$ be any 5-dimensional subspace and let $T \in A \otimes B \otimes C$ be such that

$$T(A^*) = \langle \mathrm{Id}_6, \begin{pmatrix} 0 & Z \\ 0 & 0 \end{pmatrix} \rangle.$$

The blocking for the matrix is $(3,3) \times (3,3)$. The component is the G orbit closure of such tensors.

One needs to prove the set has minimal border rank and fills out a codimension one subvariety of σ_6.

Debt: To explain: Where did this come from?

Classification Problem $\rightsquigarrow$ 4 problems

$T \in A \otimes B \otimes C = \mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$

Def T is 1_A-*generic* if $\exists \alpha \in A^*$ with $T(\alpha) \in B \otimes C$ of rank m.

1_B, 1_C generic defined similarly.

- T is 1-*generic* if it is $1_A, 1_B, 1_C$ generic
- T is *binding* if it is at least two of $1_A, 1_B, 1_C$ generic (strictly binding if exactly two)
- T is 1_*-*generic* if it is at least one of $1_A, 1_B, 1_C$ generic (strictly 1_*-generic if exactly one)
- T is 1-*degenerate* if it is not 1_*-generic, i.e., $T(A^*) \subset \{\det_m = 0\} \subset B \otimes C$ and similarly for $T(B^*)$, $T(C^*)$.

The 4 problems

Classify minimal border rank tensors that are:

- 1-generic

- strictly binding

- strictly 1_*-generic

- 1-degenerate,

Remark: The component $\overline{G\cdot(M_{\langle 1\rangle}^{\oplus m}\oplus W)}\subset\partial\sigma_m$ (for all $m\geq 2$) is such that a general element is 1-generic.

Remark: The $m=6$ component above is such that a general element is strictly 1_*-generic. (surprise?)

New tensors for the laser method?

The $m = 6$ tensors above do not appear to be good for the laser method.

The big Coppersmith-Winograd tensor CW_q is 1-generic.

Thm (Conner-Gesmundo-L-Ventura) It is the *worst* 1-generic minimal border rank tensor for the laser method.

proof: Thm. (CGLV,Hoyois-J-Nardin-Yakerson) All 1-generic (m, m, m)-tensors degenerate to CW_{m-2}, in particular all 1-generic minimal border rank tensors degenerate to CW_{m-2}.

Idea: If find new 1-generic tensors "closer" to $M_{\langle 1 \rangle}^{\oplus m}$, they might have higher $\underset{\sim}{\mathbf{Q}}$.

But: unit tensor itself is useless for the laser method. Below I describe a class of minimal border rank tensors that is promising for the laser method that includes $T_{better,6}$ from last lecture.

Binding tensors in general

Consider $T \in A \otimes B \otimes C$ as a bilinear map $T : A^* \times B^* \to C$.

E.g., $a \otimes b \otimes c(\alpha, \beta) = \alpha(a)\beta(b)c \in C$.

If T is $1_A, 1_B$-generic, have isomorphisms $T(\alpha) : B^* \to C$, $T(\beta) : A^* \to C$.

(Bläser-Lysikov) Apply these to T to get a bilinear map $T' : C \times C \to C$ isomorphic to $T : A^* \times B^* \to C$.

This gives C the structure of an *algebra*: a vector space with a multiplication (not in general abelian or even associative).

Binding tensors satisfying Strassen's equations

Recall Strassen's equations: T: 1_A-generic and minimal border rank implies

$\mathcal{E}_\alpha := T(A^*)T(\alpha)^{-1} \subset \mathrm{End}(C)$ is abelian.

$\rightsquigarrow$ algebra structure defined by T' is abelian.

Let $S = \mathbb{C}[y_1, \cdots, y_{m-1}]$ and let $I \subset S$ be an ideal such that S/I is finite dimensional of dimension m.

Algebra induced by T' is of the form S/I.

Binding tensors satisfying Strassen's equations (cont'd)

$T \rightsquigarrow$ algebra S/I

T has minimal border rank if and only if S/I is *smoothable*, i.e., a limit of algebras that are direct sums of m copies of the trivial one-dimensional algebra.

Subtlety: limit in Hilbert scheme. I.e., S/I lies in the smoothable component of the Hilbert scheme.

(Cartwright-Erman-Velasco-Viray 2009) All such algebras are smoothable $m \leq 7$.

$\rightsquigarrow$ characterization of minimal border rank binding tensors $m \leq 7$.

A strictly binding component of $\partial\sigma_m$

Thm. (GJLM) There is a component of $\partial\sigma_{10}$ consisting of the closure of tensors of the form $T_{\mathcal{A}}$ with $\mathcal{A} = Sym(V^*)/\mathcal{I}$ where $\dim V = 5$ and $\mathcal{I}$ is generated in degree two by 11 quadratic polynomials. A general element of this component is strictly binding.

Remark: This is the smallest strictly binding component that we are aware of.

1_*-generic tensors satisfying Strassen's equations

Recall Strassen's equations: $\mathcal{E}_\alpha := T(A^*)T(\alpha)^{-1} \subset \mathrm{End}(C)$ is abelian.

Let $\alpha, \alpha_1, \cdots, \alpha_{m-1}$ be a basis of A^*, let $S = \mathbb{C}[y_1, \cdots, y_{m-1}]$, and let S act on $c \in C$ by $y_j(c) = T(\alpha_j)T(\alpha)^{-1}(c)$. Extends to an action of S because $\mathcal{E}_\alpha$ is abelian.

Gives C structure of an S-module.

T has minimal border rank iff C lies in the smoothable component of the corresponding Quot scheme.

(Note if binding, module is isomorphic to S/I and is an algebra.)

1-generic tensors satisfying Strassen's equations

By binding, have algebra structure, what do we get in addition?

1-generic $\rightsquigarrow$ Gorenstein algebra.

(Casnati-Jelisiejew-Notari 2015) All Gorenstein algebras are smoothable $m \leq 13$.

$\rightsquigarrow$ characterization of minimal border rank 1-generic tensors $m \leq 13$.

What is a Gorenstein algebra?

Finite dimensional Gorenstein algebras

Let $p \in S^{\leq d}V$ be concise and have degree d. Define the *annihilator of p*,

$$\mathrm{Ann}\,(p) = \{\phi \in Sym(V^*) \mid \phi \lrcorner p = 0\}$$

where the action of $Sym(V^*)$ on $S^{\leq d}V$ is differentiation.

In particular $\mathrm{Ann}\,(p) \supset S^\delta V^*$ for all $\delta > d$ and in degree d the annihilator $\mathrm{Ann}\,(p)_d \subset S^d V^*$ is a hyperplane.

Define the *apolar algebra* of p to be

$$\mathcal{A}_p := Sym(V^*)/\mathrm{Ann}\,(p).$$

For those familiar with the terminology, apolar algebras are the *local Artinian Gorenstein algebras*.

All finite dimensional Gorenstein algebras are direct sums of apolar algebras.

Example

Let V have basis $x_1, \cdots, x_q$ and let $\{y_j\}$ denote the dual basis.

Let $p = \sum_{j=1}^q x_j^2$ and let $\Gamma \in S^2 V^*$ be such that $\Gamma \lrcorner p = 1$.

Then

$$
\begin{aligned}
(\mathcal{A}_p)_0 &= [1],\\
(\mathcal{A}_p)_1 &= \langle [y_1], \cdots, [y_q] \rangle,\\
(\mathcal{A}_p)_2 &= [\Gamma],
\end{aligned}
$$

The nontrivial multiplication is $[y_i][y_j] = \delta_{ij}[\Gamma]$. Setting $c_0 = [\Gamma]$, $c_j = [y_j]$, $c_{q+1} = [1]$, and $a_0 = b_0 = [1]^*$, $a_j = b_j = [y_j]^*$, $a_{q+1} = b_{q+1} = [\Gamma]^*$, one has

$$
\begin{aligned}
&T_{\mathcal{A}_p} =\\
&a_0 \otimes b_0 \otimes c_{q+1} + a_0 \otimes b_{q+1} \otimes c_0 + a_{q+1} \otimes b_0 \otimes c_0\\
&+ \sum_j a_j \otimes b_j \otimes c_0 + a_j \otimes b_0 \otimes c_j + a_0 \otimes b_j \otimes c_j
\end{aligned}
$$

$= CW_q$ the *big Coppersmith-Winograd tensor*.

Structure tensors of apolar algebras

CW_q is the structure tensor of the apolar algebra of a smooth quadric in q variables.

$T_{better,6}$ is the structure tensor of the apolar algebra to xyz.

Thm. (GJLM 2025) There is a component of $\partial\sigma_{16}$ consisting of the closure of the set of structure tensors of apolar algebras to cubic polynomials in 7 variables. In particular, a general element of the component is 1-generic.

This is the smallest m that we are aware of where $\partial\sigma_m$ has a component of whose general element is the structure tensor of an apolar algebra.

1-degenerate case

Previously: Friedland - if corank one, can salvage something from Strassen's equations $\rightsquigarrow$ $m \leq 4$ no 1-degenerate minimal border rank tensors.

No further progress due to lack of structure

JLP: if T has minimal border rank, then

its 111-algebra must be dimension at least m.

it must be smoothable

and its modules A, B, C must be in the principal component of the quot scheme.

Even if all this, not known if sufficient.

Considerably more difficult.

JLP classified $m = 5$ case $\rightsquigarrow$ classification of all minimal border rank tensors $m \leq 5$.

Question

Does there exist a component of $\partial\sigma_m$ where the general element of the component is 1-degenerate?

Thank you for your attention

For more on **tensors**, their geometry and applications, resp. **geometry and complexity**, resp. **asymptotic geometry**, resp. **quantum computation and information**: