Probabilistic Optimality Guarantees in a Modified A* Algorithm

Aaryan Koulavkar

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Abstract

In this document, we analyze a probabilistic variant of the A* algorithm. At each expansion step, the algorithm chooses either the node with the minimum f-value (greedy choice) with probability α, or picks a node uniformly at random from the open list with probability $1 - \alpha$. We derive the probability of selecting the optimal node at each step and illustrate the theory with an example on the 8-puzzle problem using the Manhattan distance heuristic.

1 Introduction

Traditional A* search guarantees optimality when an admissible and consistent heuristic is employed, since it always expands the node that appears most promising (i.e., with the least f-value, where $f = g + h$). In some scenarios, introducing randomness into the decision process might help in overcoming local minima or diversifying the search. In our probabilistic A* variant, at each expansion step, the node is chosen according to the following rule:

$$\text{With probability } \alpha, \text{ choose the node with the minimum } f\text{-value;} \\ \text{with probability } 1-\alpha, \text{ choose a node uniformly at random.}$$

In this document, we formalize this strategy and provide detailed analysis, including a concrete example with the 8-puzzle.

2 Problem Statement

Consider a search problem solved by A* where the open list (or frontier) contains a set of nodes, and each node n has an associated cost

$$f(n) = g(n) + h(n).$$

We assume the heuristic h is admissible and consistent. In our probabilistic variant, at each expansion step:

1. With probability α, the algorithm expands the node n^* with the minimum f-value.
2. With probability $1 - \alpha$, the algorithm picks a node uniformly at random from the open list. If the open list has N nodes, each node is chosen with probability $\frac{1}{N}$.

Assuming that the optimal path requires following a sequence of n critical expansions (i.e., the nodes along the optimal path must be expanded in order), we wish to determine the probability that *at a given step* the optimal node is chosen and then extend this analysis to the entire optimal path.

3 Theoretical Analysis

3.1 Per-Step Probability

Let N denote the number of nodes in the open list at a certain expansion step, and assume that the optimal node is present in the open list and is indeed the one with the smallest f-value. Then the probability P_{opt} that the optimal node is chosen at that step is given by:

$$P_{\text{opt}} = \underbrace{\alpha}_{\text{greedy selection}} + \underbrace{(1-\alpha)\frac{1}{N}}_{\text{random selection}},$$

which combines the two mutually exclusive cases.

3.2 Overall Optimality Probability

If the optimal solution requires n critical expansions (one per node on the optimal path), and if we denote by N_i the number of nodes in the open list at the i-th expansion where the optimal node is present, the probability $P_{\text{optimal path}}$ that the entire optimal path is followed is:

$$P_{\text{optimal path}} = \prod_{i=1}^{n} \left(\alpha + \frac{1-\alpha}{N_i}\right).$$

In the limiting case where N_i is large for all i, the contribution of the random part becomes small, and the overall probability approximates:

$$P_{\text{optimal path}} \approx \alpha^n.$$

4 Concrete Example: 8-Puzzle with Manhattan Distance

4.1 Problem Setup

Consider the classic 8-puzzle where the **goal state** is:

$$\begin{matrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & \_ \end{matrix}$$

Assume an **initial state**:

$$\begin{matrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & \_ & 8 \end{matrix}$$

where the blank (denoted by $\_$) is swapped with tile 8.

Using the Manhattan distance heuristic, the Manhattan distance for tile 8 is:

$$h(8) = |2-2| + |2-1| = 1.$$

Assuming that the cost so far is $g = 1$, we have:

$$f = g + h = 1 + 1 = 2.$$

4.2 Successor Generation

From the initial state, assume the blank is located at position $(2, 1)$. Three successor moves are possible:

1. **Move Blank Right:** Resulting state (Goal state):

$$\begin{matrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & \_ \end{matrix}$$

   Here, assume $g = 2$ and $h = 0$, so $f = 2$.
2. **Move Blank Left:** Resulting state:

$$\begin{matrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ \_ & 7 & 8 \end{matrix}$$

   For this state, suppose:
   - Tile 7, with goal position $(2, 0)$, is now at $(2, 1)$, so its Manhattan distance is 1.
   - Tile 8, with goal position $(2, 1)$, remains at $(2, 2)$, so its Manhattan distance is also 1.

   Then, $h = 1 + 1 = 2$ and with $g = 2$, $f = 2 + 2 = 4$.
3. **Move Blank Up:** Resulting state:

$$\begin{matrix} 1 & 2 & 3 \\ 4 & \_ & 6 \\ 7 & 5 & 8 \end{matrix}$$

   Here,
   - Tile 5 (goal position $(1, 1)$) is at $(2, 1)$ giving a Manhattan distance of 1.
   - Tile 8 (goal position $(2, 1)$) remains at $(2, 2)$ with a Manhattan distance of 1.

   Thus, $h = 1 + 1 = 2$ and with $g = 2$, $f = 4$.

4.3 Open List and Node Selection

After generating the successors, the open list contains three nodes with the following evaluation:

- **Node R (Right move):** $f = 2$ (Optimal, as it reaches the goal).
- **Node L (Left move):** $f = 4$.
- **Node U (Up move):** $f = 4$.

The optimal node is clearly Node R. The size of the open list is $N = 3$.

Using the probabilistic rule at this expansion step, the probability P_{opt} of picking the optimal node (Node R) is:

$$P_{\text{opt}} = \alpha + (1 - \alpha)\frac{1}{3}.$$

4.4 Numerical Example

Suppose we set $\alpha = 0.8$. Then:

- With probability 0.8, the algorithm picks the best node (Node R).
- With probability 0.2, a random node is selected. Since there are 3 nodes, the probability of selecting Node R randomly is $\frac{1}{3} \approx 0.333$.

Thus, the combined probability is:

$$P_{\text{opt}} = 0.8 + 0.2 \times \frac{1}{3} \approx 0.8 + 0.0667 \approx 0.8667.$$

There is therefore an approximately 86.67% chance that the optimal node will be expanded at this step.

5 Conclusion

We have described a probabilistic variant of the A* algorithm, where at each expansion the node selection is governed by:

$$P(\text{optimal node}) = \alpha + \frac{1-\alpha}{N},$$

with α controlling the degree of greediness and N the number of nodes in the open list. In the 8-puzzle example with the Manhattan distance heuristic, we demonstrated this by considering three possible successors. With a chosen $\alpha = 0.8$ and an open list of 3 nodes, the probability of selecting the optimal node is approximately 86.67%. This analysis provides a theoretical basis for understanding how such a modification to the A* algorithm can probabilistically guarantee the optimality of the found solution, at least in a per-step analysis.