# Introduction to Cut Finite Element Exterior Calculus

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Technical note
Stockholm, Sweden 2024

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# Contents

# Preface

*With four parameters I can fit an elephant, with five I can make him wiggle his trunk.*

John von Neumann

This text is an introduction to Cut Finite Element Methods (CutFEM) via the language of exterior calculus and differential forms. It is adapted from the PhD thesis of the author.

# Chapter 1

# Introduction

Partial differential equations (PDEs for short) are effective models for a diverse range of physical phenomena. To find an exact solution to a PDE is however typically a very difficult task, and in some cases not even possible. A numerical method is the umbrella term used to describe an algorithm which finds an approximate solution to a PDE, via the help of a computer. The finite element method (FEM), devised in the early 1940s is one such numerical method.

The finite element method is particularly well liked by mathematicians given its strong connection to the mathematical theory of PDE, as it is easily formulated in the language of functional analysis. The FEM is also popular among engineers given its versatility and robustness; it can be used to solve a wide range of problems. Indeed the FEM is today so popular that it too is a kind of umbrella term used to describe a wide spectrum of numerical methods, all of which share the same basic idea of dividing the domain of the PDE into smaller pieces, called elements, and then solving the PDE on each element separately. The solutions on the elements are then glued together to form a solution to the PDE on the entire domain.

Among the myriad variations of the FEM, cut finite element methods (CutFEM) has emerged as a useful alternative in certain situations, allowing for flexibility in handling complex geometries and dynamic interfaces undergoing changes in topology. This thesis presents a comprehensive investigation into the development and application of CutFEM on elliptic PDEs that are specifically designed to preserve a divergence condition present in the PDEs.

The emphasis on divergence preservation is motivated by the crucial role that this property plays in many physical phenomena, including fluid dynamics [GR86], electromagnetism [Bos98], and elasticity [Hug00]. Traditional CutFEM (and even many FEM) approaches often struggle to maintain this key feature, potentially leading to numerical errors and unphysical results. The research presented herein addresses this challenge by developing and analyzing novel CutFEM that inherently conserve divergence properties of the underlying physical fields.

Though CutFEM is perhaps most motivated by time-dependent PDEs in moving domains, in this thesis we have had to go back to basics to time-invariant elliptic PDEs in order to focus on key aspects of saddle-point problems, pertaining to what is essentially functional analysis. The books of Brezzi et. al. [BF12, BBF$^+$13] have been useful throughout the writing of this work. None the less, the goal going forward is to from this point build upon the foundations we have constructed, and look to divergence preserving CutFEM discretizations of time-dependent hyperbolic and parabolic PDEs, like for instance the Navier-Stokes equations.

A numerical method like FEM transforms or discretizes a given PDE into a finite dimensional linear algebra system $Ax = b$ with unknown $x$ and data $b$. A prevalent difficulty in the design of CutFEM is the control of the condition number of the matrix $A$, a quantity the magnitude of which represents how hard it is to invert the matrix $A$ to get the solution $x = A^{-1}b$. In CutFEM the condition number can get arbitrarily big depending on how the geometry of the problem, decoupled as it is to the meshing procedure, cuts through the mesh. To control the condition number of a CutFEM discretization while at the same time preserving a divergence condition is a challenging task, and the main focus of this thesis.

# Chapter 2

# Partial differential equations

A PDE can be described abstractly by an operator $D$ which maps the unknown function $u$ to a known function $f$ which we name the data. That is,

$$Du = f. \tag{2.1}$$

The task of solving the PDE amounts to finding $u$. Since the $u$ and $f$ are functions, one has to specify a domain $\Omega$ on which they are defined. The domain $\Omega$ is typically a subset of $\mathbb{R}^n$ for some $n \in \mathbb{N}$, and specifies through its boundary the geometry of the problem.

At this point let us restrict to linear operators $D$, since we (thankfully) don't consider nonlinear operators in this thesis. A linear operator satisfies the following two properties

$$D(u + v) = Du + Dv, \tag{2.2}$$

$$D(\beta u) = \beta Du, \tag{2.3}$$

for any functions $u, v$ and any scalar $\beta$. Let us also consider second order operators $D$ which contain for any one term at most two derivatives of $u$. Any such $D$ can be written as

$$Du = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i} + cu. \tag{2.4}$$

The higher order derivative terms determine the type of PDE which is usually classified as elliptic, parabolic, or hyperbolic. This classification is done as follows. Let $A = (a_{ij})_{i,j=1}^n$ be the matrix of coefficients of the second order derivative terms. Then the PDE is

- elliptic if $A$ is positive (or negative) definite,
- parabolic if exactly one eigenvalue of $A$ is 0 and all others have the same sign,
- hyperbolic if exactly one eigenvalue of $A$ has the opposite sign of the others, and all are nonzero.

The names elliptic, parabolic, and hyperbolic are inspired by the two-dimensional case

$$a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial^2 u}{\partial x \partial y} + c\frac{\partial^2 u}{\partial y^2} + (\text{lower order terms}) = 0. \tag{2.5}$$

Investigating the corresponding polynomial equation

$$ap^2 + bpq + cq^2 = 0, \tag{2.6}$$

the classification given above is seen to be analogous to the classification of conic sections to (2.6) by way of the discriminant $b^2 - 4ac$. Namely, the PDE is elliptic if $b^2 - 4ac < 0$, parabolic if $b^2 - 4ac = 0$, and hyperbolic if $b^2 - 4ac > 0$.

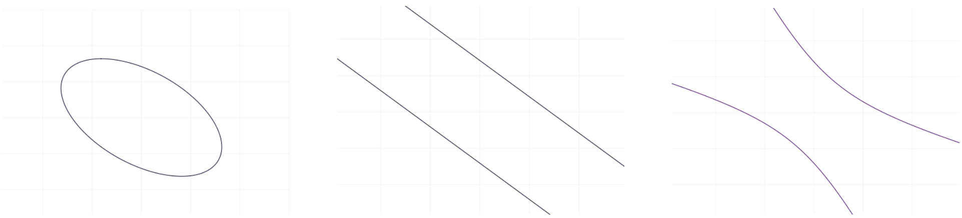

Figure 2.1: Levelsets of (2.6), elliptic, parabolic, hyperbolic.

Let us give three examples, which have each played a role in this thesis. The Poisson equation is elliptic, given by

$$-\Delta u = f, \tag{2.7}$$

where $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator. The solution $u$ is a scalar field which can represent a temperature, displacement, or pressure, depending on the context.

The Stokes equations are also elliptic, and for piecewise constant viscosity $\nu$ given by

$$-\nu\Delta u + \nabla p = f, \tag{2.8}$$

$$\nabla \cdot u = 0, \tag{2.9}$$

where $\nabla = (\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_d})$ is the gradient operator and $\cdot$ is the scalar product. The solution $u$ is a vector field representing a fluid velocity, and $p$ is a scalar field representing the pressure. Inserting (2.9) into the identity $-\Delta u = \nabla \wedge \nabla \wedge u - \nabla(\nabla \cdot u)$, where $\wedge$ is the cross product, the Stokes equations can be written as

$$\nu\nabla \wedge \nabla \wedge u + \nabla p = f, \tag{2.10}$$

$$\nabla \cdot u = 0. \tag{2.11}$$

The $e$-formulation of Maxwell's equations in vacuum is hyperbolic. Indeed, combined they take the form of a wave equation, a classic example of a linear hyperbolic equation. The equations are given by

$$\nabla \wedge \nabla \wedge u + \mu\varepsilon\frac{\partial^2 u}{\partial t^2} = f, \tag{2.12}$$

$$\nabla \cdot (\varepsilon\frac{\partial u}{\partial t}) = -\nabla \cdot f, \tag{2.13}$$

where $\mu, \varepsilon$ are permeability and permittivity, $f$ is the current density, and $u$ is a vector field representing the electric field. Note that Stokes equations and the $e$-formulation of Maxwell's equations are both examples of systems of PDEs, each consisting of two equations. For these systems one has to be more careful in applying the classification given above.

The classification into the three different types is not arbitrary, within each type the behaviour is similar. Roughly speaking, elliptic equations are typically stationary with no time-dependence, parabolic equations are typically diffusive with information (from the data) travelling at infinite speed, and hyperbolic equations propagate discontinuities forward with finite speed. When given a PDE, finding out its type can be instrumental in guiding the design of a numerical scheme.

Lastly, we must mention boundary conditions. A PDE is undetermined without information about the behaviour of its sought solution at the boundary $\partial\Omega$. As such a PDE must be supplemented with boundary conditions in order to make a satisfactory model. A given PDE can be supplemented with different types of boundary conditions, each choice given rise to a fully determined system. As has been done many times before, we will use the Poisson equation to illustrate the concepts.

First consider the Poisson problem with so called Dirichlet boundary conditions. Find $u$ such that

$$-\Delta u = f \text{ in } \Omega \tag{2.14a}$$

$$u = g \text{ on } \partial\Omega \tag{2.14b}$$

We can also consider the Poisson problem with Neumann boundary conditions. Let $\mathfrak{n}$ be the unit normal vector pointing outward from $\partial\Omega$. Find $u$ (with zero average[1]) such that

$$-\Delta u = f \text{ in } \Omega \tag{2.15a}$$

$$\nabla u \cdot \mathfrak{n} = g \text{ on } \partial\Omega \tag{2.15b}$$

This second boundary condition may seem more arbitrary than the first, what use is the knowledge of the normal gradient of $u$ on $\partial\Omega$? On the contrary, this boundary condition is more natural from the point of view of the PDE than the first. Indeed, the types of boundary conditions are classified according to the role they play in the weak formulation of the PDE. We make this explicit in Section 2.4.

## 2.1 Some motivation

PDE theory can be very confusing to navigate. There are many equations and the methods used to investigate one might not prove useful for another. This is often true even within the class of elliptic equations, for instance. The mathematician George Polya once quipped “In order to solve this differential equation you look at it until a solution occurs to you.” In extension then, numerical methods are equally varied and diversed, if not more so. For the Poisson equation alone there are maybe thousands of numerical methods which work comparably similarly.

A question immediately arises given a PDE, namely “Which method is the best to use?” Of course, it is not easy to answer. This is because ‘best’ is not

[1] More on this later!

specific enough; it can mean best in terms of computation time, convergence rates, or discrete preservation of the physical laws, or any other of a number of different aspects. That said, I have found the following design principle consistently useful: "Under the restrictions inherent in your discrete setting, follow the mathematical framework describing the physics in the PDE as much as possible."

In practice I have found that this principle leads to methods which work well apriori, that is from the start. From this point of view one can argue that doing it right from the start alleviates problems later on. Someone with another perspective would likely deem such a statement presumptuous, but that is okay.

PDE theory often starts with the theory of Sobolev spaces. We consider many Sobolev spaces in this thesis, so it is useful to consider a theory that unify their treatment. It is also worth doing since the last paper uses this language. I am talking about the theory of differential forms, and an excursion to this platonic realm is well worth the time even beyond the mental real estate it frees up regarding Sobolev spaces[2].

From this point on the material will be more advanced. It takes some time to get to the actual numerical methods. Since we are still not at liberty to produce an entirely self-contained introduction here, we refer the reader to the excellent introduction in [Arn18]. We never the less follow Arnold's guidelines which he describes: "From a practical point of view, even if we are only interested in solving boundary value problems in $R^n$, we will need differential forms not only on the domain where the differential equation is defined but also on its boundary—which is a manifold, but not a domain in Euclidean space."

## 2.2 Sobolev spaces

Let $\Omega \subset \mathbb{R}^n$ be a smooth manifold, for us $n = 2, 3$ usually. Given any $p \in \Omega$, let $V = T_p\Omega \approx \mathbb{R}^n$ be the tangent space to $\Omega$ at $p$. The exterior algebra[3] corresponding to $V$ is a graded associative algebra

$$\Lambda V = V \oplus \Lambda^1 V \oplus \Lambda^2 V \oplus \ldots \Lambda^n V = \oplus_{k=0}^{n} \Lambda^k V, \tag{2.16}$$

[2] I believe future courses in mathematical finite element theory should be taught in this language.

[3] It is also called Grassmann algebra after Hermann Grassmann. It can also be seen as the Clifford algebra of $V$ with quadratic form $Q = 0$.

where the product, called the wedge or exterior product, is alternating:

$$v \wedge v = 0, \quad \forall v \in V. \tag{2.17}$$

Since the exterior product is also associative and distributive, it is anti-symmetric:

$$v \wedge w = -w \wedge v, \quad \forall v, w \in V. \tag{2.18}$$

The dimension of $\Lambda V$ is $2^n$.

Each summand $\Lambda^k V$ is itself a vector space with dimension $\binom{n}{k}$, and elements called $k-$multivectors consisting of sums of so called $k-$blades $v_1 \wedge v_2 \wedge \cdots \wedge v_k$.

It is an exercise to show that the dual of the vector space of $k-$vectors is the $k$th exterior power of the dual of $V$, that is

$$(\Lambda^k V)^* = \Lambda^k V^*. \tag{2.19}$$

Recalling that $V = T_p\Omega$, $V^* = T_p^*\Omega$ is the cotangent space to $\Omega$ at $p$, an element $\alpha : V \to \mathbb{R}$ of which is a functional which assigns to each $v \in V$ a real number $\alpha(v) \in \mathbb{R}$ linearly. The exterior algebra of $k-$forms on $\Omega$ is then defined as

$$\Lambda V^* = \oplus_{k=0}^n \Lambda^k V^*, \tag{2.20}$$

with the "same" wedge product $\wedge$ now acting on forms but otherwise the same.

A $k-$form is a local object defined only around $p$. A differential $k-$form is a section of the $k$th exterior power of the cotangent bundle

$$\Lambda^k T^*\Omega = \bigcup_{p\in\Omega} \{p\} \times \Lambda^k T_p^*\Omega. \tag{2.21}$$

Being a section means that a differential $k-$form $\alpha$ is a continuous map $\alpha : \Omega \to \Lambda^k T^*\Omega$ such that $\alpha(p) \in \Lambda^k T_p^*\Omega \approx \Lambda^k \mathbb{R}^n$ for all $p \in \Omega$. As such it is a global object. The set of all differential $k$-forms make up another vector space which we will denote by

$$A^k\Omega = \{\text{differential } k-\text{forms on } \Omega\} = A^k, \tag{2.22}$$

the last equality being a shorthand. The exterior algebra corresponding to $A^k\Omega$ is simply

$$A\Omega = \oplus_{k=0}^n \Lambda^k A^k \Omega, \tag{2.23}$$

and we define the exterior product of a differential $k-$form $\alpha$ and a differential $l-$form $\beta$ by

$$(\alpha \wedge \beta)(p) = \alpha(p) \wedge \beta(p), \quad \forall p \in \Omega. \tag{2.24}$$

This is a differential $k + l-$form. The exterior product inherits the local anti-symmetry, and is also associative and distributive.

With respect to the dual basis $\{dx^1, \dots, dx^n\}$ of $\mathbb{R}^n$, satisfying $dx^i(e_j) = \delta_{ij}$, a basis vector of $\Lambda^k\mathbb{R}^n$ is given by

$$dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k}, \quad 1 \leq i_1 < i_2 < \cdots < i_k \leq n. \tag{2.25}$$

We define the exterior derivative of a simple differential $k-$form $\alpha = f dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k}$ by

$$d\alpha = \sum_{j=1}^n \frac{\partial f}{\partial x_j} dx^j \wedge dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k}, \tag{2.26}$$

with the general case given by extending linearly. It can be seen that the exterior derivative is a linear map

$$d : A^k\Omega \to A^{k+1}\Omega. \tag{2.27}$$

It also satisfies a Leibniz rule; for any $\alpha \in A^k\Omega, \beta \in A^l\Omega$ we have

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta. \tag{2.28}$$

Since differential $k-$forms eat $k-$vector fields and spit out real numbers, it makes sense that there should be a way to integrate them over $\Omega$. For details see for instance [Lee12]. Consider

$$(\alpha, \beta)_\Omega = \int_\Omega \alpha \wedge \star\beta, \tag{2.29}$$

$$\|\alpha\|_\Omega^2 = \int_\Omega \alpha \wedge \star\alpha \; dx^1 \wedge \cdots \wedge dx^n, \tag{2.30}$$

where $\star : \Lambda^k\mathbb{R}^n \to \Lambda^{n-k}\mathbb{R}^n$ is the Hodge star operator (defined through the Riesz representation theorem applied in $\Lambda^{n-k}\mathbb{R}^n$.) Then we can define[4]

$$L^2A^k\Omega = \{\alpha \in A^k\Omega : \|\alpha\|_\Omega^2 < \infty\}, \tag{2.31}$$

$$HA^k\Omega = \{\alpha \in L^2A^k\Omega : \|d\alpha\|_\Omega^2 < \infty\}, \tag{2.32}$$

$$= \{\alpha \in L^2A^k\Omega : d\alpha \in L^2A^{k+1}\Omega\}. \tag{2.33}$$

Inner product and norm for the latter space are defined as,

$$(\alpha,\beta)_{H\Omega} = (\alpha,\beta)_\Omega + (d\alpha, d\beta)_\Omega, \tag{2.34}$$

$$\|\alpha\|_{H\Omega}^2 = (\alpha,\alpha)_{H\Omega}. \tag{2.35}$$

In $\mathbb{R}^3$ and $\mathbb{R}^2$ each differential $k-$form has a vector proxy, and each operator on differential $k-$forms has a corresponding operator on vector proxies. This relationship is what unifies the standard theory of Sobolev spaces with the theory of differential forms; each general formula involving differential forms has a corresponding transformed formula for each vector proxy.

*Remark* 2.1 (Vector proxies). The Hodge star $\star$ reduces to the identity in terms of vector proxies.

| k | Proxy | Form | Sobolev space | Space of forms |
|---|---|---|---|---|
| 0 | $\alpha$ | $\alpha$ | $H^1(\Omega) \approx H^{\nabla\wedge}(\Omega)$ | $HA^0\Omega$ |
| 1 | $(\alpha_1,\alpha_2)$ | $\alpha_2 dx^1 - \alpha_1 dx^2$ | $H^{\nabla\cdot}(\Omega)$ | $HA^1\Omega$ |
| 2 | $\alpha$ | $\alpha dx^1 \wedge dx^2$ | $L^2(\Omega)$ | $HA^2\Omega$ |

Table 2.1: Vector proxies in $\mathbb{R}^2$.

| k | Proxy | Form | Sobolev space | Space of forms |
|---|---|---|---|---|
| 0 | $\alpha$ | $\alpha$ | $H^1(\Omega)$ | $HA^0\Omega$ |
| 1 | $(\alpha_1,\alpha_2,\alpha_3)$ | $\alpha_1 dx^1 + \alpha_2 dx^2 + \alpha^3 dx^3$ | $H^{\nabla\wedge}(\Omega)$ | $HA^1\Omega$ |
| 2 | $(\alpha_1,\alpha_2,\alpha_3)$ | $\alpha_1 dx^2 \wedge dx^3 - \alpha_2 dx^1 \wedge dx^3 + \alpha^3 dx^1 \wedge dx^2$ | $H^{\nabla\cdot}(\Omega)$ | $HA^2\Omega$ |
| 3 | $\alpha$ | $\alpha dx^1 \wedge dx^2 \wedge dx^3$ | $L^2(\Omega)$ | $HA^3\Omega$ |

Table 2.2: Vector proxies in $\mathbb{R}^3$.

[4]Formally, when $\alpha \in L^2A^k$, $d\alpha$ is defined as a weak derivative using (2.45) alongside forms with compact support [Arn18].

Given a smooth map $F : \Omega \to \Gamma$ between manifolds, we can define the pushforward $F_* : T_p\Omega \to T_{F(p)}\Gamma$ and pullback $F^* : T^*_{F(p)}\Gamma \to T^*_p\Omega$ operators by

$$F_*(v)(f) = v(f \circ F), \quad \forall v \in T_p\Omega, f \in C^\infty(\Gamma), \tag{2.36}$$

$$F^*(\alpha)(v) = \alpha(F_*(v)), \quad \forall \alpha \in T^*_{F(p)}\Gamma, v \in T_p\Omega. \tag{2.37}$$

The pullback operator $F^*$ is a linear map which extends to differential forms $k-$forms $F^* : A^k\Gamma \to A^k\Omega$ linearly and commutes with the exterior derivative, that is $dF^* = F^*d$. It also distributes over the wedge product, that is $F^*(\alpha \wedge \beta) = F^*\alpha \wedge F^*\beta$.

The trace operator $\gamma$ is defined as the pullback of the inclusion map $\iota : \Gamma \hookrightarrow \Omega$, that is $\gamma = \iota^* : A^k\Omega \to A^k\Gamma$. Without proof we can now state the generalized Stokes theorem;

$$\int_\Omega d\alpha = \int_{\partial\Omega} \gamma\alpha, \quad \forall \alpha \in A^{n-1}\Omega, \tag{2.38}$$

from which we get a myriad of integration by parts formulas in

$$\int_{\partial\Omega} \gamma\alpha \wedge \gamma\beta = \int_\Omega d(\alpha \wedge \beta) \tag{2.39}$$

$$= \int_\Omega d\alpha \wedge \beta + (-1)^{k-1}\alpha \wedge d\beta, \tag{2.40}$$

where $\alpha \in A^{k-1}\Omega, \beta \in A^{n-k}\Omega$. When context is clear we will omit the $\gamma$ and simply write for instance

$$\int_\Gamma \alpha \wedge \beta = (\alpha, \star\beta)_\Gamma. \tag{2.41}$$

For the $L^2$-theory it is sufficient for us to accept that forms in $HA^k\Omega$ have well-defined traces on $\partial\Omega$. They give rise to the boundary spaces:

$$\mathring{H}A^k\Omega := \{\alpha \in HA^k\Omega : \gamma\alpha = 0\}. \tag{2.42}$$

## 2.3 Boundary conditions

The exterior derivative can be viewed as an unbounded operator $d : L^2A^k\Omega \to L^2A^{k+1}\Omega$ with domain the Sobolev space $HA^k\Omega$, which we write as $(d, HA^k\Omega)$.

Another version of it can be seen as the unbounded operator $(d, \mathring{H}A^k\Omega)$ with domain the boundary space. As such, both versions $(d, HA^k\Omega)$ and $(d, \mathring{H}A^k\Omega)$ have a respective dual unbounded operator $\delta : L^2A^{k+1}\Omega \to L^2A^k\Omega$ with domain

$$\mathcal{H}A^{k+1}\Omega := \{\beta \in L^2A^{k+1}\Omega : \delta\beta \in L^2A^k\Omega\}, \text{ for } (d, \mathring{H}A^k\Omega), \tag{2.43}$$

$$\mathring{\mathcal{H}}A^{k+1}\Omega := \{\beta \in \mathcal{H}A^{k+1}\Omega : \gamma(\star\beta) = 0\}, \text{ for } (d, HA^k\Omega). \tag{2.44}$$

We write these operators as $(\delta, \mathcal{H}A^{k+1}\Omega)$ and $(\delta, \mathring{\mathcal{H}}A^{k+1}\Omega)$, each satisfying $\delta\delta = 0$. The operator $\delta$ is called the codifferential, and given $\alpha \in HA^k\Omega, \beta \in HA^{k+1}\Omega$ it is defined by the duality

$$(\alpha, \delta\beta)_\Omega = (d\alpha, \beta)_\Omega, \text{ which by (2.40) requires,} \tag{2.45}$$

$$\delta = (-1)^{n(k-1)+1} \star d \star . \tag{2.46}$$

The exterior derivative and the codifferential do not commute, and the degree to which they do not is in some sense measured by the Hodge Laplacian

$$d\delta + \delta d : L^2A^k\Omega \to L^2A^k\Omega. \tag{2.47}$$

The equations of interest in this thesis, i.e. Poisson (2.7), Stokes (2.10), and Maxwell (2.12), are all equations involving the Hodge Laplacian. For example, the Poisson equation (2.7) can be seen as the special cases $k = 0$ or $k = n$ of the Hodge Laplace equation for $k-$forms:

$$(d\delta + \delta d)u = f - \pi f, \tag{2.48a}$$

$$\pi u = 0, \tag{2.48b}$$

where $\pi$ is the orthogonal projection from $L^2A^k\Omega$ to the space of harmonic forms

$$\mathfrak{H}^k = \{\alpha \in L^2A^k\Omega : d\alpha = 0, \delta\alpha = 0\} = \ker d \cap \ker \delta. \tag{2.49}$$

The equation $\pi u = 0$ determines certain compatibility conditions depending on the domains of the differential operators. Indeed the dimension $\dim \mathfrak{H}^k$ is equal to the $k$th Betti number of $\Omega$, which is for

$k = 0$: the number of connected components of $\Omega$,

$$
\begin{tikzcd}
0 \arrow[r, hook] & \text{0-forms} \arrow[r, "d"] \arrow[dr, "*"] & \text{1-forms} \arrow[r, "d"] \arrow[dr, "*"] & \text{2-forms} \arrow[r, "d"] \arrow[dr, "*"] & \text{3-forms} \arrow[r] & 0 \\
0 & \text{0-forms} \arrow[l] & \text{1-forms} \arrow[l, "\delta"'] & \text{2-forms} \arrow[l, "\delta"'] & \text{3-forms} \arrow[l, "\delta"'] & 0 \arrow[l, hook']
\end{tikzcd}
$$

Figure 2.2: Duality takes $k-$forms to $(k+1)-$forms.

$k = 1$: the number of holes (or tunnels in 3D) in $\Omega$,

$k = 2$: the number of voids inside $\Omega$ in 3D, equal to 0 in 2D,

$k = 3$: equal to 0.

Boundary conditions are also determined by the domains of the unbounded differential operators. Let us elaborate on this.

Let $V^k$ be the domain corresponding to $d$ acting on $L^2 A^k\Omega$, and identify $V^k = 0$ for $k < 0$ and $k > n$. Then,

$$0 \hookrightarrow V^{n-3} \xrightarrow{d} V^{n-2} \xrightarrow{d} V^{n-1} \xrightarrow{d} V^n \to 0. \tag{2.50}$$

is an exact sequence, meaning that $dV^{k-1} = \ker d^k$ for all $k$. (While otherwise omitted, we will sometimes indicate with $d^k : V^k \to V^{k+1}$ the differential which increments $k$ to $k+1$.) The codifferential gives rise to another exact sequence going the other way

$$0 \leftarrow V_*^{n-3} \xleftarrow{\delta} V_*^{n-2} \xleftarrow{\delta} V_*^{n-1} \xleftarrow{\delta} V_*^n \hookleftarrow 0. \tag{2.51}$$

Note that $(d, V^k)^* = (\delta, V_*^{k+1})$, see Figure 2.2. Depending on if $V^k = HA^k\Omega$ or $V^k = \overset{\circ}{H}A^k\Omega$, we get different sequences. Let us write them out in 3D:

$$0 \hookrightarrow H^1(\Omega) \xrightarrow{\nabla} H^{\nabla\wedge}(\Omega) \xrightarrow{\nabla\wedge} H^{\nabla\cdot}(\Omega) \xrightarrow{\nabla\cdot} L^2(\Omega) \quad \to 0 \tag{2.52a}$$

$$0 \leftarrow L^2(\Omega) \xleftarrow{-\nabla\cdot} H_0^{\nabla\cdot}(\Omega) \xleftarrow{\nabla\wedge} H_0^{\nabla\wedge}(\Omega) \xleftarrow{-\nabla} H_0^1(\Omega) \hookleftarrow 0 \tag{2.52b}$$

$$0 \hookrightarrow H_0^1(\Omega) \xrightarrow{\nabla} H_0^{\nabla\wedge}(\Omega) \xrightarrow{\nabla\wedge} H_0^{\nabla\cdot}(\Omega) \xrightarrow{\nabla\cdot} L^2(\Omega) \quad \to 0 \tag{2.53a}$$

$$0 \leftarrow L^2(\Omega) \xleftarrow{-\nabla\cdot} H^{\nabla\cdot}(\Omega) \xleftarrow{\nabla\wedge} H^{\nabla\wedge}(\Omega) \xleftarrow{-\nabla} H^1(\Omega) \hookleftarrow 0 \tag{2.53b}$$

The sequence (2.52a) is called the de Rham cochain complex[5]. We see that the dual sequences are simply the same sequences again, up to signs

[5]A cochain complex, a term from homological algebra, is simply a sequence of vector spaces $V^k$, the index of which is incremented by a differential map $d$ satisfying $dd = 0$.

and indices. Still, we can use this to derive the boundary conditions for the Poisson equation. Henceforth let us assume $\Omega$ is contractible[6].

Consider (2.52) and the Hodge Laplace equation (2.56). (Note the important detail that $d$ and $\delta$ are different derivatives and act differently in general on the same $k$-form.) For $k = 0$ we get the Neumann problem. Indeed, we have $\delta d = -\nabla \cdot \nabla = -\Delta$ and $d\delta = 0$, and the harmonic functions are the piecewise constants, whereby $\pi f = \int_\Omega f$. The boundary condition is given by the space $H_0^{\nabla\cdot}(\Omega)$ in (2.52b), i.e. the space of functions in $H^{\nabla\cdot}(\Omega)$ with zero normal component on $\partial\Omega$. For $k = n = 3$ we get the Dirichlet problem. We have $\delta d = 0$ and $d\delta = \nabla \cdot (-\nabla) = -\Delta$, and the harmonic functions are again the piecewise constants. The boundary condition is given by the space $H_0^1(\Omega)$ in (2.52b), which is the space of functions in $H^1(\Omega)$ with zero trace on $\partial\Omega$. We will get back to (2.53).

In this first half of the thesis we will only consider homogeneous boundary conditions $\gamma u = 0$ as we have done here, but of course non homogeneous conditions $\gamma u = g$, for some form on the boundary $g$, are also possible.

## 2.4 Weak formulations; primal and mixed

Let $f_\pi = f - \pi f$, with $f \in L^2 A^k\Omega$, and consider again the Hodge Laplace equation (2.48). Let us focus on (2.52). For starters we know that $u \in L^2 A^k\Omega$ must be in the domains of each differential operator $(d, HA^k\Omega)$ and its dual $(\delta, \mathring{\mathcal{H}}A^k\Omega)$, and moreover the second derivative must be well defined also. This implies

$$u \in HA^k\Omega \cap \mathring{\mathcal{H}}A^k\Omega \text{ and } \delta u \in HA^{k-1}\Omega, du \in \mathring{\mathcal{H}}A^{k+1}\Omega \tag{2.54}$$

whence the boundary conditions on $\partial\Omega$ are

$$\star u = 0, \ \star du = 0. \tag{2.55}$$

**Strong formulation**

[6] Intuitively this means we can contract $\Omega$ to a point.

Find $u \in HA^k\Omega \cap \mathring{\mathcal{H}}A^k\Omega$ such that $\delta u \in HA^{k-1}\Omega, du \in \mathring{\mathcal{H}}A^{k+1}\Omega$ and

$$(d\delta + \delta d)u = f_\pi, \text{ in } \Omega, \tag{2.56a}$$
$$\pi u = 0, \text{ in } \Omega, \tag{2.56b}$$
$$\star u = 0, \text{ on } \partial\Omega, \tag{2.56c}$$
$$\star du = 0, \text{ on } \partial\Omega. \tag{2.56d}$$

To prove (which we won't) that this equation is well-posed, meaning to make sure that it even has a unique solution, it is convenient to use functional analysis on the so-called weak formulation[7]. The formulations which we will present now are all equivalent, a fact proven in [Arn18].

Now, multiply with a "test" function $v \in HA^k\Omega \cap \mathring{\mathcal{H}}A^k\Omega$, integrate over $\Omega$, and use the formula (2.40) to get

$$\int_\Omega f_\pi \wedge \star v = \int_\Omega d\delta u \wedge \star v + \int_\Omega \delta du \wedge \star v \tag{2.57}$$
$$= \int_\Omega \delta u \wedge \star\delta v - \int_{\partial\Omega} \delta u \wedge \star v + \int_\Omega du \wedge \star dv + \int_{\partial\Omega} du \wedge \star v \tag{2.58}$$
$$= \int_\Omega \delta u \wedge \star\delta v + \int_\Omega du \wedge \star dv. \tag{2.59}$$

Note that $\int_{\partial\Omega} du \wedge \star v = 0$ in two ways, by either $\star v = 0$ or $du \wedge \star v = v \wedge \star du = 0$. As a result we get the following weak formulation which we call primal.

**Primal weak formulation**

Find $u \in HA^k\Omega \cap \mathring{\mathcal{H}}A^k\Omega$ with $\pi u = 0$ such that

$$(\delta u, \delta v)_\Omega + (du, dv)_\Omega = (f_\pi, v)_\Omega \quad \forall v \in HA^k\Omega \cap \mathring{\mathcal{H}}A^k\Omega. \tag{2.60}$$

Alternatively, we can let every $k-$form involved be its own variable to be solved for. This way of thinking gives rise to what is called a mixed

[7]Historically, this is what was done to expand the notion of solution from just continuously differentiable functions to equivalence classes of integrable functions with weak derivatives. Regularity theory is the theory of how to show that a weak solution is in fact a continuously differentiable solution.

formulation. The philosophy is that taking derivatives is a numerical loss of accuracy, so solving for theses derivatives directly can be beneficial. Consider the strong formulation again, and let $\lambda = \pi f \in \mathfrak{H}^k$ and $\sigma = \delta u \in HA^{k-1}\Omega$. From (2.56) we get

$$d\sigma + \delta du + \lambda = f \tag{2.61}$$

which we can again multiply by a test function $v \in HA^k\Omega$, integrate over $\Omega$, and use (2.40) to get

$$\begin{aligned}
\int_\Omega f \wedge \star v &= \int_\Omega \lambda \wedge \star v + \int_\Omega d\sigma \wedge \star v + \int_\Omega \delta du \wedge \star v && (2.62)\\
&= \int_\Omega \lambda \wedge \star v + \int_\Omega d\sigma \wedge \star v + \int_\Omega du \wedge \star dv - \int_{\partial\Omega} \underbrace{du \wedge \star v}_{=v\wedge\star du} && (2.63)\\
&= \int_\Omega \lambda \wedge \star v + \int_\Omega d\sigma \wedge \star v + \int_\Omega du \wedge \star dv. && (2.64)
\end{aligned}$$

Moreover, we can symmetrize $(\lambda, v)_\Omega$ and $(d\sigma, v)_\Omega$ by using $\pi u = 0$ respectively $\sigma - \delta u = 0$:

$$\begin{aligned}
0 &= \int_\Omega u \wedge \star\rho, \quad \forall \rho \in \mathfrak{H}^k, && (2.65)\\
0 &= \int_\Omega \sigma \wedge \star\tau - \int_\Omega \delta u \wedge \star\tau && (2.66)\\
&= \int_\Omega \sigma \wedge \star\tau - \int_\Omega u \wedge \star d\tau + \int_{\partial\Omega} \underbrace{u \wedge \star\tau}_{=\tau\wedge\star u}, && (2.67)\\
&= \int_\Omega \sigma \wedge \star\tau - \int_\Omega u \wedge \star d\tau, \quad \forall \tau \in HA^{k-1}\Omega. && (2.68)
\end{aligned}$$

*Remark* 2.2. Notice that both boundary conditions appear naturally in the integration by parts in this formulations. For a given formulation, we call such boundary conditions natural, while boundary conditions which have to be enforced via the test variables are called essential.

Taken together we get the weak formulation:

**Mixed weak formulation**

Find $(\sigma, u, \lambda) \in HA^{k-1}\Omega \times HA^k\Omega \times \mathfrak{H}^k$ such that

$$(\sigma, \tau)_\Omega - (u, d\tau)_\Omega = 0 \qquad \forall \tau \in HA^{k-1}\Omega, \tag{2.69a}$$

$$(\lambda, v)_\Omega + (d\sigma, v)_\Omega + (du, dv)_\Omega = (f, v)_\Omega \qquad \forall v \in HA^k\Omega, \quad (2.69b)$$

$$(u, \rho)_\Omega = 0 \qquad \forall \rho \in \mathfrak{H}^k. \quad (2.69c)$$

This mixed weak formulation is the formulation that most of our CutFEM are based on. The only exception is Method NC inside Paper B.

*Remark* 2.3 (Darcy flow, Papers A and C). As a first example consider the case $k = n$ of (2.56) so $u \in L^2(\Omega)$ with $u|_{\partial\Omega} = 0$ and $du = 0u = 0$. Also there are no harmonic forms and $\sigma \in H^{\nabla\cdot}(\Omega)$. The equations of the mixed formulation read

$$(\sigma, \tau)_\Omega - (u, \nabla \cdot \tau)_\Omega = 0 \qquad \forall \tau \in H^{\nabla\cdot}(\Omega), \qquad (2.70a)$$

$$(\nabla \cdot \sigma, v)_\Omega = (f, v)_\Omega \qquad \forall v \in L^2(\Omega), \qquad (2.70b)$$

where the condition $u|_{\partial\Omega}$ is natural. With $\sigma$ as the velocity and $u$ as the pressure, this is the mixed formulation of Darcy's equations with natural boundary conditions. Had we started with the boundary complex (2.53a) we would have gotten the same formulation but with essential boundary conditions of $\sigma \cdot n = 0$, and $\mathring{\mathfrak{H}}^n = \mathfrak{H}^0 = \mathbb{R}$. We will keep our focus on the mixed weak formulation going forward.

## 2.5 The Lagrangian

So far we have seen strong and weak formulations as two ways to represent the same PDE. Stemming from the fact that physical phenomena in nature in general follow a principle of least action, an example being a lightray of photons, there is third equivalent perspective coming from Lagrangian mechanics, with connections to optimization theory. I choose to include this because I believe it is yet another way to glean insight into the nature of the equations we are dealing with.

For ease of presentation let us assume for this section that $\pi u = 0$. For the primal formulation we can write the action as

$$\inf_{u \in HA^k \cap \mathring{\mathcal{H}}A^k} \int_\Omega \left( \frac{1}{2}(\delta u \wedge \star \delta u + du \wedge \star du) - f_\pi \wedge \star u \right). \qquad (2.71)$$

The integrand is called the Lagrangian $\mathcal{L}$. If we view $du$ and $\delta u$ as the velocities of the $k-$form $u$, then the Lagrangian is the kinetic energy minus the potential energy. The principle of virtual work states that the action is stationary with respect to variations of the minimizer $u$. This leads to the weak primal formulation.

The mixed formulation is a little more complicated to relate, let us derive it in a somewhat formal way. Let $\mathcal{L} = K_\delta + \mathcal{L}_d$ where $K_\delta(u) = \frac{1}{2}(\delta u, \delta u)_\Omega$. In the language of [Lue97], given $\omega \in (\mathring{\mathcal{H}A}^k)^* = HA^{k-1}$ we define the convex conjugate functional to $K_\delta$ as

$$K_\delta^*(\omega) = \sup_{u \in \mathring{\mathcal{H}A}^k} \langle \omega, u \rangle - K_\delta(u) \tag{2.72}$$

$$= \sup_{u \in \mathring{\mathcal{H}A}^k} \int_\Omega d\omega \wedge \star u - \frac{1}{2} \int_\Omega \delta u \wedge \star \delta u \tag{2.73}$$

$$= \sup_{u \in \mathring{\mathcal{H}A}^k} \int_\Omega (\omega - \frac{1}{2}\delta u) \wedge \star \delta u, \tag{2.74}$$

where the last equation is due to $u \in \mathring{\mathcal{H}A}^k$. We can get back to the original functional by taking the convex conjugate again, that is

$$K_\delta(u) = \sup_{\omega \in HA^{k-1}} \langle u, \omega \rangle - K_\delta^*(\omega) \tag{2.75}$$

$$= \sup_{\omega \in HA^{k-1}} \int_\Omega d\omega \wedge \star u - K_\delta^*(\omega). \tag{2.76}$$

Applying the principle of virtual work (which amounts to adding a virtual shift $u_0 + \varepsilon v$ to the minimizer $u_0$ and differentiating with respect to $\varepsilon > 0$) to (2.74) we get $\omega = \delta u_0$ so that $K_\delta^*(\omega) = K_\delta(u) = \frac{1}{2}(\omega, \omega)_\Omega$. Hence

$$K_\delta(u) = \sup_{\omega \in HA^{k-1}} \int_\Omega d\omega \wedge \star u - \frac{1}{2}\omega \wedge \star\omega. \tag{2.77}$$

With this we can write the action functional corresponding to the mixed weak formulation as follows.

$$\mathcal{L} = K_\delta + \mathcal{L}_d \tag{2.78}$$

$$= \sup_{\omega \in HA^{k-1}} \int_\Omega \left( d\omega \wedge \star u - \frac{1}{2}\omega \wedge \star\omega \right) \tag{2.79}$$

$$+ \inf_{u \in HA^k \cap \mathring{\mathcal{H}A}^k} \int_\Omega \left( \frac{1}{2} du \wedge \star du - f \wedge \star u \right) \tag{2.80}$$

$$= \inf_{u \in HA^k \cap \overset{\circ}{\mathcal{H}}A^k} \sup_{\omega \in HA^{k-1}} \int_\Omega \left( d\omega \wedge \star u - \frac{1}{2}\omega \wedge \star\omega + \frac{1}{2} du \wedge \star du - f \wedge \star u \right) \tag{2.81}$$

$$\geq \inf_{u \in HA^k} \sup_{\omega \in HA^{k-1}} \int_\Omega \left( d\omega \wedge \star u - \frac{1}{2}\omega \wedge \star\omega + \frac{1}{2} du \wedge \star du - f \wedge \star u \right). \tag{2.82}$$

The optimization problem of the left hand side has become a saddle-point problem on the right hand side. One can see that varying the functional around the minimizer $u_0 \in HA^k$ gives (2.69b) while varying around the maximizer $\omega_0 \in HA^{k-1}$ gives (2.69a). If the minimizer $u_0 \in \overset{\circ}{\mathcal{H}}A^k$, then it is a minimizer of $\mathcal{L}$ as well. This is the case since $u_0 \in \overset{\circ}{\mathcal{H}}A^k$ is implied by equation (2.69a).

From this point one can use duality methods to derive even more weak formulations, see for instance [BF12], but we will not do this.

## 2.6 Three equations as examples involving the Hodge Laplacian

We have already seen how the Poisson equation (2.7) can be seen as a special case of the Hodge Laplace equation (2.56). We will now see how the Stokes equations (2.10) and the Maxwell equations (2.12), with some extra terms, can be seen as special cases as well.

We start by mentioning the $d-$ and $\delta-$problems. The right hand side $f$ can be such that $f = dg$ or $f = \delta g$. In the case $f = dg$ the solution to $(d\delta + \delta d)u = f$ is given by $u = dv$ where $v$ is the solution to the standard Hodge Laplace equation $(d\delta + \delta d)v = g - \pi g$. To see this formally consider,

$$f = (d\delta + \delta d)dv = d\delta dv = d(d\delta + \delta d)v = d(g - \pi g) = dg. \tag{2.83}$$

Recall by exactness that $u \in dHA^{k-1}\Omega \iff du = 0$. We thus have two variations of the Hodge Laplace problem for (2.52a):

**$d-$ and $\delta-$formulations**

Given $f \in dHA^{k-1}\Omega$, find $u \in \mathring{\mathcal{H}}A^k\Omega$ such that

$$d\delta u = f, \tag{2.84a}$$

$$du = 0. \tag{2.84b}$$

Given $f \in \delta\mathring{\mathcal{H}}A^{k+1}\Omega$, find $u \in \mathring{\mathcal{H}}A^k\Omega$ such that $du \in \mathring{\mathcal{H}}A^{k+1}\Omega$ and

$$\delta du = f, \tag{2.85a}$$

$$\delta u = 0. \tag{2.85b}$$

We can derive weak formulations for both variations, we focus on (2.84). For ease of presentation we let $n > k > 0$ so $\mathfrak{H}^k = \emptyset$ since $\Omega$ is contractible. To account for the new equation $du = 0$ we can introduce a Lagrange multiplier variable $p \in HA^{k+1}\Omega$ to the Lagrangian. The resulting mixed weak formulation is

**Mixed weak $d-$formulation**

Find $(\sigma, u, p) \in HA^{k-1}\Omega \times HA^k\Omega \times HA^{k+1}\Omega$ such that

$$(\sigma, \tau)_\Omega - (u, d\tau)_\Omega = 0 \qquad \forall \tau \in HA^{k-1}\Omega, \tag{2.86a}$$

$$(d\sigma, v)_\Omega - (p, dv)_\Omega = (f, v)_\Omega \qquad \forall v \in HA^k\Omega, \tag{2.86b}$$

$$(du, q)_\Omega = 0 \qquad \forall q \in HA^{k+1}\Omega. \tag{2.86c}$$

*Remark* 2.4 (Time-harmonic Maxwell's equations, Paper D). Let $n = 3$. Assuming the electric fields vary periodically in time with given amplitude depending only on the spatial coordinates $x$, we can write the electric field as $u(x,t) = u(x)e^{-i\beta t}$ where $\beta \in \mathbb{R}$ is a given frequency. The time-harmonic Maxwell's equations are then given by

$$\nabla \wedge \nabla \wedge u - \mu\varepsilon\beta^2 u = f, \qquad \text{in } \Omega, \tag{2.87a}$$

$$\nabla \cdot (\varepsilon u) = i/\beta \nabla \cdot f \qquad \text{in } \Omega, \tag{2.87b}$$

with perfect electric conductor (PEC) boundary conditions $n \wedge u = 0$, or magnetic wall (PMC) boundary conditions $n \wedge \nabla \wedge u = 0$. Assuming $\varepsilon$ is piecewise constant and $f \in \nabla \wedge H^{\nabla\wedge}(\Omega)$ the second bulk equation reads $\nabla \cdot u = 0$. Minus the extra term $\mu\varepsilon\beta^2 u$ this is exactly the $d-$formulation

(2.84) with $k = 2$, which gives the PEC boundary conditions. The equations of the mixed weak $d-$formulation read:

$$(\sigma, \tau)_\Omega - (u, \nabla \wedge \tau)_\Omega = 0 \qquad \forall \tau \in H^{\nabla \wedge}(\Omega), \tag{2.88a}$$

$$(\nabla \wedge \sigma, v)_\Omega - (\mu\varepsilon\beta^2 u, v)_\Omega - (p, \nabla \cdot v)_\Omega = (f, v)_\Omega \quad \forall v \in H^{\nabla \cdot}(\Omega), \tag{2.88b}$$

$$(\nabla \cdot u, q)_\Omega = 0 \qquad \forall q \in L^2(\Omega). \tag{2.88c}$$

Starting instead with the boundary complex (2.53a) we get the same equations but with PMC boundary conditions; observe that $\delta u \in \mathring{\mathcal{H}}A^{k-1}\Omega$ in that case. (Note that we should read the unknown as the displacement current $u/\varepsilon$, since physically speaking this quantity is a $2-$form while $u$ should really be viewed as a $1-$form according to Ampére's and Faraday's laws [Hip02, Sec. 2.1].)

*Remark* 2.5 (Stokes equations, Paper B). Let $n = 3$. Unsurprisingly we do a very similar thing here, and consider the $d-$formulation (2.84) with $k = 2$, where the new variable $\sigma = \nabla \wedge u$ is the vorticity. We view the pressure as a Lagrange multiplier for the divergence condition, and we integrate $(\nabla p, v)_\Omega$ by parts. The equations of the mixed weak $d-$formulation read:

$$(\sigma, \tau)_\Omega - (u, \nabla \wedge \tau)_\Omega = 0 \qquad \forall \tau \in H^{\nabla \wedge}(\Omega), \tag{2.89a}$$

$$(\nabla \wedge \sigma, v)_\Omega - (p, \nabla \cdot v)_\Omega = (f, v)_\Omega \qquad \forall v \in H^{\nabla \cdot}(\Omega), \tag{2.89b}$$

$$(\nabla \cdot u, q)_\Omega = 0 \qquad \forall q \in L^2(\Omega). \tag{2.89c}$$

We do not necessarily have that the bulk force $f$ is the curl of a vector field. If we know $du = \nabla \cdot u = 0$ in $\Omega$ however, we could use the mixed weak formulation (2.69) with $k = 2$ to get the same equations with no restriction on the bulk force; the term $(du, dv)_\Omega$ would be equal to zero. We could reinterpret the PEC and PMC boundary conditions as conditions involving the velocity and the vorticity, but one also wants to consider no-slip conditions $u|_{\partial\Omega} = 0$. We can do this using Nitsche's method, see Section 4.1, but we pay the price of falling outside the de Rham complex framework.

# Chapter 3

# Finite element methods

There are two main motivations for numerical methods. Though we have theory to guarantee existence of solutions to PDEs, in general we do not know the solution given a set of boundary conditions. At the same time, the Sobolev space $HA^k$ are infinite-dimensional; searching for a solution inside these spaces is an impossible task for a computer since it can only deal with finite amounts of data. So, we want to approximate the solution which we do not know, and we want to do it in a finite-dimensional, or discrete, setting.

Finite element methods are a very natural idea from this point of view, if one starts with a weak formulation. Consider again the primal weak formulation of the Hodge Laplace equation: Find $u \in HA^k\Omega \cap \mathring{\mathcal{H}}A^k\Omega$ with $\pi u = 0$ such that

$$(\delta u, \delta v)_\Omega + (du, dv)_\Omega = (f_\pi, v)_\Omega \quad \forall v \in HA^k\Omega \cap \mathring{\mathcal{H}}A^k\Omega. \tag{3.1}$$

If we had a space $V^k \subset HA^k\Omega \cap \mathring{\mathcal{H}}A^k\Omega$ of finite dimension $N$, then we could write $u = \sum_{i=1}^N u_i\phi_i$ and $v = \sum_{i=1}^N v_i\phi_i$ for some basis $\{\phi_i\}_{i=1}^N$ of $V^k$, where $u_i, v_i \in \mathbb{R}$. By linearity, testing the equation with each basis function is enough. Then the weak formulation would be equivalent to finding the unknowns $u_i$ such that

$$\sum_{i=1}^N u_i \left((\delta\phi_i, \delta\phi_j)_\Omega + (d\phi_i, d\phi_j)_\Omega\right) = (f_\pi, \phi_j)_\Omega \quad \forall i = 1, \ldots, N. \tag{3.2}$$

This is a linear system of equations, which can be written in matrix form as

$$A\mathbf{u} = \mathbf{b}, \tag{3.3}$$

where $A_{ji} = (\delta\phi_i, \delta\phi_j)_\Omega + (d\phi_i, d\phi_j)_\Omega$ and $b_i = (f_\pi, \phi_j)_\Omega$. (The matrix $A$ is called the stiffness matrix and $\mathbf{b}$ the load vector.)

## 3.1 Mesh

The question remains of how to choose these discrete subspaces $V^k$. We proceed heuristically so as to avoid getting bogged down in details. With the idea that our problems are intimitely tied to the geometry in which they are posed, we start by discretizing $\Omega$. We do this by partitioning (or triangulating) $\Omega$ into a finite number of similar sized elements $\{K_j\}_{j=1}^N$, the collection of which is called a mesh:

$$\mathcal{T}_h = \{K_j, j = 1, \ldots, N\}, \tag{3.4}$$

$$h = \max_{j=1,\ldots,N} \operatorname{diam} K_j, \tag{3.5}$$

$$\Omega = \bigcup_{K \in \mathcal{T}_h} K. \tag{3.6}$$

The elements are typically simplices, i.e. triangles in 2D and tetrahedra in 3D, but can be also hypercubes[1]. Let us stick with simplices for now. As we increase $N$, the mesh becomes finer, and the diameter $h$ of the elements becomes smaller. Our approximation space which we will call $V_h^k \subset HA^k$, indicating its dependence on $\mathcal{T}_h$, is then expected to grow closer to $HA^k$ as $h \to 0$.

For the mesh we use the following nomenclature. An element $K$ is an $n$-dimensional simplex, a facet is an $(n-1)$-dimensional simplex, an edge is a simplex of dimension 1, and a node is a 0-dimensional simplex. Letting $K$ be an element of our mesh $\mathcal{T}_h$, we denote by $\mathcal{F}(K)$ the set of facets of $K$, by $\mathcal{E}(K)$ its set of edges, and by $\mathcal{N}(K)$ its set of nodes. Taking unions over all elements, we obtain the sets:

$$\mathcal{F}_h := \bigcup_{K \in \mathcal{T}_h} \mathcal{F}(K), \qquad \mathcal{E}_h := \bigcup_{K \in \mathcal{T}_h} \mathcal{E}(K), \qquad \mathcal{N}_h := \bigcup_{K \in \mathcal{T}_h} \mathcal{N}(K). \tag{3.7}$$

To each facet $F \in \mathcal{F}_h$ we associate a unique normal vector $\mathfrak{n}$.

We shall assume that the mesh is shape regular. This means that there exists a constant $c > 0$ independent of $h$ such that

$$\max_{K \in \mathcal{T}_h} \frac{h_K}{\rho_K} \le c, \tag{3.8}$$

[1]Or any polytope in the case of the VEM, but we will see that later.

where $h_K$ is the diameter of $K$ and $\rho_K$ is the radius of the largest ball contained in $K$.

## 3.2 Finite element spaces

Our mesh is designed so as to constitute a pure simplicial $n-$complex; every $k-$simplex is contained in an $n$-simplex. We can denote a $k-$simplex by its nodes, e.g. $K = [x_0, x_1, x_2]$ for a triangle or $2-$simplex. The ordering of nodes determines an orientation of the simplex, we denote by $-K = [x_2, x_1, x_0]$. A given $k-$simplex can only have two orientations, regardless of $k$. A $k-$chain is a real linear combination of oriented $k-$simplices, and the set of $k-$chains forms a real (finite dimensional) vector space denoted by $C_k = C_k(\mathcal{T}_h)$. Note that in 3D

$$C_0 = \text{span } \mathcal{N}_h,\ C_1 = \text{span } \mathcal{E}_h,\ C_2 = \text{span } \mathcal{F}_h,\ C_3 = \text{span } \mathcal{T}_h.$$

The boundary operator $\partial : C_k \to C_{k-1}$ is a linear map between vector spaces defined by

$$\partial K = \sum_{i=0}^{k} (-1)^i [x_0, \ldots, \hat{x}_i, \ldots, x_k]. \tag{3.9}$$

It remarkably also satisfies $\partial \circ \partial = 0$, just like the exterior derivative does. In fact our mesh gives rise to a chain complex

$$0 \hookrightarrow C_n \xrightarrow{\partial} C_{n-1} \xrightarrow{\partial} C_{n-2} \xrightarrow{\partial} C_{n-3} \to 0. \tag{3.10}$$

The sequence is in general not exact, the Betti numbers that we discussed prior are exactly the degree to which it is not; the $k$th Betti number is the dimension of the $k$th homology group $\ker \partial_k / \partial C_{k+1}$. For our purposes therefore, given the assumptions on $\Omega$, the sequence is exact except for $b_0 = 1$, the number of connected components.

Now, since any linear map between vector spaces induces a dual map between their dual spaces, a chain complex gives rise to a dual chain complex called the simplicial cochain complex:

$$0 \hookrightarrow \mathfrak{C}^{n-3} \xrightarrow{\mathfrak{d}} \mathfrak{C}^{n-2} \xrightarrow{\mathfrak{d}} \mathfrak{C}^{n-1} \xrightarrow{\mathfrak{d}} \mathfrak{C}^{n} \to 0. \tag{3.11}$$

Here $\mathfrak{C}^k = (C_k)^*$ and the elements of which, called $k-$cochains, assign a real number to each $k-$simplex. The dual boundary operator $\mathfrak{d}^k = \partial_{k+1}^* : \mathfrak{C}^k \to \mathfrak{C}^{k+1}$ is defined by

$$\mathfrak{d}^k L = L \circ \partial_{k+1}, \quad L \in \mathfrak{C}^k. \tag{3.12}$$

Here comes the kicker. Let us restrict to forms on an $n-$simplex $K \in \mathcal{T}_h$ and $k-$cochains $\mathfrak{C}^k(K)$ with dimension $m = \binom{n+1}{k+1}$. A way to ascribe a number to a $k-$chain $c$ is to integrate a $k-$form over it, and the forms in $HA^kK$ are exactly those which have a well defined trace on $c$. This means that we can construct a map[2] $\pi : HA^kK \to \mathfrak{C}^k(K)$, and doing so for each $k$ actually makes the composite diagram of (2.50) to (3.11) commute! (This means $\pi \circ \mathfrak{d} = d \circ \pi$.)

If we view the image element $L_c(u) = \pi u(c) = \int_c \gamma u$ instead as a functional on $HA^kK$, then the collection of such functionals $L_c = \int_c \gamma \bullet$ form a set

$$\mathfrak{B}_k = \{L_{c_1}, L_{c_2}, \ldots, L_{c_m}\}, \qquad \text{where } c_1, \ldots, c_m \in C_k(K), \tag{3.13}$$

and we can ask: is there a finite dimensional subspace of $HA^kK$ such that $\mathfrak{B}_k$ is a basis of its dual space? The answer is yes, and for simplices these forms have polynomial coefficients. What's more: we can find these $k-$forms by considering the dual basis $\mathfrak{B}'_k = \{\phi_1, \phi_2, \ldots, \phi_m\}$ satisfying

$$L_{c_i}(\phi_j) = \delta_{ij}. \tag{3.14}$$

Solving these equations for a given $j$ gives a unique $k-$form $\phi_j$, called a shape function.

**Example 3.1** (Derivation of Raviart-Thomas elements, zeroth order)**.** Decide on an origin in $\mathbb{R}^3$ and consider the tetrahedron with edges coinciding with the x-, y-, and z-axes (the reference tetrahedron.) The degrees of freedom are the facet normal fluxes, of which there are four on a tetrahedron. For each facet we want a corresponding shape function. Let us label $x = x_1, y = x_2, z = x_3$. A simple $2-$form looks like

$$\omega = \beta_3 dx_1 \wedge dx_2 + \beta_2 dx_1 \wedge dx_3 + \beta_1 dx_2 \wedge dx_3. \tag{3.15}$$

[2] Called the de Rham map.

A priori each $\beta_i$ is a polynomial of first order, i.e.

$$\beta_i = c_i + c_{i1}x_1 + c_{i2}x_2 + c_{i3}x_3. \tag{3.16}$$

This gives twelve unknowns, so the system is quite underdetermined. To get four unknowns we can make an educated guess and set

$$\begin{aligned}
\beta_i &= c_i + cx_i, \text{ so that} && (3.17)\\
\omega &= (c_3 + cx_3)dx_1 \wedge dx_2 - (c_2 + cx_2)dx_1 \wedge dx_3 && (3.18)\\
&\quad + (c_1 + cx_1)dx_2 \wedge dx_3. && (3.19)
\end{aligned}$$

As of yet we have not discussed how to integrate differential $k-$forms, nor even how to evaluate them on $k-$vectors. Let us show how to do this for simple $2-$forms. We have that

$$dx_i \wedge dx_j(v,w) = \begin{vmatrix} dx_i(v) & dx_j(v) \\ dx_i(w) & dx_j(w) \end{vmatrix} = v_i w_j - v_j w_i. \tag{3.20}$$

For a parameterization $\phi : D \to \mathbb{R}^3$ of a two-dimensional surface $S = \phi(D) \subset \mathbb{R}^3$ we have

$$\begin{aligned}
\int_S dx_i \wedge dx_j = \int_D \phi^*(dx_i \wedge dx_j) &= \int_D \phi^* dx_i \wedge \phi^* dx_j && (3.21)\\
&= \int_D (\frac{\partial \phi_i}{\partial u}du + \frac{\partial \phi_i}{\partial v}dv) \wedge (\frac{\partial \phi_j}{\partial u}du + \frac{\partial \phi_j}{\partial v}dv) && (3.22)\\
&= \int_D (\frac{\partial \phi_i}{\partial u}\frac{\partial \phi_j}{\partial v} - \frac{\partial \phi_j}{\partial u}\frac{\partial \phi_i}{\partial v})du \wedge dv && (3.23)\\
&= \int_D (dx_i \wedge dx_j(\partial_u \phi, \partial_v \phi))du \wedge dv && (3.24)\\
&= \int_D (dx_i \wedge dx_j(\partial_u \phi, \partial_v \phi))dudv, && (3.25)
\end{aligned}$$

where the right most term is a standard Riemann integral. Let $D = \{(u,v) : 0 \leq v \leq 1-u, 0 \leq u \leq 1\}$ and let $F_0 = \phi_0(D), F_{12} = \phi_{12}(D), F_{13} = \phi_{13}(D), F_{23} = \phi_{23}(D)$ be the four facets of the tetrahedron, where only the variables $x_i, x_j$ vary over $F_{ij}$. We have

$$\begin{aligned}
\phi_{12}(u,v) &= (u,v,0), && (3.26)\\
\phi_{13}(u,v) &= (u,0,v) && (3.27)\\
\phi_{23}(u,v) &= (0,v,u), && (3.28)
\end{aligned}$$

$$\phi_0(u,v) = (u, v, 1-u-v). \tag{3.29}$$

Since the pullback distributes over the wedge product, and we have $dx^i(\partial_{x_j}) = \delta_{ij}$, the trace of

$$dx_2 \wedge dx_3 = dx_1 \wedge dx_3 = 0, \text{ on } F_{12}, \tag{3.30}$$
$$dx_1 \wedge dx_2 = dx_2 \wedge dx_3 = 0, \text{ on } F_{13}, \tag{3.31}$$
$$dx_1 \wedge dx_2 = dx_1 \wedge dx_3 = 0, \text{ on } F_{23}. \tag{3.32}$$

So $\omega|_{F_{ij}} = c_k dx_i \wedge dx_j$, where $k \neq i, j$.

Let us compute $\omega_0$ corresponding to $F_0$. The equations $0 = \int_{F_{ij}} \omega_0 = \int_{F_{ij}} c_k dx_i \wedge dx_j$ give $c_k = 0$ for $k \neq i, j$. We thus have

$$\omega_0 = c(x_1 dx_2 \wedge dx_3 - x_2 dx_1 \wedge dx_3 + x_3 dx_1 \wedge dx_2). \tag{3.33}$$

To determine $c$ we must calculate the result of $1 = \int_{F_0} \omega_0$. We have

$$dx_i \wedge dx_j(\partial_u \phi_0, \partial_v \phi_0) = \begin{cases} 1, \text{ for } (i,j) = (1,2), (2,3) \\ -1 \text{ for } (i,j) = (1,3) \end{cases}, \tag{3.34}$$

and some ordinary integrals give $\int_{F_0} \omega_0 = c(1/6 + 1/6 + 1/6) = c/2$ whereby $c = 2$. Thus $\omega_0 = 2(x_1 dx_2 \wedge dx_3 - x_2 dx_1 \wedge dx_3 + x_3 dx_1 \wedge dx_2) = 2(x_1, x_2, x_3)$. The other three basis functions are found similarly. □

Restricted to $\mathfrak{B}'_k$ the map $\pi : \mathfrak{B}'_k \to \mathfrak{C}_k(K)$ is an isomorphism. We make the following loose definition of finite element, inspired by the Ciarlet [Cia02] definition:

**Definition 3.2.** A finite element is a pair $(K, \mathfrak{B})$ where $K$ is a $k-$simplex and $\mathfrak{B}$ is a basis of functionals on polynomials of $K$, called degrees of freedom. The polynomial basis is given by $\mathfrak{B}'$, the dual basis of $\mathfrak{B}$, and its elements are called shape functions.

For each $k$ we thus have a finite element called the Whitney finite element

$$(K, \mathfrak{B}_k), \text{ where we write } \mathfrak{B}'_k = P_1^- A^k(K). \tag{3.35}$$

The polynomial set is denoted by $P_1^- A^k(K)$, following [Arn18], because the polynomials will be at most degree 1. The superscript sign is there because there is another larger space $P_1 A^k(K)$ of which $P_1^- A^k(K)$ is a subset. For

0−forms the set $P_1^- A^0(K)$ are exactly isomorphic to the commonly known Lagrange polynomials of lowest order, whose degrees of freedom $(P_1^- A^0(K))'$ are the $\binom{n+1}{1} = n + 1$ nodal evaluations.

By taking unions over all elements, and "gluing" by requiring that values over common degrees of freedom coincide between elements, we obtain the conforming finite element spaces

$$P_1^- A^k\Omega = \{v \in HA^k\Omega : v|_K \in P_1^- A^k(K) \text{ for all } K \in \mathcal{T}_h\} \subset HA^k\Omega. \quad (3.36)$$

In order of inceasing $k = 0, 1, 2, 3$ these are the lowest order Lagrange, Raviart-Thomas, Nédelec, and piecewise Lagrange elements. They are discrete analogues of the Sobolev spaces $HA^k\Omega$, and the approximation spaces we have sought. Moreover, they form a discrete de Rham complex under (2.50) with exactness (or lack thereof) inherited. For the 3D case under (2.52), see Figure 3.1.

One can also increase the polynomial order $r \geq 1$ to get $P_r A^k\Omega$ and $P_r^- A^k\Omega \subset P_r A^k\Omega$. We will not give the general construction here even though we consider some such spaces in our papers. For the construction see [Arn18]. An example of $P_r A^k\Omega$ is the BDM-family of elements, given by $P_r A^1\Omega$ in 2D and $P_r A^2\Omega$ in 3D.

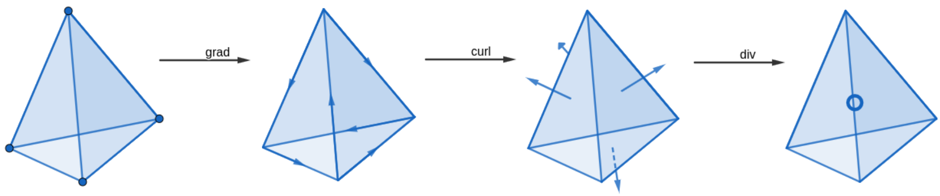


Figure 3.1: De Rham cochain complex of Whitney finite element degrees of freedom. Filled in vertices are point evaluations, arrows along edges are line integrals of tangential components, arrows pointing out of facets are surface integrals of normal components, and the hollow circle is a volume integral.

## 3.3 Approximation properties and stability

Consider the spaces of $H^1$-regular potentials of discrete functions:

$$\tilde{H}A^k\Omega = \{v \in [HA^0\Omega]^{\binom{n}{k}} : dv \in P_{r-1}^- A^{k+1}\Omega\}. \quad (3.37)$$

Let $\Pi_h^- : \tilde{H}A^k\Omega \to P_r^- A^k\Omega$, $\Pi_h : \tilde{H}A^k\Omega \to P_r A^k\Omega$ be the canonical $L^2$-projection operators (also called interpolation operators) onto the finite element spaces defined by the degrees of freedom. Along with the exterior derivative they satisfy the commuting diagram property:

$$\Pi_h^- \circ d = d \circ \Pi_h^-, \quad \Pi_h \circ d = d \circ \Pi_h. \tag{3.38}$$

This gives rise to the following theorem, whose proof is found in [AFW06, Thm. 5.3].

**Theorem 3.3.** *The following interpolation estimates hold:*

$$\|u - \Pi_h^- u\|_\Omega \lesssim h^{r+1}\|u\|_{H^{r+1}\Omega}, \quad \forall u \in H^{r+1}A^k\Omega, \tag{3.39}$$
$$\|u - \Pi_h u\|_\Omega \lesssim h^{r}\|u\|_{H^{r}\Omega}, \quad \forall u \in H^{r}A^k\Omega, \tag{3.40}$$

*and due to the commuting diagram property* (3.38) *we also have*

$$\|d(u - \Pi_h^- u)\|_\Omega \lesssim h^{r}\|du\|_{H^{r}\Omega}, \quad \forall u \in H^{r+1}A^k\Omega, \tag{3.41}$$
$$\|d(u - \Pi_h u)\|_\Omega \lesssim h^{r}\|du\|_{H^{r}\Omega}, \quad \forall u \in H^{r}A^k\Omega. \tag{3.42}$$

The spaces $H^r A^k\Omega$, $r \geq 0$, are Sobolev spaces of differential forms whose vector proxies have $r$ generalized derivatives in $L^2(\Omega)$.

Let us write the mixed discrete formulation of the Hodge Laplace problem. We focus on the reduced spaces, and let $\mathfrak{H}_h^k = \mathfrak{H}^k$ which is equal to $\mathbb{R}$ for $k = 0$ and 0 otherwise.

**Mixed discrete method**

Find $(\sigma_h, u_h, \lambda_h) \in P_r^- A^{k-1}\Omega \times P_r^- A^k\Omega \times \mathfrak{H}_h^k$ such that

$$(\sigma_h, \tau)_\Omega - (u_h, d\tau)_\Omega = 0 \qquad \forall \tau \in P_r^- A^{k-1}\Omega, \tag{3.43a}$$
$$(\lambda_h, v)_\Omega + (d\sigma_h, v)_\Omega + (du_h, dv)_\Omega = (f, v)_\Omega \quad \forall v \in P_r^- A^k\Omega, \tag{3.43b}$$
$$(u_h, \rho)_\Omega = 0 \qquad \forall \rho \in \mathfrak{H}_h^k. \tag{3.43c}$$

The discrete formulation is well posed and has a unique discrete solution. The continuous solution also solves the discrete problem - a consequence of $P_r A^k, P_r^- A^k \subset HA^k$. The mixed discrete method is as such an example of what we call a consistent method.

*Remark* 3.4. If $\mathfrak{H}^k$ is something more complicated than 0 or $\mathbb{R}$ then $\mathfrak{H}^k_h \not\subset \mathfrak{H}^k$ and the method is non conforming, and a consistency error has to be accounted for. Since our domain $\Omega$ is assumed contractible, we don't have to worry about this case.

The error converges with order $r$.

**Theorem 3.5.** *Let $(\sigma, u, \lambda) \in HA^{k-1}\Omega \times HA^k\Omega \times \mathfrak{H}^k$ be the solution to the mixed weak formulation and $(\sigma_h, u_h, \lambda_h) \in P_r^- A^{k-1}\Omega \times P_r^- A^k\Omega \times \mathfrak{H}^k_h$ the solution to the mixed discrete formulation. Then*

$$\|\sigma - \sigma_h\|_{H\Omega} + \|u - u_h\|_{H\Omega} + \|\lambda - \lambda_h\|_\Omega \lesssim h^r(\|\sigma\|_{H^r\Omega} + \|u\|_{H^r\Omega}). \tag{3.44}$$

*Proof.* See the proof of [Arn18, Thm. 5.2] and the discussion after. □

So far we have not mentioned stability of the finite element method. Stability is a measure of the continuous dependence of solution on the data; variations in $f$ cannot induce too large variations in the solution of a stable method. This property is necessarily required to have a convergent method, it measures norm of the solution operator $f \mapsto (\omega_h, u_h, \lambda_h)$ in the following sense:

$$\|\omega_h\| + \|u_h\| + \|\lambda_h\| \leq C_h \|f\|_\Omega, \tag{3.45}$$

where $\|\cdot\|$ is some norm on the discrete solution space which has to be chosen (possibly differently for each variable.) The constant $C_h$ is called the stability constant and it enters into error estimates such as Theorem 3.5 as a multiplicative factor. Thus if for instance $C_h \to \infty$ as $h \to 0$ the method may not converge.

**Definition 3.6.** A numerical method is said to be stable if there exists a constant $C$ independent of $h$ such that (3.45) holds.

From (3.45) we have

$$C_h = \sup_{f \in L^2\Omega} \frac{\|\omega_h\| + \|u_h\| + \|\lambda_h\|}{\|f\|_\Omega}, \tag{3.46}$$

which, if we view the discrete problem as a linear system $A\mathbf{u} = \mathbf{b}$, is simply the norm of the inverse of $A$. The condition number

$$\kappa(A) = \|A^{-1}\|\|A\| \tag{3.47}$$

of the system measures the complexity of the linear system, i.e. how hard it is to solve. If the condition number is large, then the system is ill-conditioned and the solution is sensitive to perturbations in the data (from either the right-hand side $\mathbf{b}$ or the matrix $A$ or both.) We can of course also see that the condition number $\kappa(A)$ will be large if the stability constant $C_h$ is large.

# Chapter 4

# The cut finite element method

CutFEM relaxes the assumption that the boundary of the mesh $\mathcal{T}_h$ coincides with the boundary of the domain $\Omega$. Instead, we allow $\partial\Omega$ to cut through the elements of the mesh. This is done by introducing a background mesh $\mathcal{T}_{0,h}$ which is boundary fitted (or fitted) with respect to some non physical domain $\Omega_0 \supset \Omega$. Then we define the active mesh

$$\mathcal{T}_h = \{K \in \mathcal{T}_{0,h} : K \cap \Omega \neq \emptyset\} \tag{4.1}$$

on which we define our finite element spaces. We also define the computational domain

$$\Omega_{\mathcal{T}_h} = \bigcup_{K \in \mathcal{T}_h} K \tag{4.2}$$

which covers the physical domain $\Omega$. (For standard FEM $\Omega_{\mathcal{T}_h} = \Omega$.) A problem with a mesh geometric setup such as this is typically called a fictitious domain problem. This means that we do not anymore have $P_r A^k \Omega_{\mathcal{T}_h}, P_r^- A^k \Omega_{\mathcal{T}_h} \subset HA^k\Omega$, so CutFEM are in this sense non conforming methods. However, suitable extension operators $E : HA^k\Omega \to HA^k\Omega_{\mathcal{T}_h}$ can be constructed [HLZ12] such that $P_r A^k \Omega_{\mathcal{T}_h}, P_r^- A^k \Omega_{\mathcal{T}_h} \subset E(HA^k\Omega)$. In addition, we also introduce the computational domain of cut elements;

$$\mathcal{T}_{\partial,h} = \{K \in \mathcal{T}_h : K \cap \Omega \neq \emptyset\}, \tag{4.3}$$

$$\Omega_{\mathcal{T}_{\partial,h}} = \bigcup_{K \in \mathcal{T}_{\partial,h}} K. \tag{4.4}$$

New difficulties arise from the unfitted setup with regards to imposition of boundary conditions and stability.

- We cannot impose essential boundary conditions strongly anymore, since the boundaries of our elements do not coincide with the boundary of the domain.

- We are also not guaranteed stability of the method, since a new source of instability is introduced in the system matrix by elements $K$ cut by the boundary $\partial\Omega$ in a way such that

$$|K \cap \Omega| \ll |K|. \tag{4.5}$$

The resolution of these issues is in a sense what characterizes CutFEM. Essential boundary conditions are enforced weakly using either an approach called Nitsche's method [HH02], or via Lagrange multiplier variables [BCH+15]. Stability and robustness of the condition number is further ensured by certain stabilization terms added to the system matrix [Bur10, BH10, HLZ14].

*Remark* 4.1. There are different ways to handle these issues, and each has loosely speaking given rise to a different method. We briefly mention some of these alternatives to CutFEM.

- XFEM enriches the finite element space to capture discontinuities. It's a popular alternative in the finite element framework for handling interface problems [MDB99].

- The Shifted Boundary Method (SBM) was proposed as an alternative to embedded/unfitted boundary methods, SBM belongs to the approximate domain methods class. A key feature of SBM is the shifting of boundary conditions from the actual boundary to a surrogate boundary. This approach allows the method to avoid the integration over cut cells [MS18].

- Phi-FEM is based on multiplying the finite element polynomials by a level-set function such that the product is zero on the boundary [DLL23].

## 4.1 Nitsche's method

Essential boundary conditions can be enforced weakly using Nitsche's method. Let us illustrate the concept using the derivation of the primal formulation of the Hodge Laplace equation (2.60). With $v \in HA^k\Omega \cap \mathring{\mathcal{H}}A^k\Omega$, we integrate by parts to get, and use $\star v = 0$ on $\partial\Omega$,

$$\int_\Omega f_\pi \wedge \star v = \int_\Omega d\delta u \wedge \star v + \int_\Omega \delta du \wedge \star v \tag{4.6}$$

$$= \int_\Omega \delta u \wedge \star\delta v - \underbrace{\int_{\partial\Omega} \delta u \wedge \star v}_{\neq 0} + \int_\Omega du \wedge \star dv. \tag{4.7}$$

From here we add a symmetric or antisymmetric term to $\int_{\partial\Omega} \delta u \wedge \star v$ which is equal to zero in the continuous setting due to $\star u = 0$ on $\partial\Omega$. Such a term is called consistent. We also add a consistent penalty term to further enforce the essential condition. Altogether we get

$$\int_\Omega f_\pi \wedge \star v = \int_\Omega \delta u \wedge \star\delta v + \int_\Omega du \wedge \star dv \tag{4.8}$$

$$- \int_{\partial\Omega} \delta u \wedge \star v + \int_{\partial\Omega} u \wedge \star\delta v + \int_{\partial\Omega} \lambda h^{-1} u \wedge \star v, \tag{4.9}$$

where $h$ is the mesh size parameter, and $\lambda \in \mathbb{R}$ is a known penalty parameter which taken large enough ensures stability of the system. Exchanges continuous space for the discrete polynomial spaces gives the resulting method.

*Remark* 4.2 (Lagrange multiplier variables). There is another way to enforce essential conditions weakly that we have made use of in Paper B and Paper C. Briefly, the idea is to mimic the strong enforcing of the fitted case by encoding the condition in the space of trial functions (i.e. the space corresponding to the unknown $u$.) So in the example above, we would make $u \in \mathring{\mathcal{H}}A^k\Omega$ explicit, and search for $u \in \{v \in HA^k\Omega \cap \mathcal{H}A^k\Omega : \gamma(\star u) = 0\}$. This condition is enforced weakly by adding a Lagrange multiplier variable $p \in HA^k\Omega$ to the Lagrangian, active only close the boundary $\partial\Omega$. The resulting weak formulation is mixed. (One can also consider a Lagrange multiplier in the trace space of $u$ on $\partial\Omega$ if $\partial\Omega$ is isoparametric.)

## 4.2 Ghost penalty stabilization

To illustrate the concept of ghost penalty stabilization, we consider the primal discrete formulation of the Hodge Laplace equation (2.60) for $k = 0$. The boundary conditions are natural, and we must assume $\int_\Omega u_h = 0$ from

$\mathfrak{H}^0 = \mathbb{R}$. In CutFEM the norm $\|\cdot\|$ of the discrete solution typically needs to taken over the entire active mesh $\Omega_{\mathcal{T}_h}$, and not just $\Omega$. To get stability one could proceed as follows:

$$\|u_h\|^2 \overset{?}{\lesssim} \|\nabla u_h\|_\Omega^2 = (f, u_h)_\Omega \leq \|f\|_\Omega \|u_h\|_\Omega \overset{?}{\lesssim} \|f\|_\Omega \|u_h\|. \tag{4.10}$$

The equality is just the equation and the first inequality is the Cauchy-Schwarz inequality; the question is whether we can perform the other two inequalities.

Let $\mathcal{T}_H$ be an initial mesh for which $\Omega_{\mathcal{T}_h} \subset \Omega_{\mathcal{T}_H}$ for all $h$. We can require that $u_h|_{\partial\Omega_{\mathcal{T}_h}} = 0$ and then extend by 0 out to $\Omega_{\mathcal{T}_H}$. Then by the Poincaré inequality we have

$$\|u_h\|_\Omega \leq \|Eu_h\|_{\Omega_{\mathcal{T}_H}} \lesssim \|\nabla E u_h\|_{\Omega_{\mathcal{T}_H}} = \|\nabla u_h\|_{\Omega_{\mathcal{T}_h}} \tag{4.11}$$

which would resolve the second unknown inequality if $\|\nabla u_h\|_{\Omega_{\mathcal{T}_h}}$ were our norm. Taking this norm as a candidate, how do we get

$$\|\nabla u_h\|_{\Omega_{\mathcal{T}_h}} \lesssim \|\nabla u_h\|_\Omega? \tag{4.12}$$

Since $\Omega_{\mathcal{T}_h} \supset \Omega$ there is no way. However, if we could constuct a term $s(\nabla u_h, \nabla u_h)$ with the property that

$$\|\nabla u_h\|_{\Omega_{\mathcal{T}_h}} \lesssim \|\nabla u_h\|_\Omega + s(\nabla u_h, \nabla u_h), \tag{4.13}$$

then a modification of the discrete formulation with $s(\nabla u_h, \nabla u_h)$ in the left hand side would solve the problem:

$$\|\nabla u_h\|_{\Omega_{\mathcal{T}_h}}^2 \lesssim \|\nabla u_h\|_\Omega^2 + s(\nabla u_h, \nabla u_h) = (f, u_h)_\Omega \tag{4.14}$$

$$\leq \|f\|_\Omega \|u_h\|_\Omega \lesssim \|f\|_\Omega \|\nabla u_h\|_{\Omega_{\mathcal{T}_h}}. \tag{4.15}$$

This is the idea of ghost penalty stabilization. The term $s(\nabla u_h, \nabla u_h)$ is called a stabilization term, and it is added to the left hand side of the discrete formulation. The stabilization term is chosen such that it is consistent, i.e. it vanishes in the continuous setting. The stabilization term is also chosen such that it is coercive, i.e. it is bounded below by a positive constant.

Let us be more concrete and discuss some actual stabilization terms satisfying the above mentioned heuristics. Introduce the set of cut facets

$$\mathcal{F}_h^* = \{F \in \mathcal{F}_h : F = T_1 \cap T_2,\ T_1, T_2 \in \mathcal{T}_h, T_1 \cap \partial\Omega \neq \emptyset\}. \tag{4.16}$$

For a given facet $F \in \mathcal{F}_h^*$ define the facet patch $M_F = T_1 \cup T_2$ consisting of the union of the elements $T_1, T_2 \in \mathcal{T}_h$ sharing the facet $F$. Let $u_h, v_h$ be some arbitrary discrete $k-$forms in $L^2(\Omega_{\mathcal{T}_h})$. On $M$ we can define the patch jump [Pre18] as

$$[u_h] = u_{h,1} - u_{h,2}, \tag{4.17}$$

where $u_{h,i} = u_h|_{T_i}$ on $T_i$ and $u_{h,i} = u_h|_{T_i}$ extended canonically on $T_j$, $j \neq i$ (e.g. the polynomial $u_h|_{T_1}$ can be evaluated on $T_2$) We can then define the stabilization term as

$$s(u_h, v_h) = \sum_{F \in \mathcal{F}_h^*} \int_{M_F} \eta [u_h] \wedge \star [v_h], \tag{4.18}$$

where $\eta$ is a positive constant. Note that $\star[v_h] = [\star v_h]$.

**Theorem 4.3.** *The following inequalities hold:*

$$\|u_h\|_{T_1}^2 \lesssim \|u_h\|_{T_2}^2 + \int_{M_F} [u_h] \wedge \star [u_h], \tag{4.19}$$

$$\|u_h\|_{\Omega_{\mathcal{T}_h}}^2 \lesssim \|u_h\|_{\Omega}^2 + s(u_h, u_h). \tag{4.20}$$

*Proof.* Let $v = u_h$. Consider a point $x \in T_1$. It holds that

$$v_1(x) = v_2(x) + [v](x), \tag{4.21}$$

so squaring (i.e. multiplying with $\star v_1$), integrating over $T_1$ and applying Young's inequality gives

$$\|v_1\|_{T_1}^2 \lesssim \|v_2\|_{T_1}^2 + \int_{T_1} [v] \wedge \star [v] \lesssim \|v_2\|_{T_1}^2 + \int_{M_F} [v] \wedge \star [v] \tag{4.22}$$

$$\lesssim \|v_2\|_{T_2}^2 + \int_{M_F} [v] \wedge \star [v]. \tag{4.23}$$

The last inequality follows from the equivalence of norms due to shape regularity of the mesh.

For proving (4.20), we decompose all elements of $\mathcal{T}_h$ into cut and uncut elements (by $\partial\Omega$.) To a face $F$ lying on the boundary of a cut element, we have two situations. Either one of $T_1, T_2$ are cut, or both of them is cut. In the first case in the inequality (4.19) above we can WLOG assume $T_2 \subset \Omega$, i.e. $T_1$ is cut. Then we it holds

$$\|u_{h,1}\|_{T_1}^2 \lesssim \|u_{h,2}\|_{\Omega}^2 + \int_{M_F} [u_h] \wedge \star [u_h]. \tag{4.24}$$

In the second case, by assumptions on the mesh, either $T_1$ or $T_2$ has a neighbour $T_3 \subset \Omega$, WLOG assume $T_3 \cap T_2 = F'$ is a facet. So for the norm over $T_2$ we can proceed as in the first case. Then by (4.19) applied two times

$$\begin{aligned}
\|u_h\|_{T_1}^2 &\lesssim \|u_h\|_{T_2}^2 + \int_{M_F} [u_h] \wedge \star [u_h] && (4.25)\\
&\lesssim \|u_h\|_{T_3}^2 + \int_{M_F} [u_h] \wedge \star [u_h] + \int_{M_{F'}} [u_h] \wedge \star [u_h] && (4.26)\\
&\lesssim \|u_h\|_{\Omega}^2 + \int_{M_F} [u_h] \wedge \star [u_h] + \int_{M_{F'}} [u_h] \wedge \star [u_h]. && (4.27)
\end{aligned}$$

Thereafter we can sum over all cut elements $T_1$ to get (4.20) from $\|u_h\|_{\Omega_{\mathcal{T}_h}}^2$. □

*Remark* 4.4. Note in the proof above that we do not need to require that a cut element is at most one cut element away from an uncut element. It is sufficient to assume that there is a finite sequence of cut elements (forming a sort of path) which eventually leads to an uncut element. Just perform the trick of applying (4.19) several times.

**Corollary 4.5.** *The norms* $\|u_h\|_{\Omega_{\mathcal{T}_h}}$ *and* $\|u_h\|_\Omega + s(u_h, u_h)^{1/2}$ *are equivalent.*

*Proof.* By Theorem 4.3 it holds that $\|u_h\|_{\Omega_{\mathcal{T}_h}} \lesssim \|u_h\|_\Omega + s(u_h, u_h)^{1/2}$. The other direction follows from again the equivalence of norms due to shape regularity of the mesh:

$$\begin{aligned}
\|[u_h]\|_{M_F}^2 &= \|u_{h,1} - u_{h,2}\|_{T_1}^2 + \|u_{h,1} - u_{h,2}\|_{T_2}^2 && (4.28)\\
&\lesssim \|u_{h,1}\|_{T_1}^2 + \|u_{h,2}\|_{T_2}^2 = \|u_h\|_{M_F}^2 \leq \|u_h\|_{\Omega_{\mathcal{T}_h}}^2. && (4.29)
\end{aligned}$$

□

Negative powers of $h$ can be added to the stabilization term, meaning $s \mapsto h^{-j} s$ for some $j > 0$, and Theorem 4.3 we of course still hold. We will however not get the norm equivalence. In that sense $j = 0$ is optimal.

*Remark* 4.6 (Other types of stabilization). There are other types of stabilization operators that can be used. For instance, the interior facet penalty term penalizes jumps in normal derivatives across interior facets. The proof of Theorem 4.3 can be adapted to show an analogous result for the interior facet penalty term, see [MLLR14]. The presently used patch extension stabilization is convenient from an implementational point of view since one does not need to implement higher order derivative operators.

The main contribution of this thesis is the development of a mixed ghost penalty stabilization term for the Hodge Laplace equation. It utilizes the fact (3.38) that $dP_rA^k\Omega \subset P_{r-1}A^{k+1}\Omega$ and $dP_r^-A^k\Omega \subset P_{r-1}^-A^{k+1}\Omega$. For $(\omega_h, u_h) \in P_rA^k\Omega \times P_rA^{k+1}\Omega$ we define it as

$$s_d(\omega_h, u_h) = \sum_{F \in \mathcal{F}_h^*} \int_{M_F} \eta[d\omega_h] \wedge \star[u_h]. \tag{4.30}$$

This is the most general form of it that the author can think of. The following result is also the most abstracted version of its kind.

**Theorem 4.7.** *The equation*

$$(d\omega_h, v_h)_\Omega + s_d(\omega_h, v_h) = 0,\ \ \forall v_h \in P_rA^{k+1}\Omega, \tag{4.31}$$

*is sufficient to guarantee* $d\omega_h = 0$ *pointwise.*

*Proof.* Pick $v_h = d\omega_h \in P_{r-1}A^{k+1}\Omega \subset P_rA^{k+1}\Omega$. Then the equation reads

$$0 = \|d\omega_h\|_\Omega^2 + s(d\omega_h, d\omega_h) \gtrsim \|d\omega_h\|_{\Omega_{\mathcal{T}_h}}^2 \tag{4.32}$$

from Theorem 4.3, which implies $d\omega_h = 0$ pointwise by definition of the norm. □

This means that the mixed ghost penalty stabilization term can be added to the mixed discrete formulation of the Hodge Laplace equation (3.43a)-(3.43b) to ensure stability (pertaining to the $u_h$ variable) and satisfaction of the conservation law $d\omega_h = 0$. For each $k = 0, 1, \ldots, n-1$ and $n = 2, 3$ we get a different result. The one used most in the papers of this thesis is $k = n-1$ for which $d = \nabla\cdot$, the divergence. The reader is encouraged to try some other values of $k$ for themselves.

**Proposition 4.8.** *Let* $(u_h, u_h)_h = \|u_h\|_h^2 = \|u_h\|_\Omega^2 + s(u_h, u_h)$. *The saddle-point action functional*

$$\inf_{u \in HA^k} \sup_{\omega \in HA^{k-1}} \left( (d\omega, u)_h - \frac{1}{2}(\omega, \omega)_h + \frac{1}{2}(du, du)_h - (f, u)_\Omega \right) \tag{4.33}$$

*gives rise to a well posed mixed CutFEM formulation, which includes the equation* (4.31)*.*

The problem is not equivalent to the problem of solving

$$\inf_{u \in HA^k} \sup_{\omega \in HA^{k-1}} \left( (d\omega, u)_{\Omega_{\mathcal{T}_h}} - \frac{1}{2}(\omega, \omega)_{\Omega_{\mathcal{T}_h}} + \frac{1}{2}(du, du)_{\Omega_{\mathcal{T}_h}} - (Ef, u)_{\Omega_{\mathcal{T}_h}} \right) \tag{4.34}$$

which requires the existence of an extension $Ef$ out to $\Omega_{\mathcal{T}_h}$. Indeed, since the terms of the sort $(u, u)_{\Omega_{\mathcal{T}_h} \setminus \Omega}$ are not weakly consistent like $s(u, u)$, the problem (4.34) is not well posed if one replaces $(Ef, u)_{\Omega_{\mathcal{T}_h}} \mapsto (f, u)_\Omega$.

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