Math 124 - Programming for Mathematical Applications

UC Berkeley, Spring 2023

Homework 12

Due Wednesday April 26

Problem 1

It is well known that $\sum_{i=1}^{n} i = \frac{1}{2} n (1+n)$. Make a table with similar formulas for $\sum_{i=1}^{n} i^k$, with *k* ranging from 1 to 8.

In[2]:=
```
Table[Sum[i^k, {i, 0, n}], {k, 1, 8}]
```

Out[2]=

$$\left\{\frac{1}{2} n (1+n), \frac{1}{6} n (1+n) (1+2n), \frac{1}{4} n^2 (1+n)^2, \frac{1}{30} n (1+n) (1+2n) \left(-1+3n+3n^2\right), \frac{1}{12} n^2 (1+n)^2 \left(-1+2n+2n^2\right), \frac{1}{42} n (1+n) (1+2n) \left(1-3n+6n^3+3n^4\right), \frac{1}{24} n^2 (1+n)^2 \left(2-4n-n^2+6n^3+3n^4\right), \frac{1}{90} n (1+n) (1+2n) \left(-3+9n-n^2-15n^3+5n^4+15n^5+5n^6\right)\right\}$$

Problem 2

Use the Factor function to prove that the product of four consecutive numbers plus one is always a squared number.

In[48]:=
```
Factor[(a * (a + 1) * (a + 2) * (a + 3)) + 1]
```

Out[48]=

$$\left(1+3a+a^2\right)^2$$

Where (1 + 3a + a^2) can be a variable, name it k. It is then clear to see that the product of four consecutive numbers plus one, or k, is always squared.

Problem 3

Show that the formula $n^2 + n + 41$ produces prime numbers for n from 0 to 39.

In[280]:=

```
j = Table[i^2 + i + 41, {i, 0, 39}]
PrimeQ[j]
```

Out[280]=

```
{41, 43, 47, 53, 61, 71, 83, 97, 113, 131, 151, 173, 197, 223,
 251, 281, 313, 347, 383, 421, 461, 503, 547, 593, 641, 691, 743, 797,
 853, 911, 971, 1033, 1097, 1163, 1231, 1301, 1373, 1447, 1523, 1601}
```

Out[281]=

```
{True, True, True, True, True, True, True, True, True, True, True, True, True,
 True, True, True, True, True, True, True, True, True, True, True, True, True, True,
 True, True, True, True, True, True, True, True, True, True, True, True, True}
```

Problem 4

11 is the first prime number with all digits equal to 1. Find the next one (using a loop).

In[101]:=

```
For[i = 111, i < Infinity, i = (i * 10) + 1,
 If[PrimeQ[i], Print[i]; Break[], Continue[]]
]
```

1 111 111 111 111 111 111

Problem 5

Define the function $f(x)$ as follows:

$$f(\mathrm{xy}) = f(x) + f(y)$$
$$f(x^n) = \mathrm{nf}(x)$$
$$f(n) = 0$$

where n is an integer. Show that

$$f\left(\prod_{k=1}^{20} k!\,(x_k)^k\right) = \sum_{k=1}^{20} k\, f(x_k)$$

In[593]:=
```
Clear[x]
v[x_ * y_] := v[x] + v[y]
v[x_^n_] := n * v[x]
v[n_Integer] := 0
lhs = v[Product[k ! * (x[[k]])^k, {k, 1, 20}]]
rhs = Sum[k * v[x[[k]]], {k, 1, 20}]
result = lhs - rhs
```

Out[597]=
```
v[x[[1]]] + 2 v[x[[2]]] + 3 v[x[[3]]] + 4 v[x[[4]]] + 5 v[x[[5]]] + 6 v[x[[6]]] + 7 v[x[[7]]] + 8 v[x[[8]]] +
 9 v[x[[9]]] + 10 v[x[[10]]] + 11 v[x[[11]]] + 12 v[x[[12]]] + 13 v[x[[13]]] + 14 v[x[[14]]] +
 15 v[x[[15]]] + 16 v[x[[16]]] + 17 v[x[[17]]] + 18 v[x[[18]]] + 19 v[x[[19]]] + 20 v[x[[20]]]
```

Out[598]=
```
v[x[[1]]] + 2 v[x[[2]]] + 3 v[x[[3]]] + 4 v[x[[4]]] + 5 v[x[[5]]] + 6 v[x[[6]]] + 7 v[x[[7]]] + 8 v[x[[8]]] +
 9 v[x[[9]]] + 10 v[x[[10]]] + 11 v[x[[11]]] + 12 v[x[[12]]] + 13 v[x[[13]]] + 14 v[x[[14]]] +
 15 v[x[[15]]] + 16 v[x[[16]]] + 17 v[x[[17]]] + 18 v[x[[18]]] + 19 v[x[[19]]] + 20 v[x[[20]]]
```

Out[599]=
```
0
```

As you can see, the difference between both the left hand side and the right hand side is 0, so they are both the same.

Problem 6

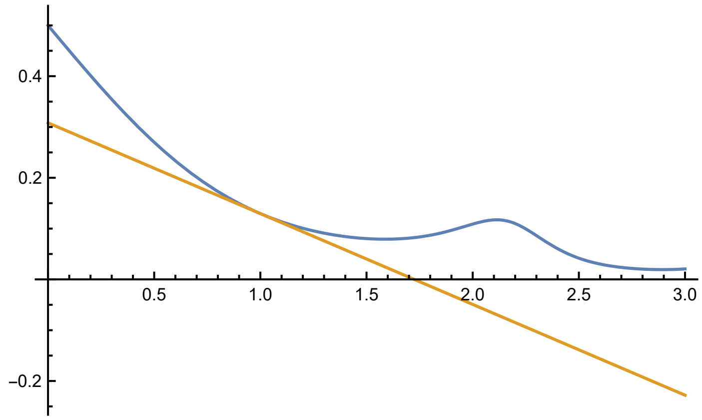
a) Plot the function $f(x) = e^{-x} / (2 + \sin(x^2))$ and its tangent line $g(x)$ at $x = 1$ for $x \in [0, 3]$.

b) Calculate the integral of $f(x) - g(x)$ between $x = 0$ and $x = 1$ numerically with 100 digits.

In[464]:=

```
L[j_] := E^(-j) / (2 + Sin[j^2])
L'[j_] := D[L[j], x]
K[j_] := L'[1] (j - 1) + L[1]
Plot[{L[x], K[x]}, {x, 0, 3}]
N[Integrate[L[x] - K[x], {x, 0, 1}], 100]
```

Out[467]=



Out[468]=

```
0.06667043442169600266471868974514408447550431370133286482763417685063844918951760745239·.
195961065844986
```

Problem 7

Define the following piecewise function:

$$f(x) = \begin{cases} -x & \text{if } |x|<1 \\ \sin(x) & \text{if } 1 \le |x| < 2 \\ \cos(x) & \text{otherwise.} \end{cases}$$

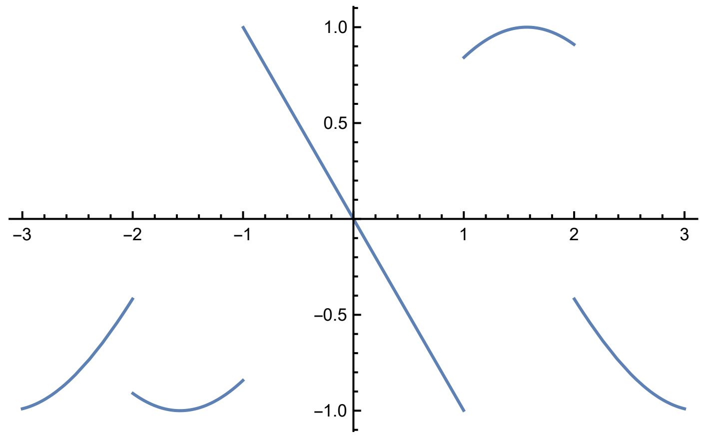
a) Plot $f(x)$ between $x = -3$ and $x = 3$.

b) Calculate the integral of $1/\left(1 + f(x)^2\right)$ between $x = -3$ and $x = 3$ (symbolically).

In[476]:=

```
f[x_] := Which[
      Abs[x] < 1, Return[-x],
      1 ≤ Abs[x] && Abs[x] < 2, Return[Sin[x]],
      True, Return[Cos[x]]
 ]
Plot[f[x], {x, -3, 3}]
```

Out[477]=



In[502]:=

```
Integrate[1 / (1 + (-x) ^2), {x, -3, -1}]
Integrate[1 / (1 + Sin[x] ^2), {x, -1, 1}]
Integrate[1 / (1 + Cos[x] ^2), {x, 1, 3}]
```

Out[502]=

$$-\frac{\pi}{4} + \text{ArcTan}[3]$$

Out[503]=

$$\sqrt{2}\ \text{ArcTan}\left[\sqrt{2}\ \text{Tan}[1]\right]$$

Out[504]=

$$\frac{\pi - \text{ArcCot}\left[\sqrt{2}\ \text{Cot}[1]\right] + \text{ArcCot}\left[\sqrt{2}\ \text{Cot}[3]\right]}{\sqrt{2}}$$