Charter 1: Linear Equations

Question: Consider a two-commodity market. When the unit prices of the products are P_1 and P_2, the quantities demanded, D_1 and D_2, and the quantities supplied, S_1 and S_2 are given by

$$D_1 = 70 - 2P_1 + P_2, \qquad S_1 = -14 + 3P_1$$
$$D_2 = 105 + P_1 - P_2, \qquad S_2 = -7 + 2P_2$$

a) What is the relationship between the two commodities? Do they compete, or do they complement one another?

b) Find the equilibrium prices for both products.

Another example, suppose we want to solve for *x*, *y*, *z*:

$$2x + 8y + 4z = 2$$
$$2x + 5y + z = 5$$
$$4x + 10y - z = 1$$

At the end we aim to get x = …, y = …, and z = …

How do we get that?

A system with infinitely many solutions:

A system without solutions:

Matrices and Gauss-Jordan Elimination

Definition: An n by m matrix is the array of numbers with n rows and m columns.

A matrix with only one column is called a *column vector*, or simply a *vector*. The entries of a vector are called its *components*. The set of all column vectors with n components is denoted by $\mathbb{R}^n$.

A matrix with only one row is called a *row vector*.

Ex. From above system of linear equations, we write the coefficients of each variable. It is called a

coefficient matrix: $$\begin{bmatrix} 2 & 8 & 4 \\ 2 & 5 & 1 \\ 4 & 10 & -1 \end{bmatrix}$$

Then we add a column of numbers on the right of equal signs in the system. We call it an *augmented matrix*.

$$\left[\begin{array}{ccc:c} 2 & 8 & 4 & 2 \\ 2 & 5 & 1 & 5 \\ 4 & 10 & -1 & 1 \end{array}\right]$$

To get a solution, we can either deal with system of equations directly or with this augmented matrix above.

However, dealing with augmented matrix requires less writing, saves time and is easier to read.

We basically want to use row reduction to get 1 on each row and each column on the left hand side of a dotted line.

In this example, the process of elimination works smoothly. We can eliminate all entries off the diagonal and can make each coefficient on the diagonal equal to 1. The process of elimination works well unless we encounter a zero along the diagonal. These zeros represent missing terms in some equations.

Definition: A matrix is in **reduced row-echelon form** if it satisfies all of the following conditions:

a. If a row has nonzero entries, then the first nonzero entry is 1, called the leading 1 in this row.

b. If a column contains a leading 1, then all other entries in that column are zero.

c. If a row contains a leading 1, then each row above contains a leading 1 further to the left.

Ex. $$\left[\begin{array}{ccccc|c} 1 & 2 & 0 & 3 & 0 & 2 \\ 0 & 0 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

This example contains 5 variables: x_1, x_2, x_3, x_4, x_5

With above matrix, we have

$$x_1 = 2 - 2x_2 - 3x_4$$
$$x_3 = 2 + x_4$$
$$x_5 = -2$$

Then we can freely choose the nonleading variables, $x_2 = s$ and $x_4 = t$, where *s* and *t* are arbitrary real numbers. The leading variables are then determined by our choices for *s* and *t*. So this system has infinitely many solutions, namely,

$$x_1 = 2 - 2s - 3t, \quad x_2 = s$$
$$x_3 = 2 + t, \quad x_4 = t, \quad x_5 = -2$$

We can represent the solutions as vectors in $\mathbb{R}^5$.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2 - 2s - 3t \\ s \\ 2 + t \\ t \\ -2 \end{bmatrix}$$

We often write as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \\ -2 \end{bmatrix} + s\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -3 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

Summary for solving systems of linear equations

Write the augmented matrix of the system. Place a cursor in top entry of the first nonzero column of this matrix.

Step 1 If the cursor entry is zero, swap the cursor row with some row below to make the cursor entry nonzero.

Step 2 Divide the cursor row by the cursor entry

Step 3 Eliminate all other entries in the cursor column, by subtracting suitable multiples of the cursor row from the other rows.

Step 4 Move the cursor down one row and over one column. If the new cursor entry and all entries below are zero, move the cursor to the next column (remaining in the same row). Repeat the last step if necessary.

Return to step 1.

The process ends when we run out of rows or columns. Then, the matrix is in reduced echelon form (rref). If the echelon form contains the equation 0=1, then there are no solutions; the system is inconsistent.

The operations performed in steps 1, 2, and 3 are called elementary row operations: Swap two rows, divide a row by a scalar, or subtract a multiple of a row from another row.

Example:

On the Solutions of Linear Systems

Fact: Number of solutions of a linear system

A linear system has either

- no solutions (it is inconsistent)
- exactly one solution (if the system is consistent and all variables are leading), or
- infinitely many solutions (if the system is consistent and there are non leading variables).

Rank: The rank of a matrix A is the number of leading 1's in rref(A).

Example: Find the rank of $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

Proposition: A linear system with fewer equations than unknowns has either no solutions or infinitely many solutions.

Proposition: A linear system of *n* equations with *n* unknowns has a unique solution if and only if the rank of its coefficient matrix *A* is *n*. This means that

$$\mathrm{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

The vector form and the matrix form of a linear system

$$\left| \begin{matrix} 3x + y = 7 \\ x + 2y = 4 \end{matrix} \right|$$

$$x\begin{bmatrix} 3 \\ 1 \end{bmatrix} + y\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \end{bmatrix}$$

This is called the **vector form** of the linear system. To solve this system means to write the vector $\begin{bmatrix} 7 \\ 4 \end{bmatrix}$ as the sum of a scalar multiple of $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and a scalar multiple of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ (see picture page 28)

Now consider the general linear system with *m* equations, *n* unknowns.

$$\left| \begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\ \vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \qquad\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m \end{array} \right|$$

We can write

$$\begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \\ \vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

or

$$x_1\vec{v}_1 + x_2\vec{v}_2 + \ldots + x_n\vec{v}_n = \vec{b}$$

where $\vec{v}_i = \begin{bmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{mi} \end{bmatrix}$ for i = 1, 2, …, n ; each $\vec{v}_i, \vec{b} \in \mathbb{R}^m$

Linear Combinations

A vector $\vec{b}$ in $\mathbb{R}^m$ is called a *linear combination* of the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ in $\mathbb{R}^m$ if there are scalars $x_1, x_2, \ldots, x_n$ such that $\vec{b} = x_1\vec{v}_1 + x_2\vec{v}_2 + \ldots + x_n\vec{v}_n$.

Solving a linear system with augmented matrix $\left[A \;\vdots\; \vec{b} \right]$ amounts to writing the vector $\vec{b}$ as a linear combination of the column vectors of A.

Another compact representation of a linear system:

$$A\vec{x} = \vec{b}$$

i.e. $$x_1\vec{v}_1 + x_2\vec{v}_2 + \ldots + x_n\vec{v}_n = \vec{b}$$

$$\text{where } A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}$$

$$\text{and } \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$$

In words, the product $A\vec{x}$ is the linear combination of the columns of A with the components of $\vec{x}$ as coefficients.

Thus we call $A\vec{x} = \vec{b}$ the *matrix form* of the linear system.

Ex. $\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} =$

Theorem: For an $m \times n$ matrix A, two vectors $\vec{x}$ and $\vec{y}$ in $\mathbb{R}^n$, and a scalar *k*,

a) $A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y}$

b) $A(k\vec{x}) = k(A\vec{x})$

Proof: (a)

Theorem: If $\vec{x}$ is a vector in $\mathbb{R}^n$ and A is an $m \times n$ matrix with *row vectors* $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_m$, then

$$A\vec{x} = \begin{bmatrix} \leftarrow & \vec{w}_1 & \rightarrow \\ \leftarrow & \vec{w}_2 & \rightarrow \\ & \vdots & \\ \leftarrow & \vec{w}_m & \rightarrow \end{bmatrix} \vec{x} = \begin{bmatrix} \vec{w}_1 \cdot \vec{x} \\ \vec{w}_2 \cdot \vec{x} \\ \vdots \\ \vec{w}_m \cdot \vec{x} \end{bmatrix}$$

(that is, the i th component of $A\vec{x}$ is the dot product of $\vec{w}_i$ and $\vec{x}$.)

Example: $\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} =$

Note: Sum of matrices and scalar multiples of matrices are simple.

Let's look at a real vector space $\mathbb{R}^n$

Definition: A real vector space $\mathbb{R}^n$ is a set endowed with a rule for addition and a rule for scalar multiplication such that these operations satisfy the following eight rules: For all vectors $\vec{u}, \vec{v}, \vec{w}$ in $\mathbb{R}^n$, and all scalar *c, k* in $\mathbb{R}$, we have

1) $(\vec{u}+\vec{v})+\vec{w}=\vec{u}+(\vec{v}+\vec{w})$ Associative

2) $\vec{u}+\vec{v}=\vec{v}+\vec{u}$ Commutative

3) There is a unique *neutral element* $\vec{0}$ such that $\vec{u}+\vec{0}=\vec{u}$, for all $\vec{u}$ in V.

4) For any vector $\vec{u}$, there is a unique inverse $(-\vec{u})$ such that $\vec{u}+(-\vec{u})=\vec{0}$

5) $c(\vec{u}+\vec{v})=c\vec{u}+c\vec{v}$ Distributive

6) $(c+k)\vec{v}=c\vec{v}+k\vec{v}$ Distributive

7) $c(k\vec{v})=(ck)\vec{v}$

8) $1\cdot\vec{v}=\vec{v}$