MATH 481, SPRING 2021

PROJECT 3
POLLUTION IN LAKES

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Abstract. We present a mathematical model of two interconnected reservoirs and solutes flowing through them. We derive a system of differential equations for the time-dependent concentrations of the solutes in the two reservoirs. We apply our model to simulate the concentrations of pollutants in lakes Erie and Ontario, and investigate the time it takes to stabilize the lakes to a lower pollution level after cutting down the inflow of pollutants.

1. Introduction

Lake Erie and Lake Ontario, two of the five Great Lakes, also known as the Great Lakes of North America, are part of a series of interconnected lakes in the upper mid-east region of North America that connect to the Atlantic Ocean. Lake Ontario is connected to Lake Erie via Niagara Falls, and both lakes have been at the center of attention for their problems with over-fishing, mercury and sewage pollution, and invasive species like cyanobacteria, eventually leading to high numbers of dead fish washing up on the shores. Since the mid - 20th century, environmental clean up efforts were initiated and eventually lead to improved conditions for both of the lakes, leading to a resurgence in their respective aquatic ecosystems. However, these efforts have not been regulated since. In this article we will determine the time it will take to clean up both lakes again, given that strict enough regulations are put into place for both lakes [2, 3, 4]. The model presented in this article is effectively similar to the one derived by Robert H. Rainey in 2009 [1].

In section 2 we describe how the system of differential equations is going to be formulated for a general model of two connected lakes, according to Figure 1, along with the assumptions we are going to make to simplify the model. Then in section 3 we actually derive the system of differential equations using the concept of conservation of mass, and we will express the solution in terms of residence times. We find the solutions for this system for the initial concentrations of pollutants along with the equilibrium concentrations. In section 4 we rework the system of differential equations given that some regulations are put in place to limit the inflow of pollutants in both lakes, which yields a new set of solutions with new equilibrium values. From this we can determine how much time it will take to reduce the level of pollution to within 95% of the equilibrium value for the second lake. Finally, in section 5 we tie this model in with the two lakes, Erie and Ontario, and how we can determine how much time it would take to clean up the lakes given reasonably stringent regulations.

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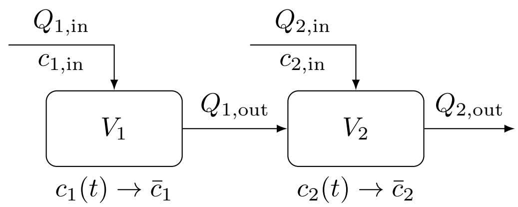

FIGURE 1. A schematic diagram of two interconnected reservoirs. If the volumes V_1 and V_2, the flow rates $Q_{1,\text{in}}$ and $Q_{2,\text{in}}$, and the input concentrations $c_{1,\text{in}}$ and $c_{2,\text{in}}$ are constants, then the concentrations $c_1(t)$ and $c_2(t)$ in the reservoirs tend to equilibrium values $\bar{c}_1$ and $\bar{c}_2$ in the long term.

2. The mathematical model

The schematic diagram in Figure 1 represents two interconnected and well-stirred reservoirs of volumes V_1 and V_2. A solute of concentration $c_{1,in}$ enters the first reservoir at the volumetric flow rate of $Q_{1,in}$. The output of the first reservoir flows into the second reservoir where it joins another inflow of the same solute at concentration $c_{2,in}$ at the flow rate of $Q_{2,in}$, then the mixture flows out at the rate of $Q_{2,out}$. The concentrations of the solutes in the two reservoirs are $c_1(t)$ and $c_2(t)$.

In this case study we will assume that the volumes V_1 and V_2, the flow rates $Q_{1,in}$ and $Q_{2,in}$, and the input concentrations $c_{1,in}$ and $c_{2,in}$ are constants, that is, they do not change with time. Consequently, $Q_{1,out} = Q_{1,in}$ and $Q_{2,out} = Q_{1,out} + Q_{2,in}$

3. The Equations of Balance of Mass

We derive the system of differential equations by considering two interconnected reservoirs of volumes V_1 and V_2, as in Figure 1. Each reservoir has their own inflow rate and outflow rate, as well their own concentration of pollutants. We seek to determine the concentrations of pollutants in both lakes at some time t.

First, we use unit analysis to find all relevant quantities. The total mass in either lake can generally be found by the product $Vc(t)$, where the rate of change of mass of pollutants in either lake is given by $d/dt(Vc(t))$. The rate of mass of pollutant entering the lake is given by Qc_{in}, and the rate of mass of pollutant leaving the lake is given by $Qc(t)$. We now use conservation of mass, which suggests that the rate of change of the mass of pollutant in the lake equals the rate that the mass of pollutant enters the lake minus the rate of mass of pollutant leaving the lake:

$$\begin{aligned} V_1 \frac{dc_1(t)}{dt} &= Q_{1,in}c_{1,in} - Q_{1,out}c_1(t) \\ V_2 \frac{dc_2(t)}{dt} &= Q_{2,in}c_{2,in} + Q_{1,out}c_1(t) - Q_{2,out}c_2(t). \end{aligned}$$

We can simplify this set of equations by including our assumptions from before that $Q_{1,out} = Q_{1,in}$ and $Q_{2,out} = Q_{1,out} + Q_{2,in}$. We can rearrange the latter equation such that $Q_{2,in} = Q_{2,out} - Q_{1,out}$. Hence, by making the necessary substitutions, factoring out where possible, and dividing both sides of each equation by their respective volumes, we arrive at:

$$\frac{dc_1(t)}{dt} = \frac{Q_{1,out}}{V_1}(c_{1,in} - c_1(t))$$
$$\frac{dc_2(t)}{dt} = \frac{Q_{2,out}}{V_2}(c_{2,in} - c_2(t)) + \frac{Q_{1,out}}{V_2}(c_1(t) - c_{2,in}).$$

We must note that $\frac{Q_{1,out}}{V_2}$ must be dealt with by ridding of the mixture of lake 1 and lake 2 information. We quickly observe that

$$\frac{Q_{1,out}}{V_2} = \frac{Q_{1,out}}{V_1}\left(\frac{V_1}{V_2}\right).$$

Again, this is done by using unit analysis. We can achieve the final form of our system of equations by introducing the term *residence time*. The residence time of a lake is a calculated value of the average amount of time that a substance, like a pollutant, spends in a particular body of water. We can think of this quantity as the amount of time it takes for an introduced substance to flow from one end of the lake to the other. We will denote these residence times as T_1 and T_2 for lakes 1 and 2, respectively. This value is calculated as $T = \frac{V}{Q}$, the ratio of the volume to the flow rate. This allows us to express our system of equations as:

$$\begin{aligned} \frac{dc_1(t)}{dt} &= \frac{1}{T_1}(c_{1,in} - c_1(t)) \\ \frac{dc_2(t)}{dt} &= \frac{1}{T_2}(c_{2,in} - c_2(t)) + \frac{1}{T_1}\left(\frac{V_1}{V_2}\right)(c_1(t) - c_{2,in}). \end{aligned} \tag{1}$$

We solve the system with initial conditions $c_1(0) = c_{1,0}$ and $c_2(0) = c_{2,0}$. Using Maple, we see that

$$c_1(t) = \left(1 - \mathrm{e}^{-\frac{t}{T_1}}\right) c_{1,\text{ in}} + c_{1,0}\mathrm{e}^{-\frac{t}{T_1}}$$

and the solution for $c_2(t)$, which is much more messy, can be simplified to be an expression that looks like:

$$c_2(t) = c_{0,1}(\cdots) + c_{1,in}(\cdots) + c_{2,in}(\cdots) + c_{2,in}(\cdots).$$

Lastly, we can find the equilibrium values, $\bar{c_1}$ and $\bar{c_2}$, by taking the limit as $t \to \infty$ of both solutions to obtain:

$$\begin{aligned} \bar{c_1} &= c_{1,in} \\ \bar{c_2} &= \frac{V_1 T_2 c_{1,\text{ in}}}{T_1 V_2} + \frac{(T_1 V_2 - V_1 T_2)\, c_{2,\text{ in}}}{T_1 V_2}. \end{aligned} \tag{2}$$

4. Disturbing The System

Suppose that we reduce the concentrations $c_{1,in}$ and $c_{2,in}$ of the solutes entering the reservoirs to $\alpha c_{1,in}$ and $\beta c_{2,in}$, respectively, where $0 \leq \alpha, \beta < 1$. Now the new system of differential equations looks almost identical to (1), except we include the new coefficients, which yields:

$$\frac{dc_1(t)}{dt} = \frac{\alpha c_{1,in}}{T_1} - \frac{c_1(t)}{T_1}$$
$$\frac{dc_2(t)}{dt} = \frac{1}{T_2}(\beta c_{2,in} - c_2(t)) + \frac{1}{T_1}(\frac{V_1}{V_2})(c_1(t) - \beta c_{2,in}).$$

We take the initial conditions to be the equilibrium values as displayed in (2). Using MAPLE again, we arrive at:

$$c_1(t) = \left(\alpha + (1-\alpha)\mathrm{e}^{-\frac{t}{T_1}}\right) c_{1,\text{ in}}$$

$$\begin{aligned} c_2(t) = &-\frac{V_1\left(T_1(\alpha-1)\mathrm{e}^{-\frac{t}{T_1}} - T_2(\alpha-1)\mathrm{e}^{-\frac{t}{T_2}} - \alpha\left(T_1 - T_2\right)\right) T_2 c_{1,\text{ in}}}{\left(T_1 - T_2\right) T_1 V_2} \\ &- \frac{(T_1V_2 - V_1T_2)((\beta-1)\mathrm{e}^{-\frac{t}{T_1}} - \beta)}{T_1V_2}. \end{aligned}$$

We now arrive at the equilibrium values for this modified system by again taking the limit, which yields:

$$\begin{aligned} \tilde{c_1} &= \alpha c_{1,in} \\ \tilde{c_2} &= \frac{-\beta\left(V_1T_2 - T_1V_2\right) c_{2,\text{ in}} + c_{1,\text{ in}}\,\alpha V_1 T_2}{T_1V_2}. \end{aligned}$$

It is not really practical to wait an infinite amount of time for the pollution in the lake to decrease to the equilibrium value. Furthermore, the values of $c_{1,in}$ and $c_{2,in}$ are not actually knowable. So, we would like to find a way to set a reasonable goal such that a realistic amount of time can be put into reducing the pollution in both of the lakes. We can set an arbitrary goal, i.e. we would be like to be within a certain percent of the actual equilibrium value, and determine how long it would take to achieve it. We can do this by defining $\delta(t) = c_2(t) - \tilde{c_2}$, which is the difference between the solution for lake 2 of our modified system of equations and the limit value of that equation, and equate it to some percent of the actual equilibrium value. Now, we will define $\rho(t) = \frac{\delta}{\tilde{c_2}}$ such that we can determine at what time t we reach that percent. We can see that by substituting all the values we have arrived at so far into ρ, we arrive at:

(3)

$$\rho(t) = \frac{\left(-\frac{V_1T_2(\alpha-1)\left(\mathrm{e}^{-\frac{t}{T_1}}T_1 - \mathrm{e}^{-\frac{t}{T_2}}T_2\right)c_{1,\text{ in}}}{(T_1-T_2)T_1V_2} - \frac{(T_1V_2-V_1T_2)(\beta-1)\mathrm{e}^{-\frac{t}{T_2}}c_{2,\text{ in}}}{T_1V_2}\right) T_1V_2}{\beta\left(T_1V_2 - V_1T_2\right) c_{2,\text{ in}} + c_{1,\text{ in}}\,\alpha V_1T_2},$$

and although this formula looks quite daunting, with enough investigation it is readily seen that (3) takes the form:

$$\rho(t) = \frac{(\cdots)(c_{1,in}) + (\cdots)(c_{2,in})}{(\cdots)(c_{1,in}) + (\cdots)(c_{2,in})} = \frac{(\cdots) + (\cdots)(\frac{c_{2,in}}{c_{1,in}})}{(\cdots) + (\cdots)(\frac{c_{2,in}}{c_{1,in}})}.$$

Thus, $\rho(t)$ depends on $c_{1,in}$ and $c_{2,in}$ only through the ratio $c_{2,in}/c_{1,in}$.

5. Lakes Erie and Ontario

Suppose the states around the Great Lakes region are considering tough regulations to clean up the lakes. As a "what if" scenario, we will look at the rather drastic measure corresponding to setting $\alpha = 0.5$ and $\beta = 0$, where α and β are defined in Section 4. We now introduce the necessary parameters to evaluate ρ for the system involving Lake Erie and Lake Ontario. According to the data readily found online, Lake Erie has a volume of $480km^3$ and a residence time of 2.6 years.

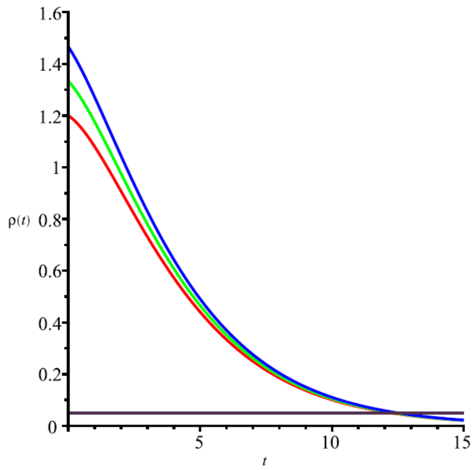

FIGURE 2. This plot represents the overall behavior of $\rho(t)$ for different choices of R. We can see that $\rho(t)$ doesn't necessarily depend on R within a reasonable approximation. The violet line represents $\rho(t) = 0.05$, and the red, green, and blue lines are for R = 0.3, 0.5, and 0.7, respectively.

Lake Ontario has a volume of $1,640km^3$ and a residence time of 6 years [3, 4]. Furthermore, we let $R = \frac{c_{2,in}}{c_{1,in}}$ be the ratio of the concentrations of pollution coming into lakes 2 and 1. We can assume that this ratio will be less than zero since Lake Erie is surrounded by more industrial activity than Lake Ontario. By substituting these parameters and $c_{2,in} = R(c_{1,in})$ into (3), we can obtain:

$$\rho(t) = \left(-2.\mathrm{e}^{-\frac{t}{6}} + 2.999999999 + 0.6666666665R\right)\mathrm{e}^{-\frac{t}{3}}.$$

This gives us an equation in terms of just time and the ratio, R. We will set $\rho(t) = 0.05$. That is to say that we will consider Lake Ontario to be stabilized when $c_2(t)$ is within 5% of $\tilde{c_2}$. It turns out that $\rho(t)$ is not very sensitive to our choice of R, as can be seen in Figure 2, and we will arbitrarily let R = 0.3, 0.5, and 0.7 to illustrate this point. Thus:

$$\rho_1(t) = \left(-2.\mathrm{e}^{-\frac{t}{6}} + 3.19999\right)\mathrm{e}^{-\frac{t}{3}}, R = 0.3$$
$$\rho_2(t) = \left(-2.\mathrm{e}^{-\frac{t}{6}} + 3.33333\right)\mathrm{e}^{-\frac{t}{3}}, R = 0.5$$
$$\rho_3(t) = \left(-2.\mathrm{e}^{-\frac{t}{6}} + 3.46667\right)\mathrm{e}^{-\frac{t}{3}}, R = 0.7$$

are the resulting equations. Setting each equal to 0.05 yields $\rho_1(t) = 12.22$, $\rho_1(t) = 12.36$, and $\rho_1(t) = 12.49$. So we can see that for any choice of R, we get that it will take about 12 years to get Lake Ontario to stabilize to a level with much lower pollution present.

Finally, we would like to see how $\tilde{c_2}$ compares to $\bar{c_2}$. That is, we would like to see how much *less* pollution is present at equilibrium in Ontario after the hypothetical regulations have been put in place. Let $F = \frac{\tilde{c_2}}{\bar{c_2}}$ be this ratio. This gives us:

$$F = \frac{-\beta \left(V_1 T_2 - T_1 V_2\right) c_{2,\text{ in}} + c_{1,\text{ in}} \alpha V_1 T_2}{T_1 V_2 \left(\frac{V_1 T_2 c_{1,\text{ in}}}{T_1 V_2} + \frac{(T_1 V_2 - V_1 T_2) c_{2,\text{ in}}}{T_1 V_2} \right)}.$$

By substituting in our parameters from before and by simplifying the expression, we obtain the simple expression

$$F = \frac{3}{6 + 2R}.$$

Again, by letting R arbitrarily range from 0.3 to 0.7, we get that F = 0.4545, 0.4286, and 0.4054, respectively. From these numbers we can conclude that the amount of pollution present in Lake Ontario after the regulation is between about 40% and 45% of what it was before the regulation was is to be put in place.

References

[1] Robert H. Rainey. "Natural Displacement of Pollution from the Great Lakes". In: *Science* 155.3767 (Mar. 10, 1967), pp. 1242–1243.

[2] Wikipedia. *Great Lakes*. Apr. 2021. URL: https://en.wikipedia.org/wiki/Great_Lakes.

[3] Wikipedia. *Lake Erie*. Apr. 2021. URL: https://en.wikipedia.org/wiki/Lake_Erie.

[4] Wikipedia. *Lake Ontario*. Apr. 2021. URL: http://en.wikipedia.org/wiki/Lake_Ontario.

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