MATH 481, SPRING 2021

PROJECT 5
POPULATION MODELS OF SHARKS AND SCALLOPS

TYRON TENTION

Abstract. We will develop upon a predator-prey model that we have previously introduced to now include human intervention. Namely, we will account for what happens to both the predator and prey populations when one or both of them are fished in order to observe the change in the equilibrium of the resulting dynamics. We will then apply this model to explain the phenomena of shark populations increasing in the Adriatic Sea during the first world war, and the virtual disappearance of scallops along the Atlantic coast in the mid-2000's.

1. Introduction

In the mid 1920's, Italian biologist Umberto D'ancona came across some data on the percentages-of-total-catch of fish species caught near different Mediterranean ports spanning the years of the first world war. The data studied specifically for the port of Fiume, Italy looked at selachians, which include sharks, rays, skates, and more. Most interestingly at the time, the trend for the data spanning the years 1914 to 1923 showed that the population of the selachians had actually increased with reduced fishing, which was of course due to the war. The data is plotted in Figure 1. D'ancona had first reasoned that reduced fishing led to more food availability to the predatory selachians. This does not directly provide a solution to what caused this sharp increase as it does not explain why the reduced fishing is actually more beneficial to the predator, only that there is more prey along with more predators [1]. We will use a modification of our previous formulation for a predator-prey model to help explain this apparently strange phenomenon.

In section 2, we will formulate a predator-prey model and modify it by including harvesting terms. This model will help us to explain Volterra's Principle and will allow us to explain a number of natural phenomena, both older and newer. We will explain how this harvesting model explains the rise in the number of sharks and other selachians in the Adriatic Sea during the first world war. In section 3, we will apply an extension of the harvesting model to a system involving sharks, rays, and scallops to help us explain mathematically why an increase in shark fishing led to a decline of the bay scallop in North Carolina.

2. Predator-Prey Model With Harvesting

We have previously explored a predator-prey model between two species, where we let $\frac{dx}{dt}$ represent the rate of growth of the prey and $\frac{dy}{dt}$ represent the rate of

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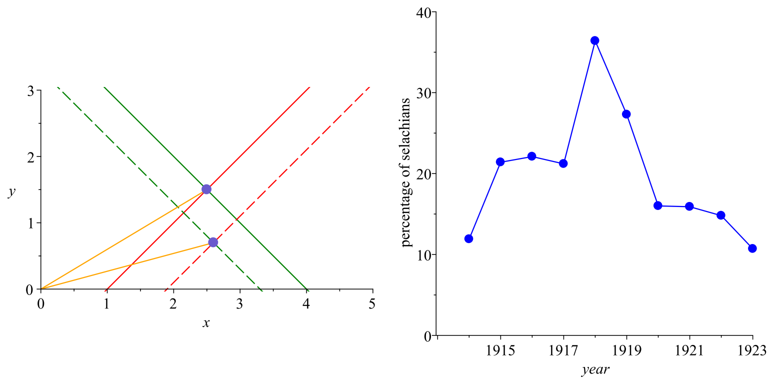

FIGURE 1. The plot on the left shows an overlay of two graphs, one for the system without harvesting, which is shown with solid lines, and the system with harvesting, shown with dashed lines. With harvesting, the number of food fish has increased on average. The plot on the left is for the data studied by D'ancona in the mid-1920's

growth of the predator. The model is as follows:

$$
\begin{aligned}
\frac{dx}{dt} &= (a_1 - b_{11}x - b_{12}y)x = f(x,y), \qquad x(0) = x_0, \\
\frac{dy}{dt} &= (-a_2 + b_{21}x - b_{22}y)y = g(x,y), \qquad y(0) = y_0.
\end{aligned} \tag{1}
$$

We have a set of coupled, non-linear differential equations, which cannot be solved analytically. The model is set up such that a_1 represents the constant growth rate of the prey, as competition increases with increasing growth rate, we again adjust the growth rate by subtracting $b_{11}x$ from the initial growth rate. Because we also have a predator, the predator will also reduce the population periodically, and so we subtract a $b_{12}y$ term, where y represents the predator population. The b_{12} constant represents population 2 acting on population 1 or y acting on x. Similarly, for the predator, let a_2 represent the natural growth rate. It is negative because the predator population actually decreases without the presence of prey. Hence, we now add a $b_{21}x$ term, which represent the increased growth rate with an increased supply of prey. Finally, we need to account for overcrowding of the predator species, which is given by the term $b_{22}y$. The growth rate declines as the predator species grows too large due to increased competition.

We now want to expand this model to account for harvesting. Specifically, we want to ultimately study the dynamics between two aquatic species, one predator and one prey, when we include mass fishing by humans of one or both species. We will assume that the harvesting rate is proportional to population growth. Let α_1 represent the constant of growth (or decline) of species 1 due to fishing, and let α_2 represent the same for species 2. We include the terms $-\alpha_1 x$ and $-\alpha_2 y$ to the system for the decline in growth rate of species x and y due to harvesting. We ultimately arrive at the following:

$$\frac{dx}{dt} = (a_1 - b_{11}x - b_{12}y)x - \alpha_1 x = f(x,y),$$
$$\frac{dy}{dt} = (-a_2 + b_{21}x - b_{22}y)y - \alpha_2 y = g(x,y).$$

We now want to see how different this modified model is to (1). We will choose arbitrary parameters to emphasize the difference. Let

$$a = \begin{pmatrix} 4 \\ 1 \end{pmatrix},\ b = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},\ \alpha = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

be our parameters. We initially let both values of $\alpha = 0$, which represents our model without harvesting. This only accounts for a predator consuming a prey, and the overcrowding leading to competition between members of the same species. Using the above parameters we can solve $\frac{f(x,y)}{x} = a_1 - b_{11}x - b_{12}y = 0$ and $\frac{g(x,y)}{y} = -a_2 + b_{21}x - b_{22}y = 0$. We find that the nullclines are given by $y = -x + 4$ and $y = x - 1$, along which $g(x,y)$ and $f(x,y) = 0$. The point of intersection is $(\frac{5}{2}, \frac{3}{2})$. Now we want to introduce harvesting. Let

$$a = \begin{pmatrix} 4 \\ 1 \end{pmatrix},\ b = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},\ \alpha = \begin{pmatrix} 0.7 \\ 0.9 \end{pmatrix}$$

be the new parameters for this system. The values chosen for α are completely arbitrary and are chosen for visual purposes. By using the same method we can find the new nullclines for the system with harvesting by including α_n, which are given by $y = 3.3 - x$ and $y = x - 1.9$. We briefly note that the slope for this system does not depend on the choice of α. This can be seen once we factor in the α terms into the parentheses of the expressions as such:

$$\frac{dx}{dt} = ((a_1 - \alpha_1) - b_{11}x - b_{12}y)x = f(x,y),$$
$$\frac{dy}{dt} = (-(a_2 + \alpha_2) + b_{21}x - b_{22}y)y = g(x,y).$$

When we set, say, $f(x,y) = 0$, we have $((a_1 - \alpha_1) - b_{11}x - b_{12}y) = 0$. Solving for y, we get $b_{12}y = (a_1 - \alpha_1) - b_{11}x$, and subsequently

$$y = \frac{a_1 - \alpha_1}{b_{12}} - \frac{b_{11}}{b_{12}}x.$$

The intersection of the nullclines occurs at the point $(2.6, 0.7)$. This point lies below the previous equilibrium point without harvesting, however, it shifts in a way which shows that the number of prey with normal harvesting rates is actually greater on average, while the number of predators decreases. Without harvesting, the number of predators increases on average, which accompanies a decrease in the number of prey. The resulting graph of this analysis can be found in Figure 1.

We can use the results of this predator-prey model, which is a modification of Volterra's initial model, to help explain the phenomenon in the Adriatic Sea. Figure 1 visually demonstrates what is known as *Volterra's Principle*. We see that *with* fishing, the number of food fish increases on average, while the number of selachians actually decreases. However, with a reduced rate of fishing, the number of selachians increases on average, along with a decreased number of fish. Volterra's Principle has widespread applications in a variety of biological contexts, from insecticide use to microbial interactions [1, 3].

3. Further Application of the Harvesting Model

In 2007, Ransom A. Myers, et al. published a paper in Science magazine exploring the top-down effects of removing sharks from the Atlantic ecosystem on elasmobranchs (rays, skates, and small sharks) and bay scallops. The article summarizes what essentially boils down to a three-fold predator-prey system with harvesting. In short, sharks eat rays, which are migratory animals that consume scallops. During the summer, rays spend their time in the cool waters near Canada and in the winter they migrate towards Florida. During this migration, they pass by the North Carolina coast, which is home to the bay scallop, and they proceed to eat the scallops. Historically, fishing sharks wasn't really desirable, so there were not as many rays as the sharks would consume many of them. Today, sharks have been increasingly fished leading to a removal of a main predator for the rays. This allowed the ray population to grow and reproduce more often, ultimately leading to a decimation of the bay scallop [2].

We can model this system using the mathematics that have been developed in the previous section. Let x represent the scallop population, y represent the population of the rays, and z represent the population of the sharks. We then arrive at a system of three coupled differential equations. We have added the necessary terms to account for the added species, and the system looks like:

$$\frac{dx}{dt} = (a_1 - b_{11}x - b_{12}y - b_{13}z)x - \alpha_1 x = f(x,y,z),$$
$$\frac{dy}{dt} = (a_2 + b_{21}x - b_{22}y - b_{23}z)y - \alpha_2 y = g(x,y,z),$$
$$\frac{dz}{dt} = (-a_3 + b_{31}x + b_{32}y - b_{33}z)z - \alpha_3 z = h(x,y,z).$$

It should be noted, however, that sharks and scallops effectively do not interact since they occupy different regions of the Atlantic ocean. Hence, we can ultimately set $b_{13} = b_{31} = 0$. The ultimate goal for this system is to mathematically show what the article summarized qualitatively: removal of the sharks from this system due to harvesting leads to a decrease in the bay scallop population.

We proceed by solving each equation, $f(x,y,z) = 0, g(x,y,z) = 0, h(x,y,z) = 0$ to find the equilibrium points. Upon computing in Maple we find that there are eight total equilibrium points, each yielding an increasingly complex expression, and for that reason they will not be included here. However, we are only interested in the last equilibrium point, and we can proceed to simplify the expressions for this point, and solving each of them with respect to either x, y, or z. Most importantly, we want to find the simplified expression for x. In Maple , we first take our expressions for $f(x,y,z), g(x,y,z), h(x,y,z)$ and divide them by x, y, and z, respectively. This yields the expressions:

$$\frac{f(x,y,z)}{x} = a_1 - b_{11}x - b_{12}y - b_{13}z - \alpha_1,$$
$$\frac{g(x,y,z)}{y} = a_2 + b_{21}x - b_{22} - b_{23}z - \alpha_2,$$
$$\frac{h(x,y,z)}{z} = -a_3 + b_{31}x + b_{32}y - b_{33} - \alpha_3.$$

We can set these new equations $= 0$, while also setting $b_{13} = b_{31} = 0$. We can then collect the expression for x alone. We want to ultimately find the partial derivative of x with respect to α_3, the harvesting rate of the shark. If $\frac{\partial x}{\partial \alpha_3} > 0$, then increased harvesting of sharks leads to an increase in the bay scallop population. We know that this statement does not match the observations. However, if $\frac{\partial x}{\partial \alpha_3} < 0$, which is what we want to show, then the increased harvesting of sharks yields a decrease in the bay scallop population. Indeed, upon differentiation we find that

$$\frac{\partial x}{\partial \alpha_3} = -\frac{b_{12}b_{23}}{b_{11}b_{22}b_{33} + b_{11}b_{23}b_{32} + b_{12}b_{21}b_{33}}.$$

Because all of the constants, $a_{mn}, b_{mn} > 0$, this expression is ultimately negative, which supports the findings in the paper. Increased fishing for sharks leads to an eventual decrease in the population of scallops, as the data by Myers et al. shows.

References

[1] Martin Braun. *Differential Equations and Their Applications*. New York: Springer-Verlag, 1975.

[2] Ransom A. Myers et al. "Cascading Effects of the Loss of Apex Predatory Sharks from a Coastal Ocean". In: *Science* 315 (Mar. 30, 2007), pp. 1846–1850.

[3] Wikipedia. *Paradox of the pesticides*. May 2021. URL: `https://en.wikipedia.org/wiki/Paradox_of_the_pestici%20des`.

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