Notes Risk Management Labs

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Abstract

We briefly recall some concepts and formulas useful for Risk Management labs. The student is invited to refer to his/her notes and the other material of the course.

1 Lab1 RM

Interest rate products

- Interest Rate Swap (IRS) with expiry in t_N, obtained setting $NPV(t_0) = 0$

$$S(t_0; t_0, t_N) = \frac{1 - B(t_0, t_N)}{\sum_{i=1}^{N} \delta(t_{i-1}, t_i)\, B(t_0, t_i)}$$

  where the denominator is known as Basis Point Value (BPV) or Annuity (old name, present in the Hull). The year fraction of IRS fixed leg $\delta(t_{i-1}, t_i)$ is generally 30/360 European for swaps denominated in Euro.

- Forward swap starting in t_n and expiring in t_N. The rate is obtained again, setting $NPV(t_0) = 0$

$$B(t_0, t_n) - B(t_0, t_N) = S(t_0; t_n, t_N) \sum_{i=n+1}^{N} B(t_0, t_i)\delta(t_{i-1}, t_i)$$

- Payer swaption with expiry in t_n and tenor $(t_N - t_n)$, implied volatility $\sigma_{n,N}$:

$$SP_{n,N}(t_0) = B(t_0, t_n) BPV_{n,N}(t_0) \left(S_{n,N}(t_0)\, N(d_1^s) - K\, N(d_2^s) \right),$$

  with

$$d_{1,2}^s = \frac{1}{\sigma_{n,N}\sqrt{t_n - t_0}} \ln\left(\frac{S_{n,N}(t_0)}{K}\right) \pm \frac{1}{2}\sigma_{n,N}\sqrt{t_n - t_0},$$

$$BPV_{n,N}(t_0) = \sum_{i=n+1}^{N} \delta(t_{i-1}, t_i)\, B(t_0; t_n, t_i)$$

- Interbank (IB) fixed coupon bond (i.e. a bond with no credit risk):

$$IB_{bond}(t_0) = \sum_{i=1}^{N} c_i \, B(t_0, t_i), \quad c_i = \begin{cases} c\,\delta(t_{i-1}, t_i), & i < N \\ 1 + c\,\delta(t_{i-1}, t_i), & i = N \end{cases}$$

Question 1. *Do you see any differences and/or similarities between an IRS and an IB fixed coupon bond with a coupon equal to the swap rate $S(t_0; t_0, t_N)$ and same year fraction and payment conventions?*

Greeks

- Macaulay duration for a fixed coupon bond:

$$DU = \frac{1}{IB_{bond}(t_0)} \sum_{i=1}^{T} (t_i - t_0) c_i B(t_0, t_i)$$

The numerator is the variation in the value of an IB coupon bond for a parallel shift of the corresponding zero-rate curve of 1 bp (i.e. the DV01$^{(z)}$ for this bond).

- Swaption delta:

$$\Delta(t_0) = \frac{\partial SP_{\alpha,\omega}(t_0)}{\partial S_{\alpha,\omega}(t_0)} = B(t_0, t_\alpha) BPV_{\alpha,\omega}(t_0) N(d_1^s)$$

Question 2. *Why is the delta formula not correct?*

- The Dollar Value of 1bp (DV01) is the measure of IR-sensitivity used in the industry:

$$DV01(t_0) = NPV_{shifted}(t_0) - NPV(t_0)$$

To compute the DV01:

1. Shift the market rate of all instruments used in the bootstrap by 1bp (pay attention to the futures);
2. Bootstrap again the curve;
3. Compute the NPV of your portfolio with the new curve and take the difference w.r.t. the NPV with the market curve.

- The DV01$^{(z)}$ is the DV01 computed shifting the zero-rate curve:

$$DV01^{(z)}(t_0) = NPV^{(z)}_{shifted}(t_0) - NPV(t_0) \ .$$

To compute the $DV01^{(z)}$:

1. Shift by one basis point the zero rate curve $R(t_0, t_i)$;

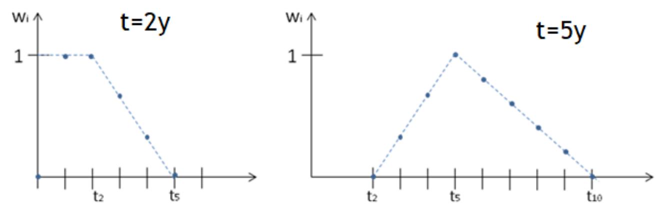

Figure 1: Course-grained Buckets: Weights example for 2 and 5 year buckets

2. Compute the NPV of your portfolio with the new curve and take the difference w.r.t. the NPV with the market curve.

Question 3. *Why do we need also this* $DV01^{(z)}$*?*

Question 4. *Why* $DV01^{(z)} = -DU$ *for an IB par fixed coupon bond?*

- Bucket DV01: the DV01 computation assumes an unrealistic parallel shift in the curve. With the bucket DV01 we can focus on the impact of the movement of each single instrument (bucket) used when bootstrapping the curve.
  For the j^{th} bucket we have:

$$DV01^{(j)}(t_0) = NPV^{(j)}_{shifted}(t_0) - NPV(t_0)$$

- Coarse-grained Bucket DV01: with the coarse-grained bucket DV01 we can focus on the impact of some main buckets of the curve.

  1. Shift by $w_i^{(j)}$ (see Figure 1) basis point the market rate of all instruments used in the bootstrap;
  2. Bootstrap again the curve;
  3. Compute the NPV of your portfolio with the shifted curve and take the difference w.r.t. the NPV computed with the market curve.

$$w_i^{(j)} = \begin{cases} \frac{t_i - \hat{t}_{j-1}}{\hat{t}_j - \hat{t}_{j-1}}, & k_{j-1} \leq i \leq k_{j+1}, \quad i \neq k_j \\ 1, & i = k_j \end{cases}$$

where k_j is the year corresponding to the j-th macro-bucket. These macro-buckets are established by the board (in agreement with bank's risk management); an example can be: $k_1 = 2y, k_2 = 5y, k_3 = 10y, k_4 = 15y, k_5 = 20y$. Recall that the first bucket have weights equal to one until year k_1 (usually $k_1 = 2y$).

Question 5. *Draw the weights for the* 10 *coarse-grained bucket and for the* 20 *one (supposing that it is the last one indicated by bank's board for IR sensitivities, as in the above example).*

Question 6. *Ineffectiveness of a parallel DV01 hedge with market curve movements. Hedge your swaption only with a* 5*y swap. Consider*

1. *first, a parallel shift of* 1*bp. What does happen to the portfolio of the hedged swaption?*

2. *second, a movement of* 1 *bp up for the 15y bucket and DV01(15y swap)/ DV01(5y swap) bps down for the 5y bucket. What does happen to the portfolio of the hedged swaption?*

Question 7. *Compute the coarse grained bucket sensitivity of the swaption with two buckets: 5y and 15y and the corresponding hedging with a 5y swap and a 15y swap. Consider the same movement as before and check what does happen to the portfolio of the hedged swaption.*

Question 8. *Can you indicate a rule of thumb for hedging a receiver swaption 10y5y (e.g. when bank systems are down)?*

2 Lab 2 RM

Intensity Assuming independence between the instantaneous risk-free rate and the default time, the price of a defaultable zero coupon bond with zero recovery is:

$$\bar{B}(t,T) = B(t,T)P(t,T) \quad ,$$

where $P(t,T)$ is the survival probability up to time T. We define the intensity $\lambda(s)$ such that

$$P(t,T) = \exp\left(-\int_t^T \lambda(s)ds\right)$$

The price of a fixed coupon bond with recovery π is approximately (we are not considering the recovery of the accruals):

$$\bar{C}(t) = \bar{c}\sum_{n=1}^{N} \delta_{n-1}\bar{B}(t,t_n) + \bar{B}(t,t_N) + \pi \sum_{n=1}^{N} e(0;t_{n-1},t_n) \quad ,$$

where $e(0;t_{n-1},t_n) = B(t,t_n)(P(t,t_{n-1}) - P(t,t_n))$ and the term $P(t,t_{n-1}) - P(t,t_n)$ is the default probability between t_n and t_{n-1} as seen in t.

Question 1. *Rewrite the price of the coupon bond in term of the intensity assuming that it is constant i.e.* $\lambda(s) = \bar{\lambda}$.

Once I have an estimation of $\bar{\lambda}$ I can compute the market-implied default probability of an issuer as $P(t,T) = \exp\left(-(T-t)\bar{\lambda}\right)$.

Historical default probabilities Another approach to estimate $P(t,T)$ is to look at historical default and migration probabilities. The baseline approach is to consider a time-homogeneous Markov chain framework, where default and migration probability at time-step t depends only on the state (rating class or default) at time t. The default is called an absorbing state (zero transition probability to other states).
Example with two rating classes and one time-step, where a firm can change rating class only between High Yield and Investment Grade or default:

$$P = \begin{array}{ccc} IG & HY & Def \\ \hline \pi_{1,1} & \pi_{1,2} & \pi_{1,3} \\ \pi_{2,1} & \pi_{2,2} & \pi_{2,3} \\ 0 & 0 & 1 \end{array}$$

π_{jk} is the probability that after one time-step (market standard one year) an issuer in class j moves to class k. The probabilities π_{jk} are estimated from historical default and migration frequencies for issuers of the same rating class.

Then, the T time-steps rating transition matrix is P^T (e.g. the two year transition matrix is P^2).

Z-spread Given that market prices are consistent with multiple sets of (λ, π) (calibration from market prices of multiple bonds issued by the same name is not practical), bond traders prefer to work with the Z-spread. The price of a defaultable coupon bond in this case is expressed as:

$$\bar{C}(t) = \bar{c} \sum_{n=1}^{N} \delta_{n-1} \hat{B}(t, t_n) + \hat{B}(t, t_N) \quad , \tag{2.1}$$

where

$$\hat{B}(t, t_n) = B(t, t_n) \exp\left(- \int_t^{t_n} z(s) ds \right) \quad ,$$

and $z(s)$ is called the Z-spread. We see that the Z-spread is the difference between the zero–coupon rate used to discount a defaultable cash–flow and the one used to discount a risk–free cash–flow.

Question 2. *Rewrite the price of the coupon bond in term of the Z-spread assuming that the spread is constant, i.e.* $z(s) = \bar{z}$.

Question 3. *Does it exist a unique solution to the Z-spread in* (2.1) *when* $z(s)$ *is constant?*

Question 4. *Why do practitioners tend to use the Z-spread instead of the intensity?*

Question 5. *If the two spreads (Z-spread and intensity) are constant and calibrated from the same bond price, which is the lowest?*

Question 6. *Given a movement of the price a corporate bond (e.g. 2% loss in 1 day for a 10y bond) is it possible to state something about the riskiness of the corporation?*

Question 7. *Have you heard of other swaps beyond IRS, CDS and ASW?*

Question 8. *Do you expect the assumption of independence between "risk-free" rate and default to be reasonable?*

3 Lab 4 RM

Rating transition matrix with two rating classes

IG	HY	Def
$\pi_{1,1}$	$\pi_{1,2}$	$\pi_{1,3}$
$\pi_{2,1}$	$\pi_{2,2}$	$\pi_{2,3}$
0	0	1

π_{jk} is the probability that after one time-step an issuer in class j moves to class k.
In one time step, one factor firm value model for defaults and downgrade each obligor i has firm value at the next time-step

$$v_i = \sqrt{\rho} y + \sqrt{1-\rho}\epsilon_i \ ,$$

where y is the market factor, ϵ_i the idiosincratic factor and ρ the correlation between v_i and y. y and ϵ_i are standard normal and thus also v_i is a standard normal. The obligor i in rating class j moves to rating class j' or defaults depending on the value of v_i and we can calibrate the threshold such that they match the rating transition matrix. For each obbligor in rating class j, $n-1$ thresholds are computed recursively starting from the default probability of each firm $\pi_{j,n}$:

1. $K_{j,n} = N^{-1}(\pi_{j,n})$;
2. $K_{j,k} = N^{-1}(\sum_{h=k}^{n} \pi_{j,h})$, $1 < k < n$.

Question: compute the thresholds for the 2 rating class matrix above.

$$\hat{B}(t, T_1, T_2) = B(t, T_2) e^{\hat{z}(T_2 - T_1)} \ ,$$

Price a zero coupon bond using the rating transition matrix. One year IG zero coupon bond (maturity t_1):

$$\bar{B}(0, t_1) = (1 - \pi_{1,3}) B(0, t_1) + \pi_{1,3} B(0, t_1/2) \eta \ ,$$

where $\eta$is the recovery. Two years IG zero coupon bond (maturity t_2):

$$\bar{B}(0, t_2) = \pi_{1,1} B(0, t_1) \bar{B}(t_1, t_2) + \pi_{1,2} B(0, t_1) \bar{B}^{HY}(t_1, t_2) + \pi_{1,3} B(0, t_1/2) \eta \ ,$$

where $\bar{B}(t_1, t_2)$ and $\bar{B}^{HY}(t_1, t_2)$ are the forward zero coupon bonds IG and HY.

Question 1. *How do you compute the $\bar{B}^{HY}(t_1, t_2)$ price?*

4 Lab 5 RM

Extended Vasicek model The Vasicek model for the short term rate r_t at time t on the interval $(t, t+dt)$ is:

$$r_t = \varphi_t + x_t,$$

where φ_t is deterministic and x_t is an OU process

$$\begin{cases} dx_t &= -ax_t dt + \sigma dW_t \\ x_{t_0} &= x_0 \end{cases}$$

where a is the process mean reversion speed and σ the interest rate volatility.
While a and σ must be calibrated from market instruments (tipically swaptions or caps/floors), the function φ_t is chosen so as to exactly fit the term structure of spot interest rates.
Question: why is it necessary to perfectly fit market data?
The solution of the OU process is:

$$x_t = x_0 e^{-a(t-t_0)} + \sigma \int_{t_0}^{t} e^{-a(t-u)} dW_u.$$

In the OU framework, at time s the forward discount factor for time T, $T+\theta$ is

$$\frac{B(s,T_1,T_2)}{B(t_0,T_1,T_2)} = \exp\left(-x_s\left(\frac{\sigma(s,T_2)-\sigma(s,T_1)}{\sigma}\right) - 1/2\int_{t_0}^{s}[\sigma^2(u,T_2)-\sigma^2(u,T_1)]du\right) \quad ,$$

where $\sigma(u,T) = \sigma\frac{1-e^{-a(T-u)}}{a}$.
Question: Compute the discount factor $B(s,\tau)$. Questions $E_0[B(t,T)]$? Questions: How to calibrate the model?
Suggestions for numerical efficency: $B(s,\tau) = A(s,\tau)e^{-x_s C(s,\tau)}$, where
$A(s,\tau) = \frac{B(t_0,\tau)}{B(t_0,s)}e^{-\frac{1}{2}\int_{t_0}^{s}(\sigma(u,\tau)^2-\sigma(u,s)^2)du}$ and
$C(s,\tau) = \frac{1-e^{a(\tau-s)}}{a}$

Simulate a portfolio value

1. Write the portfolio value at time t in terms of x_t. Question: how do you do it for a swap? .

2. Simulate x_t on the selected grid. Question: what is the law of $x_{t+\Delta t}$ given x_t?

3. Evaluate the portfolio price in the different MC scenarios.

Netting and collateral can be also considered in this framework.

Collateral To reduce counterparty credit risk (CCR), the two counterparties agree to exchange a cash amount (or other eligible collateral) whose value equals the mark-to-market (MtM) of the position at a specified frequency. This cash amount is in general revalued with the risk-free rate. The Credit Support Annex (CSA) is a legal document that regulates credit support (collateral) for derivative transactions.

Netting In the CSA it is specified that a set of derivatives with the same counterparty can be considered as a portfolio. The counterparty risk is computed at the level of this portfolio (netting set). The collateral is exchanged based on this set.

Metrics: EE, EPE, PFE, Peak-PFE Consider the process MtM_t: the mark to market of a portfolio of instruments within a netting set with the same counterparty. The credit exposure (CE) at time τ is

$$CE(\tau) = \max(MtM_\tau, 0) \ .$$

Different metrics are utilized to quantify this risk.

- Expected future exposure $EE(\tau) = E_0[MtM_\tau | MtM_\tau > 0] = E_0[CE_\tau]/\mathbb{P}[MtM_\tau > 0]$.
- Expected positive exposure on a time grid $\{t_i\}_{i=1,..,N}$, $EPE = \frac{\sum_i EE(t_i)}{N}$.
- Potential future exposure $PFE(\tau)$: it is the α quantile of MtM_τ (usually $\alpha = 95\%$ or 99%).
- Peak future exposure $Peak - PFE = \max_{0<\tau<T} PFE(\tau)$.