Scuola di Ingegneria Industriale e dell'Informazione
Corso di Laurea Magistrale in Mathematical Engineering - Quantitative Finance

Financial Engineering: Group 4
Assignment 6

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Academic Year: 2024-2025

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1 Certificate Pricing

The structured product we considered pays a 6% coupon at year 1 only if the index is below the strike, otherwise it pays 0, and a 2% coupon at year 2 if no payment took place in the first year. Hence, if at the first observation the 6% coupon has been earned, the note redeems early at 100% of par and also the Euribor-leg of the hedging swap is terminated.
Annex 1 sets out the note's payoff rules (coupons and redemption triggers).
Annex 2 details the hedging swap.
Together, they're essential: Annex 1 tells "what" the investor gets, Annex 2 tells "how" the commercial bank hedges and what happens to those hedge cash-flows upon early redemption.

1.1 Upfront under the Variance Gamma model

The upfront is set so that, under the risk-neutral measure, the present value of all future outgoing flows minus incoming flows is zero. One can write:

$$\begin{aligned}\text{Upfront} = {} & (1 - B_1) + s\,\text{BPV}_1 \\ & + P\big(S_{t_1} > K\big)\left[s\,\text{BPV}_2 - (B_2 - B_1)\right] \\ & - \Big[6\%\,\delta_1\,B_1\,P\big(S_{t_1} \leq K\big) + 2\%\,\delta_2\,B_2\,P\big(S_{t_1} > K\big)\Big].\end{aligned}$$

where $P\big(S_{t_1} > K\big)$ is the probability the index is above strike at year 1 (survival probability) and $P\big(S_{t_1} < K\big)$ is the probability the structured product is knocked out and pays the 6% coupon.
Practically, we obtain $P\big(S_{t_1} < K\big)$ pricing a Digital option as an infinitesimal bull-spread: in particular, letting $C(K)$ denote the price of a European call with strike K, we have:

$$P\big(S_{t_1} < K\big) \; = \; 1 - \text{priceDigital}$$

where

$$\text{priceDigital} \; = \; \frac{C(K) - C(K + \varepsilon)}{\varepsilon}$$

The calls are priced under the Variance Gamma model via the Lewis formula, with the transform approximated by a Fast Fourier Transform. We set the VG parameters ($\alpha = 0$) from the data provided in Assignment 4: $k = 1$, $\sigma = 20\%$ and $\eta = 3$.

Upfront VG
1.77 %

1.2 Upfront under the Black model

We decided to use Black model since it has a single volatility parameter versus three in VG, reducing model risk. We keep the same upfront structure, but replace $P\big(S_{t_1} > K\big)$ with the probability implied by Black model. Digital under Black is priced analytically and

$$P\big(S_{t_1} > K\big) = \text{priceDigital}$$

where

$$\text{priceDigital} = DF * N(d_2)$$

All other terms remain unchanged.

Upfront Black
0.61 %

We observe a much lower price, since the Black model is not the correct one to price a digital option.

1.3 Limitations of the Black model for the three-year structure

For a three-year note with early-redemption checks at the first and second year, the upfront becomes:

$$\begin{aligned}
Upfront = (1 - B_1) \ + \ & s\,\mathrm{BPV}_1 \\
& + P\big(S_{t_1} > K\big)\,\big[s\,\mathrm{BPV}_2 - (B_2 - B_1)\big] \\
& + P\big(S_{t_1} > K,\ S_{t_2} > K\big)\,\big[s\,\mathrm{BPV}_3 - (B_3 - B_2)\big] \\
& - \Big[6\%\,\delta_1\,B_1\,P\big(S_{t_1} \le K\big) + 6\%\,\delta_2\,B_2\,P\big(S_{t_1} > K,\ S_{t_2} \le K\big) \\
& \qquad + \ 2\%\,\delta_3\,B_3\,P\big(S_{t_1} > K,\ S_{t_2} > K\big)\Big].
\end{aligned}$$

In the three-year extension, the upfront formula relies on joint probabilities that are path-dependent. Black fails here because it cannot capture the path dependence between the two observations; indeed, it treats each observation independently, so it cannot capture the fact that surviving the second knockout check depends on having survived the first.

1.4 VG-model Monte Carlo simulation for the three-year Upfront

To capture the full path-dependence of the three year knock-out structure under Variance Gamma dynamics, we run Monte Carlo simulation of forward paths. At each annual reset we generated a standard normal together with an independent Gamma variate, combined them into a log-return increment and exponentiated cumulatively to obtain forwards at year 1, year 2 and year 3. In exactly the same way, the 6% coupons at years 1 and 2 are only paid on paths where the respective knock-out events occur, and the 2% coupon at maturity only on the paths that never hit the barrier. We apply the knock-out checks in sequence: if the forward ever falls below the strike at the first observation, if it survives to year 2 but then falls below the strike or remains above the strike twice. From every simulated path we therefore record four simple binary outcomes:

- survive year 1 $\rightarrow$ $P(S_{t_1} > K)$

- survive year 2 $\rightarrow$ $P(S_{t_1} > K,\, S_{t_2} > K)$
- knock out at year 1 $\rightarrow$ $P(S_{t_1} \leq K)$
- knock out at year 2 $\rightarrow$ $P(S_{t_1} > K,\, S_{t_2} \leq K)$

In conclusion, for each simulated path we plugged the recorded survival and knock-out indicators into the analytical upfront formula to compute a path-wise upfront value. We then took the arithmetic mean of all these path-wise upfront values to produce the final three-year upfront premium.

Upfront VG (MC)
6.06339%

We also assessed the precision of our estimate by computing a 95% confidence interval, the result is $[6.06331\%, 6.06344\%]$.

1.5 Digital Risk

As a final step, observing that the Black model yielded an upfront 50 basis points lower than the one obtained using the Variance Gamma model, we adjusted the Black formula to account for digital risk. Specifically, we introduced a correction factor in the pricing of the digital component of the certificate, so as not to ignore the fact that the volatility σ varies (in this case, decreases) with respect to the strike. The correction term

$$-\nu \frac{\partial \sigma(K)}{\partial K}$$

was added, where ν denotes the vega of the option and $\frac{\partial \sigma(K)}{\partial K}$ is the derivative of the implied volatility with respect to the strike. This correction lead to an upfront higher than that predicted by the Black model.

Upfront Digital Risk
1.36 %

2 Bermudian Swaption Pricing via Hull-White

2.1 HW Trinomial Tree

In the Hull-White model setting with parameters $a = 10\%$ and $\sigma = 0.8\%$, we constructed a trinomial tree to simulate the dynamics of the Ornstein-Uhlenbeck process X_t, initializing it at zero in the first node. We then considered an appropriate number of time steps (having performed various simulations with different discretizations), and computed the forward discount factors at each node using the following formula:

$$B(t_i; t_i, t_i{+}\tau) = B(t_0; t_i, t_i{+}\tau) \exp\left\{ -x_i \frac{\sigma(0,\tau)}{\sigma} - \frac{1}{2} \int_{t_0}^{t_i} \left[\sigma(u, t_i + \tau)^2 - \sigma(u, t_i)^2 \right] du \right\}$$

where the functions $\sigma(t)$ are the deterministic volatility terms typical of the Hull-White model.

Having obtained the forward discount factors, we discounted the payoffs at year nine (the last swap contract that can be entered spans from year 9 to 10). For the nodes corresponding to years 2 through 8, we compared the immediate payoff at the node with the continuation value (i.e. the discounted value of future payoffs), and selected the maximum between the two.

Bermudan Swaption Price
10.58 bp

Table 1: The price refers to a 10 days discretization of the tree

2.2 Check of Correct Implementation

To verify the correct implementation of the tree, the swaption price was analyzed as the Δt parameter varied, in order to confirm convergence. Below the table reffering to the different swaption prices; as the width of the discretization interval decreases, the price exhibits stabilization and convergence.

$\Delta t_{40/365}$	$\Delta t_{30/365}$	$\Delta t_{20/365}$	$\Delta t_{10/365}$	$\Delta t_{5/365}$	$\Delta t_{1/365}$
6.8 bp	3.03 bp	6.99 bp	10.58 bp	10.39 bp	10.41 bp

2.3 Jamshidian

Using Jamshidian's approach, we priced the eight European swaptions of interest, all with a 10-year expiry and start dates ranging from 2 to 9 years. As an upper bound for the Bermudan swaption, we considered the sum of these eight European option prices, while for the lower bound we took the maximum among them.

Lower Bound	**Upper Bound**
9.05 bp	40.66 bp

2.4 Coarse Bucket Delta Hedging with Mid-Market Swaps

The purpose of this section was to compute the coarse-grained DV01 across different maturities (2y, 5y, and 10y) and to hedge the delta using Mid Market Swaps (2y, 5y, and 10y). Initially, the weights of the different maturity buckets were determined in accordance with the methodology illustrated in the figure below. Afterward, the interest rates of the available financial instruments were shifted using the computed weights. After each shift, we recalibrated the bootstrapped discount curve to reflect the modified rate environment. The new swaption prices were subsequently calculated using the shifted rates, and the DV01s were determined as the difference between the updated prices and the original ones. These exposures are referred to as $\Delta 2y$, $\Delta 5y$, and $\Delta 10y$, respectively. To neutralize them, we employed three Mid Market Swaps payer with matching tenors of 2, 5, and 10 years. The DV01s of these swaps, under a 1bp parallel shift, are denoted by $S_{2\text{y}}$, $S_{5\text{y}}$, and $S_{10\text{y}}$. Our hedging approach focuses on determining the appropriate notional values x_2, x_5, and x_{10} for the 2y, 5y, and 10y Mid Market Swaps to ensure the combined portfolio's Delta equals zero across all buckets. This creates a triangular equation system that we can solve in sequence, beginning with the longest maturity (10 years), which aligns with interest rate risk management best practices. Longer-dated instruments typically influence a wider segment of the yield curve. By addressing the 10y bucket first, we prevent its exposure from affecting the sensitivity calculations for shorter-term buckets. We considered a Bermudan Swaption Notional of 100 Mln in order to display significant values in term of € per bp.

$$\begin{cases} S_{10\text{y}} \cdot x_{10} + \Delta_{10\text{y}} \cdot \text{Notional} = 0 \\ S_{5\text{y}} \cdot x_5 + S_{10\text{y}} \cdot x_{10} + \Delta_{5\text{y}} \cdot \text{Notional} = 0 \\ S_{2\text{y}} \cdot x_2 + S_{5\text{y}} \cdot x_5 + S_{10\text{y}} \cdot x_{10} + \Delta_{2\text{y}} \cdot \text{Notional} = 0 \end{cases}$$

	2y	**5y**	**10y**
Coarse Grained Sensitivity	-3.44 €	-2.6×10^3€	4.05×10^3€
Hedging Swap Notional	25.79 Mln	17.91 Mln	4.72 Mln
Hedging Swap Type	Receiver	Payer	Receiver

Table 2: Hedging Strategy Details

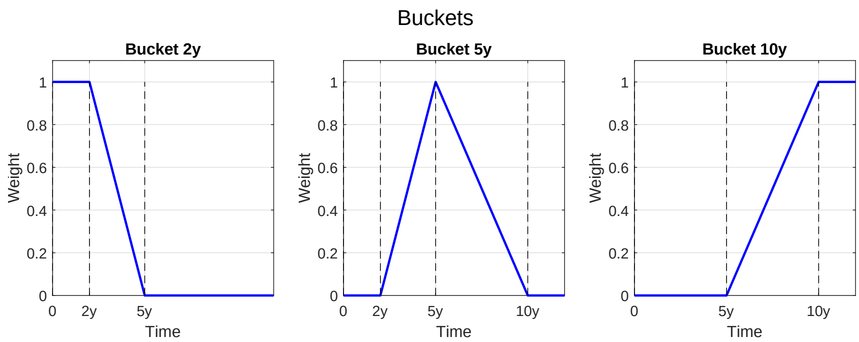

Figure 1: Weight profiles for the 2y, 5y, and 10y buckets used in the hedging