Scuola di Ingegneria Industriale e dell'Informazione
Corso di Laurea Magistrale in Mathematical Engineering - Quantitative Finance

Financial Engineering: Group 4
Assignment 5

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1 Case Study 1

1.1 LMM Calibration

In the first point of the exercise, it was required to calibrate the LMM model to obtain the values of spot volatilities every 3 months. For this purpose, the first three were imposed equal to the one-year flat (i.e., $\tilde{\sigma}_{1,2,3} = \sigma^F_{1y}$). Subsequently, we priced a Cap_{1y} using the `priceCap` function implemented by us; the price of the Cap_{2y} was also computed (considering the two-years flat volatility σ^{F}_{2y}). Afterwards we considered the difference of the two caps as function of the intermediate spot volatilities (our four unknowns):

$$\Delta C = Cap_{2y} - Cap_{1y} = \sum_{i=4}^{7} Caplet(t_i, \tilde{\sigma}_i) \tag{1}$$

Some linear constraints have been imposed in order to have a proper number of equation to solve the problem, below an example (referred to the 1y2y time horizon):

$$\frac{\tilde{\sigma_7} - \sigma_3}{T_7 - T_3} = \frac{\tilde{\sigma_i} - \sigma_3}{T_i - T_3} \qquad (i = 4, 5, 6) \tag{2}$$

To find the solutions, we wrote all the equations as functions of $\tilde{\sigma}_7$, moreover we employed the MATLAB built-in function `fzero` to determine the final unknown, and then proceeded backwards to compute the remaining variables. The above procedure was repeated for each annual interval considered (up to 10 years) and repeated further for each strike considering at each iteration the difference of the previous Cap's price and the one of the following. The linear constraints were always considered as illustrated in the first iteration but adjusted to current sigmas and times. For the years after the 10^{th} the procedure adopted is the same as the one previously illustrated, the only difference is that we are considering time horizons different from the year (10y→12y, 12y→15y, 15y→20y), therefore the number of unknowns in system to be solved will be greater (but we will also have a greater amount of equations in order to have a feasible problem).

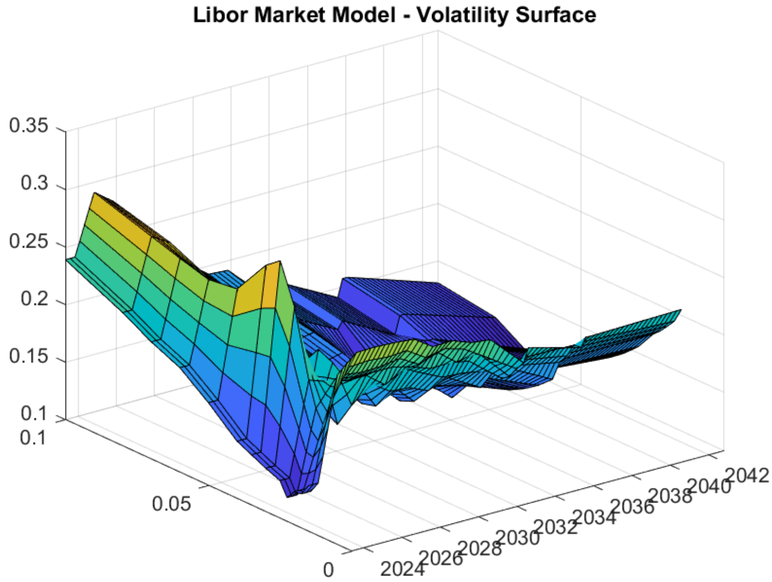

Figure 1: In the picture the Volatility Surface over the years and the strikes

1.2 Computation of the Upfront

The upfront calculation was divided into two parts, the evaluation of the Party A payments (the Bank XX) and the Party B ones (the IB). The formula that was employed to compute the required quantity is the following:

$$Upfront = 1 - B(t_0, t_N) + spol * BPV - \sum_{i=1}^{N} c_i * B(t_0, t_i) \tag{3}$$

The evaluation of the final discount and the BPV were almost immediate and straightforward. To compute the coupons we considered three different time horizons (taking in consideration the fact that the first quarter coupon is 3%):

- 0→ 3y
- 3y→ 6y
- 6y→ 10y

The corresponding coupons (paid quarterly) are:

- $1.1\% + \text{Euribor3m} - \text{Cap}(K_{\text{cap1}}) - 0.9\% \times \text{Digital}(K_{\text{dig1}})$
- $1.1\% + \text{Euribor3m} - \text{Cap}(K_{\text{cap2}}) - 0.9\% \times \text{Digital}(K_{\text{dig2}})$
- $1.1\% + \text{Euribor3m} - \text{Cap}(K_{\text{cap3}}) - 0.9\% \times \text{Digital}(K_{\text{dig3}})$

The discounts are evaluated at each reset date and to price the Cap and the Digital Option we considered three different for loop (one for every time window since at every period corresponds a different form of the coupon). The strikes for the Cap are $\mathbf{K}_{cap}$=$\{0.043, 0.046, 0.052\}$, which are the same for the Digital. The volatilities employed in the pricing were obtained via spline interpolation.

Upfront
10,7 %

1.3 Evaluation of the Delta-Bucket Sensitivities

To evaluate the Delta-bucket sensitivities for all buckets, we shifted the rates of 1 bp of the financial instruments in the Excel file of the Boostrap using the `rateShift` function, implemented by us, up to the 02-Feb-2033. We bootstrapped considering the new rates and computed the discount factors at the reset dates via `interpolation` function (also implemented by us in the Lab on the Boostrap). Subsequently we did the calibration and calculated the forward discounts, the deltas and the forward Euribors; we employed another time the `computeUpfront` function with the new data and, in order to show the sensitivities, we evaluated the difference between the new upfront and the one with the no-shifted rates. The output is a vector of 21 sensitivities (one for each target date of the Boostrap), for more clarity and understanding we report only the values at 2y, 6y and 10y:

Delta-Sens$_{2y}$	**Delta-Sens**$_{6y}$	**Delta-Sens**$_{10y}$
$-1,1 \times 10^3$	$-4,7 \times 10^3$	$-2,3 \times 10^3$

Table 1: The results refer to a Principal Amount of 50 Mln

1.4 The total Vega

In order to assess the Vega we calculated the "shocked" sigma matrix by adding 1 bp to all elements of the market flat volatilities. Afterwards we did again the calibration and estimated the new upfront with our taylor made Matlab function. The Vega is computed as the difference of the new upfront and the previous one; the result is shown below and refers to the contract notional.

Vega
$4,9 \times 10^3$

1.5 Coarse Bucket Delta Hedging with Interest Rate Swaps

This section outlines the procedure adopted to measure the structured bond's rate sensitivities and to neutralise them with appropriately selected interest rate swaps. To start the delta hedging process, we first shifted the interest rate curve using predefined weights across different maturities; after each shift, we recalibrated the bootstrapped discount curve to reflect the modified rate environment.
We computed the sensitivity of the structured product's present value, specifically its upfront value, for any of the bucketed shifts. The differences between the original upfront value and the perturbed ones can be interpreted as the bucketed DV01s for

the 2y, 6y, and 10y buckets. We refer to them respectively as $\Delta_{2\text{y}}$, $\Delta_{6\text{y}}$ and $\Delta_{10\text{y}}$. To neutralise these exposures we employ three payer IRS with the same tenors (2,6,10 years). Their DV01s under a 1-bp shift are denoted as $S_{2\text{y}}$, $S_{6\text{y}}$, and $S_{10\text{y}}$.

The core idea of the hedging strategy is to find the notional amounts x_2, x_6 and x_{10} of the 2y, 6y and 10y swaps such that the total Delta of the combined hedged portfolio is zero in every bucket. This leads to a triangular system of equations that can be solved sequentially, starting from the longest maturity (10 years), in line with best practice in interest rate risk management, as longer maturity instruments tend to affect a broader portion of the yield curve. By hedging the 10y bucket first, we ensure that the exposure it covers does not interfere with the sensitivities of shorter buckets.

$$\begin{cases} S_{10\text{y}} \cdot x_{10} + (\Delta_{2\text{y}} + \Delta_{6\text{y}} + \Delta_{10\text{y}}) \cdot \text{Notional} = 0 \\ S_{6\text{y}} \cdot x_6 + S_{10\text{y}} \cdot x_{10} + (\Delta_{2\text{y}} + \Delta_{6\text{y}}) \cdot \text{Notional} = 0 \\ S_{2\text{y}} \cdot x_2 + S_{6\text{y}} \cdot x_6 + S_{10\text{y}} \cdot x_{10} + \Delta_{2\text{y}} \cdot \text{Notional} = 0 \end{cases}$$

IRS 2y	**IRS 6y**	**IRS 10y**
-7,5 Mln	-3.9 Mln	1,7 Mln

Table 2: Notional amount of 2y - 6y - 10y IRS payer

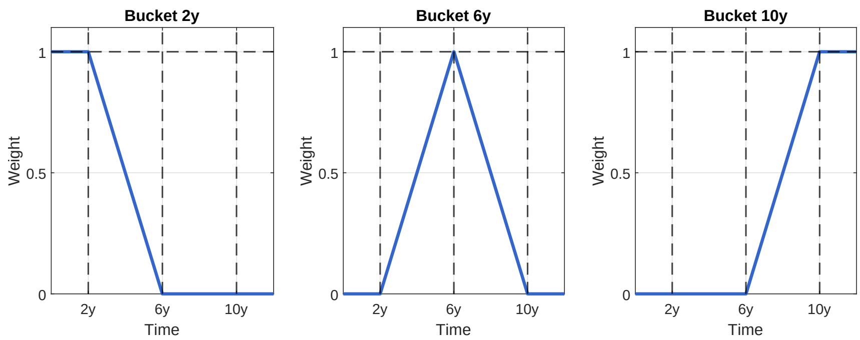

Figure 2: Weight profiles for the 2y, 6y, and 10y buckets used in the hedging

1.6 Vega Hedging with ATM Caps

To hedge the Vega exposure of the structured product, we started by perturbing the calibrated spot volatility surface using weighted shifts corresponding to two maturity buckets: 0–6 years and 6–10 years. After shifting the volatility surface in each bucket, we recalculated the upfront value and compared it with the original. The differences obtained represent the structured product's bucketed Vega sensitivities, denoted as $\text{Vega}_{6\text{y}}$ and $\text{Vega}_{10\text{y}}$.

To hedge these risks, we used two ATM-Swap Rate Cap instruments with maturities 6 and 10 years. Each Cap's Vega was computed as the sum of the Vegas of the caplets that compose it and these caplet Vegas were calculated following standard Black model. Finally, we solved a two-equation system to find the notional amounts of the 6y and 10y Caps that would offset the bucketed Vega exposures. We started by hedging the Vega in the longest maturity bucket (10y) because the 10y Cap has exposure to volatility across a broader time horizon. Just like in delta hedging, instruments with longer maturities impact both their own segment and partially overlap with shorter buckets. We avoid interference when calibrating the notional of the 6y Cap by neutralizing the 10y Vega first.

$$\begin{cases} x_{10} \cdot V_{10y}^{\text{cap}} + \text{Vega}_{10y} \cdot \text{Notional} = 0 \\ x_6 \cdot V_{6y}^{\text{cap}} + x_{10} \cdot V_{10y}^{\text{cap}} + \text{Vega}_{6y} \cdot \text{Notional} = 0 \end{cases}$$

Cap 6y	**Cap 10y**
46k	-24k

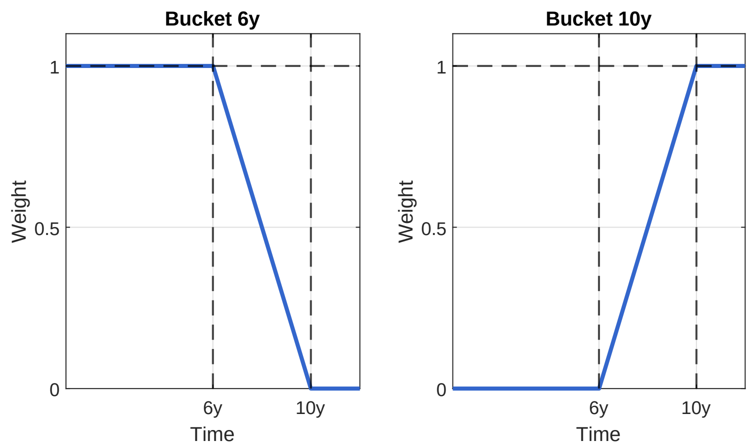

Figure 3: Weight profiles for the 6y and 10y buckets used in the hedging

1.7 Digital Risk Analysis

The structured bond contains coupon that work like digital options whenever the Euribor crosses a fixed threshold. Black approach ignores the fact that market-quoted volatilities change with strike. Since the payoff is discontinuous at the strike, the option's value is sensitive not only to the local volatility level but also to the slope of the volatility smile around that strike. We replaced the pure Black price with a smile-adjusted figure that adds a correction term proportional to the slope of the smile and to the caplet's vega.

Upfront Black	**Upfront Digital Risk**
10,7%	10,6%

Including digital risk effects reduces the upfront by about 5 bp. The reason is that, at the relevant barriers (Euribor thresholds), implied volatility increases with strike, giving the smile a positive slope. The adjustment term therefore subtracts value from each digital leg, so the issuer can charge a slightly smaller upfront in order to remain risk-neutral.

2 Case Study 2

2.1 BMM Calibration

Firstly, we calibrated the BMM model, obtaining the $v(t_i, t_{i+1})$ values for each 3-month period. We derived $v(t_1, t_2)$ by minimizing the difference between the price of the 3m/6m caplet obtained using the LMM model and the corresponding price in the BMM model, expressed as a function of the unknown. Repeating the same reasoning with the corresponding caps, we then obtained $v(t_2, t_3)$ and $v(t_3, t_4)$.

Next, we imposed a linear constraint on the four v functions corresponding to each year, expressing all of them in terms of the final one. At this stage, we minimized the following difference (for notation simplicity we refer to the 1y2y period):

$$\left(\mathrm{Cap}_{2\mathrm{y}}^{\mathrm{LMM}} - \mathrm{Cap}_{1\mathrm{y}}^{\mathrm{LMM}}\right) - \sum_{i=1}^{4} \mathrm{Caplet}_i^{\mathrm{BMM}}(v)$$

where the annual Cap prices were computed using the LMM model, and the intra-annual Caplet prices were obtained via the BMM pricing formula. We then retrieved all the relevant v values recursively and we repeated this procedure year by year, up to the tenth year.

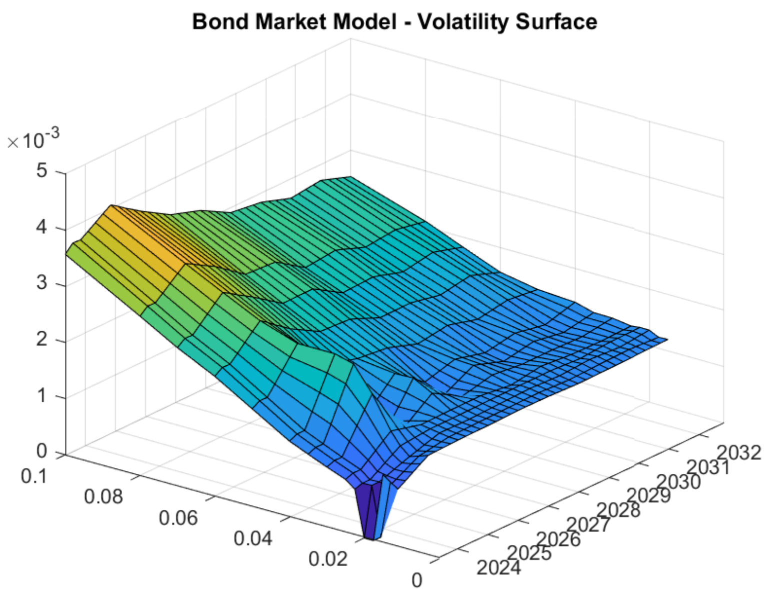

Figure 4: v values as function of market strikes and dates

2.2 BMM pricing

To price the exotic option, we used a Monte Carlo simulation, choosing to perform 10^7 simulations for each $B(t_i, t_{i+1})$. These were generated under the assumption that the forward zero-coupon bond follows a geometric Brownian motion with no drift under the forward t_i-measure in the BMM model, while also accounting for the correlation term between $B(t_{i-1}, t_i)$ and $B(t_i, t_{i+1})$.

We then directly obtained the values of $L(t_i, t_{i+1})$ for each 3-month interval, and finally discounted the resulting payoff.

Price	$\mathbf{CI_{0.99}}$
0.01387	[0.0138; 0.0139]