POLITECNICO
MILANO 1863

Scuola di Ingegneria Industriale e dell'Informazione
Corso di Laurea Magistrale in Mathematical Engineering - Quantitative Finance

Financial Engineering: Group 4 Assignment 5 Risk Management

Matteo Bovio: 272377
Alberto Busci: 278889
Matteo Campagnoli: 275975
Alice Sofia Casciani: 275720

Academic Year: 2024-2025

Index

1 The Discount Factors in the Ho Lee Model

In the Hull-White model, the short rate is given by:

$$r_t = \varphi_t + X_t$$

The stochastic discount factor is:

$$\begin{aligned} D(s,T) &= \exp\left(-\int_s^T r_t\,dt\right) \\ &= \exp\left(-\int_s^T \varphi_t\,dt - X_s\frac{\sigma(s,T)}{\sigma} - \int_s^T \sigma(t,T)\,dW_t\right) \end{aligned}$$

where

$$\sigma(t,T) = \frac{\sigma}{a}\left(1 - e^{-a(T-t)}\right)$$

Knowing that:

$$B(s,T) = \mathbb{E}_s[D(s,T)$$

we get:

$$B(s,T) = \exp\left(-\int_s^T \varphi_t\,dt - X_s\frac{\sigma(s,T)}{\sigma} - \frac{1}{2}\int_s^T \sigma^2(t,T)\,dt\right)$$

Now we compute:

$$B(s,T,T+\theta) = \frac{B(s,T+\theta)}{B(s,T)}$$

That is:

$$\begin{aligned} B(s,T,T+\theta) = \exp\Bigg(&-\int_T^{T+\theta} \varphi_t\,dt - \frac{X_s}{\sigma}\left(\sigma(s,T+\theta) - \sigma(s,T)\right) \\ &- \frac{1}{2}\int_s^{T+\theta} \sigma^2(t,T+\theta)\,dt + \frac{1}{2}\int_s^T \sigma^2(t,T)\,dt\Bigg) \end{aligned}$$

The initial condition is:

$$\begin{aligned} B(t_0,T,T+\theta) = \exp\Bigg(&-\int_T^{T+\theta} \varphi_t\,dt - \frac{1}{2}\int_{t_0}^{T+\theta} \sigma^2(t,T+\theta)\,dt \\ &+ \frac{1}{2}\int_{t_0}^T \sigma^2(t,T)\,dt\Bigg) \end{aligned}$$

Therefore:

$$\begin{aligned} B(s,T,T+\theta) = B(t_0,T,T+\theta)\cdot\exp\Bigg(&-\frac{X_s}{\sigma}\left(\sigma(s,T+\theta) - \sigma(s,T)\right) \\ &- \frac{1}{2}\int_{t_0}^s \left(\sigma^2(t,T+\theta) - \sigma^2(t,T)\right)dt\Bigg) \end{aligned}$$

Defining

$$A(t,T) = \frac{B(t_0,T)}{B(t_0,t)} \exp\left\{ -\frac{1}{2} \int_{t_0}^{t} \left(\sigma(u,T)^2 - \sigma(u,t)^2\right) du \right\}$$

$$C(t,T) = \frac{1}{\sigma} \cdot \sigma(t,T)$$

we obtain

$$B(t,T) = A(t,T) \exp\left\{-X_t \cdot C(t,T)\right\}$$

As we can observe, we have obtained an expression for the discount factors that is independent of the unknown function φ_t.

2 The Expected Exposure

At this point, we were tasked with computing the Expected Exposure (EE) profile over 32 future time points. The initial step involved determining the swap rate using the well known formula, assuming the fixed leg consists of four annual payments over a period of eight years (quarterly cash flows). Subsequently, we implemented the `affine_trick` function by expressing $A(s,\tau)$ and $C(s,\tau)$ in accordance with the theoretical framework of Hull-White model. Afterwards, we calculated the IRS cash flows, initially focusing only on the fixed leg of the 8-year payer interest rate swap. This was computed as the product of the δ_i and the swap rate, deferring the discounting of cash flows to the Mark-to-Market (MtM) valuation stage. To obtain the MtM values, it was necessary to simulate the evolution of the short rate using a Monte Carlo simulation approach on a grid of 32 time step and with 250.000 simulations. This is mandatory because x_t is used in the discount $B(s,\tau)$. By the theory we know that:

$$r_t = \varphi_t + x_t, \tag{1}$$

where φ_t is deterministic and x_t is an OU process

$$\begin{cases} dx_t = -a x_t \, dt + \sigma \, dW_t \\ x_{t_0} = x_0 \end{cases}$$

where a is the process mean reversion speed and σ the interest rate volatility. The solution of the OU process is:

$$x_t = x_0 e^{-a(t-t_0)} + \sigma \int_{t_0}^{t} e^{-a(t-u)} dW_u.$$

To simulate x_t it is necessary to deal with a stochastic integral; first of all we notice that inside this integral we have a deterministic function of time, so the stochastic part will be a gaussian random variable with suitable mean and variance. Moreover:

$$\mathbb{E}[x_t] = \mathbb{E}\left[x_0 e^{-a(t-t_0)} + \sigma \int_{t_0}^{t} e^{-a(t-u)} dW_u\right] = x_0 e^{-a(t-t_0)}$$

$$\begin{aligned}
\mathrm{Var}(x_t) &= \mathbb{E}\left[(x_t - \mathbb{E}[x_t])^2\right] \\
&= \mathbb{E}\left[\left(\sigma \int_{t_0}^{t} e^{-a(t-u)}\, dW_u\right)^2\right] \qquad \text{(thanks to Itô isometry)} \\
&= \sigma^2 \int_{t_0}^{t} e^{-2a(t-u)}\, du \\
&= \sigma^2 \left[-\frac{e^{-2a(t-u)}}{2a}\right]_{u=t_0}^{t} \\
&= \frac{\sigma^2}{2a}\left(1 - e^{-2a(t-t_0)}\right)
\end{aligned}$$

In conclusion

$$x_t = \mu + \sigma * g, \tag{2}$$

where g is a standard gaussian random variable. Finally, by cycling on each date on which there are payments, we simulated the Mark-to-Market for the remaining time of the IRS by discounting the cash flows previously described with the discount $B(s,\tau)$ and adding outside of this expression also the final discount with the corresponding x_t: this is necessary otherwise we would not have correctly evaluated all the payoffs (the final discount that describes the floating part of the financial instrument would be missing). To compute the $EE(\tau)$ we first calculated the $CE(\tau)$ according to the theory for each available data, we evaluated its expected value as an average, the probability that the MtM>0, and finally:

$$EE(\tau) = \frac{\mathbb{E}_0[CE_\tau]}{\mathbb{P}(MtM_\tau > 0)} \tag{3}$$

The distribution of the EE can be seen in Figure 2.

3 The Potential Future Exposure

To evaluate the $PFE(\tau)$ we simply considered the $\alpha = 95\%$ quantile of the MtM_τ previously obtained. In the Figure 1 at the end of the document it is possible to observe the value of the PFE. For better understanding we are plotting the charts referred to the 2 Feb 2028, the other graphs considering to the other dates follow the same distribution.

4 Expected Positive Exposure (EPE)

The expected positive exposure (EPE) is defined as the time-average of the expected exposure (EE) profile over all future simulation dates. Numerically, EPE was obtained by computing the mean of the EE curve generated by the Monte Carlo runs under the Hull–White framework.

EPE
3.6%

Because an IRS's mark-to-market is linear in its notional N, its positive exposure at any time t simply scales as $EE(t;N) = N * EE(t;1)$. Averaging over all payment dates then gives $EE(N) = N * EE(1)$, so that the ratio

$$\lambda = \frac{\text{EPE}}{N} = \text{EPE}(1)$$

depends only on model inputs and not on the absolute notional. Once λ has been computed the EPE of any homologous payer swap can be obtained instantly as $EPE = \lambda * Notional$ without rerunning MC simulations.
This shortcut, however, applies only to payer IRS. For a **receiver** IRS the exposure profile is fundamentally different. Here, $EE(t)$ derives from the negative swap value and only becomes positive when market rates fall below the fixed rate, so "in-the-money" scenarios are different than for a payer. Although the linearity relation

$$\text{EPE}(N) = N \times \text{EPE}(1)_{\text{receiver}}$$

still holds, but the numerical value of $\text{EPE}(1)_{\text{receiver}}$ differs from that of the payer. As a result, applying the payer-calibrated λ to a receiver swap systematically misestimates its true EPE (typically underestimating credit exposure). To obtain a valid scaling factor for a receiver IRS, one must therefore recalibrate λ.
An alternative, closed-form approach to recover $\text{EE}(t)$, without running MC, relies on pricing a series of ATM payer swaptions: at each future date t, the positive part of the swap's mark-to-market coincides with the payoff of a payer swaption expiring at t. Under Black, the price of each swaption today gives $\text{EE}(t)$ directly. Averaging these prices again produces the same EPE and hence the same λ. However, we are unable to implement or verify this swaption-based shortcut because we do not have access to a market data vendor (e.g. Bloomberg) for the required implied volatilities.

5 Peak Potential Future Exposure (Peak-PFE)

The Peak-PFE is defined as the maximum of the potential future exposure (PFE) profile over the entire simulation horizon.

Peak-PFE
9.9%

6 Impact of Annual Collateral

When collateral is posted once each year to eliminate the MtM, we proceed as follows:

- **Annual reset dates**: At each anniversary T_k, we compute the uncollateralized MtM, $\text{MtM}_{\text{uncoll}}(T_k)$. Every previous collateral transfer C_i posted at date

$T_i < T_k$ is grown to T_k by dividing for the forward discount factor. Summing these forward-valued amounts with $\text{MtM}_{\text{uncoll}}(T_k)$ yields the fresh collateral

$$C_k = -\Big[\text{MtM}_{\text{uncoll}}(T_k) + \sum_{i<k} C_i(T_k)\Big],$$

which brings the net MtM back to zero.

- **Intermediate dates**: Between anniversaries no new collateral is exchanged. Instead, we simply forward-value the existing collateral postings to the current date and adjust the MtM by their aggregate effect. So, each past C_i is carried forward to t and the collateralized MtM becomes

$$\text{MtM}_{\text{coll}}(t) = \text{MtM}_{\text{uncoll}}(t) + \sum_i C_i(t).$$

By offsetting the MtM exactly at each annual reset and correctly carrying forward past collateral, this scheme collapses most of the counterparty risk. As a result, both the expected positive exposure and the peak-PFE are materially reduced relative to the uncollateralized case.

EPE (no collateral)	**EPE (collateral)**
3.6%	2.48%

Peak-PFE (no collateral)	**Peak-PFE (collateral)**
9.9%	9.1%

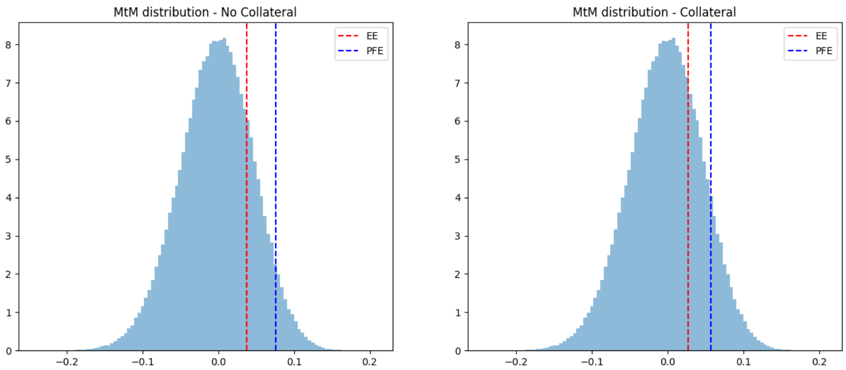

Figure 1: Distribution of simulated Mark-to-Market exposures for the IRS under (left) no collateral and (right) annual collateral posting.

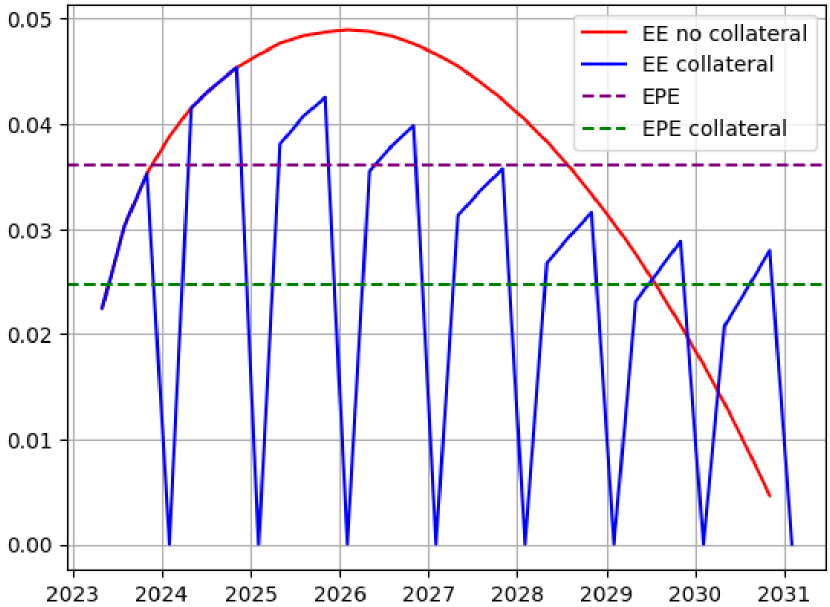

Figure 2: Time series of Expected Exposure (EE) without collateral versus with annual collateral resets. Horizontal lines indicate the corresponding average EPE levels for each case.

Appendix: Errors in the code

- `ex5_notebook`: we replaced

  ```
  swap_type = SwapType.RECEIVER
  ```

  with

  ```
  swap_type = SwapType.PAYER
  ```

- `ex5_notebook`: we replaced the MtM formula as explained in the second point