Calcoli Algoritmo

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1 Updating function

1.1 Update sigma

Function from which we update σ

$$\mathcal{L}\big(\sigma \mid X_{1:n}, \vartheta, \bar{m}_1\big) \propto \mathcal{L}\big(\sigma\big)\sigma^{k-\bar{m}_1} \prod_{i=m_1+1}^{k} \big(1-\sigma\big)_{n_i-1} \frac{\Gamma\big(\vartheta/\sigma + k - \bar{m}_1\big)}{\Gamma\big(\vartheta/\sigma\big)}$$

We know that $\mathcal{L}\big(\sigma\big) \sim Beta\big(\alpha, \beta\big)$

So we explicit $\mathcal{L}\big(\sigma\big)$:

$$\mathcal{L}\big(\sigma\big)\sigma^{k-\bar{m}_1} \prod_{i=m_1+1}^{k} \big(1-\sigma\big)_{n_i-1} \frac{\Gamma(\vartheta/\sigma + k - \bar{m}_1)}{\Gamma(\vartheta/\sigma)} =$$

$$= \frac{\sigma^{\alpha-1}(1-\sigma)^{\beta-1}}{B(\alpha,\beta)} \sigma^{k-\bar{m}_1} \prod_{i=m_1+1}^{k} \big(1-\sigma\big)_{n_i-1} \frac{\Gamma(\vartheta/\sigma + k - \bar{m}_1)}{\Gamma(\vartheta/\sigma)} =$$

$$= \frac{\sigma^{\alpha-1}(1-\sigma)^{\beta-1}\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \sigma^{k-\bar{m}_1} \prod_{i=m_1+1}^{k} \big(1-\sigma\big)_{n_i-1} \frac{\Gamma(\vartheta/\sigma + k - \bar{m}_1)}{\Gamma(\vartheta/\sigma)} =$$

$$= \frac{\sigma^{\alpha-1}(1-\sigma)^{\beta-1}\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \sigma^{k-\bar{m}_1} \prod_{i=m_1+1}^{k} \frac{\Gamma(n_i+\sigma)}{\Gamma(1-\sigma)} \frac{\Gamma(\vartheta/\sigma + k - \bar{m}_1)}{\Gamma(\vartheta/\sigma)} =$$

$$= \frac{\sigma^{\alpha-1+k-\bar{m}_1}(1-\sigma)^{\beta-1}\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \prod_{i=m_1+1}^{k} \frac{\Gamma(n_i+\sigma)}{\Gamma(1-\sigma)} \frac{\Gamma(\vartheta/\sigma + k - \bar{m}_1)}{\Gamma(\vartheta/\sigma)} = f\big(\sigma\big)$$

For Metropolis-Hasting our ratio to evaluate a new value σ^* is:

$$r(\sigma^*,\sigma) = \frac{f\big(\sigma^*\big)}{f\big(\sigma\big)} = \frac{\frac{\sigma^{*\alpha-1+k-\bar{m}_1}(1-\sigma^*)^{\beta-1}\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\prod_{i=m_1+1}^{k}\frac{\Gamma(n_i+\sigma^*)}{\Gamma(1-\sigma^*)}\frac{\Gamma(\vartheta/\sigma^*+k-\bar{m}_1)}{\Gamma(\vartheta/\sigma^*)}}{\frac{\sigma^{\alpha-1+k-\bar{m}_1}(1-\sigma)^{\beta-1}\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\prod_{i=m_1+1}^{k}\frac{\Gamma(n_i+\sigma)}{\Gamma(1-\sigma)}\frac{\Gamma(\vartheta/\sigma+k-\bar{m}_1)}{\Gamma(\vartheta/\sigma)}}$$

We can simplify some terms and we obtain:

$$r(\sigma^*,\sigma) = \frac{f\big(\sigma^*\big)}{f\big(\sigma\big)} = \frac{\sigma^{*\alpha-1+k-\bar{m}_1}(1-\sigma^*)^{\beta-1}\prod_{i=m_1+1}^{k}\frac{\Gamma(n_i+\sigma^*)}{\Gamma(1-\sigma^*)}\frac{\Gamma(\vartheta/\sigma^*+k-\bar{m}_1)}{\Gamma(\vartheta/\sigma^*)}}{\sigma^{\alpha-1+k-\bar{m}_1}(1-\sigma)^{\beta-1}\prod_{i=m_1+1}^{k}\frac{\Gamma(n_i+\sigma)}{\Gamma(1-\sigma)}\frac{\Gamma(\vartheta/\sigma+k-\bar{m}_1)}{\Gamma(\vartheta/\sigma)}}$$

If we consider the logarithm of this ratio and a new function h:

$$h(\sigma) = (\alpha-1+k-\bar{m}_1)log(\sigma)\ +\ (\beta-1)log(1-\sigma)\ +\ \sum_{i=m_1+1}^{k}\big(log\Gamma(n_i+\sigma)\ -\ log\Gamma(1-\sigma)\big)$$
$$+\ log\Gamma(\vartheta/\sigma+k-\bar{m}_1)\ -\ log\Gamma(\vartheta/\sigma)$$

we find out that our ratio is: $log(r(\sigma^*,\sigma)) = h(\sigma^*) - h(\sigma)$

In Metropolis-Hasting to decide if add σ^* to our vector we compare our ratio with u, where u is a random value generated from a uniform distribution from 0 to 1. If $u < r(\sigma^*,\sigma)$ then we add σ^* in our vector otherwise we add σ again. In this case our ratio will be $exp(h(\sigma^*) - h(\sigma))$.

Before using this function instead of σ we will use $\psi = log(\sigma) - log(1-\sigma)$, in this way we consider a variable with domain in $\mathbb{R}$ instead of $[0,1)$. For the same reason we transform also ϑ with $\lambda = log(\vartheta)$. So our function $h(\sigma)$ became:

$$h(\psi) = (\alpha-1+k-\bar{m}_1)(\psi - log(1+e^{\psi}))\ -\ (\beta-1)log(1+e^{\psi})\ +\ \sum_{i=m_1+1}^{k}\big(log\Gamma(n_i - \tfrac{e^{\psi}}{1+e^{\psi}})$$
$$-\ log\Gamma(\tfrac{1}{1+e^{\psi}})\big)\ +\ log\Gamma(\tfrac{e^{\lambda}}{e^{\psi}} + e^{\lambda} + k - \bar{m}_1)\ -\ log\Gamma(\tfrac{e^{\lambda}}{e^{\psi}} + e^{\lambda})$$

1.2 Update theta

Function from which we update ϑ

$$\mathcal{L}\big(\vartheta \mid X_{1:n},\sigma,\bar{m}_1\big) \propto \mathcal{L}\big(\vartheta\big)\frac{\Gamma\big(\vartheta\big)\Gamma\big(\vartheta/\sigma+k-\bar{m}_1\big)}{\Gamma\big(\vartheta/\sigma\big)\Gamma\big(\vartheta+n-\bar{m}_1\big)}$$

We know that $\mathcal{L}\big(\vartheta\big) \sim Gamma\big(\alpha,\beta\big)$

So we explicit $\mathcal{L}\big(\vartheta\big)$:

$$\mathcal{L}\big(\vartheta\big)\frac{\Gamma\big(\vartheta\big)\Gamma\big(\vartheta/\sigma+k-\bar{m}_1\big)}{\Gamma\big(\vartheta/\sigma\big)\Gamma\big(\vartheta+n-\bar{m}_1\big)} = \frac{\beta^{\alpha}}{\Gamma(\alpha)}\vartheta^{\alpha-1}e^{-\beta\vartheta}\frac{\Gamma\big(\vartheta\big)\Gamma\big(\vartheta/\sigma+k-\bar{m}_1\big)}{\Gamma\big(\vartheta/\sigma\big)\Gamma\big(\vartheta+n-\bar{m}_1\big)} = f(\vartheta)$$

For Metropolis-Hastings our ratio to evaluate a new value ϑ^* is:

$$r(\vartheta^*,\vartheta) = \frac{f(\vartheta^*)}{f(\vartheta)} = \frac{\frac{\beta^{\alpha}}{\Gamma(\alpha)}\vartheta^{*\alpha-1}e^{-\beta\vartheta^*}\frac{\Gamma\big(\vartheta^*\big)\Gamma\big(\vartheta^*/\sigma+k-\bar{m}_1\big)}{\Gamma\big(\vartheta^*/\sigma\big)\Gamma\big(\vartheta^*+n-\bar{m}_1\big)}}{\frac{\beta^{\alpha}}{\Gamma(\alpha)}\vartheta^{\alpha-1}e^{-\beta\vartheta}\frac{\Gamma\big(\vartheta\big)\Gamma\big(\vartheta/\sigma+k-\bar{m}_1\big)}{\Gamma\big(\vartheta/\sigma\big)\Gamma\big(\vartheta+n-\bar{m}_1\big)}}$$

We can simplify some terms and we obtain:

$$r(\vartheta^*,\vartheta) = \frac{f(\vartheta^*)}{f(\vartheta)} = \frac{\vartheta^{*\alpha-1}e^{-\beta\vartheta^*}\frac{\Gamma\big(\vartheta^*\big)\Gamma\big(\vartheta^*/\sigma+k-\bar{m}_1\big)}{\Gamma\big(\vartheta^*/\sigma\big)\Gamma\big(\vartheta^*+n-\bar{m}_1\big)}}{\vartheta^{\alpha-1}e^{-\beta\vartheta}\frac{\Gamma\big(\vartheta\big)\Gamma\big(\vartheta/\sigma+k-\bar{m}_1\big)}{\Gamma\big(\vartheta/\sigma\big)\Gamma\big(\vartheta+n-\bar{m}_1\big)}}$$

If we consider the logarithm of this ratio and a new function g:

$g(\vartheta) = (\alpha-1)log(\vartheta) \, - \, \beta\vartheta \, + \, log\Gamma(\vartheta) \, + \, log\Gamma(\vartheta/\sigma + k - \bar{m}_1) \, - \, log\Gamma(\vartheta/\sigma) \, - \, log\Gamma(\vartheta + n - \bar{m}_1)$

we find out that our ratio is: $log(r(\vartheta^*, \vartheta)) = g(\vartheta^*) - g(\vartheta)$

Now we substitute (σ, ϑ) with (ψ, λ) and we obtain:

$g(\lambda) = (\alpha-1)\lambda \, - \, \beta e^{\lambda} \, + \, log\Gamma(e^{\lambda}) \, + \, log\Gamma(\frac{e^{\lambda}}{e^{\psi}} + e^{\lambda} + k - \bar{m}_1) \, - \, log\Gamma(\frac{e^{\lambda}}{e^{\psi}} + e^{\lambda}) \, - \, log\Gamma(e^{\lambda} + n - \bar{m}_1)$

1.3 Update Beta

Function from which we update β

$$\mathcal{L}\big(\beta \mid X_{1:n}, \bar{m}_1\big) \propto \mathcal{L}\big(\beta\big)\beta^{n-\bar{m}_1}(1-\beta)^{\bar{m}_1}$$

In this case we have a conjugate prior so we don't need Metropolis-Hastings to find the posterior distribution.

$$\mathcal{L}\big(\beta\big) \sim Beta\big(\alpha, \gamma\big)$$

We explicit prior distribution and we find the posterior:

$$\mathcal{L}\big(\beta\big)\beta^{n-\bar{m}_1}(1-\beta)^{\bar{m}_1} =$$

$$= \frac{\beta^{\alpha-1}(1-\beta)^{\gamma-1}}{B(\alpha,\gamma)}\beta^{n-\bar{m}_1}(1-\beta)^{\bar{m}_1} =$$

$$= \frac{1}{B(\alpha,\gamma)}\beta^{\alpha+n-\bar{m}_1-1}(1-\beta)^{\gamma+\bar{m}_1-1} =$$

$$= \frac{B(\alpha+n-\bar{m}_1, \gamma+\bar{m}_1)}{B(\alpha,\gamma)}\mathcal{L}\big(\beta_{new}\big)$$

with

$$\mathcal{L}\big(\beta_{new}\big) \sim Beta\big(\alpha+n-\bar{m}_1, \gamma+\bar{m}_1\big)$$

1.4 Update $\bar{\mathbf{m}}_\mathbf{1}$

Function from which we update $\bar{m}_1$:

$$\mathcal{L}(\bar{m}_1 | X_{1:n}, \sigma, \vartheta, \beta) \propto \binom{m_1}{\bar{m}_1}\beta^{n-\bar{m}_1}(1-\beta)^{\bar{m}_1}\sigma^{k-\bar{m}_1}\frac{\Gamma(\vartheta/\sigma + k - \bar{m}_1)}{\Gamma(\vartheta + n - \bar{m}_1)}$$