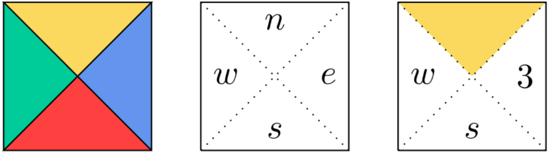
Wang tiles and undecidability problems

Matteo Cavallaro

Scuola Superiore di Catania, Scuola Superiore di Udine

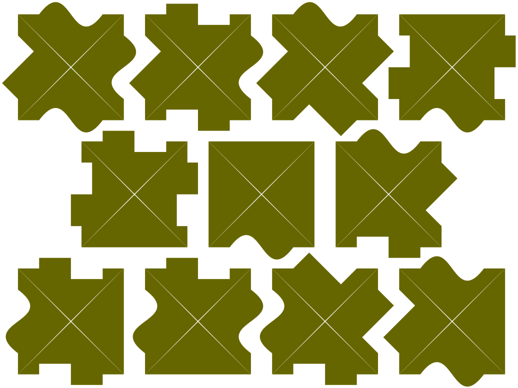
Simposio Matematico RIMSE 2022, 1-2 Ottobre 2022

Wang tiles



*

*figure from Thijs van Loenhout, *Wang Tiles and Cubes*



†

†figure from Wikipedia

Decision problem

$$\mathcal{P}\colon A \to \{0,1\}$$

Examples:

- "is x prime?"
- "is a propositional/first order/... formula ϕ satisfiable?"

Decidability

$\mathcal{P}$ is decidable if exists a Turing machine/algorithm $\mathcal{M}$ such that

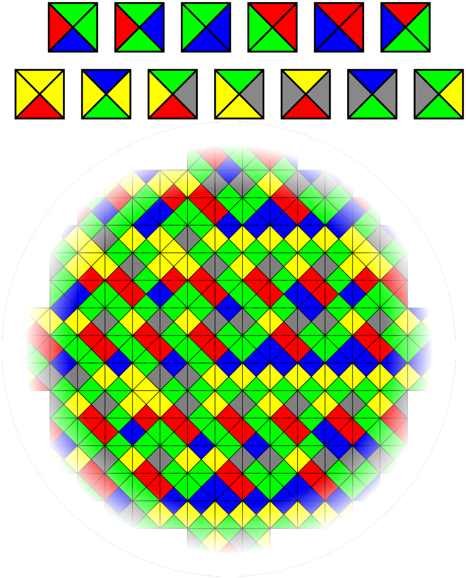
$$\mathcal{M}(a) = \mathcal{P}(a) \qquad \text{for all } a \in A$$

Decision problems on Wang tiles

Given a set of Wang tiles, does exists...

- a tiling of the plane?
- a periodic tiling of the plane?
- an aperiodic tiling of the plane?

A minimal aperiodic tiling set



‡

‡figures from Wikipedia

How to prove decidability?

By constructing an algorithm that solves the problem! e.g.:

procedure $\textsc{Sieve of Eratosthenes}(n)$
  for $i \leftarrow 2, 3, \ldots, \lfloor \sqrt{n} \rfloor$ **do**
    if $n \mod i = 0$ **then**
      return 1
  return 0

How to prove undecidability?

Usually one of two ways:

- by contradiction (*reductio ad absurdum*)
- by a reduction from another undecidable problem

Reductions

Given

$$\mathcal{A}\colon A \to \{0,1\}$$
$$\mathcal{B}\colon B \to \{0,1\}$$

a reduction from $\mathcal{A}$ to $\mathcal{B}$ is a function

$$f : A \to B$$

such that

$$\mathcal{B}(f(a)) = \mathcal{A}(a) \qquad \text{for all } a \in A$$

f must be

- total: $\mathrm{dom}\ f = A$
- computable: exists a Turing machine/algorithm that computes f

Why reductions?

$f \colon A \to B$ $\qquad$ $\mathcal{B}(f(a)) = \mathcal{A}(a)$ $\qquad$ f computable

$$\mathcal{B} \text{ decidable} \implies \mathcal{A} \text{ decidable}$$
$$\mathcal{A} \text{ undecidable} \implies \mathcal{B} \text{ undecidable}$$

Reductions from and to problems on Wang tiles

$$\mathcal{A} \leq \mathcal{W} \leq \mathcal{B}$$

Which $\mathcal{A}$ and $\mathcal{B}$?

- proof of their undecidability: reduction the from halting problem
- useful to prove the undecidability of many problems; the first of these results is the undecidability of $\forall\exists\forall$ fragment of first order logic (Wang)

formulae of $\forall\exists\forall$ fragment:

$$(\forall \ldots \forall)(\exists \ldots \exists)(\forall \ldots \forall)\phi, \qquad \phi \text{ propositional formula}$$

Reductions from and to problems on Wang tiles

$$\mathcal{A} \leq \mathcal{W} \leq \mathcal{B}$$

Which $\mathcal{A}$ and $\mathcal{B}$?

- proof of their undecidability: reduction the from halting problem
- useful to prove the undecidability of many problems; the first of these results is the undecidability of $\forall\exists\forall$ fragment of first order logic (Wang)

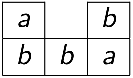
formulae of $\forall\exists\forall$ fragment:

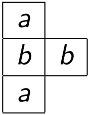
$$(\forall \ldots \forall)(\exists \ldots \exists)(\forall \ldots \forall)\phi, \qquad \phi \text{ propositional formula}$$

Undecidability of brick codes I

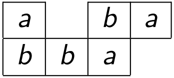
- bricks:

$k_1 = \boxed{a}$

$k_2 =$ 

$k_3 =$ 

- a brick tilable with k_1 and k_2:



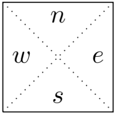
- a set of bricks X is a code if every brick has **at most one** tiling with the elements of X

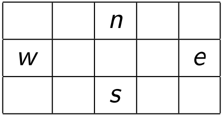
Undecidability of brick codes II

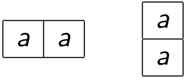
Undecidability provable by reduction from finite Wang tiling problem:

- a tile can have some edges (but not all) labelled with a blank colour
- does exists a finite tiling with blank border?

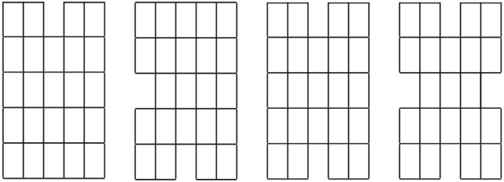
Undecidability of brick codes III

- for each tile 

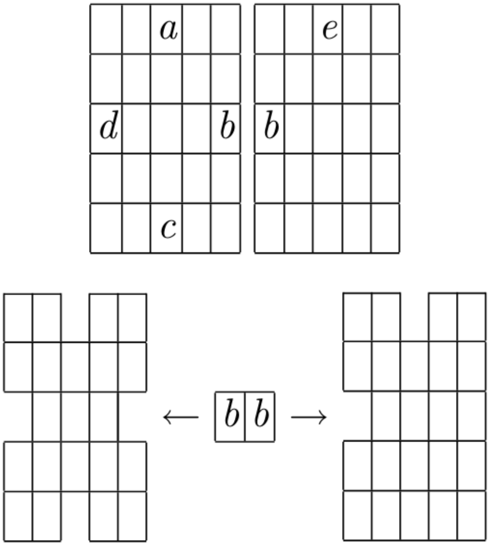
: 

- for each colour a: 

- all the blank shapes with one or more edge mid-points removed, e.g.:



Undecidability of brick codes IV



Thanks for your attention!