Algebraic aspects of the normalized Laplacian

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Abstract Spectral graph theory looks at the interplay between the structure of a graph and the eigenvalues of a matrix associated with the graph. Many interesting graphs have rich structure which can help in determining the eigenvalues associated with some particular matrix of a graph. This survey looks at some common techniques in working with and determining the eigenvalues associated with the normalized Laplacian matrix, in addition to some algebraic applications of these eigenvalues.

1 Introduction

Spectral graph theory looks at understanding the relationship between the structure of a graph and the eigenvalues, or spectrum, of some matrix (or collection of matrices) associated with the graph. There are many different matrices that are considered, including the adjacency matrix (A, whose entries indicate when two vertices are adjacent), Laplacian ($L = D - A$ where D is the diagonal degree matrix), signless Laplacian ($L = D + A$), and the normalized Laplacian ($\mathscr{L}$"="$D^{-1/2}(D-A)D^{-1/2}$). The goal of this paper is to investigate the spectrum of the normalized Laplacian.

One can roughly divide spectral graph theory into two camps. In one camp we look at the structure of graphs and subgraphs, how eigenvalues relate, how to build smaller graphs into larger graphs and what this does to the spectrum, and precise relationships. This is *algebraic spectral graph theory*. In the other camp we look at approximating the structures of a graph, and giving bounds for the spectrum using various combinatorial parameters and conversely using various combinatorial parameters to give bounds for the spectrum. This is *analytic spectral graph theory*.

While both are important in understanding and using eigenvalues, our goal in this survey is to focus on the algebraic aspects of the normalized Laplacian matrix. In

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particular, we will be looking at understanding how eigenvalues can be computed, and various graph operations that can be carried out and the corresponding result on the eigenvalues or characteristic polynomials.

We note that there is a large amount of literature on *algebraic graph theory* which involves spectral graph theory for *regular* graphs (see [3, 14]). We will for the most part avoid results that deal with regular graphs (where possible). This is for the simple reason that if a graph is regular, then knowing the spectrum of any one of the adjacency, Laplacian, signless Laplacian, or normalized Laplacian will give us the spectrum of all the others.

We proceed by first going into more detail on the formal definition of the normalized Laplacian (Section 2). We will then look at common ways to combine two graphs together, i.e., graph products, and look at how this affects the spectrum (Section 3). We also consider equitable partitions, and duplicated structures in the graph (Section 4). Finally we will consider some properties of the characteristic polynomial for the normalized Laplacian, including its relation to Kemeny's constant (Section 5). Throughout the paper we will give several examples showing how these various techniques can be applied.

More information about the normalized Laplacian, with a greater focus on the analytic results can be found in the work of Chung [12] and recent surveys [6, 8].

2 Weighted graphs and the normalized Laplacian

Most of the focus in spectral graph theory (and in this paper) is on *simple* graphs, i.e., graphs without loops or multiple edges. This allows us to reduce the edge relationships between vertices to a simple binary 'yes' or 'no'. While this is useful combinatorially, we will see that in some cases it is better to work in the larger context of weighted graphs. These are graphs where there are weight functions $w(u,v)$ on the *edges* of the graph and $w(u)$ on the *vertices* of the graph.

The weight function on the edges satisfies $w(u,v) \geq 0$ and $w(u,v) = w(v,u)$. (There has been some investigation into the case when negative edge weights are allowed which came up in the context of a variation of a "2-lift" (see [5]).) We will follow the convention that $w(u,v) > 0$ if and only if $u \sim v$, i.e., u and v are adjacent. The requirement that $w(u,v) = w(v,u)$ reflects that we want to limit our discussion to undirected graphs. With the weight function on the edges we can define the *weighted* adjacency matrix by letting $A_{uv} = w(u,v)$.

The weight function on the vertices satisfies $w(u) \geq 0$. The weight function affects the degree of a vertex u, denoted $d(u)$ and defined by

$$d(u) = w(u) + \sum_{v} w(u,v).$$

With this definition of degree we now can define the weighted degree matrix D. We note here that $w(u,u)$ and $w(u)$ are *not* the same thing. Namely $w(u,u)$ represents the weight of a loop at the vertex u (which is an edge) while $w(u)$ is some intrinsic weight to the vertex. Put in another way, $w(u,u)$ contributes both to A and D, while $w(u)$ only contributes to D.

Most applications will assume that the weight function on the vertices are all zero. However, these weights can play an important and useful role, particularly when we have several duplicate structures in a graph. They also show up when looking at random walks on a portion of the graph, i.e., a subgraph. In such a situation there are generally three options available as to what to do with the edges between the subgraph we are interested in and the remainder graph.

1. Delete the edges (i.e., work on the "induced" subgraph). This preserves all the weights on the edges of the subgraph, i.e., A, but will affect the degree of vertices incident with deleted edges, i.e., changes D.
2. Change edges to loops (i.e., we take random walks in the whole graph, but if we exit the subgraph then we go right back to the vertex we came from). This will now preserve the degree of vertices, i.e., D, but with added loops we affect A. (This is known as the Neumann boundary condition [12].)
3. Let the edges become escape routes (i.e., we take random walks in the whole graph, but if we exit the subgraph then we stop the walk). This is done by having the weights of any deleted incident edges of a vertex be added to the weight of the vertex, which helps to preserve both D and A. (This is known as the Dirichlet boundary condition [12].)

In particular, a useful way to think about degree weights are as half-edges incident to a vertex. In later sections we will give examples of how we can compute the spectrum of some of these graphs.

2.1 Formal definition of the normalized Laplacian

We can now give the formal definition for the normalized Laplacian. Namely,

$$\mathscr{L}_{uv} = \begin{cases} \frac{-w(u,v)}{\sqrt{d(u)d(v)}} & \text{if } u \neq v \text{ and } u \sim v; \\ \frac{d(u)-w(u,u)}{d(u)} & \text{if } u = v \text{ and } d(u) \neq 0; \\ 0 & \text{else.} \end{cases} \tag{1}$$

In the special case when none of the vertices has degree 0, then this is equivalent to $\mathscr{L} = D^{-1/2}(D-A)D^{-1/2}$ (and in many papers this is how it is defined). When the graph is regular of degree d then $D^{-1/2} = (1/\sqrt{d})I$ so that $\mathscr{L} = I - (1/d)A$. In particular, if we know the spectrum of the normalized Laplacian matrix, we also know the spectrum of the adjacency matrix (as well as the Laplacian and signless Laplacian). Therefore any known results for regular graphs will translate into the normalized Laplacian matrix. Further if a graph is bipartite, i.e., $V = A \cup B$ with edges going between A and B and the $d(u) = s$ for $u \in A$ and $d(u) = t$ for $u \in B$ (i.e., the graph is bi-regular), then $\mathscr{L} = I - (1/\sqrt{st})A$. So again results for such graphs translate between the normalized Laplacian and adjacency matrices.

When none of the vertices have degree 0 the normalized Laplacian is similar to the matrix $I - D^{-1}A$. The matrix $D^{-1}A$ is the probability transition matrix for random walks. Therefore the study of the normalized Laplacian is intrinsically linked to understanding random walks.

As we mentioned earlier, most of spectral graph theory is concerned with looking at simple graphs, which are the case $w(u,v) \in \{0,1\}$ and $w(u,u) = w(u) = 0$. When there are no loops and vertex weights are 0 (such as in simple graphs), then the diagonal entries of the normalized Laplacian consists of 0's and 1's where the 0's correspond to isolated vertices. In particular, we have that the trace is the number of non-isolated vertices, or put another way, we can determine the number of isolated vertices by looking at the sum of the eigenvalues compared to the number of eigenvalues. Once we identify how many isolated vertices there are we can then remove the corresponding number of 0's and have the spectrum of the part of the graph which has no isolated vertices. Therefore when working with simple graphs we can simply assume that there are no isolated vertices when working to understand the spectrum.

We conclude this part by noting the following basic properties of $\mathscr{L}$.

Theorem 1 (Chung [12]). *The matrix $\mathscr{L}$ is positive semi-definite and satisfies the following:*

1. *All eigenvalues lie in the interval* $[0,2]$.
2. *The multiplicity of* 0 *is the number of connected components of the graph.*
3. *The multiplicity of* 2 *is the number of bipartite components with at least two vertices.*
4. *A graph without isolated vertices is bipartite if and only if the spectrum is symmetric around* 1.

2.2 Scaling and the normalized Laplacian

Given a weighted graph G and a parameter $\alpha > 0$ we will let αG denote the weighted graph where $w_{\alpha G}(u,v) = \alpha w_G(u,v)$ and $w_{\alpha G}(u) = \alpha w_G(u)$, i.e., we have scaled all the weights.[1] (Here and throughout we will use subscripts to indicate which graph is being indicated if there are several graphs being considered.) If we scale all of the weights by α, that also will scale all of the degrees by α. Now considering (1) we immediately get the following.

Proposition 1. *For any weighted graph G we have $\mathscr{L}_{\alpha G} = \mathscr{L}_G$.*

In particular, scaling the weights does not affect the matrix, and hence also the spectrum. This is the "normalization" of the normalized Laplacian.

[1] This is different from the usual convention of letting αG denote α disjoint copies of G.

2.3 Harmonic eigenvectors

One of the best ways to understand what is happening with the eigenvalues associated with a matrix is to look at the eigenvectors and how they relate to the structure. For the normalized Laplacian we will find it easier to work with the following variant.

Definition 1. Given a graph with no isolated vertices and $\mathscr{L}\mathbf{x} = \lambda\mathbf{x}$, i.e., $\mathbf{x}$ is an eigenvector of $\mathscr{L}$, then the corresponding *harmonic eigenvector* associated with the eigenvalue λ is $\mathbf{y} = D^{-1/2}\mathbf{x}$.

We can now rewrite $\mathscr{L}\mathbf{x} = \lambda\mathbf{x}$ as follows:

$$\begin{aligned} D^{-1/2}(D-A)D^{-1/2}\mathbf{x} &= \lambda\mathbf{x} \\ (D-A)\mathbf{y} &= \lambda D^{1/2}(D^{1/2}\mathbf{y}) = \lambda D\mathbf{y} \\ (1-\lambda)D\mathbf{y} &= A\mathbf{y} \end{aligned}$$

This relationship becomes a *local* requirement at each vertex. Namely, $\mathbf{y}$ is a harmonic eigenvector for λ if and only if it is nonzero and at each vertex u

$$\sum_{v\sim u} w(u,v)\mathbf{y}(v) = (1-\lambda)d(u)\mathbf{y}(u). \tag{2}$$

It is instructive to compare this to what happens if we use the adjacency matrix, i.e., $A\mathbf{z} = \lambda\mathbf{z}$ if and only if

$$\sum_{v\sim u} w(u,v)\mathbf{z}(v) = \lambda\mathbf{z}(u).$$

From this we see that any eigenvector of the adjacency matrix associated with the eigenvalue 0 is also a harmonic eigenvector of the normalized Laplacian associated with the eigenvalue 1 (i.e., in both cases the right hand side is 0).

Given we have a graph with no vertex weights, then if we let $\mathbf{y} = \mathbf{1}$ (the all 1's vector), (2) becomes

$$\sum_{v\sim u} w(u,v)\mathbf{z}(v) = \sum_{v\sim u} w(u,v) = d(u) = (1-0)d(u)\mathbf{z}(u),$$

showing $\mathbf{1}$ is the harmonic eigenvector for the eigenvalue 0. This places limitations on the remaining (harmonic) eigenvectors. In particular, since the normalized Laplacian is symmetric, the remaining eigenvectors must be perpendicular to $D^{1/2}\mathbf{1}$.We will say that the harmonic eigenvectors $\mathbf{y}$ and $\mathbf{z}$ are perpendicular if and only if

$$0 = (D^{1/2}\mathbf{y})^T(D^{1/2}\mathbf{z}) = \mathbf{y}^T D\mathbf{z} = \sum_u d(u)\mathbf{y}(u)\mathbf{z}(u),$$

i.e., the corresponding eigenvectors are perpendicular.

3 Graph products

Our goal in this section is to show how to use graphs as building blocks to construct larger graphs. Before we begin, we will find it useful to have a basic collection of simple graphs for which the eigenvalues are known. We will use the complete graph K_n, the complete bipartite graph $K_{m,n}$, the cycle graph C_n, and the path graph P_n. For the normalized Laplacian these graphs have the following spectrum (see Chung [12] for more information):

$$\begin{aligned} K_n: &\quad \left\{0, \tfrac{n}{n-1}^{(n-1)}\right\} \\ K_{s,t}: &\quad \left\{0, 1^{(s+t-2)}, 2\right\} \\ C_n: &\quad \left\{1-\cos\tfrac{2k\pi}{n} : 0 \le k \le n-1\right\} \\ P_n: &\quad \left\{1-\cos\tfrac{k\pi}{n-1} : 0 \le k \le n-1\right\}. \end{aligned}$$

Here we use the notation $*^{(i)}$ to indicate that an eigenvalue is repeated i times.

From this simple list of graphs we already see that the normalized Laplacian cannot always determine the number of edges or if a graph is regular. For example $K_{1,3}$ and $K_{2,2}$ both have spectrum $\{0, 1^{(2)}, 2\}$, but they have differing number of edges and one is regular while the other is not. Note that for the other common matrices studied in spectral graph theory, i.e., adjacency, Laplacian, and Laplacian, the number of edges and whether a graph is regular can always be determined by the spectrum.

We now want to look at ways to combine our building blocks. This will be done by looking at various products and the join operation which are defined for simple graphs.

Definition 2. The *Cartesian product* of G and H, denoted $G\Box H$, the *tensor product* of G and H, denoted $G\times H$, and the *strong product* of G and H, denoted $G\boxtimes H$, all have vertex set $\{(a,b) : a \in V(G), b \in V(H)\}$ and edge sets respectively as follows:

$$\begin{aligned} E(G\Box H) &= \big\{\{(a,b),(c,d)\} : a = c \text{ and } b{\sim}d;\text{ or } a{\sim}c \text{ and } b = d\big\}, \\ E(G\times H) &= \big\{\{(a,b),(c,d)\} : a{\sim}c \text{ and } b{\sim}d\big\}, \\ E(G\boxtimes H) &= E(G\Box H) \cup E(G\times H). \end{aligned}$$

Note that the notation used for the different products reflects the result when you take the product of K_2 with itself.

Definition 3. The *join* of G and H, denoted $G\vee H$, is the graph formed by taking the disjoint union of G and H and then adding an edge between every vertex in G and every vertex in H.

Proposition 2. *There is* no *way to determine the spectrum of the normalized Laplacian for the graphs $G\Box H$, $G\boxtimes H$ or $G\vee H$ by only knowing the spectrum of the normalized Laplacian of G and H.*

This can be seen by starting with $K_{1,3}$ and $K_{2,2}$, which have the same spectrum, and then looking at the different graphs that result when combined with K_2. These graphs are shown in Figure 1.

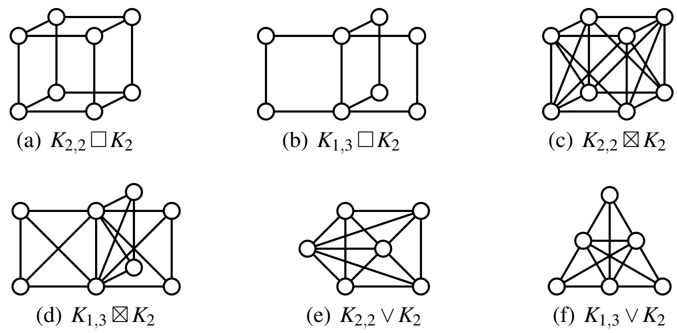


Fig. 1 Examples of graph operations

The eigenvalues for the normalized Laplacian are given below, the important thing to notice being that the various graphs differ in their spectrum.

$K_{2,2}\,\square\,K_2$:	$\{0,\frac{2}{3}^{(3)},\frac{4}{3}^{(3)},2\}$	$K_{1,3}\,\square\,K_2$:	$\{0,\frac{1}{2}^{(2)},\frac{3}{4},\frac{5}{4},\frac{3}{2}^{(2)},2\}$
$K_{2,2}\boxtimes K_2$:	$\{0,\frac{4}{5}^{(2)},\frac{6}{5}^{(4)},\frac{8}{5}\}$	$K_{1,3}\boxtimes K_2$:	$\{0,\frac{2}{3}^{(2)},\frac{8}{7},\frac{4}{3}^{(3)},\frac{32}{21}\}$
$K_{2,2}\vee K_2$:	$\{0,1^{(2)},\frac{6}{5},\frac{13}{10},\frac{3}{2}\}$	$K_{1,3}\vee K_2$:	$\{0,1^{(2)},\frac{6}{5}^{(2)},\frac{8}{5}\}$

When the graphs are regular, relationships of the spectrums can be established.

Proposition 3. *Let G be an r-regular simple graph with normalized Laplacian eigenvalues $0=\lambda_0\le\lambda_1\le\cdots\le\lambda_{n-1}$, and let H be an s-regular simple graph with normalized Laplacian eigenvalues $0=\theta_0\le\theta_1\le\cdots\le\theta_{m-1}$. Then the following hold.*

- *If at least one of r and s is nonzero, the eigenvalues of $G\,\square\,H$ are*

$$\left\{\frac{r\lambda_i+s\theta_j}{r+s}:0\le i\le n-1,0\le j\le m-1\right\}.$$

- *If at least one of r and s is nonzero, the eigenvalues of $G\boxtimes H$ are*

$$\left\{\frac{rs(\lambda_i+\theta_j-\lambda_i\theta_j)+r\lambda_i+s\theta_j}{rs+r+s}:0\le i\le n-1,0\le j\le m-1\right\}.$$

- *The eigenvalues of $G\vee H$ are*

$$\left\{0,2-\frac{r}{m+r}-\frac{s}{n+s}\right\}\cup\left\{\frac{m+r\lambda_i}{m+r}:1\le i\le n-1\right\}\cup\left\{\frac{n+s\theta_j}{n+s}:1\le j\le m-1\right\}.$$

The results for the Cartesian product and strong product readily follow by noting that the product of regular graphs is regular. Therefore we can translate the eigenvalues of the normalized Laplacian into the eigenvalues of the adjacency matrix, apply known results for these products on the adjacency matrix (see [3]), and then translate the resulting eigenvalues back to the normalized Laplacian.

The result for the join is more interesting because the join of two graphs does not have to be regular so we cannot simply translate results for the adjacency matrix.

Example 1. The eigenvalues of $W_{n+1} = K_1 \vee C_n$ (known as the wheel graph) and the friendship graph on $2k+1$ vertices (formed by taking the join of a single vertex and k disjoint copies of K_2 are) are respectively

$$\big\{0, \tfrac{4}{3}\big\} \cup \big\{1 - \tfrac{2}{3}\cos\tfrac{2i\pi}{n} : 1 \le i \le n-1\big\} \quad \text{and} \quad \big\{0, \tfrac{1}{2}^{(k-1)}, \tfrac{3}{2}^{(k+1)}\big\}.$$

(a) Wheel graph (b) Friendship graph

Fig. 2 Examples of graphs formed by joins

The proof for the join result can be found in [5]. We sketch the key idea here as it is indicative of a common theme in this area. Consider $G \vee H$ and let us look at what happens with the part that is restricted to G. In addition to the edges of G each vertex in G now has m additional edges (those that connect to the vertices of H) and so now the degree of vertices is $m+r$. Now take a harmonic eigenvector of G which is perpendicular to the harmonic eigenvector $\mathbf{1}$ (there are $n-1$ such eigenvectors). By definition we know that such a harmonic eigenvector has the following two properties in G for some λ:

$$\sum_{v \sim u} \mathbf{y}(v) = (1-\lambda) r \mathbf{y}(u) \qquad \text{and} \qquad \sum_{v} \mathbf{y}(v) = 0.$$

(The first is from (2), the second from the perpendicular requirement.) Now we take this vector and create a new vector $\widehat{\mathbf{y}}$ for $G \vee H$ by making the value $\widehat{\mathbf{y}}(w) = 0$ for all $w \in V(H)$. Then we note the following happens

$$\sum_{v \sim u} \widehat{\mathbf{y}}(v) = \begin{cases} \sum_{v \sim u} \mathbf{y}(v) = (1-\lambda) r \mathbf{y}(u) = (1-\lambda')(m+r)\widehat{\mathbf{y}}(u) & \text{if } u \in V(G); \\ \sum_{v} \mathbf{y}(v) = 0 = (1-\lambda')(n+s)\widehat{\mathbf{y}}(u) & \text{if } u \in V(H). \end{cases}$$

In particular this will be a harmonic eigenvector for the eigenvalue λ' where

$$(1-\lambda')(m+r) = (1-\lambda) r \quad \text{or} \quad \lambda' = \frac{m + r\lambda}{m+r}.$$

So therefore we can readily find all but two of the eigenvalues. On the other hand we know that 0 must be an eigenvalue and it comes from none of these, leaving us with one eigenvalue, and then knowing that the sum of the eigenvalues must be $m+n$ allows us to find the last. (Equitable partitions, discussed in Section 4, give another method to find the remaining eigenvalues by “collapsing” G and H.)

The argument boiled down to two facts. First, we knew that the sum of the entries of the harmonic eigenvector was 0 and since we “coned” over the graph we could assign all other values to 0 and still preserve the local requirements. Second, when we start with a regular graph and then perturb the degree, this will perturb the eigenvalue in a predictable way.

In the case that we have a bipartite graph where one part is regular we can cone over the regular side in a similar way and still be able to locally control the eigenvalues. This basic idea was used by Butler and Grout [9] to form cospectral graphs by gluing in such bipartite graphs into generic graphs.

It remains to now look at what happens for the tensor product of two graphs.

Theorem 2. *Let G and H be simple graphs without isolated vertices, further let $0=\lambda_0\le\lambda_1\le\cdots\le\lambda_{n-1}$ and $0=\theta_0\le\theta_1\le\cdots\le\theta_{m-1}$ be the normalized Laplacian eigenvalues for G and H, respectively. Then the normalized Laplacian eigenvalues for $G\times H$ are*

$$\{\lambda_i+\theta_j-\lambda_i\theta_j : 0\le i\le n-1, 0\le j\le m-1\}.$$

Proof. This follows from combining two simple observations. First, λ is an eigenvalue of $\mathscr{L}$ if and only if $1-\lambda$ is an eigenvalue of $D^{-1/2}AD^{-1/2}$. Second, since $d_{G\times H}\big((u,v)\big)=d_G(u)d_G(v)$ then when $(u,v)\sim(s,t)$ we have

$$\frac{1}{\sqrt{d_{G\times H}\big((u,v)\big)d_{G\times H}\big((s,t)\big)}}=\frac{1}{\sqrt{d_G(u)d_G(s)}\sqrt{d_H(v)d_H(t)}}.$$

And in particular for the tensor product we have

$$D_{G\times H}^{-1/2}A_{G\times H}D_{G\times H}^{-1/2}=\big(D_G^{-1/2}A_GD_G^{-1/2}\big)\otimes\big(D_H^{-1/2}A_HD_H^{-1/2}\big)$$

where “$\otimes$” indicates the tensor product of matrices. But the eigenvalues of a tensor product of matrices is found by taking all possible products of the eigenvalues of the initial matrices. $\square$

Example 2. The spectrum of $K_t\times P_n$ (see Figure 3) is

$$\big\{1-\cos\tfrac{k\pi}{n-1}, \big(1+\tfrac{1}{t-1}\cos\tfrac{k\pi}{n-1}\big)^{(t-1)} : 0\le k\le n-1\big\}.$$

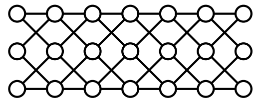


Fig. 3 $K_3 \times P_7$

4 Equitable partitions and twins

In some graphs there can be portions which have similar structure. This allows us to group vertices by in such a way that in each group the vertices behave similarly. More precisely, we have the following.

Definition 4. An *equitable partition* of G is a partition $V(G) = V_1 \cup V_2 \cup \cdots \cup V_k$ so that the sum of edge weights from a particular vertex in V_i to all the vertices in V_j does not depend on which vertex we choose, i.e., for $s,t \in V_i$ we have

$$\sum_{v \in V_j} w(s,v) = \sum_{v \in V_j} w(t,v).$$

Theorem 3 (Butler and Chung [8]). *Given an equitable partition $V = V_1 \cup \cdots \cup V_k$ of a weighted graph G with no vertex weights we can form a new graph H with vertices $v_1,\ldots,v_k$ and edge weights*

$$w_H(v_i,v_j) = \sum_{\substack{s \in V_i \\ t \in V_j}} w_G(s,t).$$

Then the eigenvalues of $\mathscr{L}_H$ are also eigenvalues of $\mathscr{L}_G$ (including multiplicity).

The basic idea is that if we have an eigenvector for H then we can lift it up to an eigenvector for G where it is constant on the parts in the equitable partition. (Note by our assumptions that the degrees of the vertices in each part are constant.)

Example 3. For the complete tripartite graph $K_{r,s,t}$ we can find $r+s+t-3$ independent eigenvectors associated with 1, i.e., choose two vertices in the same part and assign one of them the value 1, the other the value -1 and all other vertices 0. For the remaining three eigenvalues we can form an equitable partition of three parts, i.e., one for each part, this equitable partition gives a weighted triangle on three vertices with edge weights rs, rt and st, for which the eigenvalues can be computed. Therefore the spectrum of $K_{r,s,t}$ is

$$\left\{0, 1^{(r+s+t-3)}, \tfrac{3}{2} \pm \tfrac{1}{2}\sqrt{9 - 8\tfrac{(r+s+t)(rs+rt+st)}{(r+s)(r+t)(s+t)}}\right\}.$$

For example, $K_{5,7,9}$ has spectrum $\left\{0, 1^{(18)}, \frac{11}{8}, \frac{13}{8}\right\}$.

One common way to form an equitable partition is to use the orbits of an automorphism. An *automorphism* of a weighted graph G is a mapping $\pi : V(G) \to V(G)$

such that for each $u,v \in V(G)$ we have $w(u,v) = w\big(\pi(u),\pi(v)\big)$ and $w(u) = w\big(\pi(u)\big)$.

Another common method which relates to equitable partitions is to use covers. We say that a weighted graph G *covers* a weighted graph H if there is a surjective map $\pi : V(G) \to V(H)$ which preserves edge weights so that for each $v \in V(G)$ the restriction of π to the neighborhood of v is a bijection. The preimages of vertices in H give an equitable partition. A classic example of this are cycles C_n are covered by cycles C_{kn} for $k \geq 1$, and therefore the eigenvalues of C_n are a subset of the eigenvalues of C_{kn}. More information about covers as it relates to the normalized Laplacian is given by Chung and Yau [13].

4.1 Twin vertices

We can use equitable partitions to find *some* of the eigenvalues of a graph, the question then remains on how do we find the rest. This can sometimes be done locally within the parts of the partition.

Theorem 4. *Let V_i be a part in an equitable partition of a graph without vertex weights and with vertices which are regular of degree k satisfying that for each $u \notin V_i$, $w(u,s) = w(u,t)$ for all $s,t \in V_i$ (i.e., the part* cones *to the remainder of the graph). If $G[V_i]$ (the induced subgraph of G on V_i) is regular of degree d and has as eigenvalues of the normalized Laplacian $0 = \lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_{m-1}$, then*

$$\left\{ \frac{k-d+d\lambda_i}{k} : 1 \leq i \leq m-1 \right\}$$

are eigenvalues of G with eigenvectors whose support lies in V_i.

This follows from the same ideas used in determining the eigenvalues of the join, which is itself a special case of an equitable partition.

One special case of this result in simple graphs is when we have *twins*, which are vertices which have the same set of (closed) neighbors. (There is also a notion of twins for weighted graphs, see [7].)

Corollary 1. *Let V_i consist of a collection of twins in a simple graph which are mutually non-adjacent. Then there are $|V_i|-1$ eigenvalues of 1 which come from eigenvectors restricted to V_i.*

Let V_i consist of a collection of twins in a simple graph which are mutually adjacent and have common degree d. Then there are $|V_i|-1$ eigenvalues of $\frac{d+1}{d}$ which come from eigenvectors restricted to V_i.

By identifying twin vertices we can collapse a graph and deal with smaller graphs. For example we can readily identify sets of twins for the two graphs in Figure 4 so that the resulting equitable partitions correspond to weighted copies of C_6. Both graphs will have six eigenvalues of 1 from the twins, and the remaining eigenvalues from the graphs $4C_6$ and $3C_6$, respectively. By Proposition 1 the graphs $4C_6$ and $3C_6$ have the same normalized Laplacian, allowing us to conclude the original graphs are cospectral. This can also be shown by noting that these particular graphs are $C_3 \times K_{2,2}$ and $C_3 \times K_{1,3}$.

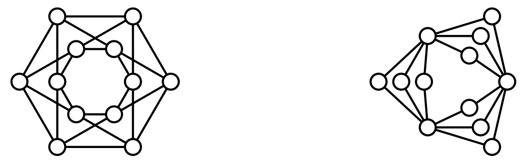


Fig. 4 An example of two cospectral graphs

More examples of this phenomenon are given in [7], including the following construction which can be used to form cospectral graphs with differing numbers of edges for the normalized Laplacian.

Theorem 5. *Given a simple bipartite graph G with $V = A \cup B$ and natural numbers s,t,s',t' satisfying $s|A|+t|B| = s'|A|+t'|B|$, then form a new graph H by replacing each vertex in A by s independent vertices, each vertex in B by t independent vertices and each edge by the complete bipartite graph $K_{s,t}$, similarly form H' but using s' and t' for the respective sizes. Then H and H' are cospectral and have $st|E(G)|$ and $s't'|E(G)|$ edges respectively.*

An example using twins that are mutually adjacent will be given later.

4.2 Twin subgraphs

We can generalize the idea of twin vertices to twin subgraphs. The key element is that we have duplicate structures in the graph which connect in a consistent manner. More precisely we have the following.

Definition 5. Let G be a simple graph and let $V_1,V_2,\ldots,V_k$ be disjoint subsets of the vertices such that $G[V_1],\ldots,G[V_k]$ (i.e., the induced subgraphs on the V_i) are isomorphic with the isomorphism $\pi_i : V_1 \to V_i$. Further, these subgraphs connect in a consistent manner, i.e., for $u \notin V_1 \cup \cdots \cup V_k$ then $u{\sim}v$ for $v \in V_1$ if and only if $u{\sim}\pi_i(v)$ for $2 \le i \le k$. Then $G[V_1],\ldots,G[V_k]$ are *twin subgraphs*.

The notion of twin subgraphs can be applied to weighted graphs. Information on how this is done can be found in [7].

Proposition 4. *Let G be a simple graph with twin subgraphs coming from $V_1,\ldots,V_k$. Then the eigenvalues of G can be found using the following weighted graphs:*

- *Multiplicity $k-1$ for each of the eigenvalues arising from the graph $G[V_1]$ with vertex weights added so that the degrees agree with the degrees in G.*

- *Multiplicity* 1 *for each of the eigenvalues arising from the graph obtained by deleting $V_2 \cup \cdots \cup V_k$ and changing the edge weight of each edge incident to a vertex in V_1 to k.*

The idea of twin subgraphs has appeared in the work of Banarjee and Jost [1] and Banarjee and Mehatari [2]. Their approach was to start with a fixed graph and then create twin subgraphs (what they called "motifs") by duplicating an induced subgraph and its connections. They then considered what happens with the eigenvalues, where particular focus was given on the eigenvalue 1.

Example 4. The graph $K_{1,t} \square K_2$ can be viewed as a collection of 4-cycles which have been glued along an edge. We can view this as a collection of t disjoint copies of K_2 that consistently glue into a single copy of K_2. Therefore the eigenvalues of $K_{1,t} \square K_2$ are found by taking $(t-1)$ copies of the eigenvalues arising from the edge K_2 where the vertices have weight 1 (which have eigenvalues $\{\frac{1}{2}, \frac{3}{2}\}$), and the graph C_4 where one edge has weight 1 and the remaining three edges have weight t (which has eigenvalues $\{0, \frac{t+3}{2t+2}, \frac{3t+1}{2t+2}, 2\}$). Therefore the spectrum is

$$\left\{0, \tfrac{1}{2}^{(t-1)}, \tfrac{t+3}{2t+2}, \tfrac{3t+1}{2t+2}, \tfrac{3}{2}^{(t-1)}, 2\right\}.$$

Example 5. 1. The path P_{2n+1} can be viewed as two copies of the path P_n (the twins) which connect to a central vertex. In particular, the eigenvalues of P_{2n+1} are the eigenvalues of $2P_{n+1}$ together with a path on n vertices with *one* end having vertex weight 1. Since we know the eigenvalues of P_{2n+1} and $2P_{n+1}$ (which has the same eigenvalues of P_{n+1}), we can conclude that the eigenvalues of P_n with one end having vertex weight 1 are

$$\left\{1 - \cos\tfrac{(2k+1)\pi}{2n} : 0 \le k \le n-1\right\}.$$

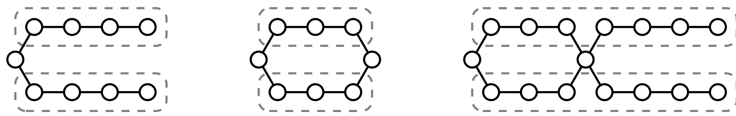
2. The cycle C_{2n} can be viewed as two copies of the path P_{n-1} (the twins) with the ends connecting to two special vertices. In particular, the eigenvalues of C_{2n} are the eigenvalues of $2P_{n+1}$ together with a path on $n-1$ vertices with *both* ends having vertex weight 1. Since we know the eigenvalues of C_{2n} and $2P_{n+1}$ we can conclude the eigenvalues of P_{n-1} with both ends having vertex weight 1 are

$$\left\{1 - \cos\tfrac{2k\pi}{2n} : n+1 \le k \le 2n-1\right\}.$$

3. Let γ_{4k} be the graph formed starting with C_{2n} and P_{2n+1} and then identifying the center vertex of the path to a point on the cycle. This can again be viewed as twin subgraphs gluing onto two special vertices so that the eigenvalues of the graph are the union of the eigenvalues of $2P_{2n+1}$, P_n with one end having vertex weight one, and P_{n-1} with both ends having vertex weight one, i.e.,

$$\begin{aligned}\{1 - \cos\tfrac{k\pi}{2n} : 0 \le k \le 2n\} \cup \left\{1 - \cos\tfrac{(2k+1)\pi}{2n} : 0 \le k \le n-1\right\}\\ \cup \left\{1 - \cos\tfrac{2k\pi}{2n} : n+1 \le k \le 2n-1\right\} = \{1 - \cos\tfrac{k\pi}{2n} : 0 \le k \le 4n-1\}.\end{aligned}$$

In particular, we can conclude that γ_{4k} is cospectral with C_{4k}.



(a) P_{2n+1} (b) C_n (c) γ_{4k}

Fig. 5 Twin subgraphs of Example 5

For the adjacency, Laplacian, and signless Laplacian it is an easy exercise to show that there are no graphs which are cospectral with the cycles. The graph γ_{4k} shows that this is *not* the case for the normalized Laplacian, though no other examples of graphs cospectral with a cycle are known.

Conjecture 1. The only graphs cospectral with a cycle are $K_{1,3}$ and γ_{4k}.

A variant of twin subgraphs comes when we can "fold" the graph in half (including folding an edge to become a loop). A discussion on how the eigenvalues can be found in this setting is given in [5].

5 The characteristic polynomial

The characteristic polynomial of the normalized Laplacian, which we will denote $\phi(x) = \det(xI - \mathscr{L})$, gives the eigenvalues of a graph. Additionally, the coefficients of the polynomial contain important information about the graph. For graphs without isolated vertices, $\mathscr{L}$ is similar to $I - D^{-1}A$ which has rational entries, so that for simple graphs $\phi(x)$ is a polynomial with rational coefficients.

Definition 6. A *decomposition*, $\mathfrak{D}$, of a graph G is a subgraph consisting of disjoint edges and cycles with no isolated vertices.

The determinant can be computed using decompositions in the following way.

Proposition 5. *For a simple graph G we have*

$$\phi(x) = \sum_{\mathfrak{D}} \frac{(-1)^{\text{even}(\mathfrak{D})} 2^{\text{long}(\mathfrak{D})} (x-1)^{n-|V(\mathfrak{D})|}}{\prod_{v \in V(\mathfrak{D})} d(v)},$$

where $\text{even}(\mathfrak{D})$ *is the number of cycles of even length (including isolated edges),* $\text{long}(\mathfrak{D})$ *is the number of cycles of length at least* 3*, and the sum ranges over all decompositions* $\mathfrak{D}$ *(including the empty decomposition).*

This proposition is not particularly useful in practice, but Butler and Heysse [10] were able to find a way to show that certain pairs of graphs were cospectral by using decompositions to show that the characteristic polynomials were equal (they also give a generalization of the decomposition formulation for weighted graphs).

Guo, Li, and Shiu [15] gave a reduction formula for the characteristic polynomial for the normalized Laplacian which involves the sum of characteristic polynomials of subgraphs found by deleting vertices but preserving degrees (i.e., by moving edge weights to vertex weights where needed). The formula itself is cumbersome to work with directly, but can be used to prove a coalescing result.

Theorem 6 (Guo, Li and Shiu [15]). *Let G and H be simple graphs with $u \in V(G)$ and $v \in V(H)$, and let $G\,{}_u{\circ}_v\,H$ be the coalescence of G and H, i.e., formed by taking the union of G and H and identifying the vertices u and v into a new vertex uv. Then*

$$\phi\big(G\,{}_u{\circ}_v\,H\big) = \frac{d_G(u)\phi(G)\phi(H') + d_H(v)\phi(G')\phi(H)}{d_G(u)+d_H(v)},$$

where G' (resp. H') is the weighted graph obtained by deleting the vertex u (resp. v) and then increasing the degree weights of any adjacent vertices by 1 *(so degrees of the remaining vertices are preserved).*

Guo, Li and Shiu also gave two non-isomorphic graphs G_1 and G_2 and vertices u_1 and u_2 for which $\phi(G_1) = \phi(G_2)$ and $\phi(G_1') = \phi(G_2')$. In particular for any graph H and vertex $v \in K$ we have $G_1\,{}_{u_1}{\circ}_v\,H$ is cospectral with $G_2\,{}_{u_2}{\circ}_v\,H$.

5.1 Spanning trees and Kemeny's coefficient

For a (simple) graph G on n vertices with eigenvalues $0 = \lambda_0 \le \lambda_1 \le \cdots \le \lambda_{n-1}$ we have

$$\begin{aligned}\phi(x) &= x(x-\lambda_1)\cdots(x-\lambda_{n-1})\\ &= x^n + c_{n-1}x^{n-1} + \cdots + c_2x^2 + c_1x.\end{aligned}$$

The coefficients of the characteristic polynomial are symmetric functions of the eigenvalues. For example, we have $c_1 = (-1)^{n-1}\lambda_1\cdots\lambda_{n-1}$.

Kirchoff's Matrix Three Theorem gives a connection between the coefficient of x in the characteristic polynomial for the Laplacian and the number of spanning trees in a graph. A similar weighted version of this result also holds for the normalized Laplacian.

Theorem 7 (Weighted Matrix Tree Theorem [12]). *Let $\tau(G)$ denote the number of spanning trees of a graph G and let $0 = \lambda_0 \le \lambda_1 \le \cdots \le \lambda_{n-1}$ denote the eigenvalues of the normalized Laplacian of G. Then*

$$(-1)^{n-1}c_1 = \lambda_1\cdots\lambda_{n-1} = \frac{\tau(G)\sum_v d(v)}{\prod_v d(v)}.$$

The coefficient c_2 also has an interesting combinatorial interpretation related to Kemeny's constant.

Definition 7. Given a graph G, *Kemeny's constant*, denoted $K(G)$, is the expected first passage time from an unknown starting point in the graph to an unknown destination point in the graph (see [17] for more information).

There is a way to compute Kemeny's constant by using the eigenvalues of the probability transition matrix. In particular, we have the following.

Proposition 6 (Levene and Loizou [18]). *Let G be a connected simple graph with $\rho_{n-1} \le \cdots \le \rho_1 \le \rho_0 = 1$ the eigenvalues for the probability transition $D^{-1}A$. Then*

$$K(G) = \sum_{i=1}^{n-1} \frac{1}{1-\rho_i}.$$

We note that $D^{-1}A$ is similar to the normalized adjacency matrix $D^{-1/2}AD^{-1/2}$, and so the eigenvalues of the probability transition matrix are connected to the eigenvalues of the normalized Laplacian by the relationship $\lambda_i = 1-\rho_i$. So we can rewrite the above expressions as

$$K(G) = \sum_{i=1}^{n-1} \frac{1}{\lambda_i}.$$

We also note

$$c_2 = (-1)^{n-2}\lambda_1 \cdots \lambda_{n-1} \sum_{i=1}^{n-1} \frac{1}{\lambda_i} = -c_1 K(G).$$

Allowing us to conclude

$$K(G) = -\frac{c_2}{c_1}.$$

Example 6. Let G be the simple graph formed by taking two disjoint copies of K_s and joining the two cliques by one edge. To find $K(G)$ we find the characteristic polynomial and take the appropriate ratio of the coefficients, but we can take some shortcuts. For example, G has an equitable partition in four parts, two with $s-1$ adjacent twins, and then two additional vertices from the connecting edge. From the twins in the cliques we get eigenvalues $\left\{\frac{s}{s-1}^{(2s-4)}\right\}$. The remaining eigenvalues come from the graph formed from the equitable partition on four vertices with edge weights $1, s-1$ (twice), and loops $(s-1)(s-2)$ on the ends. (Note that the loops are *not* $\binom{s-1}{2}$, because each edge contributes twice, once as $w(u,v)$ and once as $w(v,u)$.) The characteristic polynomial for the graph on four vertices is

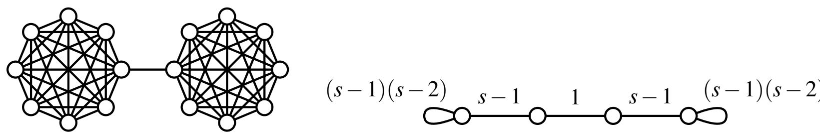
$$x^4 - \tfrac{s}{s-1}x^3 + \tfrac{s^4+s^2-1}{s^2(s-1)^2}x^2 - \tfrac{2(s^2-s+1)}{s^2(s-1)^2}.$$

Therefore the characteristic polynomial for G is

$$\begin{aligned}\phi(x) &= \big(x-\tfrac{s}{s-1}\big)^{2s-4}\big(x^4-\tfrac{s}{s-1}x^3+\tfrac{s^4+s^2-1}{s^2(s-1)^2}x^2-\tfrac{2(s^2-s+1)}{s^2(s-1)^2}x\big)\\ &= \big(\cdots-(2s-4)\big(\tfrac{s}{s-1}\big)^{2s-5}x+\big(\tfrac{s}{s-1}\big)^{2s-4}\big)\big(\cdots+\tfrac{s^4+s^2-1}{s^2(s-1)^2}x^2-\tfrac{2(s^2-s+1)}{s^2(s-1)^2}x\big)\\ &= \tfrac{s^{2s-7}}{(s-1)^{2s-2}}\big(\cdots-(2s-4)(s-1)x+s\big)\big(\cdots+(s^4+s^2-1)x^2-2(s^2-s+1)x\big)\\ &= \tfrac{s^{2s-7}}{(s-1)^{2s-2}}\big(\cdots+(s^5+4s^4-15s^3+24s^2-21s+8)x^2-2(s^3-s^2+s)x\big).\end{aligned}$$

Finally allowing us to conclude that

$$K(G)=\tfrac{(s^5+4s^4-15s^3+24s^2-21s+8)}{2(s^3-s^2+s)}.$$



(a) G with $s=8$ (b)

Fig. 6 Graph for Example 6

There is a related concept called *the degree-Kirchoff index*, denoted $K'(G)$ which is a generalization of the Kirchoff index. It is known

$$K'(G)=\bigg(\sum_v d(v)\bigg)\bigg(\sum_{i=1}^{n-1}\frac{1}{\lambda_i}\bigg)=\bigg(\sum_v d(v)\bigg)K(G).$$

For more information on this parameter see Chen and Zhang [11].

5.2 Characteristic polynomials and graph operations

Definition 8. For a simple graph G we can construct the following graphs:

- The *line graph*, denoted $\ell(G)$, whose vertex set corresponds to the edges of G and two vertices in $\ell(G)$ are adjacent if and only if the corresponding edges in G were incident to a common vertex.
- The *subdivision graph*, denoted $s(G)$, formed by adding one new vertex v_e for each edge $e=u{\sim}w$ and then deleting edge e and adding in edges $u{\sim}v_e$ and $v_e{\sim}w$.
- The graph $r(G)$ formed by adding back in the deleted edges to $s(G)$.
- The graph $q(G)$ formed by taking $s(G)$ and then adding in the corresponding edges of $\ell(G)$.

The graphs $\ell(G)$ and $s(G)$ are well-studied. When the initial graph G is regular, then these graphs are regular ($\ell(G)$) or bi-regular ($s(G)$) graphs, so the known results for the adjacency spectrums will translate to these graphs. The other two graphs are not regular. Huang and Li [16] were able to establish connections between the characteristic polynomial for all of these graphs and the characteristic polynomial of G (through the use of Schur complements and properties of incidence matrices). Using this they then were able to find information on the number of spanning trees in these graphs compared to those of G as well as look at the relationship for the degree-Kirchoff index in regards to G.

Theorem 8 (Huang and Li [16]). *Let G be a k-regular simple graph with n vertices and m edges. Then the following holds:*

- $\phi_{\ell(G)}(x) = \frac{k^n((k-1)x-k)^{m-n}}{2^n(k-1)^m}\phi_G\big(\frac{2(k-1)x}{k}\big)$.
- $\phi_{s(G)}(x) = \big(\frac{-1}{2}\big)^n(x-1)^{m-n}\phi_G\big(2x(2-x)\big)$.
- $\phi_{r(G)}(x) = \frac{(2x-3)^n(x-1)^{m-n}}{4^n}\phi_G(2x)$.
- $\phi_{q(G)}(x) = \frac{(kx-k-1)^m}{2^nk^m}\phi_G(2x)$.

From the above results on the characteristic polynomials, relationships between the eigenvalues of the graph and the graphs obtained by these operations can be obtained.

5.3 Verifying cospectral graphs

One way to show that two graphs are cospectral (have the same eigenvalues) is to use structure to find many eigenvalues which agree on both graphs and then reduce the problem of finding the remaining eigenvalues to a smaller graph which can then be checked using characteristic polynomials. We give here two such examples, which help illustrate some of the unusual aspects of the spectrum of the normalized Laplacian.

Example 7. Let G and H be the two graphs shown in Figure 7 where we have put vertices into groups with the number of vertices in each group marked as a function of k. Note that H is a subgraph of G, and the two graphs differ by the edges of the complete bipartite graph $K_{k,k}$. The indicated groups of vertices give an equitable partition of both graphs where in each case the vertices in the parts are twins and so both graphs have the eigenvalue 1 with multiplicity $3k-2$. We can now use the equitable partitions to form the graphs G' and H' which give the remaining eigenvalues of G and H. These are graphs on five vertices with edge weights as a function of k and in *both* cases the characteristic polynomials are:

$$x^5 - \frac{6k^2+8k+3}{4(k+1)^2}x^3 - \frac{1}{4(k+1)}x^2 + \frac{k(2k+1)}{4(k+1)^2}x.$$

In particular the graphs G and H are cospectral.

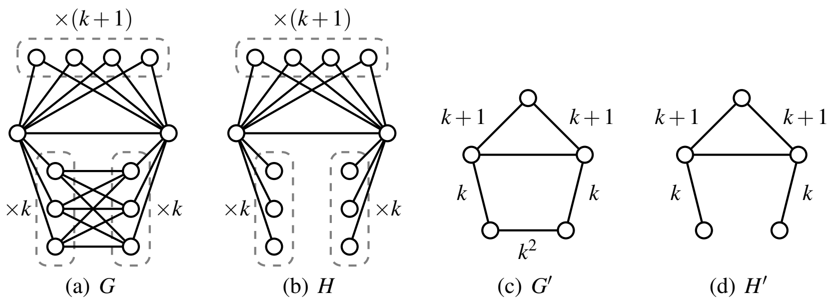


Fig. 7 The graphs from Example 7

This example illustrates that it is possible for a simple graph to be cospectral with a subgraph. More strongly, it shows that it is possible for a *dense* graph to be cospectral with a *sparse* subgraph.

Example 8. Let $B_{s,t}$ be the graph on $2s+2t+2$ vertices formed by taking two copies of K_{s+t} and two special vertices v_s and v_t where v_s connects to s vertices in each of the cliques and v_t connects to the remaining t vertices. Figure 8 shows $B_{2,2}$ and $B_{1,3}$. The graph $B_{s,t}$ has a natural equitable partition into six parts, v_s, v_t, for each clique the vertices which connect to v_s, and for each clique the vertices which connect to v_t. For the twins in the equitable partitions in the two cliques they will contribute $2s+2t-4$ eigenvalues of $\frac{s+t+1}{s+t}$. The remaining eigenvalues comes from a weighted graph on six vertices arising from the equitable partition, shown in Figure 8. This graph has characteristic polynomial

$$x^6 - \tfrac{4(s+t+1)}{(s+t)}x^5 + \tfrac{6(s+t+1)^2}{(s+t)^2}x^4 - \tfrac{(4(s+t)^2+5(s+t)+2)(s+t+2)}{(s+t)^3}x^3 + \tfrac{((s+t)^3+5(s+t)^2+3(s+t)+1)(s+t+1)}{(s+t)^4}x^2 - \tfrac{(s+t+1)^2}{(s+t)^3}x.$$

In particular we can conclude that all of the eigenvalues can be expressed in terms of $s+t$, and so the graphs $B_{k-i,k+i}$ for $0 \le i \le k-1$ are all cospectral. Note that $B_{k,k}$ is regular of degree $2k$ while for $1 \le i \le k-1$ the graphs $B_{k-i,k+i}$ are not regular.

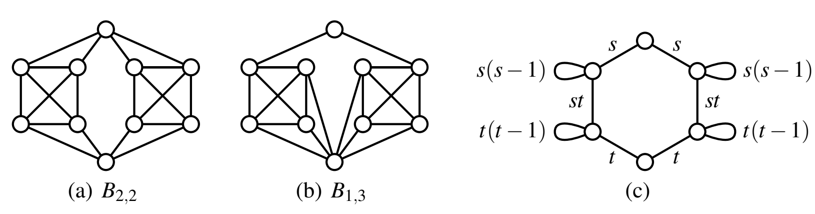


Fig. 8 The graphs from Example 8

This example illustrates that it is possible for a simple graph which is regular to be cospectral with a simple graph which is not regular (in addition to the previous biparitite examples of $K_{s,s}$ and C_{4k}). This stands in stark contrast to the other common matrices in spectral graph theory (adjacency, Laplacian, and signless Laplacian) where this is impossible. Of course this is a common theme in the subject, the normalized Laplacian behaves quite differently, and in interesting and unusual ways. There still remains much to understand for the normalized Laplacian.

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References

1. Anirban Banarjee, Jürgen Jost, *The spectrum of the normalized graph Laplacian*, Linear Algebra and its Applications, **428** (2008), 3015–3022.
2. Anirban Banerjee and Ranjit Mehatari, *Effect on normalized Graph Laplacian spectrum by motif attachment and duplication*, arXiv:1210.5125v2
3. Andries Brouwer and Willem Haemers, *Spectra of Graphs*, Springer, 2012.
4. Steve Butler, *Eigenvalues and structures of graphs*, Ph.D. Dissertation, UC San Diego, 2008
5. Steve Butler, *Eigenvalues of* 2-*edge-coverings*, Linear and Multilinear Algebra **58** (2010), 413–423.
6. Steve Butler, *A gentle introduction to the normalized Laplacian*, IMAGE, **53** (Fall 2014), 19–27.
7. Steve Butler, *Using twins and scaling to construct cospectral graphs for the normalized Laplacian*, Electronic Journal of Linear Algebra **28** (2015), 54–68.
8. Steve Butler and Fan Chung, *Spectral Graph Theory*, in Handbook of Linear Algebra, Second Edition, Ed: L. Hogben, CCR Press, 2014.
9. Steve Butler and Jason Grout, *A construction of cospectral graphs for the normalized Laplacian*, Electronic Journal of Combinatorics **18** (2011), #231, 20pp.
10. Steve Butler and Kristin Heysse, *Forming cospectral graphs by toggling*, preprint.
11. Haiyan Chen and Fuji Zhang, *Resistance distance and the normalized Laplacian spectrum*, Discrete Applied Mathematics **155** (2007), 654–661.
12. Fan Chung, *Spectral Graph Theory, Second Edition*, AMS, 1997.
13. Fan Chung and S. T. Yau, *Coverings, heat kernels and spanning trees*, Electronic Journal of combinatorics, **6** (1999) #12, 21 pp.
14. Chris Godsil and Gordon Royle, *Algebraic Graph Theory*, Springer, 2001.
15. Ji-Ming Guo, Jianxi Li and Wai Chee Shiu, *On the Laplacian, signless Laplacian and normalized Laplacian characteristic polynomials of a graph*, Czechoslovak Mathematical Journal **63** (2013), 701–720.
16. Jing Juang and Shuchao Li, *On the normalized Laplacian spectrum, degree-Kirchoff index and spanning trees of graphs*, Bulletin of the Australian Mathematical Society **91** (2015), 353–367.
17. Jeffrey Hunter, *The role of Kemeny's constant in properties of Markov chains*, Communications in Statistics–Theory and Methods **43** (2014), 1309–1321.
18. Mark Levene and George Loizou, *Kemeny's constant and the random surfer*, American Mathematical Monthly **109** (2002), 741–745.