ELSEVIER

NP-hard graph problems and boundary classes of graphs

V.E. Alekseev[a], R. Boliac[b], D.V. Korobitsyn[a], V.V. Lozin[b,*]

[a] *University of Nizhny Novgorod, Gagarina 23, Nizhny Novgorod, 603950, Russia*
[b] *RUTCOR, Rutgers University, 640 Bartholomew Road, Piscataway NJ, 08854-8003, USA*

Communicated by A. Razborov

Abstract

Any graph problem, which is NP-hard in general graphs, becomes polynomial-time solvable when restricted to graphs in special classes. When does a difficult problem become easy? To answer this question we study the notion of boundary classes. In the present paper we define this notion in its most general form, describe several approaches to identify boundary classes and apply them to various graph problems.
© 2007 Elsevier B.V. All rights reserved.

1. Introduction

We study graph problems that are NP-hard in general, i.e. in the class of all graphs. When we restrict the class of input graphs, any of these problems can become tractable, i.e. solvable in polynomial time. When does a difficult problem become easy? Is there any boundary that separates classes of graphs where the problem remains NP-hard from those where it has a polynomial-time solution? To answer these questions, we study the notion of boundary classes of graphs. Originally, this notion was introduced with respect to the maximum independent set problem [2] and then was applied to the minimum dominating set problem [4]. In the present paper, we define this notion in its most general form, describe several approaches to identify boundary classes and apply them to various graph problems.

To give an intuitive notion of boundary classes, let us observe that all classes of graphs studied in this paper are hereditary, i.e. those containing with every graph G all induced subgraphs of G. Obviously, every hereditary class X can be characterized by a set of minimal graphs that do not belong to X. A famous example of this type is the class of *perfect graphs* characterized recently in terms of the forbidden cycles of odd length at least 5 and their complements [13].

Assume now that $\mathcal{F}$ is a family of graph classes which is hereditary in the sense that with every class X it contains all subclasses of X. Our purpose is to characterize this family by a set of minimal classes that do not belong to it.

---

* Corresponding address: Mathematics Institute, University of Warwick, Coventry CV4 7AL, United Kingdom.
 E-mail addresses: ave@uic.nnov.ru (V.E. Alekseev), boliac@rutcor.rutgers.edu (R. Boliac), dvk@nts.ru (D.V. Korobitsyn),
V.Lozin@warwick.ac.uk (V.V. Lozin).

Unfortunately, in general this is not possible, as a class of graphs $Y \notin \mathcal{F}$ may contain infinitely many subclasses $Y_1 \supset Y_2 \supset Y_3 \ldots$ that do not belong to $\mathcal{F}$. We call the intersection of the sequence $Y_1 \supset Y_2 \supset Y_3 \ldots$ a limit class for $\mathcal{F}$ and a minimal limit class a boundary class for $\mathcal{F}$. Therefore, in a sense, the set of boundary classes for a hereditary family of graph classes is an analogue of the induced subgraph characterization of a hereditary class of graphs.

Various families of graph classes studied in the literature are hereditary in the sense defined above. Two important examples are classes of graphs of bounded tree- or clique-width. The notion of boundary classes proposes a uniform way to characterize such families. In the present paper, we employ this notion for the study of families of graph classes admitting polynomial-time solutions for some NP-hard problems. Throughout the paper we assume that $P \neq NP$.

The organization of the paper is as follows. In the next section, we give necessary preliminaries regarding graphs, graph classes and graph problems which deal with. Section 3 provides a formal definition of the two key notions of the paper: limit classes and boundary classes. In Section 4, we analyze several approaches to identify limit classes and apply them to various graph problems. Section 5 is devoted to the proof of minimality of some of the limit classes found in the previous section. Finally, in Section 6 we discuss many of the open problems on this topic.

2. Preliminaries

2.1. Graphs

All graphs in this paper are undirected, without loops and multiple edges. For a graph G, we denote by $V(G)$ and $E(G)$ the sets of vertices and edges of G, respectively, and by $\overline{G}$ the complement of G. A subgraph H of G is *induced* if two vertices of H are adjacent if and only if they are adjacent in G. The neighborhood $N(v)$ of a vertex $v \in V(G)$ is the set of vertices adjacent to v, and the degree of v is $|N(v)|$. As usual, C_n is the chordless cycle, P_n is the chordless path, K_n is the complete graph on n vertices, and $K_{n,m}$ is the complete bipartite graph with parts of size n and m. A *diamond* is $K_4 - e$, i.e. a K_4 without an edge. For non-negative integers i, j, k, we denote by $S_{i,j,k}$ and $T_{i,j,k}$ the graphs represented in Fig. 1.
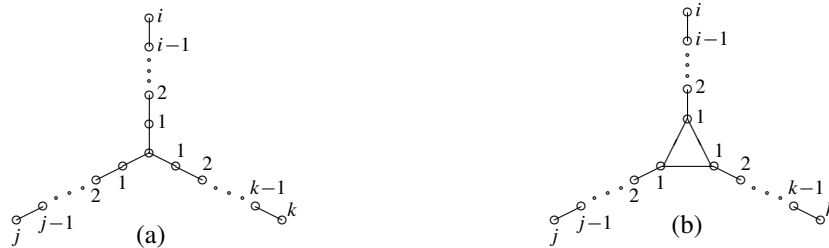

Fig. 1. The graphs $S_{i,j,k}$ (a) and $T_{i,j,k}$ (b).

2.2. Graph classes

All graph classes in this paper are *hereditary*, which means if a graph belongs to a class, then every induced subgraph of the graph belongs to the same class. In other words, deletion of a vertex from a graph results in a graph in the same class. This restriction to hereditary classes only, which may seem artificial at first sight, can be justified by the following arguments.

First, many classes of theoretical and practical importance are hereditary, which include, among others,

1. *planar graphs*;
2. *bipartite graphs*;
3. *graphs of bounded vertex degree*;
4. *graphs of bounded tree-width*;
5. *graphs of bounded clique-width*;
6. *forests*, i.e. graphs without cycles;
7. *split graphs*, i.e. graphs partitionable into a clique and an independent set [18];
8. *transitively orientable graphs* (have applications in the theory of partially ordered sets [42]);
9. *threshold graphs* (see a monograph [35] for numerous applications of these graphs);

10. *perfect graphs* (a monograph on this class passed the second edition [21]);
11. *interval graphs* (arise in molecular biology [8] and have many applications in other fields [17]);
12. *chordal bipartite graphs*, i.e. bipartite graphs without chordless cycles of length at least six (this class has applications in the study of linear programming, as the bipartite adjacency matrices of chordal bipartite graphs are totally balanced [22]);
13. *line graphs*. A graph G is a line graph if the vertices of G are in one-to-one correspondence to the edges of some other graph H, and two vertices of G are adjacent if and only if the respective edges of H have a vertex in common.

Many of those classes that are not hereditary have natural hereditary extensions. For instance, for the non-hereditary class of trees such an extension is the class of forests.

One more reason to be restricted to hereditary classes is that these and only these classes admit a uniform description in terms of forbidden induced subgraphs, which, in turn, provides a systematic way of investigation of various problems associated with graph classes. If a graph G does not contain induced subgraphs isomorphic to graphs in a set Z, we say that G is Z-free. The set of all Z-free graphs will be denoted *Free*(Z). With this notation the above statement about the induced subgraph characterization can be reformulated as follows: a class X of graphs is hereditary if and only if $X = Free(Z)$ for some set Z. The minimal set Z possessing this property is unique and will be denoted by *Forb*(X). For instance,

Forb(*forests*) $= \{C_3, C_4, C_5, C_6, C_7, \ldots\}$ (by definition);
Forb(*bipartite graphs*) $= \{C_3, C_5, C_7, \ldots\}$ [28];
Forb(*split graphs*) $= \{C_4, \overline{C}_4, C_5\}$ [18];
Forb(*chordal bipartite graphs*) $= \{C_3, C_5, C_6, C_7, \ldots\}$ (by definition);
Forb(*perfect graphs*) $= \{C_5, \overline{C}_5, C_7, \overline{C}_7, \ldots\}$ [13].

Notice that for the class of split graphs the set of minimal forbidden induced subgraphs is finite. The same is true for the line graphs [7] and graphs of vertex degree at most k (k is a constant). We shall call such classes *finitely defined*. If $X \subseteq Y$ and *Forb*$(X) - Forb(Y)$ is a finite set, we shall say that X is defined by finitely many forbidden induced subgraphs *with respect to* Y. For instance, the class of forests is defined by a single forbidden induced subgraph with respect to chordal bipartite graphs.

Among hereditary classes we shall distinguish a certain family with the property that deletion of any edge from a graph in a class results in a graph in the same class. Such classes will be called *monotone*. In the above list, classes 1, 2, 3, 4, 6 are monotone, while the others are not.

Throughout the paper we use special notations for two particular classes of graphs:

$\mathcal{S}$ is the class of graphs every connected component of which is of the form $S_{i,j,k}$ for some non-negative integers i, j, k;

$\mathcal{T}$ is the class of graphs every connected component of which is of the form $T_{i,j,k}$ for some non-negative integers i, j, k.

For a class Z of graphs we shall denote $L(Z) := \{L(G) \mid G \in Z\}$, where $L(G)$ stands for the line graph of G. It is not hard to see that $\mathcal{T} = L(\mathcal{S})$.

2.3. Graph problems

In this paper we shall refer to the following problems.

- INDEPENDENT SET: given a graph G, find a maximum cardinality subset of vertices no two of which are adjacent.
- DOMINATING SET: given a graph G, find a minimum cardinality subset U of vertices with the property that every vertex outside U has a neighbor in U.
- INDEPENDENT DOMINATING SET: given a graph G, find a minimum cardinality dominating set which is independent.
- EDGE DOMINATION: given a graph G, find a minimum cardinality subset F of edges such that every edge outside F has a vertex in common with an edge in F.
- BIPARTITE SET: given a graph G, find a maximum cardinality subset of vertices that induces a bipartite graph in G.

- DISSOCIATION SET: given a graph G, find a maximum cardinality subset of vertices that induces a graph with maximum vertex degree at most 1.
- INDUCED MATCHING: given a graph G, find a maximum cardinality subset of vertices that induces a graph every vertex of which is of degree 1.
- ALTERNATING CYCLE-FREE MATCHING: given a graph G, find a maximum cardinality matching M such that no cycle of the graph has exactly half of its edges in M.
- P_3-FACTOR: given a graph G with $3k$ vertices, determine whether G contains a P_3-factor, i.e. a partition of $V(G)$ into k subsets such that each subset contains P_3 as a subgraph (not necessarily induced).
- HAMILTONIAN CYCLE: given a graph G, determine whether there is a cycle containing every vertex of G exactly once.
- LONGEST CYCLE: given a graph G, find a simple cycle of maximum length.
- LONGEST PATH: given a graph G and two designated vertices in G, find a simple path of maximum length between the vertices.
- VERTEX COLORING: given a graph G, find a partition of its vertex set into minimum number of independent sets.
- VERTEX 3-COLORABILITY: given a graph G, determine whether the vertex set of G can be partitioned into at most 3 independent sets.
- EDGE 3-COLORABILITY: given a graph G, determine whether the edge set of G can be partitioned into at most 3 subsets each of which is a matching.

Now we present some results regarding these problems that will be needed in the course of our study.

Let Π be one of the following problems: INDEPENDENT SET, DOMINATING SET or LONGEST PATH. The following result was proved in [3].

Lemma 1. *If X is a monotone graph class such that $\mathcal{S} \not\subseteq X$, then Π is polynomial-time solvable for graphs in X.*

This lemma can be extended to Π being the HAMILTONIAN CYCLE or LONGEST CYCLE problem, since HAMILTONIAN CYCLE reduces to LONGEST CYCLE on the same graph, and LONGEST CYCLE, in turn, reduces to LONGEST PATH by considering all pairs of adjacent vertices as designated pairs. Further extensions of Lemma 1 can be derived from the following result proved in [9].

Lemma 2. *If X is a monotone graph class such that $\mathcal{S} \not\subseteq X$, then the clique-width of graphs in X is bounded by a constant.*

Among problems solvable in polynomial-time on graphs of bounded clique-width the most representative category relates to problems expressible in a certain variation of Monadic Second-Order Logic with quantification over subsets of vertices but not edges (see [14] for more information on such problems). In particular, this category includes INDEPENDENT SET, DOMINATING SET, INDEPENDENT DOMINATING SET, INDUCED MATCHING, DISSOCIATION SET, VERTEX 3-COLORABILITY. Moreover, some problems that do not belong to this category may have polynomial-time solutions on graphs of bounded clique-width, which is the case, for instance, for the HAMILTONIAN CYCLE problem [12]. Summarizing the above discussion we conclude that

Proposition 1. *Lemma 1 holds for the following problems:* INDEPENDENT SET, DOMINATING SET, INDEPENDENT DOMINATING SET, INDUCED MATCHING, DISSOCIATION SET, VERTEX 3-COLORABILITY, HAMILTONIAN CYCLE, LONGEST CYCLE, LONGEST PATH.

3. Boundary classes

In this section we define the notion of boundary class for a given NP-hard graph problem Π. To simplify our discussion we shall say that a class of graphs is Π-*hard* if the problem Π is NP-hard in this class, and Π-*easy* if Π has a polynomial-time solution for graphs in the class.

We shall define the notion of boundary class in two steps. First, let us define the notion of *limit class*.

Definition 1. Let Π be an NP-hard graph problem and Y a Π-hard class of graphs. A class X of graphs is called a *limit class* for Π with respect to Y ((Π, Y)-limit for short) if there exists a sequence $X_1 \supseteq X_2 \supseteq \ldots$ of Π-hard subclasses of Y such that $\bigcap_{n \geq 1} X_n = X$. We shall say that X is a limit class for Π (Π-limit) if there is a Π-hard class Y such that X is (Π, Y)-limit.

Remark 1. Notice that Definition 1 does not require the classes $X_1 \supseteq X_2 \supseteq \ldots$ to be distinct. Therefore, every Π-hard subclass of Y is (Π, Y)-limit. The following example shows that the converse statement is not true in general.

Example 1. Let Π be the INDEPENDENT SET problem, and X_k the class $Free(C_3, \ldots, C_k)$. It has been shown in [39] that this problem is NP-hard in the class X_k for each particular value $k \geq 3$. It is not hard to see that $X_k \supseteq X_{k+1}$ for any k, and $\bigcap_{k \geq 3} X_k$ is the class of forests. Therefore, the class of forests is a limit class for the INDEPENDENT SET problem. On the other hand, this problem is known to be polynomial-time solvable in the class of forests. This example shows that a Π-limit class is not necessarily Π-hard.

Remark 2. If X is a Π-limit subclass of a Π-hard class Z, then X is not necessarily (Π, Z)-limit (see Example 2).

Example 2. Let Π be the INDEPENDENT DOMINATING SET problem. It has been shown in [10] that this problem is NP-hard in the class $Free(C_3, \ldots, C_k)$ for each particular value $k \geq 3$. Therefore, similarly as in Example 1, the class of forests is a limit class for the problem in question. Now let Z be the class of chordal bipartite graphs. It is known that the INDEPENDENT DOMINATING SET problem is NP-hard in this class [15], and clearly forests constitute a subclass of Z. But the class of forests is not (Π, Z)-limit. Indeed, assume there is a sequence $X_1 \supseteq X_2 \supseteq \ldots$ of Π-hard subclasses of Z such that the intersection $\bigcap_{n \geq 1} X_n$ coincides with the class of forests. Then there must exist a class X_j in the sequence that does not contain the cycle C_4 on 4 vertices (otherwise the intersection would contain the cycle). But then X_j contains no cycles and hence the problem is polynomial-time solvable for graphs in X_j (see e.g. [16]). This contradiction shows that a Π-limit subclass of a Π-hard class Z is not necessarily (Π, Z)-limit.

The following two lemmas prove important properties of limit classes that will be useful in the sequel.

Lemma 3. *If X is a (Π, Y)-limit class and $X \subseteq Z \subseteq Y$, then Z also is a (Π, Y)-limit class.*

Proof. Let $X_1 \supseteq X_2 \supseteq \ldots$ be a sequence of Π-hard subclasses of Y such that $X = \bigcap_n X_n$. Then the class $Z_n := X_n \cup Z$ is Π-hard for each n, $Y \supseteq Z_1 \supseteq Z_2 \supseteq \ldots$, and $Z = \bigcap_n Z_n$. ∎

Lemma 4. *Let Y be a Π-hard class and X be a subclass of Y defined by finitely many forbidden induced subgraphs with respect to Y. Then X is a (Π, Y)-limit class if and only if it is Π-hard.*

Proof. According to Remark 1, we only have to prove the necessity. Let $Forb(X) - Forb(Y) = \{G_1, \ldots, G_k\}$, and X be a (Π, Y)-limit class, i.e. $X = \bigcap_n X_n$ for a sequence $X_1 \supseteq X_2 \supseteq \ldots$ of Π-hard subclasses of Y. Clearly, there must exist a natural number n such that X_n does not contain $G_1, \ldots, G_k$. But then $X_i = X$ for each $i \geq n$, and hence X is Π-hard. ∎

We are now in a position to define the notion of boundary class.

Definition 2. A minimal under inclusion (Π, Y)-limit class is called a *boundary class* for Π with respect to Y ((Π, Y)-boundary for short). We shall say that X is a boundary class for Π (Π-boundary) if there is a Π-hard class Y such that X is (Π, Y)-boundary.

Remark 3. Clearly if X is a (Π, Y)-limit class and $Y \subseteq Z$, then X is a (Π, Z)-limit class. However, the fact that X is a (Π, Y)-boundary class does not necessarily imply that X is (Π, Z)-boundary for $Z \supseteq Y$, since X may contain a (Π, Z)-boundary class which is not a limit class with respect to Y, according to Remark 2.

The importance of the notion of boundary class is due to the following theorem.

Theorem 1. *A problem Π is NP-hard in a class Y of graphs if and only if Y contains a (Π, Y)-boundary class.*

Proof. The if-part is trivial. Let $X \subseteq Y$ be a (Π, Y)-boundary class. Then there is a sequence $X_1 \supseteq X_2 \supseteq \ldots$ of Π-hard subclasses of Y such that $\bigcap_n X_n = X$. Therefore, Π is NP-hard in the class Y.

Conversely, let Π be NP-hard in the class Y. We assume that the graphs in Y are indexed: $Y = \{G_1, G_2, \ldots\}$. Let us define a sequence $X_1 \supseteq X_2 \supseteq \ldots$ of subclasses of Y as follows. Set $X_1 := Y$ and assume the class X_n has been defined. Let j be the minimum index such that $G_j \in X_n$ and $X_n \cap Free(G_j)$ is a (Π, Y)-limit class. If there is no such $G_j \in Y$, then we define $X_{n+1} := X_n$, otherwise $X_{n+1} := X_n \cap Free(G_j)$.

Now consider the class $X := \bigcap_n X_n$, which is clearly a subclass of Y, and let us prove that X is a (Π, Y)-boundary class. To this end, we first show that X is a (Π, Y)-limit class. Let $Forb(X) - Forb(Y) = \{H_1, H_2, \ldots\}$. For every

k, define $X^{(k)}$ to be the subclass of Y defined by forbidding graphs $H_1, \ldots, H_k$. Obviously, for every k, there exists an n such that X_n does not contain $H_1, \ldots, H_k$, and hence $X_n \subseteq X^{(k)}$. By definition, X_n is a (Π, Y)-limit class, and therefore, by Lemma 3, $X^{(k)}$ is also a (Π, Y)-limit class. Consequently, by Lemma 4, $X^{(k)}$ is Π-hard for each k. Obviously $\bigcap_k X^{(k)} = X$. Therefore, X is a (Π, Y)-limit class.

In order to prove the minimality of X, assume, to the contrary, there is a (Π, Y)-limit class Z which is a proper subclass of X. Then there must exist a graph $G_j \in Y$ that belongs to X but not to Z, and hence $Z \subseteq X \cap Free(G_j) \subseteq X_n \cap Free(G_j)$ for each n. Therefore, by Lemma 3, $X_n \cap Free(G_j)$ is a (Π, Y)-limit class for each n. Moreover, for some n, the index j is the minimum one with the property that $X_n \cap Free(G_j)$ is a (Π, Y)-limit class. But then $X_{n+1} := X_n \cap Free(G_j)$, and G_j belong to no class X_k with $k > n$, contradicting the assumption that $G_j \in X$.   ∎

In fact, the above theorem remains valid even for any subclass of Y defined by finitely many forbidden induced subgraphs with respect to Y.

Theorem 2. *A problem Π is NP-hard in a subclass $Z \subset Y$ defined by finitely many forbidden induced subgraphs with respect to Y if and only if Z contains a (Π, Y)-boundary class.*

Proof. If Z contains a (Π, Y)-boundary class, then, by Lemma 3, Z is a (Π, Y)-limit class, and by Lemma 4, Z is Π-hard.

Conversely, if Z is a Π-hard class, then Z contains a (Π, Z)-boundary class X by Theorem 1. The class X is (Π, Y)-limit and hence it contains a (Π, Y)-boundary class.   ∎

4. Approaching limit classes

In this section we identify two general tools to discover limit classes: graph transformations and reducibility between NP-hard problems.

4.1. Transformations of graphs

A productive approach to discover limit classes is based on the employment of graph transformations. Below we distinguish several useful types of transformations and show how they can be applied to reveal limit classes for various problems.

4.1.1. Stretching operations

One of the simplest stretching operations is the *subdivision* of an edge, i.e. introducing a new vertex on the edge. In case we introduce k new vertices, we refer to the operation as *k-subdivision*. If the operation applies to all edges of the graph, then we call it *total k-subdivision*. This operation increases the length of each cycle in the graph $k + 1$ times. As a trivial consequence, we conclude that

Lemma 5. *In the graph obtained from a graph G by the total k-subdivision, the length of a longest cycle is exactly $k + 1$ times as large as that in G.*

The importance of the subdivision operation is due to the fact that it permits to get rid of small induced cycles in the input graph. Perhaps more importantly, a double subdivision of an edge increases the graph independence number by exactly one, which was used in [39] to obtain some results for the maximum independent set problem on graphs with large *girth* (the length of a smallest cycle). It is also worth mentioning that if k is odd, then the total k-subdivision results in a bipartite graph.

Now let us consider a more general operation. Let G be a graph and x be a vertex in G. A *vertex stretching* with respect to x is defined to be the transformation consisting of the following steps:

(1) partition the neighborhood $N(x)$ of vertex x into two subsets Y and Z in an arbitrary way;
(2) delete vertex x from the graph together with incident edges;
(3) add a chordless path on k vertices $(y, a_1, \ldots, a_{k-2}, z)$ to the remaining graph;
(4) connect y to each vertex in Y, and connect z to each vertex in Z.

Notice that k is a parameter associated with the transformation. We shall denote the vertex stretching with parameter k by $\mathcal{P}^k$. The graph produced by this operation will be denoted $\mathcal{P}^k(G) = \mathcal{P}^k(G, x)$. Fig. 2 illustrates the operation $\mathcal{P}^4$.
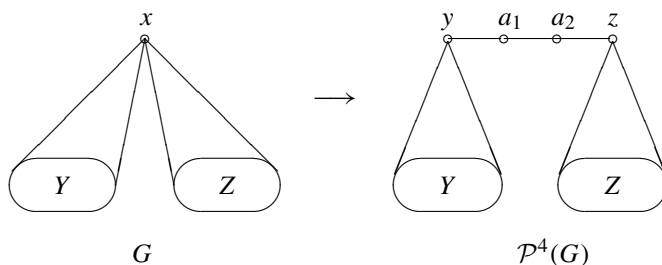
Fig. 2. Transformation $\mathcal{P}^4$.

Consider now a special type of vertex stretching when Y is of particular size, say l. This version of the operation will be denoted $\mathcal{P}_l^k$. It is not hard to verify that $\mathcal{P}_1^k$ is equivalent to a $(k-1)$-subdivision of an edge. In our study, we shall also use $\mathcal{P}_2^k$. This operation can be used to decrease the maximum vertex degree, since in the graph $\mathcal{P}_2^k(G)$ vertices $y, a_1, \ldots, a_{k-2}$ have degree at most 3, while the degree of z is exactly one less than that of x in G.

Various types of vertex stretching have been applied to different problems to achieve various goals (see e.g. [5,10]). For the purpose of our paper, it is necessary to mention that

$\mathcal{P}^3$ increases the independence number [1] and the size of a maximum alternating cycle-free matching [30] by exactly one;

$\mathcal{P}^4$ increases the domination number [29] and the size of a maximum induced matching [31] by one, and the size of a maximum dissociation set [11] by two.

Now let us propose two new results of this nature.

Lemma 6. $\mathcal{P}^4(G)$ *contains a P_3-factor if and only if G does.*

Proof. First, suppose G contains a P_3-factor $\mathcal{F}$ and let W be the triple of vertices in $\mathcal{F}$ containing x. Denote the other two vertices of the triple by c and d, and without loss of generality assume $c \in Y$. For the remaining vertex d, we consider the following two options: $d \in Z$ and $d \notin Z$. If $d \in Z$, then $(\mathcal{F} - W) \cup \{(c, y, a_1), (a_2, z, d)\}$ is a P_3-factor in $\mathcal{P}^4(G)$. If $d \notin Z$, then $(\mathcal{F} - W) \cup \{(y, c, d), (a_1, a_2, z)\}$ is a P_3-factor in $\mathcal{P}^4(G)$.

Conversely, let $\mathcal{F}'$ be a P_3-factor in $\mathcal{P}^4(G)$ and W' the triple in $\mathcal{F}'$ containing a_1. It is not hard to see that W' has one of the following three forms: $W' = (y, a_1, a_2)$ or $W' = (a_1, a_2, z)$ or $W' = (a_1, y, c)$ for some vertex $c \in Y$. In the latter case, $\mathcal{F}'$ must contain a triple $W'' = (a_2, z, d)$ with some vertex $d \in Z$. But then $(\mathcal{F}'-\{W', W''\})\cup\{(c, x, d)\}$ is a P_3-factor in G. The first two options are symmetric, so let us assume without loss of generality that $W' = (y, a_1, a_2)$. Denote by W'' the triple in $\mathcal{F}'$ containing z and by W''' a triple obtained from W'' by replacing z with x. Then $(\mathcal{F}' - W') \cup W'''$ is a P_3-factor in G. ∎

Lemma 7. *The transformation $\mathcal{P}^4$ increases the edge domination number by exactly one.*

The proof of this lemma is left to the reader as an exercise.

In spite of the wide area of applicability of vertex stretching, there are many problems that cannot be treated with this operation. In such cases, some more exotic stretching operations can be devised. In this paper we propose two examples of such operations. One of them is represented in Fig. 3. It applies to graphs with a vertex a_0 of degree three and consists of the following steps:

(1) choose arbitrarily two neighbors a_1 and a_2 of a_0;
(2) introduce two new vertices y_1 and x_k on the edge $a_0 a_1$ and two new vertices x_1 and y_k on the edge $a_0 a_2$ so that $x_1, y_1 \in N(a_0)$;
(3) connect x_1 to x_k with a chordless path of length $k-1$, and connect y_1 to y_k with a chordless path of length $k-1$.

We shall denote this transformation by $\mathcal{H}^k$ (k is a parameter) and the resulting graph by $\mathcal{H}^k(G)$. This transformation will be applied to the HAMILTONIAN CYCLE problem.

Lemma 8. $\mathcal{H}^k(G)$ *contains a Hamiltonian cycle if and only if G does.*
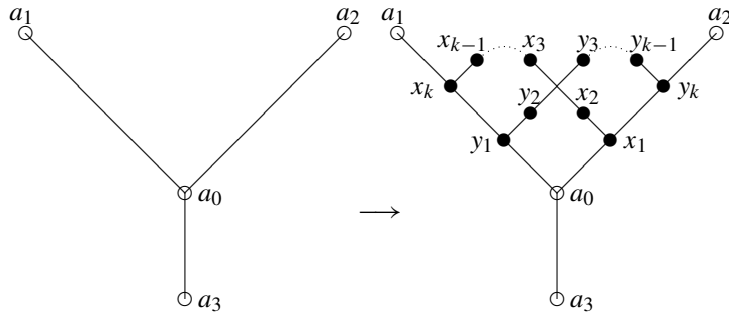
Fig. 3. Transformation $\mathcal{H}^k$.

Proof. Assume G has a Hamiltonian cycle C. If $C = (a_1, a_0, a_2, \ldots, a_1)$, then the cycle $C' = (a_1, x_k, x_{k-1}, \ldots, x_1, a_0, y_1, y_2, \ldots, y_k, a_2, \ldots, a_1)$ is Hamiltonian in the graph $\mathcal{H}^k(G)$. If $C = (a_1, a_0, a_3, \ldots, a_1)$, then the cycle $C' = (a_1, x_k, x_{k-1}, \ldots, x_1, y_k, y_{k-1}, \ldots, y_1, a_0, a_3, \ldots, a_1)$ is Hamiltonian in the graph $\mathcal{H}^k(G)$.

Conversely, if $\mathcal{H}^k$ has a Hamiltonian cycle, then the vertices $a_0, x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_k$ form an interval in the cycle, i.e. they occur in the cycle consecutively (not necessarily in the listed order). Replacing this interval by the vertex a_0, we obtain a Hamiltonian cycle in G. ∎

Another transformation that will be of use in the sequel is a *diamond implantation*. We shall distinguish between two versions of this transformation: vertex diamond implantation and edge diamond implantation. Vertex diamond implantation applies to an arbitrary vertex x and consists of the following steps (see Fig. 4):

(1) partition the neighborhood $N(x)$ of x into two subsets Y and Z in an arbitrary way;
(2) delete vertex x from the graph together with incident edges;
(3) add a diamond with vertices of degree two y and z;
(4) connect y to each vertex in Y, and connect z to each vertex in Z.

Vertex diamond implantation will be denoted $\mathcal{VD}$ and the graph produced by this transformation $\mathcal{VD}(G)$.
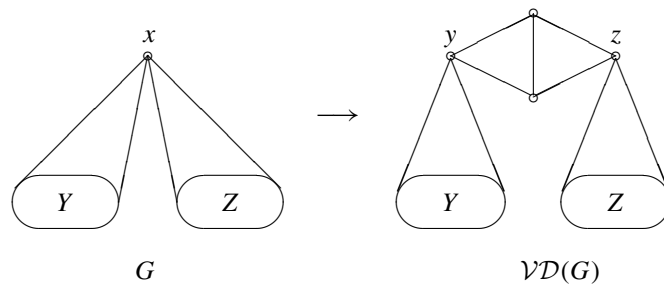

Fig. 4. Vertex diamond implantation.

Edge diamond implantation applies to an arbitrary edge and is illustrated by Fig. 5. This transformation will be denoted $\mathcal{ED}$ and the result of its application by $\mathcal{ED}(G)$.
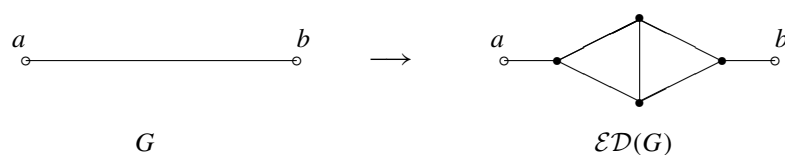

Fig. 5. Edge diamond implantation.

For the vertex variant, we prove the following lemma, where $\alpha_2(G)$ denotes the cardinality of a maximum bipartite set in G, i.e. a subset of vertices inducing a bipartite graph in G.

Lemma 9. (1) *The graph $\mathcal{VD}(G)$ has a vertex 3-coloring if and only if G does.*
(2) $\alpha_2(\mathcal{VD}(G)) = \alpha_2(G) + 2.$

Proof. Part (1) of the statement is trivial. To prove part (2), consider a maximum cardinality bipartite set $U \subseteq V(G)$. If $x \notin U$, then U together with the vertices of degree 3 in the diamond is a bipartite set in $\mathcal{VD}(G)$. If $x \in U$, then a bipartite graph in $\mathcal{VD}(G)$ is induced by $(U - x) \cup \{y, z\}$ together with any vertex of degree 3 in the diamond. Therefore, $\alpha_2(\mathcal{VD}(G)) \geq \alpha_2(G) + 2$.

Conversely, let U' be a maximum cardinality bipartite set in $\mathcal{VD}(G)$. If U' contains 3 vertices of the diamond, then replacement of these vertices by x results in a bipartite set in G. If U' contains at most 2 vertices of the diamond, then deletion of these vertices results in a bipartite set in G. Therefore, $\alpha_2(\mathcal{VD}(G)) \leq \alpha_2(G) + 2$. ∎

The edge variant of the transformation will be applied to EDGE 3-COLORABILITY. The proof of the following lemma is straightforward.

Lemma 10. *The graph $\mathcal{ED}(G)$ has an edge 3-coloring if and only if G does.*

4.1.2. Vertex splitting

Let x be a vertex of degree at least 3 in a graph G. Partition the neighborhood of x into 3 subsets A_1, A_2, A_3 in an arbitrary way, and replace x with the subgraph represented in Fig. 6. The resulting graph will be denoted $\mathcal{R}(G) = \mathcal{R}(G, x, A_1, A_2, A_3)$.
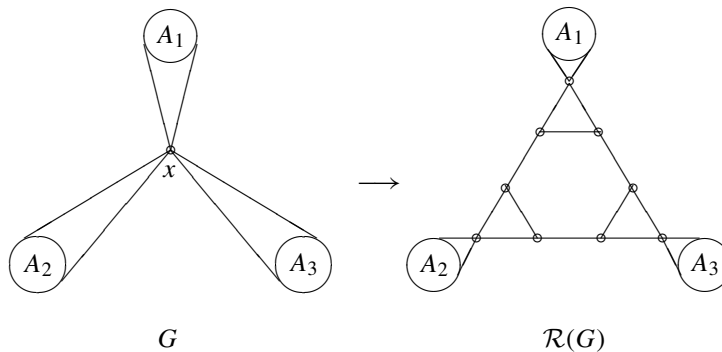

Fig. 6. Transformation $\mathcal{R}$.

In [10], it was shown that

$\mathcal{R}$ increases the independent domination number by two.

The special version of vertex splitting with $|A_1| = |A_2| = |A_3| = 1$ will be denoted $\mathcal{R}_1$, i.e. $\mathcal{R}_1$ is applicable only to vertices of degree 3. This version was used in [29], where it was shown that

$\mathcal{R}_1$ increases the domination number by two.

Now consider a simpler transformation applicable to a vertex of degree 3, which is represented in Fig. 7. This transformation will be denoted $\mathcal{R}_{1/3}$. It is not hard to see that $\mathcal{R}_1$ can be obtained by a triple application of $\mathcal{R}_{1/3}$.

We shall apply $\mathcal{R}_{1/3}$ to the HAMILTONIAN CYCLE and LONGEST CYCLE problems.

Lemma 11. *Let G be a graph with maximum vertex degree at most 3.*

(1) *The graph obtained from G by a single application of $\mathcal{R}_{1/3}$ contains a Hamiltonian cycle if and only if G does.*
(2) *If G' is the graph obtained from G by applying to every vertex of degree 3 the transformation $\mathcal{R}_{1/3}$ and to every vertex of degree 2 the transformation $\mathcal{P}_1^3$, then the length of a longest cycle in G' is 2 times greater than that in G.*
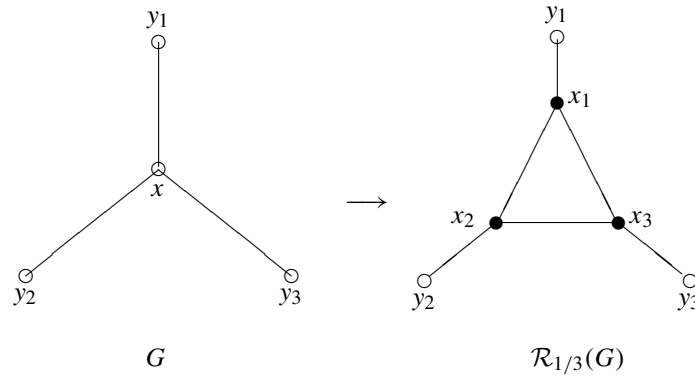
Fig. 7. Transformation $\mathcal{R}_{1/3}$.

Proof. To prove (1), consider a vertex x with three neighbors y_1, y_2, y_3 in G. Assume G contains a Hamiltonian cycle $C = (y_1, x, y_2, \ldots, y_1)$. Then the cycle $C' = (y_1, x_1, x_3, x_2, y_2, \ldots, y_1)$ is Hamiltonian in the graph $\mathcal{R}_{1/3}(G)$. Conversely, assume $\mathcal{R}_{1/3}(G)$ contains a Hamiltonian cycle. It is not hard to see that the vertices x_1, x_2, x_3 occur in the cycle consecutively. Replacing them with x we obtain a Hamiltonian cycle in G.

The proof of part (2) is similar in essence but involves more technicalities. We leave it to the reader as an exercise. ■

4.1.3. Applications of graph transformations

In this section we apply the above described transformations to reveal several limit classes for some of the problems under considerations. To this end, let us introduce the following notation:

$\mathcal{Z}_k$ is the class of $(C_3, \ldots, C_k, H_1, \ldots, H_k)$-free graphs with vertex degree at most three,

where H_i is the graph represented in Fig. 8.
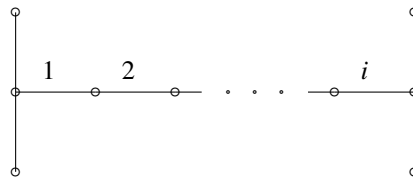
Fig. 8. Graph H_i.

Lemma 12. *For any integer $k \geq 3$, the P_3-FACTOR, EDGE DOMINATION, LONGEST CYCLE and LONGEST PATH problems are NP-hard for bipartite graphs in the class $\mathcal{Z}_k$.*

Proof. We shall prove the lemma by showing that any graph G can be transformed in polynomial time by a sequence of stretching operations into a bipartite graph in the class $\mathcal{Z}_k$. Since LONGEST CYCLE reduces to LONGEST PATH, we prove the lemma only for the first three problems.

First, observe that each of these problems is NP-hard for graphs with maximum vertex degree at most 3. For the LONGEST CYCLE problem, we refer the reader to [19]. For the other two problems we may use the reduction $\mathcal{P}_2^4$ in order to transform any graph into a graph in which every vertex has degree at most three.

Given a graph of vertex degree at most three, we apply to it total 3-subdivision. Applying the transformation sufficiently many times, we get rid of induced cycles C_i with $i \leq k$ and induced graphs of the form H_i with $i \leq k$. The resulting graph is clearly bipartite and the entire transformation can be implemented in polynomial time. Together with Lemmas 5–7 this proves the statement. ■

Similar or slightly weaker results have been previously obtained for the following problems: DOMINATING SET [29], INDEPENDENT DOMINATING SET [10], ALTERNATING CYCLE-FREE MATCHING [30], INDUCED MATCHING [31], DISSOCIATION SET [11], and INDEPENDENT SET [1] (for the INDEPENDENT SET problem the result of Lemma 12 becomes valid if we omit the phrase "for bipartite graphs" from the statement).

Now consider the sequence of graph classes $\mathcal{Z}_k$ with $k = 3, 4, 5, \ldots$ and the intersection $\bigcap_k \mathcal{Z}_k$ of this sequence. In the intersection every graph G has degree at most three, contains no cycles and no graphs of the form H_i. Therefore, every connected component of G is of the form $S_{i,j,k}$, and hence $\bigcap_k \mathcal{Z}_k = \mathcal{S}$. This conclusion is formally settled in the following theorem.

Theorem 3. *Let Π be one of the following problems:* INDEPENDENT SET, DOMINATING SET, INDEPENDENT DOMINATING SET, EDGE DOMINATION, ALTERNATING CYCLE-FREE MATCHING, INDUCED MATCHING, DISSOCIATION SET, P_3-FACTOR, LONGEST CYCLE *or* LONGEST PATH. *Then the class $\mathcal{S}$ is a limit class for the problem Π. Moreover,*

- *if Π is NP-hard in the class of bipartite graphs, then $\mathcal{S}$ is a Π-limit class with respect to bipartite graphs;*
- *if Π is NP-hard in the class of planar graphs, then $\mathcal{S}$ is a Π-limit class with respect to planar graphs;*
- *if Π is NP-hard in the class of graphs of vertex degree at most k, then $\mathcal{S}$ is a Π-limit class with respect to the class of graphs of vertex degree at most k.*

Now we proceed to some other limit classes. To this end, let us denote by Φ_i the graph represented in Fig. 9.
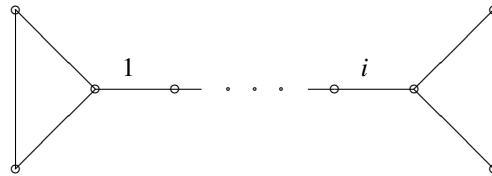

Fig. 9. Graph Φ_i.

Lemma 13. *For any $k \geq 4$, the* LONGEST CYCLE *and* LONGEST PATH *problems are NP-hard for graphs of vertex degree at most 3 in the class Free$(K_{1,3}, K_4, K_4 - e, C_4, \ldots, C_k, \Phi_1, \ldots, \Phi_k)$.*

Proof. As in Lemma 12, we prove the statement only for the LONGEST CYCLE problem. Given a graph G with vertex degree at most 3, we first apply total l-subdivision with appropriate value of l to transform G into a $(C_3, C_4, \ldots, C_k)$-free graph G' in which the distance between any two vertices of degree 3 is more than k. We then apply to G' the transformation described in Lemma 11(2). In this way we obtain a graph G'' containing no induced $K_{1,3}$. G'' may contain triangles but it is still $(K_4, K_4 - e, C_4, \ldots, C_k)$-free. Moreover, the distance between any two triangles is large enough, so G'' is $(\Phi_1, \ldots, \Phi_k)$-free. By Lemmas 5 and 11, a longest cycle in G can be found from a solution to this problem in the graph G'', which completes the proof. ∎

Similar results have been proved for the DOMINATING SET [29] and INDEPENDENT DOMINATING SET [10] problems. It is not hard to see that the intersection of graph classes mentioned in Lemma 13 over all $k \geq 4$ coincides with the class $\mathcal{T}$ and hence we obtain the following conclusion.

Theorem 4. *Let Π be one of the following problems:* DOMINATING SET, INDEPENDENT DOMINATING SET, LONGEST CYCLE *or* LONGEST PATH. *Then the class $\mathcal{T}$ is a limit class for the problem Π. Moreover,*

- *if Π is NP-hard in the class of planar graphs, then $\mathcal{T}$ is a Π-limit class with respect to planar graphs;*
- *if Π is NP-hard in the class of graphs of vertex degree at most k, then $\mathcal{T}$ is a Π-limit class with respect to the class of graphs of vertex degree at most k.*

For the DOMINATING SET and INDEPENDENT DOMINATING SET problems we sketch an alternative proof of Theorem 4 in the next subsection.

For HAMILTONIAN CYCLE, VERTEX 3-COLORABILITY, EDGE 3-COLORABILITY, and BIPARTITE SET we use graph transformations to obtain some weaker results. In other words, we reveal some limit classes for these problems that are extensions of $\mathcal{S}$ or $\mathcal{T}$.

Theorem 5. *Forests of maximum degree* 3 *constitute a limit class for the* HAMILTONIAN CYCLE *problem with respect to bipartite graphs.*

Proof. The HAMILTONIAN CYCLE problem is known to be NP-complete in the class of bipartite graphs with vertex degree at most 3 [24]. Let G be a graph in this class. If we apply the transformation $\mathcal{H}^k$ with even k, then G transforms again into a bipartite graph of maximum degree 3. Repeated applications of this transformation to vertices of degree 3 result in a graph without small induced cycles (vertices of degree 2 can be replaced by chordless paths of appropriate length). Together with Lemma 8 this proves that the HAMILTONIAN CYCLE problem is NP-complete for bipartite graphs of vertex degree at most 3 in the class $Free(C_3, C_4, \ldots, C_p)$ for any particular value of p. Now with p tending to infinity, the proposition follows. ∎

For VERTEX 3-COLORABILITY, EDGE 3-COLORABILITY, and BIPARTITE SET problems we shall use diamond implantation. To better understand the effects of application of this transformation, let us think of a diamond with two 2-degree vertices a and b as an edge of special type between a and b, a *diamond* edge. With this interpretation, the vertex diamond implantation is equivalent to the vertex stretching $\mathcal{P}^2$. As shown before, by means of vertex stretching operation any graph can be transformed into a graph in the class $\mathcal{Z}_k$, and moreover, $\bigcap_k \mathcal{Z}_k = \mathcal{S}$. By replacing in each graph $G \in \mathcal{S}$ every diamond edge with a diamond we obtain the class of graphs that will be denoted by $\widetilde{\mathcal{S}}$. More formally, let $\widetilde{S}_{i,j,k}$ denote the graph obtained from $S_{i,j,k}$ by replacement of each edge with a diamond (see Fig. 10). Then $\widetilde{\mathcal{S}}$ is the class of graphs every connected component of which is an induced subgraph of $\widetilde{S}_{i,j,k}$ with some i, j, k.
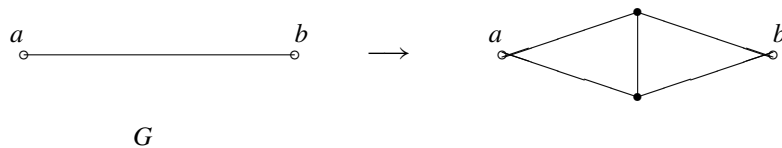

G

Fig. 10. Replacement of an edge by a diamond.

The above discussion on conjunction with Lemma 9 leads to the following conclusion.

Theorem 6. $\widetilde{\mathcal{S}}$ *is a limit class for the* BIPARTITE SET *and* VERTEX 3-COLORABILITY *problems.*

For VERTEX 3-COLORABILITY more conclusions can be derived from some known results. For instance, it is known that this problem is NP-complete in the class of $(K_{1,3}, K_4 - e, K_4)$-free graphs with vertex degree at most 4 [27]. Applying the transformation $\mathcal{P}^2$ appropriately to vertices of degree 4 in a graph G in this class, we reduce the maximum degree of G to 3. Then, subdividing the edges of G, we transform it into a graph in the class $Free(K_{1,3}, K_4, K_4 - e, C_4, \ldots, C_k, \Phi_1, \ldots, \Phi_k)$ by analogy with Lemma 13. Finally, we know that the intersection of such classes over all $k \geq 4$ is the class $\mathcal{T}$. Now all we have to do in order to obtain a new limit class $\widetilde{\mathcal{T}}$ for the VERTEX 3-COLORABILITY problem is to replace diamond edges in graphs in $\mathcal{T}$ by diamonds. Notice however that a diamond edge cannot belong to a triangle. So, we define $\widetilde{\mathcal{T}}$ as follows. Denote by $\widetilde{T}_{i,j,k}$ the graph obtained from $T_{i,j,k}$ by replacing each edge that does not belong to the only triangle by a diamond. Then $\widetilde{\mathcal{T}}$ is the class of graphs every connected component of which is an induced subgraph of $\widetilde{T}_{i,j,k}$ for some i, j, k. With this notation, we obtain the following conclusion.

Theorem 7. $\widetilde{\mathcal{T}}$ *is a limit class for the* VERTEX 3-COLORABILITY *problem.*

For the EDGE 3-COLORABILITY problem we use the edge version of the diamond implantation operation. It is known that this problem is NP-complete for graphs with vertex degree at most 3 [23]. Now by analogy with the above

discussion we can derive the following conclusion. Let $\widetilde{S}'_{i,j,k}$ denote the graph obtained from $S_{i,j,k}$ by implanting a diamond in each edge of this graph, and let $\widetilde{S}'$ denote the class of graphs every connected component of which is an induced subgraph of $\widetilde{S}'_{i,j,k}$ with some i, j, k.

Theorem 8. *The class $\widetilde{S}'$ is a limit class for the* EDGE 3-COLORABILITY *problem.*

4.2. Reducibility between NP-hard problems

Along with graph transformations, a good source of results on this topic is the reducibility between NP-hard problems. To give an example, let us observe that EDGE DOMINATION naturally reduces to the DOMINATING SET problem restricted to the class of line graphs. Moreover, DOMINATING SET and INDEPENDENT DOMINATING SET are polynomially equivalent in the class of line graphs [43]. Combining these remarks with Theorem 3 and the fact that $\mathcal{T} = L(\mathcal{S})$, we conclude that $\mathcal{T}$ is a limit class for the DOMINATING SET and INDEPENDENT DOMINATING SET problems, which was stated before in Theorem 4.

Another example follows readily from the fact that the P_3-FACTOR problem in general graphs reduces in polynomial time to the INDUCED MATCHING problem in the class of line graphs (see e.g. [26]). Since $\mathcal{S}$ is a limit class for P_3-FACTOR (Theorem 3) and $\mathcal{T} = L(\mathcal{S})$, we conclude the following.

Theorem 9. $\mathcal{T}$ *is a limit class for the* INDUCED MATCHING *problem.*

More simple conclusions follow from Theorems 6 and 7 and the obvious fact that VERTEX 3-COLORABILITY reduces to VERTEX COLORING.

Theorem 10. $\widetilde{S}$ *and* $\widetilde{\mathcal{T}}$ *are limit classes for the* VERTEX COLORING *problem.*

Now let us apply the idea of reducibility to reveal one more limit class for the VERTEX COLORING problem.

Theorem 11. *The class co-$\mathcal{T} := \{G : \overline{G} \in \mathcal{T}\}$ is a limit class for* VERTEX COLORING.

Proof. For clarity, we shall prove that $\mathcal{T}$ is a limit class for the complementary problem, i.e. the problem of finding a partition of the vertex set of a graph G into minimum number of cliques (CLIQUE PARTITION), which is equivalent to VERTEX COLORING of the complement of G. We use the reduction from INDEPENDENT SET in the class of triangle-free graphs, which is an NP-hard problem.

Let G be a triangle-free graph and $H = L(G)$ its line graph. Clearly every clique C in H corresponds to a set of edges of G incident to a same vertex $v(C)$. Therefore, if $C^1, \ldots, C^p$ are cliques that partition $V(H)$, then $V(G) - \{v(C^1), \ldots, v(C^p)\}$ is an independent set in G. This reduces the INDEPENDENT SET problem in triangle-free graphs to the CLIQUE PARTITION problem in the class of line graphs of triangle-free graphs. Now observe that $\mathcal{S}$ is a limit class for the INDEPENDENT SET problem *with respect to* triangle-free graphs, since $\mathcal{Z}_k$ is a subclass of triangle-free graphs for any $k \geq 3$ and $\bigcap_k \mathcal{Z}_k = \mathcal{S}$. Finally, we use once again the fact that $\mathcal{T} = L(\mathcal{S})$ to conclude that $\mathcal{T}$ is a limit class for the CLIQUE PARTITION problem.   ∎

Many more examples of this nature can be found in the literature. For instance, the INDEPENDENT SET problem has been reduced

- to the DOMINATING SET problem in split graphs in [4],
- to the DOMINATING SET problem in chordal bipartite graphs in [37],
- to the ALTERNATING CYCLE-FREE MATCHING problem in chordal bipartite graphs in [36].

Each of these reductions leads to a limit class for the respective problem. However, the proof of minimality of limit classes is not a simple task. In the next section we solve this task for some of the limit classes found above.

5. Proof of minimality of limit classes

The first proof of minimality was given for the class $\mathcal{S}$ with respect to the INDEPENDENT SET problem [2]. This proof relies essentially on Lemma 1 and can be extended to any problem for which this lemma holds and for which $\mathcal{S}$

is a limit class. Surprisingly enough, with the help of the same lemma one can prove that $\mathcal{T}$ is a minimal limit class whenever it is Π-limit for a problem Π validating Lemma 1, which was used in [4] to prove minimality of $\mathcal{T}$ for the DOMINATING SET problem. Below we restate these two propositions in the more general context studied in the present paper.

Theorem 12. *Let Π be a problem for which Lemma 1 holds. Then $\mathcal{S}$ is a (Π, Y)-boundary class whenever it is (Π, Y)-limit.*

Proof. Assume that $\mathcal{S}$ is a (Π, Y)-limit class for a problem Π for which Lemma 1 holds. By contradiction, let X be a (Π, Y)-limit class, which is a proper subclass of $\mathcal{S}$. Let G be a graph in $\mathcal{S} - X$. Then $X \subseteq \mathcal{S} \cap Free(G)$. If G contains a connected component without vertices of degree three, we extend G to an arbitrary graph $H \in \mathcal{S}$ in which every connected component has a vertex of degree three. Otherwise, define $H := G$. Since G is an induced subgraph of H, we have $Free(G) \subseteq Free(H)$. Now let M denote the set of graphs containing H as a spanning subgraph. In other words, every graph in M, other than H, is obtained from H by adding some edges. Obviously H is the only graph in $\mathcal{S}$ belonging to M, since the addition of any edge to H results in appearing either a cycle or a graph of the form H_i (Fig. 8). We thus obtain the following inclusions:

$$X \subseteq \mathcal{S} \cap Free(G) \subseteq \mathcal{S} \cap Free(H) \subseteq Free(M) \cap Y.$$

Clearly M is a finite set, and therefore the problem Π is NP-hard in the class $Free(M) \cap Y$ by Lemmas 3 and 4. On the other hand, $Free(M)$ is a monotone graph class and $\mathcal{S} \not\subseteq Free(M)$, since $H \in \mathcal{S} - Free(M)$. Therefore, the problem Π is polynomially solvable in the class $Free(M)$ by Lemma 1. This contradicts the assumption that $P \neq NP$ and proves the theorem. ∎

Theorem 13. *Let Π be a problem for which Lemma 1 holds. Then $\mathcal{T}$ is a (Π, Y)-boundary class whenever it is (Π, Y)-limit.*

Proof. Let $\mathcal{T}$ be (Π, Y)-limit. Assume to the contrary that X is a (Π, Y)-limit class, which is properly contained in $\mathcal{T}$, and let G be a graph in $\mathcal{T} - X$. Then $X \subseteq \mathcal{T} \cap Free(G)$. Now we extend G to a graph $H \in \mathcal{T}$ such that every connected component of H is of the form $T_{j-1, j-1, j}$ for some value $j > 1$. Since G is an induced subgraph of H, we have $Free(G) \subseteq Free(H)$. It is easy to see that one can delete an edge from each connected component of H so that in the resulting graph, denoted by H', every connected component has the form $S_{j, j, j}$, i.e. H' belongs to $\mathcal{S}$. Now let M be the set of all graphs containing H' as a spanning subgraph. It is not hard to verify that $M \cap \mathcal{T} = \{H\}$. Hence, the following inclusions hold:

$$X \subseteq \mathcal{T} \cap Free(G) \subseteq \mathcal{T} \cap Free(H) \subseteq Free(M) \cap Y.$$

These inclusions in conjunction with Lemmas 3 and 4 imply NP-hardness of the problem Π in the class $Free(M) \cap Y$, since M is a finite set. On the other hand, $Free(M)$ is a monotone graph class and $\mathcal{S} \not\subseteq Free(M)$, because $H' \in \mathcal{S} - Free(M)$. Therefore, the problem in question is polynomially solvable in the class $Free(M)$ by Lemma 1. This contradiction proves the theorem. ∎

The above two theorems and the results of the preceding section lead to the following conclusion.

Theorem 14. *Let Π be one of the following problems:* INDEPENDENT SET, DOMINATING SET, INDEPENDENT DOMINATING SET, INDUCED MATCHING, DISSOCIATION SET, LONGEST CYCLE, LONGEST PATH, EDGE DOMINATION *or* P_3-FACTOR. *Then the class $\mathcal{S}$ is (Π, Y)-boundary whenever it is (Π, Y)-limit.*

Let Π be one of the following problems: DOMINATING SET, INDEPENDENT DOMINATING SET, INDUCED MATCHING, LONGEST CYCLE *or* LONGEST PATH. *Then the class $\mathcal{T}$ is (Π, Y)-boundary whenever it is (Π, Y)-limit.*

Proof. For problems different from EDGE DOMINATION and P_3-FACTOR, the result follows from Theorems 3, 4, 9, 12, 13 and Proposition 1.

For the EDGE DOMINATION and P_3-FACTOR problems we again use reducibility between NP-hard problems. Let Π be the EDGE DOMINATION or P_3-FACTOR problem, and Y be a class of graphs such that $\mathcal{S}$ is (Π, Y)-limit. We will show that $\mathcal{S}$ is a (Π, Y)-boundary class. By contradiction, assume that $\mathcal{S}$ properly contains a (Π, Y)-limit class X. Then $L(X)$ is properly contained in $\mathcal{T}$ and they both are subclasses of $L(Y)$. Since EDGE DOMINATION reduces to

DOMINATING SET and P_3-FACTOR reduces to INDUCED MATCHING on line graphs, we conclude that $L(X)$ and $\mathcal{T}$ are limit classes for the DOMINATING SET and INDUCED MATCHING problems with respect to $L(Y)$, which contradicts the minimality of $\mathcal{T}$. ∎

Remark 4. For Π being the INDEPENDENT SET and DOMINATING SET problems and Y being the class of all graphs, the results of Theorem 14 have been originally obtained in [2] and [4], respectively.

6. Concluding remarks and open problems

In this paper, we studied the notion of boundary classes of graphs with respect to various NP-hard graph problems. Unfortunately, none of the problems received a complete characterization in terms of the boundary classes. One of the most explored problems in this respect is DOMINATING SET. Three boundary classes for this problem have been completely described in [4]: these are $\mathcal{S}$, $\mathcal{T}$ and a certain subclass of split graphs. The existence of one more boundary class for this problem follows from NP-hardness of the problem in the class of chordal bipartite graphs [37] and its polynomial-time solvability in the class of forests. The same conclusion is true for the INDEPENDENT DOMINATING SET problem (see Example 2 for more information). However, unlike DOMINATING SET, the INDEPENDENT DOMINATING SET problem is polynomial-time solvable in the class of split graphs. On the other hand, this problem has been recently shown to be NP-hard in a certain extension of split graphs, the so-called $(2, 1)$-polar graphs, i.e. graphs partitionable into a clique and a graph of vertex degree at most 1 [45]. Therefore, there must exist a boundary class for INDEPENDENT DOMINATION with respect to $(2, 1)$-polar graphs. Finding such a class is a challenging research problem.

Returning to the DOMINATING SET problem, let us remark that the third boundary class for this problem identified in [4] has an intriguing relationship with the class $\mathcal{S}$: it is the subclass of split graphs that can be obtained from graphs in $\mathcal{S}$ by partitioning them into two independent sets and replacing one of the sets with a clique. Together with the fact that $\mathcal{T}$ is the class of line graphs of graphs in $\mathcal{S}$, and several limit classes identified in this paper have a close connection to $\mathcal{S}$, this suggests the idea of an exceptional role of the class $\mathcal{S}$ in the theory of boundary classes. Clarifying this role is another question we leave for future research.

Taking into account the particular role of the class $\mathcal{S}$ in this topic, it would be natural to expect that $\mathcal{S}$ is a boundary class for the HAMILTONIAN CYCLE problem, as there must exist a boundary class for this problem which is a (sub)class of forests of vertex degree at most 3 (Theorem 5). However, this is not true. Let us show that any boundary class for the HAMILTONIAN CYCLE problem contained in the class of forests is different from $\mathcal{S}$.

Observation 1. *If X is a class of forests, which is boundary for the* HAMILTONIAN CYCLE *problem, then $H_1 \in X$.*

Proof. First, observe that the HAMILTONIAN CYCLE problem has a polynomial-time solution for graphs in the class *Free*(H_1, C_3, C_4). Indeed, if a graph G in this class has a vertex v of degree at least 3, then every vertex in the neighborhood of v has degree at most 2, since otherwise one of the forbidden subgraphs arises. As a result, G has no Hamiltonian cycle in this case. For graphs of vertex degree at most 2, the problem is trivial. Therefore, if $H_1 \notin X$, then, on the one hand, *Free*(H_1, C_3, C_4) contains the boundary class X, and, on the other hand, the problem in question is polynomial-time solvable for graphs in the class *Free*(H_1, C_3, C_4), which contradicts Theorem 2. ∎

HAMILTONIAN CYCLE is one of the most stubborn problems in terms of boundary classes: none of them has been found so far. On the other hand, we can claim the existence of at least five boundary classes for this problem. One of them was mentioned above, i.e., a subclass of forests of vertex degree at most 3. The existence of one more boundary class arises from the fact that HAMILTONIAN CYCLE is NP-hard in the class of chordal bipartite graphs [38]. Since the problem is solvable in polynomial time in the class of forests, no subclass of forests can be boundary for this problem with respect to chordal bipartite graphs (see Example 2 for argumentation). Some hints regarding the structure of graphs in any boundary class with respect to chordal bipartite graphs are given in the following two observations.

Observation 2. *Let Π be a problem, which is solvable in polynomial time on graphs of bounded clique-width and which is NP-hard in the class of chordal bipartite graphs. Then any Π-boundary class with respect to chordal bipartite graphs contains a domino, i.e. a graph obtained from a chordless cycle C_6 by adding an edge connecting two vertices of distance 3.*

Proof. Every connected domino-free chordal bipartite graph is distance-hereditary [6]. The clique-width of distance-hereditary graphs is at most 3 [20] and hence the problem Π can be solved for distance-hereditary graphs in polynomial time. Therefore, by Theorem 2, the class of domino-free chordal bipartite graphs contains no Π-boundary class with respect to chordal bipartite graphs. In other words, every Π-boundary class with respect to chordal bipartite graphs contains a domino. ∎

Observation 3. *Let Π be a problem, which is solvable in polynomial time on graphs of bounded clique-width and which is NP-hard in the class of chordal bipartite graphs. Then the maximum vertex degree of graphs in any Π-boundary class X with respect to chordal bipartite graphs is unbounded.*

Proof. For contradiction, assume there is a constant k bounding the maximum vertex degree for any graph in X. Consider a sequence $X_1 \supseteq X_2 \supseteq \ldots$ of Π-hard subclasses of chordal bipartite graphs such that $X = \bigcap_j X_j$. It has been proven in [33] that the clique-width of chordal bipartite graphs of bounded vertex degree is bounded. So, the problem Π can be solved for chordal bipartite graphs of bounded vertex degree in polynomial time. Therefore, for each j, the maximum vertex degree of graphs in X_j is unbounded. Since each X_j is a hereditary class, it contains a graph G on $k + 2$ vertices one of which is of degree $k + 1$. Clearly there are finitely many such graphs, and hence there must exist a graph G with this property contained in each class X_j. But then G belongs to X, which contradicts our assumption on the degree bound for graphs in X. ∎

Our next observation leads to the conclusion of the existence of two more boundary classes for the HAMILTONIAN CYCLE problem.

Observation 4. *Let G be a connected bipartite graph, and G' a split graph obtained by creating a clique in one of the parts of G. Then G has a Hamiltonian cycle if and only if G' does.*

Proof. It is not hard to see that both G and G' have a Hamiltonian cycle only if the parts of G are of equal size. Therefore, a cycle is Hamiltonian in G if and only if it is Hamiltonian in G'. ∎

From this observation it follows that if X is a boundary class for the HAMILTONIAN CYCLE problem with respect to bipartite graphs, then there must exist a corresponding boundary class with respect to split graphs. Together with the above discussion this doubles the number of boundary classes for the HAMILTONIAN CYCLE problem. The fifth class arises from the following observation, which is a direct consequence of the first part of Lemma 11.

Observation 5. *The HAMILTONIAN CYCLE problem is NP-hard in the class of ($K_{1,3}$, K_4, $K_4 - e$)-free graphs with vertex degree at most three.*

One more stubborn problem is VERTEX COLORING. For this problem, also no boundary class is available so far, and again we claim the existence of at least 5 such classes. Indeed, three of them are (sub)classes of $\widetilde{S}$, $\widetilde{T}$ and co-T. For the other two, we refer to the reducibility of VERTEX COLORING to VERTEX 3-COLORABILITY and the following facts. First, VERTEX 3-COLORABILITY is NP-complete in the class of K_3-free graphs with maximum degree four [34] and in the class of ($K_{1,3}$, K_4, $K_4 - e$)-free graphs with maximum degree four [27] and these classes contain none of the three limit classes mentioned above. Second, any limit class with respect to K_3-free graphs is not limit with respect to $K_{1,3}$-free graphs and vice versa, since VERTEX 3-COLORABILITY is polynomial-time solvable for (K_3, $K_{1,3}$)-free graphs [40]. Therefore, there are at least two more boundary classes for VERTEX 3-COLORABILITY and hence for VERTEX COLORING. Let us also mention a helpful observation regarding any boundary class for VERTEX 3-COLORABILITY. It is known that this problem is polynomial-time solvable in the class of P_6-free graphs [41]. Therefore,

Observation 6. *Every boundary class for VERTEX 3-COLORABILITY contains a P_6.*

The long list of candidates for boundary classes for HAMILTONIAN CYCLE and VERTEX COLORING leads to the following important question: how many boundary classes may exist for an NP-hard graph problem? More specifically: can the list of boundary classes for an NP-hard problem be infinite? We believe the answer to this question is positive, but we do not think the problems studied in the present paper belong to this category.

In the examples found so far, all boundary classes are easy for their respective problems. It is natural to ask whether there is a problem Π with a Π-hard boundary class. We believe the answer to this question is positive and justify this

conclusion as follows. There are several graph problems that remain difficult on the class of unit interval graphs (see e.g. [44]). The clique-width of unit interval graphs is unbounded [20] and it was conjectured in [25] that in any hereditary subclass of unit interval graphs the clique-width is bounded. If this conjecture is true, then for any problem Π which is NP-hard on unit interval graphs and solvable in polynomial time for graphs of bounded clique-width, the class of unit interval graphs is a Π-hard boundary class.

To conclude the paper let us propose a conjecture.

Conjecture. *For any problem Π solvable in polynomial time on graphs of bounded clique-width, $\mathcal{S}$ and $\mathcal{T}$ are the only Π-boundary classes with respect to any Π-hard class X of bounded vertex degree.*

This conjecture is suggested by the following result proved in [32].

Theorem 15. *If X is a class of graphs of bounded vertex degree such that $\mathcal{S} \nsubseteq X$ and $\mathcal{T} \nsubseteq X$, then the clique-width of graphs in the class X is bounded by a constant.*

In view of Remark 3, this theorem does not necessarily imply the conjecture. However, we believe the conjecture is true and leave its proof for future research.

Acknowledgement

References

[1] V.E. Alekseev, On the local restrictions effect on the complexity of finding the graph independence number, in: Combinatorial-Algebraic Methods in Applied Mathematics, Gorkiy Univ. Press, Gorky, 1983, pp. 3–13 (in Russian).

[2] V.E. Alekseev, On easy and hard hereditary classes of graphs with respect to the independent set problem, Discrete Appl. Math. 132 (2003) 17–26.

[3] V.E. Alekseev, D.V. Korobitsyn, On the complexity of some problems on hereditary classes of graphs, Diskret. Mat. 4 (4) (1992) 34–40 (in Russian).

[4] V.E. Alekseev, D.V. Korobitsyn, V.V. Lozin, Boundary classes of graphs for the dominating set problem, Discrete Math. 285 (2004) 1–6.

[5] P. Alimonti, V. Kann, Some APX-completeness results for cubic graphs, Theoret. Comput. Sci. 237 (2000) 123–134.

[6] H.J. Bandelt, H.M. Mulder, Distance-hereditary graphs, J. Combin. Theory Ser. B 41 (2) (1986) 182–208.

[7] L.W. Beineke, Characterizations of derived graphs, J. Combin. Theory 9 (1970) 129–135.

[8] S. Benzer, On the topology of the genetic fine structure, Proc. Natl. Acad. Sci. USA 45 (1959) 1607–1620.

[9] R. Boliac, V. Lozin, On the clique-width of graphs in hereditary classes, Lecture Notes in Comput. Sci. 2518 (2002) 44–54.

[10] R. Boliac, V. Lozin, Independent domination in finitely defined classes of graphs, Theoret. Comput. Sci. 301 (2003) 271–284.

[11] R. Boliac, K. Cameron, V. Lozin, On the computing the dissociation number of bipartite graphs, Ars Combin. 72 (2004) 241–253.

[12] R.B. Borie, R.G. Parker, C.A. Tovey, Solving problems on recursively constructed graphs, ACM Comput. Surveys (in press) (available online at http://cs.ua.edu/691Borie/ComputingSurveys.pdf).

[13] M. Chudnovsky, N. Robertson, P.D. Seymour, R. Thomas, The Strong Perfect Graph Theorem, Ann. of Math. 164 (2006) 51–229.

[14] B. Courcelle, J.A. Makowsky, U. Rotics, Linear time solvable optimization problems on graphs of bounded clique-width, Theory Comput. Syst. 33 (2000) 125–150.

[15] P. Damaschke, H. Müller, D. Kratsch, Domination in convex and chordal bipartite graphs, Inform. Process. Lett. 36 (1990) 231–236.

[16] M. Farber, Independent domination in chordal graphs, Oper. Res. Lett. 1 (1982) 131–138.

[17] P.C. Fishburn, Interval Orders and Interval Graphs, John Wiley & Sons, Ltd., Chichester, 1985.

[18] S. Foldes, P.L. Hammer, Split graphs, Congr. Numer. 19 (1977) 311–315.

[19] M.G. Garey, D.S. Johnson, Computers and Intractability: A Guide to the Theory of NP-Completeness, in: A Series of Books in the Mathematical Sciences, W.H. Freeman and Co., San Francisco, Calif., 1979.

[20] M.C. Golumbic, U. Rotics, On the clique-width of some perfect graph classes, Internat. J. Found. Comput. Sci. 11 (2000) 423–443.

[21] M.C. Golumbic, Algorithmic Graph Theory and Perfect Graphs, second ed., in: Annals of Discrete Mathematics, vol. 57, Elsevier Science B.V., Amsterdam, 2004.

[22] A.J. Hoffman, A.W.J. Kolen, M. Sakarovitch, Totally-balanced and greedy matrices, SIAM J. Algebr. Discrete Methods 6 (1985) 721–730.

[23] I. Holyer, The NP-completeness of edge-coloring, SIAM J. Comput. 10 (1981) 718–720.

[24] A. Itai, C.H. Papadimitriou, J.L. Szwarcfiter, Hamilton paths in grid graphs, SIAM J. Comput. 11 (1982) 676–686.

[25] M. Kaminski, V. Lozin, M. Milanic, Recent developments on graphs of bounded clique-width, RUTCOR Research Report 6-2007, Rutgers University, 2007 (available online at http://rutcor.rutgers.edu/~rrr/2007.html).

[26] D. Kobler, U. Rotics, Finding maximum induced matchings in subclasses of claw-free and P_5-free graphs, and in graphs with matching and induced matching of equal maximum size, Algorithmica 37 (2003) 327–346.

[27] M. Kochol, V. Lozin, B. Randerath, The 3-colorability problem on graphs with maximum degree four, SIAM J. Comput. 32 (2003) 1128–1139.

[28] D. König, Über Graphen und ihre Anwendungen auf Determinantentheorie und Mengenlehre, Math. Ann. 77 (1916) 435–465.

[29] D.V. Korobitsyn, On the complexity of determining the domination number in monogenic classes of graphs, Diskret. Mat. 2 (3) (1990) 90–96 (in Russian). Translation in Discrete Math. Appl. 2 (2) (1992) 191–199.

[30] V. Lozin, M. Gerber, On the jump number problem in hereditary classes of bipartite graphs, Order 17 (2000) 377–385.

[31] V. Lozin, On maximum induced matchings in bipartite graphs, Inform. Process. Lett. 81 (2002) 7–11.

[32] V. Lozin, D. Rautenbach, On the band-, tree- and clique-width of graphs with bounded vertex degree, SIAM J. Discrete Math. 18 (2004) 195–206.

[33] V. Lozin, D. Rautenbach, Chordal bipartite graphs of bounded tree- and clique-width, Discrete Math. 283 (2004) 151–158.

[34] F. Maffray, M. Preissmann, On the NP-completeness of the k-colorability problem for triangle-free graphs, Discrete Math. 162 (1996) 313–317.

[35] N.V.R. Mahadev, U.N. Peled, Threshold graphs and related topics, in: Annals of Discrete Mathematics, vol. 56, North-Holland Publishing Co., Amsterdam, 1995.

[36] H. Müller, Alternating cycle-free matchings, Order 7 (1990) 11–21.

[37] H. Müller, A. Brandstädt, The NP-completeness of steiner tree and dominating set for chordal bipartite graphs, Theoret. Comput. Sci. 53 (1987) 257–265.

[38] H. Müller, Hamiltonian circuits in chordal bipartite graphs, Discrete Math. 156 (1996) 291–298.

[39] O.J. Murphy, Computing independent sets in graphs with large girth, Discrete Appl. Math. 35 (1992) 167–170.

[40] B. Randerath, I. Schiermeyer, M. Tewes, Three-colourability and forbidden subgraphs. II. Polynomial algorithms, Discrete Math. 251 (2002) 137–153.

[41] B. Randerath, I. Schiermeyer, 3-colorability $\in \mathcal{P}$ for P_6-free graphs, Discrete Appl. Math. 136 (2004) 299–313.

[42] W.T. Trotter Jr., Combinatorics and Partially Ordered Sets. Dimension Theory, Johns Hopkins University Press, Baltimore, MD, 1992.

[43] M. Yannakakis, F. Gavril, Edge dominating sets in graphs, SIAM J. Appl. Math. 38 (1980) 364–372.

[44] M. Yannakakis, Computing the minimum fill-in is NP-complete, SIAM J. Algebr. Discrete Methods 2 (1981) 77–79.

[45] I. Zverovich, Satgraphs and independent domination, Theoret. Comput. Sci. 352 (2006) 47–56.