1.4 Birkhoff Interpolation

Consider the following situation: We have a moving object and the times $t_0, t_1, \ldots, t_m$. For some of these nodes, we know the *distances* traveled $d_i = d(t_i), i \in I \subset \{0, 1, \ldots, m\}$, for others, the *velocities* $v_j = d'(t_j), j \in \tilde{I} \subset \{0, 1, \ldots, m\}$ and for others, the *accelerations* $a_k = d''(t_k), k \in I^* \subset \{0, 1, \ldots, m\}$. Having all these data, can we find a polynomial approximation of the distance function $d = d(t)$ on the entire interval containing the points $t_0, \ldots, t_m$?

Obviously, this is *not* a Lagrange interpolation problem, because we do not have the values of the function at all the nodes. We *cannot* find a Hermite polynomial, either, because at some nodes, only the value of the derivative (or the second derivative) is given (without the values of the function). This is a **Birkhoff interpolation** problem, also known as *lacunary Hermite interpolation* (because not *all* the functional or derivative values for all points are provided) and it is more general than Hermite interpolation.

1.4.1 Birkhoff interpolation polynomial

Birkhoff interpolation problem. Let $x_k \in [a, b], k = \overline{0, m}$, be $m + 1$ distinct nodes, $r_k \in \mathbb{N}$ and $I_k \subseteq \{0, , \ldots, r_k\},\ k = 0, \ldots, m$. Consider the function $f : [a, b] \to \mathbb{R}$ whose derivatives $f^{(j)}(x_k), k = 0, \ldots, m, j \in I_k$ exist. Find a polynomial $P(x)$ of minimum degree, satisfying the interpolation conditions

$$P^{(j)}(x_k) \quad = \quad f^{(j)}(x_k),\ k = \overline{0, m},\ j \in I_k. \tag{1.1}$$

Denote by $n + 1 = |I_0| + \cdots + |I_m|$, where $|I_k|$ is the cardinality (number of elements) of I_k. There are $n + 1$ interpolation conditions in (1.1), so we seek a polynomial of degree at most n,

$$P_n(x) \quad = \quad a_0 + a_1 x + \cdots + a_n x^n,$$

whose coefficients are found from the linear system generated by the interpolation conditions (1.1). If the determinant of this system is not equal to zero, then the Birkhoff interpolation problem has a unique solution.

Remark 1.1. If $I_k = \{0, 1, \ldots, r_k\}$, for every $k = 0, \ldots, m$, then the Birkhoff interpolation problem is reduced to Hermite interpolation (which, in turn, is reduced to Lagrange interpolation when $r_k = 0, k = 0, \ldots, m$). Hence, Birkhoff interpolation is more general.

Unlike Lagrange and Hermite interpolation, the Birkhoff interpolation problem (1.1) *does not* always have a solution. When such a polynomial, denoted by B_nf, exists, it has the form

$$B_nf(x) = \sum_{k=0}^{m}\sum_{j\in I_k} b_{kj}(x)f^{(j)}(x_k). \tag{1.2}$$

The terms $b_{kj}(x)$ are called **Birkhoff fundamental polynomials** and they satisfy the relations:

$$\begin{aligned} b_{kj}^{(p)}(x_\nu) &= 0,\ \nu \neq k,\ p \in I_\nu, \\ b_{kj}^{(p)}(x_k) &= \delta_{jp},\ p \in I_k,\ \text{for}\ \ j \in I_k \ \text{and}\ \ \nu, k \ = \ 0, 1, \ldots, m, \end{aligned} \tag{1.3}$$

where

$$\delta_{jp} = \begin{cases} 0, & j \neq p \\ 1, & j = p \end{cases}$$

Kronecker's symbol.

Remark 1.2. Because some of the functional (or derivative) values are missing, finding mathematical expressions for the Birkhoff fundamental polynomials $b_{kj}, k = 0, \ldots, m; j \in I_k$, is, in general, difficult. They can be determined (when possible) directly from din conditions (1.1).

Example 1.3. Let $f \in C^2[0,1]$ and consider the nodes $x_0 = 0,\ x_1 = 1$, for which the values $f(0) = 1$ and $f'(1) = 2$ are given. Find the Birkhoff polynomial that interpolates these data.

Solution.
We have $m = 1$, two nodes, with $I_0 = \{0\}, I_1 = \{1\}$, so $n = 1 + 1 - 1 = 1$. We want a polynomial of degree 1,

$$P(x) = a_0 + a_1 x,$$

satisfying the conditions

$$\begin{aligned} P(0) &= f(0), \\ P'(1) &= f'(1). \end{aligned}$$

From here, we have the linear system

$$\begin{cases} a_0 & & = f(0) \\ & a_1 & = f'(1) \end{cases}.$$

The determinant of this system is

$$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0,$$

so this problem has a unique solution

$$\begin{cases} a_0 &= f(0) \\ a_1 &= f'(1) \end{cases},$$

i.e. the polynomial we seek is

$$P(x) \;=\; f(0) + f'(1)x \;=\; 1 + 2x.$$

On the other hand, by (1.2), the Birkhoff polynomial is of the form

$$B_1 f(x) \;=\; b_{00}(x)f(0) + b_{11}(x)f'(1).$$

Let us find the fundamental polynomials $b_{00}(x)$ şi $b_{11}(x)$. Both have degree 1, hence,

$$\begin{aligned} b_{00}(x) &= ax + b, \\ b_{11}(x) &= cx + d. \end{aligned}$$

By conditions (1.3), for $b_{00,}$ we have

$$\begin{cases} b_{00}(x_0) &= 1 \\ b'_{00}(x_1) &= 0 \end{cases} \Longleftrightarrow \begin{cases} b_{00}(0) &= 1 \\ b'_{00}(1) &= 0 \end{cases} \Longleftrightarrow \begin{cases} b &= 1 \\ a &= 0 \end{cases},$$

thus,

$$b_{00}(x) \;=\; 1.$$

Similarly, for b_{11}, we have

$$\begin{cases} b_{11}(x_0) &= 0 \\ b'_{11}(x_1) &= 1 \end{cases} \Longleftrightarrow \begin{cases} b_{11}(0) &= 0 \\ b'_{11}(1) &= 1 \end{cases} \Longleftrightarrow \begin{cases} d &= 0 \\ c &= 1 \end{cases},$$

so we get

$$b_{11}(x) \;=\; x.$$

Thus,

$$B_1 f(x) \;=\; f(0) + x f'(1) \;=\; 1 + 2x.$$

■

Example 1.4. Find a polynomial of smallest degree (if it exists) satisfying the conditions

$$P(-1) \;=\; P(1) \;=\; 0,\; P'(0) \;=\; 1.$$

Solution.

Here, we have $m = 2$, so 3 nodes, for which $I_0 = \{0\}, I_1 = \{1\}, I_2 = \{0\}$. Hence, we seek a polynomial of degree $n = 1 + 1 + 1 - 1 = 2$. This is of the form

$$P(x) \;=\; a_0 + a_1 x + a_2 x^2$$

and must satisfy the relations

$$\begin{aligned} P(-1) &= 0, \\ P'(0) &= 1, \\ P(1) &= 0. \end{aligned}$$

We obtain the linear system

$$\begin{cases} a_0 - a_1 + a_2 = 0 \\ \quad\;\; a_1 \quad\;\;\; = 1 \\ a_0 + a_1 + a_2 = 0 \end{cases}.$$

Subtracting the first equation from the third, we get $a_1 = 0$, which contradicts the second equation. The system is incompatible and, thus, this interpolation problem *does not have* a solution.

■

Example 1.5. [Abel-Goncearov interpolation] Let $f \in C^{n+1}[0, nh]$, with $h > 0,\; n \in \mathbb{N}$. Find a polynomial of smallest degree satisfying the relations

$$\begin{aligned} P(0) &= f(0), \\ P'(h) &= f'(h), \\ &\dots \\ P^{(n)}(nh) &= f^{(n)}(nh). \end{aligned}$$

This problem has a unique solution for every $h > 0,\ n \in \mathbb{N}$.

1.4.2 Peano's theorem and the error for Birkhoff interpolation

To find an error formula for Birkhoff interpolation (when the Birkhoff polynomial exists), we need an important result from linear operator theory.

Let us recall some notions and properties:

- Let $n \in \mathbb{N}^*$. We define the space

$$H^n[a,b] = \{f : [a,b] \to \mathbb{R} \mid f \in C^{n-1}[a,b], f^{(n-1)} \text{ absolutely continuous on } [a,b]\}.$$

- A function $f : [a,b] \to \mathbb{R}$ is *absolutely continuous* on $[a,b]$, if, for instance, it has a derivative f' almost everywhere, the derivative is Lebesgue integrable, and

$$f(x) = f(a) + \int_a^x f'(t)dt,\ \forall x \in [a,b].$$

- $H^n[a,b]$ is linear space.
- Any function $f \in H^n[a,b]$ has a Taylor-type representation, with the remainder in integral form

$$f(x) = \sum_{k=0}^{n-1} \frac{(x-a)^k}{k!} f^{(k)}(a) + \int_a^x \frac{(x-t)^{n-1}}{(n-1)!} f^{(n)}(t)\, dt.$$

- The function

$$z_+ = \begin{cases} z, & z \geq 0 \\ 0, & z < 0 \end{cases}$$

is called the *positive part* of z, and z_+^n is called a *truncated power*.

• For $m \in \mathbb{N}$, $\mathbb{P}_m$ denotes the space of all polynomials of degree at most m. Obviously, $\mathbb{P}_m \subset C^\infty[a,b], \forall m \geq 0$.

• The *kernel* of a linear map $L : V \to W$ between two vector spaces V and W, is the set of all vectors in V that are mapped to zero:

$$\ker L \;=\; \{v \in V \mid L(v) = \mathbf{0}_{\boldsymbol{W}}\},$$

where $\mathbf{0}_{\boldsymbol{W}}$ is null vector in W.

The next theorem is paramount in Numerical Analysis. It gives a representation of real linear functionals defined on $H^n[a,b]$. This result provides means for expressing the errors in many approximating procedures.

Theorem 1.6. [Peano]
Let $L : H^n[a,b] \to \mathbb{R}$ be a linear functional that commutes with the definite integral operator. If $\ker L = \mathbb{P}_{n-1}$, *then*

$$Lf \;=\; \int\limits_a^b K_n(t) f^{(n)}(t)\,dt, \tag{1.4}$$

where

$$K_n(t) \;=\; \frac{1}{(n-1)!} L\Big[(\cdot - t)_+^{n-1}\Big] \tag{1.5}$$

*is called the **Peano kernel**. The notation above means that the operator L is applied to f with respect to the variable $(\cdot)$.*

Proof. As $f \in H^n[a,b]$, we know that it has a Taylor-type representation, with the remainder in integral form

$$f(x) \;=\; (T_{n-1}f)(x) + (R_{n-1}f)(x), \tag{1.6}$$

where

$$(T_{n-1}f)(x) \;=\; \sum_{k=0}^{n-1} \frac{(x-a)^k}{k!} f^{(k)}(a) \tag{1.7}$$

and

$$(R_{n-1}f)(x) = \int_a^x \frac{(x-t)^{n-1}}{(n-1)!} f^{(n)}(t)\, dt = \frac{1}{(n-1)!} \int_a^x (x-t)^{n-1} f^{(n)}(t)\, dt. \tag{1.8}$$

So,

$$(Lf)(x) = \big(L(T_{n-1}f)\big)(x) + \big(L(R_{n-1}f)\big)(x). \tag{1.9}$$

Since $\ker L = \mathbb{P}_{n-1}$ and $(T_{n-1}f) \in \mathbb{P}_{n-1}$, we have

$$\big(L(T_{n-1}f)\big)(x) = 0, \forall x \in [a,b]. \tag{1.10}$$

Now, the positive part of $(x-t)$, for $x, t \in [a,b]$ is

$$(x-t)_+ = \begin{cases} x-t, & t \le x \\ 0, & t > x \end{cases}.$$

Thus, we can write

$$\begin{aligned} \int_a^b (x-t)_+^{n-1} f^{(n)}(t)\, dt &= \int_a^x (x-t)_+^{n-1} f^{(n)}(t)\, dt + \int_x^b (x-t)_+^{n-1} f^{(n)}(t)\, dt \\ &= \int_a^x (x-t)^{n-1} f^{(n)}(t)\, dt. \end{aligned}$$

Substituting this into (1.8), we get

$$(R_{n-1}f)(x) = \frac{1}{(n-1)!} \int_a^b (x-t)_+^{n-1} f^{(n)}(t)\, dt. \tag{1.11}$$

Since L commutes with the definite integral operator, we obtain

$$\begin{aligned}\big(L(R_{n-1}f)\big)(x) &= \frac{1}{(n-1)!}L\Big(\int\limits_a^b (x-t)_+^{n-1}f^{(n)}(t)\,dt\Big)\\ &= \frac{1}{(n-1)!}\int\limits_a^b L\Big((x-t)_+^{n-1}\Big)f^{(n)}(t)\,dt \qquad (1.12)\\ &= \int\limits_a^b \big(K_n(t)\big)(x)f^{(n)}(t)\,dt\end{aligned}$$

By relations (1.9), (1.10) and (1.12), it follows that

$$Lf = \int\limits_a^b K_n(t)f^{(n)}(t)dt.$$

□

Corollary 1.7. If the kernel K has constant sign on $[a, b]$ and $f^{(n)}$ is continuous on $[a, b]$, then there exists $\xi \in [a, b]$ such that

$$Lf = \frac{1}{n!}f^{(n)}(\xi)Le_n, \qquad (1.13)$$

where $e_k(x) = x^k,\ k \in \mathbb{N}$.

Proof. If the kernel K has constant sign on $[a, b]$, we can apply the mean value theorem in (1.4):

$$Lf = f^{(n)}(\xi)\int\limits_a^b K_n(t)\,dt. \qquad (1.14)$$

Notice that the kernel K *does not* depend on f and the relation above is true *regardless* of the function f. Then, taking $f = e_n$, we get

$$\begin{aligned} Le_n &= e_n^{(n)}(\xi) \int_a^b K_n(t)\, dt \\ &= n! \int_a^b K_n(t)\, dt, \end{aligned}$$

from which we get

$$\int_a^b K_n(t)\, dt = \frac{1}{n!} Le_n.$$

Using this in (1.14), we obtain (1.13).

□

This corollary is the one that is mostly used in applications (to assess the approximation error).

Example 1.8. Let us find a formula for the rest of the Birkhoff polynomial in Example 1.3.

Solution. We found the Birkhoff polynomial

$$B_1 f(x) = f(0) + f'(1)x,\ x \in [0,1],$$

so, we have

$$f(x) = B_1 f(x) + R_1 f(x).$$

We apply Peano's theorem to the rest operator,

$$Lf = R_1 f = f - B_1 f.$$

We have

$$\begin{aligned} R_1 e_0(x) &= e_0(x) - B_1 e_0(x) = e_0(x) - \big(e_0(0) + e_0'(1)x\big) = 1 - (1+0) = 0, \\ R_1 e_1(x) &= e_1(x) - B_1 e_1(x) = e_1(x) - \big(e_1(0) + e_1'(1)x\big) = x - (0 + 1 \cdot x) = 0, \\ R_1 e_2(x) &= e_2(x) - B_1 e_2(x) = e_2(x) - \big(e_2(0) + e_2'(1)x\big) = x^2 - 2x \neq 0, \end{aligned}$$

(the first two were obvious, since B_1 is polynomial of degree 1, but it is a good computational exercise). Thus,

$$R_1 f(x) \;=\; \int\limits_0^1 K_2(x,t) f''(t)\,dt,$$

with

$$\begin{aligned} K_2(x,t) &= R_1\left(\frac{(x-t)_+^1}{1!}\right) \\ &= \underbrace{(x-t)_+}_{f(x)} - \Big(\underbrace{(0-t)_+}_{f(0)} + \underbrace{1\cdot x}_{f'(1)x}\Big) \\ &= (x-t)_+ - (-t)_+ - x. \end{aligned}$$

Since $x,t \in [0,1]$, we have $(-t)_+ = 0$. Fix an arbitrary $x \in [0,1]$.
If $0 \le t \le x$, then $(x-t)_+ \;=\; x-t$ and

$$K_2(x,t) \;=\; x-t-x \;=\; -t \;\le\; 0.$$

If $x \le t \le 1$, then $(x-t)_+ \;=\; 0$ and

$$K_2(x,t) \;=\; 0-x \;=\; -x \;\le\; 0.$$

So, either way, K_2 has constant sign on $[0,1]$. By Corollary 1.7, it follows that

$$\begin{aligned} R_1 f(x) &= \frac{1}{2!} f''(\xi) R_1 e_2(x) \\ &= \frac{x^2-2x}{2!} f''(\xi),\ \xi \in [0,1]. \end{aligned}$$

As $|x(x-2)| \le 1$, for $x \in [0,1]$, we have the following estimate for the interpolation error:

$$|R_1 f(x)| \;\le\; \frac{1}{2} \|f''\|_\infty,\ \forall x \in [0,1].$$

■

Example 1.9. Let $f \in C^3[0,2]$ be a function for which the values $f'(0) = 1$, $f(1) = 2$ and $f'(2) = 1$ are known. Approximate $f(1/2)$ using Birkhoff interpolation and estimate the error.

Solution.

We have $m+1=3$ nodes, with $I_0=\{1\}, I_1=\{0\}, I_2=\{1\}$, so $n=1+1+1-1=2$. The Birkhoff polynomial of degree (at most) 2,

$$\begin{aligned} P(x) &= ax^2+bx+c, \\ P'(x) &= 2ax+b, \end{aligned}$$

must satisfy the relations

$$\begin{aligned} P'(0) &= b \;=\; f'(0), \\ P(1) &= a+b+c \;=\; f(1) \\ P'(2) &= 4a+b \;=\; f'(2). \end{aligned}$$

The determinant of the corresponding linear system is

$$\begin{vmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 4 & 1 & 0 \end{vmatrix} = -\begin{vmatrix} 1 & 1 \\ 4 & 0 \end{vmatrix} = 4 \neq 0,$$

hence, there exists a unique Birkhoff interpolation polynomial, of the form

$$B_2 f(x) \;=\; b_{01}(x)f'(0)+b_{10}(x)f(1)+b_{21}(x)f'(2).$$

The fundamental polynomials (of degree at most 2) must satisfy the conditions

$$\left\{\begin{aligned} b'_{01}(0) &= 1 \\ b_{01}(1) &= 0 \\ b'_{01}(2) &= 0 \end{aligned}\right., \quad \left\{\begin{aligned} b'_{10}(0) &= 0 \\ b_{10}(1) &= 1 \\ b'_{10}(2) &= 0 \end{aligned}\right. \quad \text{and} \quad \left\{\begin{aligned} b'_{21}(0) &= 0 \\ b_{21}(1) &= 0 \\ b'_{21}(2) &= 1 \end{aligned}\right. .$$

From these, we find the interpolation polynomial

$$\begin{aligned} B_2 f(x) &= \frac{1}{4}(x-1)(3-x)f'(0)+f(1)+\frac{1}{4}(x^2-1)f'(2) \\ \Big(&= \frac{1}{4}(x-1)(3-x)+2+\frac{1}{4}(x^2-1) \;=\; x+1\Big), \end{aligned}$$

with derivative

$$\begin{aligned}(B_2f)'(x) &= \frac{1}{2}(2-x)f'(0) + \frac{1}{2}xf'(2) \\ \Bigg(&= \frac{1}{2}(2-x) + \frac{1}{2}x \;=\; 1\Bigg).\end{aligned}$$

The remainder is computed as

$$R_2f(x) = f(x) - B_2f(x) \;=\; f(x) - \left[\frac{1}{4}(x-1)(3-x)f'(0) + f(1) + \frac{1}{4}(x^2-1)f'(2)\right].$$

We have

$$\begin{aligned}R_2e_2(x) &= x^2 - \left[0 + 1 + \frac{1}{4}(x^2-1)\cdot 4\right] \;=\; x^2 - 1 - (x^2-1) \;=\; 0, \\ R_2e_3(x) &= x^3 - \left[0 + 1 + \frac{1}{4}(x^2-1)\cdot 12\right] \;=\; (x-1)(x^2-2x-2) \;\neq\; 0.\end{aligned}$$

(again, the first relation *needed not* be checked, but it is important to see that it *does hold*, even though, for this particular data, the polynomial B_2f has actually degree *less than* 2).
The remainder is given by

$$R_2f(x) \;=\; \int_0^2 K_3(x,t)f'''(t)\,dt.$$

where

$$\begin{aligned}K_3(x,t) &= R_2\left(\frac{(x-t)_+^2}{2!}\right) \\ &= \frac{(x-t)_+^2}{2} - \left[\frac{1}{4}(x-1)(3-x)(0-t)_+ + \frac{(1-t)_+^2}{2} + \frac{1}{4}(x^2-1)(2-t)_+\right].\end{aligned}$$

We compute

$$\begin{aligned}K_3(1/2,t) &= \frac{(1/2-t)_+^2}{2} \\ &- \left[\frac{1}{4}(1/2-1)(3-1/2)(0-t)_+ + \frac{(1-t)_+^2}{2} + \frac{1}{4}((1/2)^2-1)(2-t)_+\right] \\ &= \frac{1}{2}\left(\frac{1}{2}-t\right)_+^2 - \frac{1}{2}(1-t)_+^2 \;+ \frac{3}{16}(2-t).\end{aligned}$$

We have the following cases:

1. $0 \leq t \leq \dfrac{1}{2}$,

$$K_3(1/2,t) \;=\; \frac{1}{8}(1-2t)^2 - \frac{1}{2}(1-t)^2 + \frac{3}{16}(2-t) \;=\; \frac{5}{16}t \;\geq\; 0.$$

2. $\dfrac{1}{2} \leq t \leq 1$,

$$K_3(1/2,t) \;=\; -\frac{1}{2}(1-t)^2 + \frac{3}{16}(2-t) \;=\; -\frac{1}{16}(8t^2-13t+2) \;\geq\; 0,$$

   because the roots of the quadratic polynomial above, $\dfrac{13 \pm \sqrt{105}}{16} = \{0.1721, 1.4529\}$, lie *outside* the interval $\left[\dfrac{1}{2}, 1\right]$.

3. $1 \leq t \leq 2$,

$$K_3(1/2,t) \;=\; \frac{3}{16}(2-t) \;\geq\; 0.$$

So, K_2 has constant on $[0,2]$ and, thus,

$$R_2 f(1/2) \;=\; \frac{1}{3!} f'''(\xi) R_2 e_3(1/2) \;=\; \frac{1}{6}\cdot\frac{11}{8} f'''(\xi) \;=\; \frac{11}{48} f'''(\xi),\; \xi \in [0,2].$$

In the end, we have the approximation

$$f(1/2) \;\approx\; B_2 f(1/2) \;=\; 3/2,$$

with the error

$$|R_2 f(1/2)| \;\leq\; \frac{11}{48} ||f'''||_\infty.$$

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