4 Examples for Lecture 11 (Nonlinear equations in $\mathbb{R}$)

Example 4.1 *Consider the equation $x^3 - 2x^2 = 5$. Give the next two iterations for approximating the solution of this equation using:*

1. Newton's method starting with $x_0 = 2$.
   Considering the equation $f(x) = 0$ (in our case we will have $f(x) = x^3 - 2x^2 - 5$), for Newton's method the next iteration is obtained from

   $$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

   for a starting value x_0. This method is a *one-step method*, i.e., to obtain an approximation for x_k we need only the previous approximation x_{k-1}, in particular we only need one starting value.

   $$f'(x) = (x^3 - 2x^2 - 5)' = 3x^2 - 4x$$

   The next two iterations are:

   $$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{f(2)}{f'(2)} = 2 - \frac{-5}{4} = \frac{13}{4} = 3.25$$

   and

   $$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 3.25 - \frac{f(3.25)}{f'(3.25)} = 3.25 - \frac{8.2}{18.68} \approx 2.81.$$

   Remark 4.2 See another **example** in Lecture 11, pp. 10–11 and the **theory** on pp. 10–13.

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2. secant method starting with $x_0 = 1$ and $x_1 = 3$.
   The secant method is a *two-step method* (we need two previous approximations to get a new one) and it has the formula

   $$x_{k+1} = x_k - \frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})} f(x_k)$$

   for two starting values x_0 and x_1. In our case we have:

   $$x_2 = x_1 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_1) = 3 - \frac{3-1}{f(3) - f(1)} f(3) = 3 - \frac{2 \cdot 4}{4 - (-6)} = 3 - 0.8 = 2.2$$

   and the second approximation

   $$x_3 = x_2 - \frac{x_2 - x_1}{f(x_2) - f(x_1)} f(x_2) = 2.2 - \frac{2.2 - 3}{f(2.2) - f(3)} f(2.2) = 2.2 - \frac{-0.8 \cdot (-4.03)}{-4.03 - 4} = 2.2 + \frac{0.8 \cdot 4.03}{8.03} \approx 2.60.$$

   Remark 4.3 See another **example** in Lecture 11, pp. 8 and the **theory** on pp. 7–9.

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3. bisection method starting with $a_0 = 1$ and $b_0 = 3$.
   The bisection method is based on the following algorithm: **If $f(a) \cdot f(b) < 0$ with f continuous on $[a, b]$, then there is at least one root of f in (a, b).**

   Let $[a, b] = [a_0, b_0]$. First we check if $f(a_0) \cdot f(b_0) < 0$ and then we compute the middle of the interval $[a, b]$, i.e., $c_0 = \frac{a_0 + b_0}{2}$ and we check:

   - if $f(c_0) \cdot f(b_0) < 0$, then the root is in the interval $[c_0, b_0]$ and we consider $a_1 = c_0$ and $b_1 = b_0$;
   - otherwise (if $f(a_0) \cdot f(c_0) < 0$), then the root is in $[a_0, c_0]$ and we consider $a_1 = a_0$ and $b_1 = c_0$.

We apply then the same steps on the new interval $[a_1, b_1]$ and so on.
In our case, we have :

$$f(a_0) \cdot f(b_0) = f(1) \cdot f(3) = (1^3 - 2 \cdot 1^2 - 5) \cdot (3^3 - 2 \cdot 3^2 - 5) = (-6) \cdot 4 = -24 < 0,$$

so

$$c_0 = \frac{a_0 + b_0}{2} = \frac{1+3}{2} = 2.$$

Next, we check:

$$f(c_0) \cdot f(b_0) = f(2) \cdot f(3) = (-5) \cdot 4 = -20 < 0$$

and the root must be in the interval $[c_0, b_0] = [2, 3]$. This is why we set $a_1 = c_0$ and $b_1 = b_0$ and we move in the interval $[a_1, b_1] = [2, 3]$. This time we have

$$c_1 = \frac{a_1 + b_1}{2} = \frac{2+3}{2} = 2.5$$

$$f(c_1) \cdot f(b_1) = f(2.5) \cdot f(3) = (-1.875) \cdot 4 < 0,$$

so the root must be in the interval $[c_1, b_1] = [2.5, 3]$. We set $a_2 = c_1$ and $b_2 = b_1$ and we move in the interval $[a_2, b_2] = [2.5, 3]$, with $c_2 = \frac{2.5+3}{2} = 2.75$.

Remark 4.4 See another **example** in Lecture 11, pp. 4–5 and the **theory** on pp. 3–6.

Example 4.5 *Show that $g(x) = \pi + \frac{1}{2}\sin(\frac{x}{2})$ has a unique fixed point on $[0, 2\pi]$. Estimate the number of iterations required to achieve 10^{-2} accuracy for the solution of $g(x) = x$. Compute the first two iterates.*

Remark 4.6 The Banach Theorem can be found on pp. 15. We will use a little modification for it (easier to apply) which can be found on pp. 16 (Th. 3.5.).

We have $g'(x) = \frac{1}{4}\cos(\frac{x}{2})$, so it is obvious that $g \in C^1([0, 2\pi])$ (= derivable, which implies continuous, and with the first derivative also continuous on $[0, 2\pi]$). Now we have to show that $g([0, 2\pi]) \subseteq [0, 2\pi]$. We compute the table of variation for g.

$$g'(x) = 0 \Longrightarrow \frac{1}{4}\cos(\frac{x}{2}) = 0 \Longrightarrow \cos(\frac{x}{2}) = 0 \xRightarrow{x \in [0,2\pi]} x = \pi.$$

x	0	π	2π
$g'(x)$	+	0	-
$g(x)$	↗		↘

$$g(0) = \pi, \; g(\pi) = \pi + \frac{1}{2}, \; g(2\pi) = \pi$$

and together with the continuity of g, we have that Img $= [\pi, \pi + \frac{1}{2}] \subseteq [0, 2\pi]$.

Now we have to check that $\lambda = \max_{x \in [0, 2\pi]} |g'(x)| < 1$.

$$|g'(x)| = \left|\frac{1}{4}\cos\left(\frac{x}{2}\right)\right| = \frac{1}{4}\left|\cos\left(\frac{x}{2}\right)\right| \le \frac{1}{4} \cdot 1 = \frac{1}{4} \Longrightarrow \lambda = \frac{1}{4} < 1.$$

So all the conditions of Banach Theorem are fulfilled. So g has a unique fixed point $\alpha \in [0, 2\pi]$ (i.e., the eq. $g(x) = x$ has a unique solution). Another result is that **the sequence**

$$x_{n+1} = g(x_n)$$

will converge to the solution α of the equation $g(x) = x$ as $n \to \infty$, for any choice of initial approximation $x_0 \in [0, 2\pi]$.

We can take for example $x_0 = \pi$. Then the first two iterates are

$$x_1 = g(x_0) = \pi + \frac{1}{2}\sin\frac{\pi}{2} = \pi + \frac{1}{2}$$

and

$$x_2 = g(x_1) = \pi + \frac{1}{2}\sin\frac{\pi + \frac{1}{2}}{2} = \ldots$$

To obtain the accuracy of 10^{-2}, we use the result:

$$|x_n - \alpha| \le \frac{\lambda^n}{1-\lambda}|x_1 - x_0|,\ n \ge 1$$

and we impose that

$$\frac{\lambda^n}{1-\lambda}|x_1 - x_0| \le 10^{-2}$$

which obviously will impose that $|x_n - \alpha| \le 10^{-2}$.

$$\frac{\lambda^n}{1-\lambda}|x_1 - x_0| \le 10^{-2} \iff \frac{\frac{1}{4^n}}{1-\frac{1}{4}}\left|\pi + \frac{1}{2} - \pi\right| \le \frac{1}{100} \iff$$

$$\iff \frac{1}{3\cdot 4^{n-1}}\cdot\frac{1}{2} \le \frac{1}{100} \iff 6\cdot 4^{n-1} \ge 100 \iff 4^{n-1} \ge \frac{100}{6} = 16.(6) \implies n = 4.$$

Remark 4.7 The **theory** is in Lecture 11, pp. 14–16. **Examples** are in Lecture 11, pp. 17.

Example 4.8 *Approximate $\sqrt{10}$ using two iterations of the Newton's method.*
If we let $x = \sqrt{10}$, then $x^2 = 10$ and $x^2 - 10 = 0$ so we can consider the equation $f(x) = 0$ with $f(x) = x^2 - 10$ and $f'(x) = 2x$. $f(3) = -1 < 0$, $f(4) = 6 > 0$ so we can take the interval $[3,4]$.

Let the first approximation be 4, so $x_0 = 4$. We have then

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 4 - \frac{6}{8} = \frac{26}{8} = 3.25$$

and

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 3.25 - \frac{0.5625}{6.5} \approx 3.16.$$

Remark 4.9 If you have to solve the equation $f(x) = 0$, but you do not have the interval, you can use the following result:

If f is continuous on $[a,b]$ and $f(a)\cdot f(b) < 0$, then there is at least one point $c \in (a,b)$ such that $f(c) = 0$.

So, first try to find two values of the function f of opposite signs to determine the interval. Then you can use a suitable method to solve the nonlinear equation $f(x) = 0$. You can use other results from Calculus (Rolle's Theorem, monotony of function - table of variation, etc.) to find more about the roots.

Some notions are found in Lecture 11, pp. 1–2.

Remark 4.10 See Lecture 12 for **Newton's method for nonlinear systems** (pp. 8–10), **Numerical approximation for multiple roots** (pp. 4–8).