1 Examples for Lecture 6

Example 1.1 *For the points* $(1,1)$, $(2,1)$, $(3,0)$ *construct a natural cubic spline that passes through them.*

- For theory, see Lecture 6, pp. 5–9. For another example, see pp. 10–11.

We have $x_1 = 1, x_2 = 2, x_3 = 3$ and $f_1 = f(x_1) = 1, f_2 = f(x_2) = 1, f_3 = f(x_3) = 0$. So, $n = 3$, $h_1 = x_2 - x_1 = 1, h_2 = x_3 - x_2 = 1$. Our spline will have the form

$$s(x) = \begin{cases} s_1(x), & x \in [1,2] \\ s_2(x), & x \in [2,3] \end{cases}$$

A property is that the spline, its first derivative and its second derivative are continuous at the nodes. Another one is that it interpolates the function at the nodes. Since we work with cubic splines, s_1 and s_2 need to be polynomials of degree 3, so they will have the form

$$s_i(x) = c_{i,0} + c_{i,1}(x - x_i) + c_{i,2}(x - x_i)^2 + c_{i,3}(x - x_i)^3,\ x \in [x_i, x_{i+1}],\ i = 1, 2.$$

The coefficients $c_{i,0}$, $c_{i,1}$, $c_{i,2}$, $c_{i,3}$ are given from (see Lect. 6, eq. (2.9))

$$\begin{aligned} c_{i,0} &= f_i \\ c_{i,1} &= m_i \\ c_{i,2} &= \frac{3f[x_i, x_{i+1}] - 2m_i - m_{i+1}}{h_i} \\ c_{i,3} &= \frac{m_{i+1} - 2f[x_i, x_{i+1}] + m_i}{h_i^2} \end{aligned}$$

where

$$f[x_i, x_{i+1}] = \frac{f_{i+1} - f_i}{x_{i+1} - x_i}$$

The unknowns here are m_i, which can be found from the following system (see Lect. 6, eq. (2.13), (2.14))

$$h_i m_{i-1} + 2(h_{i-1} + h_i)m_i + h_{i-1}m_{i+1} = 3(h_i f[x_{i-1}, x_i] + h_{i-1} f[x_i, x_{i+1}]),\ i = 2, \ldots, n-1. \tag{1.1}$$

Some information about them is known from the type of the spline (natural, clamped, de Boor, etc.) (see Lect. 6, pp. 7–8).

In our case, since we have to construct a natural spline, we have

$$\begin{aligned} 2m_1 + m_2 &= 3f[x_1, x_2] \\ m_{n-1} + 2m_n &= 3f[x_{n-1}, x_n] \end{aligned} \tag{1.2}$$

Remark 1.2 m_i *are computed only once and then used to compute the coefficients of the spline on each subinterval.*

The conditions (1.2) become in our case

$$\begin{aligned} 2m_1 + m_2 &= 3\frac{f_2 - f_1}{x_2 - x_1} = 3 \cdot 0 = 0 \\ m_2 + 2m_3 &= 3\frac{f_3 - f_2}{x_3 - x_2} = 3 \cdot (-1) = -3 \end{aligned}$$

From equation (1.1), since $n = 3$, we only have to write the eq. for $i = 2$, so:

$$h_2 m_1 + 2(h_1 + h_2)m_2 + h_1 m_3 = 3(h_2 \frac{f_2 - f_1}{x_2 - x_1} + h_1 \frac{f_3 - f_2}{x_3 - x_2}) \iff$$

$$m_1 + 4m_2 + m_3 = -3$$

We obtain the system

$$\begin{cases} 2m_1 + m_2 & = 0 \\ m_1 + 4m_2 + m_3 & = -3 \\ m_2 + 2m_3 & = -3 \end{cases}$$

with solutions

$$m_1 = \frac{1}{4}, \quad m_2 = -\frac{1}{2}, \quad m_3 = -\frac{5}{4}.$$

We now construct the coefficients $c_{i,j}$ for s_1 and s_2.

- $s_1(x)$:

$$\begin{aligned} c_{1,0} &= f_1 = 1 \\ c_{1,1} &= m_1 = \frac{1}{4} \\ c_{1,2} &= \frac{3f[x_1,x_2] - 2m_1 - m_2}{h_1} = 0 \\ c_{1,3} &= \frac{m_2 - 2f[x_1,x_2] + m_1}{h_1^2} = -\frac{1}{4} \end{aligned}$$

$$\implies s_1(x) = 1 + \frac{1}{4}(x - x_1) + 0(x - x_1)^2 - \frac{1}{4}(x - x_1)^3 = 1 + \frac{1}{4}(x-1) - \frac{1}{4}(x-1)^3, \;\; x \in [1,2]$$

- $s_2(x)$:

$$\begin{aligned} c_{2,0} &= f_2 = 1 \\ c_{2,1} &= m_2 = -\frac{1}{2} \\ c_{2,2} &= \frac{3f[x_2,x_3] - 2m_2 - m_3}{h_2} = -\frac{3}{4} \\ c_{2,3} &= \frac{m_3 - 2f[x_2,x_3] + m_2}{h_2^2} = \frac{1}{4} \end{aligned}$$

$$\implies s_2(x) = 1 - \frac{1}{2}(x - x_2) - \frac{3}{4}(x - x_2)^2 + \frac{1}{4}(x - x_2)^3 = 1 - \frac{1}{2}(x-2) - \frac{3}{4}(x-2)^2 + \frac{1}{4}(x-2)^3, \;\; x \in [2,3]$$

In conclusion, $s(x) = \begin{cases} 1 + \frac{1}{4}(x-1) - \frac{1}{4}(x-1)^3, & x \in [1,2] \\ 1 - \frac{1}{2}(x-2) - \frac{3}{4}(x-2)^2 + \frac{1}{4}(x-2)^3, & x \in [2,3] \end{cases}$

Another method to construct the splines is by imposing the continuity of the spline, of the first derivative and of the second derivative on the nodes, together with the condition of a natural spline ($s''(x_1) = 0$ and $s''(x_n) = 0$) and the condition of interpolating the function on the nodes. We obtain

$$\begin{aligned} & s_1(x_2) = s_2(x_2) \;\; (\text{ continuity at the nodes}) \\ & s_1'(x_2) = s_2'(x_2) \;\; (\text{ continuity of first der. at the nodes}) \\ & s_1''(x_2) = s_2''(x_2) \;\; (\text{ continuity of second der. at the nodes}) \\ & s_1(x_1) = f_1 \\ & s_1(x_2) = f_2 \\ & s_2(x_2) = f_2 \\ & s_2(x_3) = f_3 \;\; (\text{ interpolation conditions}) \\ & s_1''(x_1) = 0 \\ & s_2''(x_3) = 0 \;\; (\text{natural spline conditions}) \end{aligned}$$

If we write again s_1 and s_2 as cubic polynomials:

$$s_1(x) = a_1 + b_1(x - x_1) + c_1(x - x_1)^2 + d_1(x - x_1)^3 = a_1 + b_1(x - 1) + c_1(x - 1)^2 + d_1(x - 1)^3$$
$$s_2(x) = a_2 + b_2(x - x_2) + c_2(x - x_2)^2 + d_2(x - x_2)^3 = a_2 + b_2(x - 2) + c_2(x - 2)^2 + d_2(x - 2)^3$$

and compute their first and second derivatives

$$\begin{aligned} s_1'(x) &= b_1 + 2c_1(x-1) + 3d_1(x-1)^2 \\ s_2'(x) &= b_2 + 2c_2(x-2) + 3d_2(x-2)^2 \\ s_1''(x) &= 2c_1 + 6d_1(x-1) \\ s_2''(x) &= 2c_2 + 6d_2(x-2) \end{aligned}$$

we obtain the following system from the above conditions:

$$\begin{cases} a_1 + b_1 + c_1 + d_1 & = a_2 \\ b_1 + 2c_1 + 3d_1 & = b_2 \\ 2c_1 + 6d_1 & = 2c_2 \\ a_1 & = 1 \\ a_1 + b_1 + c_1 + d_1 & = 1 \\ a_2 & = 1 \\ a_2 + b_2 + c_2 + d_2 & = 0 \\ 2c_1 & = 0 \\ 2c_2 + 6d_2 & = 0 \end{cases} \iff \begin{cases} a_1 & = 1 \\ c_1 & = 0 \\ a_2 & = 1 \\ b_1 + d_1 & = 0 \\ b_1 + 3d_1 - b_2 & = 0 \\ 3d_1 - c_2 & = 0 \\ b_2 + c_2 + d_2 & = -1 \\ c_2 + 3d_2 & = 0 \end{cases}$$

which has the solution

$$a_1 = 1,\ b_1 = \frac{1}{4},\ c_1 = 0,\ d_1 = -\frac{1}{4}$$
$$a_2 = 1,\ b_2 = -\frac{1}{2},\ c_2 = -\frac{3}{4},\ d_2 = \frac{1}{4}$$

as we previously obtained.

Example 1.3 *Construct a complete cubic spline s that passes through the points $(1,2)$, $(2,3)$, $(3,5)$ and has $s'(1) = 2,\ s'(3) = 1$.*

$$x_1 = 1,\ x_2 = 2,\ x_3 = 3,\ f_1 = 2,\ f_2 = 3,\ f_3 = 5,\ n = 3,\ h_1 = 1,\ h_2 = 1.$$

The information we know in this case (complete/clamped spline) for m_i is $m_1 = f'(x_1)$ and $m_n = f'(x_n)$. So we have

$$m_1 = 2,\ m_3 = 1.$$

Writing again eq. (1.1), for $i = 2$, we get:

$$h_2 m_1 + 2(h_1 + h_2)m_2 + h_1 m_3 = 3(h_2 \frac{f_2 - f_1}{x_2 - x_1} + h_1 \frac{f_3 - f_2}{x_3 - x_2}) \iff$$
$$2 + 4m_2 + 1 = 3(1 + 2) \implies m_2 = \frac{3}{2}.$$

We now construct the coefficients $c_{i,j}$ for s_1 and s_2.

- $s_1(x)$:

$$\begin{aligned} c_{1,0} &= f_1 = 2 \\ c_{1,1} &= m_1 = 2 \\ c_{1,2} &= \frac{3f[x_1,x_2] - 2m_1 - m_2}{h_1} = -\frac{5}{2} \\ c_{1,3} &= \frac{m_2 - 2f[x_1,x_2] + m_1}{h_1^2} = \frac{3}{2} \end{aligned}$$

$$\implies s_1(x) = 2 + 2(x - x_1) - \frac{5}{2}(x - x_1)^2 + \frac{3}{2}(x - x_1)^3 = 2 + 2(x-1) - \frac{5}{2}(x-1)^2 + \frac{3}{2}(x-1)^3,\ \ x \in [1,2]$$

- $s_2(x)$:

$$\begin{aligned}
c_{2,0} &= f_2 = 3 \\
c_{2,1} &= m_2 = \frac{3}{2} \\
c_{2,2} &= \frac{3f[x_2,x_3] - 2m_2 - m_3}{h_2} = 2 \\
c_{2,3} &= \frac{m_3 - 2f[x_2,x_3] + m_2}{h_2^2} = -\frac{3}{2}
\end{aligned}$$

$$\implies s_2(x) = 3 + \frac{3}{2}(x-x_2) + 2(x-x_2)^2 - \frac{3}{2}(x-x_2)^3 = 3 + \frac{3}{2}(x-2) + 2(x-2)^2 - \frac{3}{2}(x-2)^3, \;\; x \in [2,3]$$

In conclusion, $s(x) = \begin{cases} 2 + 2(x-1) - \frac{5}{2}(x-1)^2 + \frac{3}{2}(x-1)^3, & x \in [1,2] \\ 3 + \frac{3}{2}(x-2) + 2(x-2)^2 - \frac{3}{2}(x-2)^3, & x \in [2,3] \end{cases}$

Remark 1.4 *We can use here method 2 as in the previous example, but know we will not have the last two conditions (for the natural spline). Instead we will have the conditions for the clamped spline:* $s'(x_1) = s_1'(x_1) = 2$ *and* $s'(x_3) = s_2'(x_3) = 1$.

Remark 1.5 *Another method is presented in Lect. 6, pp. 9–10 (Finding cubic splines using the second derivatives).*

Example 1.6 *A clamped cubic spline s for a function f is defined as*

$$s(x) = \begin{cases} s_1(x) & = 1 + ax + 2x^2 - 2x^3, \; x \in [0,1] \\ s_2(x) & = 1 + b(x-1) - 4(x-1)^2 + 7(x-1)^3, \; x \in [1,2] \end{cases}$$

Compute $f'(0)$ and $f'(2)$.

We need to determine first a and b. We don't know the values of the function on the nodes 0, 1, 2 so the interpolation condition won't help us. We will use the conditions for continuity of the spline, its first der. and second der. at the nodes. Let us first compute s', s''.

$$s'(x) = \begin{cases} s_1'(x) & = a + 4x - 6x^2, \; x \in [0,1] \\ s_2'(x) & = b - 8(x-1) + 21(x-1)^2, \; x \in [1,2] \end{cases}$$

$$s''(x) = \begin{cases} s_1''(x) & = 4 - 12x, \; x \in [0,1] \\ s_2''(x) & = -8 + 42(x-1), \; x \in [1,2] \end{cases}$$

We obtain

$$\begin{cases} s_1(1) = s_2(1) \\ s_1'(1) = s_2'(1) \\ s_1''(1) = s_2''(1) \end{cases}$$

which is equivalent to

$$\begin{cases} 1 + a = 1 \implies a = 0 \\ a - 2 = b \implies b = -2 \\ -8 = -8, \; ''True'' \end{cases}$$

We get:

$$\begin{aligned}
f'(0) &= s'(0) = s_1'(0) = 0 \\
f'(2) &= s'(2) = s_2'(2) = 11
\end{aligned}$$

Example 1.7 • For theory, see Lecture 6, pp. 14–16. For another example, see pp. 17.

Fit the data from the table with

x	-1	0	1
y	1	1	2

a) the best least squares line

The best least squares line is obtained by considering the linear polynomial (degree = 1) $P(x) = ax + b$, with the unknowns a and b. We have to minimize the error

$$E(a,b) = \sum_{i=1}^{n} (y_i - P(x_i))^2$$

where $(y_i - P(x_i))$ is called residual. This minimization occurs when

$$\frac{\partial E}{\partial a} = 0 \text{ and } \frac{\partial E}{\partial b} = 0.$$

Remark 1.8 **If** $P(x) = a_0 + a_1 x + \ldots + a_m x^m$**, then each** $\frac{\partial E}{\partial a_i} = 0,\ i = 0, \ldots, m$**.**

In the general case, for n points and linear polynomial, we have

$$E(a,b) = \sum_{i=1}^{n} (y_i - ax_i - b)^2.$$

We obtain

$$\frac{\partial E}{\partial a} = 2\sum_{i=1}^{n} (y_i - ax_i - b)\cdot(-x_i) = -2\sum_{i=1}^{n} (x_i y_i - ax_i^2 - bx_i)$$
$$\frac{\partial E}{\partial b} = 2\sum_{i=1}^{n} (y_i - ax_i - b)\cdot(-1) = -2\sum_{i=1}^{n} (y_i - ax_i - b)$$

$$\frac{\partial E}{\partial a} = 0 \Longrightarrow = -2\sum_{i=1}^{n} (x_i y_i - ax_i^2 - bx_i) = 0 \xRightarrow{:(-2)} \sum_{i=1}^{n} ax_i^2 + \sum_{i=1}^{n} bx_i = \sum_{i=1}^{n} x_i y_i$$
$$\frac{\partial E}{\partial b} = 0 \Longrightarrow = -2\sum_{i=1}^{n} (y_i - ax_i - b) = 0 \xRightarrow{:(-2)} \sum_{i=1}^{n} ax_i + \sum_{i=1}^{n} b = \sum_{i=1}^{n} y_i$$

and finally

$$\begin{cases} a\sum_{i=1}^{n} x_i^2 + b\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} x_i y_i \\ a\sum_{i=1}^{n} x_i + bn = \sum_{i=1}^{n} y_i \end{cases} \tag{1.3}$$

In our case we have

	x_i	y_i	x_i^2	$x_i y_i$
	-1	1	1	-1
	0	1	0	0
	1	2	1	2
Σ	0	4	2	1

so the system becomes

$$\begin{cases} 2\cdot a + 0\cdot b = 1 \\ 0\cdot a + 3\cdot b = 4 \end{cases} \Longrightarrow \begin{cases} a = \frac{1}{2} \\ b = \frac{4}{3} \end{cases}$$

so $P(x) = \frac{1}{2}x + \frac{4}{3}$.

b) the best least squares polynomial of degree 2

Our polynomial will have the form $P(x) = ax^2 + bx + c$, with the unknowns a, b, c that will be obtain in a similar way as in the linear case. We have to minimize the error

$$E(a,b,c) = \sum_{i=1}^{n} (y_i - P(x_i))^2$$

which occurs when

$$\frac{\partial E}{\partial a} = 0, \ \frac{\partial E}{\partial b} = 0 \text{ and } \frac{\partial E}{\partial c} = 0.$$

In the general case, for n points and quadratic polynomial, we have

$$E(a,b,c) = \sum_{i=1}^{n} (y_i - ax_i^2 - bx_i - c)^2.$$

We obtain

$$\begin{aligned}
\frac{\partial E}{\partial a} &= 2\sum_{i=1}^{n} (y_i - ax_i^2 - bx_i - c) \cdot (-x_i^2) = -2\sum_{i=1}^{n} (x_i^2 y_i - ax_i^4 - bx_i^3 - cx_i^2) \\
\frac{\partial E}{\partial b} &= 2\sum_{i=1}^{n} (y_i - ax_i^2 - bx_i - c) \cdot (-x_i) = -2\sum_{i=1}^{n} (x_i y_i - ax_i^3 - bx_i^2 - cx_i) \\
\frac{\partial E}{\partial c} &= 2\sum_{i=1}^{n} (y_i - ax_i^2 - bx_i - c) \cdot (-1) = -2\sum_{i=1}^{n} (y_i - ax_i^2 - bx_i - c)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial E}{\partial a} = 0 &\implies -2\sum_{i=1}^{n} (x_i^2 y_i - ax_i^4 - bx_i^3 - cx_i^2) = 0 \xRightarrow{:(-2)} \sum_{i=1}^{n} ax_i^4 + \sum_{i=1}^{n} bx_i^3 + \sum_{i=1}^{n} cx_i^2 = \sum_{i=1}^{n} x_i^2 y_i \\
\frac{\partial E}{\partial b} = 0 &\implies = -2\sum_{i=1}^{n} (x_i y_i - ax_i^3 - bx_i^2 - cx_i) = 0 \xRightarrow{:(-2)} \sum_{i=1}^{n} ax_i^3 + \sum_{i=1}^{n} bx_i^2 + \sum_{i=1}^{n} cx_i = \sum_{i=1}^{n} x_i y_i \\
\frac{\partial E}{\partial c} = 0 &\implies = -2\sum_{i=1}^{n} (y_i - ax_i^2 - bx_i - c) = 0 \xRightarrow{:(-2)} \sum_{i=1}^{n} ax_i^2 + \sum_{i=1}^{n} bx_i + \sum_{i=1}^{n} c = \sum_{i=1}^{n} y_i
\end{aligned}$$

and finally

$$\begin{cases}
a\sum_{i=1}^{n} x_i^4 + b\sum_{i=1}^{n} x_i^3 + c\sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n} x_i^2 y_i \\
a\sum_{i=1}^{n} x_i^3 + b\sum_{i=1}^{n} x_i^2 + c\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} x_i y_i \\
a\sum_{i=1}^{n} x_i^2 + b\sum_{i=1}^{n} x_i + cn = \sum_{i=1}^{n} y_i
\end{cases} \tag{1.4}$$

In our case, we have

	x_i	y_i	x_i^2	$x_i y_i$	x_i^3	x_i^4	$x_i^2 y_i$
	-1	1	1	-1	-1	1	1
	0	1	0	0	0	0	0
	1	2	1	2	1	1	2
Σ	0	4	2	1	0	2	3

so the system becomes

$$\begin{cases}
2 \cdot a + 0 \cdot b + 3 \cdot c = 3 \\
0 \cdot a + 2 \cdot b + 0 \cdot c = 1 \\
2 \cdot a + 0 \cdot b + 3 \cdot c = 4
\end{cases} \implies \begin{cases}
c = 1 \\
b = \dfrac{1}{2} \\
a = \dfrac{1}{2}
\end{cases}$$

so $P(x) = \frac{1}{2}x^2 + \frac{1}{2}x + 1$.

Example 1.9 *Fit the data from the table with*

x	-2	-1	1	2	3
y	0	-3	2	2	5

a) a linear least squares polynomial

As in the previous example, we have $P(x) = ax + b$, with the unknowns a and b. They can be found from

$$\frac{\partial E}{\partial a} = 0, \quad \frac{\partial E}{\partial b} = 0$$

We will use the system (1.3), and for this we first compute

	x_i	y_i	x_i^2	$x_i y_i$
	-2	0	4	0
	-1	-3	1	3
	1	2	1	2
	2	2	4	4
	3	5	9	15
$\sum$	3	6	19	24

so the system becomes

$$\begin{cases} 19 \cdot a + 3 \cdot b = 24 \\ 3 \cdot a + 5 \cdot b = 6 \end{cases} \Longrightarrow \begin{cases} a = \dfrac{51}{43} \\ b = \dfrac{21}{43} \end{cases}$$

so $P(x) = \frac{51}{43}x + \frac{21}{43}$.

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b) the least squares polynomial of degree 2

$P(x) = ax^2 + bx + c$, with the unknowns a, b, c They can be found from

$$\frac{\partial E}{\partial a} = 0, \quad \frac{\partial E}{\partial b} = 0, \quad \frac{\partial E}{\partial c} = 0.$$

We will use the system (1.4), and for this we first compute

	x_i	y_i	x_i^2	$x_i y_i$	x_i^3	x_i^4	$x_i^2 y_i$
	-2	0	4	0	-8	16	0
	-1	-3	1	3	-1	1	-3
	1	2	1	2	1	1	2
	2	2	4	4	8	16	8
	3	5	9	15	27	81	45
$\sum$	3	6	19	24	27	115	52

so the system becomes

$$\begin{cases} 115 \cdot a + 27 \cdot b + 19 \cdot c = 52 \\ 27 \cdot a + 19 \cdot b + 3 \cdot c = 24 \\ 19 \cdot a + 3 \cdot b + 5 \cdot c = 6 \end{cases} \Longrightarrow \begin{cases} a = \dfrac{115}{308} \\ b = \dfrac{261}{308} \\ c = -\dfrac{8}{11} \end{cases}$$

so $P(x) = \frac{115}{308}x^2 + \frac{261}{308}x - \frac{8}{11}$.