## 1.3 Hermite Interpolation

Consider the following situation: For a moving object, we know the distances traveled $d_0, d_1, \dots, d_m$, at times $t_0, t_1, \dots, t_m$, and we want a polynomial approximation of the distance function $d = d(t)$ on the entire interval containing the points $t_0, \dots, t_m$. Obviously, this is a Lagrange interpolation problem and we already know how to find the interpolation polynomial.

Now, assume that, in addition, we also know the values of the *velocities* $v_i$ of the object at times $t_i, i = \overline{0, m}$. We would expect that this additional information helps us find an *even better* approximation of the function $d$. However, from what we know about Lagrange interpolation, there is *no way* to include this data into our approximation. Since the velocity is the derivative with respect to time of the distance traveled, this means that we also have information about the *derivatives* of the function we want to interpolate. This is a **Hermite interpolation** problem. The ideas and computational formulas are similar to the ones we used to determine the Lagrange interpolation polynomial.

### 1.3.1 Interpolation with double nodes

For a variety of applications, as the one described above, it is convenient to consider polynomials $P(x)$ that interpolate a function $f(x)$ and in addition have the derivative polynomial $P'(x)$ also interpolate the derivative function $f'(x)$.

**Hermite interpolation problem with double nodes.** Given $m + 1$ distinct nodes $x_i, i = \overline{0, m}$ and the values $f(x_i), f'(x_i)$ of an unknown function $f$ and its derivative, find a polynomial $P(x)$ of minimum degree, satisfying the interpolation conditions

$$\begin{aligned} P(x_i) &= f(x_i), \\ P'(x_i) &= f'(x_i),\ i = \overline{0, m}. \end{aligned} \tag{1.1}$$

Since for each node there are two values (of the function and of its derivative) given, we call them *double* nodes.

There are $2m + 2$ conditions in (1.1), so we seek a polynomial of degree (at most) $n = 2m + 1$. We determine this polynomial in a similar way to the construction of the Lagrange polynomial. Recall the notations:

$$\begin{aligned} \psi_m(x) &= (x - x_0) \dots (x - x_{m-1})(x - x_m), \\ l_i(x) &= \frac{(x - x_0) \dots (x - x_{i-1})(x - x_{i+1}) \dots (x - x_m)}{(x_i - x_0) \dots (x_i - x_{i-1})(x_i - x_{i+1}) \dots (x_i - x_m)} = \frac{\psi_m(x)}{(x - x_i)\psi'_m(x_i)}, \end{aligned} \tag{1.2}$$

for $i = 0, 1, \ldots, m$.

**Theorem 1.1.** *There is a unique polynomial $H_n f$ of degree at most $n$, satisfying the interpolation conditions* (1.1)*. This polynomial can be written as*

$$H_n f(x) = \sum_{i=0}^{m} \Big[ h_{i0}(x) f(x_i) + h_{i1}(x) f'(x_i) \Big], \tag{1.3}$$

*where*

$$\begin{aligned} h_{i0}(x) &= \big[1 - 2l_i'(x_i)(x - x_i)\big] \big[l_i(x)\big]^2, \\ h_{i1}(x) &= (x - x_i)\big[l_i(x)\big]^2,\ i = 0, \ldots, m. \end{aligned} \tag{1.4}$$

*$H_n f$ is called the **Hermite interpolation polynomial** of $f$ at the double nodes $x_0, x_1, \ldots, x_m$. The functions $h_{i0}(x), h_{i1}(x),\ i = \overline{0, m}$ are called **Hermite fundamental polynomials** associated with these points.*

*Proof.* The degree of polynomials $l_i$ from (1.2) is $m$, so the degree of $h_{i0}, h_{i1}$ and $H_n f$ is $2m+1 = n$. The derivatives of the Hermite fundamental polynomials are

$$\begin{aligned} h_{i0}'(x) &= -2l_i'(x_i)\big(l_i(x)\big)^2 + 2\big(1 - 2l_i'(x_i)(x - x_i)\big) l_i'(x) l_i(x), \\ h_{i1}'(x) &= \big(l_i(x)\big)^2 + 2(x - x_i) l_i'(x) l_i(x). \end{aligned}$$

Notice that $l_i(x),\ i = \overline{0, m}$ are the Lagrange fundamental polynomials, thus,

$$l_i(x_j) = \delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}.$$

Then,

$$\begin{aligned} h_{i0}(x_j) &= 0,\ j \neq i, \\ h_{i0}(x_i) &= 1 \cdot \big(l_i(x_i)\big)^2 = 1, \\ h_{i1}(x_j) &= 0,\ j \neq i, \\ h_{i1}(x_i) &= 0. \end{aligned}$$

The values of the derivatives at the nodes are

$$\begin{aligned} h'_{i0}(x_j) &= 0,\ j \neq i, \\ h'_{i0}(x_i) &= -2l'_i(x_i) + 2l'_i(x_i)l_i(x_i) \ = \ 0, \\ h'_{i1}(x_j) &= 0,\ j \neq i, \\ h'_{i1}(x_i) &= 1 + 0 \ = \ 1. \end{aligned}$$

It follows that

$$\begin{aligned} (H_n f)(x_i) &= f(x_i), \\ (H_n f)'(x_i) &= f'(x_i),\ i = \overline{0, m}, \end{aligned}$$

hence, the polynomial $H_n f$ given in (1.3) satisfies the interpolation conditions (1.1).

To prove uniqueness, assume there exists another polynomial $G_n$ (of degree at most $n = 2m+1$) satisfying relations (1.1) and consider

$$Q_n \ = \ H_n - G_n.$$

From the interpolation conditions, it follows that

$$Q_n(x_i) \ = \ Q'_n(x_i) \ = \ 0,\ i = 0, \ldots, m.$$

So, $Q_n$, a polynomial of degree at most $2m + 1$, has $m + 1$ *double* roots. By the Fundamental Theorem of Algebra, $Q_n$ must be identically zero, thus proving the uniqueness of $H_n$.

$\square$

**Example 1.2.** One of the most widely used form of Hermite interpolation is the cubic Hermite polynomial, which solves the interpolation problem with two double nodes $a < b$,

$$\begin{aligned} P(a) &= f(a),\ P(b) \ = \ f(b), \\ P'(a) &= f'(a),\ P'(b) \ = \ f'(b). \end{aligned} \tag{1.5}$$

**Solution.** Letting $x_0 = a,\ x_1 = b$, with our previous notations and formulas, we have

$$\begin{aligned}
\psi_1(x) &= (x-a)(x-b), \\
l_0(x) &= \frac{x-b}{a-b},\ \ l_0'(x) \ = \ \frac{1}{a-b}, \\
l_1(x) &= \frac{x-a}{b-a},\ \ l_1'(x) \ = \ \frac{1}{b-a}.
\end{aligned}$$

The Hermite fundamental polynomials are given by

$$\begin{aligned}
h_{00}(x) &= \big(1 - 2l_0'(a)(x-a)\big)\big(l_0(x)\big)^2 \ = \ \left[1 + 2\frac{x-a}{b-a}\right]\left[\frac{b-x}{b-a}\right]^2, \\
h_{10}(x) &= \big(1 - 2l_1'(b)(x-b)\big)\big(l_1(x)\big)^2 \ = \ \left[1 + 2\frac{b-x}{b-a}\right]\left[\frac{x-a}{b-a}\right]^2, \\
h_{01}(x) &= (x-a)\big(l_0(x)\big)^2 \ = \ \frac{(x-a)(b-x)^2}{(b-a)^2}, \\
h_{11}(x) &= (x-b)\big(l_1(x)\big)^2 \ = \ -\frac{(x-a)^2(b-x)}{(b-a)^2}.
\end{aligned}$$

So the cubic Hermite polynomial is

$$\begin{aligned}
H_3f(x) &= \left[1 + 2\frac{x-a}{b-a}\right]\left[\frac{b-x}{b-a}\right]^2 \cdot f(a) \ + \ \left[1 + 2\frac{b-x}{b-a}\right]\left[\frac{x-a}{b-a}\right]^2 \cdot f(b) \\
&+ \ \frac{(x-a)(b-x)^2}{(b-a)^2} \cdot f'(a) \ - \ \frac{(x-a)^2(b-x)}{(b-a)^2} \cdot f'(b).
\end{aligned}$$

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### 1.3.2 Newton's divided difference form

Just as in the case of Lagrange interpolation, Newton's divided differences provide a more easily computable form of the Hermite interpolation polynomial.

Consider $2m+2$ distinct nodes $z_0, z_1, \ldots, z_{2m}, z_{2m+1}$ and the Newton polynomial interpolating a function $f$ at these nodes.

$$N_{2m+1}(x) \ = \ f(z_0) + f[z_0, z_1](x - z_0) + \cdots + f[z_0, \ldots, z_{2m+1}](x - z_0) \ldots (x - z_{2m}),$$

with the error given by

$$R_{2m+1}(x) \;=\; f(x) - N_{2m+1}(x) \;=\; f[x, z_0, \ldots, z_{2m+1}](x - z_0) \ldots (x - z_{2m+1}).$$

We take the limits in the two relations above

$$z_0, z_1 \to x_0, \quad z_2, z_3 \to x_1, \quad \ldots, \quad z_{2i}, z_{2i+1} \to x_i, \quad \ldots \quad z_{2m}, z_{2m+1} \to x_m.$$

Denoting by $n = 2m + 1$, we get

$$\begin{aligned} N_n(x) &= f(x_0) + f[x_0, x_0](x - x_0) + f[x_0, x_0, x_1](x - x_0)^2 \\ &+ f[x_0, x_0, x_1, x_1](x - x_0)^2(x - x_1) + \ldots \\ &+ f[x_0, x_0, \ldots, x_m, x_m](x - x_0)^2 \ldots (x - x_{m-1})^2(x - x_m) \end{aligned} \tag{1.6}$$

and for the remainder,

$$f(x) - N_n(x) \;=\; f[x, x_0, x_0, \ldots, x_m, x_m](x - x_0)^2 \ldots (x - x_m)^2. \tag{1.7}$$

**Proposition 1.3.** *Let $[a, b] \subset \mathbb{R}$ be the smallest interval containing the distinct nodes $x_0, \ldots, x_m$ and $f : [a, b] \to \mathbb{R}$ be a function of class $C^{2m+2}[a, b]$. Then, for the two polynomials in* (1.3) *and* (1.6)*, we have*

$$H_n f(x) \;=\; N_n(x), \forall x \in [a, b], \tag{1.8}$$

*with the interpolation error*

$$R_n(x) \;=\; f(x) - H_n f(x) \;=\; \big[\psi_m(x)\big]^2 \frac{f^{(n+1)}(\xi_x)}{(n+1)!}, \; \xi_x \in (a, b). \tag{1.9}$$

*Proof.* By the way it was constructed (in (1.6)), obviously the polynomial $N_n$ has degree at most $n$. Then, by the uniqueness of the Hermite interpolation polynomial, it suffices to show that $N_n$ satisfies the interpolation conditions (1.1).
From (1.7), it follows that

$$f(x_i) - N_n(x_i) \;=\; 0, \; i = 0, \ldots, m.$$

Also, by the same relation, we have for the derivatives,

$$\begin{aligned} f'(x) - N_n'(x) &= (x-x_0)^2 \dots (x-x_m)^2 \frac{\partial}{\partial x} f[x, x_0, x_0, \dots, x_m, x_m] \\ &+ 2f[x, x_0, x_0, \dots, x_m, x_m] \sum_{i=0}^{m} \left[ (x-x_i) \prod_{\substack{j=0 \\ j \neq i}}^{m} (x-x_j)^2 \right], \end{aligned}$$

hence,

$$f'(x_i) - N_n'(x_i) \;=\; 0,\; i = 0, \dots, m.$$

Thus,

$$H_n f(x) \;=\; N_n(x), \forall x \in [a, b]$$

and the error formula (1.9) follows directly from (1.7) and the mean-value formula for divided differences.

$\square$

**Example 1.4.** Let us find the polynomial and the remainder for the Hermite interpolation problem with two double nodes $a < b$, from Example 1.2.

**Solution.** We have

$$\begin{aligned} H_3 f(x) &= f(a) + f[a, a](x-a) + f[a, a, b](x-a)^2 \\ &+ f[a, a, b, b](x-a)^2(x-b). \end{aligned}$$

The divided difference table for two double nodes is

$$\begin{array}{llclclcl}
z_0 = a & f(a) & \longrightarrow & f[a, a] = f'(a) & \longrightarrow & f[a, a, b] & \longrightarrow & f[a, a, b, b] \\
 & & \nearrow & & \nearrow & & \nearrow & \\
z_1 = a & f(a) & \longrightarrow & f[a, b] = \dfrac{f(b) - f(a)}{b - a} & \longrightarrow & f[a, b, b] & & \\
 & & \nearrow & & \nearrow & & & \\
z_2 = b & f(b) & \longrightarrow & f[b, b] = f'(b) & & & & \\
 & & \nearrow & & & & & \\
z_3 = b & f(b), & & & & & &
\end{array}$$

where

$$
\begin{aligned}
f[a,a,b] &= \frac{f[a,b]-f'(a)}{b-a}, \\
f[a,b,b] &= \frac{f'(b)-f[a,b]}{b-a}, \\
f[a,a,b,b] &= \frac{f[a,b,b]-f[a,a,b]}{b-a} = \frac{f'(b)-2f[a,b]+f'(a)}{(b-a)^2}.
\end{aligned}
$$

The interpolation error is given by

$$
\begin{aligned}
f(x)-H_3f(x) &= (x-a)^2(x-b)^2 f[x,a,a,b,b] \\
&= \frac{(x-a)^2(x-b)^2}{24} f^{(4)}(\xi_x),
\end{aligned}
$$

with $\xi_x$ belonging to the smallest interval that contains the points $a, b$ and $x$.
We can find a bound for the error. Considering that on $[a,b]$, the maximum of the function $|(x-a)(x-b)|$ occurs at the midpoint of the interval, $\dfrac{a+b}{2}$, and that the maximum value is $\dfrac{(b-a)^2}{4}$, we have

$$
\max_{x\in[a,b]} \left|f(x)-H_3f(x)\right| \le \frac{(b-a)^4}{384} \max_{t\in[a,b]} \left|f^{(4)}(t)\right|.
$$

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**Example 1.5** (Continuation of Example 1.1 in Lecture 2)**.** Consider the function $f : [0.5, 5] \to \mathbb{R},\ f(x) = \sqrt{x}$ and the nodes $a = 1, b = 4$. Let us compare Lagrange and Hermite approximations.

**Solution.** For the *simple* nodes $a = 1, b = 4$, we have the interpolation conditions

$$
\begin{aligned}
L_1f(a) &= f(a) = 1, \\
L_1f(b) &= f(b) = 2,
\end{aligned}
$$

satified by the Lagrange polynomial of degree 1

$$
L_1f(x) = \frac{1}{3}x + \frac{2}{3}.
$$

If the nodes are *double*, the interpolation conditions are

$$
\begin{aligned}
H_3 f(a) &= f(a) = 1, \\
H_3 f(b) &= f(b) = 2, \\
H_3 f'(a) &= f'(a) = 1/(2\sqrt{1}) = 1/2, \\
H_3 f'(b) &= f'(b) = 1/(2\sqrt{4}) = 1/4.
\end{aligned}
$$

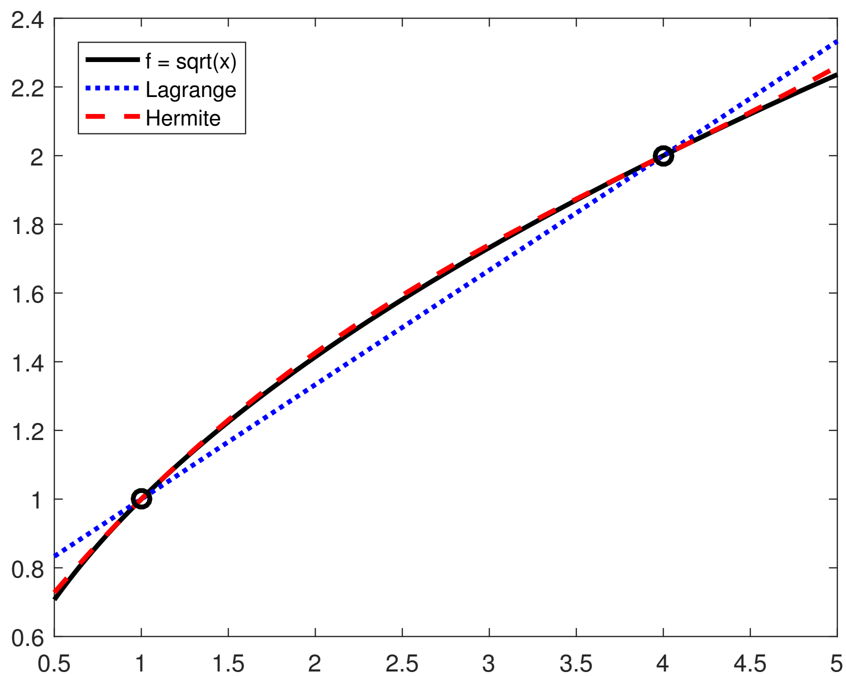


Fig. 1: Lagrange and Hermite interpolation with 2 nodes of the function $\sqrt{x}$

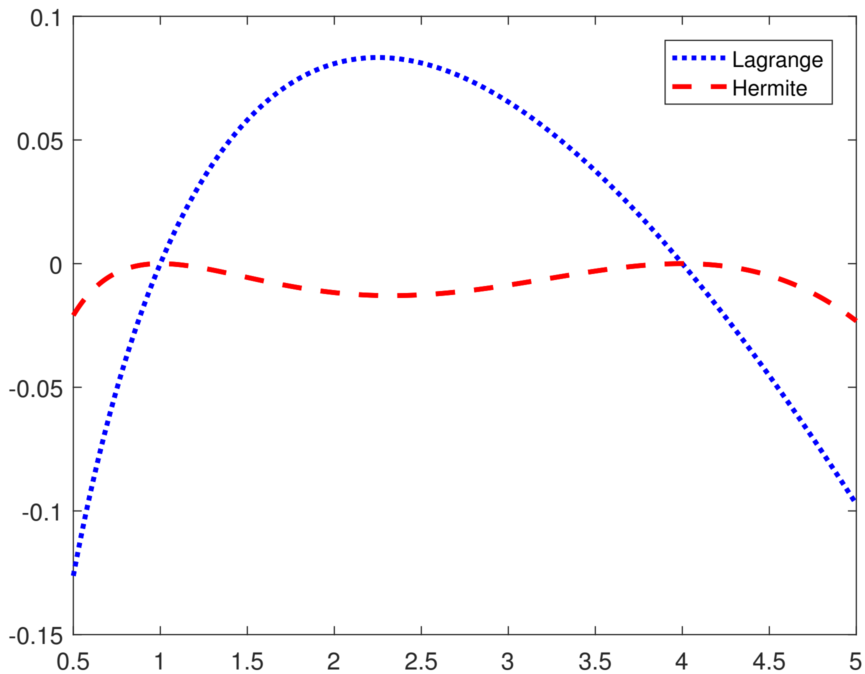


Fig. 2: Error of Lagrange and Hermite interpolation with 2 nodes of the function $\sqrt{x}$

The divided difference table is

$$
\begin{array}{llclclcl}
z_0 = 1 & f(1) = 1 & \longrightarrow & f'(1) = 1/2 & \longrightarrow & f[1,1,4] = -1/18 & \longrightarrow & f[1,1,4,4] = 1/108 \\
 & & \nearrow & & \nearrow & & \nearrow & \\
z_1 = 1 & f(1) = 1 & \longrightarrow & f[1,4] = 1/3 & \longrightarrow & f[1,4,4] = -1/36 & & \\
 & & \nearrow & & \nearrow & & & \\
z_2 = 4 & f(4) = 2 & \longrightarrow & f'(4) = 1/4 & & & & \\
 & & \nearrow & & & & & \\
z_3 = 4 & f(4) = 2, & & & & & &
\end{array}
$$

The corresponding cubic Hermite interpolation polynomial is given by

$$H_3 f(x) \;=\; 1 + \frac{1}{2}(x-1) - \frac{1}{18}(x-1)^2 + \frac{1}{108}(x-1)^2(x-4).$$

The graphs of $f$ and the two interpolation polynomials, $L_1, H_3$, on the interval $[0.5, 5]$, are shown in Figure 1. The interpolation errors are plotted in Figure 2.

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### 1.3.3 General case

**Hermite interpolation problem.** Given $m+1$ distinct nodes $x_i \in [a,b], i=\overline{0,m}$,

$$\begin{aligned} x_0 &\text{ of multiplicity } r_0+1,\\ x_1 &\text{ of multiplicity } r_1+1,\\ &\dots\\ x_i &\text{ of multiplicity } r_i+1,\\ &\dots\\ x_m &\text{ of multiplicity } r_m+1,\end{aligned}$$

and the values $f^{(j)}(x_i),\ i=0,1,\dots,m,\ j=0,\dots,r_i$, of an unknown function $f:[a,b]\to\mathbb{R}$ whose derivatives of order up to $r_i$ exist at $x_i, i=\overline{0,m}$, find a polynomial $P(x)$ of minimum degree, satisfying the interpolation conditions

$$P^{(j)}(x_i) = f^{(j)}(x_i),\ i=\overline{0,m},\ j=\overline{0,r_i}. \tag{1.10}$$

Above, there are

$$n+1 \overset{\text{not}}{=} \sum_{i=0}^{m}(r_i+1)$$

conditions, so the polynomial satisfying these relations will have degree at most $n$.

**Theorem 1.6.** *There is a unique polynomial $H_nf$ of degree at most $n$, satisfying the interpolation conditions* (1.10)*. This polynomial is called the **Hermite interpolation polynomial** of the function $f$, relative to the nodes $x_0,x_1,\dots,x_m$ and the integers $r_0,r_1,\dots,r_m$, and it can be written as*

$$H_nf(x) = \sum_{i=0}^{m}\sum_{j=0}^{r_i} h_{ij}(x)f^{(j)}(x_i). \tag{1.11}$$

**Remark 1.7.**
1. The functions $h_{ij}(x),\ i=\overline{0,m}, j=\overline{0,r_i}$, are called **Hermite fundamental polynomials**.
2. If we denote by

$$\begin{aligned} u(x) &= \prod_{i=0}^{m}(x-x_i)^{r_i+1},\\ u_i(x) &= \prod_{\substack{j=0\\ j\neq i}}^{m}(x-x_j)^{r_j+1} = \frac{u(x)}{(x-x_i)^{r_i+1}},\end{aligned} \tag{1.12}$$

then the fundamental polynomials $h_{ij}(x)$ din (1.11) can be written as

$$h_{ij}(x) \;=\; \frac{(x-x_i)^j}{j!}\Big[\sum_{k=0}^{r_i-j}\frac{(x-x_i)^k}{k!}\Big[\frac{1}{u_i(x)}\Big]^{(k)}_{x=x_i}\Big]u_i(x). \tag{1.13}$$

2. A more computable form can be found using Newton divided differences. Re-indexing the nodes according to their multiplicity,

$$\begin{aligned}
z_0 &= x_0,\ \ldots,\ z_{r_0} = x_0,\\
z_{r_0+1} &= x_1,\ \ldots,\ z_{(r_0+1)+r_1} = x_1,\\
z_{(r_0+1)+(r_1+1)} &= x_2,\ \ldots,\ z_{(r_0+1)+(r_1+1)+r_2} = x_2,\\
&\ldots\\
z_{n-r_m} &= x_m,\ \ldots,\ z_n = x_m,
\end{aligned}$$

the Hermite polynomial can be written in Newton's form as

$$\begin{aligned}
N_nf(x) &= f(z_0)+f[z_0,z_1](x-z_0)+\ldots\\
&+ f[z_0,\ldots,z_n](x-z_0)\ldots(x-z_{n-1}),
\end{aligned} \tag{1.14}$$

with interpolation error

$$\begin{aligned}
R_n(x) &= f(x)-N_n(x) = f[x,z_0,\ldots,z_n](x-z_0)\ldots(x-z_n)\\
&= \frac{u(x)}{(n+1)!}f^{(n+1)}(\xi_x),\ \xi_x\in(a,b).
\end{aligned} \tag{1.15}$$

**Special cases**

1. If all $r_i = 0, i = \overline{0,m}$, the nodes are simple and we have the Lagrange interpolation formula.

2. If we consider one single node, $x_0$, of multiplicity $n+1$, the Hermite interpolation formula is reduced to Taylor's formula.

$$\begin{aligned}
f(x) &= f(x_0)+\frac{x-x_0}{1!}f'(x_0)+\frac{(x-x_0)^2}{2!}f''(x_0)+\ldots\\
&+ \frac{(x-x_0)^n}{n!}f^{(n)}(x)+R_n(f),
\end{aligned} \tag{1.16}$$

with

$$R_n(x) = \frac{(x-x_0)^{n+1}}{(n+1)!} f^{(n+1)}(\xi_x). \tag{1.17}$$

**3.** Consider the simple node $x_0$ and the double node $x_1$.

We have

$$\begin{array}{l} \text{multiplicity of the node } x_0: \quad r_0 + 1 = 1, \\ \text{multiplicity of the node } x_1: \quad r_1 + 1 = 2, \end{array}$$

so $n + 1 = 3$ and the polynomial has degree $n = 2$. Thus, it is of the form

$$H_2 f(x) \;=\; ax^2 + bx + c.$$

Coefficients $a, b$ and $c$ are found from the interpolation conditions:

$$\begin{cases} H_2 f(x_0) &= f(x_0) \\ H_2 f(x_1) &= f(x_1) \\ H_2' f(x_1) &= f'(x_1) \end{cases},$$

i.e., from the linear system

$$\begin{cases} x_0^2 a + x_0 b + c = f(x_0) \\ x_1^2 a + x_1 b + c = f(x_1) \\ 2x_1 a + b = f'(x_1) \end{cases}. \tag{1.18}$$

The matrix of this system,

$$V \;=\; \begin{bmatrix} 1 & x_0 & x_0^2 \\ 1 & x_1 & x_1^2 \\ 0 & 1 & 2x_1 \end{bmatrix},$$

is called a *generalized Vandermonde matrix*. It is invertible and the elements of its inverse are the coefficients of $h_{00}, h_{10}$ and $h_{11}$ from the expression in (1.11) for the Hermite polynomial

$$H_2 f(x) \;=\; h_{00} f(x_0) + h_{10} f(x_1) + h_{11} f'(x_1).$$

The solution of the system (1.18) is given by

$$\begin{aligned} a &= \frac{f[x_1,x_1]-f[x_0,x_1]}{x_1-x_0} = f[x_0,x_1,x_1], \\ b &= f[x_0,x_1]-(x_1+x_0)f[x_0,x_1,x_1], \\ c &= f(x_0)-x_0f[x_0,x_1]+x_0x_1f[x_0,x_1,x_1], \end{aligned}$$

from which we get the Newton form of the Hermite polynomial

$$H_2f(x) = f(x_0)+(x-x_0)f[x_0,x_1]+(x-x_0)(x-x_1)f[x_0,x_1,x_1]. \tag{1.19}$$

The formula for the remainder follows from (1.15):

$$R_2(x) = \frac{f^{(3)}(\xi_x)}{3!}(x-x_0)(x-x_1)^2. \tag{1.20}$$

By symmetry, if the node $x_0$ is double and $x_1$ is simple, the corresponding Hermite polynomial and its error are given by

$$\begin{aligned} H_2(x) &= f(x_1)+(x-x_1)f[x_0,x_1]+(x-x_0)(x-x_1)f[x_0,x_0,x_1], \\ R_2(x) &= \frac{f^{(3)}(\xi_x)}{3!}(x-x_0)^2(x-x_1). \end{aligned}$$

**4.** Consider two nodes, $x_0=a$, of multiplicity $m+1$ and $x_1=b$, of multiplicity $n+1$.

The Hermite polynomial has degree

$$(m+1)+(n+1)-1 = m+n+1.$$

With the notations from Remark 1.7, we have

$$\begin{aligned} u(x) &= (x-a)^{m+1}(x-b)^{n+1}, \\ u_0(x) &= (x-b)^{n+1}, \\ u_1(x) &= (x-a)^{m+1}. \end{aligned}$$

the Hermite polynomial is of the form

$$H_{m+n+1}f(x) \;=\; \sum_{j=0}^{m} h_{0j}(x)f^{(j)}(a) + \sum_{i=0}^{n} h_{1i}(x)f^{(i)}(b) \tag{1.21}$$

and the fundamental polynomials are given by

$$\begin{aligned} h_{0j}(x) &= \frac{(x-a)^j}{j!}\left[\sum_{k=0}^{m-j}\frac{(x-a)^k}{k!}\left[\frac{1}{(x-b)^{n+1}}\right]^{(k)}_{x=a}\right](x-b)^{n+1}, \\ h_{1i}(x) &= \frac{(x-b)^i}{i!}\left[\sum_{k=0}^{n-i}\frac{(x-b)^k}{k!}\left[\frac{1}{(x-a)^{m+1}}\right]^{(k)}_{x=b}\right](x-a)^{m+1}. \end{aligned}$$

In Newton's form (1.14),

$$\begin{aligned} H_{m+n+1}f(x) &= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(m)}(a)}{m!}(x-a)^m \\ &+ f[\underbrace{a,\ldots,a}_{m+1}, b](x-a)^{m+1} + f[\underbrace{a,\ldots,a}_{m+1}, b, b](x-a)^{m+1}(x-b) \\ &+ \cdots + f[\underbrace{a,\ldots,a}_{m+1}, \underbrace{b,\ldots,b}_{n+1}](x-a)^{m+1}(x-b)^n, \end{aligned}$$

with remainder

$$\begin{aligned} R_{m+n+1} &= f[x, \underbrace{a,\ldots,a}_{m+1}, \underbrace{b,\ldots,b}_{n+1}](x-a)^{m+1}(x-b)^{n+1} \\ &= \frac{f^{(m+n+2)}(\xi_x)}{(m+n+2)!}(x-a)^{m+1}(x-b)^{n+1},\ \xi_x \in (a,b). \end{aligned}$$