1 Examples for Lectures 1 – 5

- These are some additional exercises to what we have done and explained at the laboratory.
- I did not include all the theory, in many parts I just wrote where you can find it in the lectures - please read it before solving these examples.
- These exercises focus on the algorithms you should use. I did not include many examples regarding the approximation errors, remainder, etc. You can find these in your lectures.
- You also have other similar exercises solved in the lectures - I indicated where you can find them.

Example 1.1 *Write the* ***Taylor polynomial of order*** n *around* $x_0 = 0$*, for the function* $f : (-1, \infty) \to \mathbb{R},\ f(x) = \ln(1 + x)$*. What is the* ***error of the Taylor approximation****?*

- For theory, see Lecture 1, pp. 1–3. For Taylor's formula in two dimensions, see pp. 4–5.

The Taylor polynomial of order n is defined as

$$T_n f(x) = \sum_{k=0}^{n} \frac{(x - x_0)^k}{k!} f^{(k)}(x_0) = f(x_0) + \frac{(x - x_0)}{1!} f'(x_0) + \frac{(x - x_0)^2}{2!} f''(x_0) + \cdots + \frac{(x - x_0)^n}{n!} f^{(n)}(x_0). \quad (1.1)$$

The error of approximation is

$$R_{n+1}(x) = \frac{(x - x_0)^{n+1}}{(n + 1)!} f^{(n+1)}(\xi), \text{ with } \xi \text{ between } x \text{ and } x_0. \quad (1.2)$$

We first need to find the kth order derivative of $f(x) = \ln(1 + x)$.

$$\begin{aligned}
f'(x) &= \frac{1}{1 + x} = (1 + x)^{-1} \\
f''(x) &= (-1) \cdot (1 + x)^{-2} \\
f'''(x) &= (-1)(-2) \cdot (1 + x)^{-3} \\
f^{(4)}(x) &= (-1)(-2)(-3) \cdot (1 + x)^{-4} \\
&\dots \\
f^{(n)}(x) &= (-1)^{n-1}(n - 1)! \cdot (1 + x)^{-n},\ n \geq 1.
\end{aligned}$$

In our formula, we need the values of the derivatives at $x_0 = 0$, so it becomes

$$f^{(n)}(0) = (-1)^{n-1}(n - 1)!,\ n \geq 1 \text{ and } f(0) = \ln(1) = 0.$$

We have the polynomial:

$$\begin{aligned}
T_n f(x) &= 0 + \frac{x}{1!} \cdot 0! - \frac{x^2}{2!} \cdot 1! + \frac{x^3}{3!} \cdot 2! - \frac{x^4}{4!} \cdot 3! + \ldots + (-1)^{n-1} \frac{x^n}{n!} \cdot (n - 1)! \\
&= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots + (-1)^{n-1} \frac{x^n}{n}
\end{aligned}$$

and the error of approximation:

$$R_{n+1}(x) = (-1)^n \frac{x^{n+1}}{(n + 1)!} \cdot n!(1 + \xi)^{-(n+1)} = (-1)^n \frac{x^{n+1}}{n + 1} \cdot (1 + \xi)^{-(n+1)} \text{ with } \xi \text{ between } x \text{ and } 0.$$

Example 1.2 *Compute the* ***finite difference table*** *for the following given data*

x	1	2	3	4	5
y	4	13	34	73	136

- For theory, see Lecture 2, pp. 8–9.

Since nothing is specified, we consider '**forward differences**' (Lect. 2, Remark 4.11, pp. 9).
The "maximum order" for the finite difference table is nr. points -1, so, in our case it is 4. (we have 5 data). On the *first column*, we should put *the function's values*, in our case y.

y	$\Delta^1 y$	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
y_0 =4	$y_1 - y_0 = 13 - 4 = 9$	$\Delta^1 y_1 - \Delta^1 y_0 = 21 - 9 = 12$	$\Delta^2 y_1 - \Delta^2 y_0 = 18 - 12 = 6$	$\Delta^3 y_1 - \Delta^3 y_0 = 6 - 6 = 0$
y_1 =13	$y_2 - y_1 = 34 - 13 = 21$	$\Delta^1 y_2 - \Delta^1 y_1 = 39 - 21 = 18$	$\Delta^2 y_2 - \Delta^2 y_1 = 24 - 18 = 6$	
y_2 = 34	$y_3 - y_2 = 73 - 34 = 39$	$\Delta^1 y_3 - \Delta^1 y_2 = 63 - 39 = 24$		
y_3 =73	$y_4 - y_3 = 136 - 73 = 63$			
y_4 =136				

Example 1.3 *Compute the* ***backward difference table*** *for the following given data*

x	1	2	3	4	5
y	4	13	34	73	136

- For theory, see Lecture 2, pp. 8–9.

The "maximum order" for the finite difference table is nr. points -1, so, in our case it is 4. (we have 5 data). On the *first column*, we should put *the function's values*, in our case y.

y	$\nabla^1 y$	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
4				
13	13-4=9			
34	34-13=21	21-9 = 12		
73	73-34=39	39-21 = 18	18-12=6	
136	136-73=63	63-39 = 24	24-18=6	6-6=0

Remark 1.4 For finite differences, the nodes should be **equidistant**.

Example 1.5 *Compute the* ***divided difference table*** *for the following given data*

x	0	1	3	4
f	1	3	49	129

- For theory, see Lecture 2, pp. 1–2.

The "maximum order" for the divided difference table is nr. points -1, so, in our case it is 3. (we have 4 data). On the *first column*, we should put *the nodes* (x), on the *second column*, we should put *the values of the function on the nodes* (f).

x_i	f_i	$\mathcal{D}_1 f_i$	$\mathcal{D}_2 f_i$	$\mathcal{D}_3 f_i$
$x_0 = 0$	1	$\frac{f_1 - f_0}{x_1 - x_0} = \frac{3-1}{1-0} = 2$	$\frac{\mathcal{D}_1 f_1 - \mathcal{D}_1 f_0}{x_2 - x_0} = \frac{23-2}{3-0} = 7$	$\frac{\mathcal{D}_2 f_1 - \mathcal{D}_2 f_0}{x_3 - x_0} = \frac{19-7}{4-0} = 3$
$x_1 = 1$	3	$\frac{f_2 - f_1}{x_2 - x_1} = \frac{49-3}{3-1} = 23$	$\frac{\mathcal{D}_1 f_2 - \mathcal{D}_1 f_1}{x_3 - x_1} = \frac{80-23}{4-1} = 19$	
$x_2 = 3$	49	$\frac{f_3 - f_2}{x_3 - x_2} = \frac{129-49}{4-3} = 80$		
$x_3 = 4$	129			

Remark 1.6 For the 1st order ($\mathcal{D}_1$), you divide by the difference of consecutive nodes ($x_1 - x_0$, $x_2 - x_1$ and so on...). For the 2nd order, you "skip" a node ($x_2 - x_0$, $x_3 - x_1$,...). Then you "skip" 2 nodes ($x_3 - x_0$), and so on...

Example 1.7 *Compute the divided difference table for the triple nodes $x_0 = x_1 = x_2 = 0$, the double nodes $x_3 = x_4 = 1$ and the function $f(x) = arctan(x)$.*

From Lecture 2, pp. 2, Divided difference with multiple nodes, you have the following property:
The **divided difference of order n** at the node x_0, of multiplicity $(n+1)$, is defined as

$$f[x_0, ..., x_0] = \frac{f^{(n)}(x_0)}{n!}, \text{ where } x_0 \text{ appears (n+1) times in [...].}$$

In our case, since the first three nodes are equal, we will use for them f' and f'' (they have order of multiplicity 3). For the other two nodes, we will use f' (their order of multiplicity is 2). The derivatives of f are:

$$f'(x) = \frac{1}{x^2+1}, \quad f''(x) = \frac{-2x}{(x^2+1)^2}.$$

Our table becomes:

x_i	f_i	D_1	D_2	D_3	D_4
$x_0 = 0$	$f(0) = 0$	$f'(0) = 1$	$\frac{f''(0)}{2!} = 0$	$\frac{\frac{\pi}{4} - 1 - 0}{1-0} = \frac{\pi}{4} - 1$	$\frac{\frac{3-\pi}{2} - \frac{\pi}{4} + 1}{1-0} = \frac{10-3\pi}{4}$
$x_1 = 0$	$f(0) = 0$	$f'(0) = 1$	$\frac{\frac{\pi}{4} - 1}{1-0} = \frac{\pi}{4} - 1$	$\frac{\frac{1}{2} - \frac{\pi}{4} - \frac{\pi}{4} + 1}{1-0} = \frac{3-\pi}{2}$	
$x_2 = 0$	$f(0) = 0$	$\frac{\frac{\pi}{4} - 0}{1-0} = \frac{\pi}{4}$	$\frac{\frac{1}{2} - \frac{\pi}{4}}{1-0} = \frac{1}{2} - \frac{\pi}{4}$		
$x_3 = 1$	$f(1) = \frac{\pi}{4}$	$f'(1) = \frac{1}{2}$			
$x_4 = 1$	$f(1) = \frac{\pi}{4}$				

Remark 1.8 The denominator is computed here as usual in a divided difference table (with the rule of skipping nodes). Even if here we have always $1 - 0$, these are not the same nodes - this happens only because the nodes are multiple.

Example 1.9 *Compute the* ***Lagrange polynomial*** *for the following data using*

x	3	4	5
y	1	2	4

1. *the fundamental formula*

   - For theory, see Lecture 2, pp. 11–15.

   The fundamental formula for the Lagrange polynomial is:

$$L_n f(x) = \sum_{i=0}^{n} l_i(x) f(x_i) \tag{1.3}$$

   considering $n+1$ interpolation nodes given x_i, $i = 0, ..., n$, and the values of an unknown function f on these nodes, $f(x_i)$, $i = 0, ..., n$.

Remark 1.10 ***The values of the polynomial*** $L_n f$ ***on the nodes should be the same as the values of the function!!*** (this is what interpolation means). So, $L_n f(x_i) = f(x_i)$, $i = 0, \dots, n$. (this is how you could check if your computations were correct.)

The fundamental interpolation polynomials l_i are defined as

$$l_i(x) = \prod_{j=0, j\neq i}^{n} \frac{x - x_j}{x_i - x_j} = \frac{(x-x_0)(x-x_1)\cdot \ldots \cdot (x-x_{i-1})(x-x_{i+1})\cdot \ldots \cdot (x-x_n)}{(x_i-x_0)(x_i-x_1)\cdot \ldots \cdot (x_i-x_{i-1})(x_i-x_{i+1})\cdot \ldots \cdot (x_i-x_n)}$$

Remark 1.11 On the numerator, the term $(x-x_i)$ is missing, on the denominator the term (x_i-x_i) is missing!

So, in our case $x_0 = 3$, $x_1 = 4$, $x_2 = 5$ and $f(x_0) = 1$, $f(x_1) = 2$, $f(x_2) = 4$.
For $l_0(x)$, the terms that contain x_0 will be missing from the numer. and denom.

$$l_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{(x-4)(x-5)}{(3-4)(3-5)} = \frac{x^2 - 9x + 20}{2}$$

For $l_1(x)$, the terms that contain x_1 will be missing from the numer. and denom.

$$l_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{(x-3)(x-5)}{(4-3)(4-5)} = \frac{x^2 - 8x + 15}{-1} = -x^2 + 8x - 15$$

For $l_2(x)$, the terms that contain x_2 will be missing from the numer. and denom.

$$l_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{(x-3)(x-4)}{(5-3)(5-4)} = \frac{x^2 - 7x + 12}{2}$$

Now we substitute them in the eq. (1.3), for $n = 2$ and get

$$\begin{aligned} L_2 f(x) &= l_0(x)\cdot f(x_0) + l_1(x)\cdot f(x_1) + l_2(x)\cdot f(x_2) = \\ &= \frac{x^2 - 9x + 20}{2}\cdot 1 + (-x^2 + 8x - 15)\cdot 2 + \frac{x^2 - 7x + 12}{2}\cdot 4 = \\ &= \frac{x^2 - 9x + 20 - 4x^2 + 32x - 60 + 4x^2 - 28x + 48}{2} = \\ &= \frac{x^2 - 5x + 8}{2}. \end{aligned}$$

Remark 1.12 To approximate $f(3.5)$ using this polynomial, you simply have to compute

$$L_2 f(3.5) = \frac{(3.5)^2 - 5\cdot(3.5) + 8}{2} = \ldots$$

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2. *the barycentric formula*

   - For theory, see Lecture 3, pp. 1–2.

The first barycentric formula for the Lagrange polynomial is

$$L_n f(x) = u(x) \sum_{i=0}^{n} \frac{w_i}{x - x_i} f(x_i), \tag{1.4}$$

with

$$u(x) = \prod_{j=0}^{n} (x - x_j) \text{ and } w_i = \frac{1}{u_i(x_i)} = \frac{1}{\prod\limits_{j=0, j\neq i}^{n} (x_i - x_j)}.$$

First, let us compute $u(x)$:

$$u(x) = (x-x_0)(x-x_1)(x-x_2) = (x-3)(x-4)(x-5);$$

Now, let us compute the weights w_0, w_1, w_2:

$$w_0 = \frac{1}{(x_0 - x_1)(x_0 - x_2)} \text{ (the term } (x_0 - x_0) \text{ is missing), so } w_0 = \frac{1}{(3-4)(3-5)} = \frac{1}{2}.$$

$$w_1 = \frac{1}{(x_1 - x_0)(x_1 - x_2)} \text{ (the term } (x_1 - x_1) \text{ is missing), so } w_1 = \frac{1}{(4-3)(4-5)} = -1.$$

$$w_2 = \frac{1}{(x_2 - x_0)(x_2 - x_1)} \text{ (the term } (x_2 - x_2) \text{ is missing), so } w_2 = \frac{1}{(5-3)(5-4)} = \frac{1}{2}.$$

We also need $\frac{w_i}{x - x_i}$, so

$$\frac{w_0}{x - x_0} = \frac{\frac{1}{2}}{x-3} = \frac{1}{2(x-3)}, \quad \frac{w_1}{x - x_1} = -\frac{1}{x-4}, \quad \frac{w_2}{x - x_2} = \frac{1}{2(x-5)}.$$

We obtain:

$$\begin{aligned} L_2 f(x) &= (x-3)(x-4)(x-5)\left[\frac{1}{2(x-3)} \cdot 1 - \frac{1}{x-4} \cdot 2 + \frac{1}{2(x-5)} \cdot 4\right] \\ &= \frac{(x-4)(x-5)}{2} - \frac{2(x-3)(x-5)}{1} + \frac{4(x-3)(x-4)}{2} \\ &= \frac{x^2 - 9x + 20 - 4x^2 + 32x - 60 + 4x^2 - 28x + 48}{2} = \frac{x^2 - 5x + 8}{2} \end{aligned}$$

as we obtained with the fundamental formula.

The second barycentric formula for the Lagrange polynomial is

$$L_n f(x) = \frac{\sum_{i=0}^{n} \frac{w_i}{x - x_i} f(x_i)}{\sum_{i=0}^{n} \frac{w_i}{x - x_i}}, \tag{1.5}$$

with

$$w_i = \frac{1}{u_i(x_i)} = \frac{1}{\prod_{j=0, j \neq i}^{n} (x_i - x_j)}.$$

First, let us compute w_0, w_1, w_2:

$$w_0 = \frac{1}{(x_0 - x_1)(x_0 - x_2)} \text{ (the term } (x_0 - x_0) \text{ is missing), so } w_0 = \frac{1}{(3-4)(3-5)} = \frac{1}{2}.$$

$$w_1 = \frac{1}{(x_1 - x_0)(x_1 - x_2)} \text{ (the term } (x_1 - x_1) \text{ is missing), so } w_1 = \frac{1}{(4-3)(4-5)} = -1.$$

$$w_2 = \frac{1}{(x_2 - x_0)(x_2 - x_1)} \text{ (the term } (x_2 - x_2) \text{ is missing), so } w_2 = \frac{1}{(5-3)(5-4)} = \frac{1}{2}.$$

We also need $\frac{w_i}{x - x_i}$, so

$$\frac{w_0}{x - x_0} = \frac{\frac{1}{2}}{x-3} = \frac{1}{2(x-3)}, \quad \frac{w_1}{x - x_1} = -\frac{1}{x-4}, \quad \frac{w_2}{x - x_2} = \frac{1}{2(x-5)}.$$

The numerator is

$$\begin{aligned} N &= \frac{w_0}{x - x_0} \cdot f(x_0) + \frac{w_1}{x - x_1} \cdot f(x_1) + \frac{w_2}{x - x_2} \cdot f(x_2) = \\ &= \frac{1}{2(x-3)} - \frac{2}{x-4} + \frac{4}{2(x-5)} = \frac{(x-4)(x-5) - 4(x-3)(x-5) + 4(x-3)(x-4)}{2(x-3)(x-4)(x-5)} = \\ &= \frac{x^2 - 5x + 8}{2(x-3)(x-4)(x-5)}. \end{aligned}$$

The denominator is

$$M = \frac{w_0}{x - x_0} + \frac{w_1}{x - x_1} + \frac{w_2}{x - x_2} = \frac{1}{2(x-3)} - \frac{1}{x-4} + \frac{1}{2(x-5)} =$$
$$= \frac{(x-4)(x-5) - 2(x-3)(x-5) + (x-3)(x-4)}{2(x-3)(x-4)(x-5)} =$$
$$= \frac{2}{2(x-3)(x-4)(x-5)}.$$

So, with the second barycentric formula $L_2 f(x)$ is

$$L_2 f(x) = \frac{N}{M} = \frac{\frac{x^2-5x+8}{2(x-3)(x-4)(x-5)}}{\frac{2}{2(x-3)(x-4)(x-5)}} = \frac{x^2 - 5x + 8}{2},$$

as we previously obtained using the fundamental formula and the first barycentric formula. ☺

- For the limit of error of the Lagrange polynomial, see Lecture 2, pp. 15–16.
- For optimal choice of nodes (= roots of Chebyshev polynomial of first kind), see Lecture 2, pp. 16–18.
- For an example in which the Lagrange polynomial does not converge, see Lecture 2, pp. 18–19 (Runge's example).

Example 1.13 *Use the **Neville's method** to approximate* $f(3)$ *using the data*

x_k	0	1	2	4
$f(x_k)$	1	1	2	5

- For theory, see Lecture 3, pp. 11–14 and for another example, see Lecture 3, pp. 15–16 .

We have to construct the table

$$\begin{array}{c|c|c|c|c}
x_0 & P_{00} & & & \\
x_1 & P_{10} & P_{11} & & \\
x_2 & P_{20} & P_{21} & P_{22} & \\
x_3 & P_{30} & P_{31} & P_{32} & \mathbf{P_{33}}
\end{array}$$

where $P_{i,0} = f(x_i),\ i = 0, \ldots, n$ and $P_{i,j} = \frac{1}{x_i - x_{i-j}} \begin{vmatrix} x - x_{i-j} & P_{i-1,j-1} \\ x - x_i & P_{i,j-1} \end{vmatrix} \quad i \geq j > 0.$

We have $x_0 = 0,\ x_1 = 1,\ x_2 = 2,\ x_3 = 4$.

In our case, since we want to approx. f(3), x=3. The table becomes:

$$\begin{array}{c|c|c|c|c}
x_0 = 0 & P_{00} = 1 & & & \\
x_1 = 1 & P_{10} = 1 & P_{11} = 1 & & \\
x_2 = 2 & P_{20} = 2 & P_{21} = \frac{5}{2} & P_{22} = 4 & \\
x_3 = 4 & P_{30} = 5 & P_{31} = 4 & P_{32} = 3 & \mathbf{P_{33} = \frac{7}{2}}
\end{array}$$

The computations are

$$P_{00} = f(x_0) = 1, \quad P_{10} = f(x_1) = 1, \quad P_{20} = f(x_2) = 2, \quad P_{30} = f(x_3) = 5,$$

$$P_{11} = \frac{1}{x_1 - x_0}\begin{vmatrix} x - x_0 & P_{00} \\ x - x_1 & P_{10} \end{vmatrix} = 1 \cdot \begin{vmatrix} 3 & 1 \\ 2 & 1 \end{vmatrix} = 1$$

$$P_{21} = \frac{1}{x_2 - x_1}\begin{vmatrix} x - x_1 & P_{10} \\ x - x_2 & P_{20} \end{vmatrix} = 1 \cdot \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3$$

$$P_{31} = \frac{1}{x_3 - x_2}\begin{vmatrix} x - x_2 & P_{20} \\ x - x_3 & P_{30} \end{vmatrix} = \frac{1}{2} \cdot \begin{vmatrix} 1 & 2 \\ -1 & 5 \end{vmatrix} = \frac{7}{2}$$

$$P_{22} = \frac{1}{x_2 - x_0}\begin{vmatrix} x - x_0 & P_{11} \\ x - x_2 & P_{21} \end{vmatrix} = \frac{1}{2} \cdot \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} = 4$$

$$P_{32} = \frac{1}{x_3 - x_1}\begin{vmatrix} x - x_1 & P_{21} \\ x - x_3 & P_{31} \end{vmatrix} = \frac{1}{3} \cdot \begin{vmatrix} 2 & 3 \\ -1 & \frac{7}{2} \end{vmatrix} = \frac{10}{3}$$

$$P_{33} = \frac{1}{x_3 - x_0}\begin{vmatrix} x - x_0 & P_{22} \\ x - x_3 & P_{32} \end{vmatrix} = \frac{1}{4} \cdot \begin{vmatrix} 3 & 4 \\ -1 & \frac{10}{3} \end{vmatrix} = \frac{7}{2}.$$

The approximation for $f(3)$ will be P_{33}, so, $\frac{7}{2}$.

- The Lagrange polynomial of degree n, at the given point x ($=L_n f(x)$) will be the element P_{nn}.
- You can add a new node, case in which you have to add another row (the previous computations remain the same).

Example 1.14 *Use the* ***Aitken's method*** *to approximate* $f(3)$ *using the data*

x_k	0	1	2	4
$f(x_k)$	1	1	2	5

- For theory, see Lecture 3, pp. 14 and for another example, see see Lecture 3, pp. 16

We have to construct the table

$$\begin{array}{c|c|c|c|c} x_0 & P_{00} & & & \\ x_1 & P_{10} & P_{11} & & \\ x_2 & P_{20} & P_{21} & P_{22} & \\ x_3 & P_{30} & P_{31} & P_{32} & \mathbf{P_{33}} \end{array}$$

where $P_{i0} = f(x_i),\ i = 0, ..., n$ and $P_{i,j+1} = \frac{1}{x_i - x_j}\begin{vmatrix} x - x_j & P_{jj} \\ x - x_i & P_{ij} \end{vmatrix} \quad i > j \geq 0.$

We have $x_0 = 0,\ x_1 = 1,\ x_2 = 2,\ x_3 = 4.$

In our case, since we want to approx. f(3), x=3. The table becomes: The computations are:

$$\begin{array}{c|c|c|c|c} x_0 = 0 & P_{00} = 1 & & & \\ x_1 = 1 & P_{10} = 1 & P_{11} = 1 & & \\ x_2 = 2 & P_{20} = 2 & P_{21} = \frac{5}{2} & P_{22} = 4 & \\ x_3 = 4 & P_{30} = 5 & P_{31} = 4 & P_{32} = 3 & \mathbf{P_{33} = \frac{7}{2}} \end{array}$$

$$P_{00} = f(x_0) = 1, \quad P_{10} = f(x_1) = 1, \quad P_{20} = f(x_2) = 2, \quad P_{30} = f(x_3) = 5.$$

$$P_{11} = P_{1,0+1} = \frac{1}{x_1 - x_0}\begin{vmatrix} x - x_0 & P_{00} \\ x - x_1 & P_{10} \end{vmatrix} = 1 \cdot \begin{vmatrix} 3 & 1 \\ 2 & 1 \end{vmatrix} = 1$$

$$P_{21} = P_{2,0+1} = \frac{1}{x_2 - x_0}\begin{vmatrix} x - x_0 & P_{00} \\ x - x_2 & P_{20} \end{vmatrix} = \frac{1}{2} \cdot \begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} = \frac{5}{2}$$

$$P_{31} = P_{3,0+1} = \frac{1}{x_3 - x_0}\begin{vmatrix} x - x_0 & P_{00} \\ x - x_3 & P_{30} \end{vmatrix} = \frac{1}{4} \cdot \begin{vmatrix} 3 & 1 \\ -1 & 5 \end{vmatrix} = 4$$

$$P_{22} = P_{2,1+1} = \frac{1}{x_2 - x_1}\begin{vmatrix} x - x_1 & P_{11} \\ x - x_2 & P_{21} \end{vmatrix} = 1 \cdot \begin{vmatrix} 2 & 1 \\ 1 & \frac{5}{2} \end{vmatrix} = 4$$

$$P_{32} = P_{3,1+1} = \frac{1}{x_3 - x_1}\begin{vmatrix} x - x_1 & P_{11} \\ x - x_3 & P_{31} \end{vmatrix} = \frac{1}{3} \cdot \begin{vmatrix} 2 & 1 \\ -1 & 4 \end{vmatrix} = 3$$

$$P_{33} = P_{3,2+1} = \frac{1}{x_3 - x_2}\begin{vmatrix} x - x_2 & P_{22} \\ x - x_3 & P_{32} \end{vmatrix} = \frac{1}{2} \cdot \begin{vmatrix} 1 & 4 \\ -1 & 3 \end{vmatrix} = \frac{7}{2}.$$

The approximation for $f(3)$ will be P_{33}, so, $\frac{7}{2}$.

- The Lagrange polynomial of degree n, at the given point x ($=L_n f(x)$) will be the element P_{nn}.
- You can add a new node, case in which you have to add another row (the previous computations remain the same).
- You can see that P_{ii} have the same values in both Aitken's and Neville's methods, only the other elements are different. This happens because P_{ii} is the Lagrange polynomial of degree i and the Lagrange polynomial is unique. **All the above methods are just different ways to compute the polynomial, but the result should always be the same!**

Example 1.15 *Construct the Lagrange polynomial in the* ***Newton form, using divided difference formula*** *for the data* $\begin{array}{c|ccc} x & 3 & 4 & 5 \\ \hline f & 1 & 2 & 4 \end{array}$.

- For theory, see Lecture 3, pp. 3–5. For the remainder's formula and another example, see see Lecture 3, pp. 6–7.

The Newton form of the Lagrange pol. is

$$L_n f(x) = f(x_0) + (x - x_0)\mathcal{D}_1 + (x - x_0)(x - x_1)\mathcal{D}_2 + \ldots + (x - x_0)(x - x_1)\ldots(x - x_{n-1})\mathcal{D}_n, \tag{1.6}$$

where $\mathcal{D}_i = f[x_0, \ldots, x_i]$ is the divided difference of order i at the nodes $x_0, \ldots, x_i$ (in our table $\mathcal{D}_i = \mathcal{D}_i f_0$).

You will use only the first row of the divided diff. table to get the required values.
First, we have to construct the div. diff. table. So, we have

x_i	f_i	$\mathcal{D}_1 f_i$	$\mathcal{D}_2 f_i$
$x_0 = 3$	1	$\frac{f_1 - f_0}{x_1 - x_0} = \frac{2-1}{4-3} = 1$	$\frac{\mathcal{D}_1 f_1 - \mathcal{D}_1 f_0}{x_2 - x_0} = \frac{2-1}{5-3} = \frac{1}{2}$
$x_1 = 4$	2	$\frac{f_2 - f_1}{x_2 - x_1} = \frac{4-2}{5-4} = 2$	
$x_2 = 5$	4		

$$\begin{aligned} L_2(f) &= f(x_0) + (x - x_0) \cdot \mathcal{D}_1 + (x - x_0)(x - x_1) \cdot \mathcal{D}_2 \\ &= 1 + (x - 3) \cdot 1 + (x - 3)(x - 4) \cdot \frac{1}{2} = 1 + x - 3 + \frac{1}{2}x^2 - \frac{7}{2}x + 6 = \\ &= \frac{1}{2}x^2 - \frac{5}{2}x + 4. \end{aligned}$$

Example 1.16 *Construct the Lagrange polynomial in the* ***Newton form, using finite differences*** *in both cases -* ***forward and backward****, for the data*

x	3	4	5
f	1	2	4

- For theory and another example, see Lecture 3, pp. 8–11.

a) forward differences

Newton's forward difference formula is:

$$L_n f(x) = f_0 + \binom{s}{1}\Delta f_0 + \binom{s}{2}\Delta^2 f_0 + \ldots + \binom{s}{n}\Delta^n f_0 \tag{1.7}$$

with

$$f_0 = f(x_0), \quad \Delta^k f_0 = \Delta^{k-1} f_1 - \Delta^{k-1} f_0, \quad x_i = x_0 + ih, \quad i = 0, \ldots, n, \quad s = \frac{x - x_0}{h}, \quad \binom{s}{k} = \frac{s(s-1)\ldots(s-k+1)}{k!}.$$

Note that the **nodes** should be **equidistant** since we have finite differences!

We construct the finite difference table (forward):

f	$\Delta^1 f$	$\Delta^2 f$
1	2-1 = 1	2-1= 1
2	4-2 = 2	
4		

$$x_0 = 3, \quad h = 1 \implies s = \frac{x-3}{1} = x - 3, \quad n = 2$$

and obtain

$$\begin{aligned} L_2 f(x) &= 1 + \binom{s}{1} \cdot 1 + \binom{s}{2} \cdot 1 = 1 + \frac{s}{1!} + \frac{s(s-1)}{2!} \\ &= 1 + x - 3 + \frac{(x-3)(x-4)}{2} = \frac{x^2 - 5x + 8}{2}. \end{aligned}$$

---

b) backward differences

Newton's backward difference formula is:

$$L_n f(x) = f_n + \binom{s}{1}\nabla f_n + \binom{s+1}{2}\nabla^2 f_n + \ldots + \binom{s+n-1}{n}\nabla^n f_n \tag{1.8}$$

with

$$f_n = f(x_n), \quad \nabla^k f_n = \nabla^{k-1} f_n - \nabla^{k-1} f_{n-1}, \quad x_i = x_0 + ih, \quad i = 0, \ldots, n, \quad s = \frac{x - x_n}{h}, \quad \binom{s}{k} = \frac{s(s-1)\ldots(s-k+1)}{k!}.$$

Note that the **nodes** again should be **equidistant** to apply this method.

We construct the finite difference table (backward):

f	$\Delta^1 f$	$\Delta^2 f$
1		
2	2-1 = 1	
4	4-2 = 2	2-1 =1

$$n = 2, \quad x_2 = 5, \quad h = 1 \implies s = \frac{x-5}{1} = x - 5$$

and obtain

$$\begin{aligned} L_2 f(x) &= 4 + \binom{s}{1} \cdot 2 + \binom{s+1}{2} \cdot 1 = 1 + \frac{s}{1!} + \frac{s(s+1)}{2!} \\ &= 4 + 2x - 10 + \frac{(x-5)(x-4)}{2} = \frac{x^2 - 5x + 8}{2}. \end{aligned}$$

Example 1.17 *Consider the double nodes $x_0 = -1$ and $x_1 = 1$. Consider also $f(-1) = -3$, $f'(-1) = 10, f(1) = 1$, $f'(1) = 2$. Find the* ***Hermite interpolation polynomial using the divided difference table for double nodes****.*

- For theory, other examples and estimation of the error, see Lecture 4, pp. 1–9. We will use the Newton's divided difference formula here (pp. 4–5).

Remark 1.18 Hermite interpolation for double nodes can be used only when you know the values of f and f' **for all the nodes**!

First, we should compute the divided difference table with double nodes.
$z_0 = x_0,\ z_1 = x_0,\ z_2 = x_1,\ z_3 = x_1$. You should also double the values of f. The difference appears when you compute the divided difference of first order. At the odd positions you have to put the derivative of f at the corresponding node. The other entries are computed in the usual manner.

z_i	f_i	$\mathcal{D}_1 f_i$	$\mathcal{D}_2 f_i$	$\mathcal{D}_3 f_i$
z_0 = -1	-3	$f'(-1) = 10$	$\frac{\mathcal{D}_1 f_1 - \mathcal{D}_1 f_0}{z_2 - z_0} = \frac{2-10}{1-(-1)} = -4$	$\frac{\mathcal{D}_2 f_1 - \mathcal{D}_2 f_0}{z_3 - z_0} = \frac{0-(-4)}{1-(-1)} = 2$
z_1 = - 1	-3	$\frac{f_2 - f_1}{z_2 - z_1} = \frac{1-(-3)}{1-(-1)} = 2$	$\frac{\mathcal{D}_1 f_2 - \mathcal{D}_1 f_1}{z_3 - z_1} = \frac{2-2}{1-(-1)} = 0$	
z_2 = 1	1	$f'(1) = 2$		
z_3 = 1	1			

Next, we will use the Newton form:

$$H_{2m+1} f(x) = f(z_0) + \sum_{i=1}^{2m+1} (x - z_0) \cdot \ldots \cdot (x - z_{i-1}) \mathcal{D}_i, \tag{1.9}$$

with $\mathcal{D}_i = \mathcal{D}_i f_0 = f[x_0, \ldots, x_i]$, that is in our case

$$\begin{aligned} H_3 f(x) &= -3 + (x - z_0) \cdot \mathcal{D}_1 + (x - z_0)(x - z_1) \cdot \mathcal{D}_2 + (x - z_0)(x - z_1)(x - z_2) \cdot \mathcal{D}_3 = \\ &= -3 + (x+1) \cdot 10 + (x+1)^2 \cdot (-4) + (x+1)^2 (x-1) \cdot 2 \\ &= -3 + 10x + 10 - 4x^2 - 8x - 4 + 2x^3 - 2x^2 + 4x^2 - 4x + 2x - 2 = \\ &= 2x^3 - 2x^2 + 1. \end{aligned}$$

Remark 1.19 A good way to check if your computations were right is by checking the interpolation conditions. Here, the polynomial $H_3 f$ should satisfy the following conditions:

$$H_3 f(-1) = f(-1) = -3,\ \ H_3 f(1) = f(1) = 1,\ \ H_3' f(-1) = f'(-1) = 10,\ \ H_3' f(1) = f(1) = 2.$$

Indeed, they are all satisfied.

Example 1.20 *Construct the interpolation polynomial that approximates the data*

x	y	y'
-1	2	-4
1	2	–

We first should decide what interpolation problem we have (Lagrange, Hermite, Birkhoff).

We have $x_0 = -1,\ x_1 = 1,\ f(x_0) = 2,\ f(x_1) = 2,\ f'(x_0) = -4$. We have some information about the derivative of f, we cannot use Lagrange interpolation, so we are left with either Hermite or Birkhoff. Since no derivative's order is skipped for the 2 nodes (we have f(-1), f'(-1) -> max. order for derivative is 1 and we have f(1) -> max. order is 0), we have to use **Hermite interpolation**. (The derivative's maximum order doesn't have to be the same for each node. It is important not to skip any order from 0 to the max.).

We can still use the method of Newton's divided difference, but in this case only the first node is doubled (for the first node we know the values of f and f').

z_i	f_i	$\mathcal{D}_1 f_i$	$\mathcal{D}_2 f_i$
$z_0 = -1$	f(-1) = 2	$f'(-1) = -4$	$\frac{D_1 f_1 - D_1 f_0}{z_2 - z_0} = \frac{0-(-4)}{1-(-1)} = 2$
$z_1 = -1$	f(-1) = 2	$\frac{f(1)-f(-1)}{z_2-z_1} = \frac{2-2}{1-(-1)} = 0$	
$z_2 = 1$	f(1) = 2		

Using Newton's form we get:

$$H_2 f(x) = 2 + (x - z_0) \cdot (-4) + (x - z_0)(x - z_1) \cdot 2 = 2 - 4(x+1) + 2(x+1)^2 = 2x^2$$

Indeed, this polynomial satisfies the interpolation properties:

$$H_2 f(-1) = 2 = f(-1), \quad H_2' f(-1) = -4 = f'(-1), \quad H_2 f(1) = 2 = f(1).$$

Remark 1.21 To approximate $f(0)$ and $f'(0)$, you compute $H_2 f(0) = 2 \cdot 0^2 = 0$ and $H_2' f(0) = 4 \cdot 0 = 0$.

Remark 1.22 Newton form can also be used for other multiplicities of the nodes - pay attention at how you compute the divided difference table in that case (see Example 1.7).

Remark 1.23 Hermite polynomial is unique!

For other examples and estimations of the errors, see Lecture 4. There are other ways to compute Hermite polynomials:

- general case - see Lecture 4, Theorem 1.6, Remark 1.7, pp. 10–11 - with Hermite fundamental polynomials, given by relations (1.11), (1.12), (1.13);
- double nodes - see Lecture 4, Theorem 1.1, pp. 2 - with Hermite fundamental polynomials, given by relations (1.3), (1.4);
- cubic Hermite polynomial - see Lecture 4, Example 1.2, pp. 3–4 - for 2 double nodes.

Example 1.24 *Approximate $f(\frac{1}{2})$ knowing the following information: $x_0 = 0$, $x_1 = 1$, $f(x_0) = 1$, $f'(x_0) = 2$ and $f'(x_1) = -1$.*

Again, let's decide the interpolation problem we have. Since for x_1 the value of f is missing, we cannot use Hermite interpolation, so we have to use **Birkhoff interpolation**.

In general, for $m+1$ given nodes, x_k, $k = 0, ..., m$, the max. degree of the polynomial is $n = |I_0| + |I_1| + ... + |I_m| - 1$, where $|I_k|$ is the number of elements of I_k and I_k is the set that contains all the derivative's orders that are known for the node x_k.

In our case, $I_0 = \{0, 1\}$ (for x_0 we know the val. for f and f') $I_1 = \{1\}$ (for x_1 we know only f'). So, the **max. degree of the Birkhoff pol.** is

$$n = |I_0| + |I_1| - 1 = 2 + 1 - 1 = 2.$$

This means that we will have a polynomial P of max. degree $n = 2$. **First, in the Birkhoff case, we should check that the problem has a unique solution** (Lagrange and Hermite polynomials have a unique solution, but Birkhoff does not have a solution all the time).

For this, let us denote $P(x) = ax^2 + bx + c$ our polynomial. We will also need its derivative, $P'(x) = 2ax + b$. The polynomial P should satisfy the interpolation conditions

$$P(x_0) = f(x_0),\;\; P'(x_0) = f'(x_0),\;\; P'(x_1) = f'(x_1) \tag{1.10}$$

which gives us

$$\begin{cases} a \cdot 0^2 + b \cdot 0 + c & = 1 \\ 2a \cdot 0 + b & = 2 \\ 2a \cdot 1 + b & = -1 \end{cases} \tag{1.11}$$

The determinant below tells us if we have a unique solution ($\neq 0$). It is constructed using the coefficients of the unknowns a, b, c, of the system (1.11) (as usual in a problem for linear systems)

$$\begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 1 & 0 \end{vmatrix} = -2 \neq 0$$

so we have a unique solution.

Now, the first method to obtain the Birkhoff polynomial is by solving the above system to find the coefficients a, b, c. From the second eq. we get $b = 2$, from the third eq. we get $a = -\frac{3}{2}$ and from the first eq. we get $c = 1$, so our polynomial is

$$P(x) = -\frac{3}{2}x^2 + 2x + 1.$$

Indeed, all the interpolation conditions (1.10) are satisfied. To approximate $f(\frac{1}{2})$, we have to compute $P(\frac{1}{2}) = -\frac{3}{8} + 1 + 1 = \frac{13}{8}$.

Another method to solve the problem is by computing the Birkhoff polynomial in a more general way. For this, consider the following expression

$$B_n f(x) = \sum_{k=0}^{m} \sum_{j \in I_k} b_{kj}(x) \cdot f^{(j)}(x_k)$$

for $m+1$ nodes x_k, $k = 0, ..., m$, with $f^{(j)}(x_k)$ denoting the derivative of order j of function f at the node x_k and the Birkhoff fundamental polynomials b_{kj} to be determined from the following properties:

$$\begin{aligned} b_{kj}^{(p)}(x_\nu) &= 0, \;\text{ when }\; k \neq \nu,\; p \in I_\nu \\ b_{kj}^{(p)}(x_k) &= \delta_{jp}, \;\text{ when }\; p \in I_k,\; j \in I_k,\; \text{ and }\; \nu, k = 0, ..., m \end{aligned}$$

$\delta_{jp} = 0$ $(j \neq p)$ and 1 $(j = p)$.

For example (not for our problem, but in general):
$b_{01}(x_1) = 0$ (because $0 \neq 1$).
$b_{11}(x_1) = 0$ (we have $1 = 1$, but $b_{11}(x_1) = b_{11}^{(0)}(x_1)$ and $0 \neq 1$).
$b'_{11}(x_1) = 1$ (because 1=1 and 1=1).

In our case, we have

$$B_2 f(x) = \sum_{k=0}^{1} \sum_{j \in I_k} b_{kj}(x) \cdot f^{(j)}(x_k) = b_{00}(x) \cdot f(x_0) + b_{01}(x) \cdot f'(x_0) + b_{11}(x) \cdot f'(x_1) \tag{1.12}$$

The unknowns here are the polynomials b_{00}, b_{01} and b_{11}. Since the Birkhoff pol. should have the max. degree 2, we will write each of these 3 polynomials as a second degree pol. $(ax^2 + bx + c)$ and determine the coeffs. a, b, c in each case, by using the fundamental properties:

- $b_{00}(x) = ax^2 + bx + c$ and $b'_{00}(x) = 2ax + b$

$$\begin{cases} b_{00}(x_0) = 1 \\ b'_{00}(x_0) = 0 \\ b'_{00}(x_1) = 0 \end{cases} \implies \begin{cases} b_{00}(0) = c = 1 \\ b'_{00}(0) = b = 0 \\ b'_{00}(1) = 2a + b = 0 \implies a = 0 \end{cases}$$

$$\implies b_{00}(x) = 1.$$

- $b_{01}(x) = ax^2 + bx + c$ and $b'_{01}(x) = 2ax + b$

$$\begin{cases} b_{01}(x_0) = 0 \\ b'_{01}(x_0) = 1 \\ b'_{01}(x_1) = 0 \end{cases} \implies \begin{cases} b_{01}(0) = c = 0 \\ b'_{01}(0) = b = 1 \\ b'_{01}(1) = 2a + b = 0 \implies a = -\dfrac{1}{2} \end{cases}$$

$$\implies b_{01}(x) = -\frac{1}{2}x^2 + x.$$

- $b_{11}(x) = ax^2 + bx + c$ and $b'_{11}(x) = 2ax + b$

$$\begin{cases} b_{11}(x_0) = c = 0 \\ b'_{11}(x_0) = b = 0 \\ b'_{11}(x_1) = 2a + b = 1 \implies a = \dfrac{1}{2} \end{cases}$$

$$\implies b_{11}(x) = \frac{1}{2}x^2.$$

Going back at eq. (1.12), we obtain

$$\begin{aligned} B_2(f) &= 1 \cdot 1 + 2 \cdot (-\frac{1}{2}x^2 + x) - 1 \cdot \frac{1}{2}x^2 == 1 - x^2 + 2x - \frac{1}{2}x^2 = \\ &= -\frac{3}{2}x^2 + 2x + 1 \end{aligned}$$

Indeed, this polynomial is the same as what we obtain with the first method.

For other examples and estimations of the errors, see Lecture 5.
