3 Examples for Lectures 9 – 10 (Numerical methods for solving linear systems)

3.1 Direct methods

Example 3.1 *Solve the system*

$$\begin{cases} 2x_1 + 4x_3 + x_4 = 7 \\ 2x_2 + 4x_3 + x_4 = 7 \\ 2x_1 + 4x_2 + 3x_3 = 9 \\ x_1 + 2x_2 + 2x_4 = 5 \end{cases}$$

using the Gauss method with partial pivoting.

We start by writing the matrix A that contains the coefficients of the unknowns (x_1, x_2, x_3, x_4). We also write $\overline{A}$ which contains also the column vector b (the result of each equation), since the modifications should be performed on this column too.

$$A = \begin{pmatrix} 2 & 0 & 4 & 1 \\ 0 & 2 & 4 & 1 \\ 2 & 4 & 3 & 0 \\ 1 & 2 & 0 & 2 \end{pmatrix} \quad \text{and} \quad \overline{A} = \begin{pmatrix} \mathbf{2} & 0 & 4 & 1 & | & 7 \\ 0 & 2 & 4 & 1 & | & 7 \\ 2 & 4 & 3 & 0 & | & 9 \\ 1 & 2 & 0 & 2 & | & 5 \end{pmatrix}.$$

On the first column of $\overline{A}$, the pivot (maximum element in absolute value) is $a_{11} = 2$, so we do not interchange any rows. $a_{21} = 0$ so we let it the same, and to obtain $a_{31} = 0$ and $a_{41} = 0$, we have to perform $R_3 - R_1$ and $R_4 - \frac{1}{2}R_1$. (Don't forget to change also the column of free term b!)

$$\overline{A} \sim \begin{pmatrix} 2 & 0 & 4 & 1 & | & 7 \\ 0 & 2 & 4 & 1 & | & 7 \\ 0 & 4 & -1 & -1 & | & 2 \\ 0 & 2 & -2 & \frac{3}{2} & | & \frac{3}{2} \end{pmatrix}$$

On the second column (below the main diagonal - and including it), the maximum element in absolute value is $a_{32} = 4$, so we interchange R_2 and R_3.

$$\overline{A} \sim \begin{pmatrix} 2 & 0 & 4 & 1 & | & 7 \\ 0 & \mathbf{4} & -1 & -1 & | & 2 \\ 0 & 2 & 4 & 1 & | & 7 \\ 0 & 2 & -2 & \frac{3}{2} & | & \frac{3}{2} \end{pmatrix}$$

The pivot is now $a_{22} = 4$. Now, to obtain 0 below the main diagonal on the second column, we need $a_{32} = 0$ and $a_{42} = 0$, so we perform $R_3 - \frac{1}{2}R_2$ and $R_4 - \frac{1}{2}R_2$, obtaining

$$\overline{A} \sim \begin{pmatrix} 2 & 0 & 4 & 1 & | & 7 \\ 0 & 4 & -1 & -1 & | & 2 \\ 0 & 0 & \mathbf{\frac{9}{2}} & \frac{3}{2} & | & 6 \\ 0 & 0 & -\frac{3}{2} & 2 & | & \frac{1}{2} \end{pmatrix}$$

On the third column, the maximum element in absolute value below the main diagonal (including it) is $a_{33} = \frac{9}{2}$, so we don't interchange anything. To obtain 0 below the main diagonal, we need $a_{43} = 0$, so we have to compute $R_4 + \frac{1}{3}R_3$, obtaining

$$\overline{A} \sim \begin{pmatrix} 2 & 0 & 4 & 1 & | & 7 \\ 0 & 4 & -1 & -1 & | & 2 \\ 0 & 0 & \frac{9}{2} & \frac{3}{2} & | & 6 \\ 0 & 0 & 0 & \frac{5}{2} & | & \frac{5}{2} \end{pmatrix}$$

Now, using **backward substitution**, we obtain

$$
\begin{aligned}
\frac{5}{2}x_4 &= \frac{5}{2} \implies x_4 = 1\\
\frac{9}{2}x_3 + \frac{3}{2}\cdot 1 &= 6 \implies x_3 = 1\\
4x_2 - 1\cdot 1 - 1\cdot 1 &= 2 \implies x_2 = 1\\
2x_1 + 0\cdot 1 + 4\cdot 1 + 1\cdot 1 &= 7 \implies x_1 = 1
\end{aligned}
$$

Remark 3.2 The **theory** can be found in Lecture 9, pp. 1–9. Other **examples** using the **Gauss elimination with partial pivoting** are in Lecture 9, pp. 7–8. **Examples** using the **Gauss elimination with scaled partial pivoting and total pivoting** are in Lecture 9, pp. 8–9.

Example 3.3 *Solve the previous system using an LUP decomposition.*

Starting with the matrix

$$A = \begin{pmatrix} 2 & 0 & 4 & 1\\ 0 & 2 & 4 & 1\\ 2 & 4 & 3 & 0\\ 1 & 2 & 0 & 2 \end{pmatrix}$$

we see that the pivot on the first column is 2, so we do not interchange any rows. In this case, P has the form $P = \begin{pmatrix} 1\\ 2\\ 3\\ 4 \end{pmatrix}$ (the order of the rows in the matrix - we change it when we interchange rows.)

1.

$$A \sim \left(\begin{array}{c|ccc} 2 & 0 & 4 & 1\\ \hline 0 & 2 & 4 & 1\\ 2 & 4 & 3 & 0\\ 1 & 2 & 0 & 2 \end{array}\right)$$

2. A will look like this at the next step

$$A \sim \left(\begin{array}{c|ccc} 2 & 0 & 4 & 1\\ \hline \frac{0}{2} & & & \\ \frac{2}{2} & & & \\ \frac{1}{2} & & & \end{array}\right) \sim \left(\begin{array}{c|ccc} 2 & 0 & 4 & 1\\ \hline 0 & & & \\ 1 & & & \\ \frac{1}{2} & & & \end{array}\right)$$

3. To fill the empty space we compute the Schur complement from the coloured part:

$$\begin{pmatrix} 2 & 4 & 1\\ 4 & 3 & 0\\ 2 & 0 & 2 \end{pmatrix} - \frac{1}{2}\cdot \begin{pmatrix} 0\\ 2\\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 4 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 4 & 1\\ 4 & -1 & -1\\ 2 & -2 & \frac{3}{2} \end{pmatrix}$$

and add it to our matrix

$$A \sim \left(\begin{array}{c|ccc} 2 & 0 & 4 & 1\\ \hline 0 & 2 & 4 & 1\\ 1 & 4 & -1 & -1\\ \frac{1}{2} & 2 & -2 & \frac{3}{2} \end{array}\right)$$

4. On the remaining part, second column, the pivot is 4, so we interchange R_2 and R_3. Now P is changed to $P = \begin{pmatrix} 1 \\ 3 \\ 2 \\ 4 \end{pmatrix}$.

$$A \sim \left(\begin{array}{c|c|cc} 2 & 0 & 4 & 1 \\ \hline 1 & 4 & -1 & -1 \\ \cline{3-4} 0 & 2 & 4 & 1 \\ \frac{1}{2} & 2 & -2 & \frac{3}{2} \end{array}\right)$$

A will have the form

$$A \sim \left(\begin{array}{c|c|cc} 2 & 0 & 4 & 1 \\ \hline 1 & 4 & -1 & -1 \\ \cline{2-4} 0 & \frac{2}{4} & & \\ \frac{1}{2} & \frac{2}{4} & & \end{array}\right) \sim \left(\begin{array}{c|c|cc} 2 & 0 & 4 & 1 \\ \hline 1 & 4 & -1 & -1 \\ \cline{2-4} 0 & \frac{1}{2} & & \\ \frac{1}{2} & \frac{1}{2} & & \end{array}\right)$$

5. Next Schur complement:

$$\begin{pmatrix} 4 & 1 \\ -2 & \frac{3}{2} \end{pmatrix} - \frac{1}{4} \cdot \begin{pmatrix} 2 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} -1 & -1 \end{pmatrix} = \begin{pmatrix} \frac{9}{2} & \frac{3}{2} \\ -\frac{3}{2} & 2 \end{pmatrix}$$

We add it to our matrix

$$A \sim \left(\begin{array}{c|c|cc} 2 & 0 & 4 & 1 \\ \hline 1 & 4 & -1 & -1 \\ \cline{2-4} 0 & \frac{1}{2} & \frac{9}{2} & \frac{3}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{3}{2} & 2 \end{array}\right)$$

6. On the remaining part the pivot is $\frac{9}{2}$, so we do not make any interchanges, P remains the same.

$$A \sim \left(\begin{array}{c|c|c|c} 2 & 0 & 4 & 1 \\ \hline 1 & 4 & -1 & -1 \\ \cline{2-4} 0 & \frac{1}{2} & \frac{9}{2} & \frac{3}{2} \\ \cline{3-4} \frac{1}{2} & \frac{1}{2} & -\frac{3}{2} & 2 \end{array}\right)$$

Finally, A will have the form:

$$A \sim \left(\begin{array}{c|c|c|c} 2 & 0 & 4 & 1 \\ \hline 1 & 4 & -1 & -1 \\ \cline{2-4} 0 & \frac{1}{2} & \frac{9}{2} & \frac{3}{2} \\ \cline{3-4} \frac{1}{2} & \frac{1}{2} & -\frac{3}{2} \cdot \frac{2}{9} & \end{array}\right)$$

the missing part being computed from the last Schur complement

$$2 - \frac{2}{9} \cdot \left(-\frac{3}{2}\right) \cdot \frac{3}{2} = \frac{5}{2}$$

7. Finally,

$$A \sim \begin{pmatrix} 2 & 0 & 4 & 1 \\ 1 & 4 & -1 & -1 \\ 0 & \frac{1}{2} & \frac{9}{2} & \frac{3}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{3} & \frac{5}{2} \end{pmatrix}$$

8. L will be the lower triangular part of A, with 1 on the main diagonal, U will be the upper triangular part of A and P will be the permutation matrix of the rows. Since $P = \begin{pmatrix} 1 \\ 3 \\ 2 \\ 4 \end{pmatrix}$, we have

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & \frac{1}{2} & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{3} & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 2 & 0 & 4 & 1 \\ 0 & 4 & -1 & -1 \\ 0 & 0 & \frac{9}{2} & \frac{3}{2} \\ 0 & 0 & 0 & \frac{5}{2} \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

9. We have

$$PA = LU$$

Multiplying at right with x, we have

$$PAx = LUx$$

Knowing that $Ax = b$ and denoting $Ux = y$, we have first to solve the system

$$Pb = Ly$$

which will be solve using forward substitution:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & \frac{1}{2} & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{3} & 1 \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} 7 \\ 9 \\ 7 \\ 5 \end{pmatrix}$$

$$\Longrightarrow y_1 = 7,\ y_2 = 2,\ y_3 = 6,\ y_4 = \frac{5}{2}.$$

To find x, we have to solve $Ux = y$, by backward substitution:

$$\begin{pmatrix} 2 & 0 & 4 & 1 \\ 0 & 4 & -1 & -1 \\ 0 & 0 & \frac{9}{2} & \frac{3}{2} \\ 0 & 0 & 0 & \frac{5}{2} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 7 \\ 2 \\ 6 \\ \frac{5}{2} \end{pmatrix}$$

that implies

$$x_4 = 1,\ x_3 = 1,\ x_2 = 1,\ x_1 = 1.$$

The solution is $x = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ as we obtained at the previous exercise.

Remark 3.4 For **theory** about **LU** and **LUP decompositions**, see Lecture 9, pp. 10–15. Other **examples** are found in Lecture 9, pp. 12–13 (LU), 14–15 (LUP).

Remark 3.5 There are two other factorization methods discussed in class (**QR decomposition** and **Cholesky factorization**). You can find the **theory** and **examples** in Lecture 10, pp. 1–3.

Example 3.6 *Find the Cholesky decomposition of the matrix*

$$A = \begin{pmatrix} 10 & 5 & 2 \\ 5 & 3 & 2 \\ 2 & 2 & 3 \end{pmatrix}$$

First, we can see that $A = A^t$, so the matrix is symmetric. Its eigenvalues (=solutions λ of the equation $det(A - \lambda I_3) = 0$) are $\approx 0.09,\ 2.6,\ 13.31$ so real and positive, hence the matrix is positive definite. The algorithm is similar to the LU decomposition.

1.
$$A \sim \left(\begin{array}{c|cc} 10 & 5 & 2 \\ \hline 5 & 3 & 2 \\ 2 & 2 & 3 \end{array}\right)$$

2. A will look at the next step as
$$A \sim \left(\begin{array}{c|cc} \sqrt{10} & \frac{5}{\sqrt{10}} & \frac{2}{\sqrt{10}} \\ \hline \frac{5}{\sqrt{10}} & & \\ \frac{2}{\sqrt{10}} & & \end{array}\right) \sim \left(\begin{array}{c|cc} \sqrt{10} & \frac{\sqrt{10}}{2} & \frac{\sqrt{10}}{5} \\ \hline \frac{\sqrt{10}}{2} & & \\ \frac{\sqrt{10}}{5} & & \end{array}\right)$$
with the empty part computed by
$$\begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} - \begin{pmatrix} \frac{\sqrt{10}}{2} \\ \frac{\sqrt{10}}{5} \end{pmatrix} \cdot \begin{pmatrix} \frac{\sqrt{10}}{2} & \frac{\sqrt{10}}{5} \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} - \begin{pmatrix} \frac{5}{2} & 1 \\ 1 & \frac{2}{5} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 1 \\ 1 & \frac{13}{5} \end{pmatrix}$$
We add it to our matrix
$$A \sim \left(\begin{array}{c|cc} \sqrt{10} & \frac{\sqrt{10}}{2} & \frac{\sqrt{10}}{5} \\ \hline \frac{\sqrt{10}}{2} & \frac{1}{2} & 1 \\ \frac{\sqrt{10}}{5} & 1 & \frac{13}{5} \end{array}\right)$$

3.
$$A \sim \left(\begin{array}{c|c|c} \sqrt{10} & \frac{\sqrt{10}}{2} & \frac{\sqrt{10}}{5} \\ \hline \frac{\sqrt{10}}{2} & \frac{1}{2} & 1 \\ \cline{2-3} \frac{\sqrt{10}}{5} & 1 & \frac{13}{5} \end{array}\right)$$
At the next step, A will have the form
$$A \sim \left(\begin{array}{c|c|c} \sqrt{10} & \frac{\sqrt{10}}{2} & \frac{\sqrt{10}}{5} \\ \hline \frac{\sqrt{10}}{2} & \sqrt{\frac{1}{2}} & \frac{1}{\sqrt{\frac{1}{2}}} \\ \cline{2-3} \frac{\sqrt{10}}{5} & \frac{1}{\sqrt{\frac{1}{2}}} & \end{array}\right) \sim \left(\begin{array}{c|c|c} \sqrt{10} & \frac{\sqrt{10}}{2} & \frac{\sqrt{10}}{5} \\ \hline \frac{\sqrt{10}}{2} & \frac{\sqrt{2}}{2} & \sqrt{2} \\ \cline{2-3} \frac{\sqrt{10}}{5} & \sqrt{2} & \end{array}\right)$$
with the empty part computed by
$$\frac{13}{5} - \sqrt{2} \cdot \sqrt{2} = \frac{3}{5}$$
We put in the empty part: $\sqrt{\frac{3}{5}}$ (!)

4. So, finally
$$A \sim \begin{pmatrix} \sqrt{10} & \frac{\sqrt{10}}{2} & \frac{\sqrt{10}}{5} \\ \frac{\sqrt{10}}{2} & \frac{\sqrt{2}}{2} & \sqrt{2} \\ \frac{\sqrt{10}}{5} & \sqrt{2} & \sqrt{\frac{3}{5}} \end{pmatrix}$$

5. Since Cholesky decomposition consists in writing $A = R^t R$, with R upper triangular, we have that (in A we put zeros below the main diagonal)
$$R = \begin{pmatrix} \sqrt{10} & \frac{\sqrt{10}}{2} & \frac{\sqrt{10}}{5} \\ 0 & \frac{\sqrt{2}}{2} & \sqrt{2} \\ 0 & 0 & \sqrt{\frac{3}{5}} \end{pmatrix}.$$

Remark 3.7 *At the lab we have solved a system using its Cholesky decomposition.*

3.2 Iterative methods

Example 3.8 *Determine the approximate solution for the system*

$$\begin{cases} 5x_1 + x_2 - x_3 = 7 \\ x_1 + 5x_2 + x_3 = 7 \\ x_1 + x_2 + 5x_3 = 7 \end{cases}$$

with the initial approximation $x^{(0)} = (0,0,0)^T$ *using*

a) Jacobi method in 3 steps ;
We can see that the matrix

$$A = \begin{pmatrix} 5 & 1 & -1 \\ 1 & 5 & 1 \\ 1 & 1 & 5 \end{pmatrix}$$

is diagonally dominant since

$$|a_{11}| = |5| > |a_{12}| + |a_{13}| = |1| + |-1| = 2$$
$$|a_{22}| = |5| > |a_{21}| + |a_{23}| = |1| + |1| = 2$$
$$|a_{3}| = |5| > |a_{31}| + |a_{32}| = |1| + |1| = 2$$

hence the method will converge (no matter what the initial approximation $x^{(0)}$ is).
To apply the method, we have to express the unknown x_k from the equation k with respect to the other unknowns. So, we have

$$\begin{cases} x_1 = \dfrac{7 - x_2 + x_3}{5} \\ x_2 = \dfrac{7 - x_1 - x_3}{5} \\ x_3 = \dfrac{7 - x_1 - x_2}{5} \end{cases} \tag{3.1}$$

Now, the Jacobi method consists in expressing $x^{(k)}$ (the unknown x at step k) using the previous approximations $x^{(k-1)}$. We have:

$$\begin{cases} x_1^{(1)} = \dfrac{7 - x_2^{(0)} + x_3^{(0)}}{5} = \dfrac{7 - 0 + 0}{5} = \dfrac{7}{5} = 1.4 \\ x_2^{(1)} = \dfrac{7 - x_1^{(0)} - x_3^{(0)}}{5} = \dfrac{7 - 0 - 0}{5} = \dfrac{7}{5} = 1.4 \\ x_3^{(1)} = \dfrac{7 - x_1^{(0)} - x_2^{(0)}}{5} = \dfrac{7 - 0 - 0}{5} = \dfrac{7}{5} = 1.4 \end{cases}$$

Next, on the second iteration we have:

$$\begin{cases} x_1^{(2)} = \dfrac{7 - x_2^{(1)} + x_3^{(1)}}{5} = \dfrac{7 - \frac{7}{5} + \frac{7}{5}}{5} = \dfrac{7}{5} = 1.4 \\ x_2^{(2)} = \dfrac{7 - x_1^{(1)} - x_3^{(1)}}{5} = \dfrac{7 - \frac{7}{5} - \frac{7}{5}}{5} = \dfrac{21}{25} = 0.84 \\ x_3^{(2)} = \dfrac{7 - x_1^{(1)} - x_2^{(1)}}{5} = \dfrac{7 - \frac{7}{5} - \frac{7}{5}}{5} = \dfrac{21}{25} = 0.84 \end{cases}$$

and the last one:

$$\begin{cases} x_1^{(3)} = \dfrac{7 - x_2^{(2)} + x_3^{(2)}}{5} = \dfrac{7 - \frac{21}{25} + \frac{21}{25}}{5} = \dfrac{7}{5} = 1.4 \\ x_2^{(3)} = \dfrac{7 - x_1^{(2)} - x_3^{(2)}}{5} = \dfrac{7 - \frac{7}{5} - \frac{21}{25}}{5} = \dfrac{119}{125} = 0.952 \cdot \\ x_3^{(3)} = \dfrac{7 - x_1^{(2)} - x_2^{(2)}}{5} = \dfrac{7 - \frac{7}{5} - \frac{21}{25}}{5} = \dfrac{119}{125} = 0.952 \end{cases}$$

Remark 3.9 Another way to compute is if we consider $A = D - L - U$, with

$$D = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix}, \quad -L = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \quad -U = \begin{pmatrix} 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

then we have

$$x^{(k+1)} = D^{-1}(L+U)x^{(k)} + D^{-1}b.$$

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b) Gauss-Seidel method in 2 steps ;

The method converges because A is diagonally dominant. The difference between Jacobi and Gauss-Seidel is that in this case, we have to replace the unknowns with their most recent approximations. So, if we are at the step k, when we compute $x_3^{(k)}$, we won't use x_1 and x_2 from the previous step $(x_1^{(k-1)}, x_2^{(k-1)})$, but instead we will use their values from the current step, since we have already determined them. Using again (3.1), we obtain

$$\begin{cases} x_1^{(1)} = \dfrac{7 - x_2^{(0)} + x_3^{(0)}}{5} = \dfrac{7 - 0 + 0}{5} = \dfrac{7}{5} = 1.4 \\ x_2^{(1)} = \dfrac{7 - x_1^{(1)} - x_3^{(0)}}{5} = \dfrac{7 - \frac{7}{5} - 0}{5} = \dfrac{28}{25} = 1.12 \\ x_3^{(1)} = \dfrac{7 - x_1^{(1)} - x_2^{(1)}}{5} = \dfrac{7 - \frac{7}{5} - \frac{28}{25}}{5} = \dfrac{112}{125} = 0.896 \end{cases}$$

Next, we have:

$$\begin{cases} x_1^{(2)} = \dfrac{7 - x_2^{(1)} + x_3^{(1)}}{5} = \dfrac{7 - \frac{28}{25} + \frac{112}{125}}{5} = \dfrac{847}{625} = 1.3552 \\ x_2^{(2)} = \dfrac{7 - x_1^{(2)} - x_3^{(1)}}{5} = \dfrac{7 - \frac{847}{625} - \frac{112}{125}}{5} = \dfrac{2968}{3125} = 0.94976 \\ x_3^{(2)} = \dfrac{7 - x_1^{(2)} - x_2^{(2)}}{5} = \dfrac{7 - \frac{847}{625} - \frac{2968}{3125}}{5} = \dfrac{14672}{15625} = 0.939008 \end{cases}$$

Remark 3.10 Again if we consider $A = D - L - U$, with

$$D = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix}, \quad -L = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \quad -U = \begin{pmatrix} 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

we have

$$x^{(k+1)} = (D-L)^{-1}Ux^{(k)} + (D-L)^{-1}b.$$

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c) SOR method for $\omega = \frac{1}{2}$ in 2 steps .

It is similar to Gauss-Seidel method. First, we compute an intermediary point $\tilde{x}^{(k)}$ as in Gauss-

Seidel and then $x^{(k)} = \omega\tilde{x}^{(k)} + (1-\omega)x^{(k-1)}$. So, for (3.1), we have:

$$\begin{cases} \tilde{x_1}^{(1)} = \dfrac{7 - x_2^{(0)} + x_3^{(0)}}{5} = \dfrac{7-0+0}{5} = \dfrac{7}{5} = 1.4 \\ x_1^{(1)} = \omega\tilde{x_1}^{(1)} + (1-\omega)x_1^{(0)} = \dfrac{1}{2}\cdot 1.4 + \dfrac{1}{2}\cdot 0 = 0.7 \\ \tilde{x_2}^{(1)} = \dfrac{7 - x_1^{(1)} - x_3^{(0)}}{5} = \dfrac{7-0.7-0}{5} = 1.26 \\ x_2^{(1)} = \omega\tilde{x_2}^{(1)} + (1-\omega)x_2^{(0)} = \dfrac{1}{2}\cdot 1.26 + \dfrac{1}{2}\cdot 0 = 0.63 \\ \tilde{x_3}^{(1)} = \dfrac{7 - x_1^{(1)} - x_2^{(1)}}{5} = \dfrac{7-0.7-0.63}{5} = 1.134 \\ x_3^{(1)} = \omega\tilde{x_3}^{(1)} + (1-\omega)x_3^{(0)} = \dfrac{1}{2}\cdot 1.134 + \dfrac{1}{2}\cdot 0 = 0.567 \end{cases}$$

And the second iteration is

$$\begin{cases} \tilde{x_1}^{(2)} = \dfrac{7 - x_2^{(1)} + x_3^{(1)}}{5} = \dfrac{7-0.63+0.567}{5} = 1.3874 \\ x_1^{(2)} = \omega\tilde{x_1}^{(2)} + (1-\omega)x_1^{(1)} = \dfrac{1}{2}\cdot 1.3874 + \dfrac{1}{2}\cdot 0.7 = 1.0437 \\ \tilde{x_2}^{(2)} = \dfrac{7 - x_1^{(2)} - x_3^{(1)}}{5} = \dfrac{7-1.0437-0.567}{5} = 1.07786 \\ x_2^{(2)} = \omega\tilde{x_2}^{(2)} + (1-\omega)x_2^{(1)} = \dfrac{1}{2}\cdot 1.07786 + \dfrac{1}{2}\cdot 0.63 = 0.85393 \\ \tilde{x_3}^{(2)} = \dfrac{7 - x_1^{(2)} - x_2^{(2)}}{5} = \dfrac{7-1.0437-0.85393}{5} = 1.020474 \\ x_3^{(2)} = \omega\tilde{x_3}^{(2)} + (1-\omega)x_3^{(1)} = \dfrac{1}{2}\cdot 1.020474 + \dfrac{1}{2}\cdot 0.567 = 0.793737 \end{cases}$$

Remark 3.11 The exact solution is (1.4; 0.9(3); 0.9(3)).

Remark 3.12 See the **theory** and other **examples** for the three Iterative methods in Lecture 10, pp. 4–12.