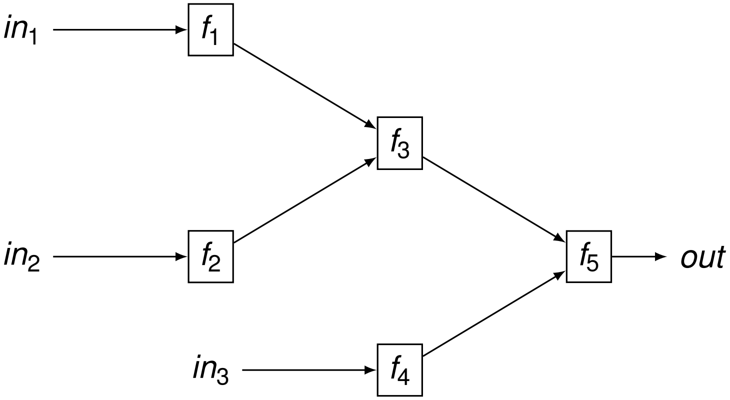
Part I

Introduction

Functional programming features

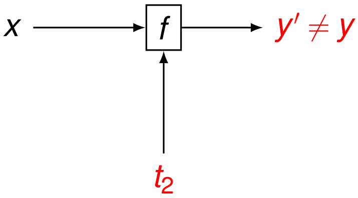
- Mathematical functions, as value transformers
- Functions as first-class values
- No side effects or state

Functional flow



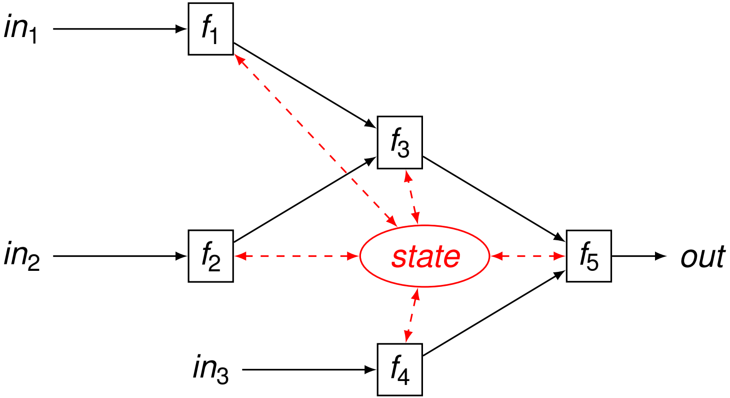
Stateful computation

Output dependent on input and time:



Functional flow

Impure



Functional programming features

- Mathematical functions, as value transformers
- Functions as first-class values
- No side effects or state
- Immutability
- Referential transparency
- Recursion
- Higher-order functions
- Lazy evaluation

Why functional programming?

- Simple evaluation model; equational reasoning
- Declarative
- Modularity, composability, reuse (lazy evaluation as glue)
- Exploration of huge or formally infinite search spaces
- Embedded Domain Specific Languages (EDSLs)
- Massive parallelization
- Type systems and logic, inextricably linked
- Automatic program verification and synthesis

Part II

Untyped Lambda Calculus

Untyped lambda calculus

- Model of computation — Alonzo Church, 1932
- Equivalent to the Turing machine (see the Church-Turing thesis)
- Main building block: the function
- Computation: evaluation of function applications, through textual substitution
- Evaluate = obtain a value (a *function*)!
- No side effects or state

Applications

- Theoretical basis of numerous languages:
  - LISP
  - Scheme
  - Haskell
  - ML
  - F#
  - Clean
  - Clojure
  - Scala
  - Erlang

- Formal program verification, due to its simple execution model

Definition 4.1 (λ-expression).

- Variable: a variable x is a λ-expression
- Function: if x is a variable and E is a λ-expression, then $\lambda x.E$ is a λ-expression, which stands for an anonymous, unary function, with the formal parameter x and the body E
- Application: if E and A are λ-expressions, then $(E \; A)$ is a λ-expression, which stands for the application of the expression E onto the actual argument A.

λ-expressions

Examples

Example 4.2 (λ-expressions).

- $x \rightarrow$ variable x
- $\lambda x.x$: the identity function
- $\lambda x.\lambda y.x$: a function with another function as body!
- $(\lambda x.x \ \ y)$: the application of the identity function onto the actual argument y
- $(\lambda x.(x \ \ x) \ \ \lambda x.x)$

Intuition on application evaluation

$$(\lambda\, x\, .\; x \quad y) \rightarrow y$$

Variable occurrences

Definitions

Definition 4.3 (Bound occurrence).

An occurrence x_n of a variable x is bound in the expression E iff:

- $E = \lambda x.F$ or
- $E = \dots \lambda x_n.F \dots$ or
- $E = \dots \lambda x.F \dots$ and x_n appears in F.

Definition 4.4 (Free occurrence).

A variable occurrence is free in an expression iff it is not bound in that expression.

Bound/ free occurrence w.r.t. a given expression!

Variable occurrences

Examples

Example 4.6 (Bound and free variables).

In the expression $E = (\lambda x.\lambda z.(z\ \ x)\ \ (z\ \ y))$, we emphasize the occurrences of x, y, z:

$$E = (\lambda x_1.\overbrace{\lambda z_1.(z_2\ \ x_2)}^{F}\ \ (z_3\ \ y_1)).$$

- x_1, x_2, z_1, z_2 bound in E
- y_1, z_3 free in E
- z_1, z_2 bound in F
- x_2 free in F

Definition 4.7 (Bound variable).

A variable is bound in an expression iff all its occurrences are bound in that expression.

Definition 4.8 (Free variable).

A variable is free in an expression iff it is not bound in that expression i.e., iff at least one of its occurrences is free in that expression.

Bound/ free variable w.r.t. a given expression!

Variable occurrences

Examples

Example 4.6 (Bound and free variables).

In the expression $E = (\lambda x.\lambda z.(z\ x)\ (z\ y))$, we emphasize the occurrences of x, y, z:

$$E = (\lambda x_1.\overbrace{\lambda z_1.(z_2\ x_2)}^{F}\ (z_3\ y_1)).$$

- x_1, x_2, z_1, z_2 bound in E
- y_1, z_3 free in E
- z_1, z_2 bound in F
- x_2 free in F
- x bound in E, but free in F
- y free in E
- z free in E, but bound in F

Free and bound variables

Free variables

- $FV(x) = \{x\}$
- $FV(\lambda x.E) = FV(E) \setminus \{x\}$
- $FV((E_1 \;\; E_2)) = FV(E_1) \cup FV(E_2)$

Bound variables

- $BV(x) = \emptyset$
- $BV(\lambda x.E) = BV(E) \cup \{x\}$
- $BV((E_1 \;\; E_2)) = BV(E_1) \setminus FV(E_2) \cup BV(E_2) \setminus FV(E_1)$

Closed expressions

Definition 4.9 (Closed expression).

An expression that does not contain any free variables.

Example 4.10 (Closed expressions).

- $(\lambda x.x \;\; \lambda x.\lambda y.x)$: closed
- $(\lambda x.x \;\; a)$: open, since a is free

Remarks:

- Free variables may stand for other λ-expressions, as in $\lambda x.((+ \;\; x) \;\; 1)$.
- Before evaluation, an expression must be brought to the closed form.
- The substitution process must terminate.

Definition 5.1 (β-reduction).

The evaluation of the application $(\lambda x.E\ A)$, by substituting every free occurrence of the formal argument, x, in the function body, E, with the actual argument, A:
$(\lambda x.E\ A) \rightarrow_\beta E_{[A/x]}$.

Definition 5.2 (β-redex).

The application $(\lambda x.E\ A)$.

β-reduction

Examples

Example 5.3 (β-reduction).

- $(\lambda x.x \;\; y) \rightarrow_\beta x_{[y/x]} \rightarrow y$
- $(\lambda x.\lambda x.x \;\; y) \rightarrow_\beta \lambda x.x_{[y/x]} \rightarrow \lambda x.x$
- $(\lambda x.\lambda y.x \;\; y) \rightarrow_\beta \lambda y.x_{[y/x]} \rightarrow \lambda y.y$

  Wrong! The free variable y becomes bound, changing its meaning!

β-reduction

Collisions

- Problem: within the expression $(\lambda x.E\ \ A)$:
  - $FV(A) \cap BV(E) = \emptyset \Rightarrow$ correct reduction always
  - $FV(A) \cap BV(E) \neq \emptyset \Rightarrow$ potentially wrong reduction
- Solution: rename the bound variables in E, that are free in A

Example 5.4 (Bound variable renaming).

$(\lambda x.\lambda y.x\ \ y) \rightarrow (\lambda x.\lambda z.x\ \ y) \rightarrow_\beta \lambda z.x_{[y/x]} \rightarrow \lambda z.y$

Definition 5.5 (α-conversion).

Systematic relabeling of bound variables in a function: $\lambda x.E \rightarrow_\alpha \lambda y.E_{[y/x]}$. Two conditions must be met.

Example 5.6 (α-conversion).

- $\lambda x.y \rightarrow_\alpha \lambda y.y_{[y/x]} \rightarrow \lambda y.y$: Wrong!
- $\lambda x.\lambda y.x \rightarrow_\alpha \lambda y.\lambda y.x_{[y/x]} \rightarrow \lambda y.\lambda y.y$: Wrong!

Conditions:

- y is not free in E
- a free occurrence in E stays free in $E_{[y/x]}$

α-conversion

Examples

Example 5.7 (α-conversion).

- $\lambda x.(x\ \ y) \rightarrow_\alpha \lambda z.(z\ \ y)$: Correct!
- $\lambda x.\lambda x.(x\ \ y) \rightarrow_\alpha \lambda y.\lambda x.(x\ \ y)$: Wrong!
  y is free in $\lambda x.(x\ \ y)$.
- $\lambda x.\lambda y.(y\ \ x) \rightarrow_\alpha \lambda y.\lambda y.(y\ \ y)$: Wrong!
  The free occurrence of x in $\lambda y.(y\ \ x)$ becomes bound, after substitution, in $\lambda y.(y\ \ y)$.
- $\lambda x.\lambda y.(y\ \ y) \rightarrow_\alpha \lambda y.\lambda y.(y\ \ y)$: Correct!

Definition 5.8 (Reduction step).

A sequence made of a possible α-conversion, followed by a β-reduction, such that the second produces no collisions: $E_1 \rightarrow E_2 \;\equiv\; E_1 \rightarrow_\alpha E_3 \rightarrow_\beta E_2$.

Definition 5.9 (Reduction sequence).

A string of zero or more reduction steps: $E_1 \rightarrow^* E_2$. It is an element of the reflexive transitive closure of relation $\rightarrow$.

Example 5.10 (Reduction).

- $((\lambda x.\lambda y.(y\ \ x)\ \ y)\ \ \lambda x.x)$
  $\rightarrow (\lambda z.(z\ \ y)\ \ \lambda x.x)$
  $\rightarrow (\lambda x.x\ \ y)$
  $\rightarrow y$

- $((\lambda x.\lambda y.(y\ \ x)\ \ y)\ \ \lambda x.x) \rightarrow^{*} y$

Reduction

Properties

- Reduction step = reduction sequence:

$$E_1 \rightarrow E_2 \Rightarrow E_1 \rightarrow^* E_2$$

- Reflexivity:

$$E \rightarrow^* E$$

- Transitivity:

$$E_1 \rightarrow^* E_2 \wedge E_2 \rightarrow^* E_3 \Rightarrow E_1 \rightarrow^* E_3$$

Questions

1. When does the computation terminate?
   Does it always?

2. Does the answer depend on the reduction sequence?

3. If the computation terminates for distinct reduction sequences, do we always get the same result?

4. If the result is unique, how do we safely obtain it?

Normal forms

Definition 6.1 (Normal form).

The form of an expression that cannot be reduced i.e., that contains no β-redexes.

Definition 6.2 (Functional normal form, FNF).

$\lambda x.E$, even if E contains β-redexes.

Example 6.3 (Normal forms).

$(\lambda x.\lambda y.(x\ \ y)\ \lambda x.x) \rightarrow_{\text{FNF}} \lambda y.(\lambda x.x\ \ y) \rightarrow_{\text{NF}} \lambda y.y$

FNF is used in programming, where the function body is evaluated only when the function is effectively applied.

Reduction termination (reducibility)

Example 6.4.

$\Omega \equiv (\lambda x.(x\ \ x)\ \ \lambda x.(x\ \ x)) \to (\lambda x.(x\ \ x)\ \ \lambda x.(x\ \ x)) \to^{*} \ldots$

Ω does not have a terminating reduction sequence.

Definition 6.5 (Reducible expression).

An expression that has a terminating reduction sequence.

Ω is irreducible.

Questions

1. When does the computation terminate?
   Does it always?

2. Does the answer depend on the reduction sequence?

3. If the computation terminates for distinct reduction sequences, do we always get the same result?

4. If the result is unique, how do we safely obtain it?

Reduction sequences

Example 6.6 (Reduction sequences).

$$E = (\lambda x.y\ \Omega)$$

- $\xrightarrow{1} y$
- $\xrightarrow{2} E \xrightarrow{1} y$
- $\xrightarrow{2} E \xrightarrow{2} E \xrightarrow{1} y$
- $\ldots$
- $\xrightarrow{2^n 1}{}^{*}\ y,\ n \geq 0$
- $\xrightarrow{2^\infty}{}^{*}\ \ldots$

- E has a nonterminating reduction sequence, but still has a normal form, y. E is reducible, Ω is not.
- The length of terminating reduction sequences is unbounded.

Normal form uniqueness

Results

Theorem 6.7 (Church-Rosser / diamond).

If $E \rightarrow^* E_1$ and $E \rightarrow^* E_2$, then there is *an E_3 such that $E_1 \rightarrow^* E_3$ and $E_2 \rightarrow^* E_3$.*

$$\begin{tikzcd}
 & E_1 \arrow[dr, "*"] & \\
E \arrow[ur, "*"] \arrow[dr, "*"'] & & E_3 \\
 & E_2 \arrow[ur, "*"'] &
\end{tikzcd}$$

Corollary 6.8 (Normal form uniqueness).

If an expression is reducible, its normal form is unique*. It corresponds to the* value *of that expression.*

Normal form uniqueness

Examples

Example 6.9 (Normal form uniqueness).

$$(\lambda x.\lambda y.(x\ y)\ (\lambda x.x\ y))$$

- $\rightarrow \lambda z.((\lambda x.x\ y)\ z) \rightarrow \lambda z.(y\ z) \rightarrow_\alpha \lambda a.(y\ a)$
- $\rightarrow (\lambda x.\lambda y.(x\ y)\ y) \rightarrow \lambda w.(y\ w) \rightarrow_\alpha \lambda a.(y\ a)$

- Normal form: class of expressions, equivalent under systematic relabeling
- Value: distinguished member of this class

Structural equivalence

Definition 6.10 (Structural equivalence).

Two expressions are structurally equivalent iff they both reduce to the same expression.

Example 6.11 (Structural equivalence).

$\lambda z.((\lambda x.x \ \ y) \ \ z)$ and $(\lambda x.\lambda y.(x \ \ y) \ \ y)$ in Example 6.9.

Computational equivalence

Definition 6.12 (Computational equivalence).

Two expressions are computationally equivalent iff they the behave in the same way when applied onto the same arguments.

Example 6.13 (Computational equivalence).

$$E_1 = \lambda y.\lambda x.(y\ \ x)$$
$$E_2 = \lambda x.x$$

- $((E_1\ \ a)\ \ b) \rightarrow^* (a\ \ b)$
- $((E_2\ \ a)\ \ b) \rightarrow^* (a\ \ b)$
- $E_1 \not\rightarrow^* E_2$ and $E_2 \not\rightarrow^* E_1$ (not structurally equivalent)

Reduction order

Definitions and examples

Definition 6.14 (Left-to-right reduction step).

The reduction of the outermost leftmost β-redex.

Example 6.15 (Left-to-right reduction).

$$(\underline{(\lambda x.x\ \lambda x.y)}\ (\lambda x.(x\ x)\ \lambda x.(x\ x))) \rightarrow \underline{(\lambda x.y\ \Omega)} \rightarrow y$$

Definition 6.16 (Right-to-left reduction step).

The reduction of the innermost rightmost β-redex.

Example 6.17 (Right-to-left reduction).

$$((\lambda x.x\ \lambda x.y)\ \underline{(\lambda x.(x\ x)\ \lambda x.(x\ x))}) \rightarrow (\lambda x.y\ \underline{\Omega}) \rightarrow \ldots$$

Questions

1. When does the computation terminate?
   Does it always?
   - NO
2. Does the answer depend on the reduction sequence?
   - YES
3. If the computation terminates for distinct reduction sequences, do we always get the same result?
   - YES
4. If the result is unique, how do we safely obtain it?
   - Left-to-right reduction

Evaluation order

Definition 7.1 (Applicative-order evaluation).
Corresponds to right-to-left reduction. Function arguments are evaluated before the function is applied.

Definition 7.2 (Strict function).
A function that uses applicative-order evaluation.

Definition 7.3 (Normal-order evaluation).
Corresponds to left-to-right reduction. Function arguments are evaluated when needed.

Definition 7.4 (Non-strict function).
A function that uses normal-order evaluation.

In practice I

Applicative-order evaluation employed in most programming languages, due to efficiency — one-time evaluation of arguments: C, Java, Scheme, PHP, etc.

Example 7.5 (Applicative-order evaluation in Scheme).

$$\texttt{((}\lambda\texttt{ (x) (+ x x)) }\underline{\texttt{(+ 2 3)}}\texttt{)}$$
$$\to \underline{\texttt{((}\lambda\texttt{ (x) (+ x x)) 5)}}$$
$$\to \underline{\texttt{(+ 5 5)}}$$
$$\to \texttt{10}$$

In practice II

Lazy evaluation (a kind of normal-order evaluation) in Haskell: on-demand evaluation of arguments, allowing for interesting constructions

Example 7.6 (Lazy evaluation in Haskell).

$\underline{\texttt{((\textbackslash x -> x + x) (2 + 3))}}$

$\rightarrow \underline{\texttt{(2 + 3)}}\ \texttt{+}\ \underline{\texttt{(2 + 3)}}$

$\rightarrow \underline{\texttt{5 + 5}}$

$\rightarrow \texttt{10}$

Need for non-strict functions, even in applicative languages: `if`, `and`, `or`, etc.

Part III

Lambda Calculus as a Programming Language

Purpose

- Proving the expressive power of lambda calculus
- Hypothetical λ-machine
- Machine code: λ-expressions — the λ_0 language
- Instead of
  - bits
  - bit operations,

  we have
  - structured strings of symbols
  - reduction — textual substitution

λ_0 features

- Instructions:
  - λ-expressions
  - top-level variable bindings: *variable* $\equiv_{\text{def}}$ *expression*
    e.g., *true* $\equiv_{\text{def}} \lambda x.\lambda y.x$
- Values represented as functions
- Expressions brought to the closed form, prior to evaluation
- Normal-order evaluation
- Functional normal form (see Definition 6.2)
- No predefined types!

Shorthands

- $\lambda x_1.\lambda x_2.\lambda \ldots.\lambda x_n.E \rightarrow \lambda x_1 x_2 \ldots x_n.E$

- $((\ldots((E\ \ A_1)\ \ A_2)\ \ \ldots)\ \ A_n) \rightarrow (E\ \ A_1\ \ A_2\ \ \ldots\ \ A_n)$

Purpose of types

- Way of expressing the programmer's intent
- Documentation: which operators act onto which objects
- Particular representation for values of different types: `1`, "`Hello`", `#t`, etc.
- Optimization of specific operations
- Error prevention
- Formal verification

No types

How are objects represented?

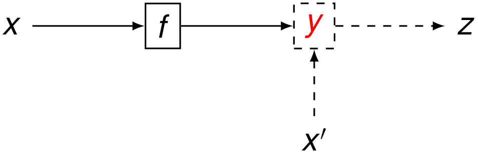
- A number, list or tree potentially designated by the same value e.g.,

$$\text{number } 3 \rightarrow \lambda x.\lambda y.x \leftarrow \text{list } (()\ ()\ ())$$

- Both values and operators represented by functions — context-dependent meaning

$$\text{number } 3 \rightarrow \lambda x.\lambda y.x \leftarrow \text{operator } car$$

- Value applicable onto another value, as an operator!



No types

How is correctness affected?

- Inability of the λ machine to
  - interpret the meaning of expressions
  - ensure their correctness
- Every operator applicable onto every value
- Both aspects above delegated to the programmer
- Erroneus constructs accepted without warning, but computation ended with
  - values with no meaning or
  - expressions that are neither values, nor reducible e.g., $(x\ \ x)$

No types

Consequences

- Enhanced representational flexibility
- Useful when the uniform representation of objects, as lists de symbols, is convenient
- Increased error-proneness
- Program instability
- Difficulty of verification and maintenance

So...

- How do we employ the λ_0 language in everyday programming?

- How do we represent usual values — numbers, booleans, lists, etc. — and their corresponding operators?

Definition

Definition 9.1 (Abstract data type, ADT).
Mathematical model of a set of values and their corresponding operations.

Example 9.2 (ADTs).
Natural, *Bool*, *List*, *Set*, *Stack*, *Tree*, ... λ-expression!

Components:

- base constructors: how are values built
- operators: what can be done with these values
- axioms: how

The *Natural* ADT

Base constructors and operators

- Base constructors:
  - $zero : \to Natural$
  - $succ : Natural \to Natural$
- Operators:
  - $zero? : Natural \to Bool$
  - $pred : Natural \setminus \{zero\} \to Natural$
  - $add : Natural^2 \to Natural$

The *Natural* ADT

Axioms

- $zero?$
  - $(zero?\ zero) = T$
  - $(zero?\ (succ\ n)) = F$
- $pred$
  - $(pred\ (succ\ n)) = n$
- add
  - $(add\ zero\ n) = n$
  - $(add\ (succ\ m)\ n) = (succ\ (add\ m\ n))$

Providing axioms

- One axiom for each (operator, base constructor) pair
- More — useless
- Less — insufficient for completely specifying the operators

From ADTs to functional programming
Exemple

- Axiome:
  - $add(zero, n) = n$
  - $add(succ(m), n) = succ(add(m, n))$
- Scheme:

```scheme
(define add
  (lambda (m n)
    (if (zero? m) n
        (+ 1 (add (- m 1) n)))))
```

- Haskell:

```haskell
add 0 n = n
add (m + 1) n = 1 + (add m n)
```

From ADTs to functional programming

Discussion

- Proving ADT correctness
  — structural induction
- Proving properties of λ-expressions, seen as values of an ADT with 3 base constructors!
- Functional programming
  — reflection of mathematical specifications
- Recursion
  — natural instrument, inherited from axioms
- Applying formal methods on the recursive code, taking advantage of the lack of side effects

The *Bool* ADT

Base contrsuctors and operators

- Base constructors:
  - $T : \to Bool$
  - $F : \to Bool$
- Operators:
  - $not : Bool \to Bool$
  - $and : Bool^2 \to Bool$
  - $or : Bool^2 \to Bool$
  - $if : Bool \times T \times T \to T$

The *Bool* ADT

Axioms

- *not*
  - $(not\ \ T) = F$
  - $(not\ \ F) = T$
- *and*
  - $(and\ \ T\ \ a) = a$
  - $(and\ \ F\ \ a) = F$
- *or*
  - $(or\ \ T\ \ a) = T$
  - $(or\ \ F\ \ a) = a$
- *if*
  - $(if\ \ T\ \ a\ \ b) = a$
  - $(if\ \ F\ \ a\ \ b) = b$

The *Bool* ADT

Base constructor implementation

- Intuition: selecting one of the two values, *true* or *false*
- $T \equiv_{\text{def}} \lambda xy.x$
- $F \equiv_{\text{def}} \lambda xy.y$
- Selector-like behavior:
  - $(T\ \ a\ \ b) \rightarrow (\lambda xy.x\ \ a\ \ b) \rightarrow a$
  - $(F\ \ a\ \ b) \rightarrow (\lambda xy.y\ \ a\ \ b) \rightarrow b$

The *Bool* ADT

Operator implementation

- $not \equiv_{\text{def}} \lambda x.(x\ \ F\ \ T)$
  - $(not\ \ T) \rightarrow (\lambda x.(x\ \ F\ \ T)\ \ T) \rightarrow (T\ \ F\ \ T) \rightarrow F$
  - $(not\ \ F) \rightarrow (\lambda x.(x\ \ F\ \ T)\ \ F) \rightarrow (F\ \ F\ \ T) \rightarrow T$
- $and \equiv_{\text{def}} \lambda xy.(x\ \ y\ \ F)$
  - $(and\ \ T\ \ a) \rightarrow (\lambda xy.(x\ \ y\ \ F)\ \ T\ \ a) \rightarrow (T\ \ a\ \ F) \rightarrow a$
  - $(and\ \ F\ \ a) \rightarrow (\lambda xy.(x\ \ y\ \ F)\ \ F\ \ a) \rightarrow (F\ \ a\ \ F) \rightarrow F$
- $or \equiv_{\text{def}} \lambda xy.(x\ \ T\ \ y)$
  - $(or\ \ T\ \ a) \rightarrow (\lambda xy.(x\ \ T\ \ y)\ \ T\ \ a) \rightarrow (T\ \ T\ \ a) \rightarrow T$
  - $(or\ \ F\ \ a) \rightarrow (\lambda xy.(x\ \ T\ \ y)\ \ F\ \ a) \rightarrow (F\ \ T\ \ a) \rightarrow a$
- $if \equiv_{\text{def}} \lambda cte.(c\ \ t\ \ e)$ non-strict!
  - $(if\ \ T\ \ a\ \ b) \rightarrow (\lambda cte.(c\ \ t\ \ e)\ \ T\ \ a\ \ b) \rightarrow (T\ \ a\ \ b) \rightarrow a$
  - $(if\ \ F\ \ a\ \ b) \rightarrow (\lambda cte.(c\ \ t\ \ e)\ \ F\ \ a\ \ b) \rightarrow (F\ \ a\ \ b) \rightarrow b$

The *Pair* ADT

Specification

- Base constructors:
  - $pair : A \times B \to Pair$
- Operators:
  - $fst : Pair \to A$
  - $snd : Pair \to B$
- Axioms:
  - $(fst\ \ (pair\ \ a\ \ b)) = a$
  - $(snd\ \ (pair\ \ a\ \ b)) = b$

The *Pair* ADT

Implementation

- Intuition: a pair = a function that expects a selector, in order to apply it onto its components
- $pair \equiv_{\text{def}} \lambda xys.(s\ x\ y)$
  - $(pair\ a\ b) \rightarrow (\lambda xys.(s\ x\ y)\ a\ b) \rightarrow \lambda s.(s\ a\ b)$
- $fst \equiv_{\text{def}} \lambda p.(p\ T)$
  - $(fst\ (pair\ a\ b)) \rightarrow (\lambda p.(p\ T)\ \lambda s.(s\ a\ b)) \rightarrow (\lambda s.(s\ a\ b)\ T) \rightarrow (T\ a\ b) \rightarrow a$
- $snd \equiv_{\text{def}} \lambda p.(p\ F)$
  - $(snd\ (pair\ a\ b)) \rightarrow (\lambda p.(p\ F)\ \lambda s.(s\ a\ b)) \rightarrow (\lambda s.(s\ a\ b)\ F) \rightarrow (F\ a\ b) \rightarrow b$

The *List* ADT

Base constructors and operators

- Base constructors:
  - $null : \to List$
  - $cons : A \times List \to List$

- Operators:
  - $car : List \setminus \{null\} \to A$
  - $cdr : List \setminus \{null\} \to List$
  - $null? : List \to Bool$
  - $append : List^2 \to List$

The *List* ADT
Axioms

- *car*
  - $(car\ (cons\ e\ L)) = e$
- *cdr*
  - $(cdr\ (cons\ e\ L)) = L$
- *null?*
  - $(null?\ null) = T$
  - $(null?\ (cons\ e\ L)) = F$
- *append*
  - $(append\ null\ B) = B$
  - $(append\ (cons\ e\ A)\ B) = (cons\ e\ (append\ A\ B))$

The *List* ADT

Implementation

- Intuition: a list = a (head, tail) pair
- $null \equiv_{\text{def}} \lambda x.T$
- $cons \equiv_{\text{def}} pair$
- $car \equiv_{\text{def}} fst$
- $cdr \equiv_{\text{def}} snd$
- $null? \equiv_{\text{def}} \lambda L.(L\ \lambda xy.F)$
  - $(null?\ null) \rightarrow (\lambda L.(L\ \lambda xy.F)\ \lambda x.T) \rightarrow (\lambda x.T\ \ldots) \rightarrow T$
  - $(null?\ (cons\ e\ L)) \rightarrow (\lambda L.(L\ \lambda xy.F)\ \lambda s.(s\ e\ L)) \rightarrow$
    $(\lambda s.(s\ e\ L)\ \lambda xy.F) \rightarrow (\lambda xy.F\ e\ L) \rightarrow F$
- $append \equiv_{\text{def}}$ ... no closed form
  $\lambda AB.(if\ (null?\ A)\ B\ (cons\ (car\ A)\ (append\ (cdr\ A)\ B)))$

The *Natural* ADT

Axioms

- *zero?*
  - $(zero?\ zero) = T$
  - $(zero?\ (succ\ n)) = F$
- *pred*
  - $(pred\ (succ\ n)) = n$
- *add*
  - $(add\ zero\ n) = n$
  - $(add\ (succ\ m)\ n) = (succ\ (add\ m\ n))$

The *Natural* ADT

Implementation

- Intuition: a number = a list having the number value as its length
- $zero \equiv_{\text{def}} null$
- $succ \equiv_{\text{def}} \lambda n.(cons\ \ null\ \ n)$
- $zero? \equiv_{\text{def}} null?$
- $pred \equiv_{\text{def}} cdr$
- $add \equiv_{\text{def}} append$

Functions

- Several possible definitions of the identity function:
  - $id(n) = n$
  - $id(n) = n + 1 - 1$
  - $id(n) = n + 2 - 2$
  - $\dots$
- Infinitely many textual representations for the same function
- Then... what is a function? A relation between inputs and outputs, independent of any textual representation e.g., $id = \{(0,0),(1,1),(2,2),\dots\}$

Perspectives on recursion

- Textual: a function that refers itself, using its name
- Constructivist: recursive functions as values of an ADT, with specific ways of building them
- Semantic: the mathematical object designated by a recursive function

Implementing *length*

Problem

- Length of a list:

  $$length \equiv_{\text{def}} \lambda L.(if\ (null?\ L)\ zero\ (succ\ (\underline{length}\ (cdr\ L))))$$

- What do we replace the underlined area with, to avoid textual recursion?
- Rewrite the definition as a fixed-point equation

  $$Length \equiv_{\text{def}} \lambda f L.(if\ (null?\ L)\ zero\ (succ\ (f\ (cdr\ L))))$$

  $$(Length\ length) \rightarrow length$$

- How do we compute the fixed point? (see code archive)

Axiomatization benefits

- Disambiguation
- Proof of properties
- Implementation skeleton

Syntax

- Variable:

$$Var ::= \text{ any symbol distinct from } \lambda, ., (,)$$

- Expression:

$$\begin{aligned} Expr ::= \; & Var \\ | \; & \lambda Var.Expr \\ | \; & (Expr \;\; Expr) \end{aligned}$$

- Value:

$$Val ::= \lambda Var.Expr$$

Evaluation rules

Rule name:

$$\frac{\textit{precondition}_1, \ldots, \textit{precondition}_n}{\textit{conclusion}}$$

Semantics for normal-order evaluation

Evaluation

- *Reduce*:

$$(\lambda x.e \;\; e') \rightarrow e_{[e'/x]}$$

- *Eval*:

$$\frac{e \rightarrow e'}{({\color{red}e} \;\; e'') \rightarrow ({\color{red}e'} \;\; e'')}$$

Semantics for normal-order evaluation

Substitution

- $x_{[e/x]} = e$
- $y_{[e/x]} = y, \quad y \neq x$
- $\langle \lambda x.e \rangle_{[e'/x]} = \lambda x.e$
- $\langle \lambda y.e \rangle_{[e'/x]} = \lambda y.e_{[e'/x]}, \quad y \neq x \land y \notin FV(e')$
- $\langle \lambda y.e \rangle_{[e'/x]} = \lambda z.e_{[z/y][e'/x]},$
  $y \neq x \land y \in FV(e') \land z \notin FV(e) \cup FV(e')$
- $(e' \; e'')_{[e/x]} = (e'_{[e/x]} \; e''_{[e/x]})$

Semantics for normal-order evaluation

Free variables

- $FV(x) = \{x\}$
- $FV(\lambda x.e) = FV(e) \setminus \{x\}$
- $FV((e' \; e'')) = FV(e') \cup FV(e'')$

Semantics for normal-order evaluation

Example

Example 12.1 (Evaluation rules).

$$((\lambda x.\lambda y.y \;\; a) \;\; b)$$

$$\dfrac{(\lambda x.\lambda y.y \;\; a) \rightarrow \lambda y.y \quad (\mathit{Reduce})}{((\lambda x.\lambda y.y \;\; a) \;\; b) \rightarrow (\lambda y.y \;\; b)} \quad (\mathit{Eval})$$

$$(\lambda y.y \;\; b) \rightarrow b \quad (\mathit{Reduce})$$

Semantics for applicative-order evaluation

- *Reduce* ($v \in Val$):

$$(\lambda x.e \;\; v) \to e_{[v/x]}$$

- $Eval_1$:

$$\frac{e \to e'}{(e \;\; e'') \to (e' \;\; e'')}$$

- $Eval_2$ ($e \notin Val$):

$$\frac{e \to e'}{(\lambda x.e'' \;\; e) \to (\lambda x.e'' \;\; e')}$$

Formal proof

Proposition 12.2 (Free and bound variables).

$$\forall e \in Expr \bullet BV(e) \cap FV(e) = \emptyset$$

Proof.
Structural induction, according to the different forms of λ-expressions (see the lecture notes). $\square$

Summary

- Practical usage of the untyped lambda calculus, as a programming language
- Formal specifications, for different evaluation semantics

Part IV

Typed Lambda Calculus

Drawbacks of the absence of types

- Meaningless expressions e.g., $(car\ 3)$
- No canonical representation for the values of a given type e.g., both a tree and a set having the same representation
- Impossibility of translating certain expressions into certain typed languages e.g., $(x\ x)$, Ω, *Fix*
- Potential irreducibility of expressions — inconsistent representation of equivalent values

$$\lambda x.(Fix\ x) \rightarrow \lambda x.(x\ (Fix\ x)) \rightarrow \lambda x.(x\ (x\ (Fix\ x))) \rightarrow \ldots$$

Solution

- Restricted ways of constructing expressions, depending on the types of their parts
- Sacrificed expressivity in change for soundness

Desired properties

Definition 13.1 (Progress).
A well-typed expression is either a value or is subject to at least one reduction step.

Definition 13.2 (Preservation).
The result obtained by reducing a well-typed expression is well-typed. Usually, the type is the same.

Definition 13.3 (Strong normalization).
The evaluation of a well-typed expression terminates.

Base and simple types

Definition 14.1 (Base type).

An atomic type e.g., numbers, booleans etc.

Definition 14.2 (Simple type).

A type built from existing types e.g., $\sigma \to \tau$, where σ and τ are types.

Notation:

- $e : \tau$: "expression e has type τ"
- $v \in \tau$: "v is a value of type τ"
- $e \in \tau \Rightarrow e : \tau$
- $e : \tau \not\Rightarrow e \in \tau$

Typed λ-expressions

Definition 14.3 (λ_t-expression).

- Base value: a base value $b \in \tau_b$ is a λ_t-expression.
- Typed variable: an (explicitly) typed variable $x : \tau$ is a λ_t-expression.
- Function: if $x : \sigma$ is a typed variable and $e : \tau$ is a λ_t-expression, then $\lambda x : \sigma . e : \sigma \rightarrow \tau$ is a λ_t-expression, which stands for ....
- Application: if $f : \sigma \rightarrow \tau$ and $a : \sigma$ are λ_t-expressions, then $(f \;\; a) : \tau$ is a λ_t-expression, which stands for ....

Relation to untyped lambda calculus

Similarities

- β-reduction
- α-conversion
- normal forms
- Church-Rosser theorem

Differences

- $(x : \tau \;\; x : \tau)$ invalid
- some fixed-point combinators are invalid

Syntax

Expressions

- Variables:

$$Var \quad ::= \quad \ldots$$

- Expressions:

$$\begin{array}{rcl} Expr & ::= & Val \\ & | & Var \\ & | & (Expr\ \ Expr) \end{array}$$

- Values:

$$\begin{array}{rcl} Val & ::= & \color{red}{BaseVal} \\ & | & \lambda Var \color{red}{:} \color{red}{Type}.Expr \end{array}$$

Syntax
Types

- Types:

$$
\begin{array}{lcl}
\mathit{Type} & ::= & \mathit{BaseType} \\
 & | & (\mathit{Type} \rightarrow \mathit{Type})
\end{array}
$$

- Typing contexts:
  - include variable-type associations
    i.e., *typing hypotheses*

$$
\begin{array}{lcl}
\mathit{TypingContext} & ::= & \emptyset \\
 & | & \mathit{TypingContext}, \mathit{Var} : \mathit{Type}
\end{array}
$$

Semantics for normal-order evaluation

Evaluation

- *Reduce*:

$$(\lambda x : \tau . e \;\; e') \rightarrow e_{[e'/x]}$$

- *Eval*:

$$\frac{e \rightarrow e'}{(e \;\; e'') \rightarrow (e' \;\; e'')}$$

The type annotations are ignored, since typing precedes evaluation.

Semantics

Typing

- *TBaseVal*:

$$\frac{v \in \tau_b}{\Gamma \vdash v : \tau_b}$$

- *TVar*:

$$\frac{x : \tau \in \Gamma}{\Gamma \vdash x : \tau}$$

- *TAbs*:

$$\frac{\Gamma, x : \tau \vdash e : \tau'}{\Gamma \vdash \lambda x : \tau . e : (\tau \rightarrow \tau')}$$

- *TApp*:

$$\frac{\Gamma \vdash e : (\tau' \rightarrow \tau) \qquad \Gamma \vdash e' : \tau'}{\Gamma \vdash (e\ e') : \tau}$$

Typing example

Example 14.4 (Typing).

$$\lambda x : \tau_1 . \lambda y : \tau_2 . x : (\tau_1 \rightarrow (\tau_2 \rightarrow \tau_1))$$

Blackboard!

Type systems

Definition 14.5 (Type system).

The set of rules and mechanisms used in a programming language to organize, build and handle the types accepted in the language.

Definition 14.6 (Soundness).

The type system of a language is *sound* if any well-typed expression in the language has the progress and preservation properties.

Proposition 14.7.

STLC is sound and possesses the strong normalization property.

Ways of extending STLC

1. Particular base types

2. n-ary type constructors, $n \geq 1$, which build simple types

The product type

Algebraic specification

- Base constructors i.e., canonical values:
  - $\tau * \tau' ::= (\tau, \tau')$
- Operators:
  - $\mathit{fst} : \tau * \tau' \to \tau$
  - $\mathit{snd} : \tau * \tau' \to \tau'$
- Axioms ($e : \tau$, $e' : \tau'$):
  - $(\mathit{fst}\ (e, e')) \to e$
  - $(\mathit{snd}\ (e, e')) \to e'$

The product type
Syntax

$$
\begin{array}{rcl}
\mathit{Expr} & ::= & \ldots \\
& \mid & (\mathit{fst}\ \ \mathit{Expr}) \\
& \mid & (\mathit{snd}\ \ \mathit{Expr}) \\
& \mid & (\mathit{Expr}, \mathit{Expr})
\end{array}
$$

$$
\begin{array}{rcl}
\mathit{BaseVal} & ::= & \ldots \\
& \mid & \mathit{ProductVal}
\end{array}
$$

$$
\mathit{ProductVal} \quad ::= \quad (\mathit{Val}, \mathit{Val})
$$

$$
\begin{array}{rcl}
\mathit{Type} & ::= & \ldots \\
& \mid & (\mathit{Type} * \mathit{Type})
\end{array}
$$

The product type

Evaluation

- *EvalFst*:

$$(\mathit{fst}\ \ (e, e')) \to e$$

- *EvalSnd*:

$$(\mathit{snd}\ \ (e, e')) \to e'$$

- *EvalFstApp*:

$$\frac{e \to e'}{(\mathit{fst}\ \ e) \to (\mathit{fst}\ \ e')}$$

- *EvalSndApp*:

$$\frac{e \to e'}{(\mathit{snd}\ \ e) \to (\mathit{snd}\ \ e')}$$

The product type

Typing

- *TProduct*:

$$\frac{\Gamma \vdash e : \tau \qquad \Gamma \vdash e' : \tau'}{\Gamma \vdash (e, e') : (\tau * \tau')}$$

- *TFst*:

$$\frac{\Gamma \vdash e : (\tau * \tau')}{\Gamma \vdash (\mathit{fst}\ \ e) : \tau}$$

- *TSnd*:

$$\frac{\Gamma \vdash e : (\tau * \tau')}{\Gamma \vdash (\mathit{snd}\ \ e) : \tau'}$$

The product type

Typing example

Example 15.1 (Typing).

$$\begin{aligned}\Gamma \vdash \; & \lambda x : ((\rho * \tau) \to \sigma).\lambda y : \rho.\lambda z : \tau.(x \; (y, z)) \\ & : ((\rho * \tau) \to \sigma) \to \rho \to \tau \to \sigma\end{aligned}$$

Blackboard!

The *Bool* type

Algebraic specification

- Base constructors i.e., canonical values:
  - $\mathit{Bool} ::= \mathit{True} \mid \mathit{False}$
- Operators:
  - $\mathit{not} : \mathit{Bool} \rightarrow \mathit{Bool}$
  - $\mathit{and} : \mathit{Bool}^2 \rightarrow \mathit{Bool}$
  - $\mathit{or} : \mathit{Bool}^2 \rightarrow \mathit{Bool}$
  - $\mathit{if} : \mathit{Bool} \times \tau \times \tau \rightarrow \tau$
- Axioms: see slide 81

The *Bool* type

Syntax

$$
\begin{array}{rcl}
\textit{Expr} & ::= & \ldots \\
 & | & (\textit{if}\ \ \textit{Expr}\ \ \textit{Expr}\ \ \textit{Expr})
\end{array}
$$

$$
\begin{array}{rcl}
\textit{BaseVal} & ::= & \ldots \\
 & | & \textit{BoolVal}
\end{array}
$$

$$
\textit{BoolVal} \quad ::= \quad \textit{True} \mid \textit{False}
$$

$$
\begin{array}{rcl}
\textit{BaseType} & ::= & \ldots \\
 & | & \textit{Bool}
\end{array}
$$

The *Bool* type
Evaluation

- *EvalIfT*:
$$(\mathit{if}\ \ \mathit{True}\ \ e\ \ e') \rightarrow e$$
- *EvalIfF*:
$$(\mathit{if}\ \ \mathit{False}\ \ e\ \ e') \rightarrow e'$$
- *EvalIf*:
$$\frac{e \rightarrow e'}{(\mathit{if}\ \ e\ \ e_1\ \ e_2) \rightarrow (\mathit{if}\ \ e'\ \ e_1\ \ e_2)}$$

The *Bool* type

Typing

- *TTrue*:

$$\Gamma \vdash \mathit{True} : \mathit{Bool}$$

- *TFalse*:

$$\Gamma \vdash \mathit{False} : \mathit{Bool}$$

- *TIf*:

$$\frac{\Gamma \vdash e : \mathit{Bool} \qquad \Gamma \vdash e_1 : \tau \qquad \Gamma \vdash e_2 : \tau}{\Gamma \vdash (\mathit{if}\ \ e\ \ e_1\ \ e_2) : \tau}$$

The *Bool* type

Top-level variable bindings

- $not \equiv \lambda x : Bool.(if \;\; x \;\; False \;\; True)$
- $and \equiv \lambda x : Bool.\lambda y : Bool.(if \;\; x \;\; y \;\; False)$
- $or \equiv \lambda x : Bool.\lambda y : Bool.(if \;\; x \;\; True \;\; y)$

The $\mathbb{N}$ type

Algebraic specification

- Base constructors i.e., canonical values:
  - $\mathbb{N} ::= 0 \mid (\textit{succ}\ \mathbb{N})$
- Operators:
  - $+ : \mathbb{N}^2 \to \mathbb{N}$
  - $\textit{zero}? : \mathbb{N} \to \textit{Bool}$
- Axioms ($m, n \in \mathbb{N}$):
  - $(+\ 0\ \ n) = n$
  - $(+\ (\textit{succ}\ \ m)\ \ n) = (\textit{succ}\ \ (+\ \ m\ \ n))$
  - $(\textit{zero}?\ \ 0) = \textit{True}$
  - $(\textit{zero}?\ \ (\textit{succ}\ \ n)) = \textit{False}$

The $\mathbb{N}$ type

Operator semantics

- How to avoid defining evaluation and typing rules for each operator of $\mathbb{N}$?
- Introduce the primitive recursor for $\mathbb{N}$, $prec_{\mathbb{N}}$, which allows for defining any primitive recursive function on natural numbers
- Define the operators using the primitive recursor

The $\mathbb{N}$ type

Syntax

$$
\begin{array}{rcl}
\mathit{Expr} & ::= & \ldots \\
& | & (\mathit{succ}\ \mathit{Expr}) \\
& | & (\mathit{prec}_{\mathbb{N}}\ \mathit{Expr}\ \mathit{Expr}\ \mathit{Expr})
\end{array}
$$

$$
\begin{array}{rcl}
\mathit{BaseVal} & ::= & \ldots \\
& | & \mathit{NVal}
\end{array}
$$

$$
\begin{array}{rcl}
\mathit{NVal} & ::= & 0 \\
& | & (\mathit{succ}\ \mathit{NVal})
\end{array}
$$

$$
\begin{array}{rcl}
\mathit{BaseType} & ::= & \ldots \\
& | & \mathbb{N}
\end{array}
$$

The $\mathbb{N}$ type

Evaluation

- *EvalSucc*:

$$\frac{e \to e'}{(succ\ \ e) \to (succ\ \ e')}$$

- *EvalPrec*$_{\mathbb{N}0}$:

$$(prec_{\mathbb{N}}\ \ e_0\ \ f\ \ 0) \to e_0$$

- *EvalPrec*$_{\mathbb{N}1}$ $(n \in \mathbb{N})$:

$$(prec_{\mathbb{N}}\ \ e_0\ \ f\ \ (succ\ \ n)) \to (f\ \ n\ \ (prec_{\mathbb{N}}\ \ e_0\ \ f\ \ n))$$

- *EvalPrec*$_{\mathbb{N}2}$:

$$\frac{e \to e'}{(prec_{\mathbb{N}}\ \ e_0\ \ f\ \ e) \to (prec_{\mathbb{N}}\ \ e_0\ \ f\ \ e')}$$

The $\mathbb{N}$ type

Typing

- *TZero*:

$$\Gamma \vdash 0 : \mathbb{N}$$

- *TSucc*:

$$\frac{\Gamma \vdash e : \mathbb{N}}{\Gamma \vdash (succ\ \ e) : \mathbb{N}}$$

- *TPrec*$_{\mathbb{N}}$:

$$\frac{\Gamma \vdash e_0 : \tau \qquad \Gamma \vdash f : \mathbb{N} \to \tau \to \tau \qquad \Gamma \vdash e : \mathbb{N}}{\Gamma \vdash (prec_{\mathbb{N}}\ \ e_0\ \ f\ \ e) : \tau}$$

The $\mathbb{N}$ type

Top-level variable bindings

- $zero? \equiv \lambda n : \mathbb{N}.(prec_{\mathbb{N}}\ \ True\ \ \lambda x : \mathbb{N}.\lambda y : Bool.False\ \ n)$

- $+ \equiv \lambda m : \mathbb{N}.\lambda n : \mathbb{N}.(prec_{\mathbb{N}}\ \ n\ \ \lambda x : \mathbb{N}.\lambda y : \mathbb{N}.(succ\ \ y)\ \ m)$

The $(\mathit{List}\ \tau)$ type

Algebraic specification

- Base constructors i.e., canonical values:
  - $(\mathit{List}\ \tau) \quad ::= \quad []_\tau \quad | \quad (\mathit{cons}\ \tau\ (\mathit{List}\ \tau))$
- Operators:
  - $\mathit{head} : (\mathit{List}\ \tau) \setminus \{[]\} \to \tau$
  - $\mathit{tail} : (\mathit{List}\ \tau) \setminus \{[]\} \to (\mathit{List}\ \tau)$
  - $\mathit{length} : (\mathit{List}\ \tau) \to \mathbb{N}$
- Axioms ($h \in \tau$, $t \in (\mathit{List}\ \tau)$):
  - $(\mathit{head}\ (\mathit{cons}\ h\ t)) = h$
  - $(\mathit{tail}\ (\mathit{cons}\ h\ t)) = t$
  - $(\mathit{length}\ []) = 0$
  - $(\mathit{length}\ (\mathit{cons}\ h\ t)) = (\mathit{succ}\ (\mathit{length}\ t))$

The $(\mathit{List}\ \tau)$ type

Syntax

$$
\begin{array}{rcl}
\mathit{Expr} & ::= & \ldots \\
& | & (\mathit{cons}\ \mathit{Expr}\ \mathit{Expr}) \\
& | & (\mathit{prec}_L\ \mathit{Expr}\ \mathit{Expr}\ \mathit{Expr})
\end{array}
$$

$$
\begin{array}{rcl}
\mathit{BaseVal} & ::= & \ldots \\
& | & \mathit{ListVal}
\end{array}
$$

$$
\begin{array}{rcl}
\mathit{ListVal} & ::= & [] \\
& | & (\mathit{cons}\ \mathit{Value}\ \mathit{ListVal})
\end{array}
$$

$$
\begin{array}{rcl}
\mathit{Type} & ::= & \ldots \\
& | & (\mathit{List}\ \mathit{Type})
\end{array}
$$

The ($List\ \ \tau$) type

Evaluation

- *EvalCons*:

$$\frac{e \rightarrow e'}{(cons\ \ e\ \ e'') \rightarrow (cons\ \ e'\ \ e'')}$$

- $EvalPrec_{L0}$:

$$(prec_L\ \ e_0\ \ f\ \ []) \rightarrow e_0$$

- $EvalPrec_{L1}$ ($v \in Value$):

$$(prec_L\ \ e_0\ \ f\ \ (cons\ \ v\ \ e)) \rightarrow (f\ \ v\ \ e\ \ (prec_L\ \ e_0\ \ f\ \ e))$$

- $EvalPrec_{L2}$:

$$\frac{e \rightarrow e'}{(prec_L\ \ e_0\ \ f\ \ e) \rightarrow (prec_L\ \ e_0\ \ f\ \ e')}$$

The $(List\ \ \tau)$ type

Typing

- *TEmpty*:

$$\Gamma \vdash []_\tau : (List\ \ \tau)$$

- *TCons*:

$$\frac{\Gamma \vdash e : \tau \qquad \Gamma \vdash e' : (List\ \ \tau)}{\Gamma \vdash (cons\ \ e\ \ e') : (List\ \ \tau)}$$

- $TPrec_L$:

$$\frac{\Gamma \vdash e_0 : \tau' \quad \Gamma \vdash f : \tau \to (List\ \ \tau) \to \tau' \to \tau' \quad \Gamma \vdash e : (List\ \ \tau)}{\Gamma \vdash (prec_L\ \ e_0\ \ f\ \ e) : \tau'}$$

The ($List\ \ \tau$) type

Top-level variable bindings

- $empty? \equiv \lambda l : (List\ \ \tau).(prec_L\ \ True\ \ f\ \ l)$,
  $f \equiv \lambda h : \tau.\lambda t : (List\ \ \tau).\lambda r : Bool.False$
- $length \equiv \lambda l : (List\ \ \tau).(prec_L\ \ 0\ \ f\ \ l)$,
  $f \equiv \lambda h : \tau.\lambda t : (List\ \ \tau).\lambda r : \mathbb{N}.(succ\ \ r)$

General recursion

- Primitive recursion
  - induces *strong normalization*
  - insufficient for capturing effectively computable functions
- Introduce the operator *fix* i.e., a fixed-point combinator
- Gain computation power at the expense of strong normalization

fix
Evaluation

- *EvalFix*:

$$(fix\ \ \lambda x : \tau . e) \rightarrow e_{[(fix\ \ \lambda x : \tau . e)/x]} \qquad = (f\ \ (fix\ \ f))$$

- *EvalFix′*:

$$\frac{e \rightarrow e'}{(fix\ \ e) \rightarrow (fix\ \ e')}$$

Example 15.2 (The *remainder* function).

$$
\begin{aligned}
&\mathit{remainder} = \lambda m : \mathbb{N}.\lambda n : \mathbb{N}. \\
&\qquad ((\mathit{fix}\ \ \lambda f : (\mathbb{N} \to \mathbb{N}).\lambda p : \mathbb{N}. \\
&\qquad\qquad (\mathit{if}\ \ p < n\ \ \mathit{then}\ \ p\ \ \mathit{else}\ \ (f\ \ (p - n))))\ \ m)
\end{aligned}
$$

The evaluation of (*remainder* 3 0) does not terminate.

Monomorphism

- Within the types $(\tau * \tau')$ and $(\mathit{List}\ \tau)$, τ and τ' designate specific types e.g., *Bool*, $\mathbb{N}$, $(\mathit{List}\ \mathbb{N})$, etc.
- Dedicated operators for each simple type
- $\mathit{fst}_{\mathbb{N},\mathit{Bool}}$, $\mathit{fst}_{\mathit{Bool},\mathbb{N}}$, $\ldots$
- $[]_{\mathbb{N}}$, $[]_{\mathit{Bool}}$, $\ldots$
- $\mathit{empty}?_{\mathbb{N}}$, $\mathit{empty}?_{\mathit{Bool}}$, $\ldots$

Idea

- Monomorphic identity function for type $\mathbb{N}$:

$$id_{\mathbb{N}} \equiv \lambda x : \mathbb{N}.x : (\mathbb{N} \to \mathbb{N})$$

- Polymorphic identity function — type variables:

$$id \equiv \lambda X.\lambda x : X.x : \forall X.(X \to X)$$

- Type coercion prior to function application:

$$(id[\mathbb{N}]\ 5) \to (id_{\mathbb{N}}\ 5) \to 5$$

Syntax

- Program variables: stand for program values

$$\mathit{Var} \quad ::= \quad \ldots$$

- Type variables: stand for types

$$\mathit{TypeVar} \quad ::= \quad \ldots$$

Syntax

- Expressions:

$$
\begin{array}{rcl}
\mathit{Expr} & ::= & \mathit{Value} \\
& | & \mathit{Var} \\
& | & (\mathit{Expr}\ \ \mathit{Expr}) \\
& | & \mathit{Expr}[\mathit{Type}]
\end{array}
$$

- Values:

$$
\begin{array}{rcl}
\mathit{Value} & ::= & \mathit{BaseValue} \\
& | & \lambda\,\mathit{Var} : \mathit{Type}.\mathit{Expr} \\
& | & \lambda\,\mathit{TypeVar}.\mathit{Expr}
\end{array}
$$

Syntax

- Types:

$$\begin{array}{rcl} \mathit{Type} & ::= & \mathit{BaseType} \\ & | & \color{red}{\mathit{TypeVar}} \\ & | & (\mathit{Type} \to \mathit{Type}) \\ & | & \color{red}{\forall \mathit{TypeVar}.\mathit{Type}} \end{array}$$

- Typing contexts:

$$\begin{array}{rcl} \mathit{TypingContext} & ::= & \emptyset \\ & | & \mathit{TypingContext}, \mathit{Var} : \mathit{Type} \\ & | & \color{red}{\mathit{TypingContext}, \mathit{TypeVar}} \end{array}$$

Semantics

Evaluation

- *Reduce*$_1$:

$$(\lambda x : \tau.e \;\; e') \rightarrow e_{[e'/x]}$$

- *Reduce*$_2$:

$$\lambda X.e[\tau] \rightarrow e_{[\tau/X]}$$

- *Eval*$_1$:

$$\frac{e \rightarrow e'}{(e \;\; e'') \rightarrow (e' \;\; e'')}$$

- *Eval*$_2$:

$$\frac{e \rightarrow e'}{e[\tau] \rightarrow e'[\tau]}$$

Semantics
Typing

- *TBaseValue*:

$$\frac{v \in \tau_b}{\Gamma \vdash v : \tau_b}$$

- *TVar*:

$$\frac{x : \tau \in \Gamma}{\Gamma \vdash x : \tau}$$

- *TAbs*$_1$:

$$\frac{\Gamma, x : \tau \vdash e : \tau'}{\Gamma \vdash \lambda x : \tau . e : (\tau \rightarrow \tau')}$$

- *TApp*$_1$:

$$\frac{\Gamma \vdash e : (\tau' \rightarrow \tau) \qquad \Gamma \vdash e' : \tau'}{\Gamma \vdash (e \; e') : \tau}$$

Semantics

Typing

- *TAbs*$_2$ — polymorphic expressions have universal types:

$$\frac{\Gamma, X \vdash e : \tau}{\Gamma \vdash \lambda X.e : \forall X.\tau}$$

- *TApp*$_2$:

$$\frac{\Gamma \vdash e : \forall X.\tau}{\Gamma \vdash e[\tau'] : \tau_{[\tau'/X]}}$$

Semantics

Substitution and free variables

- $Expr_{[Expr\,/\,Var]}$
- $Expr_{[Type\,/\,TypeVar]}$
- $Type_{[Type\,/\,TypeVar]}$
- Free program variables
- Free type variables

Typing example

Example 16.1 (Typing).

$$\Gamma \vdash \lambda f : \forall X.(X \to X).\lambda Y.\lambda x : Y.(f[Y]\ x)$$
$$: (\forall X.(X \to X) \to \forall Y.(Y \to Y))$$

Monomorphic function
with polymorphic argument and result!

Blackboard!

Examples of polymorphic expressions

Example 16.2 (Doubling a computation).

$$\begin{aligned} \mathit{double} &\equiv \lambda X.\lambda f:(X\to X).\lambda x:X.(f\ \ (f\ \ x)) \\ &: \forall X.((X\to X)\to(X\to X)) \end{aligned}$$

Example 16.3 (Quadrupling a computation).

$$\begin{aligned} \mathit{quadruple} &\equiv \lambda X.(\mathit{double}[X\to X]\ \ \mathit{double}[X]) \\ &: \forall X.((X\to X)\to(X\to X)) \end{aligned}$$

Examples of polymorphic expressions

Example 16.4 (Reflexive computation).

$$
\begin{array}{rcl}
\mathit{reflexive} & \equiv & \lambda f : \forall X.(X \to X).(f[\forall X.(X \to X)]\ f) \\
 & : & (\forall X.(X \to X) \to \forall X.(X \to X))
\end{array}
$$

Example 16.5 (Fixed-point combinator).

$$
\begin{array}{rcl}
\mathit{Fix} & \equiv & \lambda X.\lambda f : (X \to X).(f\ (\mathit{Fix}[X]\ f)) \\
 & : & \forall X.((X \to X) \to X)
\end{array}
$$

Problem

- Polymorphic identity function, on objects of a type built using 1-ary type constructors e.g., *List*:

$$f \equiv \lambda C.\lambda X.\lambda x : (C \;\; X).x : \forall C.\forall X.((C \;\; X) \to (C \;\; X))$$

- C stands for a 1-ary type constructor, X stands for a type of program values i.e., a *proper type*

- Monomorphic identity function for type ($List \;\; \mathbb{N}$):

$$f[List][\mathbb{N}] \to \lambda x : (List \;\; \mathbb{N}).x : ((List \;\; \mathbb{N}) \to (List \;\; \mathbb{N}))$$

- How do we prevent erroneous situations e.g., $f[\mathbb{N}][\mathbb{N}]$, $f[List][List]$?

Solution

- Two categories of types: proper types, and type constructors i.e., $\lambda \mathit{TypeVar}.\mathit{Type}$
- Type not only program variables, but also type variables
- The type of a type: *kind*

Kinds

Notation

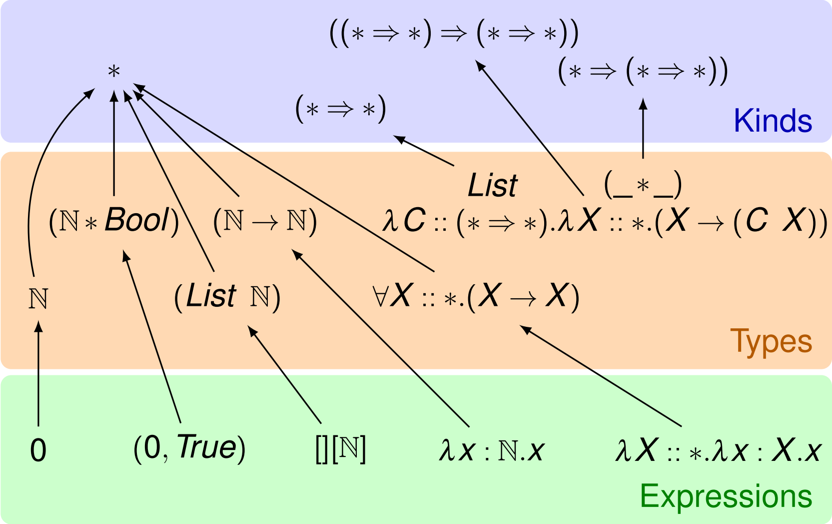
- The kind of a proper type: $*$
- The kind of a 1-ary type constructor: $(* \Rightarrow *)$
- The kind of an *n*-ary type constructor, $n \geq 1$: $k_1 \Rightarrow k_2$
- The kind k of a type τ: $\tau :: k$

Kinds
Examples

Example 18.1 (Kinds).

- $\mathbb{N} :: *$
- $\mathit{List} :: (* \Rightarrow *)$
- $f \equiv \lambda C :: (* \Rightarrow *).\lambda X :: *.\lambda x : (C\ X).x$
  $f : \forall C :: (* \Rightarrow *).\forall X :: *.((C\ X) \rightarrow (C\ X))$

Levels of expressions



Type equivalence

- Two syntactically distinct types:

$$\tau_1 \equiv ((\mathit{List}\ \mathbb{N}) \rightarrow (\mathit{List}\ \mathbb{N}))$$
$$\tau_2 \equiv (\lambda X :: *.((\mathit{List}\ X) \rightarrow (\mathit{List}\ X))\ \mathbb{N})$$

- Semantically, they denote the same type i.e., they are equivalent: $\tau_1 \equiv \tau_2$

Syntax

- Types:

$$
\begin{array}{rcl}
\mathit{Type} & ::= & \mathit{BaseType} \\
& | & \mathit{TypeVar} \\
& | & (\mathit{Type} \to \mathit{Type}) \\
& | & \forall \mathit{TypeVar} :: \mathit{Kind}.\mathit{Type} \\
& | & \lambda \mathit{TypeVar} :: \mathit{Kind}.\mathit{Type} \\
& | & (\mathit{Type}\ \ \mathit{Type})
\end{array}
$$

- Typing contexts:

$$
\begin{array}{rcl}
\mathit{TypingContext} & ::= & \emptyset \\
& | & \mathit{TypingContext}, \mathit{Var} : \mathit{Type} \\
& | & \mathit{TypingContext}, \mathit{TypeVar} :: \mathit{Kind}
\end{array}
$$

Syntax

- Kinds:

$$\begin{array}{lcl} \mathit{Kind} & ::= & * \\ & \mid & (\mathit{Kind} \Rightarrow \mathit{Kind}) \end{array}$$

Semantics

Evaluation

- *Reduce*$_1$:

$$(\lambda x : \tau.e\ \ e') \to e_{[e'/x]}$$

- *Reduce*$_2$:

$$\lambda X :: K.e[\tau] \to e_{[\tau/X]}$$

- *Eval*$_1$:

$$\frac{e \to e'}{(e\ \ e'') \to (e'\ \ e'')}$$

- *Eval*$_2$:

$$\frac{e \to e'}{e[\tau] \to e'[\tau]}$$

Semantics

Typing

- *TBaseValue*:

$$\frac{v \in \tau_b}{\Gamma \vdash v : \tau_b}$$

- *TVar*:

$$\frac{x : \tau \in \Gamma}{\Gamma \vdash x : \tau}$$

- *TAbs*$_1$:

$$\frac{\Gamma, x : \tau \vdash e : \tau'}{\Gamma \vdash \lambda x.e : (\tau \to \tau')}$$

- *TApp*$_1$:

$$\frac{\Gamma \vdash e : (\tau' \to \tau) \qquad \Gamma \vdash e' : \tau'}{\Gamma \vdash (e\ e') : \tau}$$

Semantics

Typing

- *TAbs*$_2$:

$$\frac{\Gamma, X :: K \vdash e : \tau}{\Gamma \vdash \lambda X :: K.e : \forall X :: K.\tau}$$

- *TApp*$_2$:

$$\frac{\Gamma \vdash e : \forall X :: K.\tau \qquad \Gamma \vdash \tau' :: K}{\Gamma \vdash e[\tau'] : \tau_{[\tau'/X]}}$$

Semantics

Kinding

- *KBaseType*:

$$\Gamma \vdash \tau_b :: *$$

- *KTypeVar*:

$$\frac{X :: K \in \Gamma}{\Gamma \vdash X :: K}$$

- *KTypeAbs*:

$$\frac{\Gamma, X :: K \vdash \tau :: K'}{\Gamma \vdash \lambda X :: K.\tau :: (K \Rightarrow K')}$$

- *KTypeApp*:

$$\frac{\Gamma \vdash \tau :: (K' \Rightarrow K) \qquad \Gamma \vdash \tau' :: K'}{\Gamma \vdash (\tau\ \tau') :: K}$$

Semantics

Kinding

- *KAbs*$_1$:

$$\frac{\Gamma \vdash \tau :: * \qquad \Gamma \vdash \tau' :: *}{\Gamma \vdash (\tau \rightarrow \tau') :: *}$$

- *KAbs*$_2$:

$$\frac{\Gamma, X :: K \vdash \tau :: *}{\Gamma \vdash \forall X :: K.\tau :: *}$$

Semantics

Type equivalence

- *EqReflexivity*:

$$\tau \equiv \tau$$

- *EqSymmetry*:

$$\frac{\tau \equiv \tau'}{\tau' \equiv \tau}$$

- *EqTransitivity*:

$$\frac{\tau \equiv \tau' \qquad \tau' \equiv \tau''}{\tau \equiv \tau''}$$

- *EqTypeReduce*:

$$(\lambda X :: K.\tau \ \ \tau') \equiv \tau_{[\tau'/X]}$$

Semantics

Type equivalence

- *EqTypeAbs*:

$$\frac{\tau \equiv \tau'}{\lambda X :: K.\tau \equiv \lambda X :: K.\tau'}$$

- *EqTypeApp*:

$$\frac{\tau \equiv \tau' \qquad \sigma \equiv \sigma'}{(\tau\ \ \sigma) \equiv (\tau'\ \ \sigma')}$$

- *EqAbs*$_1$:

$$\frac{\tau \equiv \tau' \qquad \sigma \equiv \sigma'}{(\tau \to \sigma) \equiv (\tau' \to \sigma')}$$

- *EqAbs*$_2$:

$$\frac{\tau \equiv \tau'}{\forall X :: K.\tau \equiv \forall X :: K.\tau'}$$

Semantics

Type equivalence

- *TypeEquivalence*:

$$\frac{\Gamma \vdash e : \tau \qquad \tau \equiv \tau'}{\Gamma \vdash e : \tau'}$$

Kinding example

Example 18.2 (Kinding).

$$\forall X :: *.(X \to ((List\ \ X) \to (Tree\ \ X))) :: *$$

Blackboard!

Part V

Constructive Type Theory

Classical logic

- Example: prove $\exists x.P(x)$
- Perhaps, proof by contradiction: assume $\neg\exists x.P(x)$ and reach a contradiction
- Assumption: $\exists x.P(x) \;\vee\; \neg\exists x.P(x)$ (law of excluded middle)
- Problem: possibly no actual evidence regarding either sentence i.e., some a s.t. either $P(a)$ or $\neg P(a)$ is true

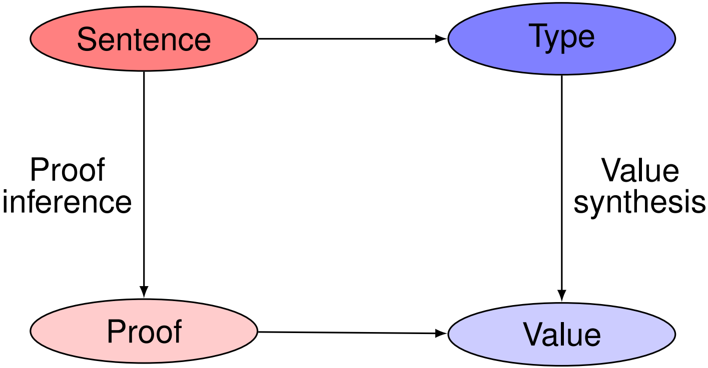
Constructive logic

- Prove $\exists x.P(x)$ by computing an object a s.t. $P(a)$ is true
- Not always possible
- However, not being able to compute a does not mean that $\exists x.P(x)$ is false
- Law of excluded middle — not an axiom in constructive logic

Constructive type theory

- Bridge between constructive logic and typed lambda calculus
- Correspondences:
  - sentence $\leftrightarrow$ type
  - logical connective $\leftrightarrow$ type constructor
  - proof $\leftrightarrow$ function with that type
- Application: synthesize a program by proving the sentence that corresponds to its specification

The Curry-Howard isomorphism



Two views

$$a : A$$

- Type-theoretic: "a is a value of type A"
- Logical: "a is a proof of sentence A"

Definitional rules

Rule	Logical view	Type-theoretic view
Formation	How a connective relates two sentences	How a type constructor is used
Introduction/ elimination	How a proof is derived	How a value is constructed
Computation	How a proof is simplified	How an expression is evaluated

Other logic-type correspondences

Logical view	Type-theoretic view
Truth ($\top$)	One-element type, containing the trivial proof
Falsity ($\perp$)	No-element type, containing no proof
Proof by induction	Definition by recursion

Logical conjunction / product type constructor I

- Formation rule ($\wedge F$):

$$\frac{A \text{ is a sentence/ type} \qquad B \text{ is a sentence/ type}}{A \wedge B \text{ is a sentence/ type}}$$

- Introduction rule ($\wedge I$):

$$\frac{a : A \qquad b : B}{(a, b) : A \wedge B}$$

Logical conjunction / product type constructor II

- Elimination rules ($\wedge E_{1,2}$):

$$\frac{p : A \wedge B}{\mathit{fst}\;\; p : A} \qquad\qquad \frac{p : A \wedge B}{\mathit{snd}\;\; p : B}$$

- Computation rules:

$$\begin{aligned} \mathit{fst}\;\; (a,b) &\to a \\ \mathit{snd}\;\; (a,b) &\to b \end{aligned}$$

Logical implication / function type constructor I

- Formation rule ($\Rightarrow F$):

$$\frac{A \text{ is a sentence/ type} \qquad B \text{ is a sentence/ type}}{A \Rightarrow B \text{ is a sentence/ type}}$$

- Introduction rule ($\Rightarrow I$)
  (square brackets = discharged assumption):

$$\frac{\begin{array}{c}[x : A] \\ \vdots \\ b : B\end{array}}{\lambda x : A.b : A \Rightarrow B}$$

Logical implication / function type constructor II

- Elimination rule ($\Rightarrow E$):

$$\frac{a : A \qquad f : A \Rightarrow B}{(f \;\; a) : B}$$

- Computation rule:

$$(\lambda x : A.b \;\; a) \rightarrow b_{[a/x]}$$

Logical disjunction / sum type constructor I

- Formation rule ($\vee F$):

$$\frac{A \text{ is a sentence/ type} \qquad B \text{ is a sentence/ type}}{A \vee B \text{ is a sentence/ type}}$$

- Introduction rules ($\vee I_{1,2}$):

$$\frac{a : A}{inl\ \ a : A \vee B} \qquad\qquad \frac{b : B}{inr\ \ b : A \vee B}$$

Logical disjunction / sum type constructor II

- Elimination rule ($\vee E$):

$$\frac{p : A \vee B \qquad f : A \Rightarrow C \qquad g : B \Rightarrow C}{cases\ p\ f\ g : C}$$

- Computation rules:

$$cases\ (inl\ a)\ f\ g \rightarrow f\ a$$
$$cases\ (inr\ b)\ f\ g \rightarrow g\ b$$

Absurd sentence / empty type I

- Formation rule ($\bot F$):

$$\frac{}{\bot \text{ is a sentence/ type}}$$

- Introduction rule: none — there is no proof of the absurd sentence

Absurd sentence / empty type II

- Elimination rule ($\bot E$)
  (a proof of the absurd sentence can prove anything):

$$\frac{p : \bot}{abort_A\ p : A}$$

- Computation rule: none

Logical negation and equivalence

- Logical negation:

$$\neg A \;\equiv\; A \Rightarrow \bot$$

- Logical equivalence:

$$A \Leftrightarrow B \;\equiv\; (A \Rightarrow B) \land (B \Rightarrow A)$$

Example proofs

- $A \Rightarrow A$
- $A \Rightarrow \neg\neg A$ (converse?)
- $((A \wedge B) \Rightarrow C) \Rightarrow A \Rightarrow B \Rightarrow C$
- $(A \Rightarrow B) \Rightarrow (B \Rightarrow C) \Rightarrow (A \Rightarrow C)$
- $(A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A)$
- $(A \vee B) \Rightarrow \neg(\neg A \wedge \neg B)$

Universal quantification / generalized function type constructor I

- Formation rule ($\forall F$)
  (square brackets = discharged assumption):

$$\dfrac{A \text{ is a sentence/ type} \qquad \begin{matrix} {\color{red}[x : A]} \\ \vdots \\ B \text{ is a sentence/ type} \end{matrix}}{(\forall x : A).B \text{ is a sentence/ type}}$$

- Introduction rule ($\forall I$):

$$\dfrac{\begin{matrix} [x : A] \\ \vdots \\ b : B \end{matrix}}{(\lambda x : A).b : (\forall x : A).B}$$

Universal quantification / generalized function type constructor II

- Elimination rule ($\forall E$):

$$\frac{a : A \qquad f : (\forall x : A).B}{(f\ \ a) : B_{[a/x]}}$$

- Computation rule:

$$((\lambda x : A).b\ \ a) \to b_{[a/x]}$$

Existential quantification / generalized product type constructor I

- Formation rule ($\exists F$)
  (square brackets = discharged assumption):

$$\frac{A \text{ is a sentence/ type} \qquad \begin{array}{c}[x : A] \\ \vdots \\ B \text{ is a sentence/ type}\end{array}}{(\exists x : A).B \text{ is a sentence/ type}}$$

- Introduction rule ($\exists I$):

$$\frac{a : A \qquad b : B_{[a/x]}}{(a, b) : (\exists x : A).B}$$

Existential quantification / generalized product type constructor II

- Elimination rules ($\exists E_{1,2}$):

$$\frac{p : (\exists x : A).B}{Fst\ \ p : A} \qquad \frac{p : (\exists x : A).B}{Snd\ \ p : B_{[Fst\ \ p/x]}}$$

- Computation rules:

$$\begin{aligned} Fst\ \ (a, b) &\to a \\ Snd\ \ (a, b) &\to b \end{aligned}$$

Example proofs

- $(\forall x : A).(B \Rightarrow C) \Rightarrow (\forall x : A).B \Rightarrow (\forall x : A).C$
- $(\exists x : X).\neg P \Rightarrow \neg(\forall x : X).P$ (converse?)
- $(\exists y : Y).(\forall x : X).P \Rightarrow (\forall x : X).(\exists y : Y).P$ (converse?)