Decomposition of complete 3-manifolds of positive scalar curvature with quadratic decay

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Introduction

(M^n, g) Riemannian manifold.

Scalar curvature at $x \in M$:

$$\mathrm{vol}(B(x,R)) = b_n R^n \left(1 - \frac{\mathrm{scal}_g(x)}{6(n+2)} R^2 + O(R^3)\right)$$

or equivalently:

orthonormal frame

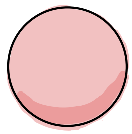
$$\mathrm{scal}_g(x) = \sum_{i \neq j} \mathrm{sect}_x(e_i, e_j)$$

? Which 3-manifolds do admit a complete metric g with $\mathrm{scal}_g > 0$?

PSC 3-manifolds

Closed PSC 3-manifolds

- Examples:



(I). Sphere: $S^3 \subseteq \mathbb{R}^4$

(II). Spherical manifolds: S^3/Γ, $\Gamma < SO(4)$ acting freely by isometries S^3

e.g. $\mathbb{R}P^3$, $L(p,q)$, ...

(III). $S^2 \times S^1$

Theorem [Gromov–Lawson '80–83] + [Perelman '03]

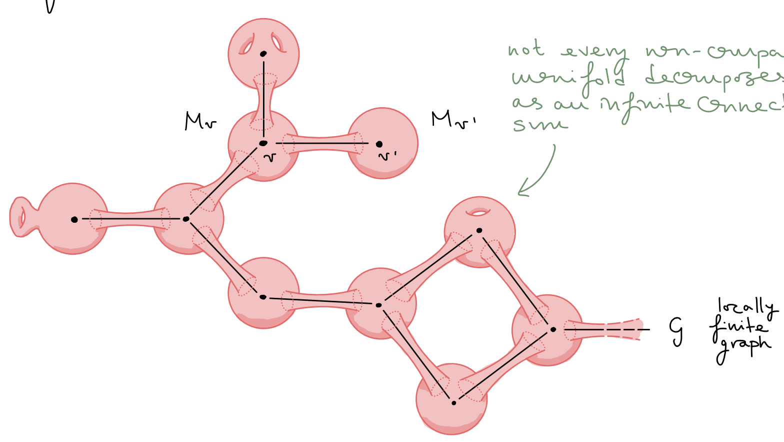
M^3 orientable, closed.

$$M \text{ PSC} \iff M \simeq \#_{i=1}^{p} S^3/\Gamma_i \; \#_{j=1}^{q} S^2 \times S^1$$

(finite)

Non-compact PSC 3-manifolds

- Infinite connected sum [Scott '75]:



not every non-compact manifold decomposes as an infinite connected sum

Theorem [Gromov '23 – Wang '23]

M^3 orientable, complete. If $\mathrm{scal} \geq s_0 > 0$, then:

$$M \simeq \#_i S^3/\Gamma_i \; \#_j S^2 \times S^1$$

(possibly infinite)

↳ μ-bubbles [Gromov '23]

Theorem [B-G-S]

M^3 orientable, complete, $x \in M$, $r := d(x, \cdot)$.

If $\mathrm{scal} > 0$ and $\exists C > 4^2 \cdot \frac{8}{3}\pi^2$ such that

$$\mathrm{scal} \geq \frac{C}{r^2} \quad (r \text{ large enough})$$

then:

$$M \simeq \#_i S^3/\Gamma_i \; \#_j S^2 \times S^1$$

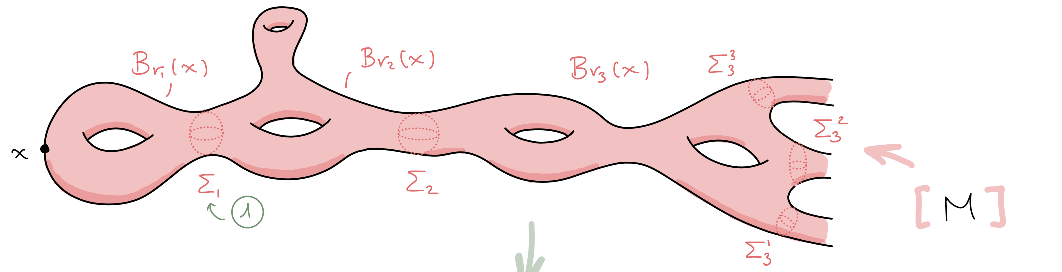
(possibly infinite)

↳ Optimal: $\mathbb{R}^2 \times S^1$ admits

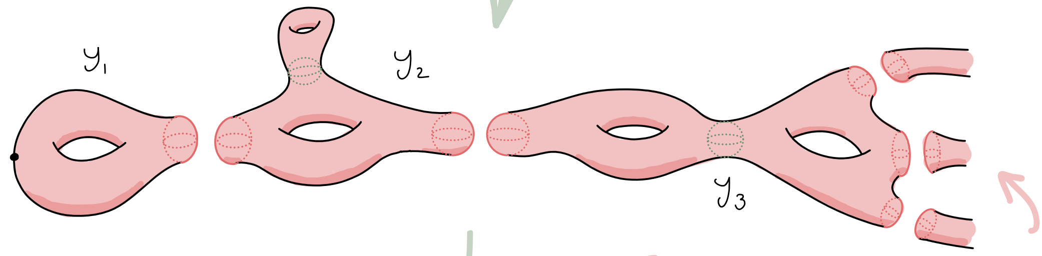
$$g = dr^2 + \sqrt{r}\, d\theta^2 + dt^2 \longrightarrow \mathrm{scal}_g = \frac{1}{2}\frac{1}{r^2}$$

but $\mathbb{R}^2 \times S^1 \not\simeq \#_i S^3/\Gamma_i \; \#_j S^2 \times S^1$

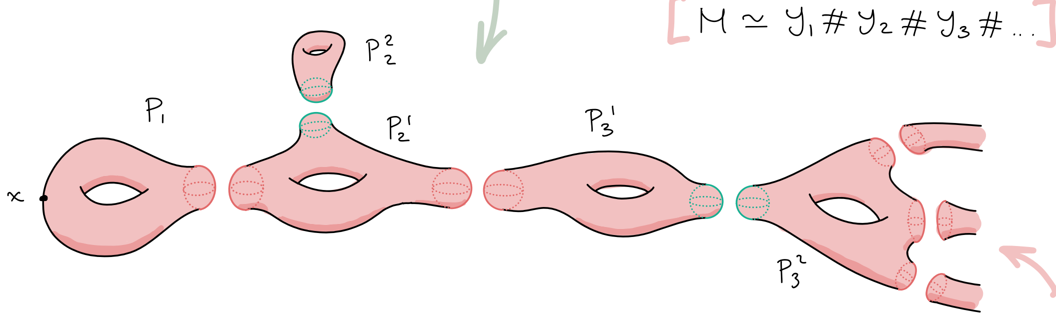
Proof



[M]



$[M \simeq Y_1 \# Y_2 \# Y_3 \# \ldots]$



$$\left[\begin{array}{l} M \simeq Y_1 \# Y_2 \# Y_3 \# \ldots \\ Y_i \simeq P_i^1 \# \ldots \# P_i^{k_i} \end{array}\right]$$

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Prime decomposition Theorem [Kneser '29] + [Milnor '62]

Every Y^3 compact, $Y \not\simeq S^3$, decomposes:

$$Y \simeq P_1 \# \ldots \# P_N$$

with P_i prime.

↳ P^3 closed prime $\simeq \begin{cases} S^3/\Gamma & \text{[Perelman '03]} \\ S^2 \times S^1 \\ \text{aspherical} \end{cases}$

We use $\mathrm{scal} \geq \frac{C}{r^2}$ to prove:

1. $\Sigma_i^j \simeq S^2$
2. $P_{k_i}^i$ not aspherical

Theorem [Gromov – Lawson '83]

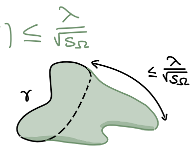
Ω^3 orientable, complete, $\mathrm{scal}_\Omega \geq s_\Omega > 0$.

Then every $\gamma \subseteq \Omega$ such that $\gamma \sim 0$ verifies either:

(I). γ bounds a disc in $\mathcal{U}_{\frac{\lambda}{\sqrt{s_\Omega}}}(\gamma)$, or

(II). $d(\gamma, \partial\Omega) \leq \frac{\lambda}{\sqrt{s_\Omega}}$

$\Big\}$ $\mathrm{fillrad}(\gamma) \leq \frac{\lambda}{\sqrt{s_\Omega}}$



with $\lambda = \sqrt{\frac{8}{3}}\pi$ [Wolfson '12]

↳ $\mathrm{scal}_M \geq s_0 > 0 \implies \mathrm{fillrad}(\gamma) \leq \frac{\lambda}{\sqrt{s_0}}$

↳ $\mathrm{scal}_M \geq C/r^2 \implies \mathrm{fillrad}(\gamma) \leq \kappa r$

$\Omega = B(x,r)$

[Ramachandran – Wolfson '10] closed case

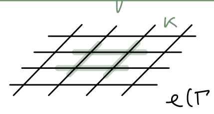
Theorem

M^3 orientable, complete.

If $\mathrm{fillrad}_{\tilde{M}}(\gamma) \leq \kappa r$, then every finitely generated $G \leq \pi_1 M$ has

$$e(G) \neq 1$$

Number of ends



$e(\Gamma) = 1$

$e(G) := e(\Gamma(G))$

Cayley graph of G

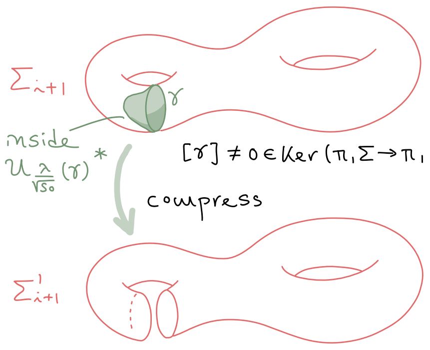
↳ 2 $P_{k_i}^i \simeq K$ aspherical $\implies \pi_1 K \leq \pi_1 M$ and $e(\pi_1 K) = 1$!

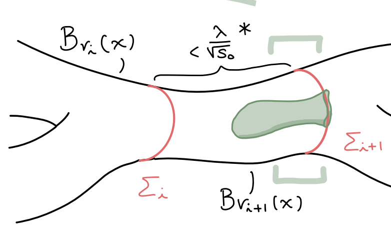
↳ 1 (I). $\pi_1 \Sigma \longrightarrow \pi_1 M$ injective $\implies \pi_1 \Sigma \leq \pi_1 M$

$\implies e(\pi_1 \Sigma) \neq 1 \implies \Sigma \simeq S^2$

(II). $\pi_1 \Sigma \longrightarrow \pi_1 M$ non-injective (Σ compressible)

Loop Theorem [Papakyriakopoulos '57]





* for $\mathrm{scal} \geq s_0 > 0$; can be adapted for $\mathrm{scal} \geq \frac{C}{r^2}$