Decomposition of complete 3-manifolds of positive scalar curvature with quadratic decay

F. Balacheff[1], T. Gil Moreno de Mora Sardà[1,2], S. Sabourau[2]

[1] Universitat Autònoma de Barcelona, [2] Université Paris-Est Créteil

Introduction

(M^n, g) Riemannian manifold.
Scalar curvature at $x \in M$:

$$\mathrm{vol}(B(x,R)) = b_n R^n \left(1 - \frac{\mathrm{scal}_g(x)}{6(n+2)} R^2 + O(R^3)\right)$$

or equivalently:

orthonormal frame

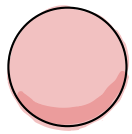
$$\mathrm{scal}_g(x) = \sum_{i \neq j} \mathrm{sect}_x(e_i, e_j)$$

? Which 3-manifolds do admit a complete metric g with $\mathrm{scal}_g > 0$?

PSC 3-manifolds

Closed PSC 3-manifolds

- Examples:
  - (I). Sphere: $S^3 \subseteq \mathbb{R}^4$
  - (II). Spherical manifolds: S^3/Γ, $\Gamma < SO(4)$ acting freely by isometries S^3 e.g. $\mathbb{RP}^3$, $L(p,q)$, …
  - (III). $S^2 \times S^1$

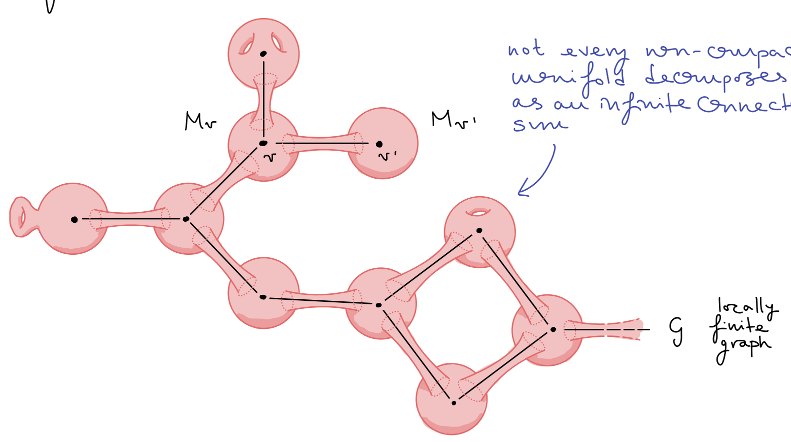


Theorem [Gromov–Lawson '80–83] + [Perelman '03]

M^3 orientable, closed.
M PSC $\iff M \simeq \#_i S^3/\Gamma_i \#_j S^2 \times S^1$ (finite)

Non-compact PSC 3-manifolds

- Infinite connected sum [Scott '75]:



not every non-compact manifold decomposes as an infinite connected sum

Theorem [Gromov '23 – Wang '23]

M^3 orientable, complete. If $\mathrm{scal} \geq s_0 > 0$, then: $M \simeq \#_i S^3/\Gamma_i \#_j S^2 \times S^1$ (possibly infinite)

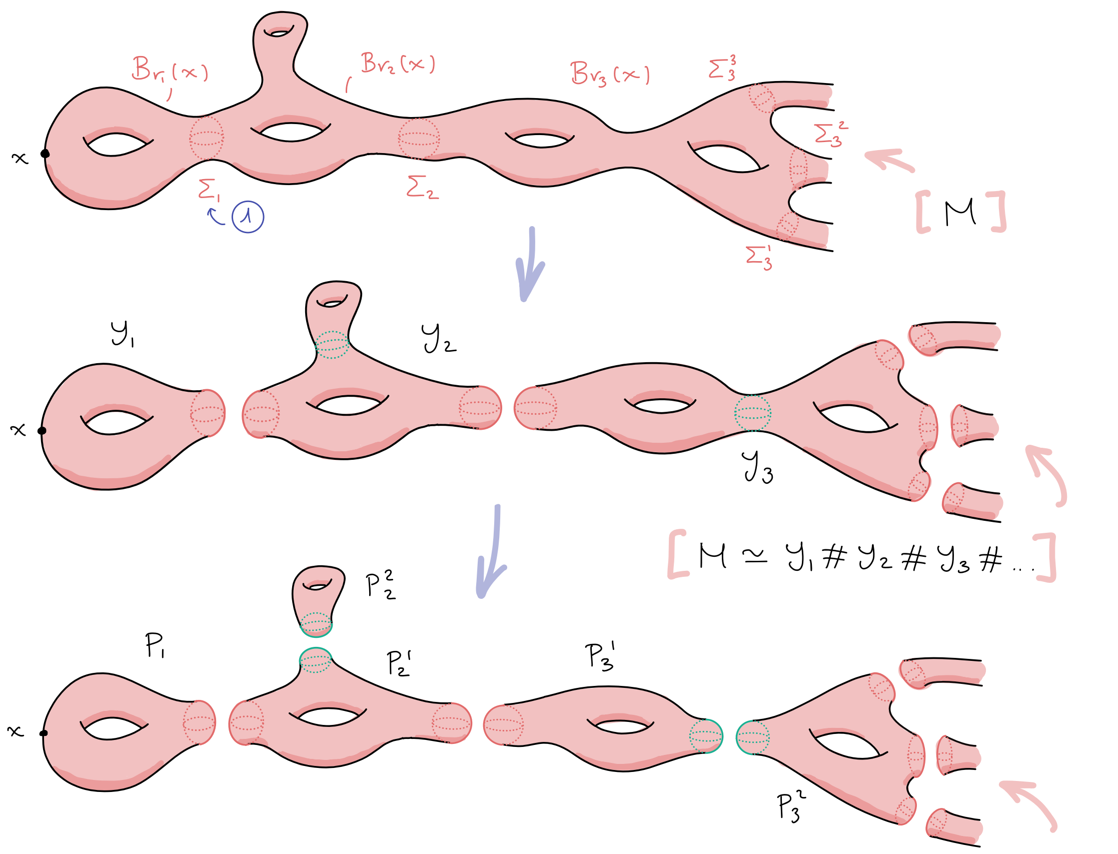
↳ μ-bubbles [Gromov '23]

Theorem [B-G-S]

M^3 orientable, complete, $x \in M$, $r := d(x, \cdot)$.
If $\mathrm{scal} > 0$ and:
$\mathrm{scal} \geq \frac{C}{r^2}$, $r \geq R_0$, for $C > 4^2 \cdot \frac{8}{3}\pi^2$;
then:
$M \simeq \#_i S^3/\Gamma_i \#_j S^2 \times S^1$ (possibly infinite)

↳ Optimal: $\mathbb{R}^2 \times S^1$ admits
$g = dr^2 + \sqrt{r}\, d\theta^2 + dt^2 \longrightarrow \mathrm{scal}_g = \frac{1}{2}\frac{1}{r^2}$
but $\mathbb{R}^2 \times S^1 \not\simeq \#_i S^3/\Gamma_i \#_j S^2 \times S^1$

Proof



$\left[\begin{array}{l} M \simeq Y_1 \# Y_2 \# Y_3 \# \ldots \\ Y_i \simeq P_i^1 \# \ldots \# P_i^{K_i} \end{array}\right]$

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Prime decomposition Theorem [Kneser '29] + [Milnor '62]

Every Y^3 compact, $Y \not\simeq S^3$ decomposes:
$Y \simeq P_1 \# \ldots \# P_N$
with P_i prime.

↳ P^3 closed prime $\simeq \begin{cases} S^3/\Gamma \text{ [Perelman '03]} \\ S^2 \times S^1 \\ \text{aspherical} \end{cases}$

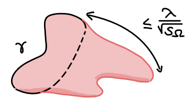
We use $\mathrm{scal} \geq \frac{C}{r^2}$ to prove:

1. $\Sigma_i^t \simeq S^2$
2. $P_{K_i}^i$ not aspherical

Theorem [Gromov–Lawson '83]

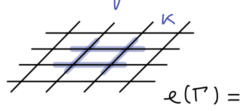
M^3 orientable, complete, $\mathrm{scal} > 0$.
$\Omega \subseteq M^3$ compact domain, $s_\Omega = \min_\Omega \mathrm{scal}$, then every $\gamma \subseteq \Omega$
(I). every $\gamma \sim 0$ bounds a disc in $\mathcal{U}_{\frac{\lambda}{\sqrt{s_\Omega}}}(\gamma)$, or
(II). $d(\gamma, \partial\Omega) \leq \frac{\lambda}{\sqrt{s_\Omega}}$
with $\lambda = \sqrt{\frac{8}{3}}\pi$ [Wolfson '12]

} $\mathrm{fillrad}(\gamma) \leq \frac{\lambda}{\sqrt{s_\Omega}}$



more generally
$\mathrm{fillrad}(\tilde{\gamma} \subseteq \tilde{M}) \leq Kr$

Number of ends



$e(G) := e(\Gamma(G))$
Cayley graph of G

Theorem [Ramachandran–Wolfson '10] closed case

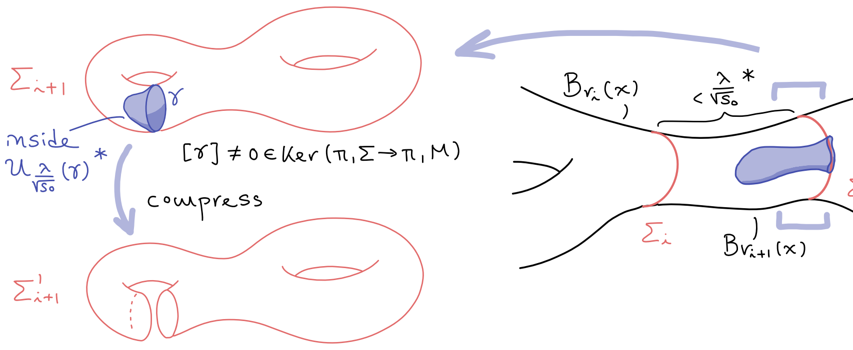
M^3 orientable, complete. If $\mathrm{scal} \geq \frac{C}{r^2}$, $C > 4^2 \cdot \frac{8}{3}\pi^2$, then every fn. gen. $G \leq \pi_1 M$ has $e(G) \neq 1$

↳ 2 $P_{K_i}^i \simeq K$ aspherical $\Rightarrow \pi_1 K \leq \pi_1 M$ and $e(\pi_1 K) = 1$!

↳ 1 (I). $\pi_1 \Sigma \longrightarrow \pi_1 M$ injective $\Rightarrow \pi_1 \Sigma \leq \pi_1 M$
$\Rightarrow e(\pi_1 \Sigma) \neq 1 \Rightarrow \Sigma \simeq S^2$

(II). $\pi_1 \Sigma \longrightarrow \pi_1 M$ non-injective (Σ compressible)

Loop Theorem [Papakyriakopoulos '57]



* for $\mathrm{scal} \geq s_0 > 0$; can be adapted for $\mathrm{scal} \geq \frac{C}{r^2}$