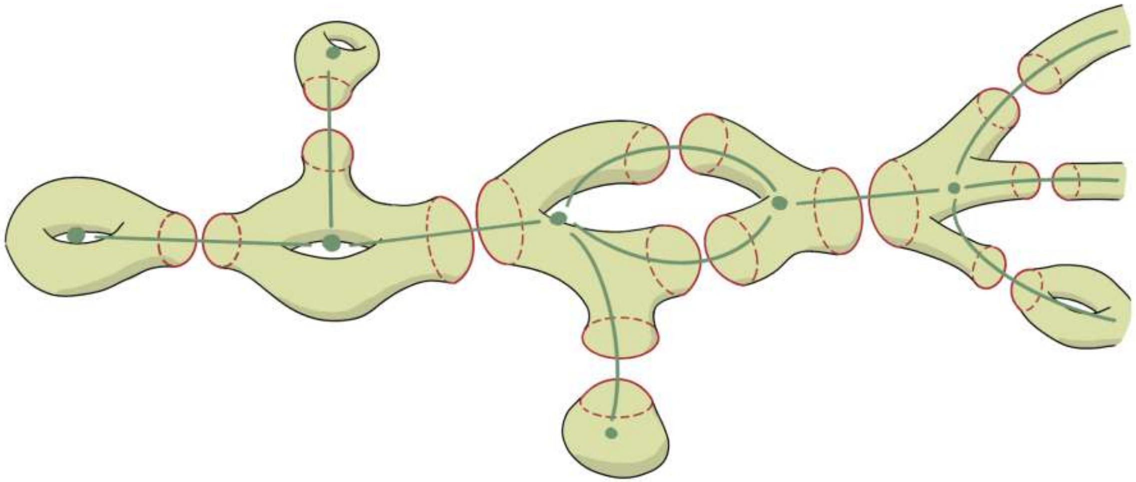
Decomposition of complete 3-manifolds of PSC with quadratic decay



Teo Gil Moreno de Mora Sardà

joint work with Florent Balacheff & Stéphane Sabourau

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$\to$ PSC 3-manifolds

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Yau's 27th Problem

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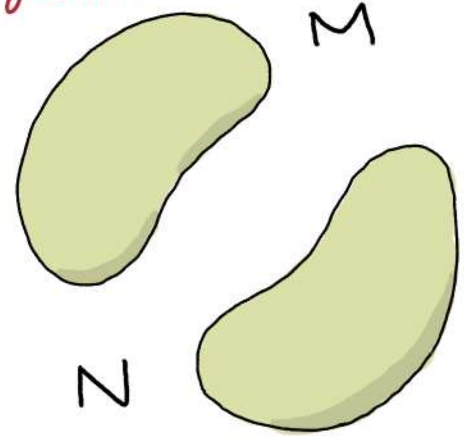

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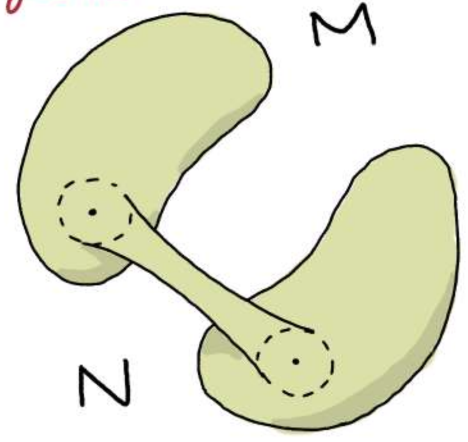

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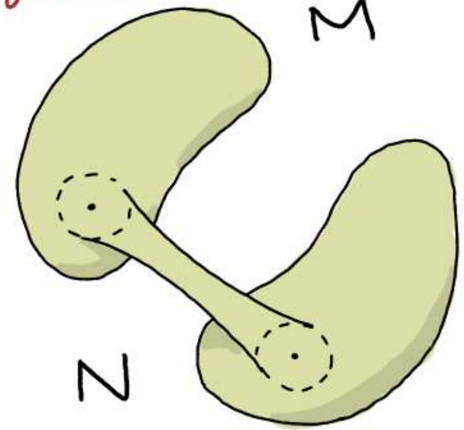

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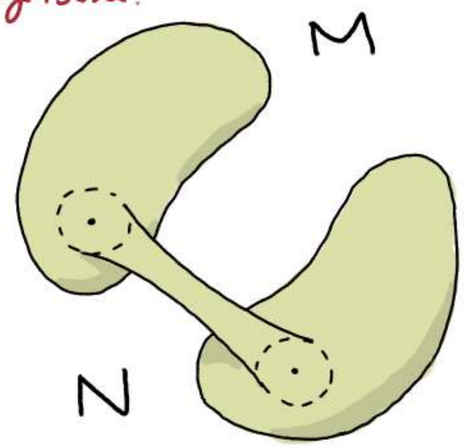

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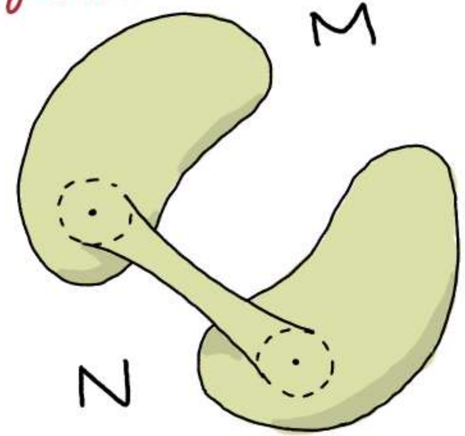

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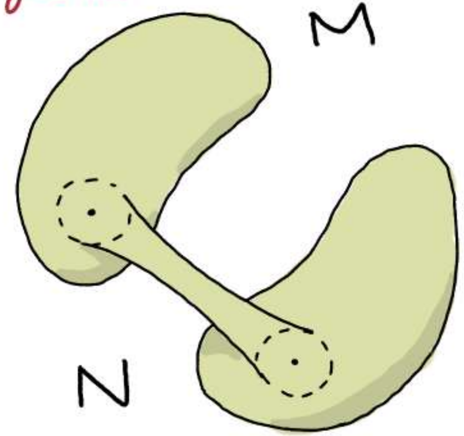

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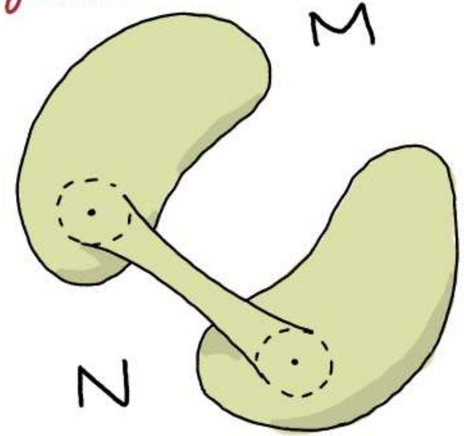

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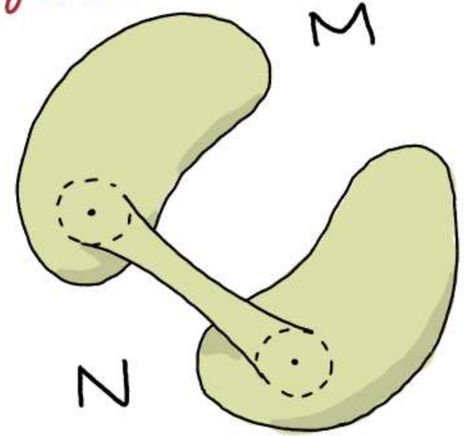

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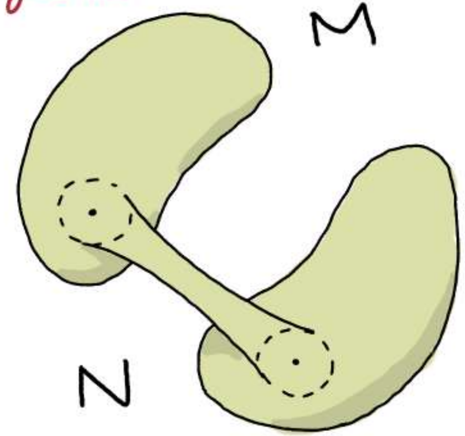

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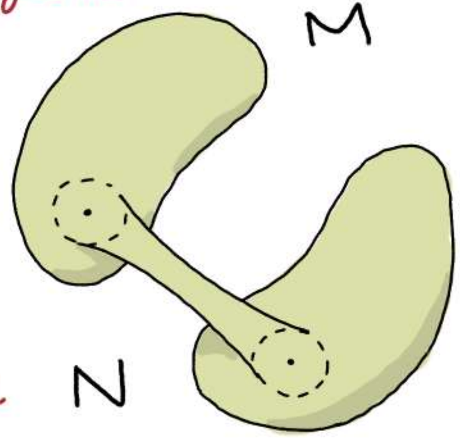

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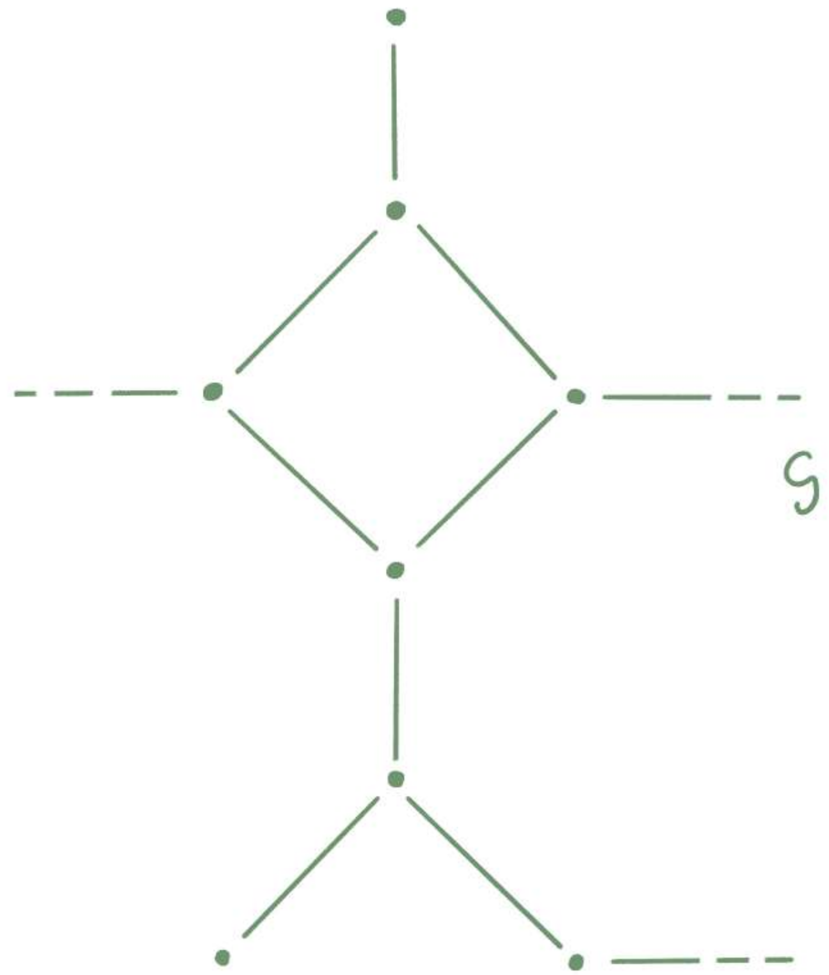

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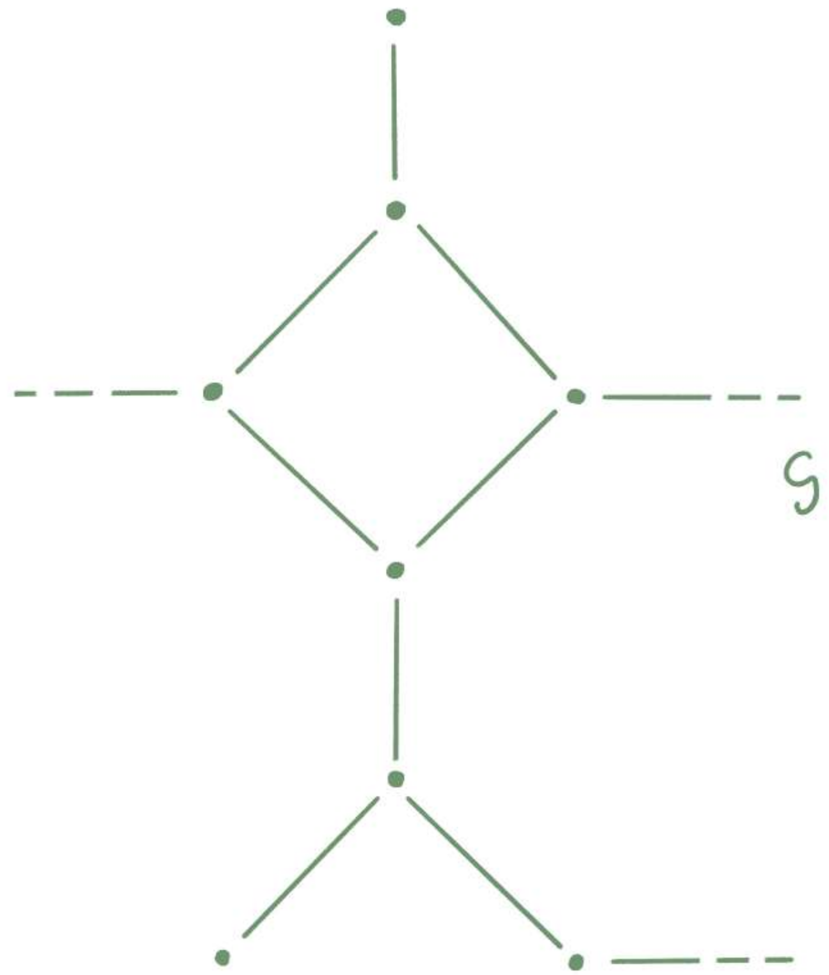

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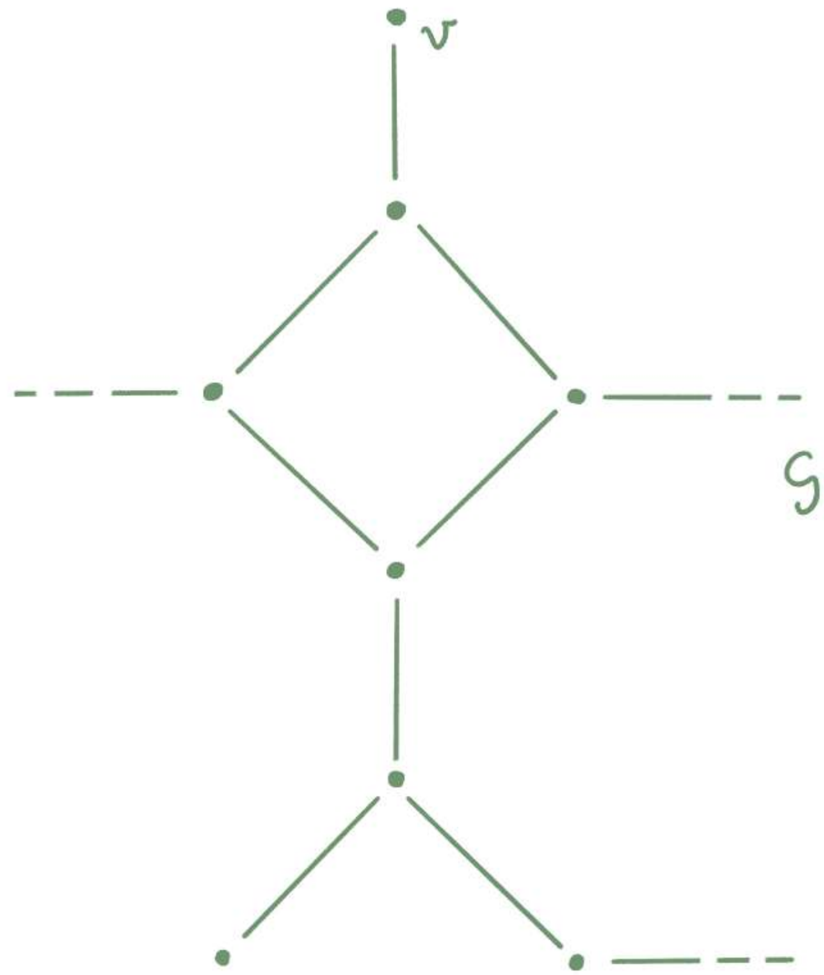

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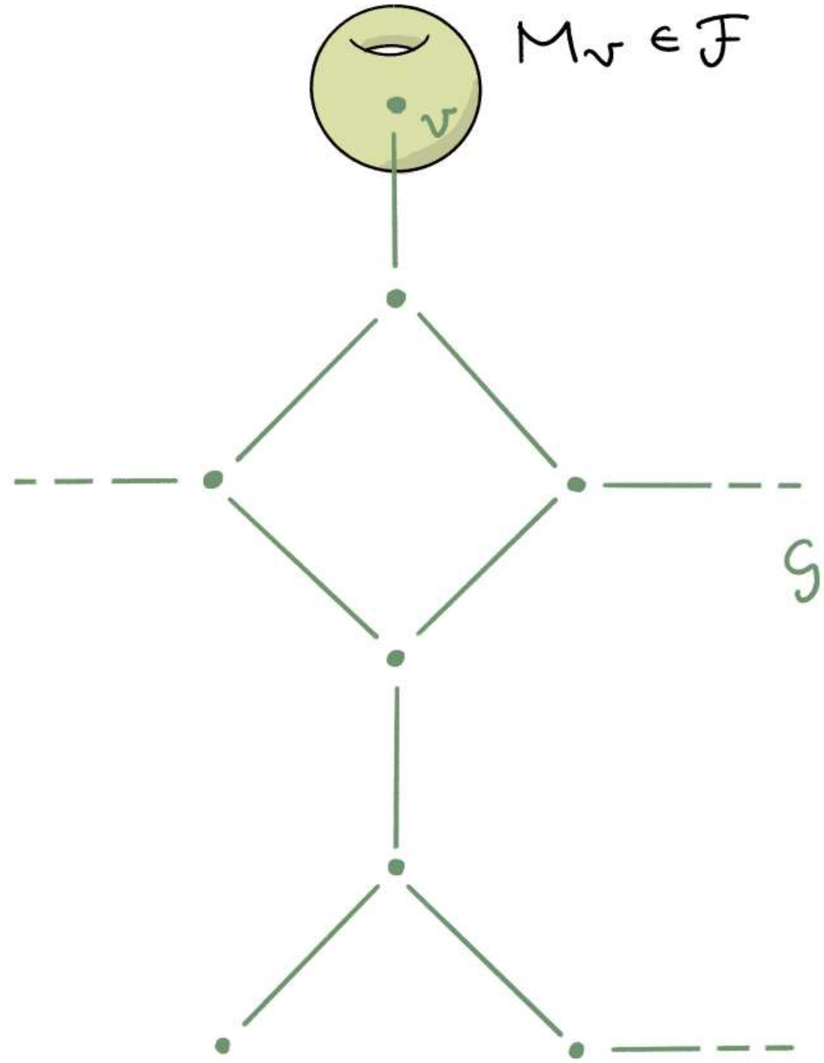

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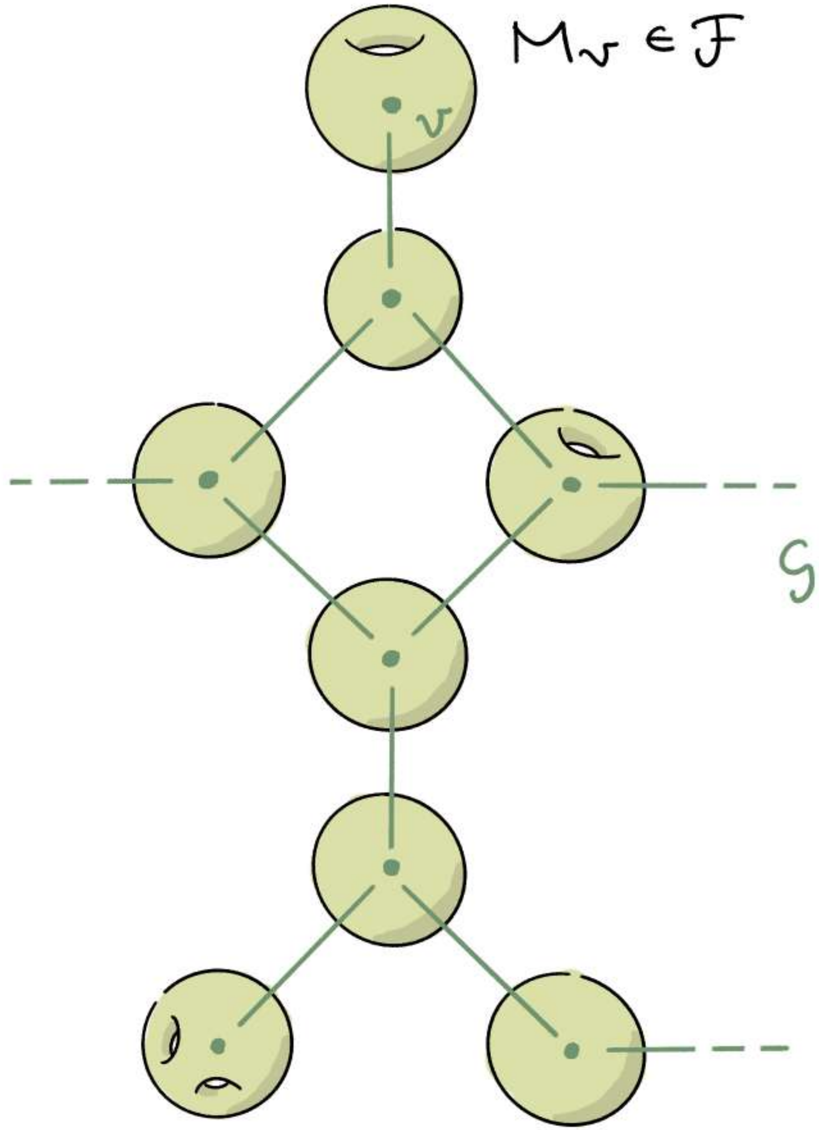

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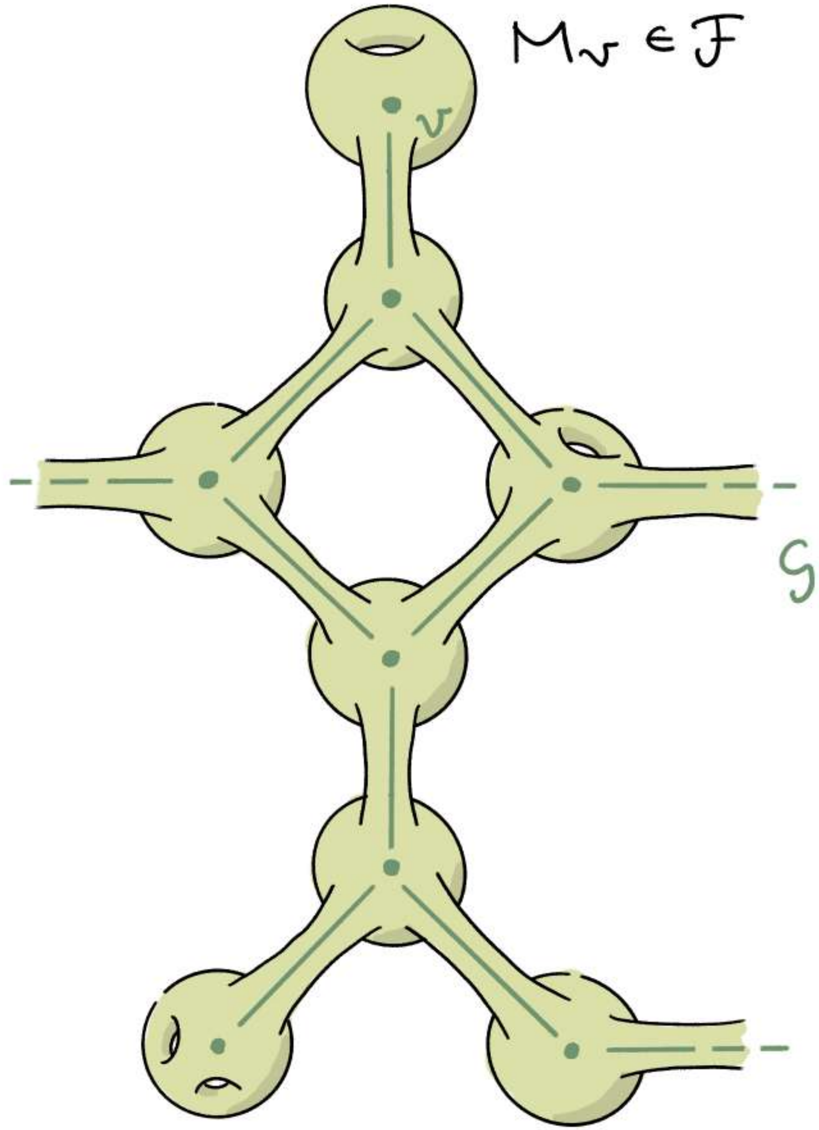

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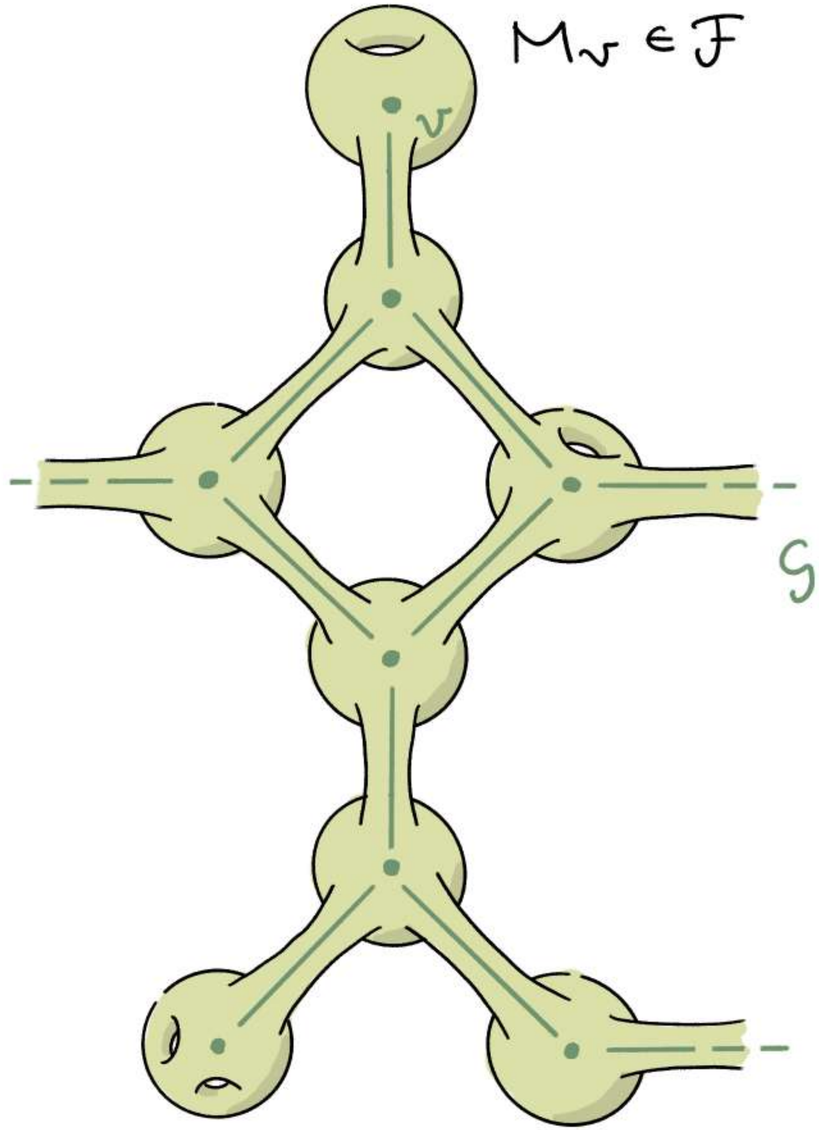

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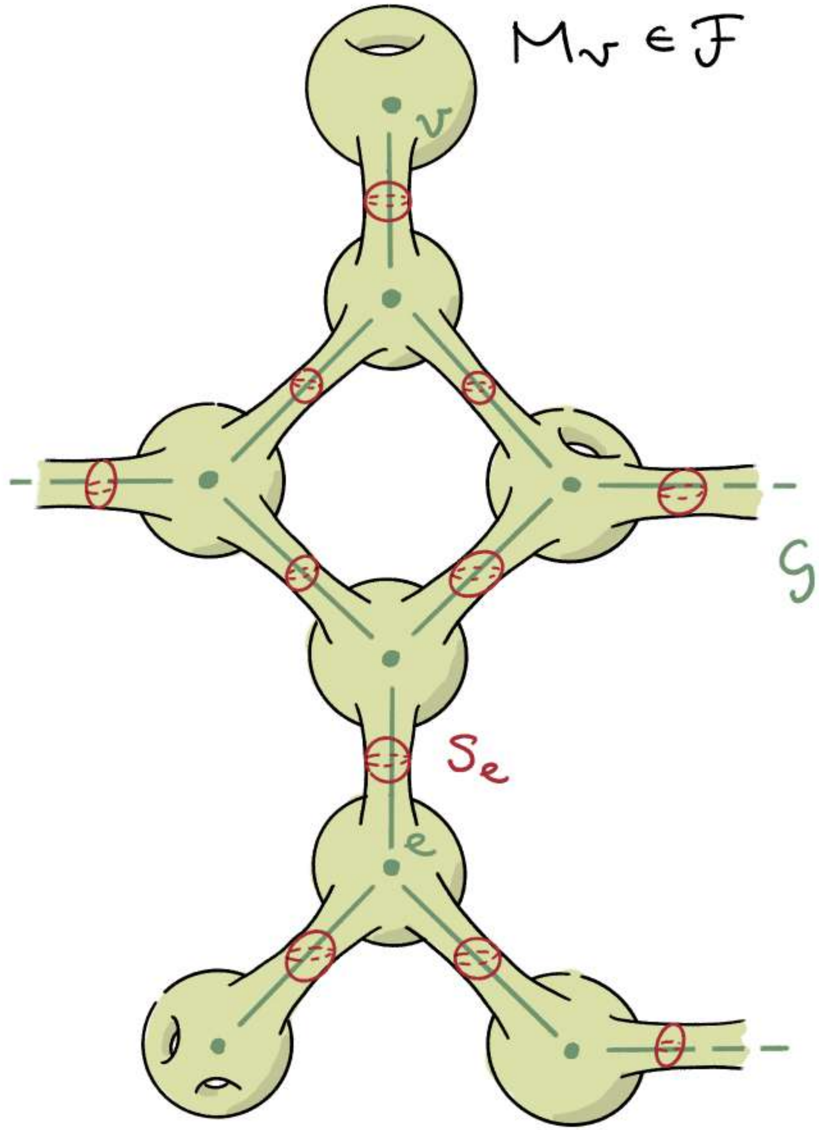

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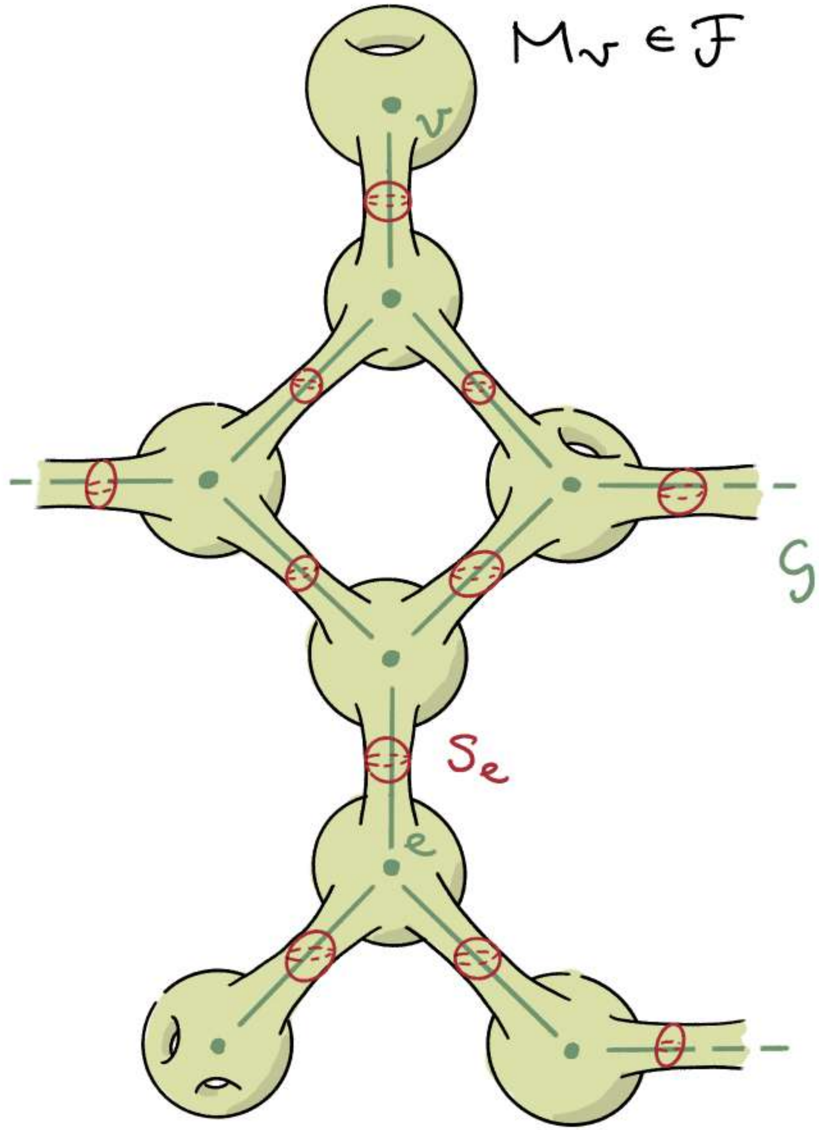

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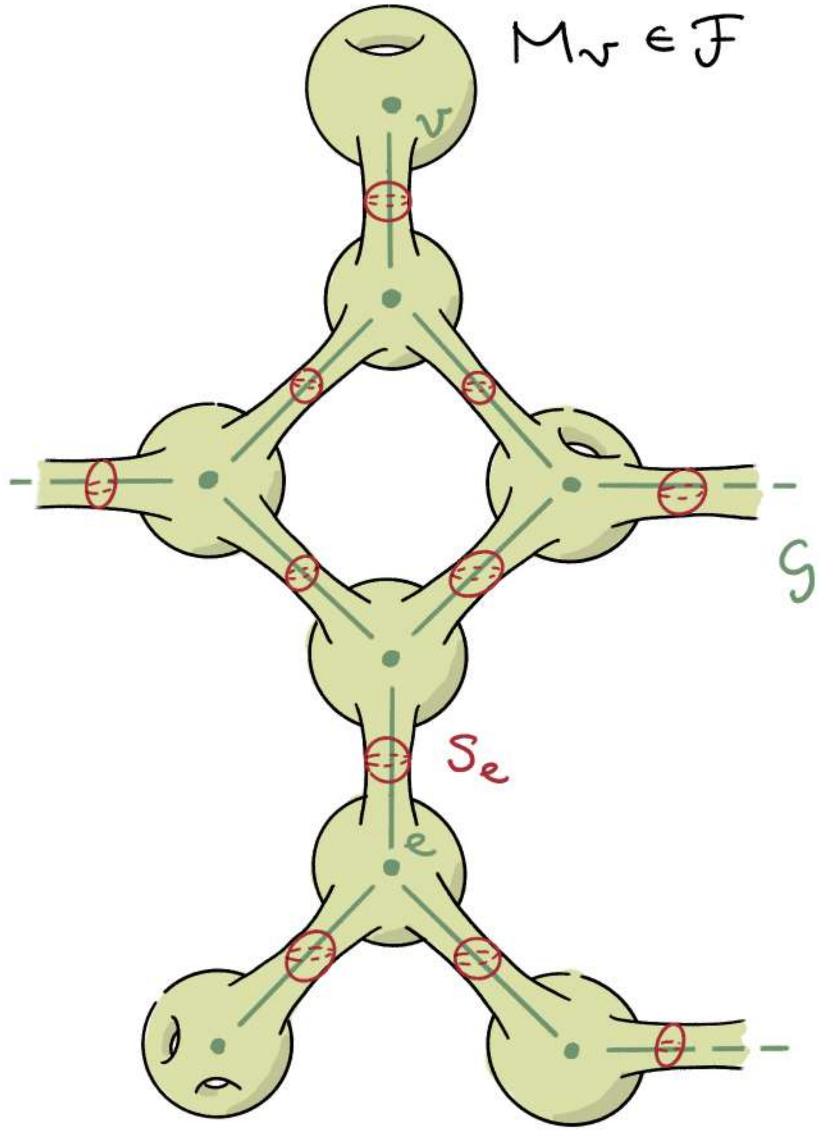

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(ii). Adding $S^2 \times S^1$ to $\mathcal{F}$: $\mathcal{G}$ locally finite tree

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- Existence?

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- Uniqueness? It's subtle

  ↳ Depends on $\mathcal{G}$

  ↳ $P_1 \# P_2 \# P_1 \# P_2 \# \dots \simeq P_1 \# P_2 \# P_2 \# P_1 \# P_2 \# P_2 \# \dots$

  ↳ $\dots \# S^3 \# S^3 \# P_1 \# P_2 \# P_3 \# \dots \not\simeq \dots \# P_5 \# P_3 \# P_1 \# P_2 \# P_4 \# \dots$

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but admits $g = dr^2 + r\, d\theta^2 + dt^2$

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$\mathbb{R}^2 \times S^1 \not\simeq \#_i S^3/\Gamma_i \#_j S^2 \times S^1$

but admits $g = dr^2 + r\, d\theta^2 + dt^2 \longrightarrow \mathrm{scal} = \frac{1}{2}\frac{1}{r^2}$

Proof

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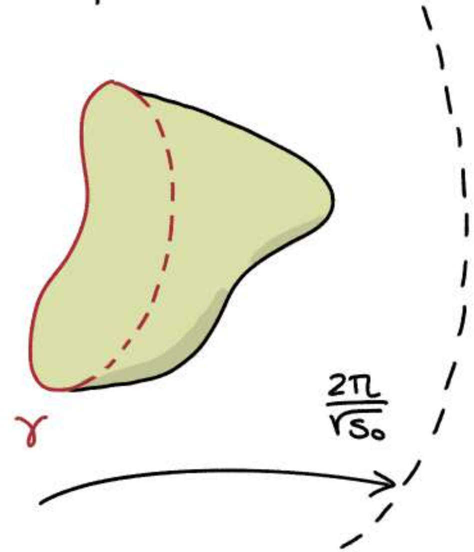

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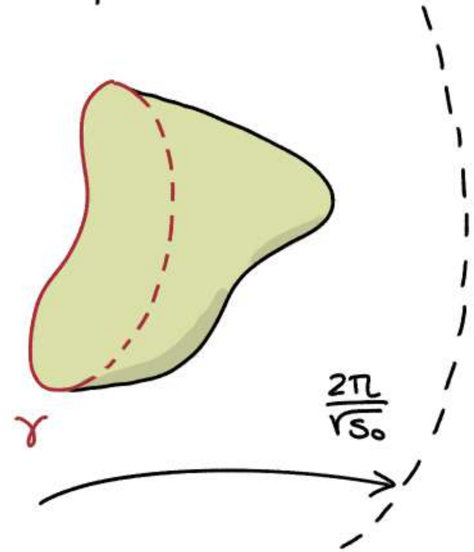

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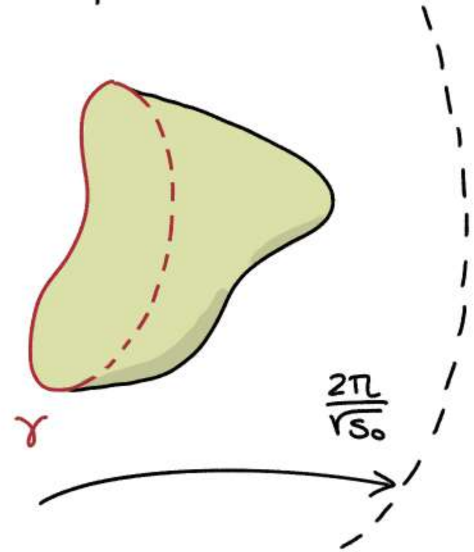
$$\mathrm{fillrad}_M \leq \frac{2\pi}{\sqrt{s_0}}$$



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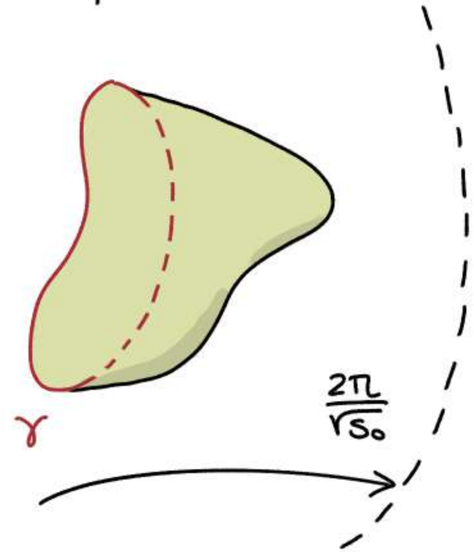

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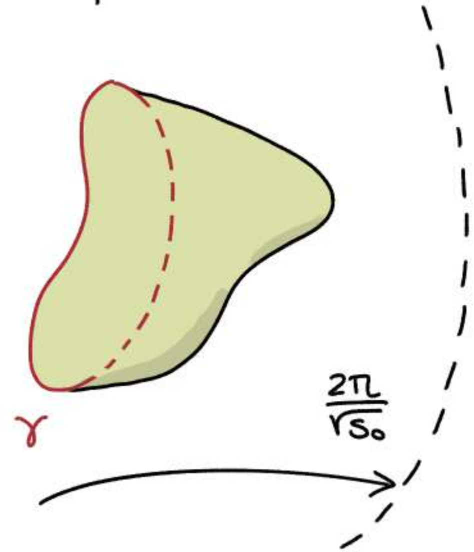
$$\mathrm{scal} \geq s_0 > 0 \implies \mathrm{fillrad}_{\tilde{M}} \leq \frac{2\pi}{\sqrt{s_0}}$$

$\searrow\ \mathrm{scal}_{\tilde{M}} \geq s_0 > 0\ \nearrow$



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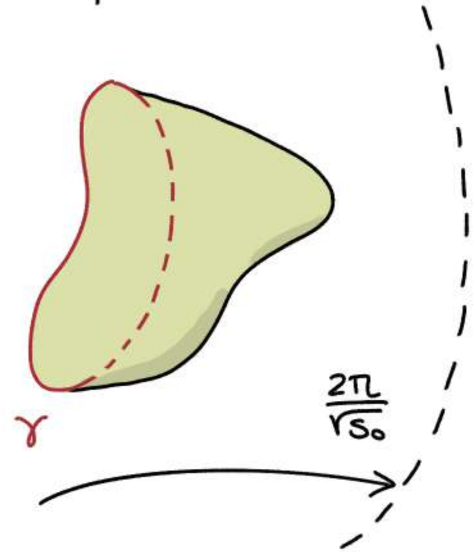

$$\mathrm{scal} \geq s_0 > 0 \implies \mathrm{fillrad}_{\tilde{M}} \leq \frac{2\pi}{\sqrt{s_0}}$$

$$\mathrm{scal} > \frac{C}{r^2}$$

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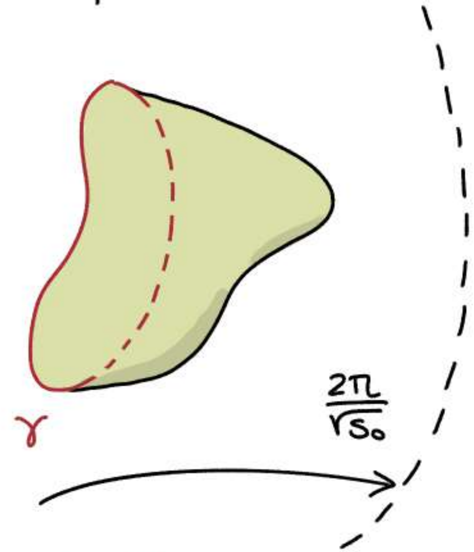
$$\mathrm{scal} \geq s_0 > 0 \implies \mathrm{fillrad}\, \tilde{M} \leq \frac{2\pi}{\sqrt{s_0}}$$

$\Big\downarrow$ applying it to metric balls

$$\mathrm{scal} > \frac{c}{r^2} \implies \mathrm{fillrad}\, \tilde{M} < c' r$$



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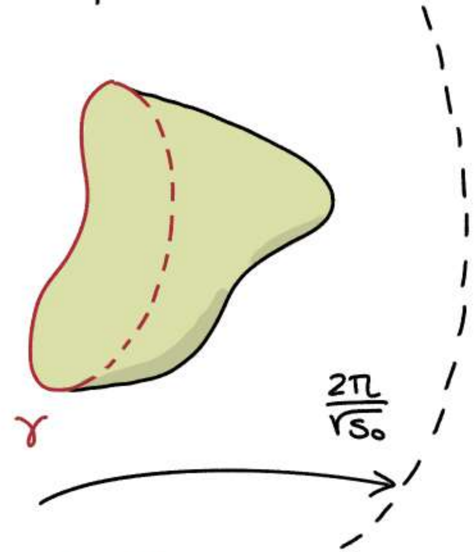

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$$\mathrm{scal} > \frac{c}{r^2} \ (c > 4\pi^2) \implies \mathrm{fillrad}\, \tilde{M} < c' r$$

$$c' = \frac{2\pi}{\sqrt{c} - 2\pi}$$

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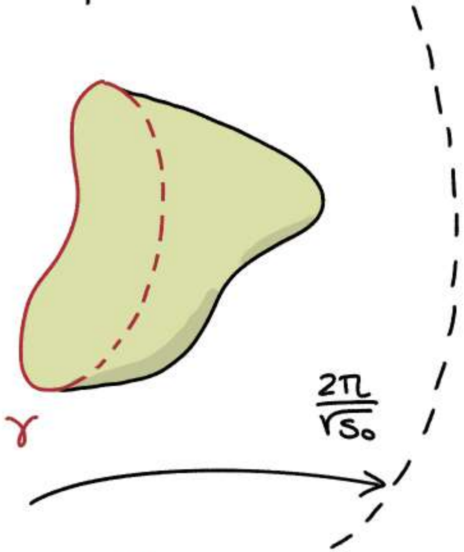

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$(c > 4\pi^2)$

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Def: every $\gamma \subseteq B(x,r)$ s.t. $\gamma \sim 0$ in M:

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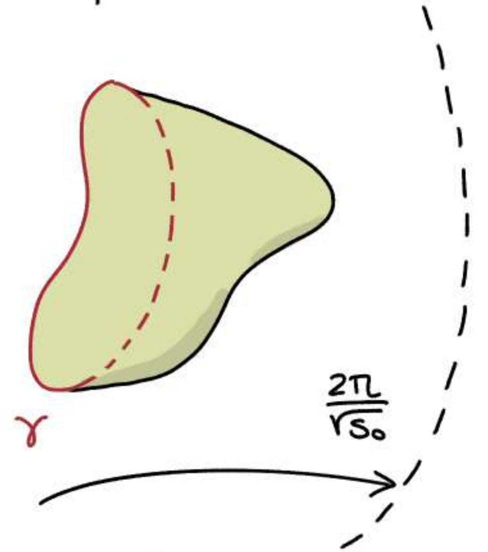

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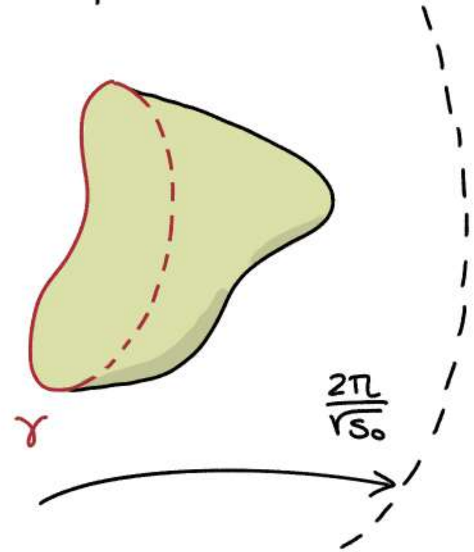

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$\mathrm{scal} \geq s_0 > 0 \implies \mathrm{fillrad}_{\tilde{M}} \leq \frac{2\pi}{\sqrt{s_0}}$

$\mathrm{scal} > \frac{c}{r^2} \implies \mathrm{fillrad}_{\tilde{M}} < c' r$ $(c > 4\pi^2)$ ← synthetic notion for PSC (with quadratic decay)

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$\mathrm{fillrad}\,\tilde{M} < c'r$ ← synthetic notion for PSC (with quadratic decay)

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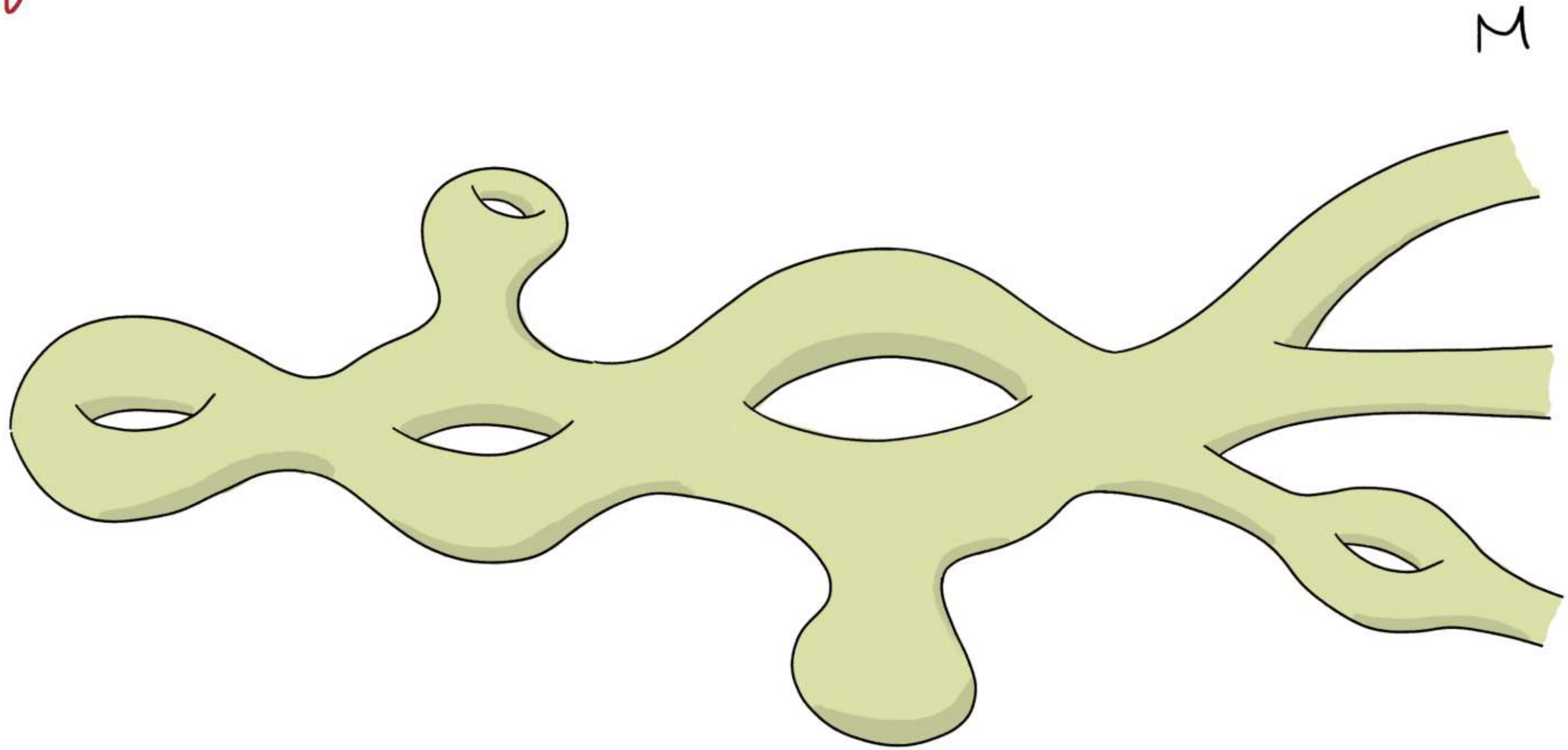

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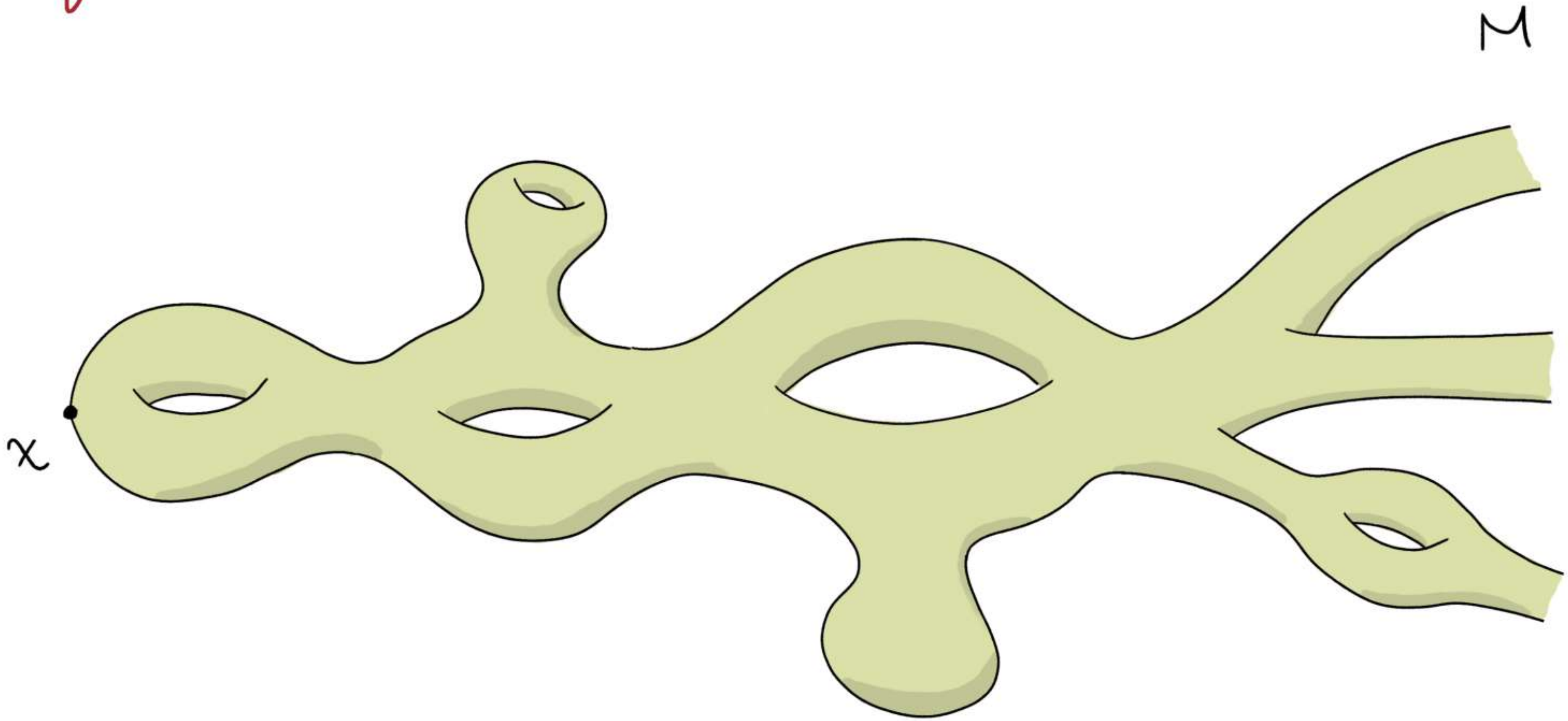

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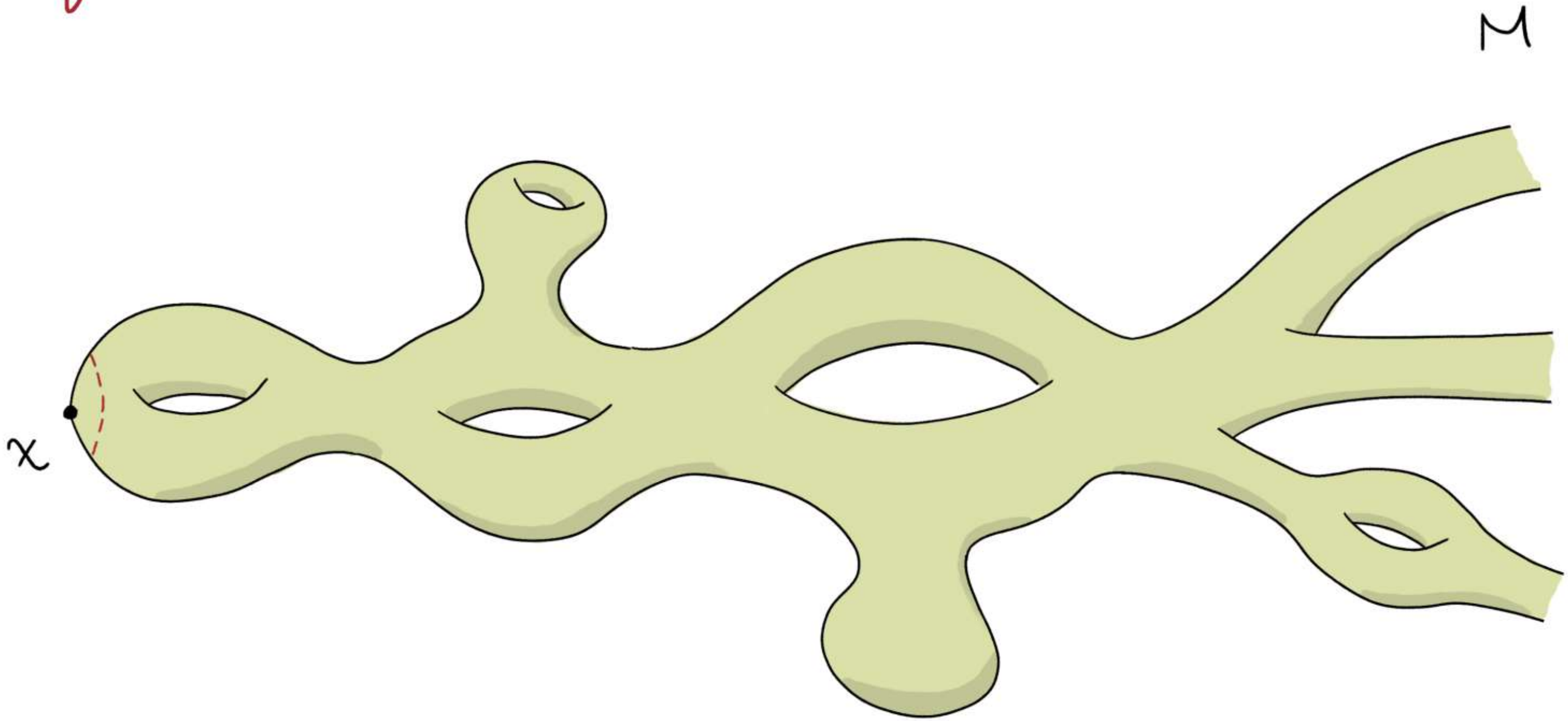

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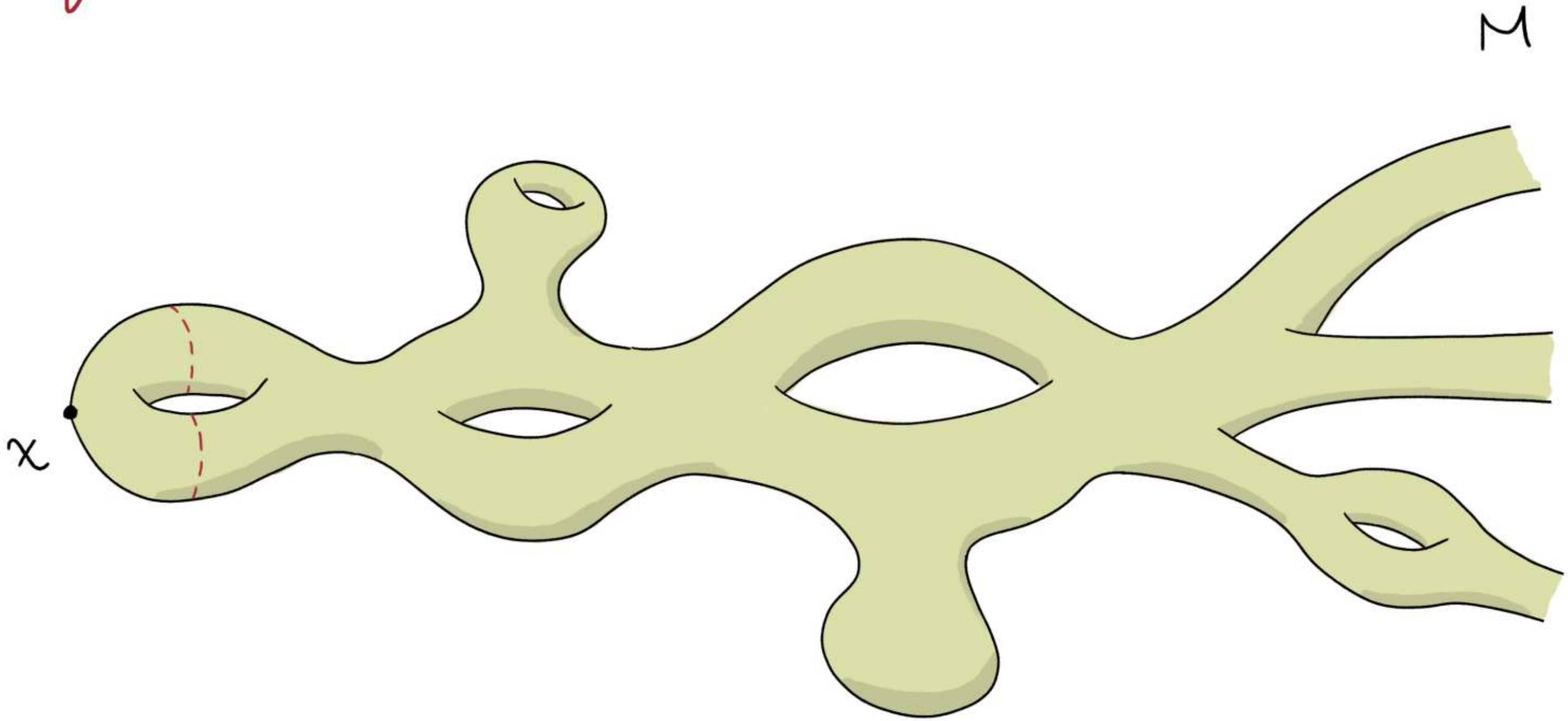

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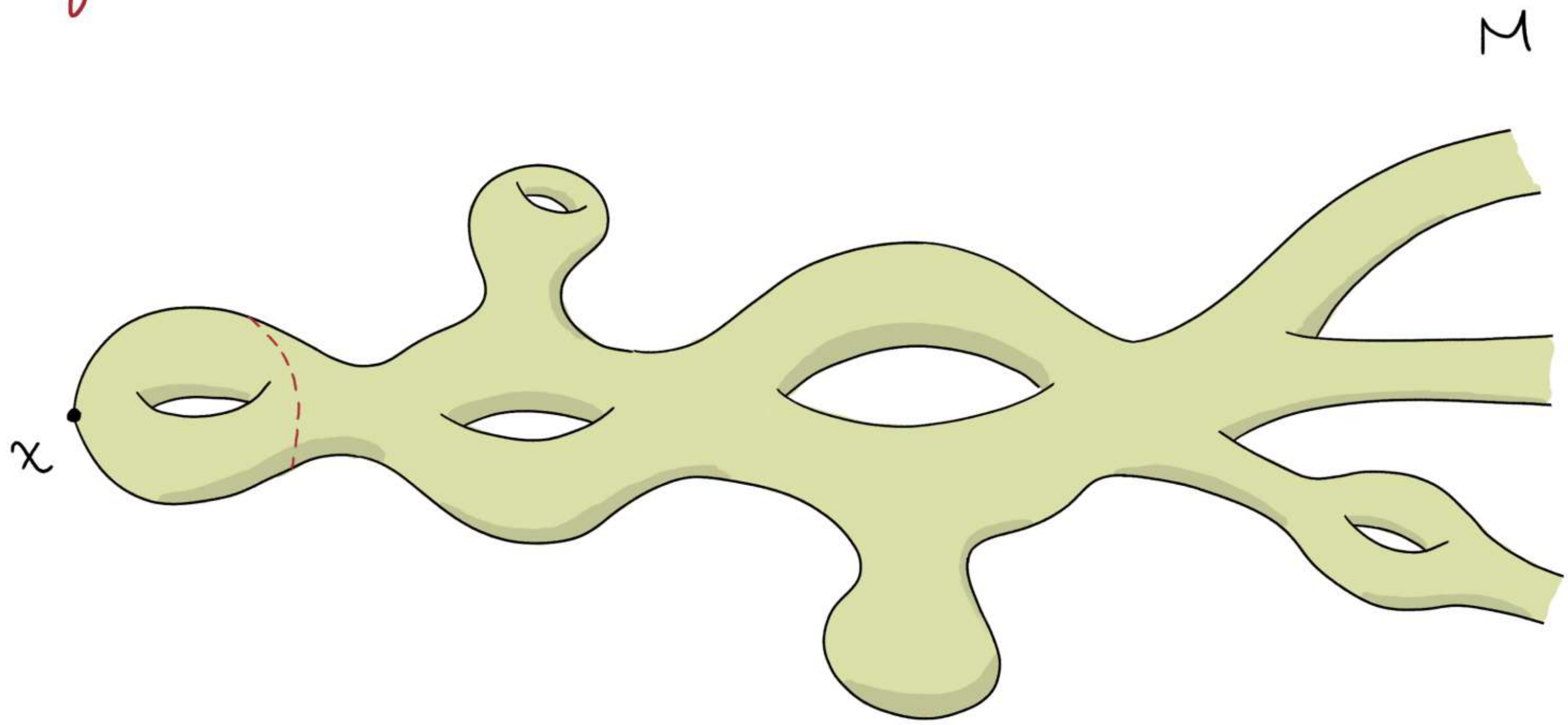

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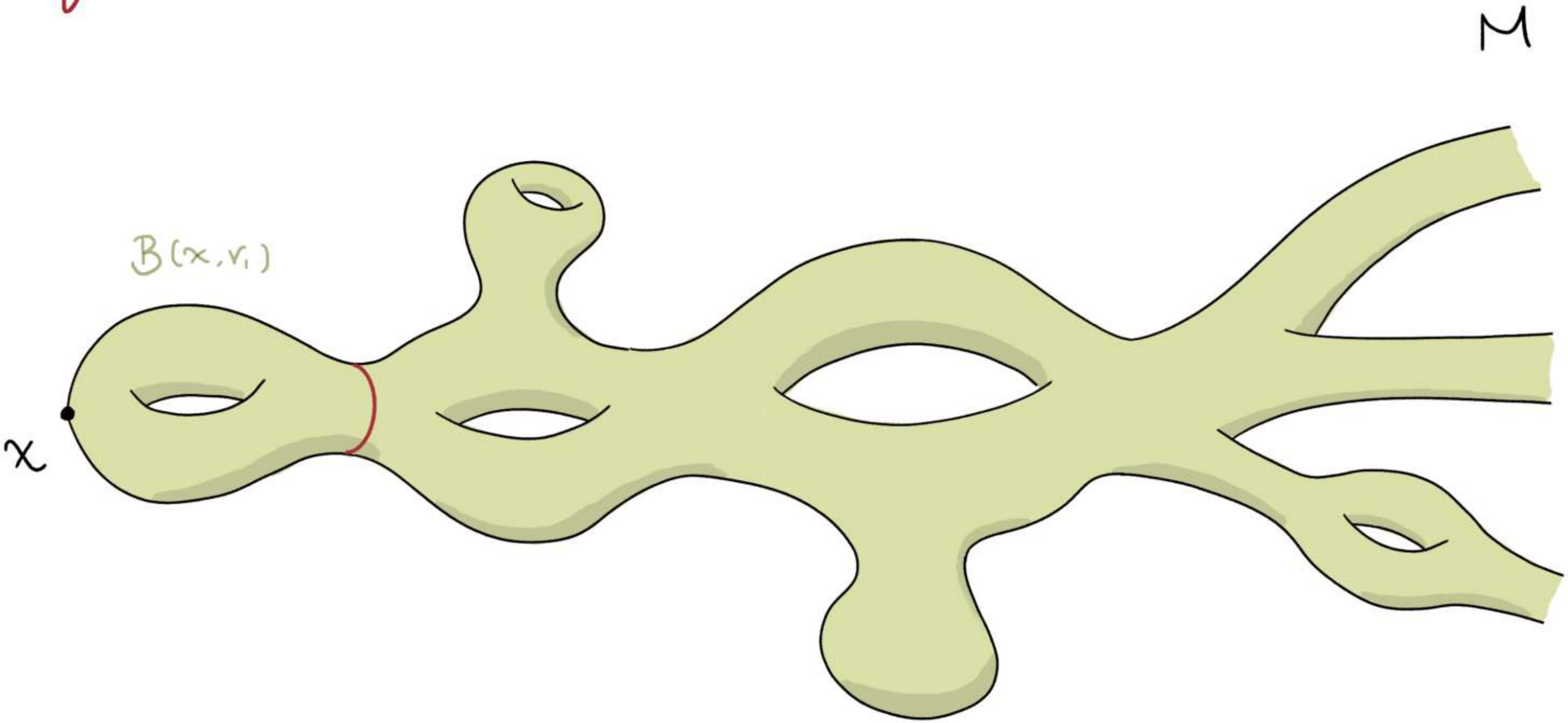

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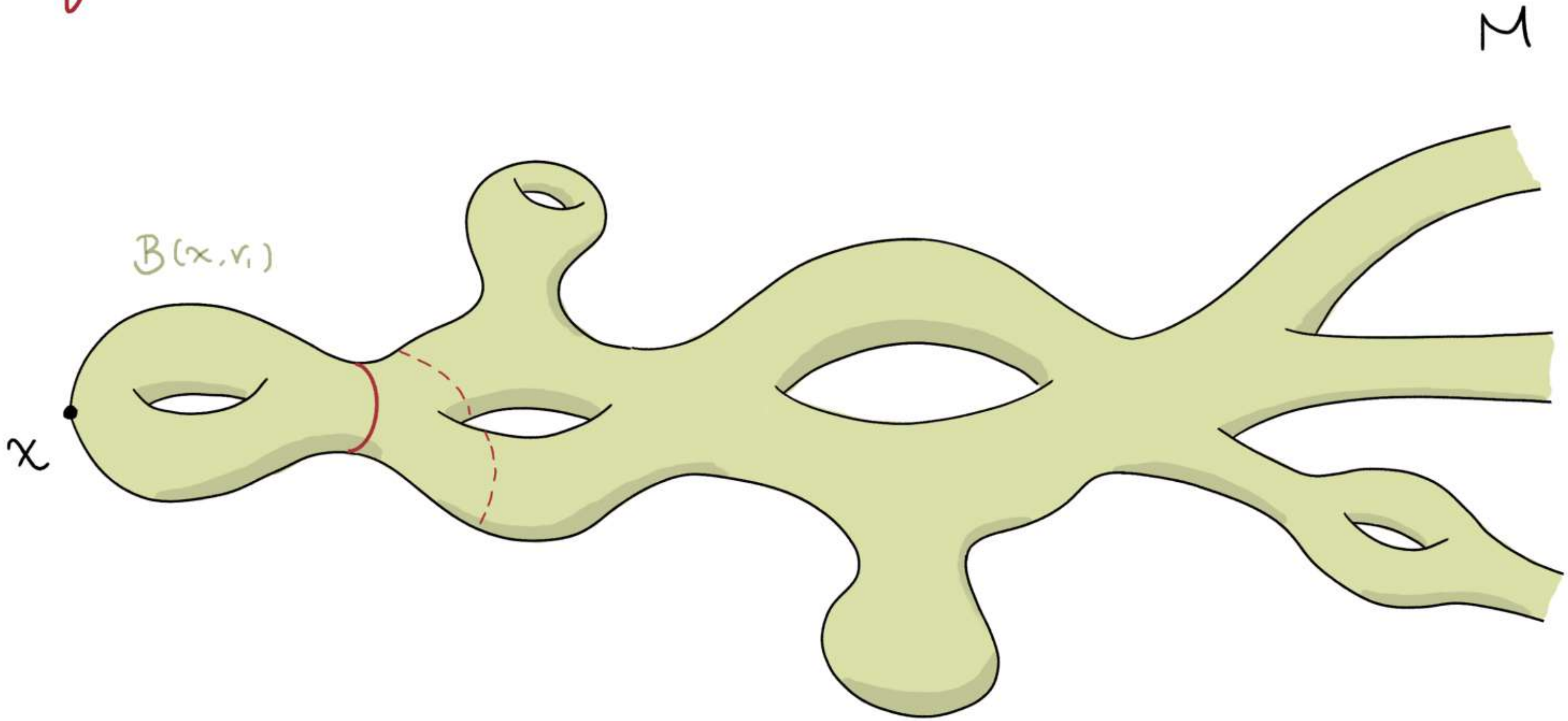

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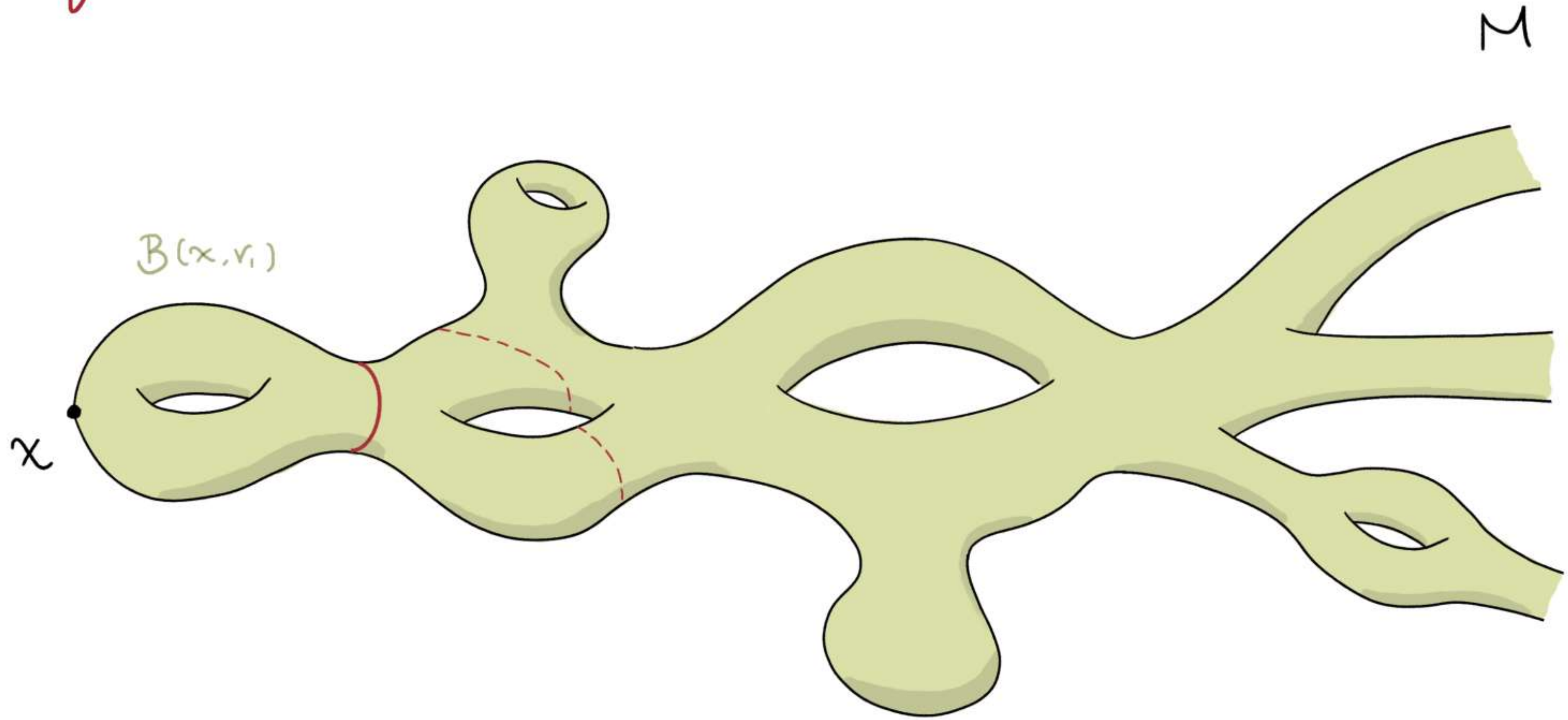

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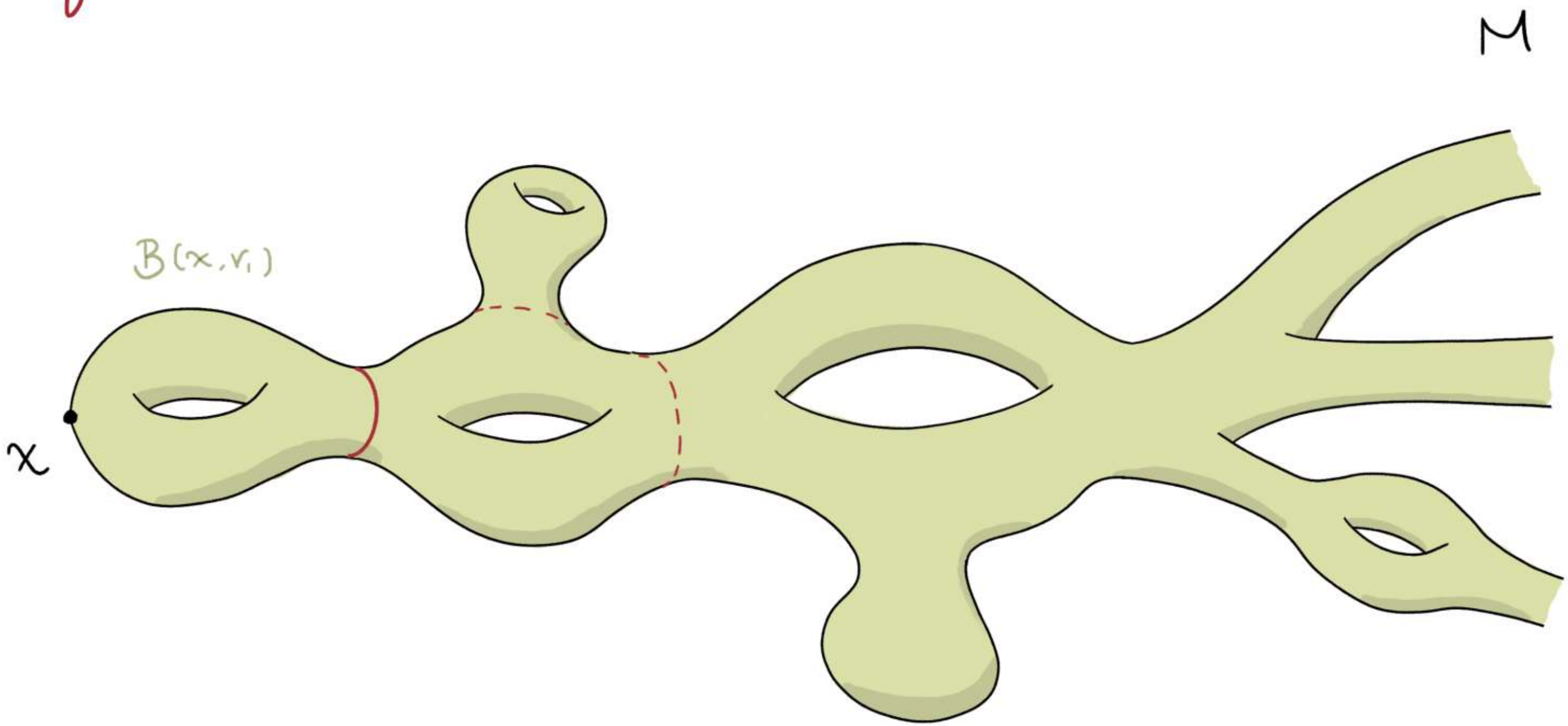

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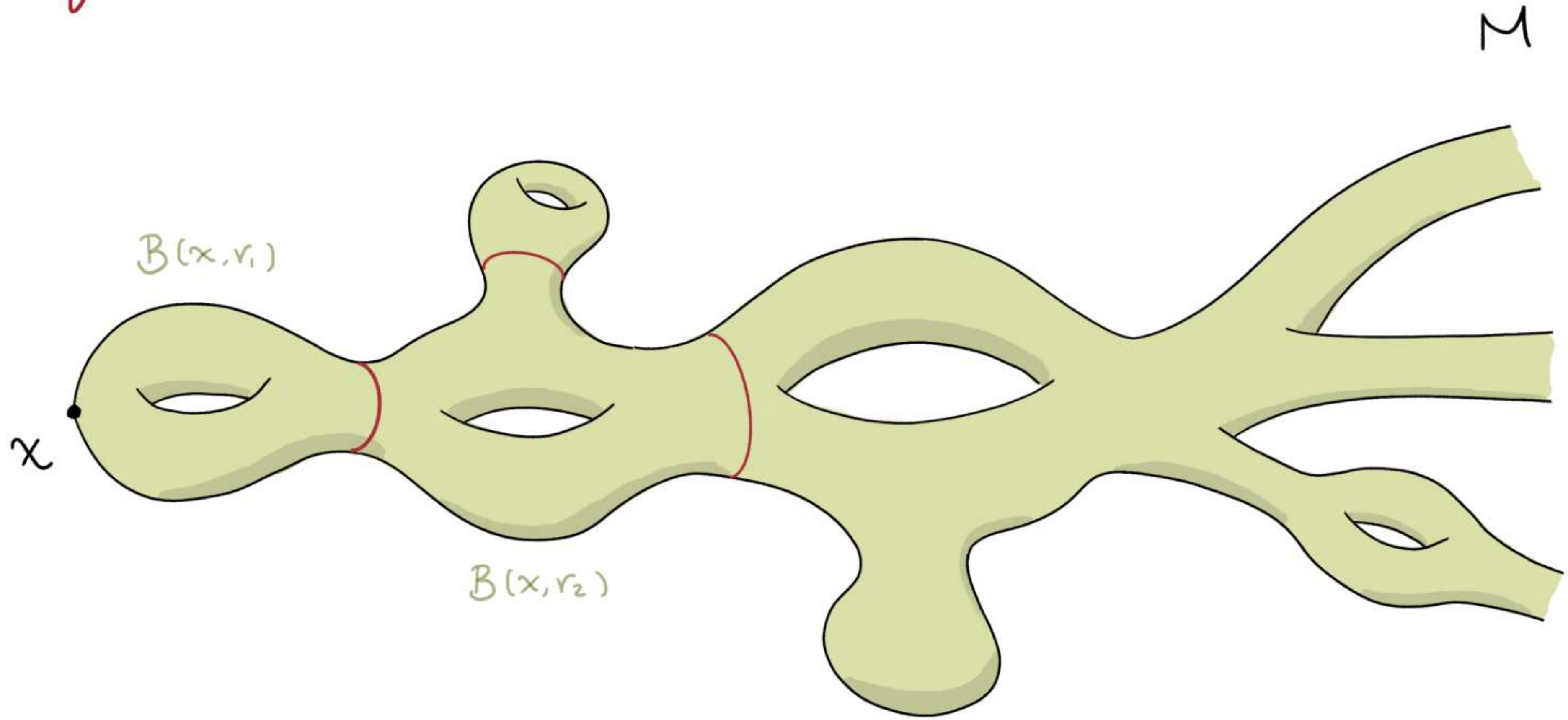

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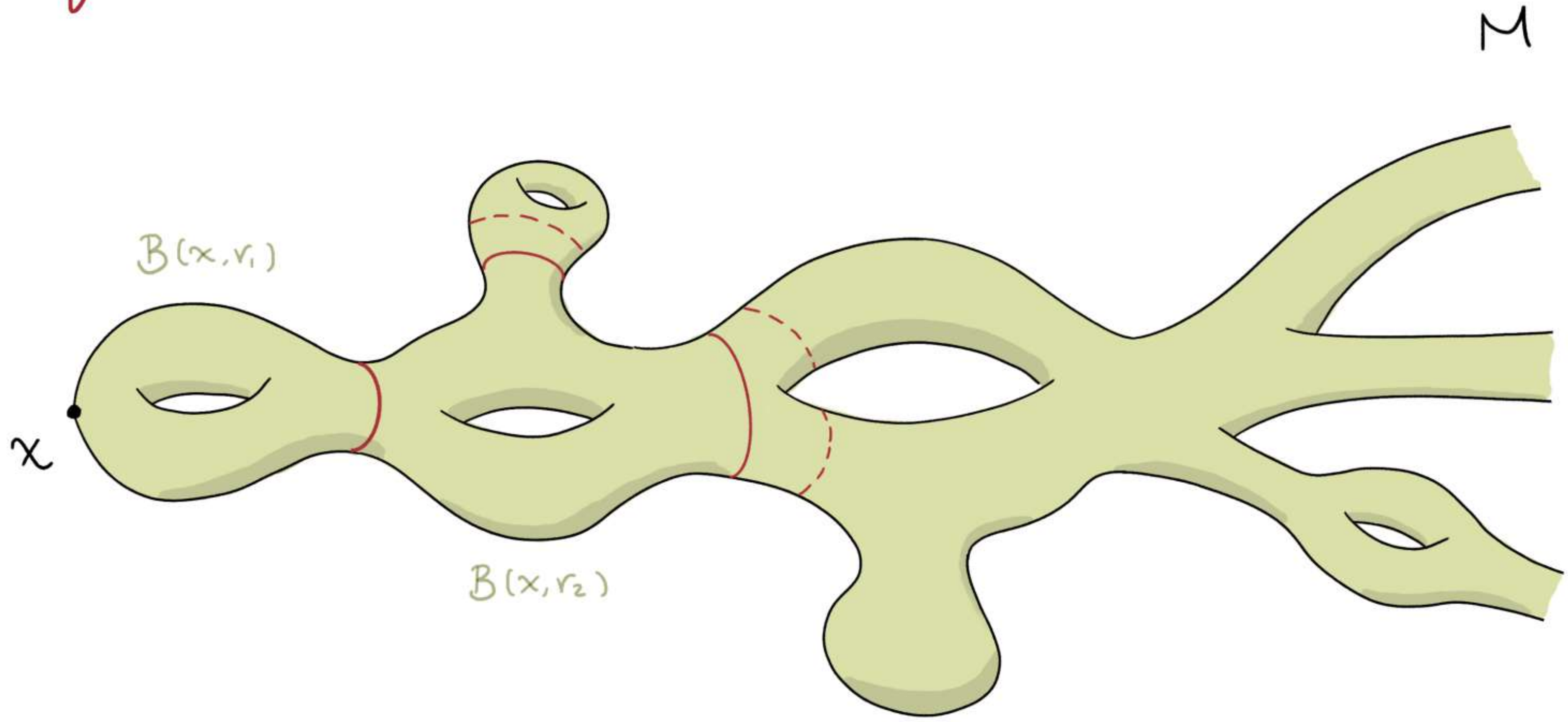

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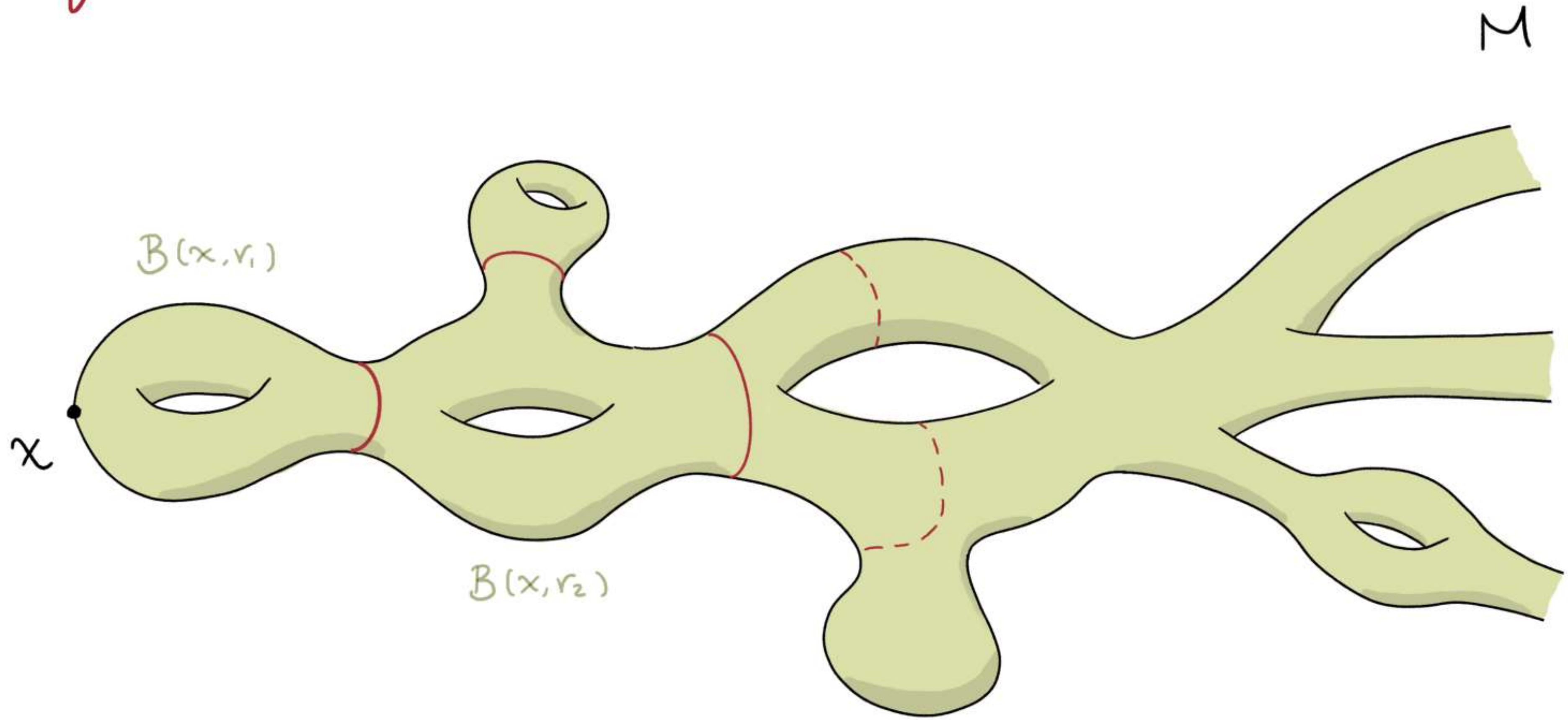

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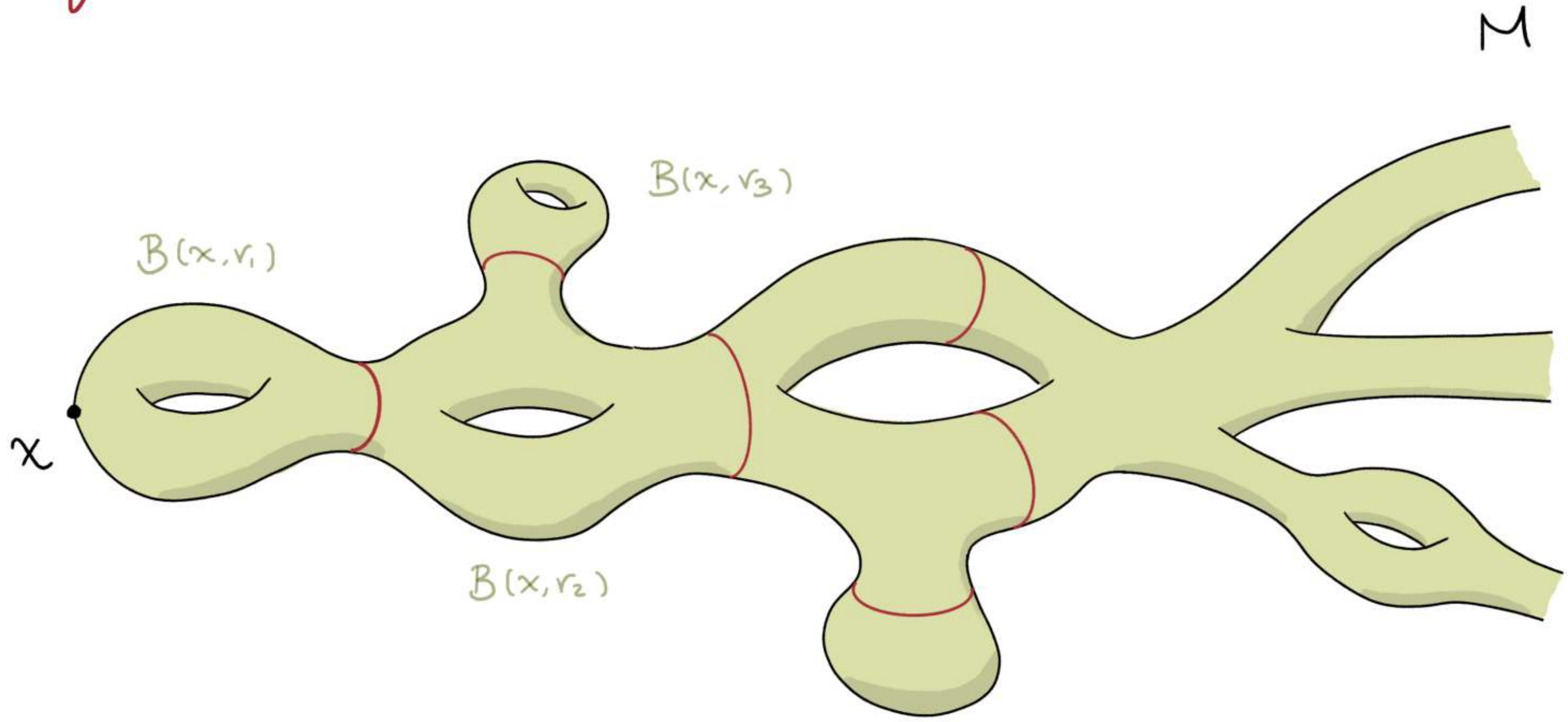

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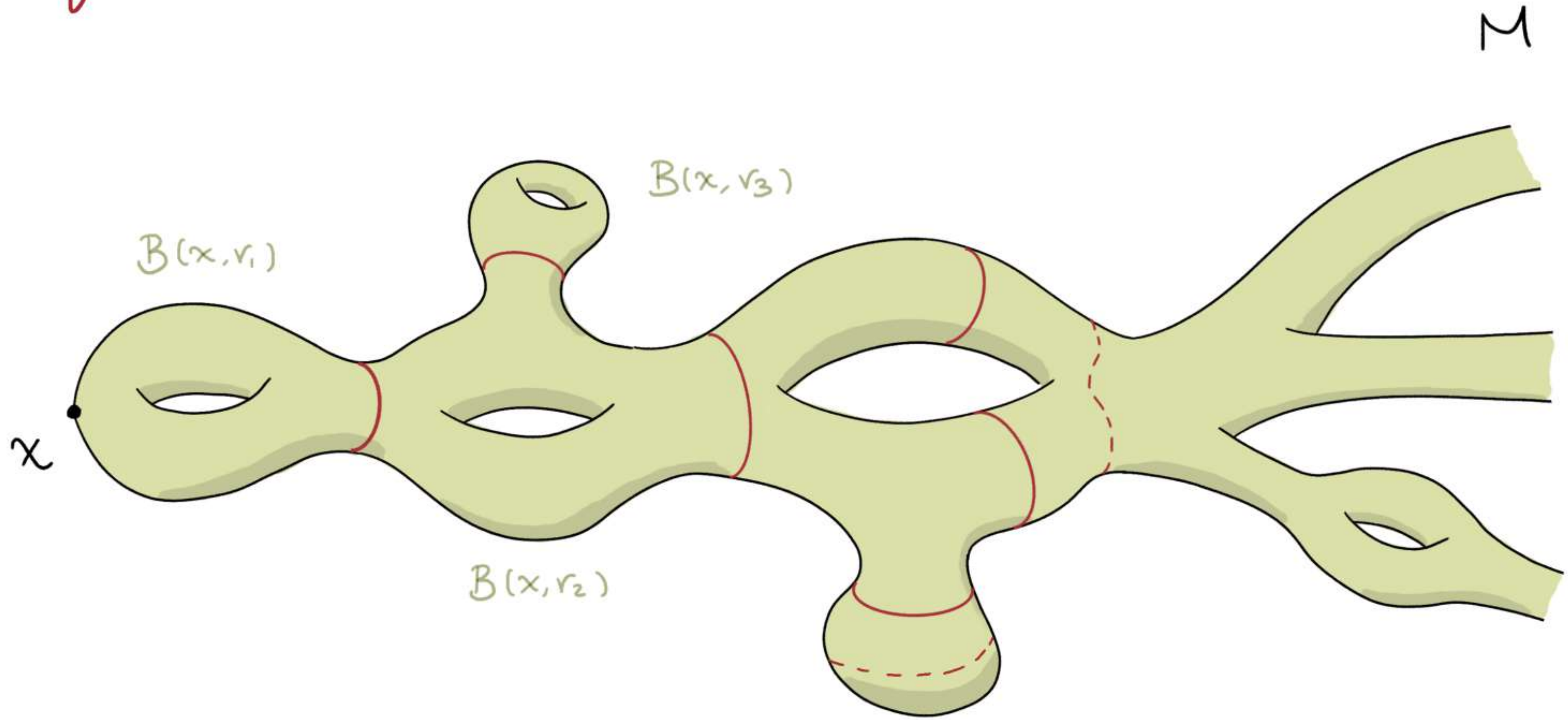

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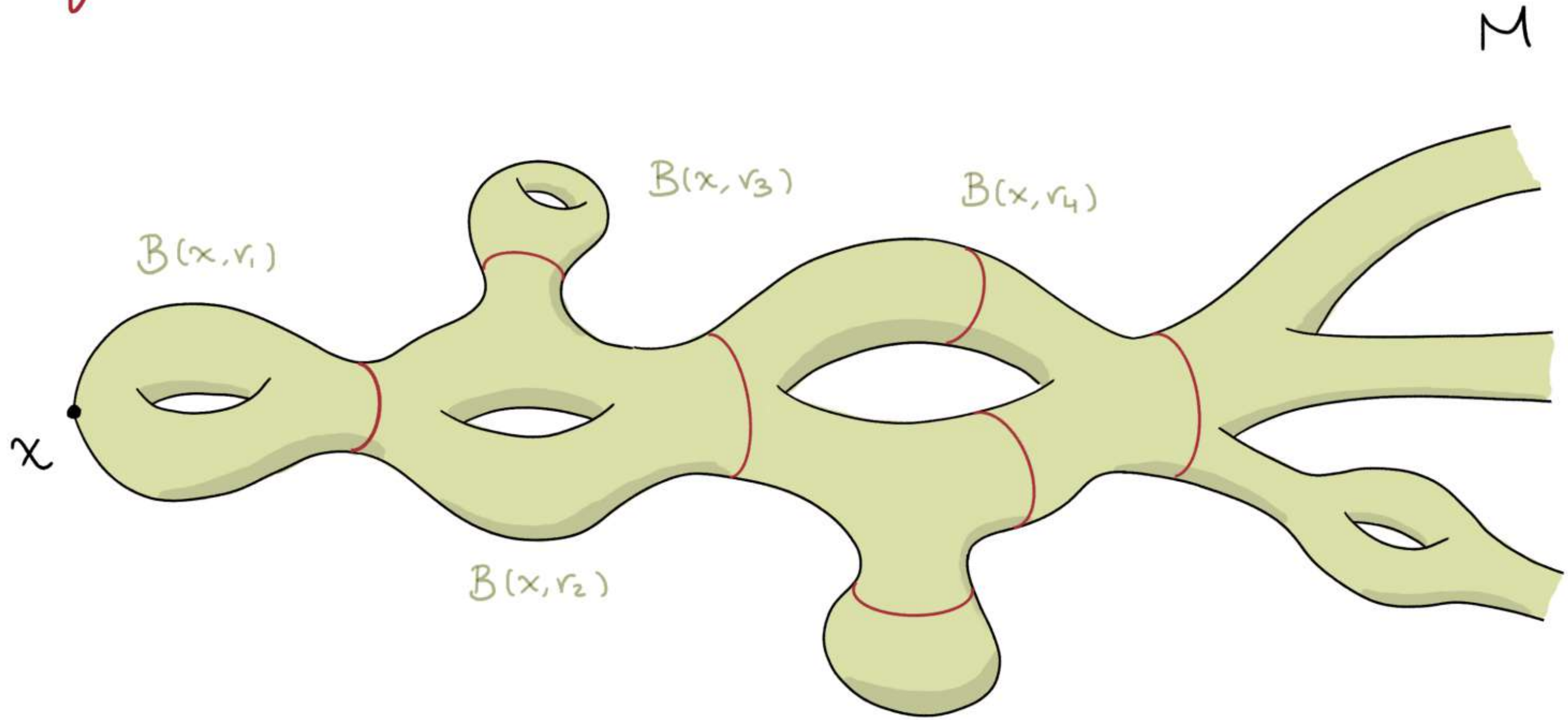

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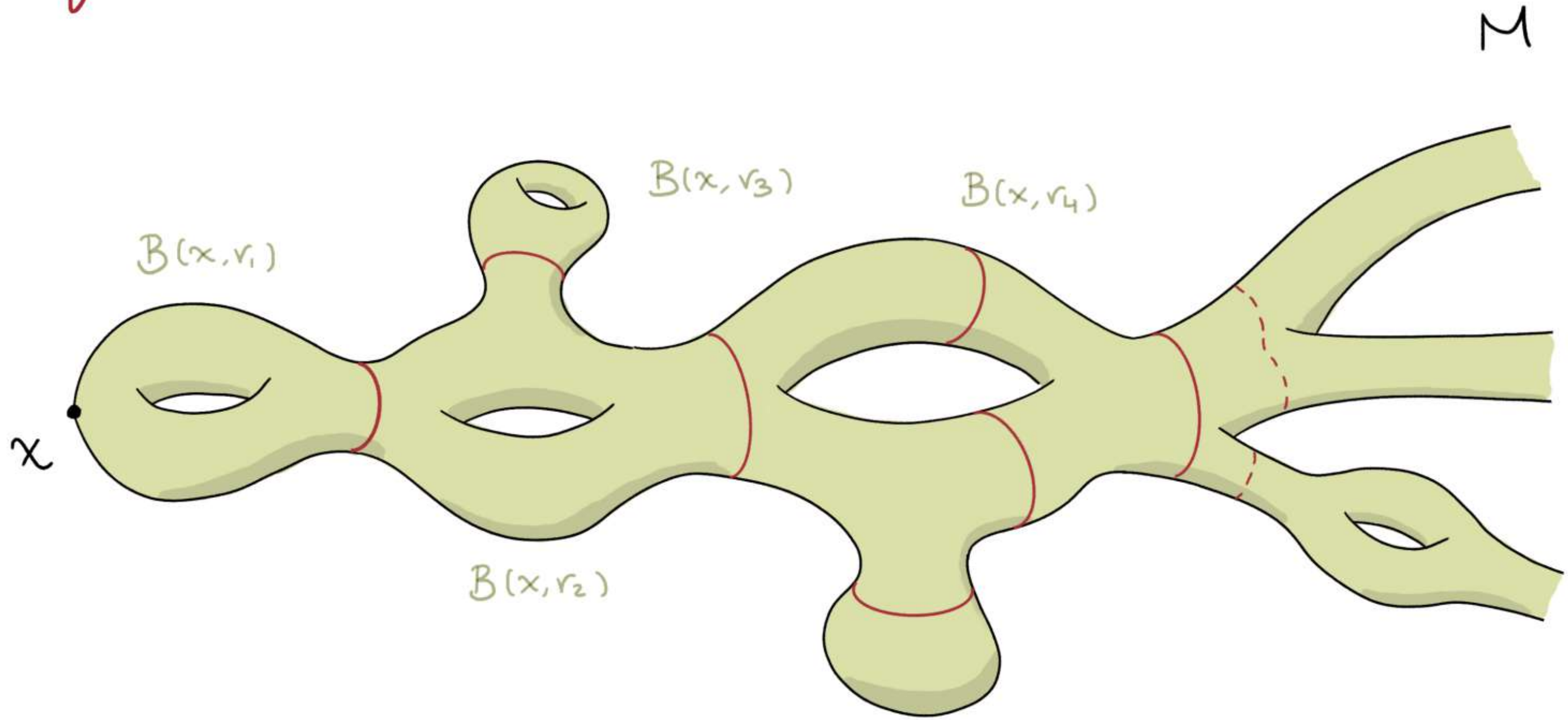

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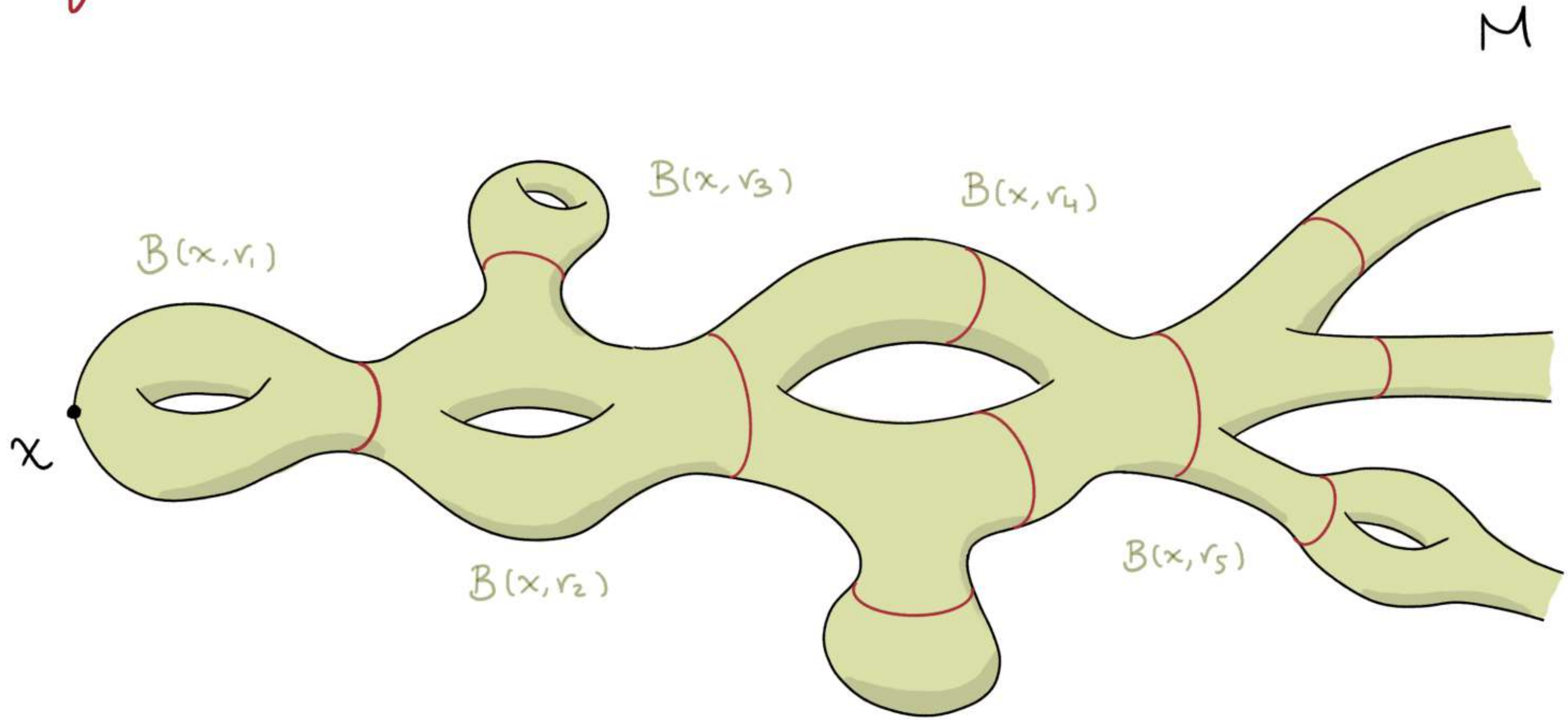

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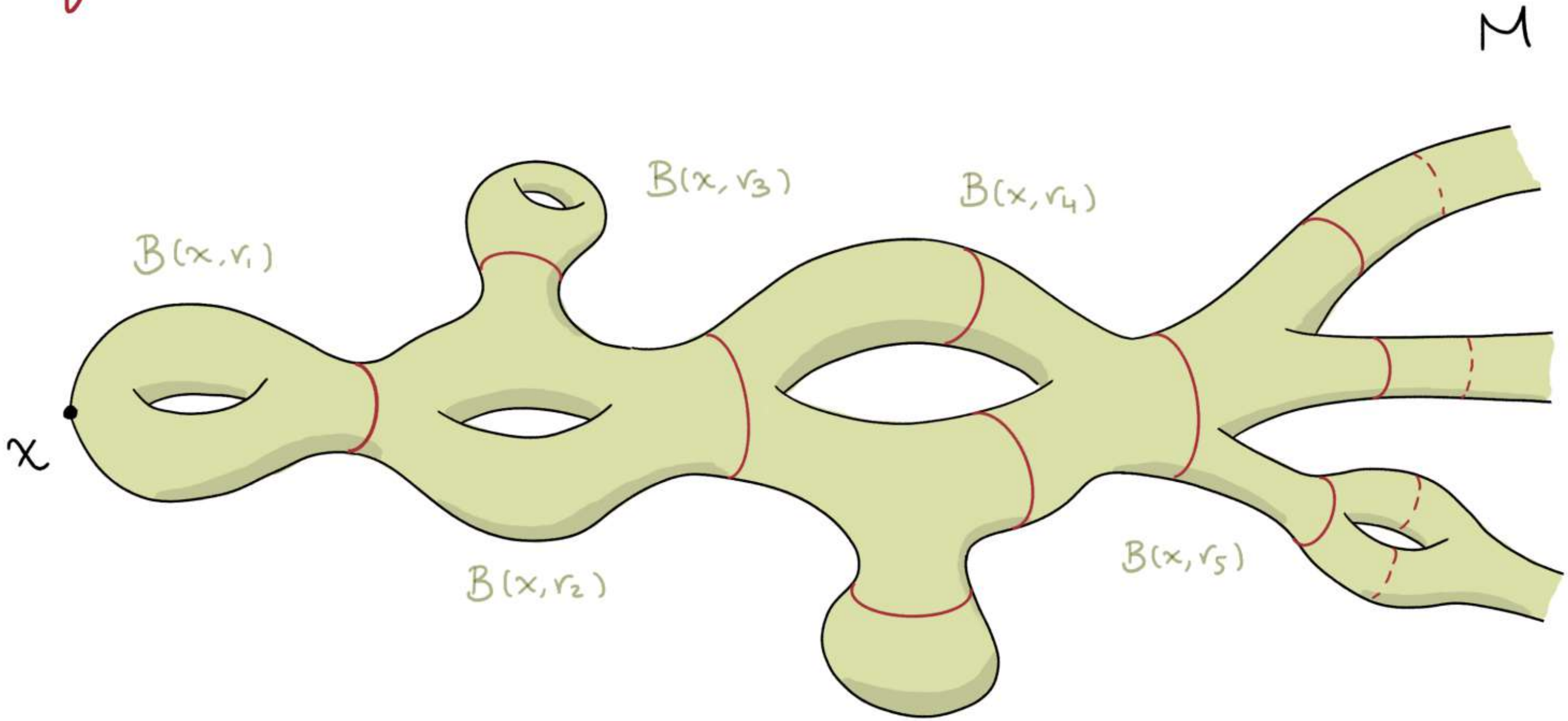

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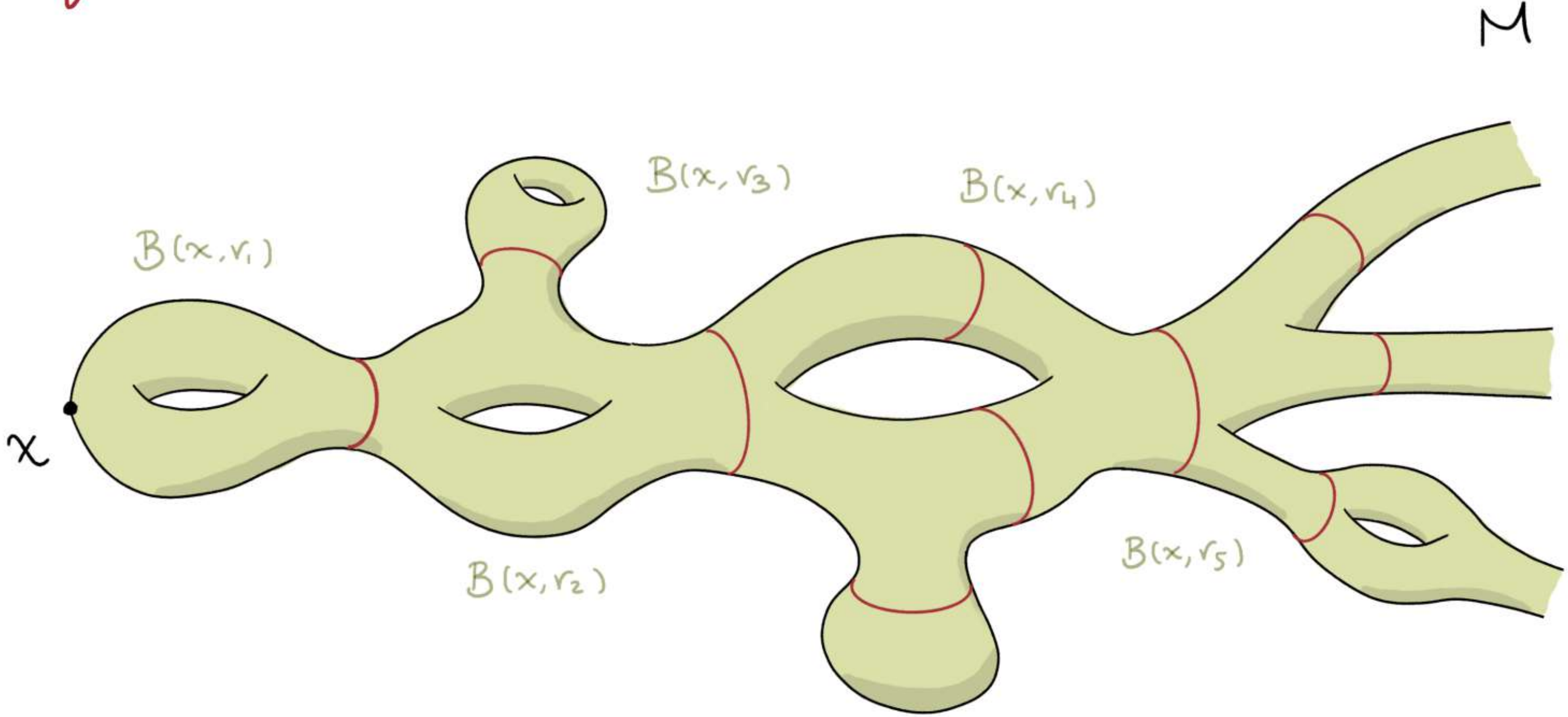

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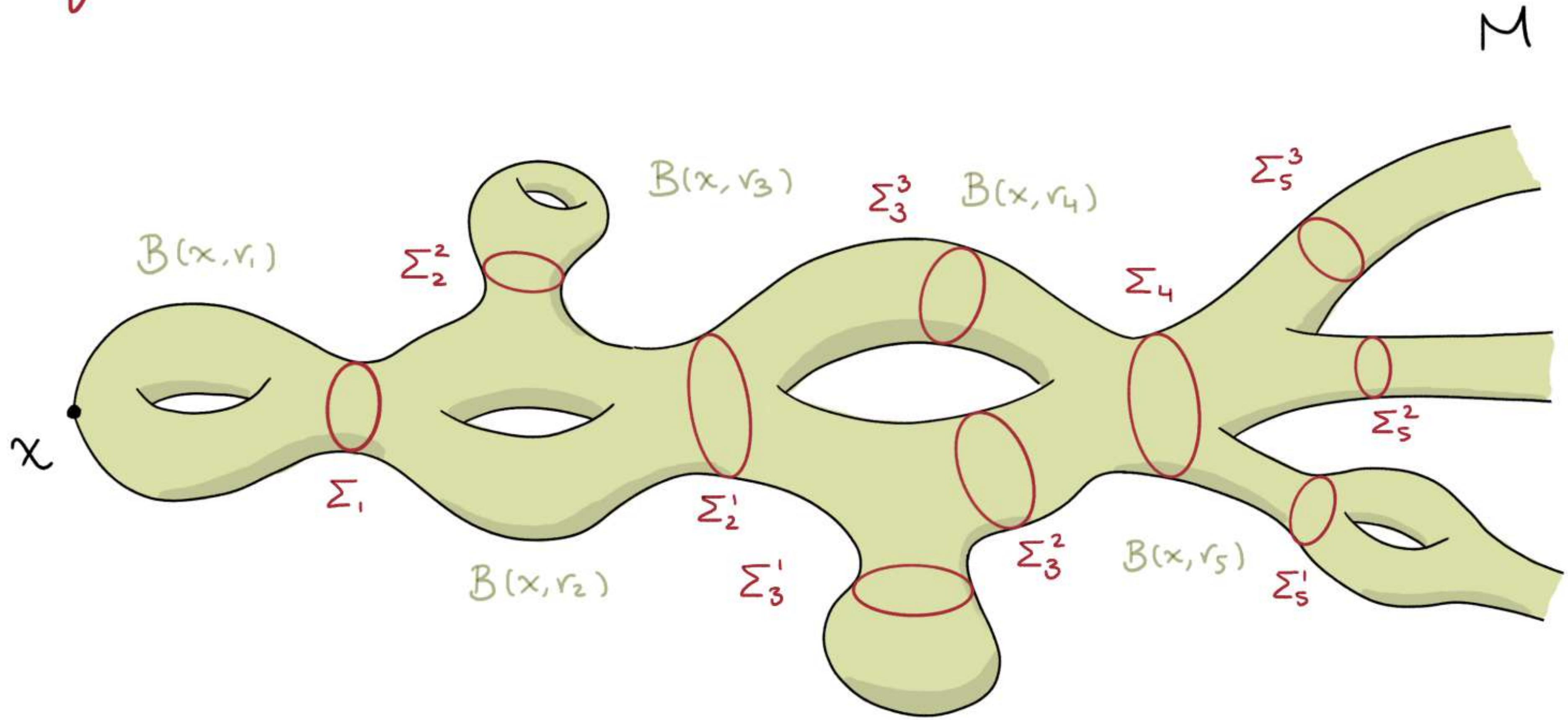
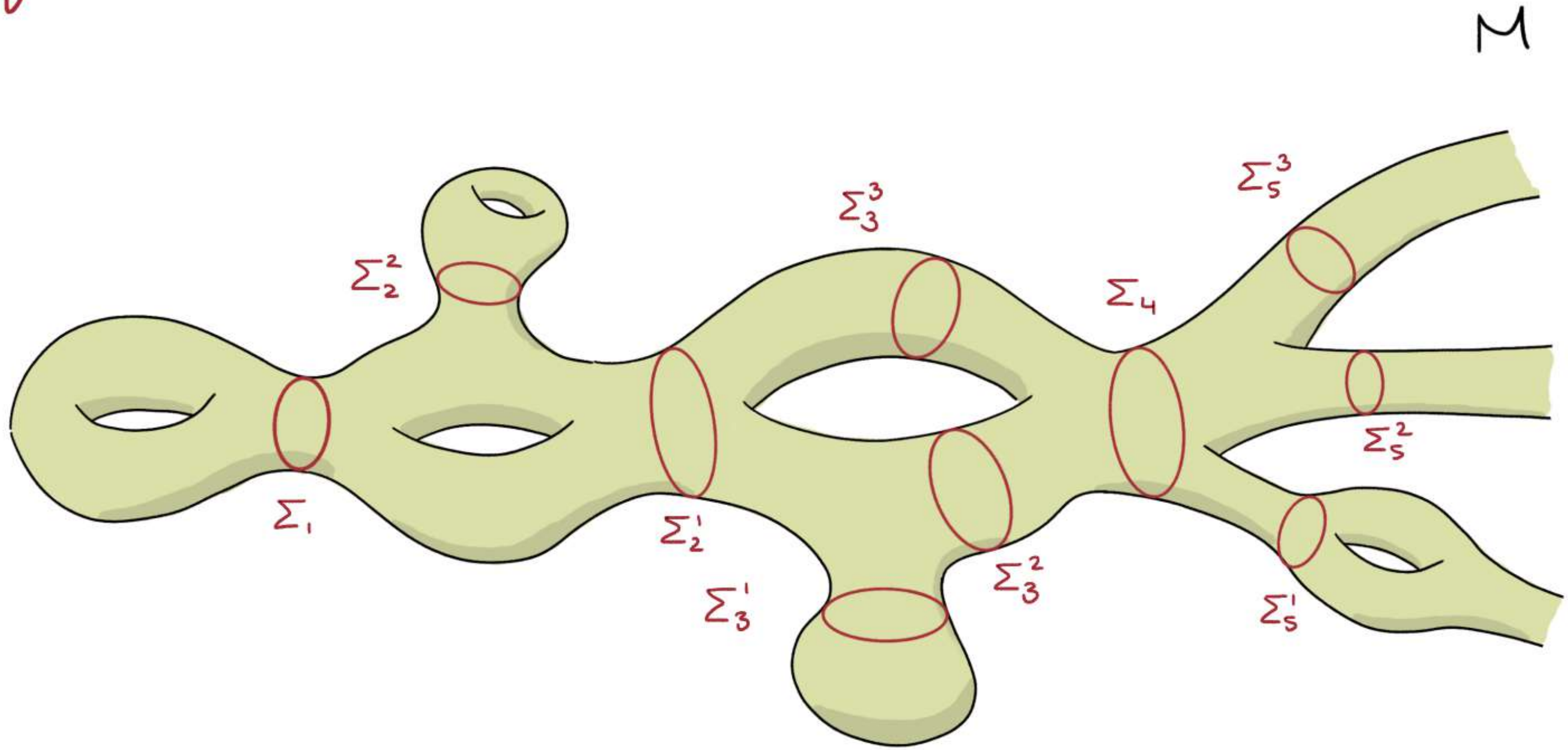
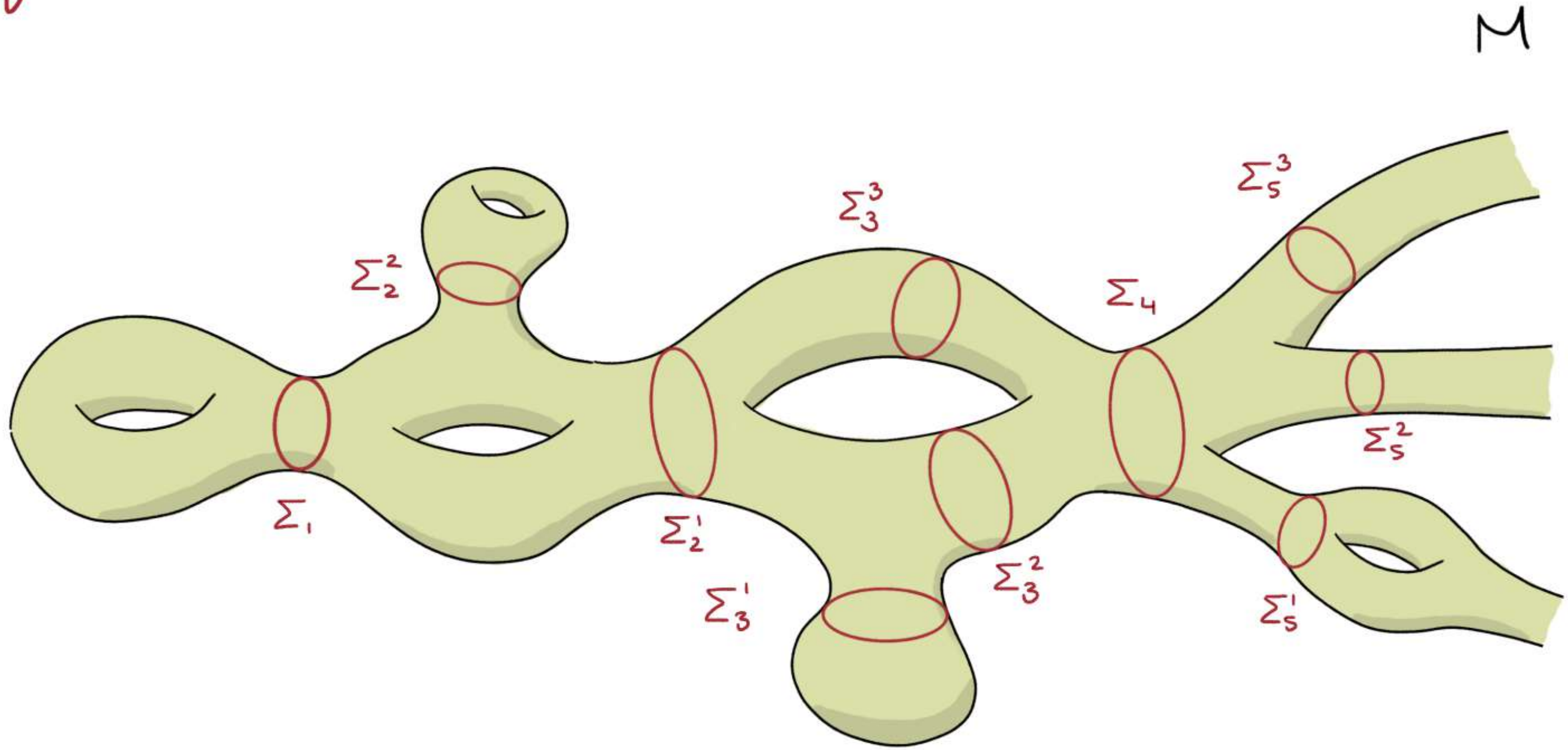


M
x
$B(x, r_1)$
$B(x, r_2)$
$B(x, r_3)$
$B(x, r_4)$
$B(x, r_5)$
Σ_1
Σ_2^1
Σ_2^2
Σ_3^1
Σ_3^2
Σ_3^3
Σ_4
Σ_5^1
Σ_5^2
Σ_5^3

Proof

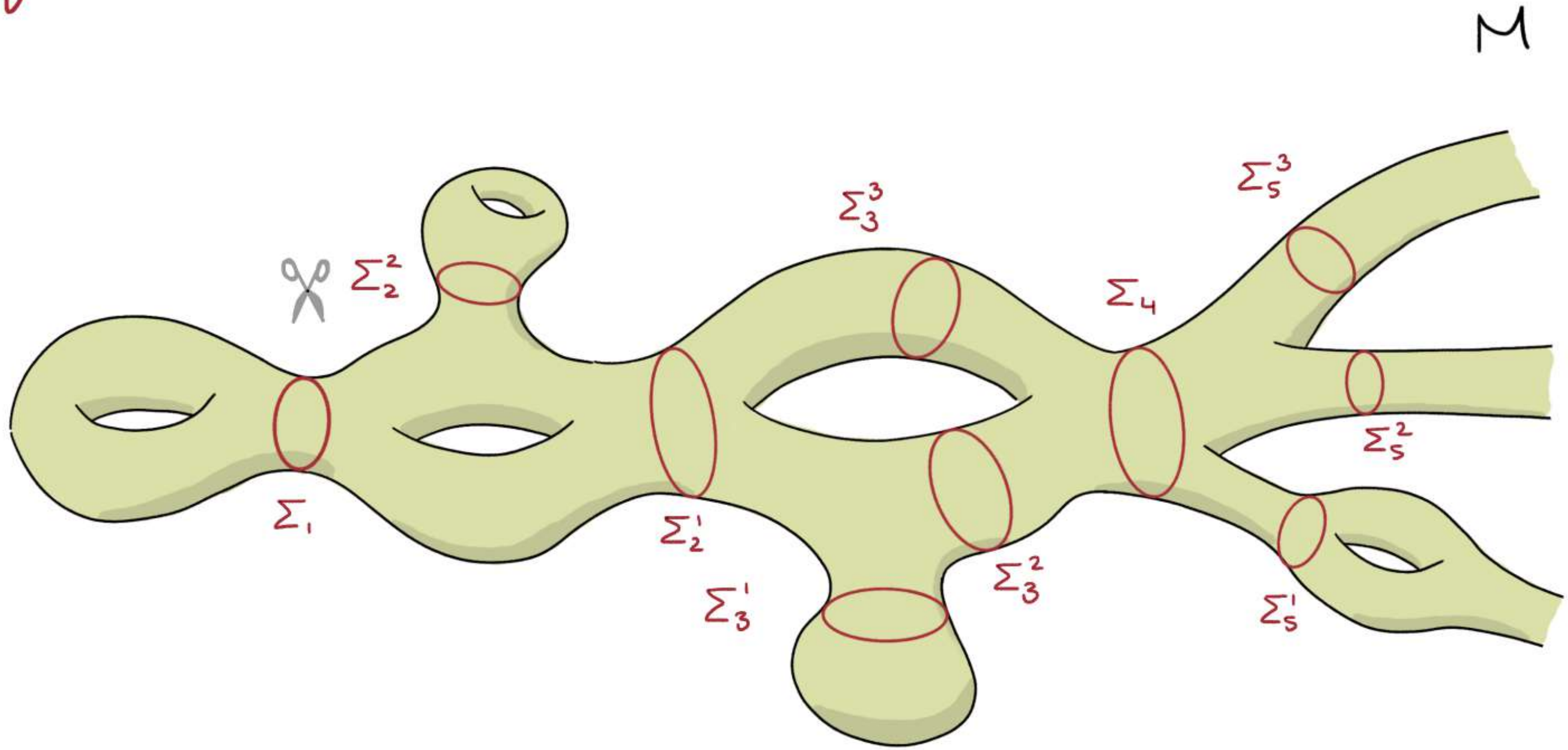

M
Σ_5^3
Σ_3^3
Σ_2^2
Σ_4
Σ_5^2
Σ_1
Σ_2^1
Σ_3^2
Σ_3^1
Σ_5^1

Proof



①. Suppose $\Sigma_i^j \simeq S^2$

Proof



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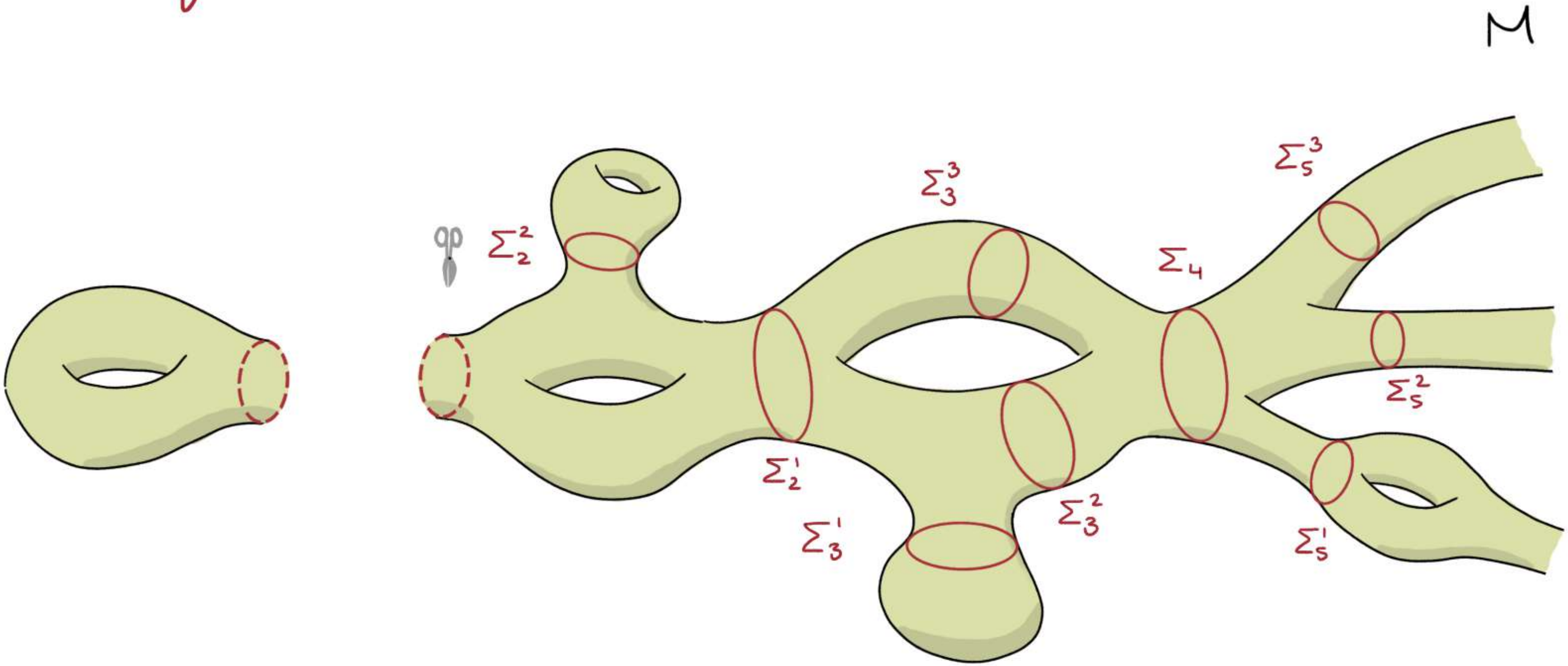

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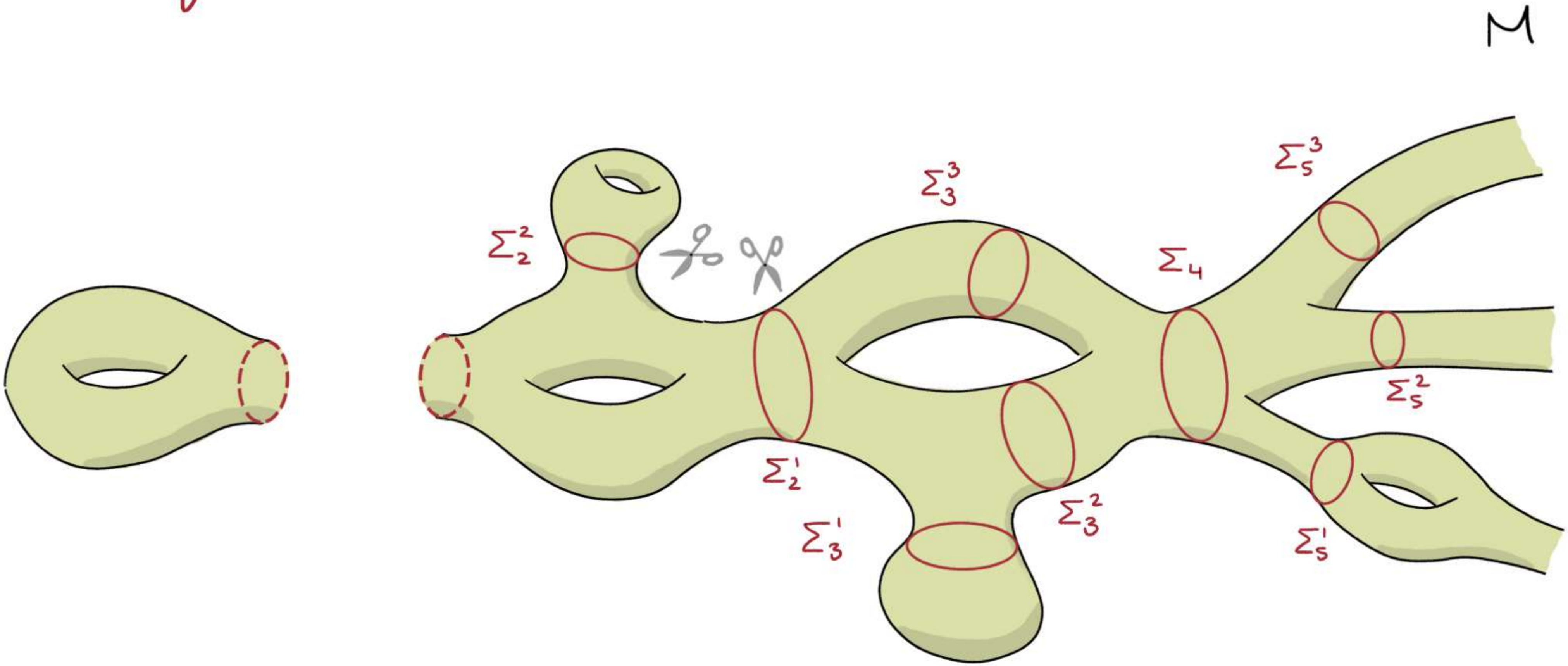

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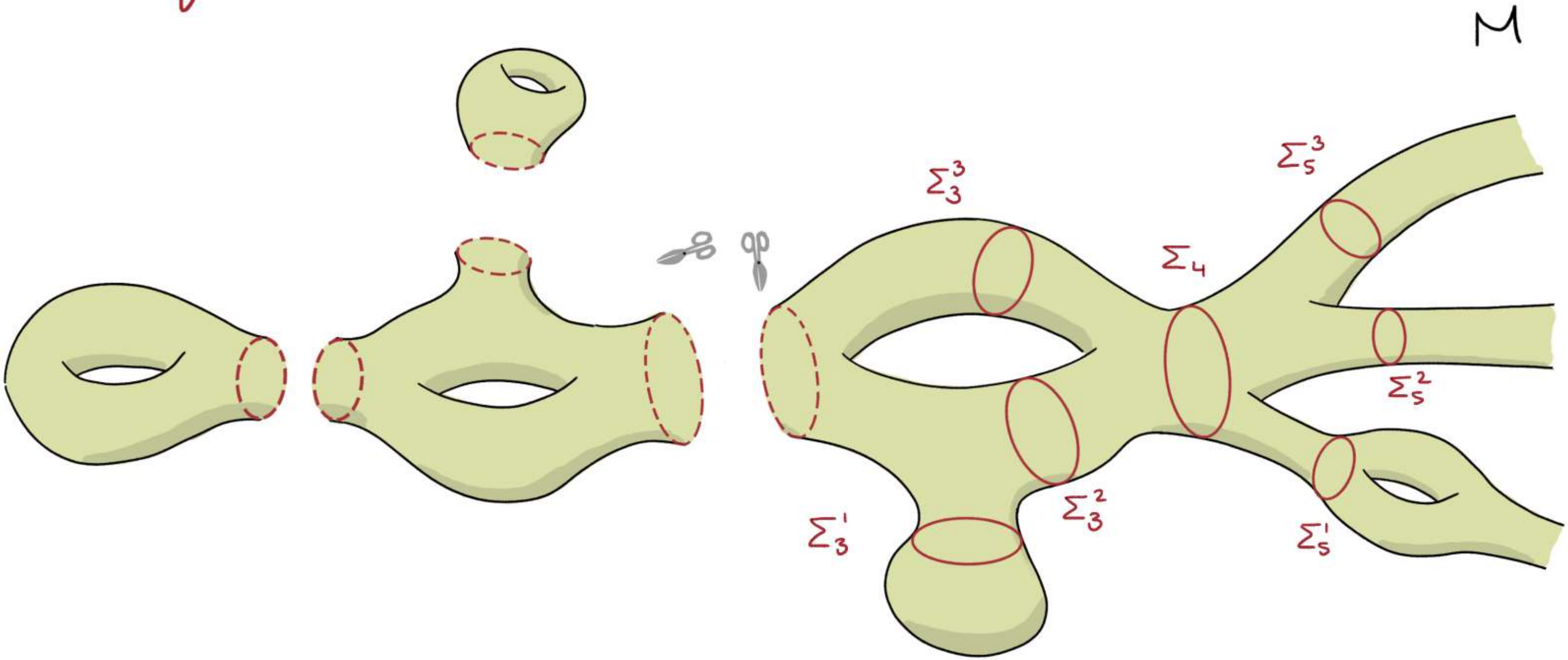

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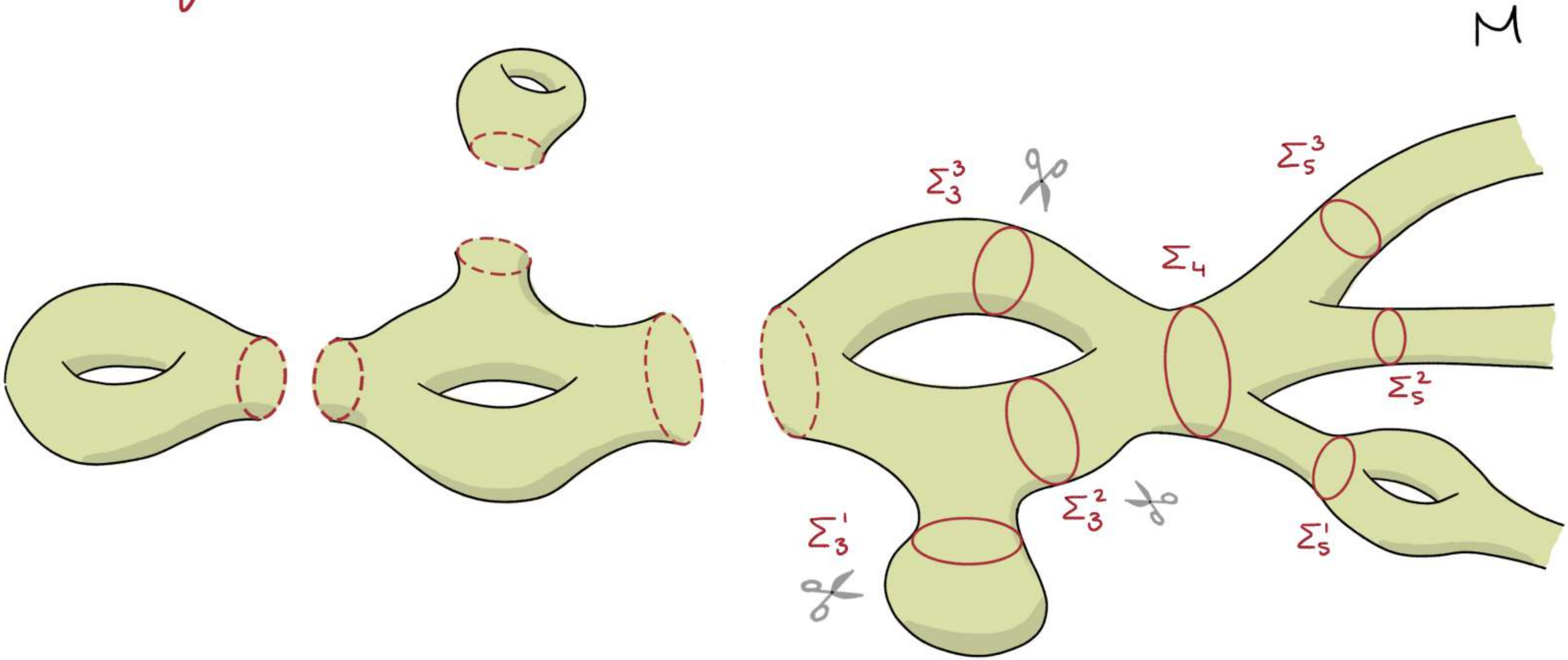

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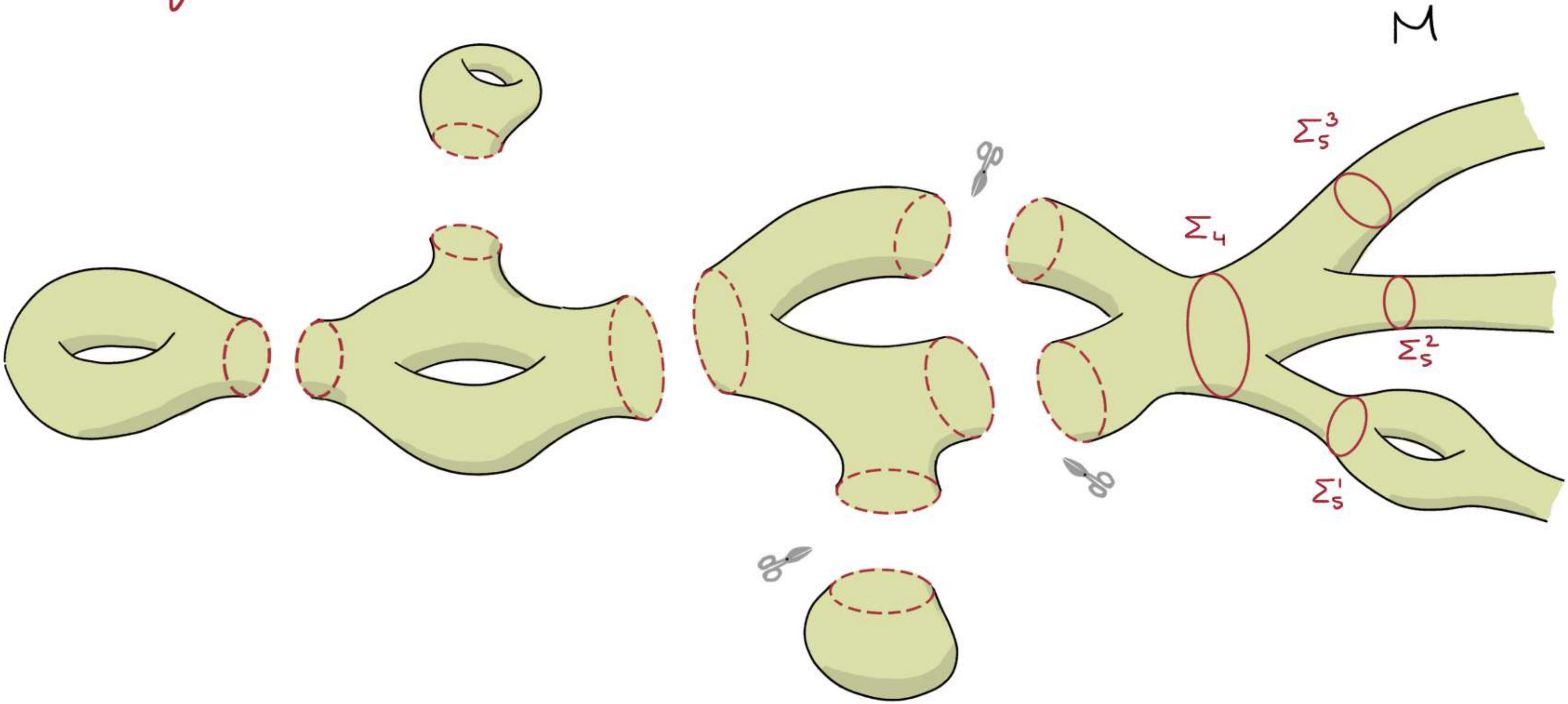

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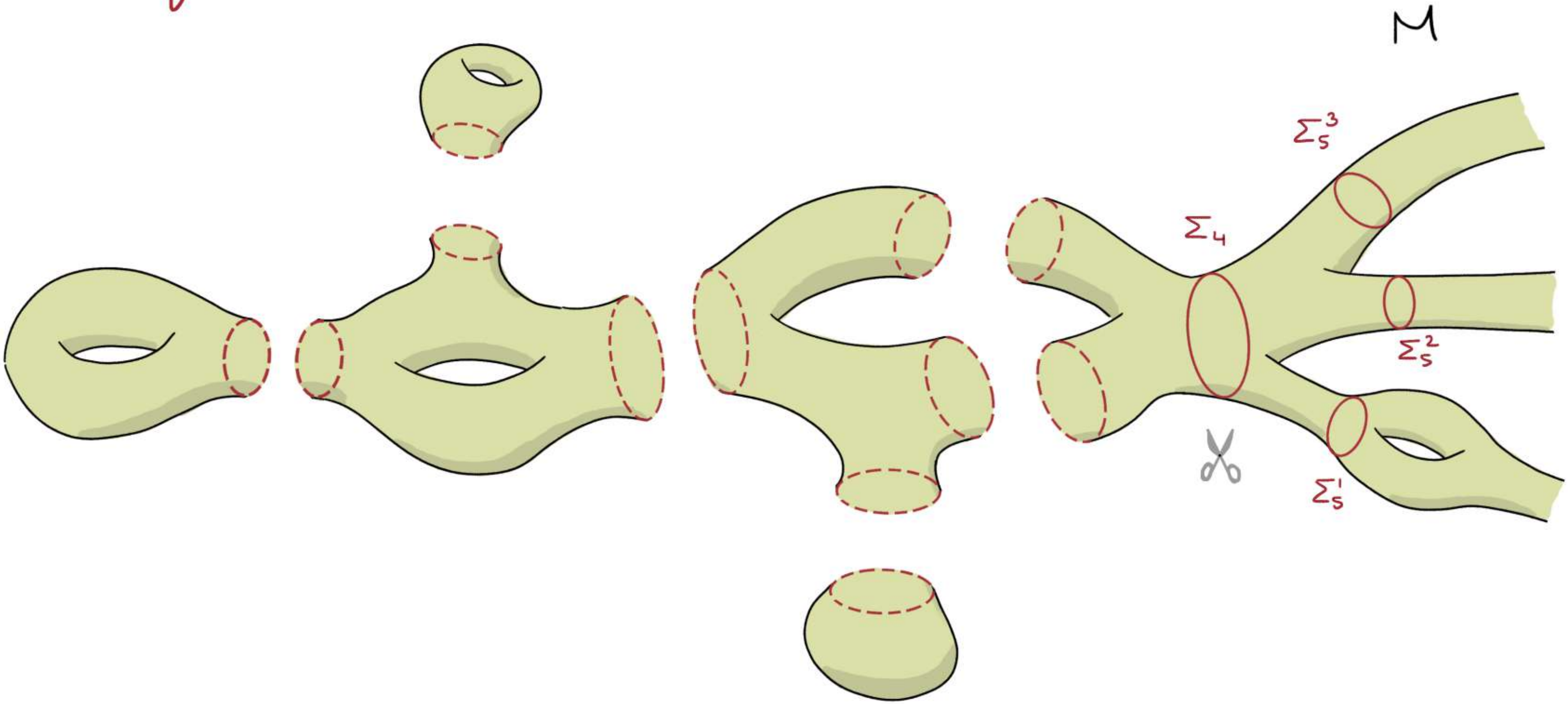

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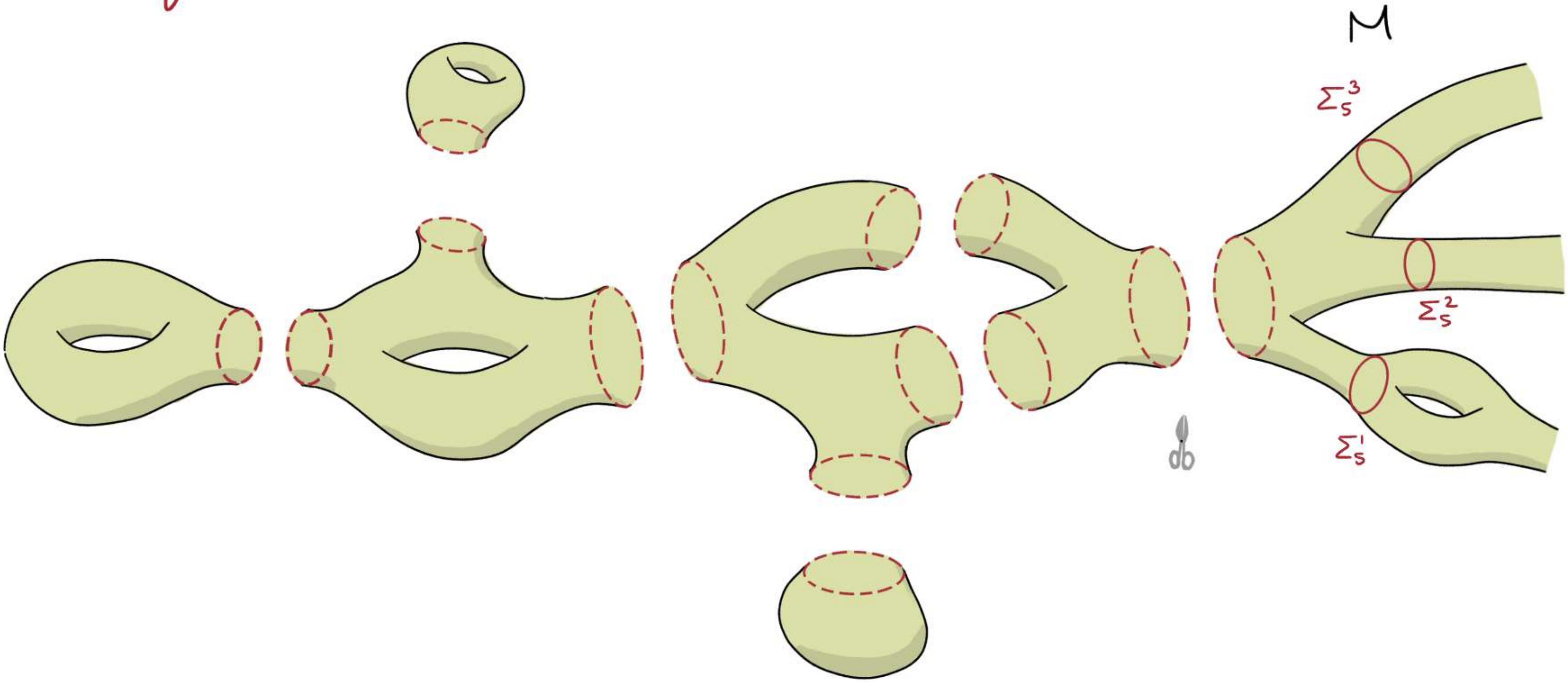

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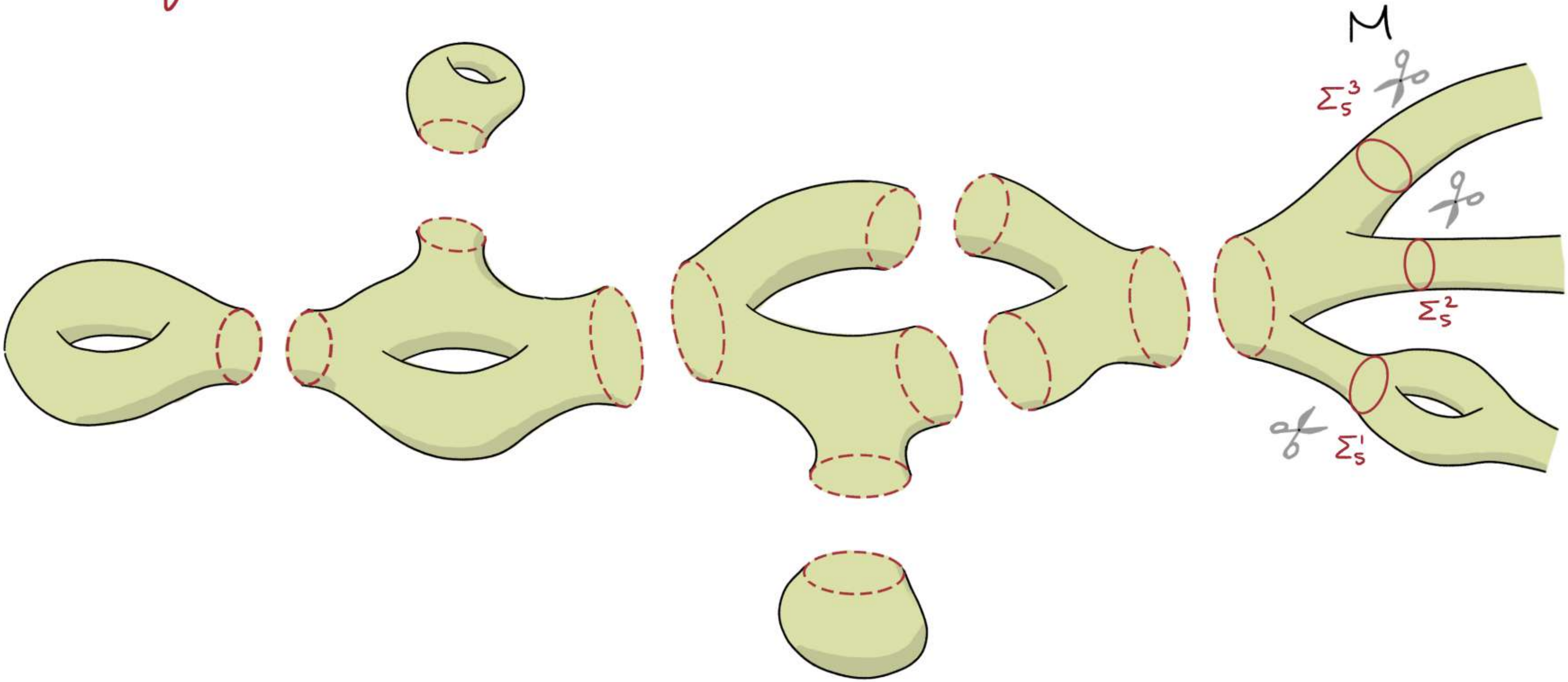

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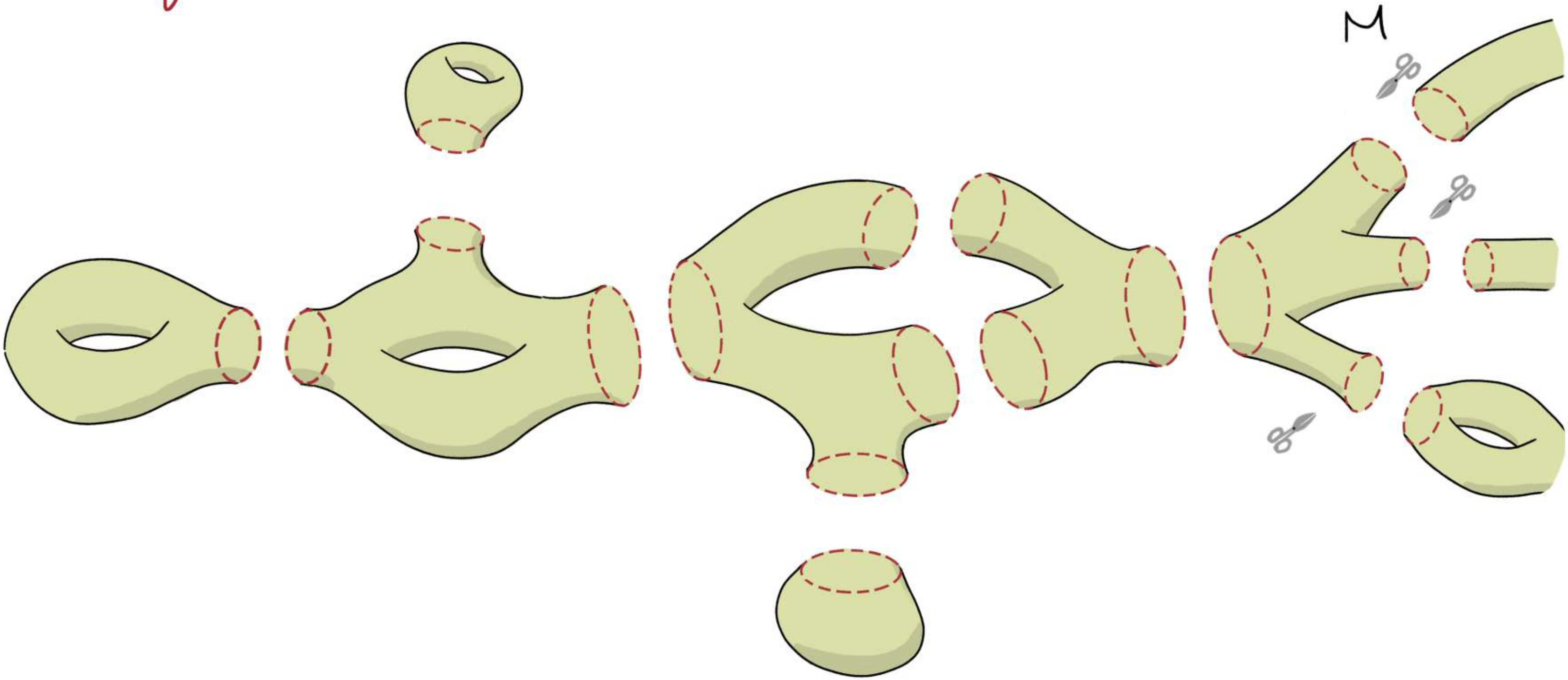

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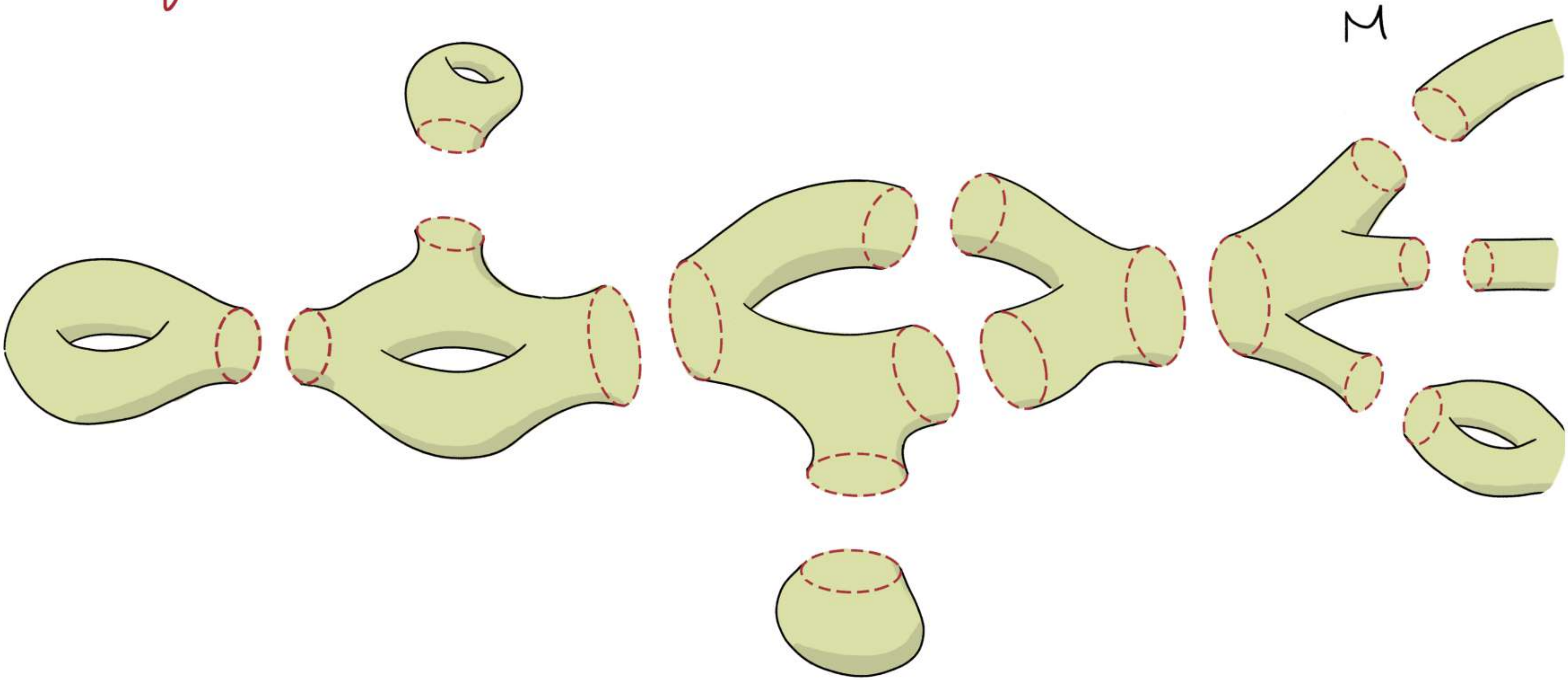

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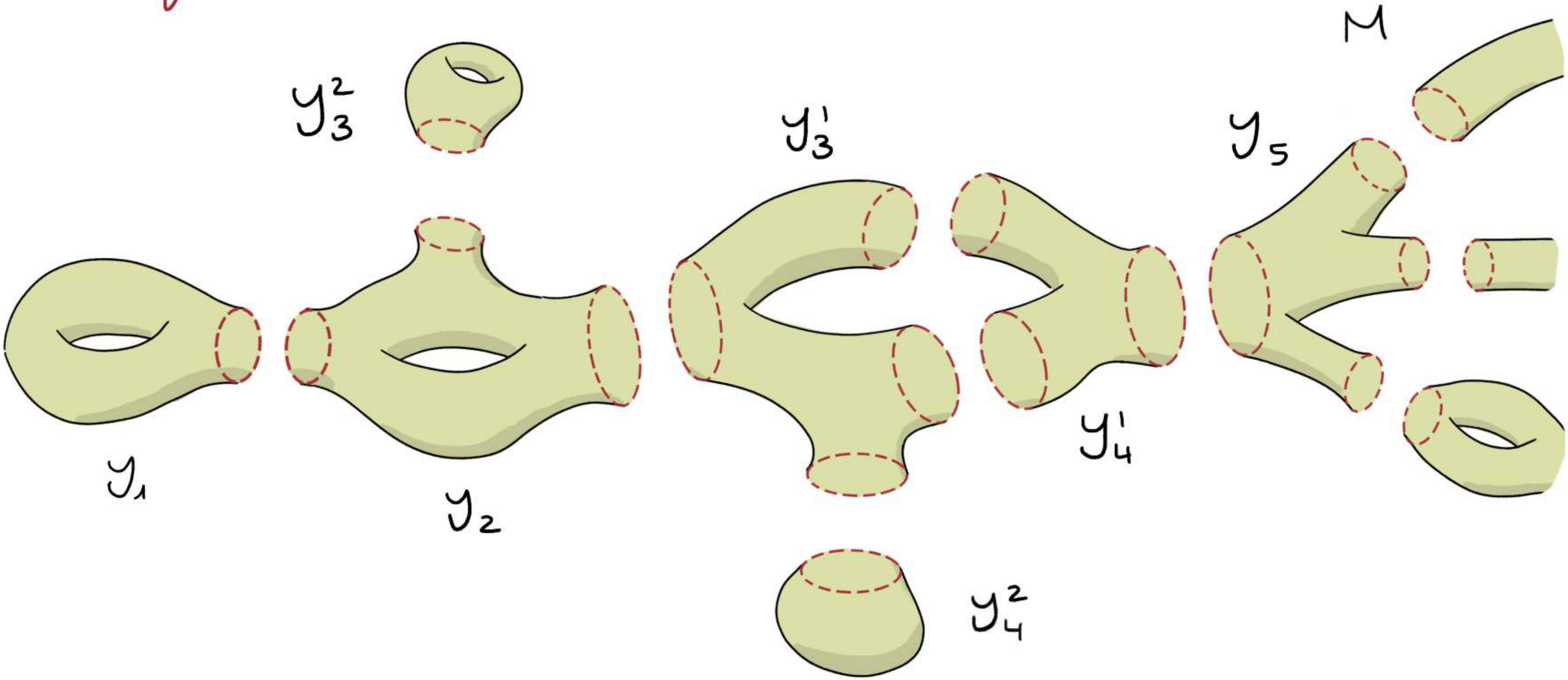

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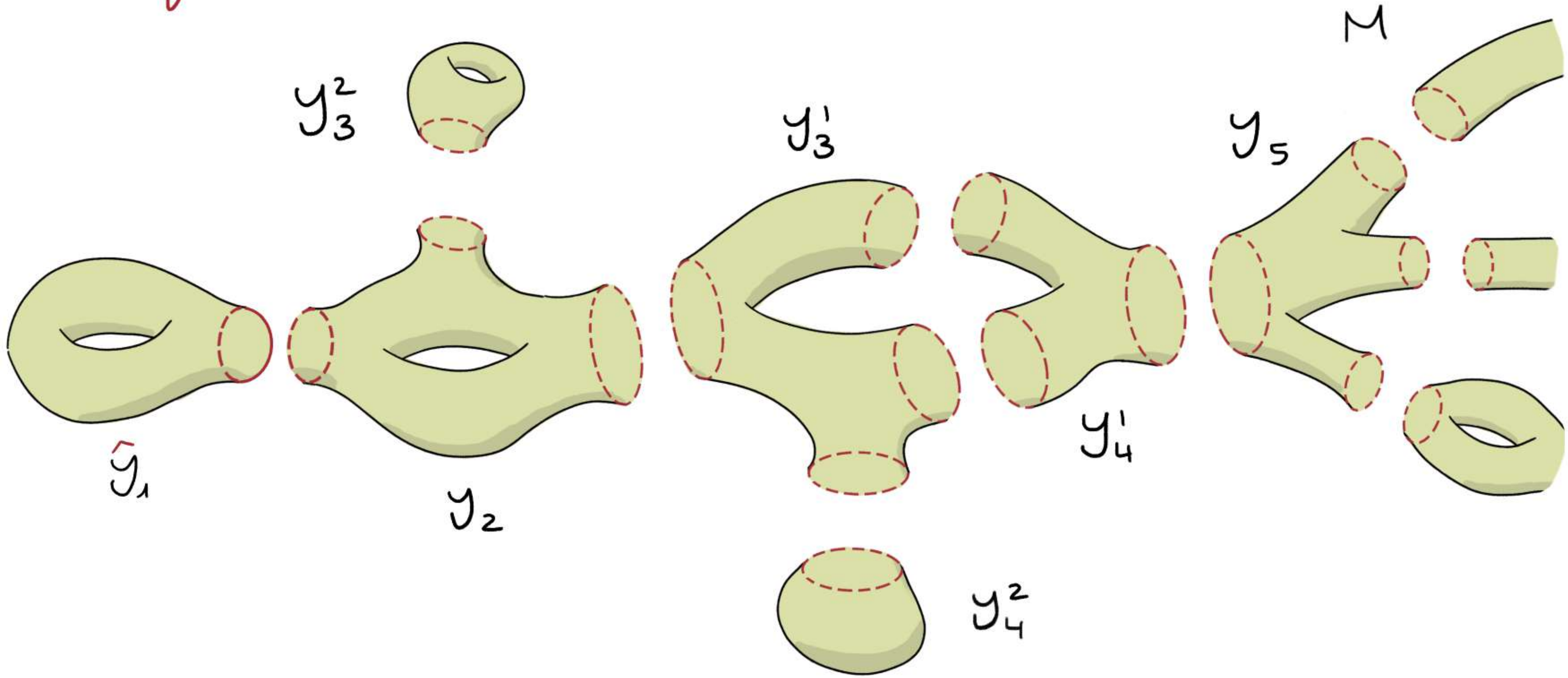

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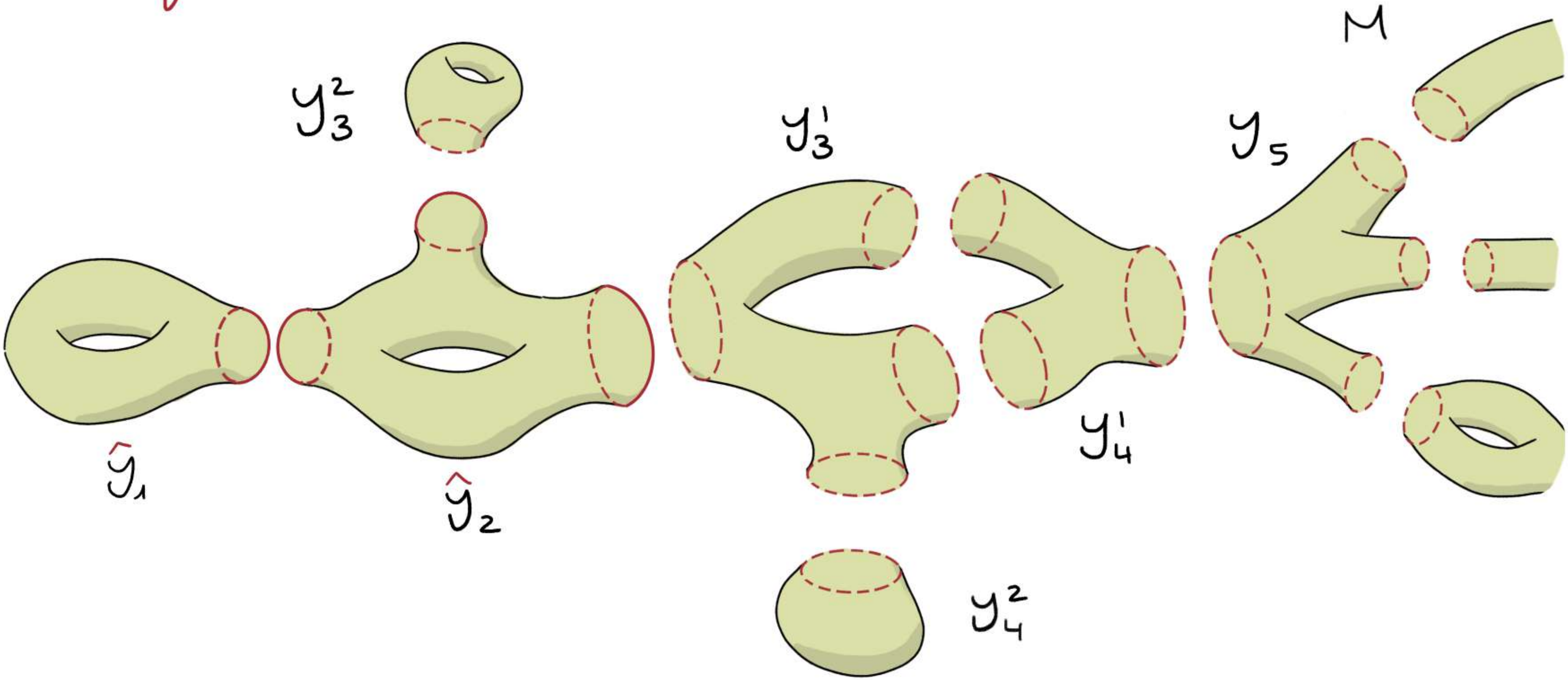

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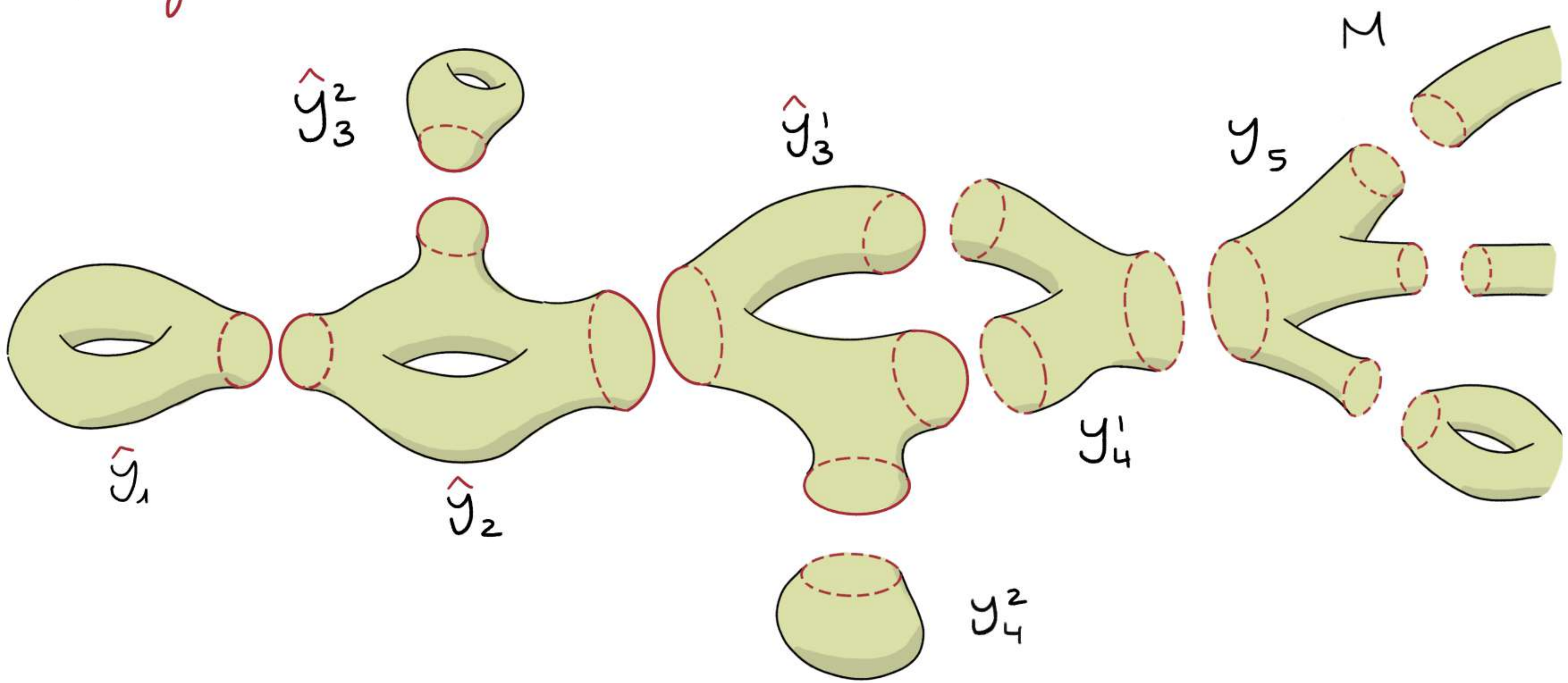

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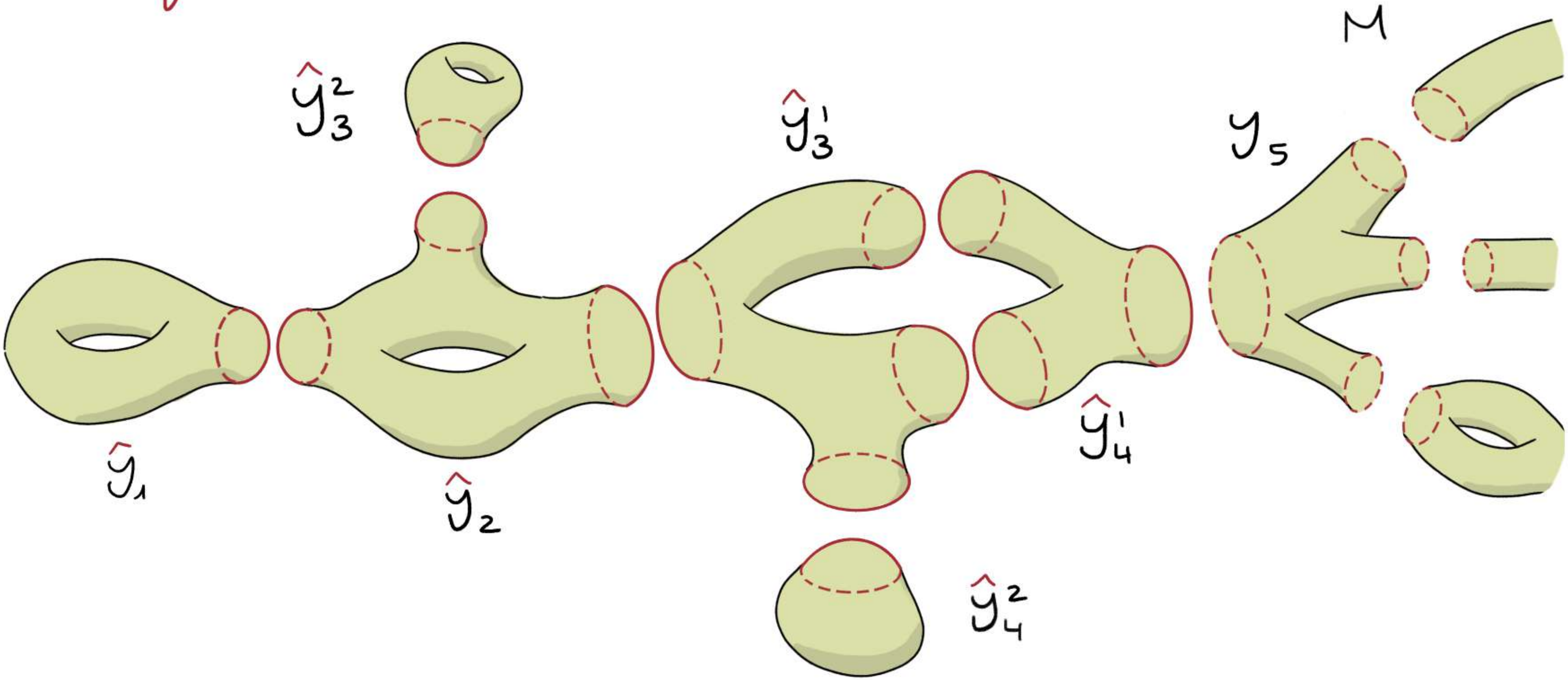
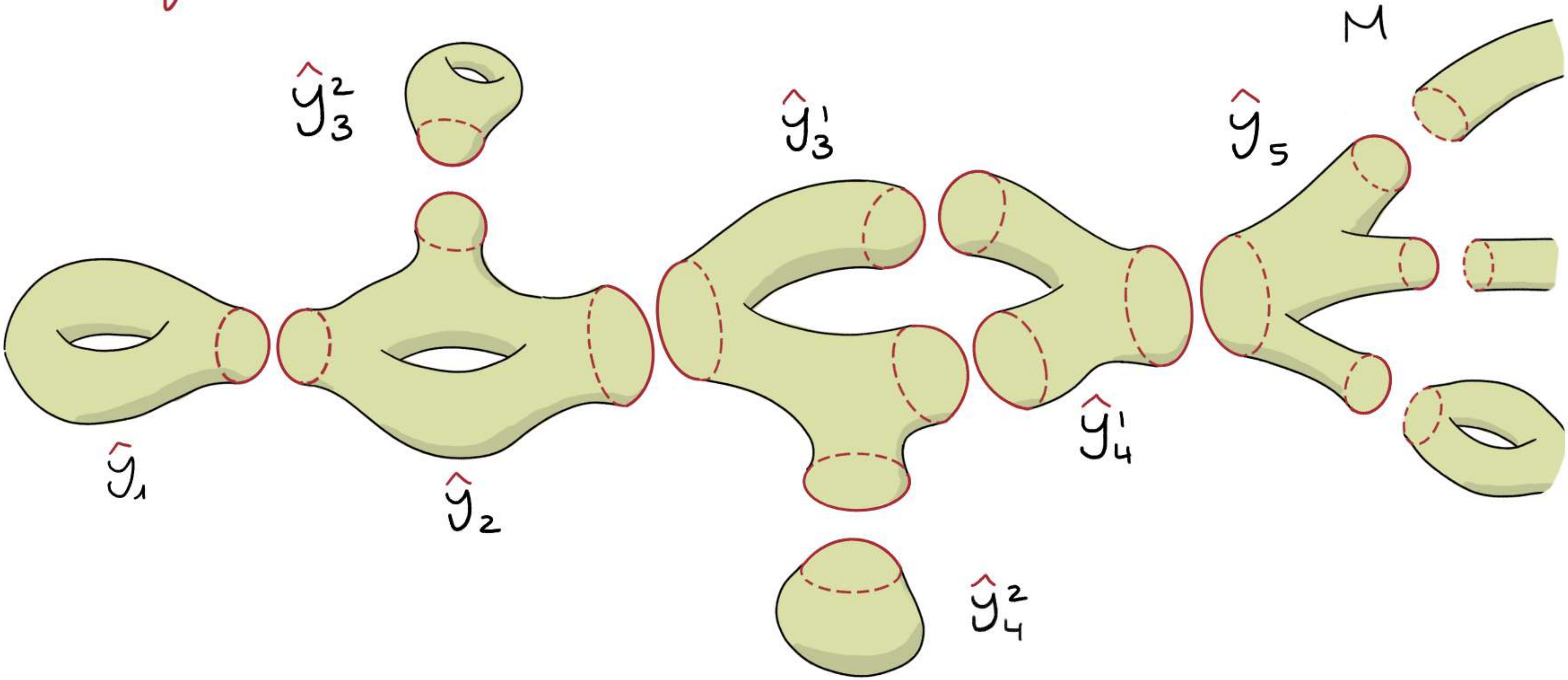


1. Suppose $\Sigma_i^+ \simeq S^2$

Proof



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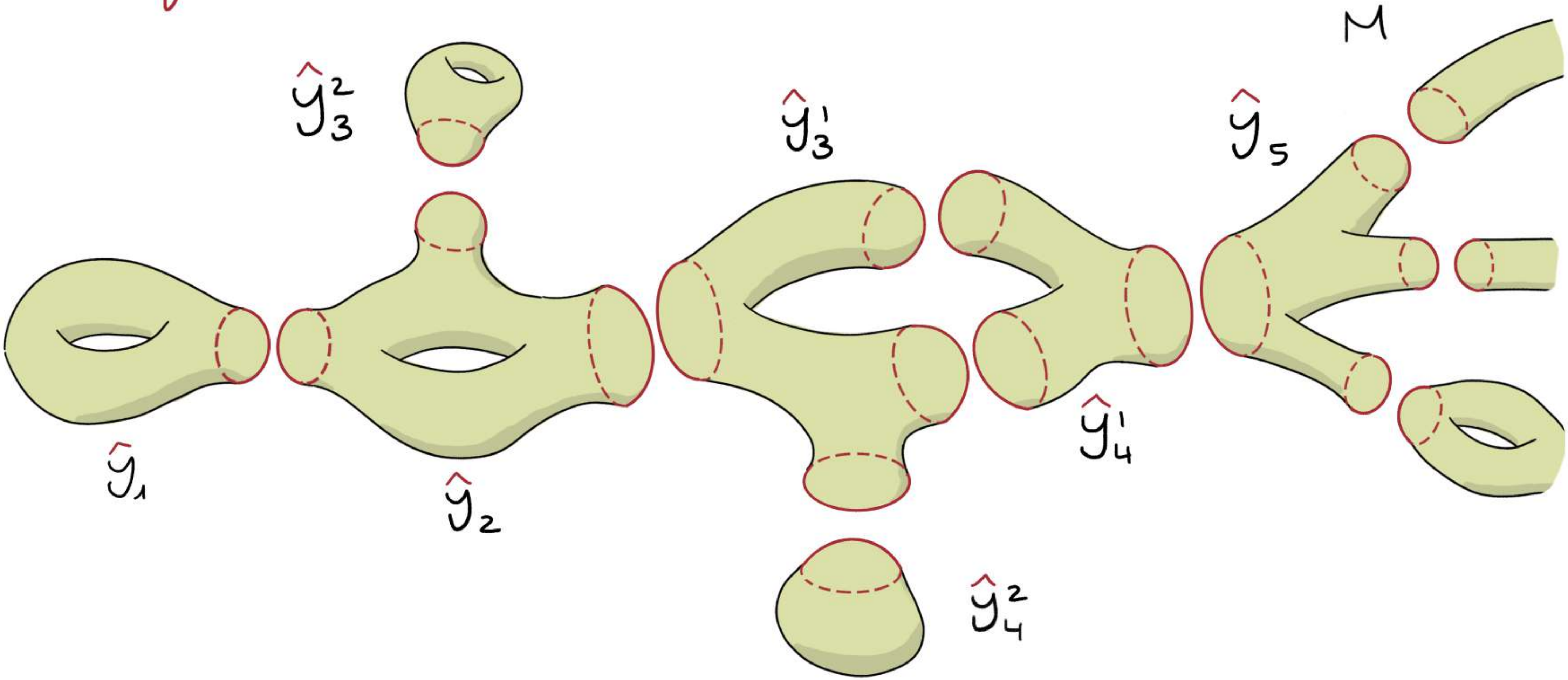

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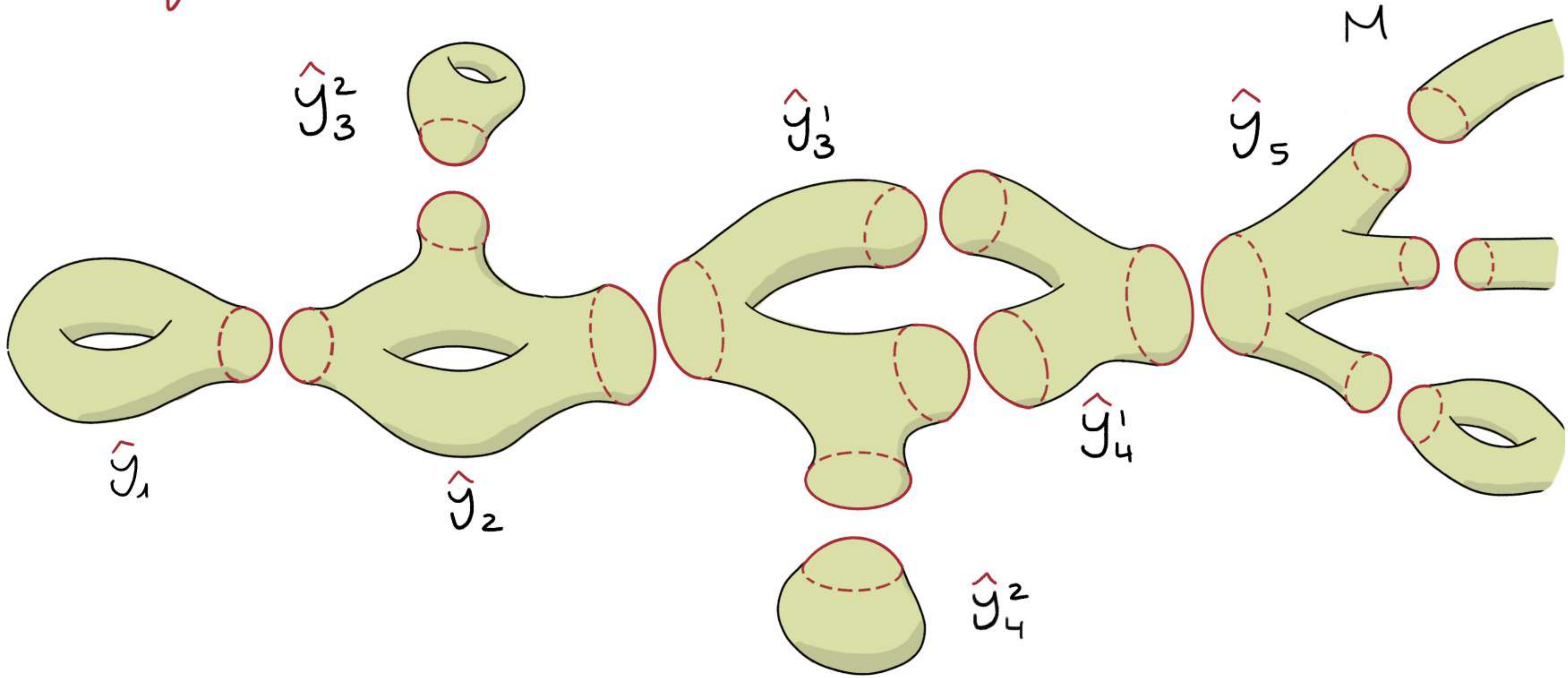
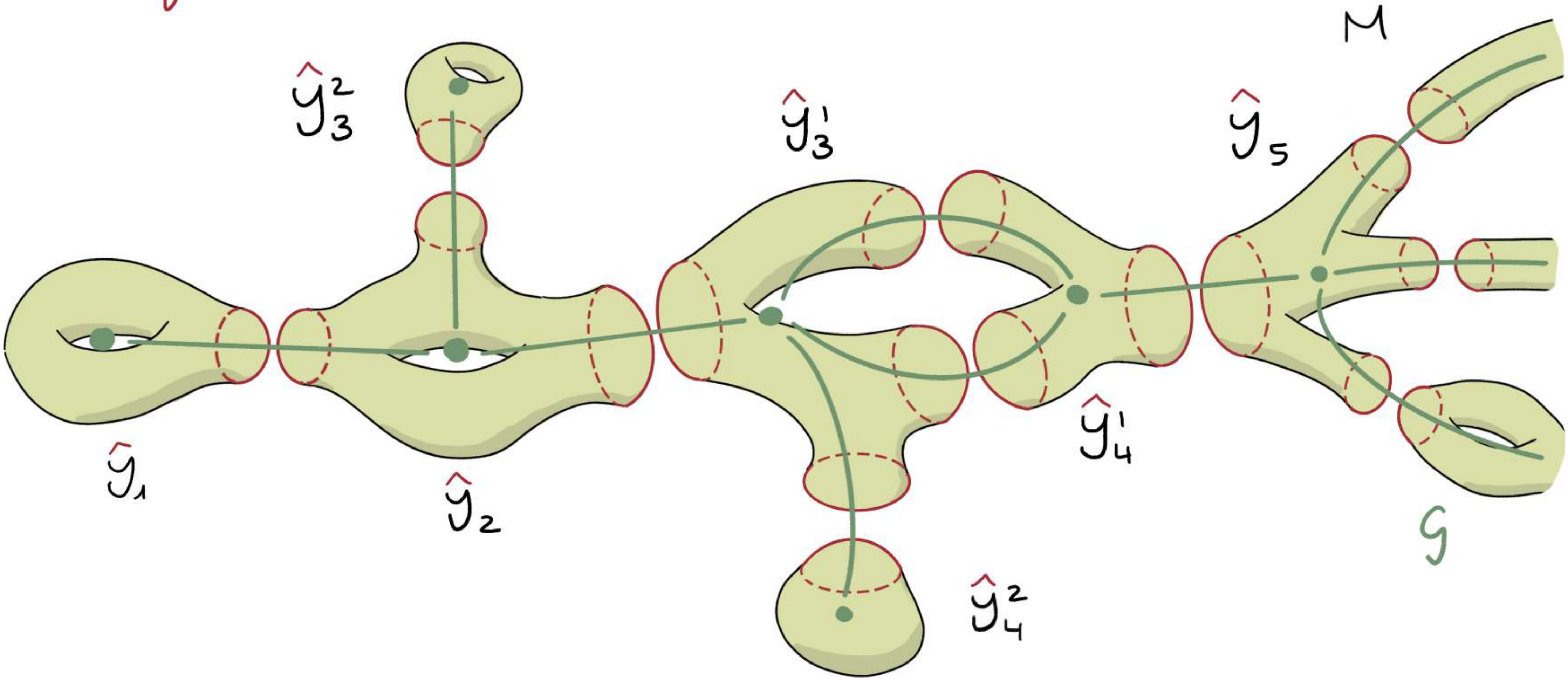


1. Suppose $\Sigma_i^t \simeq S^2 \implies M \simeq \hat{Y}_1 \# \hat{Y}_2 \# \hat{Y}_3^{i_3} \# \hat{Y}_4^{i_4} \# \hat{Y}_5^{i_5} \# \dots$

Proof



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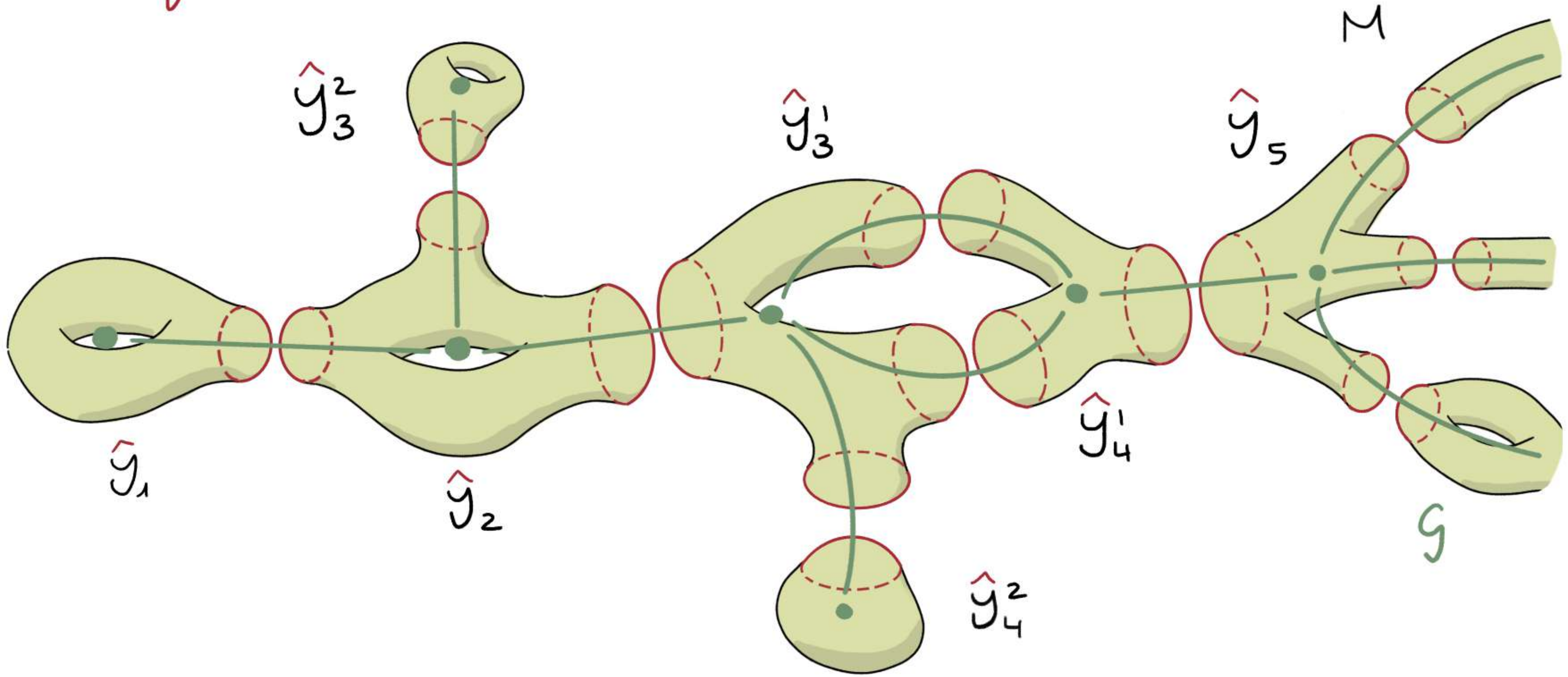
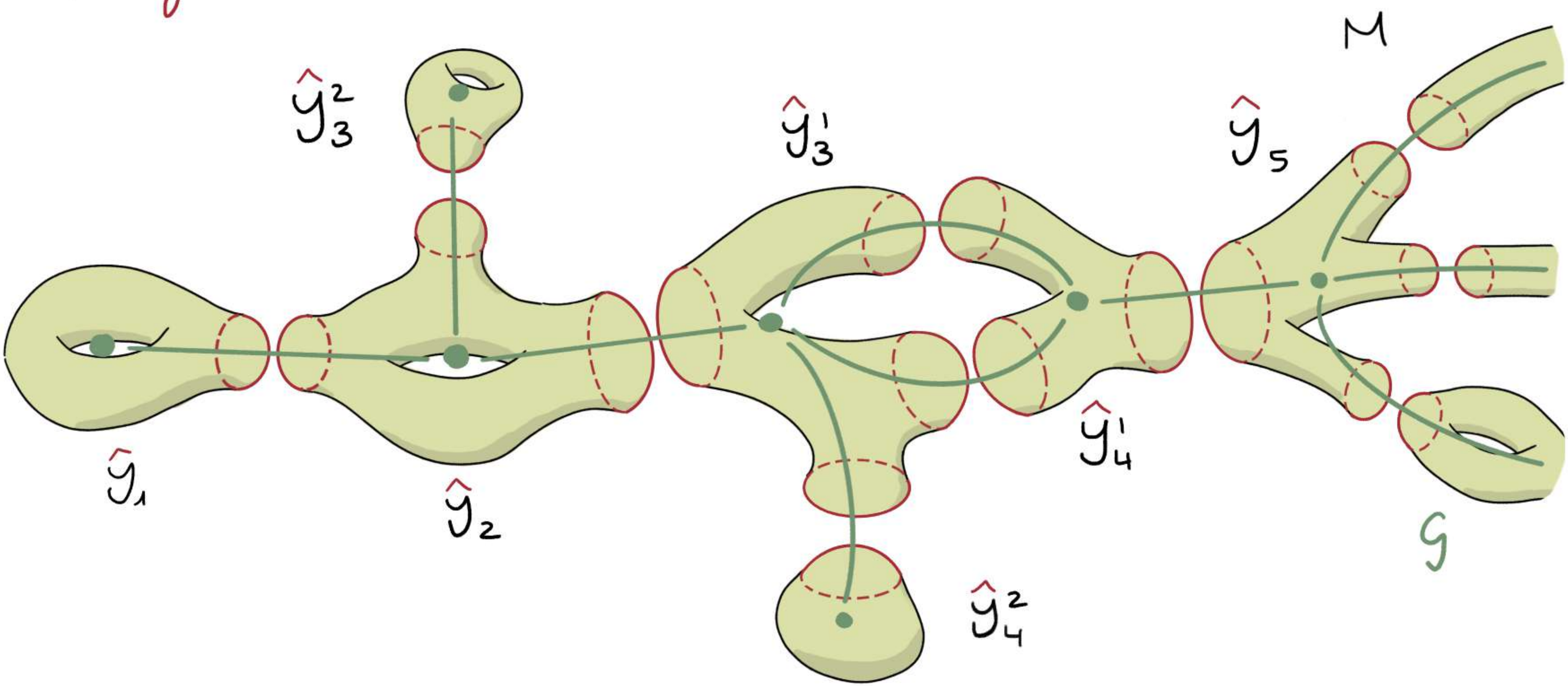


①. Suppose $\Sigma_i^j \simeq S^2 \implies M \simeq \hat{Y}_1 \# \hat{Y}_2 \# \hat{Y}_3^{i_3} \# \hat{Y}_4^{i_4} \# \hat{Y}_5^{i_5} \# \dots$ along $\mathcal{G}$

$\hat{Y}_i^j \simeq \underbrace{P_1 \# \dots \# P_{N_{ij}}}_{\text{prime}}$

Proof



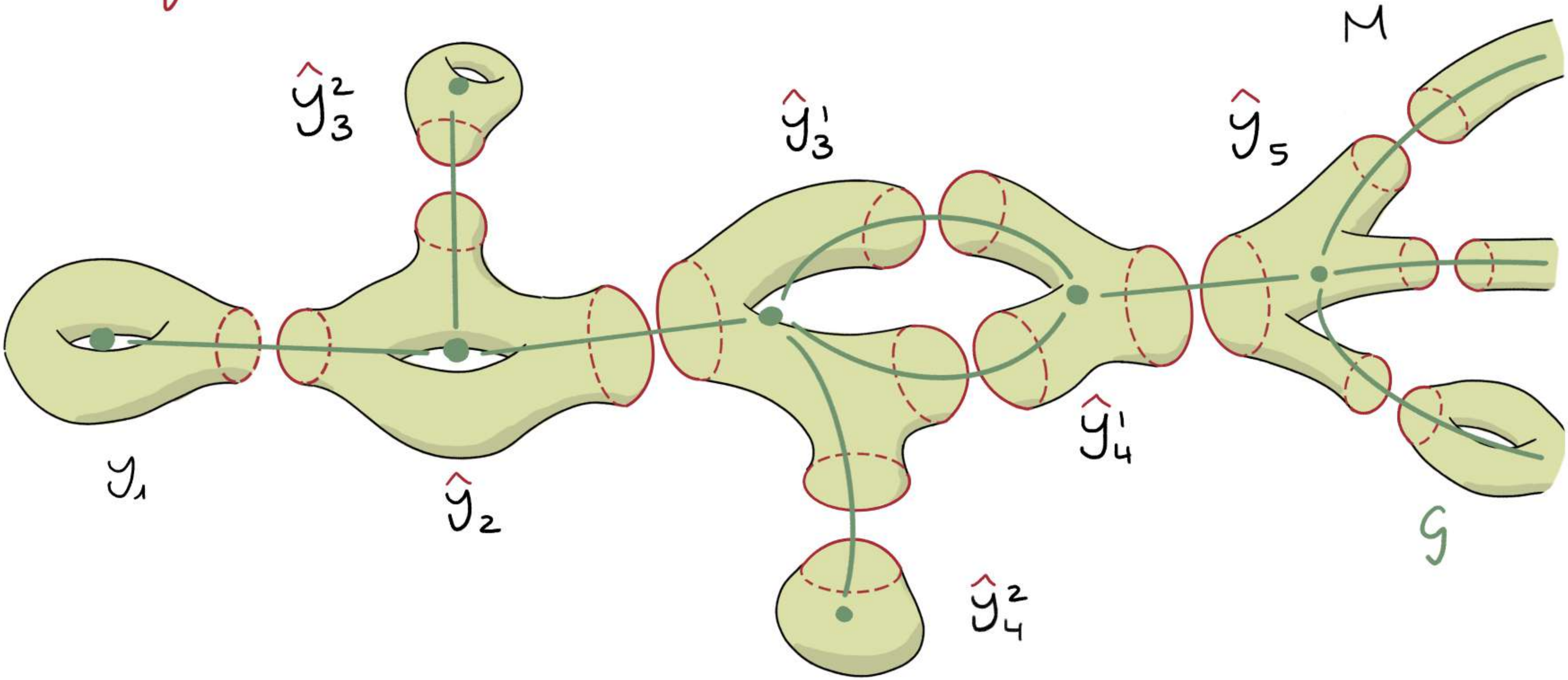
①. Suppose $\Sigma_i^j \simeq S^2$ $\Longrightarrow$ $M \simeq \hat{Y}_1 \# \hat{Y}_2 \# \hat{Y}_3^{i_3} \# \hat{Y}_4^{i_4} \# \hat{Y}_5^{i_5} \# \dots$ along $\mathcal{G}$

$\hat{Y}_i^j \simeq P_1 \# \dots \# P_{N_{ij}}$ ②. $\hat{Y}_i^j$ contains no aspherical terms

prime

$(P_\alpha \neq K(\pi, 1))$

Proof



$\text{fillrad}\,\tilde{M} < c'r \implies \begin{cases} ①.\ \text{Suppose } \Sigma_i^t \simeq S^2 \\ ②.\ \hat{y}_i^t \text{ contains no aspherical terms} \end{cases}$

Proof

- Thm [BGS'24]: M^n complete Riemannian manifold.

  $\operatorname{fillrad} \tilde{M} < c' r \quad (c' < \frac{1}{3}) \implies$ every $G \leq \pi_1 M$ fin. gen. has

  $$e(G) \neq 1$$

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Number of ends

Proof

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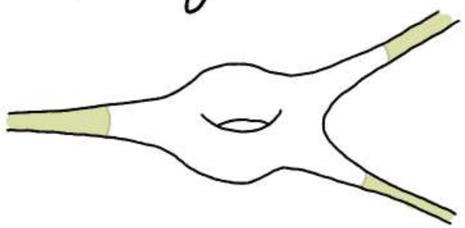
Number of ends of ...

... a topological space



$e = 3$

Proof

- Thm [BGS'24]: M^n complete Riemannian manifold.
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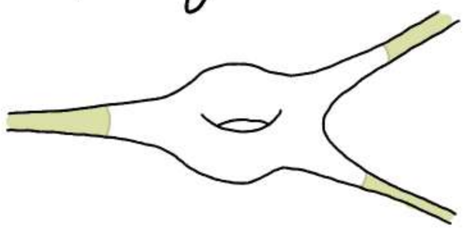

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... a (fin. gen.) group G

$$e(G) := e(\Gamma(G))$$

Proof

- Thm [BGS'24]: M^n complete Riemannian manifold. $\operatorname{fillrad}\tilde{M} < c'r$ $(c' < \frac{1}{3})$ $\Longrightarrow$ every $G \leq \pi_1 M$ fin. gen. has $e(G) \neq 1$

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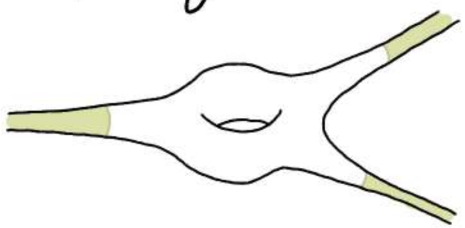

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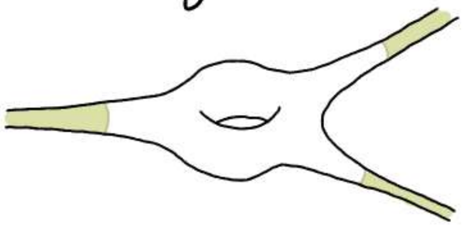

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Proof

- Thm [BGS'24]: M^n complete Riemannian manifold.
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Number of ends of ...

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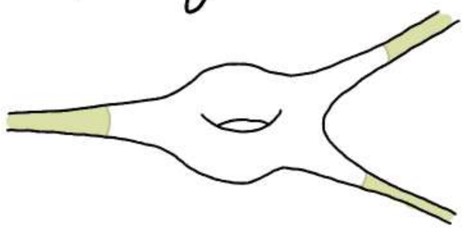

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if M^n closed: $e(\pi_1 M) = e(\widetilde{M})$

Proof

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↳ M^n closed, $\operatorname{fillrad} \tilde{M} \leq \rho_0$ [Ramachandran - Wolfson'09]

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↳ Proof:

If $G \leq \pi_1 M$ fin. gen. has $e(G) = 1$,

$\forall R > 0$, we can construct $\eta_R \subset \tilde{M}$

$$\frac{1}{3} R \leq \operatorname{fillrad}(\eta_R)$$

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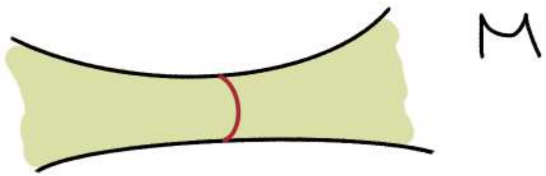

$[\gamma] \in G$
(order $= \infty$)

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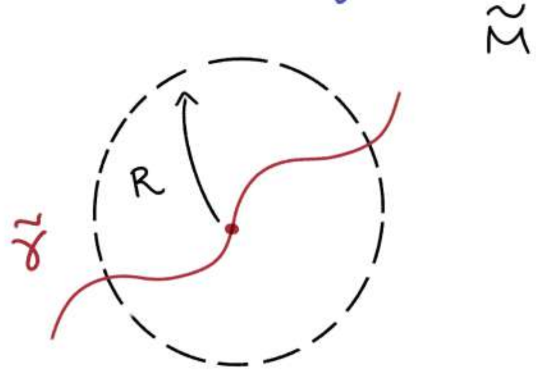

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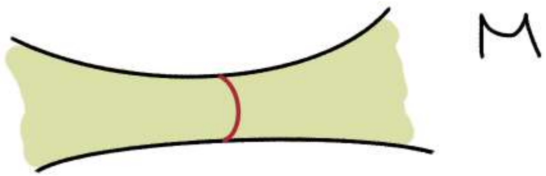

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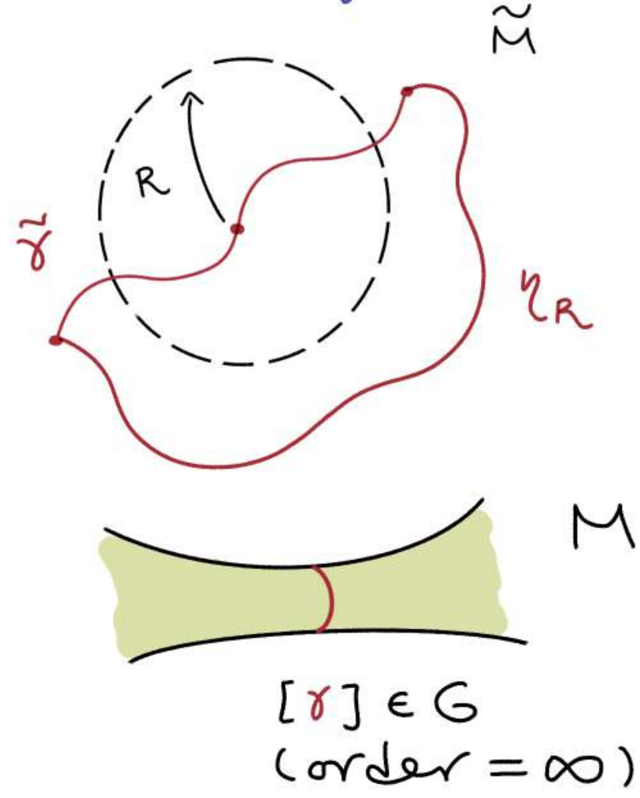

Proof

M^n complete Riemannian manifold

$\operatorname{fillrad} \tilde{M} < c'r$ $\Longrightarrow$ $G \leq \pi_1 M$ fin. gen.

$(c' < \frac{1}{3})$ $\qquad$ $e(G) \neq 1$

Proof

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$$e(G) \neq 1$$

$$\Downarrow$$

$\pi_1 M$ virtually free

contains a free subgroup of finite index

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$\swarrow$ $(n \geq 2)$

$M \neq N \# K$

any n-manifold

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$(e(\pi_1 K) = 1)$

Proof

M^n complete Riemannian manifold

$$\operatorname{fillrad} \tilde{M} < c' r \quad (c' < \tfrac{1}{3}) \implies G \leq \pi_1 M \text{ fin. gen. } \quad e(G) \neq 1$$

$\Downarrow$

$(n \geq 2)$

$M \neq N \# K$ — N: any n-manifold; K: closed aspherical

$\pi_1 M$ virtually free — contains a free subgroup of finite index

$(e(\pi_1 K) = 1)$

- Conjecture: M^n closed aspherical is not PSC

Proof

M^n complete Riemannian manifold

$\operatorname{fillrad} \tilde{M} < c'r \Longrightarrow G \leq \pi_1 M$ fin. gen.

$(c' < \frac{1}{3})$ $\qquad e(G) \neq 1$

$\Downarrow$

contains a free subgroup of finite index

$(n \geq 2)$

$M \neq N \# K$ $\qquad \pi_1 M$ virtually free

any n-manifold; K closed aspherical $(e(\pi_1 K) = 1)$

dim 3 [Gromov-Lawson'83], dim 4, 5 [Chodosh-Li'24]

- Conjecture: M^n closed aspherical is not PSC

Proof

M^n complete Riemannian manifold

$$\begin{array}{ccc} \operatorname{fillrad} \tilde{M} < c' r & \Longrightarrow & G \leq \pi_1 M \text{ fin. gen.} \\ (c' < \tfrac{1}{3}) & & e(G) \neq 1 \\ & \swarrow & \Downarrow \\ \overset{(n \geq 2)}{M \neq N \# K} & & \pi_1 M \text{ virtually free} \end{array}$$

contains a free subgroup of finite index

N: any n-manifold

K: closed aspherical $(e(\pi_1 K) = 1)$

dim 3 [Gromov-Lawson'83], dim 4, 5 [Chodosh-Li'24]

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  M^n closed $\mathbb{Q}$-essential is not PSC

Proof

M^n complete Riemannian manifold

$\operatorname{fillrad} \tilde{M} < c'r$ $(c' < \frac{1}{3})$ $\Longrightarrow$ $G \leq \pi_1 M$ fin. gen. $e(G) \neq 1$

contains a free subgroup of finite index

$\Downarrow$

$(n \geq 2)$

$M \neq N \# K$ $\quad$ $\pi_1 M$ virtually free

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M^n closed $\mathbb{Q}$-essential is not PSC

$f : M \to K(\pi_1 M, 1)$ satisfies $f_*[M] \neq 0 \in H_n(K; \mathbb{Q})$

Proof

M^n complete Riemannian manifold

$$\operatorname{fillrad} \tilde{M} < c' r \quad \Longrightarrow \quad G \leq \pi_1 M \text{ fin. gen.}$$
$(c' < \frac{1}{3})$

$$e(G) \neq 1$$

contains a free subgroup of finite index

$(n \geq 2)$

$$M \neq N \# K \qquad\qquad \pi_1 M \text{ virtually free}$$

any n-manifold

closed aspherical $(e(\pi_1 K) = 1)$

dim 3 [Gromov-Lawson'83], dim 4,5 [Chodosh-Li'24]

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M^n complete Riemannian manifold

$\operatorname{fillrad} \tilde{M} < c'r$ $\Longrightarrow$ $G \leq \pi_1 M$ fin. gen.
$(c' < \frac{1}{3})$

$e(G) \neq 1$

$\Downarrow$ contains a free subgroup of finite index

$\pi_1 M$ virtually free

$(n \geq 2)$

$M \neq N \# K$

any n-manifold $\lrcorner$ $\llcorner$ closed aspherical $(e(\pi_1 K) = 1)$

dim 3 [Gromov–Lawson'83], dim 4,5 [Chodosh–Li'24]

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M^n closed $\mathbb{Q}$-essential is not PSC

$\llcorner$ $f: M \to K(\pi_1 M, 1)$ satisfies

$f_*[M] \neq 0 \in H_n(K; \mathbb{Q})$

- Cor [BGS'24]: M^n closed.

$\operatorname{fillrad} \tilde{M} \leq \rho_0 \Longrightarrow M$ not $\mathbb{Q}$-essential

Proof

$$\mathrm{fillrad}\,\tilde{M} < c'r \quad (c' < \tfrac{1}{3}) \overset{?}{\Longrightarrow} \begin{cases} ① \ \Sigma_i^+ \simeq S^2 \\ ② \ \hat{Y}_i^+ \text{ contains no aspherical terms} \end{cases}$$

Proof

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$$\Downarrow$$

every $G \leq \pi_1 M$ fin. gen. has $e(G) \neq 1$

Proof

$\mathrm{fillrad}\,\tilde{M} < c'r \overset{?}{\Longrightarrow} \begin{cases} ①.\ \Sigma_i^+ \simeq S^2 \\ ②.\ \hat{y}_i^+ \text{ contains no aspherical terms} \end{cases}$

$(c' < \frac{1}{3})$

$\Downarrow$

every $G \leq \pi_1 M$ fin. gen. has $e(G) \neq 1$

②.

Proof

$$\operatorname{fillrad} \tilde{M} < c'r \quad (c' < \tfrac{1}{3}) \overset{?}{\Longrightarrow} \begin{cases} ①.\ \Sigma_i^t \simeq S^2 \\ ②.\ \hat{\mathcal{Y}}_i^t \text{ contains no aspherical terms} \end{cases}$$

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every $G \leq \pi_1 M$ fin. gen. has $e(G) \neq 1$

②. $M \simeq \hat{\mathcal{Y}}_1 \# \hat{\mathcal{Y}}_2 \# \hat{\mathcal{Y}}_3 \# \ldots$

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every $G \leq \pi_1 M$ fin. gen. has $e(G) \neq 1$

②. $M \simeq \hat{Y}_1 \# \hat{Y}_2 \# \hat{Y}_3 \# \ldots \not\simeq N \# \underset{\text{aspherical}}{K}$

Proof

$\mathrm{fillrad}\,\tilde{M} < c'r \overset{?}{\Longrightarrow} \begin{cases} ①.\ \Sigma_i^t \simeq S^2 \\ ②.\ \hat{\mathcal{Y}}_i^t \text{ contains no aspherical terms} \end{cases}$

$(c' < \frac{1}{3})$

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②. $M \simeq \hat{\mathcal{Y}}_1 \# \hat{\mathcal{Y}}_2 \# \hat{\mathcal{Y}}_3 \# \ldots \not\simeq N \# K$

$\llcorner$ aspherical ✓

Proof

$$\mathrm{fillrad}\,\tilde{M} < c'r \quad \stackrel{?}{\Longrightarrow} \quad \begin{cases} ①.\ \Sigma_i^t \simeq S^2 \\ ②.\ \hat{Y}_i^t \text{ contains no aspherical terms} \end{cases}$$

$$(c' < \tfrac{1}{3})$$

$$\Downarrow$$

every $G \leq \pi_1 M$ fin. gen. has $e(G) \neq 1$

①.

Proof

$\mathrm{fillrad}\, \tilde{M} < c' r \overset{?}{\Longrightarrow} \begin{cases} ①.\ \Sigma_i^+ \simeq S^2 \\ ②.\ \hat{\mathcal{Y}}_i^+ \text{ contains no aspherical terms} \end{cases}$

$(c' < \frac{1}{3})$

$\Downarrow$

every $G \leq \pi_1 M$ fin. gen. has $e(G) \neq 1$

①. (i). Σ incompressible

Proof

$$\operatorname{fillrad} \tilde{M} < c'r \stackrel{?}{\Longrightarrow} \begin{cases} ①.\ \Sigma_i^{\dagger} \simeq S^2 \\ ②.\ \hat{\mathcal{Y}}_i^{\dagger} \text{ contains no aspherical terms} \end{cases}$$

$(c' < \frac{1}{3})$

$\Downarrow$

every $G \leq \pi_1 M$ fin. gen. has $e(G) \neq 1$

① (i). Σ incompressible : $i_* : \pi_1 \Sigma \longrightarrow \pi_1 M$ injective

Proof

$$\operatorname{fillrad} \tilde{M} < c' r \quad (c' < \tfrac{1}{3}) \overset{?}{\Longrightarrow} \begin{cases} ①.\ \Sigma_i^t \simeq S^2 \\ ②.\ \hat{\mathcal{Y}}_i^t \text{ contains no aspherical terms} \end{cases}$$

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every $G \leq \pi_1 M$ fin. gen. has $e(G) \neq 1$

① (i). Σ incompressible : $i_* : \pi_1 \Sigma \longrightarrow \pi_1 M$ injective

$$\Longrightarrow \pi_1 \Sigma \leq \pi_1 M$$

Proof

$$\begin{array}{l} \text{fillrad}\,\tilde{M} < c'r \\ \quad (c' < \tfrac{1}{3}) \end{array} \overset{?}{\Longrightarrow} \begin{cases} ①.\ \Sigma_i^{+} \simeq S^2 \\ ②.\ \hat{\mathcal{Y}}_i^{+} \text{ contains no aspherical terms} \end{cases}$$

$$\Downarrow$$

every $G \leq \pi_1 M$ fin. gen. has $e(G) \neq 1$

① (i). Σ incompressible : $i_* : \pi_1 \Sigma \longrightarrow \pi_1 M$ injective

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Proof

$$\text{fillrad}\,\tilde{M} < c'r \overset{?}{\Longrightarrow} \begin{cases} ① \ \Sigma_i^+ \simeq S^2 \\ ② \ \hat{\mathcal{Y}}_i^+ \text{ contains no aspherical terms} \end{cases}$$

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① (i). Σ incompressible: $i_* : \pi_1 \Sigma \longrightarrow \pi_1 M$ injective

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Proof

$$\left.\begin{array}{c}\mathrm{fillrad}\,\tilde{M} < c'r \\ \left(c' < \tfrac{1}{3}\right)\end{array}\right. \overset{?}{\Longrightarrow} \left\{\begin{array}{l} \text{①. } \Sigma_i^+ \simeq S^2 \\ \text{②. } \hat{\mathcal{Y}}_i^+ \text{ contains no aspherical terms}\end{array}\right.$$

$$\Downarrow$$

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①. (i). Σ incompressible: $i_* : \pi_1 \Sigma \longrightarrow \pi_1 M$ injective

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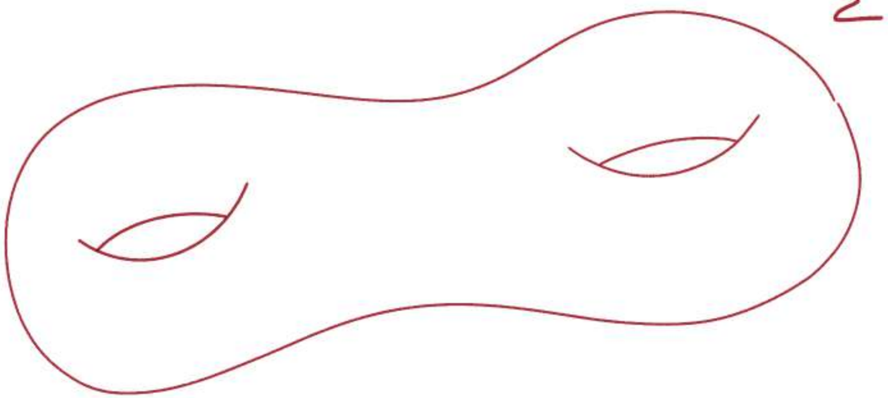
(ii). Σ compressible

Σ



Proof

$$\text{fillrad}\, \tilde{M} < c'r \quad (c' < \tfrac{1}{3}) \overset{?}{\Longrightarrow} \begin{cases} \text{①. } \Sigma_i^+ \simeq S^2 \\ \text{②. } \hat{\mathcal{Y}}_i^+ \text{ contains no aspherical terms} \end{cases}$$

$$\Downarrow$$

every $G \leq \pi_1 M$ fin. gen. has $e(G) \neq 1$

① (i). Σ incompressible: $i_* : \pi_1 \Sigma \longrightarrow \pi_1 M$ injective

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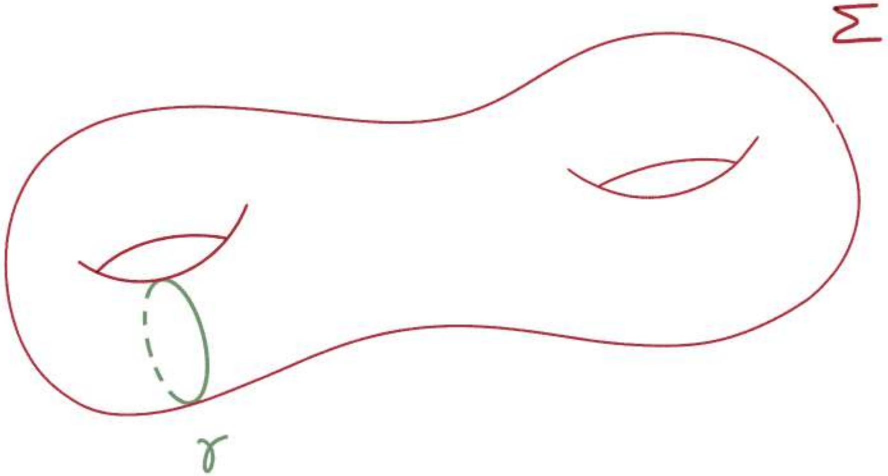

Proof

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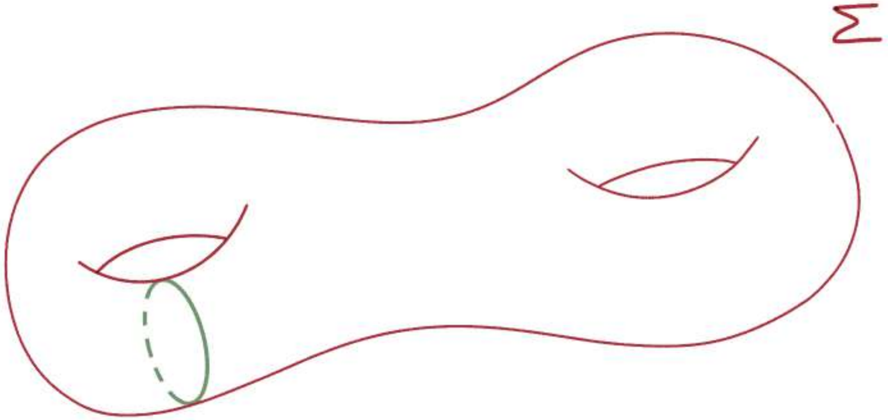

$[\gamma] \neq 0 \in \operatorname{Ker}(\pi_1 \Sigma \to \pi_1 M)$

Proof

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$$(c' < \tfrac{1}{3})$$

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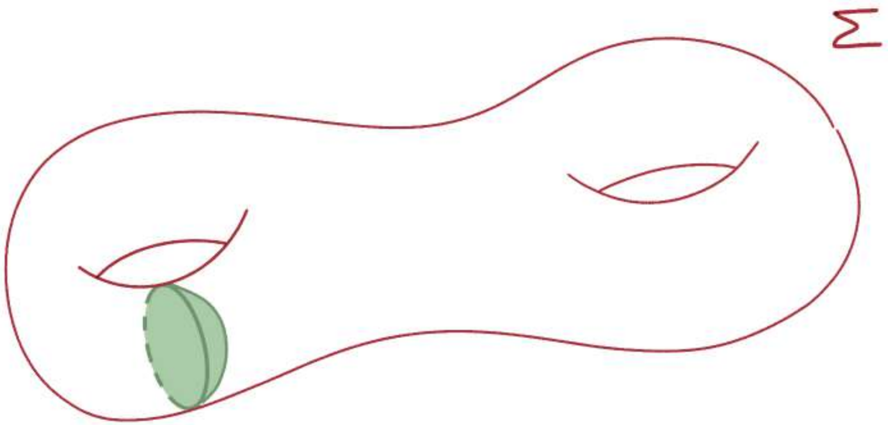

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Proof

$$\text{fillrad}\,\tilde{M} < c' r \quad (c' < \tfrac{1}{3}) \overset{?}{\Longrightarrow} \begin{cases} ①.\ \Sigma_i^{+} \simeq S^2 \\ ②.\ \hat{\mathcal{Y}}_i^{+} \text{ contains no aspherical terms} \end{cases}$$

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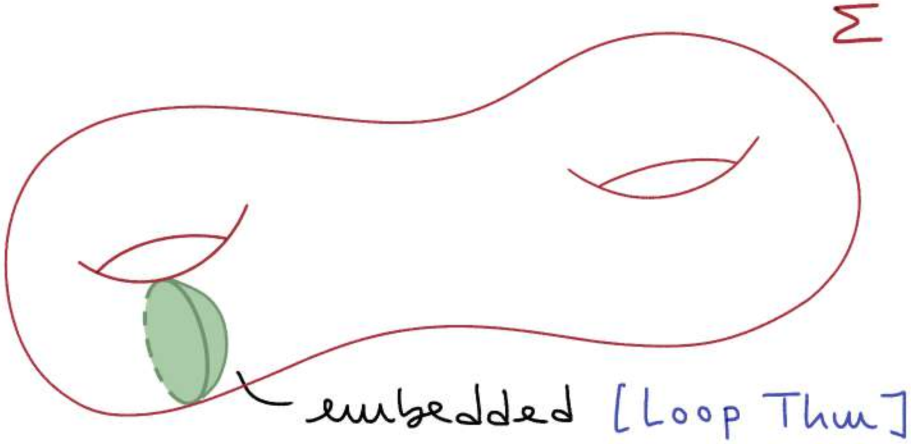

$[\gamma] \neq 0 \in \operatorname{Ker}(\pi_1 \Sigma \to \pi_1 M)$

Proof

$$\operatorname{fillrad}\tilde{M} < c'r \quad \left(c' < \tfrac{1}{3}\right) \overset{?}{\Longrightarrow} \begin{cases} ①.\ \Sigma_i^{+} \simeq S^2 \\ ②.\ \hat{y}_i^{+} \text{ contains no aspherical terms} \end{cases}$$

$\Downarrow$

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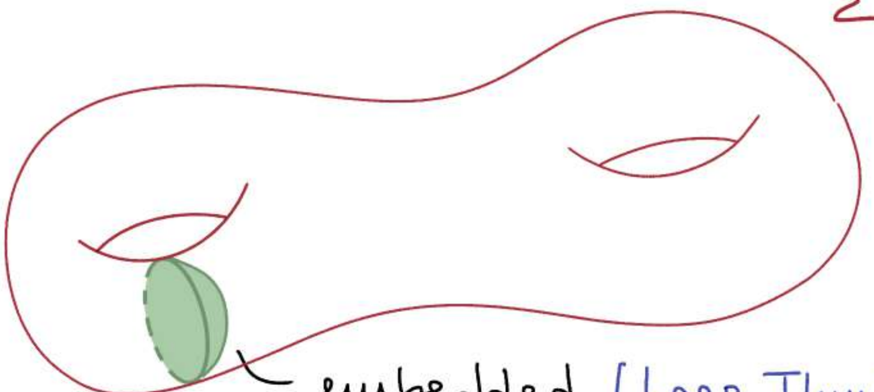
(ii). Σ compressible

Σ



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Proof

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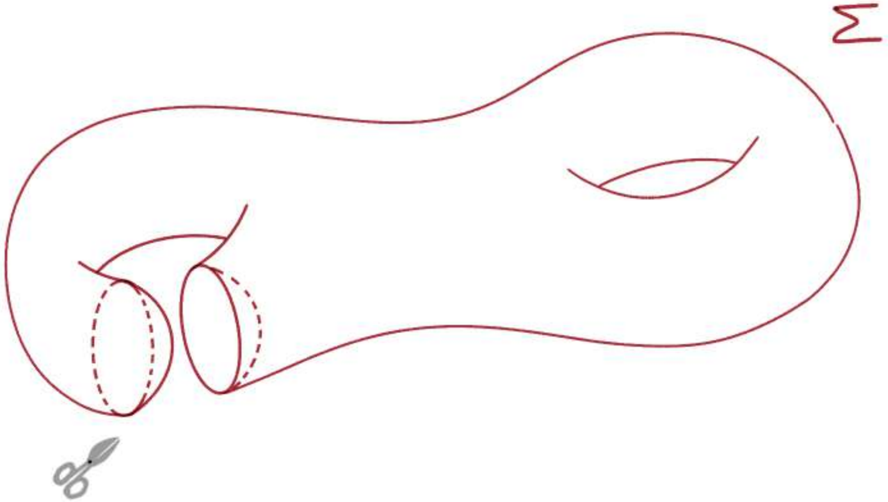

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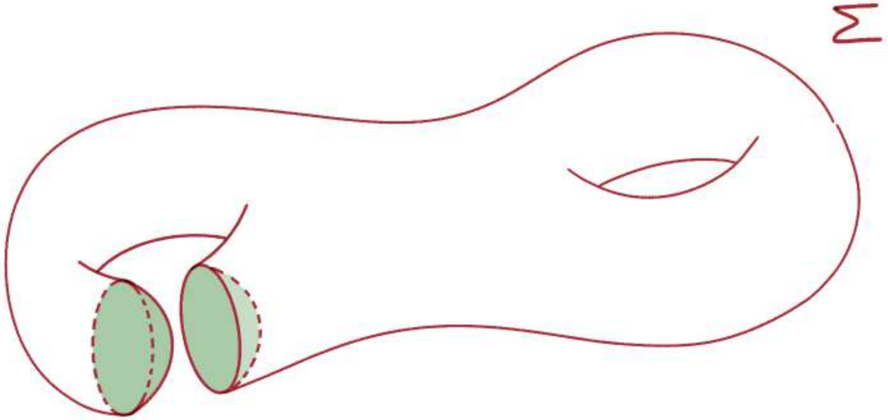

Proof

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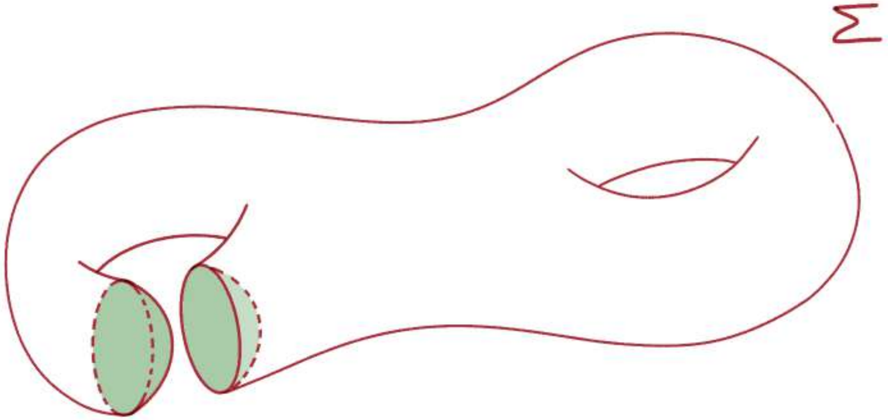

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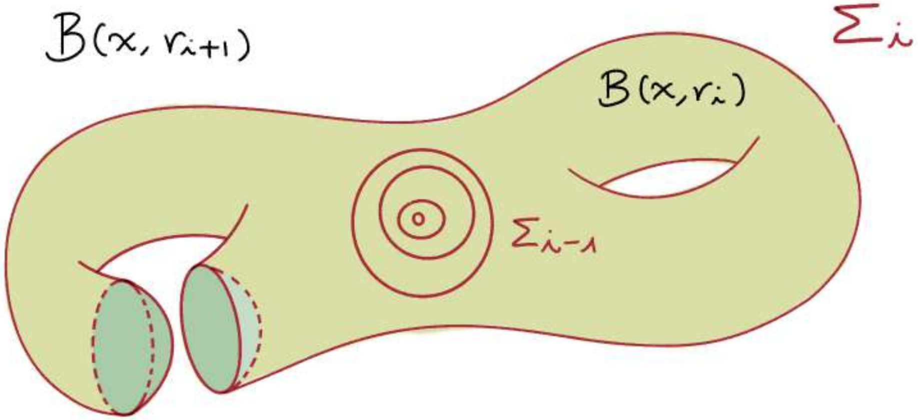

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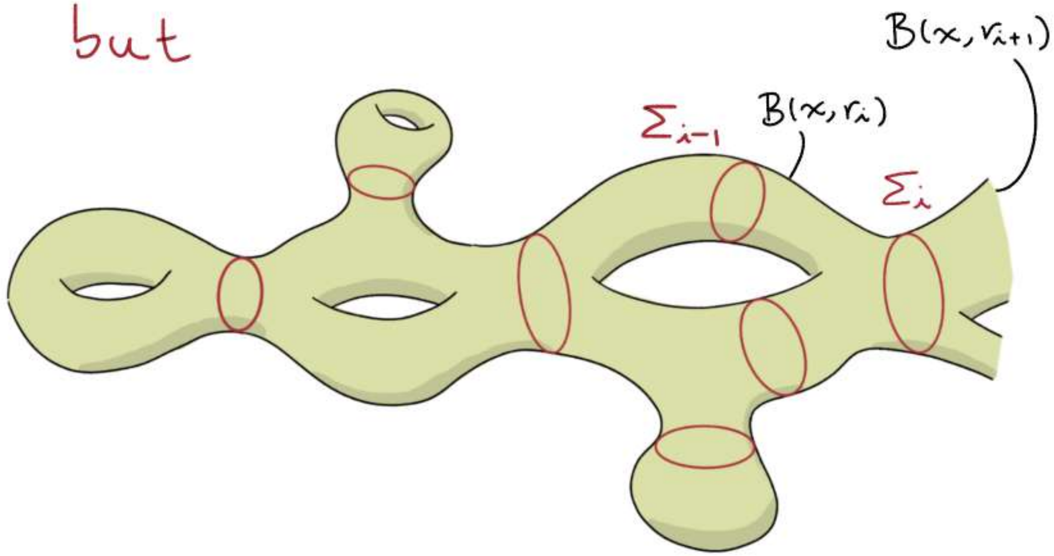

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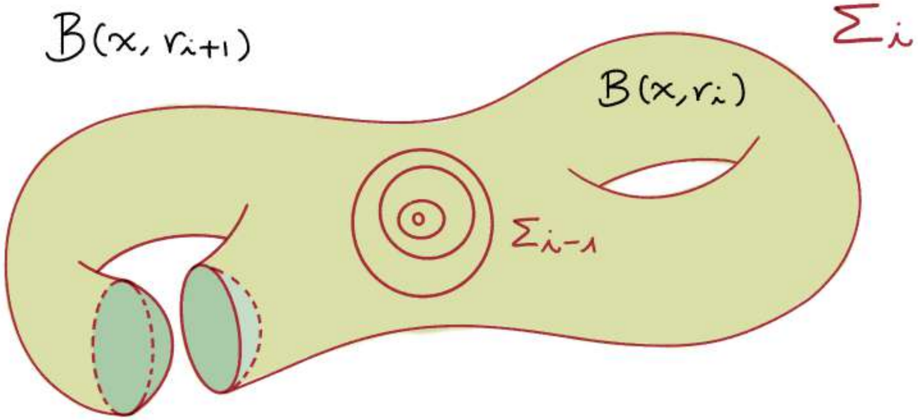

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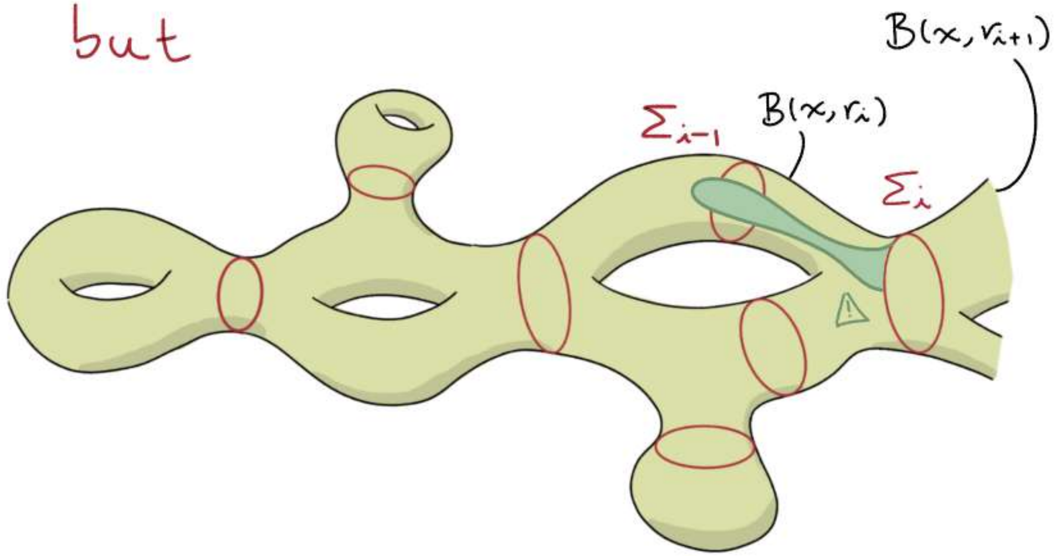

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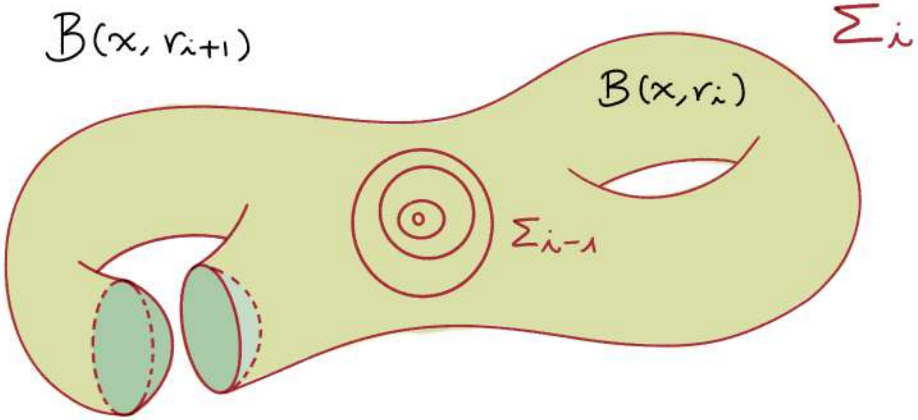

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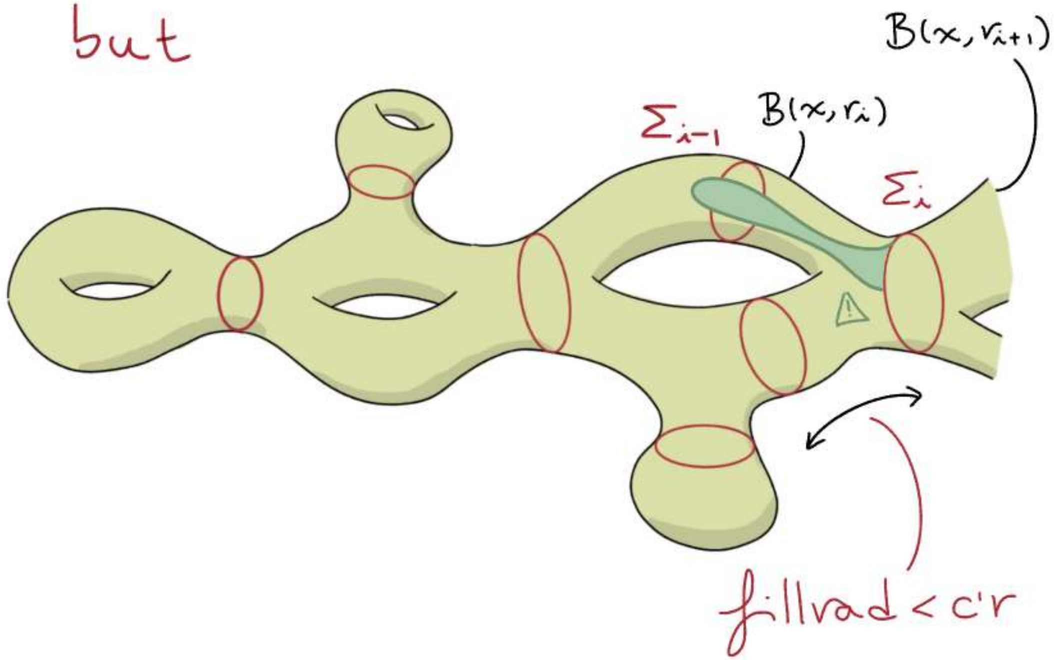

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