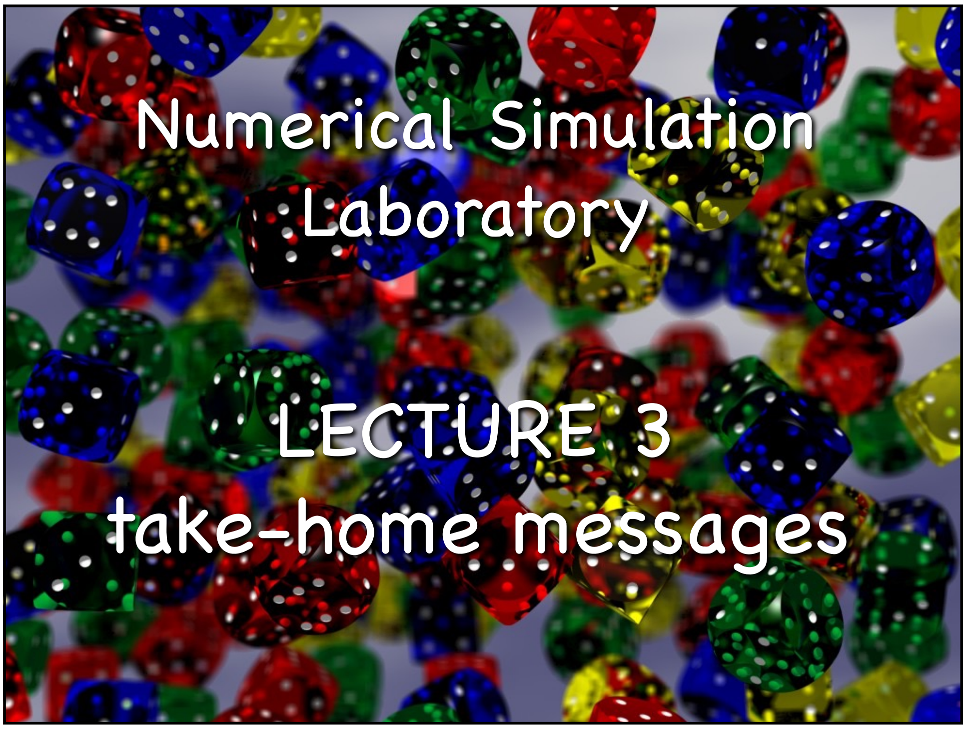

1

Path properties

- The Brownian motion has continuous sample paths:

$$Prob\left[\left|W(t+\Delta t)-W(t)\right|>k\right]=2\int_k^\infty dx\,\frac{1}{\sqrt{2\pi\Delta t}}\exp\left[-\frac{x^2}{2\Delta t}\right]$$

$$\Rightarrow \lim_{\Delta t\to 0} Prob\left[\left|W(t+\Delta t)-W(t)\right|>k\right]=0 \qquad \forall k$$

- The sample paths are, with probability one, nowhere differentiable:

$$Prob\left[\left|\frac{W(t+\Delta t)-W(t)}{\Delta t}\right|>k\right]=2\int_{k\Delta t}^\infty dx\,\frac{1}{\sqrt{2\pi\Delta t}}\exp\left[-\frac{x^2}{2\Delta t}\right]$$

$$\lim_{\Delta t\to 0} Prob\left[\left|\frac{W(t+\Delta t)-W(t)}{\Delta t}\right|>k\right]=1 \qquad \forall k$$

- This opens a problem in the interpretation of quantities like:

$$\frac{dW(t)}{dt};\ dW(t)$$

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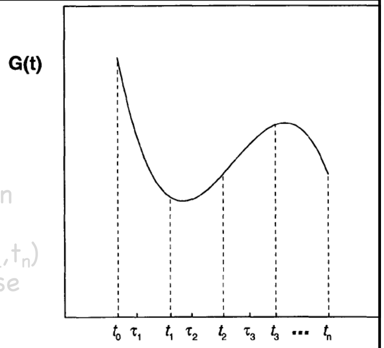
Mean-square limit



- Let us consider a function $G(t)$ over an interval (t_0,t_n)
- In the same way as is done for the Riemann integral, we consider a partitioning of this interval into subintervals $(t_0,t_1),(t_1,t_2),\ldots,(t_{n-1},t_n)$ and a set of points τ_i, $t_{i-1} < \tau_i < t_i$ inside these subintervals
- We define the mean-square limit in the following way:

$$\underset{n\to\infty}{\text{ms-}\lim} X_n = X \Leftrightarrow \lim_{n\to\infty} \left\langle (X_n - X)^2 \right\rangle = 0$$

- We then define: $$\int_{t_0}^{t_n} G(t')\,dW(t') := \underset{n\to\infty}{\text{ms-}\lim} \sum_{i=1}^{n} G(\tau_i)\left[W(t_i) - W(t_{i-1})\right]$$
- Convergence in the mean-square limit is a weaker requirement than the point-wise convergence used for the Riemann integral. This equation defines a stochastic integral, but not yet uniquely, because its value depends on the choice of points τ_i at which the integrand is evaluated

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Itô stochastic integral

- Setting $\tau_i = \alpha t_i + (1-\alpha)t_{i-1}$, with $0 \le \alpha \le 1$, the most used choice for α is: $\alpha = 0$, defining the Itô stochastic integral: $\tau_i = t_{i-1}$
- For example, with this choice one can show that (see supplementary material):

$$\int_{t_0}^{t_n} W(t')\,dW(t') = 0$$

- The Itô stochastic integral has the advantage that it is a martingale (see supplementary material); this fact leads to many helpful mathematical properties and makes usage of the Ito stochastic integral ubiquitous in the literature
- Definition: A sequence Y_n is a martingale iff $E\left[Y_{n+1} \mid Y_1,\ldots,Y_n\right] = Y_n$
- This means that for a martingale the best estimator for the next value, given all the information obtained through the past values, is the actual value.
- If the process is a martingale, its changes (increment process) are a "fair game", meaning that the expectation value of the increments is zero.

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Mnemonic equations

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- For an arbitrary non-anticipating G(t') we have (see supplementary information):

$$\text{ms-}\lim_{n\to\infty}\left(\sum_{i=1}^{n}\Delta W_i^2 G(t_{i-1})-\sum_{i=1}^{n}\Delta t_i G(t_{i-1})\right)=0 \quad\Rightarrow \int_{t_0}^{t}\left[dW(t')\right]^2 G(t')=\int_{t_0}^{t}dt'\,G(t')$$

$$\Rightarrow \boxed{\left[dW(t)\right]^2=dt}$$

- One can prove also that:

$$\text{ms-}\lim_{n\to\infty}\left(\sum_{i=1}^{n}\Delta W_i^m G(t_{i-1})\right)_{m\geq 3}=\int_{t_0}^{t}\left[dW(t')\right]^m G(t')=0$$

$$\Rightarrow \boxed{\left[dW(t)\right]^m=0\,,\ m\geq 3}$$

$$\text{ms-}\lim_{n\to\infty}\sum_{i=1}^{n}\Delta W_i\Delta t_i G(t_{i-1})=\int_{t_0}^{t}dW(t')\,dt'\,G(t')=0 \quad\Rightarrow \boxed{dW(t)dt=0}$$

- One can interpret these equations to mean: $\boxed{dW(t)=O\left(dt^{1/2}\right)}$

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Itô formula

- We obtain …

$$df=\frac{\partial f}{\partial t}dt+\frac{\partial f}{\partial x}\left[a\,dt+b\,dW\right]+\tfrac{1}{2}\frac{\partial^2 f}{\partial x^2}\left[a\,dt+b\,dW\right]^2+\frac{\partial^2 f}{\partial t\partial x}dt\left[a\,dt+b\,dW\right]=$$

$$=\frac{\partial f}{\partial t}dt+a\frac{\partial f}{\partial x}dt+b\frac{\partial f}{\partial x}dW+\tfrac{1}{2}b^2\frac{\partial^2 f}{\partial x^2}dW^2=\cdots$$

… the Itô formula: $d f[x(t),t]=$

$$=\left\{\frac{\partial f[x(t),t]}{\partial t}+a[x(t),t]\frac{\partial f[x(t),t]}{\partial x}+\tfrac{1}{2}b^2[x(t),t]\frac{\partial^2 f[x(t),t]}{\partial x^2}\right\}dt+$$

$$+\,b[x(t),t]\frac{\partial f[x(t),t]}{\partial x}dW(t)$$

- e.g. consider the SDE: $dx = a\cdot x\ dt + b\cdot x\ dW$ and $f(x) = \ln(x)$, it follows

$$d\ln[x(t)]=\left\{a\cdot x(t)\cdot\frac{1}{x(t)}+\tfrac{1}{2}b^2x^2(t)\frac{-1}{x^2(t)}\right\}dt+b\cdot x(t)\frac{1}{x(t)}dW(t)$$

$$\Rightarrow\quad d\ln[x(t)]=\left(a-\tfrac{1}{2}b^2\right)dt+b\,dW(t)$$

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Sampling Brownian motion

- Because Brownian motion has independent normally distributed increments, simulating the $W(t_i)$ or $X(t_i)$ from their increments is straightforward
- Let $Z_1, \ldots, Z_n$ be independent standard normal random variables, for a standard Brownian motion set $t_0 = 0$ and $W(0) = 0$. Subsequent values can be generated as follows:

$$W(t_{i+1}) = W(t_i) + Z_{i+1}\sqrt{t_{i+1} - t_i} \qquad i = 0, \cdots, n-1$$

- For $X(t) \sim BM(\mu, \sigma^2)$ with constant μ and $\sigma > 0$ and given $X(0)=X_0$ the path can be obtained as:

$$X(t_{i+1}) = X(t_i) + \mu(t_{i+1} - t_i) + \sigma Z_{i+1}\sqrt{t_{i+1} - t_i} \qquad i = 0, \cdots, n-1$$

- Whereas with time-dependent coefficients, the recursion becomes

$$X(t_{i+1}) = X(t_i) + \int_{t_i}^{t_{i+1}} \mu(t')\,dt' + Z_{i+1}\sqrt{\int_{t_i}^{t_{i+1}} \sigma^2(t')dt'} \qquad i = 0, \cdots, n-1$$

- These methods are exact in the sense that the joint distribution of the simulated values coincides with the joint distribution of the corresponding Brownian motion

Sampling a geometric Brownian motion

- Moreover, since the increments of W are independent and normally distributed, this provides a simple recursive procedure for sampling values of $S \sim GBM(\mu, \sigma^2)$ at $0 = t_0 < t_1 < \ldots < t_n$:

$$\boxed{S(t_{i+1}) = S(t_i)\exp\left\{(\mu - \tfrac{1}{2}\sigma^2)(t_{i+1} - t_i) + \sigma Z_{i+1}\sqrt{t_{i+1} - t_i}\right\} \qquad i = 0, \cdots, n-1}$$

with $Z_1, Z_2, \ldots, Z_n$ independent standard normals.

- In fact, this is equivalent to exponential version of the algorithmic sampling equation for $X \sim BM(\mu, \sigma^2)$ with μ replaced by $\mu - \sigma^2/2$
- This method is exact (no discretization error) in the sense that the sequence it produces has the joint distribution of the process $S \sim GBM(\mu, \sigma^2)$ at t_0 , t_1 , …, t_n
- Time-dependent parameters can be incorporated by taking the exponential of the relative algorithmic sampling equation of the Brownian motion with time dependent drift and volatility

Options

- Options are contracts in which only one partner assumes the obligation, whereas the other obtains a right. There are many different kinds of options. The simplest are European options
- A European option is a contract between two parties in which:
  - the seller of the option, known as the writer,
  - grants the buyer of the option, known as the holder,
  - the right to purchase (= call option) from the writer or to sell (= put option) to him an underlying (with a current spot price S(t))
  - for a prescribed price K, called the exercise or strike price
  - at the expiry date T in the future.
- The key property of an option is that only the writer has an obligation. He must sell or buy the underlying asset for the strike price at time T.
- The holder will only exploit his right if he gains a profit, i.e., if S(T) > K for a call option. Otherwise, he can buy the underlying for a cheaper price, S(T) < K, on the market:

  profit=max[0,S(T)-K] for a call; profit=max[0,K-S(T)] for a put
- Of course, the writer does not incur this financial obligation without requiring a compensation

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Numerical options pricing

- As soon as we abandon Black-Scholes assumptions, e.g. S(t) is not a geometric Brownian motion with constant drift and volatility, or if we want the fair price of an exotic option ⇒ no more analytic solution is available; as we shall see, we can use Monte Carlo methods
- Consider an asset with price S(t) and an European call option related to it; assume that the evolution of S(t) is given by $S \sim GBM(r,\sigma^2)$
- The profit at expiry (t=T) is given by: $\left(S(T)-K\right)^+ := \max\left[0, S(T)-K\right]$
- To obtain the present value of this profit, i.e. at time t, we have to discount it by a factor exp(-rT) due to the interest that a Bank would have guaranteed with a deposit at time $t_0=0$, in fact:

  $t=t_0$ ⇒ initial value: $\exp(-rT)\,(S(T)-K)^+$

  $t=T$ ⇒ final value: $\exp(rT)\exp(-rT)\,(S(T)-K)^+ = (S(T)-K)^+$
- Now, the calculation of the option price corresponds to the calculation of the expectation (average) value for the discounted profit on the distribution of the prices at expiry:

$$E\left[e^{-rT}\left(S(T)-K\right)^+\right] = \left\langle e^{-rT}\left(S(T)-K\right)^+\right\rangle$$

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- This quantity is, in fact, the estimate of what the holder of the option have to pay at t=0 on the basis of the expected profit at time t=T
- This average value can be easily obtained via a Monte Carlo simulation; algorithm:

```
do i=1,N
  generate Zi~N(0,1)
  Si(T)=S(0)exp[(r-σ²/2)T+σZi√T]
  Ci=exp(-rT)max[0,Si(T)-K]
enddo
CN=(∑i=1,N Ci)/N
end
```

```
do j=1,M
  S(ti+1)= …
enddo
S(Ti)=S(tM)
```

- In this way we will have that: $C_N \underset{N\to\infty}{\to} E\left[e^{-rT}\left(S(T)-K\right)^+\right]$
- An estimate for the statistical uncertainty can be obtained via: $\Delta C_N = \left[\frac{1}{N(N-1)}\sum_{i=1}^{N}\left(C_i - C_N\right)^2\right]^{1/2}$
- By simulating S(t) when we do not know the solution of a SDE, Monte Carlo can be extended to compute any exotic option price