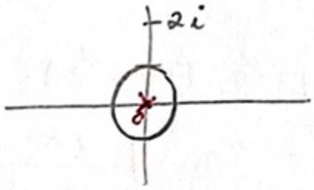
Frequência 2021/2022

1. a) $$\frac{z+1}{z^{2022}(z-2i)} = \frac{(z+1)}{z^{2022}} \times \frac{1}{2i\times(\frac{z}{2i}-1)} = \frac{z+1}{z^{2022}\times 2i} \times \frac{-1}{1-\frac{z}{2i}}$$

$$= -\frac{z+1}{z^{2022}\times 2i} \times \sum_{m=0}^{+\infty} \frac{z^m}{(2i)^m} = (z+1)\times \sum_{m=0}^{+\infty} -\frac{z^{m-2022}}{(2i)^{m+1}}$$

$$= \sum_{n=0}^{+\infty} -\frac{z^{m-2021}}{(2i)^{m+1}} + \sum_{m=0}^{+\infty} -\frac{z^{m-2022}}{(2i)^{m+1}}, \quad |z|<2$$

b) $z_0 = 0$ e $R = 1$



$$\oint_{C(0,1)} \frac{z+1}{z^{2022}(z-2i)}\,dz = 2\pi i \times (\operatorname{Res}_{z\to 0} g_z)$$

O Resíduo vamos buscar à alínea a)

- $m-2021 = -1 \Leftrightarrow m = 2020$

$$b_1' = -\frac{1}{(2i)^{2021}}$$

- $m-2022 = -1 \Leftrightarrow m = 2021$

$$b_1'' = -\frac{1}{(2i)^{2022}}$$

$$b_1 = b_1' + b_1'' = -\frac{1}{(2i)^{2021}} - \frac{1}{(2i)^{2022}} = -\frac{(2i)}{(2i)^{2022}} - \frac{1}{(2i)^{2022}} = -\frac{(2i+1)}{(2i)^{2022}}$$

Logo:

$$\oint_{C(0,1)} \frac{z+1}{(z^{2022})(z-2i)}\,dz = -\frac{2\pi i\times(2i+1)}{(2i)^{2022}}$$

2.

a) $X(z) = \dfrac{z}{(z+1)(z-2)^2}$

Seja $\{x_k\}_{k\geq 0} = Z^{-1}\{X(z)\}$

$$\{x_k\}_{k\geq 0} = \frac{1}{2\pi i}\oint_{C(0,5)} \frac{z \times z^{k-1}}{(z+1)(z-2)^2}\,dz = \frac{1}{2\pi i}\oint_{C(0,5)} \frac{z^k}{(z+1)(z-2)^2}\,dz$$

$$= \operatorname{Res}_{z\to -1} g_z + \operatorname{Res}_{z\to 2} g_z = \frac{1}{9}(-1)^k + \frac{k 2^{k-1} \times 3 - 2^k}{9}$$

C.aux:

- $z_0 = -1$ é polo simples

$$\operatorname{Res}_{z_0 \to -1} g_z = \lim_{z\to -1}\left[\frac{z^k}{(z-2)^2}\right] = \frac{(-1)^k}{9}$$

- $z_0 = 2$ é polo duplo

$$\operatorname{Res}_{z\to 2} g_z = \frac{1}{1!}\lim_{z\to 2}\frac{d}{dz}\left[\frac{z^k}{z+1}\right] = \lim_{z\to 2}\left[\frac{(k z^{k-1})(z+1) - z^k}{(z+1)^2}\right]$$

$$= \frac{k 2^{k-1} \times 3 - 2^k}{9}$$

b) $x_{k+2} - x_{k+1} - 2x_k = 2^{k+2}$, $x_0 = 0, x_1 = 0$

Seja $X(z) = Z\{x_k\}$

$$Z\{x_{k+2} - x_{k+1} - 2x_k\} = Z\{2^{k+2}\} \overset{\text{linearidade}}{\Leftrightarrow} Z\{x_{k+2}\} - Z\{x_{k+1}\} - 2Z\{x_k\} = Z\{2^{k+2}\}$$

$$\Leftrightarrow z^2 X(z) \underbrace{- x_0 z^2}_{0} \underbrace{- x_1 z}_{0} - (zX(z) \underbrace{- x_0 z}_{0}) - 2X(z) = Z\{4\, 2^k\}$$

$$\Leftrightarrow z^2 X(z) - zX(z) - 2X(z) = 4\frac{z}{z-2} \Leftrightarrow X(z) \times (z^2 - z - 2) = 4 \times \frac{z}{z-2}$$

$$\Leftrightarrow X(z) = 4 \times \frac{z}{(z-2)^2(z+1)}$$

$$\{x_k\}_{k\geq 0} = Z^{-1}\{X(z)\} = 4 \times \left(\frac{(-1)^k + k 2^{k-1} \times 3 - 2^k}{9}\right)$$

3.

$$g(t) = \begin{cases} t, & 0 \leq t < 1 \\ 1, & t \geq 1 \end{cases}$$

Darei jeito termos da forma $(t-1)H(t-1)$ para usarmos o teorema de Heaviside

a) Sabemos que:

$$g(t) = t \times (H(t) - H(t-1)) + 1 \times H(t-1)$$

$$\mathcal{L}\{g(t)\} = \mathcal{L}\{tH(t)\} - \mathcal{L}\{tH(t-1)\} + \mathcal{L}\{H(t-1)\}$$

$$= \frac{1}{s^2} - \mathcal{L}\{(t-1+1)H(t-1)\} + \mathcal{L}\{H(t-1)\}$$

$$= \frac{1}{s^2} - \mathcal{L}\{((t-1)+1)H(t-1)\} + \mathcal{L}\{H(t-1)\}$$

e^x, e^{-x}

e^{x-x}

$$= \frac{1}{s^2} - \mathcal{L}\{(t-1)H(t-1) + H(t-1)\} + \mathcal{L}\{H(t-1)\}$$

$$= \frac{1}{s^2} - (\mathcal{L}\{(t-1)H(t-1)\} + \mathcal{L}\{H(t-1)\}) + \mathcal{L}\{H(t-1)\}$$

$$= \frac{1}{s^2} - \mathcal{L}\{(t-1)H(t-1)\} - \cancel{\mathcal{L}\{H(t-1)\}} + \cancel{\mathcal{L}\{H(t-1)\}}$$

$$= \frac{1}{s^2} - \frac{1}{s^2}e^{-s}, \ \text{Re } s > 0, \ \text{c.q.d}$$

b) $$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2} \times \frac{1}{(s+1)}\right\}$$

$$= (t * e^{-t}) = \int_0^t e^{-(t-u)} \times u \, du = \int_0^t e^{-t+u} \times u \, du$$

$$= e^{-t}\int_0^t e^u \times u \, du = e^{-t} \times \left([e^u \times u]_0^t - \int_0^t e^u \, du\right)$$

$$= e^{-t} \times (e^t \times t - [e^u]_0^t) = e^{-t} \times (e^t \times t - (e^t - 1))$$

$$= e^{-t} \times e^t \times t - e^{-t} \times (e^t - 1) = t - 1 + e^{-t}$$

c) $y(0)=0$

Seja $\mathcal{L}\{y(t)\} = F(s)$:

$$\mathcal{L}\{y'(t)\} + \mathcal{L}\{y(t)\} = \mathcal{L}\{g(t)\}$$

$$(=) sF(s) - \underbrace{y(0)}_{0} + F(s) = \frac{1}{s^2} - \frac{1}{s^2}e^{-s}$$

$$(=) F(s) \times (s+1) = \frac{1}{s^2} \times (1-e^{-s})$$

$$(=) F(s) = \frac{1}{s^2(s+1)} \times (1-e^{-s})$$

$$(=) F(s) = \frac{1}{s^2(s+1)} - \frac{e^{-s}}{s^2(s+1)}$$

$$y(t) = \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2(s+1)}\right\} - \mathcal{L}^{-1}\left\{\frac{e^{-s}}{s^2(s+1)}\right\}$$

$$= t-1+e^{-t} - \mathcal{L}^{-1}\left\{e^{-s} \times \frac{1}{s^2(s+1)}\right\}$$

Teorema de Heaviside com $k=1$

$$= t-1+e^{-t} - \mathcal{L}^{-1}\{e^{-s}F(s)\} = t-1+e^{-t} - \left((t-1)-1+e^{-(t-1)}\right)$$

$$= t-1+e^{-t} - (t-1)+1-e^{-t+1}$$

$$= \cancel{t}-\cancel{1}+e^{-t}-\cancel{t}+\cancel{1}+1-e^{-t+1} = e^{-t+1}+e^{-t}+1 = e^{-t}\times(e+1)+1$$

4.

a) $w = 2\ rad/s$, $[-1,1]$ $\qquad w = \frac{2\pi}{T} (=) T = \frac{2\pi}{2} = \pi$

$$\sum_{n=-\infty}^{+\infty} c_n e^{inwt} = \sum_{n=\infty}^{+\infty} c_n e^{in2t}$$

• $n=0$

$$c_0 = \frac{1}{\pi}\int_{1/2}^{1} 2022 \times e^{-in2t}\,dt = \frac{1}{\pi}\left(\int_{-1}^{-1/2} 2022 \times e^{-in2t} + \int_{1/2}^{1} 2022 \times e^{-in2t}\right)$$

$$= \frac{1}{\pi}\left(2022[t]_{-1}^{-1/2} + 2022[t]_{1/2}^{1}\right) = \frac{1}{\pi}\left[2022 \times \left(\frac{1}{2}\right) + 2022 \times \frac{1}{2}\right]$$

$$= \frac{1}{\pi} \times 2022 = \frac{2022}{\pi}$$

$\cdot\, n \neq 0:$

$$C_n = \frac{1}{\pi}\left(\int_{-1}^{-\frac{1}{2}} 2022\, e^{-in2t}\, dt + \int_{1/2}^{1} 2022\, e^{-in2t}\, dt \right)$$

$$= \frac{1}{\pi}\left(2022 \times \frac{1}{-in2} \left[e^{-in2t} \right]_{-1}^{-\frac{1}{2}} + 2022 \times \frac{1}{-in2} \left[e^{-in2t} \right]_{1/2}^{1} \right)$$

$$= \frac{1}{\pi} \times \left(2022 \times \frac{1}{-in2} \left[e^{in} - e^{in2} \right] + 2022 \times \frac{1}{-in2} \times \left(e^{-in2} - e^{-in} \right) \right)$$

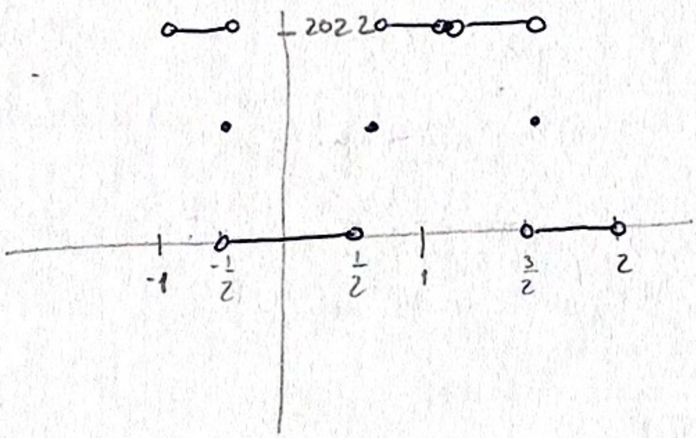
$$= \frac{1}{\pi} \times \frac{2022}{-in2} \times \left(e^{in} - e^{in2} + e^{-in2} - e^{-in} \right)$$

$$= \frac{1}{\pi} \times \frac{2022}{-n} \times \left(\frac{e^{in} - e^{-in}}{2i} - \frac{e^{in2} - e^{-in2}}{2i} \right)$$

$$= \frac{1}{\pi} \times \frac{2022}{-n} \times \left(\sin(n) - \sin(2n) \right)$$

$$= -\frac{2022}{n\pi} \left(\sin(n) - \sin(2n) \right)$$

b)



5. $f(t) = e^{-3|t|}$

a)
$$F\{f(t)\} = \int_{-\infty}^{+\infty} e^{-3|t|} e^{-iwt}\, dt$$

$$= \int_{-\infty}^{0} e^{-3(-t)} e^{-iwt}\, dt + \int_{0}^{+\infty} e^{-3t} e^{-iwt}$$

$$= \int_{-\infty}^{0} e^{3t} e^{-iwt} dt + \int_{0}^{+\infty} e^{-3t} e^{-iwt}$$

$$= \int_{-\infty}^{0} e^{t\times(3-iw)} dt + \int_{0}^{+\infty} e^{t(-3-iw)}$$

$$= \lim_{A\to-\infty} \int_{A}^{0} e^{t(3-iw)} dt + \lim_{A\to+\infty} \int_{0}^{A} e^{-t(3+iw)} dt$$

$$= \frac{1}{3-iw} \lim_{A\to-\infty} \left[e^{t(3-iw)}\right]_{A}^{0} - \frac{1}{3+iw} \lim_{A\to+\infty} \left[e^{-t(3+iw)}\right]_{0}^{A}$$

$$= \frac{1}{3-iw} \times \lim_{A\to-\infty} \left[1 - \overset{0}{e^{A(3-iw)}}\right] - \frac{1}{3+iw} \lim_{A\to+\infty} \left[\overset{0}{e^{-A(3+iw)}} - 1\right]$$

$$= \frac{1}{3-iw} + \frac{1}{3+iw} = \frac{3+\cancel{iw}+3-\cancel{iw}}{(3-iw)(3+iw)} = \frac{6}{9+w^2}$$

b)

Formula simetrie

$$F\left\{\frac{1}{9+t^2}\right\} = \frac{1}{6} F\left\{\frac{6}{9+t^2}\right\} = \frac{1}{6} 2\pi f(-w)$$

$$= \frac{\pi}{3} e^{-3|-w|} = \frac{\pi}{3} e^{-3|w|}, \quad w \in \mathbb{R}$$