Frequência 12.01.2024

1. a)

$$f(z) = \underbrace{\frac{1}{(z+3)^{2024}}}_{\text{É isto em série de Laurent de potências } z+3} \times \frac{1}{z-4i} = \frac{1}{(z+3)^{2024}} \times \frac{1}{z-4i+3-3}$$

$$= \frac{1}{(z+3)^{2024}} \times \frac{1}{z+3-4i-3} = \frac{1}{(z+3)^{2024}} \times \frac{1}{z+3-(4i+3)}$$

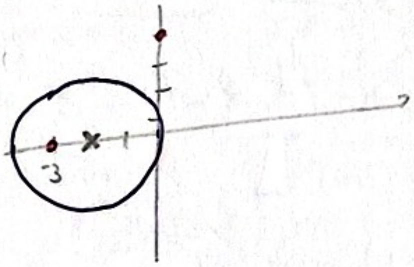
$$= \frac{1}{(z+3)^{2024}} \times \frac{1}{4i+3 \times \left(\frac{z+3}{4i+3} - 1\right)} = \frac{1}{(z+3)^{2024}} \times \frac{-1}{4i+3} \times \frac{1}{1-\frac{z+3}{4i+3}}$$

$$= \frac{-1}{(z+3)^{2024} \times (4i+3)} \times \sum_{n=0}^{+\infty} \left(\frac{z+3}{4i+3}\right)^n = -\sum^{+\infty} \frac{(z+3)^{n-2024}}{(4i+3)^{n+1}}, \quad \left|\frac{z+3}{4i+3}\right| < 1$$

$$= -\sum_{n=0}^{+\infty} \frac{(z+3)^{n-2024}}{(4i+3)^{n+1}}, \quad |z+3| < 5$$

$\sqrt{4^2+3^2} = 5$

b) Singularidades: -3 e $4i$



$z_0 = -3$ está dentro da circunferência, logo pelo teorema dos resíduos:

$$\oint_\gamma \frac{1}{(z+3)^{2024} \times (z-4i)} = 2\pi i \times \operatorname*{Res}_{z \to -3} f = 2\pi i \times \frac{-1}{(4i+3)^{2024}} = \frac{-2\pi i}{(4i+3)^{2024}}$$

C.Aux:

Na alínea a) Temos o desenvolvimento em série de Laurent em $z+3$, logo podemos ir lá buscar o resíduo, ou seja, o b_1, que corresponde ao coeficiente de $(z+3)^{-1}$:

$n - 2024 = -1 \Leftrightarrow n = 2023$

$$a_{2023} = \frac{-1}{(4i+3)^{2024}}$$

2.

a) $X(z) = \dfrac{z}{(z^2+4)(z-2)^2}$, $|z|>2$

C. Aux:

$z^2+4=0 \Leftrightarrow z^2=-4$
$\Leftrightarrow z = \pm\sqrt{-4}$
$\Leftrightarrow z = \pm\sqrt{-1}\times\sqrt{4}$
$\Leftrightarrow z = \pm 2i$

Seja $\{x_k\}_{k\geq 0} = Z^{-1}\{X(z)\}$

$\{x_k\}_{k\geq 0} = Z^{-1}\left\{\dfrac{z}{(z^2+4)(z-2)^2}\right\}$

$= \dfrac{1}{2\pi i}\oint_{C(0,10)} \dfrac{z\times z^{k-1}}{(z-2i)(z+2i)(z-2)^2}\,dz$

$= \dfrac{1}{2\pi i}\oint_{C(0,10)} \dfrac{z^k}{(z-2i)(z+2i)(z-2)^2}\,dz = \underset{z\to 2i}{\operatorname{Res}}\, g_z + \underset{z\to -2i}{\operatorname{Res}}\, g_z + \underset{z\to 2}{\operatorname{Res}}\, g_z$

C. Aux:

$z_0 = 2i$ é pólo simples
$z_0 = -2i$ é pólo simples
$z_0 = 2$ é pólo duplo

• $\underset{z\to 2i}{\operatorname{Res}}\, g_z = \lim_{z\to 2i}\left[\dfrac{\cancel{(z-2i)}\, z^k}{\cancel{(z-2i)}(z+2i)(z-2)^2}\right] = \lim_{z\to 2i}\left[\dfrac{z^k}{(z+2i)(z-2)^2}\right]$

$= \dfrac{(2i)^k}{4i\times(2i-2)^2} = \dfrac{2^k i^k}{4i\times(\cancel{-4}-8i+\cancel{4})} = \dfrac{2^k\times i^k}{-32i^2} = \dfrac{2^k i^k}{32} = \dfrac{(2i)^k}{32} = \dfrac{2^k\times e^{ik\frac{\pi}{2}}}{32}$

• $\underset{z\to -2i}{\operatorname{Res}}\, g_z = \lim_{z\to -2i}\left[\dfrac{\cancel{(z+2i)}\, z^k}{(z-2i)\cancel{(z+2i)}(z-2)^2}\right] = \lim_{z\to -2i}\left[\dfrac{z^k}{(z-2i)(z-2)^2}\right]$

$= \dfrac{(-2i)^k}{-4i\times(-2i-2)^2} = \dfrac{(-2i)^k}{-4i\times(\cancel{-4}+8i+\cancel{4})} = \dfrac{(-2i)^k}{-32i^2} = \dfrac{(-1)^k\times(2i)^k}{32} = \dfrac{2^k\times e^{-ik\frac{\pi}{2}}}{32}$

• $\underset{z\to 2}{\operatorname{Res}}\, g_z = \dfrac{1}{1!}\times\lim_{z\to 2}\dfrac{d}{dz}\left[\dfrac{\cancel{(z-2)^2}\, z^k}{(z^2+4)\cancel{(z-2)^2}}\right] = \lim_{z\to 2}\dfrac{d}{dz}\left[\dfrac{z^k}{z^2+4}\right]$

$= \lim_{z\to 2}\left[\dfrac{(kz^{k-1})(z^2+4) - z^k\times 2z}{(z^2+4)^2}\right] = \dfrac{k2^{k-1}\times 8 - 2^k\times 4}{64}$

$= \dfrac{k\times 2^k\times(\frac{1}{2})\times 8 - 2^k\times 4}{64} = \dfrac{k2^k\times 4 - 2^k\times 4}{64} = \dfrac{4\times(k2^k - 2^k)}{64} = \dfrac{k2^k - 2^k}{16}$

$= \dfrac{2^k\times(k-1)}{16}$

Logo: (3)

$$\{x_k\}_{k\geq 0} = \operatorname*{Res}_{z\to 2i} g_z + \operatorname*{Res}_{z\to -2i} g_z + \operatorname*{Res}_{z\to 2} g_z$$

$$= \frac{2^k}{32}\times e^{ik\frac{\pi}{2}} + \frac{2^k}{32}\times e^{-ik\frac{\pi}{2}} + \frac{2^k}{16}\times(k-1)$$

$$= 2^{k-5}\times\left(\cos\left(k\frac{\pi}{2}\right)+i\sin\left(k\frac{\pi}{2}\right)\right) + 2^{k-5}\times\left(\cos\left(-k\frac{\pi}{2}\right)+i\sin\left(-k\frac{\pi}{2}\right)\right) + 2^{k-4}(k-1)$$

$$= 2^{k-5}\times i\sin\left(k\frac{\pi}{2}\right) + 2^{k-5}\times i\sin\left(-k\frac{\pi}{2}\right) + 2^{k-4}(k-1)$$

$$= 2^{k-4}(k-1) + \cancel{2^{k-5}\times i\times(-1)^k} + \cancel{2^{k-5}\times i\times(-1)^k}$$

$$= 2^{k-4}(k-1)$$

b) $x_{k+2} - 4x_{k+1} + 4x_k = 2^k\sin\left(k\frac{\pi}{2}\right)$, $x_0 = x_1 = 0$

Seja $X(z) = Z\{x_k\}$

$$Z\{x_{k+2} - 4x_{k+1} + 4x_k\} = Z\left\{2^k\sin\left(k\frac{\pi}{2}\right)\right\}$$

$$\Leftrightarrow Z\{x_{k+2}\} - 4Z\{x_{k+1}\} + 4Z\{x_k\} = \frac{2z\sin\left(\frac{\pi}{2}\right)}{z^2 - 2\times 2\times z\cos\left(\frac{\pi}{2}\right) + 2^2}$$

Linearidade

$$\Leftrightarrow z^2X(z) - \underbrace{x_0z^2}_{0} - \underbrace{x_1z}_{0} - 4\times\left(zX(z) - x_0z\right) + 4X(z) = \frac{2z}{z^2+4}$$

Deslocamento à direita

$$\Leftrightarrow z^2X(z) - 4zX(z) + \underbrace{4x_0z}_{0} + 4X(z) = \frac{2z}{z^2+4}$$

$$\Leftrightarrow X(z)\times(z^2-4z+4) = \frac{2z}{z^2+4} \Leftrightarrow X(z) = \frac{2z}{(z^2+4)(z^2-4z+4)}$$

$$\Leftrightarrow X(z) = \frac{2z}{(z^2+4)(z-2)^2}$$

Logo:

$$\{x_k\}_{k\geq 0} = Z^{-1}\{X(z)\} = Z^{-1}\left\{\frac{2z}{(z^2+4)(z-2)^2}\right\} = 2Z^{-1}\left\{\frac{z}{(z^2+4)(z-2)^2}\right\}$$

$$= 2\times 2^{k-4}(k-1) = 2^{k-3}\times(k-1)$$

alínea a)

3.

a) $\mathcal{L}^{-1}\left\{ \frac{1}{(s+1)^2(s-1)} \right\}$

Seja $f(t) = \mathcal{L}^{-1}\left\{ \frac{1}{(s+1)^2(s-1)} \right\}$

$f(t) = \mathcal{L}^{-1}\left\{ \frac{1}{(s+1)^2} \times \frac{1}{s-1} \right\} = (e^{-t}t * e^{t})$

$= \int_0^t f(t-u)\,g(u)\,du = \int_0^t e^{(t-u)}\, e^{-u} \times u\,du$

$= \int_0^t e^{t} \times e^{-u} \times e^{-u} \times u\,du = e^{t}\int_0^t e^{-2u} \times u\,du$

$= e^{t}\left(\left[-\frac{e^{-2u}}{2} \times u \right]_0^t - \int_0^t -\frac{e^{-2u}}{2}\,du \right) = e^{t} \times \left(-\frac{e^{-2t}}{2} \times t + \frac{1}{2}\int_0^t e^{-2u}\,du \right)$

$= e^{t} \times \left(-\frac{e^{-2t}}{2} \times t + \frac{1}{2}\left[\frac{e^{-2u}}{-2} \right]_0^t \right) = -\frac{e^{-t}}{2} \times t + \frac{1}{2}\left(-\frac{e^{-2t}}{2} - \left(-\frac{1}{2} \right) \right)$

$= -\frac{1}{2} \times (e^{-t}t) - \frac{1}{4}\, e^{-2t} \times e^{t} + \frac{1}{4}\, e^{t}$

$= -\frac{1}{2}e^{-t}t - \frac{1}{4}e^{-t} + \frac{1}{4}e^{t}$

b) $y''(t) - y(t) = e^{-t}$

Seja $F(s) = \mathcal{L}\{y(t)\}$

$\mathcal{L}\{y''(t) - y(t)\} = \mathcal{L}\{e^{-t}\} \Leftrightarrow \mathcal{L}\{y''(t)\} - \mathcal{L}\{y(t)\} = \frac{1}{s+1}$

$\Leftrightarrow s^2F(s) - \underbrace{s\,y(0)}_{0} - \underbrace{y'(0)}_{0} - F(s) = \frac{1}{s+1}$

$\Leftrightarrow F(s) \times (s^2 - 1) = \frac{1}{s+1} \Leftrightarrow F(s) = \frac{1}{(s+1)^2(s-1)}$

$y(t) = \mathcal{L}^{-1}\{F(s)\} = -\frac{1}{2}e^{-t}t - \frac{1}{4}e^{-t} + \frac{1}{4}e^{t}$

alínea a)

4. $w = 2\ rad/s$ $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right[$

$v(t) = t$

a) A série de Fourier é dada por, na forma complexa:

$$\sum_{n=-\infty}^{+\infty} c_n e^{inwt} = \sum_{n=-\infty}^{+\infty} c_n e^{in2t} \qquad w = \frac{2\pi}{T} \Leftrightarrow 2 = \frac{2\pi}{T} \Leftrightarrow T = \frac{2\pi}{2} = \pi$$

• $n = 0$:

$$c_0 = \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} v(t)\, e^{-in2t}\, dt = \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} t \times e^0\, dt = \frac{1}{\pi} \left[\frac{t^2}{2}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= \frac{1}{\pi} \times \left(\frac{\left(\frac{\pi}{2}\right)^2}{2} - \frac{\left(-\frac{\pi}{2}\right)^2}{2} \right) = 0$$

• $n \neq 0$:

$$c_n = \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} t \times e^{-in2t}\, dt = \frac{1}{\pi} \times \frac{1}{-in2} \times \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (-in2)\, e^{-in2t} \times t\, dt$$

$$= -\frac{1}{2\pi in} \times \left(\left[e^{-in2t} \times t \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - \int_{-\frac{\pi}{2}}^{\pi/2} e^{-in2t}\, dt \right)$$

$$= -\frac{1}{2\pi in} \times \left(e^{-i\pi n} \times \frac{\pi}{2} - e^{i\pi n} \times \left(-\frac{\pi}{2}\right) + \frac{1}{in2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (-in2)\, e^{-in2t}\, dt \right)$$

$$= -\frac{1}{2\pi in} \times \left(e^{-i\pi n} \times \frac{\pi}{2} + e^{i\pi n} \times \frac{\pi}{2} + \frac{1}{in2} \left[e^{-in2t} \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \right)$$

$$= -\frac{1}{2\pi in} \left(e^{-i\pi n} \times \frac{\pi}{2} + e^{i\pi n} \frac{\pi}{2} + \frac{1}{in2} \left(e^{-in\pi} - e^{in\pi} \right) \right)$$

$$= -\frac{1}{2\pi in} \left(\pi \times \left(\frac{e^{i\pi n} + e^{-i\pi n}}{2} \right) - \frac{1}{n} \left(\frac{e^{i\pi n} - e^{-in\pi}}{2i} \right) \right)$$

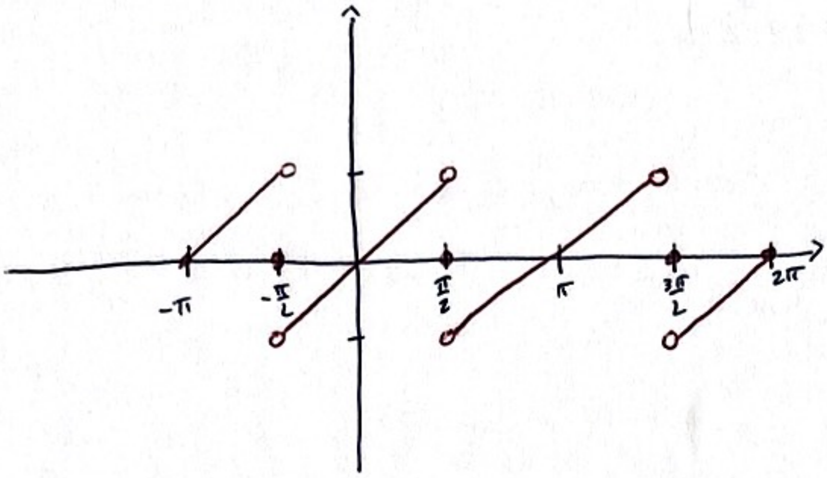
$$= -\frac{1}{2\cancel{\pi} in} \times \left(\cancel{\pi} \cos(\pi n) - \underbrace{\frac{1}{n} \sin(n\pi)}_{0} \right)$$

$$= -\frac{1}{2ni} \times (-1)^n$$

Série de Fourier:

$$\sum_{\substack{n=-\infty \\ n \setminus \{0\}}}^{+\infty} \frac{(-1)^{n+1}}{2ni} \times e^{in2t}$$

b)



c) Função soma da série de Fourier $\rightarrow \tilde{v}(t)$

$$\tilde{v}(t) = \begin{cases} 0, & t = \frac{\pi}{2} \vee t = \frac{3\pi}{2} \\ v(t-\pi), & \frac{\pi}{2} < t < \frac{3\pi}{2} \end{cases} = \begin{cases} 0, & t = \frac{\pi}{2} \vee t = \frac{3\pi}{2} \\ t-\pi, & \frac{\pi}{2} < t < \frac{3\pi}{2} \end{cases}$$

5.

$$F(w) = \int_{-\infty}^{+\infty} f(t)\, e^{-iwt}\, dt = \int_{-\infty}^{+\infty} e^{-|t|} H(-t)\, e^{iwt}$$

$$H(-t) = \begin{cases} 0, & -t < 0 \\ 1, & -t \geq 0 \end{cases}$$

$$H(-t) = \begin{cases} 0, & t > 0 \\ 1, & t \leq 0 \end{cases}$$

$$= \int_{-\infty}^{0} e^{-|t|}\, e^{-iwt}\, dt = \int_{-\infty}^{0} e^{t}\, e^{iwt}\, dt$$

$$= \int_{-\infty}^{0} e^{t-iwt}\, dt = \int_{-\infty}^{0} e^{t\times(1-iw)}\, dt = \frac{1}{1-iw} \int_{-\infty}^{0} (1-iw)\, e^{t(1-iw)}\, dt$$

$$= \lim_{A\to-\infty} \frac{1}{1-iw} \int_{A}^{0} (1-iw)\, e^{t(1-iw)}\, dt = \lim_{A\to-\infty} \frac{1}{1-iw} \left[e^{t(1-iw)} \right]_{A}^{0}$$

$$= \lim_{A\to-\infty} \frac{1}{1-iw} \left(e^{0} - e^{A(1-iw)} \right) = \frac{1}{1-iw} \times (1-0) = \frac{1}{1-iw}$$

C. Aux:

$$\lim_{A\to-\infty} \left| e^{(1-iw)A} \right| = \lim_{A\to-\infty} \left| e^{A}\, \overbrace{e^{iwA}}^{1} \right| = \lim_{A\to-\infty} e^{A} = 0$$