Frequência 2018/2019

1. $\operatorname*{Res}_{z\to z_0} f = \frac{1}{(2-1)!}\lim_{z\to z_0}\frac{d}{dz}\left[(z-z_0)^2 f(z)\right]$

$= \lim_{z\to z_0}\frac{d}{dz}\left[(z-z_0)^2 f(z)\right]$

2.

a) $f(z) = \frac{1}{(-z-1)^9(z+3)} = \frac{1}{(-1)^9\times(z+1)^9}\times\frac{1}{z+3}$

$= -\frac{1}{(z+1)^9}\times\frac{1}{z+3+1-1} = -\frac{1}{(z+1)^9}\times\frac{1}{(z+1)+2}$

$= -\frac{1}{(z+1)^9}\times\frac{1}{2\times\left(\frac{(z+1)}{2}+1\right)} = -\frac{1}{2(z+1)^9}\times\frac{1}{1-\left(-\frac{(z+1)}{2}\right)}$

$= -\frac{1}{2(z+1)^9}\times\sum_{n=0}^{+\infty}\left(-\frac{(z+1)}{2}\right)^n$

$= -\frac{1}{2(z+1)^9}\times\sum_{n=0}^{+\infty}(-1)^n\times\frac{(z+1)^n}{2^n} = \sum_{n=0}^{+\infty}(-1)^{n+1}\times\frac{(z+1)^{n-9}}{2^{n+1}}, \quad |z+1|<2$

b) $n-9=-1 \iff n=8$

$\operatorname*{Res}_{z=-1} f = \frac{(-1)^9}{2^9} = -\frac{1}{2^9}$

3. $x''(t)+4x(t) = e^{-t}t, \quad x(0)=x'(0)=0$

$\mathcal{L}\{x''(t)+4x(t)\} = \mathcal{L}\{e^{-t}t\}$

$\iff \mathcal{L}\{x''(t)\}+4F(s) = \frac{1}{(s+1)^2} \iff s^2F(s) - s\underbrace{x(0)}_{0} - \underbrace{x'(0)}_{0} + 4F(s) = \frac{1}{(s+1)^2}$

$\iff F(s)\times(s^2+4) = \frac{1}{(s+1)^2} \iff F(s) = \frac{1}{(s+1)^2(s^2+4)}$

$x(t) = \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{2}{25}\frac{1}{s+1} + \frac{A}{(s+1)^2} - \frac{1}{25}\frac{2s+3}{s^2+4}\right\}$

$$A = \frac{1}{5} = \frac{5}{25}$$

$$x(t) = \frac{2}{25}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + \frac{5}{25}\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\} - \frac{1}{25}\mathcal{L}^{-1}\left\{\frac{2s+3}{s^2+4}\right\}$$

$$= \frac{2}{25}e^{-t}H(t) + \frac{5}{25}e^{-t}tH(t) - \frac{1}{25}\left(\mathcal{L}^{-1}\left\{\frac{2s}{s^2+4}\right\} + 3\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\}\right)$$

$$= \frac{2}{25}e^{-t}H(t) + \frac{5}{25}e^{-t}tH(t) - \frac{2}{25}\cos(2t)H(t) - \frac{3}{25}\times\frac{1}{2}\times\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\}$$

$$= \frac{2}{25}e^{-t}H(t) + \frac{5}{25}e^{-t}tH(t) - \frac{2}{25}\cos(2t)H(t) - \frac{3}{50}\sin(2t)H(t)$$

4. a) $F(s) = \dfrac{1}{(s+1)(s-1)^2}$

$$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s+1)(s-1)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s+1}\times\frac{1}{(s-1)^2}\right\}$$

$$= \left(e^{-t}H(t) * e^{t}t\right) = \int_0^t e^{-(t-u)}e^{u}u\,du$$

$$= \int_0^t e^{-t}e^{u}e^{u}u\,du = e^{-t}\int_0^t e^{2u}u\,du$$

$$= e^{-t}\times\left(\left[\frac{e^{2u}}{2}u\right]_0^t - \int_0^t \frac{e^{2u}}{2}\right) = e^{-t}\times\left(\frac{e^{2t}}{2}t - \frac{1}{2}\left[\frac{e^{2u}}{2}\right]_0^t\right)$$

$$= \frac{e^{t}t}{2} - \frac{1}{2}e^{-t}\times\left(\frac{e^{2t}}{2} - \frac{1}{2}\right) = \frac{e^{t}t}{2} - \frac{1}{2}\frac{e^{t}}{2} + \frac{1}{2}\times\frac{1}{2}\times e^{-t}$$

$$= \frac{1}{2}e^{t}t - \frac{1}{4}e^{t} + \frac{1}{4}e^{-t}$$

b) $y''(t) - 2y'(t) + y(t) = e^{-(t-4)} H(t-4) \ , \ y(0) = y'(0) = 0$

$\Leftrightarrow \mathcal{L}\{y''(t) - 2y'(t) + y(t)\} = \mathcal{L}\{e^{-(t-4)} H(t-4)\}$

$\Leftrightarrow \mathcal{L}\{y''(t)\} - 2\mathcal{L}\{y'(t)\} + \mathcal{L}\{y(t)\} = e^{-4s} \dfrac{1}{s+1}$

$\Leftrightarrow s^2 F(s) - \underbrace{s\,y(0)}_{0} - \underbrace{y'(0)}_{0} - 2(s F(s) - \underbrace{y(0)}_{0}) + F(s) = \dfrac{e^{-4s}}{s+1}$

$\Leftrightarrow s^2 F(s) - 2s F(s) + F(s) = \dfrac{e^{-4s}}{s+1}$

$\Leftrightarrow F(s) \times (s^2 - 2s + 1) = \dfrac{e^{-4s}}{s+1} \Leftrightarrow F(s) = \dfrac{e^{-4s}}{(s+1)(s^2-2s+1)}$

$\Leftrightarrow F(s) = \dfrac{e^{-4s}}{(s+1)(s-1)^2}$

Teorema Heaviside

$y(t) = \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{e^{-4s} \times \dfrac{1}{(s+1)(s-1)^2}\right\} =$

$= f(t-4)\, H(t-4)$

$= \dfrac{1}{2} e^{(t-4)}(t-4) H(t-4) - \dfrac{1}{4} e^{(t-4)} H(t-4) + \dfrac{1}{4} e^{-(t-4)} H(t-4)$

5.

a) $Z\{x_{k+2}\} = \sum\limits_{k=0}^{+\infty} \dfrac{x_{k+2}}{z^k} = z^2 \sum\limits_{k=0}^{+\infty} \dfrac{x_{k+2}}{z^{k+2}}$

$= z^2 \sum\limits_{k=2}^{+\infty} \dfrac{x_k}{z^k} = z^2 \left(\sum\limits_{k=0}^{+\infty} \dfrac{x_k}{z^k} - \sum\limits_{k=0}^{2-1} \dfrac{x_k}{z^k} \right)$

$= z^2 \left(X(z) - \sum\limits_{k=0}^{1} \dfrac{x_k}{z^k} \right) = z^2 X(z) - \sum\limits_{k=0}^{1} z^{2-k} x_k$

$= z^2 X(z) - z^2 x_0 - z x_1 \ , \ |z| > R$

6)

b) $X(z) = \dfrac{z}{(z+1)(z+2)(z-1)^2}$

Seja $\{x_k\}_{k\geq 0} = Z^{-1}\{X(z)\}$

$\{x_k\}_{k\geq 0} = Z^{-1}\left\{\dfrac{z}{(z+1)(z+2)(z-1)^2}\right\}$

$= \dfrac{1}{2\pi i}\oint_{C(0,5)} \dfrac{z \times z^{k-1}}{(z+1)(z+2)(z-1)^2}\,dz = \dfrac{1}{2\pi i}\oint_{C(0,5)} \dfrac{z^k}{(z+1)(z+2)(z-1)^2}\,dz$

$= \operatorname{Res}_{z\to -1} g_z + \operatorname{Res}_{z\to -2} g_z + \operatorname{Res}_{z\to 1} g_z = \frac{1}{4}(-1)^k - \frac{1}{9}(-2)^k + \dfrac{6k-5}{36}$

C. Aux:

- $z_0 = -1$ é polo simples

$\operatorname{Res}_{z\to -1} g_z = \lim_{z\to -1}\left[\dfrac{z^k}{(z+2)(z-1)^2}\right] = \dfrac{(-1)^k}{4}$

- $z_0 = -2$ é polo simples

$\operatorname{Res}_{z\to -2} g_z = \lim_{z\to -2}\left[\dfrac{z^k}{(z+1)(z-1)^2}\right] = \dfrac{(-2)^k}{-1\times(-3)^2} = \dfrac{(-2)^k}{-9} = -\dfrac{(-2)^k}{9}$

- $z_0 = 1$ é polo duplo

$\operatorname{Res}_{z\to 1} g_z = \lim_{z\to 1}\dfrac{d}{dz}\left[\dfrac{z^k}{(z+1)(z+2)}\right] = \lim_{z\to 1}\left[\dfrac{k z^{k-1}\overbrace{(z+1)(z+2)}^{2\quad 3} - z^k(2z+3)}{\underbrace{((z+1)(z+2))^2}_{2\times 3}}\right]$

$= \dfrac{6k-5}{36}$

e) $x_{k+2} + 3x_{k+1} + 2x_k = k$, $x_0 = 0$, $x_1 = 0$, $X(z) = Z\{x_k\}$

$Z\{x_{k+2}\} + 3Z\{x_{k+1}\} + 2Z\{x_k\} = \left(-z\dfrac{d}{dz}\right)\left(\dfrac{z}{z-1}\right)$

$(\Rightarrow) z^2X(z) - \underbrace{x_0 z^2}_{0} - \underbrace{x_1 z}_{0} + 3(zX(z) - \underbrace{x_0 z}_{0}) + 2X(z) = -z\times\left(\dfrac{1(z-1)-z}{(z-1)^2}\right)$

$(\Rightarrow) z^2X(z) + 3zX(z) + 2X(z) = \dfrac{z}{(z-1)^2}$

$(\Rightarrow) X(z)\times(z^2+3z+2) = \dfrac{z}{(z-1)^2} \Leftrightarrow X(z) = \dfrac{z}{(z-1)^2(z+1)(z+2)}$

$(\Rightarrow) \{x_k\}_{k\geq 0} = Z^{-1}\{X(z)\} = \frac{1}{4}(-1)^k - \frac{1}{9}(-2)^k + \dfrac{6k-5}{36}$ //