Frequência 2022/2023

1. a)

$$f(z) = \frac{z}{(z+1)^{2023}(z-3i)} = \frac{z}{(z+1)^{2023}} \times \frac{1}{z-3i+1-1}$$

$$= \frac{z}{(z+1)^{2023}} \times \frac{1}{z+1-3i-1} = \frac{z}{(z+1)^{2023}} \times \frac{1}{z+1-(3i+1)} = \frac{z}{(z+1)^{2023}} \times \frac{1}{(3i+1)\times\left(\frac{z+1}{3i+1}-1\right)}$$

$$= -\frac{z}{(z+1)^{2023}(3i+1)} \times \frac{1}{1-\left(\frac{z+1}{3i+1}\right)} = -z \times \sum_{n=0}^{+\infty} \frac{(z+1)^{n-2023}}{(3i+1)^{n+1}} = z \times \sum_{n=0}^{+\infty} -\frac{(z+1)^{n-2023}}{(3i+1)^{n+1}}$$

$$= ((z+1)-1) \times \sum_{n=0}^{+\infty} -\frac{(z+1)^{n-2023}}{(3i+1)^{n+1}} = \sum_{n=0}^{+\infty} \frac{-(z+1)^{n-2023}(z+1)}{(3i+1)^{n+1}} + \sum_{n=0}^{+\infty} \frac{(z+1)^{n-2023}}{(3i+1)^{n+1}}$$

$$= \sum_{n=0}^{+\infty} -\frac{(z+1)^{n-2022}}{(3i+1)^{n+1}} + \sum_{n=0}^{+\infty} \frac{(z+1)^{n-2023}}{(3i+1)^{n+1}} \quad , \ 0 < |z+1| < \sqrt{10}$$

b)

$$\oint_{C(0,2)} \frac{z}{(z+1)^{2023}(z-3i)}\, dz = 2\pi i \times \operatorname*{Res}_{z \to -1} f = 2\pi i \times \left(-\frac{3i}{(3i+1)^{2023}}\right)$$

$$= \frac{6\pi}{(3i+1)^{2023}}$$

C.Aux:

$\operatorname*{Res}_{z\to -1} f = b_1 \Rightarrow$ Temos de ir à alínea a) e ir buscar o b_1, ou seja, o coeficiente da potência $(z+1)^{-1}$. Para isto temos de calcular o b_1 para cada uma das parcelas da expressão e somar para termos o b_1 da expressão toda:

- $n - 2022 = -1 \Leftrightarrow n = 2021$

$$b_1' = -\frac{1}{(3i+1)^{2022}}$$

- $n - 2023 = -1 \Leftrightarrow n = 2022$

$$b_1'' = \frac{1}{(3i+1)^{2023}}$$

$$b_1 = b_1' + b_1'' = -\frac{1}{(3i+1)^{2022}} + \frac{1}{(3i+1)^{2023}} = -\frac{(3i+1)}{(3i+1)^{2023}} + \frac{1}{(3i+1)^{2023}} = \frac{-3i-1+1}{(3i+1)^{2023}}$$

$$= \frac{-3i}{(3i+1)^{2023}}$$

2.

a) $X(z) = \dfrac{z}{(z+1)^2(z+4)(z-2)}$, $|z|>4$

Seja $\{x_k\}_k = Z^{-1}\{X(z)\}$:

$$\{x_k\}_k = Z^{-1}\left\{\frac{z}{(z+1)^2(z+4)(z-2)}\right\} = \frac{1}{2\pi i}\oint_{C(0,10)} \frac{z \times z^{k-1}}{(z+1)^2(z+4)(z-2)}\,dz$$

$$= \frac{1}{2\pi i}\oint_{C(0,10)} \frac{z^k}{(z+1)^2(z+4)(z-2)}\,dz = \operatorname*{Res}_{z\to -1} g_z + \operatorname*{Res}_{z\to -4} g_z + \operatorname*{Res}_{z\to 2} g_z$$

$$= \frac{k(-1)^k}{9} - \frac{(-4)^k}{54} + \frac{2^k}{54}$$

• C. Aux:

$z_0 = -1$ é pólo duplo (zero duplo em $q(z)$ e não-zero em $p(z)$)

• $$\operatorname*{Res}_{z\to -1} g_z = \frac{1}{1!}\lim_{z\to -1}\frac{d}{dz}\left[\frac{z^k}{(z+4)(z-2)}\right] = \lim_{z\to -1}\frac{d}{dz}\left[\frac{z^k}{z^2+2z-8}\right]$$

$$= \lim_{z\to -1}\left[\frac{(kz^{k-1})(z+4)(z-2) - z^k(2z+2)}{((z+4)(z-2))^2}\right]$$

$$= \frac{k(-1)^{k-1}\times(-9)}{81} = \frac{k(-1)^{k-1}\times(-1)\times 9}{81} = \frac{k(-1)^k}{9}$$

$z_0 = -4$ é pólo simples

• $$\operatorname*{Res}_{z\to -4} g_z = \lim_{z\to -4}\left[\frac{z^k}{(z+1)^2(z-2)}\right] = \frac{(-4)^k}{(-3)^2(-6)} = \frac{(-4)^k}{-54} = -\frac{(-4)^k}{54}$$

$z_0 = 2$ é pólo simples

• $$\operatorname*{Res}_{z\to 2} g_z = \lim_{z\to 2}\left[\frac{z^k}{(z+1)^2(z+4)}\right] = \frac{2^k}{54}$$

b) $x_{k+2} + 2x_{k+1} - 8x_k = k(-1)^k$, $x_0 = 0$, $x_1 = 0$

Seja $X(z) = Z\{x_k\}$:

$$Z\{x_{k+2} + 2x_{k+1} - 8x_k\} = Z\{k(-1)^k\} \Leftrightarrow Z\{x_{k+2}\} + 2Z\{x_{k+1}\} - 8Z\{x_k\} = Z\{k(-1)^k\}$$

$$\Leftrightarrow z^2X(z) - \underbrace{x_0 z^2}_{0} - \underbrace{x_1 z}_{0} + 2\times(zX(z) - x_0 z) - 8X(z) = (-1)\left(z\frac{d}{dz}\right)\left(\frac{z}{z+1}\right)$$

$$\Leftrightarrow z^2X(z) + 2zX(z) - 8X(z) = -1\times z\times\left(\frac{\cancel{z}+1-\cancel{z}}{(z+1)^2}\right)$$

$$\Leftrightarrow X(z)\times(z^2+2z-8) = -\frac{z}{(z+1)^2} \Leftrightarrow X(z) = -\frac{z}{(z+1)^2(z+4)(z-2)}$$

Assim, $\{x_k\}_k = Z^{-1}\{X(z)\} = -\dfrac{k(-1)^k}{9} + \dfrac{(-4)^k}{54} - \dfrac{2^k}{54}$ //

3.

a) $\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2(s+2)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2}\times\frac{1}{s+2}\right\}$ $\quad \frac{e^{3u}}{3}$

$= (e^t t * e^{-2t}) = \int_0^t e^{-2(t-u)}\times e^u\times u\,du$

$= \int_0^t e^{-2t+2u}\times e^u\times u\,du = e^{-2t}\int_0^t e^{2u}\times e^u\times u\,du$

$= e^{-2t}\int_0^t e^{3u}\times u\,du = e^{-2t}\times\left(\left[\frac{e^{3u}}{3}\times u\right]_0^t - \int_0^t \frac{e^{3u}}{3}\,du\right)$

$= e^{-2t}\times\left(\frac{e^{3t}}{3}\times t - \frac{1}{3}\left[\frac{e^{3u}}{3}\right]_0^t\right)$

$= \frac{e^t}{3}\times t - e^{-2t}\times\frac{1}{3}\times\left(\frac{e^{3t}}{3}-\frac{1}{3}\right) = \frac{e^t\times t}{3} - \frac{e^{-2t}\times e^{3t}}{9} + \frac{e^{-2t}}{9}$

$= \frac{e^t\times t}{9} - \frac{e^t}{9} + \frac{e^{-2t}}{9}$

b) $y''(t) - 2y'(t) + y(t) = e^{-2t}$, $y(0)=1$, $y'(0)=0$

Seja $F(s) = \mathcal{L}\{y(t)\}$:

$\mathcal{L}\{y''(t) - 2y'(t) + y(t)\} = \mathcal{L}\{e^{-2t}\}$

$\Leftrightarrow \mathcal{L}\{y''(t)\} - 2\mathcal{L}\{y'(t)\} + \mathcal{L}\{y(t)\} = \frac{1}{s+2}$

$\Leftrightarrow s^2F(s) - s\,y(0) - y'(0) - 2\times(sF(s) - y(0)) + F(s) = \frac{1}{s+2}$

$\Leftrightarrow s^2F(s) - s - 2sF(s) + 2 + F(s) = \frac{1}{s+2}$

$\Leftrightarrow s^2F(s) - 2sF(s) + F(s) = \frac{1}{s+2} + s - 2 \Leftrightarrow$

$\Leftrightarrow F(s)\times(s^2-2s+1) = \frac{1}{s+2} + s - 2$

$\Leftrightarrow F(s) = \frac{1}{(s+2)(s-1)^2} + \frac{s}{(s-1)^2} - \frac{2}{(s-1)^2}$

$y(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{e^t t}{9} - \frac{e^t}{9} + \frac{e^{-2t}}{9} + \mathcal{L}^{-1}\left\{\frac{s}{(s-1)^2}\right\} - 2\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2}\right\}$

$= \frac{e^t t}{9} - \frac{e^t}{9} + \frac{e^{-2t}}{9} + e^t\times t + e^t - 2t\,e^t = \frac{-6e^t t + 8e^t + e^{-2t}}{9}$

C.Aux

• $\frac{s}{(s-1)^2} = \frac{A_1}{(s-1)^2} + \frac{A_2}{(s-1)} \Rightarrow s = A_1 + A_2(s-1) \Leftrightarrow s = 1 + A_2(s-1) \Leftrightarrow s = 1 + A_2 s - A_2 \Rightarrow A_2 = 1$

$= \frac{1}{(s-1)^2} + \frac{1}{(s-1)}$ • $\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2}\right\} = F(s-1) = e^t\times t$

4. $w = 2$ rad/s , $[0, \pi[$ $v(t) = t$ $w = \frac{2\pi}{T} \Leftrightarrow T = \frac{2\pi}{2} = \pi$

a) $\sum_{m=-\infty}^{+\infty} c_m e^{im2t}$

• $m = 0$:

$$c_0 = \frac{1}{\pi} \int_0^{\pi} t e^{-im2t} = \frac{1}{\pi} \int_0^{\pi} t \, dt = \frac{1}{\pi} \left[\frac{t^2}{2} \right]_0^{\pi} =$$

$$= \frac{1}{\pi} \times \left(\frac{\pi}{2}^2 - 0 \right) = \frac{\pi}{2}$$

• $m \neq 0$:

$$c_m = \frac{1}{\pi} \int_0^{\pi} t e^{-im2t} dt = \frac{1}{\pi} \times \frac{1}{-im2} \int_0^{\pi} t \times (-im2) e^{-im2t} dt$$

$$= -\frac{1}{im2\pi} \times \left(\left[e^{-im2t} \times t \right]_0^{\pi} - \int_0^{\pi} e^{-im2t} dt \right)$$

$$= -\frac{1}{im2\pi} \times \left(\left(e^{-im2\pi} \times \pi - 0 \right) - \frac{1}{-im2} \int_0^{\pi} (-im2) e^{-im2t} dt \right)$$

$$= -\frac{1}{im2\pi} \times \left(e^{-im2\pi} \pi + \frac{1}{im2} \left[e^{-im2t} \right]_0^{\pi} \right)$$

$$= -\frac{1}{im2\pi} \times \left(e^{-im2\pi} \times \pi + \frac{1}{im2} \left[e^{-im2\pi} - 1 \right] \right)$$

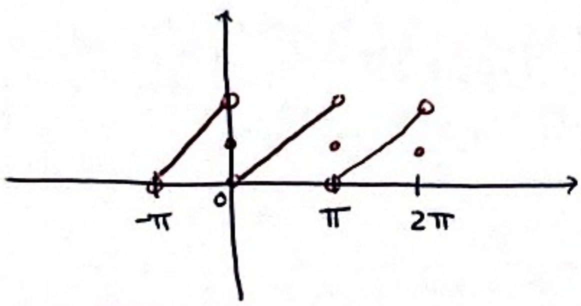
$$= -\frac{1}{im2\pi} \times \left(-e^{-im2\pi} \pi + \frac{e^{-im2\pi}}{im2} - \frac{1}{im2} \right)$$

$$= -\frac{e^{-im2\pi} \pi}{im2\pi} - \frac{e^{-im2\pi}}{im2 \times (im2\pi)} + \frac{1}{im2(im2\pi)}$$

Série:

$$\frac{\pi}{2} + \sum_{m=-\infty}^{+\infty} \left(-\frac{e^{-im2\pi} \times \pi}{im2\pi} - \frac{e^{-im2\pi}}{im2 \times (im2\pi)} + \frac{1}{im2(im2\pi)} \right) e^{-im2t}$$

b)



c) $\tilde{v}(t) \to$ Função soma

$$\tilde{v}(t) = \begin{cases} \frac{\pi}{2}, & t = 2\pi \vee t = \pi \\ t-\pi, & \pi < t < 2\pi \end{cases}$$

5. $f(t) = e^{-2|t|} H(t)$

a) $$F\{e^{-2|t|}H(t)\} = \int_{-\infty}^{+\infty} e^{-2|t|}H(t)e^{-i\omega t}\,dt \qquad H(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

$$= \int_0^{+\infty} e^{-2t}e^{-i\omega t}\,dt = \int_0^{+\infty} e^{(-2-i\omega)t}\,dt$$

$$= \lim_{A\to+\infty} \int_0^A e^{(-2-i\omega)t} = \lim_{A\to+\infty} \frac{1}{-2-i\omega}\int_0^A (-2-i\omega)\,e^{(-2-i\omega)t}\,dt$$

$$= \lim_{A\to+\infty} \frac{1}{-2-i\omega}\left[e^{(-2-i\omega)t}\right]_0^A$$

$$= \lim_{A\to+\infty} \frac{1}{-2-i\omega} \times \left(e^{-A(2+i\omega)} - 1\right) = \frac{1}{2+i\omega}$$

C.Aux:

$$\lim_{A\to+\infty} \left|e^{-A(2+i\omega)}\right| = \lim_{A\to+\infty} \left|e^{-A\times 2} \times \overset{\to 1}{e^{-i\omega A}}\right| = \lim_{A\to+\infty} e^{-2A} = 0$$

b) $$F\left\{\frac{2023}{2i-t}\right\} = 2023 \times F\left\{\frac{1}{2i-t}\right\} = 2023 \times F\left\{\frac{1}{2i+i^2t}\right\}$$

$$= \frac{2023}{i} \times \underbrace{F\left\{\frac{1}{2+it}\right\}}_{\text{Propriedade 7)}} = \frac{2023}{i} \times 2\pi f(-\omega)$$

$$= \frac{2023}{i} \times 2\pi \times e^{-2|-\omega|}H(-\omega) = \frac{2023}{i} \times 2\pi \times e^{-2|\omega|}H(\omega),\ \omega \in \mathbb{R}$$