Numerical Approach to Water Flowing Systems

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Abstract

In this study a model for water-flowing systems on uneven slope landscapes has been designed and used for solving not only theoretical examples but also a real flood occurred last summer in the municipality of Esplugues del Francolí, Spain. Firstly, a one-dimensional case has been designed and applied over a mathematical function, which further on has been extended in two dimensions. Finally, a three-dimensional model of the desired landscape has been rendered and a storm has been applied over it in order to study water flow in real terrain. To do so, diffusion and convection differential equations have been normalized and the non-linear Lax-Wendroff method has been used to solve the problem numerically.

I. INTRODUCTION

Understanding how water flows in landscapes is key in several fields of study: not only issues involving flood prevention can be analyzed, but also water supplies, river formation and others.

Water flow on real landscapes depends on a large number of geological factors. Despite this, it can be simplified into a model of a liquid flowing on a surface and therefore, dependency on water flow lies only on the local terrain slope.

A scientific computer model for this kind of water flowing systems is developed in this project, which further on is applied on a real case: a heavy rain in Esplugues del Francolí, Spain (a potentially flooded area).

II. MATHEMATICAL MODELING

Water-flowing systems may be regarded mathematically as a volume function $V(\vec{x},t)$ depending on space coordinates and time. Convection-diffusion differential equation might be used to deal with these types of systems. Let's consider first the one-dimension scenario, where the volume has $(space)^2$ dimensions. Firstly, a diffusion term may be defined as:

$$\frac{\partial V_{dif.}}{\partial t} = \frac{\partial}{\partial x}\left(\mathbf{D}\frac{\partial V}{\partial x}\right) = \frac{\partial}{\partial x}\left(\alpha f_1\left(\frac{\partial h}{\partial x}\right)\frac{\partial V}{\partial x}\right). \quad (1)$$

Term $\alpha f_1(\frac{\partial h}{\partial x})$ is the diffusivity $\mathbf{D}$, which is the term that will define how influent the diffusion at each point is. Let's set $f_1(\frac{\partial h}{\partial x})$ as a non-linear slope and point-depending equation such as

$$f_1\left(\beta, \frac{\partial h}{\partial x}\right) = e^{-\beta|\frac{\partial h}{\partial x}|}, \quad (2)$$

which establishes the relation $\mathbf{D}(\alpha, \beta, \frac{\partial h}{\partial x})$. α modulates the total function's height and β the slope influence in diffusion decay. Diffusivity over α as function of slope can be seen for three different values of β in (Fig. 1).

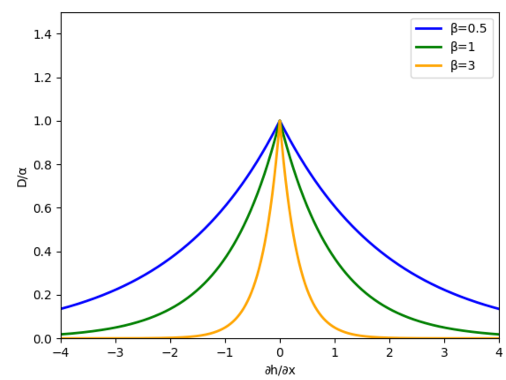

Fig.1: *$\boldsymbol{D}/\alpha$, as function of the local slope, for* $\beta = 0.5$, 1, 3.

Diffusivity is considered to decay exponentially at higher terrain slopes since water flow by convection will become the most important term as will be seen further on. Hence, liquid diffusivity will be more influent in flat terrain. These diffusivity function properties may be modulated further on via α and β. Setting

$$f_{dif.}\left(\alpha, \beta, V, \frac{\partial h}{\partial x}\right) = \alpha f_1\left(\beta, \frac{\partial h}{\partial x}\right)\frac{\partial V}{\partial x}, \quad (3)$$

equation (1) can be displayed as

$$\frac{\partial V_{dif.}}{\partial t} - \frac{\partial (f_{dif.}(\alpha, \beta, V, \frac{\partial h}{\partial x}))}{\partial x} = 0. \quad (4)$$

Secondly, a convection term may be defined in one dimension as

$$\frac{\partial V_{conv.}}{\partial t} = \frac{\partial}{\partial x}(\mathbf{c}V) = \frac{\partial}{\partial x}\left(\gamma\frac{\partial h}{\partial x}f_2(V)V\right), \quad (5)$$

where $\mathbf{c} = \gamma \frac{\partial h}{\partial x} f_2(V)$ is the drift velocity through which the volume is convected, proportional in a γ factor to the local height derivative and a non-linear volume function $f(V)$. For example, let

$$f_2(\delta, \xi, V) = (1 - \xi e^{-\delta V}), \tag{6}$$

which gives the relation $\mathbf{c}(\gamma, \delta, \xi, \frac{\partial h}{\partial x}, V)$. Again, γ modulates the function's height, ξ the relation between minimum and maximum convection, and δ the volume influence in maximum convection convergence. $\mathbf{c}/\alpha$ as function of volume can be seen for different values of ξ and δ in (Fig.2).

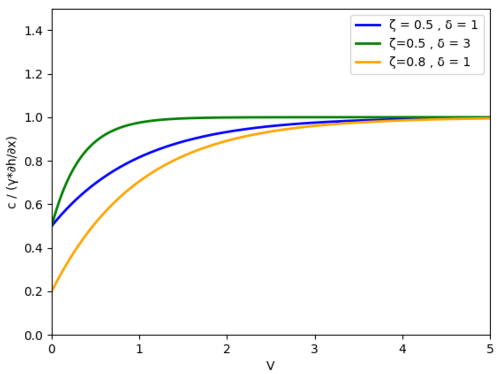

Fig.2: *$c/(\gamma \frac{\partial h}{\partial x})$ as volume function for different ξ and δ values.*

Higher volumes will be more affected by convection than lower ones, with a maximum limit drift speed. This drift speed can be modulated through the mentioned variables. As can be seen in the previous plot, ξ modulates the relation between the minimum value of drift speed and its maximum, δ modulates how fast this change is, and γ modulates the maximum convection convergence value. It is important to highlight that convection is also proportional to the height derivative, which will also establish the drift speed direction through its sign (positive or negative). By defining

$$f_{conv.}\left(\gamma, \delta, \xi, V, \frac{\partial h}{\partial x}\right) = \gamma \frac{\partial h}{\partial x} f_2(V) V, \tag{7}$$

the differential equation ends up to be

$$\frac{\partial V_{conv.}}{\partial t} - \frac{\partial f_{conv.}(\delta, \gamma, \xi, V, \frac{\partial h}{\partial x})}{\partial x} = 0. \tag{8}$$

Finally, an accretion term can be included in the differential equation. It will have the role as a volume generator, as if it was rain:

$$\frac{\partial V_{acr.}}{\partial t} = \epsilon \;\; f_{acr.}(x, t). \tag{9}$$

The whole volume variation in time will be the sum of the three terms mentioned above:

$$\frac{\partial V}{\partial t} = \frac{\partial V_{dif.}}{\partial t} + \frac{\partial V_{conv..}}{\partial t} + \frac{\partial V_{acr.}}{\partial t}. \tag{10}$$

Let's move on to system normalization. Constant values dimensions are:

$$dim[\alpha] = (space)^2(time)^{-1} \quad , \quad dim[\beta] = \varnothing$$
$$dim[\gamma] = (space)(time)^{-1} \quad , \quad dim[\delta] = (space)^{-2}$$
$$dim[\epsilon] = (space)^2(time)^{-1} \quad , \quad dim[\xi] = \varnothing.$$

It is important to highlight that ϵ and δ values are in area units in the 1-dimensional case, not in volumetric ones. Normalized variables (*) in a l_x x-ranged space can be expressed as:

$$x^* = \frac{x}{l_x} \quad , \quad t^* = \frac{\gamma^2}{\alpha} t$$
$$h^* = \frac{\gamma}{\alpha} h \quad , \quad V^* = \delta V. \tag{11}$$

Now, normalized equations needed to be solved are:

$$\frac{\partial V^*_{dif.}}{\partial t^*} = \frac{\alpha^2}{\gamma^2 l_x^2} \frac{\partial}{\partial x^*} \left(f_1 \left(\beta, \frac{\alpha}{l_x \gamma} \frac{\partial h^*}{\partial x^*} \right) \frac{\partial V^*}{\partial x^*} \right), \tag{12}$$

$$\frac{\partial V^*_{conv.}}{\partial t^*} = \frac{\alpha^2}{\gamma^2 {l_x}^2} \frac{\partial}{\partial x^*} \left(\frac{\partial h^*}{\partial x^*} f_2 \left(\frac{V^*}{\delta} \right) V^* \right), \tag{13}$$

$$\frac{\partial V^*_{acr.}}{\partial t^*} = \frac{\alpha \delta \epsilon}{\gamma^2} \;\; f_{acr.} \left(l_x x^*, \frac{\alpha}{\gamma^2} t^* \right). \tag{14}$$

Equation (10) is the same in normalized variables t^* and V_i^*, therefore, differential equations needed to be solved are (12), (13) and (14). Equation (14) will be solved by Euler's explicit method. Equations (13) and (14) are a little more tricky: they are hyperbolic partial differential equations, and through the process, water must be conserved by these two terms. Euler explicit or implicit methods can't be used as they neither show volume conservation nor high accuracy.

Let's introduce the Lax-Wendroff method [1], which appears to be most suitable. Not only it is a conservative method that will enable volume conservation by diffusion and convection, but it also gives a second-order approximation which proves to be more accurate.

Hence, this problem will be entirely solved through Lax-Wendroff conservative method as it fits perfectly to the problem needs. The method is developed in the following section.

III. LAX-WENDROFF CONSERVATIVE METHOD

As said, diffusion and convection terms will be calculated through the Lax-Wendroff conservative method, as it enables to calculate non-linear equations (12) and (13) in a volume-conservative way. In particular, using Richtmyer two-step Lax–Wendroff method [3] will avoid Jacobian calculations at solving a second-order approximation of the problem.

Firstly, volumes at intermediate adjacent spaces at an intermediate time location will be calculated through

$$V_{x+1/2}^{t+1/2} = \frac{1}{2}(V_{x+1}^t + V_x^t) - \frac{\Delta t}{2\Delta x}(f_i(V_{x+1}^t) - f_i(V_x^t)), \quad (15)$$

$$V_{x-1/2}^{t+1/2} = \frac{1}{2}(V_x^t + V_{x-1}^t) - \frac{\Delta t}{2\Delta x}(f_i(V_x^t) - f_i(V_{x-1}^t)), \quad (16)$$

where f_i is $f_{conv.}$ or $f_{dif.}$, depending on the differential equation dealing with. Finally, the updated volume is calculated as:

$$V_x^{t+1} = V_x^t - \frac{\Delta t}{\Delta x}(f_i(V_{x+1/2}^{t+1/2}) - f_i(V_{x-1/2}^{t+1/2})). \quad (17)$$

Therefore, a point's update can be calculated in second order through its adjacent ones. Frontier points will have a different evaluation, since they only have one adjacent point. In this case, their updates have been considered to be equal to the adjacent ones. Eventually this would lead to significant errors, but since volume calculation will mainly be done in intermediate positions, it won't have a big influence.

Let's try a one-dimensional case example by setting space dimensions in hectometers (hm) and time dimensions in hours (h). Let a height function be

$$h(x) = -\frac{1}{2}\sin\left(\frac{9x}{l_x}\pi\right) + \frac{1}{2}, \quad (18)$$

which visual interpretation can be seen in (Fig.3).

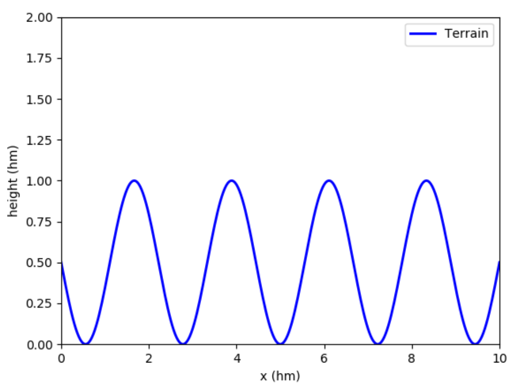

Fig.3: *Example landscape height as function of x/l_x.*

Again, it is important to remember that in one dimension $dim[V] = (space)^2$: volume has area units, so δ and ϵ constants are affected by so. Numerical values to the constants, such as:

$$\alpha = 2\cdot 10^{-3}\ \ hm^2h^{-1} \quad , \quad \beta = 1$$

$$\gamma = 0.5\ \ hm\ h^{-1} \quad , \quad \delta = 1\ \ hm^{-3}$$

$$\epsilon = 5\cdot 10^{-2}\ \ hm^2h^{-1} \quad , \quad \xi = 0.5$$

$$l_x = 10\ \ hm \quad .$$

are given. Accretion function will correspond to the following function:

$$f_{acr.} = \sin\left(\frac{x}{l_x}\pi\right)\ \sin\left(\frac{t}{t_{max.}}\pi\right) \quad , \quad if \quad t \le t_{max},$$

$$f_{acr.} = 0 \quad , \quad if \quad t > t_{max},$$

which corresponds to a sinusoidal rain, both in space and time, with maximum values at $x = l_x/2$ and $t = t_{max}/2$. Set a partition $\Delta t = 5\cdot 10^{-4}\ \ h$, $\Delta x = l_x/301$ hm, and total precipitation time $t_{max} = 1\ h$, volume as a function of x at different instances of time is represented in (Fig.4).

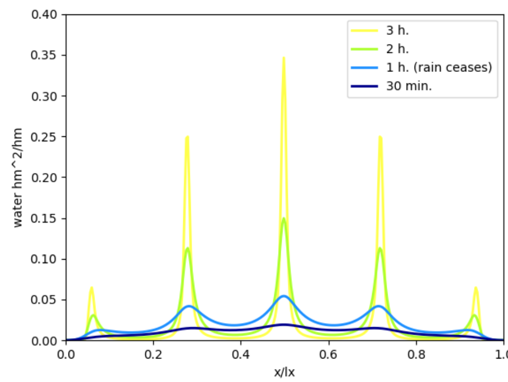

Fig.4: *Area liquid as a function of space at times 30 min,1h,2h and 3h.*

As seen, water flows correctly towards minimums. It is interesting to watch the animation *Ani_1D.mp4* attached.

Let's study Lax-Wendroff accuracy from this previus case. Since landscape shape suggests water flow inwards the system, after $t = 1h$ and rain stops, water volume should be constant. Considering volume at $t = 1h$ as reference volume, relative error accumulating in the volume as a function of time can be seen in (Fig.5).

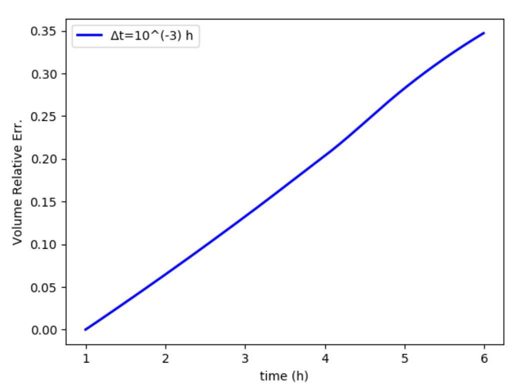

Fig.5: *Relative error in volume as function of time after rain ceases, for a time partition $\Delta t = 10^{-3}h$.*

Checking (Fig.5) clearly suggests that despite the high accuracy Lax-Wendroff method has, it still accumulates a small error. This error is relatively small in our case (about 0.35% in 5 hours), but in large-exposure systems it might diverge, making it impossible to solve them with normal spatial and time partitions. In fact, an accurate convergence criteria for this non-linear case is complicated to show in an explicit way for Δx and Δt, as it will depend on each point's f_1 or f_2 spatial derivatives and constants.

IV. EXTENSION TO THE 2-DIMENSIONAL CASE

Extending our mathematical model to a two-dimensional case is simple. Now volume will have the $(space)^3$ dimensions it should have, and constant δ and ϵ dimensions are affected by so: $dim[\delta] = (space)^{-3}$, $dim[\epsilon] = (space)^3 (time)^{-1}$. Terms of convection, diffusion and accretion are modified as:

$$\frac{\partial V_{dif.}}{\partial t} = \frac{\partial}{\partial x}\left(\alpha f_1\left(\frac{\partial h}{\partial x}\right)\frac{\partial V}{\partial x}\right) + \frac{\partial}{\partial y}\left(\alpha f_1\left(\frac{\partial h}{\partial y}\right)\frac{\partial V}{\partial y}\right) \quad (19)$$

$$\frac{\partial V_{conv.}}{\partial t} = \frac{\partial}{\partial x}\left(\gamma \frac{\partial h}{\partial x} f_2(V) V\right) + \frac{\partial}{\partial y}\left(\gamma \frac{\partial h}{\partial y} f_2(V) V\right) \quad (20)$$

$$\frac{\partial V_{acr.}}{\partial t} = f_{acr.}(x, y, t) \quad (21)$$

Normalization is done in a similar way as in the one-dimensional case. Considering now l_y as the range on the y dimension, and therefore $y^* = \frac{y}{l_y}$. Equation (13) is satisfied, but now:

$$\frac{\partial V^*_{dif.}}{\partial t^*} = \nabla^*\left(\frac{\alpha^2}{\gamma^2 l_i^2} f_1\left(\beta, \frac{\alpha}{l_i \gamma}\frac{\partial h^*}{\partial x_i^*}\right)\frac{\partial V^*}{\partial x_i^*}\right) \quad (22)$$

$$\frac{\partial V^*_{conv.}}{\partial t^*} = \nabla^*\left(\frac{\alpha^2}{\gamma^2 {l_i}^2}\frac{\partial h^*}{\partial x_i^*} f_2\left(\frac{V^*}{\delta}\right) V^*\right) \quad (23)$$

$$\frac{\partial V^*_{acr.}}{\partial t^*} = \frac{\alpha\delta\epsilon}{\gamma^2}\ f_{acr.}(l_x x^*, l_y y^*, \frac{\alpha}{\gamma^2} t^*) \quad (24)$$

Where ∇^* is the nabla operator in normalized variables x^* and y^*, and ${x_i}^*$ is the normalized dimension (x^* or y^*) in which ∇^* is applied. Setting the same numerical values as in the 1D-case for α , ξ , l_x, but now:

$$\gamma = 1\ \ hm\ \ h^{-1} \quad , \quad \epsilon = 10^{-5}\ \ hm^3 h^{-1}$$

$$\beta = 5 \quad , \quad \delta = 4 \cdot 10^{-4}\ \ hm^{-3}$$

$$l_x = 10\ \ hm \quad , \quad l_y = 10\ \ hm$$

Let $\Delta x = 0.1\ hm$, $\Delta y = 0.1\ hm$, $\Delta t = 0.025\ h$. ϵ value corresponds to a maximum of $10^4 \frac{L}{h}$ per point, which with the conditions set, each has an area of 100 m^2. Therefore, ϵ value corresponds to $100 \frac{L}{h \cdot m^3}$. Let now the accretion function in space and time be:

$$f_{acr.} = \sin\left(\frac{x}{l_x}\pi\right)\sin\left(\frac{y}{l_y}\pi\right)\sin\left(\frac{t}{t_{max.}}\pi\right), \quad if \quad t \leq t_{max}$$

$$f_{acr.} = 0 \qquad\qquad , \quad if \quad t > t_{max}$$

Since the sine terms will reach their maximum value of 1 at $t = t_{max.}/2$, $x = l_x/2$ and $y = l_y/2$, this means that ϵ value corresponds to the maximum instantaneous precipitation, which is in accordance with a heavy rain. Defining $t_{max} = 3h$ and setting a height function as in equation (26), plotted in (Fig.6)

$$h(x, y) = \frac{1}{2}\sin\left(\frac{3x}{l_x}\pi\right)\sin\left(\frac{3y}{l_y}\pi\right) + \frac{1}{2} \quad (25)$$

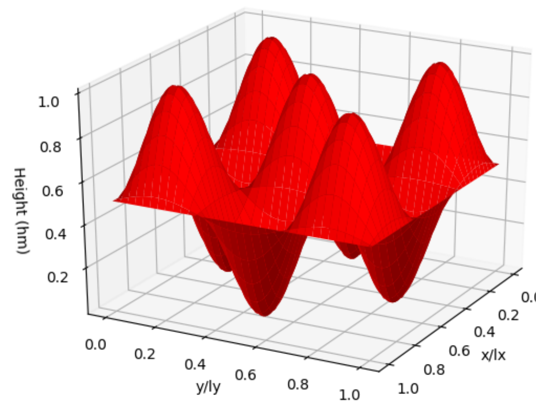

Fig.6: *2-dimensional example terrain corresponding to equation (25). There are four local minimums and five local maximums.*

The system's evolution can be seen in (Fig.7). It's of special interest checking the animation *Ani_Surface.mp4* attached.

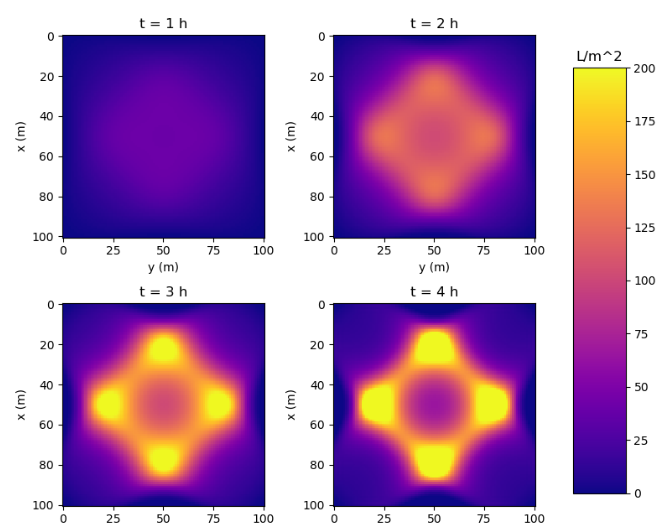

Fig.7: *Flow evolution of the 2-dimensional example system as seen from above. Times include 1, 2, 3 and 4 hours.*

V. REAL FLOOD IN ESPLUGUES DEL FRANCOLÍ

Last summer, Esplugues del Francolí municipality (Tarragona, Spain) was flooded due to summer rains. Since the two-dimensional model for water-flow is already working, it will be applied on the real landscape around this town to simulate a flood.

The map via satellite image that will be used for the simulation can be seen in (Fig.8), where Esplugues del Francolí municipality has been marked in red.

Fig.8: *Satellite image of the area around Esplugues. The municipality location has been indicated in red.*

Using the program QGIS3, and topological data extracted from the National Spanish Geographical Institute [3], a grey scale height image has been acquired. The image has been simplified to a 144 x 258 height matrix to shorten compiling time, and a 3-dimensional topological map has been rendered in python (Fig.9).

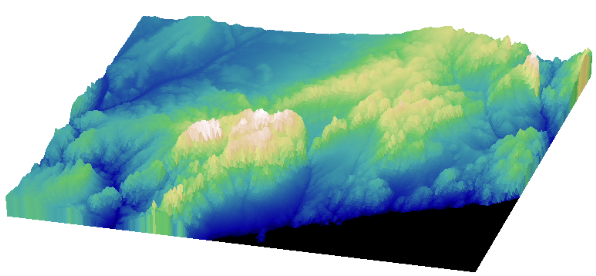

Fig.9: *Rendered 3-dimensional landscape around Esplugues*

This landscape can be visualized in three dimensions by executing the attached code *Landscape.py*. A convection-diffusion-accretion function will be applied over the landscape in order to evaluate rain effects in the area.

In this case $l_x = 76\ km$, $l_y = 136\ km$, and constants values are set to

$$\alpha = 2 \cdot 10^{-5}\ km^2h^{-1} \quad , \quad \beta = 10\ ,$$

$$\gamma = 0.2\ km\ h^{-1} \quad , \quad \delta = 2 \cdot 10^{5}\ km^{-3}\ ,$$

$$\epsilon = 2.78 \cdot 10^{-5}\ km^3h^{-1} \quad , \quad \xi = 0.5\ ,$$

so they are suitable for the problem. With the conditions set, $\Delta x = \Delta y = 0.528 km$, ϵ value corresponds to a maximum instantaneous precipitation of $100 \frac{L}{h \cdot m^2}$.

A more elaborated accretion-moving precipitation has been added to the system. It simulates 3-hour heavy rain approaching from the sea in diagonal, which again is a sinusoidal function both in space and time. For further details of this storm, the code may be checked.

Sea area has been considered flat terrain for simplicity. Water concentration at those points is so high that the diffusivity included would break down the system. Therefore, water concentration included in sea areas must be ignored.

Final conditions after 5 hours, can be seen in (Fig.10). The full animation for this rain can be seen in the *Ani_Esplugues.mp4* document joined.

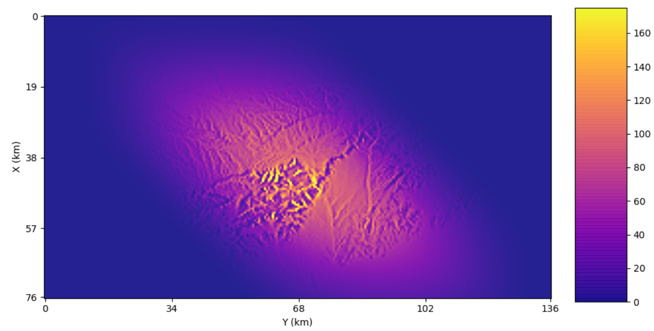

Fig.10: *Water concentration in L/m^2 after 5 hours rain has begun.*

As can be seen, after several hours water is mainly retained in low slope terrains as water falling over high terrain slopes tend to convect. Around Prades mountains, the ones in white in (Fig.8), water coming from the top is accumulated.

No solid scientific information has been acquired from the flood, but the important fact is that the flowing model does work out in real landscapes.

When looking at $\gamma = 0.2 km/s$, it can be seen that liquid convection in only 6 hours time won't be very significant. As seen in the simulation, water only develops the first steps of its trajectory. No more time than 6 hours has been calculated since numerical error gains significant influence after this time. This simulation has been done with 144 X dimension partitions, 258 Y dimension partitions, and 240-time iterations: compilation time for this kind of systems is huge. If more computer performance was available, a finer terrain partition could be done and hence a more accurate simulation could be rendered. As done, each point has an area of $0.2782\ km^2$, so terrain slope is hard to detect at small scales. The approach to real terrains could be significantly improved, but it is quite accurate for the first system hours.

VI . CONCLUSIONS

In this project, mathematical modeling for water flowing systems has been developed. A model involving a non-linear convection-diffusion-accretion differential equation has been suggested, which further on has been numerically solved through the Richtmyer two-stem Lax-Wendroff conservative method. This model has been firstly applied to a one-dimensional sinusoidal-height case scenario, and secondly, it has been expanded into two dimensions. Finally, the model has been applied over the real landscape of Esplugues del Francolí (Tarragona, Spain), to understand water-flow in the area, since it is a potentially hazardous flooded area.

The model for water-flowing systems on uneven terrain slopes has been proven to work successfully. With some improvements that could be done in the convection and diffusion constants **D** and **c** as a function of height derivative and volume, and more exact values of the constants used in the problem, the model would represent an accurate approximation of a real system. Other geological factors such as soil permeability would also improve the method.

The non-linear Lax-Wendroff conservative method has proved to be accurate enough in our range of study. As mentioned above, the Euler explicit and implicit methods were not useful in the studied case as they were not always convergent and did not show the required accuracy for this type of simulation.

While looking at the animation of the rain around Esplugues, high water concentrations are seen around the municipality after 3 hours the rain ceases. Therefore, the inundations taken place last summer make perfect sense with the animations presented. A better study of the flood could be done with a more accurate terrain height map (which would be thanks to a finer terrain partition): terrain slopes would be detected with more ease, and water would be more fluent. Flooded areas could possibly be detected as well as river formation and evolution.

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