Analisi della risposta libera (caso TC), autovalori multipli

In[∘]:=
```
ClearAll["Global`*"]
```

In[∘]:=
```
A = {{-4, -2, 2, -4}, {-1, -3, 1, -4}, {-1, -1, -1, -2}, {1, 1, -1, 1}}
```

Out[∘]=
```
{{-4, -2, 2, -4}, {-1, -3, 1, -4}, {-1, -1, -1, -2}, {1, 1, -1, 1}}
```

Determiniamo lo spettro di A

In[∘]:=
```
λ = Eigenvalues[A]
```

Out[∘]=
```
{-2, -2, -2, -1}
```

Posso servirmi del polinomio minimo per capire se la matrice e' diagonalizzabile o meno

In[∘]:=
```
MatrixMinimalPolynomial[a_List?MatrixQ, x_] :=
 Module[{i, n = 1, qu = {}, mnm = {Flatten[IdentityMatrix[Length[a]]]}},
  While[Length[qu] == 0, AppendTo[mnm, Flatten[MatrixPower[a, n]]];
   qu = NullSpace[Transpose[mnm]];
   n++];
  First[qu].Table[x^i, {i, 0, n - 1}]]
```

In[∘]:=
```
Factor[MatrixMinimalPolynomial[A, x]]
```

Out[∘]=

$$(1 + x)\ (2 + x)^2$$

In[∘]:=
```
Factor[CharacteristicPolynomial[A, x]]
```

Out[∘]=

$$(1 + x)\ (2 + x)^3$$

Poiche' nel polinomio minimo (x+2), legato all'autovalore multiplo non ha "esponente" unitario, deduco che la matrice NON E' diagonalizzabile. Devo quindi ricorrere alla forma di Jordan

In[∘]:=
```
{T, Λ} = JordanDecomposition[A]
```

Out[∘]=
```
{{{1, 2, -3, 0}, {0, -1, 0, -2}, {1, 1, 0, 0}, {0, 0, 1, 1}},
 {{-2, 0, 0, 0}, {0, -2, 1, 0}, {0, 0, -2, 0}, {0, 0, 0, -1}}}
```

In[∘]:=
```
Λ // MatrixForm
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Per "vedere" i modi naturali, mi conviene calcolare l'esponenziale di matrice della forma canonica di Jordan di A

```
In[•]:= Simplify[MatrixExp[Λ t]] // MatrixForm
```

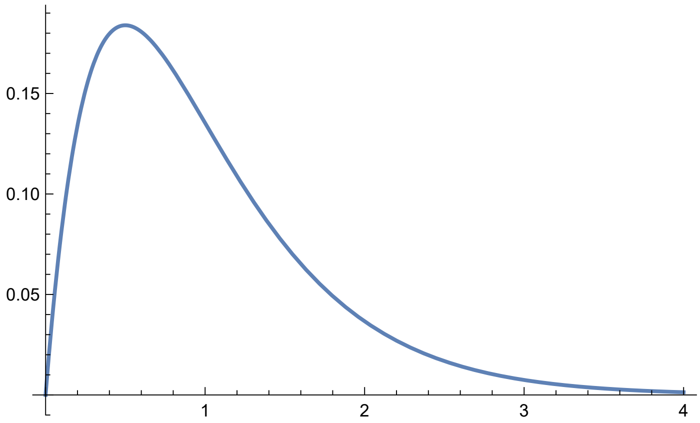
Out[•]//MatrixForm=

$$\begin{pmatrix} e^{-2t} & 0 & 0 & 0 \\ 0 & e^{-2t} & e^{-2t} t & 0 \\ 0 & 0 & e^{-2t} & 0 \\ 0 & 0 & 0 & e^{-t} \end{pmatrix}$$

Analizziamo il grafico del modo polinomial-esponenziale

```
In[•]:= Plot[t Exp[-2 t], {t, 0, 4}, PlotRange → All]
```

Out[•]=




```
In[•]:= Λ // MatrixForm
```

Out[•]//MatrixForm=

$$\begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

```
In[•]:= T // MatrixForm
```

Out[•]//MatrixForm=

$$\begin{pmatrix} 1 & 2 & -3 & 0 \\ 0 & -1 & 0 & -2 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

Mi calcolo la risposta libera (inserisco x0)

```
In[•]:= x0 = {{1}, {1}, {0}, {0}}
```

```
Out[•]=
{{1}, {1}, {0}, {0}}
```

```
In[•]:= z0 = Inverse[T].x0
```

```
Out[•]=
{{5}, {-5}, {-2}, {2}}
```

```
In[•]:= x1[t_] := T.MatrixExp[Λ t].z0
```

RIsposta libera nell’uscita

In[∘]:= `C1 = {0, 1, 0, 0}`

Out[∘]=

`{0, 1, 0, 0}`

In[∘]:= `y1[t_] := C1.x1[t]`

In[∘]:= `x1[t] // MatrixForm`

Out[∘]//MatrixForm=

$$\begin{pmatrix} -5\,e^{-2t} - 2\left(-3\,e^{-2t} + 2\,e^{-2t}\,t\right) \\ 5\,e^{-2t} - 4\,e^{-t} + 2\,e^{-2t}\,t \\ -2\,e^{-2t}\,t \\ -2\,e^{-2t} + 2\,e^{-t} \end{pmatrix}$$

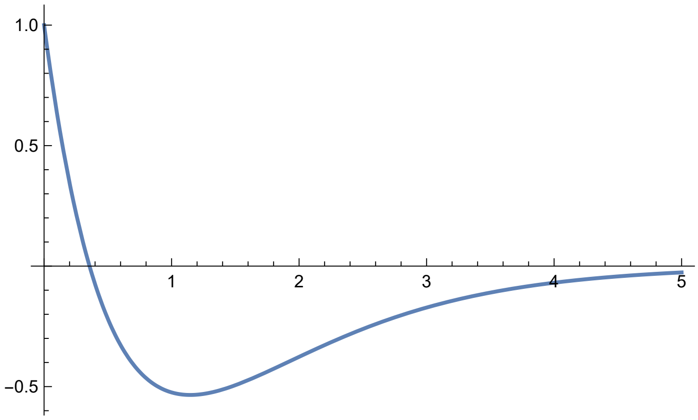
In[∘]:= `y1[t]`

Out[∘]=

$$\left\{5\,e^{-2t} - 4\,e^{-t} + 2\,e^{-2t}\,t\right\}$$

In[∘]:= `Plot[y1[t], {t, 0, 5}, PlotRange → All]`

Out[∘]=



In[∘]:= `T // MatrixForm`

Out[∘]//MatrixForm=

$$\begin{pmatrix} 1 & 2 & -3 & 0 \\ 0 & -1 & 0 & -2 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

In[∘]:= `MatrixExp[Λ t] // MatrixForm`

Out[∘]//MatrixForm=

$$\begin{pmatrix} e^{-2t} & 0 & 0 & 0 \\ 0 & e^{-2t} & e^{-2t}\,t & 0 \\ 0 & 0 & e^{-2t} & 0 \\ 0 & 0 & 0 & e^{-t} \end{pmatrix}$$

In[∘]:= `x0 = {{2}, {-1}, {1}, {0}}`

Out[∘]=

`{{2}, {-1}, {1}, {0}}`

In[∘]:= `z0 = Inverse[T].x0`

Out[∘]=

`{{0}, {1}, {0}, {0}}`

In[∘]:= `x1[t_] := T.MatrixExp[Λ t].z0`

In[∘]:= `x1[t] // MatrixForm`

Out[∘]//MatrixForm=

$$\begin{pmatrix} 2\,e^{-2t} \\ -e^{-2t} \\ e^{-2t} \\ 0 \end{pmatrix}$$

In[∘]:= `x0 = {{1}, {0}, {1}, {0}}`

Out[∘]=

{{1}, {0}, {1}, {0}}

In[∘]:= `z0 = Inverse[T].x0`

Out[∘]=

{{1}, {0}, {0}, {0}}

In[∘]:= `x1[t_] := T.MatrixExp[Λ t].z0`

In[∘]:= `x1[t] // MatrixForm`

Out[∘]//MatrixForm=

$$\begin{pmatrix} e^{-2t} \\ 0 \\ e^{-2t} \\ 0 \end{pmatrix}$$

In[∘]:= `x0 = {{-3}, {0}, {0}, {1}}`

Out[∘]=

{{-3}, {0}, {0}, {1}}

In[∘]:= `z0 = Inverse[T].x0`

Out[∘]=

{{0}, {0}, {1}, {0}}

In[∘]:= `x1[t_] := T.MatrixExp[Λ t].z0`

In[∘]:= `x1[t] // MatrixForm`

Out[∘]//MatrixForm=

$$\begin{pmatrix} -3\,e^{-2t} + 2\,e^{-2t}\,t \\ -e^{-2t}\,t \\ e^{-2t}\,t \\ e^{-2t} \end{pmatrix}$$

"Cambio" il sistema

In[∘]:= `ClearAll["Global`*"]`

In[∘]:= `A = {{2, 1, -4, -4}, {22, 7, -25, -28}, {12, 5, -15, -14}, {-1, -1, 1, -1}}`

Out[∘]=

{{2, 1, -4, -4}, {22, 7, -25, -28}, {12, 5, -15, -14}, {-1, -1, 1, -1}}

Calcolo lo spettro

In[•]:=
```
λ = Eigenvalues[A]
```
Out[•]=

$\{-2, -2, -2, -1\}$

In[•]:=
```
MatrixMinimalPolynomial[a_List?MatrixQ, x_] :=
 Module[{i, n = 1, qu = {}, mnm = {Flatten[IdentityMatrix[Length[a]]]}},
  While[Length[qu] == 0, AppendTo[mnm, Flatten[MatrixPower[a, n]]];
   qu = NullSpace[Transpose[mnm]];
   n++];
  First[qu].Table[x^i, {i, 0, n - 1}]]
```

Test del polinomio minimo per capire se la matrice e' diagonalizzabile o meno

In[•]:=
```
Factor[MatrixMinimalPolynomial[A, x]]
```
Out[•]=

$(1 + x)\ (2 + x)^3$

In[•]:=
```
Factor[CharacteristicPolynomial[A, x]]
```
Out[•]=

$(1 + x)\ (2 + x)^3$

La matrice non e' diagonalizzabile e quindi devo ricorrere alla Forma di Jordan

In[•]:=
```
{T, Λ} = JordanDecomposition[A]
```
Out[•]=

$\{\{\{-1, -3, -5, 2\}, \{0, -1, 1, -2\}, \{-2, -3, -4, 0\}, \{1, 0, 0, 1\}\},$
$\{\{-2, 1, 0, 0\}, \{0, -2, 1, 0\}, \{0, 0, -2, 0\}, \{0, 0, 0, -1\}\}\}$

In[•]:=
```
Λ // MatrixForm
```
Out[•]//MatrixForm=

$$\begin{pmatrix} -2 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

In[•]:=
```
T // MatrixForm
```
Out[•]//MatrixForm=

$$\begin{pmatrix} -1 & -3 & -5 & 2 \\ 0 & -1 & 1 & -2 \\ -2 & -3 & -4 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

Per “vedere” i modi mi calcolo l’esponenziale di Matrice della forma canonica

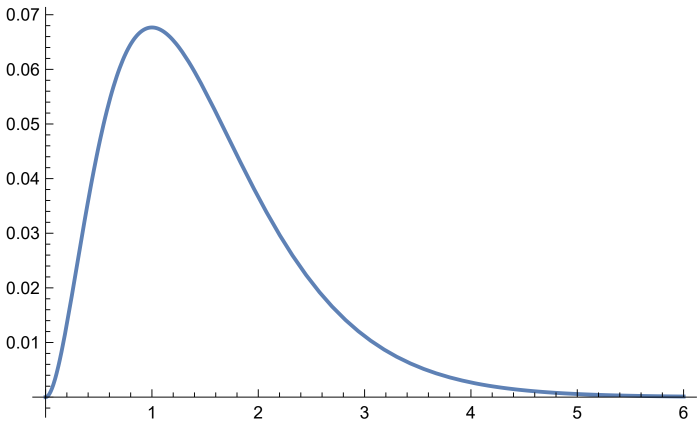
In[•]:=
```
MatrixExp[Λ t] // MatrixForm
```
Out[•]//MatrixForm=

$$\begin{pmatrix} e^{-2t} & e^{-2t}\,t & \frac{1}{2}\,e^{-2t}\,t^2 & 0 \\ 0 & e^{-2t} & e^{-2t}\,t & 0 \\ 0 & 0 & e^{-2t} & 0 \\ 0 & 0 & 0 & e^{-t} \end{pmatrix}$$

Analisi del grafico del modo polinomial-esponenziale con polinomio “parabolico”

In[∘]:= Plot$\left[\frac{1}{2} e^{-2t} t^2, \{t, 0, 6\}, \text{PlotRange} \to \text{All}\right]$

Out[∘]=



In[∘]:= T // MatrixForm

Out[∘]//MatrixForm=

$$\begin{pmatrix} -1 & -3 & -5 & 2 \\ 0 & -1 & 1 & -2 \\ -2 & -3 & -4 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

Calcoliamo ora la risposta libera “senza preoccuparci” di direzioni privilegiate (colonne di T)

In[∘]:= x_0 = {{1}, {0}, {1}, {0}}

Out[∘]=

{{1}, {0}, {1}, {0}}

In[∘]:= z_0 = Inverse[T].x_0

Out[∘]=

{{-1}, {-1}, {1}, {1}}

In[∘]:= C1 = {2, 0, 1, 0}

Out[∘]=

{2, 0, 1, 0}

In[∘]:= T // MatrixForm

Out[∘]//MatrixForm=

$$\begin{pmatrix} -1 & -3 & -5 & 2 \\ 0 & -1 & 1 & -2 \\ -2 & -3 & -4 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

In[∘]:= x_1[t_] := T.MatrixExp[Λ t].z_0

In[∘]:= y_1[t_] := Simplify[C1.x_1[t]]

In[∘]:= x_1[t] // MatrixForm

Out[∘]//MatrixForm=

$$\begin{pmatrix} -e^{-2t} + 2 e^{-t} - 2 e^{-2t} t - \frac{1}{2} e^{-2t} t^2 \\ 2 e^{-2t} - 2 e^{-t} - e^{-2t} t \\ e^{-2t} - e^{-2t} t - e^{-2t} t^2 \\ -e^{-2t} + e^{-t} - e^{-2t} t + \frac{1}{2} e^{-2t} t^2 \end{pmatrix}$$

In[ ]:= `Expand[y1[t]]`

Out[ ]=

$$\left\{-e^{-2t}+4e^{-t}-5e^{-2t}t-2e^{-2t}t^2\right\}$$

In[ ]:= `MatrixExp[Λ t] // MatrixForm`

Out[ ]//MatrixForm=

$$\begin{pmatrix} e^{-2t} & e^{-2t}t & \frac{1}{2}e^{-2t}t^2 & 0 \\ 0 & e^{-2t} & e^{-2t}t & 0 \\ 0 & 0 & e^{-2t} & 0 \\ 0 & 0 & 0 & e^{-t} \end{pmatrix}$$

In[ ]:= `T // MatrixForm`

Out[ ]//MatrixForm=

$$\begin{pmatrix} -1 & -3 & -5 & 2 \\ 0 & -1 & 1 & -2 \\ -2 & -3 & -4 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

In[ ]:= `x0 = {{2}, {-2}, {0}, {1}}`

Out[ ]=

{{2}, {-2}, {0}, {1}}

In[ ]:= `z0 = Inverse[T].x0`

Out[ ]=

{{0}, {0}, {0}, {1}}

In[ ]:= `x1[t_] := T.MatrixExp[Λ t].z0`

In[ ]:= `x1[t] // MatrixForm`

Out[ ]//MatrixForm=

$$\begin{pmatrix} 2e^{-t} \\ -2e^{-t} \\ 0 \\ e^{-t} \end{pmatrix}$$

In[ ]:= `x0 = Transpose[T[[All, 1]]]`

Out[ ]=

{-1, 0, -2, 1}

In[ ]:= `z0 = Inverse[T].x0`

Out[ ]=

{1, 0, 0, 0}

In[ ]:= `x1[t_] := T.MatrixExp[Λ t].z0`

In[ ]:= `x1[t] // MatrixForm`

Out[ ]//MatrixForm=

$$\begin{pmatrix} -e^{-2t} \\ 0 \\ -2e^{-2t} \\ e^{-2t} \end{pmatrix}$$

In[ ]:= x_0

Out[ ]=

{-1, 0, -2, 1}

In[ ]:= x_0 = Transpose[T[[All, 2]]]

Out[ ]=

{-3, -1, -3, 0}

In[ ]:= z_0 = Inverse[T].x_0

Out[ ]=

{0, 1, 0, 0}

In[ ]:= x_1[t_] := T.MatrixExp[Λ t].z_0

In[ ]:= x_1[t] // MatrixForm

Out[ ]//MatrixForm=

$$\begin{pmatrix} -3\, e^{-2t} - e^{-2t}\, t \\ -e^{-2t} \\ -3\, e^{-2t} - 2\, e^{-2t}\, t \\ e^{-2t}\, t \end{pmatrix}$$

In[ ]:= x_0 = Transpose[T[[All, 3]]]

Out[ ]=

{-5, 1, -4, 0}

In[ ]:= z_0 = Inverse[T].x_0

Out[ ]=

{0, 0, 1, 0}

In[ ]:= x_1[t_] := T.MatrixExp[Λ t].z_0

In[ ]:= x_1[t] // MatrixForm

Out[ ]//MatrixForm=

$$\begin{pmatrix} -5\, e^{-2t} - 3\, e^{-2t}\, t - \frac{1}{2}\, e^{-2t}\, t^2 \\ e^{-2t} - e^{-2t}\, t \\ -4\, e^{-2t} - 3\, e^{-2t}\, t - e^{-2t}\, t^2 \\ \frac{1}{2}\, e^{-2t}\, t^2 \end{pmatrix}$$