▪ Tre matrici “strane”

Consideriamo le seguenti tre matrici:

In[∘]:=

```
ClearAll["Global`*"]
```

In[∘]:=

```
A0 = {{-2, 9, -3, 7, -1}, {1, -4, 0, -4, 1}, {1, 3, -4, -1, 1}, {-1, 3, 0, 3, -1}, {-2, 0, 3, 5, -3}}
```

Out[∘]=

$$\{\{-2, 9, -3, 7, -1\}, \{1, -4, 0, -4, 1\}, \{1, 3, -4, -1, 1\}, \{-1, 3, 0, 3, -1\}, \{-2, 0, 3, 5, -3\}\}$$

In[∘]:=

$$A_1 = \left\{\left\{-4, -\frac{1}{5}, -\frac{3}{5}, -\frac{6}{5}, -\frac{2}{5}\right\}, \{8, 3, 2, 2, 2\}, \left\{-5, -\frac{19}{5}, -\frac{12}{5}, -\frac{4}{5}, -\frac{12}{5}\right\}, \left\{-3, -\frac{11}{5}, -\frac{3}{5}, -\frac{11}{5}, -\frac{8}{5}\right\}, \left\{-5, -\frac{24}{5}, -\frac{7}{5}, -\frac{4}{5}, -\frac{22}{5}\right\}\right\}$$

Out[∘]=

$$\left\{\left\{-4, -\frac{1}{5}, -\frac{3}{5}, -\frac{6}{5}, -\frac{2}{5}\right\}, \{8, 3, 2, 2, 2\}, \left\{-5, -\frac{19}{5}, -\frac{12}{5}, -\frac{4}{5}, -\frac{12}{5}\right\}, \left\{-3, -\frac{11}{5}, -\frac{3}{5}, -\frac{11}{5}, -\frac{8}{5}\right\}, \left\{-5, -\frac{24}{5}, -\frac{7}{5}, -\frac{4}{5}, -\frac{22}{5}\right\}\right\}$$

In[∘]:=

$$A_2 = \left\{\left\{\frac{31}{3}, -\frac{14}{3}, \frac{8}{3}, 6, 8\right\}, \left\{\frac{91}{6}, -\frac{25}{3}, \frac{7}{3}, 7, 9\right\}, \left\{\frac{19}{2}, -5, 0, 4, 5\right\}, \{-9, 5, -2, -6, -5\}, \{-5, 1, -2, -3, -6\}\right\}$$

Out[∘]=

$$\left\{\left\{\frac{31}{3}, -\frac{14}{3}, \frac{8}{3}, 6, 8\right\}, \left\{\frac{91}{6}, -\frac{25}{3}, \frac{7}{3}, 7, 9\right\}, \left\{\frac{19}{2}, -5, 0, 4, 5\right\}, \{-9, 5, -2, -6, -5\}, \{-5, 1, -2, -3, -6\}\right\}$$

In[∘]:=

```
MatrixMinimalPolynomial[a_List?MatrixQ, x_] := Module[{i, n = 1, qu = {}, mnm = {Flatten[IdentityMatrix[Length[a]]]}},
  While[Length[qu] == 0, AppendTo[mnm, Flatten[MatrixPower[a, n]]];
   qu = NullSpace[Transpose[mnm]];
   n++];
  First[qu].Table[x^i, {i, 0, n-1}]]
```

▪ Mi calcolo gli autovalori delle tre matrici

Calcolo autovalori

In[∘]:=

```
Eigenvalues[A0]
```

Out[∘]=

$$\{-4, -3, -1, -1, -1\}$$

In[∘]:=

```
Eigenvalues[A1]
```

Out[∘]=

$$\{-4, -3, -1, -1, -1\}$$

In[∘]:=

```
Eigenvalues[A2]
```

Out[∘]=

$$\{-4, -3, -1, -1, -1\}$$

Calcolo della molteplicita’ geometrica nel caso delle tre matrici. Il test va solamente effettuato sugli autovalori multipli, quelli semplici possono essere “messi da parte”.

In[ ]:=
```
NullSpace[A0 - (-1) IdentityMatrix[5]]
```
Out[ ]=
```
{{-1, 0, 0, 0, 1}, {5, -1, 0, 2, 0}, {3, 1, 2, 0, 0}}
```

La molteplicita’ geometrica dell’autovalore multiplo e’ pari alla sua moltemplicita’ algebrica, deduco che A0 e’ diagonalizzabile.

In[ ]:=
```
T0 = Transpose[Eigenvectors[A0]]
```
Out[ ]=
```
{{-1, 1, -1, 5, 3}, {0, -1, 0, -1, 1}, {-1, -1, 0, 0, 2}, {0, 1, 0, 2, 0}, {1, 2, 1, 0, 0}}
```

In[ ]:=
```
T0 // MatrixForm
```
Out[ ]//MatrixForm=

$$\begin{pmatrix} -1 & 1 & -1 & 5 & 3 \\ 0 & -1 & 0 & -1 & 1 \\ -1 & -1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 & 0 \\ 1 & 2 & 1 & 0 & 0 \end{pmatrix}$$

In[ ]:=
```
Eigenvalues[A0]
```
Out[ ]=
```
{-4, -3, -1, -1, -1}
```

In[ ]:=
```
Inverse[T0].A0.T0 // MatrixForm
```
Out[ ]//MatrixForm=

$$\begin{pmatrix} -4 & 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

In[ ]:=
```
Factor[MatrixMinimalPolynomial[A0, x]]
```
Out[ ]=

$$(1 + x)\ (3 + x)\ (4 + x)$$

Calcolo ora la molteplicita’ geometrica dell’autovalore multiplo nel caso A1

In[ ]:=
```
NullSpace[A1 - (-1) IdentityMatrix[5]]
```
Out[ ]=
```
{{0, -1, 1, 0, 1}, {-1, 0, 3, 1, 0}}
```

In[ ]:=
```
Factor[MatrixMinimalPolynomial[A1, x]]
```
Out[ ]=

$$(1 + x)^2\ (3 + x)\ (4 + x)$$

Calcolo ora la molteplicita’ geometrica dell’autovalore multiplo nel caso A2

In[ ]:=
```
NullSpace[A2 - (-1) IdentityMatrix[5]]
```
Out[ ]=

$$\left\{\left\{-1, -1, -\frac{1}{2}, 0, 1\right\}\right\}$$

In[∘]:=

```
Factor[MatrixMinimalPolynomial[A2, x]]
```

Out[∘]=

$$(1+x)^3\ (3+x)\ (4+x)$$

Poiche' A1 e A2 non sono diagonalizzabili, sono costretto ad identificare le rispettive forme di Jordan

In[∘]:=

```
{T1, Λ1} = JordanDecomposition[A1]
```

Out[∘]=

$$\left\{\left\{\left\{-2, -\frac{1}{2}, 0, \frac{1}{2}, -1\right\}, \{2, 0, -1, 0, 0\}, \left\{0, \frac{1}{2}, 1, -\frac{5}{2}, 3\right\}, \left\{0, \frac{1}{2}, 0, 0, 1\right\}, \{1, 1, 1, 0, 0\}\right\}, \{\{-4, 0, 0, 0, 0\}, \{0, -3, 0, 0, 0\}, \{0, 0, -1, 1, 0\}, \{0, 0, 0, -1, 0\}, \{0, 0, 0, 0, -1\}\}\right\}$$

In[∘]:=

```
Λ1 // MatrixForm
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} -4 & 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

In[∘]:=

```
{T2, Λ2} = JordanDecomposition[A2]
```

Out[∘]=

$$\left\{\left\{\left\{0, -1, -1, -\frac{1}{2}, \frac{1}{4}\right\}, \left\{1, -\frac{1}{2}, -1, -\frac{1}{2}, -\frac{1}{4}\right\}, \left\{1, 0, -\frac{1}{2}, -\frac{1}{4}, \frac{9}{8}\right\}, \left\{-1, \frac{1}{2}, 0, \frac{1}{2}, -\frac{5}{4}\right\}, \{1, 1, 1, 0, 0\}\right\}, \{\{-4, 0, 0, 0, 0\}, \{0, -3, 0, 0, 0\}, \{0, 0, -1, 1, 0\}, \{0, 0, 0, -1, 1\}, \{0, 0, 0, 0, -1\}\}\right\}$$

In[∘]:=

```
Λ2 // MatrixForm
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} -4 & 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

In[∘]:=

```
Λ1 // MatrixForm
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} -4 & 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

In[•]:=
```
T₁ // MatrixForm
```

Out[•]//MatrixForm=

$$\begin{pmatrix} -2 & -\frac{1}{2} & 0 & \frac{1}{2} & -1 \\ 2 & 0 & -1 & 0 & 0 \\ 0 & \frac{1}{2} & 1 & -\frac{5}{2} & 3 \\ 0 & \frac{1}{2} & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \end{pmatrix}$$

Verifico che la prima, seconda e quinta colonna sono autovettori “standard” di A1

In[•]:=
```
(A₁ - (-4) IdentityMatrix[5]).T₁[[All, 1]]
```

Out[•]=

{0, 0, 0, 0, 0}

In[•]:=
```
(A₁ - (-3) IdentityMatrix[5]).T₁[[All, 2]]
```

Out[•]=

{0, 0, 0, 0, 0}

In[•]:=
```
(A₁ - (-1) IdentityMatrix[5]).T₁[[All, 5]]
```

Out[•]=

{0, 0, 0, 0, 0}

Mi soffermo sulla terza e la quarta colonna, la terza (verifica) e’ ancora un autovettore standard di -1

In[•]:=
```
(A₁ - (-1) IdentityMatrix[5]).T₁[[All, 3]]
```

Out[•]=

{0, 0, 0, 0, 0}

In[•]:=
```
T₁[[All, 3]]
```

Out[•]=

{0, -1, 1, 0, 1}

In[•]:=
```
(A₁ - (-1) IdentityMatrix[5]).T₁[[All, 4]]
```

Out[•]=

{0, -1, 1, 0, 1}

In[•]:=
```
(A₁ - (-1) IdentityMatrix[5]).(A₁ - (-1) IdentityMatrix[5]).T₁[[All, 4]]
```

Out[•]=

{0, 0, 0, 0, 0}

Analizzo ora il caso A2

In[•]:=
```
Λ₂ // MatrixForm
```

Out[•]//MatrixForm=

$$\begin{pmatrix} -4 & 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

In[∘]:=
```
T₂ // MatrixForm
```
Out[∘]//MatrixForm=

$$\begin{pmatrix} 0 & -1 & -1 & -\frac{1}{2} & \frac{1}{4} \\ 1 & -\frac{1}{2} & -1 & -\frac{1}{2} & -\frac{1}{4} \\ 1 & 0 & -\frac{1}{2} & -\frac{1}{4} & \frac{9}{8} \\ -1 & \frac{1}{2} & 0 & \frac{1}{2} & -\frac{5}{4} \\ 1 & 1 & 1 & 0 & 0 \end{pmatrix}$$

Verifica delle “catene”

In[∘]:=
```
(A₂ - (-4) IdentityMatrix[5]).T₂[[All, 1]]
```
Out[∘]=

$\{0, 0, 0, 0, 0\}$

In[∘]:=
```
(A₂ - (-3) IdentityMatrix[5]).T₂[[All, 2]]
```
Out[∘]=

$\{0, 0, 0, 0, 0\}$

In[∘]:=
```
(A₂ - (-1) IdentityMatrix[5]).T₂[[All, 3]]
```
Out[∘]=

$\{0, 0, 0, 0, 0\}$

In[∘]:=
```
T₂[[All, 3]]
```
Out[∘]=

$\left\{-1, -1, -\frac{1}{2}, 0, 1\right\}$

In[∘]:=
```
(A₂ - (-1) IdentityMatrix[5]).T₂[[All, 4]]
```
Out[∘]=

$\left\{-1, -1, -\frac{1}{2}, 0, 1\right\}$

In[∘]:=
```
T₂[[All, 4]]
```
Out[∘]=

$\left\{-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{4}, \frac{1}{2}, 0\right\}$

In[∘]:=
```
(A₂ - (-1) IdentityMatrix[5]).T₂[[All, 5]]
```
Out[∘]=

$\left\{-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{4}, \frac{1}{2}, 0\right\}$

In[∘]:=
```
(A₂ - (-1) IdentityMatrix[5]).(A₂ - (-1) IdentityMatrix[5]).T₂[[All, 4]]
```
Out[∘]=

$\{0, 0, 0, 0, 0\}$

In[ ]:=

```
(A₂ - (-1) IdentityMatrix[5]).(A₂ - (-1) IdentityMatrix[5]).(A₂ - (-1) IdentityMatrix[5]).T₂〚All, 5〛
```

Out[ ]=

```
{0, 0, 0, 0, 0}
```