SCRITTURA DELLA RISPOSTA LIBERA

$$\begin{cases} x(k+1) = A\,x(k) \\ x(0) = x_0 \end{cases}$$

$$\begin{cases} \dot{x}(t) = A\,x(t) \\ x(0) = x_0 \end{cases}$$

A PRESENTA AUTOVALORI $\boxed{\text{REALI}}$ E DISTINTI

TEOREMA FONDAMENTALE DELL'ALGEBRA

UN' EQ. ALGEBRICA DI GRADO n A COEFF. REALI AMMETTE n RADICI COMPLESSE E CONIUGATE.

$$\det(A - \lambda I) = \det(\lambda I - A) = 0$$

$$\lambda^n + a_1 \lambda^{n-1} + \dots + a_n = 0$$

A PRESENTA n AUTOVALORI DISTINTI $\Rightarrow$

È DIAGONALIZZABILE? SI

$$A \overset{T}{\sim} \Lambda$$

$$\Lambda = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

TD

$\lambda_i \in \mathbb{R} \Rightarrow \lambda_i^k$

$v_i \in \mathbb{R}^n$

TC

$\lambda_i \in \mathbb{R} \Rightarrow e^{\lambda_i t}$

$v_i \in \mathbb{R}^n$

$\lambda \in \mathbb{C}$ $\quad \bar{\lambda} \in \mathbb{C}$ => MODI NATURALI ?

TD

λ^k , $\bar{\lambda}^k$

TC

$e^{\lambda t}$, $e^{\bar{\lambda} t}$

SE LA MATRICE A È DI ORDINE n
E PRESENTA n AUTOVALORI DISTINTI
ALLORA IL SISTEMA AVRÀ n MODI

$$x(k+1) = A\,x(k) \qquad x(k) \in \mathbb{R}^2$$

IPOTIZZO CHE A ABBIA UNA COPPIA
DI AUTOVALORI COMPLESSI E CONIUGATI

$$\lambda, \bar{\lambda} \in \mathbb{C}$$

$$A \overset{T}{\sim} \begin{bmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{bmatrix}$$

$$A v = \lambda v \qquad v \in \mathbb{C}^2$$

$$(A v = \lambda v \implies v \in \mathbb{C}^n)$$

$$A \hat{v} = \bar{\lambda} \hat{v}$$

$$\hat{v} = \overline{v}$$

$$\overline{A v} = \overline{\lambda v}$$

$$\bar{A} \bar{v} = \bar{\lambda} \bar{v}$$

$$A \bar{v} = \bar{\lambda} \bar{v}$$

$$T = [v \quad \bar{v}]$$

$$A\,T = T\,\Lambda$$

$$A\,[v \quad \bar{v}] = [v \quad \bar{v}] \begin{bmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{bmatrix}$$

$$T = [v \quad \bar{v}] \in \mathbb{C}^{2\times 2}$$

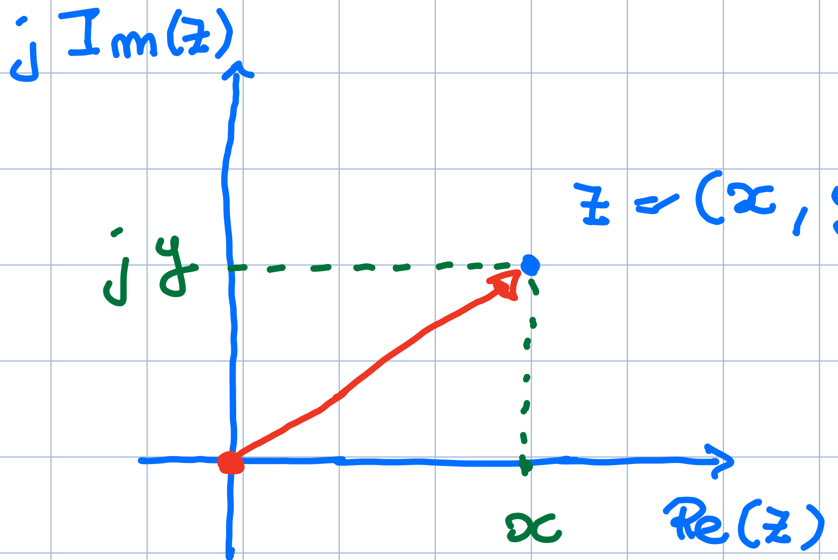
$$\hat{T} = [Re(v)\ Im(v)] \in \mathbb{R}^{2\times 2}$$

$$T = \begin{bmatrix} 1+2j & 1-2j \\ 3-j & 3+j \end{bmatrix}$$

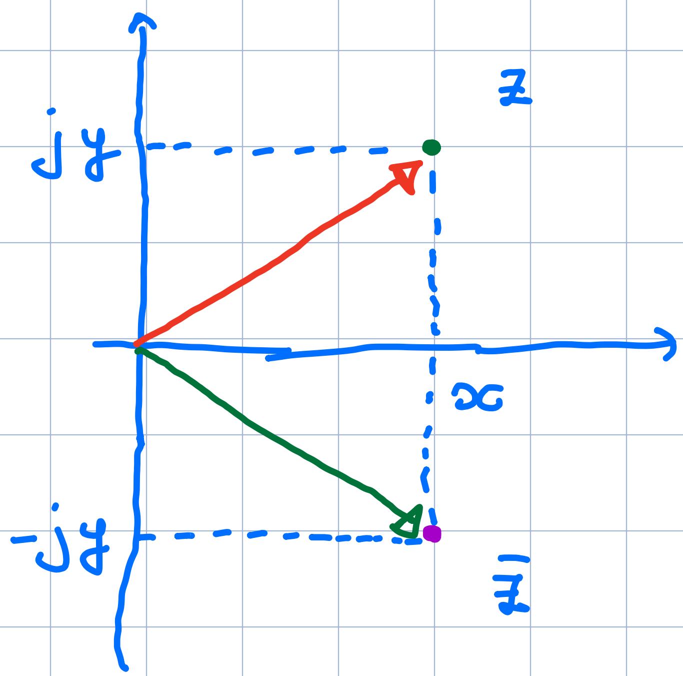
$$\hat{T} = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$$

QUALE È L'OPERAZIONE, NEL SENSO DELL'ALGEBRA LINEARE, CHE MI CONSENTE DI PASSARE DA $T \in \mathbb{C}^{2\times 2}$ A $\hat{T} \in \mathbb{R}^{2\times 2}$

$z = x + jy$ RAPPRESENTAZIONE CARTESIANA DI $z \in \mathbb{C}$



$$\bar{z} = x - jy$$



$$z = x + jy = \mathrm{Re}(z) + j\,\mathrm{Im}(z)$$

$$\bar{z} = x - jy = \mathrm{Re}(z) - j\,\mathrm{Im}(z)$$

$$z + \bar{z} = 2x = 2\,\mathrm{Re}(z)$$

$$\mathrm{Re}(z) = \frac{z + \bar{z}}{2}$$

$$z = x + jy = \mathrm{Re}(z) + j\,\mathrm{Im}(z)$$

$$\bar{z} = x - jy = \mathrm{Re}(z) - j\,\mathrm{Im}(z).$$

$$z - \bar{z} = jy + jy = j\,\mathrm{Im}(z) + j\,\mathrm{Im}(z)$$

$$z - \bar{z} = 2jy = 2j\,\mathrm{Im}(z).$$

$$\mathrm{Im}(z) = \frac{z - \bar{z}}{2j}$$

$$T = \begin{bmatrix} v & \bar{v} \end{bmatrix}$$

$$\hat{T} = \begin{bmatrix} \mathrm{Re}(v) & \mathrm{Im}(v) \end{bmatrix} =$$

$$= \begin{bmatrix} \dfrac{v + \bar{v}}{2} & \dfrac{v - \bar{v}}{2j} \end{bmatrix}$$

$$\hat{T} = T \cdot \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix} = [v \quad \bar{v}] \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix} =$$

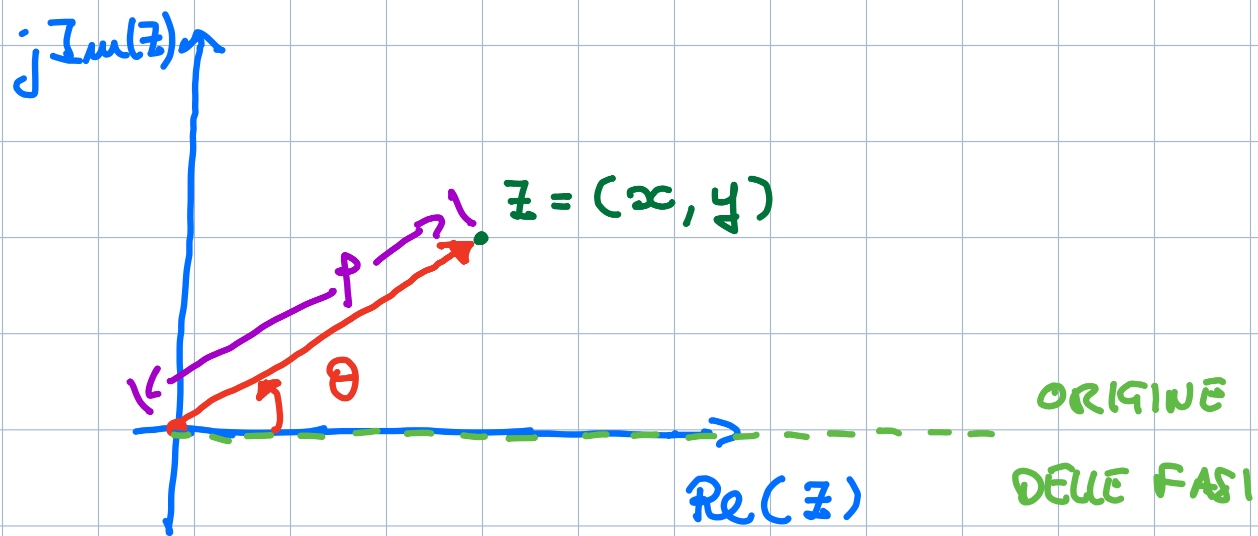
$$= \left[\frac{v+\bar{v}}{2} \quad \frac{v-\bar{v}}{2j} \right] = \left[\frac{1}{2} v + \frac{1}{2} \bar{v} \quad \frac{1}{2j} v - \frac{1}{2j} \bar{v} \right]$$

$$\begin{bmatrix} 1/2 & 1/2j \\ 1/2 & -1/2j \end{bmatrix}$$

$$\hat{T} = [Re(v) \quad Im(v)] = \left[\frac{v+\bar{v}}{2} \quad \frac{v-\bar{v}}{2j} \right] = .$$

$$= [v \quad \bar{v}] \begin{bmatrix} \frac{1}{2} & \frac{1}{2j} \\ \frac{1}{2} & -\frac{1}{2j} \end{bmatrix}$$

FORMA TRIGONOMETRICA DI UN NUMERO COMPLESSO



$$\rho = |z| = \sqrt{x^2 + y^2}$$

UN NUMERO REALE POSITIVO È UN PARTICOLARE NUMERO COMPLESSO A FASE (O ARGOMENTO) NULLA

θ FASE DEL NUMERO COMPLESSO z

θ SENSO ANTIORARIO $(\theta > 0)$

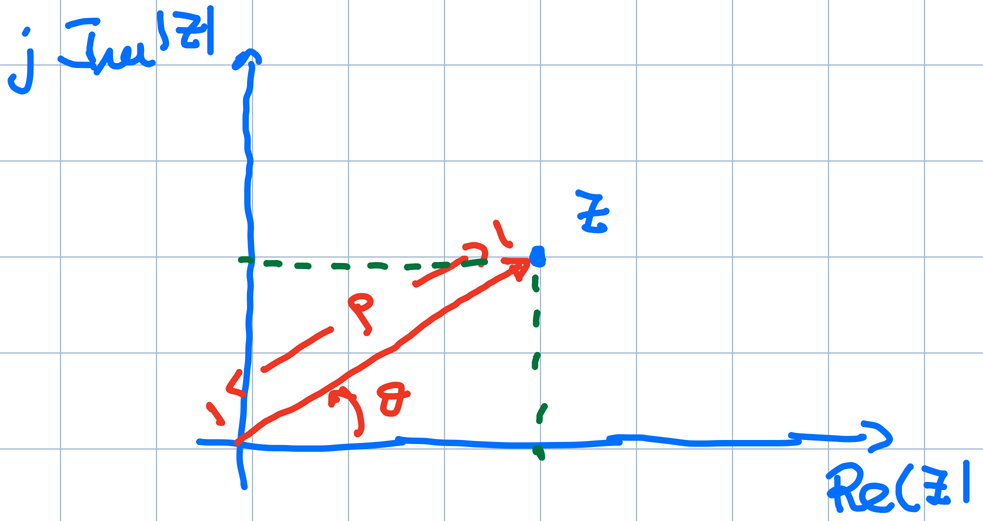
θ SENSO ORARIO $(\theta < 0)$

ANTIORARIO $\Rightarrow$ ANTICIPO DI FASE

ORARIO $\Rightarrow$ RITARDO DI FASE

(ρ, θ) FORMA TRIGONOMETRICA DI Z.

DA FORMA TRIGONOMETRICA A FORMA CARTESIANA (SEMPLICE)

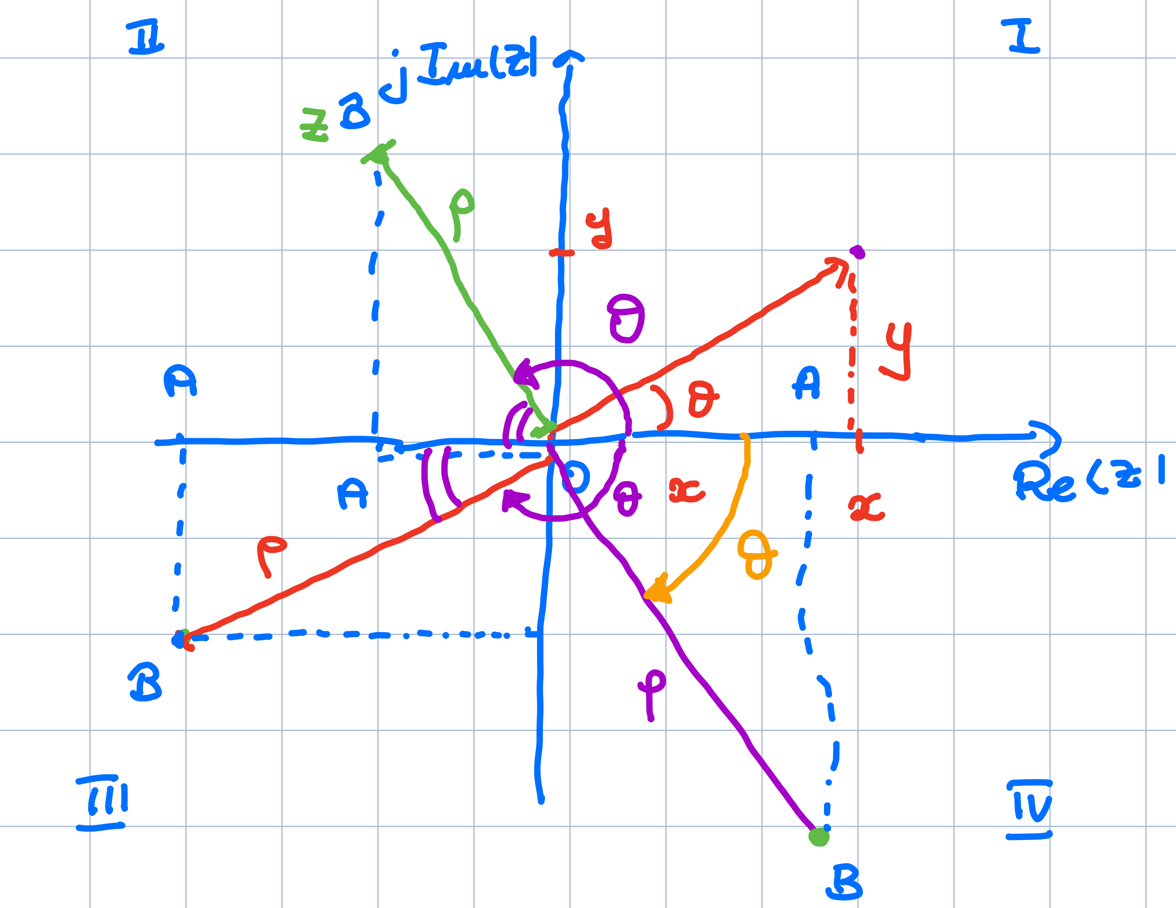


$$\mathrm{Re}(Z) = \rho \cos(\theta)$$

$$\mathrm{Im}(Z) = \rho \sin(\theta)$$

$$z = \rho \cos(\theta) + j \sin(\theta)$$

DA FORMA CARTESIANA A FORMA TRIGONOMETRICA (AMBIGUO)



$$z = x + jy \Rightarrow (\rho, \theta)$$

$$\rho = \sqrt{x^2 + y^2}$$

$$\theta = tg^{-1}\left(\frac{y}{x}\right)$$

$\theta \in [0, 2\pi)$ $\theta \in [-\pi, +\pi)$

$$\theta = \pi - Tg^{-1}\left(\frac{\overline{AB}}{\overline{OA}}\right)$$ II° QUADRANTE

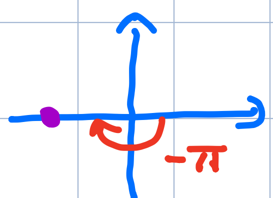
$$\theta = -\pi + Tg\left(\frac{\overline{AB}}{\overline{OA}}\right)$$ III° QUADRANTE

$$\theta = - Tg\left(\frac{\overline{AB}}{\overline{OA}}\right)$$ IV° QUADRANTE

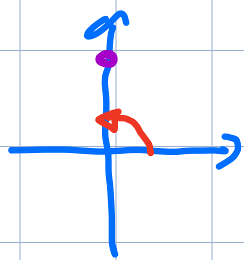
$\theta \in [-\pi, +\pi)$

NUMERO REALE POSITIVO $\theta = 0$

NUMERO REALE NEGATIVO $\theta = (\pm)\pi = -\pi$

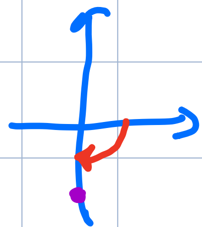


NUMERO IMMAGINARIO CON PARTE IMMAGINARIA POSITIVA



$\theta = \frac{\pi}{2}$

NUMERO IMMAGINARIO CON PARTE IMMAGINARIA NEGATIVA



$\theta = -\frac{\pi}{2}$

$atan(y, x)$ $atan(x, y)$

$e^{j\varphi}$ SICURAMENTE È UN NUMERO COMPLESSO

$e^{j\varphi} = \cos(\varphi) + j\sin(\varphi)$

$$f(\varphi) = e^{-j\varphi}\left(\cos(\varphi) + j\sin(\varphi)\right)$$

$$\frac{d}{d\varphi}\left(f(\varphi)\right) = (-j)\, e^{-j\varphi}\left(\cos(\varphi) + j\sin(\varphi)\right) +$$

$$+ e^{-j\varphi}\left(-\sin(\varphi) + j\cos(\varphi)\right) =$$

$$= -j\, e^{-j\varphi}\cos(\varphi) + e^{-j\varphi}\sin(\varphi) -$$

$$- e^{-j\varphi}\sin(\varphi) + j\, e^{-j\varphi}\cos(\varphi) = 0$$

HO UNA FUNZIONE LA CUI

DERIVATA È NULLA $\forall \varphi$

$$\frac{d}{dt} f(\varphi) = 0 \qquad \forall \varphi$$

IMPLICA CHE

$$f(\varphi) = \text{COSTANTE} \qquad \forall \varphi$$

$$F(0)$$

$$F(\varphi) = e^{-j\varphi}\left(\cos(\varphi) + j\sin(\varphi)\right)\Big|_{\varphi=0} = 1$$

$$F(\varphi) = 1 \quad \forall \varphi$$

$$e^{-j\varphi}\left(\cos(\varphi) + j\sin(\varphi)\right) = 1 \quad \forall \varphi$$

$$\frac{\cos(\varphi) + j\sin(\varphi)}{e^{j\varphi}} = 1 \quad \forall \varphi$$

$$e^{j\varphi} = \cos(\varphi) + j\sin(\varphi)$$

FORMULA DI EULERO, IN PRIMA BATTUTA

LEGA IL NUMERO COMPLESSO

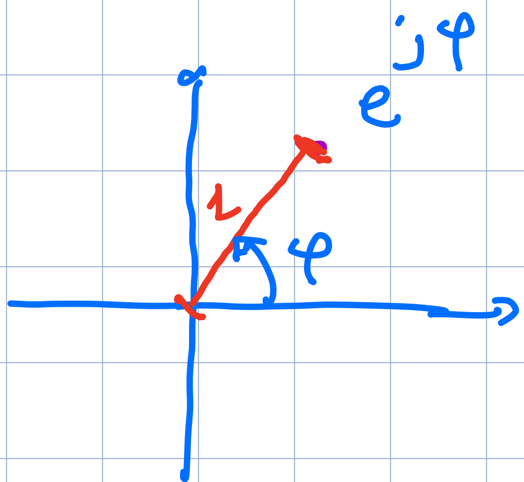
$\cos(\varphi) + j\sin(\varphi)$

IN FORMA TRIGONOMETRICA IL CUI MODULO È UNITARIO

$$\sqrt{\cos^2(\varphi) + \sin^2(\varphi)} = 1$$

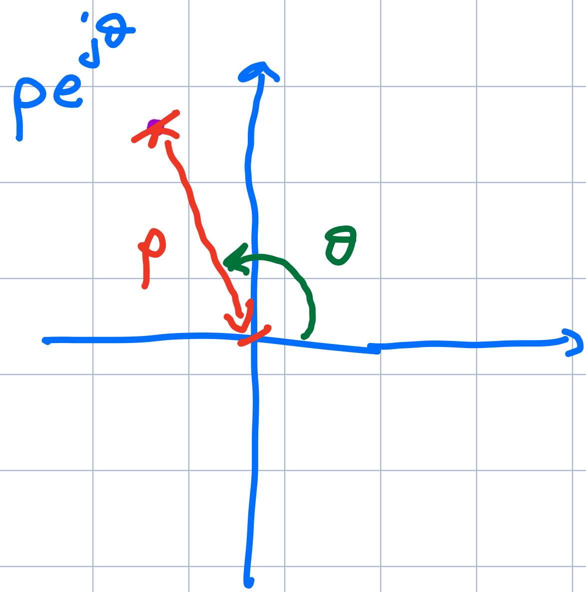
ALL' ESPONENZIALE

$$e^{j\varphi}$$



$$\rho\cos(\theta) + j\rho\sin(\theta) =$$

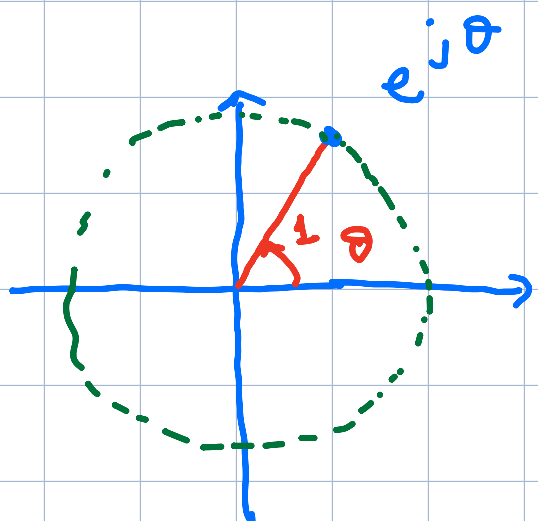
$$= \rho(\cos(\theta) + j\sin(\theta)) = \rho e^{j\theta}$$



$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

QUALE È IL LUOGO DEI PUNTI SUL PIANO DI GAUSS

$$e^{j\theta} \quad \forall\ \theta \in [-\pi, +\pi)$$



È LA CIRCONFERENZA UNITARIA DI CENTRO PARI ALL'ORIGINE.

$$|e^{j\theta}| = 1 \quad \forall \theta$$

CHI È

$$e^{-j\theta}$$

$$e^{-j\theta} e^{j\theta} = 1$$

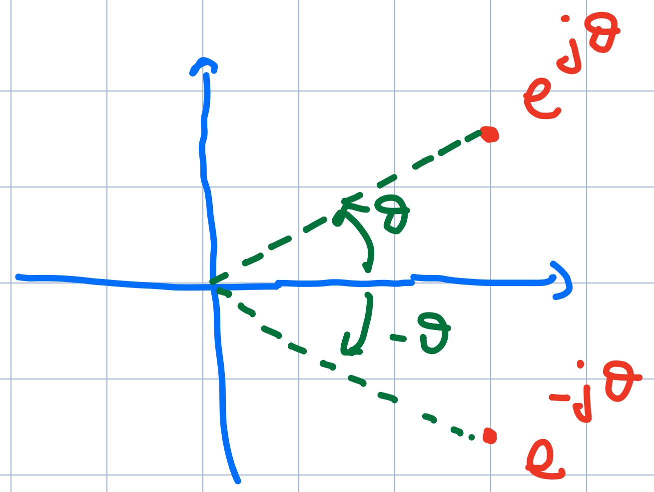
È L'INVERSO DI $e^{j\theta}$

$$e^{-j\theta} = e^{j(-\theta)} = \cos(-\theta) + j\sin(-\theta) =$$

$$= \cos(\theta) - j\sin(\theta)$$

$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

$e^{-j\theta}$ È IL CONIUGATO DI $e^{j\theta}$



FORMULE DI EULERO INVERSE

$$e^{j\vartheta} = \cos(\vartheta) + j\sin(\vartheta)$$

$$e^{-j\vartheta} = \cos(\vartheta) - j\sin(\vartheta)$$

$$e^{j\vartheta} + e^{-j\vartheta} = 2\cos(\vartheta)$$

$$\cos(\vartheta) = \frac{e^{j\vartheta} + e^{-j\vartheta}}{2} = \mathrm{Re}(e^{j\vartheta})$$

$$e^{j\vartheta} - e^{-j\vartheta} = j\sin\vartheta + j\sin(\vartheta) = 2j\sin(\vartheta)$$

$$\sin(\vartheta) = \frac{e^{j\vartheta} - e^{-j\vartheta}}{2j} = \mathrm{Im}(e^{j\vartheta})$$

$$x(k+1) = A\,x(k)$$

$$A\begin{bmatrix} v & \bar{v} \end{bmatrix} = \begin{bmatrix} v & \bar{v} \end{bmatrix} \begin{bmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{bmatrix}$$

$$A\begin{bmatrix} Re(v) & Im(v) \end{bmatrix} = \begin{bmatrix} Re(v) & Im(v) \end{bmatrix} \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix}$$

$[v \quad \bar{v}]$ È NON-SINGOLARE

$[Re(v) \quad Im(v)]$ È NON-SINGOLARE ?

$$\begin{bmatrix} Re(v) & Im(v) \end{bmatrix} = \begin{bmatrix} \dfrac{v+\bar{v}}{2} & \dfrac{v-\bar{v}}{2j} \end{bmatrix} =$$

$$= \begin{bmatrix} v & \bar{v} \end{bmatrix} \underbrace{\begin{bmatrix} \frac{1}{2} & \frac{1}{2j} \\ \frac{1}{2} & -\frac{1}{2j} \end{bmatrix}}_{\det = -\frac{1}{4j} - \frac{1}{4j} \neq 0} .$$

$$A \begin{bmatrix} v & \bar{v} \end{bmatrix} = \begin{bmatrix} v & \bar{v} \end{bmatrix} \begin{bmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{bmatrix}$$

$$A \begin{bmatrix} v & \bar{v} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2j} \\ \frac{1}{2} & -\frac{1}{2j} \end{bmatrix} = \begin{bmatrix} v & \bar{v} \end{bmatrix} \begin{bmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{bmatrix} .$$

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2j} \\ \frac{1}{2} & -\frac{1}{2j} \end{bmatrix}$$

$$A\,[\,Re(v)\quad Im(v)\,] = [\,v\quad \bar{v}\,]\underbrace{\begin{bmatrix}\lambda & 0\\ 0 & \bar{\lambda}\end{bmatrix}}_{I}$$

$$\begin{bmatrix}1/2 & 1/2j\\ 1/2 & -1/2j\end{bmatrix}$$

$$[\,v\quad \bar{v}\,]\begin{bmatrix}1/2 & 1/2j\\ 1/2 & -1/2j\end{bmatrix}\begin{bmatrix}1 & 1\\ j & -j\end{bmatrix}\begin{bmatrix}\lambda & 0\\ 0 & \bar{\lambda}\end{bmatrix}\begin{bmatrix}1/2 & 1/2j\\ 1/2 & -1/2j\end{bmatrix}$$

$$[\,Re(v)\quad Im(v)\,]\begin{bmatrix}1 & 1\\ j & -j\end{bmatrix}\begin{pmatrix}\lambda & 0\\ 0 & \bar{\lambda}\end{pmatrix}\begin{bmatrix}1/2 & 1/2j\\ 1/2 & -1/2j\end{bmatrix}$$

LA MATRICE $A \in \mathbb{R}^{2\times 2}$ CHE PRESENTA UNA COPPIA DI AUTOVALORI COMPLESSI E CONIUGATI $\lambda, \bar{\lambda} \in \mathbb{C}$

È SIMILE, SECONDO

$$\hat{T} = [\,\mathrm{Re}(\sigma) \quad \mathrm{Im}(\sigma)\,]$$

A

$$\begin{bmatrix} 1 & 1 \\ j & -j \end{bmatrix} \begin{bmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2j} \\ \frac{1}{2} & -\frac{1}{2j} \end{bmatrix}$$