5B) 8.26

estimate by MLE

$$P(X=k) = \frac{\binom{100}{k}\binom{N-100}{50-k}}{\binom{N}{50}}$$

$$lk(N) = \frac{\binom{100}{20}\binom{N-100}{30}}{\binom{N}{50}}$$

$$\frac{l(N)}{l(N-1)} = \frac{(N-100)(N-50)}{N(N-130)} \geq 1$$

$$\Rightarrow (N-100)(N-50) \geq N(N-130)$$

$$\Rightarrow N \leq 250$$

$$\boxed{\hat{N}_{MLE} = 250}$$

5c) 8.36

a) $lik(\lambda) = (\lambda e^{-5\lambda})(\lambda e^{-3\lambda})(e^{-10\lambda})$

$$\boxed{= \lambda^2 e^{-18\lambda}}$$

b) $\ell(\lambda) = \log(lik(\lambda)) = 2\log\lambda - 18\lambda$

$$\ell'(\lambda) = \frac{2}{\lambda} - 18 = 0$$

$$\boxed{\frac{1}{9} = \hat{\lambda}_{ML}}$$

8.32

5D) a) $\mu = \bar{x} \leftarrow \frac{(5.3299 + 4.2537 + \ldots +)}{n}$

$\sigma^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \hat{\mu})^2$

$\boxed{\mu \approx 3.61 \quad \sigma^2 \approx 3.20}$

b) 90% CI

$\hat{\mu} \pm \frac{S}{\sqrt{n}} t_{n-1}(0.05) \doteq \boxed{[2.8, 4.4]}$

$S = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \hat{\mu})^2} = 1.8488$

$\left[\frac{n\hat{\sigma}^2}{\chi^2_{n-1}(0.5 \pm 0.45)}\right] = \boxed{[2.05, 7.06]}$

c) 90% CI = sqrt(σ^2)

$[\sqrt{2.05}, \sqrt{7.06}] = \boxed{[1.43, 2.66]}$

d) $\frac{1}{2}$ len(CI for μ), would need to 4x it because its a sqrt

5E) 8.48

δ-method

estimate λ

$$\hat{\lambda} = g(Y) = -\log\left(\frac{Y}{n}\right)$$

$$Y \sim \text{Bin}(n, e^{-\lambda}), \quad g(x) = -\ln\left(\frac{x}{n}\right)$$

$$\mathbb{E}[\hat{\lambda}] \approx g(\mathbb{E}[Y]) + \frac{1}{2}\text{Var}(Y) * g''(\mathbb{E}(Y))$$

$$\text{Var}(Y) \approx \text{Var}(Y) \cdot g''((\mathbb{E}[Y]))^2$$

$$\mathbb{E}(Y) = ne^{-\lambda}, \quad \text{Var}(Y) = ne^{-\lambda}(1-e^{-\lambda})$$

$$g''(x) = \frac{1}{x^2}$$

$$\mathbb{E}(\hat{\lambda}) \approx \lambda + \frac{1}{2}\underbrace{\frac{1-e^{-\lambda}}{ne^{-\lambda}}}$$

$$\text{Var}(\hat{\lambda}) = \frac{1-e^{-\lambda}}{ne^{-\lambda}}$$

bias of estimate

$$\mathbb{E}(\hat{\lambda}) - \mathbb{E}(\lambda) = \frac{1}{2}\frac{1-e^{-\lambda}}{ne^{-\lambda}}$$

$$\cancel{\lambda - \lambda}$$

$$\text{efficiency}(\hat{\lambda}_{ML}, \hat{\lambda}) = \frac{\frac{\lambda}{n}}{\frac{1-e^{-\lambda}}{ne^{-\lambda}}} = \frac{\lambda e^{-\lambda}}{1-e^{-\lambda}}$$

λ	1	2	5
efficiency	0.58	0.31	0.03

the $\hat{\lambda}_{ML}$ has a lower variance than $\hat{\lambda}$, meaning that it is a better estimator

5f) Bias - Variance

$$MSE(\hat{\theta}) = \mathbb{E}(\hat{\theta} - \theta)^2 \quad , \quad \text{prove } \checkmark$$

$$MSE(\hat{\theta}) = Var(\hat{\theta}) + Bias(\hat{\theta}))^2$$

$$MSE(\hat{\theta}) = \mathbb{E}(\hat{\theta} - \theta)^2$$

dist

$$= \mathbb{E}\left[\left(\left[\hat{\theta} - \mathbb{E}(\hat{\theta})\right] + \mathbb{E}(\hat{\theta}) - \theta)\right)^2\right]$$

$$= \mathbb{E}\left[(\hat{\theta} - E(\hat{\theta}))^2\right] + \mathbb{E}\left[E(\hat{\theta}) - \theta)^2\right] + \cancel{2E((\hat{\theta} - E(\hat{\theta}))(E(\hat{\theta} - \theta))}$$

$$\boxed{= var(\theta) + \underbrace{\left[E(\hat{\theta}) - \theta\right]^2}_{[Bias(\hat{\theta})]^2}}$$

5k)

$$\frac{f(x_1, x_2, \ldots x_n|\theta)}{f(y_1, f y_1 \ldots y_n|\theta)} = \prod_{i=1}^{n} \frac{\theta e^{-(\theta+1)\ln(1+x_i)}}{\theta e^{-(\theta+1)\ln(1+y_i)}}$$

$$= e^{-(\theta+1)\sum_{i=1}^{n}[\ln(1+x_i)-\ln(1+y_i)]}$$

independant iff $\Longleftrightarrow$

$$\sum_{i=1}^{n} \ln(1+x_i) = \sum_{i=1}^{n} \ln(1+y_i)$$

$$T = \sum_{i=1}^{n} \ln(1+x_i)$$

is sufficient statistic