Estimate tools

Terence Tao

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Chapter 1

Introduction

Introduction goes here.

Chapter 2

Orders of magnitude

The hyperreals ${}^*\mathbb{R}$ are already defined in Mathlib, using a Lean-canonically chosen ultrafilter on $\mathbb{N}$. One could consider generalizing the hyperreal construction to other filters or ultrafilters, but given the extensive library support for the Mathlib hyperreals, and the fact that it already enjoys enough of a transfer principle for most applications, we will base our theory here on the Mathlib hyperreals.

Definition 1 (Positive hyperreals)**.** The positive hyperreals ${}^*\mathbb{R}^+$ are the set of hyperreals $X \in {}^*\mathbb{R}$ such that $X > 0$. (Note that X could be infinitesimal or infinite).

If X, Y are positive hyperreals, we write $X \lesssim Y$ if $X \leq CY$ for some real $C > 0$. We write $X \asymp Y$ if $X \lesssim Y \lesssim X$. We write $X \ll Y$ if $X \leq \varepsilon Y$ for all real $\varepsilon > 0$.

Lemma 2 (Lesssim order)**.** *$\lesssim$ is a pre-order on ${}^*\mathbb{R}^+$, with $\asymp$ the associated equivalence relation, and $\ll$ the associated strict order. Any two positive hyperreals are comparable under $\lesssim$.*

Proof. Easy. □

Definition 3 (Orders of magnitude)**.** The space $\mathcal{O}$ of orders of magnitude is defined to be the quotient space ${}^*\mathbb{R}^+/\asymp$ of the positive hyperreals by the relation of asymptotic equivalence. For a positive hyperreal X, we let $\Theta(X)$ denote the order of magnitude of X; this is a surjection from ${}^*\mathbb{R}^+$ to $\mathcal{O}$.

Lemma 4 (Theta kernel)**.** *For positive hyperreals X, Y, we have $\Theta(X) = \Theta(Y)$ if and only if $X \asymp Y$. Similarly, we have $\Theta(X) < \Theta(Y)$ if and only if $X \ll Y$, and $\Theta(X) \leq \Theta(Y)$ if and only if $X \lesssim Y$.*

Proof. Easy. □

Definition 5 (Ordering on magnitudes)**.** We linearly order $\mathcal{O}$ by the requirement that $\Theta(X) \leq \Theta(Y)$ if and only if $X \lesssim Y$, and $\Theta(X) < \Theta(Y)$ if and only if $X \ll Y$.

Definition 6 (One)**.** We define $1 := \Theta(1)$.

Lemma 7 (Constants trivial)**.** *We have $\Theta(C) = 1$ for all real $C > 0$.*

Proof. Easy. □

Definition 8 (Arithmetic on magnitudes)**.** We define addition on $\mathcal{O}$ by the requirement that $\Theta(X) + \Theta(Y) = \Theta(X + Y)$ for positive hyperreals X, Y. Similarly we define multiplication, inverse, and division. We define real exponentiation by requiring that $\Theta(X^\alpha) = \Theta(X)^\alpha$ for positive hyperreals X and real α.

Lemma 9 (Addition is tropical)**.** *For all orders of magnitude $\Theta(X), \Theta(Y)$, we have $\Theta(X) + \Theta(Y) = \max(\Theta(X), \Theta(Y))$.*

Proof. Easy. □

Corollary 10 (Additive commutative monoid)**.** *$\mathcal{O}$ is an ordered additive idempotent commutative monoid.*

Proof. Easy. □

Lemma 11 (Commutative semiring)**.** *$\mathcal{O}$ is a multiplicative ordered commutative group that distributes over addition. (It is not a semiring in the Mathlib sense because it does not have a zero element.)*

Proof. Easy. □

Lemma 12 (Power laws)**.** *Let $\Theta(X), \Theta(Y)$ be orders of magnitude, and α, β be real numbers.*

(i) We have $\Theta(XY)^\alpha = \Theta(X^\alpha Y^\alpha)$ and $\Theta(X/Y)^\alpha = \Theta(X^\alpha/Y^\alpha)$.

(ii) We have $\Theta(X^{\alpha\beta}) = \Theta(X^\alpha)^\beta$.

(iii) We have $\Theta(X)^0 = 1$, $\Theta(X)^1 = \Theta(X)$, and $\Theta(X)^{-1} = 1/\Theta(X)$.

(iv) We have $\Theta(1)^\alpha = 1$.

(v) We have $\Theta(X+Y)^\alpha = \Theta(X)^\alpha + \Theta(Y)^\alpha$ for $\alpha \geq 0$.

(vi) If $\alpha \geq 0$ and $\Theta(X) \leq \Theta(Y)$, then $\Theta(X)^\alpha \leq \Theta(Y)^\alpha$.

(vii) If $\alpha > 0$, then $\Theta(X) \leq \Theta(Y)$, if and only if $\Theta(X)^\alpha \leq \Theta(Y)^\alpha$, and $\Theta(X) < \Theta(Y)$ if and only if $\Theta(X)^\alpha < \Theta(Y)^\alpha$.

(viii) If $\alpha \leq 0$ and $\Theta(X) \leq \Theta(Y)$, then $\Theta(X)^\alpha \geq \Theta(Y)^\alpha$.

(ix) If $\alpha < 0$, then $\Theta(X) \leq \Theta(Y)$, if and only if $\Theta(X)^\alpha \geq \Theta(Y)^\alpha$, and $\Theta(X) < \Theta(Y)$ if and only if $\Theta(X)^\alpha > \Theta(Y)^\alpha$.

Proof. Straightforward. □

Definition 13 (Log-order of magnitude)**.** We define $\log \mathcal{O}$ to be the collection of formal logarithms of magnitude $\log(\Theta(X))$ of orders of magnitude $\Theta(X)$. 1, multiplication, and exponentiation of orders of magnitude become 0, addition, and scalar multiplication; and order is preserved. By definition, $\log\colon \mathcal{O} \to \log \mathcal{O}$ is a order isomorphism.

Lemma 14 (Log of addition)**.** *For orders of magnitude $\Theta(X), \Theta(Y)$, we have $\log(\Theta(X)+\Theta(Y)) = \max(\log(\Theta(X)), \log(\Theta(Y)))$.*

Proof. Immediate from Lemma 9. □

Lemma 15 (Logarithm of multiplication and exponentiation)**.** *For orders of magnitude $\Theta(X), \Theta(Y)$, we have $\log(1) = 0$ (and more generally $\log(C) = 0$ for any real $C > 0$), $\log(\Theta(X)\Theta(Y)) = \log(\Theta(X)) + \log(\Theta(Y))$, and $\log(\Theta(X)/\Theta(Y)) = \log(\Theta(X)) - \log(\Theta(Y))$. For any real α, we have $\log(\Theta(X)^\alpha) = \alpha \log(\Theta(X))$.*

Proof. . Straightforward from Lemma 8 and Lemma 12. □

Lemma 16 (Ordered vector space)**.** $\log \mathcal{O}$ *is a linearly ordered real vector space.*

Proof. Straightforward from Lemma 15. $\square$

Lemma 17 (Countable saturation)**.** *Let $I_1, I_2, \ldots$ be a sequence of internal sets in the hyperreals. If every finite intersection of these sets is non-empty, then $\bigcap_{n=1}^{\infty} I_n$ is non-empty.*

Proof. Write each I_n as an ultraproduct of $I_{n,m}$. Then, we can find a nested sequence $E_1 \supset E_2 \supset \ldots$ of large sets in the ultrafilter such that $\bigcap_{n \leq n_0} I_{n,m}$ is non-empty for all $m \in E_{n_0}$ and all n_0, and also $n_0 \notin E_{n_0}$. If we then pick x_m to lie in $\bigcap_{n \leq n_0} E_{n,m}$ where n_0 is the largest number for which $m \in E_{n_0}$, the ultralimit of the x_m will have the required properties. $\square$

Lemma 18 (Completeness)**.** *Let $I_1 \supset I_2 \supset \ldots$ be a decreasing sequence of non-empty open intervals (possibly unbounded) in $\mathcal{O}$. Then $\bigcap_{n=1}^{\infty} I_n$ is non-empty.*

Proof. Pull back each of the I_n to the hyperreals as a countable intersection of internal sets, and apply Lemma 17. $\square$

Lemma 19 (Completeness, II)**.** *Let $I_1 \supset I_2 \supset \ldots$ be a decreasing sequence of non-empty intervals (possibly unbounded or closed) in $\mathcal{O}$. Then $\bigcap_{n=1}^{\infty} I_n$ is non-empty.*

Proof. If one of the I_n is a singleton, the claim is straightforward. Otherwise, replace each I_n by its interior and use Lemma 18. $\square$