

PROOF OF MACLAURIN INEQUALITY

1. INTRODUCTION

Given n real numbers $y = (y_1, \dots, y_n) \in \mathbb{R}^n$ and $0 \leq k \leq n$, let $s_k(y)$ denote the elementary symmetric means

$$s_k(y) := \frac{1}{\binom{n}{k}} \sum_{1 \leq i_1 < \dots < i_k \leq n} y_{i_1} \dots y_{i_k}$$

(thus for instance $s_0(y) = 1$). We call a $n+1$ -tuple (s_0, \dots, s_n) of real numbers *attainable* if it is of the form $(s_0(y), \dots, s_n(y))$ for some y , or equivalently if the polynomial $\sum_{k=0}^n (-1)^k \binom{n}{k} s_k z^{n-k}$ is monic with all roots real. We have previously proven

Proposition 1.1 (Newton's inequality). *If (s_0, \dots, s_n) is attainable, then $s_k s_{k+2} \leq s_{k+1}^2$ for all $0 \leq k \leq n$.*

We now prove

Theorem 1.2 (Maclaurin inequality). *Suppose that (s_0, \dots, s_n) is an attainable tuple with all s_i positive. Then $s_\ell^{1/\ell} \leq s_k^{1/k}$ for all $1 \leq k \leq \ell \leq n$.*

Proof. By induction on ℓ it suffices to verify the case $\ell = k + 1$.

Now write $d_i = s_i / s_{i-1}$ for $1 \leq i \leq k + 1$, then the d_i are positive. The Newton inequality can then be rewritten as

$$d_{i+1} \leq d_i$$

for all $1 \leq i \leq k$, while the Maclaurin inequality

$$s_{k+1}^{\frac{1}{k+1}} \leq s_k^{\frac{1}{k}}$$

is equivalent to

$$s_{k+1}^k \leq s_k^{k+1}$$

which expands to

$$(d_1 \dots d_{k+1})^k \leq (d_1 \dots d_k)^{k+1}$$

which simplifies to

$$d_{k+1}^k \leq d_1 \dots d_k$$

so it suffices to show that $d_{k+1} \leq d_i$ for all $1 \leq i \leq k$. But this follows from the monotone decreasing nature of the d_i . \square

One can now obtain the Maclaurin inequality $s_{\ell}^{1/\ell}(y) \leq s_k^{1/k}(y)$ for all non-negative y by a standard limiting argument, replacing y_i by $y_i + \varepsilon$ for $\varepsilon > 0$, applying the above inequality, and sending ε to zero.