Doing Category Theory within HoTT

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Our goal is to formulate Category Theory inside Homotopy Type Theory and look at certain advantages this point of view offers.

Before we begin recall that a type A

- is called a (-1)**-type** or a **mere proposition** if for all $p, q : A$, $p =_A q$; if it is also inhabited, it is called **contractible**, i.e. there is $a : A$ so that $a = x$ for all $x : A$,
- is called a 0**-type** or a **set** if for all $x, y : A$ and $p, q : x =_A y$ we have $p =_{x =_A y} q$, and
- is called an 1**-type** if for all $x, y : A$ and $f, g : x =_A y$ and $p, q : f =_{x =_A y} g$ we have $p = q$.

Also, if A is an m-type $(-1 \leqslant m \leqslant \infty)$ and $-1 \leqslant n \leqslant m$ then $\|A\|_n$ is an n-type obtained by A by **truncating** at n.

1 Isomorphic objects *are* equal

2 Equivalent categories

3 The Rezk completion

To talk about categories inside HoTT, we begin from the notion of a "classical", set-theoretic category and replace sets with types so that we consider *types* of objects and *types* of arrows.

We shall further impose the condition that the type of arrows between two objects is Set so that the resulting theory remains along the same lines as fundamental Category Theory, without the need for higher categories, essentially dealing with 1-categories only.

We call the resulting structure a *precategory*.

A **precategory** $\mathscr{A}$ consists of a *type* $\mathscr{A}_0$ of **objects**, for every $a, b : \mathscr{A}_0$ (or, informally, $a, b : \mathscr{A}$) a *set* $\mathscr{A}(a, b)$ of **arrows** so that

- For each object $a : \mathscr{A}$ there is an arrow $1_a : \mathscr{A}(a, a)$.
- For each $a, b, c : \mathscr{A}$ there is a *function*

$$\mathscr{A}(b, c) \to \mathscr{A}(a, b) \to \mathscr{A}(a, c)$$

  called **composition**, denoted by $g \mapsto f \mapsto gf$.

These data are subject to the following axioms

1. For all $a, b : \mathscr{A}$ and $f : \mathscr{A}(a, b)$, $f 1_a = f$ and $1_b f = f$.
2. For all $a, b, c, d : \mathscr{A}$ and

$$f : \mathscr{A}(a, b), \quad g : \mathscr{A}(b, c), \quad h : \mathscr{A}(c, d),$$

   we have $h(gf) = (hg)f$.

Examples. All examples are relative to a fixed universe $\mathcal{U}$.

1. **Set** is the precategory whose objects are sets, that is

$$\mathbf{Set}_0 :\equiv \{X : \mathcal{U} \mid \mathsf{isSet}(X)\},$$

and for any $X, Y : \mathbf{Set}_0$,

$$\mathbf{Set}(X, Y) :\equiv (X \to Y).$$

$1_X : X \to X$ is the identity function $1_X :\equiv \lambda(x : X).\ x$ and, for every $X, Y, Z : \mathbf{Set}_0$, function composition is

$$\lambda(f : \mathbf{Set}(Y, Z)).\ \lambda(g : \mathbf{Set}(X, Y)).\ \lambda(x : X).\ g\big(f(x)\big).$$

We shall not bother to be so formal from now on though.

Examples. (continued)

2. Suppose G is a group. There is a precategory $\mathscr{G}$ (or $\mathsf{B}G$) where $\mathscr{G}_0 :\equiv \mathbf{1}$, the unit type with $\star : \mathbf{1}$ and $\mathscr{G}(\star, \star) :\equiv G$. Composition is the group multiplication and 1_G is the neutral element of the group G.
3. For any type X we can form a precategory $\mathscr{A}$ whose type of objects is
$$\mathscr{A}_0 :\equiv X$$
and
$$\mathscr{A}(x, y) :\equiv \|x = y\|_0$$
for any $x, y : X$. This precategory is called the **fundamental pregroupoid** of X. If X is a set, no truncation is needed and we call $\mathscr{A}$ the **discrete precategory** on X.

In precategories, as in set-theoretic Category Theory, we have two notions of "sameness" for objects $a, b : \mathscr{A}_0$ in a precategory $\mathscr{A}$:

1. *the usual equality*: a, b are the "same" if $a =_{\mathscr{A}_0} b$
2. *category-theoretic isomorphism*: a, b are *for all intents and purposes* the "same" if they are **isomorphic**: $a \cong_{\mathscr{A}_0} b$.

Here, the type $a \cong_{\mathscr{A}_0} b$ (or simply $a \cong b$) is defined in the obvious way:

$$a \cong b :\equiv \left\{ f : \mathscr{A}(a,b) \,\middle|\, \sum_{g:\mathscr{A}(b,a)} (fg = 1_b) \times (gf = 1_a) \right\}.$$

Classical, set-theoretic foundations, fail to precisely connect the two notions. Let's see how Homotopy Type Theory handles it.

We want to formally be able to identify equal and isomorphic objects. The only claim we can make in a precategory, like for categories within Set Theory, is that

"Equal ⇒ isomorphic"

Given a precategory $\mathscr{A}$ and $a, b : \mathscr{A}$, $\mathsf{idtoiso} : (a = b) \to (a \cong b)$.

But contrary to Set Theory, the phrase "isomorphic objects are equal" is meaningful inside HoTT. It means: "*idtoiso is an equivalence*".

We want our type-theoretic notion of a category to encompass the previous idea, we therefore define

Definition

A **category** is a precategory for which idtoiso is an equivalence.

Now, we can retrieve a good part of set-based Category Theory just by invoking the

Univalence Axiom

For any $A, B : \mathcal{U}$, $\mathsf{idtoeqv} : (A =_{\mathcal{U}} B) \to (A \simeq B)$ is an equivalence.

Examples. (all relative to a *univalent* universe $\mathcal{U}$)

1. The precategory **Set** is a category by the Univalence Axiom (specializing $\mathsf{idtoeqv}$ to sets gives us $\mathsf{idtoiso}$).
2. (**Structure Identity Principle**) Many familiar mathematical structures, such as groups, rings, topological spaces etc, form a category (like in set-theoretic frameworks) as follows: we start with a precategory of sets, endow these sets with the desired structure, obtaining a precategory of structures and apply the Structure Identity Principle to prove that they form categories.

Some examples of precategories that are *not* categories are

1. For a group G, $\mathscr{G}$ is not a category unless G is trivial; $G = G$ contains only refl_G while *all* arrows $G \to G$ are isomorphisms.
2. The fundamental pregroupoid is not necessarily a category.

"In set-theoretic foundations, the statement “every fully faithful and essentially surjective functor is an equivalence of categories” is equivalent to the axiom of choice. But with the univalence axiom, it is just *true*."

—The HoTT book.

Without dwelling on notions like the precategory of precategories or the category of categories, we will examine the various notions of "sameness" between (pre)categories within HoTT. For this, we may fix a universe $\mathcal{U}$.

The simplest and most restrictive notion of "sameness" between (pre)categories is equality. Naturally, this is over the top restrictive, so we immediately move to better notions of sameness.

The next best notion is, naturally, isomorphism which is characterized by the following

Lemma

Given precategories $\mathscr{A}, \mathscr{B}$ and a functor $F : \mathscr{A} \to \mathscr{B}$, the following are equivalent:

1. *F is fully faithful and $F_0 : \mathscr{A}_0 \to \mathscr{B}_0$ is an equivalence of types.*
2. *There **merely** exists a functor $G : \mathscr{B} \to \mathscr{A}$ and $\eta : GF = 1_{\mathscr{A}}$ and $\varepsilon : FG = 1_{\mathscr{B}}$.*

If F satisfies any of the equivalent conditions above we say that F is an **isomorphism of (pre)categories** and that $\mathscr{A}, \mathscr{B}$ are **isomorphic**, symbolically $\mathscr{A} \cong \mathscr{B}$. Two (pre)categories are *for all intents and purposes* the "same" if they are isomorphic.

For (pre)categories, isomorphism is, by extension of the Univalence Axiom, equivalent to equality.

Proposition

If $\mathscr{A}, \mathscr{B}$ are precategories, the function

$$(\mathscr{A} = \mathscr{B}) \to (\mathscr{A} \cong \mathscr{B}),$$

which is defined by induction from the identity functor, is an equivalence of types.

From a set-theoretic viewpoint, isomorphism is still very restrictive. A better notion of sameness for set-based categories is equivalence "$\simeq$":

Two (pre)categories are *essentially* the "same" if they are equivalent.

But, simply translating the set-theoretic equivalence does not yield a mere proposition; two (pre)categories can be equivalent in more than one ways. That's why we use adjoint equivalences.

Definition

A functor $F : \mathscr{A} \to \mathscr{B}$ is an **equivalence of (pre)categories** if it is left adjoint whose unit and counit η, ε are isomorphisms.

Adjointness and isomorphism are both mere propositions, therefore so is being an equivalence.

For precategories, we have the desired result

Proposition

Let $F : \mathscr{A} \to \mathscr{B}$ be a functor between precategories. The following **types** are equivalent

1. F is an equivalence of precategories.
2. F is fully faithful and **split essentially surjective**, i.e. for every $b : \mathscr{B}$ there exists some $a : \mathscr{A}$ so that $Fa \cong b$.

The fine point here lies with the existential quantifier which implies that such $a : \mathscr{A}$ can be *constructed*, hence the adjective "split".

When $\mathscr{A}$ is *not* a category however, F being split essentially surjective may not be a mere proposition; for any witness (a, f) and any $g : a \cong a'$, $(a', f \circ Fg)$ is also a witness. If we want a *mere* property, we are forced to define a functor $F : \mathscr{A} \to \mathscr{B}$ between (pre)categories to be **essentially surjective** if for all $b : \mathscr{B}$ there *merely* exists some $a : \mathscr{A}$ so that $Fa \cong b$.
Luckily, for categories we have that

Lemma

For a category $\mathscr{A}$, a precategory $\mathscr{B}$ and $F : \mathscr{A} \to \mathscr{B}$ a fully faithful functor, the type $\sum_{(a:\mathscr{A})} Fa \cong b$ is a mere proposition for any $b : \mathscr{B}$.

Therefore we obtain that

Corollary (choice free!)

A functor between categories is an equivalence if and only if it is fully faithful and essentially surjective.

It is worth noting that

Lemma

A functor $F : \mathscr{A} \to \mathscr{B}$ between categories is an equivalence of categories if and only if it is an isomorphism.

Therefore we get the next important result

Theorem

For categories $\mathscr{A}$, $\mathscr{B}$, the function

$$(\mathscr{A} = \mathscr{B}) \to (\mathscr{A} \simeq \mathscr{B}),$$

defined by induction on the identity functor, is an equivalence of types.

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The Yoneda lemma carries over to precategories only with the appropriate transitions from sets to types. In particular, for a precategory $\mathscr{A}$ we denote the **Yoneda embedding** as

$$\mathbf{y} : \mathscr{A} \to \mathbf{Set}^{\mathscr{A}^{\mathrm{op}}} : \quad a \mapsto \mathscr{A}(-,a)$$
$$(a \xrightarrow{f} b) \mapsto \left(\mathscr{A}(-,a) \xrightarrow{g \mapsto fg} \mathscr{A}(-,b)\right)$$

And we have that

The Yoneda Lemma

For any precategory $\mathscr{A}$, any $a : \mathscr{A}$ and any functor $F : \mathscr{A}^{\mathrm{op}} \to \mathbf{Set}$, we have an isomorphism $\mathbf{Set}^{\mathscr{A}^{\mathrm{op}}}(\mathbf{y}a, F) \cong Fa$. This is natural in both a and F.

Corollary

The Yoneda embedding $\mathbf{y}$ *is fully faithful.*

As we saw, there are precategories which are *not* categories. But a precategory always admits a fully faithful and essentially surjective functor into a category. This is called the **Rezk completion** of $\mathscr{A}$.

Theorem

For any precategory $\mathscr{A}$ there is a category $\hat{\mathscr{A}}$ and a fully faithful and essentially surjective functor $\mathscr{A} \to \hat{\mathscr{A}}$.

Proof. Define

$$\hat{\mathscr{A}}_0 :\equiv \left\{ F : \mathbf{Set}^{\mathscr{A}^{\mathrm{op}}} \;\middle|\; \text{there is some } a : \mathscr{A}.\, (\mathbf{y}a \cong F) \right\}$$

with arrows inherited from $\mathbf{Set}^{\mathscr{A}^{\mathrm{op}}}$. The inclusion $\hat{\mathscr{A}} \to \mathbf{Set}^{\mathscr{A}^{\mathrm{op}}}$ is fully faithful and an embedding on objects. Since **Set** is a category, so is $\mathbf{Set}^{\mathscr{A}^{\mathrm{op}}}$ and therefore $\hat{\mathscr{A}}$ is also a category.

The Yoneda embedding $\mathscr{A} \to \hat{\mathscr{A}}$ is fully faithful and essentially surjective by the definition of $\hat{\mathscr{A}}_0$. ⋄

Examples.

1. The Rezk completion of $\mathscr{G}$, G being a group, is the category of G-torsors or principal G-bundles.
2. The Rezk completion of the fundamental pregroupoid of a type X is the **fundamental groupoid** of X.

The importance of the Rezk completion lies in the following result.

Theorem

Given two precategories $\mathscr{A}$, $\mathscr{B}$, a category $\mathscr{C}$ and a fully faithful and essentially surjective functor $H : \mathscr{A} \to \mathscr{B}$, the functor

$$- \circ H : \mathscr{C}^{\mathscr{A}} \to \mathscr{C}^{\mathscr{B}}$$

is an isomorphism.

For a precategory $\mathscr{A}$, take $\mathscr{B}$ to be its Rezk completion $\hat{\mathscr{A}}$ in the above theorem and H to be the Yoneda embedding $\mathscr{A} \to \hat{\mathscr{A}}$. The above isomorphism states that any functor from $\mathscr{A}$ to a category $\hat{\mathscr{A}}$ factors *essentially* uniquely through $\hat{\mathscr{A}}$.

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