Grothendieck's Galois Theory

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April 4, 2024

Fundamental Theorem of Galois Theory for finite extensions; 1

Let $k \leqslant E$ be a finite Galois extension. The Galois correspondence

$$\{L : k \leqslant L \leqslant E\} \underset{\mathrm{Fix}_E(H) \leftarrow\!\shortmid H}{\overset{L \mapsto \mathrm{Aut}(E/L)}{\rightleftarrows}} \{H : H \leqslant \mathrm{Gal}(E/k)\}$$

is order reversing and bijective.

Fundamental Theorem of Galois Theory for finite extensions; 2

Suppose $k \leqslant E$ is a finite Galois extension as before and L is a subextension, i.e. $k \leqslant L \leqslant E$. Then $L \leqslant E$ is Galois. Furthermore

$$[E:L] = |\operatorname{Gal}(E/L)| \quad \text{and} \quad [L:k] = [\operatorname{Gal}(E/k) : \operatorname{Aut}(E/L)].$$

The extension $k \leqslant L$ is normal (hence Galois) if and only if $\operatorname{Aut}(E/L)$ is a normal subgroup of $\operatorname{Gal}(E/k)$ in which case

$$\operatorname{Gal}(L/k) \cong \operatorname{Gal}(E/k)/\operatorname{Gal}(E/L).$$

For infinite Galois extensions, the theory breaks: the Galois group **is no longer finite** and therefore the correspondence is **not** bijective.

Since the Galois group is infinite, it has **too many subgroups** and not all subgroups can therefore arise as Galois groups of subextensions.

This can be fixed by endowing the Galois group with an appropriate topology, the **Krull topology** (name after Wolfang Krull) which can distinguish between subgroups than can arise as Galois groups and subgroups that cannot. In particular we have

Proposition

The subgroups (of the Galois group of an extension) that can arise as Galois groups of subextensions are precisely the **closed** subgroups with respect to the Krull topology.

The Krull topology is a rather natural one. The important thing to notice is that if $k \leqslant E$ is an infinite extension, then $\mathrm{Gal}(E/k)$ is a profinite topological group. In particular,

$$\mathrm{Gal}(E/k) \cong \varprojlim_{L} \mathrm{Gal}(L/k)$$

through the map (which is both a group isomorphism and a homeomorphism)

$$\vartheta : \mathrm{Gal}(E/k) \to \varprojlim_{L} \mathrm{Gal}(L/k) : \sigma \mapsto (\sigma|_L)_L$$

where L runs through all subextensions $k \leqslant L \leqslant E$ with $k \leqslant L$ being a finite Galois extension. If we endow the finite groups $\mathrm{Gal}(L/k)$ with the **discrete** topology then $\varprojlim_{L} \mathrm{Gal}(L/k)$ has a natural topology, the Krull topology. **The Krull topology for a finite extension is the discrete topology**.

Fundamental Theorem of Galois Theory for infinite extensions; 1

Let $k \leqslant E$ be a (possibly infinite) Galois extension. The Galois correspondence

$$\{L : k \leqslant L \leqslant E\} \underset{\mathrm{Fix}_E(H) \leftarrow H}{\overset{L \mapsto \mathrm{Aut}(E/L)}{\rightleftarrows}} \{H : H \leqslant \mathrm{Gal}(E/k) \text{ closed}\}$$

is order reversing and bijective.

But the Krull topology does not only fix the correspondence. It makes the Galois group a **compact**, **Hausdorff**, **totally disconnected** space, properties which enable us to prove...

Fundamental Theorem of Galois Theory for infinite extensions; 2

Suppose $k \leqslant E$ is a (possibly infinite) Galois extension as before, with Galois group $G = \mathrm{Gal}(E/k)$ endowed with the Krull topology, and L a subextension, i.e. $k \leqslant L \leqslant E$. Then $L \leqslant E$ is Galois, with Galois group $H = \mathrm{Gal}(E/L)$ and

$$[G : H] < \infty \Leftrightarrow H \text{ is open} \Leftrightarrow [L : k] < \infty.$$

in which case $[G : H] = [L : k]$. Moreover, the extension $k \leqslant L$ is normal (hence Galois) if and only if $\mathrm{Aut}(E/L)$ is a normal subgroup of $\mathrm{Gal}(E/k)$ in which case

$$\mathrm{Gal}(L/k) \cong \mathrm{Gal}(E/k)/\,\mathrm{Gal}(E/L),$$

i.e. are both isomorphic as groups and homeomorphic as topological spaces.

Classification of Covering Spaces; 1

If X is path connected, locally path connected and semilocally simply connected topological space and $x_0 \in X$, then the correspondence

$$\left\{ \begin{array}{c} \text{path connected,} \\ \text{locally path connected} \\ \text{covering spaces of } X \end{array} \right\} \rightleftarrows \left\{ \begin{array}{c} \text{conjugacy classes of} \\ \text{subgroups of } \pi_1(X, x_0) \end{array} \right\}$$

$$\widetilde{X} \mapsto \pi_1(\widetilde{X}, \widetilde{x}_0)$$

between path connected and locally path connected coverings of X and conjugacy classes of subgroups of $\pi_1(X, x_0)$ is bijective.

Classification of Covering Spaces; 2

Suppose X is a path connected, locally path connected and semilocally simply connected space and $p : (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ a path connected, locally path connected covering of X. Set $H = p_*\big(\pi_1(\widetilde{X}, \widetilde{x}_0)\big)$. Then

(i) The covering is Galois if and only if H is normal in $G = \pi_1(X, x_0)$.

(ii) $A(\widetilde{X}, p)$ is isomorphic to $N(H)/H$, where $N(H)$ is the normalizer of H in G.

(iii) If $(\widetilde{X}, p)$ is Galois then $A(\widetilde{X}, p)$ is isomorphic to G/H. In particular, $A(Y, q)$ is isomorphic to G; here (Y, q) is the universal cover of X.

Grothendieck's perspective of the Fundamental Theorem

Let k be a field and $G = \mathrm{Gal}(k_{\mathrm{sep}}/k)$ its absolute Galois group. The category $\mathbf{Sep}_k^f$ of finite separable k-algebras is antiequivalent to the category $G\text{-}\mathbf{set}_f^t$ of sets on which G acts continuously and transitively. Galois extensions of k correspond to G-sets that are isomorphic to finite quotients of G.

But Grothendieck's theory takes things one step further.

A finite k-algebra A is called **étale** if it is the finite product of separable extensions of k.

Grothendieck's formulation of Galois Theory

Let k be a field and $G = \mathrm{Gal}(k_{\mathrm{sep}}/k)$ its absolute Galois group. The category $\mathbf{FEt}_k$ of finite étale k-algebras is antiequivalent to the category $G\text{-}\mathbf{set}_f$ of finite sets equipped with a continuous G-action.

Infinite Galois Theory – Grothendieck's approach

Let $k \leqslant E$ be a (possibly infinite) Galois extension. Then, the category $\mathbf{Split}_k(E)$ of k-algebras split by E is antiequivalent to the category $\mathrm{Gal}(E/k)\textbf{-set}_{\mathrm{prof}}$ of profinite $\mathrm{Gal}(E/k)$-sets.