Towards fibrant double categories

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Abstract

These are some *elementary* notes I took on fibrant double categories for the purposes of our group seminar. Any feedback will be very much appreciated.

Contents

1 Categories and functors

1.1 2-categories

There are examples of categories where, together with objects and arrows, we have *arrows between arrows* as well. Such an example is of course the category of (small) categories with functors and natural transformations between functors. Another example is the category of topological spaces with continuous maps and (homotopy classes of) homotopies.

The definition of a 2-category captures such structures. The main point is that arrows and arrows between arrows also form a category!

Definition. A 2-**category** $\mathscr{A}$ consists of

- A class $\mathscr{A}_0$ of 0-**cells** (objects of $\mathscr{A}$) which we will denote with capital letters as $X, Y, Z, \ldots$.
- For any two 0-cells X, Y, a *small category* $\mathscr{A}(X, Y)$ whose objects are called 1-**cells** (arrows in $\mathscr{A}$) and we will denote traditionally as $f, g, h, \ldots$ and arrows are called 2-**cells** (arrows between arrows in $\mathscr{A}$) which we denote like natural transformations as $\alpha, \beta, \gamma, \ldots$.

- For each triple of 0-cells X, Y, Z, a *functor* (of two variables)

$$\circ_{XYZ} : \mathscr{A}(X,Y) \times \mathscr{A}(Y,Z) \to \mathscr{A}(X,Z)$$

  (the composition of arrows in $\mathscr{A}$ which now has to respect the category structure in contrast to an ordinary category).

- For each 0-cell X, a *functor*

$$u_X : \mathbf{1} \to \mathscr{A}(X,X),$$

  (which gives the unit arrows of $\mathscr{A}$ and now also has to respect the category structure) where $\mathbf{1}$ is the category with one object and one (identity) arrow.

These data obey the following axioms

1. (*associativity*) For any 0-cells X, Y, Z, W the following square commutes

$$\begin{tikzcd}
\mathscr{A}(X,Y) \times \mathscr{A}(Y,Z) \times \mathscr{A}(Z,W) \arrow[r, "\circ_{XYZ} \times 1"] \arrow[d, "1 \times \circ_{YZW}"'] & \mathscr{A}(X,Z) \times \mathscr{A}(Z,W) \arrow[d, "\circ_{XZW}"] \\
\mathscr{A}(X,Y) \times \mathscr{A}(Y,W) \arrow[r, "\circ_{XYW}"] & \mathscr{A}(X,W)
\end{tikzcd}$$

2. (*units*) For any 0-cells X, Y the following diagram commutes

$$\begin{tikzcd}
\mathbf{1} \times \mathscr{A}(X,Y) \arrow[d, "u_X \times 1"'] & \mathscr{A}(X,Y) \arrow[l, "\cong"'] \arrow[r, "\cong"] \arrow[d, equal] & \mathscr{A}(X,Y) \times \mathbf{1} \arrow[d, "1 \times u_Y"] \\
\mathscr{A}(X,X) \times \mathscr{A}(X,Y) \arrow[r, "\circ_{XXY}"'] & \mathscr{A}(X,Y) & \mathscr{A}(X,Y) \times \mathscr{A}(Y,Y) \arrow[l, "\circ_{XYY}"]
\end{tikzcd}$$

Before we see some examples, we need to make a

Remark 1.1. Having a category inside a category suggests that we have two kinds of compositions. Actually, since $\circ_{XYZ}$ are functors, we have 3! Let's break this down.

First of all, we have the usual composition of (compatible) 1-cells under the functors $\circ_{XYZ}$ which we shall denote as $f \circ g$, omitting the subscript when no confusion arises.

Secondly, we have two kinds of composition for 2-cells. The first is the (vertical) composition of 2-cells inside each category $\mathscr{A}(X,Y)$ which we shall denote as $\beta \odot \alpha$. The second, is a (horizontal) composition of compatible 2-cells: Given $\alpha : f_1 \Rightarrow f_2$ in $\mathscr{A}(X,Y)$ and $\beta : g_1 \Rightarrow g_2$ in $\mathscr{A}(Y,Z)$, we get the image of (α, β) under $\circ_{XYZ}$, which is a 2-cell in $\mathscr{A}(X,Z)$, and will be denoted as $\beta * \alpha$; that is, $\beta * \alpha : g_1 \circ f_1 \Rightarrow g_2 \circ f_2$.

Diagrammatically, this means that 2-cells can be composed vertically

$$\begin{tikzcd}
X \arrow[r, bend left=60, "f"{name=F}] \arrow[r, "g" description, ""{name=G}] \arrow[r, bend right=60, "h"'{name=H}] & Y \arrow[r, rightsquigarrow] & X \arrow[r, bend left=60, "f"{name=F2}] \arrow[r, bend right=60, "h"'{name=H2}] & Y
\arrow[Rightarrow, from=F, to=G, "\alpha"]
\arrow[Rightarrow, from=G, to=H, "\beta"]
\arrow[Rightarrow, from=F2, to=H2, "\beta \odot \alpha"]
\end{tikzcd}$$

and horizontally as

$$\begin{tikzcd}
X \arrow[r, bend left, "f_1"{name=F1}] \arrow[r, bend right, "f_2"'{name=F2}] & Y \arrow[r, bend left, "g_1"{name=G1}] \arrow[r, bend right, "g_2"'{name=G2}] & Z \arrow[r, rightsquigarrow] & X \arrow[r, bend left, "g_1 \circ f_1"{name=A}] \arrow[r, bend right, "g_2 \circ f_2"'{name=B}] & Z
\arrow[Rightarrow, from=F1, to=F2, "\alpha"]
\arrow[Rightarrow, from=G1, to=G2, "\beta"]
\arrow[Rightarrow, from=A, to=B, "\beta * \alpha"]
\end{tikzcd}$$

The two compositions of 2-cells are compatible in the following sense.

Proposition 1.2 (Interchange Law). *Given a diagram in a 2-category*

$$\begin{tikzcd}
X \arrow[rr, bend left=50, "f_1"] \arrow[rr, "f_2" description] \arrow[rr, bend right=50, "f_3"'] & \arrow[phantom, u, shift left=0, "\Downarrow\alpha"] \arrow[phantom, d, "\Downarrow\beta"] & Y \arrow[rr, bend left=50, "g_1"] \arrow[rr, "g_2" description] \arrow[rr, bend right=50, "g_3"'] & \arrow[phantom, u, "\Downarrow\phi"] \arrow[phantom, d, "\Downarrow\psi"] & Z
\end{tikzcd}$$

the two possible composites $g_1 f_1 \Rightarrow g_3 f_3$ are equal: $(\psi * \beta) \odot (\varphi * \alpha) = (\psi \odot \varphi) * (\beta \odot \alpha)$.

The next examples will hopefully make the definition clearer. As usual, we will not bother checking all the details, we just describe the three kinds of cells.

Examples 1.3.
1. Categories, functors and natural transformations constitute a 2-category. This is the prime example to have in mind.
2. As we mentioned in the beginning, topological spaces, continuous maps and *homotopy classes of* homotopies also form a 2-category.
3. From a monoidal category $(\mathcal{V}, \otimes, I)$ we can form the category $\mathsf{Mon}(\mathcal{V})$ of monoids and monoid morphisms in $\mathcal{V}$. Given two monoid morphisms

$$\begin{tikzcd}
(A, m_A, \eta_A) \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & (B, m_B, \eta_B)
\end{tikzcd}$$

   we can define 2-cells $\alpha : f \Rightarrow g$ as arrows $\alpha : A \to B$ inside $\mathcal{V}$ such that the diagrams

$$\begin{tikzcd}
A \arrow[r, "\alpha \otimes f"] \arrow[d, "g \otimes \alpha"'] & B \otimes B \arrow[d, "m_B"] \\
B \otimes B \arrow[r, "m_B"'] & B
\end{tikzcd}$$

   commute. We thus obtain the 2-category of monoids in $\mathcal{V}$. Dually, one can take the 2-category $\mathsf{Comon}(\mathcal{V})$ of comonoids in $\mathcal{V}$.
4. The morphisms of the category Ord of preordered sets and monotone mappings have a natural preorder $\leq$ defined for any $f, g \in \mathsf{Ord}[(X, \leqslant_X), (Y, \leqslant_Y)]$ as

$$f \leq g \iff f(x) \leqslant_Y g(x) \ \forall x \in X,$$

   which makes Ord into a 2-category.
5. In the category Grp of groups, we can define a 2-cell between any two homomorphisms of groups $f, g : G \rightrightarrows H$ as follows: $\alpha : f \Rightarrow g$ if $\alpha \in H$ so that $g(x) = \alpha^{-1} f(x) \alpha$ for all $x \in G$, thus forming a 2-category.

1.2 Bicategories

Weakening the notion of a 2-category in the appropriate way, i.e. by replacing the over-restricting equalities by isomorphisms, we get the notion of a *bicategory*.

Definition. A **bicategory** $\mathscr{A}$ consists of

- A class $\mathscr{A}_0$ of 0-**cells**.

- For any two 0-cells X, Y, a *small category* $\mathscr{A}(X,Y)$ whose objects are called 1-**cells** and arrows are called 2-**cells**.
- For each triple of 0-cells X, Y, Z, a *functor* (of two variables)

$$\circ_{XYZ} : \mathscr{A}(X,Y) \times \mathscr{A}(Y,Z) \to \mathscr{A}(X,Z).$$

- For each 0-cell X, a *functor*

$$u_X : \mathbf{1} \to \mathscr{A}(X,X).$$

These data obey the following axioms

1. (*associativity*) For any 0-cells X, Y, Z, W there is a *natural isomorphism*

$$\begin{tikzcd}
\mathscr{A}(X,Y) \times \mathscr{A}(Y,Z) \times \mathscr{A}(Z,W) \arrow[r, "\circ_{XYZ} \times 1"] \arrow[d, "1 \times \circ_{YZW}"'] & \mathscr{A}(X,Z) \times \mathscr{A}(Z,W) \arrow[d, "\circ_{XZW}"] \\
\mathscr{A}(X,Y) \times \mathscr{A}(Y,W) \arrow[r, "\circ_{XYW}"'] \arrow[ur, phantom, "\overset{\cong}{\underset{\alpha}{\nearrow}}"] & \mathscr{A}(X,W)
\end{tikzcd}$$

   whose components are $\alpha_{h,g,f} : (h \circ g) \circ f \overset{\cong}{\Rightarrow} h \circ (g \circ f)$

2. (*units*) For any 0-cells X, Y, two *natural isomorphisms*

$$\begin{tikzcd}
\mathbf{1} \times \mathscr{A}(X,Y) \arrow[d, "u_X \times 1"'] & \mathscr{A}(X,Y) \arrow[l, "\cong"'] \arrow[r, "\cong"] \arrow[d, equal] \arrow[dl, phantom, "\overset{\cong}{\underset{\lambda}{\Leftarrow}}"] \arrow[dr, phantom, "\overset{\cong}{\underset{\rho}{\Rightarrow}}"] & \mathscr{A}(X,Y) \times \mathbf{1} \arrow[d, "1 \times u_Y"] \\
\mathscr{A}(X,X) \times \mathscr{A}(X,Y) \arrow[r, "\circ_{XXY}"'] & \mathscr{A}(X,Y) & \mathscr{A}(X,Y) \times \mathscr{A}(Y,Y) \arrow[l, "\circ_{XYY}"]
\end{tikzcd}$$

   with components $\lambda_f : f \overset{\cong}{\Rightarrow} 1_Y \circ f$ and $\rho_f : f \overset{\cong}{\Rightarrow} f \circ 1_X$ respectively.

and satisfy the following *coherence conditions* (due to the lack of equalities): for any 1-cells $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W \xrightarrow{\ell} T$, the following diagrams commute

$$\begin{tikzcd}
((\ell \circ h) \circ g) \circ f \arrow[rr, "\alpha_{\ell,h,g} * 1_f"] \arrow[d, "\alpha_{\ell \circ h,g,f}"'] & & (\ell \circ (h \circ g)) \circ f \arrow[d, "\alpha_{\ell,h \circ g,f}"] \\
(\ell \circ h) \circ (g \circ f) \arrow[dr, "\alpha_{\ell,h,g \circ f}"'] & & \ell \circ ((h \circ g) \circ f) \arrow[dl, "1_\ell * \alpha_{h,g,f}"] \\
& \ell \circ (h \circ (g \circ f)) &
\end{tikzcd}$$

$$\begin{tikzcd}
(g \circ 1_Y) \circ f \arrow[rr, "\alpha_{g,1_Y,f}"] \arrow[dr, "\rho_g * 1_f"'] & & g \circ (1_Y \circ f) \arrow[dl, "1_g * \lambda_f"] \\
& g \circ f &
\end{tikzcd}$$

Remark 1.4. The same remark we did on 2-categories applies here as well. Namely, we have composition of 1-cells as well as two kind of compositions for 2-cells which we will also denote as $\circ$, $\odot$ and $*$. The interchange law holds for bicategories as well.

Examples 1.5. 1. Every 2-category is a bicategory.

2. Recall that in a category $\mathscr{C}$, a **span** from an object X to an object Y is a triple of the form $(Z \in \mathscr{C}, f : Z \to X, g : Z \to Y)$ depicted as

$$\begin{tikzcd} & Z \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X & & Y \end{tikzcd}$$

and a **morphism of spans** $(Z_1, f_1, g_1) \to (Z_2, f_2, g_2)$ is an arrow $h : Z_1 \to Z_2$ in $\mathscr{C}$ so that

$$\begin{tikzcd} & Z_1 \arrow[dl, "f_1"'] \arrow[dr, "g_1"] \arrow[dd, "h"] & \\ X & & Y \\ & Z_2 \arrow[ul, "g_2"] \arrow[ur, "f_2"'] & \end{tikzcd}$$

commutes. Now suppose $\mathscr{C}$ has *pullbacks*. Then we can compose spans by taking pullbacks:

$$\begin{tikzcd} Z_1 \times_Y Z_2 \arrow[r] \arrow[d] & Z_2 \arrow[r] \arrow[d] & W \\ Z_1 \arrow[r] \arrow[d] & Y & \\ X & & \end{tikzcd}$$

Thus we obtain the bicategory $\mathsf{Span}(\mathscr{C})$ of spans whose 0-cells are the objects of $\mathscr{C}$, 1-cells are spans and 2-cells are morphisms of spans. Note that composition (pullbacks) of 1-cells (spans) is unique only *up to a 2-isomorphism* (morphism of spans) and not unique in general.

Dually, one can also form the bicategory $\mathsf{Cospan}(\mathscr{C})$ of **cospans** (provided of course that $\mathscr{C}$ has enough *pushouts*).

3. There's the bicategory Bim of rings and bimodules.

The 0-cells are unital rings $R, S, T, \ldots$

For any two 0-cells (rings) R, S, $\mathsf{Bim}(R, S)$ is the category of (R, S)-bimodules. What this means is that the 1-cells of Bim from a 0-cell R to a 0-cell S is an (R, S)-bimodule M, which we shall denote as $M : R \multimap S$, and given two 1-cells $M, N : R \multimap S$, a 2-cell $f : M \Rightarrow N$ is just an (R, S)-linear map.

Composition is defined naturally as

$$\mathsf{Bim}(R, S) \times \mathsf{Bim}(S, T) \to \mathsf{Bim}(S, T) : ({}_R M_S, {}_S N_T) \mapsto M \otimes_S N.$$

And it is known that tensoring is functorial. Observe that this composition is associative *up to isomorphism* since the two possible composites are isomorphic

$$(L \otimes M) \otimes N \cong L \otimes (M \otimes N)$$

but not equal.

Lastly, for any 0-cell R, R is itself an (R,R)-bimodule and tensoring some M (either right or left R-module) with R (on the appropriate side) over R produces something *isomorphic* to M. Hence R is the unit in $\mathsf{Bim}(R,R)$.

4. From any monoidal category $(\mathscr{C},\otimes,I)$ we get a bicategory $\mathsf{B}\mathscr{C}$, its **delooping**.

   $\mathsf{B}\mathscr{C}$ has only one 0-cell, let's name it $\bullet$. $\mathsf{B}\mathscr{C}(\bullet,\bullet)$ is then the category $\mathscr{C}$ itself. That is, 1-cells of $\mathsf{B}\mathscr{C}$ are the objects of $\mathscr{C}$ and 2-cells of $\mathsf{B}\mathscr{C}$ are the arrows of $\mathscr{C}$.

   Composition of arrows in given by the tensor product while the units are given by the tensor unit. Associativity and unitality carry over from $\mathscr{C}$.

1.3 Appropriate functors

Along with 2-categories, there is a proper definition for 2-functors.

Definition. Suppose $\mathscr{A},\mathscr{B}$ are 2-categories. A **2-functor** $F:\mathscr{A}\to\mathscr{B}$ assigns

- to each 0-cell $X\in\mathscr{A}$ an 0-cell $FX\in\mathscr{B}$
- to every pair of 0-cells X,Y of $\mathscr{A}$, *functor*

$$F_{X,Y}:\mathscr{A}(X,Y)\to\mathscr{B}(FX,FY)$$

so that the structure of the 2-category is preserved. Namely,

1. (composition) For any 0-cells $X,Y,Z\in\mathscr{A}_0$, the following square commutes

$$\begin{tikzcd}
\mathscr{A}(X,Y)\times\mathscr{A}(Y,Z) \arrow[r, "\circ_{XYZ}"] \arrow[d, "F_{XY}\times F_{YZ}"'] & \mathscr{A}(X,Z) \arrow[d, "F_{XZ}"] \\
\mathscr{B}(FX,FY)\times\mathscr{B}(FY,FZ) \arrow[r, "\circ_{FX,FY,FZ}"'] & \mathscr{B}(FX,FZ)
\end{tikzcd}$$

2. (units) For any 0-cell $X\in\mathscr{A}_0$, the following triangle commutes

$$\begin{tikzcd}
\mathbf{1} \arrow[r, "u_X"] \arrow[dr, "u_{FX}"'] & \mathscr{A}(X,X) \arrow[d, "F_{XX}"] \\
& \mathscr{B}(FX,FX)
\end{tikzcd}$$

Actually there are weaker analogues of 2-functors that have been proven most fruitful. For the next two definitions take $\mathscr{A},\mathscr{B}$ to be 2- or bicategories.

Definition. A **pseudofunctor** $F:\mathscr{A}\to\mathscr{B}$ assigns

- to each 0-cell $X\in\mathscr{A}$ an 0-cell $FX\in\mathscr{B}$
- to every pair of 0-cells X,Y of $\mathscr{A}$, *functor*

$$F_{X,Y}:\mathscr{A}(X,Y)\to\mathscr{B}(FX,FY)$$

so that the structure of the 2-category is preserved *up to isomorphism*. Namely,

1. (composition) For any 0-cells $X, Y, Z \in \mathscr{A}_0$, there is a *natural isomorphism*

$$\begin{tikzcd} \mathscr{A}(X,Y) \times \mathscr{A}(Y,Z) \arrow[rr, "\circ_{XYZ}"] \arrow[d, "F_{XY} \times F_{YZ}"'] & & \mathscr{A}(X,Z) \arrow[d, "F_{XZ}"] \\ \mathscr{B}(FX,FY) \times \mathscr{B}(FY,FZ) \arrow[rr, "\circ_{FX,FY,FZ}"'] \arrow[urr, phantom, "\overset{\gamma_{XYZ}}{\Nearrow}\cong"] & & \mathscr{B}(FX,FZ) \end{tikzcd}$$

2. (units) For any 0-cell $X \in \mathscr{A}_0$, there is a *natural isomorphism*

$$\begin{tikzcd} \mathbf{1} \arrow[r, "u_X"] \arrow[dr, "u_{FX}"'] \arrow[r, phantom, shift right=4ex, "\overset{\delta_X}{\Nearrow}\cong"] & \mathscr{A}(X,X) \arrow[d, "F_{XX}"] \\ & \mathscr{B}(FX,FX) \end{tikzcd}$$

In many cases, it is best to consider even more relaxed functors; functors that preserve the structure only up to a 2-cell.

Definition. A **lax functor** $F : \mathscr{A} \to \mathscr{B}$ assigns

- to each 0-cell $X \in \mathscr{A}$ an 0-cell $FX \in \mathscr{B}$
- to every pair of 0-cells X, Y of $\mathscr{A}$, *functor*

$$F_{X,Y} : \mathscr{A}(X,Y) \to \mathscr{B}(FX,FY)$$

so that the structure of the 2-category is preserved *up to natural transformation*. Namely,

1. (composition) For any 0-cells $X, Y, Z \in \mathscr{A}_0$, there is a *natural transformation*

$$\begin{tikzcd} \mathscr{A}(X,Y) \times \mathscr{A}(Y,Z) \arrow[rr, "\circ_{XYZ}"] \arrow[d, "F_{XY} \times F_{YZ}"'] & & \mathscr{A}(X,Z) \arrow[d, "F_{XZ}"] \\ \mathscr{B}(FX,FY) \times \mathscr{B}(FY,FZ) \arrow[rr, "\circ_{FX,FY,FZ}"'] \arrow[urr, phantom, "\overset{\gamma_{XYZ}}{\Nearrow}"] & & \mathscr{B}(FX,FZ) \end{tikzcd}$$

2. (units) For any 0-cell $X \in \mathscr{A}_0$, there is a *natural transformation*

$$\begin{tikzcd} \mathbf{1} \arrow[r, "u_X"] \arrow[dr, "u_{FX}"'] \arrow[r, phantom, shift right=4ex, "\overset{\delta_X}{\Nearrow}"] & \mathscr{A}(X,X) \arrow[d, "F_{XX}"] \\ & \mathscr{B}(FX,FX) \end{tikzcd}$$

The above natural transformations γ_{XYZ} and δ_A need not be invertible. They associate the composites in one way, perhaps not the other. This leads naturally to the notion of an **oplax functor** where the above transformations are the other way around.

Of course any 2-functor is a pseudofunctor and every pseudofunctor is (op)lax.

Examples 1.6. 1. For a small 2-category $\mathscr{A}$ and some 0-cell $X \in \mathscr{A}$ we get the representable 2-functor

$$\mathscr{A}(X,-) : \mathscr{A} \to \mathsf{Cat}.$$

Any 0-cell Y is sent to the category $\mathscr{A}(X,Y)$ and any 1-cell $f : Y \to Z$ is sent to the functor

$$\mathscr{A}(X,f) \equiv f_* : \mathscr{A}(X,Y) \to \mathscr{A}(X,Z)$$

where $f_*(g) = f \circ g$ for any 1-cell $g : X \to Y$ and $f_*(\alpha) = i_f * \alpha$ for any 2-cell $\alpha : g \Rightarrow h$.

2. For a bicategory $\mathscr{A}$, a(n) (op)lax functor $\mathbf{1} \to \mathscr{A}$ is equivalent to a (co)monad in $\mathscr{A}$.

2 Grothendieck Fibrations

2.1 Fibrations

There are many occasions when we work with slice categories $\mathscr{C}/I$ over an object. For example, when $\mathscr{C} = \mathsf{Set}$, an object of Set/I, $f : X \to I$, is just the usual notion of a family of sets indexed by I, $(X_i)_{i\in I}$, where $X_i := \{x \in X : f(x) = i\}$. We will "generalize" that in the sense that we will consider families of objects of a category indexed by another category.

For the rest of this section fix two categories $\mathscr{A}, \mathscr{B}$ and a functor $P : \mathscr{A} \to \mathscr{B}$.

Definition. Suppose $I \in \mathscr{B}_0$. The **fiber $\mathscr{A}_I$ of P over I** consists of those objects $X \in \mathscr{A}_0$ for which $PX = I$ and those arrows $f \in \mathscr{A}_1$ for which $Pf = 1_I$.

A simple implication of the definition is that

Lemma 2.1. *With the notation of the above definition, $\mathscr{A}_I$ is a subcategory of $\mathscr{A}$.*

Definition. An arrow $f : X \to Y$ in $\mathscr{A}$ is **cartesian over** $\alpha : I \to J$ in $\mathscr{B}$ if $Pf = \alpha$ and for all g into Y and every factorization of Pg through α as $Pg = \alpha \circ v$ there is a unique factorization of g through f as $g = f \circ u$ with $Pu = v$.

$$\begin{tikzcd}
\mathscr{A} \arrow[dd, "P"'] & Z \arrow[dr, dashed, "\exists ! u"'] \arrow[drr, bend left, "\forall g"] & & \\
& & X \arrow[r, "f"'] & Y \\
\mathscr{B} & PZ \arrow[dr, "\forall v"'] \arrow[drr, bend left, "Pg"] & & \\
& & I \arrow[r, "\alpha"'] & J
\end{tikzcd}$$

In classical homotopy theory, a *fibration* is a continuous function $p : E \to B$ between topological spaces that has certain *lifting properties* such as the lifting of paths and homotopies. The definition of a Grothendieck fibration vaguely resembles this.

Definition. $P : \mathscr{A} \to \mathscr{B}$ is a (**Grothendieck**) **fibration** if for any $\alpha : I \to J$ and $Y \in \mathscr{A}_J$, α has a cartesian lift with codomain Y.

Remarks 2.2. 1. An arrow $\alpha : I \to J$ may have more than one cartesian liftings to a given codomain; the collection of all cartesian lifts with a given codomain is called a **cleavage**.

For $P : \mathscr{A} \to \mathscr{B}$ to be a fibration means that for each such arrow α (whose codomain is in the image of P) we have *chosen* a specific cartesian lift of α for every $Y \in \mathscr{A}_J$. Such a lift will be denoted as $\mathrm{Cart}(\alpha, Y)$.

2. If $P : \mathscr{A} \to \mathscr{B}$ is a fibration, then any $\alpha : I \to J$ whose codomain is in the image of P induces a functor between the corresponding fibers

$$\alpha^* : \mathscr{A}_J \to \mathscr{A}_I,$$

which is called the **reindexing functor** and is defined as follows. For any object $Y \in \mathscr{A}_J$ take $\alpha^* Y$ to be the domain of $\mathrm{Cart}(\alpha, Y)$. If $u : Y \to Z$ inside $\mathscr{A}_J$ then take $\alpha^* u$ to be the unique (from the cartesian property of $\mathrm{Cart}(\alpha, Z)$) arrow $\alpha^* Y \to \alpha^* Z$ that makes the following square commutative.

$$\begin{tikzcd}
\mathscr{A} \arrow[ddd, "P"'] & \alpha^* Y \arrow[r, "\mathrm{Cart}(\alpha, Y)"] \arrow[d, dashed, "\exists!\, \alpha^* u"'] & Y \arrow[d, "u"] \\
 & \alpha^* Z \arrow[r, "\mathrm{Cart}(\alpha, Z)"'] & Z \\
 & & \\
\mathscr{B} & I \arrow[r, "\alpha"] & J
\end{tikzcd}$$

Examples 2.3. 1. For any category $\mathscr{C}$, the identity functor $1_{\mathscr{C}} : \mathscr{C} \to \mathscr{C}$ and the unique functor $\mathscr{C} \to \mathbf{1}$ are (trivially) fibrations.

2. Given any category $\mathscr{C}$, the category $\mathsf{Fam}(\mathscr{C})$ has objects families $(X_i)_{i \in I}$ of objects of $\mathscr{C}$ indexed by some set I. A morphism between two families $(X_i)_{i \in I} \to (Y_j)_{j \in J}$ consists of a function $\alpha : I \to J$ together with a family of arrows $f_i : X_i \to Y_{\alpha(i)}$ in $\mathscr{C}$.

Then, the functor

$$\varphi : \mathsf{Fam}(\mathscr{C}) \to \mathsf{Set}.$$

that sends a family to the indexing set and a morphism to the underlying set function is a fibration. Indeed, for any change of indices $\alpha : I \to J$ and any family $(Y_j)_{j \in J}$ indexed by J, every arrow

$$(\beta, f_\ell) : (Z_\ell)_{\ell \in L} \to (Y_j)_{j \in J} \quad \text{in } \mathsf{Fam}(\mathscr{C})$$

can be factored uniquely through the family $(Y_{\alpha(i)})_{i \in I}$.

3. Suppose $\mathscr{C}$ is a category with finite limits. The codomain functor

$$\mathsf{cod} : \mathscr{C}^{\mathbf{2}} \to \mathscr{C}$$

($\mathbf{2}$ being the category $0 \to 1$) that sends an arrow to its codomain and a morphism of arrows to its component on the codomains is a fibration.

Indeed, for any $\alpha : X \to Y$ in $\mathscr{C}$ and any $f : Z \to Y$ in $(\mathscr{C}^{\mathbf{2}})_Y$, form the pullback

$$\begin{tikzcd}
X \times_Y Z \arrow[r, "p_Z"] \arrow[d, "p_X"'] & Z \arrow[d, "f"] \\
X \arrow[r, "\alpha"'] & Y
\end{tikzcd}$$

in $\mathscr{C}$ and take $\mathrm{Cart}(\alpha, Z \xrightarrow{f} Y)$ to be (p_Z, α) and $\alpha^*(Z \xrightarrow{f} Y)$ to be $X \times_Y Z \xrightarrow{p_X} X$. By universality of the pullback, this is indeed a cartesian lifting of α.

4. For any categories $\mathscr{A}, \mathscr{B}$, the projection functor

$$\pi : \mathscr{A} \times \mathscr{B} \to \mathscr{B}$$

is a fibration since for any $\alpha : B \to B'$ in $\mathscr{B}$ and any $(A,B') \in (\mathscr{A} \times \mathscr{B})_{B'}$,

$$(A,B) \xrightarrow{(1_A,\alpha)} (A,B')$$

is a cartesian lift of α to (A,B').

2.2 Fibrations and pseudofunctors

Given a fibration $F : \mathscr{A} \to \mathscr{B}$, we noted that for any $I \in \mathscr{B}$, $\mathscr{A}_I$ is a subcategory of $\mathscr{A}$. Moreover, for any arrow $\alpha : I \to J$ in $\mathscr{B}$ there is an induced reindexing pseudofunctor $\alpha^* : \mathscr{A}_J \to \mathscr{A}_I$. This suggests the existence of a functor

$$\mathscr{B}^{op} \to \mathsf{Cat} : I \mapsto \mathscr{A}_I,\ \alpha \mapsto \alpha^*.$$

Actually, this is a pseudofunctor since $(g \circ f)^* \cong g^* \circ f^*$ and $1_{\mathscr{A}_X} \cong 1_X^*$. If these isomorphisms are equalities then we have indeed a functor and the fibration is called **split**.

But the inverse is also true. That is, given a (pseudo)functor $F : \mathscr{B}^{op} \to \mathsf{Cat}$ we can get a fibration over $\mathscr{B}$; this is called the **Grothendieck construction** and goes as follows. From F form the **Grothendieck category** $\int F$ whose elements are pairs $(X \in \mathscr{B}, A \in (FX)_0)$ and arrows $(X,A) \to (Y,B)$ are pairs of the form $(f : X \to Y, \phi : A \to f^*B)$. Then, define a functor

$$\int F \to \mathscr{B} : (X,A) \mapsto X,\ \ (f,\phi) \mapsto f.$$

This is a fibration.

This "analogy" actually goes deeper and reveals a 2-equivalence of 2-categories (which we shall not exhibit).

2.3 Fibrations and limits

We shall present a result on how fibrations and limits interact.

Proposition 2.4. *Let $\mathcal{I}$ be a small category and $P : \mathscr{A} \to \mathscr{B}$ a fibration whose codomain $\mathscr{B}$ has all $\mathcal{I}$-limits. The following are equivalent:*

1. *$\mathscr{A}$ has all $\mathcal{I}$-limits and these are preserved by P.*
2. *All fibers $\mathscr{A}_X$ have $\mathcal{I}$-limits and these are preserved by the reindexing functors.*

Proof. We shall present the proof for the case of products. That is, take $\mathcal{I}$ to be the discrete category with 2 objects, say a and b.

(1) $\implies$ (2) : Fix some arbitrary $X \in \mathscr{B}$ and take a functor $G : \mathcal{I} \to \mathscr{A}_X$. Since $\mathscr{A}_X$ is a subcategory of $\mathscr{A}$ and the latter has all $\mathcal{I}$-limits by definition, the composite diagram

$$\mathcal{I} \xrightarrow{G} \mathscr{A}_X \overset{i}{\hookrightarrow} \mathscr{A}$$

has a limit, say

$$\begin{tikzcd} & G_b \\ \lim(iG) \arrow[ur, "\lambda_B"] \arrow[dr, "\lambda_a"'] & \\ & G_a \end{tikzcd}$$

Since P preserves $\mathcal{I}$-limits, $P\lim(iG) = \lim(PiG)$, that is

$$\begin{tikzcd}
 & PG_b \\
P\lim(iG) \arrow[ur, "P\lambda_B"] \arrow[dr, "P\lambda_a"'] & \\
 & PG_a
\end{tikzcd}$$

is a limit cone for PiG. But G takes values in $\mathscr{A}_X$, therefore $PiG_a = PiG_b = X$. Since $(X, 1_X, 1_X)$ is another cone, there is a unique $\Delta : X \to P\lim(iG)$.

$$\begin{tikzcd}
 & & G_b \\
\mathscr{A} \arrow[dddd, "P"'] & \lim(iG) \arrow[ur, "\lambda_B"] \arrow[dr, "\lambda_a"'] & \\
 & & G_a \\
 & & PG_a \arrow[d, equal, "1_X"] \\
 & & \\
\mathscr{B} & P\lim(iG) \arrow[uur, "P\lambda_a"] \arrow[ddr, "P\lambda_b"'] & X \arrow[l, dashed, "\exists!\Delta"'] \arrow[dd, equal, "1_X"] \\
 & & \\
 & & PG_b
\end{tikzcd}$$

Now P being a fibration implies that there is a cartesian lift of Δ to $\lim(iG)$,

$$\Delta^* \lim(iG) \xrightarrow{\mathrm{Cart}(\Delta, \lim(iG))} \lim(iG).$$

We will show that

$$(\Delta^* \lim(iG) \xrightarrow{\lambda_a \circ \mathrm{Cart}(\Delta, \lim(iG))} G_a, \quad \Delta^* \lim(iG) \xrightarrow{\lambda_b \circ \mathrm{Cart}(\Delta, \lim(iG))} G_b)$$

is a limit cone for G in $\mathscr{A}_X$.

$$
\begin{tikzcd}
 & & G_a \\
\mathscr{A} \arrow[dddd, "P"'] & \lim(iG) \arrow[ur, bend left, "\lambda_a"] \arrow[dr, bend right, "\lambda_B"'] & \Delta^*\lim(iG) \arrow[l, "{\mathrm{Cart}(\Delta, \lim(iG))}"'] \\
 & & G_b \\
 & & PG_a \arrow[d, equal, "1_X"] \\
\mathscr{B} & P\lim(iG) \arrow[ur, bend left, "P\lambda_a"] \arrow[dr, bend right, "P\lambda_b"'] & X \arrow[l, "\Delta"'] \arrow[d, equal, "1_X"] \\
 & & PG_b
\end{tikzcd}
$$

It is trivially a cone (by the lack of arrows in the initial diagram) and by construction $\Delta^*\lim(iG)$ is in $\mathscr{A}_X$. We need only show that it is a limit cone. For this take any other cone (C, f_a, f_b) in $\mathscr{A}_X$. By universality, there is a unique $f : C \to \lim(iG)$ which translates through P to an arrow $Pf : PC \to P\lim(iG)$ in $\mathscr{B}$ that has a factorization through Δ as $Pf = \Delta$.

$$
\begin{tikzcd}
 & & G_a & \\
\mathscr{A} \arrow[dddd, "P"'] & \lim(iG) \arrow[ur, bend left, "\lambda_a"] \arrow[dr, bend right, "\lambda_B"'] & \Delta^*\lim(iG) \arrow[l, "{\mathrm{Cart}(\Delta, \lim(iG))}"'] & C \arrow[ul, bend right, "f_a"'] \arrow[dl, bend left, "f_b"] \arrow[l, dashed, "\exists!\hat{f}"'] \arrow[ll, dashed, bend left, "f"] \\
 & & G_b & \\
 & & PG_a \arrow[d, equal] & \\
\mathscr{B} & P\lim(iG) \arrow[ur, bend left, "P\lambda_a"] \arrow[dr, bend right, "P\lambda_b"'] & X \arrow[l, "\Delta"'] \arrow[d, equal] \arrow[r, equal] & PC \arrow[ul, bend right, "Pf_a"'] \arrow[dl, bend left, "Pf_b"] \\
 & & PG_b &
\end{tikzcd}
$$

By the property of the cartesian lift $\mathrm{Cart}(\Delta, \lim(iG))$ there is a unique $\hat{f} : C \to \Delta^*\lim(iG)$ with $P\hat{f} = 1_X$ so that $f = \mathrm{Cart}(\Delta, \lim(iG)) \circ \hat{f}$. This $\hat{f}$ is the required arrow.

(2) $\implies$ (1): For the contrary, take a functor $G : \mathcal{I} \to \mathscr{A}$. By hypothesis, $\mathscr{B}$ has all $\mathcal{I}$-limits and therefore the diagram $P \circ G$ has a limit, say $(\lim(PG), k_a, k_b)$. Since P is a fibration, we can

take the cartesian lifts of k_a and k_b and by definition their domains $k_a^*G_a, k_b^*G_b$ are in the same fiber $\mathscr{A}_{\lim(PG)}$ of P.

$$
\begin{tikzcd}
\mathscr{A} \arrow[dddd, "P"'] & & k_a^*G_a \arrow[dl, "\mathrm{Cart}(k_a, G_a)"'] & & \\
& G_a & k_b^*G_b \arrow[dr, "\mathrm{Cart}(k_b, G_b)"] & & \\
& & & G_b & \\
& & \lim(PG) \arrow[dl, "k_a"'] \arrow[dr, "k_b"] & & \\
\mathscr{B} & PG_a & & PG_b &
\end{tikzcd}
$$

But fibers have all $\mathcal{I}$-limits hence (by taking the diagram $G' : \mathcal{I} \to \mathscr{A}$ with $a \mapsto k_a^*G_a, b \mapsto k_b^*G_b$) there exists their limit, say $(L, \lambda_a, \lambda_b)$. We will show that

$$\big(L, \lambda_a \circ \mathrm{Cart}(k_a, G_a), \lambda_b \circ \mathrm{Cart}(K_b, G_b)\big)$$

is a limit cone for the diagram G. For this, take another cone (C, c_a, c_b) of G. Then, (PC, Pc_a, Pc_b) is a cone for the diagram $PG_a \quad PG_b$ which gives a unique arrow $f : PC \to \lim(PG)$.

$$
\begin{tikzcd}
\mathscr{A} \arrow[dddddd, "P"'] & & & L \arrow[d, "\lambda_a"'] \arrow[dr, bend left, "\lambda_b"] & & \\
& & & k_a^*G_a \arrow[dl, "\mathrm{Cart}(k_a, G_a)"'] & k_b^*G_b \arrow[dr, "\mathrm{Cart}(k_b, G_b)"] & \\
& & G_a & & & G_b \\
& C \arrow[ur, "c_a"] \arrow[urrrr, bend right, "c_b"'] & & & & \\
& & & \lim(PG) \arrow[dl, "k_a"'] \arrow[drr, "k_b"] & & \\
& & PG_a & & & PG_b \\
\mathscr{B} & PC \arrow[ur, "Pc_a"] \arrow[uurr, dashed, bend right, "f"] \arrow[urrrr, bend right, "Pc_b"'] & & & &
\end{tikzcd}
$$

By the cartesianess of $\mathrm{Cart}(k_a, G_a)$ and $\mathrm{Cart}(k_b, G_b)$, there are unique arrows $C \to k_a^*G_a$ and $C \to k_b^*G_b$. If C is in the same fiber as L then by the universal property of L as a limit, there is a unique arrow $C \to L$. If C is not in the same fiber, we can use the reindexing functor $f^* : \mathscr{A}_{\lim(PG)} \to \mathscr{A}_{PC}$ to obtain the necessary arrow $PC \to \lim(PG)$. $\diamond$

3 Double stuff

3.1 Double categories

An ordinary category is a category object internal to Set; a *double* category is a category internal to Cat (and this is a definition!). What does this mean exactly? Well, for starters, this amounts to the collections of objects and arrows being *categories* instead of plain sets.

Definition. A **pseudo double category** $\mathbb{D}$ consists of

- A *category* $\mathbb{D}_0$ whose objects are 0**-cells** and arrows are **vertical** 1**-cells**.
- A *category* $\mathbb{D}_1$ whose objects are **horizontal** 1**-cells** and arrows are 2**-cells**.
- An **identity** *functor*
$$1 : \mathbb{D}_0 \to \mathbb{D}_1,$$
sending each 0-cell to the horizontal identity and each vertical 1-cell to the 2-cell identity.
- **Source** and **target** *functors*
$$\mathfrak{s}, \mathfrak{t} : \mathbb{D}_1 \rightrightarrows \mathbb{D}_0,$$
sending horizontal 1-cells to their domains/codomains and 2-cells to their left/right vertical 1-cell.
- A **composition** *functor*
$$\odot : \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \to \mathbb{D}_1$$
for composing horizontally, horizontal 1-cells.
- *Natural isomoprhisms*[1]
$$\begin{aligned} &\alpha : (M \odot N) \odot P \overset{\cong}{\Rightarrow} M \odot (N \odot P) && \text{(associator)} \\ &\lambda : 1_{\mathfrak{s}(M)} \odot M \overset{\cong}{\Rightarrow} M && \text{(left unitor)} \\ &\rho : M \odot 1_{\mathfrak{t}(M)} \overset{\cong}{\Rightarrow} M && \text{(right unitor)} \end{aligned}$$

Satisfying the following axioms

1. $\mathfrak{s}(1_X) = \mathfrak{t}(1_X) = X$, $\mathfrak{s}(M \odot N) = \mathfrak{s}(M)$ and $\mathfrak{t}(M \odot N) = \mathfrak{t}(N)$.
2. $\mathfrak{s}(\alpha), \mathfrak{t}(\alpha), \mathfrak{s}(\lambda), \mathfrak{t}(\lambda), \mathfrak{s}(\rho)$ and $\mathfrak{t}(\rho)$ are all identities.

and subject to the same coherence conditions for the associator and the unitors as a bicategory.

Let's unpack the definition slowly and establish some notation. First of all, we have the category $\mathbb{D}_0$ of 0-cells which we shall denote as ordinary objects $X, Y, Z, \ldots$ and *vertical* 1-cells which we shall denote as

$$\begin{tikzcd} X \arrow[d, "f"'] \\ Y \end{tikzcd}$$

or just $f : X \to Y$ when writing them in line. $\mathbb{D}_0$ being a category entails, first and foremost, that we can compose vertical arrows $f : X \to Y$ and $g : Y \to Z$ as $g \circ f : X \to Z$ and that there is an identity vertical arrow $1_X : X \to X$ for any 0-cell X.

[1] Hence the adjective "pseudo" double category.

Then, there is the category $\mathbb{D}_1$ of *horizontal* 1-cells which we shall denote as

$$X \overset{A}{\longrightarrow\!\!\!\!\!\!\bullet\;\;} Y$$

or $A : X \to\!\!\!\!\!\bullet\;\; Y$ when writing them in line, and 2-cells which will be denoted as

$$\begin{tikzcd} X \arrow[r, "A", "\bullet"{marking}] \arrow[d, phantom, "\Downarrow\alpha"] & Y \\ Z \arrow[r, "B"', "\bullet"{marking}] & W \end{tikzcd}$$

As before, we can compose two 2-cells $\alpha : A \Rightarrow B$ and $\beta : B \Rightarrow C$ as $\beta \circ \alpha : A \Rightarrow C$ and there are identity 2-cells $1_A : A \Rightarrow A$ for any horizontal 1-cell $A : X \to\!\!\!\!\!\bullet\;\; Y$.

$\mathbb{D}_0$ is the intended category of objects and $\mathbb{D}_1$ the intended category of arrows we want to study. 0-cells are in most cases our objects, vertical 1-cells are usually some kind of "strict" arrows between the objects while horizontal 1-cells usually account for more "loose"" kinds of arrows (see examples below). The 2-cells are our arrows between arrows.

The identity functor gives the horizontal 1-cell identity[2] $1_X : X \to\!\!\!\!\!\bullet\;\; X$ for any 0-cell X. The source and target functors assign to each horizontal 1-cell $X \to\!\!\!\!\!\bullet\;\; Y$ its source X and target Y, both 0-cells, hence the notation $X \to\!\!\!\!\!\bullet\;\; Y$ for horizontal 1-cells. They also assign a source and a target vertical 1-cell to every 2-cell. Thus, 2-cells are actually of the form

$$\begin{tikzcd} X \arrow[r, "A", "\bullet"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\alpha"] & Y \arrow[d, "g"] \\ Z \arrow[r, "B"', "\bullet"{marking}] & W \end{tikzcd}$$

and we shall denote them as ${}^f\alpha^g$ when we want to emphasize that. We must point out that the source and target of a 2-cell as above do *not* refer to A and B but to f and g.

Lastly, the composition gives for any two horizontal 1-cells $X \to\!\!\!\!\!\bullet\;\; Y$ and $Y \to\!\!\!\!\!\bullet\;\; Z$ a horizontal 1-cell $X \to\!\!\!\!\!\bullet\;\; Z$ and for another kind (horizontal) of composition of 2-cells (contrary to the previous vertical composition of 2-cells).

Remark 3.1. From any double category $\mathbb{D}$, we get its **horizontal bicategory** $\mathcal{H}(\mathbb{D})$ if we disregard its vertical 1-cells, that is the bicategory consisting of the 0-cells, horizontal 1-cells and **globular** (i.e. with the same source and target) 2-cells of $\mathbb{D}$. Many interesting bicategories arise as horizontal bicategories of some double category.

Examples 3.2. 1. Any category can be viewed as a double category where the only 2-cells are the identities. Similarly, any 2-category can be viewed as a double category in two trivial ways, by taking either horizontal or vertical 1-cells to be identities. This way, both categories and 2-categories can be studied through double categories.

2. Given a 2-category $\mathscr{A}$, we can form a double category $\mathbb{Q}\mathscr{A}$ of **quintets** of $\mathscr{A}$. The 0-cells of $\mathbb{Q}\mathscr{A}$ are the 0-cells of $\mathscr{A}$, the *horizontal and vertical* 1-cells of $\mathbb{Q}\mathscr{A}$ are the 1-cells of $\mathscr{A}$ and the 2-cells of $\mathbb{Q}\mathscr{A}$ are the 2-cells of $\mathscr{A}$.

3. $\mathbb{C}\mathrm{at}$ is the double category of categories (0-cells), functors (vertical 1-cells), profunctors (horizontal 1-cells) and natural transformations (2-cells). Identity, source and target are given in the obvious way and composition of profunctors is given by the coend

$$(G \odot F)(x,z) = \int^y F(y,z) \times G(x,y).$$

[2] We have already used the same symbol 1 with some subscript to denote identities of vertical 1-cells and 2-cells and use it again for identities of horizontal 1-cells. I kindly ask the reader to tolerate this abuse of notation which will be of lesser importance for the rest of this short exposition.

4. Given a category $\mathscr{C}$ with pullbacks, $\mathbb{S}\mathsf{pan}(\mathscr{C})$ is the double category whose 0-cells are objects of $\mathscr{C}$, vertical 1-cells are arrows of $\mathscr{C}$, horizontal 1-cells are spans and 2-cells are morphisms of spans. Identities, source and target functors are evident. Composition $\odot$ of horizontal 1-cells is given by taking pullbacks. Observe that $\mathcal{H}(\mathbb{S}\mathsf{pan}(\mathscr{C})) = \mathsf{Span}(\mathscr{C})$.

   Dual arguments can be made for $\mathbb{C}\mathsf{ospan}(\mathscr{C})$.

5. $\mathbb{R}\mathsf{el}(\mathsf{Set})$ is the double category of sets (0-cells), functions (vertical 1-cells) and relations (horizontal 1-cells). A 2-cell

$$\begin{tikzcd} X \arrow[r, "R", "\bullet"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\alpha"] & Y \arrow[d, "g"] \\ Z \arrow[r, "S"', "\bullet"{marking}] & W \end{tikzcd}$$

   exists if and only if $xRy \implies f(x)\,S\,g(y)$. Again, identities, source and target are evident. Composition $\odot$ is given by the usual composition of relations.

   We can replace Set with any category that we can compose relations, i.e. any *regular* category. $\mathcal{H}(\mathbb{R}\mathsf{el}(\mathsf{Set}))$ is the bicategory $\mathsf{Rel}(\mathsf{Set})$ of relations.

6. Suppose $(\mathcal{V}, \otimes, I)$ is a monoidal category. Then, $\mathcal{V}$-$\mathbb{M}\mathsf{at}$ is the double category with 0-cells being objects of $\mathcal{V}$ and vertical 1-cells arrows in $\mathcal{V}$, horizontal 1-cells from some X to some Y are **$\mathcal{V}$-matrices** $S : X \mathrel{-\!\!\bullet\!\!\to} Y$, which are families (S_{yx}) of objects of $\mathcal{V}$ indexed by $Y \times X$ or, equivalently, profunctors $Y \times X \to \mathcal{V}$ and the 2-cells between two $\mathcal{V}$-matrices $S : X \mathrel{-\!\!\bullet\!\!\to} Y$ and $T : W \mathrel{-\!\!\bullet\!\!\to} Z$ with vertical 1-cells $f : X \to W$ and $g : Y \to Z$ are families of arrows

$$^{f}\alpha^{g} = \{\alpha_{yx} : S_{yx} \to T_{gy,fx}\}$$

   that is, natural transformations

$$\begin{tikzcd} Y \times X \arrow[rr, bend left, "R"] \arrow[dr, "g \times f"'] \arrow[rr, phantom, "\Downarrow\alpha"] & & \mathcal{V} \\ & Z \times W \arrow[ur, "T"'] & \end{tikzcd}$$

   Composition is given by

$$S_{yx} \odot T_{zy} = (T \odot S)_{zx} := \sum_{y} T_{zy} \otimes S_{yx}.$$

7. The double category $\mathbb{B}\mathsf{im}$ has rings as 0-cells and ring homomorphisms as vertical 1-cells. Its horizontal 1-cells are bimodules and 2-cells are bimodule homomorphisms. $\mathcal{H}(\mathbb{B}\mathsf{im}) = \mathsf{Bim}$.

3.2 Fibrant double categories

There are certain double categories whose structure is richer due to an interplay between their vertical and their horizontal 1-cells.

Proposition 3.3. *Let $\mathbb{D}$ be a double category. The following are equivalent.*

1. *Every vertical 1-cell $f : X \to Y$ has a* ***companion****, i.e. some horizontal 1-cell $\hat{f} : X \nrightarrow Y$ and 2-cells p_1, p_2 such that*

$$\begin{tikzcd} X \arrow[r, "\hat{f}"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow p_1"] & Y \arrow[d, equal] \\ Y \arrow[r, "1_Y"'] & Y \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "1_X"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow p_2"] & X \arrow[d, "f"] \\ X \arrow[r, "\hat{f}"'] & Y \end{tikzcd}$$

*and a **cojoint**, i.e. some horizontal* 1*-cell* $\check{f}: Y \nrightarrow X$ *and* 2*-cells* q_1, q_2 *such that*

$$\begin{tikzcd} Y \arrow[r, "\check{f}"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow q_1"] & X \arrow[d, "f"] \\ Y \arrow[r, "1_Y"'] & Y \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "1_X"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow q_2"] & X \arrow[d, equal] \\ Y \arrow[r, "\check{f}"'] & X \end{tikzcd}$$

2a. *The functor* $\langle \mathfrak{s}, \mathfrak{t} \rangle : \mathbb{D}_1 \to \mathbb{D}_0 \times \mathbb{D}_0$ *is a fibration.*

2b. *The functor* $\langle \mathfrak{s}, \mathfrak{t} \rangle : \mathbb{D}_1 \to \mathbb{D}_0 \times \mathbb{D}_0$ *is an opfibration.*

2c. *The functor* $\langle \mathfrak{s}, \mathfrak{t} \rangle : \mathbb{D}_1 \to \mathbb{D}_0 \times \mathbb{D}_0$ *is a bifibration.*

Proof. If we unpack the definition of $\mathfrak{s}$ or $\mathfrak{t}$ being (op)fibrations we will retrieve (1). So the only thing we really need to check is why these two being fibrations and opfibrations are equivalent conditions. ⋄

Definition. A double category which satisfies the above conditions is called **fibrant**.

A great number of the categories encountered in practice are fibrant.

Examples 3.4.

1. $\mathbb{B}$im is fibrant. For any ring homomorphism $f : R \to S$, the (R, S)-bimodule ${}_RS_S$ is a companion of f while the (S, R)-bimodule ${}_SS_R$ is a cojoint (both R-actions are given by f).
2. $\mathbb{S}$pan($\mathscr{C}$) is fibrant ($\mathscr{C}$ being a category with pulbacks). For any arrow $f : X \to Y$ the span $X \xleftarrow{1_X} X \xrightarrow{f} Y$ is a companion and $Y \xleftarrow{f} X \xrightarrow{1_X} X$ is a cojoint.
3. $\mathbb{R}$el is fibrant. For any function $f : X \to Y$, its companion $\hat{f}$ is simply its graph and its cojoint $\check{f}$ the opposite graph.
4. $\mathcal{V}$-$\mathbb{M}$at is fibrant. For any arrow $f : X \to Y$, its companion and cojoint are defined as

$$\hat{f}_{y,x} = \check{f}_{x,y} = \begin{cases} I, & \text{if } y = f(x) \\ 0, & \text{otherwise} \end{cases}$$

References

[1] V. Aravantinos-Sotiropoulos, C. Vasilakopoulou, *Sweedler Theory for Double Categories*.

[2] F. Borceux, *Handbook of Categorical Algebra 1 & 2*.

[3] M. Grandis, *Higher Dimensional Categories*.