Spatiotemporal Modeling and Simulation

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Solutions 1

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Question 1: Dimensional Analysis

Recall the set up depicted in Figure 1.

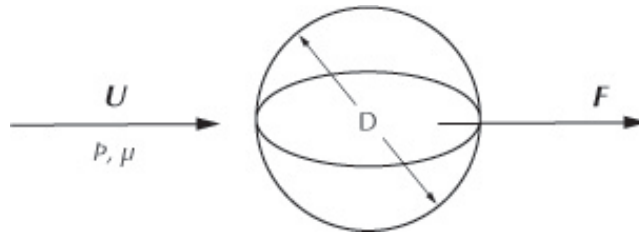

Figure 1: Viscous flow past a sphere

a) Assuming the sphere is not heated or cooled and that gravitational effects (or any other external forces) are of no consequence, there are a total of five dimensional quantities in the problem:

- F = force on the sphere $[MLT^{-2}]$

- μ = viscosity of the fluid $[ML^{-1}T^{-1}]$

- ρ = density of the fluid $[ML^{-3}]$

- U = velocity of the fluid $[LT^{-1}]$

- D = diameter of the sphere $[L]$

b) According to Taylor's method the initial matrix is given by:

$$\begin{array}{c|ccc} & \text{M} & \text{L} & \text{T} \\ \hline \mu & 1 & -1 & -1 \\ \rho & 1 & -3 & 0 \\ U & 0 & 1 & -1 \\ D & 0 & 1 & 0 \\ F & 1 & 1 & -2 \end{array}$$

We now choose the simplest looking column, namely the first one, and the simplest looking row with non-zero entries in the first column, namely the ρ row. Hence we divide all columns rows by this row. Division is accomplished by substracting all elements of this row from those of another. We arrive at:

$$\begin{array}{c|ccc} & \text{M} & \text{L} & \text{T} \\ \hline \frac{\mu}{\rho} & 0 & 2 & -1 \\ \rho & 1 & -3 & 0 \\ U & 0 & 1 & -1 \\ D & 0 & 1 & 0 \\ \frac{F}{\rho} & 0 & 4 & -2 \end{array}$$

We can delete the mass M column now and the ρ row and get:

$$\begin{array}{c|cc} & \text{L} & \text{T} \\ \hline \frac{\mu}{\rho} & 2 & -1 \\ U & 1 & -1 \\ D & 1 & 0 \\ \frac{F}{\rho} & 4 & -2 \end{array}$$

We now eliminate the time dimension T in all quantities except $\frac{\mu}{\rho}$ and arrive at:

$$\begin{array}{c|cc} & \text{L} & \text{T} \\ \hline \frac{\mu}{\rho} & 2 & -1 \\ \frac{\rho U}{\mu} & -1 & 0 \\ D & 1 & 0 \\ \frac{\rho F}{\mu^2} & 0 & 0 \end{array}$$

We can delete the time T column now and the $\frac{\mu}{\rho}$ row and get:

$$\begin{array}{c|c} & \text{L} \\ \hline \frac{\rho U}{\mu} & -1 \\ D & 1 \\ \frac{\rho F}{\mu^2} & 0 \end{array}$$

Adding the second row D to the first one yield the two identified dimensionless groupings:

$$\Pi_1 = \frac{\rho F}{\mu^2}, \qquad \Pi_2 = \frac{\rho U D}{\mu} \tag{1}$$

where Π_2 is the famous Reynolds number Re.

c) The dimension analysis reveals the following relationship for F.

$$F = \frac{\mu^2}{\rho} f(Re) \tag{2}$$

where f is a function of the Reynolds number. Accurate measurements would reveal for $Re \ll 1$ that

$$f(Re) \approx 3\pi Re. \tag{3}$$

This is the theoretical limiting case known as **Stokes Flow**. Plugging in yields

$$F = \frac{\mu^2}{\rho} f(Re) \approx \frac{\mu^2}{\rho} \cdot 3\pi Re = 3\pi \frac{\mu^2}{\rho} \frac{\rho U D}{\mu} \tag{4}$$

$$F \approx 3\pi \mu U D \tag{5}$$

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where the last expression is usually found in textbooks.

For high Reynolds numbers measurements suggest for F:

$$f(Re) \approx 0.16\pi Re^2 \tag{6}$$

and hence

$$F \approx \frac{\mu^2}{\rho} \cdot 0.16\pi \left(\frac{\rho U D}{\mu} \right)^2 = 0.16\pi \, \rho U^2 D^2. \tag{7}$$

This is described in textbooks as the turbulent regime. The prefactor varies depending on the object shape.

Question 2: Dynamic similitude

Recall the Navier Stokes equation in 1D

$$\rho\frac{\partial u}{\partial t} + \rho u\frac{\partial u}{\partial x} = -\frac{\partial p}{\partial x} + \mu\frac{\partial^2 u}{\partial x^2} \tag{8}$$

and a small one-dimensional sound source of size L moving at a constant velocity U emits a tone with frequency ω. The density of the fluid far away from the source is ρ_∞.

a) The dimensionless variables can be defined as follows:

- $\bar{x} = \frac{x}{L}$

- $\bar{t} = \omega t$

- $\bar{u} = \frac{u}{U}$

- $\bar{\rho} = \frac{\rho}{\rho_\infty}$

- $\bar{p} = \frac{p}{\rho_\infty U^2}$

Expressing all quantities in (8) by the dimensionless quantities gives:

$$\bar{\rho}\rho_\infty\frac{\partial(\bar{u}U)}{\partial(\frac{\bar{t}}{\omega})} + \bar{\rho}\rho_\infty\bar{u}U\frac{\partial(\bar{u}U)}{\partial(\bar{x}L)} = -\frac{\partial(\bar{p}\rho_\infty U^2)}{\partial(\bar{x}L)} + \mu\frac{\partial^2(\bar{u}U)}{\partial(\bar{x}L)^2} \tag{9}$$

Rearranging the terms in Equations (9) holds:

$$\rho_\infty U\omega\bar{\rho}\frac{\partial\bar{u}}{\partial\bar{t}} + \frac{\rho_\infty U^2}{L}\bar{\rho}\bar{u}\frac{\partial\bar{u}}{\partial\bar{x}} = -\frac{\rho_\infty U^2}{L}\frac{\partial\bar{p}}{\partial\bar{x}} + \frac{\mu U}{L^2}\frac{\partial^2\bar{u}}{\partial\bar{x}^2} \tag{10}$$

b) To express Equation(10) in terms of the Strouhal and the Reynolds number we can multiply the equation by the term $\frac{L}{\rho_\infty U^2}$:

$$St\,\bar{\rho}\frac{\partial\bar{u}}{\partial\bar{t}} + \bar{\rho}\bar{u}\frac{\partial\bar{u}}{\partial\bar{x}} = -\frac{\partial\bar{p}}{\partial\bar{x}} + \frac{1}{Re}\frac{\partial^2\bar{u}}{\partial\bar{x}^2} \tag{11}$$

where $St = \frac{\omega L}{U}$ is the Strouhal number and $Re = \frac{\rho_\infty UL}{\mu}$ is the Reynolds number.

c) Models usually have a different size than the reality they represent. Similitude is achieved when the model is built s.t. the test results are applicable to reality, e.g. in engineering when the test results are applicable to the real design. The term dynamic similitude is often used as a catch-all because it implies that geometric and kinematic similitude have already been met. The following criteria are required to achieve similitude:

- Geometric similarity - The model is the same shape as the application, usually scaled.
- Kinematic similarity - Fluid flow of both the model and real application must undergo similar time rates of change motions. (fluid streamlines are similar)

- Dynamic similarity - Ratios of all forces acting on corresponding fluid particles and boundary surfaces in the two systems are constant.

Assuming in our case a sound source with $\hat{U} = 10U$. Given the dimensionless groupings of Equation (11) to ensure dynamic similitude of both systems we must e.g. higher the frequency to $\hat{\omega} = 10\omega$ **and** lower the density to $\hat{\rho} = \frac{1}{10}\rho_\infty$ or higher the viscosity to $\hat{\mu} = 10\mu$. Many other possible combination are possible involving the length L as well.

Question 3: Continuum assumption

```matlab
% Solution of Exercise 1, Question 3

% Homogeneous part
% Create 10,000 particles in a 3D box of side length 10 (vol = 10^3 = 1000)
% => rho = 10
partBoxLen = 5;
% produces a 10,000 x 3 matrix with random values in the range (−5, 5)
particles = −partBoxLen + 2 * partBoxLen * rand(10000, 3); % rand(a, b) returns axb matrix

figure
subplot(2, 2, 1)
plot3(particles (:, 1), particles(:, 2), particles(:, 3), 'r.', 'MarkerSize', 1)
axis([−5 5 −5 5 −5 5])
grid on

numAvBoxes = 100;
maxAvBoxLen = 5; % may not exceed partBoxLen, otherwise extremely inhomogeneous case

avBoxLen = transpose(linspace(0, maxAvBoxLen, numAvBoxes));
avBoxVol = (2 * avBoxLen).^3;
numPartAvBox = zeros(numAvBoxes, 1);

for i = 1 : numAvBoxes
    currParticles = find(abs(particles(:, 1)) < avBoxLen(i) & ...
                    abs(particles(:, 2)) < avBoxLen(i) & ...
                    abs(particles(:, 3)) < avBoxLen(i));
    numPartAvBox(i) = size(currParticles, 1);
end

subplot(2, 2, 2)
plot(avBoxVol, numPartAvBox ./ avBoxVol,'r*','MarkerSize', 3)
xlabel('averaging volume {\it V}_{av}')
ylabel('density \rho')
grid on
% zoom into the plot to see the jumps at the beginning
xlim([0, 10])
pause(10)
xlim([0, 1000])

% Inhomogeneous part
particles = [−partBoxLen + 2 * partBoxLen * rand(8000,3); 2 + randn(2000, 3)];

subplot(2,2,3)
plot3(particles(:, 1), particles(:, 2), particles(:, 3), 'b.', 'MarkerSize', 1)
axis([−5 5 −5 5 −5 5])
```

```
grid on

numAvBoxes=100;
maxAvBoxLen=5;

avBoxLen=transpose(linspace(0, maxAvBoxLen, numAvBoxes));
avBoxVol= (2 * avBoxLen).^3;
numPartAvBox = zeros(numAvBoxes, 1);

for i = 1 : numAvBoxes
    currParticles = find(abs(particles(:, 1)) < avBoxLen(i) & ...
                    abs(particles(:, 2)) < avBoxLen(i) & ...
                    abs(particles(:, 3)) < avBoxLen(i));
    numPartAvBox(i) = size(currParticles, 1);
end

subplot(2, 2, 4)
plot(avBoxVol, numPartAvBox ./ avBoxVol, 'b*', 'MarkerSize', 3)
xlabel('averaging volume {\it V}_{av}')
ylabel('density \rho')
grid on
xlim([0, 1000])
```

Question 4: Quorum sensing

A dimensional analysis of the Quorum Sensing model is not possible *per se*. We would need a certain set-up, say a certain number of cells of known diameter, the distance between the cells. We will come back to this at a later point of this course when we begin to implement the model equations numerically.