# 1 Vector Calculus

## Scalar Fields

$f : \mathbb{R}^n \to \mathbb{R}$

## Vector Fields

$\underline{v} : \mathbb{R}^n \to \mathbb{R}^m$

## Differentiation of Vector Fields

Let $\underline{a}, \underline{b}, \underline{c}$ be vector fields and $\varphi$ be a scalar field, then

$$\begin{aligned}
\frac{d}{dt}(\varphi \underline{a}) &= \frac{d\varphi}{dt}\underline{a} + \varphi \frac{d\underline{a}}{dt} \\
\frac{d}{dt}(\underline{a} \cdot \underline{b}) &= \frac{d\underline{a}}{dt} \cdot \underline{b} + \underline{a} \cdot \frac{d\underline{b}}{dt} \\
\frac{d}{dt}(\underline{a} \times \underline{b}) &= \frac{d\underline{a}}{dt} \times \underline{b} + \underline{a} \times \frac{d\underline{b}}{dt} \\
\frac{d}{dt}(\underline{a}(\varphi(t))) &= \frac{d\underline{a}}{d\varphi}\frac{d\varphi}{dt}
\end{aligned}$$

## Differential Operators

For a scalar field $f : \mathbb{R}^n \to \mathbb{R}, \underline{x} \mapsto f(\underline{x})$ the gradient is

$$\operatorname{grad} f(\underline{x}) = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{pmatrix}$$

and gives the direction of steepest ascent

For a vector field $\underline{v} : \mathbb{R}^n \to \mathbb{R}^n$ the divergence is defined as

$$\operatorname{div} \underline{v}(\underline{x}) = \sum_{j=1}^{n} \frac{\partial v_j}{\partial x_j}(\underline{x})$$

and gives the source density at a point

For a vector field $\underline{v} : \mathbb{R}^3 \to \mathbb{R}^3$ the curl is defined as

$$\operatorname{curl}(\underline{v}(x,y,z)) = \begin{pmatrix} \frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z} \\ \frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x} \\ \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} \end{pmatrix}$$

and gives the vortex strength

The Laplace operator for a scalar field $f : \mathbb{R}^n \to \mathbb{R}, \underline{x} \mapsto f(\underline{x})$ gives

$$\Delta f(\underline{x}) = \nabla \cdot \nabla f(\underline{x}) = \sum_{j=1}^{n} \frac{\partial^2 f}{\partial^2 x_j}(\underline{x})$$

Expressed with the $\nabla$ operator we get the representations

$$\begin{aligned}
\operatorname{grad} f &= \nabla f \\
\operatorname{div} \underline{v} &= \nabla \cdot \underline{v} \\
\operatorname{curl} \underline{v} &= \nabla \times \underline{v} \\
\Delta &= \nabla \cdot \nabla
\end{aligned}$$

## Computation Rules for Differential Operators

The three operators grad, div and curl are linear. Rules for one active operator:

$$\begin{aligned}
\operatorname{grad}(f_1 f_2) &= f_1 \operatorname{grad} f_2 + f_2 \operatorname{grad} f_1 \\
\Leftrightarrow \nabla(f_1 f_2) &= f_1 \nabla f_2 + f_2 \nabla f_1
\end{aligned}$$

$$\begin{aligned}
\operatorname{grad} F(f) &= F'(f) \operatorname{grad} f \\
\Leftrightarrow \nabla F(f) &= F'(f) \nabla f
\end{aligned}$$

$$\begin{aligned}
\operatorname{div}(\underline{v}_1 \times \underline{v}_2) &= \underline{v}_2 \cdot \operatorname{curl} \underline{v}_1 - \underline{v}_1 \cdot \operatorname{curl} \underline{v}_2 \\
\Leftrightarrow \nabla \cdot (\underline{v}_1 \times \underline{v}_2) &= \underline{v}_2 \cdot (\nabla \times \underline{v}_1) - \underline{v}_1 \cdot (\nabla \times \underline{v}_2)
\end{aligned}$$

Rules for the interactions between vector- and scalarfields:

$$\begin{aligned}
\operatorname{div} f\underline{v} &= \underline{v} \operatorname{grad} f + f \operatorname{div} \underline{v} \\
\Leftrightarrow \nabla \cdot (f\underline{v}) &= \underline{v} \nabla f + f(\nabla \cdot \underline{v})
\end{aligned}$$

$$\begin{aligned}
\operatorname{curl} f\underline{v} &= f \operatorname{curl} \underline{v} - \underline{v} \times \operatorname{grad} f \\
\Leftrightarrow \nabla \times (f\underline{v}) &= f(\nabla \times \underline{v}) - \underline{v} \times \nabla f
\end{aligned}$$

Rules for the concatenation of the operators:

$$\begin{aligned}
\operatorname{div} \operatorname{curl} \underline{v} &= 0 \\
\Leftrightarrow \nabla \cdot (\nabla \times \underline{v}) &= 0
\end{aligned}$$

$$\begin{aligned}
\operatorname{curl} \operatorname{grad} f &= \underline{0} \\
\Leftrightarrow \nabla \times \nabla f &= \underline{0}
\end{aligned}$$

$$\begin{aligned}
\operatorname{div} \operatorname{grad} f &= \Delta f \\
\Leftrightarrow \nabla \cdot \nabla f &= \Delta f
\end{aligned}$$

$$\begin{aligned}
\operatorname{curl} \operatorname{curl} \underline{v} &= \operatorname{grad} \operatorname{div} \underline{v} - \Delta \underline{v} \\
\Leftrightarrow \nabla \times (\nabla \times \underline{v}) &= \nabla(\nabla \cdot \underline{v}) - \Delta \underline{v}
\end{aligned}$$

## Flux ($\Phi$)

**Q**: Let $\underline{v}$ be the velocity field of a flow; how much fluid flows through a surface $S$ per unit time in direction $\underline{n}$?

**A**: Split $S$ into infinitesimal area elements $dS$. Because they are infinitesimal, the $dS$ are flat and $\underline{v}$ is homogeneous within a single $dS$. The total flow through a single $dS$ is

$$d\Phi = \underline{v} \cdot \underline{n} dS$$

The flux is the "sum" over all infinitesimal $dS$ and thus given by the integral

$$\Phi = \int_S \underline{v} \cdot \underline{n} dS$$

## Work ($W$)

Another fundamental quantity in modeling is the work (a flow of energy) done by a vector field $\underline{v}$ along a line path $L$ with beginning $A$ and end $B$

**Q**: Let $\underline{v}$ be a force field; how much work is done by $\underline{v}$ moving a point mass along L from A to B?

**A**: Split $L$ into infinitesimal line segments $d\underline{r}$. Because they are infinitesimal, the $d\underline{r}$ are straight and $\underline{v}$ is homogeneous within a single $d\underline{r}$. The work done by the force field $\underline{v}$ per line segment is

$$dW = \underline{v} \cdot d\underline{r}$$

and the total work correspondingly is

$$W = \int_L \underline{v} \cdot d\underline{r}$$

For parametric curves

$$L : t \mapsto c(t) \quad \Rightarrow \quad d\underline{r} \simeq \dot{c}(t)dt$$

## Gauss Theorem

Consider a vector field $\underline{v}$ that is defined and continuously differentiable in a closed region $B$ with boundary $\partial B$ and outer unit normal $\underline{n}$.

$$\oint_{\partial B} \underline{v} \cdot \underline{n} \, dS = \int_B \operatorname{div} \underline{v} \, dV = \int_B \nabla \cdot \underline{v} \, dV$$

## Stokes Theorem

Consider a vector field $\underline{v}$ that is defined and continuously differentiable in a region $D$. Consider further a bounded surface S that is entirely contained in D and has the border line $\partial S$ $C$ is a path along $\partial S$ such that its sense forms a right-hand screw with the normal onto $S$.

$$\oint_C \underline{v} \cdot d\underline{r} = \int_S \operatorname{curl} \underline{v} \cdot \underline{n} \, dS = \int_S (\nabla \times \underline{v}) \cdot \underline{n} \, dS$$

## Greens Theorems

Obtained by applying Gauss to $\underline{v} = f_1 \nabla f_2$

$$\int_B (f_1 \Delta f_2 + \nabla f_1 \nabla f_2) \, dV = \oint_{\partial B} f_1 \nabla f_2 \cdot \underline{n} \, dS$$

$$\int_B (f_1 \Delta f_2 - f_2 \Delta f_1) \, dV = \oint_{\partial B} (f_1 \nabla f_2 - f_2 \nabla f_1) \cdot \underline{n} \, dS$$

## Conservative Fields

A vector field $\underline{v}$ is called conservative if and only if the work along all possible paths from $P$ to $Q$ is equal, for all $P, Q \in \mathbb{R}^n$.

- Each gradient field is conservative and vice versa, and thus also called potential field.
- Each gradient field is vortex free and vice versa, i.e. $\nabla \times \nabla f \equiv 0$ .

The definition is equivalent to

$$\begin{aligned}
&\underline{v} \text{ conservative} \Leftrightarrow \operatorname{curl} \underline{v} = \nabla \times v = 0 \\
&\underline{v} \text{ conservative} \Leftrightarrow \exists f : \mathbb{R}^n \to \mathbb{R} : \nabla f = \underline{v}
\end{aligned}$$

# 2 Modeling Spatial Effects

## Reynolds transport theorem

$$\begin{aligned}
\frac{D\varphi}{Dt} &= \frac{D}{Dt}\int_{V(t)} f dV(t) \\
&= \int_{V(t)} \frac{Df}{Dt} dV(t) + \int_{V(t)} f \underbrace{\frac{D}{Dt}[dV(t)]}_{\underbrace{\frac{\partial dV(t)}{\partial t}}_{=0} + (\underline{v}\cdot\nabla)dV} \\
&= \int_{V(t)} \left[\frac{Df}{Dt} + f(\underline{v} \cdot \nabla)\right] dV(t) \\
&= \int_{V(t)} [\frac{\partial f}{\partial t} + \underbrace{\underline{v} \cdot (\nabla f) + f(\nabla \cdot \underline{v})}_{=\nabla\cdot(f\underline{v})\text{: see compute rules}} ]dV(t \\
&= \int_{V(t)} \left[\frac{\partial f}{\partial t} + \nabla \cdot (f\underline{v})\right] dV(t).
\end{aligned}$$

## Vector Fields

$\underline{v} : \mathbb{R}^n \to \mathbb{R}^m$