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Exercise 4

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Question 1: Eulerian versus Lagrangian description

In the lecture you have encountered the Eulerian and Lagrangian description of particles. To illustrate the difference between them, consider the following. Imagine that in a one-dimensional flow, fluid particles are located all along the x-axis. Let us assume there are infinitely many with an infinitesimal small distance between each other. Each particle carries a quantity u. At time $t^0 := 0$, fluid particles at each point x have the quantity $u(x, t^0) := f(x)$ and begin motion in the x-direction with constant velocity $v(x, t^0) := Kx$.

a) Please derive the expressions for $x(t)$, $v(x(t), t)$ and the quantity $u(x(t), t)$ in the Lagrangian framework, where one control-volume corresponds to one fluid particle.

b) Please derive the expressions for $x(t)$, $v(x, t)$ and the quantity $u(x, t)$ in the Eulerian framework. What is the asymptotic quantity for large t and why?

Question 2: Conservation laws, diffusion and advection

We recapitulate what we have learned on conservation laws and control volumes in this exercise. For simplicity, let us first restrict ourselves to a problem with variations only into one dimension. Consider a chemical species C whose concentration $c((x, y, z), t) = c(x, t)$ varies in time and space (variations only along dimension x!). The situation is illustrated in Figure 1 where the chemical species C is contained in a long, thin tube with constant cross-section A.

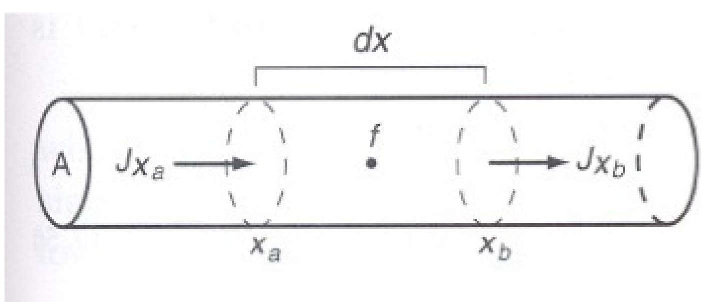

Figure 1: Conservation in thin long tube (3D), with variables varying only in one dimension (from C.P. Fall, 2005)

Let us think of the slice between x_a and x_b as our "control volume" V.

a) What are the extensive and the intensive quantities in the set-up? Let J denote the flux density rate, what dimensions does it have? Let f be the net reaction rate of increase of C (source or sink!). What dimensions does it have?

b) Now, let us derive the conservation of the total amount of the C in a more pictorial way using the following steps:

- Express the total amount of C in the control volume in mathematical terms.
- Express the net rate of entry of C in mathematical terms.
- Express the net rate of production of C in mathematical terms.
- Express now the conservation of C and derive the differential form.

Link the terms in this derivation with the terms from the Reynolds transport theorem. What additional term do we have and what does it mean? Please state your conservation law in differential form for three dimensions as well.

c) In the lecture you have been introduced to the term constitutive equation. Explain this expression! Write down Fick's law in one dimension using the notation of this exercise and embed it into the conservation law. You have now derived a reaction-diffusion equation!

d) Suppose that you have in addition a uniform macroscopic flow of the solvent, with speed ν along the x-axis. What is the advective flux for our problem? Embed this into your conservation law. What is the three dimensional expression of the flux? State the final three dimensional form of your equation. You have now derived a reaction-advection-diffusion equation!

Question 3: The Cable equation

One of the most important equations in neural sciences is the Cable equation that is the fundamental model for describing the electrical potential of nerve axons. Let us derive this equation now. Consider that our tube is bounded by a membrane, as in nerve axons! In this case, we wish to keep track of the electrical potential across the membrane, rather than the chemical species within the tube! The total current along the interior of the axon is $I(x)$, positive from left (x_a) to right (x_b), and the transmembrane current per unit area I_T, positive outward.

a) Express the conservation of current in the control volume where S is the circumference of the tube!

b) The total transmembrane current I_T consists of two components, a capacity and an ion current. Find these components in classical text books and embed these components into the equation assuming that V is the transmembrane potential.

c) The constitutive relationship for this problem is Ohm's law which is given by

$$I = -\frac{A}{R_c}\frac{\partial V}{\partial x} \tag{1}$$

under the assumption that the extracelluar electric potential is a constant. R_c is the cytoplasmic resistance (dimension [Ohm L]). Please embed this relationship into the equation. Now you have derived the famous Cable equation.

Question 4: Conservation in the QS system

Please check the QS system for conservation laws. How do they derive the governing equations? What kind of system do we have here?

Question 5: Something about numerics: Finite Volume methods

Familiarize yourself with the term Finite Volume method. What is the basic idea behind this numerical method?
Please sketch a Finite Volume method for the diffusion equation from the lecture in a unit cube with a given initial concentration $c(x, y, z, 0) = c_0(x, y, z)$.