# 1 Vector Calculus

## Scalar Fields

$f : \mathbb{R}^n \to \mathbb{R}$

## Vector Fields

$\underline{v} : \mathbb{R}^n \to \mathbb{R}^m$

## Differentiation of Vector Fields

Let $\underline{a}, \underline{b}, \underline{c}$ be vector fields and $\varphi$ be a scalar field, then

$$\begin{aligned}\frac{d}{dt}(\varphi\underline{a}) &= \frac{d\varphi}{dt}\underline{a} + \varphi\frac{d\underline{a}}{dt}\\ \frac{d}{dt}(\underline{a}\cdot\underline{b}) &= \frac{d\underline{a}}{dt}\cdot\underline{b} + \underline{a}\cdot\frac{d\underline{b}}{dt}\\ \frac{d}{dt}(\underline{a}\times\underline{b}) &= \frac{d\underline{a}}{dt}\times\underline{b} + \underline{a}\times\frac{d\underline{b}}{dt}\\ \frac{d}{dt}(\underline{a}(\varphi(t))) &= \frac{d\underline{a}}{d\varphi}\frac{d\varphi}{dt}\end{aligned}$$

## Differential Operators

For a scalar field $f : \mathbb{R}^n \to \mathbb{R}, \underline{x} \mapsto f(\underline{x})$ the gradient is

$$\operatorname{grad} f(\underline{x}) = \begin{pmatrix}\frac{\partial f}{\partial x_1}\\ \vdots \\ \frac{\partial f}{\partial x_n}\end{pmatrix}$$

and gives the direction of steepest ascent

For a vector field $\underline{v} : \mathbb{R}^n \to \mathbb{R}^n$ the divergence is defined as

$$\operatorname{div}\underline{v}(\underline{x}) = \sum_{j=1}^{n}\frac{\partial v_j}{\partial x_j}(\underline{x})$$

and gives the source density at a point

For a vector field $\underline{v} : \mathbb{R}^3 \to \mathbb{R}^3$ the curl is defined as

$$\operatorname{curl}(\underline{v}(x,y,z)) = \begin{pmatrix}\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\\ \frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\\ \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\end{pmatrix}$$

and gives the vortex strength

The Laplace operator for a scalar field $f : \mathbb{R}^n \to \mathbb{R}, \underline{x} \mapsto f(\underline{x})$ gives

$$\Delta f(\underline{x}) = \nabla\cdot\nabla f(\underline{x}) = \sum_{j=1}^{n}\frac{\partial^2 f}{\partial^2 x_j}(\underline{x})$$

Expressed with the $\nabla$ operator we get the representations

$$\begin{aligned}\operatorname{grad} f &= \nabla f\\ \operatorname{div}\underline{v} &= \nabla\cdot\underline{v}\\ \operatorname{curl}\underline{v} &= \nabla\times\underline{v}\\ \Delta &= \nabla\cdot\nabla\end{aligned}$$

## Computation Rules for Differential Operators

The three operators grad, div and curl are linear. Rules for one active operator:

$$\begin{aligned}\operatorname{grad}(f_1 f_2) &= f_1 \operatorname{grad} f_2 + f_2 \operatorname{grad} f_1\\ \Leftrightarrow \nabla(f_1 f_2) &= f_1\nabla f_2 + f_2\nabla f_1\end{aligned}$$

$$\begin{aligned}\operatorname{grad} F(f) &= F'(f)\operatorname{grad} f\\ \Leftrightarrow \nabla F(f) &= F'(f)\nabla f\end{aligned}$$

$$\begin{aligned}\operatorname{div}(\underline{v}_1\times\underline{v}_2) &= \underline{v}_2\cdot\operatorname{curl}\underline{v}_1 - \underline{v}_1\cdot\operatorname{curl}\underline{v}_2\\ \Leftrightarrow \nabla\cdot(\underline{v}_1\times\underline{v}_2) &= \underline{v}_2\cdot(\nabla\times\underline{v}_1) - \underline{v}_1\cdot(\nabla\times\underline{v}_2)\end{aligned}$$

Rules for the interactions between vector- and scalarfields:

$$\begin{aligned}\operatorname{div} f\underline{v} &= \underline{v}\operatorname{grad} f + f\operatorname{div}\underline{v}\\ \Leftrightarrow \nabla\cdot(f\underline{v}) &= \underline{v}\nabla f + f(\nabla\cdot\underline{v})\end{aligned}$$

$$\begin{aligned}\operatorname{curl} f\underline{v} &= f\operatorname{curl}\underline{v} - \underline{v}\times\operatorname{grad} f\\ \Leftrightarrow \nabla\times(f\underline{v}) &= f(\nabla\times\underline{v}) - \underline{v}\times\nabla f\end{aligned}$$

Rules for the concatenation of the operators:

$$\begin{aligned}\operatorname{div}\operatorname{curl}\underline{v} &= 0\\ \Leftrightarrow \nabla\cdot(\nabla\times\underline{v}) &= 0\end{aligned}$$

$$\begin{aligned}\operatorname{curl}\operatorname{grad} f &= \underline{0}\\ \Leftrightarrow \nabla\times\nabla f &= \underline{0}\end{aligned}$$

$$\begin{aligned}\operatorname{div}\operatorname{grad} f &= \Delta f\\ \Leftrightarrow \nabla\cdot\nabla f &= \Delta f\end{aligned}$$

$$\begin{aligned}\operatorname{curl}\operatorname{curl}\underline{v} &= \operatorname{grad}\operatorname{div}\underline{v} - \Delta\underline{v}\\ \Leftrightarrow \nabla\times(\nabla\times\underline{v}) &= \nabla(\nabla\cdot\underline{v}) - \Delta\underline{v}\end{aligned}$$

## Flux ($\Phi$)

**Q**: Let be $\underline{v}$ the velocity field of a flow; how much fluid flows through a surface $S$ per unit time in direction $\underline{n}$?

**A**: Split $S$ into infinitesimal area elements $dS$. Because they are infinitesimal, the $dS$ are flat and $\underline{v}$ is homogeneous within a single $dS$. The total flow through a single $dS$ is

$$d\Phi = \underline{v}\cdot\underline{n}dS$$

The flux is the "sum" over all infinitesimal $dS$ and thus given by the integral

$$\Phi = \int_S \underline{v}\cdot\underline{n}dS$$

# 2 Modeling Spatial Effects

## Reynolds transport theorem

$$\begin{aligned}\frac{D\varphi}{Dt} &= \frac{D}{Dt}\int_{V(t)} f dV(t)\\ &= \int_{V(t)}\frac{Df}{Dt}dV(t) + \int_{V(t)} f \underbrace{\frac{D}{Dt}[dV(t)]}_{\underbrace{\frac{\partial dV(t)}{\partial t}}_{=0} + (\underline{v}\cdot\nabla)dV(t)}\\ &= \int_{V(t)}\left[\frac{Df}{Dt} + f(\underline{v}\cdot\nabla)\right]dV(t)\\ &= \int_{V(t)}[\frac{\partial f}{\partial t} + \underbrace{\underline{v}\cdot(\nabla f) + f(\nabla\cdot\underline{v})}_{=\nabla\cdot(f\underline{v})\text{: see compute rules}}]dV(t)\\ &= \int_{V(t)}\left[\frac{\partial f}{\partial t} + \nabla\cdot(f\underline{v})\right]dV(t).\end{aligned}$$

## Vector Fields

$\underline{v} : \mathbb{R}^n \to \mathbb{R}^m$