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# Solution 3

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## Question 1: Calculations with operators

Let $\mathbf{v}(x,y,z) = \begin{pmatrix} v_1(x,y,z) \\ v_2(x,y,z) \\ v_3(x,y,z) \end{pmatrix}$ be a vector field and $f = f(x,y,z)$ a scalar field.
The question implicitly assumed sufficient smoothness of $f$ and $\mathbf{v}$ as stated in the answers to the subquestions.

a) Show that: $div\,(f\mathbf{v}) = \mathbf{v}\cdot\mathbf{grad}f + f\ div\,\mathbf{v}$.
It is necessary that $f$ and $\mathbf{v}$ are both (at least once) differentiable.
Application of the product rule for differentiation of scalar functions to each of the components yields

$$\begin{aligned}
div\,(f\mathbf{v}) =& \frac{\partial}{\partial x}(f v_1) + \frac{\partial}{\partial y}(f v_2) + \frac{\partial}{\partial z}(f v_3) \\
=& f\frac{\partial}{\partial x}(v_1) + v_1\frac{\partial}{\partial x}(f) + f\frac{\partial}{\partial y}(v_2) + v_2\frac{\partial}{\partial y}(f) + f\frac{\partial}{\partial z}(v_3) + v_3\frac{\partial}{\partial z}(f) \\
=& v_1\frac{\partial}{\partial x}(f) + v_2\frac{\partial}{\partial y}(f) + v_3\frac{\partial}{\partial z}(f) + f\frac{\partial}{\partial x}(v_1) + f\frac{\partial}{\partial y}(v_2) + f\frac{\partial}{\partial z}(v_3) \\
=& \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{pmatrix} + f\left(\frac{\partial}{\partial x}(v_1) + \frac{\partial}{\partial y}(v_2) + \frac{\partial}{\partial z}(v_3)\right) \\
=& \mathbf{v}\cdot\mathbf{grad}f + f\ div\,\mathbf{v}
\end{aligned}$$

Note that the statement holds true for any differentiable $n$-dimensional vector field $\mathbf{v}$, the proof then simply includes $n$ summands, one for each coordinate direction.

b) Show that: $\mathit{div}\,\mathbf{curl}\,\mathbf{v} = 0$. ("The curl is source-free.")
It is necessary that $\mathbf{v}$ is twice differentiable and second order derivatives can be interchanged in order. The latter is satisfied p.e. when $\mathbf{v}$ is twice *continuously* differentiable (Schwarz's theorem).

$$
\begin{aligned}
\mathit{div}\,\mathbf{curl}\,\mathbf{v} &= \mathit{div}\begin{pmatrix} \frac{\partial}{\partial y} v_3 - \frac{\partial}{\partial z} v_2 \\ \frac{\partial}{\partial z} v_1 - \frac{\partial}{\partial x} v_3 \\ \frac{\partial}{\partial x} v_2 - \frac{\partial}{\partial y} v_1 \end{pmatrix} \\
&= \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y} v_3 - \frac{\partial}{\partial z} v_2\right) + \frac{\partial}{\partial y}\left(\frac{\partial}{\partial z} v_1 - \frac{\partial}{\partial x} v_3\right) + \frac{\partial}{\partial z}\left(\frac{\partial}{\partial x} v_2 - \frac{\partial}{\partial y} v_1\right) \\
&= \frac{\partial^2}{\partial x \partial y} v_3 - \frac{\partial^2}{\partial x \partial z} v_2 + \frac{\partial^2}{\partial y \partial z} v_1 - \frac{\partial^2}{\partial y \partial x} v_3 + \frac{\partial^2}{\partial z \partial x} v_2 - \frac{\partial^2}{\partial z \partial y} v_1 \\
&= 0
\end{aligned}
$$

c) Show that: $\mathbf{curl}\,\mathbf{curl}\,\mathbf{v} = \mathbf{grad}\,div\,\mathbf{v} - \Delta\mathbf{v}$

Again, it is necessary that $\mathbf{v}$ is twice differentiable and second order derivatives can be interchanged in order. The latter is satisfied p.e. when $\mathbf{v}$ is twice *continuously* differentiable (Schwarz's theorem).
First, expand both **curl** operators into the coordinate form, then add a clever zero like $+\frac{\partial^2}{\partial x \partial x}v_1 - \frac{\partial^2}{\partial x^2}v_1$ in the $x$-coordinate to get $\Delta v_1$.

Note: Here $\Delta\mathbf{v}$ is the *vector Laplace operator* applied to the vector field $\mathbf{v}$. In the orthogonal Cartesian coordinates, $\Delta\mathbf{v}$ simply returns a vector field equal to the *scalar Laplacian* applied *to each vector component*.

$$
\begin{aligned}
\mathbf{curl}\,\mathbf{curl}\,\mathbf{v} =& \mathbf{curl}\begin{pmatrix} \frac{\partial}{\partial y}v_3 - \frac{\partial}{\partial z}v_2 \\ \frac{\partial}{\partial z}v_1 - \frac{\partial}{\partial x}v_3 \\ \frac{\partial}{\partial x}v_2 - \frac{\partial}{\partial y}v_1 \end{pmatrix} \\
=& \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \times \begin{pmatrix} \frac{\partial}{\partial y}v_3 - \frac{\partial}{\partial z}v_2 \\ \frac{\partial}{\partial z}v_1 - \frac{\partial}{\partial x}v_3 \\ \frac{\partial}{\partial x}v_2 - \frac{\partial}{\partial y}v_1 \end{pmatrix} \\
=& \begin{pmatrix} \left(\frac{\partial^2}{\partial x \partial y}v_2 - \frac{\partial^2}{\partial y^2}v_1\right) - \left(\frac{\partial^2}{\partial z^2}v_1 - \frac{\partial^2}{\partial x \partial z}v_3\right) \\ \left(\frac{\partial^2}{\partial z \partial y}v_3 - \frac{\partial^2}{\partial z^2}v_2\right) - \left(\frac{\partial^2}{\partial x^2}v_2 - \frac{\partial^2}{\partial x \partial y}v_1\right) \\ \left(\frac{\partial^2}{\partial x \partial z}v_1 - \frac{\partial^2}{\partial x^2}v_3\right) - \left(\frac{\partial^2}{\partial y^2}v_3 - \frac{\partial^2}{\partial y \partial z}v_2\right) \end{pmatrix} \\
=& \begin{pmatrix} \frac{\partial^2}{\partial x \partial y}v_2 + \frac{\partial^2}{\partial x \partial z}v_3 - \frac{\partial^2}{\partial y^2}v_1 - \frac{\partial^2}{\partial z^2}v_1 + \frac{\partial^2}{\partial x \partial x}v_1 - \frac{\partial^2}{\partial x^2}v_1 \\ \frac{\partial^2}{\partial z \partial y}v_3 + \frac{\partial^2}{\partial x \partial y}v_1 - \frac{\partial^2}{\partial y^2}v_2 - \frac{\partial^2}{\partial z^2}v_2 + \frac{\partial^2}{\partial y \partial y}v_2 - \frac{\partial^2}{\partial y^2}v_2 \\ \frac{\partial^2}{\partial x \partial z}v_1 + \frac{\partial^2}{\partial y \partial z}v_2 - \frac{\partial^2}{\partial y^2}v_3 - \frac{\partial^2}{\partial z^2}v_3 + \frac{\partial^2}{\partial z \partial z}v_3 - \frac{\partial^2}{\partial z^2}v_3 \end{pmatrix} \\
=& \begin{pmatrix} \frac{\partial^2}{\partial x \partial y}v_2 + \frac{\partial^2}{\partial x \partial z}v_3 + \frac{\partial^2}{\partial x^2}v_1 - \Delta v_1 \\ \frac{\partial^2}{\partial z \partial y}v_3 + \frac{\partial^2}{\partial x \partial y}v_1 + \frac{\partial^2}{\partial y^2}v_2 - \Delta v_2 \\ \frac{\partial^2}{\partial x \partial z}v_1 + \frac{\partial^2}{\partial y \partial z}v_2 + \frac{\partial^2}{\partial z^2}v_3 - \Delta v_3 \end{pmatrix} \\
=& \begin{pmatrix} \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}v_1 + \frac{\partial}{\partial y}v_2 + \frac{\partial}{\partial z}v_3\right) \\ \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}v_1 + \frac{\partial}{\partial y}v_2 + \frac{\partial}{\partial z}v_3\right) \\ \frac{\partial}{\partial z}\left(\frac{\partial}{\partial x}v_1 + \frac{\partial}{\partial y}v_2 + \frac{\partial}{\partial z}v_3\right) \end{pmatrix} - \Delta\mathbf{v} \\
=& \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \left(\frac{\partial}{\partial x}v_1 + \frac{\partial}{\partial y}v_2 + \frac{\partial}{\partial z}v_3\right) - \Delta\mathbf{v} \\
=& \mathbf{grad}\,div\,\mathbf{v} - \Delta\mathbf{v}
\end{aligned}
$$

d) Show that: $div\,(\mathbf{v_1} \times \mathbf{v_2}) = \mathbf{v_2} \cdot \mathbf{curl}\,\mathbf{v_1} - \mathbf{v_1} \cdot \mathbf{curl}\,\mathbf{v_2}$.
To define the **curl**'s, $\mathbf{v_1}, \mathbf{v_2}$ need to be (at least once) differentiable.

Let $\mathbf{v_1}(x,y,z) = \begin{pmatrix} v_{11}(x,y,z) \\ v_{12}(x,y,z) \\ v_{13}(x,y,z) \end{pmatrix}$ and $\mathbf{v_2}(x,y,z) = \begin{pmatrix} v_{21}(x,y,z) \\ v_{22}(x,y,z) \\ v_{23}(x,y,z) \end{pmatrix}$.

$$
\begin{aligned}
div\,(\mathbf{v_1} \times \mathbf{v_2}) =& div \begin{pmatrix} v_{12}v_{23} - v_{13}v_{22} \\ v_{13}v_{21} - v_{11}v_{23} \\ v_{11}v_{22} - v_{12}v_{21} \end{pmatrix} \\
=& \frac{\partial}{\partial x}\left(v_{12}v_{23} - v_{13}v_{22}\right) + \frac{\partial}{\partial y}\left(v_{13}v_{21} - v_{11}v_{23}\right) + \frac{\partial}{\partial z}\left(v_{11}v_{22} - v_{12}v_{21}\right) \\
=& v_{12}\frac{\partial}{\partial x}v_{23} + v_{23}\frac{\partial}{\partial x}v_{12} - v_{13}\frac{\partial}{\partial x}v_{22} - v_{22}\frac{\partial}{\partial x}v_{13} \\
&+ v_{21}\frac{\partial}{\partial y}v_{13} + v_{13}\frac{\partial}{\partial y}v_{21} - v_{11}\frac{\partial}{\partial y}v_{23} - v_{23}\frac{\partial}{\partial y}v_{11} \\
&+ v_{22}\frac{\partial}{\partial z}v_{11} + v_{11}\frac{\partial}{\partial z}v_{22} - v_{12}\frac{\partial}{\partial z}v_{21} - v_{21}\frac{\partial}{\partial z}v_{12} \\
=& \left(v_{21}\left(\frac{\partial}{\partial y}v_{13} - \frac{\partial}{\partial z}v_{12}\right) + v_{22}\left(\frac{\partial}{\partial z}v_{11} - \frac{\partial}{\partial x}v_{13}\right) + v_{23}\left(\frac{\partial}{\partial x}v_{12} - \frac{\partial}{\partial y}v_{11}\right)\right) \\
&- \left(v_{11}\left(\frac{\partial}{\partial y}v_{23} - \frac{\partial}{\partial z}v_{22}\right) + v_{12}\left(\frac{\partial}{\partial z}v_{21} - \frac{\partial}{\partial x}v_{23}\right) + v_{13}\left(\frac{\partial}{\partial x}v_{22} - \frac{\partial}{\partial y}v_{21}\right)\right) \\
=& \mathbf{v_2} \cdot \mathbf{curl}\,\mathbf{v_1} - \mathbf{v_1} \cdot \mathbf{curl}\,\mathbf{v_2}
\end{aligned}
$$

## Aside: The Levi-Civita symbol

Cross products may also be written with the *Levi-Civita symbol* or *Epsilon tensor* $\epsilon_{ijk}$. It is defined as

$$
\epsilon_{ijk} = \begin{cases} 1 & \text{if } ijk \text{ is an even permutation, e.g. } 123 \\ -1 & \text{if } ijk \text{ is an odd permutation, e.g. } 321 \\ 0 & \text{if any indices repeat} \end{cases}
$$

For example,

$$
\epsilon_{231} = -\epsilon_{132} = -(-\epsilon_{123}) = 1, \tag{1}
$$

so by switching indices, one can quickly arrive at a known value for $\epsilon$. It can be generalized to $n$ dimensions as

$$
\epsilon_{a_1 a_2 \ldots a_n} = \prod_{1 \leq i \leq j \leq n} \operatorname{sgn}(a_j - a_i), \tag{2}
$$

although the index switching described above might be faster for calculations.

With $\epsilon_{ijk}$, a cross product of two vectors $\vec{a}, \vec{b}$ can be written as

$$
(\vec{a} \times \vec{b})_i = \sum_{j=1}^{3} \sum_{k=1}^{3} \epsilon_{ijk} \vec{e}_i a^j b^k, \tag{3}
$$

where $\vec{e}_i$ is the i'th orthogonal basis vector. Equipped with this knowledge, some identities involving the $\nabla$ operator are quite quick to prove:

Let's take $\nabla \cdot (\nabla \times \vec{v})$ as an example:

$$\begin{aligned} \nabla \cdot (\nabla \times \vec{v}) &= \partial_i \vec{e}_i (\epsilon_{ijk} \partial_j v_k \vec{e}_k) && (4) \\ &= \partial_i \epsilon_{ijk} \partial_j v_k \vec{e}_i \vec{e}_k && (5) \\ &= 0, && (6) \end{aligned}$$

as $\vec{e}_i \vec{e}_k = \delta_{ik}$, meaning that their scalar product equals 0 (they're orthogonal!) if they are not the same two vectors. Now $\epsilon_{ijk}$ vanishes when two indices coindice while $\delta_{ik}$ is only non-zero if they do. Therefore, their product has to be zero. This is by the way a very general result: The product of any anti-symmetric tensor with a symmetric one is bound to be zero. In this example we have also used the Einstein summation convention, where a sum over repeated indices is implied.

## Question 2: Rotation of a rigid body

Let us consider a rotating rigid body with rotation axis in the origin $O$. Let the position vector be $\mathbf{r} = (x, y, z)$ and the angular velocity $\omega = (\omega_1, \omega_2, \omega_3)$.

a) Angular velocity of $\omega = (\omega_1, \omega_2, \omega_3)$ implies by the right-hand rule that the velocity field $\mathbf{v}$ of the rigid body is:

$$\mathbf{v} = \omega \times \mathbf{r} = \begin{pmatrix} \omega_2 z - \omega_3 y \\ \omega_3 x - \omega_1 z \\ \omega_1 y - \omega_2 x \end{pmatrix}$$

b)

$$\begin{aligned} \mathbf{curl}\,\mathbf{v} =& \mathbf{curl} \begin{pmatrix} \omega_2 z - \omega_3 y \\ \omega_3 x - \omega_1 z \\ \omega_1 y - \omega_2 x \end{pmatrix} \\ =& \begin{pmatrix} \frac{\partial}{\partial y}(\omega_1 y - \omega_2 x) - \frac{\partial}{\partial z}(\omega_3 x - \omega_1 z) \\ \frac{\partial}{\partial z}(\omega_2 z - \omega_3 y) - \frac{\partial}{\partial x}(\omega_1 y - \omega_2 x) \\ \frac{\partial}{\partial x}(\omega_3 x - \omega_1 z) - \frac{\partial}{\partial y}(\omega_2 z - \omega_3 y) \end{pmatrix} \\ =& \begin{pmatrix} \omega_1 + \omega_1 \\ \omega_2 + \omega_2 \\ \omega_3 + \omega_3 \end{pmatrix} \end{aligned}$$

So **curl** actually measures the angular velocity. In the case of the rigid body, it is twice the angular velocity.
Moreover for the rotation of a rigid body, the **curl** of the velocity field which is itself a vector field, turns out to be constant in space.

## Question 3: Flux in a Coulomb field

Consider an electric point charge $e$ in the origin $O$ of a cartesian coordinate system. Let $\mathbf{v}(\mathbf{r})$ be the corresponding electric Coulomb field with

$$\mathbf{v}(\mathbf{r}) = C\frac{e}{|\mathbf{r}|^3}\mathbf{r}$$

with $\mathbf{r} = (x, y, z)^{\mathrm{T}}$ and $|\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$.
Then, the flux $\phi$ of a point charge through a sphere with radius $R$ and surface $S$ is given by:

$$\phi = \int_S C\frac{e}{R^3}\mathbf{r} \cdot \mathbf{n}\, dS$$

With $\mathbf{r}$ and $\mathbf{n}$ parallel we get $\mathbf{r} \cdot \mathbf{n} = R$ on the sphere. Hence,

$$\phi = C\frac{e}{R^2}\int_S dS = C\frac{e}{R^2}4\pi R^2 = 4\pi Ce$$

So the flux is proportional to the charge $e$ and independent of the radius.

Note that we may *not* use Gauss' theorem, because $\mathbf{v}(\mathbf{r})$ is not continuously differentiable in the origin. In fact, $\mathbf{v}(\mathbf{r})$ is not continuous in the origin, hence it cannot be differentiable. However, off the origin we have

$$\begin{aligned}
div\,\mathbf{v}(\mathbf{r}) &= Ce\left[\frac{\partial}{\partial x}\frac{x}{|\mathbf{r}|^3} + \frac{\partial}{\partial y}\frac{y}{|\mathbf{r}|^3} + \frac{\partial}{\partial z}\frac{z}{|\mathbf{r}|^3}\right] \\
&= Ce\left[\left(\frac{1}{|\mathbf{r}|^3} - \frac{x\frac{3}{2}\cdot 2x}{|\mathbf{r}|^5}\right) + \left(\frac{1}{|\mathbf{r}|^3} - \frac{y\frac{3}{2}\cdot 2y}{|\mathbf{r}|^5}\right) + \left(\frac{1}{|\mathbf{r}|^3} - \frac{z\frac{3}{2}\cdot 2z}{|\mathbf{r}|^5}\right)\right] \\
&= Ce\left[\frac{|\mathbf{r}|^3 - x\frac{3}{2}|\mathbf{r}|\cdot 2x}{|\mathbf{r}|^6} + \frac{|\mathbf{r}|^3 - y\frac{3}{2}|\mathbf{r}|\cdot 2y}{|\mathbf{r}|^6} + \frac{|\mathbf{r}|^3 - z\frac{3}{2}|\mathbf{r}|\cdot 2z}{|\mathbf{r}|^6}\right] \\
&= Ce\left[\frac{1}{|\mathbf{r}|^6}\left(3|\mathbf{r}|^3 - 3\left(x^2+y^2+z^2\right)|\mathbf{r}|\right)\right] \\
&= 0
\end{aligned}$$

So for any region $B$ not containing the origin with sufficiently smooth boundary $\partial B$ we conclude by Gauss' theorem

$$\int_{\partial B} C\frac{e}{R^3}\mathbf{r}\cdot\mathbf{n}\,dS = \int_{\partial B}\mathbf{v}(\mathbf{r})\cdot\mathbf{n}\,dS = \int_B div\,\mathbf{v}(\mathbf{r})\,dV = \int_B 0\,dV = 0.$$

For example, this applies to the region obtained by removing an inner ball of radius $R_i$ from a ball of outer radius $R_o$. The flux through this region's boundary is zero by Gauss' theorem, and also zero because the influx at the inner surface ($4\pi Ce$ by the direct calculation) is exactly balanced by the outflux at the outer surface ($4\pi Ce$ as well).

## Question 4: Potential fields

Let $\mathbf{v}$ be a potential field with potential $f$. This implies that

$$\mathbf{v} = \mathbf{grad} f$$

a) In the self-test question, you showed that $\mathbf{curl}\,\mathbf{grad} f = 0$. Hence, it is easy to show that $\mathbf{v}$ is vortex-free, namely:

$$\mathbf{curl}\,\mathbf{v} = \mathbf{curl}\,\mathbf{grad} f = 0$$

This is stated in words as "Potential fields are vortex-free".

b) Let $\mathbf{v}(\mathbf{r})$ be a Coulomb field with

$$\mathbf{v}(\mathbf{r}) = -C\frac{\mathbf{r}}{|\mathbf{r}|^3}$$

with $\mathbf{r} = (x, y, z)^{\mathrm{T}}$ and $|\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$ in cartesian coordinates. So $\mathbf{v}$ is also given by:

$$\mathbf{v}(\mathbf{r}) = -C\left(\frac{x}{|\mathbf{r}|^3}, \frac{y}{|\mathbf{r}|^3}, \frac{z}{|\mathbf{r}|^3}\right)^T$$

Then

$$\begin{aligned}
\mathbf{curl}\,\mathbf{v}(\mathbf{r}) &= -C\,\mathbf{curl}\begin{pmatrix} \frac{x}{|\mathbf{r}|^3} \\ \frac{y}{|\mathbf{r}|^3} \\ \frac{z}{|\mathbf{r}|^3} \end{pmatrix} \\
&= -C\begin{pmatrix} \frac{\partial}{\partial y}\frac{z}{|\mathbf{r}|^3} - \frac{\partial}{\partial z}\frac{y}{|\mathbf{r}|^3} \\ \frac{\partial}{\partial z}\frac{x}{|\mathbf{r}|^3} - \frac{\partial}{\partial x}\frac{z}{|\mathbf{r}|^3} \\ \frac{\partial}{\partial x}\frac{y}{|\mathbf{r}|^3} - \frac{\partial}{\partial y}\frac{x}{|\mathbf{r}|^3} \end{pmatrix} \\
&= -C\begin{pmatrix} \frac{-z3|\mathbf{r}|y+y3|\mathbf{r}|z}{|\mathbf{r}|^6} \\ \frac{-x3|\mathbf{r}|z+z3|\mathbf{r}|x}{|\mathbf{r}|^6} \\ \frac{-y3|\mathbf{r}|x+x3|\mathbf{r}|y}{|\mathbf{r}|^6} \end{pmatrix} \\
&= 0
\end{aligned}$$