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# Exercise 3

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## Question 1: Calculations with operators

To train your skills in calculating with the operators in vector analysis you have been introduced to in the lecture, please prove the following statements. Let $\mathbf{v}$ be a vector field and $f$ a scalar field

a)
$$div\,(f\mathbf{v}) = \mathbf{v} \cdot \mathbf{grad} f + f\; div\,\mathbf{v} \tag{1}$$

b)
$$div\,\mathbf{curl}\,\mathbf{v} = \mathbf{0} \tag{2}$$

c)
$$\mathbf{curl}\,\mathbf{curl}\,\mathbf{v} = \mathbf{grad}\; div\,\mathbf{v} - \boldsymbol{\Delta}\mathbf{v} \tag{3}$$

d)
$$div\,(\mathbf{v_1} \times \mathbf{v_2}) = \mathbf{v_2} \cdot \mathbf{curl}\,\mathbf{v_1} - \mathbf{v_1} \cdot \mathbf{curl}\,\mathbf{v_2} \tag{4}$$

## Question 2: Rotation of a rigid body

Consider a rotating rigid body with rotation axis in the origin $O$. Let the position vector be $\mathbf{r} = (x, y, z)$ and the angular velocity $\omega = (\omega_1, \omega_2, \omega_3)$.

a) What is the velocity field $\mathbf{v}$ of the rigid body?

b) Compute $\mathbf{curl}\,\mathbf{v}$. Given your result can you tell what quantity the operator $\mathbf{curl}$ is actually measuring?

## Question 3: Flux in a Coulomb field

Consider an electric point charge $e$ in the origin $O$ of a cartesian coordinate system. Let $\mathbf{v}(\mathbf{r})$ be the corresponding electric Coulomb field with

$$\mathbf{v}(\mathbf{r}) = C\frac{e}{|\mathbf{r}|^3}\mathbf{r} \tag{5}$$

with $\mathbf{r} = (x, y, z)^{\mathrm{T}}$ and $|\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$.
Calculate the flux $\phi$ of a point charge through a sphere with radius $R$ and origin $O$.

## Question 4: Potential fields

Let $\mathbf{v}$ be a potential field with potential $f$.

a) Show that $\mathbf{v}$ is vortex-free.
*Hint:* Recall the definition of a potential field and your calculations in the self-test questions.

b) Let $\mathbf{v}(\mathbf{r})$ be a Coulomb field with

$$\mathbf{v}(\mathbf{r}) = -C\frac{\mathbf{r}}{|\mathbf{r}|^3} \tag{6}$$

with $\mathbf{r} = (x, y, z)^{\mathrm{T}}$ and $|\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$ in cartesian coordinates.
Show that $\mathbf{curl}\,\mathbf{v} = \mathbf{0}$.