# Induction in Trees

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# 1 Language of Arithmetic

## 1.1 Formalizing Arithmetic

By introducing a name and functions, we can develop arithmetic in $L_3$. What we need are the following.

**Name**

$0$**:** A name for Zero

**Functions**

$x'$**:** The successor of $x$
$x + y$**:** The sum of $x$ and $y$
$x \cdot y$**:** The product of $x$ and $y$

First, we define the natural numbers.

**Definition**

1. 0 is a natural number.
2. If $x$ is a natural number, so is $x'$.
3. Nothing else is a natural number.

By the clause 1 of the definition, 0 is a natural number. And by the clause 2, $0'$ is a natural number. Since $0'$ is a natrual number, so is $0''$. Repeating this process, we can create *all* natural numbers.[1]

[1] $0'$ and $0''$ are usually denoted as 1 and 2. Howevr, we stick to this notation.

Next, we need to make clear what kind of properties we want the natural numbers to have. We do this by providing *axioms*.[2]

**Axioms**

1. $\forall x \forall y (x \neq y \to x' \neq y')$
2. $\forall x\ 0 \neq x'$
3. $\forall x (x \neq 0 \to \exists y x = y')$
4. $\forall x\ x + 0 = x$
5. $\forall x \forall y\ x + y' = (x + y)'$
6. $\forall x\ x \cdot 0 = 0$
7. $\forall x \forall y\ x \cdot y' = x \cdot y + y$

The system of arithmetic which is determined by those axioms is usually called *Robinson arithmetic* and abbreviated as Q.

We can use the axioms of Robinson arithmetic for proving various arithmeic statements. For example, we can prove "$2 + 3 = 5$" as follows.

[2]Axioms are sentences the turths of which we take for granted. As such, we needn't to prove their truths. The reason why we take the truths of the axioms for granted is in the meanings of the axioms. The followings are English translations of the axioms 1–3.

1. If two numbers are distinct, so are their successors.
2. 0 is not the successor of any number.
3. If a number is not 0, that number is the successor of some other number.

These statements agree with our naïve intuition about the natural numbers. Mainly for this reason, we take these axioms for granted.

The axioms 4–7 define addition and multiplication, and as such, they agree with our naïve conception of addition and multiplication.

| | | |
|---|---|---|
| 1. | $\forall x\ x+0=x$ | Axiom |
| 2. | $\forall x \forall y\ x+y'=(x+y)'$ | Axiom |
| 3. | $0''+0''' \neq 0'''''$ | The negation of what we want to prove |
| 4. | $0''+0'''=(0''+0'')'$ | UI to Line 2 with $0''$ to $x$ and $y$ |
| 5. | $0''+0''=(0''+0')'$ | UI to Line 2 with $0''$ to $x$ and $0'$ to $y$ |
| 6. | $0''+0'=(0''+0)'$ | UI to Line 2 with $0''$ to $x$ and 0 to $y$ |
| 7. | $0''+0=0''$ | UI to Line 1 with $0''$ to $x$ |
| 8. | $0''+0'=0'''$ | Rule 16 to Lines 6 and 7 |
| 9. | $0''+0''=0''''$ | Rule 16 to Lines 5 and 8 |
| 10. | $0''+0'''=0'''''$ | Rule 16 to Lines 4 and 9 |
| | $\underset{3,10}{\times}$ | |

In a similar fashion, we can prove $3+2=5$ as well; so, by applying the identity rule to these sentences ($2+3=5$ and $3+2=5$), we can also prove $2+3=3+2$. However, with the axioms 1–7, we *can't* prove its generalized version $\forall x \forall y\ x+y=y+x$. To prove this, we need something more.

## 1.2 Mathematical Induction

As is seen above, the arithmetic system Q is strong enough to prove particular arithmetical statements ($2+3=5$, $2+3=3+2$, etc.) but not strong enough to prove general arithmetical statements like $\forall x \forall y\ x+y=y+x$. To prove such general arithmetical statements, we need the following axiom of *mathematical induction*.[3]

8. $((P_0 \wedge \forall x(P_x \rightarrow P_{x'})) \rightarrow \forall x P_x)$

The meaning of this axiom is this: If 0 has a property $P$, and if each natural number $x$ has the property $P$ then its successor also has it, then *all* natural numbers has the property $P$.[4] So, if we want to prove that all natural numbers have some property $P$, first we prove that the number 0 has the property $P$, and prove that $x'$ has the property $P$ from the assumption

[3]Strictly speaking, this should be called an *axiom scheme* because for each property $P$ (whatever it is), we need an instance of this axiom scheme.

[4]To convince yourself why this axiom (scheme) holds, just think about how we defined the natural numbers.

that $x$ has it. The arithmetic system Q supplemented with mathematical induction is usually called *Peano arithmetic*.

A paradigmatic example to illustrate how mathematical induction works is to prove the following formula for the sum from 0 to $n$.

$$\sum_{i=0}^{n} i = \frac{n \cdot (n+1)}{2}$$

First, like I said, we need to prove that the above formula holds when $n = 0$. This is rather an easy task because

$$\sum_{i=0}^{0} i = \frac{0 \cdot (0+1)}{2} = 0.$$

Now we gonna prove that, from the assumption that the sum from 0 to $n$ can be expressed as $\frac{n \cdot (n+1)}{2}$, the sum from 0 to $n+1$ can be expressed as $\frac{(n+1) \cdot (n+2)}{2}$.

Since we're assuming the truth of $\frac{n \cdot (n+1)}{2}$, we can express the sum from 0 to $n+1$ as $\frac{n \cdot (n+1)}{2} + (n+1)$ (the sum from 0 to $n$ *plus* $n+1$). And this expression can be transformed as follows.

$$\begin{aligned} \frac{n \cdot (n+1)}{2} + (n+1) &= \frac{n \cdot (n+1)}{2} + \frac{2 \cdot (n+1)}{2} \\ &= \frac{n \cdot (n+1) + 2 \cdot (n+1)}{2} \\ &= \frac{(n+1) \cdot (n+2)}{2} \end{aligned}$$

In the above, the last expression $\frac{(n+1) \cdot (n+2)}{2}$ is nothing but the case for $n+1$; we proved that the sum from 0 to $n+1$ can be expressed as $\frac{(n+1) \cdot (n+2)}{2}$ from the assumption that $\frac{n \cdot (n+1)}{2}$. Therefore, by mathematical induction, the formula for the sum from 0 to $n$ ($\frac{n \cdot (n+1)}{2}$) holds for *every* natural number.

## 2 Implementing Mathematical Induction as a Tree Rule

Now we need to implement mathematical induction in our tree rules. A tree-rule version of mathematical induction is:

18. Mathematical Induction (MI)[5]

$$
\begin{array}{c}
P_0 \\
\vdots \\
-P_m \\
P_n \\
-P_{n'}
\end{array}
$$

What this rule allows you to do is: If you have $P_0$ and $-P_m$ (0 has a property $P$ but some name $m$ doesn't. The order of appearances of $P_0$ and $-P_m$ doesn't matter; $-P_m$ can appear before $P_0$ appears in the path) in your path, then, with a new name $n$, you can add $P_n$ and $-P_{n'}$ to the path. With this rule (together with the other rules of the tree of course), we can prove general statements like $\forall x \forall y\ x+y=y+x$. So let's prove $\forall x \forall y\ x+y=y+x$.

To prove $\forall x \forall y\ x+y=y+x$, however, we need to prove $\forall x\ 0+x=x$ and $\forall x \forall y\ x'+y=(x+y)'$ first. A tree proof of $\forall x\ 0+x=x$ is as follows.

| | | |
|---|---|---|
| 1. | $\forall x\ x+0=x$ | Axiom 4 of PA |
| 2. | $\forall x \forall y\ x+y'=(x+y)'$ | Axiom 5 of PA |
| 3. | $\checkmark\ -\forall x\ 0+x=x$ | The negation of what we want to prove |
| 4. | $\checkmark\ \exists x\ 0+x\neq x$ | NEQ to Line 3 |
| 5. | $0+a\neq a$ | EI to Line 4 |
| 6. | $0+0=0$ | UI to Line 1 |
| 7. | $0+b=b$ | MI to Lines 5 and 6 with $P_x$: $0+x=x$ |
| 8. | $0+b'\neq b'$ | MI to Lines 5 and 6 with $P_x$: $0+x=x$ |
| 9. | $0+b'=(0+b)'$ | UI to Line 2 with 0 to $x$ and $b$ to $y$ |
| 10. | $0+b'=b'$ | Rule 16 to Lines 7 and 9 |
| | $\underset{8,10}{\times}$ | |

[5]This implementation is based on mathematical induction negatively stated. Mathematical induction negatively stated is: If 0 has a property $P$ and some natural number $m$ *doesn't* have it, there has to be a natural number $n$ such that $n$ has the property $P$ and $n'$ doesn't have it; that is, there's the maximum natural number which has the property $P$.

Next, we prove $\forall x \forall y\ x' + y = (x + y)'$.

| | | |
|---|---|---|
| 1. | $\forall x\ x + 0 = x$ | Axiom 4 of PA |
| 2. | $\forall x \forall y\ x + y' = (x + y)'$ | Axiom 5 of PA |
| 3. | $\checkmark\ -\forall x \forall y\ x' + y = (x + y)'$ | The negation of what we want to prove |
| 4. | $\checkmark\ \exists x \exists y\ x' + y \neq (x + y)'$ | NEQ twice to Line 3 |
| 5. | $a' + b \neq (a + b)'$ | EI to Line 4 with $a$ to $x$ and $b$ to $y$ |
| 6. | $a' + 0 = a'$ | UI to Line 1 with $a'$ to $x$ |
| 7. | $a + 0' = (a + 0)'$ | UI to Line 2 with $a$ to $x$ and 0 to $y$ |
| 8. | $a + 0 = a$ | UI to Line 1 with $a$ to $x$ |
| 9. | $a + 0' = a'$ | Rule 16 to Lines 7 and 8 |
| 10. | $(a + 0)' = a'$ | Rule 16 to Lines 7 and 9 |
| 11. | $a' + 0 = (a + 0)'$ | Rule 17 to Lines 6 and 10 |
| 12. | $a' + c = (a + c)'$ | MI to Lines 5 and 11 with $P_x$: $a' + x = (a + x)'$ |
| 13. | $a' + c' \neq (a + c')'$ | MI to Lines 5 and 11 with $P_x$: $a' + x = (a + x)'$ |
| 14. | $a' + c' = (a' + c)'$ | UI to Line 2 with $a'$ to $x$ and $c$ to $y$ |
| 15. | $a' + c' = (a + c)''$ | Rule 16 to Lines 11 and 14 |
| 16. | $a + c' = (a + c)'$ | UI to Line 2 with $a$ to $x$ and $c$ to $y$ |
| 17. | $a' + c' \neq (a + c)''$ | Rule 16 to Lines 13 and 16 |
| | $\underset{15,17}{\times}$ | |

Once we proved sentences, we can use them as premises. Let's prove (finally!) $\forall x \forall y\ x + y = y + x$ with what we've just proved ($\forall x\ 0 + x = x$ and $\forall x \forall y\ x' + y = (x + y)'$, together with $\forall x\ x + 0 = x$ and $\forall x \forall y\ x + y' = (x + y)'$).

| | | |
|---|---|---|
| 1. | $\forall x\ x+0=x$ | Axiom 4 of PA |
| 2. | $\forall x\ 0+x=x$ | What we've just proved |
| 3. | $\forall x \forall y\ x+y'=(x+y)'$ | Axiom 5 of PA |
| 4. | $\forall x \forall y\ x'+y=(x+y)'$ | What we've just proved |
| 5. | $-\forall x \forall y\ x+y=y+x$ | The negation of what we want to prove |
| 6. | $\exists x \exists y\ x+y \neq y+x$ | NEQ twice to Line 5 |
| 7. | $a+b \neq b+a$ | EI to Line 6 with $a$ to $x$ and $b$ to $y$ |
| 8. | $a+0=a$ | UI to Line 1 with $a$ to $x$ |
| 9. | $0+a=a$ | UI to Line 2 with $a$ to $x$ |
| 10. | $a+0=0+a$ | Rule 17 to Lines 8 and 9 |
| 11. | $a+c=c+a$ | MI to Lines 7 and 10 with $P_x$: $a+x=x+a$ |
| 12. | $a+c' \neq c'+a$ | MI to Lines 7 and 10 with $P_x$: $a+x=x+a$ |
| 13. | $a+c'=(a+c)'$ | UI to Line 3 with $a$ to $x$ and $c$ to $y$ |
| 14. | $c'+a=(c+a)'$ | UI to Line 4 with $c$ to $x$ and $a$ to $y$ |
| 15. | $c'+a=(a+c)'$ | Rule 16 to Lines 11 and 14 |
| 16. | $a+c' \neq (a+c)'$ | Rule 16 to Lines 12 and 15 |
| | $\underset{13,16}{\times}$ | |

Hence, $\forall x \forall y\ x+y=y+x$ (for all natural numbers, addition is commutative).