Solving Actuarial Math with Python

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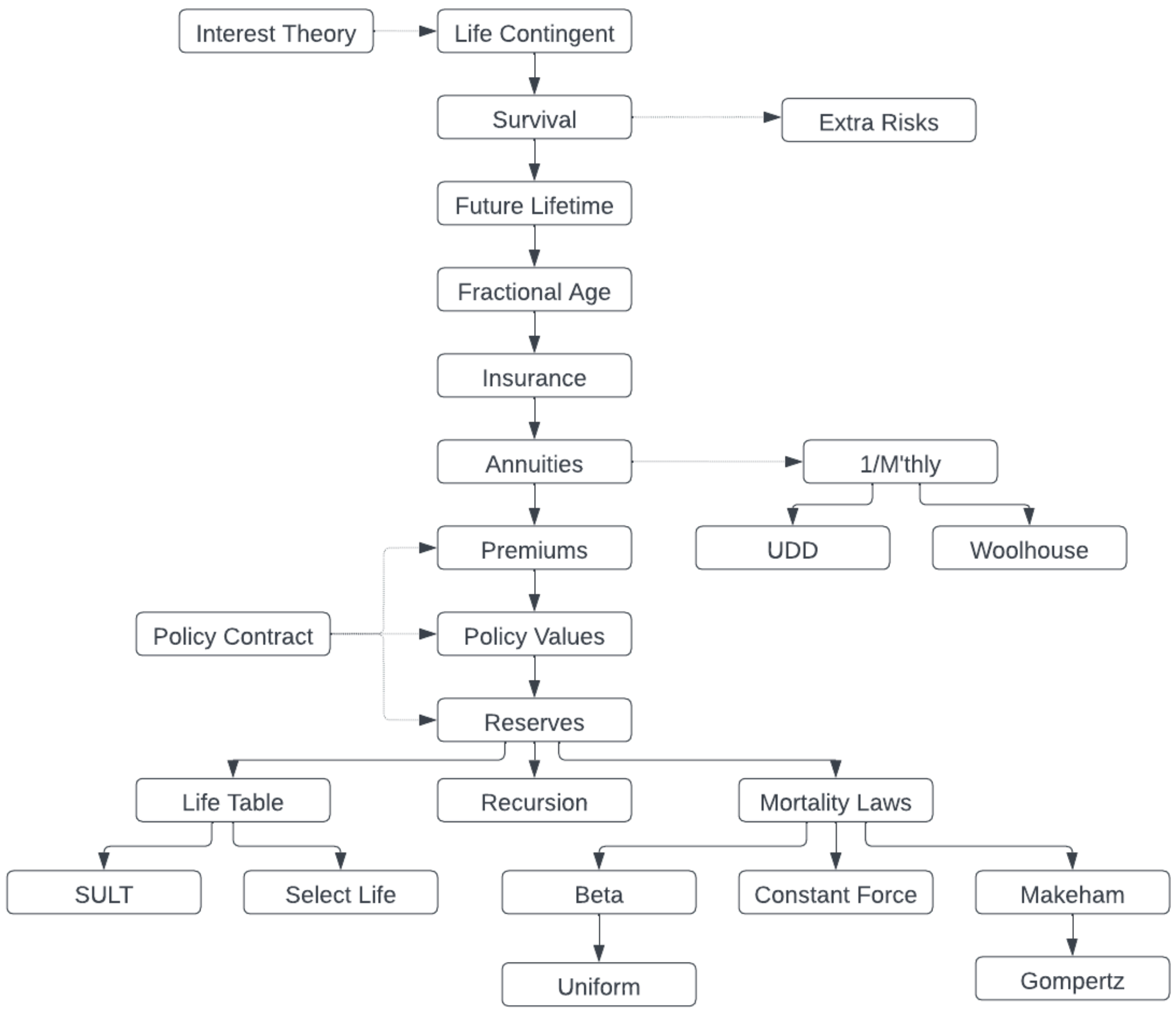
CONTENTS

actuarialmath – Life Contingent Risks with Python

This Python package implements fundamental methods for modeling life contingent risks, and closely follows the coverage of traditional topics in actuarial exams and standard texts such as the "Fundamentals of Actuarial Math - Long-term" exam syllabus by the Society of Actuaries, and "Actuarial Mathematics for Life Contingent Risks" by Dickson, Hardy and Waters. The actuarial concepts, and corresponding Python classes, are introduced and modeled hierarchically.



Quick Start

1. `pip install actuarialmath`
   - also requires `numpy`, `scipy`, `matplotlib` and `pandas`.
2. Start Python (version >= 3.10) or Jupyter-notebook
   - Select and import a suitable subclass to initialize with your actuarial assumptions, such as `MortalityLaws` (or a special law like `ConstantForce`), `LifeTable`, `SULT`, `SelectLife` or `Recursion`.
   - Call appropriate methods to compute intermediate or final results, or to `solve` parameter values implicitly.
   - Adjust the answers with `ExtraRisk` or `Mthly` (or its `UDD` or `Woolhouse`) classes

Examples

SOA FAM-L sample question 5.7:

Given $A_{35} = 0.188$, $A_{65} = 0.498$, $S_{35}(30) = 0.883$, calculate the EPV of a temporary annuity $\ddot{a}^{(2)}_{35:\overline{30|}}$ paid half-yearly using the Woolhouse approximation.

```
from actuarialmath import Recursion, Woolhouse
# initialize Recursion class with actuarial inputs
life = Recursion().set_interest(i=0.04)\
                  .set_A(0.188, x=35)\
                  .set_A(0.498, x=65)\
                  .set_p(0.883, x=35, t=30)
# modfy the standard results with Woolhouse mthly approximation
mthly = Woolhouse(m=2, life=life, three_term=False)
# compute the desired temporary annuity value
print(1000 * mthly.temporary_annuity(35, t=30)) #   solution = 17376.7
```

SOA FAM-L sample question 7.20:

For a fully discrete whole life insurance of 1000 on (35), you are given

- First year expenses are 30% of the gross premium plus 300
- Renewal expenses are 4% of the gross premium plus 30
- All expenses are incurred at the beginning of the policy year
- Gross premiums are calculated using the equivalence principle
- The gross premium policy value at the end of the first policy year is R
- Using the Full Preliminary Term Method, the modified reserve at the end of the first policy year is S
- Mortality follows the Standard Ultimate Life Table
- $i = 0.05$

Calculate R – S

```
from actuarialmath import SULT, Contract
life = SULT()
# compute the required FPT policy value
S = life.FPT_policy_value(35, t=1, b=1000)  # is always 0 in year 1!
# input the given policy contract terms
contract = Contract(benefit=1000,
                    initial_premium=.3,
                    initial_policy=300,
                    renewal_premium=.04,
                    renewal_policy=30)
# compute gross premium using the equivalence principle
G = life.gross_premium(A=life.whole_life_insurance(35), **contract.premium_terms)
# compute the required policy value
R = life.gross_policy_value(35, t=1, contract=contract.set_contract(premium=G))
print(R-S)    # solution = -277.19
```

Resources

1. Colab or Jupyter notebook, to solve all sample SOA FAM-L exam questions
2. Online User Guide, or download pdf
3. API reference
4. Github repo and issues

Sources

- SOA FAM-L Sample Questions: copy retrieved Aug 2022
- SOA FAM-L Sample Solutions: copy retrieved Aug 2022
- Actuarial Mathematics for Life Contingent Risks, by David Dickson, Mary Hardy and Howard Waters, published by Cambridge University Press.

Contact

Github: https://terence-lim.github.io

CHAPTER

ONE

ACTUARIAL PYTHON

The `actuarialmath` package is written in and requires Python (currently: version 3.10). Though the comparable R language possesses other desirable qualities, object-oriented programming is more straightforward in Python: since our sequence of actuarial concepts logically build upon each other, they are naturally developed as a hieararchy of Python classes with inherited methods and properties.

1.1 Installation

Install either by using pip:

- `pip install actuarialmath`

or cloning from github:

- `git clone https://github.com/terence-lim/actuarialmath.git`

1.2 Overview

Each section of this document introduces a class, along with the actuarial concepts it implements, arranged logically in three groups. To use the package, a suitable subclass should first be selected from the last group to load the given actuarial assumptions. Then the appropriate computational methods can be called, which may be inherited from the other general classes or make use of any shortcut formulas that can be obtained from the specific survival distribution assumed.

1. Implement general actuarial methods
    - Basic interest theory and probability laws
    - Survival functions, expected future lifetimes and fractional ages
    - Insurance, annuity, premiums, policy values, and reserves calculations
2. Adjust results for
    - Extra mortality risks
    - 1/mthly payment frequency using UDD or Woolhouse approaches
3. Specify a particular form of actuarial assumptions
    - Mortality laws, such as constant force of maturity, beta and uniform distributions, or Makeham's and Gompertz's laws
    - Recursion inputs
    - Life table, select life table, or standard ultimate life table

1.3 License

1.4 Methods

The `Actuarial` base class provides some common helpful utility functions and definitions of constants, that are needed by other classes in the package.

```python
from actuarialmath import Actuarial
import describe
describe.methods(Actuarial)
```

```
class Actuarial - Define constants and common utility functions

    Constants:
      VARIANCE : select variance as the statistical moment to calculate
      WHOLE : indicates that term of insurance or annuity is Whole Life

    Methods:
    --------

    solve(fun, target, grid, mad):
      Solve root, or parameter that minimizes absolute value, of a function

    add_term(t, n):
      Add two terms, either term may be Whole Life

    max_term(x, t, u):
      Decrease term t if adding deferral period u to (x) exceeds maxage
```

1.5 Examples

The constant `WHOLE` indicates that the contract term of a insurance or annuity policy is *whole life*, whenever we need to add or subtract time periods.

```
actuarial = Actuarial()

def as_term(t): return "WHOLE_LIFE" if t == Actuarial.WHOLE else t

for a,b in [(3, Actuarial.WHOLE), (Actuarial.WHOLE, -1), (3, 2), (3, -1)]:
    print(f"{as_term(a)} + {as_term(b)} =", as_term(actuarial.add_term(a, b)))
```

```
3 + WHOLE_LIFE = WHOLE_LIFE
WHOLE_LIFE + -1 = WHOLE_LIFE
3 + 2 = 5
3 + -1 = 2
```

The `solve()` method returns the zero (i.e root), or alternatively the value that minimizes the absolute value, of a function, hence can be useful for imputing the value of the parameter which sets the output of a formula equal to a required target.

```
Actuarial.solve(fun=lambda omega: 1/omega, target=1/20, grid=[1, 100])
```

```
20.0
```

CHAPTER

TWO

INTEREST THEORY

Interest theory functions, that are in common actuarial and financial use, are reviewed. Interest rates are generally assumed to be fixed and constant.

2.1 Interest rates

i is the amount earned on $1 after one year

- effective annual *interest rate*
- $i^{(m)}$ denotes the nominal interest rate, stated on annual basis, compounded *m* times per year

$d = \frac{i}{1+i}$

- annual *discount rate* of interest
- $d^{(m)}$ denotes the nominal discount rate, stated on annual basis, compounded *m* times per year

$v = \frac{1}{1+i}$

- annual *discount factor*

$\delta = \log(1+i)$

- *continuosly-compounded rate* of interest, or *force of interest per year*

Relationships between interest rates

$$\begin{aligned}(1+i)^t &= (1-d)^{-t}\\ &= (1+\frac{i^{(m)}}{m})^{mt}\\ &= (1-\frac{d^{(m)}}{m})^{-mt}\\ &= e^{\delta t}\\ &= v^{-t}\end{aligned}$$

Doubling the force of interest

$\delta' \leftarrow 2\delta$

$i' \leftarrow 2i + i^2$

$d' \leftarrow 2d - d^2$

$v' \leftarrow v^2$

Annuity certain

Pays \$1 per year for n years

$$\ddot{a}_{\overline{n|}} = \frac{1 - v^n}{d}$$

- Annuity certain due: pays \$1 at the beginning of the year

$$a_{\overline{n|}} = \frac{1 - v^n}{i} = \ddot{a}_{\overline{n+1|}} - 1$$

- Immediate annuity certain: pays \$1 at the end of the year

$$\bar{a}_{\overline{n|}} = \frac{1 - v^n}{\delta}$$

- Continuous annuity certain: pays at a rate of \$1 per year continuously.

2.2 Methods

The `Interest` class implements methods to convert between nominal, discount, continuously-compounded and 1/m'thy rates of interest, and compute the value of an annuity certain.

```
from actuarialmath import Interest
import describe
describe.methods(Interest)
```

```
class Interest - Converts interest rates, and computes value of annuity certain

    Args:
      i : assumed annual interest rate
      d : or assumed discount rate
      v : or assumed discount factor
      delta : or assumed continuously compounded interest rate
      v_t : or assumed discount rate as a function of time
      i_m : or assumed monthly interest rate
      d_m : or assumed monthly discount rate
      m : m'thly frequency, if i_m or d_m are given

    Examples:
      >>> interest = Interest(v=0.75)

    Methods:
    --------

    annuity(t, m, due):
      Compute value of the annuity certain factor

    mthly(m, i, d, i_m, d_m):
      Convert to or from m'thly interest rates

    double_force(i, delta, d, v):
      Double the force of interest
```

2.3 Examples

SOA Question 3.10:

A group of 100 people start a Scissor Usage Support Group. The rate at which members enter and leave the group is dependent on whether they are right-handed or left-handed. You are given the following:

- The initial membership is made up of 75% left-handed members (L) and 25% right-handed members (R)
- After the group initially forms, 35 new (L) and 15 new (R) join the group at the start of each subsequent year
- Members leave the group only at the end of each year
- $q^L = 0.25$ for all years
- $q^R = 0.50$ for all years Calculate the proportion of the Scissor Usage Support Group's expected membership that is left-handed at the start of the group's6 th year, before any new members join for that year.

```
print("SOA Question 3.10:  (C) 0.86")
interest = Interest(v=0.75)
L = 35 * interest.annuity(t=4, due=False) + 75 * interest.v_t(t=5)
interest = Interest(v=0.5)
R = 15 * interest.annuity(t=4, due=False) + 25 * interest.v_t(t=5)
print(L / (L + R))
```

```
SOA Question 3.10:  (C) 0.86
0.8578442833761983
```

Example for doubling the force of interest:

```
print("Example: double the force of interest i=0.05")
i = 0.05
d = Interest(i=i).d                 # convert interest rate to discount rate
print('i:', i, 'd:', d)
i2 = Interest.double_force(i=i)    # interest rate after doubling force
d2 = Interest.double_force(d=d)    # discount rate after doubling force
print('i:', round(i2, 6), round(Interest(d=d2).i, 6))
print('d:', round(d2, 6), round(Interest(i=i2).d, 6))
```

```
Example: double the force of interest i=0.05
i: 0.05 d: 0.047619047619047616
i: 0.1025 0.1025
d: 0.092971 0.092971
```

CHAPTER

THREE

LIFE CONTINGENT RISKS

By modeling future lifetimes as random variables, probabilities of survival or death can be calculated. This section reviews essential probability relationships and the moments of common distributions.

3.1 Probability

$Var(aX + bY) = a^2\ Var(X) + b^2\ Var(Y) + 2\ a\ b\ Cov(X, Y)$

$Cov(X, Y) = E[XY] - E[X] \cdot E[Y]$

Bernoulli (p) distribution

$Y \in \{a,\ b\}$ w.p. $(p,\ 1-p)$

- $E[Y] = a\ p + b\ (1-p)$
- $Var[Y] = (a-b)^2\ p\ (1-p)$

Binomial $(N,\ p)$ distribution

Y is sum of N i.i.d. 0-1 Bernoulli(p)

- $E[Y] = N\ p$
- $Var[Y] = N\ p\ (1-p)$

Mixture

A mixture distribution is a random variable Y whose distribution function can be expressed as a weighted average of the distribution functions of n random variables $Y_1, \cdots, Y_n$

e.g. Y is mixture of two Binomial distributions $(p',\ N)$, where $p' \in (p_1,\ p_2)$ w.p. $(p,\ 1-p)$

- $E[Y] = p\ N\ p_1 + (1-p)\ N\ p_2$
- $Var[Y] = E[Y^2] - E[Y]^2 = E[Var(Y \mid p') + E(Y \mid p')^2] - E[Y]^2$

Conditional Variance

Alternative calculation of the variance of a mixture by using conditional variance formula

- $Var[Y] = Var(E[Y \mid p']) + E[Var(Y \mid p')]$

3.2 Portfolio Percentile

Normal Approximation

A portfolio, Y, which is the sum of N iid random variables each with mean μ and variance σ^2, has a normal distribution with

- mean $E[Y] = N\mu$ and
- variance $Var[Y] = N\sigma^2$

Percentiles

Percentiles are essentially an inverse function of the cumulative probability distribution. If $F(y)$ is the cdf for Y, then the p'th quantile is a number y_p such that $F(y) == p$.

$Y_p = E[Y] + z_p \sqrt{Var[Y]}$

- value of Y at percentile p

3.3 Methods

The `Life` class implements methods for computing moments and probabilities of random variables.

```
import math
from actuarialmath import Life
import describe
describe.methods(Life)
```

```
class Life - Compute moments and probabilities

    Methods:
    --------

    variance(a, b, var_a, var_b, cov_ab):
      Variance of weighted sum of two r.v.

    covariance(a, b, ab):
      Covariance of two r.v.

    bernoulli(p, a, b, variance):
      Mean or variance of bernoulli r.v. with values {a, b}

    binomial(p, N, variance):
      Mean or variance of binomial r.v.

    mixture(p, p1, p2, N, variance):
      Mean or variance of binomial mixture

    conditional_variance(p, p1, p2, N):
      Conditional variance formula

    portfolio_percentile(mean, variance, prob, N):
      Probability percentile of the sum of N iid r.v.'s

    set_interest(interest):
```

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```
      Set interest rate, which can be given in any form

    quantiles_frame(quantiles):
      Display selected quantile values from Normal distribution table
```

3.4 Examples

SOA Question 2.2:

Scientists are searching for a vaccine for a disease. You are given:

- 100,000 lives age x are exposed to the disease
- Future lifetimes are independent, except that the vaccine, if available, will be given to all at the end of year 1
- The probability that the vaccine will be available is 0.2
- For each life during year 1, $q_x = 0.02$
- For each life during year 2, $q_{x+1} = 0.01$ if the vaccine has been given and $q_{x+1} = 0.02$ if it has not been given

Calculate the standard deviation of the number of survivors at the end of year 2.

```
print("SOA Question 2.2: (D) 400")
p1 = (1. - 0.02) * (1. - 0.01)  # 2_p_x if vaccine given
p2 = (1. - 0.02) * (1. - 0.02)  # 2_p_x if vaccine not given
print(math.sqrt(Life.conditional_variance(p=.2, p1=p1, p2=p2, N=100000)))
print(math.sqrt(Life.mixture(p=.2, p1=p1, p2=p2, N=100000, variance=True)))
```

```
SOA Question 2.2: (D) 400
396.5914603215815
396.5914603237804
```

Normal distribution table:

Generate extract of normal distribution table (for SOA FAM-L exam)

```
print("Values of z for selected values of Pr(Z<=z)")
print("----------------------------------------")
print(Life.quantiles_frame().to_string(float_format=lambda x: f"{x:.3f}"))
```

```
Values of z for selected values of Pr(Z<=z)
-------------------------------------------
z          0.842  1.036  1.282  1.645  1.960  2.326  2.576
Pr(Z<=z)   0.800  0.850  0.900  0.950  0.975  0.990  0.995
```

CHAPTER

FOUR

SURVIVAL MODELS

The future lifetime of an individual is represented as a random variable. Under this framework, probabilities of death or survival, as well as an important quantity known as the force of mortality, can be calculated. It is shown how these quantitites are related, along with some actuarial notation.

4.1 Lifetime distribution

Let (x) denotes a life aged x, where $x \geq 0$, and T_x is time-to-death, or future lifetime, of (x). This means that $x + T_x$ represents the age-at-death random variable for (x).

$F_x(t) = Pr[T_x \leq t] = \int_0^t f_x(t)ds$

- probability that (x) does not survive beyond age $x + t$.

Lifetime density function

$f_x(t) = \frac{\partial}{\partial t} F_x(t) = \dfrac{f_0(x+t)}{S_0(x)}$

- probability density function for the random variable T_x

4.2 Survival function

In life insurance problems we may be interested in the probability of survival rather than death

$S_x(t) \equiv {}_tp_x = Pr[T_x > t] = 1 - F_x(t)$

- the probability that (x) survives for at least t years

$S_x(t) = \dfrac{S_0(x+t)}{S_0(x)}$

- by assumption that $Pr[T_x \leq t] = Pr[T_0 \leq x + t | T_0 > x]$

$S_x(t) = \int_t^\infty f_x(s)ds$

- since $\int_0^t f_x(s))ds + \int_t^\infty f_x(s)ds = 1$

$S_x(t) = \dfrac{l_{x+t}}{l_x}$

- relate survivor function to the number of lives in life table

$S_x(t) = e^{-\int_0^t \mu_{x+t} ds}$

- relate survival function to the force of mortality

$S_x(0) = 1,\ S_x(\infty) = 0,\ S'_x(t) \leq 0$

- conditions to be a valid survival function

4.3 Force of mortality

$$\mu_{x+t} = \frac{f_x(t)}{S_x(t)} = \frac{-\frac{\partial}{\partial t}\,{}_tp_x}{{}_tp_x} - \frac{\partial}{\partial t} \ln\ {}_tp_x$$

- is what actuaries call the force of mortality (and known as the hazard rate in survival analysis and the failure rate in reliability theory)

$\mu_x dx \approx Pr[T_0 \leq x + dx | T_0 > x]$

- can be interpreted as the probability that (x) dies before attaining age $x + dx$

$f_x(t) =\ {}_tp_x\ \mu_{x+t}$

- can be related to the lifetime density function

$\int_0^\infty \mu_{x+s} ds = \infty$

- since $S_x(\infty) = 0$

4.4 Actuarial notation

Survival probability

${}_tp_x = Pr[T_x > t] \equiv S_x(t)$

- probability that (x) survives to at least age $x + t$

Expected number of survivors

$l_x = l_{x_0}\ {}_{x-x_0}p_{x_0}$

- is the expected number of surviving lives at age x from l_{x_0} independent individuals aged x_0.

Mortality rate

${}_tq_x = 1 -\ {}_tp_x \equiv F_x(t)$

- probability that (x) dies before age $x + t$.

Deferred mortality probability

${}_{u|t}q_x = Pr[u < T_x \leq u + t] = \int_u^{u+t}\ {}_sp_x \mu_{x+s} ds$

- probability that (x) survives u years, and then dies in the subsequent t years.

$${}_{u|t}q_x = \frac{l_{x+u} - l_{x+u+t}}{l_x}$$

- can be related to number of lives in the life table

${}_{u|t}q_x =\ {}_up_x\, t q_{x+u} =\ {}_{u+t}q_x -\ {}_uq_x =\ {}_up_x -\ {}_{u+t}p_x$

- can be computed as either (1) deferred mortality (2) limited mortality or (3) complement of survival functions.

4.5 Methods

The `Survival` class implements methods to compute and apply relationships between the various forms of survival and mortality functions. The force of mortality function fully describes the lifetime distribution, just as the survival function does.

```
import math
from actuarialmath import Survival
import describe
describe.methods(Survival)
```

```
class Survival - Set and derive basic survival and mortality functions

    Examples:
      >>> B, c = 0.00027, 1.1
      >>> def S(x,s,t): return (math.exp(-B * c**(x+s) * (c**t - 1)/math.log(c)))
      >>> f = Survival().set_survival(S=S).f_x(x=50, t=10)
      >>> def ell(x,s): return (1 - (x+s) / 60)**(1 / 3)
      >>> mu = Survival().set_survival(l=ell).mu_x(35) * 1000
      >>> def ell(x,s): return (1 - ((x+s)/250)) if x+s < 40 else (1 - ((x+s)/
↪100)**2)
      >>> q = Survival().set_survival(l=ell).q_x(30, t=20)
      >>> def fun(k):
      >>>     return Survival().set_survival(l=lambda x,s: 100*(k - (x+s)/2)**(2/
↪3))      >>>                      .mu_x(50)
      >>> k = int(Survival.solve(fun, target=1/48, grid=50))

    Methods:
    --------

    set_survival(S, f, l, mu, maxage, minage):
      Construct the basic survival and mortality functions given any one form

    l_x(x, s):
      Number of lives at integer age [x]+s: l_[x]+s

    d_x(x, s):
      Number of deaths at integer age [x]+s: d_[x]+s

    p_x(x, s, t):
      Probability that [x]+s lives another t years: : t_p_[x]+s

    q_x(x, s, t, u):
      Probability that [x]+s lives for u, but not t+u years: u|t_q_[x]+s

    f_x(x, s, t):
      Lifetime density function of [x]+s after t years: f_[x]+s(t)

    mu_x(x, s, t):
      Force of mortality of [x] at  s+t years: mu_[x](s+t)
```

4.6 Examples

SOA Question 2.3:

You are given that mortality follows Gompertz Law with B = 0.00027 and c = 1.1. Calculate $f_{50}(10)$.

```
print("SOA Question 2.3: (A) 0.0483")
B, c = 0.00027, 1.1
def S(x,s,t): return (math.exp(-B * c**(x+s) * (c**t - 1)/math.log(c)))
life = Survival().set_survival(S=S)
print(life.f_x(x=50, t=10))
```

```
SOA Question 2.3: (A) 0.0483
0.048327399045049846
```

SOA Question 2.6

You are given the survival function:

$S_0(x) = \left(1 - \frac{x}{60}\right)^{\frac{1}{3}}, \quad 0 \leq x \leq 60$

Calculate $1000\mu_{35}$.

```
print("# SOA Question 2.6: (C) 13.3")
life = Survival().set_survival(l=lambda x,s: (1 - (x+s) / 60)**(1 / 3))
print(1000*life.mu_x(35))
```

```
# SOA Question 2.6: (C) 13.3
13.340451278922776
```

SOA Question 2.7

You are given the following survival function of a newborn:

$$\begin{aligned} S_0(x) &= 1 - \frac{x}{250}, \quad 0 \leq x < 40 \\ &= 1 - (\frac{x}{100})^2, \quad 40 \leq x \leq 100 \end{aligned}$$

Calculate the probability that (30) dies within the next 20 years

```
print("SOA Question 2.7: (B) 0.1477")
def S(x,s):
    return 1 - ((x+s) / 250) if (x+s) < 40 else 1 - ((x+s) / 100)**2
print(Survival().set_survival(l=S).q_x(30, t=20))
```

```
SOA Question 2.7: (B) 0.1477
0.1477272727272727
```

SOA Question 2.8

In a population initially consisting of 75% females and 25% males, you are given:

- For a female, the force of mortality is constant and equals μ
- For a male, the force of mortality is constant and equals 1.5μ

- At the end of 20 years, the population is expected to consist of 85% females and 15% males.

Calculate the probability that a female survives one year.

```
print("SOA Question 2.8: (C) 0.938")
def fun(mu):  # Solve first for mu, given ratio of start and end proportions
    male = Survival().set_survival(mu=lambda x,s: 1.5 * mu)
    female = Survival().set_survival(mu=lambda x,s: mu)
    return (75 * female.p_x(0, t=20)) / (25 * male.p_x(0, t=20))
mu = Survival.solve(fun, target=85/15, grid=[0.89, 0.99])
p = Survival().set_survival(mu=lambda x,s: mu).p_x(0, t=1)
print(p)
```

```
SOA Question 2.8: (C) 0.938
0.9383813306903799
```

CAS41-F99:12

You are given the following survival function:

$S(x) = 100(k - \frac{x}{2})^{\frac{2}{3}}$

Find k, given that $\mu_{50} = \frac{1}{48}$

```
print("CAS41-F99:12: k = 41")
def fun(k):
    return Survival().set_survival(l=lambda x,s: 100*(k - (x+s)/2)**(2/3))\
                     .mu_x(50)
print(Survival.solve(fun, target=1/48, grid=50))
```

```
CAS41-F99:12: k = 41
41.005207994280646
```

CHAPTER FIVE

EXPECTED FUTURE LIFETIMES

In many insurance applications we are interested not only in the future lifetime of an individual, but also the individual's *curtate* future lifetime, defined as the integer part of future lifetime. For some lifetime distributions we are able to integrate for the mean and standard deviations of future lifetimes directly, without using numerical techniques.

5.1 Complete expectation of life

$\mathring{e}_x = E[T_x] = \int_0^\infty t \; {}_tp_x \; \mu_{x+t} ds = \int_0^\infty \; {}_tp_x \; dt$

- is the complete expectation of life, or the expected future lifetime

$E[T_x^2] = \int_0^\infty t^2 \; {}_tp_x \; \mu_{x+t} \; ds = \int_0^\infty \; 2 \; t \; {}_tp_x \; dt$

- Second moment of future lifetime

$Var[T_x] = E[T_x^2] - (\mathring{e}_x)^2$

- Variance of future lifetime

5.2 Curtate expectation of life

$K_x = \lfloor T_x \rfloor$

- is the curtate future lifetime random variable, representing the number of completed whole future years by (x) prior to death

$e_x = E[K_x] = \sum_{k=0}^\infty k \; {}_{k|}q_x \; = \sum_{k=1}^\infty \; {}_kp_x \; dt$

- Is the curtate expectation of life, representing the expected curtate lifetime

$E[K_x^2] = \sum_{k=0}^\infty k^2 \; {}_{k|}q_x \; = \sum_{k=1}^\infty (2k-1) \; {}_kq_x \; dt$

- Second moment of curtate future lifetime

$Var[K_x] = E[K_x^2] - (e_x)^2$

- Variance of curtate future lifetime

5.3 Temporary expectation of life

We are sometimes interested in the future lifetime random variable subject to a cap of n years, which is represented by the random variable $min(T_x, n)$.

$\overset{\circ}{e}_{x:\overline{n|}} = \int_0^n t~_tp_x~\mu_{x+t}~ds + n~_np_x = \int_0^n~_tp_x~dt$

- term complete expectation of life

$e_{x:\overline{n|}} = \sum_{k=0}^{n-1} k~_{k|}q_x~+ n~_np_x = \sum_{k=1}^{n}~_kp_x$

- temporary curtate expectation of life, limited at n years

5.4 Methods

The `Lifetime` class implements methods to compute curtate and complete expectations and second moments of future lifetime

```
from actuarialmath import Lifetime
import describe
describe.methods(Lifetime)
```

```
class Lifetime - Computes expected moments of future lifetime

    Methods:
    --------

    e_x(x, s, t, curtate, moment):
      Compute curtate or complete expectations and moments of life
```

5.5 Examples

AMLCR Exericse 2.1

Let $F_0(t) = 1 - (1 - t/105)^{1/5}$ for $0 \leq t \leq 105$. Calculate

(a) the probability that a newborn life dies before age 60 [0.1559]

(b) the probability that a life aged 30 survives to at least age 70 [0/8586]

(c) the probability that a life aged 20 dies between ages 90 and 100 [0.1394]

(d) the force of mortality at age 50 [0.0036]

(e) the median future lifetime at age 50 [53.28]

(f) the complete expectation of life at age 50 [45.83]

(g) the curtate expectation of life at age 50 [45.18]

```
def F(t): return (1 - (1 - t/105)**(1/5) if t <= 105 else 1)
life = Lifetime().set_survival(S=lambda x,s,t: (1-F(x+s+t)) / (1-F(x+s)))
print("(a)", life.q_x(x=0, t=60))
print("(b)", life.p_x(x=30, t=70-30))
```

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```
print("(c)", life.q_x(x=20, u=90-20, t=100-90))
print("(d)", life.mu_x(x=50))
print("(e)", life.solve(fun=lambda t: life.p_x(x=50, t=t), target=0.5, grid=[0,55]))
print("(f)", life.e_x(x=50, curtate=False))
print("(g)", life.e_x(x=50))
```

```
(a) 0.1558791201558899
(b) 0.8586207034844889
(c) 0.13943444007155992
(d) 0.003636764417087519
(e) 53.28124999999994
(f) 45.833333333333336
(g) 45.17675143564247
```

SOA Question 2.1

You are given:

- $S0(t) = (1 - \frac{t}{\omega})^{\frac{1}{4}}, \quad 0 \leq t \leq \omega$
- $\mu_{65} = \frac{1}{180}$

Calculate e_{106}, the curtate expectation of life at age 106.

```
print("SOA Question 2.1: (B) 2.5")
def fun(omega):  # Solve first for omega, given mu_65 = 1/180
    return Lifetime().set_survival(l=lambda x,s: (1-(x+s)/omega)**0.25,
                                   maxage=omega).mu_x(65)
omega = int(Lifetime.solve(fun, target=1/180, grid=100))  # solve for omega
e = Lifetime().set_survival(l=lambda x,s: (1 - (x+s)/omega)**0.25,
                            maxage=omega).e_x(106)
print(e)
```

```
SOA Question 2.1: (B) 2.5
2.4786080555423604
```

SOA Question 2.4

You are given ${}_tq_0 = \frac{t^2}{10{,}000} \quad 0 < t < 100$. Calculate $\mathring{e}_{75:\overline{10}|}$.

```
print("SOA Question 2.4: (E) 8.2")
def S(x,s) : return 0. if (x+s) >= 100 else 1 - ((x+s)**2)/10000.
e = Lifetime().set_survival(l=S).e_x(75, t=10, curtate=False)
print(e)
```

```
SOA Question 2.4: (E) 8.2
8.20952380952381
```

CHAPTER

SIX

FRACTIONAL AGES

Given values of l_x at integer ages only, we need to make some assumption about the probability distribution for the future lifetime random variable between integer ages, in order to calculate probabilities for non-integer ages or durations. Such fractional age assumptions may be specified in terms of the force of mortality function (e.g. constant) or the survival or mortality probabilities (e.g. uniform distribution of deaths).

6.1 Uniform distribution of deaths

$T_x = K_x + R_x$

- The UDD assumptions defines a new random variable $R_x \sim U(0,1)$ which is independent of curtate lifetime K_x.

${}_rq_x = r\ q_x$, for integer x and $0 \leq s < 1$

- is an equivalent way of formulating the UDD assumption

$l_{x+r} = (1-r)\ l_x + r\ l_{x+1} = l_x - r\ d_x$

- UDD is linear interpolation of lives between integer ages

${}_rq_{x+s} = \dfrac{r\ q_x}{1 - s\ q_x}, \quad$ for $0 \leq s + r < 0$

- mortality rate at a fractional age over a fractional duration, under UDD

$\mu_{x+r} = \dfrac{1}{1 - r\ q_x}$

- applying the UDD approximation over successive ages implies a discontinuous function for the force of mortality, with discontinuities occurring at integer ages.

$f_x(r) = \ {}_rp_x \mu_{x+r} = \ q_x$

- lifetime density is constant between integer ages, which also follows from the UDD assumption for R_x.

$\mathring{e}_x = q_x \dfrac{1}{2} + p_x(1 + \mathring{e}_{x+1})$

- recursive expression for complete expectation of life obtained with UDD assumption

$\mathring{e}_{x:\overline{1}|} = 1 - q_x \dfrac{1}{2} = q_x \dfrac{1}{2} + \ p_x$

- 1-year limited complete expectation under UDD

$\mathring{e}_x \approx e_x + 0.5$

- This exact result under UDD is often used as an approximation of complete and curtate expectations.

6.2 Constant force of mortality

$\mu_{x+r} = \mu_x = -\ln p_x, \quad$ for $0 \leq r < 1$

- force of mortality is constant between integer ages, which leads to a step function over successive years of age

$l_{x+r} = (l_x)^{1-r} \cdot (l_{x+1})^r$

- constant force of mortality is exponential interpolation of lives

${}_rp_x = e^{-\mu_x r} = (p_x)^r$

- since $p_x = e^{-\int_0^1 \mu_{x+u} du} = e^{-\mu_x}$

${}_rp_{x+s} = e^{-\int_0^r \mu_{x+s+u} du} = (p_x)^r, \quad$ for $0 \leq r + s < 1$

- the probability of surviving for period of s from age $x + t$ is independent of t under constant force of mortality

$f_x(r) = {}_rp_x \, \mu_{x+r} = e^{-\mu_x r} \cdot \mu_x, \quad$ for $0 \leq r < 1$

- relate lifetime density of (x) to constant force of mortality assumption

6.3 Methods

The Fractional class implements methods to compute survival and mortality functions between integer ages, assuming either uniform distribution of deaths or constant force of mortality

```
from actuarialmath import Fractional
import describe
describe.methods(Fractional)
```

```
class Fractional - Compute survival functions at fractional ages and durations

    Args:
      udd : select UDD (True, default) or CFM (False) between integer ages

    Methods:
    --------

    l_r(x, s, r):
      Number of lives at fractional age: l_[x]+s+r

    p_r(x, s, r, t):
      Probability of survival from and through fractional age: t_p_[x]+s+r

    q_r(x, s, r, t, u):
      Deferred mortality rate within fractional ages: u|t_q_[x]+s+r

    mu_r(x, s, r):
      Force of mortality at fractional age: mu_[x]+s+r

    f_r(x, s, r, t):
      mortality function at fractional age: f_[x]+s+r (t)

    E_r(x, s, r, t):
      Pure endowment at fractional age: t_E_[x]+s+r
```

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```
    e_r(x, s, t):
      Temporary expected future lifetime at fractional age: e_[x]+s:t

    e_approximate(e_complete, e_curtate):
      Convert between curtate and complete expectations assuming UDD shortcut
```

6.4 Examples

Compare fractional age assumptions:

```
life1 = Fractional(udd=False).set_survival(l=lambda x,t: 50-x-t)
life2 = Fractional(udd=True).set_survival(l=lambda x,t: 50-x-t)
print('           Constant Force of Mortality           UDD')
print('           ---------------------------          -----')
print('mortality rate    ', life1.q_r(40, t=0.5), life2.q_r(40, t=0.5))
print('force of mortality', life1.mu_r(40, r=0.5), life2.mu_r(40, r=0.5))
print('lifetime density  ', life1.f_r(40, r=0.5), life2.f_r(40, r=0.5))
```

```
           Constant Force of Mortality           UDD
           ---------------------------          -----
mortality rate     0.05131670194948623 0.04999999999999999
force of mortality 0.10536051565782628 0.10526315789473682
lifetime density   0.10536051565782628 0.09999999999999998
```

CHAPTER

SEVEN

INSURANCE

For a life insurance policy, the time at which the benefit will be paid is unknown until the policyholder actually dies and the policy becomes a claim. In this section, formulas are developed for the valuation of traditional insurance benefits (such as whole life, term and endowment insurance) and their second moments, as well as varying insurances.

7.1 Present value of life insurance r.v. Z

Valuation functions for the present value of insurance benefits, denoted by Z, are based on the continuous future lifetime random variable, T_x, or the curtate future lifetime random variable, K_x. The interest rate is generally assumed to be fixed and constant. Given a survival model and an interest rate we can derive the distribution of the present value random variable for a life contingent benefit, and can therefore compute quantities such as the mean and variance of the present value. Valuation functions are developed per dollar of sum assured, which can be multiplied by the actual sum insured for difference benefit amounts. The expected present value of insurance benefit, denoted and solved by $EPV(Z)$, is sometimes referred to as the actuarial value or actuarial present value.

Whole life insurance:

$Z = v^{T_x}$

- continuous insurance, where benefit is payable immediately on death

$\overline{A}_x = E[v^{T_x}] = \int_{t=0}^{\infty}~ v^t~ {}_tp_x~ \mu_{x+t}~ dt$

- EPV continuous whole life insurance

$Z = v^{K_x+1}$

- annual insurance, where benefit is payable at end of year of death

$A_x = E[v^{K_x+1}] = \sum_{k=0}^{\infty}~ v^{k+1}~ {}_{k|}q_x$

- EPV of annual whole life insurance

Term insurance:

$Z = 0$ when $T_x > t$, else v^{T_x}

- death benefit is payable immediately only if the policy-holder dies within t years

$\overline{A}^{~1}_{x:\overline{t|}} = \int_{s=0}^{t}~ v^s~ {}_sp_x~ \mu_{x+s}~ ds = \overline{A}_x - ~{}_tE_x~ \overline{A}_{x+t}$

- continous term insurance as the difference of continous whole life and deferred continuous whole life

$Z = 0$ when $K_x \ge t$, else v^{K_x+1}

- death benefit is payable at the end of the year of death provided this occurs within t years

$A^{~1}_{x:\overline{t|}} = \sum_{k=0}^{t-1}~ v^{k+1}~ {}_{k|}q_x = A_x - ~{}_tE_x~ A_{x+t}$

- annual term insurance as the difference of annual whole life and deferred annual whole life

Deferred whole life insurance:

Does not begin to offer death benefit cover until the end of a deferred period

$Z = 0$ when $T_x < u$, else v^{T_x}

- benefit is payable immediately on the death of (x) provided that (x) dies after the age of $x + u$

${}_{u|}\overline{A}_{x:\overline{t|}} = \ {}_uE_x \ \overline{A}_{x+u:\overline{t|}}$

- continuous deferred insurance as the EPV of whole life insurance starting at the end of deferred period.

$Z = 0$ when $K_x < u$, else v^{K_x+1}

- annual deferred insurance where the death benefit is payable at the end of the year of death

${}_{u|}A_{x:\overline{t|}} = \ {}_uE_x \ A_{x+u:\overline{t|}}$

- annual deferred insurance as the EPV of whole life insurance starting at the end of deferred period.

Endowment insurance:

An endowment insurance provides a combination of a term insurance and a pure endowment.

$Z = v^t$ when $T_x \geq t$, else v^{T_x}

- benefit is payable on the death of (x) should (x) die within t years, but if (x) survives for t years, the sum insured is payable at the end of the t-th year.

$\overline{A}_{x:\overline{t|}} = \overline{A}^{1}_{x:\overline{t|}} + \ {}_tE_x$

- continuous endowment insurance as continuous term insurance plus a pure endowment

$Z = v^t$ when $K_x \geq t$, else v^{K_x+1}

- annual endowment insurance where the death benefit is payable at the end of the year of death

$A_{x:\overline{t|}} = A^{1}_{x:\overline{t|}} + \ {}_tE_x$

- annual endowment insurance as annual term insurance plus a pure endowment

7.2 Pure endowment

$Z = 0$ when $T_x < t$, else $Z = v^t$

- benefit payable in t years if (x) is still alive at that time, but pays nothing if (x) dies before then.

${}_nE_x = A_{x:\overline{n|}}^{\ \ 1} = v^n \ {}_np_x$

- Because the pure endowment will be paid only at time t, assuming the life survives, there is only the discrete time version

7.3 Variances

The variance of life insurance benefits is computed as the difference of the second moment and square of the first moment.

$^2\overline{A}_x = \int_{t=0}^{\infty} v^{2t} \ {}_tp_x \ \mu_{x+t} \ dt$

- second moment of continuous insurance equal to $\overline{A}_x$ at double the force of interest.

$^2A_x = \sum_{k=0}^{\infty} v^{2(k+1)} \ {}_{k|}q_x$

- second moment of annual insurance is equal to A_x at double the force of interest.

Life insurance:

$Var(\overline{A}_x) = {}^2\overline{A}_x - (\overline{A}_x)^2$

- variance of continuous whole life insurance

$Var(A_x) = {}^2A_x - \ (A_x)^2$

- variance of annual whole life insurance

$Var(\overline{A}^{1}_{x:\overline{t|}}) = {}^2\overline{A}^{1}_{x:\overline{t|}} - (\overline{A}^{1}_{x:\overline{t|}})^2$

- variance of continuous term life insurance

$Var(A^{1}_{x:\overline{t|}}) = {}^2A^{1}_{x:\overline{t|}} - \ (A^{1}_{x:\overline{t|}})^2$

- variance of annual term life insurance

$Var({}_{u|}\overline{A}_{x:\overline{t|}}) = {}^2_{u|}\overline{A}_{x:\overline{t|}} - ({}_{u|}\overline{A}_{x:\overline{t|}})^2$

- variance of continuous deferred life insurance

$Var({}_{u|}A_{x:\overline{t|}}) = {}^2_{u|}A_{x:\overline{t|}} - \ ({}_{u|}A_{x:\overline{t|}})^2$

- variance of annual deferred life insurance

$Var(\overline{A}_{x:\overline{t|}}) = {}^2\overline{A}_{x:\overline{t|}} - (\overline{A}_{x:\overline{t|}})^2$

- variance of continous endowment insurance

$Var(A_{x:\overline{t|}}) = {}^2A_{x:\overline{t|}} - \ (A_{x:\overline{t|}})^2$

- variance of annual endowment insurance

Pure Endowment:

${}^2_tE_x = v^{2t} \ {}_tp_x = v_t \ {}_tE_x$

- second moment of pure endowment by discounting ${}_tE_x$

$Var({}_tE_x) = v^{2t} \ {}_tp_x \ {}_tq_x = v^{2t} \ {}_tp_x - (v^t \ {}_tp_x)^2$

- variance of pure endowment is the variance of a Bernoulli random variable

7.4 Varying insurance

Increasing insurance:

Amount of death benefit increases arithmetically at a rate of \$1 per year.

$(\overline{IA})_x = \int_{t=0}^{\infty}~t~v^t~{}_tp_x~\mu_{x+t}~dt$

- increasing continuous whole life insurance

$(IA)_x = \sum_{k=0}^{\infty}~(k+1)~v^{k+1}~{}_{k|}q_x$

- increasing annual whole life insurance

$(\overline{IA})^1_{x:\overline{t|}} = \int_{s=0}^{t}~s~v^s~{}_sp_x~\mu_{x+s}~ds$

- increasing continuous term insurance

$(IA)^1_{x:\overline{t|}} = \sum_{k=0}^{t-1}~(k+1)~v^{k+1}~{}_{k|}q_x$

- increasing annual term insurance

Decreasing insurance:

Amount of death benefit increasing arithmetically at a rate of \$1 per year.

$(\overline{DA})^1_{x:\overline{t|}} = \int_{s=0}^{t}~(t-s)~v^s~{}_sp_x~\mu_{x+s}~ds$

- decreasing continuous term insurance (not defined for whole life)

$(DA)^1_{x:\overline{t|}} = \sum_{k=0}^{t-1}~(t-k)~v^{k+1}~{}_{k|}q_x$

- decreasing annual term insurance (not defined for whole life)

Identity relationship:

$(\overline{DA})^1_{x:\overline{t|}} + (\overline{IA})^1_{x:\overline{t|}} = t~\overline{A}^1_{x:\overline{t|}}$

- relates continuous increasing and decreasing insurances to level insurance

$(DA)^1_{x:\overline{t|}} + (IA)^1_{x:\overline{t|}} = (t+1)~A^1_{x:\overline{t|}}$

- relates annual increasing and decreasing insurances to level insurance

7.5 Methods

The `Insurance` class implements methods to compute the expected present value of life insurance

```
import math
import numpy as np
import matplotlib.pyplot as plt
from actuarialmath import Insurance
import describe
describe.methods(Insurance)
```

```
class Insurance - Compute expected present values of life insurance

    Methods:
    --------
```

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```
E_x(x, s, t, endowment, moment):
  Pure endowment: t_E_x

A_x(x, s, t, u, benefit, endowment, moment, discrete):
  Numerically compute EPV of insurance from basic survival functions

insurance_variance(A2, A1, b):
  Compute variance of insurance given moments and benefit

whole_life_insurance(x, s, moment, b, discrete):
  Whole life insurance: A_x

term_insurance(x, s, t, b, moment, discrete):
  Term life insurance: A_x:t^1

deferred_insurance(x, s, u, t, b, moment, discrete):
  Deferred insurance n|_A_x:t^1 = discounted term or whole life

endowment_insurance(x, s, t, b, endowment, moment, discrete):
  Endowment insurance: A_x^1:t = term insurance + pure endowment

increasing_insurance(x, s, t, b, discrete):
  Increasing life insurance: (IA)_x

decreasing_insurance(x, s, t, b, discrete):
  Decreasing life insurance: (DA)_x

Z_t(x, prob, discrete):
  T_x given percentile of the PV of WL or Term insurance, i.e. r.v. Z(t)

Z_from_t(t, discrete):
  PV of insurance payment Z(t), given T_x (or K_x if discrete)

Z_to_t(Z):
  T_x s.t. PV of insurance payment is Z

Z_from_prob(x, prob, discrete):
  Percentile of insurance PV r.v. Z, given probability

Z_to_prob(x, Z):
  Cumulative density of insurance PV r.v. Z, given percentile value

Z_x(x, s, t, discrete):
  EPV of year t insurance death benefit for life aged [x]+s: b_x[s]+s(t)

Z_plot(x, s, stop, benefit, T, discrete, ax, title, color):
  Plot of PV of insurance r.v. Z vs t
```

7.6 Examples

SOA Question 6.33

An insurance company sells 15-year pure endowments of 10,000 to 500 lives, each age x, with independent future lifetimes. The single premium for each pure endowment is determined by the equivalence principle.

You are given:

- i = 0.03
- $\mu_x(t) = 0.02t, \quad t \geq 0$
- ${}_0L$ is the aggregate loss at issue random variable for these pure endowments.

Using the normal approximation without continuity correction, calculate $Pr({}_0L) > 50,000)$.

```
print("SOA Question 6.33:  (B) 0.13")
life = Insurance().set_survival(mu=lambda x,t: 0.02*t).set_interest(i=0.03)
var = life.E_x(x=0, t=15, moment=life.VARIANCE, endowment=10000)
p = 1 - life.portfolio_cdf(mean=0, variance=var, value=50000, N=500)
print(p)
```

```
SOA Question 6.33:  (B) 0.13
0.12828940905648634
```

SOA Question 4.18

You are given that T, the time to first failure of an industrial robot, has a density f(t) given by

$$\begin{aligned} f(t) &= 0.1, \quad 0 \leq t < 2 \\ &= 0.4t^{-2}, \quad t \leq t < 10 \end{aligned}$$

with $f(t)$ undetermined on $[10, \infty)$.

Consider a supplemental warranty on this robot that pays 100,000 at the time T of its first failure if $2 \leq T \leq 10$, with no benefits payable otherwise. You are also given that $\delta = 5\%$. Calculate the 90th percentile of the present value of the future benefits under this warranty.

```
print("SOA Question 4.18  (A) 81873 ")
def f(x,s,t): return 0.1 if t < 2 else 0.4*t**(-2)
life = Insurance().set_interest(delta=0.05)\
                  .set_survival(f=f, maxage=10)
def benefit(x,t): return 0 if t < 2 else 100000
prob = 0.9 - life.q_x(x=0, t=2)
T = life.Z_t(x=0, prob=prob)
Z = life.Z_from_t(T) * benefit(0, T)
print(Z)
```

```
SOA Question 4.18  (A) 81873
81873.07530779815
```

SOA Question 4.10

The present value random variable for an insurance policy on (x) is expressed as: $$

Determine which of the following is a correct expression for $E[Z]$.

(A) ${}_{10|}\overline{A}_x + {}_{20|}\overline{A}_x - {}_{30|}\overline{A}_x$

(B) $\overline{A}_x +_{20} E_x \overline{A}_{x+20} - 2\ _{30}E_x \overline{A}_{x+30}$

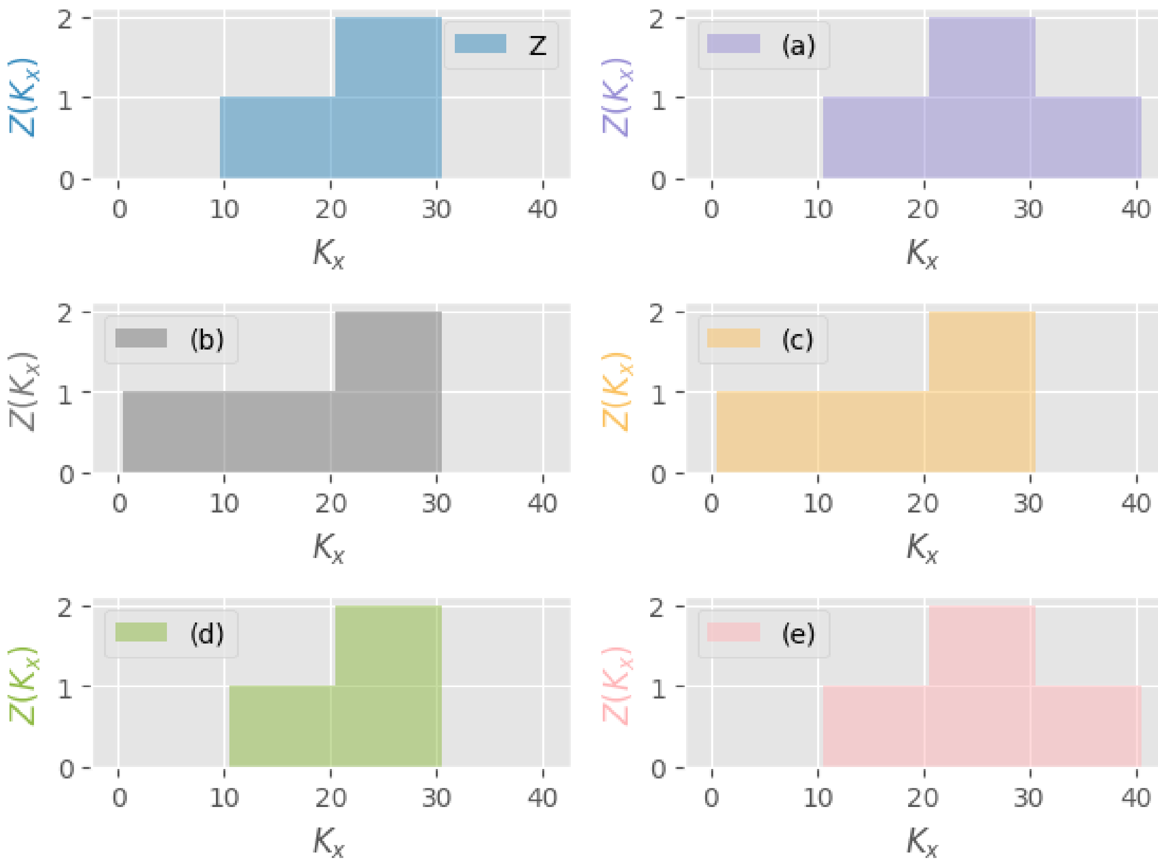
(C) $_{10}E_x \overline{A}_x +_{20} E_x \overline{A}_{x+20} - 2\ _{30}E_x \overline{A}_{x+30}$

(D) $_{10}E_x \overline{A}_{x+10} +_{20} E_x \overline{A}_{x+20} - 2\ _{30}E_x \overline{A}_{x+30}$

(E) $_{10}E_x[\overline{A}_x +_{10} E_{x+10} + \overline{A}_{x+20} -_{10} E_{x+20} + \overline{A}_{x+30}]$

```
print("SOA Question 4.10:  (D)")
life = Insurance().set_interest(i=0)\
                    .set_survival(S=lambda x,s,t: 1, maxage=40)
def fun(x, t):
    if 10 <= t <= 20: return life.interest.v_t(t)
    elif 20 < t <= 30: return 2 * life.interest.v_t(t)
    else: return 0
def A(x, t):  # Z_x+k (t-k)
    return life.interest.v_t(t - x) * (t > x)
x = 0
benefits=[lambda x,t: (life.E_x(x, t=10) * A(x+10, t)
                            + life.E_x(x, t=20)* A(x+20, t)
                            - life.E_x(x, t=30) * A(x+30, t)),
            lambda x,t: (A(x, t)
                            + life.E_x(x, t=20) * A(x+20, t)
                            - 2 * life.E_x(x, t=30) * A(x+30, t)),
            lambda x,t: (life.E_x(x, t=10) * A(x, t)
                            + life.E_x(x, t=20) * A(x+20, t)
                            - 2 * life.E_x(x, t=30) * A(x+30, t)),
            lambda x,t: (life.E_x(x, t=10) * A(x+10, t)
                            + life.E_x(x, t=20) * A(x+20, t)
                            - 2 * life.E_x(x, t=30) * A(x+30, t)),
            lambda x,t: (life.E_x(x, t=10)
                            * (A(x+10, t)
                            + life.E_x(x+10, t=10) * A(x+20, t)
                            - life.E_x(x+20, t=10) * A(x+30, t)))]
fig, ax = plt.subplots(3, 2)
ax = ax.ravel()
for i, b in enumerate([fun] + benefits):
    life.Z_plot(0, benefit=b, ax=ax[i], color=f"C{i+1}", title='')
    ax[i].legend(["(" + "abcde"[i-1] + ")" if i else "Z"])
z = [sum(abs(b(0, t) - fun(0, t)) for t in range(40)) for b in benefits]
print("ABCDE"[np.argmin(z)])
```

```
SOA Question 4.10:  (D)
D
```



SOA Question 4.12

For three fully discrete insurance products on the same (x), you are given:

- Z_1 is the present value random variable for a 20-year term insurance of 50
- Z_2 is the present value random variable for a 20-year deferred whole life insurance of 100
- Z_3 is the present value random variable for a whole life insurance of 100.
- $E[Z_1] = 1.65$ and $E[Z_2] = 10.75$
- $Var(Z_1) = 46.75$ and $Var(Z_2) = 50.78$

Calculate $Var(Z_3)$.

```
print("SOA Question 4.12:  (C) 167")
cov = Insurance.covariance(a=1.65, b=10.75, ab=0) # Z1 and Z2 nonoverlapping
var = Insurance.variance(a=2, b=1, var_a=46.75, var_b=50.78, cov_ab=cov)
print(var)
```

```
SOA Question 4.12:  (C) 167
166.82999999999998
```

SOA Question 4.11

You are given:

- Z_1 is the present value random variable for an n-year term insurance of 1000 issued to (x)

- Z_2 is the present value random variable for an n-year endowment insurance of 1000 issued to (x)
- For both Z_1 and Z_2 the death benefit is payable at the end of the year of death
- $E[Z_1] = 528$
- $Var(Z_2) = 15,000$
- $A_{\overset{\ \ 1}{x:\overline{n|}}} = 0.209$
- ${}^2A_{\overset{\ \ 1}{x:\overline{n|}}} = 0.136$

Calculate $Var(Z_1)$.

```
print("SOA Question 4.11:  (A) 143385")
A1 = 528/1000   # E[Z1]  term insurance
C1 = 0.209      # E[pure_endowment]
C2 = 0.136      # E[pure_endowment^2]
def fun(A2):
    B1 = A1 + C1   # endowment = term + pure_endowment
    B2 = A2 + C2   # double force of interest
    return Insurance.insurance_variance(A2=B2, A1=B1)
A2 = Insurance.solve(fun, target=15000/(1000*1000), grid=[143400, 279300])
var = Insurance.insurance_variance(A2=A2, A1=A1, b=1000)
print(var)
```

```
SOA Question 4.11:  (A) 143385
143384.99999999997
```

SOA Question 4.15

For a special whole life insurance on (x), you are given:

- Death benefits are payable at the moment of death
- The death benefit at time t is $b_t = e^{0.02t}$, for $t \geq 0$
- $\mu_{x+t} = 0.04$, for $t \geq 0$
- $\delta = 0.06$
- Z is the present value at issue random variable for this insurance.

Calculate $Var(Z)$.

```
print("SOA Question 4.15  (E) 0.0833 ")
life = Insurance().set_survival(mu=lambda *x: 0.04)\
                  .set_interest(delta=0.06)
benefit = lambda x,t: math.exp(0.02*t)
A1 = life.A_x(0, benefit=benefit, discrete=False)
A2 = life.A_x(0, moment=2, benefit=benefit, discrete=False)
var = A2 - A1**2
print(var)
```

```
SOA Question 4.15  (E) 0.0833
0.08334849338238598
```

SOA Question 4.4

For a special increasing whole life insurance on (40), payable at the moment of death, you are given:

- The death benefit at time t is $b_t = 1 + 0.2t, \quad t \geq 0$
- The interest discount factor at time t is $v(t) = (1 + 0.2t) - 2, \quad t \geq 0$
- ${}_tp_{40}\mu_{40+t} = 0.025$ if $0 \leq t < 40$, otherwise 0
- Z is the present value random variable for this insurance Calculate Var(Z).

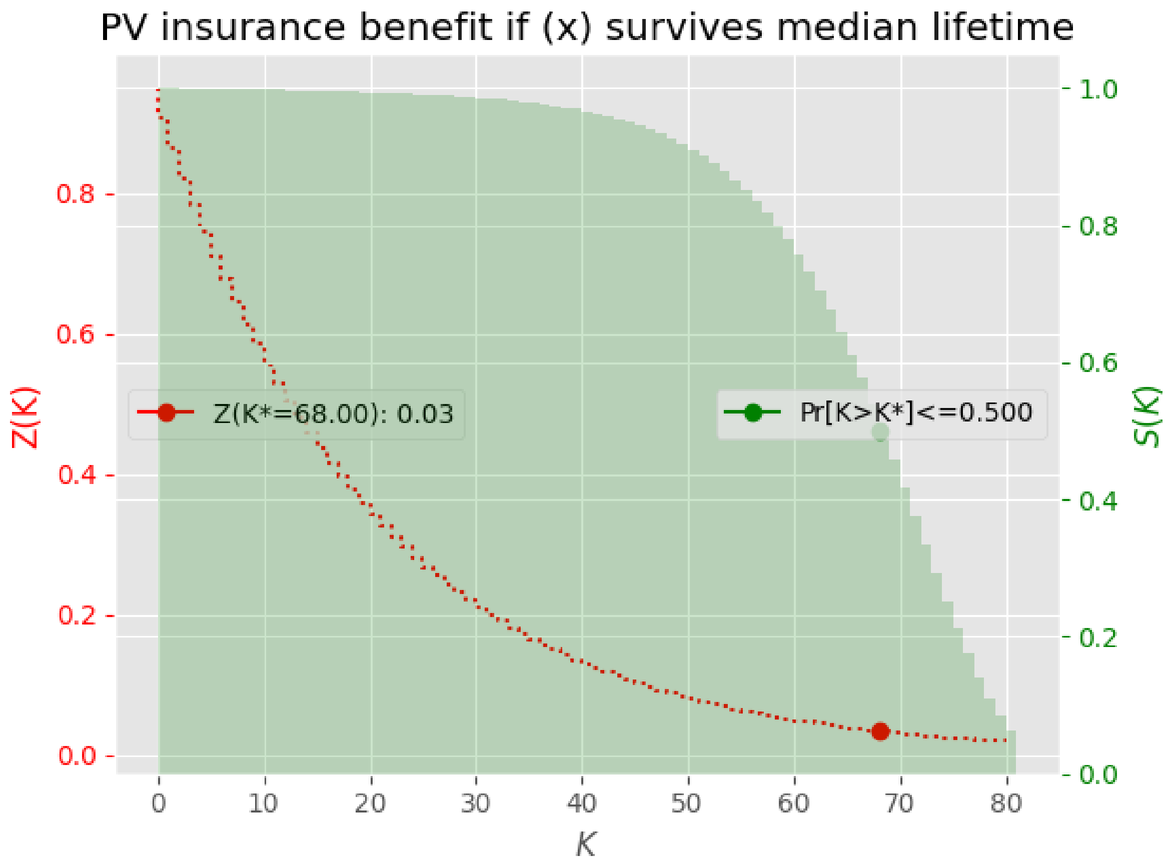
```
print("SOA Question 4.4  (A) 0.036")
x = 40
life = Insurance().set_survival(f=lambda *x: 0.025, maxage=x+40)\
                  .set_interest(v_t=lambda t: (1 + .2*t)**(-2))
benefit = lambda x,t: 1 + .2 * t
A1 = life.A_x(x, benefit=benefit, discrete=False)
A2 = life.A_x(x, moment=2, benefit=benefit, discrete=False)
var = A2 - A1**2
print(var)
```

```
SOA Question 4.4  (A) 0.036
0.03567680106032681
```

Plot insurance present value r.v Z

Plot the insurance present value r.v. Z and survival probability of (x), and indicate median survival lifetime. Assume mortality follows Standard Ultimate Life Table

```
from actuarialmath.sult import SULT
life = SULT()
life.Z_curve(x=20, stop=80, T=life.Z_t(x=20, prob=0.5),
             title="PV insurance benefit if (x) survives median lifetime")
```


PV insurance benefit if (x) survives median lifetime
Z(K)
S(K)
K
Z(K*=68.00): 0.03
Pr[K>K*]<=0.500
0.0
0.2
0.4
0.6
0.8
1.0
0
10
20
30
40
50
60
70
80

CHAPTER EIGHT

ANNUITIES

A life annuity is a regular sequence of payments as long as the annuitant is alive on the payment date.

8.1 Present value of life annuity r.v. Y

Valuation functions for the present value of annuity benefits, denoted by Y, are based on the continuous future lifetime random variable, T_x, or the curtate future lifetime random variable, K_x. The expected present value of annuity benefits is denoted and solved by EPV(Y)

Whole life annuity:

$Y = \overline{a}_{\overline{T_x|}} = \dfrac{1 - v^{T_x}}{\delta}$ = present value of continuous annuity certain through T_x

- present value random variable for continuous life annuity

$\overline{a}_x = EPV[\overline{a}_{\overline{T_x|}}] = \int_{t=0}^{\infty} v^t \ {}_tp_x \ dt$

- continuous whole life annuity, benefit is payable up to the moment of death

$\overline{a}_x = \mathring{e}_x$

- special case when interest rate is 0

$Y = \ddot{a}_{\overline{K_x+1|}} = \dfrac{1 - v^{K_x+1}}{d}$ = present value of certain annuity due through $K_x + 1$

- present value random variable for annual life annuity due

$\ddot{a}_x = EPV[\ddot{a}_{\overline{K_x+1|}}] = \sum_{k=0}^{\infty} v^k \ {}_{k|}p_x$

- annual whole life annuity due, benefit is payable up to the beginning of the year of death

$\ddot{a}_x = 1 + e_x$

- special case when interest rate is 0

Temporary annuity:

$Y = \overline{a}_{\overline{t|}}$ when or $T_x > t$, else $Y = \overline{a}_{\overline{T_x|}}$

- present value random variable for continuous life annuity

$\overline{a}_{x:\overline{t|}} = \int_{s=0}^{t} v^s \ {}_sp_x \ ds = \overline{a}_x - \ {}_tE_x \ \overline{a}_{x+t}$

- continuous temporary life annuity

$\overline{a}_{x:\overline{t|}} = \mathring{e}_{x:\overline{t|}}$

- when interest rate is 0

$Y = \ddot{a}_{\overline{t|}}$ when $K_x \geq t$, else $Y = \ddot{a}_{\overline{K_x+1|}}$

- present value random variable for annual life annuity

$\ddot{a}_{x:\overline{t|}} = \sum_{k=0}^{t-1} v^k \; _{k|}p_x = \ddot{a}_x - \; _tE_x \; \ddot{a}_{x+t}$

- annual temporary life annuity due

$\ddot{a}_{x:\overline{t|}} = 1 + e_{x:\overline{t|}} - \; _tp_x$

- when interest rate is 0

Deferred whole life annuity:

$_{u|}\overline{a}_x = \overline{a}_x - \overline{a}_{x+u}$

- continuous deferred life annuity as the difference of whole life annuities

$_{u|}\ddot{a}_x = \ddot{a}_x - \ddot{a}_{x+u}$

- annual deferred annuity due as the difference of annual whole life annuities due

Certain and life annuity:

A common feature of pension benefits is that the pension annuity is guaranteed to be paid for some period even if the annuitant dies before the end of the period.

$Y = \; \overline{a}_{\overline{n|}}$ when $T_x \leq n$, else $\overline{a}_{\overline{T_x|}}$

- present value random variable for continuous annuity

$\overline{a}_{\overline{\overline{x:n|}}} = \overline{a}_{\overline{n|}} + \; _{n|}\overline{a}_x$

- is a continuous temporary certain annuity plus a deferred continuous whole life annuity

$Y = \ddot{a}_{\overline{n|}}$ when $K_x < n$, else $\ddot{a}_{\overline{K_x+1|}}$

- present value random variable for annual annuity due

$\ddot{a}_{\overline{\overline{x:n|}}} = \ddot{a}_{\overline{n|}} + \; _{n|}\ddot{a}_x$

- is an annual temporary certain annuity due, plus a deferred annual life annuity due

8.2 Life insurance twin

Whole and *Temporary Life Annuities* (and *Whole Life* and *Endowment Insurance*) **ONLY**:

$\overline{a}_x = \dfrac{1 - \overline{A}_x}{\delta}$

$\overline{A}_x = 1 - \delta \, \overline{a}_x$

- continuous whole life insurance twin for continuous whole life annuity

$\ddot{a}_x = \dfrac{1 - A_x}{d}$

$A_x = 1 - d \, \ddot{a}_x$

- annual whole life insurance twin for annual life annuity due

$\overline{a}_{x:\overline{t|}} = \dfrac{1 - \overline{A}_{x:\overline{t|}}}{\delta}$

$\overline{A}_{x:\overline{t|}} = 1 - \delta \, \overline{a}_{x:\overline{t|}}$

- continuous endowment insurance twin for continuous temporary life annuity

$$\ddot{a}_{x:\overline{t|}} = \frac{1 - A_{x:\overline{t|}}}{d}$$

$$A_{x:\overline{t|}} = 1 - d\ \ddot{a}_{x:\overline{t|}}$$

- annual endowment insurance twin for annual temporary life annuity due

8.3 Variances

Whole life annuity

$$^2\overline{a}_x = \frac{1 - {}^2\overline{A}_x}{2\delta}$$

$$^2\overline{A}_x = 1 - (2\delta)\ {}^2\overline{a}_x$$

$$Var(\overline{a}_x) = \frac{{}^2\overline{A}_x - (\overline{A}_x)^2}{d^2}$$

- from doubling the force of interest of continuous whole life insurance

$$^2\ddot{a}_x = \frac{1 - {}^2A_x}{2d - d^2}$$

$$^2A_x = 1 - (2d - d^2)\ {}^2\ddot{a}_x$$

$$Var(\ddot{a}_x) = \frac{{}^2\ddot{A}_x -\ (A_x)^2}{\delta^2}$$

- from doubling the force of interest of annual whole life insurance

Temporary life annuity

$$^2\overline{a}_{x:\overline{t|}} = \frac{1 - {}^2\overline{A}_{x:\overline{t|}}}{2\delta}$$

$$^2\overline{A}_{x:\overline{t|}} = 1 - (2\delta)\ {}^2\overline{a}_{x:\overline{t|}}$$

$$Var(\overline{a}_{x:\overline{t|}}) = \frac{{}^2\overline{A}_{x:\overline{t|}} - (\overline{A}_{x:\overline{t|}})^2}{d^2}$$

- from doubling the force of interest of continuous endowment insurance

$$^2\ddot{a}_{x:\overline{t|}} = \frac{1 - {}^2A_{x:\overline{t|}}}{2d - d^2}$$

$$^2A_{x:\overline{t|}} = 1 - (2d - d^2)\ {}^2\ddot{a}_{x:\overline{t|}}$$

$$Var(\ddot{a}_{x:\overline{t|}}) = \frac{{}^2A_{x:\overline{t|}} -\ (A_{x:\overline{t|}})^2}{\delta^2}$$

- from doubling the force of interest of annual endowment insurance

8.4 Immediate life annuity

Benefit is paid at the *end of the year*, as long as the annuitant is alive, and can be relate to the value of an annual life annuity due:

$a_x = \ddot{a}_x - 1$

- whole life annuities

$a_{x:\overline{t|}} = \ddot{a}_{x:\overline{t|}} - 1 + \ {}_tE_x$

- temporary life annuities

8.5 Varying life annuities

Increasing annuity:

The amount of the annuity payment increases arithmetically with time.

$(\overline{Ia})_x = \int_{t=0}^{\infty} \ t \ v^t \ {}_tp_x \ dt$

- increasing continuous whole life annuity

$(I\ddot{a})_x = \sum_{k=0}^{\infty} \ (k+1) \ v^{k+1} \ {}_kp_x$

- increasing annual whole life annuity due

$(\overline{Ia})_{x:\overline{t|}} = \int_{s=0}^{t} \ s \ v^s \ {}_sp_x \ ds$

- increasing continuous temporary life annuity

$(I\ddot{a})_{x:\overline{t|}} = \sum_{k=0}^{t-1} \ (k+1) \ v^{k+1} \ {}_kp_x$

- increasing annual temporary life annuity due

Decreasing annuity:

The amount of the annuity payment decreases arithmetically with time.

$(\overline{Da})_{x:\overline{t|}} = \int_{s=0}^{t} \ (t-s) \ v^s \ {}_sp_x \ ds$

- decreasing continuous temporary life annuity (not defined for whole life)

$(D\ddot{a})_{x:\overline{t|}} = \sum_{k=0}^{t-1} \ (t-k) \ v^{k+1} \ {}_kp_x$

- decreasing annual temporary life annuity due (not defined for whole life)

Identity relationships:

$(\overline{Da})_{x:\overline{t|}} + (\overline{Ia})_{x:\overline{t|}} = t \ \overline{a}_{x:\overline{t|}}$

- relating continuous decreasing and increasing life annuities

$(D\ddot{a})_{x:\overline{t|}} + (I\ddot{a})_{x:\overline{t|}} = (t+1) \ \ddot{a}_{x:\overline{t|}}$

- relating annual decreasing and increasing life annuities due

8.6 Methods

The `Annuity` class implements methods to compute the expected present value of life annuities

```
from actuarialmath import Annuity
import describe
describe.methods(Annuity)
```

```
class Annuity - Compute present values and relationships of life annuities

    Methods:
    --------

    a_x(x, s, t, u, benefit, discrete):
      Compute EPV of life annuity from survival function

    immediate_annuity(x, s, t, b, variance):
      Compute EPV of immediate life annuity

    annuity_twin(A, discrete):
      Returns annuity from its WL or Endowment Insurance twin"

    insurance_twin(a, moment, discrete):
      Returns WL or Endowment Insurance twin from annuity

    annuity_variance(A2, A1, b, discrete):
      Compute variance from WL or endowment insurance twin

    whole_life_annuity(x, s, b, variance, discrete):
      Whole life annuity: a_x

    temporary_annuity(x, s, t, b, variance, discrete):
      Temporary life annuity: a_x:t

    deferred_annuity(x, s, u, t, b, discrete):
      Deferred life annuity n|t_a_x =  n+t_a_x - n_a_x

    certain_life_annuity(x, s, u, t, b, discrete):
      Certain and life annuity = certain + deferred

    increasing_annuity(x, s, t, b, discrete):
      Increasing annuity

    decreasing_annuity(x, s, t, b, discrete):
      Identity (Da)_x:n + (Ia)_x:n = (n+1) a_x:n temporary annuity

    Y_t(x, prob, discrete):
      T_x given percentile of the r.v. Y = PV of WL or Temporary Annuity

    Y_from_t(t, discrete):
      PV of insurance payment Y(t), given T_x (or K_x if discrete)

    Y_from_prob(x, prob, discrete):
      Percentile of annuity PV r.v. Y, given probability

    Y_to_prob(x, Y):
```

(continues on next page)

(continued from previous page)

```
      Cumulative density of insurance PV r.v. Y, given percentile value

    Y_x(x, s, t, discrete):
      EPV of year t annuity benefit for life aged [x]+s: b_x[s]+s(t)

    Y_plot(x, s, stop, benefit, T, discrete, ax, title, color):
      Plot PV of annuity r.v. Y vs T
```

8.7 Examples

SOA Question 5.6

For a group of 100 lives age x with independent future lifetimes, you are given:

- Each life is to be paid 1 at the beginning of each year, if alive
- $A_x = 0.45$
- ${}^2A_x = 0.22$
- $i = 0.05$
- Y is the present value random variable of the aggregate payments.

Using the normal approximation to Y, calculate the initial size of the fund neededt o be 95% certain of being able to make the payments for these life annuities.

```
print("SOA Question 5.6:  (D) 1200")
life = Annuity().set_interest(i=0.05)
var = life.annuity_variance(A2=0.22, A1=0.45)
mean = life.annuity_twin(A=0.45)
print(life.portfolio_percentile(mean=mean, variance=var, prob=.95, N=100))
```

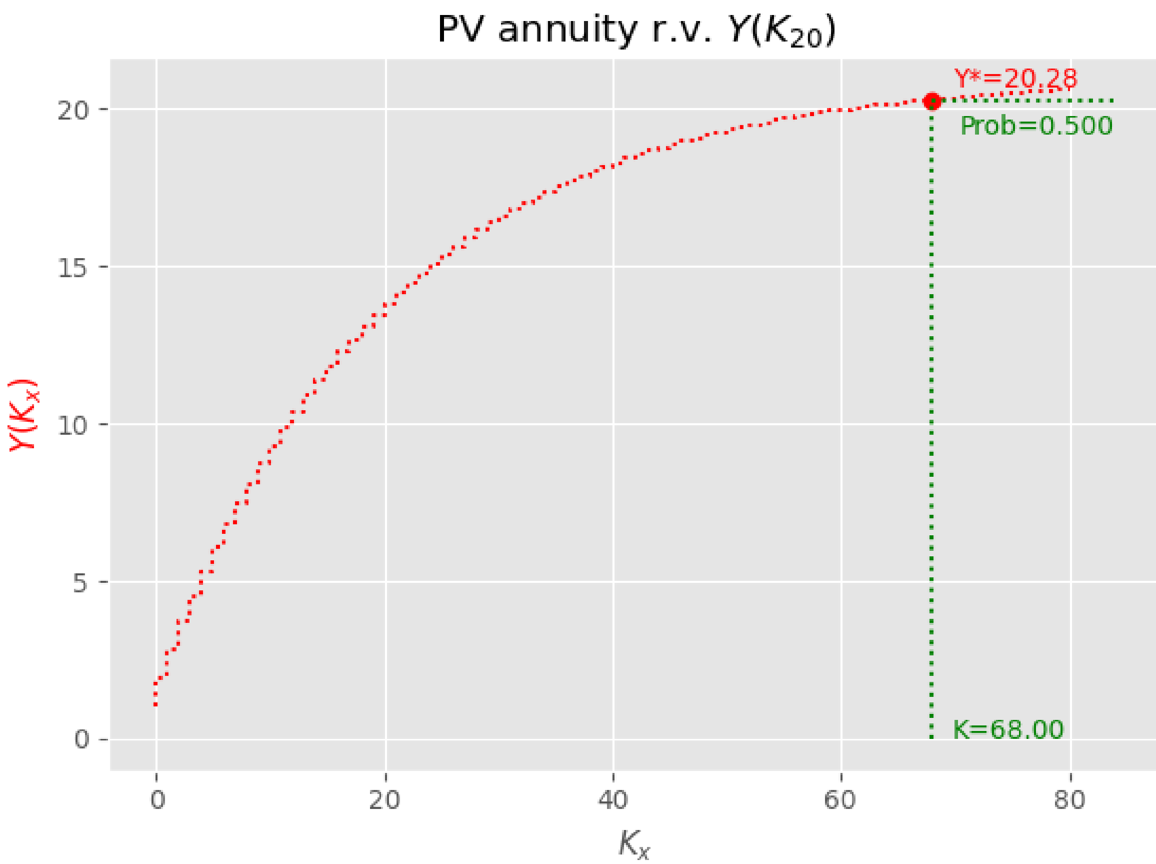
```
SOA Question 5.6:  (D) 1200
1200.6946732201702
```

Plot annuity present value r.v. Y:

Mortality follows Standard Ultimate Life Table. Indicate median lifetime of (20).

```
from actuarialmath.sult import SULT
life = SULT()
life.Y_plot(x=20, stop=80, T=life.Y_t(x=20, prob=0.5))
```

```
20.27530100523103
```


PV annuity r.v. $Y(K_{20})$
Y*=20.28
Prob=0.500
K=68.00
$Y(K_x)$
K_x
0
5
10
15
20
0
20
40
60
80

CHAPTER NINE

PREMIUMS

When insurance company agrees to pay some life contingent benefits, the policyholder agrees to pay premiums to the insurance company to secure these benefits. The premiums will also need to reimburse the insurance company for the expenses associated with the policy. The calculation of the premium may not explicitly allow for the insurance company's expenses. In this case we refer to a net premium (also called a risk premium or benefit premium). If the calculation does explicitly allow for expenses, the premium is called a gross premium (also called expense-loaded premium).

Premiums for life insurance are payable in advance, with the first premium payable when the policy is purchased.

9.1 Present value of future loss r.v. L

${}_0L$ = PV of loss at issue = PV of future benefits - PV of future premiums

9.2 Equivalence principle

Under this principle, premiums are set such that expected loss at issue equals zero:

$E[{}_0L] = EPV_0(\text{future benefits}) - EPV_0(\text{future premiums}) = 0$

- Fully continuous: both benefits and premiums are payable continuously
- Fully discrete: benefits are paid at the end of the year, premiums are paid at the beginning of the year
- Semi-continuous: benefits are paid at moment of death, premiums are paid at the beginning of the year

9.3 Net premium

For net premiums, we take into consideration outgoing benefit payments only: expenses are not a part of the calculation. The benefit may be a death benefit or a survival benefit or a combination. Under the equivalence principle, the net premium is set such that the expected value of the future loss is zero at the start of the contract, $E[L_0] = 0$.

$P_x = \dfrac{A_x}{\ddot{a}_x}$

- fully discrete whole life insurance net premium

$P_x = \dfrac{\overline{A}_x}{\bar{a}_x}$

- fully continuous whole life insurance net premium

$P^1_{x:\overline{t|}} = \dfrac{A^1_{x:\overline{t|}}}{\ddot{a}_{x:\overline{t|}}}$

- fully discrete term life net premium

$P^1_{x:\overline{t|}} = \dfrac{\overline{A}^1_{x:\overline{t|}}}{\overline{a}_{x:\overline{t|}}}$

- fully continuous term life net premium

$P_{x:\overline{t|}}^{\ \ 1} = \dfrac{{}_tE_x}{\ddot{a}_{x:\overline{t|}}}$

- fully discrete pure endowment net premium

$P_{x:\overline{t|}}^{\ \ 1} = \dfrac{{}_tE_x}{\overline{a}_{x:\overline{t|}}}$

- fully continuous pure endowment net premium

$P_{x:\overline{t|}} = \dfrac{A_{x:\overline{t|}}}{\ddot{a}_{x:\overline{t|}}}$

- fully discrete endowment insurance net premium

$P_{x:\overline{t|}} = \dfrac{\overline{A}_{x:\overline{t|}}}{\overline{a}_{x:\overline{t|}}}$

- fully continuous endowment insurance net premium

Shortcuts for whole life and endowment insurance only:

For whole life and endowment insurance only, by plugging in the insurance or annuity twin, the following shortcuts are available for calculating net premiums from only the life insurance or annuity factor.

$P = b\,(\dfrac{1}{\ddot{a}_x} - d) = b\,(\dfrac{dA_x}{1 - A_x})$

- fully discrete whole life shortcut

$P = b\,(\dfrac{1}{\overline{a}_x} - \delta) = b\,(\dfrac{d\overline{A}_x}{1 - \overline{A}_x})$

- fully continuous whole life shortcut

$P = b\,(\dfrac{1}{\ddot{a}_{x:\overline{n|}}} - d) = b\,(\dfrac{dA_{x:\overline{n|}}}{1 - A_{x:\overline{n|}}})$

- fully discrete endowment insurance shortcut

$P = b\,(\dfrac{1}{\overline{a}_{x:\overline{n|}}} - \delta) = b\,(\dfrac{d\overline{A}_{x:\overline{n|}}}{1 - \overline{A}_{x:\overline{n|}}})$

- fully continuous endowment insurance shortcut

9.4 Portfolio Percentile Premium

The portfolio percentile premium principle is an alternative to the equivalence premium principle. Using the mean and variance of the future loss random variable, the portfolio percentile premium principle can be used to determine a premium. We assume a large portfolio of N identical and independent policies. The present value of the total future loss $\overline{L}$ of the portfolio can be approximated by a normal distribution over the sum of the individual losses L_i

$$\begin{aligned}\overline{L} &= L_1 + L_2 + \cdots + L_N \\ E[\overline{L}] &= N\ E[L] \\ Var[\overline{L}] &= N\ Var[L]\end{aligned}$$

Note $E[\overline{L}]$ and $Var[\overline{L}]$ are functions of the unspecified premium P. A probability percentile q (say, 95% confidence) and a threshold L^* (say, 0) are chosen, then P is solved for implicitly from the following equation, such that the probability of $\overline{L}$ not exceeding L^* is q

$$Pr[\overline{L} < L^*] = Pr\left[\frac{\overline{L} - E[\overline{L}]}{\sqrt{Var[\overline{L}]}} < \frac{L^* - E[\overline{L}]}{\sqrt{Var[\overline{L}]}}\right] = \Phi\left(\frac{L^* - E[\overline{L}]}{\sqrt{Var[\overline{L}]}}\right) = q$$

9.5 Gross premium

When we calculate a gross premium for an insurance policy or an annuity, we take account of the expenses the insurer incurs. There are three main types of expense associated with policies – initial expenses, renewal expenses and termination or claim expenses.

Expenses:

$e_i =$ initial_per_policy + initial_per_premium $\times$ gross_premium

- initial expenses at the beginning of year 1 when a policy is issued, which may be both proportional to premiums or may be 'per policy', meaning that the amount is fixed for all policies, and is not related to the size of the contract.

$e_r =$ renewal_per_policy + renewal_per_premium $\times$ gross_premium

- renewal expenses in the beginning of each year 2+, may be both per policy or percent of premium.

$E =$ settlement expense

- is paid with death benefit (b); hence

  claim cost $= b + E =$ death benefit + settlement expense.

Return of premiums paid without interest upon death:

$EPV_0(\text{return of premiums paid}) = \sum_{k=0}^{t-1}\ P(k+1)\ v^{k+1}\ {}_{k|}q_x = P \cdot (IA)^{1}_{x:\overline{t|}}$

- an additional benefit in some insurance policies, whose EPV can be calculated using an increasing insurance factor

Equivalence principle:

If gross premiums are set under equivalence principle, then expected gross future loss at issue equals zero:

$E[{}_0L^g] = EPV_0(\text{future benefits}) + EPV_0(\text{future expenses}) - EPV_0(\text{future premiums}) = 0$

9.6 Methods

The `Premiums` class implements methods for computing net and gross premiums under the equivalence principle

```
import numpy as np
from actuarialmath import Premiums
import describe
describe.methods(Premiums)
```

```
class Premiums - Compute et and gross premiums under equivalence principle

    Methods:
    --------

    net_premium(x, s, t, u, n, b, endowment, discrete, return_premium, annuity,␣
 ↪initial_cost):
      Net level premium for special n-pay, u-deferred t-year term insurance

    insurance_equivalence(premium, b, discrete):
      Compute whole life or endowment insurance factor, given net premium

    annuity_equivalence(premium, b, discrete):
      Compute whole life or temporary annuity factor, given net premium

    premium_equivalence(A, a, b, discrete):
      Compute premium from whole life or endowment insurance and annuity factors

    gross_premium(a, A, IA, discrete, benefit, E, endowment, settlement_policy,␣
 ↪initial_policy, initial_premium, renewal_policy, renewal_premium):
      Gross premium by equivalence principle
```

9.7 Examples

SOA Question 5.6:

For a group of 100 lives age x with independent future lifetimes, you are given:

- Each life is to be paid 1 at the beginning of each year, if alive
- $A_x = 0.45$
- ${}_2A_x = 0.22$
- $i = 0.05$
- Y is the present value random variable of the aggregate payments.

Using the normal approximation to Y, calculate the initial size of the fund needed in order to be 95% certain of being able to make the payments for these life annuities.

```
print("SOA Question 5.6:  (D) 1200")
life = Premiums().set_interest(i=0.05)
var = life.annuity_variance(A2=0.22, A1=0.45)
mean = life.annuity_twin(A=0.45)
fund = life.portfolio_percentile(mean, var, prob=.95, N=100)
print(fund)
```

```
SOA Question 5.6:  (D) 1200
1200.6946732201702
```

SOA Question 6.29

(35) purchases a fully discrete whole life insurance policy of 100,000. You are given:

- The annual gross premium, calculated using the equivalence principle, is 1770
- The expenses in policy year 1 are 50% of premium and 200 per policy
- The expenses in policy years 2 and later are 10% of premium and 50 per policy
- All expenses are incurred at the beginning of the policy year
- $i = 0.035$

Calculate $\ddot{a}_{35}$.

```
print("SOA Question 6.29  (B) 20.5")
life = Premiums().set_interest(i=0.035)
def fun(a):
    return life.gross_premium(A=life.insurance_twin(a=a),
                                a=a,
                                benefit=100000,
                                initial_policy=200,
                                initial_premium=.5,
                                renewal_policy=50,
                                renewal_premium=.1)
print(life.solve(fun, target=1770, grid=[20, 22]))
```

```
SOA Question 6.29  (B) 20.5
20.480268314431726
```

SOA Question 6.2

For a fully discrete 10-year term life insurance policy on (x), you are given:

- Death benefits are 100,000 plus the return of all gross premiums paid without interest
- Expenses are 50% of the first year's gross premium, 5% of renewal gross premiums and 200 per policy expenses each year
- Expenses are payable at the beginning of the year
- $A^{1}_{x:\overline{10|}} = 0.17094$
- $(IA)^{1}_{x:\overline{10|}} = 0.96728$
- $\ddot{a}_{x:\overline{10|}} = 6.8865$

Calculate the gross premium using the equivalence principle.

```
print("SOA Question 6.2: (E) 3604")
life = Premiums()
A, IA, a = 0.17094, 0.96728, 6.8865
print(life.gross_premium(a=a,
                         A=A,
                         IA=IA,
                         benefit=100000,
```

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```
                             initial_premium=0.5,
                             renewal_premium=.05,
                             renewal_policy=200,
                             initial_policy=200))
```

```
SOA Question 6.2: (E) 3604
3604.229940320728
```

SOA Question 6.16

For a fully discrete 20-year endowment insurance of 100,000 on (30), you are given:

- d = 0.05
- Expenses, payable at the beginning of each year, are:

	First Year	First Year	Renewal Years	Renewal Years
	Percent of Premium	Per Policy	Percent of Premium	Per Policy
Taxes	4%	0	4%	0
Sales Commission	35%	0	2%	0
Policy Maintenance	0%	250	0%	50

- The net premium is 2143

Calculate the gross premium using the equivalence principle.

```
print("SOA Question 6.16: (A) 2408.6")
life = Premiums().set_interest(d=0.05)
A = life.insurance_equivalence(premium=2143, b=100000)
a = life.annuity_equivalence(premium=2143, b=100000)
p = life.gross_premium(A=A,
                       a=a,
                       benefit=100000,
                       settlement_policy=0,
                       initial_policy=250,
                       initial_premium=.04+.35,
                       renewal_policy=50,
                       renewal_premium=.04+.02)
print(A, a, p)
```

```
SOA Question 6.16: (A) 2408.6
0.3000139997200056 13.999720005599887 2408.575206281868
```

SOA Question 6.20

For a special fully discrete 3-year term insurance on (75), you are given:

- The death benefit during the first two years is the sum of the net premiums paid without interest
- The death benefit in the third year is 10,000

x	p_x
75	0.90
76	0.88
77	0.85

- $i = 0.04$

Calculate the annual net premium.

```
print("SOA Question 6.20:  (B) 459")
l = lambda x,s: dict(zip([75, 76, 77, 78],
                         np.cumprod([1, .9, .88, .85]))).get(x+s, 0)
life = Premiums().set_interest(i=0.04).set_survival(l=l)
a = life.temporary_annuity(75, t=3)
IA = life.increasing_insurance(75, t=2)
A = life.deferred_insurance(75, u=2, t=1)
print(life.solve(lambda P: P*IA + A*10000 - P*a, target=0, grid=100))
```

```
SOA Question 6.20:  (B) 459
458.83181728297285
```

Other examples

```
life = Premiums().set_interest(delta=0.06)\
                 .set_survival(mu=lambda x,s: 0.04)
print(life.net_premium(0))
```

```
0.03692697915432344
```

CHAPTER

TEN

POLICY VALUES

Policy value at time t is the present value, at time t, of the future loss random variable

Net future loss at issue

For net future loss at time $t=0$, we consider benefit payments and net premiums only.

$_0L = b\, v^{K_x+1} - P\ddot{a}_{\overline{K_x+1|}} = (b + \frac{P}{d})\, v^{K_x+1} - \frac{P}{d}$

- net future loss at issue of fully discrete whole life insurance

$_0L = b\, v^{T_x} - P\overline{a}_{\overline{T_x|}} = (b + \frac{P}{\delta})\, v^{T_x} - \frac{P}{\delta}$ (continuous)

- net future loss at issue of fully continuous whole life insurance

Gross future loss at issue

For gross future loss at time $t=0$, expenses are included along with benefits payments and gross premiums.

$_0L = (b + E + \frac{G - e_r}{d})\, v^{K_x+1} - \frac{G - e_r}{d} + (e_i - e_r)$

- gross future loss at issue of fully discrete whole life insurance

10.1 Net policy value

The amount needed at time t to cover the shortfall of expected future benefits greater than the EPV of future premiums is called the policy value for the policy at time t, denoted $_tV$,

$_tV = E[_tL] = EPV_t(\text{future benefits}) - EPV_t(\text{future premiums})$

- net policy value at time t is the expected net future loss of benefits less premiums after time t

$_0V = 0$

- net policy value at issue is 0 because of the equivalence principle

$_nV = 0$

- net policy value at year n is 0 for a n-year term insurance

$_nV = \text{endowment benefit}$

- net policy value at year n is equal to the endowment benefit for a n-year endowment insurance

Shortcuts for whole life and endowment insurance:

$_tV = b[1 - \frac{\ddot{a}_{x+t}}{\ddot{a}_x}]$ or $b[\frac{A_{x+t} - A_x}{1 - A_x}]$

- net policy value at time t of fully-discrete whole life insurance*

$${}_tV = b[1 - \frac{\overline{a}_{x+t}}{\overline{a}_x}] \text{ or } b[\frac{\overline{A}_{x+t} - \overline{A}_x}{1 - \overline{A}_x}]$$

- net policy value at time t of fully-continuous whole life insurance*

*Add $:\overline{n|}$ to A's and a's for endowment insurance

10.2 Gross policy value

Gross premium policy values explicitly allow for expenses and for the full gross premium, whereas net premium policy values exclude expenses from cash flows, and only the net premium is counted.

$${}_tV^g = E[{}_tL^g] = EPV_t(\text{future benefits}) + EPV_t(\text{future expenses}) - EPV_t(\text{future premiums})$$

- gross policy value at time t is the expected net future loss of benefits and expenses less premiums after time t

10.3 Variance of future loss

These formulas apply to *whole life* and *endowment insurance* **only**:

Net future loss

$$Var[{}_tL] = (b + \frac{P}{d})^2\ [{}^2A_{x+t:\overline{n-t|}} - (A_{x+t:\overline{n-t|}})^2]$$

- fully discrete endowment insurance*

$$Var[{}_tL] = (b + \frac{P}{\delta})^2\ [{}^2\overline{A}_{x+t:\overline{n-t|}} - (\overline{A}_{x+t:\overline{n-t|}})^2]$$

- fully continuous endowment insurance*

Gross future loss

$$Var[{}_tL] = (b + E + \frac{G - e_r}{d})^2\ [{}^2A_{x+t:\overline{n-t|}} - (A_{x+t:\overline{n-t|}})^2]$$

- fully discrete endowment insurance*

Shortcuts for variance of net future loss

When net premiums are set under equivalence principle, these shortcuts are available without explicitly specifying the value of net premiums, again, for *whole life* and *endowment insurance* **only**:

$$Var[{}_tL] = b^2[\frac{{}^2A_{x+t:\overline{n-t|}} - (A_{x+t:\overline{n-t|}})^2}{(1 - A_{x:\overline{n|}})^2}]$$

- variance of net future loss for fully-discrete endowment insurance*

$$Var[{}_tL] = b^2[\frac{{}^2\overline{A}_{x+t:\overline{n-t|}} - (\overline{A}_{x+t:\overline{n-t|}})^2}{(1 - \overline{A}_{x:\overline{n|}})^2}]$$

- variance of net future loss for fully-continuous endowment insurance*

*For whole life insurance, remove the $\overline{:n|}$ and $\overline{:n-t|}$ notations.

10.4 Expense reserve

$P^e = P^g - P^n$

- expense premium (sometimes, expense loading) is defined as the difference of gross premium and net premium: if expenses are weighted to the start of the contract, as is normally the case, then P^e will be greater than the renewal expense as it must fund both the renewal and initial expenses.

${}_tV^e = {}_tV^g - {}_tV = EPV_t(\text{future expenses}) - EPV_t(\text{future expense loadings})$

- expense reserves, defined as the difference between gross reserves and net reserves, also equals the expected present value of future expenses less the expected present value of future expense loadings (or expense premiums)

Generally:

- ${}_tV^e < 0$
- ${}_tV > {}_tV^g > 0 > {}_tV^e$

10.5 Methods

The `PolicyValues` class implements methods for computing net and gross future losses, and policy values (expected present values). The `Contract` class is used store and retrieve policy contract terms, such as benefit amounts and the various types of expenses.

```
from actuarialmath.policyvalues import PolicyValues, Contract
import describe
describe.methods(PolicyValues)
```

```
class PolicyValues - Compute net and gross future losses and policy values

    Methods:
    --------

    net_future_loss(A, A1, b):
      Shortcuts for WL or Endowment Insurance net loss

    net_variance_loss(A1, A2, A, b):
      Shortcuts for variance of net loss of WL or Endowment Insurance

    net_policy_variance(x, s, t, b, n, endowment, discrete):
      Variance of future loss for WL or Endowment Ins assuming equivalence

    gross_future_loss(A, a, contract):
      Shortcut for WL or Endowment Insurance gross future loss

    gross_policy_variance(x, s, t, n, contract):
      Variance of gross policy value for WL and Endowment Insurance

    gross_policy_value(x, s, t, n, contract):
      Gross policy values for insurance: t_V = E[L_t]

    L_from_t(t, contract):
      PV of Loss L(t) at time of death t = T_x (or K_x if discrete)
```

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```
    L_to_t(L, contract):
      Compute time of death T_x s.t. PV future loss is L

    L_from_prob(x, prob, contract):
      Compute PV of future loss at given percentile prob

    L_to_prob(x, L, contract):
      Compute percentile of L on the PV of future loss curve"

    L_plot(x, s, stop, T, contract, ax, title, color):
      Plot PV of future loss r.v. L vs time of death T_x
```

```
describe.methods(Contract)
```

```
class Contract - Set and retrieve policy contract terms

    Args:
      premium : level premium amount
      benefit : insurance death benefit amount
      settlement_policy : settlement expense per policy
      endowment : endowment benefit amount
      initial_policy : first year total expense per policy
      initial_premium : first year total premium per $ of gross premium
      renewal_policy : renewal expense per policy
      renewal_premium : renewal premium per $ of gross premium
      discrete : annuity due (True) or continuous (False)
      T : term of insurance
      discrete : annuity due (True) or continuous (False)

    Methods:
    --------

    set_contract(terms):
      Update any existing policy contract terms

    premium_terms():
      Getter returns dict of terms required for calculating gross premiums

    renewal_profit():
      Getter returns renewal dollar profit (premium less renewal expenses)

    initial_cost():
      Getter returns total initial cost (excludes renewal expenses)

    claims_cost():
      Getter returns total claims cost (death benefit + settlement expense)

    renewals(t):
      Returns contract object with initial terms set to renewal terms
```

10.6 Examples

SOA Question 6.24

For a fully continuous whole life insurance of 1 on (x), you are given:

- L is the present value of the loss at issue random variable if the premium rate is determined by the equivalence principle
- L^* is the present value of the loss at issue random variable if the premium rate is 0.06
- $\delta = 0.07$
- $\overline{A}_x = 0.30$
- $Var(L) = 0.18$

Calculate $Var(L^*)$.

```
print("SOA Question 6.24:  (E) 0.30")
life = PolicyValues().set_interest(delta=0.07)
x, A1 = 0, 0.30   # Policy for first insurance
P = life.premium_equivalence(A=A1, discrete=False)  # Need its premium
contract = Contract(premium=P, discrete=False)
def fun(A2):  # Solve for A2, given Var(Loss)
    return life.gross_variance_loss(A1=A1, A2=A2, contract=contract)
A2 = life.solve(fun, target=0.18, grid=0.18)
contract = Contract(premium=0.06, discrete=False)     # Solve second insurance
variance = life.gross_variance_loss(A1=A1, A2=A2, contract=contract)
print(variance)
```

```
SOA Question 6.24:  (E) 0.30
0.30419999999999975
```

SOA Question 6.30

For a fully discrete whole life insurance of 100 on (x), you are given:

- The first year expense is 10% of the gross annual premium
- Expenses in subsequent years are 5% of the gross annual premium
- The gross premium calculated using the equivalence principle is 2.338
- $i = 0.04$
- $\ddot{a}_x = 16.50$
- ${}^2A_x = 0.17$

Calculate the variance of the loss at issue random variable.

```
print("SOA Question 6.30:  (A) 900")
life = PolicyValues().set_interest(i=0.04)
contract = Contract(premium=2.338, benefit=100, initial_premium=.1,
                    renewal_premium=0.05)
var = life.gross_variance_loss(A1=life.insurance_twin(16.50),
                                A2=0.17, contract=contract)
print(var)
```

```
SOA Question 6.30:  (A) 900
908.141412994607
```

SOA Question 7.32

For two fully continuous whole life insurance policies on (x), you are given:

	Death Benefit	Annual Premium Rate	Variance of the PV of Future Loss at t
Policy A	1	0.10	0.455
Policy B	2	0.16	-

- $\delta = 0.06$

Calculate the variance of the present value of future loss at t for Policy B.

```
print("SOA Question 7.32:  (B) 1.4")
life = PolicyValues().set_interest(i=0.06)
contract = Contract(benefit=1, premium=0.1)
def fun(A2):
    return life.gross_variance_loss(A1=0, A2=A2, contract=contract)
A2 = life.solve(fun, target=0.455, grid=0.455)
contract = Contract(benefit=2, premium=0.16)
var = life.gross_variance_loss(A1=0, A2=A2, contract=contract)
print(var)
```

```
SOA Question 7.32:  (B) 1.4
1.3848168384380901
```

SOA Question 6.12

For a fully discrete whole life insurance of 1000 on (x), you are given:

- The following expenses are incurred at the beginning of each year:

	Year 1	Years 2+
Percent of premium	75%	10%
Maintenance expenses	10	2

- An additional expense of 20 is paid when the death benefit is paid
- The gross premium is determined using the equivalence principle
- $i = 0.06$
- $\ddot{a}_x = 12.0$
- ${}^2A_x = 0.14$

Calculate the variance of the loss at issue random variable.

```
print("SOA Question 6.12:  (E) 88900")
life = PolicyValues().set_interest(i=0.06)
a = 12
```

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```
A = life.insurance_twin(a)
contract = Contract(benefit=1000, settlement_policy=20,
                    initial_policy=10, initial_premium=0.75,
                    renewal_policy=2, renewal_premium=0.1)
contract.premium = life.gross_premium(A=A, a=a, **contract.premium_terms)
print(A, contract.premium)
L = life.gross_variance_loss(A1=A, A2=0.14, contract=contract)
print(L)
```

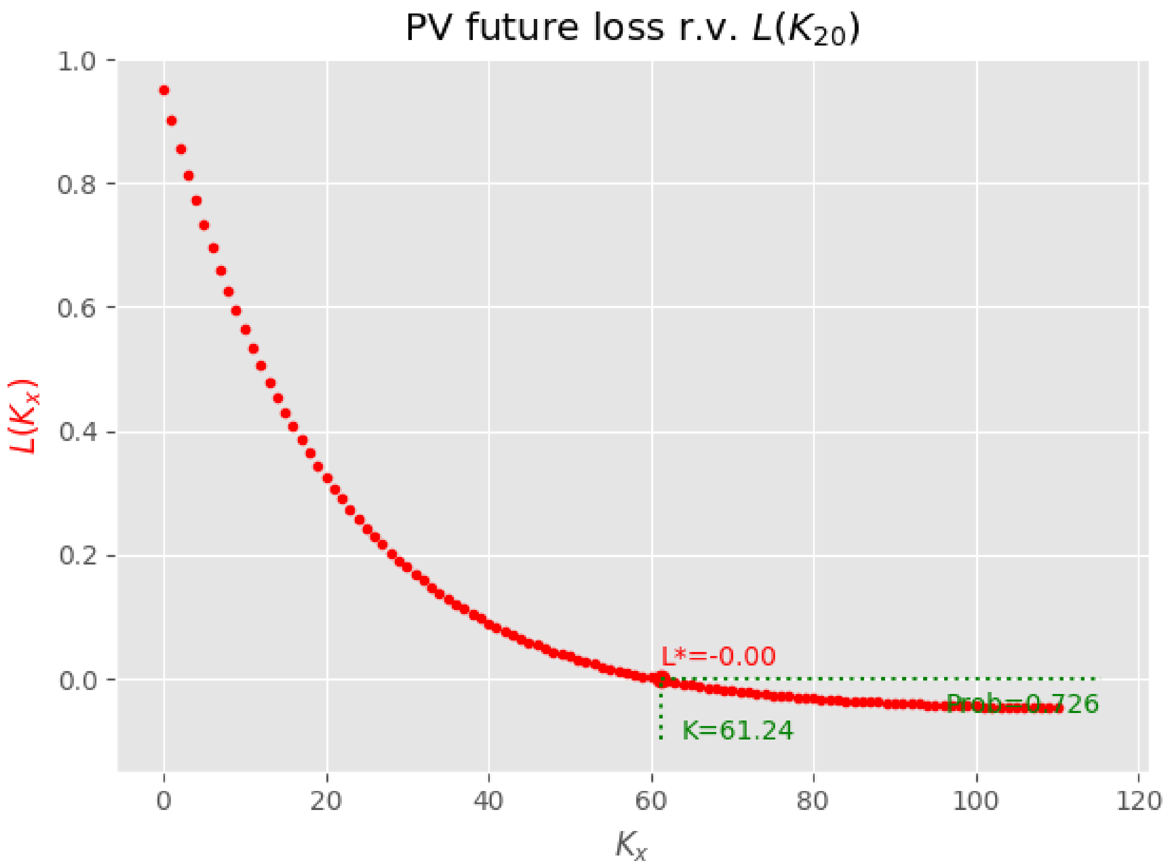
```
SOA Question 6.12:  (E) 88900
0.3207547169811321 35.38618830746352
88862.59592874818
```

Plot present value of future loss r.v L:

Assume mortality follows Standard Ultimate Life Table. Indicate breakeven lifetime.

```
from actuarialmath.sult import SULT
life = SULT()
x = 20
P = life.net_premium(x=x)
contract = Contract(premium=P, discrete=True)
T = life.L_to_t(L=0, contract=contract)  # breakeven T
life.L_plot(x=x, T=T, contract=contract)
```

```
-0.000695265147726408
```


PV future loss r.v. $L(K_{20})$
1.0
0.8
0.6
0.4
0.2
0.0
$L(K_x)$
L*=-0.00
Prob=0.726
K=61.24
0
20
40
60
80
100
120
K_x

CHAPTER

ELEVEN

RESERVES

The term reserves is sometimes used in place of policy values. AMLCR uses policy value to mean the expected value of the future loss random variable, and restricts reserve to mean the actual capital held in respect of a policy, which may be greater than or less than the policy value.

11.1 Recursion

The following recursive formulae relating ${}_tV$ to ${}_{t+1}V$ for policy values can be derived for policies with discrete cash flows.

Gross reserves

$({}_tV^g + G - e)(1+i) = q_{x+t}\,(b+E) + p_{x+t}\;{}_{t+1}V^g$

- recursion for gross reserves

Net reserves

$({}_tV + P)(1+i) = q_{x+t}\,b + p_{x+t}\;{}_{t+1}V$

- recursion for net reserves

Expense reserves

$({}_tV^e + P^e - e)(1+i) = q_{x+t}\,E + p_{x+t}\;{}_{t+1}V^e$

- recursion for expense reserves

11.2 Interim reserves

Recursive formulae for interim reserves ${}_{t+r}V$ where $0 \leq r \leq 1$ can be similarly obtained:

$({}_tV + P)(1+i)^r = \;{}_rq_{x+t}\,b\,v^{1-r} + \;{}_rp_{x+t}\;{}_{t+r}V$

- forward recursion for interim net reserves

${}_{t+r}V\;(1+i)^{1-r} = \;{}_{1-r}q_{x+t+r}\,b + \;{}_{1-r}p_{x+t+r}\;{}_{t+1}V$

- backward recursion for interim net reserves

11.3 Modified reserves

Because acquisition expenses are large relative to the renewal and claims expenses, accounting with level net premiums typically results in large negative values for expense reserves (called deferred acquisition costs or DAC) particularly at issue. Modified premium reserves are computed without expenses, and modifies the net premium method to assume a lower initial net premium that allow implicitly for the DAC.

Full Preliminary Term

FPT is the most common method for modifying net premium policy value. It treats the insurance policy as one-year term insurance combined with a policy as if it were issued one year later.

$\alpha = A^{1}_{x:\overline{1|}} = v\, q_x$

- initial FPT premium

$\beta = \dfrac{A_{x+1}}{\ddot{a}_{x+1}}$

- renewal FPT premium

${}_0V^{FPT} \;=\; {}_1V^{FPT} \;= 0$

- since renewal premium set year 1 policy value to 0, while initial premium set to equal year 1 expected benefits.

${}_tV^{FPT}$ for $(x) \;=\; {}_{t-1}V$ for $(x+1)$

- since renewal FPT premium for (x) is net premium for (x+1) with term lengths adjusted

11.4 Methods

The `Reserves` class implements methods to solve reserves by recursion, and compute interim and modified reserves.

```python
from actuarialmath import Reserves, Contract
import describe
describe.methods(Reserves)
```

```
class Reserves - Compute recursive, interim or modified reserves

    Methods:
    --------

    set_reserves(T, endowment, V):
      Set values of the reserves table and the endowment benefit amount

    fill_reserves(x, s, reserve_benefit, contract):
      Iteratively fill in missing values in reserves table

    t_V_forward(x, s, t, premium, benefit, per_premium, per_policy, reserve_
↪benefit):
      Forward recursion (with optional reserve benefit)

    t_V_backward(x, s, t, premium, benefit, per_premium, per_policy, reserve_
↪benefit):
      Backward recursion (with optional reserve benefit)

    t_V(x, s, t, premium, benefit, reserve_benefit, per_premium, per_policy):
```

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```
      Solve year-t reserves by forward or backward recursion

    r_V_forward(x, s, r, premium, benefit):
      Forward recursion for interim reserves

    r_V_backward(x, s, r, benefit):
      Backward recursion for interim reserves

    FPT_premium(x, s, n, b, first):
      Initial or renewal Full Preliminary Term premiums

    FPT_policy_value(x, s, t, b, n, endowment, discrete):
      Compute Full Preliminary Term policy value at time t

    V_plot(ax, color, title):
      Plot values from reserves tables

    V_t():
      Returns reserves table as a DataFrame
```

11.5 Examples

SOA Question 7.31

For a fully discrete 3-year endowment insurance of 1000 on (x), you are given:

- Expenses, payable at the beginning of the year, are:

Year(s)	Percent of Premium	Per Policy
1	20%	15
2 and 3	8%	5

- The expense reserve at the end of year 2 is –23.64
- The gross annual premium calculated using the equivalence principle is G = 368.
- $G = 1000 P_{x:\overline{3|}} + P^e$, where P^e is the expense loading

Calculate $P_{x:\overline{3|}}$.

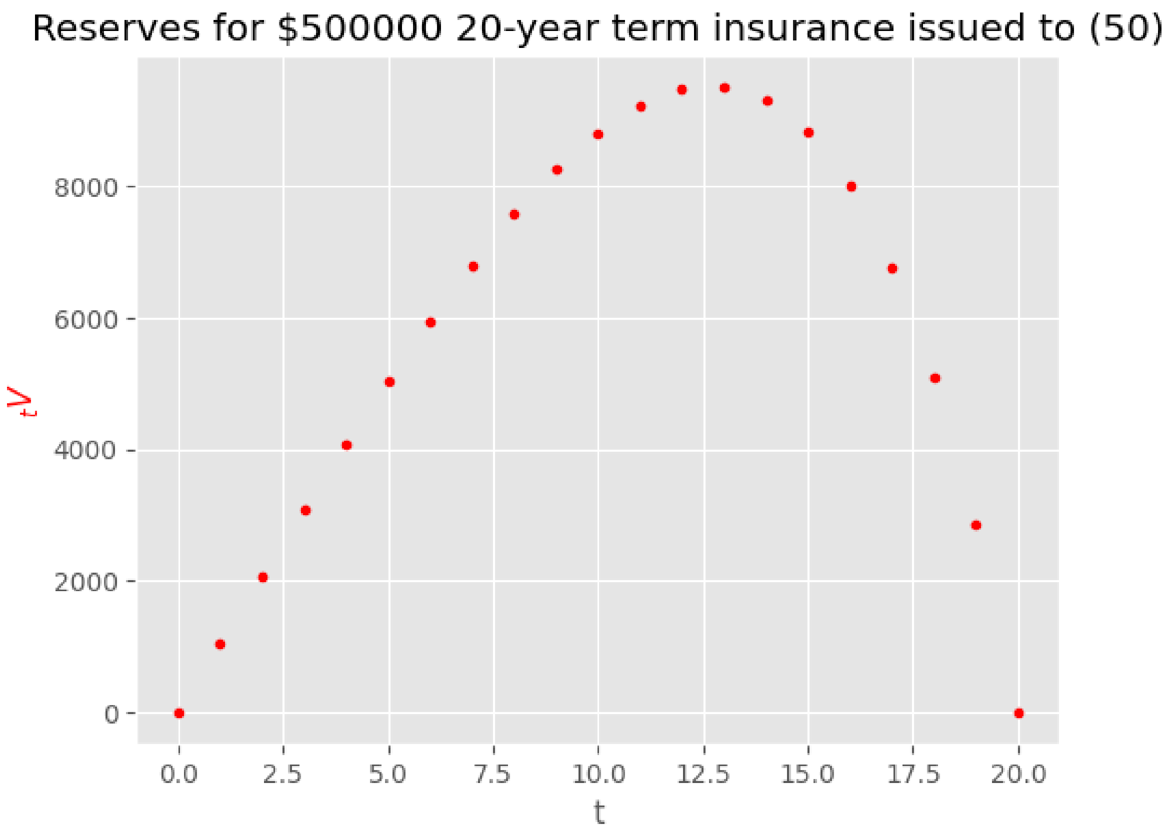
```
print("SOA Question 7.31:  (E) 0.310")
x = 0
life = Reserves().set_reserves(T=3)
G = 368.05
def fun(P):  # solve net premium from expense reserve equation
    return life.t_V(x=x, t=2, premium=G-P, benefit=lambda t: 0,
                    per_policy=5 + .08*G)
P = life.solve(fun, target=-23.64, grid=[.29, .31]) / 1000
print(P)
```

```
SOA Question 7.31:  (E) 0.310
0.309966
```

AMLCR2 Figure 7.4:

Policy values for each year of a 20-year term insurance, sum insured $500,000, issuedto(50).MortalityfollowstheStandardUltimateLi$

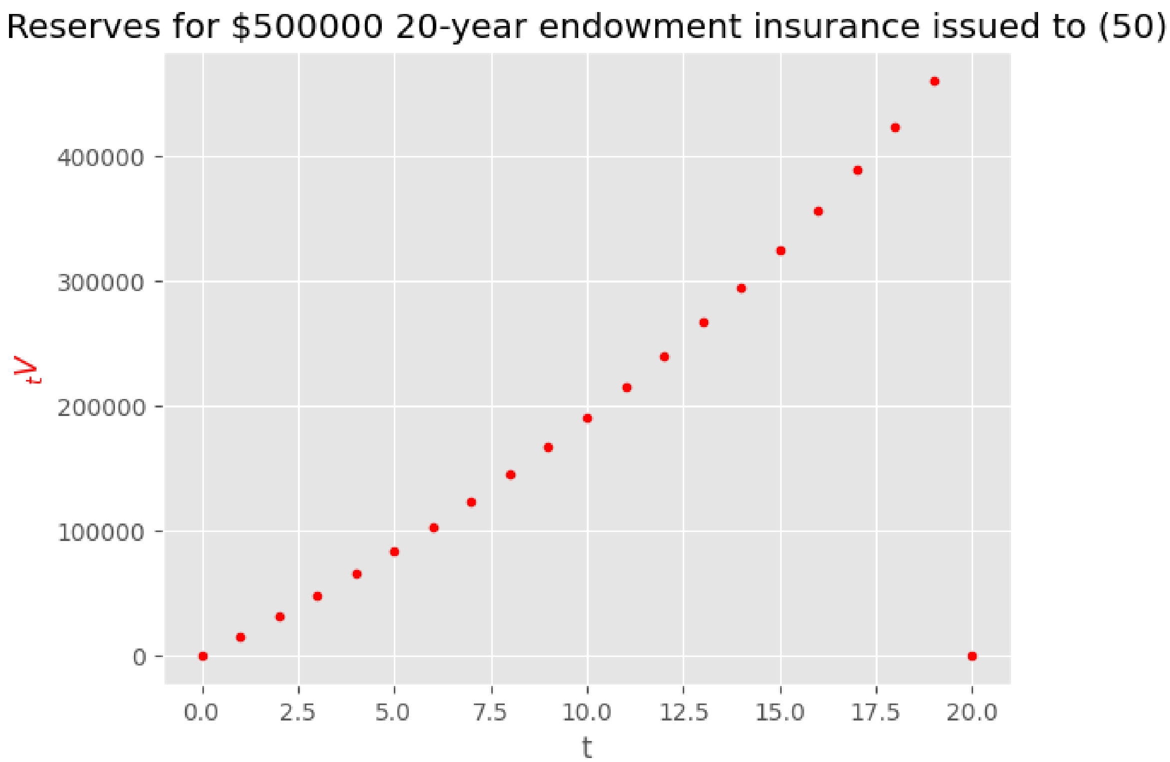
```
from actuarialmath.sult import SULT
life = SULT()
x, T, b = 50, 20, 500000     # $500K 20-year term insurance for (50)
P = life.net_premium(x=x, t=T, b=b)
life.set_reserves(T=T)\
    .fill_reserves(x=x, contract=Contract(premium=P, benefit=b))
life.V_plot(title=f"Reserves for ${b} {T}-year term insurance issued to ({x})")
```



AMLCR2 Figure 7.3:

Policy values for each year of a 20-year endowment insurance, sum insured $500,000, issuedto(50).MortalityfollowstheStandardUltimateLifeTable(noteAMLCR2usedStandardSelectTable), withint$

```
from actuarialmath.sult import SULT
life = SULT()
x, T, b = 50, 20, 500000     # $500K 20-year term insurance for (50)
P = life.net_premium(x=x, t=T, b=b, endowment=b)
life.set_reserves(T=T)\
    .fill_reserves(x=x, contract=Contract(premium=P, benefit=b, endowment=b))
life.V_plot(title=f"Reserves for ${b} {T}-year endowment insurance issued to ({x})")
```


Reserves for $500000 20-year endowment insurance issued to (50)
400000
300000
200000
100000
0
tV
0.0
2.5
5.0
7.5
10.0
12.5
15.0
17.5
20.0
t

CHAPTER

TWELVE

RECURSION

Using annual values provided we can calculate other values at other ages and durations by applying recursion formulas and other actuarial identities.

12.1 Chain rule

$_{t+n}p_x = \ _np_x \cdot \ _tp_{x+n}$

- survival probability chain rule

$_{t+n}E_x = \ _nE_x \cdot \ _tE_{x+n}$

- pure endowment chain rule

12.2 Expected future lifetime

$e_x = e_{x:\overline{m|}} + \ _mp_x \ e_{x+m}$

- curtate expectation of lifetime

$\mathring{e}_x = \mathring{e}_{x:\overline{m|}} + \ _mp_x \ \mathring{e}_{x+m}$

- complete expectation of lifetime

$e_{x:\overline{m+n|}} = e_{x:\overline{m|}} + \ _mp_x \ e_{x:\overline{n|}}$

- limited curtate expectation of lifetime

$\mathring{e}_{x:\overline{m+n|}} = \mathring{e}_{x:\overline{m|}} + \ _mp_x \ \mathring{e}_{x+m:\overline{n|}}$

- limited complete expectation of lifetime

$e_x = p_x(1 + e_{x+1})$

- one-year recursion for curtate expectaion of lifetime

$\mathring{e}_x = \mathring{e}_{x:\overline{1|}} + p_x \ \mathring{e}_{x+1}$

- one-year recursion for complete expectation of lifetime

$e_{x:\overline{1|}} = p_x$

- shortcut for one-year limited curtate expectaion of lifetime

12.3 Life insurance

$A_x = v\,q_x + v\,p_x\,A_{x+1} \Rightarrow A_{x+1} = \dfrac{A_x - v\,q_x}{v\,p_x}$

- whole life insurance recursion

$A^1_{x:\overline{t|}} = v\,q_x + v\,p_x\,A^1_{x+1:\overline{t-1|}}$

- term life insurance recursion

$A^1_{x:\overline{1|}} = v\,q_x$

- shortcut for one-year term life insurance

${}^2A^1_{x:\overline{1|}} = v^2\,q_x$

- shortcut for second moment of one-year term life insurance

$A_{x:\overline{0|}} = {}_0E_x = 1$

- endowment insurance at end of term is pure endowment

$A_{x:\overline{1|}} = q_x\,v + p_x\,v = v$

- shortcut for one-year endowment insurance

${}^2A_{x:\overline{1|}} = v^2$

- shortcut for second moment of one-year endowment insurance

$IA^1_{x:\overline{t|}} = v\,q_x + v\,p_x\,(A_{x+1} + IA^1_{x+1:\overline{t-1|}})$

- increasing insurance recursion

$DA^1_{x:\overline{t|}} = t\,v\,q_x + v\,p_x\,(DA^1_{x+1:\overline{t-1|}})$

- decreasing insurance recursion

12.4 Life annuities

$\ddot{a}_x = 1 + v\,p_x\,\ddot{a}_{x+1} \Rightarrow \ddot{a}_{x+1} = \dfrac{\ddot{a}_x - 1}{v\,p_x}$

- whole life annuity recursion

$\ddot{a}_{x:\overline{t|}} = 1 + v\,p_x\,\ddot{a}_{x+1:\overline{t-1|}}$

- temporary annuity recursion

$\ddot{a}_{x:\overline{1|}} = 1$

- shortcut for one-year temporary annuity

12.5 Methods

The `Recursion` class implements methods to apply recursive, shortcut and actuarial formulas, and traces the steps taken to find the solution.

Caveats:

1. Not all possible recursion rules and actuarial equations have (yet) been implemented in the present version of the package.
2. You may set the recursion depth to a larger limit than the default of 3 (with the keyword argument `depth` when initializing a Recursion class object).
3. But generally, the current implementation may be fragile if the solution is not available within a relatively shallow search.

Notes:

- If a colab or jupyter notebook is auto-detected, the steps are displayed in latex format; else as raw text.
- These display options can be changed by calling the `blog_options` method

```
from actuarialmath import Recursion, ConstantForce, Contract
import describe
describe.methods(Recursion)
```

```
class Recursion - Solve by appling recursive, shortcut and actuarial formulas␣
↪repeatedly

    Args:
      depth : maximum depth of recursions (default is 3)
      verbose : whether to echo recursion steps (True, default)

    Notes:
      7 types of information can be loaded and calculated in recursions:

      - 'q' : (deferred) probability (x) dies in t years
      - 'p' : probability (x) survives t years
      - 'e' : (temporary) expected future lifetime, and moments
      - 'A' : deferred, term, endowment or whole life insurance, and moments
      - 'IA' : decreasing life insurance of t years
      - 'DA' : increasing life insurance of t years
      - 'a' : deferred, temporary or whole life annuity of t years, and moments

    Examples:
      >>> x = 0
      >>> life = Recursion().set_interest(i=0.06).set_a(7, x=x+1).set_q(0.05, x=x)
      >>> a = life.whole_life_annuity(x)
      >>> A = 110 * a / 1000
      >>> print(a, A)
      >>> life = Recursion().set_interest(i=0.06).set_A(A, x=x).set_q(0.05, x=x)
      >>> A1 = life.whole_life_insurance(x+1)
      >>> P = life.gross_premium(A=A1 / 1.03, a=7) * 1000

    Methods:
    --------

    set_q(val, x, s, t, u):
```

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```
      Set mortality rate u|t_q_[x+s] to given value

    set_p(val, x, s, t):
      Set survival probability t_p_[x+s] to given value

    set_e(val, x, s, t, curtate, moment):
      Set expected future lifetime e_[x+s]:t to given value

    set_E(val, x, s, t, endowment, moment):
      Set pure endowment t_E_[x+s] to given value

    set_A(val, x, s, t, u, b, moment, endowment, discrete):
      Set insurance u|_A_[x+s]:t to given value

    set_IA(val, x, s, t, b, discrete):
      Set increasing insurance IA_[x+s]:t to given value

    set_DA(val, x, s, t, b, discrete):
      Set decreasing insurance DA_[x+s]:t to given value

    set_a(val, x, s, t, u, b, variance, discrete):
      Set annuity u|_a_[x+s]:t to given value

    blog_options(latex, notebook):
      Static method to change display options for tracing the recursion steps taken
```

12.6 Examples

AMLCR2 Exercise 2.6

Given $P_x = 0.99$, $P_{x+1} = 0.985$, ${}_3P_{x+1} = 0.95$, $q_{x+3} = 0.02$,

Calculate (a) P_{x+3}, (b) ${}_2P_x$, (c) ${}_2P_{x+1}$, (d) ${}_3P_x$, (e) ${}_{1|2}q_x$.

```
from actuarialmath.recursion import Recursion
x = 0
life = Recursion(depth=3).set_interest(i=0.06)\
                         .set_p(0.99, x=x)\
                         .set_p(0.985, x=x+1)\
                         .set_p(0.95, x=x+1, t=3)\
                         .set_q(0.02, x=x+3)

print(life.p_x(x=x+3), 0.98)
print(life.p_x(x=x, t=2),  0.97515)
print(life.p_x(x=x+1, t=2), 0.96939)
print(life.p_x(x=x, t=3), 0.95969)
print(life.q_x(x=x, t=2, u=1),  0.03031)
```

Survival p_{x+3} :=
$p_{x+3} = 1 - q_{x+3}$ complement of mortality

```
0.98 0.98
```

$$\begin{aligned}
&\texttt{Survival}\ {}_2p_x\ := \\
&{}_2p_x = {}_3p_x /\ p_{x+2} && \texttt{survival chain rule} \\
&p_{x+2} = {}_2p_{x+1} /\ p_{x+1} && \texttt{survival chain rule} \\
&{}_2p_{x+1} = {}_3p_{x+1} /\ p_{x+3} && \texttt{survival chain rule} \\
&{}_3p_x = {}_4p_x /\ p_{x+3} && \texttt{survival chain rule} \\
&{}_4p_x = {}_3p_{x+1} *\ p_x && \texttt{survival chain rule} \\
&p_{x+3} = 1 -\ q_{x+3} && \texttt{complement of mortality}
\end{aligned}$$

```
0.9751500000000001 0.97515
```

$$\begin{aligned}
&\texttt{Survival}\ {}_2p_{x+1}\ := \\
&{}_2p_{x+1} = {}_3p_x /\ p_x && \texttt{survival chain rule} \\
&{}_3p_x = {}_4p_x /\ p_{x+3} && \texttt{survival chain rule} \\
&{}_4p_x = {}_3p_{x+1} *\ p_x && \texttt{survival chain rule} \\
&p_{x+3} = 1 -\ q_{x+3} && \texttt{complement of mortality}
\end{aligned}$$

```
0.9693877551020409 0.96939
```

$$\begin{aligned}
&\texttt{Survival}\ {}_3p_x\ := \\
&{}_3p_x = {}_4p_x /\ p_{x+3} && \texttt{survival chain rule} \\
&{}_4p_x = {}_3p_{x+1} *\ p_x && \texttt{survival chain rule} \\
&p_{x+3} = 1 -\ q_{x+3} && \texttt{complement of mortality}
\end{aligned}$$

```
0.9596938775510204 0.95969
```

$$\begin{aligned}
&\texttt{Mortality}\ {}_{1|2}q_x\ := \\
&{}_{1|2}q_x = p_x -\ {}_3p_x && \texttt{complement survival} \\
&{}_3p_x = {}_4p_x /\ p_{x+3} && \texttt{survival chain rule} \\
&{}_4p_x = {}_3p_{x+1} *\ p_x && \texttt{survival chain rule} \\
&p_{x+3} = 1 -\ q_{x+3} && \texttt{complement of mortality}
\end{aligned}$$

```
0.030306122448979567 0.03031
```

SOA Question 6.40

For a special fully discrete whole life insurance, you are given:

- The death benefit is $1000(1.03)^k$ for death in policy year k, for $k = 1, 2, 3...$
- $q_x = 0.05$
- $i = 0.06$
- $\ddot{a}_{x+1} = 7.00$
- The annual net premium for this insurance at issue age x is 110

Calculate the annual net premium for this insurance at issue age $x + 1$.

```
print("SOA Question 6.40: (C) 116 ")
x = 0
life = Recursion().set_interest(i=0.06).set_a(7, x=x+1).set_q(0.05, x=x)
a = life.whole_life_annuity(x)
```

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```
A = 110 * a / 1000
print(a, A)
life = Recursion().set_interest(i=0.06).set_A(A, x=x).set_q(0.05, x=x)
A1 = life.whole_life_insurance(x+1)
P = life.gross_premium(A=A1 / 1.03, a=7) * 1000
print(P)
```

```
SOA Question 6.40: (C) 116
```

Whole Life Annuity $\ddot{a}_x$:=	
$\ddot{a}_x = 1 + E_x * \ddot{a}_{x+1}$	backward recursion
$E_x = p_x * v$	pure endowment
$p_x = 1 - q_x$	complement of mortality

```
7.2735849056603765 0.8000943396226414
```

Whole Life Insurance A_{x+1} :=	
$A_{x+1} = [\, A_x/v - q_x * b]/\, p_x$	forward recursion
$p_x = 1 - q_x$	complement of mortality

```
116.51945397474269
```

SOA Question 6.10 : (D) 0.91

For a fully discrete 3-year term insurance of 1000 on (x), you are given:

1. $p_x = 0.975$
2. $i = 0.06$
3. The actuarial present value of the death benefit is 152.85
4. The annual net premium is 56.05

Calculate p_{x+2}.

```
print("SOA Question 6.10: (D) 0.91")
x = 0
life = Recursion(depth=5).set_interest(i=0.06)\
                         .set_p(0.975, x=x)\
                         .set_a(152.85/56.05, x=x, t=3)\
                         .set_A(152.85, x=x, t=3, b=1000)
p = life.p_x(x=x+2)
print(p)
```

```
SOA Question 6.10: (D) 0.91
```

Survival p_{x+2} :=

$p_{x+2} = E_{x+2}/v$	one-year pure endowment
$E_{x+2} = A_{x+2:\overline{1}\vert} - A^{1}_{x+2:\overline{1}\vert}$	endowment insurance minus term
$A^{1}_{x+2:\overline{1}\vert} = [A^{1}_{x+1:\overline{2}\vert}/v - q_{x+1} * b]/p_{x+1}$	forward recursion
$p_{x+1} = [\ddot{a}_{x+1:\overline{2}\vert} - 1]/[v * \ddot{a}_{x+2:\overline{1}\vert}]$	annuity recursion
$\ddot{a}_{x+1:\overline{2}\vert} = [\ddot{a}_{x:\overline{3}\vert} - 1]/E_x$	forward recursion
$A^{1}_{x+1:\overline{2}\vert} = [A^{1}_{x:\overline{3}\vert}/v - q_x * b]/p_x$	forward recursion
$E_x = p_x * v$	pure endowment

```
0.9097382950525702
```

SOA Question 6.48

For a special fully discrete 5-year deferred 3-year term insurance of 100,000 on (x) you are given:

- There are two premium payments, each equal to P . The first is paid at the beginning of the first year and the second is paid at the end of the 5-year deferral period
- $p_x = 0.95$
- $q_{x+5} = 0.02$
- $q_{x+6} = 0.03$
- $q_{x+7} = 0.04$
- $i = 0.06$

Calculate P using the equivalence principle.

```
print("SOA Question 6.48:  (A) 3195")
life = Recursion(depth=3).set_interest(i=0.06)
x = 0
life.set_p(0.95, x=x, t=5)
life.set_q(0.02, x=x+5)
life.set_q(0.03, x=x+6)
life.set_q(0.04, x=x+7)
a = 1 + life.E_x(x, t=5)
A = life.deferred_insurance(x, u=5, t=3)
P = life.gross_premium(A=A, a=a, benefit=100000)
print(P)
```

```
SOA Question 6.48:  (A) 3195
```

Pure Endowment ${}_5E_x$:=

${}_5E_x = {}_5p_x * v^5$	pure endowment

Pure Endowment ${}_5E_x$:=

${}_5E_x = {}_5p_x * v^5$	pure endowment

Term Insurance $A^{1}_{x+5:\overline{3|}}$:=

$A^{1}_{x+5:\overline{3\vert}} = A_{x+5:\overline{3\vert}} - {}_{3}E_{x+5}$	endowment insurance - pure
${}_{3}E_{x+5} = E_{x+5} * {}_{2}E_{x+6}$	pure endowment chain rule
${}_{2}E_{x+6} = E_{x+6} * E_{x+7}$	pure endowment chain rule
$E_{x+7} = p_{x+7} * v$	pure endowment
$E_{x+6} = p_{x+6} * v$	pure endowment
$E_{x+5} = p_{x+5} * v$	pure endowment
$p_{x+7} = 1 - q_{x+7}$	complement of mortality
$A^{1}_{x+5:\overline{3\vert}} = v * [\, q_{x+5} * b + p_{x+5} * A^{1}_{x+6:\overline{2\vert}}]$	backward recursion
$A^{1}_{x+6:\overline{2\vert}} = v * [\, q_{x+6} * b + p_{x+6} * A^{1}_{x+7:\overline{1\vert}}]$	backward recursion
$p_{x+6} = 1 - q_{x+6}$	complement of mortality
$p_{x+5} = 1 - q_{x+5}$	complement of mortality

Term Insurance $A^{1}_{x+5:\overline{3|}}$:=

$A^{1}_{x+5:\overline{3\vert}} = A_{x+5:\overline{3\vert}} - {}_{3}E_{x+5}$	endowment insurance - pure
${}_{3}E_{x+5} = E_{x+5} * {}_{2}E_{x+6}$	pure endowment chain rule
${}_{2}E_{x+6} = E_{x+6} * E_{x+7}$	pure endowment chain rule
$E_{x+7} = p_{x+7} * v$	pure endowment
$E_{x+6} = p_{x+6} * v$	pure endowment
$E_{x+5} = p_{x+5} * v$	pure endowment
$p_{x+7} = 1 - q_{x+7}$	complement of mortality
$A^{1}_{x+5:\overline{3\vert}} = v * [\, q_{x+5} * b + p_{x+5} * A^{1}_{x+6:\overline{2\vert}}]$	backward recursion
$A^{1}_{x+6:\overline{2\vert}} = v * [\, q_{x+6} * b + p_{x+6} * A^{1}_{x+7:\overline{1\vert}}]$	backward recursion
$p_{x+6} = 1 - q_{x+6}$	complement of mortality
$p_{x+5} = 1 - q_{x+5}$	complement of mortality

```
3195.118917658744
```

SOA Question 6.17

An insurance company sells special fully discrete two-year endowment insurance policies to smokers (S) and non-smokers (NS) age x. You are given:

- The death benefit is 100,000; the maturity benefit is 30,000
- The level annual premium for non-smoker policies is determined by the equivalence principle
- The annual premium for smoker policies is twice the non-smoker annual premium
- $\mu^{NS}_{x+t} = 0.1. \quad t > 0$
- $q^{S}_{x+k} = 1.5 q^{NS}_{x+k}$, for $k = 0, 1$
- $i = 0.08$

Calculate the expected present value of the loss at issue random variable on a smoker policy.

```
print("SOA Question 6.17:  (A) -30000")
x = 0
life = ConstantForce(mu=0.1).set_interest(i=0.08)
A = life.endowment_insurance(x, t=2, b=100000, endowment=30000)
a = life.temporary_annuity(x, t=2)
P = life.gross_premium(a=a, A=A)
print(A, a, P)

life1 = Recursion().set_interest(i=0.08)\
```

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```
                    .set_q(life.q_x(x, t=1) * 1.5, x=x, t=1)\
                    .set_q(life.q_x(x+1, t=1) * 1.5, x=x+1, t=1)
contract = Contract(premium=P * 2, benefit=100000, endowment=30000)
L = life1.gross_policy_value(x, t=0, n=2, contract=contract)
print(L)
```

```
SOA Question 6.17:  (A) -30000
37251.49857703497 1.8378124241073728 20269.478042694187
```

$$\begin{array}{ll}
\texttt{Term Insurance } A^1_{x:\overline{2|}} := & \\
A^1_{x:\overline{2|}} = A_{x:\overline{2|}} - {}_2E_x & \texttt{endowment insurance - pure} \\
{}_2E_x = {}_2p_x * v^2 & \texttt{pure endowment} \\
{}_2p_x = p_{x+1} * p_x & \texttt{survival chain rule} \\
A^1_{x:\overline{2|}} = v * [\, q_x * b + p_x * A^1_{x+1:\overline{1|}}] & \texttt{backward recursion} \\
p_{x+1} = 1 - q_{x+1} & \texttt{complement of mortality} \\
E_x = p_x * v & \texttt{pure endowment} \\
p_x = 1 - q_x & \texttt{complement of mortality}
\end{array}$$

$$\begin{array}{ll}
\texttt{Temporary Annuity } \ddot{a}_{x:\overline{2|}} := & \\
\ddot{a}_{x:\overline{2|}} = 1 + E_x * \ddot{a}_{x+1:\overline{1|}} & \texttt{backward recursion} \\
E_x = p_x * v & \texttt{pure endowment} \\
p_x = 1 - q_x & \texttt{complement of mortality}
\end{array}$$

$$\begin{array}{ll}
\texttt{Pure Endowment } {}_2E_x := & \\
{}_2E_x = {}_2p_x * v^2 & \texttt{pure endowment} \\
{}_2p_x = p_{x+1} * p_x & \texttt{survival chain rule} \\
p_x = 1 - q_x & \texttt{complement of mortality} \\
p_{x+1} = 1 - q_{x+1} & \texttt{complement of mortality}
\end{array}$$

```
-30107.42633581125
```

SOA Question 2.5 : (B) 37.1

You are given the following:

1. $e_{40:20} = 18$
2. $e_{60} = 25$
3. ${}_{20}q_{40} = 0.2$
4. $q_{40} = 0.003$

Calculate e_{41}.

hints:

- solve for e_{40} from limited lifetime formula
- compute e_{41} using backward recursion

```
print("SOA Question 2.5:  (B) 37.1")
life = Recursion(verbose=True).set_e(25, x=60, curtate=True)\
                              .set_q(0.2, x=40, t=20)\
```

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```
                                .set_q(0.003, x=40)\
                                .set_e(18, x=40, t=20, curtate=True)
e = life.e_x(41, curtate=True)
print(e)
```

```
SOA Question 2.5:  (B) 37.1
```

$$\begin{array}{ll}
\texttt{Lifetime}\ e_{x+41} \ := & \\
e_{x+41} = [\, e_{x+40} - \ e_{x+40:\overline{1|}}] /\ p_{x+40} & \texttt{forward recursion} \\
e_{x+40} = \ e_{x+40:\overline{20|}} + \ {}_{20}p_{x+40} * \ e_{x+60} & \texttt{backward recursion} \\
{}_{20}p_{x+40} = 1 - \ {}_{20}q_{x+40} & \texttt{complement of mortality} \\
e_{x+40:\overline{1|}} = \ p_{x+40} & \texttt{1-year curtate shortcut} \\
p_{x+40} = 1 - \ q_{x+40} & \texttt{complement of mortality}
\end{array}$$

```
37.11434302908726
```

CHAPTER

THIRTEEN

LIFE TABLE

A life table, from some initial age x_0 to a maximum age ω, represents a survival model with probabilities ${}_tp_x$. Let l_{x_0} be an arbitrary positive number of lives at age x_0, called the radix, and $l_{x_0+t} = l_{x_0}\ {}_tp_{x_0}$ A life table is typically tabulated at integer ages only – fractional age assumptions would be needed to calculate survival probabilities for non-integer ages and durations.

$d_x = l_x - l_{x+1}$

- it is usual for a life table to also show the values of d_x, the expected of deaths in the year of age x to $x + 1$.

$q_x = \dfrac{d_x}{l_x}$

- the mortality rate can then be derived, which is the probability that a life aged x dies within one year.

13.1 Methods

The `LifeTable` class implements methods to define and calculate life table entries

```
from actuarialmath import LifeTable
import describe
describe.methods(LifeTable)
```

```
class LifeTable - Calculate life table, and iteratively fill in missing values

    Args:
      udd : assume UDD or constant force of mortality for fractional ages
      verbose : whether to echo update steps

    Notes:
      4 types of columns can be loaded and calculated in the life table:

      - 'q' : probability (x) dies in one year
      - 'l' : number of lives aged x
      - 'd' : number of deaths of age x
      - 'p' : probability (x) survives at least one year

    Methods:
    --------

    set_table(fill, minage, maxage, l, d, p, q):
      Update life table
```

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```
    fill_table(radix):
      Iteratively fill in missing table cells (does not check consistency)

    frame():
      Return life table columns and values in a DataFrame

    __getitem__(col):
      Returns a column of the life table
```

13.2 Examples

AMLCR2 Exercise 3.2

You are given the following life table extract.

Age,x	l_x
52	89948
53	89089
54	88176
55	87208
56	86181
57	85093
58	83940
59	82719
60	81429

Calculate

- ${}_{0.2}q_{52.4}$ assuming UDD (fractional age assumption),
- ${}_{0.2}q_{52.4}$ assuming constant force of mortality (fractional age assumption),
- ${}_{5.7}p_{52.4}$ assuming UDD,
- ${}_{5.7}p_{52.4}$ assuming constant force of mortality,
- ${}_{3.2|2.5}q_{52.4}$ assuming UDD, and
- ${}_{3.2|2.5}q_{52.4}$ assuming constant force of mortality.

```
table = {x:l for x,l in zip(range(52, 61),
                            [89948, 89089, 88176, 87208, 86181,
                             85093, 83940, 82719, 81429])}
life1 = LifeTable(udd=True, verbose=True).set_table(l=table)
life2 = LifeTable(udd=False).set_table(l=table)
results = [life1.q_r(x=52, r=0.4, t=0.2),    # 0.001917
           life2.q_r(x=52, r=0.4, t=0.2),    # 0.001917
           life1.p_r(x=52, r=0.4, t=5.7),    # 0.935422
           life2.p_r(x=52, r=0.4, t=5.7),    # 0.935423
```

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```
            life1.q_r(x=52, r=0.4, u=3.2, t=2.5),   # 0.030957
↪
            life2.q_r(x=52, r=0.4, u=3.2, t=2.5)]  # 0.030950
print([round(r, 6) for r in results])
```

```
1 d(x=52) = 859
2 d(x=53) = 913
3 d(x=54) = 968
4 d(x=55) = 1027
5 d(x=56) = 1088
6 d(x=57) = 1153
7 d(x=58) = 1221
8 d(x=59) = 1290
9 q(x=52) = 0.009549962200382444
10 q(x=53) = 0.01024817878750463
11 q(x=54) = 0.010978043912175649
12 q(x=55) = 0.011776442528208421
13 q(x=56) = 0.01262459242756524
14 q(x=57) = 0.013549880718743022
15 q(x=58) = 0.014546104360257326
16 q(x=59) = 0.015594966090015596
17 p(x=52) = 0.99045
18 p(x=53) = 0.9897518
19 p(x=54) = 0.989022
20 p(x=55) = 0.9882236
21 p(x=56) = 0.9873754
22 p(x=57) = 0.9864501
23 p(x=58) = 0.9854539
24 p(x=59) = 0.984405
[0.001917, 0.001917, 0.935422, 0.935423, 0.030957, 0.03095]
```

SOA Question 6.53

A warranty pays 2000 at the end of the year of the first failure if a washing machine fails within three years of purchase. The warranty is purchased with a single premium, G, paid at the time of purchase of the washing machine. You are given:

- 10% of the washing machines that are working at the start of each year fail by the end of that year
- i = 0.08
- The sales commission is 35% of G
- G is calculated using the equivalence principle

Calculate G.

```
print("SOA Question 6.53:  (D) 720")
x = 0
life = LifeTable().set_interest(i=0.08)\
                  .set_table(q={x: 0.1, x+1: 0.1, x+2: 0.1})
A = life.term_insurance(x, t=3)
G = life.gross_premium(a=1, A=A, benefit=2000, initial_premium=0.35)
print(A, G)
print(life.frame())
```

```
SOA Question 6.53:  (D) 720
0.23405349794238678 720.1646090534978
          l        d    q    p
0  100000.0  10000.0  0.1  0.9
1   90000.0   9000.0  0.1  0.9
2   81000.0   8100.0  0.1  0.9
3   72900.0      NaN  NaN  NaN
```

SOA Question 6.41

For a special fully discrete 2-year term insurance on (x), you are given:

- $q_x = 0.01$
- $q_{x+1} = 0.02$
- $i = 0.05$
- The death benefit in the first year is 100,000
- Both the benefits and premiums increase by 1% in the second year

Calculate the annual net premium in the first year.

```
print("SOA Question 6.41:  (B) 1417")
x = 0
life = LifeTable().set_interest(i=0.05)\
                  .set_table(q={x:.01, x+1:.02})
P = 1416.93
a = 1 + life.E_x(x, t=1) * 1.01
A = (life.deferred_insurance(x, u=0, t=1)
     + 1.01 * life.deferred_insurance(x, u=1, t=1))
print(a, A)
P = 100000 * A / a
print(P)
print(life.frame())
```

```
SOA Question 6.41:  (B) 1417
1.9522857142857144 0.027662585034013608
1416.9332301924137
          l       d     q     p
0  100000.0  1000.0  0.01  0.99
1   99000.0  1980.0  0.02  0.98
2   97020.0     NaN   NaN   NaN
```

SOA Question 3.11

For the country of Bienna, you are given:

- Bienna publishes mortality rates in biennial form, that is, mortality rates are of the form: ${}_2q_{2x}$, for $x = 0, 1, 2, ...$
- Deaths are assumed to be uniformly distributed between ages $2x$ and $2x + 2$, for $x = 0, 1, 2, ...$
- ${}_2q_{50} = 0.02$
- ${}_2q_{52} = 0.04$ Calculate the probability that (50) dies during the next 2.5 years.

```
print("SOA Question 3.11:  (B) 0.03")
life = LifeTable(udd=True).set_table(q={50//2: .02, 52//2: .04})
```

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```
print(life.q_r(50//2, t=2.5/2))
print(life.frame())
```

```
SOA Question 3.11:  (B) 0.03
0.0298
           l       d     q     p
25  100000.0  2000.0  0.02  0.98
26   98000.0  3920.0  0.04  0.96
27   94080.0     NaN   NaN   NaN
```

SOA Question 3.5

You are given:

x	60	61	62	63	64	65	66	67
l_x	99,999	88,888	77,777	66,666	55,555	44,444	33,333	22,222

$a = {}_{3.4|2.5}q_{60}$ assuming a uniform distribution of deaths over each year of age

$b = {}_{3.4|2.5}q_{60}$ assuming a constant force of mortality over each year of age

Calculate $100,000(a - b)$

```
print("SOA Question 3.5:  (E) 106")
l = {60+x: n*11111 for x,n in enumerate([9, 8, 7, 6, 5, 4, 3, 2])}
a, b = (LifeTable(udd=udd).set_table(l=l).q_r(60, u=3.4, t=2.5)
        for udd in [True, False])
print(100000 * (a - b))
```

```
SOA Question 3.5:  (E) 106
106.16575827938624
```

SOA Question 3.14

You are given the following information from a life table:

x	l_x	d_x	p_x	q_x
95	–	–	–	0.40
96	–	–	0.20	–
97	–	72	–	1.00

You are also given:

- $l_{90} = 1000$ and $l_{93} = 825$
- Deaths are uniformly distributed over each year of age.

Calculate the probability that (90) dies between ages 93 and 95.5.

```
print("SOA Question 3.14:  (C) 0.345")
life = LifeTable(udd=True).set_table(l={90: 1000, 93: 825},
                                     d={97: 72},
```

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```
                                            p={96: .2},
                                            q={95: .4, 97: 1})
print(life.q_r(90, u=93-90, t=95.5-93))
print(life.frame())
```

```
SOA Question 3.14:  (C) 0.345
0.345
         l      d    q    p
90  1000.0    NaN  NaN  NaN
93   825.0    NaN  NaN  NaN
95   600.0  240.0  0.4  0.6
96   360.0  288.0  0.8  0.2
97    72.0   72.0  1.0  0.0
98     0.0    NaN  NaN  NaN
```

CHAPTER

FOURTEEN

SULT

14.1 Standard ultimate life table

This tabulates single net premiums and basic functions (whole life and endowment insurances, whole life and temporary annuities and pure endowments) for several time periods at integer ages between 20 and 100 years. According to the SOA's "Excel Workbook for FAM-L Tables", this table was developed from the following assumptions:

- constant interest rate $i = 0.05$
- radix of 100000 initial lives aged 20
- incorporates Makeham's Law as its survival model with $A = 0.00022,\ B = 0.0000027,\ c = 1.124$

14.2 Pure endowment

Pure endowment functions can be calculated from numbers of lives survived and compounded interest rates.

$$ {}_tE_x = v^t\ \frac{l_{x+t}}{l_x} $$

$$ {}^2_tE_x = v^{2t}\ \frac{l_{x+t}}{l_x} = v^t\ {}_tE_x $$

14.3 Term life insurance

Term life insurance functions can be calculated from whole life insurance and pure endowment table columns.

$$ A^{1}_{x:\overline{t|}} = A_x -\ {}_tE_x\ A_{x+t} = A_{x:\overline{t|}} -\ {}_tE_x $$

$$ {}^2A^{1}_{x:\overline{t|}} = {}^2A_x -\ {}^2_tE_x\ {}^2A_{x+t} = {}^2A_x -\ v^t\ {}_tE_x\ {}^2A_{x+t} $$

14.4 Methods

The SULT class generates and uses the standard ultimate life table, which is constructed, by default, from Makeham's Law and parameters in SOA's "Excel Workbook for FAM-L Tables"

```
import math
from actuarialmath import SULT
import describe
describe.methods(SULT)
```

```
class SULT - Generates and uses a standard ultimate life table

    Args:
      i : interest rate
      radix : initial number of lives
      minage : minimum age
      maxage : maximum age
      S : survival function, default is Makeham with SOA FAM-L parameters

    Methods:
    --------

    frame(minage, maxage):
      Derive FAM-L exam table columns of SULT as a DataFrame

    __getitem__(col):
      Returns a column of the sult table
```

14.5 Examples

SOA Question 6.52

For a fully discrete 10-payment whole life insurance of H on (45), you are given:

- Expenses payable at the beginning of each year are as follows:

Expense Type	First Year	Years 2-10	Years 11+
Per policy	100	20	10
% of Premium	105%	5%	0%

- Mortality follows the Standard Ultimate Life Table
- i = 0.05
- The gross annual premium, calculated using the equivalence principle, is of the form $G = gH + f$, where g is the premium rate per 1 of insurance and f is the per policy fee

Calculate f.

```
print("SOA Question 6.52:  (D) 50.80")
sult = SULT()
a = sult.temporary_annuity(45, t=10)
other_cost = 10 * sult.deferred_annuity(45, u=10)
```

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```
P = sult.gross_premium(a=a,
                       A=0,
                       benefit=0,
                       initial_premium=1.05,
                       renewal_premium=0.05,
                       initial_policy=100 + other_cost,
                       renewal_policy=20)
print(a, P)
```

```
SOA Question 6.52:  (D) 50.80
8.0750937741422 50.80135534704229
```

SOA Question 6.47

For a 10-year deferred whole life annuity-due with payments of 100,000 per year on (70), you are given:

- Annual gross premiums of G are payable for 10 years
- First year expenses are 75% of premium
- Renewal expenses for years 2 and later are 5% of premium during the premium paying period
- Mortality follows the Standard Ultimate Life Table
- i = 0.05

Calculate G using the equivalence principle.

```
print("SOA Question 6.47:  (D) 66400")
sult = SULT()
a = sult.temporary_annuity(70, t=10)
A = sult.deferred_annuity(70, u=10)
P = sult.gross_premium(a=a, A=A, benefit=100000, initial_premium=0.75,
                        renewal_premium=0.05)
print(P)
```

```
SOA Question 6.47:  (D) 66400
66384.13293704337
```

SOA Question 6.43

For a fully discrete, 5-payment 10-year term insurance of 200,000 on (30), you are given:

- Mortality follows the Standard Ultimate Life Table
- The following expenses are incurred at the beginning of each respective year:

	Percent of Premium	Per Policy	Percent of Premium	Per Policy
	Year 1	Year 1	Years 2 - 10	Years 2 - 10
Taxes	5%	0	5%	0
Commissions	30%	0	10%	0
Maintenance	0%	8	0%	4

- i = 0.05

- $\ddot{a}_{30:\overline{5}|} = 4.5431$

Calculate the annual gross premium using the equivalence principle.

```
print("SOA Question 6.43:  (C) 170")
sult = SULT()
a = sult.temporary_annuity(30, t=5)
A = sult.term_insurance(30, t=10)
other_expenses = 4 * sult.deferred_annuity(30, u=5, t=5)
P = sult.gross_premium(a=a, A=A, benefit=200000, initial_premium=0.35,
                        initial_policy=8 + other_expenses, renewal_policy=4,
                        renewal_premium=0.15)
print(P)
```

```
SOA Question 6.43:  (C) 170
171.22371939459944
```

SOA Question 6.39

XYZ Insurance writes 10,000 fully discrete whole life insurance policies of 1000 on lives age 40 and an additional 10,000 fully discrete whole life policies of 1000 on lives age 80.

XYZ used the following assumptions to determine the net premiums for these policies:

- Mortality follows the Standard Ultimate Life Table
- i = 0.05

During the first ten years, mortality did follow the Standard Ultimate Life Table.

Calculate the average net premium per policy in force received at the beginning of the eleventh year.

```
print("SOA Question 6.39:  (A) 29")
sult = SULT()
P40 = sult.premium_equivalence(sult.whole_life_insurance(40), b=1000)
P80 = sult.premium_equivalence(sult.whole_life_insurance(80), b=1000)
p40 = sult.p_x(40, t=10)
p80 = sult.p_x(80, t=10)
P = (P40 * p40 + P80 * p80) / (p80 + p40)
print(P)
```

```
SOA Question 6.39:  (A) 29
29.033866427845496
```

SOA Question 6.37

For a fully discrete whole life insurance policy of 50,000 on (35), with premiums payable for a maximum of 10 years, you are given:

- Expenses of 100 are payable at the end of each year including the year of death
- Mortality follows the Standard Ultimate Life Table
- i = 0.05

Calculate the annual gross premium using the equivalence principle.

```
print("SOA Question 6.37:  (D) 820")
sult = SULT()
```

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```
benefits = sult.whole_life_insurance(35, b=50000 + 100)
expenses = sult.immediate_annuity(35, b=100)
a = sult.temporary_annuity(35, t=10)
print(benefits, expenses, a)
print((benefits + expenses) / a)
```

```
SOA Question 6.37:  (D) 820
4836.382819496279 1797.2773668474615 8.092602358383987
819.7190338249138
```

SOA Question 6.35

For a fully discrete whole life insurance policy of 100,000 on (35), you are given:

- First year commissions are 19% of the annual gross premium
- Renewal year commissions are 4% of the annual gross premium
- Mortality follows the Standard Ultimate Life Table
- i = 0.05

Calculate the annual gross premium for this policy using the equivalence principle.

```
print("SOA Question 6.35:  (D) 530")
sult = SULT()
A = sult.whole_life_insurance(35, b=100000)
a = sult.whole_life_annuity(35)
print(sult.gross_premium(a=a, A=A, initial_premium=.19, renewal_premium=.04))
```

```
SOA Question 6.35:  (D) 530
534.4072234303344
```

SOA Question 5.8

For an annual whole life annuity-due of 1 with a 5-year certain period on (55), you are given:

- Mortality follows the Standard Ultimate Life Table
- i = 0.05

Calculate the probability that the sum of the undiscounted payments actually made under this annuity will exceed the expected present value, at issue, of the annuity.

```
print("SOA Question 5.8: (C) 0.92118")
sult = SULT()
a = sult.certain_life_annuity(55, u=5)
print(sult.p_x(55, t=math.floor(a)))
```

```
SOA Question 5.8: (C) 0.92118
0.9211799771029529
```

SOA Question 5.3

You are given:

- Mortality follows the Standard Ultimate Life Table
- Deaths are uniformly distributed over each year of age

- i = 0.05

Calculate $\frac{d}{dt}(\overline{I}\overline{a})_{40:\overline{t}|}$ at $t = 10.5$.

```
print("SOA Question 5.3:  (C) 6.239")
sult = SULT()
t = 10.5
print(t * sult.E_r(40, t=t))
```

```
SOA Question 5.3:  (C) 6.239
6.23871918627528
```

SOA Question 4.17

For a special whole life policy on (48), you are given:

- The policy pays 5000 if the insured's death is before the median curtate future lifetime at issue and 10,000 if death is after the median curtate future lifetime at issue
- Mortality follows the Standard Ultimate Life Table
- Death benefits are paid at the end of the year of death
- i = 0.05

Calculate the actuarial present value of benefits for this policy.

```
print("SOA Question 4.17:  (A) 1126.7")
sult = SULT()
median = sult.Z_t(48, prob=0.5, discrete=False)
benefit = lambda x,t: 5000 if t < median else 10000
print(sult.A_x(48, benefit=benefit))
```

```
SOA Question 4.17:  (A) 1126.7
1126.774772894844
```

SOA Question 4.14

A fund is established for the benefit of 400 workers all age 60 with independent future lifetimes. When they reach age 85, the fund will be dissolved and distributed to the survivors.

The fund will earn interest at a rate of 5% per year.

The initial fund balance, F, is determined so that the probability that the fund will pay at least 5000 to each survivor is 86%, using the normal approximation.

Mortality follows the Standard Ultimate Life Table.

Calculate F.

```
print("SOA Question 4.14:  (E) 390000    ")
sult = SULT()
p = sult.p_x(60, t=85-60)
mean = sult.bernoulli(p)
var = sult.bernoulli(p, variance=True)
F = sult.portfolio_percentile(mean=mean, variance=var, prob=.86, N=400)
print(F * 5000 * sult.interest.v_t(85-60))
```

```
SOA Question 4.14:  (E) 390000
389322.86778416135
```

SOA Question 4.5

For a 30-year term life insurance of 100,000 on (45), you are given:

- The death benefit is payable at the moment of death
- Mortality follows the Standard Ultimate Life Table
- $\delta = 0.05$
- Deaths are uniformly distributed over each year of age

Calculate the 95th percentile of the present value of benefits random variable for this insurance

```
print("SOA Question 4.5:  (C) 35200")
sult = SULT(udd=True).set_interest(delta=0.05)
Z = 100000 * sult.Z_from_prob(45, prob=0.95, discrete=False)
print(Z)
```

```
SOA Question 4.5:  (C) 35200
35187.952037196534
```

SOA Question 3.9

A father-son club has 4000 members, 2000 of which are age 20 and the other 2000 are age 45. In 25 years, the members of the club intend to hold a reunion. You are given:

- All lives have independent future lifetimes.
- Mortality follows the Standard Ultimate Life Table.

Using the normal approximation, without the continuity correction, calculate the 99th percentile of the number of surviving members at the time of the reunion.

```
print("SOA Question 3.9:  (E) 3850")
sult = SULT()
p1 = sult.p_x(20, t=25)
p2 = sult.p_x(45, t=25)
mean = sult.bernoulli(p1) * 2000 + sult.bernoulli(p2) * 2000
var = (sult.bernoulli(p1, variance=True) * 2000
        + sult.bernoulli(p2, variance=True) * 2000)
print(sult.portfolio_percentile(mean=mean, variance=var, prob=.99))
```

```
SOA Question 3.9:  (E) 3850
3850.144345130047
```

SOA Question 3.8

A club is established with 2000 members, 1000 of exact age 35 and 1000 of exact age 45. You are given:

- Mortality follows the Standard Ultimate Life Table
- Future lifetimes are independent
- N is the random variable for the number of members still alive 40 years after the club is established

Using the normal approximation, without the continuity correction, calculate the smallest n such that $Pr(N \geq n) \leq 0.05$.

```
print("SOA Question 3.8:  (B) 1505")
sult = SULT()
p1 = sult.p_x(35, t=40)
p2 = sult.p_x(45, t=40)
mean = sult.bernoulli(p1) * 1000 + sult.bernoulli(p2) * 1000
var = (sult.bernoulli(p1, variance=True) * 1000
        + sult.bernoulli(p2, variance=True) * 1000)
print(sult.portfolio_percentile(mean=mean, variance=var, prob=.95))
```

```
SOA Question 3.8:  (B) 1505
1504.8328375406456
```

SOA Question 3.4

The SULT Club has 4000 members all age 25 with independent future lifetimes. The mortality for each member follows the Standard Ultimate Life Table.

Calculate the largest integer N, using the normal approximation, such that the probability that there are at least N survivors at age 95 is at least 90%.

```
print("SOA Question 3.4:  (B) 815")
sult = SULT()
mean = sult.p_x(25, t=95-25)
var = sult.bernoulli(mean, variance=True)
print(sult.portfolio_percentile(N=4000, mean=mean, variance=var, prob=.1))
```

```
SOA Question 3.4:  (B) 815
815.0943255167722
```

Generate SULT Table:

```
print("Standard Ultimate Life Table at i=0.05")
sult.frame()
```

```
Standard Ultimate Life Table at i=0.05
```

```
          l_x       q_x      a_x      A_x     2A_x   a_x:10   A_x:10   a_x:20  
20   100000.0  0.000250  19.9664  0.04922  0.00580  8.0991  0.61433  13.0559   \
21    99975.0  0.000253  19.9197  0.05144  0.00614  8.0990  0.61433  13.0551   
22    99949.7  0.000257  19.8707  0.05378  0.00652  8.0988  0.61434  13.0541   
23    99924.0  0.000262  19.8193  0.05622  0.00694  8.0986  0.61435  13.0531   
24    99897.8  0.000267  19.7655  0.05879  0.00739  8.0983  0.61437  13.0519   
..        ...       ...      ...      ...      ...     ...      ...      ...   
96    17501.8  0.192887   3.5597  0.83049  0.69991  3.5356  0.83164   3.5597   
97    14125.9  0.214030   3.3300  0.84143  0.71708  3.3159  0.84210   3.3300   
98    11102.5  0.237134   3.1127  0.85177  0.73356  3.1050  0.85214   3.1127   
99     8469.7  0.262294   2.9079  0.86153  0.74930  2.9039  0.86172   2.9079   
100    6248.2  0.289584   2.7156  0.87068  0.76427  2.7137  0.87078   2.7156   

      A_x:20    5_E_x    10_E_x   20_E_x  
20   0.37829  0.78252  0.61224  0.37440  
21   0.37833  0.78250  0.61220  0.37429  
22   0.37837  0.78248  0.61215  0.37417  
23   0.37842  0.78245  0.61210  0.37404  
```

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```
24   0.37848  0.78243  0.61205  0.37390
..       ...      ...      ...      ...
96   0.83049  0.19872  0.01330  0.00000
97   0.84143  0.16765  0.00827  0.00000
98   0.85177  0.13850  0.00485  0.00000
99   0.86153  0.11173  0.00266  0.00000
100  0.87068  0.08777  0.00136  0.00000

[81 rows x 12 columns]
```

CHAPTER

FIFTEEN

SELECT LIFE TABLE

15.1 Select and ultimate life model

A newly selected policyholder is in the best health condition possible, compared to the general population with the same age. The life table can be expanded to tabulate the select period when selection has an effect on mortality. Since this selection process wears off after a few years, the ultimate part of the table can be then be used when select age is assumed to no longer have an effect on mortality.

- Future survival probabilities depend on the individual's current age and on the age at which the individual joined the group (i.e. was *selected*). Current age is written $[x] + s$, where x is the selected age and s is the number of years after selection.
- If an individual joined the group more than d years ago (called the *select period*), future survival probabilities (called the *ultimate mortality*) depend only on current age. The initial selection effect is assumed to have worn off after d years. Current age can be written as $x + s$ after the select period $s \geq d$

Select life tables reflect duration as well as age during the select period.

Notation for select survival models:

${}_tp_{[x]+s}$ = Pr(a life aged $x + s$, selected at age x, survives to age $x + s + t$)

- defines survival probability in the select period

${}_tq_{[x]+s}$ = Pr(a life aged $x + s$, selected at age x, dies before age $x + s + t$)

- defines mortality rate in the select period

$l_{[x]+s} = \dfrac{l_{x+d}}{{}_{d-s}p_{[x]+s}}$ = number of lives, selected at age x, who are aged $x + s$, given that l_{x+d} survived to age $x + d$.

- defines the life table within the select period, by working backwards from the value of l_{x+d} in the ultimate part of the table which only depends on current age.

With a select period d and for $s \geq d$ (i.e. durations beyond the select period) the values of $p_{[x-s]+s}, q_{[x-s]+s}, l_{[x-s]+s}$ depend only on current age x and not on s. So for $s \geq d$, these terms are all equal to and can be written simply as p_x, q_x, l_x respectively.

15.2 Methods

The `SelectLife` class implements methods to construct and fill in a select life table

```
from actuarialmath import SelectLife
import describe
describe.methods(SelectLife)
```

```
class SelectLife - Calculate select life table, and iteratively fill in missing␣
↪values

    Args:
      periods : number of select period years
      verbose : whether to echo update steps

    Notes:
      6 types of columns can be loaded and calculated in the select table:

      - 'q' : probability [x]+s dies in one year
      - 'l' : number of lives aged [x]+s
      - 'd' : number of deaths of age [x]+s
      - 'A' : whole life insurance
      - 'a' : whole life annuity
      - 'e' : expected future curtate lifetime of [x]+s

    Methods:
    --------

    set_table(fill, l, d, q, A, a, e):
      Update from table, every age has row for all select durations

    set_select(s, age_selected, fill, l, d, q, A, a, e):
      Update a table column, for a particular duration s in the select period

    fill_table(radix):
      Fills in missing table values (does not check for consistency)

    __getitem__(table):
      Returns values from a select and ultimate table

    frame(table):
      Returns select and ultimate table values as a DataFrame

    l_x(x, s):
      Returns number of lives aged [x]+s computed from select table

    p_x(x, s, t):
      t_p_[x]+s by chain rule: prod(1_p_[x]+s+y) for y in range(t)

    q_x(x, s, t, u):
      t|u_q_[x]+s = [x]+s survives u years, does not survive next t

    e_x(x, s, t, curtate):
      Returns expected life time computed from select table

    A_x(x, s, moment, discrete, kwargs):
```

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```
      Returns insurance value computed from select table

    a_x(x, s, moment, discrete, kwargs):
      Returns annuity value computed from select table
```

15.3 Examples

SOA Question 3.2:

You are given:

- The following extract from a mortality table with a one-year select period:

x	$l_{[x]}$	$d_{[x]}$	l_{x+1}	$x+1$
65	1000	40	–	66
66	955	45	–	67

- Deaths are uniformly distributed over each year of age

$\mathring{e}_{[65]} = 15.0$

Calculate $\mathring{e}_{[66]}$.

```
print("SOA Question 3.2:  (D) 14.7")
e_curtate = SelectLife.e_approximate(e_complete=15)
life = SelectLife(udd=True).set_table(l={65: [1000, None,],
                                         66: [955, None]},
                                      e={65: [e_curtate, None]},
                                      d={65: [40, None,],
                                         66: [45, None]})
print(life.e_r(66))
print(life.frame('e'))
```

```
SOA Question 3.2:  (D) 14.7
14.67801047120419
e_[x]+s:         0          1
Age
65        14.50000  14.104167
66        14.17801  13.879121
```

SOA Question 4.16

You are given the following extract of ultimate mortality rates from a two-year select and ultimate mortality table:

x	q_x
50	0.045
51	0.050
52	0.055
53	0.060

The select mortality rates satisfy the following:

- $q_{[x]} = 0.7q_x$
- $q_{[x]+1} = 0.8q_{x+1}$

You are also given that $i = 0.04$.

Calculate $A^{1}_{[50]:\overline{3}|}$.

```
print("SOA Question 4.16:  (D) .1116")
q = [.045, .050, .055, .060]
q_ = {50+x: [0.7 * q[x] if x < 4 else None,
             0.8 * q[x+1] if x+1 < 4 else None,
             q[x+2] if x+2 < 4 else None]
      for x in range(4)}
life = SelectLife().set_table(q=q_).set_interest(i=.04)
print(life.term_insurance(50, t=3))
```

```
SOA Question 4.16:  (D) .1116
0.1115661982248521
```

SOA Question 4.13

For a 2-year deferred, 2-year term insurance of 2000 on [65], you are given:

- The following select and ultimate mortality table with a 3-year select period:

x	$q_{[x]}$	$q_{[x]+1}$	$q_{[x]+2}$	q_{x+3}	$x+3$
65	0.08	0.10	0.12	0.14	68
66	0.09	0.11	0.13	0.15	69
67	0.10	0.12	0.14	0.16	70
68	0.11	0.13	0.15	0.17	71
69	0.12	0.14	0.16	0.18	72

- $i = 0.04$
- The death benefit is payable at the end of the year of death

Calculate the actuarial present value of this insurance.

```
print("SOA Question 4.13:  (C) 350 ")
life = SelectLife().set_interest(i=0.04)\
                   .set_table(q={65: [.08, .10, .12, .14],
                                 66: [.09, .11, .13, .15],
                                 67: [.10, .12, .14, .16],
                                 68: [.11, .13, .15, .17],
                                 69: [.12, .14, .16, .18]})
print(life.deferred_insurance(65, t=2, u=2, b=2000))
```

```
SOA Question 4.13:  (C) 350
351.0578236056159
```

SOA Question 3.13

A life is subject to the following 3-year select and ultimate table:

$[x]$	$\ell_{[x]}$	$\ell_{[x]+1}$	$\ell_{[x]+2}$	ℓ_{x+3}	$x+3$
55	10,000	9,493	8,533	7,664	58
56	8,547	8,028	6,889	5,630	59
57	7,011	6,443	5,395	3,904	60
58	5,853	4,846	3,548	2,210	61

You are also given:

- $e_{60} = 1$
- Deaths are uniformly distributed over each year of age

Calculate $\mathring{e}_{[58]+2}$.

```
print("SOA Question 3.13:  (B) 1.6")
life = SelectLife().set_table(l={55: [10000, 9493, 8533, 7664],
                                 56: [8547, 8028, 6889, 5630],
                                 57: [7011, 6443, 5395, 3904],
                                 58: [5853, 4846, 3548, 2210]},
                              e={57: [None, None, None, 1]})
print(life.e_r(58, s=2))
```

```
SOA Question 3.13:  (B) 1.6
1.6003382187147688
```

SOA Question 3.12

X and Y are both age 61. X has just purchased a whole life insurance policy. Y purchased a whole life insurance policy one year ago.

Both X and Y are subject to the following 3-year select and ultimate table:

x	$\ell_{[x]}$	$\ell_{[x]+1}$	$\ell_{[x]+2}$	ℓ_{x+3}	$x+3$
60	10,000	9,600	8,640	7,771	63
61	8,654	8,135	6,996	5,737	64
62	7,119	6,549	5,501	4,016	65
63	5,760	4,954	3,765	2,410	66

The force of mortality is constant over each year of age.

Calculate the difference in the probability of survival to age 64.5 between X and Y.

```
print("SOA Question 3.12: (C) 0.055 ")
life = SelectLife(udd=False).set_table(l={60: [10000, 9600, 8640, 7771],
                                          61: [8654, 8135, 6996, 5737],
                                          62: [7119, 6549, 5501, 4016],
                                          63: [5760, 4954, 3765, 2410]})
print(life.q_r(60, s=1, t=3.5) - life.q_r(61, s=0, t=3.5))
```

```
SOA Question 3.12: (C) 0.055
0.05465655938591829
```

SOA Question 3.7

For a mortality table with a select period of two years, you are given:

x	$q_{[x]}$	$q_{[x]+1}$	q_{x+2}	$x+2$
50	0.0050	0.0063	0.0080	52
51	0.0060	0.0073	0.0090	53
52	0.0070	0.0083	0.0100	54
53	0.0080	0.0093	0.0110	55

The force of mortality is constant between integral ages.

Calculate $1000\ {}_{2.5}q_{[50]+0.4}$.

```
print("SOA Question 3.7: (b) 16.4")
life = SelectLife().set_table(q={50: [.0050, .0063, .0080],
                                 51: [.0060, .0073, .0090],
                                 52: [.0070, .0083, .0100],
                                 53: [.0080, .0093, .0110]})
print(1000*life.q_r(50, s=0, r=0.4, t=2.5))
```

```
SOA Question 3.7: (b) 16.4
16.420207214428586
```

SOA Question 3.6

You are given the following extract from a table with a 3-year select period:

x	$q_{[x]}$	$q_{[x]+1}$	$q_{[x]+2}$	q_{x+3}	$x+3$
60	0.09	0.11	0.13	0.15	63
61	0.10	0.12	0.14	0.16	64
62	0.11	0.13	0.15	0.17	65
63	0.12	0.14	0.16	0.18	66
64	0.13	0.15	0.17	0.19	67

$e_{64} = 5.10$

Calculate $e_{[61]}$.

```
print("SOA Question 3.6:  (D) 5.85")
life = SelectLife().set_table(q={60: [.09, .11, .13, .15],
                              61: [.1, .12, .14, .16],
                              62: [.11, .13, .15, .17],
                              63: [.12, .14, .16, .18],
                              64: [.13, .15, .17, .19]},
                           e={61: [None, None, None, 5.1]})
print(life.e_x(61))
```

```
SOA Question 3.6:  (D) 5.85
5.846832
```

SOA Question 3.3

You are given:

- An excerpt from a select and ultimate life table with a select period of 2 years:

x	$\ell_{[x]}$	$\ell_{[x]+1}$	ℓ_{x+2}	$x+2$
50	99,000	96,000	93,000	52
51	97,000	93,000	89,000	53
52	93,000	88,000	83,000	54
53	90,000	84,000	78,000	55

- Deaths are uniformly distributed over each year of age

Calculate $10,000\ {}_{2.2}q_{[51]+0.5}$.

```
print("SOA Question 3.3:  (E) 1074")
life = SelectLife().set_table(l={50: [99, 96, 93],
                                 51: [97, 93, 89],
                                 52: [93, 88, 83],
                                 53: [90, 84, 78]})
print(10000*life.q_r(51, s=0, r=0.5, t=2.2))
```

```
SOA Question 3.3:  (E) 1074
1073.684210526316
```

SOA Question 3.1

You are given:

- An excerpt from a select and ultimate life table with a select period of 3 years:

x	$\ell_{[x]}$	$\ell_{[x]+1}$	$\ell_{[x]+2}$	ℓ_{x+3}	$x+3$
60	80,000	79,000	77,000	74,000	63
61	78,000	76,000	73,000	70,000	64
62	75,000	72,000	69,000	67,000	65
63	71,000	68,000	66,000	65,000	66

- Deaths follow a constant force of mortality over each year of age

Calculate $1000\ {}_{23}q_{[60]+0.75}$.

```
print("SOA Question 3.1:  (B) 117")
life = SelectLife().set_table(l={60: [80000, 79000, 77000, 74000],
                                    61: [78000, 76000, 73000, 70000],
                                    62: [75000, 72000, 69000, 67000],
                                    63: [71000, 68000, 66000, 65000]})
print(1000*life.q_r(60, s=0, r=0.75, t=3, u=2))
```

```
SOA Question 3.1:  (B) 117
116.7192429022082
```

show verbose calculations:

```
table={21: [0.00120, 0.00150, 0.00170, 0.00180],
       22: [0.00125, 0.00155, 0.00175, 0.00185],
       23: [0.00130, 0.00160, 0.00180, 0.00195]}
life = SelectLife(verbose=True).set_table(q=table)
print(life.p_x(x=21, s=1, t=4))   # 0.9931
```

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```
life.frame('l')
```

```
1 l(x=21, s=1) = 99880.0
2 l(x=21, s=2) = 99730.18000000001
3 l(x=21, s=3) = 99560.63869400001
4 l(x=22, s=3) = 99381.4295443508
5 l(x=23, s=3) = 99197.57389969376
6 d(x=21, s=0) = 120.0
7 d(x=21, s=1) = 149.81999999999243
8 d(x=21, s=2) = 169.54130599999917
9 d(x=21, s=3) = 179.20914964920667
10 d(x=22, s=3) = 183.85564465704374
11 l(x=22, s=2) = 99555.65193523747
12 l(x=23, s=2) = 99376.45151241611
13 d(x=22, s=2) = 174.22239088667266
14 d(x=23, s=2) = 178.87761272235366
15 l(x=22, s=1) = 99710.2027494992
16 l(x=23, s=1) = 99535.70864625012
17 d(x=22, s=1) = 154.55081426173274
18 d(x=23, s=1) = 159.2571338340058
19 l(x=22, s=0) = 99834.9964951181
20 l(x=23, s=0) = 99665.27350180245
21 d(x=22, s=0) = 124.79374561889563
22 d(x=23, s=0) = 129.56485555233667
0.9931675400449915
```

```
l_[x]+s:             0            1            2            3
Age
21       100000.000000  99880.000000  99730.180000  99560.638694
22        99834.996495  99710.202749  99555.651935  99381.429544
23        99665.273502  99535.708646  99376.451512  99197.573900
```

CHAPTER

SIXTEEN

MORTALITY LAWS

When using special mortality laws for lifetime distribution, shortcut formulas may be available.

16.1 Beta distribution

With two parameters α, ω:

$l_x \sim (\omega - x)^\alpha$

$\mu_x = \frac{\alpha}{\omega - x}$

${}_tp_x = (\frac{\omega - (x+t)}{\omega - x})^\alpha$

$\mathring{e}_x = \frac{\omega - x}{\alpha + 1}$

16.2 Uniform distribution

Is Beta distribution with $\alpha = 1$

$l_x \sim \omega - x$

$\mu_x = \frac{1}{\omega - x}$

${}_tp_x = \frac{\omega - (x+t)}{\omega - x}$

$\mathring{e}_x = \frac{\omega - x}{2}$

$\mathring{e}_{x:\overline{n|}} = \ {}_np_x \ n + \ {}_nq_x \ \frac{n}{2}$

${}_nE_x = v^n \frac{\omega - (x+n)}{\omega - x}$

$\bar{A}_x = \frac{\bar{a}_{\overline{\omega - x|}}}{\omega - x}$

$\bar{A}^1_{x:\overline{n|}} = \frac{\bar{a}_{\overline{n|}}}{\omega - x}$

16.3 Makeham's Law

With three parameters $c > 1,\ B > 0,\ A \geq -B$, it includes an element in the force of mortality that does not depend on age.

$\mu_x = A + Bc^x$

${}_tp_x = e^{\frac{Bc^x}{\ln c}(c^t - 1) - At}$

16.4 Gompertz's Law

Is Makeham's Law with $A = 0$

$\mu_x = Bc^x$

${}_tp_x = e^{\frac{Bc^x}{\ln c}(c^t - 1)}$

16.5 Methods

The `MortalityLaws` class, and `Beta`, `Uniform`, `Makeham` and `Gompertz` subclasses, implement methods to apply shortcut equations that are available when assuming these special mortality laws for the distribution of future lifetime.

```
from actuarialmath import MortalityLaws, Uniform, Beta, Makeham, Gompertz
import describe
describe.methods(MortalityLaws)
describe.methods(Beta)
describe.methods(Uniform)
describe.methods(Makeham)
describe.methods(Gompertz)
```

```
class MortalityLaws - Apply shortcut formulas for special mortality laws

    Methods:
    --------

    l_r(x, s, r):
      Fractional lives given special mortality law: l_[x]+s+r

    p_r(x, s, r, t):
      Fractional age survival probability given special mortality law

    q_r(x, s, r, t, u):
      Fractional age deferred mortality given special mortality law

    mu_r(x, s, r):
      Fractional age force of mortality given special mortality law

    f_r(x, s, r, t):
      fractional age lifetime density given special mortality law

    e_r(x, s, r, t):
      Fractional age future lifetime given special mortality law
```

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```
class Beta - Shortcuts with beta distribution of deaths (is Uniform when alpha = 1)

    Args:
      omega : maximum age
      alpha : alpha paramter of beta distribution
      lives : assumed starting number of lives for survival function

    Examples:
      >>> life = Beta(omega=100, alpha=0.5)
      >>> print(life.q_x(25, t=1, u=10))     # 0.0072
      >>> print(life.e_x(25))                # 50
      >>> print(Beta(omega=60, alpha=1/3).mu_x(35) * 1000)


class Uniform - Shortcuts with uniform distribution of deaths aka DeMoivre's Law

    Args:
      omega : maximum age
      udd : assume UDD (True, default) or CFM (False) between integer ages

    Examples:
      >>> uniform = Uniform(80).set_interest(delta=0.04)
      >>> print(uniform.whole_life_annuity(20, discrete=False))        # 15.53
      >>> print(uniform.temporary_annuity(20, t=5, discrete=False))   # 4.35
      >>> print(Uniform(161).p_x(70, t=1)) # 0.98901
      >>> print(Uniform(95).e_x(30, t=40, curtate=False)) # 27.692


class Makeham - Includes element in force of mortality that does not depend on age

    Args:
      A, B, c : parameters of Makeham distribution

    Examples:
      >>> life = Makeham(A=0.00022, B=2.7e-6, c=1.124)
      >>> print(life.mu_x(60) * 0.9803)  # 0.00316


class Gompertz - Is Makeham's Law with A = 0

    Args:
      B, c : parameters of Gompertz distribution

    Examples:
      >>> life = Gompertz(B=0.000005, c=1.10)
      >>> p = life.p_x(80, t=10)  # 869.4
      >>> print(life.portfolio_percentile(N=1000, mean=p, variance=p*(1-p), prob=0.
↪99))
      >>> print(Gompertz(B=0.00027, c=1.1).f_x(50, t=10)) # 0.04839
```

16.6 Examples

SOA Question 2.3:

You are given that mortality follows Gompertz Law with B = 0.00027 and c = 1.1. Calculate $f_{50}(10)$.

```
print("SOA Question 2.3: (A) 0.0483")
print(Gompertz(B=0.00027, c=1.1).f_x(x=50, t=10))
```

```
SOA Question 2.3: (A) 0.0483
0.048389180223511644
```

SOA Question 2.6

You are given the survival function:

$S_0(x) = \left(1 - \frac{x}{60}\right)^{\frac{1}{3}}, \quad 0 \leq x \leq 60$

Calculate $1000\mu_{35}$.

```
print("# SOA Question 2.6: (C) 13.3")
print(Beta(omega=60, alpha=1/3).mu_x(35) * 1000)
```

```
# SOA Question 2.6: (C) 13.3
13.333333333333332
```

Beta distribution:

```
life = Beta(omega=100, alpha=0.5)
print(life.q_x(25, t=1, u=10))                      # 0.0072
print(life.e_x(25))                                 # 50
print(Beta(omega=60, alpha=1/3).mu_x(35) * 1000)   # 13.33
```

```
0.007188905547861446
50.0
13.333333333333332
```

Uniform distribution:

```
print('Uniform')
uniform = Uniform(80).set_interest(delta=0.04)
print(uniform.whole_life_annuity(20))             # 15.53
print(uniform.temporary_annuity(20, t=5))         # 4.35
print(Uniform(161).p_x(70, t=1))                  # 0.98901
print(Uniform(95).e_x(30, t=40, curtate=False)) # 27.692
print()

uniform = Uniform(omega=80).set_interest(delta=0.04)
print(uniform.E_x(20, t=5))   # .7505
print(uniform.whole_life_insurance(20, discrete=False))       # .3789
print(uniform.term_insurance(20, t=5, discrete=False))        # .0755
print(uniform.endowment_insurance(20, t=5, discrete=False))   # .8260
print(uniform.deferred_insurance(20, u=5, discrete=False))    # .3033
```

```
Uniform
16.03290804858584
4.47503070125663
0.989010989010989
32.30769230769231

0.7505031903214833
0.378867519462745
0.07552885288417432
0.8260320432056576
0.30333866657857067
```

Makeham's and Gompertz's Laws:

```
life = Gompertz(B=0.000005, c=1.10)
p = life.p_x(80, t=10)  # 869.4
print(life.portfolio_percentile(N=1000, mean=p, variance=p*(1-p), prob=0.99))

print(Gompertz(B=0.00027, c=1.1).f_x(50, t=10)) # 0.04839
life = Makeham(A=0.00022, B=2.7e-6, c=1.124)
print(life.mu_x(60) * 0.9803)  # 0.00316
```

```
869.3908338193208
0.048389180223511644
0.0031580641631654026
```

CHAPTER

SEVENTEEN

CONSTANT FORCE OF MORTALITY

Shortcut formulas by assuming an exponential distribution (constant force of mortality) for future lifetime.

$_tp_x = e^{-\mu t}$

- survival functions do not depend on age x

17.1 Expected future lifetime

$\mathring{e}_x = \dfrac{1}{\mu}$

- expected future lifetime does not depend on age x

$\mathring{e}_{x:\overline{n}|} = \dfrac{1}{\mu}(1 - e^{-\mu n})$

- expected temporary future life time does not depend on age

$Var(T_x) = \dfrac{1}{\mu^2}$

- variance of future lifetime

17.2 Pure endowment

$_nE_x = e^{-(\mu+\delta)n}$

17.3 Life insurance

$\bar{A}_x = \dfrac{\mu}{\mu + \delta}$

- whole life insurance

$\bar{A}_{x:\overline{t}|} = \dfrac{\mu}{\mu + \delta}(1 - e^{-\mu t})$

- term life insurance

17.4 Life annuities

$$\bar{a}_x = \frac{1}{\mu + \delta}$$

- whole life annuity

$$\bar{a}_{x:\overline{t|}} = \frac{1}{\mu + \delta}(1 - e^{-\mu t})$$

- temporary life annuity

17.5 Methods

The `ConstantForce` class implements methods that apply shortcut formulas available when assuming constant force of mortality for the distribution of future lifetime

```
from actuarialmath import ConstantForce
import describe
describe.methods(ConstantForce)
```

```
class ConstantForce - Constant force of mortality - memoryless exponential␣
↪distribution of lifetime

    Args:
      mu : constant value of force of mortality
      udd : assume UDD (True) or CFM (False, default) between integer ages

    Examples:
      >>> life = ConstantForce(mu=0.01).set_interest(delta=0.05)
      >>> life.term_insurance(35, t=35, discrete=False) + life.E_x(35, t=35)*0.
↪51791

    Methods:
    --------

    e_x(x, s, t, curtate, moment):
      Expected lifetime E[T_x] is memoryless: does not depend on (x)

    E_x(x, s, t, endowment, moment):
      Shortcut for pure endowment: does not depend on age x

    whole_life_insurance(x, s, moment, b, discrete):
      Shortcut for APV of whole life: does not depend on age x

    temporary_annuity(x, s, t, b, variance, discrete):
      Shortcut for temporary life annuity: does not depend on age x

    term_insurance(x, s, t, b, moment, discrete):
      Shortcut for APV of term life: does not depend on age x

    Z_t(x, prob, discrete):
      Shortcut for T_x (or K_x) given survival probability for insurance
```

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```
    Y_t(x, prob, discrete):
      Shortcut for T_x (or K_x) given survival probability for annuity
```

17.6 Examples

SOA Question 6.36

For a fully continuous 20-year term insurance policy of 100,000 on (50), you are given:

- Gross premiums, calculated using the equivalence principle, are payable at an annual rate of 4500
- Expenses at an annual rate of R are payable continuously throughout the life of the policy
- $\mu_{50+t} = 0.04$, for $t > 0$
- $\delta = 0.08$

Calculate R.

```python
print("SOA Question 6.36:  (B) 500")
life = ConstantForce(mu=0.04).set_interest(delta=0.08)
a = life.temporary_annuity(50, t=20, discrete=False)
A = life.term_insurance(50, t=20, discrete=False)
def fun(R):
    return life.gross_premium(a=a, A=A, initial_premium=R/4500,
                              renewal_premium=R/4500, benefit=100000)
R = life.solve(fun, target=4500, grid=[400, 800])
print(R)
```

```
SOA Question 6.36:  (B) 500
500.0
```

SOA Question 6.31

For a fully continuous whole life insurance policy of 100,000 on (35), you are given:

- The density function of the future lifetime of a newborn: $$
- $\delta = 0.05$
- $\overline{A}_{70} = 0.51791$

Calculate the annual net premium rate for this policy.

```python
print("SOA Question 6.31:  (D) 1330")
life = ConstantForce(mu=0.01).set_interest(delta=0.05)
A = life.term_insurance(35, t=35) + life.E_x(35, t=35) * 0.51791 # A_35
A = (life.term_insurance(35, t=35, discrete=False)
        + life.E_x(35, t=35) * 0.51791)     # A_35
P = life.premium_equivalence(A=A, b=100000, discrete=False)
print(P)
```

```
SOA Question 6.31:  (D) 1330
1326.5406293909457
```

SOA Question 6.27

For a special fully continuous whole life insurance on (x), you are given:

- Premiums and benefits:

	First 20 years	After 20 years
Premium Rate	3P	P
Benefit	1,000,000	500,000

- $\mu_{x+t} = 0.03, \quad t \geq 0$
- $\delta = 0.06$

Calculate P using the equivalence principle.

```
print("SOA Question 6.27:  (D) 10310")
life = ConstantForce(mu=0.03).set_interest(delta=0.06)
x = 0
payments = (3 * life.temporary_annuity(x, t=20, discrete=False)
            + life.deferred_annuity(x, u=20, discrete=False))
benefits = (1000000 * life.term_insurance(x, t=20, discrete=False)
            + 500000 * life.deferred_insurance(x, u=20, discrete=False))
P = benefits / payments
print(P)
```

```
SOA Question 6.27:  (D) 10310
10309.617799001708
```

SOA Question 5.4

(40) wins the SOA lottery and will receive both:

- A deferred life annuity of K per year, payable continuously, starting at age $40 + \mathring{e}_{40}$ and
- An annuity certain of K per year, payable continuously, for $\mathring{e}_{40}$ years

You are given:

- $\mu = 0.02$
- $\delta = 0.01$
- The actuarial present value of the payments is 10,000

Calculate K.

```
print("SOA Question 5.4:  (A) 213.7")
life = ConstantForce(mu=0.02).set_interest(delta=0.01)
P = 10000 / life.certain_life_annuity(40, u=life.e_x(40, curtate=False),
                                       discrete=False)
print(P)
```

```
SOA Question 5.4:  (A) 213.7
213.74552118275955
```

SOA Question 5.1

You are given:

- $\delta_t = 0.06, \quad t \geq 0$
- $\mu_x(t) = 0.01, \quad t \geq 0$
- Y is the present value random variable for a continuous annuity of 1 per year, payable for the lifetime of (x) with 10 years certain

Calculate $Pr(Y > E[Y])$.

```
print("SOA Question 5.1: (A) 0.705")
life = ConstantForce(mu=0.01).set_interest(delta=0.06)
EY = life.certain_life_annuity(0, u=10, discrete=False)
print(life.p_x(0, t=life.Y_to_t(EY)))  # 0.705
```

```
SOA Question 5.1: (A) 0.705
0.7053680433746505
```

CHAPTER

EIGHTEEN

EXTRA RISK

If underwriting determines that an individual should offered insurance but at above standard rates, there are different ways in which we can model the extra mortality risk in a premium calculation.

18.1 Age rating

$(x) \leftarrow (x + k)$

- add years to age, referred to as age rating: the insurer may compensate for extra risk by treating the individual as being older, for example, an impaired life aged 40 might be asked to pay the same premium paid by a non- impaired life aged 45.

18.2 Multiple of mortality rate

$q_x \leftarrow q_x \cdot k$

- multiply mortality rate by a constant, which assumes that lives are subject to mortality rates that are higher than the standard lives' mortality rates.

18.3 Force of mortality

$\mu_{x+t} \leftarrow \mu_{x+t} + k \Rightarrow \ {}_tp_x \leftarrow \ {}_tp_x \ e^{-kt}$

- add constant to force of mortality, when the extra risk is largely independent of age

$\mu_{x+t} \leftarrow \mu_{x+t} \cdot k \Rightarrow \ {}_tp_x \leftarrow \ ({}_tp_x)^k$

- multiply force of mortality by constant

18.4 Methods

The `ExtraRisk` class implements methods to adjust the survival or mortality function by extra risks.

```
from actuarialmath import ExtraRisk, SelectLife, SULT
import describe
describe.methods(ExtraRisk)
```

```
class ExtraRisk - Adjust mortality by extra risk

    Args:
      life : contains original survival and mortality rates
      extra : amount of extra risk to adjust
      risk : adjust by {"ADD_FORCE", "MULTIPLY_FORCE", "ADD_AGE" or "MULTIPLY_RATE
↪"}

    Methods:
    --------

    q_x(x, s):
      Return q_[x]+s after adding age rating or multipliying mortality rate

    p_x(x, s):
      Return p_[x]+s after adding or multiplying force of mortality

    __getitem__(col):
      Returns survival function values adjusted by extra risk
```

18.5 Examples

SOA Question 5.5

For an annuity-due that pays 100 at the beginning of each year that (45) is alive, you are given:

- Mortality for standard lives follows the Standard Ultimate Life Table
- The force of mortality for standard lives age 45 + t is represented as μ_{45+t}^{SULT}
- The force of mortality for substandard lives age 45 + t, μ_{45+t}^{S}, is defined as:

$$\begin{aligned}\mu_{45+t}^{S} &= \mu_{45+t}^{SULT} + 0.05, \quad 0 \leq t < 1\\ &= \mu_{45+t}^{SULT}, \quad t \geq 1\end{aligned}$$

- $i = 0.05$

Calculate the actuarial present value of this annuity for a substandard life age 45.

```
print("SOA Question 5.5: (A) 1699.6")
life = SULT()
extra = ExtraRisk(life=life, extra=0.05, risk="ADD_FORCE")
select = SelectLife(periods=1)\
    .set_interest(i=.05)\
    .set_select(s=0, age_selected=True, q=extra['q'])\
    .set_select(s=1, age_selected=False, a=life['a'])\
    .fill_table()
print(100*select['a'][45][0])
```

```
SOA Question 5.5: (A) 1699.6
1699.6076593190103
```

SOA Question 4.19

(80) purchases a whole life insurance policy of 100,000. You are given:

- The policy is priced with a select period of one year
- The select mortality rate equals 80% of the mortality rate from the Standard Ultimate Life Table
- Ultimate mortality follows the Standard Ultimate Life Table
- $i = 0.05$

Calculate the actuarial present value of the death benefits for this insurance

```
print("SOA Question 4.19:  (B) 59050")
life = SULT()
extra = ExtraRisk(life=life, extra=0.8, risk="MULTIPLY_RATE")
select = SelectLife(periods=1)\
    .set_interest(i=.05)\
    .set_select(s=0, age_selected=True, q=extra['q'])\
    .set_select(s=1, age_selected=False, q=life['q'])\
    .fill_table()
print(100000*select.whole_life_insurance(80, s=0))
```

```
SOA Question 4.19:  (B) 59050
59050.59973285648
```

Other examples

```
life = SULT()
extra = ExtraRisk(life=life, extra=2, risk="MULTIPLY_FORCE")
print(life.p_x(45), extra.p_x(45))
```

```
0.9992288829941123 0.9984583606096613
```

CHAPTER

NINETEEN

1/M'THLY

A new lifetime random variable is introduced to value benefits which depend on the number of complete periods of length $1/m$ years lived by a life (x).

$K_x^{(m)} = \dfrac{1}{m}\lfloor mT_x \rfloor$

- 1/mthly curtate future lifetime random variable, where $m > 1$ is an integer, is the future lifetime of (x) in years rounded to the lower $\frac{1}{m}$ th of a year.

$Pr[K_x^{(m)} = k] = Pr[k \le T_x \le k + \dfrac{1}{m}] = \ _{k|\frac{1}{m}}q_x = \ _kp_x - \ _{k+\frac{1}{m}}p_x$

- the probability function for $K_x^{(m)}$ can be derived from the associated probabilities for T_x.

19.1 Life Insurance

Whole life insurance

$Z = v^{K_x^{(k)}+1/m}$

- present value random variable of 1/mthly whole life insurance

$A_x^{(m)} = E[Z] = \sum_{k=0}^{\infty} v^{\frac{k+1}{m}} \ _{\frac{k}{m}|\frac{1}{m}}q_x$

- 1/m'thly whole life insurance

$E[Z^2] = E[(v^2)^{K_x^{(k)}+1/m}] = \ ^2A_x^{(m)}$

- second moment is also obtained from $A_x^{(m)}$ at double the force of interest

$Var(Z) = E[Z^2] - E[Z]^2 = \ ^2A_x^{(m)} - (A_x^{(m)})^2$

- the variance of the present value of the $1/m$'thly insurance benefit can be derived by adjusting the interest rate in the first time

Term life insurance

$Z = 0$ if $K_x^{(m)} \ge t$, else $v^{K_x^{(k)}+1/m}$

- death benefit is payable at the end of the 1/m-th year of death , provided this occurs within t years.

$A_{x:\overline{t|}}^{1^{(m)}} = \sum_{k=0}^{mt-1} v^{\frac{k+1}{m}} \ _{\frac{k}{m}|\frac{1}{m}}q_x$

- EPV of 1/m-thly term insurance benefits

19.2 Life Annuity

$\ddot{a}_x^{(m)} = \sum_{k=0}^{\infty} \dfrac{1}{m}\, v^{\frac{k}{m}} \,\, _{\frac{k}{m}}p_x$

- 1/mthly whole life annuity

$\ddot{a}_{x:\overline{t|}}^{(m)} = \sum_{k=0}^{mt-1} \dfrac{1}{m}\, v^{\frac{k}{m}} \,\, _{\frac{k}{m}}p_x$

- 1/mthly temporary life annuity

$a_x^{(m)} = \ddot{a}_x^{(m)} - \dfrac{1}{m}$

- immediate 1/m'thly whole life annuity

$a_{x:\overline{t|}}^{(m)} = \ddot{a}_{x:\overline{t|}}^{(m)} - \dfrac{1}{m}(1 - \, _tE_x)$

- immediate 1/m'thly temporary life annuity

19.3 Life Insurance Twin

Whole and *Temporary Life Annuities* (and *Whole Life* and *Endowment Insurance*) **ONLY**:

$A_x^{(m)} = 1 - d^{(m)}\, \ddot{a}_x^{(m)} \iff \ddot{a}_x^{(m)} = \dfrac{1 - A_x^{(m)}}{d^{(m)}}$

- 1/m'thly whole life annuity due

$A_{x:\overline{t|}}^{(m)} = 1 - d^{(m)}\, \ddot{a}_{x:\overline{t|}}^{(m)} \iff \ddot{a}_{x:\overline{t|}}^{(m)} = \dfrac{1 - A_{x:\overline{t|}}^{(m)}}{d^{(m)}}$

- 1/m'thly temporary annuity due and endowment insurance

19.4 Methods

This `Mthly` class implements methods to compute life insurance and annuity values with 1/mthly benefits.

```
from actuarialmath import Mthly, Premiums, LifeTable
import describe
describe.methods(Mthly)
```

```
class Mthly - Compute 1/M'thly insurance and annuities

    Args:
      m : number of payments per year
      life : original survival and life contingent functions

    Methods:
    --------

    v_m(k):
      Compute discount rate compounded over k m'thly periods
```

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```
p_m(x, s_m, t_m):
  Compute survival probability over m'thly periods

q_m(x, s_m, t_m, u_m):
  Compute deferred mortality over m'thly periods

Z_m(x, s, t, benefit, moment):
  Return PV of insurance r.v. Z and probability of death at mthly intervals

E_x(x, s, t, moment, endowment):
  Compute pure endowment factor

A_x(x, s, t, u, benefit, moment):
  Compute insurance factor with m'thly benefits

whole_life_insurance(x, s, moment, b):
  Whole life insurance: A_x

term_insurance(x, s, t, b, moment):
  Term life insurance: A_x:t^1

deferred_insurance(x, s, n, b, t, moment):
  Deferred insurance n|_A_x:t^1 = discounted whole life

endowment_insurance(x, s, t, b, endowment, moment):
  Endowment insurance: A_x:t = term insurance + pure endowment

immediate_annuity(x, s, t, b):
  Immediate m'thly annuity

insurance_twin(a):
  Return insurance twin of m'thly annuity

annuity_twin(A):
  Return value of annuity twin of m'thly insurance

annuity_variance(A2, A1, b):
  Variance of m'thly annuity from m'thly insurance moments

whole_life_annuity(x, s, b, variance):
  Whole life m'thly annuity: a_x

temporary_annuity(x, s, t, b, variance):
  Temporary m'thly life annuity: a_x:t

deferred_annuity(x, s, u, t, b):
  Deferred m'thly life annuity due n|t_a_x =  n+t_a_x - n_a_x

immediate_annuity(x, s, t, b):
  Immediate m'thly annuity
```

19.5 Examples

SOA Question 6.4

For whole life annuities-due of 15 per month on each of 200 lives age 62 with independent future lifetimes, you are given:

- $i = 0.06$
- $A_{62}^{(12)} = 0.2105$ and ${}^2A_{62}^{(12)} = 0.4075$
- π is the single premium to be paid by each of the 200 lives
- S is the present value random variable at time 0 of total payments made to the 200 lives

Using the normal approximation, calculate π such that $Pr(200\pi > S) = 0.90$.

```
print("SOA Question 6.4:  (E) 1893.9")
mthly = Mthly(m=12, life=Premiums().set_interest(i=0.06))
A1, A2 = 0.4075, 0.2105
mean = mthly.annuity_twin(A1)*15*12
var = mthly.annuity_variance(A1=A1, A2=A2, b=15 * 12)
S = Premiums.portfolio_percentile(mean=mean, variance=var, prob=.9, N=200)
print(S / 200)
```

```
SOA Question 6.4:  (E) 1893.9
1893.912859650868
```

SOA Question 4.2

For a special 2-year term insurance policy on (x), you are given:

- Death benefits are payable at the end of the half-year of death
- The amount of the death benefit is 300,000 for the first half-year and increases by 30,000 per half-year thereafter
- $q_x = 0.16$ and $q_{x+1} = 0.23$
- $i^{(2)} = 0.18$
- Deaths are assumed to follow a constant force of mortality between integral ages
- Z is the present value random variable for this insurance

Calculate $Pr(Z > 277,000)$.

```
print("SOA Question 4.2:  (D) 0.18")
life = LifeTable(udd=False).set_table(q={0: 0.16, 1: 0.23})\
                           .set_interest(i_m=0.18, m=2)
mthly = Mthly(m=2, life=life)
Z = mthly.Z_m(0, t=2, benefit=lambda x,t: 300000 + t*30000*2)
print(Z)
print(Z[Z['Z'] >= 277000]['q'].sum())
```

```
SOA Question 4.2:  (D) 0.18
               Z         q
m
1  275229.357798  0.083485
2  277754.397778  0.076515
3  277986.052822  0.102903
```

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```
4  276285.832315  0.090297
0.17941813045022975
```

CHAPTER

TWENTY

UDD M'THLY

With the UDD fractional age assumption, we can work with annual insurance and annuity factors A_x and $\ddot{a}_x$, then adjust for a more appropriate frequency $A_x^{(m)}$ and $\ddot{a}_x^{(m)}$ using the following relationships.

20.1 Life insurance

Under UDD, the values of annual A_x can be used to derive exact results for 1/mthly insurance $A_x^{(m)}$.

$A_x^{(m)} = \dfrac{i}{i^{(m)}} A_x$

- discrete whole life insurance

$A^{1\ (m)}_{x:\overline{t|}} = \dfrac{i}{i^{(m)}} A^{1}_{x:\overline{t|}}$

- discrete term insurance

$A^{(m)}_{x:\overline{t|}} = \dfrac{i}{i^{(m)}} A^{1}_{x:\overline{t|}} + \ _tE_x$

- endowment insurance combines the death and survival benefits, so we need to split off the death benefit to apply the derivations.

$_{u|}A_x^{(m)} = \ _uE_x \dfrac{i}{i^{(m)}} A_{x+u}$

- discrete deferred insurance

Double the force of interest

$^2A_x^{(m)} = \dfrac{i^2 - 2i}{(i^{(m)})^2 - 2i^{(m)}} \ ^2A_x$

- relate to doubling the force of interest for annual whole life insurance

20.2 Continuous Life Insurance

Under UDD, continuous life insurance can also be related to annual life insurance factors

$\overline{A}_x = \dfrac{i}{\delta} A_x$

- whole life insurance

$\overline{A}^{1}_{x:\overline{t|}} = \dfrac{i}{\delta} A^{1}_{x:\overline{t|}}$

- term life insurance

$\overline{A}_{x:\overline{t|}} = \dfrac{i}{\delta} A^{1}_{x:\overline{t|}} + {}_tE_x$

- endowment insurance

${}_{u|}\overline{A}_x = {}_uE_x \, \dfrac{i}{\delta} A_{x+u}$

- deferred life insurance

Double the force of interest

${}^2\overline{A}_x = \dfrac{i^2 - 2i}{2\delta} \, {}^2A_x$

- relate to doubling the force of interest for annual whole life insurance

20.3 Interest functions

It can be shown that by subtituting in annuity twins in the above relationships under UDD, values of 1/mthly life annuities can be adjusted from annual life annuity factors using interest rate functions $\alpha(m)$ and $\beta(m)$

$\alpha(m) = \dfrac{id}{i^{(m)} \, d^{(m)}}$

$\beta(m) = \dfrac{i - i^{(m)}}{i^{(m)} \, d^{(m)}}$

20.4 Life annuities

Using the values of $\ddot{a}_x$, and the interest rate functions, to obtain $\ddot{a}^{(m)}_x$ under UDD.

$\ddot{a}^{(m)}_x = \alpha(m) \, \ddot{a}_x - \beta(m)$

- whole life annuity

$\ddot{a}^{(m)}_{x:\overline{n|}} = \alpha(m) \, \ddot{a}_{x:\overline{n|}} - \beta(m)(1 - {}_tE_x)$

- temporary life annuity

${}_{u|}\ddot{a}^{(m)}_x = \alpha(m) \, {}_{u|}\ddot{a}_x - \beta(m) \, {}_uE_x$

- deferred whole life annuity

20.5 Methods

The `UDD` class implements methods to compute life insurance and annuities with 1/mthly benefits assuming uniform distribution of deaths (UDD) between integer ages.

```python
from actuarialmath import UDD, SULT, Recursion, Contract
import describe
describe.methods(UDD)
```

```
class UDD - 1/mthly shortcuts with UDD assumption

    Args:
      m : number of payments per year
      life : original fractional survival and mortality functions

    Methods:
    --------

    alpha(m, i):
      Derive 1/mthly UDD interest rate beta function value

    beta(m, i):
      Derive 1/mthly UDD interest rate alpha function value

    interest_frame(i):
      Display 1/mthly UDD interest function values
```

20.6 Examples

SOA Question 6.38

For an n-year endowment insurance of 1000 on (x), you are given:

- Death benefits are payable at the moment of death
- Premiums are payable annually at the beginning of each year
- Deaths are uniformly distributed over each year of age
- $i = 0.05$
- ${}_nE_x = 0.172$
- $\overline{A}_{x:\overline{n}|} = 0.192$

Calculate the annual net premium for this insurance.

```python
print("SOA Question 6.38:  (B) 11.3")
x, n = 0, 10
life = Recursion().set_interest(i=0.05)\
                  .set_A(0.192, x=x, t=n, endowment=1, discrete=False)\
                  .set_E(0.172, x=x, t=n)
a = life.temporary_annuity(x, t=n, discrete=False)
print(a)
def fun(a):      # solve for discrete annuity, given continuous
    life = Recursion().set_interest(i=0.05)\
                      .set_a(a, x=x, t=n)\
                      .set_E(0.172, x=x, t=n)
    return UDD(m=0, life=life).temporary_annuity(x, t=n)
a = life.solve(fun, target=a, grid=a)  # discrete annuity
P = life.gross_premium(a=a, A=0.192, benefit=1000)
print(a, P)
```

```
SOA Question 6.38:  (B) 11.3
 *Temporary Annuity a_0(t=10) <--
```

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```
   a_0(t=10) = [ 1 - A_0(t=10) ] / d(t=10)                      ~annuity twin
      a_0(t=1) = 1                                      ~one-year discrete annuity
      a_1(t=1) = 1                                      ~one-year discrete annuity
16.560714925944584
16.978162620976775 11.308644185253657
```

SOA Question 6.32

For a whole life insurance of 100,000 on (x), you are given:

- Death benefits are payable at the moment of death
- Deaths are uniformly distributed over each year of age
- Premiums are payable monthly
- $i = 0.05$
- $\ddot{a}_x = 9.19$

Calculate the monthly net premium.

```
print("SOA Question 6.32:  (C) 550")
x = 0
life = Recursion().set_interest(i=0.05).set_a(9.19, x=x)
benefits = UDD(m=0, life=life).whole_life_insurance(x)
payments = UDD(m=12, life=life).whole_life_annuity(x)
print(benefits, payments)
print(life.gross_premium(a=payments, A=benefits, benefit=100000)/12)
```

```
SOA Question 6.32:  (C) 550
 *Whole Life Insurance A_0(t=WL) <--
   A_x = 1 - d * a_x                                            ~annuity twin
0.5763261529803323 8.72530251348809
550.4356936711871
```

SOA Question 6.22

For a whole life insurance of 100,000 on (45) with premiums payable monthly for a period of 20 years, you are given:

- The death benefit is paid immediately upon death
- Mortality follows the Standard Ultimate Life Table
- Deaths are uniformly distributed over each year of age
- $i = 0.05$

Calculate the monthly net premium.

```
print("SOA Question 6.22:  (C) 102")
life = SULT(udd=True)
a = UDD(m=12, life=life).temporary_annuity(45, t=20)
A = UDD(m=0, life=life).whole_life_insurance(45)
print(life.gross_premium(A=A, a=a, benefit=100000)/12)
```

```
SOA Question 6.22:  (C) 102
102.40668704849178
```

SOA Question 7.9

For a semi-continuous 20-year endowment insurance of 100,000 on (45), you are given:

- Net premiums of 253 are payable monthly
- Mortality follows the Standard Ultimate Life Table
- Deaths are uniformly distributed over each year of age
- $i = 0.05$

Calculate ${}_{10}V$, the net premium policy value at the end of year 10 for this insurance.

```
print("SOA Question 7.9:  (A) 38100")
sult = SULT(udd=True)
x, n, t = 45, 20, 10
a = UDD(m=12, life=sult).temporary_annuity(x+10, t=n-10)
A = UDD(m=0, life=sult).endowment_insurance(x+10, t=n-10)
print(a, A)
contract = Contract(premium=253*12, endowment=100000, benefit=100000)
print(A*100000 - a*12*253, sult.gross_future_loss(A=A, a=a, contract=contract))
```

```
SOA Question 7.9:  (A) 38100
7.831075686716718 0.6187476755196442
38099.62176709247 38099.62176709246
```

SOA Question 6.49

For a special whole life insurance of 100,000 on (40), you are given:

- The death benefit is payable at the moment of death
- Level gross premiums are payable monthly for a maximum of 20 years
- Mortality follows the Standard Ultimate Life Table
- $i = 0.05$
- Deaths are uniformly distributed over each year of age
- Initial expenses are 200
- Renewal expenses are 4% of each premium including the first
- Gross premiums are calculated using the equivalence principle

Calculate the monthly gross premium.

```
print("SOA Question 6.49:  (C) 86")
sult = SULT(udd=True)
a = UDD(m=12, life=sult).temporary_annuity(40, t=20)
A = sult.whole_life_insurance(40, discrete=False)
P = sult.gross_premium(a=a, A=A, benefit=100000, initial_policy=200,
                        renewal_premium=0.04, initial_premium=0.04)
print(P/12)
```

```
SOA Question 6.49:  (C) 86
85.99177833261696
```

Generate table of interest functions (for FAM-L exam):

```
print("Interest Functions at i=0.05")
print("----------------------------")
print(UDD.interest_frame())
```

```
Interest Functions at i=0.05
----------------------------
       i(m)     d(m)   i/i(m)   d/d(m)  alpha(m)  beta(m)
1   0.05000  0.04762  1.00000  1.00000   1.00000  0.00000
2   0.04939  0.04820  1.01235  0.98795   1.00015  0.25617
4   0.04909  0.04849  1.01856  0.98196   1.00019  0.38272
12  0.04889  0.04869  1.02271  0.97798   1.00020  0.46651
0   0.04879  0.04879  1.02480  0.97600   1.00020  0.50823
```

CHAPTER

TWENTYONE

WOOLHOUSE M'THLY

Woolhouse's formula is a method of approximating 1/mthly life annuities from annual factors that does not depend on a fractional age assumption. It is based on the Euler-Maclaurin series expansion for the integral of a function. Life insurances may then be computed from twin relationships.

21.1 Life Annuities

$$\ddot{a}_x^{(m)} \approx \ddot{a}_x - \frac{m-1}{2m} - \frac{m^2-1}{12m^2}(\mu_x + \delta)$$

- 1/m'thly whole life annuity using the three-term Woolhouse approximation. The third term is often omitted in practice, which leads to poor approximations in some cases.

$$\ddot{a}_{x:\overline{t|}}^{(m)} \approx \ddot{a}_x^{(m)} - ~_tE_x ~\ddot{a}_{x+t}^{(m)} = \ddot{a}_{x:\overline{t|}} - \frac{m-1}{2m}(1 - ~_tE_x) - \frac{m^2-1}{12m^2}(\mu_x + \delta - ~_tE_x(\mu_{x+t} + \delta))$$

- 1/m'thly temporary life annuity from the difference of whole life Woolhouse approximations

$$\overline{a}_x \approx \ddot{a}_x - \frac{1}{2} - \frac{1}{12}(\mu_x + \delta)$$

- continuous life annuity with Woolhouse approximation when we let $m \to \infty$.

$$\mu_x \approx -\frac{1}{2}(\ln p_{x-1} + \ln p_x)$$

- if the force of mortality μ is not provided for the third Woolhouse term, it can be approximated from survival probabilities at integer ages.

21.2 Methods

Ths `Woolhouse` class implements methods to compute m'thly-pay annuity values using the Woolhouse assumption with either two or three terms.

```
from actuarialmath import Woolhouse, SULT, Recursion, UDD, Contract
import describe
describe.methods(Woolhouse)
```

```
class Woolhouse - 1/m'thly shortcuts with Woolhouse approximation

    Args:
      m : number of payments per year
```

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```
      life : original fractional survival and mortality functions
      three_term : whether to include (True) or ignore (False) third term
      approximate_mu : exact (False), approximate (True) or function for third term

    Methods:
    --------

    mu_x(x, s):
      Computes mu_x or calls approximate_mu for third term
```

21.3 Examples

SOA Question 7.7

For a whole life insurance of 10,000 on (x), you are given:

- Death benefits are payable at the end of the year of death
- A premium of 30 is payable at the start of each month
- Commissions are 5% of each premium
- Expenses of 100 are payable at the start of each year
- $i = 0.05$
- $1000 A_{x+10} = 400$
- ${}_{10}V$ is the gross premium policy value at the end of year 10 for this insurance

Calculate ${}_{10}V$ using the two-term Woolhouse formula for annuities.

```
print("SOA Question 7.7:  (D) 1110")
x = 0
life = Recursion().set_interest(i=0.05).set_A(0.4, x=x+10)
a = Woolhouse(m=12, life=life).whole_life_annuity(x+10)
print(a)
contract = Contract(premium=0, benefit=10000, renewal_policy=100)
V = life.gross_future_loss(A=0.4, contract=contract.renewals())
contract = Contract(premium=30*12, renewal_premium=0.05)
V1 = life.gross_future_loss(a=a, contract=contract.renewals())
print(V, V1, V+V1)
```

```
SOA Question 7.7:  (D) 1110
```

Whole Life Annuity $\ddot{a}_{x+10}$:=

$\ddot{a}_{x+10} = [1 - A_{x+10}]/d$ insurance twin

```
12.141666666666666
5260.0 -4152.028174603174 1107.9718253968258
```

SOA Question 6.25:

For a fully discrete 10-year deferred whole life annuity-due of 1000 per month on (55), you are given:

- The premium, G, will be paid annually at the beginning of each year during the deferral period
- Expenses are expected to be 300 per year for all years, payable at the beginning of the year
- Mortality follows the Standard Ultimate Life Table
- $i = 0.05$
- Using the two-term Woolhouse approximation, the expected loss at issue is -800

Calculate G.

```
print("SOA Question 6.25:  (C) 12330")
life = SULT()
woolhouse = Woolhouse(m=12, life=life)
benefits = woolhouse.deferred_annuity(55, u=10, b=1000 * 12)
expenses = life.whole_life_annuity(55, b=300)
payments = life.temporary_annuity(55, t=10)
print(benefits + expenses, payments)
def fun(G):
    return life.gross_future_loss(A=benefits + expenses, a=payments,
                                  contract=Contract(premium=G))
G = life.solve(fun, target=-800, grid=[12110, 12550])
print(G)
```

```
SOA Question 6.25:  (C) 12330
98042.52569470297 8.019169307712845
12325.781125438532
```

SOA Question 6.15

For a fully discrete whole life insurance of 1000 on (x) with net premiums payable quarterly, you are given:

- $i = 0.05$
- $\ddot{a}_x = 3.4611$
- $P^{(W)}$ and $P^{(UDD)}$ are the annualized net premiums calculated using the 2-term Woolhouse (W) and the uniform distribution of deaths (UDD) assumptions, respectively

Calculate $\frac{P^{(UDD)}}{P^{(W)}}$.

```
print("SOA Question 6.15:  (B) 1.002")
x = 0
life = Recursion().set_interest(i=0.05).set_a(3.4611, x=0)
A = life.insurance_twin(3.4611)
udd = UDD(m=4, life=life)
a1 = udd.whole_life_annuity(x=x)
woolhouse = Woolhouse(m=4, life=life)
a2 = woolhouse.whole_life_annuity(x=x)
print(life.gross_premium(a=a1, A=A)/life.gross_premium(a=a2, A=A))
```

```
SOA Question 6.15:  (B) 1.002
1.0022973504113772
```

SOA Question 5.7

You are given:

- $A_{35} = 0.188$

- $A_{65} = 0.498$
- ${}_{30}p_{35} = 0.883$
- $i = 0.04$

Calculate $1000\ddot{a}^{(2)}_{35:\overline{30|}}$ using the two-term Woolhouse approximation.

```
print("SOA Question 5.7:  (C) 17376.7")
life = Recursion().set_interest(i=0.04)
life.set_A(0.188, x=35)
life.set_A(0.498, x=65)
life.set_p(0.883, x=35, t=30)
mthly = Woolhouse(m=2, life=life, three_term=False)
print(mthly.temporary_annuity(35, t=30))
print(1000 * mthly.temporary_annuity(35, t=30))
```

```
SOA Question 5.7:  (C) 17376.7
```

Whole Life Annuity $\ddot{a}_{x+35}$:= $\ddot{a}_{x+35} = [1 - A_{x+35}]/d$ insurance twin

Whole Life Annuity $\ddot{a}_{x+65}$:= $\ddot{a}_{x+65} = [1 - A_{x+65}]/d$ insurance twin

Pure Endowment ${}_{30}E_{x+35}$:= ${}_{30}E_{x+35} = {}_{30}p_{x+35} * v^{30}$ pure endowment

```
17.37671459632958
```

Whole Life Annuity $\ddot{a}_{x+35}$:= $\ddot{a}_{x+35} = [1 - A_{x+35}]/d$ insurance twin

Whole Life Annuity $\ddot{a}_{x+65}$:= $\ddot{a}_{x+65} = [1 - A_{x+65}]/d$ insurance twin

Pure Endowment ${}_{30}E_{x+35}$:= ${}_{30}E_{x+35} = {}_{30}p_{x+35} * v^{30}$ pure endowment

```
17376.71459632958
```

CHAPTER

TWENTYTWO

SAMPLE SOLUTIONS AND HINTS

actuarialmath – Life Contingent Risks with Python

This package implements fundamental methods for modeling life contingent risks, and closely follows traditional topics covered in actuarial exams and standard texts such as the "Fundamentals of Actuarial Math - Long-term" exam syllabus by the Society of Actuaries, and "Actuarial Mathematics for Life Contingent Risks" by Dickson, Hardy and Waters. These code chunks demonstrate how to solve each of the sample FAM-L exam questions released by the SOA.

Sources:

- SOA FAM-L Sample Solutions: copy retrieved Aug 2022
- SOA FAM-L Sample Questions: copy retrieved Aug 2022
- Online User Guide, or download pdf
- API reference
- Github repo and issues

```
#! pip install actuarialmath==0.0.11
```

```
"""Solutions code and hints for SOA FAM-L sample questions

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"""
import math
import pandas as pd
import matplotlib.pyplot as plt
import numpy as np
from actuarialmath import Interest
from actuarialmath import Life
from actuarialmath import Survival
from actuarialmath import Lifetime
from actuarialmath import Fractional
from actuarialmath import Insurance
from actuarialmath import Annuity
from actuarialmath import Premiums
from actuarialmath import PolicyValues, Contract
from actuarialmath import Reserves
from actuarialmath import Recursion
from actuarialmath import LifeTable
from actuarialmath import SULT
from actuarialmath import SelectLife
from actuarialmath import MortalityLaws, Beta, Uniform, Makeham, Gompertz
from actuarialmath import ConstantForce
```

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```
from actuarialmath import ExtraRisk
from actuarialmath import Mthly
from actuarialmath import UDD
from actuarialmath import Woolhouse
```

Helper to compare computed answers to expected solutions

```
class IsClose:
    """Helper class for testing and reporting if two values are close"""
    def __init__(self, rel_tol : float = 0.01, score : bool = False,
                 verbose: bool = False):
        self.den = self.num = 0
        self.score = score       # whether to count INCORRECTs instead of assert
        self.verbose = verbose   # whether to run silently
        self.incorrect = []      # to keep list of messages for INCORRECT
        self.tol = rel_tol

    def __call__(self, solution, answer, question="", rel_tol=None):
        """Compare solution to answer within relative tolerance

        Args:
          solution (str | numeric) : gold label
          answer (str | numeric) : computed answer
          question (str) : label to associate with this test
          rel_tol (float) : relative tolerance to be considered close
        """
        if isinstance(solution, str):
            isclose = (solution == answer)
        else:
            isclose = math.isclose(solution, answer, rel_tol=rel_tol or self.tol)
        self.den += 1
        self.num += isclose
        msg = f"{question} {solution}: {answer}"
        if self.verbose:
            print("-----", msg, "[OK]" if isclose else "[INCORRECT]", "-----")
        if not self.score:
            assert isclose, msg
        if not isclose:
            self.incorrect.append(msg)
        return isclose

    def __str__(self):
        """Display cumulative score and errors"""
        return f"Passed: {self.num}/{self.den}\n" + "\n".join(self.incorrect)
isclose = IsClose(0.01, score=False, verbose=True)
```

22.1 1 Tables

These tables are provided in the FAM-L exam

- Interest Functions at i=0.05
- Normal Distribution Table
- Standard Ultimate Life Table

but you actually do not need them here!

```
print("Interest Functions at i=0.05")
UDD.interest_frame()
```

```
Interest Functions at i=0.05
```

```
      i(m)     d(m)     i/i(m)   d/d(m)   alpha(m)  beta(m)
1   0.05000  0.04762  1.00000  1.00000   1.00000  0.00000
2   0.04939  0.04820  1.01235  0.98795   1.00015  0.25617
4   0.04909  0.04849  1.01856  0.98196   1.00019  0.38272
12  0.04889  0.04869  1.02271  0.97798   1.00020  0.46651
0   0.04879  0.04879  1.02480  0.97600   1.00020  0.50823
```

```
print("Values of z for selected values of Pr(Z<=z)")
print(Life.quantiles_frame().to_string(float_format=lambda x: f"{x:.3f}"))
```

```
Values of z for selected values of Pr(Z<=z)
z          0.842  1.036  1.282  1.645  1.960  2.326  2.576
Pr(Z<=z)   0.800  0.850  0.900  0.950  0.975  0.990  0.995
```

```
print("Standard Ultimate Life Table at i=0.05")
SULT().frame()
```

```
Standard Ultimate Life Table at i=0.05
```

```
          l_x        q_x       a_x       A_x      2A_x    a_x:10    A_x:10    a_x:20
20   100000.0  0.000250  19.9664  0.04922  0.00580  8.0991  0.61433  13.0559  \
21    99975.0  0.000253  19.9197  0.05144  0.00614  8.0990  0.61433  13.0551
22    99949.7  0.000257  19.8707  0.05378  0.00652  8.0988  0.61434  13.0541
23    99924.0  0.000262  19.8193  0.05622  0.00694  8.0986  0.61435  13.0531
24    99897.8  0.000267  19.7655  0.05879  0.00739  8.0983  0.61437  13.0519
..        ...       ...      ...      ...      ...     ...      ...      ...
96    17501.8  0.192887   3.5597  0.83049  0.69991  3.5356  0.83164   3.5597
97    14125.9  0.214030   3.3300  0.84143  0.71708  3.3159  0.84210   3.3300
98    11102.5  0.237134   3.1127  0.85177  0.73356  3.1050  0.85214   3.1127
99     8469.7  0.262294   2.9079  0.86153  0.74930  2.9039  0.86172   2.9079
100    6248.2  0.289584   2.7156  0.87068  0.76427  2.7137  0.87078   2.7156

      A_x:20    5_E_x   10_E_x   20_E_x
20   0.37829  0.78252  0.61224  0.37440
21   0.37833  0.78250  0.61220  0.37429
22   0.37837  0.78248  0.61215  0.37417
```

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```
23   0.37842  0.78245  0.61210  0.37404
24   0.37848  0.78243  0.61205  0.37390
..       ...      ...      ...      ...
96   0.83049  0.19872  0.01330  0.00000
97   0.84143  0.16765  0.00827  0.00000
98   0.85177  0.13850  0.00485  0.00000
99   0.86153  0.11173  0.00266  0.00000
100  0.87068  0.08777  0.00136  0.00000

[81 rows x 12 columns]
```

22.2 2 Survival models

SOA Question 2.1 : (B) 2.5

You are given:

1. $S_0(t) = (1 - \frac{t}{\omega})^{\frac{1}{4}}, \quad 0 \le t \le \omega$
2. $\mu_{65} = \frac{1}{180}$

Calculate e_{106}, the curtate expectation of life at age 106.

hints:

- derive formula for μ from given survival function
- solve for ω given μ_{65}
- calculate e by summing survival probabilities

```
life = Lifetime()
def mu_from_l(omega):    # first solve for omega, given mu_65 = 1/180
    return life.set_survival(l=lambda x,s: (1 - (x+s)/omega)**0.25).mu_x(65)
omega = int(life.solve(mu_from_l, target=1/180, grid=100))
e = life.set_survival(l=lambda x,s:(1 - (x + s)/omega)**.25, maxage=omega)\
        .e_x(106)        # then solve expected lifetime from omega
isclose(2.5, e, question="Q2.1")
```

```
----- Q2.1 2.5: 2.4786080555423604 [OK] -----
```

```
True
```

SOA Question 2.2 : (D) 400

Scientists are searching for a vaccine for a disease. You are given:

1. 100,000 lives age x are exposed to the disease
2. Future lifetimes are independent, except that the vaccine, if available, will be given to all at the end of year 1
3. The probability that the vaccine will be available is 0.2
4. For each life during year 1, q_x = 0.02
5. For each life during year 2, q_{x+1} = 0.01 if the vaccine has been given and q_{x+1} = 0.02 if it has not been given

Calculate the standard deviation of the number of survivors at the end of year 2.

hints:

- calculate survival probabilities for the two scenarios
- apply conditional variance formula (or mixed distribution)

```
p1 = (1. - 0.02) * (1. - 0.01)  # 2_p_x if vaccine given
p2 = (1. - 0.02) * (1. - 0.02)  # 2_p_x if vaccine not given
std = math.sqrt(Life.conditional_variance(p=.2, p1=p1, p2=p2, N=100000))
isclose(400, std, question="Q2.2")
```

```
----- Q2.2 400: 396.5914603215815 [OK] -----
```

```
True
```

SOA Question 2.3 : (A) 0.0483

You are given that mortality follows Gompertz Law with B = 0.00027 and c = 1.1.

Calculate $f_{50}(10)$.

hints:

- Derive formula for f given survival function

```
B, c = 0.00027, 1.1
S = lambda x,s,t: math.exp(-B * c**(x+s) * (c**t - 1)/math.log(c))
life = Survival().set_survival(S=S)
f = life.f_x(x=50, t=10)
isclose(0.0483, f, question="Q2.3")
```

```
----- Q2.3 0.0483: 0.048327399045049846 [OK] -----
```

```
True
```

SOA Question 2.4 : (E) 8.2

You are given ${}_tq_0 = \frac{t^2}{10{,}000} \quad 0 < t < 100$. Calculate $\mathring{e}_{75:\overline{10}|}$.

hints:

- derive survival probability function ${}_tp_x$ given ${}_tq_0$
- compute $\mathring{e}$ by integration

```
def l(x, s): return 0. if (x+s) >= 100 else 1 - ((x + s)**2) / 10000.
e = Lifetime().set_survival(l=l).e_x(75, t=10, curtate=False)
isclose(8.2, e, question="Q2.4")
```

```
----- Q2.4 8.2: 8.20952380952381 [OK] -----
```

```
True
```

SOA Question 2.5 : (B) 37.1

You are given the following:

1. $e_{40:20} = 18$
2. $e_{60} = 25$
3. ${}_{20}q_{40} = 0.2$
4. $q_{40} = 0.003$

Calculate e_{41}.

hints:

- solve for e_{40} from limited lifetime formula
- compute e_{41} using forward recursion

```
life = Recursion(verbose=True).set_e(25, x=60, curtate=True)\
                              .set_q(0.2, x=40, t=20)\
                              .set_q(0.003, x=40)\
                              .set_e(18, x=40, t=20, curtate=True)
e = life.e_x(41, curtate=True)
isclose(37.1, e, question="Q2.5")
```

Lifetime e_{x+41} :=

$e_{x+41} = [\, e_{x+40} - \, e_{x+40:\overline{1}|}]/\, p_{x+40}$	forward recursion
$e_{x+40} = \, e_{x+40:\overline{20}|} + \, {}_{20}p_{x+40} * \, e_{x+60}$	backward recursion
${}_{20}p_{x+40} = 1 - \, {}_{20}q_{x+40}$	complement of mortality
$e_{x+40:\overline{1}|} = \, p_{x+40}$	1-year curtate shortcut
$p_{x+40} = 1 - \, q_{x+40}$	complement of mortality

```
----- Q2.5 37.1: 37.11434302908726 [OK] -----
```

```
True
```

SOA Question 2.6 : (C) 13.3

You are given the survival function:

$S_0(x) = \left(1 - \frac{x}{60}\right)^{\frac{1}{3}}, \quad 0 \leq x \leq 60$

Calculate $1000\mu_{35}$.

hints:

- derive force of mortality function μ from given survival function

```
life = Survival().set_survival(l=lambda x,s: (1 - (x+s)/60)**(1/3))
mu = 1000 * life.mu_x(35)
isclose(13.3, mu, question="Q2.6")
```

```
----- Q2.6 13.3: 13.340451278922776 [OK] -----
```

```
True
```

SOA Question 2.7 : (B) 0.1477

You are given the following survival function of a newborn:

$$\begin{aligned} S_0(x) &= 1 - \frac{x}{250}, \quad 0 \leq x < 40 \\ &= 1 - \left(\frac{x}{100}\right)^2, \quad 40 \leq x \leq 100 \end{aligned}$$

Calculate the probability that (30) dies within the next 20 years.

hints:

- calculate from given survival function

```
l = lambda x,s: (1-((x+s)/250) if (x+s)<40 else 1-((x+s)/100)**2)
q = Survival().set_survival(l=l).q_x(30, t=20)
isclose(0.1477, q, question="Q2.7")
```

```
----- Q2.7 0.1477: 0.1477272727272727 [OK] -----
```

```
True
```

SOA Question 2.8 : (C) 0.94

In a population initially consisting of 75% females and 25% males, you are given:

1. For a female, the force of mortality is constant and equals μ
2. For a male, the force of mortality is constant and equals 1.5 μ
3. At the end of 20 years, the population is expected to consist of 85% females and 15% males

Calculate the probability that a female survives one year.

hints:

- relate p_{male} and p_{female} through the common term μ and the given proportions

```
def fun(mu):  # Solve first for mu, given ratio of start and end proportions
    male = Survival().set_survival(mu=lambda x,s: 1.5 * mu)
    female = Survival().set_survival(mu=lambda x,s: mu)
    return (75 * female.p_x(0, t=20)) / (25 * male.p_x(0, t=20))
mu = Survival.solve(fun, target=85/15, grid=[0.89, 0.99])
p = Survival().set_survival(mu=lambda x,s: mu).p_x(0, t=1)
isclose(0.94, p, question="Q2.8")
```

```
----- Q2.8 0.94: 0.9383813306903799 [OK] -----
```

```
True
```

22.3 3 Life tables and selection

SOA Question 3.1 : (B) 117

You are given:

1. An excerpt from a select and ultimate life table with a select period of 3 years:

x	$\ell_{[x]}$	$\ell_{[x]+1}$	$\ell_{[x]+2}$	ℓ_{x+3}	$x+3$
60	80,000	79,000	77,000	74,000	63
61	78,000	76,000	73,000	70,000	64
62	75,000	72,000	69,000	67,000	65
63	71,000	68,000	66,000	65,000	66

2. Deaths follow a constant force of mortality over each year of age

Calculate $1000\ {}_{23}q_{[60]+0.75}$.

hints:

- interpolate with constant force of maturity

```
life = SelectLife().set_table(l={60: [80000, 79000, 77000, 74000],
                                 61: [78000, 76000, 73000, 70000],
                                 62: [75000, 72000, 69000, 67000],
                                 63: [71000, 68000, 66000, 65000]})
q = 1000 * life.q_r(60, s=0, r=0.75, t=3, u=2)
isclose(117, q, question="Q3.1")
```

```
----- Q3.1 117: 116.7192429022082 [OK] -----
```

```
True
```

SOA Question 3.2 : (D) 14.7

You are given:

1. The following extract from a mortality table with a one-year select period:

x	$l_{[x]}$	$d_{[x]}$	l_{x+1}	$x+1$
65	1000	40	–	66
66	955	45	–	67

2. Deaths are uniformly distributed over each year of age

$\mathring{e}_{[65]} = 15.0$

Calculate $\mathring{e}_{[66]}$.

hints:

- UDD $\Rightarrow \mathring{e}_x = e_x + 0.5$
- fill select table using curtate expectations

```
e_curtate = Fractional.e_approximate(e_complete=15)
life = SelectLife(udd=True).set_table(l={65: [1000, None,],
                                         66: [955, None]},
                                      e={65: [e_curtate, None]},
                                      d={65: [40, None,],
                                         66: [45, None]})
e = life.e_r(66)
isclose(14.7, e, question="Q3.2")
```

```
----- Q3.2 14.7: 14.67801047120419 [OK] -----
```

```
True
```

SOA Question 3.3 : (E) 1074

You are given:

1. An excerpt from a select and ultimate life table with a select period of 2 years:

x	$\ell_{[x]}$	$\ell_{[x]+1}$	ℓ_{x+2}	$x+2$
50	99,000	96,000	93,000	52
51	97,000	93,000	89,000	53
52	93,000	88,000	83,000	54
53	90,000	84,000	78,000	55

2. Deaths are uniformly distributed over each year of age

Calculate $10,000\ {}_{2.2}q_{[51]+0.5}$.

hints:

- interpolate lives between integer ages with UDD

```
life = SelectLife().set_table(l={50: [99, 96, 93],
                                 51: [97, 93, 89],
                                 52: [93, 88, 83],
                                 53: [90, 84, 78]})
q = 10000 * life.q_r(51, s=0, r=0.5, t=2.2)
isclose(1074, q, question="Q3.3")
```

```
----- Q3.3 1074: 1073.684210526316 [OK] -----
```

```
True
```

SOA Question 3.4 : (B) 815

The SULT Club has 4000 members all age 25 with independent future lifetimes. The mortality for each member follows the Standard Ultimate Life Table.

Calculate the largest integer N, using the normal approximation, such that the probability that there are at least N survivors at age 95 is at least 90%.

hints:

- compute portfolio percentile with N=4000, and mean and variance from binomial distribution

```
sult = SULT()
mean = sult.p_x(25, t=95-25)
var = sult.bernoulli(mean, variance=True)
pct = sult.portfolio_percentile(N=4000, mean=mean, variance=var, prob=0.1)
isclose(815, pct, question="Q3.4")
```

```
----- Q3.4 815: 815.0943255167722 [OK] -----
```

```
True
```

SOA Question 3.5 : (E) 106

You are given:

x	60	61	62	63	64	65	66	67
l_x	99,999	88,888	77,777	66,666	55,555	44,444	33,333	22,222

$a = {}_{3.4|2.5}q_{60}$ assuming a uniform distribution of deaths over each year of age

$b = {}_{3.4|2.5}q_{60}$ assuming a constant force of mortality over each year of age

Calculate $100,000(a - b)$

hints:

- compute mortality rates by interpolating lives between integer ages, with UDD and constant force of mortality assumptions

```
l = [99999, 88888, 77777, 66666, 55555, 44444, 33333, 22222]
a = LifeTable(udd=True).set_table(l={age:l for age,l in zip(range(60, 68), l)})\
                       .q_r(60, u=3.4, t=2.5)
b = LifeTable(udd=False).set_table(l={age:l for age,l in zip(range(60, 68), l)})\
                        .q_r(60, u=3.4, t=2.5)
isclose(106, 100000 * (a - b), question="Q3.5")
```

```
----- Q3.5 106: 106.16575827938624 [OK] -----
```

```
True
```

SOA Question 3.6 : (D) 15.85

You are given the following extract from a table with a 3-year select period:

x	$q_{[x]}$	$q_{[x]+1}$	$q_{[x]+2}$	q_{x+3}	$x+3$
60	0.09	0.11	0.13	0.15	63
61	0.10	0.12	0.14	0.16	64
62	0.11	0.13	0.15	0.17	65
63	0.12	0.14	0.16	0.18	66
64	0.13	0.15	0.17	0.19	67

$e_{64} = 5.10$

Calculate $e_{[61]}$.

hints:

- apply recursion formulas for curtate expectation

```
e = SelectLife().set_table(q={60: [.09, .11, .13, .15],
                              61: [.1, .12, .14, .16],
                              62: [.11, .13, .15, .17],
                              63: [.12, .14, .16, .18],
                              64: [.13, .15, .17, .19]},
                           e={61: [None, None, None, 5.1]})\
                .e_x(61)
isclose(5.85, e, question="Q3.6")
```

```
----- Q3.6 5.85: 5.846832 [OK] -----
```

```
True
```

SOA Question 3.7 : (b) 16.4

For a mortality table with a select period of two years, you are given:

x	$q_{[x]}$	$q_{[x]+1}$	q_{x+2}	$x+2$
50	0.0050	0.0063	0.0080	52
51	0.0060	0.0073	0.0090	53
52	0.0070	0.0083	0.0100	54
53	0.0080	0.0093	0.0110	55

The force of mortality is constant between integral ages.

Calculate $1000\ {}_{2.5}q_{[50]+0.4}$.

hints:

- use deferred mortality formula
- use chain rule for survival probabilities,
- interpolate between integer ages with constant force of mortality

```
life = SelectLife().set_table(q={50: [.0050, .0063, .0080],
                                 51: [.0060, .0073, .0090],
                                 52: [.0070, .0083, .0100],
                                 53: [.0080, .0093, .0110]})
q = 1000 * life.q_r(50, s=0, r=0.4, t=2.5)
isclose(16.4, q, question="Q3.7")
```

```
----- Q3.7 16.4: 16.420207214428586 [OK] -----
```

```
True
```

SOA Question 3.8 : (B) 1505

A club is established with 2000 members, 1000 of exact age 35 and 1000 of exact age 45. You are given:

1. Mortality follows the Standard Ultimate Life Table
2. Future lifetimes are independent
3. N is the random variable for the number of members still alive 40 years after the club is established

Using the normal approximation, without the continuity correction, calculate the smallest n such that $Pr(N \geq n) \leq 0.05$.

hints:

- compute portfolio means and variances from sum of 2000 independent members' means and variances of survival.

```
sult = SULT()
p1 = sult.p_x(35, t=40)
p2 = sult.p_x(45, t=40)
mean = sult.bernoulli(p1) * 1000 + sult.bernoulli(p2) * 1000
var = (sult.bernoulli(p1, variance=True) * 1000
       + sult.bernoulli(p2, variance=True) * 1000)
pct = sult.portfolio_percentile(mean=mean, variance=var, prob=.95)
isclose(1505, pct, question="Q3.8")
```

```
----- Q3.8 1505: 1504.8328375406456 [OK] -----
```

```
True
```

SOA Question 3.9 : (E) 3850

A father-son club has 4000 members, 2000 of which are age 20 and the other 2000 are age 45. In 25 years, the members of the club intend to hold a reunion.

You are given:

1. All lives have independent future lifetimes.
2. Mortality follows the Standard Ultimate Life Table.

Using the normal approximation, without the continuity correction, calculate the 99th percentile of the number of surviving members at the time of the reunion.

hints:

- compute portfolio means and variances as sum of 4000 independent members' means and variances (of survival)
- retrieve normal percentile

```
sult = SULT()
p1 = sult.p_x(20, t=25)
p2 = sult.p_x(45, t=25)
mean = sult.bernoulli(p1) * 2000 + sult.bernoulli(p2) * 2000
var = (sult.bernoulli(p1, variance=True) * 2000
       + sult.bernoulli(p2, variance=True) * 2000)
pct = sult.portfolio_percentile(mean=mean, variance=var, prob=.99)
isclose(3850, pct, question="Q3.9")
```

```
----- Q3.9 3850: 3850.144345130047 [OK] -----
```

```
True
```

SOA Question 3.10 : (C) 0.86

A group of 100 people start a Scissor Usage Support Group. The rate at which members enter and leave the group is dependent on whether they are right-handed or left-handed.

You are given the following:

1. The initial membership is made up of 75% left-handed members (L) and 25% right-handed members (R)
2. After the group initially forms, 35 new (L) and 15 new (R) join the group at the start of each subsequent year
3. Members leave the group only at the end of each year
4. q_L = 0.25 for all years
5. q_R = 0.50 for all years

Calculate the proportion of the Scissor Usage Support Group's expected membership that is left-handed at the start of the group's 6th year, before any new members join for that year.

hints:

- reformulate the problem by reversing time: survival to year 6 is calculated in reverse as discounting by the same number of years.

```
interest = Interest(v=0.75)
L = 35*interest.annuity(t=4, due=False) + 75*interest.v_t(t=5)
interest = Interest(v=0.5)
R = 15*interest.annuity(t=4, due=False) + 25*interest.v_t(t=5)
isclose(0.86, L / (L + R), question="Q3.10")
```

```
----- Q3.10 0.86: 0.8578442833761983 [OK] -----
```

```
True
```

SOA Question 3.11 : (B) 0.03

For the country of Bienna, you are given:

1. Bienna publishes mortality rates in biennial form, that is, mortality rates are of the form: ${}_2q_{2x}$, for $x = 0, 1, 2, ...$
2. Deaths are assumed to be uniformly distributed between ages $2x$ and $2x + 2$, for $x = 0, 1, 2, ...$
3. ${}_2q_{50} = 0.02$
4. ${}_2q_{52} = 0.04$

Calculate the probability that (50) dies during the next 2.5 years.

hints:

- calculate mortality rate by interpolating lives assuming UDD

```
life = LifeTable(udd=True).set_table(q={50//2: .02, 52//2: .04})
q = life.q_r(50//2, t=2.5/2)
isclose(0.03, q, question="Q3.11")
```

```
----- Q3.11 0.03: 0.0298 [OK] -----
```

```
True
```

SOA Question 3.12 : (C) 0.055

X and Y are both age 61. X has just purchased a whole life insurance policy. Y purchased a whole life insurance policy one year ago.

Both X and Y are subject to the following 3-year select and ultimate table:

x	$\ell_{[x]}$	$\ell_{[x]+1}$	$\ell_{[x]+2}$	ℓ_{x+3}	$x+3$
60	10,000	9,600	8,640	7,771	63
61	8,654	8,135	6,996	5,737	64
62	7,119	6,549	5,501	4,016	65
63	5,760	4,954	3,765	2,410	66

The force of mortality is constant over each year of age.

Calculate the difference in the probability of survival to age 64.5 between X and Y.

hints:

- compute survival probability by interpolating lives assuming constant force

```
life = SelectLife(udd=False).set_table(l={60: [10000, 9600, 8640, 7771],
                                          61: [8654, 8135, 6996, 5737],
                                          62: [7119, 6549, 5501, 4016],
                                          63: [5760, 4954, 3765, 2410]})
q = life.q_r(60, s=1, t=3.5) - life.q_r(61, s=0, t=3.5)
isclose(0.055, q, question="Q3.12")
```

```
----- Q3.12 0.055: 0.05465655938591829 [OK] -----
```

```
True
```

SOA Question 3.13 : (B) 1.6

A life is subject to the following 3-year select and ultimate table:

$[x]$	$\ell_{[x]}$	$\ell_{[x]+1}$	$\ell_{[x]+2}$	ℓ_{x+3}	$x+3$
55	10,000	9,493	8,533	7,664	58
56	8,547	8,028	6,889	5,630	59
57	7,011	6,443	5,395	3,904	60
58	5,853	4,846	3,548	2,210	61

You are also given:

1. $e_{60} = 1$
2. Deaths are uniformly distributed over each year of age

Calculate $\mathring{e}_{[58]+2}$.

hints:

- compute curtate expectations using recursion formulas
- convert to complete expectation assuming UDD

```
life = SelectLife().set_table(l={55: [10000, 9493, 8533, 7664],
                                 56: [8547, 8028, 6889, 5630],
                                 57: [7011, 6443, 5395, 3904],
                                 58: [5853, 4846, 3548, 2210]},
                              e={57: [None, None, None, 1]})
e = life.e_r(58, s=2)
isclose(1.6, e, question="Q3.13")
```

```
----- Q3.13 1.6: 1.6003382187147688 [OK] -----
```

```
True
```

SOA Question 3.14 : (C) 0.345

You are given the following information from a life table:

x	l_x	d_x	p_x	q_x
95	–	–	–	0.40
96	–	–	0.20	–
97	–	72	–	1.00

You are also given:

1. $l_{90} = 1000$ and $l_{93} = 825$
2. Deaths are uniformly distributed over each year of age.

Calculate the probability that (90) dies between ages 93 and 95.5.

hints:

- compute mortality by interpolating lives between integer ages assuming UDD

```
life = LifeTable(udd=True).set_table(l={90: 1000, 93: 825},
                                     d={97: 72},
                                     p={96: .2},
                                     q={95: .4, 97: 1})
q = life.q_r(90, u=93-90, t=95.5 - 93)
isclose(0.345, q, question="Q3.14")
```

```
----- Q3.14 0.345: 0.345 [OK] -----
```

```
True
```

22.4 4 Insurance benefits

SOA Question 4.1 : (A) 0.27212

For a special whole life insurance policy issued on (40), you are given:

1. Death benefits are payable at the end of the year of death
2. The amount of benefit is 2 if death occurs within the first 20 years and is 1 thereafter
3. *Z* is the present value random variable for the payments under this insurance
4. *i* = 0.03
5.

x	A_x	${}_{20}E_x$
40	0.36987	0.51276
60	0.62567	0.17878

6. $E[Z^2] = 0.24954$

Calculate the standard deviation of *Z*.

hints:

- solve EPV as sum of term and deferred insurance
- compute variance as difference of second moment and first moment squared

```
life = Recursion().set_interest(i=0.03)
life.set_A(0.36987, x=40).set_A(0.62567, x=60)
life.set_E(0.51276, x=40, t=20).set_E(0.17878, x=60, t=20)
Z2 = 0.24954
A = (2 * life.term_insurance(40, t=20) + life.deferred_insurance(40, u=20))
std = math.sqrt(life.insurance_variance(A2=Z2, A1=A))
isclose(0.27212, std, question="Q4.1")
```

```
----- Q4.1 0.27212: 0.2721117749374753 [OK] -----
```

```
True
```

SOA Question 4.2 : (D) 0.18

or a special 2-year term insurance policy on (*x*), you are given:

1. Death benefits are payable at the end of the half-year of death
2. The amount of the death benefit is 300,000 for the first half-year and increases by 30,000 per half-year thereafter
3. q_x = 0.16 and q_{x+1} = 0.23
4. $i^{(2)}$ = 0.18
5. Deaths are assumed to follow a constant force of mortality between integral ages
6. *Z* is the present value random variable for this insurance

Calculate Pr(*Z* > 277,000) .

hints:

- calculate Z(t) and deferred mortality for each half-yearly t
- sum the deferred mortality probabilities for periods when PV > 277000

```
life = LifeTable(udd=False).set_table(q={0: .16, 1: .23})\
                           .set_interest(i_m=.18, m=2)
mthly = Mthly(m=2, life=life)
Z = mthly.Z_m(0, t=2, benefit=lambda x,t: 300000 + t*30000*2)
p = Z[Z['Z'] >= 277000]['q'].sum()
isclose(0.18, p, question="Q4.2")
```

```
----- Q4.2 0.18: 0.17941813045022975 [OK] -----
```

```
True
```

SOA Question 4.3 : (D) 0.878

You are given:

1. $q_{60} = 0.01$
2. Using $i = 0.05$, $A_{60:\overline{3|}} = 0.86545$
3. Using $i = 0.045$ calculate $A_{60:\overline{3|}}$

hints:

- solve q_{61} from endowment insurance EPV formula
- solve $A_{60:\overline{3|}}$ with new $i = 0.045$ as EPV of endowment insurance benefits.

```
life = Recursion(verbose=True).set_interest(i=0.05)\
                              .set_q(0.01, x=60)\
                              .set_A(0.86545, x=60, t=3, endowment=1)
q = life.q_x(x=61)
A = Recursion(verbose=True).set_interest(i=0.045)\
                           .set_q(0.01, x=60)\
                           .set_q(q, x=61)\
                           .endowment_insurance(60, t=3)
isclose(0.878, A, question="Q4.3")
```

Mortality q_{x+61} :=

$q_{x+61} = 1 - p_{x+61}$ complement survival

$p_{x+61} = [v - A_{x+61:\overline{2|}}]/[v * [1 - A_{x+62:\overline{1|}}]]$ insurance recursion

$A^1_{x+61:\overline{2|}} = [A^1_{x+60:\overline{3|}}/v - q_{x+60} * b]/ p_{x+60}$ forward recursion

$p_{x+60} = 1 - q_{x+60}$ complement of mortality

Endowment Insurance $A_{x+60:\overline{3|}}$:=

$A^1_{x+60:\overline{3|}} = v * [q_{x+60} * b + p_{x+60} * A^1_{x+61:\overline{2|}}]$ backward recursion

$A^1_{x+61:\overline{2|}} = v * [q_{x+61} * b + p_{x+61} * A^1_{x+62:\overline{1|}}]$ backward recursion

$p_{x+61} = 1 - q_{x+61}$ complement of mortality

$E_{x+60} = p_{x+60} * v$ pure endowment

$p_{x+60} = 1 - q_{x+60}$ complement of mortality

```
----- Q4.3 0.878: 0.8777667236003878 [OK] -----
```

```
True
```

SOA Question 4.4 : (A) 0.036

For a special increasing whole life insurance on (40), payable at the moment of death, you are given :

1. The death benefit at time t is $b_t = 1 + 0.2t, \quad t \geq 0$
2. The interest discount factor at time t is $v(t) = (1 + 0.2t) - 2, \quad t \geq 0$
3. ${}_tp_{40}\ \mu_{40+t} = 0.025$ if $0 \leq t < 40$, otherwise 0
4. Z is the present value random variable for this insurance

Calculate Var(Z).

hints:

- integrate to find EPV of Z and Z^2
- variance is difference of second moment and first moment squared

```
x = 40
life = Insurance().set_survival(f=lambda *x: 0.025, maxage=x+40)\
                  .set_interest(v_t=lambda t: (1 + .2*t)**(-2))
def benefit(x,t): return 1 + .2 * t
A1 = life.A_x(x, benefit=benefit, discrete=False)
A2 = life.A_x(x, moment=2, benefit=benefit, discrete=False)
var = A2 - A1**2
isclose(0.036, var, question="Q4.4")
```

```
----- Q4.4 0.036: 0.03567680106032681 [OK] -----
```

```
True
```

SOA Question 4.5 : (C) 35200

For a 30-year term life insurance of 100,000 on (45), you are given:

1. The death benefit is payable at the moment of death
2. Mortality follows the Standard Ultimate Life Table
3. $\delta = 0.05$
4. Deaths are uniformly distributed over each year of age

Calculate the 95th percentile of the present value of benefits random variable for this insurance

hints:

- interpolate between integer ages with UDD, and find lifetime that mortality rate exceeded
- compute PV of death benefit paid at that time.

```
sult = SULT(udd=True).set_interest(delta=0.05)
Z = 100000 * sult.Z_from_prob(45, 0.95, discrete=False)
isclose(35200, Z, question="Q4.5")
```

```
----- Q4.5 35200: 35187.952037196534 [OK] -----
```

```
True
```

SOA Question 4.6 : (B) 29.85

For a 3-year term insurance of 1000 on (70), you are given:

1. q_{70+k}^{SULT} is the mortality rate from the Standard Ultimate Life Table, for k = 0,1,2
2. q_{70+k} is the mortality rate used to price this insurance, for k = 0,1, 2
3. $q_{70+k} = (0.95)^k q_{70+k}^{SULT}$, for k = 0,1, 2
4. *i* = 0.05

Calculate the single net premium.

hints:

- calculate adjusted mortality rates
- compute term insurance as EPV of benefits

```
sult = SULT()
life = LifeTable().set_interest(i=0.05)\
                  .set_table(q={70+k: .95**k * sult.q_x(70+k) for k in range(3)})
A = life.term_insurance(70, t=3, b=1000)
isclose(29.85, A, question="Q4.6")
```

```
----- Q4.6 29.85: 29.84835110355902 [OK] -----
```

```
True
```

SOA Question 4.7 : (B) 0.06

For a 25-year pure endowment of 1 on (*x*), you are given:

1. *Z* is the present value random variable at issue of the benefit payment
2. *Var (Z)* = 0.10 *E[Z]*
3. ${}_{25}p_x = 0.57$

Calculate the annual effective interest rate.

hints:

- use Bernoulli shortcut formula for variance of pure endowment Z
- solve for i, since p is given.

```
def fun(i):
    life = Recursion(verbose=False).set_interest(i=i)\
                                   .set_p(0.57, x=0, t=25)
    return 0.1*life.E_x(0, t=25) - life.E_x(0, t=25, moment=life.VARIANCE)
i = Recursion.solve(fun, target=0, grid=[0.058, 0.066])
isclose(0.06, i, question="Q4.7")
```

```
----- Q4.7 0.06: 0.06008023738770262 [OK] -----
```

```
True
```

SOA Question 4.8 : (C) 191

For a whole life insurance of 1000 on (50), you are given :

1. The death benefit is payable at the end of the year of death
2. Mortality follows the Standard Ultimate Life Table
3. $i = 0.04$ in the first year, and $i = 0.05$ in subsequent years

Calculate the actuarial present value of this insurance.

hints:

- use insurance recursion with special interest rate $i = 0.04$ in first year.

```
def v_t(t): return 1.04**(-t) if t < 1 else 1.04**(-1) * 1.05**(-t+1)
A = SULT().set_interest(v_t=v_t).whole_life_insurance(50, b=1000)
isclose(191, A, question="Q4.8")
```

```
----- Q4.8 191: 191.1281281882354 [OK] -----
```

```
True
```

SOA Question 4.9 : (D) 0.5

You are given:

1. $A_{35:\overline{15|}} = 0.39$
2. $A^1_{35:\overline{15|}} = 0.25$
3. $A_{35} = 0.32$

Calculate A_{50}.

hints:

- solve ${}_{15}E_{35}$ from endowment insurance minus term insurance
- solve implicitly from whole life as term plus deferred insurance

```
E = Recursion().set_A(0.39, x=35, t=15, endowment=1)\
               .set_A(0.25, x=35, t=15)\
               .E_x(35, t=15)
life = Recursion(verbose=False).set_A(0.32, x=35)\
                                .set_E(E, x=35, t=15)
def fun(A): return life.set_A(A, x=50).term_insurance(35, t=15)
A = life.solve(fun, target=0.25, grid=[0.35, 0.55])
isclose(0.5, A, question="Q4.9")
```

Pure Endowment ${}_{15}E_{x+35}$:=
${}_{15}E_{x+35} = A_{x+35:\overline{15|}} - A^1_{x+35:\overline{15|}}$ endowment insurance minus term

```
----- Q4.9 0.5: 0.5 [OK] -----
```

```
True
```

SOA Question 4.10 : (D)

The present value random variable for an insurance policy on (x) is expressed as: $$

Determine which of the following is a correct expression for $E[Z]$.

(A) $_{10|}\overline{A}_x +_{20|} \overline{A}_x -_{30|} \overline{A}_x$

(B) $\overline{A}_x +_{20} E_x\overline{A}_{x+20} - 2\ _{30}E_x\overline{A}_{x+30}$

(C) $_{10}E_x\overline{A}_x +_{20} E_x\overline{A}_{x+20} - 2\ _{30}E_x\overline{A}_{x+30}$

(D) $_{10}E_x\overline{A}_{x+10} +_{20} E_x\overline{A}_{x+20} - 2\ _{30}E_x\overline{A}_{x+30}$

(E) $_{10}E_x[\overline{A}_x +_{10} E_{x+10} + \overline{A}_{x+20} -_{10} E_{x+20} + \overline{A}_{x+30}]$

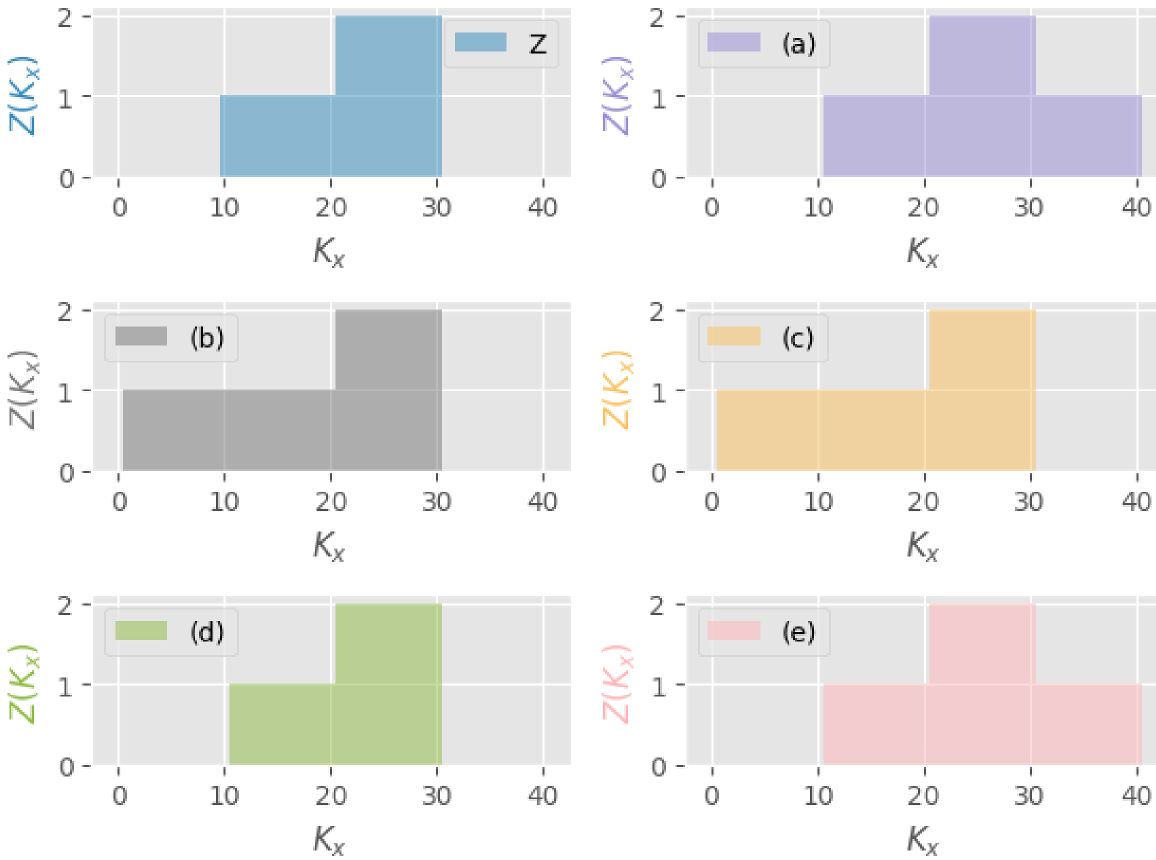
hints:

- draw and compare benefit diagrams

```
life = Insurance().set_interest(i=0.0).set_survival(S=lambda x,s,t: 1, maxage=40)
def fun(x, t):
    if 10 <= t <= 20: return life.interest.v_t(t)
    elif 20 < t <= 30: return 2 * life.interest.v_t(t)
    else: return 0
def A(x, t):  # Z_x+k (t-k)
    return life.interest.v_t(t - x) * (t > x)
x = 0
benefits=[lambda x,t: (life.E_x(x, t=10) * A(x+10, t)
                          + life.E_x(x, t=20)* A(x+20, t)
                          - life.E_x(x, t=30) * A(x+30, t)),
          lambda x,t: (A(x, t)
                     + life.E_x(x, t=20) * A(x+20, t)
                     - 2 * life.E_x(x, t=30) * A(x+30, t)),
          lambda x,t: (life.E_x(x, t=10) * A(x, t)
                     + life.E_x(x, t=20) * A(x+20, t)
                     - 2 * life.E_x(x, t=30) * A(x+30, t)),
          lambda x,t: (life.E_x(x, t=10) * A(x+10, t)
                     + life.E_x(x, t=20) * A(x+20, t)
                     - 2 * life.E_x(x, t=30) * A(x+30, t)),
          lambda x,t: (life.E_x(x, t=10)
                     * (A(x+10, t)
                        + life.E_x(x+10, t=10) * A(x+20, t)
                        - life.E_x(x+20, t=10) * A(x+30, t)))]
fig, ax = plt.subplots(3, 2)
ax = ax.ravel()
for i, b in enumerate([fun] + benefits):
    life.Z_plot(0, benefit=b, ax=ax[i], color=f"C{i+1}", title='')
    ax[i].legend(["(" + "abcde"[i-1] + ")" if i else "Z"])
z = [sum(abs(b(0, t) - fun(0, t)) for t in range(40)) for b in benefits]
ans = "ABCDE"[np.argmin(z)]
isclose('D', ans, question="Q4.10")
```

```
----- Q4.10 D: D [OK] -----
```

```
True
```



SOA Question 4.11 : (A) 143385

You are given:

1. Z_1 is the present value random variable for an n-year term insurance of 1000 issued to (x)
2. Z_2 is the present value random variable for an n-year endowment insurance of 1000 issued to (x)
3. For both Z_1 and Z_2 the death benefit is payable at the end of the year of death
4. $E[Z_1] = 528$
5. $Var(Z_2) = 15,000$
6. $A_{\substack{1\\x:\overline{n|}}} = 0.209$
7. ${}^2A_{\substack{1\\x:\overline{n|}}} = 0.136$

Calculate $Var(Z_1)$.

hints:

- compute endowment insurance = term insurance + pure endowment
- apply formula of variance as the difference of second moment and first moment squared.

```
A1 = 528/1000    # E[Z1]  term insurance
C1 = 0.209       # E[pure_endowment]
C2 = 0.136       # E[pure_endowment^2]
B1 = A1 + C1     # endowment = term + pure_endowment
def fun(A2):
    B2 = A2 + C2    # double force of interest
    return Insurance.insurance_variance(A2=B2, A1=B1)
A2 = Insurance.solve(fun, target=15000/(1000*1000), grid=[143400, 279300])
var = Insurance.insurance_variance(A2=A2, A1=A1, b=1000)
isclose(143385, var, question="Q4.11")
```

```
----- Q4.11 143385: 143384.99999999997 [OK] -----
```

```
True
```

SOA Question 4.12 : (C) 167

For three fully discrete insurance products on the same (x), you are given:

- Z_1 is the present value random variable for a 20-year term insurance of 50
- Z_2 is the present value random variable for a 20-year deferred whole life insurance of 100
- Z_3 is the present value random variable for a whole life insurance of 100.
- $E[Z_1] = 1.65$ and $E[Z_2] = 10.75$
- $Var(Z_1) = 46.75$ and $Var(Z_2) = 50.78$

Calculate $Var(Z_3)$.

hints:

- since $Z_1,\ Z_2$ are non-overlapping, $E[Z_1\ Z_2] = 0$ for computing $Cov(Z_1, Z_2)$
- whole life is sum of term and deferred, hence equals variance of components plus twice their covariance

```
cov = Life.covariance(a=1.65, b=10.75, ab=0)  # E[Z1 Z2] = 0 nonoverlapping
var = Life.variance(a=2, b=1, var_a=46.75, var_b=50.78, cov_ab=cov)
isclose(167, var, question="Q4.12")
```

```
----- Q4.12 167: 166.82999999999998 [OK] -----
```

```
True
```

SOA Question 4.13 : (C) 350

For a 2-year deferred, 2-year term insurance of 2000 on [65], you are given:

1. The following select and ultimate mortality table with a 3-year select period:

x	$q_{[x]}$	$q_{[x]+1}$	$q_{[x]+2}$	q_{x+3}	$x+3$
65	0.08	0.10	0.12	0.14	68
66	0.09	0.11	0.13	0.15	69
67	0.10	0.12	0.14	0.16	70
68	0.11	0.13	0.15	0.17	71
69	0.12	0.14	0.16	0.18	72

2. $i = 0.04$
3. The death benefit is payable at the end of the year of death

Calculate the actuarial present value of this insurance.

hints:

- compute term insurance as EPV of benefits

```
life = SelectLife().set_table(q={65: [.08, .10, .12, .14],
                                 66: [.09, .11, .13, .15],
                                 67: [.10, .12, .14, .16],
                                 68: [.11, .13, .15, .17],
                                 69: [.12, .14, .16, .18]})\
                   .set_interest(i=.04)
A = life.deferred_insurance(65, t=2, u=2, b=2000)
isclose(350, A, question="Q4.13")
```

```
----- Q4.13 350: 351.0578236056159 [OK] -----
```

```
True
```

SOA Question 4.14 : (E) 390000

A fund is established for the benefit of 400 workers all age 60 with independent future lifetimes. When they reach age 85, the fund will be dissolved and distributed to the survivors.

The fund will earn interest at a rate of 5% per year.

The initial fund balance, F, is determined so that the probability that the fund will pay at least 5000 to each survivor is 86%, using the normal approximation.

Mortality follows the Standard Ultimate Life Table.

Calculate F.

hints:

- discount (by interest rate $i = 0.05$) the value at the portfolio percentile, of the sum of 400 bernoulli r.v. with survival probability ${}_{25}p_{60}$

```
sult = SULT()
p = sult.p_x(60, t=85-60)
mean = sult.bernoulli(p)
var = sult.bernoulli(p, variance=True)
F = sult.portfolio_percentile(mean=mean, variance=var, prob=.86, N=400)
F *= 5000 * sult.interest.v_t(85-60)
isclose(390000, F, question="Q4.14")
```

```
----- Q4.14 390000: 389322.86778416135 [OK] -----
```

```
True
```

SOA Question 4.15 : (E) 0.0833

For a special whole life insurance on (x), you are given :

- Death benefits are payable at the moment of death

- The death benefit at time t is $b_t = e^{0.02t}$, for $t \geq 0$
- $\mu_{x+t} = 0.04$, for $t \geq 0$
- $\delta = 0.06$
- Z is the present value at issue random variable for this insurance.

Calculate $Var(Z)$.

hints:

- this special benefit function has effect of reducing actuarial discount rate to use in constant force of mortality shortcut formulas

```
life = Insurance().set_survival(mu=lambda *x: 0.04).set_interest(delta=0.06)
benefit = lambda x,t: math.exp(0.02*t)
A1 = life.A_x(0, benefit=benefit, discrete=False)
A2 = life.A_x(0, moment=2, benefit=benefit, discrete=False)
var = life.insurance_variance(A2=A2, A1=A1)
isclose(0.0833, var, question="Q4.15")
```

```
----- Q4.15 0.0833: 0.08334849338238598 [OK] -----
```

```
True
```

SOA Question 4.16 : (D) 0.11

You are given the following extract of ultimate mortality rates from a two-year select and ultimate mortality table:

x	q_x
50	0.045
51	0.050
52	0.055
53	0.060

The select mortality rates satisfy the following:

1. $q_{[x]} = 0.7q_x$
2. $q_{[x]+1} = 0.8q_{x+1}$

You are also given that $i = 0.04$.

Calculate $A^1_{[50]:\overline{3}|}$.

hints:

- compute EPV of future benefits with adjusted mortality rates

```
q = [.045, .050, .055, .060]
q = {50 + x: [q[x] * 0.7 if x < len(q) else None,
              q[x+1] * 0.8 if x + 1 < len(q) else None,
              q[x+2] if x + 2 < len(q) else None]
     for x in range(4)}
life = SelectLife().set_table(q=q).set_interest(i=.04)
A = life.term_insurance(50, t=3)
isclose(0.1116, A, question="Q4.16")
```

```
----- Q4.16 0.1116: 0.1115661982248521 [OK] -----
```

```
True
```

SOA Question 4.17 : (A) 1126.7

For a special whole life policy on (48), you are given:

1. The policy pays 5000 if the insured's death is before the median curtate future lifetime at issue and 10,000 if death is after the median curtate future lifetime at issue
2. Mortality follows the Standard Ultimate Life Table
3. Death benefits are paid at the end of the year of death
4. i = 0.05

Calculate the actuarial present value of benefits for this policy.

hints:

- find future lifetime with 50% survival probability
- compute EPV of special whole life as sum of term and deferred insurance, that have different benefit amounts before and after median lifetime.

```
sult = SULT()
median = sult.Z_t(48, prob=0.5, discrete=False)
def benefit(x,t): return 5000 if t < median else 10000
A = sult.A_x(48, benefit=benefit)
isclose(1130, A, question="Q4.17")
```

```
----- Q4.17 1130: 1126.774772894844 [OK] -----
```

```
True
```

SOA Question 4.18 : (A) 81873

You are given that T, the time to first failure of an industrial robot, has a density f(t) given by

$$\begin{aligned} f(t) &= 0.1, \quad 0 \leq t < 2 \\ &= 0.4t^{-2}, \quad t \leq t < 10 \end{aligned}$$

with $f(t)$ undetermined on $[10, \infty)$.

Consider a supplemental warranty on this robot that pays 100,000 at the time T of its first failure if $2 \leq T \leq 10$, with no benefits payable otherwise. You are also given that $\delta = 5\%$. Calculate the 90th percentile of the present value of the future benefits under this warranty.

hints:

- find values of limits such that integral of lifetime density function equals required survival probability

```
def f(x,s,t): return 0.1 if t < 2 else 0.4*t**(-2)
life = Insurance().set_interest(delta=0.05)\
                  .set_survival(f=f, maxage=10)
def benefit(x,t): return 0 if t < 2 else 100000
prob = 0.9 - life.q_x(0, t=2)
```

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```
T = life.Z_t(0, prob=prob)
Z = life.Z_from_t(T) * benefit(0, T)
isclose(81873, Z, question="Q4.18")
```

```
----- Q4.18 81873: 81873.07530779815 [OK] -----
```

```
True
```

SOA Question 4.19 : (B) 59050

(80) purchases a whole life insurance policy of 100,000. You are given:

1. The policy is priced with a select period of one year
2. The select mortality rate equals 80% of the mortality rate from the Standard Ultimate Life Table
3. Ultimate mortality follows the Standard Ultimate Life Table
4. $i = 0.05$

Calculate the actuarial present value of the death benefits for this insurance

hints:

- calculate adjusted mortality for the one-year select period
- compute whole life insurance using backward recursion formula

```
life = SULT()
q = ExtraRisk(life=life, extra=0.8, risk="MULTIPLY_RATE")['q']
select = SelectLife(periods=1).set_select(s=0, age_selected=True, q=q)\
                              .set_select(s=1, age_selected=False, q=life['q'])\
                              .set_interest(i=.05)\
                              .fill_table()
A = 100000 * select.whole_life_insurance(80, s=0)
isclose(59050, A, question="Q4.19")
```

```
----- Q4.19 59050: 59050.59973285648 [OK] -----
```

```
True
```

22.5 5 Annuities

SOA Question 5.1 : (A) 0.705

You are given:

1. $\delta_t = 0.06, \quad t \geq 0$
2. $\mu_x(t) = 0.01, \quad t \geq 0$
3. Y is the present value random variable for a continuous annuity of 1 per year, payable for the lifetime of (x) with 10 years certain

Calculate $Pr(Y > E[Y])$.

hints:

- sum annuity certain and deferred life annuity with constant force of mortality shortcut
- apply equation for PV annuity r.v. Y to infer lifetime
- compute survival probability from constant force of mortality function.

```
life = ConstantForce(mu=0.01).set_interest(delta=0.06)
EY = life.certain_life_annuity(0, u=10, discrete=False)
p = life.p_x(0, t=life.Y_to_t(EY))
isclose(0.705, p, question="Q5.1")   # 0.705
```

```
----- Q5.1 0.705: 0.7053680433746505 [OK] -----
```

```
True
```

SOA Question 5.2 : (B) 9.64

You are given:

1. $A_x = 0.30$
2. $A_{x+n} = 0.40$
3. $A_{\overset{1}{x}:\overline{n|}} = 0.35$
4. *i* = 0.05

Calculate $a_{x:\overline{n|}}$.

hints:

- compute term life as difference of whole life and deferred insurance
- compute twin annuity-due, and adjust to an immediate annuity.

```
x, n = 0, 10
a = Recursion().set_interest(i=0.05)\
               .set_A(0.3, x)\
               .set_A(0.4, x+n)\
               .set_E(0.35, x, t=n)\
               .immediate_annuity(x, t=n)
isclose(9.64, a, question="Q5.2")
```

Whole Life Annuity $\ddot{a}_x$:=
$\ddot{a}_x = [1 - A_x]/d$ insurance twin

Whole Life Annuity $\ddot{a}_{x+10}$:=
$\ddot{a}_{x+10} = [1 - A_{x+10}]/d$ insurance twin

```
----- Q5.2 9.64: 9.639999999999999 [OK] -----
```

```
True
```

SOA Question 5.3 : (C) 6.239

You are given:

- Mortality follows the Standard Ultimate Life Table
- Deaths are uniformly distributed over each year of age
- i = 0.05

Calculate $\frac{d}{dt}(\overline{I}\overline{a})_{40:\overline{t}|}$ at $t = 10.5$.

hints:

- Differential reduces to be the EPV of the benefit payment at the upper time limit.

```
t = 10.5
E = t * SULT().E_r(40, t=t)
isclose(6.239, E, question="Q5.3")
```

```
----- Q5.3 6.239: 6.23871918627528 [OK] -----
```

```
True
```

SOA Question 5.4 : (A) 213.7

(40) wins the SOA lottery and will receive both:

- A deferred life annuity of K per year, payable continuously, starting at age $40 + \mathring{e}_{40}$ and
- An annuity certain of K per year, payable continuously, for $\mathring{e}_{40}$ years

You are given:

1. $\mu = 0.02$
2. $\delta = 0.01$
3. The actuarial present value of the payments is 10,000

Calculate K.

hints:

- compute certain and life annuity factor as the sum of a certain annuity and a deferred life annuity.
- solve for amount of annual benefit that equals given EPV

```
life = ConstantForce(mu=0.02).set_interest(delta=0.01)
u = life.e_x(40, curtate=False)
P = 10000 / life.certain_life_annuity(40, u=u, discrete=False)
isclose(213.7, P, question="Q5.4") # 213.7
```

```
----- Q5.4 213.7: 213.74552118275955 [OK] -----
```

```
True
```

SOA Question 5.5 : (A) 1699.6

For an annuity-due that pays 100 at the beginning of each year that (45) is alive, you are given:

1. Mortality for standard lives follows the Standard Ultimate Life Table

2. The force of mortality for standard lives age 45 + t is represented as μ_{45+t}^{SULT}
3. The force of mortality for substandard lives age 45 + t, μ_{45+t}^{S}, is defined as:

$$\mu_{45+t}^{S} = \mu_{45+t}^{SULT} + 0.05, \quad 0 \le t < 1$$
$$= \mu_{45+t}^{SULT}, \quad t \ge 1$$

4. $i = 0.05$

Calculate the actuarial present value of this annuity for a substandard life age 45.

hints:

- adjust mortality rate for the extra risk
- compute annuity by backward recursion.

```
life = SULT()    # start with SULT life table
q = ExtraRisk(life=life, extra=0.05, risk="ADD_FORCE")['q']
select = SelectLife(periods=1).set_select(s=0, age_selected=True, q=q)\
                              .set_select(s=1, age_selected=False, a=life['a'])\
                              .set_interest(i=0.05)\
                              .fill_table()
a = 100 * select['a'][45][0]
isclose(1700, a, question="Q5.5")
```

```
----- Q5.5 1700: 1699.6076593190103 [OK] -----
```

```
True
```

SOA Question 5.6 : (D) 1200

For a group of 100 lives age x with independent future lifetimes, you are given:

1. Each life is to be paid 1 at the beginning of each year, if alive
2. $A_x = 0.45$
3. ${}^2A_x = 0.22$
4. $i = 0.05$
5. Y is the present value random variable of the aggregate payments.

Using the normal approximation to Y, calculate the initial size of the fund needed to be 95% certain of being able to make the payments for these life annuities.

hints:

- compute mean and variance of EPV of whole life annuity from whole life insurance twin and variance identities.
- portfolio percentile of the sum of $N = 100$ life annuity payments

```
life = Annuity().set_interest(i=0.05)
var = life.annuity_variance(A2=0.22, A1=0.45)
mean = life.annuity_twin(A=0.45)
fund = life.portfolio_percentile(mean, var, prob=.95, N=100)
isclose(1200, fund, question="Q5.6")
```

```
----- Q5.6 1200: 1200.6946732201702 [OK] -----
```

```
True
```

SOA Question 5.7 : (C)

You are given:

1. $A_{35} = 0.188$
2. $A_{65} = 0.498$
3. ${}_{30}p_{35} = 0.883$
4. $i = 0.04$

Calculate $1000\ddot{a}^{(2)}_{35:\overline{30|}}$ using the two-term Woolhouse approximation.

hints:

- compute endowment insurance from relationships of whole life, temporary and deferred insurances.
- compute temporary annuity from insurance twin
- apply Woolhouse approximation

```
life = Recursion().set_interest(i=0.04)\
                  .set_A(0.188, x=35)\
                  .set_A(0.498, x=65)\
                  .set_p(0.883, x=35, t=30)
mthly = Woolhouse(m=2, life=life, three_term=False)
a = 1000 * mthly.temporary_annuity(35, t=30)
isclose(17376.7, a, question="Q5.7")
```

Whole Life Annuity $\ddot{a}_{x+35}$:=
$\ddot{a}_{x+35} = [1 - A_{x+35}]/d$ insurance twin

Whole Life Annuity $\ddot{a}_{x+65}$:=
$\ddot{a}_{x+65} = [1 - A_{x+65}]/d$ insurance twin

Pure Endowment ${}_{30}E_{x+35}$:=
${}_{30}E_{x+35} = {}_{30}p_{x+35} * v^{30}$ pure endowment

```
----- Q5.7 17376.7: 17376.71459632958 [OK] -----
```

```
True
```

SOA Question 5.8 : (C) 0.92118

For an annual whole life annuity-due of 1 with a 5-year certain period on (55), you are given:

1. Mortality follows the Standard Ultimate Life Table
2. i = 0.05

Calculate the probability that the sum of the undiscounted payments actually made under this annuity will exceed the expected present value, at issue, of the annuity.

hints:

- calculate EPV of certain and life annuity.

- find survival probability of lifetime s.t. sum of annual payments exceeds EPV

```
sult = SULT()
a = sult.certain_life_annuity(55, u=5)
p = sult.p_x(55, t=math.floor(a))
isclose(0.92118, p, question="Q5.8")
```

```
----- Q5.8 0.92118: 0.9211799771029529 [OK] -----
```

```
True
```

SOA Question 5.9 : (C) 0.015

hints:

- express both EPV's expressed as forward recursions
- solve for unknown constant k.

```
x, p = 0, 0.9  # set arbitrary p_x = 0.9
a = Recursion().set_a(21.854, x=x)\
               .set_p(p, x=x)\
               .whole_life_annuity(x+1)
life = Recursion(verbose=False).set_a(22.167, x=x)
def fun(k): return a - life.set_p((1 + k) * p, x=x).whole_life_annuity(x + 1)
k = life.solve(fun, target=0, grid=[0.005, 0.025])
isclose(0.015, k, question="Q5.9")
```

Whole Life Annuity $\ddot{a}_{x+1}$:=

$\ddot{a}_{x+1} = [\,\ddot{a}_x - 1]/\ E_x$	forward recursion
$E_x = \ p_x * v$	pure endowment

```
----- Q5.9 0.015: 0.015009110961925157 [OK] -----
```

```
True
```

22.6 6 Premium Calculation

SOA Question 6.1 : (D) 35.36

6.1. You are given the following information about a special fully discrete 2-payment, 2-year term insurance on (80):

(i) Mortality follows the Standard Ultimate Life Table

(ii) $i = 0.03$

(iii) The death benefit is 1000 plus a return of all premiums paid without interest

(iv) Level premiums are calculated using the equivalence principle

Calculate the net premium for this special insurance.

[A modified version of Question 22 on the Fall 2012 exam]

hints:

- solve net premium such that EPV annuity = EPV insurance + IA factor for returns of premiums without interest

```
P = SULT().set_interest(i=0.03)\
          .net_premium(80, t=2, b=1000, return_premium=True)
isclose(35.36, P, question="Q6.1")
```

```
----- Q6.1 35.36: 35.35922286190033 [OK] -----
```

```
True
```

SOA Question 6.2 : (E) 3604

6.2. For a fully discrete 10-year term life insurance policy on (*x*), you are given:

(i) Death benefits are 100,000 plus the return of all gross premiums paid without interest

(ii) Expenses are 50% of the first year's gross premium, 5% of renewal gross premiums and 200 per policy expenses each year

(iii) Expenses are payble at the beginnig of the year

(iv) $A^1_{x:\overline{10|}} = 0.17094$

(v) $(IA)^1_{x:\overline{10|}} = 0.96728$

(vi) $\ddot{a}^1_{x:\overline{10|}} = 6.8865$

Calculate the gross premium using the equivalence principle.

[Question 25 on the Fall 2012 exam]

hints:

- EPV return of premiums without interest = Premium $\times$ IA factor
- solve for gross premiums such that EPV premiums = EPV benefits and expenses

```
life = Premiums()
A, IA, a = 0.17094, 0.96728, 6.8865
P = life.gross_premium(a=a, A=A, IA=IA, benefit=100000,
                       initial_premium=0.5, renewal_premium=.05,
                       renewal_policy=200, initial_policy=200)
isclose(3604, P, question="Q6.2")
```

```
----- Q6.2 3604: 3604.229940320728 [OK] -----
```

```
True
```

SOA Question 6.3 : (C) 0.390

S, now age 65, purchased a 20-year deferred whole life annuity-due of 1 per year at age 45. You are given:

1. Equal annual premiums, determined using the equivalence principle, were paid at the beginning of each year during the deferral period
2. Mortality at ages 65 and older follows the Standard Ultimate Life Table
3. *i* = 0.05
4. *Y* is the present value random variable at age 65 for S's annuity benefits

Calculate the probability that Y is less than the actuarial accumulated value of S's premiums.

hints:

- solve lifetime t such that PV annuity certain = PV whole life annuity at age 65
- calculate mortality rate through the year before curtate lifetime

```
life = SULT()
t = life.Y_to_t(life.whole_life_annuity(65))
q = 1 - life.p_x(65, t=math.floor(t) - 1)
isclose(0.39, q, question="Q6.3")
```

```
----- Q6.3 0.39: 0.39039071872030084 [OK] -----
```

```
True
```

SOA Question 6.4 : (E) 1890

For whole life annuities-due of 15 per month on each of 200 lives age 62 with independent future lifetimes, you are given:

1. $i = 0.06$
2. $A_{62}^{12} = 0.4075$ and ${}^2A_{62}^{(12)} = 0.2105$
3. π is the single premium to be paid by each of the 200 lives
4. S is the present value random variable at time 0 of total payments made to the 200 lives

Using the normal approximation, calculate π such at $Pr(200\pi > S) = 0.90$

```
mthly = Mthly(m=12, life=Annuity().set_interest(i=0.06))
A1, A2 = 0.4075, 0.2105
mean = mthly.annuity_twin(A1) * 15 * 12
var = mthly.annuity_variance(A1=A1, A2=A2, b=15 * 12)
S = Annuity.portfolio_percentile(mean=mean, variance=var, prob=.9, N=200) / 200
isclose(1890, S, question="Q6.4")
```

```
----- Q6.4 1890: 1893.912859650868 [OK] -----
```

```
True
```

SOA Question 6.5 : (D) 33

For a fully discrete whole life insurance of 1000 on (30), you are given:

1. Mortality follows the Standard Ultimate Life Table
2. $i = 0.05$
3. The premium is the net premium

Calculate the first year for which the expected present value at issue of that year's premium is less than the expected present value at issue of that year's benefit.

```
life = SULT()
P = life.net_premium(30, b=1000)
def gain(k): return life.Y_x(30, t=k) * P - life.Z_x(30, t=k) * 1000
```

(continues on next page)

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```
k = min([k for k in range(20, 40) if gain(k) < 0])
isclose(33, k, question="Q6.5")
```

```
----- Q6.5 33: 33 [OK] -----
```

```
True
```

SOA Question 6.6 : (B) 0.79

For fully discrete whole life insurance policies of 10,000 issued on 600 lives with independent future lifetimes, each age 62, you are given:

1. Mortality follows the Standard Ultimate Life Table
2. $i = 0.05$
3. Expenses of 5% of the first year gross premium are incurred at issue
4. Expenses of 5 per policy are incurred at the beginning of each policy year
5. The gross premium is 103% of the net premium.
6. ${}_0L$ is the aggregate present value of future loss at issue random variable

Calculate $Pr({}_0L < 40,000)$, using the normal approximation.

```
life = SULT()
P = life.net_premium(62, b=10000)
contract = Contract(premium=1.03*P,
                    renewal_policy=5,
                    initial_policy=5,
                    initial_premium=0.05,
                    benefit=10000)
L = life.gross_policy_value(62, contract=contract)
var = life.gross_policy_variance(62, contract=contract)
prob = life.portfolio_cdf(mean=L, variance=var, value=40000, N=600)
isclose(.79, prob, question="Q6.6")
```

```
----- Q6.6 0.79: 0.7914321142683509 [OK] -----
```

```
True
```

SOA Question 6.7 : (C) 2880

For a special fully discrete 20-year endowment insurance on (40), you are given:

1. The only death benefit is the return of annual net premiums accumulated with interest at 5% to the end of the year of death
2. The endowment benefit is 100,000
3. Mortality follows the Standard Ultimate Life Table
4. $i = 0.05$

Calculate the annual net premium.

```
life = SULT()
a = life.temporary_annuity(40, t=20)
A = life.E_x(40, t=20)
IA = a - life.interest.annuity(t=20) * life.p_x(40, t=20)
G = life.gross_premium(a=a, A=A, IA=IA, benefit=100000)
isclose(2880, G, question="Q6.7")
```

```
----- Q6.7 2880: 2880.2463991134578 [OK] -----
```

```
True
```

SOA Question 6.8 : (B) 9.5

For a fully discrete whole life insurance on (60), you are given:

1. Mortality follows the Standard Ultimate Life Table
2. $i = 0.05$
3. The expected company expenses, payable at the beginning of the year, are:

- 50 in the first year
- 10 in years 2 through 10
- 5 in years 11 through 20
- 0 after year 20

Calculate the level annual amount that is actuarially equivalent to the expected company expenses.

hints:

- calculate EPV of expenses as deferred life annuities
- solve for level premium

```
life = SULT()
initial_cost = (50 + 10 * life.deferred_annuity(60, u=1, t=9)
                + 5 * life.deferred_annuity(60, u=10, t=10))
P = life.net_premium(60, initial_cost=initial_cost)
isclose(9.5, P, question="Q6.8")
```

```
----- Q6.8 9.5: 9.526003201821927 [OK] -----
```

```
True
```

SOA Question 6.9 : (D) 647

```
life = SULT()
a = life.temporary_annuity(50, t=10)
A = life.term_insurance(50, t=20)
initial_cost = 25 * life.deferred_annuity(50, u=10, t=10)
P = life.gross_premium(a=a, A=A, benefit=100000,
                       initial_premium=0.42, renewal_premium=0.12,
                       initial_policy=75 + initial_cost, renewal_policy=25)
isclose(647, P, question="Q6.9")
```

```
----- Q6.9 647: 646.8608151974504 [OK] -----
```

```
True
```

SOA Question 6.10 : (D) 0.91

For a fully discrete 3-year term insurance of 1000 on (x), you are given:

1. $p_x = 0.975$
2. $i = 0.06$
3. The actuarial present value of the death benefit is 152.85
4. The annual net premium is 56.05

Calculate p_{x+2}.

```
x = 0
life = Recursion(depth=5).set_interest(i=0.06)\
                         .set_p(0.975, x=x)\
                         .set_a(152.85/56.05, x=x, t=3)\
                         .set_A(152.85, x=x, t=3, b=1000)
p = life.p_x(x=x+2)
isclose(0.91, p, question="Q6.10")
```

Survival p_{x+2} :=

$p_{x+2} = E_{x+2}/v$	one-year pure endowment
$E_{x+2} = A_{x+2:\overline{1}\|} - A^{1}_{x+2:\overline{1}\|}$	endowment insurance minus term
$A^{1}_{x+2:\overline{1}\|} = [\,A^{1}_{x+1:\overline{2}\|}/v - q_{x+1} * b]/\,p_{x+1}$	forward recursion
$p_{x+1} = [\,\ddot{a}_{x+1:\overline{2}\|} - 1]/[v * \ddot{a}_{x+2:\overline{1}\|}]$	annuity recursion
$\ddot{a}_{x+1:\overline{2}\|} = [\,\ddot{a}_{x:\overline{3}\|} - 1]/\,E_x$	forward recursion
$A^{1}_{x+1:\overline{2}\|} = [\,A^{1}_{x:\overline{3}\|}/v - q_x * b]/\,p_x$	forward recursion
$E_x = p_x * v$	pure endowment

```
----- Q6.10 0.91: 0.9097382950525702 [OK] -----
```

```
True
```

SOA Question 6.11 : (C) 0.041

```
life = Recursion().set_interest(i=0.04)
A = life.set_A(0.39788, 51)\
        .set_q(0.0048, 50)\
        .whole_life_insurance(50)
P = life.gross_premium(A=A, a=life.annuity_twin(A=A))
A = life.set_q(0.048, 50).whole_life_insurance(50)
loss = A - life.annuity_twin(A) * P
isclose(0.041, loss, question="Q6.11")
```

Whole Life Insurance A_{x+50} :=

$A_{x+50} = v * [\,q_{x+50} * b + p_{x+50} * A_{x+51}]$	backward recursion
$p_{x+50} = 1 - q_{x+50}$	complement of mortality

$$\begin{array}{ll} \texttt{Whole Life Insurance } A_{x+50} := & \\ A_{x+50} = v * [\, q_{x+50} * b + \, p_{x+50} * \, A_{x+51}] & \texttt{backward recursion} \\ p_{x+50} = 1 - \, q_{x+50} & \texttt{complement of mortality} \end{array}$$

```
----- Q6.11 0.041: 0.04069206883563675 [OK] -----
```

```
True
```

SOA Question 6.12 : (E) 88900

For a fully discrete whole life insurance of 1000 on (x), you are given:

1. The following expenses are incurred at the beginning of each year:

	Year 1	Years 2+
Percent of premium	75%	10%
Maintenance expenses	10	2

2. An additional expense of 20 is paid when the death benefit is paid
3. The gross premium is determined using the equivalence principle
4. $i = 0.06$
5. $\ddot{a}_x = 12.0$
6. ${}^2A_x = 0.14$

Calculate the variance of the loss at issue random variable.

```
life = PolicyValues().set_interest(i=0.06)
a = 12
A = life.insurance_twin(a)
contract = Contract(benefit=1000, settlement_policy=20,
                        initial_policy=10, initial_premium=0.75,
                        renewal_policy=2, renewal_premium=0.1)
contract.premium = life.gross_premium(A=A, a=a, **contract.premium_terms)
L = life.gross_variance_loss(A1=A, A2=0.14, contract=contract)
isclose(88900, L, question="Q6.12")
```

```
----- Q6.12 88900: 88862.59592874818 [OK] -----
```

```
True
```

SOA Question 6.13 : (D) -400

For a fully discrete whole life insurance of 10,000 on (45), you are given:

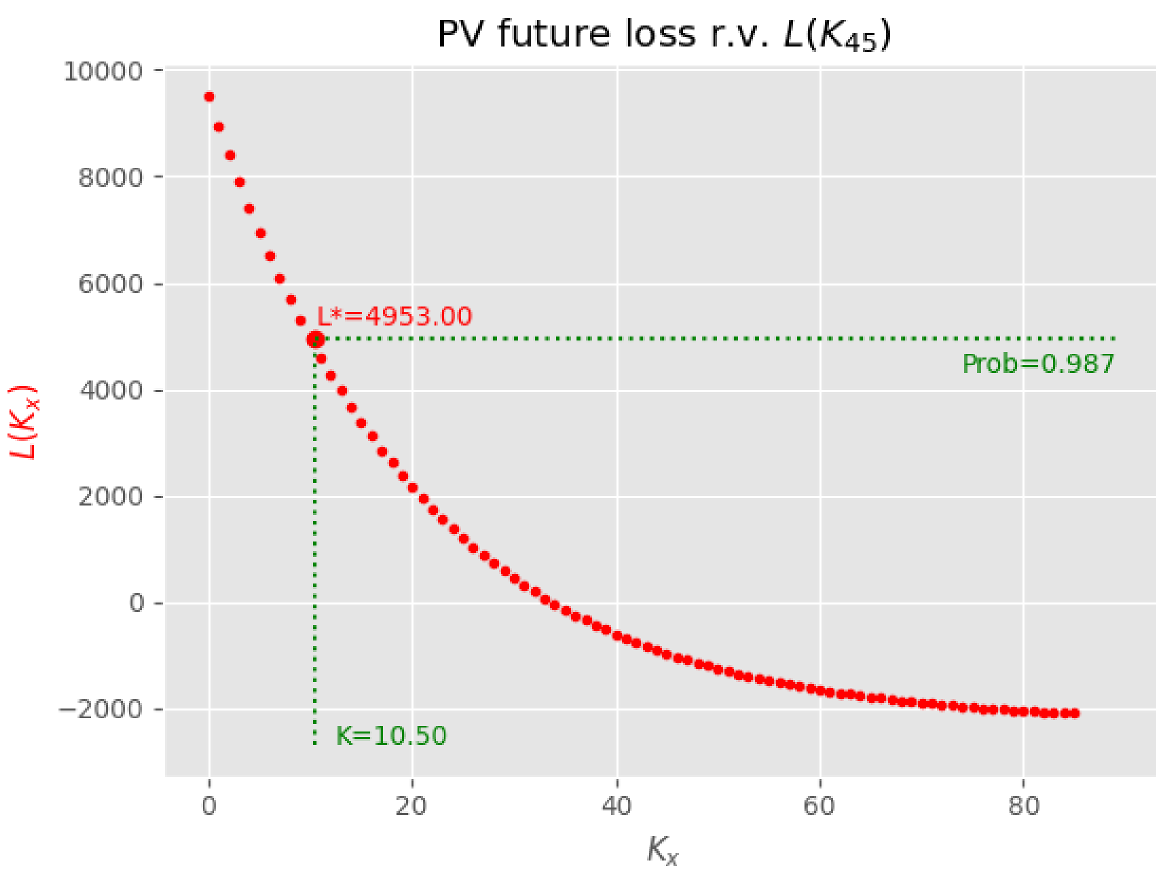
1. Commissions are 80% of the first year premium and 10% of subsequent premiums. There are no other expenses
2. Mortality follows the Standard Ultimate Life Table
3. $i = 0.05$
4. ${}_0L$ denotes the loss at issue random variable
5. If $T_{45} = 10.5$, then ${}_0L = 4953$

Calculate $E[{}_0L]$.

```
life = SULT().set_interest(i=0.05)
A = life.whole_life_insurance(45)
contract = Contract(benefit=10000, initial_premium=.8, renewal_premium=.1)
def fun(P):   # Solve for premium, given Loss(t=0) = 4953
    return life.L_from_t(t=10.5, contract=contract.set_contract(premium=P))
contract.set_contract(premium=life.solve(fun, target=4953, grid=100))
L = life.gross_policy_value(45, contract=contract)
life.L_plot(x=45, T=10.5, contract=contract)
isclose(-400, L, question="Q6.13")
```

```
----- Q6.13 -400: -400.94447599879277 [OK] -----
```

```
True
```



SOA Question 6.14 : (D) 1150

For a special fully discrete whole life insurance of 100,000 on (40), you are given:

1. The annual net premium is P for years 1 through 10, $0.5P$ for years 11 through 20, and 0 thereafter
2. Mortality follows the Standard Ultimate Life Table
3. $i = 0.05$

Calculate P.

```
life = SULT().set_interest(i=0.05)
a = life.temporary_annuity(40, t=10) + 0.5*life.deferred_annuity(40, u=10, t=10)
A = life.whole_life_insurance(40)
P = life.gross_premium(a=a, A=A, benefit=100000)
isclose(1150, P, question="Q6.14")
```

```
----- Q6.14 1150: 1148.5800555155263 [OK] -----
```

```
True
```

SOA Question 6.15 : (B) 1.002

For a fully discrete whole life insurance of 1000 on (x) with net premiums payable quarterly, you are given:

1. $i = 0.05$
2. $\ddot{a}_x = 3.4611$
3. $P^{(W)}$ and $P^{(UDD)}$ are the annualized net premiums calculated using the 2-term Woolhouse (W) and the uniform distribution of deaths (UDD) assumptions, respectively

Calculate $\dfrac{P^{(UDD)}}{P^{(W)}}$.

```
life = Recursion().set_interest(i=0.05).set_a(3.4611, x=0)
A = life.insurance_twin(3.4611)
udd = UDD(m=4, life=life)
a1 = udd.whole_life_annuity(x=x)
woolhouse = Woolhouse(m=4, life=life)
a2 = woolhouse.whole_life_annuity(x=x)
P = life.gross_premium(a=a1, A=A)/life.gross_premium(a=a2, A=A)
isclose(1.002, P, question="Q6.15")
```

```
----- Q6.15 1.002: 1.0022973504113772 [OK] -----
```

```
True
```

SOA Question 6.16 : (A) 2408.6

For a fully discrete 20-year endowment insurance of 100,000 on (30), you are given:

1. d = 0.05
2. Expenses, payable at the beginning of each year, are:

	First Year	First Year	Renewal Years	Renewal Years
	Percent of Premium	Per Policy	Percent of Premium	Per Policy
Taxes	4%	0	4%	0
Sales Commission	35%	0	2%	0
Policy Maintenance	0%	250	0%	50

3. The net premium is 2143

Calculate the gross premium using the equivalence principle.

```
life = Premiums().set_interest(d=0.05)
A = life.insurance_equivalence(premium=2143, b=100000)
a = life.annuity_equivalence(premium=2143, b=100000)
p = life.gross_premium(A=A, a=a, benefit=100000, settlement_policy=0,
                       initial_policy=250, initial_premium=0.04 + 0.35,
                       renewal_policy=50, renewal_premium=0.04 + 0.02)
isclose(2410, p, question="Q6.16")
```

```
----- Q6.16 2410: 2408.575206281868 [OK] -----
```

```
True
```

SOA Question 6.17 : (A) -30000

An insurance company sells special fully discrete two-year endowment insurance policies to smokers (S) and non-smokers (NS) age x. You are given:

1. The death benefit is 100,000; the maturity benefit is 30,000
2. The level annual premium for non-smoker policies is determined by the equivalence principle
3. The annual premium for smoker policies is twice the non-smoker annual premium
4. $\mu_{x+t}^{NS} = 0.1. \quad t > 0$
5. $q_{x+k}^{S} = 1.5 q_{x+k}^{NS}$, for $k = 0, 1$
6. $i = 0.08$

Calculate the expected present value of the loss at issue random variable on a smoker policy.

```
x = 0
life = ConstantForce(mu=0.1).set_interest(i=0.08)
A = life.endowment_insurance(x, t=2, b=100000, endowment=30000)
a = life.temporary_annuity(x, t=2)
P = life.gross_premium(a=a, A=A)
life1 = Recursion().set_interest(i=0.08)\
                   .set_q(life.q_x(x, t=1) * 1.5, x=x, t=1)\
                   .set_q(life.q_x(x+1, t=1) * 1.5, x=x+1, t=1)
contract = Contract(premium=P*2, benefit=100000, endowment=30000)
L = life1.gross_policy_value(x, t=0, n=2, contract=contract)
isclose(-30000, L, question="Q6.17")
```

Term Insurance $A^{1}_{x:\overline{2|}}$:=

$A^{1}_{x:\overline{2\|}} = A_{x:\overline{2\|}} - {}_2E_x$	endowment insurance - pure
${}_2E_x = {}_2p_x * v^2$	pure endowment
${}_2p_x = p_{x+1} * p_x$	survival chain rule
$A^{1}_{x:\overline{2\|}} = v * [\, q_x * b + p_x * A^{1}_{x+1:\overline{1\|}}]$	backward recursion
$p_{x+1} = 1 - q_{x+1}$	complement of mortality
$E_x = p_x * v$	pure endowment
$p_x = 1 - q_x$	complement of mortality

Temporary Annuity $\ddot{a}_{x:\overline{2|}}$:=

$\ddot{a}_{x:\overline{2\|}} = 1 + E_x * \ddot{a}_{x+1:\overline{1\|}}$	backward recursion
$E_x = p_x * v$	pure endowment
$p_x = 1 - q_x$	complement of mortality

```
Pure Endowment $_2E_x$ :=
 $_2E_x = \ _2p_x * v^2$            pure endowment
  $_2p_x = \ p_{x+1} * \ p_x$       survival chain rule
   $p_x = 1 - \ q_x$                complement of mortality
    $p_{x+1} = 1 - \ q_{x+1}$       complement of mortality
```

```
----- Q6.17 -30000: -30107.42633581125 [OK] -----
```

```
True
```

SOA Question 6.18 : (D) 166400

For a 20-year deferred whole life annuity-due with annual payments of 30,000 on (40), you are given:

1. The single net premium is refunded without interest at the end of the year of death if death occurs during the deferral period
2. Mortality follows the Standard Ultimate Life Table
3. $i = 0.05$

Calculate the single net premium for this annuity.

```
life = SULT().set_interest(i=0.05)
def fun(P):
    A = (life.term_insurance(40, t=20, b=P)
         + life.deferred_annuity(40, u=20, b=30000))
    return life.gross_premium(a=1, A=A) - P
P = life.solve(fun, target=0, grid=[162000, 168800])
isclose(166400, P, question="Q6.18")
```

```
----- Q6.18 166400: 166362.83871487685 [OK] -----
```

```
True
```

SOA Question 6.19 : (B) 0.033

```
life = SULT()
contract = Contract(initial_policy=.2, renewal_policy=.01)
a = life.whole_life_annuity(50)
A = life.whole_life_insurance(50)
contract.premium = life.gross_premium(A=A, a=a, **contract.premium_terms)
L = life.gross_policy_variance(50, contract=contract)
isclose(0.033, L, question="Q6.19")
```

```
----- Q6.19 0.033: 0.03283273381910885 [OK] -----
```

```
True
```

SOA Question 6.20 : (B) 459

For a special fully discrete 3-year term insurance on (75), you are given:

1. The death benefit during the first two years is the sum of the net premiums paid without interest
2. The death benefit in the third year is 10,000

x	p_x
75	0.90
76	0.88
77	0.85

3. $i = 0.04$

Calculate the annual net premium.

```
life = LifeTable().set_interest(i=.04).set_table(p={75: .9, 76: .88, 77: .85})
a = life.temporary_annuity(75, t=3)
IA = life.increasing_insurance(75, t=2)
A = life.deferred_insurance(75, u=2, t=1)
def fun(P): return life.gross_premium(a=a, A=P*IA + A*10000) - P
P = life.solve(fun, target=0, grid=[449, 489])
isclose(459, P, question="Q6.20")
```

```
----- Q6.20 459: 458.83181728297353 [OK] -----
```

```
True
```

SOA Question 6.21 : (C) 100

```
life = Recursion(verbose=False).set_interest(d=0.04)
life.set_A(0.7, x=75, t=15, endowment=1)
life.set_E(0.11, x=75, t=15)
def fun(P):
    return (P * life.temporary_annuity(75, t=15) -
            life.endowment_insurance(75, t=15, b=1000, endowment=15*float(P)))
P = life.solve(fun, target=0, grid=(80, 120))
isclose(100, P, question="Q6.21")
```

```
----- Q6.21 100: 100.85470085470084 [OK] -----
```

```
True
```

SOA Question 6.22 : (C) 102

For a whole life insurance of 100,000 on (45) with premiums payable monthly for a period of 20 years, you are given:

1. The death benefit is paid immediately upon death
2. Mortality follows the Standard Ultimate Life Table
3. Deaths are uniformly distributed over each year of age
4. $i = 0.05$

Calculate the monthly net premium.

```
life=SULT(udd=True)
a = UDD(m=12, life=life).temporary_annuity(45, t=20)
A = UDD(m=0, life=life).whole_life_insurance(45)
P = life.gross_premium(A=A, a=a, benefit=100000) / 12
isclose(102, P, question="Q6.22")
```

```
----- Q6.22 102: 102.40668704849178 [OK] -----
```

```
True
```

SOA Question 6.23 : (D) 44.7

```
x = 0
life = Recursion().set_a(15.3926, x=x)\
                  .set_a(10.1329, x=x, t=15)\
                  .set_a(14.0145, x=x, t=30)
def fun(P):
    per_policy = 30 + (30 * life.whole_life_annuity(x))
    per_premium = (0.6 + 0.1*life.temporary_annuity(x, t=15)
                   + 0.1*life.temporary_annuity(x, t=30))
    a = life.temporary_annuity(x, t=30)
    return (P * a) - (per_policy + per_premium * P)
P = life.solve(fun, target=0, grid=[30.3, 49.5])
isclose(44.7, P, question="Q6.23")
```

```
----- Q6.23 44.7: 44.70806635781144 [OK] -----
```

```
True
```

SOA Question 6.24 : (E) 0.30

For a fully continuous whole life insurance of 1 on (x), you are given:

1. L is the present value of the loss at issue random variable if the premium rate is determined by the equivalence principle
2. L^* is the present value of the loss at issue random variable if the premium rate is 0.06
3. $\delta = 0.07$
4. $\overline{A}_x = 0.30$
5. $Var(L) = 0.18$

Calculate $Var(L^*)$.

```
life = PolicyValues().set_interest(delta=0.07)
x, A1 = 0, 0.30   # Policy for first insurance
P = life.premium_equivalence(A=A1, discrete=False)  # Need its premium
contract = Contract(premium=P, discrete=False)
def fun(A2):  # Solve for A2, given Var(Loss)
    return life.gross_variance_loss(A1=A1, A2=A2, contract=contract)
A2 = life.solve(fun, target=0.18, grid=0.18)

contract = Contract(premium=0.06, discrete=False) # Solve second insurance
var = life.gross_variance_loss(A1=A1, A2=A2, contract=contract)
isclose(0.304, var, question="Q6.24")
```

```
----- Q6.24 0.304: 0.30419999999999975 [OK] -----
```

```
True
```

SOA Question 6.25 : (C) 12330

For a fully discrete 10-year deferred whole life annuity-due of 1000 per month on (55), you are given:

1. The premium, G, will be paid annually at the beginning of each year during the deferral period
2. Expenses are expected to be 300 per year for all years, payable at the beginning of the year
3. Mortality follows the Standard Ultimate Life Table
4. $i = 0.05$
5. Using the two-term Woolhouse approximation, the expected loss at issue is -800

Calculate G.

```
life = SULT()
woolhouse = Woolhouse(m=12, life=life)
benefits = woolhouse.deferred_annuity(55, u=10, b=1000 * 12)
expenses = life.whole_life_annuity(55, b=300)
payments = life.temporary_annuity(55, t=10)
def fun(P):
    return life.gross_future_loss(A=benefits + expenses, a=payments,
                                  contract=Contract(premium=P))
P = life.solve(fun, target=-800, grid=[12110, 12550])
isclose(12330, P, question="Q6.25")
```

```
----- Q6.25 12330: 12325.781125438532 [OK] -----
```

```
True
```

SOA Question 6.26 : (D) 180

For a special fully discrete whole life insurance policy of 1000 on (90), you are given:

1. The first year premium is 0
2. *P* is the renewal premium
3. Mortality follows the Standard Ultimate Life Table
4. $i = 0.05$
5. Premiums are calculated using the equivalence principle

Calculate *P*.

```
life = SULT().set_interest(i=0.05)
def fun(P):
    return P - life.net_premium(90, b=1000, initial_cost=P)
P = life.solve(fun, target=0, grid=[150, 190])
isclose(180, P, question="Q6.26")
```

```
----- Q6.26 180: 180.03164891315885 [OK] -----
```

```
True
```

SOA Question 6.27 : (D) 10310

For a special fully continuous whole life insurance on (x), you are given:

1. Premiums and benefits:

	First 20 years	After 20 years
Premium Rate	3P	P
Benefit	1,000,000	500,000

2. $\mu_{x+t} = 0.03, \quad t \geq 0$
3. $\delta = 0.06$

Calculate P using the equivalence principle.

```
life = ConstantForce(mu=0.03).set_interest(delta=0.06)
x = 0
payments = (3 * life.temporary_annuity(x, t=20, discrete=False)
            + life.deferred_annuity(x, u=20, discrete=False))
benefits = (1000000 * life.term_insurance(x, t=20, discrete=False)
            + 500000 * life.deferred_insurance(x, u=20, discrete=False))
P = benefits / payments
isclose(10310, P, question="Q6.27")
```

```
----- Q6.27 10310: 10309.617799001708 [OK] -----
```

```
True
```

SOA Question 6.28 : (B) 36

```
life = SULT().set_interest(i=0.05)
a = life.temporary_annuity(40, t=5)
A = life.whole_life_insurance(40)
P = life.gross_premium(a=a, A=A, benefit=1000,
                       initial_policy=10, renewal_premium=.05,
                       renewal_policy=5, initial_premium=.2)
isclose(36, P, question="Q6.28")
```

```
----- Q6.28 36: 35.72634219391481 [OK] -----
```

```
True
```

SOA Question 6.29 : (B) 20.5

(35) purchases a fully discrete whole life insurance policy of 100,000. You are given:

1. The annual gross premium, calculated using the equivalence principle, is 1770
2. The expenses in policy year 1 are 50% of premium and 200 per policy
3. The expenses in policy years 2 and later are 10% of premium and 50 per policy
4. All expenses are incurred at the beginning of the policy year
5. $i = 0.035$

Calculate $\ddot{a}_{35}$.

```
life = Premiums().set_interest(i=0.035)
def fun(a):
    return life.gross_premium(A=life.insurance_twin(a=a), a=a,
                              initial_policy=200, initial_premium=.5,
                              renewal_policy=50, renewal_premium=.1,
                              benefit=100000)
a = life.solve(fun, target=1770, grid=[20, 22])
isclose(20.5, a, question="Q6.29")
```

```
----- Q6.29 20.5: 20.480268314431726 [OK] -----
```

```
True
```

SOA Question 6.30 : (A) 900

For a fully discrete whole life insurance of 100 on (x), you are given:

1. The first year expense is 10% of the gross annual premium
2. Expenses in subsequent years are 5% of the gross annual premium
3. The gross premium calculated using the equivalence principle is 2.338
4. $i = 0.04$
5. $\ddot{a}_x = 16.50$
6. ${}^2A_x = 0.17$

Calculate the variance of the loss at issue random variable.

```
life = PolicyValues().set_interest(i=0.04)
contract = Contract(premium=2.338,
                    benefit=100,
                    initial_premium=.1,
                    renewal_premium=0.05)
var = life.gross_variance_loss(A1=life.insurance_twin(16.50),
                               A2=0.17, contract=contract)
isclose(900, var, question="Q6.30")
```

```
----- Q6.30 900: 908.141412994607 [OK] -----
```

```
True
```

SOA Question 6.31 : (D) 1330

For a fully continuous whole life insurance policy of 100,000 on (35), you are given:

1. The density function of the future lifetime of a newborn: $$
2. $\delta = 0.05$
3. $\overline{A}_{70} = 0.51791$

Calculate the annual net premium rate for this policy.

```
life = ConstantForce(mu=0.01).set_interest(delta=0.05)
A = (life.term_insurance(35, t=35, discrete=False)
     + life.E_x(35, t=35)*0.51791)      # A_35
P = life.premium_equivalence(A=A, b=100000, discrete=False)
isclose(1330, P, question="Q6.31")
```

```
----- Q6.31 1330: 1326.5406293909457 [OK] -----
```

```
True
```

SOA Question 6.32 : (C) 550

For a whole life insurance of 100,000 on (x), you are given:

1. Death benefits are payable at the moment of death
2. Deaths are uniformly distributed over each year of age
3. Premiums are payable monthly
4. $i = 0.05$
5. $\ddot{a}_x = 9.19$

Calculate the monthly net premium.

```
x = 0
life = Recursion().set_interest(i=0.05).set_a(9.19, x=x)
benefits = UDD(m=0, life=life).whole_life_insurance(x)
payments = UDD(m=12, life=life).whole_life_annuity(x)
P = life.gross_premium(a=payments, A=benefits, benefit=100000)/12
isclose(550, P, question="Q6.32")
```

Whole Life Insurance A_x := $\ddot{a}_x = [1 - A_x]/d$ annuity twin

```
----- Q6.32 550: 550.4356936711871 [OK] -----
```

```
True
```

SOA Question 6.33 : (B) 0.13

An insurance company sells 15-year pure endowments of 10,000 to 500 lives, each age x, with independent future lifetimes. The single premium for each pure endowment is determined by the equivalence principle.

You are given:

1. $i = 0.03$
2. $\mu_x(t) = 0.02t, \quad t \geq 0$
3. ${}_0L$ is the aggregate loss at issue random variable for these pure endowments.

Using the normal approximation without continuity correction, calculate $Pr({}_0L) > 50,000)$.

```
life = Insurance().set_survival(mu=lambda x,t: 0.02*t).set_interest(i=0.03)
x = 0
var = life.E_x(x, t=15, moment=life.VARIANCE, endowment=10000)
p = 1- life.portfolio_cdf(mean=0, variance=var, value=50000, N=500)
isclose(0.13, p, question="Q6.33", rel_tol=0.02)
```

```
----- Q6.33 0.13: 0.12828940905648634 [OK] -----
```

```
True
```

SOA Question 6.34 : (A) 23300

For a fully discrete whole life insurance policy on (61), you are given:

1. The annual gross premium using the equivalence principle is 500
2. Initial expenses, incurred at policy issue, are 15% of the premium
3. Renewal expenses, incurred at the beginning of each year after the first, are 3% of the premium
4. Mortality follows the Standard Ultimate Life Table
5. $i = 0.05$

Calculate the amount of the death benefit.

```
life = SULT()
def fun(benefit):
    A = life.whole_life_insurance(61)
    a = life.whole_life_annuity(61)
    return life.gross_premium(A=A, a=a, benefit=benefit,
                              initial_premium=0.15, renewal_premium=0.03)
b = life.solve(fun, target=500, grid=[23300, 23700])
isclose(23300, b, question="Q6.34")
```

```
----- Q6.34 23300: 23294.288659265632 [OK] -----
```

```
True
```

SOA Question 6.35 : (D) 530

For a fully discrete whole life insurance policy of 100,000 on (35), you are given:

1. First year commissions are 19% of the annual gross premium
2. Renewal year commissions are 4% of the annual gross premium
3. Mortality follows the Standard Ultimate Life Table
4. $i = 0.05$

Calculate the annual gross premium for this policy using the equivalence principle.

```
sult = SULT()
A = sult.whole_life_insurance(35, b=100000)
a = sult.whole_life_annuity(35)
P = sult.gross_premium(a=a, A=A, initial_premium=.19, renewal_premium=.04)
isclose(530, P, question="Q6.35")
```

```
----- Q6.35 530: 534.4072234303344 [OK] -----
```

```
True
```

SOA Question 6.36 : (B) 500

```
life = ConstantForce(mu=0.04).set_interest(delta=0.08)
a = life.temporary_annuity(50, t=20, discrete=False)
A = life.term_insurance(50, t=20, discrete=False)
def fun(R):
    return life.gross_premium(a=a, A=A, initial_premium=R/4500,
                              renewal_premium=R/4500, benefit=100000)
R = life.solve(fun, target=4500, grid=[400, 800])
isclose(500, R, question="Q6.36")
```

```
----- Q6.36 500: 500.0 [OK] -----
```

```
True
```

SOA Question 6.37 : (D) 820

For a fully discrete whole life insurance policy of 50,000 on (35), with premiums payable for a maximum of 10 years, you are given:

1. Expenses of 100 are payable at the end of each year including the year of death
2. Mortality follows the Standard Ultimate Life Table
3. $i = 0.05$

Calculate the annual gross premium using the equivalence principle.

```
sult = SULT()
benefits = sult.whole_life_insurance(35, b=50000 + 100)
expenses = sult.immediate_annuity(35, b=100)
a = sult.temporary_annuity(35, t=10)
P = (benefits + expenses) / a
isclose(820, P, question="Q6.37")
```

```
----- Q6.37 820: 819.7190338249138 [OK] -----
```

```
True
```

SOA Question 6.38 : (B) 11.3

For an n-year endowment insurance of 1000 on (x), you are given:

1. Death benefits are payable at the moment of death
2. Premiums are payable annually at the beginning of each year
3. Deaths are uniformly distributed over each year of age
4. $i = 0.05$
5. ${}_nE_x = 0.172$

6. $\overline{A}_{x:\overline{n|}} = 0.192$

Calculate the annual net premium for this insurance.

```
x, n = 0, 10
life = Recursion().set_interest(i=0.05)\
                  .set_A(0.192, x=x, t=n, endowment=1, discrete=False)\
                  .set_E(0.172, x=x, t=n)
a = life.temporary_annuity(x, t=n, discrete=False)

def fun(a):   # solve for discrete annuity, given continuous
    life = Recursion(verbose=False).set_interest(i=0.05)\
                                   .set_a(a, x=x, t=n)\
                                   .set_E(0.172, x=x, t=n)
    return UDD(m=0, life=life).temporary_annuity(x, t=n)
a = life.solve(fun, target=a, grid=a)  # discrete annuity
P = life.gross_premium(a=a, A=0.192, benefit=1000)
isclose(11.3, P, question="Q6.38")
```

Temporary Annuity $a_{x:\overline{10|}}$:=
$\ddot{a}_{x:\overline{10|}} = [1 - A_{x:\overline{10|}}]/d$ annuity twin

```
----- Q6.38 11.3: 11.308644185253657 [OK] -----
```

```
True
```

SOA Question 6.39 : (A) 29

XYZ Insurance writes 10,000 fully discrete whole life insurance policies of 1000 on lives age 40 and an additional 10,000 fully discrete whole life policies of 1000 on lives age 80.

XYZ used the following assumptions to determine the net premiums for these policies:

1. Mortality follows the Standard Ultimate Life Table
2. i = 0.05

During the first ten years, mortality did follow the Standard Ultimate Life Table.

Calculate the average net premium per policy in force received at the beginning of the eleventh year.

```
sult = SULT()
P40 = sult.premium_equivalence(sult.whole_life_insurance(40), b=1000)
P80 = sult.premium_equivalence(sult.whole_life_insurance(80), b=1000)
p40 = sult.p_x(40, t=10)
p80 = sult.p_x(80, t=10)
P = (P40 * p40 + P80 * p80) / (p80 + p40)
isclose(29, P, question="Q6.39")
```

```
----- Q6.39 29: 29.033866427845496 [OK] -----
```

```
True
```

SOA Question 6.40 : (C) 116

For a special fully discrete whole life insurance, you are given:

1. The death benefit is $1000(1.03)^k$ for death in policy year k, for $k = 1, 2, 3...$
2. $q_x = 0.05$
3. $i = 0.06$
4. $\ddot{a}_{x+1} = 7.00$
5. The annual net premium for this insurance at issue age x is 110

Calculate the annual net premium for this insurance at issue age $x + 1$.

```
# - standard formula discounts/accumulates by too much (i should be smaller)
x = 0
life = Recursion().set_interest(i=0.06).set_a(7, x=x+1).set_q(0.05, x=x)
a = life.whole_life_annuity(x)
A = 110 * a / 1000
life = Recursion().set_interest(i=0.06).set_A(A, x=x).set_q(0.05, x=x)
A1 = life.whole_life_insurance(x+1)
P = life.gross_premium(A=A1 / 1.03, a=7) * 1000
isclose(116, P, question="Q6.40")
```

Whole Life Annuity $\ddot{a}_x$:=

$\ddot{a}_x = 1 + E_x * \ddot{a}_{x+1}$ backward recursion

$E_x = p_x * v$ pure endowment

$p_x = 1 - q_x$ complement of mortality

Whole Life Insurance A_{x+1} :=

$A_{x+1} = [\,A_x/v - q_x * b]/\,p_x$ forward recursion

$p_x = 1 - q_x$ complement of mortality

```
----- Q6.40 116: 116.51945397474269 [OK] -----
```

```
True
```

SOA Question 6.41 : (B) 1417

For a special fully discrete 2-year term insurance on (x), you are given:

1. $q_x = 0.01$
2. $q_{x+1} = 0.02$
3. $i = 0.05$
4. The death benefit in the first year is 100,000
5. Both the benefits and premiums increase by 1% in the second year

Calculate the annual net premium in the first year.

```
x = 0
life = LifeTable().set_interest(i=0.05).set_table(q={x:.01, x+1:.02})
a = 1 + life.E_x(x, t=1) * 1.01
A = life.deferred_insurance(x, u=0, t=1) + 1.01*life.deferred_insurance(x, u=1, t=1)
P = 100000 * A / a
isclose(1417, P, question="Q6.41")
```

```
----- Q6.41 1417: 1416.9332301924137 [OK] -----
```

```
True
```

SOA Question 6.42 : (D) 0.113

```
x = 0
life = ConstantForce(mu=0.06).set_interest(delta=0.06)
contract = Contract(discrete=True, premium=315.8,
                    T=3, endowment=1000, benefit=1000)
L = [life.L_from_t(t, contract=contract) for t in range(3)]     # L(t)
Q = [life.q_x(x, u=u, t=1) for u in range(3)]               # prob(die in year t)
Q[-1] = 1 - sum(Q[:-1])    # follows SOA Solution: incorrectly treats endowment!
p = sum([q for (q, l) in zip (Q, L) if l > 0])
isclose(0.113, p, question="Q6.42")
```

```
----- Q6.42 0.113: 0.11307956328284252 [OK] -----
```

```
True
```

SOA Question 6.43 : (C) 170

For a fully discrete, 5-payment 10-year term insurance of 200,000 on (30), you are given:

1. Mortality follows the Standard Ultimate Life Table
2. The following expenses are incurred at the beginning of each respective year:

	Percent of Premium	Per Policy	Percent of Premium	Per Policy
	Year 1	Year 1	Years 2 - 10	Years 2 - 10
Taxes	5%	0	5%	0
Commissions	30%	0	10%	0
Maintenance	0%	8	0%	4

3. i = 0.05
4. $\ddot{a}_{30:\overline{5|}} = 4.5431$

Calculate the annual gross premium using the equivalence principle.

- although 10-year term, premiums only paid first first years: separately calculate the EPV of per-policy maintenance expenses in years 6-10 and treat as additional initial expense

```
sult = SULT()
a = sult.temporary_annuity(30, t=5)
A = sult.term_insurance(30, t=10)
other_expenses = 4 * sult.deferred_annuity(30, u=5, t=5)
P = sult.gross_premium(a=a, A=A, benefit=200000, initial_premium=0.35,
                       initial_policy=8 + other_expenses, renewal_policy=4,
                       renewal_premium=0.15)
isclose(170, P, question="Q6.43")
```

```
----- Q6.43 170: 171.22371939459944 [OK] -----
```

```
True
```

SOA Question 6.44 : (D) 2.18

```
life = Recursion().set_interest(i=0.05)\
                  .set_IA(0.15, x=50, t=10)\
                  .set_a(17, x=50)\
                  .set_a(15, x=60)\
                  .set_E(0.6, x=50, t=10)
A = life.deferred_insurance(50, u=10)
IA = life.increasing_insurance(50, t=10)
a = life.temporary_annuity(50, t=10)
P = life.gross_premium(a=a, A=A, IA=IA, benefit=100)
isclose(2.2, P, question="Q6.44")
```

Whole Life Insurance A_{x+60} := $\ddot{a}_{x+60} = [1 - A_{x+60}]/d$ annuity twin

Whole Life Insurance A_{x+60} := $\ddot{a}_{x+60} = [1 - A_{x+60}]/d$ annuity twin

Whole Life Insurance A_{x+60} := $\ddot{a}_{x+60} = [1 - A_{x+60}]/d$ annuity twin

```
----- Q6.44 2.2: 2.183803457688809 [OK] -----
```

```
True
```

SOA Question 6.45 : (E) 690

For a fully continuous whole life insurance of 100,000 on (35), you are given:

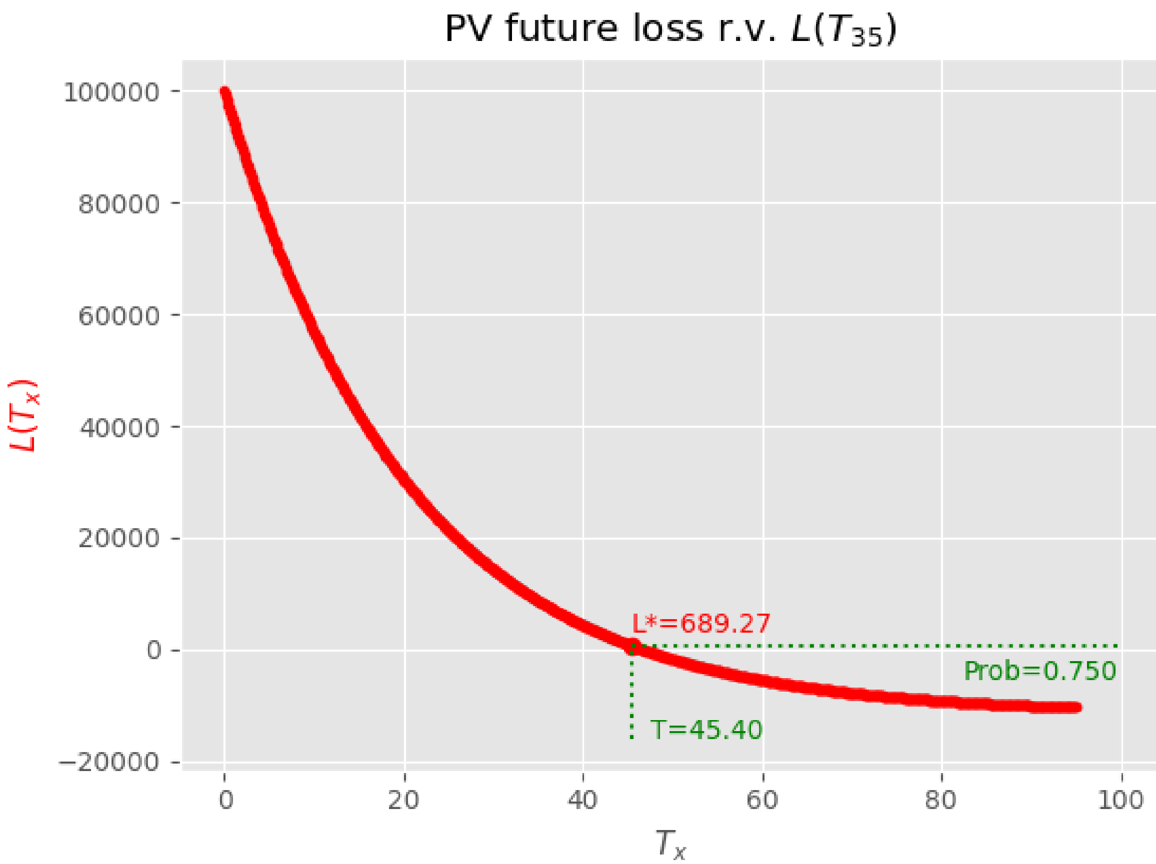
1. The annual rate of premium is 560
2. Mortality follows the Standard Ultimate Life Table
3. Deaths are uniformly distributed over each year of age
4. $i = 0.05$

Calculate the 75th percentile of the loss at issue random variable for this policy.

```
life = SULT(udd=True)
contract = Contract(benefit=100000, premium=560, discrete=False)
L = life.L_from_prob(x=35, prob=0.75, contract=contract)
life.L_plot(x=35, contract=contract,
            T=life.L_to_t(L=L, contract=contract))
isclose(690, L, question="Q6.45")
```

```
----- Q6.45 690: 689.2659416264196 [OK] -----
```

```
True
```



SOA Question 6.46 : (E) 208

```
life = Recursion().set_interest(i=0.05)\
                   .set_IA(0.51213, x=55, t=10)\
                   .set_a(12.2758, x=55)\
                   .set_a(7.4575, x=55, t=10)
A = life.deferred_annuity(55, u=10)
IA = life.increasing_insurance(55, t=10)
a = life.temporary_annuity(55, t=10)
P = life.gross_premium(a=a, A=A, IA=IA, benefit=300)
isclose(208, P, question="Q6.46")
```

```
----- Q6.46 208: 208.12282139036515 [OK] -----
```

```
True
```

SOA Question 6.47 : (D) 66400

For a 10-year deferred whole life annuity-due with payments of 100,000 per year on (70), you are given:

1. Annual gross premiums of G are payable for 10 years
2. First year expenses are 75% of premium

3. Renewal expenses for years 2 and later are 5% of premium during the premium paying period
4. Mortality follows the Standard Ultimate Life Table
5. i = 0.05

Calculate G using the equivalence principle.

```
sult = SULT()
a = sult.temporary_annuity(70, t=10)
A = sult.deferred_annuity(70, u=10)
P = sult.gross_premium(a=a, A=A, benefit=100000, initial_premium=0.75,
                        renewal_premium=0.05)
isclose(66400, P, question="Q6.47")
```

```
----- Q6.47 66400: 66384.13293704337 [OK] -----
```

```
True
```

SOA Question 6.48 : (A) 3195

For a special fully discrete 5-year deferred 3-year term insurance of 100,000 on (x) you are given:

1. There are two premium payments, each equal to P . The first is paid at the beginning of the first year and the second is paid at the end of the 5-year deferral period
2. $p_x = 0.95$
3. $q_{x+5} = 0.02$
4. $q_{x+6} = 0.03$
5. $q_{x+7} = 0.04$
6. $i = 0.06$

Calculate P using the equivalence principle.

```
x = 0
life = Recursion(depth=5).set_interest(i=0.06)\
                         .set_p(.95, x=x, t=5)\
                         .set_q(.02, x=x+5)\
                         .set_q(.03, x=x+6)\
                         .set_q(.04, x=x+7)
a = 1 + life.E_x(x, t=5)
A = life.deferred_insurance(x, u=5, t=3)
P = life.gross_premium(A=A, a=a, benefit=100000)
isclose(3195, P, question="Q6.48")
```

Pure Endowment ${}_5E_x$:=

${}_5E_x = {}_5p_x * v^5$ pure endowment

Pure Endowment ${}_5E_x$:=

${}_5E_x = {}_5p_x * v^5$ pure endowment

Term Insurance $A^{1}_{x+5:\overline{3|}}$:=

$A^{1}_{x+5:\overline{1|}} = A_{x+5:\overline{1|}} - E_{x+5}$	endowment insurance - pure
$A_{x+5:\overline{3|}} = A^{1}_{x+5:\overline{3|}} + {}_3E_{x+5}$	term plus pure endowment
${}_3E_{x+5} = {}_3p_{x+5} * v^3$	pure endowment
${}_3p_{x+5} = {}_2p_{x+6} * p_{x+5}$	survival chain rule
${}_2p_{x+6} = p_{x+7} * p_{x+6}$	survival chain rule
$A^{1}_{x+5:\overline{3|}} = A_{x+5:\overline{3|}} - {}_3E_{x+5}$	endowment insurance - pure
${}_3E_{x+5} = E_{x+5} * {}_2E_{x+6}$	pure endowment chain rule
${}_2E_{x+6} = E_{x+6} * E_{x+7}$	pure endowment chain rule
$E_{x+7} = p_{x+7} * v$	pure endowment
$E_{x+6} = p_{x+6} * v$	pure endowment
$E_{x+5} = p_{x+5} * v$	pure endowment
$p_{x+7} = 1 - q_{x+7}$	complement of mortality
$A^{1}_{x+5:\overline{3|}} = v * [\, q_{x+5} * b + p_{x+5} * A^{1}_{x+6:\overline{2|}}]$	backward recursion
$A^{1}_{x+6:\overline{2|}} = v * [\, q_{x+6} * b + p_{x+6} * A^{1}_{x+7:\overline{1|}}]$	backward recursion
$p_{x+6} = 1 - q_{x+6}$	complement of mortality
$p_{x+5} = 1 - q_{x+5}$	complement of mortality

Term Insurance $A^{1}_{x+5:\overline{3|}}$:=

$A^{1}_{x+5:\overline{1|}} = A_{x+5:\overline{1|}} - E_{x+5}$	endowment insurance - pure
$A_{x+5:\overline{3|}} = A^{1}_{x+5:\overline{3|}} + {}_3E_{x+5}$	term plus pure endowment
${}_3E_{x+5} = {}_3p_{x+5} * v^3$	pure endowment
${}_3p_{x+5} = {}_2p_{x+6} * p_{x+5}$	survival chain rule
${}_2p_{x+6} = p_{x+7} * p_{x+6}$	survival chain rule
$A^{1}_{x+5:\overline{3|}} = A_{x+5:\overline{3|}} - {}_3E_{x+5}$	endowment insurance - pure
${}_3E_{x+5} = E_{x+5} * {}_2E_{x+6}$	pure endowment chain rule
${}_2E_{x+6} = E_{x+6} * E_{x+7}$	pure endowment chain rule
$E_{x+7} = p_{x+7} * v$	pure endowment
$E_{x+6} = p_{x+6} * v$	pure endowment
$E_{x+5} = p_{x+5} * v$	pure endowment
$p_{x+7} = 1 - q_{x+7}$	complement of mortality
$A^{1}_{x+5:\overline{3|}} = v * [\, q_{x+5} * b + p_{x+5} * A^{1}_{x+6:\overline{2|}}]$	backward recursion
$A^{1}_{x+6:\overline{2|}} = v * [\, q_{x+6} * b + p_{x+6} * A^{1}_{x+7:\overline{1|}}]$	backward recursion
$p_{x+6} = 1 - q_{x+6}$	complement of mortality
$p_{x+5} = 1 - q_{x+5}$	complement of mortality

```
----- Q6.48 3195: 3195.118917658744 [OK] -----
```

```
True
```

SOA Question 6.49 : (C) 86

For a special whole life insurance of 100,000 on (40), you are given:

1. The death benefit is payable at the moment of death
2. Level gross premiums are payable monthly for a maximum of 20 years
3. Mortality follows the Standard Ultimate Life Table
4. $i = 0.05$
5. Deaths are uniformly distributed over each year of age
6. Initial expenses are 200

7. Renewal expenses are 4% of each premium including the first
8. Gross premiums are calculated using the equivalence principle

Calculate the monthly gross premium.

```
sult = SULT(udd=True)
a = UDD(m=12, life=sult).temporary_annuity(40, t=20)
A = sult.whole_life_insurance(40, discrete=False)
P = sult.gross_premium(a=a, A=A, benefit=100000, initial_policy=200,
                       renewal_premium=0.04, initial_premium=0.04) / 12
isclose(86, P, question="Q6.49")
```

```
----- Q6.49 86: 85.99177833261696 [OK] -----
```

```
True
```

SOA Question 6.50 : (A) -47000

On July 15, 2017, XYZ Corp buys fully discrete whole life insurance policies of 1,000 on each of its 10,000 workers, all age 35. It uses the death benefits to partially pay the premiums for the following year.

You are given:

1. Mortality follows the Standard Ultimate Life Table
2. $i = 0.05$
3. The insurance is priced using the equivalence principle

Calculate XYZ Corp's expected net cash flow from these policies during July 2018.

```
life = SULT()
P = life.premium_equivalence(a=life.whole_life_annuity(35), b=1000)
a = life.deferred_annuity(35, u=1, t=1)
A = life.term_insurance(35, t=1, b=1000)
cash = (A - a * P) * 10000 / life.interest.v
isclose(-47000, cash, question="Q6.50")
```

```
----- Q6.50 -47000: -46948.2187697819 [OK] -----
```

```
True
```

SOA Question 6.51 : (D) 34700

```
life = Recursion().set_DA(0.4891, x=62, t=10)\
                  .set_A(0.0910, x=62, t=10)\
                  .set_a(12.2758, x=62)\
                  .set_a(7.4574, x=62, t=10)
IA = life.increasing_insurance(62, t=10)
A = life.deferred_annuity(62, u=10)
a = life.temporary_annuity(62, t=10)
P = life.gross_premium(a=a, A=A, IA=IA, benefit=50000)
isclose(34700, P, question="Q6.51")
```

$$\texttt{Increasing Insurance } (IA)_{x+62:\overline{10|}} :=$$
$$(IA)_{x+62:\overline{10|}} = 11\ A^{1}_{x+62:\overline{10|}} - (DA)_{x+62:\overline{10|}} \quad \texttt{varying insurance identity}$$

```
----- Q6.51 34700: 34687.207544453246 [OK] -----
```

```
True
```

SOA Question 6.52 : (D) 50.80

For a fully discrete 10-payment whole life insurance of H on (45), you are given:

1. Expenses payable at the beginning of each year are as follows:

Expense Type	First Year	Years 2-10	Years 11+
Per policy	100	20	10
% of Premium	105%	5%	0%

2. Mortality follows the Standard Ultimate Life Table
3. i = 0.05
4. The gross annual premium, calculated using the equivalence principle, is of the form $G = gH + f$, where g is the premium rate per 1 of insurance and f is the per policy fee

Calculate f.

- set face value benefits to 0

```
sult = SULT()
a = sult.temporary_annuity(45, t=10)
other_cost = 10 * sult.deferred_annuity(45, u=10)
P = sult.gross_premium(a=a, A=0, benefit=0,     # set face value H = 0
                       initial_premium=1.05, renewal_premium=0.05,
                       initial_policy=100 + other_cost, renewal_policy=20)
isclose(50.8, P, question="Q6.52")
```

```
----- Q6.52 50.8: 50.80135534704229 [OK] -----
```

```
True
```

SOA Question 6.53 : (D) 720

A warranty pays 2000 at the end of the year of the first failure if a washing machine fails within three years of purchase. The warranty is purchased with a single premium, G, paid at the time of purchase of the washing machine. You are given:

1. 10% of the washing machines that are working at the start of each year fail by the end of that year
2. *i* = 0.08
3. The sales commission is 35% of G
4. G is calculated using the equivalence principle

Calculate G.

```
x = 0
life = LifeTable().set_interest(i=0.08).set_table(q={x:.1, x+1:.1, x+2:.1})
A = life.term_insurance(x, t=3)
P = life.gross_premium(a=1, A=A, benefit=2000, initial_premium=0.35)
isclose(720, P, question="Q6.53")
```

```
----- Q6.53 720: 720.1646090534978 [OK] -----
```

```
True
```

SOA Question 6.54 : (A) 25440

For a fully discrete whole life insurance of 200,000 on (45), you are given:

1. Mortality follows the Standard Ultimate Life Table.
2. $i = 0.05$
3. The annual premium is determined using the equivalencep rinciple.

Calculate the standard deviation of ${}_0L$, the present value random variable for the loss at issue.

[A modified version of Question 12 on the Fall 2017 exam]

```
life = SULT()
std = math.sqrt(life.net_policy_variance(45, b=200000))
isclose(25440, std, question="Q6.54")
```

```
----- Q6.54 25440: 25441.694847703857 [OK] -----
```

```
True
```

22.7 7 Policy Values

SOA Question 7.1 : (C) 11150

For a special fully discrete whole life insurance on (40), you are given:

1. The death benefit is 50,000 in the first 20 years and 100,000 thereafter
2. Level net premiums of 875 are payable for 20 years
3. Mortality follows the Standard Ultimate Life Table
4. $i = 0.05$

Calculate ${}_{10}V$ the net premium policy value at the end of year 10 for this insurance.

```
life = SULT()
x, n, t = 40, 20, 10
A = (life.whole_life_insurance(x+t, b=50000)
     + life.deferred_insurance(x+t, u=n-t, b=50000))
a = life.temporary_annuity(x+t, t=n-t, b=875)
L = life.gross_future_loss(A=A, a=a)
isclose(11150, L, question="Q7.1")
```

```
----- Q7.1 11150: 11152.108749338717 [OK] -----
```

```
True
```

SOA Question 7.2 : (C) 1152

```
x = 0
life = Recursion(verbose=False).set_interest(i=.1)\
                               .set_q(0.15, x=x)\
                               .set_q(0.165, x=x+1)\
                               .set_reserves(T=2, endowment=2000)

def fun(P):  # solve P s.t. V is equal backwards and forwards
    policy = dict(t=1, premium=P, benefit=lambda t: 2000, reserve_benefit=True)
    return life.t_V_backward(x, **policy) - life.t_V_forward(x, **policy)
P = life.solve(fun, target=0, grid=[1070, 1230])
isclose(1152, P, question="Q7.2")
```

```
----- Q7.2 1152: 1151.5151515151515 [OK] -----
```

```
True
```

SOA Question 7.3 : (E) 730

```
x = 0  # x=0 is (90) and interpret every 3 months as t=1 year
life = LifeTable().set_interest(i=0.08/4)\
                  .set_table(l={0:1000, 1:898, 2:800, 3:706})\
                  .set_reserves(T=8, V={3: 753.72})
V = life.t_V_backward(x=0, t=2, premium=60*0.9, benefit=lambda t: 1000)
V = life.set_reserves(V={2: V})\
        .t_V_backward(x=0, t=1, premium=0, benefit=lambda t: 1000)
isclose(730, V, question="Q7.3")
```

```
----- Q7.3 730: 729.998398765594 [OK] -----
```

```
True
```

SOA Question 7.4 : (B) -74

For a special fully discrete whole life insurance on (40), you are given:

1. The death benefit is 1000 during the first 11 years and 5000 thereafter
2. Expenses, payable at the beginning of the year, are 100 in year 1 and 10 in years 2 and later
3. π is the level annual premium, determined using the equivalence principle
4. $G = 1.02 \times \pi$ is the level annual gross premium
5. Mortality follows the Standard Ultimate Life Table
6. $i = 0.05$
7. ${}_{11}E_{40} = 0.57949$

Calculate the gross premium policy value at the end of year 1 for this insurance.

hints:

- split benefits into two policies

```
life = SULT()
P = life.gross_premium(a=life.whole_life_annuity(40),
                       A=life.whole_life_insurance(40),
                       initial_policy=100, renewal_policy=10,
                       benefit=1000)
P += life.gross_premium(a=life.whole_life_annuity(40),
                        A=life.deferred_insurance(40, u=11),
                        benefit=4000)    # for deferred portion
contract = Contract(benefit=1000, premium=1.02*P,
                    renewal_policy=10, initial_policy=100)
V = life.gross_policy_value(x=40, t=1, contract=contract)
contract = Contract(benefit=4000, premium=0)
A = life.deferred_insurance(41, u=10)
V += life.gross_future_loss(A=A, a=0, contract=contract)   # for deferred portion
isclose(-74, V, question="Q7.4")
```

```
----- Q7.4 -74: -73.942155695248 [OK] -----
```

```
True
```

SOA Question 7.5 : (E) 1900

For a fully discrete whole life insurance of 10,000 on (x), you are given:

1. Deaths are uniformly distributed over each year of age
2. The net premium is 647.46
3. The net premium policy value at the end of year 4 is 1405.08
4. q_{x+4} = 0.04561
5. i = 0.03

Calculate the net premium policy value at the end of 4.5 years.

```
x = 0
life = Recursion(udd=True).set_interest(i=0.03)\
                          .set_q(0.04561, x=x+4)\
                          .set_reserves(T=3, V={4: 1405.08})
V = life.r_V_forward(x, s=4, r=0.5, benefit=10000, premium=647.46)
isclose(1900, V, question="Q7.5")
```

```
----- Q7.5 1900: 1901.766021537228 [OK] -----
```

```
True
```

Answer 7.6: (E) -25.4

```
life = SULT()
P = life.net_premium(45, b=2000)
contract = Contract(benefit=2000, initial_premium=.25, renewal_premium=.05,
                    initial_policy=2*1.5 + 30, renewal_policy=2*.5 + 10)
```

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```
G = life.gross_premium(a=life.whole_life_annuity(45), **contract.premium_terms)
gross = life.gross_policy_value(45, t=10, contract=contract.set_contract(premium=G))
net = life.net_policy_value(45, t=10, b=2000)
V = gross - net
isclose(-25.4, V, question="Q7.6")
```

```
----- Q7.6 -25.4: -25.44920289521204 [OK] -----
```

```
True
```

SOA Question 7.7 : (D) 1110

For a whole life insurance of 10,000 on (x), you are given:

1. Death benefits are payable at the end of the year of death
2. A premium of 30 is payable at the start of each month
3. Commissions are 5% of each premium
4. Expenses of 100 are payable at the start of each year
5. $i = 0.05$
6. $1000A_{x+10} = 400$
7. ${}_{10}V$ is the gross premium policy value at the end of year 10 for this insurance

Calculate ${}_{10}V$ using the two-term Woolhouse formula for annuities.

```
x = 0
life = Recursion().set_interest(i=0.05).set_A(0.4, x=x+10)
a = Woolhouse(m=12, life=life).whole_life_annuity(x+10)
contract = Contract(premium=0, benefit=10000, renewal_policy=100)
V = life.gross_future_loss(A=0.4, contract=contract.renewals())
contract = Contract(premium=30*12, renewal_premium=0.05)
V += life.gross_future_loss(a=a, contract=contract.renewals())
isclose(1110, V, question="Q7.7")
```

Whole Life Annuity $\ddot{a}_{x+10}$:=
$\ddot{a}_{x+10} = [1 - A_{x+10}]/d$ insurance twin

```
----- Q7.7 1110: 1107.9718253968258 [OK] -----
```

```
True
```

SOA Question 7.8 : (C) 29.85

```
sult = SULT()
x = 70
q = {x: [sult.q_x(x+k)*(.7 + .1*k) for k in range(3)] + [sult.q_x(x+3)]}
life = Recursion().set_interest(i=.05)\
                  .set_q(sult.q_x(70)*.7, x=x)\
```

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```
                   .set_reserves(T=3)
V = life.t_V(x=70, t=1, premium=35.168, benefit=lambda t: 1000)
isclose(29.85, V, question="Q7.8")
```

Survival p_{x+70} :=
$p_{x+70} = 1 - q_{x+70}$ complement of mortality

```
----- Q7.8 29.85: 29.85469179271202 [OK] -----
```

```
True
```

SOA Question 7.9 : (A) 38100

For a semi-continuous 20-year endowment insurance of 100,000 on (45), you are given:

1. Net premiums of 253 are payable monthly
2. Mortality follows the Standard Ultimate Life Table
3. Deaths are uniformly distributed over each year of age
4. $i = 0.05$

Calculate ${}_{10}V$, the net premium policy value at the end of year 10 for this insurance.

```
sult = SULT(udd=True)
x, n, t = 45, 20, 10
a = UDD(m=12, life=sult).temporary_annuity(x=x+10, t=n-t)
A = UDD(m=0, life=sult).endowment_insurance(x=x+10, t=n-t)
contract = Contract(premium=253*12, endowment=100000, benefit=100000)
V = sult.gross_future_loss(A=A, a=a, contract=contract)
isclose(38100, V, question="Q7.9")
```

```
----- Q7.9 38100: 38099.62176709246 [OK] -----
```

```
True
```

SOA Question 7.10 : (C) -970

For a fully discrete whole life insurance of 100,000 on (45), you are given:

1. Mortality follows the Standard Ultimate Life Table
2. i = 0.05
3. Commission expenses are 60% of the first year's gross premium and 2% of renewal gross premiums
4. Administrative expenses are 500 in the first year and 50 in each renewal year
5. All expenses are payable at the start of the year
6. The gross premium, calculated using the equivalence principle, is 977.60

Calculate ${}_{5}V^{e}$, the expense reserve at the end of year 5 for this insurance.

```
life = SULT()
G = 977.6
P = life.net_premium(45, b=100000)
contract = Contract(benefit=0, premium=G-P, renewal_policy=.02*G + 50)
V = life.gross_policy_value(45, t=5, contract=contract)
isclose(-970, V, question="Q7.10")
```

```
----- Q7.10 -970: -971.8909301877826 [OK] -----
```

```
True
```

SOA Question 7.11 : (B) 1460

```
life = Recursion().set_interest(i=0.05).set_a(13.4205, x=55)
contract = Contract(benefit=10000)
def fun(P):
    return life.L_from_t(t=10, contract=contract.set_contract(premium=P))
P = life.solve(fun, target=4450, grid=400)
V = life.gross_policy_value(45, t=10, contract=contract.set_contract(premium=P))
isclose(1460, V, question="Q7.11")
```

Whole Life Insurance A_{x+55} :=
$\ddot{a}_{x+55} = [1 - A_{x+55}]/d$ annuity twin

```
----- Q7.11 1460: 1459.9818035330218 [OK] -----
```

```
True
```

SOA Question 7.12 : (E) 4.09

For a special fully discrete 25-year endowment insurance on (44), you are given:

1. The death benefit is ($26-k$) for death in year ,k for $k = 1,2,3,\ldots,25$
2. The endowment benefit in year 25 is 1
3. Net premiums are level
4. $q_{55} = 0.15$
5. $i = 0.04$
6. ${}_{11}V$ the net premium policy value at the end of year 11, is 5.00
7. ${}_{24}V$ the net premium policy value at the end of year 24, is 0.60

Calculate ${}_{12}V$ the net premium policy value at end of year 12.

```
benefit = lambda k: 26 - k
x = 44
life = Recursion().set_interest(i=0.04)\
                  .set_q(0.15, x=55)\
                  .set_reserves(T=25, endowment=1, V={11: 5.})
def fun(P):  # solve for net premium, from final year recursion
```

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```
    return life.t_V(x=x, t=24, premium=P, benefit=benefit)
P = life.solve(fun, target=0.6, grid=0.5)     # solved net premium
V = life.t_V(x, t=12, premium=P, benefit=benefit)  # recursion formula
isclose(4.09, V, question="Q7.12")
```

Survival p_{x+55} :=
$p_{x+55} = 1 - q_{x+55}$ complement of mortality

```
----- Q7.12 4.09: 4.089411764705883 [OK] -----
```

```
True
```

SOA Question 7.13 : (A) 180

```
life = SULT()
V = life.FPT_policy_value(40, t=10, n=30, endowment=1000, b=1000)
isclose(180, V, question="Q7.13")
```

```
----- Q7.13 180: 180.1071785904076 [OK] -----
```

```
True
```

SOA Question 7.14 : (A) 2200

For a fully discrete whole life insurance of 100,000 on (45), you are given:

1. The gross premium policy value at duration 5 is 5500 and at duration 6 is 7100
2. $q_{50} = 0.009$
3. $i = 0.05$
4. Renewal expenses at the start of each year are 50 plus 4% of the gross premium.
5. Claim expenses are 200.

Calculate the gross premium.

```
x = 45
life = Recursion(verbose=False).set_interest(i=0.05)\
                               .set_q(0.009, x=50)\
                               .set_reserves(T=10, V={5: 5500})
def fun(P):  # solve for net premium,
    return life.t_V(x=x, t=6, premium=P*0.96 - 50, benefit=lambda t: 100000+200)
P = life.solve(fun, target=7100, grid=[2200, 2400])
isclose(2200, P, question="Q7.14")
```

```
----- Q7.14 2200: 2197.8174603174602 [OK] -----
```

```
True
```

SOA Question 7.15 : (E) 50.91

For a fully discrete whole life insurance of 100 on (x), you are given:

1. $q_{x+15} = 0.10$
2. Deaths are uniformly distributed over each year of age
3. $i = 0.05$
4. ${}_tV$ denotes the net premium policy value at time t
5. ${}_{16}V = 49.78$

Calculate 15.6.

```
x = 0
V = Recursion(udd=True).set_interest(i=0.05)\
                       .set_q(0.1, x=x+15)\
                       .set_reserves(T=3, V={16: 49.78})\
                       .r_V_backward(x, s=15, r=0.6, benefit=100)
isclose(50.91, V, question="Q7.15")
```

```
----- Q7.15 50.91: 50.91362826922369 [OK] -----
```

```
True
```

SOA Question 7.16 : (D) 380

```
life = SelectLife().set_interest(v=.95)\
                   .set_table(A={86: [683/1000]},
                              q={80+k: [.01*(k+1)] for k in range(6)})
x, t, n = 80, 3, 5
A = life.whole_life_insurance(x+t)
a = life.temporary_annuity(x+t, t=n-t)
V = life.gross_future_loss(A=A, a=a, contract=Contract(benefit=1000, premium=130))
isclose(380, V, question="Q7.16")
```

```
----- Q7.16 380: 381.6876905200001 [OK] -----
```

```
True
```

SOA Question 7.17 : (D) 1.018

```
x = 0
life = Recursion().set_interest(v=math.sqrt(0.90703))\
                  .set_q(0.02067, x=x+10)\
                  .set_A(0.52536, x=x+11)\
                  .set_A(0.30783, x=x+11, moment=2)
A1 = life.whole_life_insurance(x+10)
A2 = life.whole_life_insurance(x+10, moment=2)
ratio = (life.insurance_variance(A2=A2, A1=A1)
         / life.insurance_variance(A2=0.30783, A1=0.52536))
isclose(1.018, ratio, question="Q7.17")
```

$$\text{Whole Life Insurance } A_{x+10} :=$$
$$A_{x+10} = v * [\, q_{x+10} * b + \, p_{x+10} * \, A_{x+11}] \quad \text{backward recursion}$$
$$p_{x+10} = 1 - \, q_{x+10} \quad \text{complement of mortality}$$

$$\text{Whole Life Insurance } {}^2A_{x+10} :=$$
$${}^2A_{x+10} = v^2 * [\, q_{x+10} * b^2 + \, p_{x+10} * \, {}^2A_{x+11}] \quad \text{backward recursion}$$
$$p_{x+10} = 1 - \, q_{x+10} \quad \text{complement of mortality}$$

```
----- Q7.17 1.018: 1.0182465434445054 [OK] -----
```

```
True
```

SOA Question 7.18 : (A) 17.1

For a fully discrete whole life insurance of 1 on (x), you are given:

1. The net premium policy value at the end of the first year is 0.012
2. $q_x = 0.009$
3. $i = 0.04$

Calculate $\ddot{a}_x$

```
x = 10
life = Recursion(verbose=False).set_interest(i=0.04).set_q(0.009, x=x)
def fun(a):
    return life.set_a(a, x=x).net_policy_value(x, t=1)
a = life.solve(fun, target=0.012, grid=[17.1, 19.1])
isclose(17.1, a, question="Q7.18")
```

```
----- Q7.18 17.1: 17.07941929974385 [OK] -----
```

```
True
```

SOA Question 7.19 : (D) 720

For a fully discrete whole life insurance of 100,000 on (40) you are given:

1. Expenses incurred at the beginning of the first year are 300 plus 50% of the first year premium
2. Renewal expenses, incurred at the beginning of the year, are 10% of each of the renewal premiums
3. Mortality follows the Standard Ultimate Life Table
4. $i = 0.05$
5. Gross premiums are calculated using the equivalence principle

Calculate the gross premium policy value for this insurance immediately after the second premium and associated renewal expenses are paid.

```
life = SULT()
contract = Contract(benefit=100000,
                    initial_policy=300,
                    initial_premium=.5,
```

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```
                    renewal_premium=.1)
P = life.gross_premium(A=life.whole_life_insurance(40), **contract.premium_terms)
A = life.whole_life_insurance(41)
a = life.immediate_annuity(41)    # after premium and expenses are paid
V = life.gross_future_loss(A=A,
                           a=a,
                           contract=contract.set_contract(premium=P).renewals())
isclose(720, V, question="Q7.19")
```

```
----- Q7.19 720: 722.7510208759086 [OK] -----
```

```
True
```

SOA Question 7.20 : (E) -277.23

For a fully discrete whole life insurance of 1000 on (35), you are given:

1. First year expenses are 30% of the gross premium plus 300
2. Renewal expenses are 4% of the gross premium plus 30
3. All expenses are incurred at the beginning of the policy year
4. Gross premiums are calculated using the equivalence principle
5. The gross premium policy value at the end of the first policy year is R
6. Using the Full Preliminary Term Method, the modified reserve at the end of the first policy year is S
7. Mortality follows the Standard Ultimate Life Table
8. $i = 0.05$

Calculate $R–S$.

```
life = SULT()
S = life.FPT_policy_value(35, t=1, b=1000)   # is 0 for FPT at t=0,1
contract = Contract(benefit=1000,
                    initial_premium=.3,
                    initial_policy=300,
                    renewal_premium=.04,
                    renewal_policy=30)
G = life.gross_premium(A=life.whole_life_insurance(35), **contract.premium_terms)
R = life.gross_policy_value(35, t=1, contract=contract.set_contract(premium=G))
isclose(-277.23, R - S, question="Q7.20")
```

```
----- Q7.20 -277.23: -277.19303323929216 [OK] -----
```

```
True
```

SOA Question 7.21 : (D) 11866

```
life = SULT()
x, t, u = 55, 9, 10
P = life.gross_premium(IA=0.14743,
                       a=life.temporary_annuity(x, t=u),
```

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```
                        A=life.deferred_annuity(x, u=u),
                        benefit=1000)
contract = Contract(initial_policy=life.term_insurance(x+t, t=1, b=10*P),
                    premium=P,
                    benefit=1000)
a = life.temporary_annuity(x+t, t=u-t)
A = life.deferred_annuity(x+t, u=u-t)
V = life.gross_future_loss(A=A, a=a, contract=contract)
isclose(11866, V, question="Q7.21")
```

```
----- Q7.21 11866: 11866.30158100453 [OK] -----
```

```
True
```

SOA Question 7.22 : (C) 46.24

```
life = PolicyValues().set_interest(i=0.06)
contract = Contract(benefit=8, premium=1.250)
def fun(A2):
    return life.gross_variance_loss(A1=0, A2=A2, contract=contract)
A2 = life.solve(fun, target=20.55, grid=20.55/8**2)
contract = Contract(benefit=12, premium=1.875)
var = life.gross_variance_loss(A1=0, A2=A2, contract=contract)
isclose(46.2, var, question="Q7.22")
```

```
----- Q7.22 46.2: 46.2375 [OK] -----
```

```
True
```

SOA Question 7.23 : (D) 233

```
life = Recursion().set_interest(i=0.04).set_p(0.995, x=25)
A = life.term_insurance(25, t=1, b=10000)
def fun(beta):  # value of premiums in first 20 years must be equal
    return beta * 11.087 + (A - beta)
beta = life.solve(fun, target=216 * 11.087, grid=[140, 260])
isclose(233, beta, question="Q7.23")
```

$$\text{Term Insurance } A^{1}_{x+25:\overline{1|}} * 10000 :=$$
$$A^{1}_{x+25:\overline{1|}} = A_{x+25:\overline{1|}} - E_{x+25} \qquad \text{endowment insurance - pure}$$
$$E_{x+25} = p_{x+25} * v \qquad \text{pure endowment}$$

```
----- Q7.23 233: 232.64747466274176 [OK] -----
```

```
True
```

SOA Question 7.24 : (C) 680

For a fully discrete whole life insurance policy of 1,000,000 on (50), you are given:

1. The annual gross premium, calculated using the equivalence principle, is 11,800
2. Mortality follows the Standard Ultimate Life Table
3. $i = 0.05$

Calculate the expense loading, P for this policy.

```
life = SULT()
P = life.premium_equivalence(A=life.whole_life_insurance(50), b=1000000)
isclose(680, 11800 - P, question="Q7.24")
```

```
----- Q7.24 680: 680.291823645397 [OK] -----
```

```
True
```

SOA Question 7.25 : (B) 3947.37

```
life = SelectLife().set_interest(i=.04)\
                   .set_table(A={55: [.23, .24, .25],
                                 56: [.25, .26, .27],
                                 57: [.27, .28, .29],
                                 58: [.20, .30, .31]})
V = life.FPT_policy_value(55, t=3, b=100000)
isclose(3950, V, question="Q7.25")
```

```
----- Q7.25 3950: 3947.3684210526353 [OK] -----
```

```
True
```

SOA Question 7.26 : (D) 28540

- backward = forward reserve recursion

```
x = 0
life = Recursion(verbose=False).set_interest(i=.05)\
                               .set_p(0.85, x=x)\
                               .set_p(0.85, x=x+1)\
                               .set_reserves(T=2, endowment=50000)
def benefit(k): return k * 25000
def fun(P):  # solve P s.t. V is equal backwards and forwards
    policy = dict(t=1, premium=P, benefit=benefit, reserve_benefit=True)
    return life.t_V_backward(x, **policy) - life.t_V_forward(x, **policy)
P = life.solve(fun, target=0, grid=[27650, 28730])
isclose(28540, P, question="Q7.26")
```

```
----- Q7.26 28540: 28542.392566782808 [OK] -----
```

```
True
```

SOA Question 7.27 : (B) 213

```
x = 0
life = Recursion(verbose=False).set_interest(i=0.03)\
                               .set_q(0.008, x=x)\
                               .set_reserves(V={0: 0})
def fun(G):  # Solve gross premium from expense reserves equation
    return life.t_V(x=x, t=1, premium=G - 187, benefit=lambda t: 0,
                    per_policy=10 + 0.25*G)
G = life.solve(fun, target=-38.70, grid=[200, 252])
isclose(213, G, question="Q7.27")
```

```
----- Q7.27 213: 212.970355987055 [OK] -----
```

```
True
```

SOA Question 7.28 : (D) 24.3

```
life = SULT()
PW = life.net_premium(65, b=1000)   # 20_V=0 => P+W is net premium for A_65
P = life.net_premium(45, t=20, b=1000)  # => P is net premium for A_45:20
isclose(24.3, PW - P, question="Q7.28")
```

```
----- Q7.28 24.3: 24.334725400123975 [OK] -----
```

```
True
```

SOA Question 7.29 : (E) 2270

For a fully discrete whole life insurance of B on (x), you are given:

1. Expenses, incurred at the beginning of each year, equal 30 in the first year and 5 in subsequent years
2. The net premium policy value at the end of year 10 is 2290
3. Gross premiums are calculated using the equivalence principle
4. $i = 0.04$
5. $\ddot{a}_x = 14.8$
6. $\ddot{a}_{x+10} = 11.4$

Calculate ${}_{10}V^g$, the gross premium policy value at the end of year 10.

```
x = 0
life = Recursion(verbose=False).set_interest(i=0.04)\
                               .set_a(14.8, x=x)\
                               .set_a(11.4, x=x+10)
def fun(B):
    return life.net_policy_value(x, t=10, b=B)
B = life.solve(fun, target=2290, grid=2290*10)  # Solve benefit B given net 10_V
contract = Contract(initial_policy=30, renewal_policy=5, benefit=B)
G = life.gross_premium(a=life.whole_life_annuity(x), **contract.premium_terms)
V = life.gross_policy_value(x, t=10, contract=contract.set_contract(premium=G))
isclose(2270, V, question="Q7.29")
```

```
----- Q7.29 2270: 2270.743243243244 [OK] -----
```

```
True
```

SOA Question 7.30 : (E) 9035

Ten years ago J, then age 25, purchased a fully discrete 10-payment whole life policy of 10,000.

All actuarial calculations for this policy were based on the following:

1. Mortality follows the Standard Ultimate Life Table
2. $i = 0.05$
3. The equivalence principle

In addition:

1. L_{10} is the present value of future losses random variable at time 10
2. At the end of policy year 10, the interest rate used to calculate L_{10} is changed to 0%

Calculate the increase in $E[L_{10}]$ that results from this change.

```
b = 10000  # premiums=0 after t=10
L = SULT().set_interest(i=0.05).whole_life_insurance(x=35, b=b)
V = SULT().set_interest(i=0).whole_life_insurance(x=35, b=b)
isclose(9035, V - L, question="Q7.30")
```

```
----- Q7.30 9035: 9034.654127845053 [OK] -----
```

```
True
```

SOA Question 7.31 : (E) 0.310

For a fully discrete 3-year endowment insurance of 1000 on (x), you are given:

1. Expenses, payable at the beginning of the year, are:

Year(s)	Percent of Premium	Per Policy
1	20%	15
2 and 3	8%	5

2. The expense reserve at the end of year 2 is –23.64
3. The gross annual premium calculated using the equivalence principle is G = 368.
4. $G = 1000P_{x:\overline{3}|} + P^e$, where P^e is the expense loading

Calculate $P_{x:\overline{3}|}$.

```
x = 0
life = Reserves().set_reserves(T=3)
G = 368.05
def fun(P):  # solve net premium expense reserve equation
    return life.t_V(x, t=2, premium=G-P, benefit=lambda t:0, per_policy=5+0.08*G)
P = life.solve(fun, target=-23.64, grid=[.29, .31]) / 1000
isclose(0.310, P, question="Q7.31")
```

```
----- Q7.31 0.31: 0.309966 [OK] -----
```

```
True
```

SOA Question 7.32 : (B) 1.4

For two fully continuous whole life insurance policies on (x), you are given:

	Death Benefit	Annual Premium Rate	Variance of the PV of Future Loss at t
Policy A	1	0.10	0.455
Policy B	2	0.16	-

- $\delta = 0.06$

Calculate the variance of the present value of future loss at t for Policy B.

```
life = PolicyValues().set_interest(i=0.06)
contract = Contract(benefit=1, premium=0.1)
def fun(A2):
    return life.gross_variance_loss(A1=0, A2=A2, contract=contract)
A2 = life.solve(fun, target=0.455, grid=0.455)
contract = Contract(benefit=2, premium=0.16)
var = life.gross_variance_loss(A1=0, A2=A2, contract=contract)
isclose(1.39, var, question="Q7.32")
```

```
----- Q7.32 1.39: 1.3848168384380901 [OK] -----
```

```
True
```

Final Score

```
from datetime import datetime
print(datetime.now())
print(isclose)
```

```
2023-07-21 17:30:21.404239
Passed: 136/136
```