Solving Actuarial Math with Python

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CONTENTS

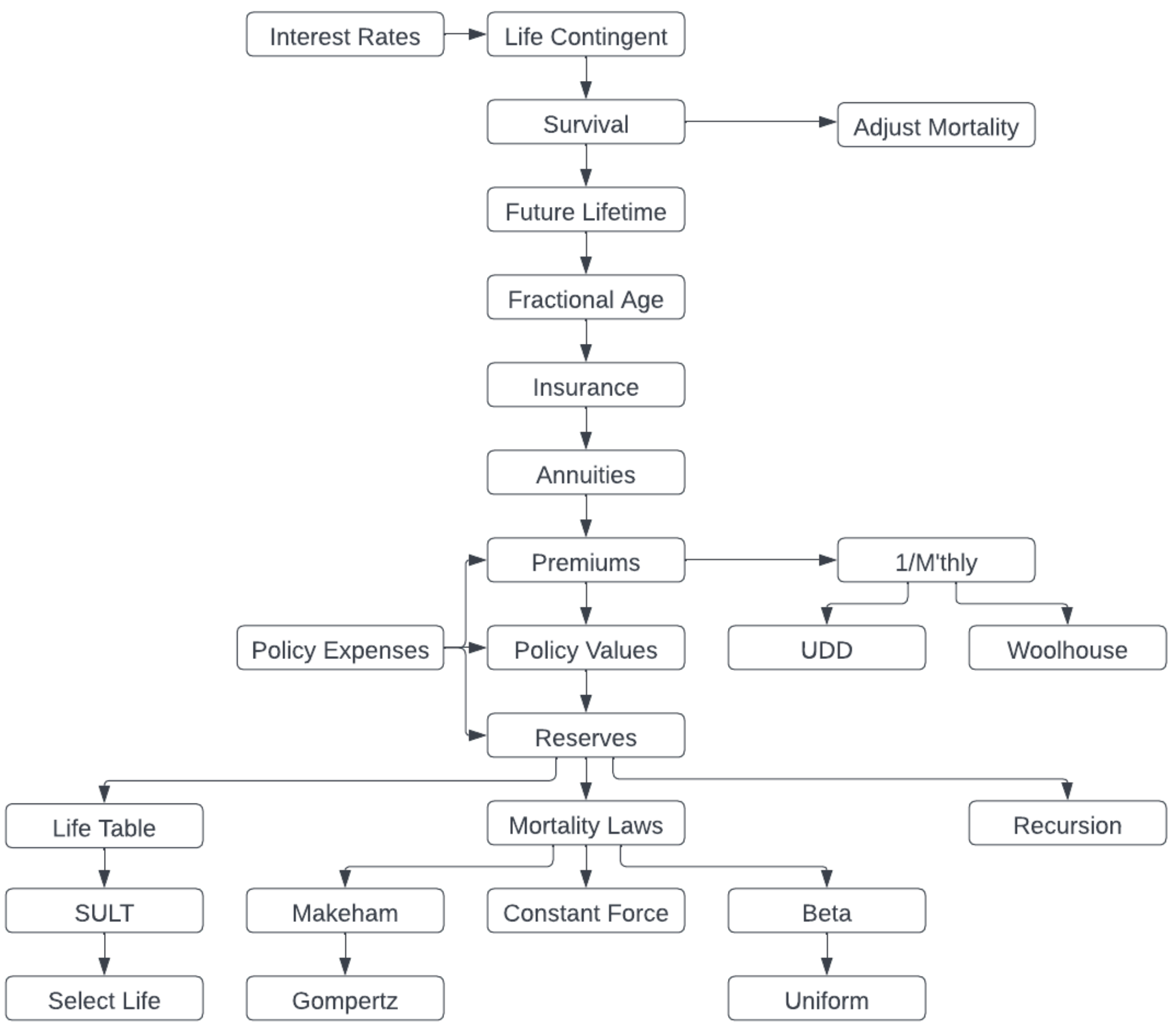
Actuarial Math - Life Contingent Risks

This `actuarialmath` package implements the general formulas, recursive relationships and shortcut equations for Fundamentals of Long Term Actuarial Mathematics, to solve the SOA sample FAM-L exam questions, and more, with Python.

- The concepts are developed hierarchically in object-oriented Python.
- Each module introduces the formulas (incrementally) used, with usage examples.
- The SOA sample questions released in August 2022 are solved in an executable Google Colab Notebook.

Enjoy!

Terence Lim



Sources:

- Documentation and formulas: actuarialmath.pdf
- Executable Colab Notebook: faml.ipynb
- Github repo: https://github.com/terence-lim/actuarialmath.git
- SOA FAM-L Sample Solutions: copy retrieved Aug 2022
- SOA FAM-L Sample Questions: copy retrieved Aug 2022
- Actuarial Mathematics for Life Contingent Risks (Dickson, Hardy and Waters), Institute and Faculty of Actuaries, published by Cambridge University Press

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CHAPTER

ONE

LIFE CONTINGENT RISKS

1.1 Interest rates

$d = \dfrac{i}{1+i}$

$v = \dfrac{1}{1+i}$

$\delta = \log(1+i)$

$(1+i)^t = (1-d)^{-t} = (1+\dfrac{i^{(m)}}{m})^{mt} = (1-\dfrac{d^{(m)}}{m})^{-mt} = e^{\delta t} = v^{-t}$

Doubling the force of interest:

- $i' \leftarrow 2i + i^2$
- $d' \leftarrow 2d - d^2$
- $v' \leftarrow v^2$
- $\delta' \leftarrow 2\delta$

Annuity certain:

- Due: $\ddot{a}_{\overline{n|}} = \dfrac{1-v^n}{d}$
- Immediate: $a_{\overline{n|}} = \dfrac{1-v^n}{i} = \ddot{a}_{\overline{n+1|}} - 1$
- Continuous: $\overline{a}_{\overline{n|}} = \dfrac{1-v^n}{\delta}$

1.2 Probability

$Var(aX, bY) = a^2\, Var(X) + b^2\, Var(Y) + 2\, a\, b\, Cov(X, Y)$

$Cov(X, Y) = E[XY] - E[X] \cdot E[Y]$

Bernoulli $(p) : Y \in (a,\ b)$ w.p. $(p,\ 1-p) \Rightarrow$

- $E[Y] = a\, p + b\, (1-p)$
- $Var[Y] = (a-b)^2\, p\, (1-p)$

Binomial $(N,\ p) : Y$ is sum of N i.i.d. 0-1 Bernoulli(p) $\Rightarrow$

- $E[Y] = N\, p$
- $Var[Y] = N\, p\, (1-p)$

Mixture $(p,\ p_1,\ p_2)$: Y is Binomial $(p',\ N)$, where $p' \in (p_1,\ p_2)$ w.p. $(p,\ 1-p) \Rightarrow$

- $E[Y] = p\, N\, p_1 + (1-p)\, N\, p_2$
- $Var[Y] = E[Y^2] - E[Y]^2 = E[Var(Y \mid p') + E(Y \mid p')^2] - E[Y]^2$

Conditional Variance shortcut:

- $Var[Y] = Var(E[Y \mid p'] + E[Var(Y \mid p')])$

Portfolio Percentile (p): Y is sum of N i.i.d. r.v. each with mean μ and variance $\sigma^2 \Rightarrow$

- $Y \sim$ Normal with mean $E[Y] = N\mu$ and variance $Var[Y] = N\sigma^2$
- $Y_p = E[y] + z_p\, \sqrt{Var[Y]}$

```
"""Base class for Life Contingent Risks

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"""
from actuarialmath.life import Life
print(Life.help())
```

```
Base class for Life Contingent Risks
------------------------------------

variance():
  Compute variance of weighted sum of two r.v.

covariance():
  Compute covariance of two r.v.

bernoulli():
  Compute mean or variance of bernoulli r.v. with range (a, b)

binomial():
  Compute mean or variance of binomial r.v.

mixture():
  Mean and variance of mixture of two binomial r.v.
```

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```
conditional_variance():
  Conditional variance formula

portfolio_percentile():
  Percentile of a cumulative probability in the sum of N iid r.v.

solve():
  Solve root of equation f(arg) = target

add_term():
  Add two terms, where either term may be whole life

max_term():
  Adjust term if adding term and deferral to (x) exceeds maxage

Interest():
  Class for interest rate conversion and math
```

1.3 Examples

```
import math

print("SOA Question 2.2: (D) 400")
p1 = (1. - 0.02) * (1. - 0.01)  # 2_p_x if vaccine given
p2 = (1. - 0.02) * (1. - 0.02)  # 2_p_x if vaccine not given
print(math.sqrt(Life.conditional_variance(p=.2, p1=p1, p2=p2, N=100000)))
print(math.sqrt(Life.mixture(p=.2, p1=p1, p2=p2, N=100000, variance=True)))
print()

print("SOA Question 3.10:  (C) 0.86")
interest = Life.Interest(v=0.75)
L = 35 * interest.annuity(t=4, due=False) + 75 * interest.v_t(t=5)
interest = Life.Interest(v=0.5)
R = 15 * interest.annuity(t=4, due=False) + 25 * interest.v_t(t=5)
print(L / (L + R))
print()


print("Example: double the force of interest i=0.05")
i = 0.05
i2 = Life.Interest.double_force(i=i)
d2 = Life.Interest.double_force(d=i/(1+i))
print('i:', round(i2, 6), round(Life.Interest(d=d2).i, 6))
print('d:', round(d2, 6), round(Life.Interest(i=i2).d, 6))
print()

print()
print("Values of z for selected values of Pr(Z<=z)")
print("-----------------------------------------")
print(Life.frame().to_string(float_format=lambda x: f"{x:.3f}"))
Life.frame()
```

```
SOA Question 2.2: (D) 400
396.5914603215815
396.5914603237804

SOA Question 3.10:  (C) 0.86
0.8578442833761983

Example: double the force of interest i=0.05
i: 0.1025 0.1025
d: 0.092971 0.092971


Values of z for selected values of Pr(Z<=z)
-------------------------------------------
z          0.842  1.036  1.282  1.645  1.960  2.326  2.576
Pr(Z<=z)  0.800  0.850  0.900  0.950  0.975  0.990  0.995
```

```
z          0.842  1.036  1.282  1.645  1.960  2.326  2.576
Pr(Z<=z)    0.8   0.85    0.9   0.95  0.975   0.99  0.995
```

CHAPTER

TWO

SURVIVAL

2.1 Survival and mortality functions

T_x is time-to-death, or future lifetime, of (x)

Survival:

$$S_x(t) = {}_tp_x = \text{Prob}\ (T_x > t) = \frac{S_0(x+t)}{S_0(x)} = 1 - F_x(t) = e^{-\int_0^t \mu_{x+t} ds} = \int_t^\infty f_x(t) ds = \frac{l_{x+t}}{l_x}$$

Mortality:

$$f_x(t) = {}_tp_x \mu_{x+t}$$

$${}_{u|t}q_x = \int_u^{u+t} {}_sp_x \mu_{x+s} ds = {}_up_x - {}_{u+t}p_x = \frac{l_{x+u} - l_{x+u+t}}{l_x}$$

$${}_tp_x + {}_tq_x = 1$$

2.2 Force of mortality

$$\mu_{x+t} = \frac{f_x(t)}{S_x(t)} = \frac{-\frac{\partial}{\partial t} {}_tp_x}{{}_tp_x} = -\frac{\partial}{\partial t} \ln\ {}_tp_x$$

Note limits: $S_x(0) = 1,\ S_x(\infty) = 0,\ \int_0^\infty \mu_{x+s} ds = \infty$

```
"""Survival and Mortality probability functions

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"""
from actuarialmath.survival import Survival
print(Survival.help())
```

```
Fundamental Survival Functions
------------------------------

l_x():
```

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```
  Number of lives age ([x]+s): l_[x]+s

p_x():
  Probability that (x) lives t years: t_p_x

q_x():
  Probability that (x) lives for u, but not for t+u: u|t_q_[x]+s

f_x():
  probability density function of mortality

mu_x():
  Force of mortality of (x+t): mu_x+t

survival_curve():
  Construct curve of survival probabilities at integer ages
```

2.3 Examples

```python
import math

print("SOA Question 2.3: (A) 0.0483")
B, c = 0.00027, 1.1
life = Survival(S=lambda x,s,t: (math.exp(-B * c**(x+s)
                                    * (c**t - 1)/math.log(c))))
print(life.f_x(x=50, t=10))
print()

print("# SOA Question 2.6: (C) 13.3")
life = Survival(l=lambda x,s: (1 - (x+s) / 60)**(1 / 3))
print(1000*life.mu_x(35))
print()

print("SOA Question 2.7: (B) 0.1477")
life = Survival(l=lambda x,s: (1 - ((x+s) / 250) if (x+s)<40
                                else 1 - ((x+s) / 100)**2))
print(life.q_x(30, t=20))
print()

print("CAS41-F99:12: k = 41")
fun = (lambda k:
        Survival(l=lambda x,s: 100*(k - .5*(x+s))**(2/3)).mu_x(50))
print(int(Survival.solve(fun, target=1/48, guess=50)))
```

```
SOA Question 2.3: (A) 0.0483
0.048327399045049846

# SOA Question 2.6: (C) 13.3
13.340451278922776

SOA Question 2.7: (B) 0.1477
0.14772727272727
```

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```
CAS41-F99:12: k = 41
41
```

CHAPTER

THREE

FUTURE LIFETIME

3.1 Complete expectation of life

First moment: $\mathring{e}_x = \int_0^\infty t \ {}_tp_x \ \mu_{x+t} ds = \int_0^\infty \ {}_tp_x \ dt$

Second moment: $E[T_x^2] = \int_0^\infty t^2 \ {}_tp_x \ \mu_{x+t} \ ds = \int_0^\infty \ 2 \ t \ {}_tp_x \ dt$

- Variance: $Var[T_x] = E[T_x^2] - (\mathring{e}_x)^2$

3.2 Curtate expectation of life

Curtate future lifetime: K_x is number of completed future years by (x) prior to death $= \lfloor T_x \rfloor$

First moment: $e_x = \sum_{k=0}^{\infty} k \ {}_{k|}q_x \ = \sum_{k=1}^{\infty} \ {}_kp_x \ dt$

Second moment: $E[K_x^2] = \sum_{k=0}^{\infty} k^2 \ {}_{k|}q_x \ = \sum_{k=1}^{\infty} (2k-1) \ {}_kq_x \ dt$

- Variance: $Var[K_x] = E[K_x^2] - (e_x)^2$

3.3 Temporary expectation of life

$\mathring{e}_{x:\overline{n|}} = \int_0^n t \ {}_tp_x \ \mu_{x+t} \ ds + n \ {}_np_x = \int_0^n \ {}_tp_x \ dt$

$e_{x:\overline{n|}} = \sum_{k=0}^{n-1} k \ {}_{k|}q_x \ + n \ {}_np_x = \sum_{k=1}^{n} \ {}_kp_x$

```
"""Expected future lifetimes

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"""
from actuarialmath.lifetime import Lifetime
print(Lifetime.help())
```

```
Expected Future Lifetime
------------------------

e_x():
  Compute moments of expected future lifetime
```

3.4 Examples

```python
import math

print("SOA Question 2.1: (B) 2.5")
def fun(omega):  # Solve first for omega, given mu_65 = 1/180
    life = Lifetime(l=lambda x,s: (1 - (x+s)/omega)**0.25)
    return life.mu_x(65)
omega = int(Lifetime.solve(fun, target=1/180, guess=100))  # solve for omega
life = Lifetime(l=lambda x,s: (1 - (x+s)/omega)**0.25, maxage=omega)
print(life.e_x(106))
print()

print("SOA Question 2.4: (E) 8.2")
life = Lifetime(l=lambda x,s: 0. if (x+s) >= 100 else 1 - ((x+s)**2)/10000.)
print(life.e_x(75, t=10, curtate=False))
print()

print("SOA Question 2.8: (C) 0.938")
def fun(mu):  # Solve first for mu, given start and end proportions
    male = Lifetime(mu=lambda x,s: 1.5 * mu)
    female = Lifetime(mu=lambda x,s: mu)
    return (75 * female.p_x(0, t=20)) / (25 * male.p_x(0, t=20))
mu = Lifetime.solve(fun, target=85/15, guess=-math.log(0.94))
life = Lifetime(mu=lambda x,s: mu)
print(life.p_x(0, t=1))
print()
```

```
SOA Question 2.1: (B) 2.5
2.4786080555423604

SOA Question 2.4: (E) 8.2
8.20952380952381

SOA Question 2.8: (C) 0.938
0.9383813306903798
```

CHAPTER

FOUR

FRACTIONAL AGES

4.1 Uniform distribution of deaths

$l_{x+r} = (1-r)\ l_x + r\ l_{x+1}$ (i.e. linear interpolation)

${}_r q_x = r\ q_x$

${}_r q_{x+s} = \dfrac{r\ q_x}{1 - s\ q_x}$

$\mu_{x+r} = \dfrac{1}{1 - r\ q_x}$

$f_x(r) = q_x$ (i.e. constant within age)

$\mathring{e}_x = q_x \dfrac{1}{2} + p_x(1 + \mathring{e}_{x+1})$

$\mathring{e}_{x:\overline{1|}} = 1 - q_x \dfrac{1}{2} = q_x \dfrac{1}{2} +\ p_x$

$\mathring{e}_x \approx e_x + 0.5$

4.2 Constant force of mortality

$l_{x+r} = (l_x)^{1-r} \cdot (l_{x+1})^r$ (i.e. exponential interpolation)

${}_r p_x = (p_x)^r$

$\mu_{x+r} = -\ln\ p_x$ (i.e. constant within age)

$f_x(r) = e^{-\mu t} \cdot \mu$

```
"""Fractional ages

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"""
from actuarialmath.fractional import Fractional
print(Fractional.help())
```

```
Compute Survival and Lifetimes with Fractional Ages
---------------------------------------------------

E_r():
  Pure endowment through fractional age: t_E_[x]+s+r

l_r():
  Lifes at fractional age: l_[x]+s+r

p_r():
  Survival from and through fractional age: t_p_[x]+s+r

q_r():
  Deferred mortality within fractional ages: u|t_q_[x]+s+r

mu_r():
  Force of mortality at fractional age: mu_[x]+s+r

f_r():
  probability distribution function at fractional age: f_[x]+s+r (t)

e_r():
  Expectation of future lifetime through fractional age: e_[x]+s:t

e_curtate():
  Convert curtate and complete lifetime assuming UDD within age
```

4.3 Examples

CHAPTER

FIVE

INSURANCE

5.1 Pure endowment

${}_nE_x = A_{\overset{}{x}:\overset{1}{\overline{n|}}} = v^n \ {}_np_x$

5.2 Life insurance

Whole life insurance:

$\overline{A}_x = \int_{t=0}^{\infty} \ v^t \ {}_tp_x \ \mu_{x+t} \ dt$

$A_x = \sum_{k=0}^{\infty} \ v^{k+1} \ {}_{k|}q_x$

Term insurance:

$\overline{A}^{1}_{x:\overline{t|}} = \int_{s=0}^{t} \ v^s \ {}_sp_x \ \mu_{x+s} \ ds = \overline{A}_x - \ {}_tE_x \ \overline{A}_{x+t}$

$A^{1}_{x:\overline{t|}} = \sum_{k=0}^{t-1} \ v^{k+1} \ {}_{k|}q_x = A_x - \ {}_tE_x \ A_{x+t}$

Endowment insurance:

$\overline{A}_{x:\overline{t|}} = \overline{A}^{1}_{x:\overline{t|}} + \ {}_tE_x$

$A_{x:\overline{t|}} = A^{1}_{x:\overline{t|}} + \ {}_tE_x$

Deferred insurance:

${}_{u|}\overline{A}_{x:\overline{t|}} = \ {}_uE_x \ \overline{A}_{x+u:\overline{t|}}$

${}_{u|}A_{x:\overline{t|}} = \ {}_uE_x \ A_{x+u:\overline{t|}}$

5.3 Annuity Twin

Whole life and **Endowment Insurance** ONLY:

$\overline{A}_x = 1 - \delta\, \overline{a}_x$

$A_x = 1 - d\, \ddot{a}_x$

$\overline{A}_{x:\overline{t|}} = 1 - \delta\, \overline{a}_{x:\overline{t|}}$

$A_{x:\overline{t|}} = 1 - d\, \ddot{a}_{x:\overline{t|}}$

Double the force of interest:

${}^2\overline{A}_x = 1 - (2\delta)\, {}^2\overline{a}_x$

${}^2A_x = 1 - (2d - d^2)\, {}^2\ddot{a}_x$

${}^2\overline{A}_{x:\overline{t|}} = 1 - (2\delta)\, {}^2\overline{a}_{x:\overline{t|}}$

${}^2A_{x:\overline{t|}} = 1 - (2d - d^2)\, {}^2\ddot{a}_{x:\overline{t|}}$

5.4 Variances

Notationally: ${}^2\overline{A}_x$ and ${}^2A_x = \overline{A}_x$ and A_x, respectively, at double the force of interest.

${}^2_tE_x = v^{2t}\ {}_tp_x = v_t\ {}_tE_x$

Pure Endowment:

$Var({}_tE_x) = v^{2t}\ {}_tp_x\ {}_tq_x = v^{2t}\ {}_tp_x - (v^t\ {}_tp_x)^2$

Whole life insurance:

$Var(\overline{A}_x) = {}^2\overline{A}_x - (\overline{A}_x)^2$

$Var(A_x) = {}^2A_x - \ (A_x)^2$

Term insurance:

$Var(\overline{A}^{\,1}_{x:\overline{t|}}) = {}^2\overline{A}^{\,1}_{x:\overline{t|}} - (\overline{A}^{\,1}_{x:\overline{t|}})^2$

$Var(A^1_{x:\overline{t|}}) = {}^2A^1_{x:\overline{t|}} - \ (A^1_{x:\overline{t|}})^2$

Deferred insurance:

$Var({}_{u|}\overline{A}_{x:\overline{t|}}) = {}^2_{u|}\overline{A}_{x:\overline{t|}} - ({}_{u|}\overline{A}_{x:\overline{t|}})^2$

$Var({}_{u|}A_{x:\overline{t|}}) = {}^2_{u|}A_{x:\overline{t|}} - \ ({}_{u|}A_{x:\overline{t|}})^2$

Endowment insurance:

$Var(\overline{A}_{x:\overline{t|}}) = {}^2\overline{A}_{x:\overline{t|}} - (\overline{A}_{x:\overline{t|}})^2$

$Var(A_{x:\overline{t|}}) = {}^2A_{x:\overline{t|}} - \ (A_{x:\overline{t|}})^2$

5.5 Varying insurance

Increasing insurance:

$(\overline{IA})_x = \int_{t=0}^{\infty}~ t ~ v^t ~ _tp_x ~ \mu_{x+t} ~ dt$

$(IA)_x = \sum_{k=0}^{\infty}~ (k+1) ~ v^{k+1} ~ _{k|}q_x$

$(\overline{IA})^1_{x:\overline{t|}} = \int_{s=0}^{t}~ s ~ v^s ~ _sp_x ~ \mu_{x+s} ~ ds$

$(IA)^1_{x:\overline{t|}} = \sum_{k=0}^{t-1}~ (k+1) ~ v^{k+1} ~ _{k|}q_x$

Decreasing insurance:

$(\overline{DA})^1_{x:\overline{t|}} = \int_{s=0}^{t}~ (t-s) ~ v^s ~ _sp_x ~ \mu_{x+s} ~ ds$

$(DA)^1_{x:\overline{t|}} = \sum_{k=0}^{t-1}~ (t-k) ~ v^{k+1} ~ _{k|}q_x$

$(\overline{DA})^1_{x:\overline{t|}} + (\overline{IA})^1_{x:\overline{t|}} = t ~ \overline{A}^1_{x:\overline{t|}}$

$(DA)^1_{x:\overline{t|}} + (IA)^1_{x:\overline{t|}} = (t+1) ~ A^1_{x:\overline{t|}}$

5.6 Present value random variable Z

Expected preseent value of insurance benefits = EPV(Z)

Whole life insurance:

$Z = v^{K_x + 1}$ (discrete) or $~v^{T_x}$ (continuous)

Term insurance:

$Z = 0$ when $K_x \ge t$ or $T_x > t$, else whole life

Deferred whole life insurance:

$Z = 0$ when $K_x < t$ or $T_x \le t$, else whole life

Endowment insurance:

$Z = v^t$ when $K_x \ge t$ or $T_x > t$, else whole life

```python
"""Life insurance functions

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"""
from actuarialmath.insurance import Insurance
print(Insurance.help())
```

```
Life Insurance
--------------

E_x():
  Pure endowment: t_E_x
```

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```
A_x():
  Numerically compute APV of insurance from survival functions

insurance_variance():
  Compute variance of insurance given its two moments and benefit

insurance_twin():
  Returns WL or Endowment Insurance twin from annuity

whole_life_insurance():
  Whole life insurance: A_x

term_insurance():
  Term life insurance: A_x:t^1

deferred_insurance():
  Deferred insurance n|_A_x:t^1 = discounted term or whole life

endowment_insurance():
  Endowment insurance: A_x^1:t = term insurance + pure endowment

increasing_insurance():
  Increasing life insurance: (IA)_x

decreasing_insurance():
  Decreasing life insurance: (DA)_x

Z_t():
  T_x given percentile of the r.v. Z: PV of WL or Term insurance

Z_from_t():
  PV of insurance payment Z(t), given T_x (or K_x if discrete)

Z_to_t():
  T_x s.t. PV of insurance payment is Z

Z_from_prob():
  Percentile of insurance PV r.v. Z, given probability

Z_to_prob():
  Cumulative density of insurance PV r.v. Z, given percentile value

Z_x():
  APV of year t insurance death benefit

Z_plot():
  Plot PV of insurance r.v. Z vs T
```

5.7 Examples

```
import matplotlib.pyplot as plt
import numpy as np
import math

print("SOA Question 6.33:  (B) 0.13")
life = Insurance(mu=lambda x,t: 0.02*t, interest=dict(i=0.03))
x = 0
print(life.p_x(x, t=15))
var = life.E_x(x, t=15, moment=life.VARIANCE, endowment=10000)
print(var)
p = 1- life.portfolio_cdf(mean=0, variance=var, value=50000, N=500)
print(p)
print()

print("SOA Question 4.18  (A) 81873 ")
life = Insurance(interest=dict(delta=0.05),
                 maxage=10,
                 f=lambda x,s,t: .1 if t < 2 else .4*t**(-2))
benefit = lambda x,t: 0 if t < 2 else 100000
prob = 0.9 - life.q_x(0, t=2)
x, y = life.survival_curve()
T = life.Z_t(0, prob=prob)
life.Z_plot(0, T=T, benefit=benefit, discrete=False, curve=(x,y))
print(life.Z_from_t(T) * benefit(0, T))
print()

print("SOA Question 4.10:  (D)")
life = Insurance(interest=dict(i=0.01), S=lambda x,s,t: 1, maxage=40)
def fun(x, t):
    if 10 <= t <= 20: return life.interest.v_t(t)
    elif 20 < t <= 30: return 2 * life.interest.v_t(t)
    else: return 0
def A(x, t):  # Z_x+k (t-k)
    return life.interest.v_t(t - x) * (t > x)
x = 0
benefits=[lambda x,t: (life.E_x(x, t=10) * A(x+10, t)
                            + life.E_x(x, t=20)* A(x+20, t)
                            - life.E_x(x, t=30) * A(x+30, t)),
          lambda x,t: (A(x, t)
                            + life.E_x(x, t=20) * A(x+20, t)
                            - 2 * life.E_x(x, t=30) * A(x+30, t)),
          lambda x,t: (life.E_x(x, t=10) * A(x, t)
                            + life.E_x(x, t=20) * A(x+20, t)
                            - 2 * life.E_x(x, t=30) * A(x+30, t)),
          lambda x,t: (life.E_x(x, t=10) * A(x+10, t)
                            + life.E_x(x, t=20) * A(x+20, t)
                            - 2 * life.E_x(x, t=30) * A(x+30, t)),
          lambda x,t: (life.E_x(x, t=10)
                            * (A(x+10, t)
                            + life.E_x(x+10, t=10) * A(x+20, t)
                            - life.E_x(x+20, t=10) * A(x+30, t)))]
fig, ax = plt.subplots(3, 2)
ax = ax.ravel()
for i, b in enumerate([fun] + benefits):
```

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```
    life.Z_plot(0, benefit=b, ax=ax[i], verbose=False, color=f"C{i+1}")
    ax[i].legend(["(" + "abcde"[i-1] + ")" if i else "Z"])
z = [sum(abs(b(0, t) - fun(0, t)) for t in range(40)) for b in benefits]
print("ABCDE"[np.argmin(z)])
print()

print("SOA Question 4.12:  (C) 167")
cov = Insurance.covariance(a=1.65, b=10.75, ab=0)  # E[Z1 Z2] = 0 nonoverlapping
print(Insurance.variance(a=2, b=1, var_a=46.75, var_b=50.78, cov_ab=cov))
print()

print("SOA Question 4.11:  (A) 143385")
A1 = 528/1000   # E[Z1]  term insurance
C1 = 0.209      # E[pure_endowment]
C2 = 0.136      # E[pure_endowment^2]
def fun(A2):
    B1 = A1 + C1   # endowment = term + pure_endowment
    B2 = A2 + C2   # double force of interest
    return Insurance.insurance_variance(A2=B2, A1=B1)
A2 = Insurance.solve(fun, target=15000/(1000*1000), guess=[143400, 279300])
print(Insurance.insurance_variance(A2=A2, A1=A1, b=1000))
print()

print("SOA Question 4.15  (E) 0.0833 ")
life = Insurance(mu=lambda *x: 0.04,
                 interest=dict(delta=0.06))
benefit = lambda x,t: math.exp(0.02*t)
A1 = life.A_x(0, benefit=benefit, discrete=False)
A2 = life.A_x(0, moment=2, benefit=benefit, discrete=False)
print(A2 - A1**2)
print()

print("SOA Question 4.4  (A) 0.036")
life = Insurance(f=lambda *x: 0.025,
                 maxage=40+40,
                 interest=dict(v_t=lambda t: (1 + .2*t)**(-2)))
benefit = lambda x,t: 1 + .2 * t
A1 = life.A_x(40, benefit=benefit, discrete=False)
A2 = life.A_x(40, moment=2, benefit=benefit, discrete=False)
print(A2 - A1**2)
print()

# Example: plot Z vs T
life = Insurance(interest=dict(delta=0.06), mu=lambda *x: 0.04)
prob = 0.8
x = 0
discrete = False
t = life.Z_t(0, prob, discrete=discrete)
Z = life.Z_from_prob(x, prob=prob, discrete=discrete)
print(t, life.Z_to_t(Z))
print(Z, life.Z_from_t(t, discrete=discrete))
print(prob, life.Z_to_prob(x, Z=Z))
life.Z_plot(0, T=t, discrete=discrete)

print("Other examples of usage")
life = Insurance(interest=dict(delta=0.06), mu=lambda *x: 0.04)
```

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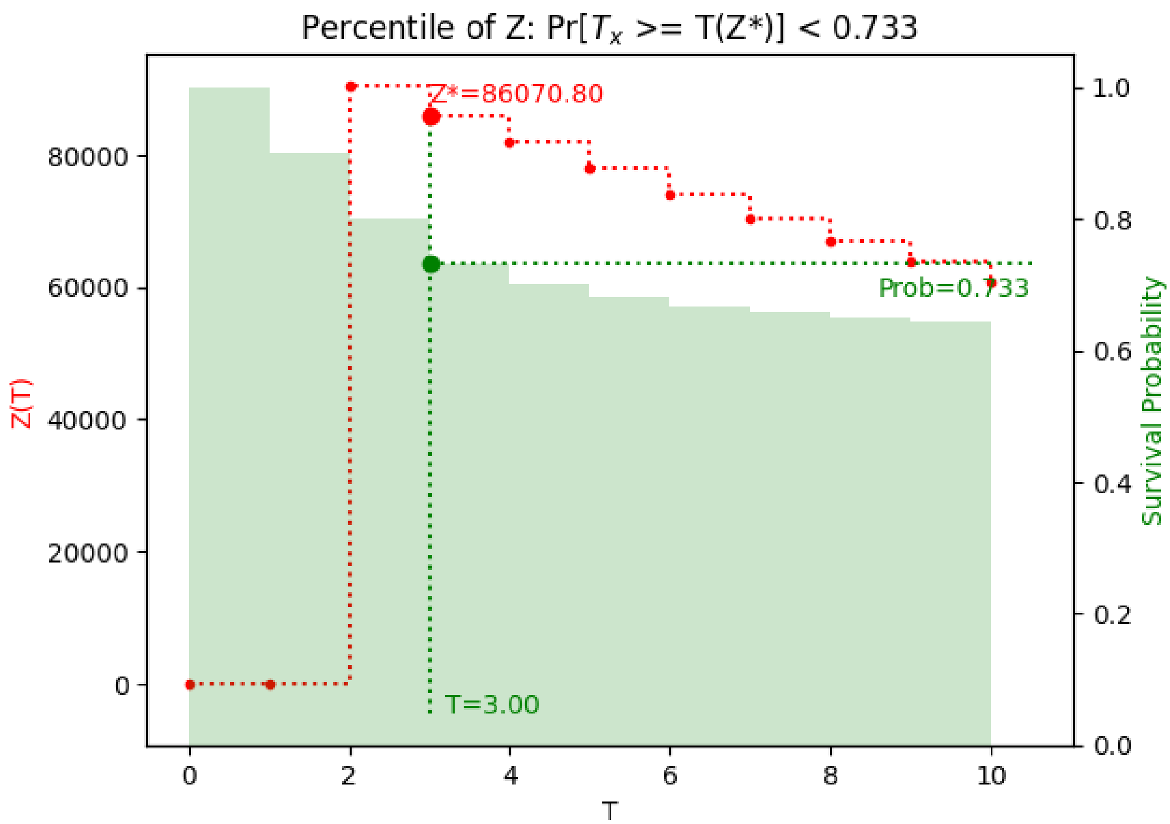
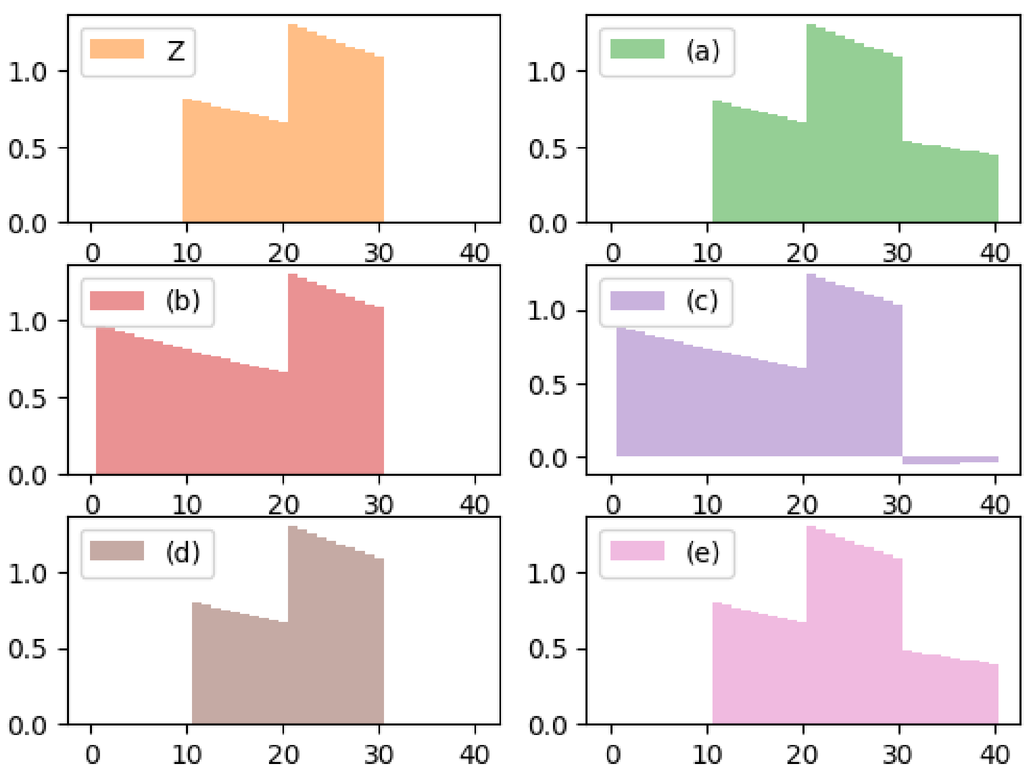
```
benefit = lambda x,t: math.exp(0.02 * t)
A1 = life.A_x(0, benefit=benefit)
A2 = life.A_x(0, moment=2, benefit=benefit)
print(A1, A2, A2 - A1**2)  # 0.0833

life = Insurance(interest=dict(delta=0.05), mu=lambda x,s: 0.03)
benefit = lambda x,t: math.exp(0.04 * t)
print(life.A_x(0, benefit=benefit))  #0.75
print(life.A_x(0, moment=2, benefit=benefit)) #0.60

life = Insurance(interest=dict(delta=0.08),
                 maxage=25,
                 S=lambda x,s,t: 1 - (0.02*t + 0.0008*(t**2)))
print(life.A_x(0)*10000)  #3647
print()
```

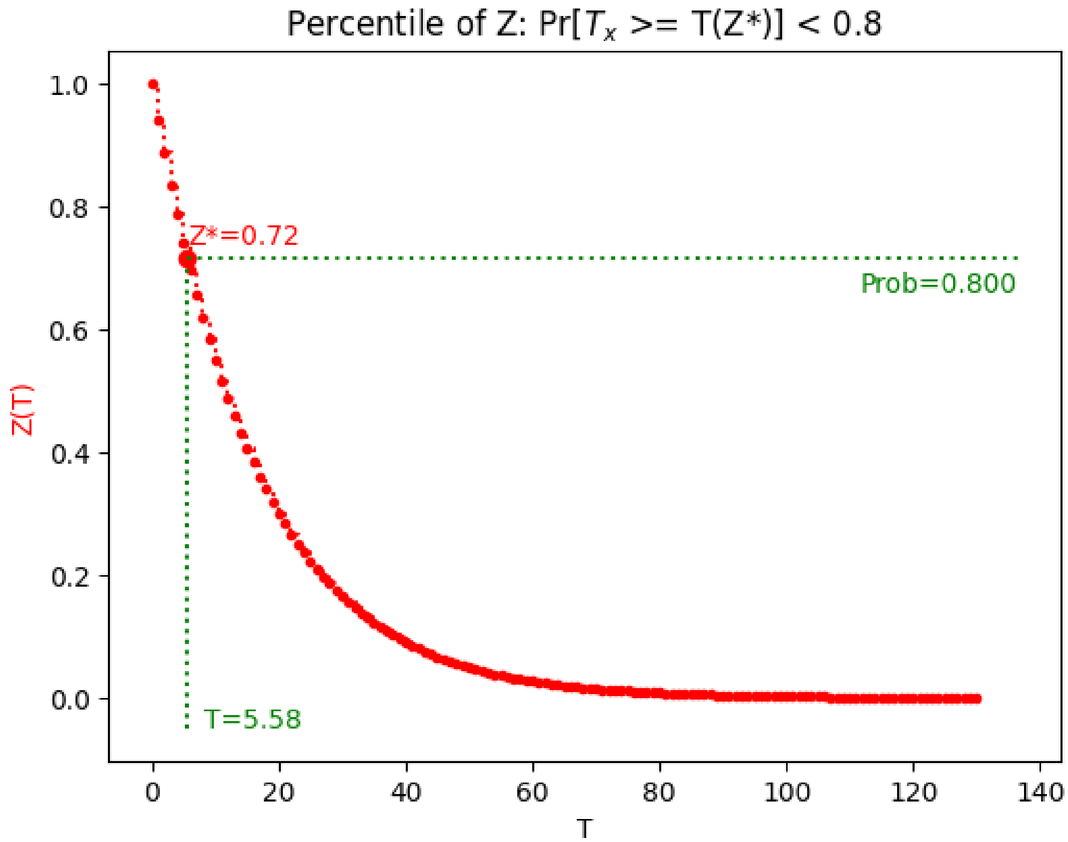
```
SOA Question 6.33:  (B) 0.13
0.10539922456186429
3884632.549746798
0.12828940905648634

SOA Question 4.18  (A) 81873
81873.07530779815

SOA Question 4.10:  (D)
D

SOA Question 4.12:  (C) 167
166.82999999999998

SOA Question 4.11:  (A) 143385
143384.99999999997

SOA Question 4.15  (E) 0.0833
0.08334849338238598

SOA Question 4.4  (A) 0.036
0.03567680106032681

5.578588782855243 5.000000000000001
0.7408182206817179 0.7155417527999328
0.8 0.8187307530779818
Other examples of usage
0.48998642116279045 0.32009235393201546 0.080005661008096
0.7421209265054034
0.5930972723997999
3503.833219537252
```



Percentile of Z: Pr[T_x >= T(Z*)] < 0.733
Z*=86070.80
Prob=0.733
T=3.00
Z(T)
Survival Probability
T



Z
(a)
(b)
(c)
(d)
(e)


Percentile of Z: Pr[T_x >= T(Z*)] < 0.8
Z*=0.72
Prob=0.800
T=5.58
Z(T)
T
1.0
0.8
0.6
0.4
0.2
0.0
0
20
40
60
80
100
120
140

CHAPTER

SIX

ANNUITIES

6.1 Life annuities

Whole life annuity:

$\overline{a}_x = \int_{t=0}^{\infty} v^t \ {}_tp_x \ dt$

$\ddot{a}_x = \sum_{k=0}^{\infty} v^k \ {}_{k|}p_x$

Temporary annuity:

$\overline{a}_{x:\overline{t|}} = \int_{s=0}^{t} v^s \ {}_sp_x \ ds = \overline{A}_x - \ {}_tE_x \ \overline{a}_{x+t}$

$\ddot{a}_{x:\overline{t|}} \sum_{k=0}^{t} v^k \ {}_{k|}p_x = \ddot{a}_x - \ {}_tE_x \ \ddot{a}_{x+t}$

Deferred whole life annuity:

${}_{u|}\overline{a}_x = \overline{a}_x - \overline{a}_{x+u}$

${}_{u|}\ddot{a}_x = \ddot{a}_x - \ddot{a}_{x+u}$

Certain and life annuity:

$\overline{a}_{\overline{x:\overline{n|}}} = \overline{a}_{\overline{n|}} + \ {}_{n|}\overline{a}_x$

$\ddot{a}_{\overline{x:\overline{n|}}} = \ddot{a}_{\overline{n|}} + \ {}_{n|}\ddot{a}_x$

6.2 Insurance twin

Whole life and **Temporary Annuities** ONLY:

$\overline{a}_x = \dfrac{1 - \overline{A}_x}{\delta}$

$\ddot{a}_x = \dfrac{1 - A_x}{d}$

$\overline{a}_{x:\overline{t|}} = \dfrac{1 - \overline{A}_{x:\overline{t|}}}{\delta}$ (continuous endowment insurance twin)

$\ddot{a}_{x:\overline{t|}} = \dfrac{1 - A_{x:\overline{t|}}}{d}$ (discrete endowment insurance twin)

Double the force of interest:

$$^2\overline{a}_x = \frac{1 - {}^2\overline{A}_x}{2\delta}$$

$$^2\ddot{a}_x = \frac{1 - {}^2A_x}{2d - d^2}$$

$$^2\overline{a}_{x:\overline{t|}} = \frac{1 - {}^2\overline{A}_{x:\overline{t|}}}{2\delta}$$

$$^2\ddot{a}_{x:\overline{t|}} = \frac{1 - {}^2A_{x:\overline{t|}}}{2d - d^2}$$

6.3 Immediate annuities

$$a_x = \ddot{a}_x - 1$$

$$a_{x:\overline{t|}} = \ddot{a}_{x:\overline{t|}} - 1 + \ {}_tE_x$$

6.4 Variances

Whole life annuity:

$$Var(\overline{a}_x) = \frac{^2\overline{A}_x - (\overline{A}_x)^2}{d^2}$$

$$Var(\ddot{a}_x) = \frac{^2\ddot{a}_x - \ (A_x)^2}{\delta^2}$$

Temporary annuity:

$$Var(\overline{a}_{x:\overline{t|}}) = \frac{^2\overline{A}_{x:\overline{t|}} - (\overline{A}_{x:\overline{t|}})^2}{d^2}$$

$$Var(\ddot{a}_{x:\overline{t|}}) = \frac{^2A_{x:\overline{t|}} - \ (A_{x:\overline{t|}})^2}{\delta^2}$$

6.5 Varying annuities

Increasing annuity:

$$(\overline{Ia})_x = \int_{t=0}^{\infty} \ t \ v^t \ {}_tp_x \ dt$$

$$(I\ddot{a})_x = \sum_{k=0}^{\infty} \ (k+1) \ v^{k+1} \ {}_kp_x$$

$$(\overline{Ia})_{x:\overline{t|}} = \int_{s=0}^{t} \ s \ v^s \ {}_sp_x \ ds$$

$$(I\ddot{a})_{x:\overline{t|}} = \sum_{k=0}^{t-1} \ (k+1) \ v^{k+1} \ {}_kp_x$$

Decreasing annuity:

$$(\overline{Da})_{x:\overline{t|}} = \int_{s=0}^{t} \ (t-s) \ v^s \ {}_sp_x \ ds$$

$$(D\ddot{a})_{x:\overline{t|}} = \sum_{k=0}^{t-1} \ (t-k) \ v^{k+1} \ {}_kp_x$$

$$(\overline{Da})_{x:\overline{t|}} + (\overline{Ia})_{x:\overline{t|}} = t \ \overline{a}_{x:\overline{t|}}$$

$(D\ddot{a})_{x:\overline{t|}} + (I\ddot{a})_{x:\overline{t|}} = (t+1)\,\ddot{a}_{x:\overline{t|}}$

6.6 Present value random variable Y

Expected present value of a life annuity = EPV(Y)

Whole life annuity:

$Y = \ddot{a}_{\overline{K_x+1|}}$ (discrete) or $\overline{a}_{\overline{T_x|}}$ (continuous)

Temporary insurance:

$Y = \ddot{a}_{\overline{t|}}$ (discrete) or $\overline{a}_{\overline{t|}}$ (continuous) when $K_x \geq t$ or $T_x > t$, else whole life

Certain and life annuity:

$Y = \ddot{a}_{\overline{n|}}$ (discrete) or $\overline{a}_{\overline{n|}}$ (continuous) when $K_x < n$ or $T_x \leq n$, else whole life

```
"""Life annuity functions

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"""
from actuarialmath.annuity import Annuity
print(Annuity.help())
```

```
Life Annuities
--------------

a_x():
  Numerically compute APV of annuities from survival functions

immediate_annuity():
  Compute APV of immediate life annuity

annuity_twin():
  Returns annuity from its WL or Endowment Insurance twin

whole_life_annuity():
  Whole life annuity: a_x

temporary_annuity():
  Temporary life annuity: a_x:t

deferred_annuity():
  Deferred life annuity n|t_a_x =  n+t_a_x - n_a_x

certain_life_annuity():
  Certain and life annuity = certain + deferred

increasing_annuity():
  Increasing annuity

decreasing_annuity():
  Identity (Da)_x:n + (Ia)_x:n = (n+1) a_x:n temporary annuity
```

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```
Y_t():
  T_x given percentile of the r.v. Y = PV of WL or Temporary Annuity

Y_from_t():
  PV of insurance payment Y(t), given T_x (or K_x if discrete)

Y_from_prob():
  Percentile of annuity PV r.v. Y, given probability

Y_to_prob():
  Cumulative density of insurance PV r.v. Y, given percentile value

Y_x():
  APV of t'th year's annuity benefit

Y_plot():
  Plot PV of annuity r.v. Y vs T
```

6.7 Examples

```
if __name__ == "__main__":
    import matplotlib.pyplot as plt

    print("SOA Question 5.6:  (D) 1200")
    life = Annuity(interest=dict(i=0.05))
    var = life.annuity_variance(A2=0.22, A1=0.45)
    mean = life.annuity_twin(A=0.45)
    print(life.portfolio_percentile(mean=mean, variance=var, prob=.95, N=100))
    print()

    print("Plot example")
    life = Annuity(interest=dict(delta=0.06), mu=lambda *x: 0.04)
    prob = 0.8
    x = 0
    discrete = True
    t = life.Y_t(0, prob, discrete=discrete)
    Y = life.Y_from_prob(x, prob=prob, discrete=discrete)
    print(t, life.Y_to_t(Y))
    print(Y, life.Y_from_t(t, discrete=discrete))
    print(prob, life.Y_to_prob(x, Y=Y))
    life.Y_plot(0, T=t, discrete=discrete)
    plt.show()

    print("Other usage")
    mu = 0.04
    delta = 0.06
    life = Annuity(interest=dict(delta=delta), mu=lambda *x: mu)
    print(life.temporary_annuity(50, t=20, b=10000, discrete=False))
    print(life.endowment_insurance(50, t=20, b=10000, discrete=False))
    print(life.E_x(50, t=20))
    print(life.whole_life_annuity(50, b=10000, discrete=False))
    print(life.whole_life_annuity(70, b=10000, discrete=False))
```

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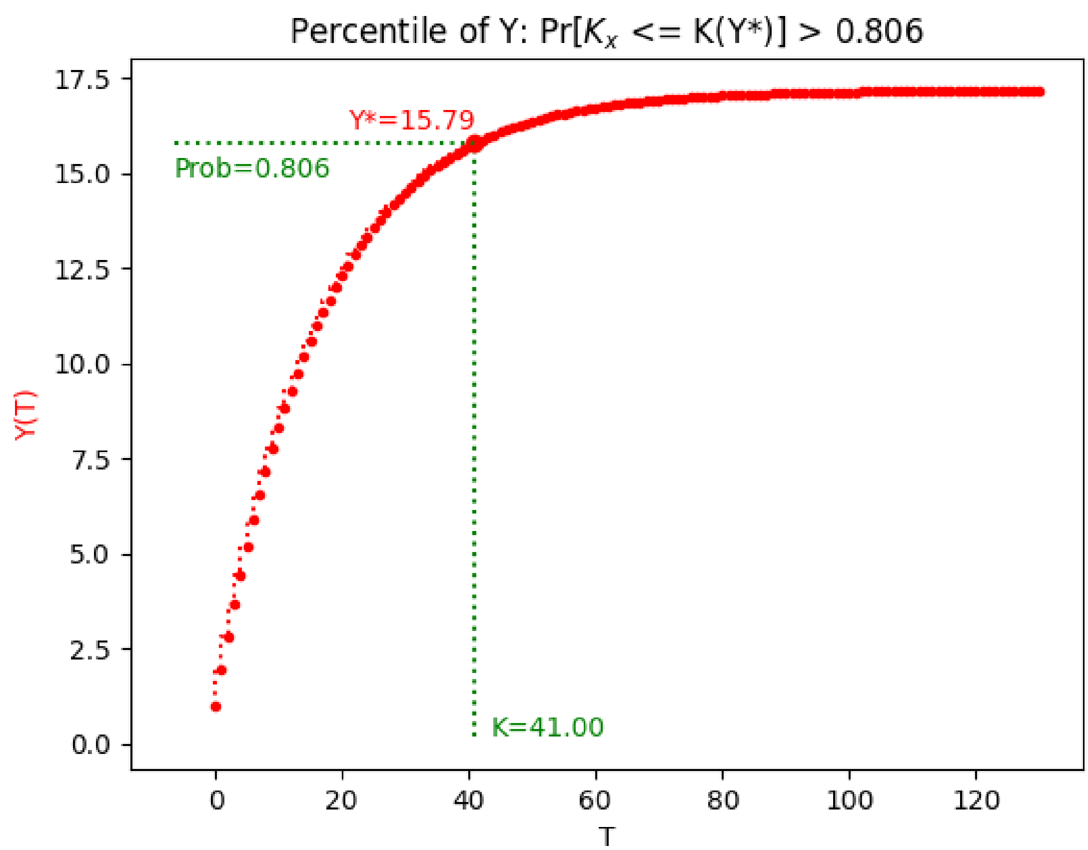
```
mu = 0.07
delta = 0.02
life = Annuity(interest=dict(delta=delta), mu=lambda *x: mu)
print(life.whole_life_annuity(0, discrete=False) * 30)    # 333.33
print(life.temporary_annuity(0, t=10, discrete=False) * 30)   # 197.81
print(life.interest.annuity(5, m=0))   # 4.7581
print(life.deferred_annuity(0, u=5, discrete=False)) # 7.0848
print(life.certain_life_annuity(0, u=5, discrete=False))   # 11.842

mu = 0.02
delta = 0.05
life = Annuity(interest=dict(delta=delta), mu=lambda *x: mu)
print(life.decreasing_annuity(0, t=5, discrete=False))   # 6.94
```

```
SOA Question 5.6:  (D) 1200
1200.6946732201702

Plot example
41 49.084762499581856
15.790040843594133 15.790040843594133
0.8 0.8596183508486661
```




```
Other usage
86466.4716763387
```

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```
4812.011699419677
0.13533528323661273
100022.36417519346
100165.25014511104
333.3430094556871
197.81011341979988
4.758129098202016
7.085079777653729
11.843208875855744
6.94209306102519
```

CHAPTER

SEVEN

PREMIUMS

7.1 Equivalence Principle

Set premiums s.t. expected loss at issue equals zero:

$E[{}_0L] = EPV_0(\text{future benefits}) - EPV_0(\text{future premiums}) = 0$

- Fully continuous: both benefits and premiums are payable continuously
- Fully discrete: benefits are paid at the end of the year, premiums are paid at the beginning of the year
- Semi-continuous: benefits are paid at moment of death, premiums are paid at the beginning of the year

7.2 Net premium

Is the premium is determined excluding expenses under the equivalence principle

- Whole life insurance: $P_x = \dfrac{A_x}{\ddot{a}_x}$ or $\dfrac{\overline{A}_x}{\overline{a}_x}$
- Term life insurance: $P^1_{x:\overline{t|}} = \dfrac{A^1_{x:\overline{t|}}}{\ddot{a}_{x:\overline{t|}}}$ or $\dfrac{\overline{A}^1_{x:\overline{t|}}}{\overline{a}_{x:\overline{t|}}}$
- Pure endowment: $P_{x:\overline{t|}}^{\ \ 1} = \dfrac{{}_tE_x}{\ddot{a}_{x:\overline{t|}}}$ or $\dfrac{{}_tE_x}{\overline{a}_{x:\overline{t|}}}$
- Endowment insurance: $P_{x:\overline{t|}} = \dfrac{A_{x:\overline{t|}}}{\ddot{a}_{x:\overline{t|}}}$ or $\dfrac{\overline{A}_{x:\overline{t|}}}{\overline{a}_{x:\overline{t|}}}$

Shortcuts for whole life and endowment insurance only:

$P = b\,(\dfrac{1}{\ddot{a}_x} - d) = b\,(\dfrac{dA_x}{1 - A_x})$ (fully discrete)

$P = b\,(\dfrac{1}{\overline{a}_x} - \delta) = b\,(\dfrac{d\overline{A}_x}{1 - \overline{A}_x})$ (fully continuous)

7.3 Gross premium

Accounts for expenses. If set under equivalence principle, then expected loss at issue equals zero:

$$E[{}_0L^g] = EPV_0(\text{future benefits}) + EPV_0(\text{future expenses}) - EPV_0(\text{future premiums}) = 0$$

Return of premiums paid without interest upon death:

- $EPV_0 = \sum_{k=0}^{t-1} P(k+1)\, v^{k+1}\ {}_{k|}q_x = P \cdot (IA)^1_{x:\overline{t|}}$

Expenses:

- per policy and/or percent of premium initial expenses in year 1:

  $e_i =$ initial_per_policy + initial_per_premium $\times$ gross_premium

- per policy and/or percent of premium renewal expenses in year 2+:

  $e_r =$ renewal_per_policy + renewal_per_premium $\times$ gross_premium

- settlement expense paid with death benefit: E

  $b + E =$ death benefit + settlement expense = claim_costs

```python
"""Premiums

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"""
from actuarialmath.premiums import Premiums
print(Premiums.help())
```

```
Premiums
--------

net_premium():
  Net level premium for n-pay, u-deferred t-year term insurance

gross_premium():
  Gross premium by equivalence principle

insurance_equivalence():
  Whole life or endowment insurance factor, given net premium

annuity_equivalence():
  Whole life or temporary annuity factor, given net premium

premium_equivalence():
  Premium given whole life or temporary/endowment annuity/insurance
```

7.4 Examples

```
import numpy as np

print("SOA Question 6.29  (B) 20.5")
life = Premiums(interest=dict(i=0.035))
def fun(a):
    return life.gross_premium(A=life.insurance_twin(a=a),
                              a=a, benefit=100000,
                              initial_policy=200, initial_premium=.5,
                              renewal_policy=50, renewal_premium=.1)
print(life.solve(fun, target=1770, guess=[20, 22]))
print()

print("SOA Question 6.2: (E) 3604")
life = Premiums()
A, IA, a = 0.17094, 0.96728, 6.8865
print(life.gross_premium(a=a, A=A, IA=IA, benefit=100000,
                         initial_premium=0.5, renewal_premium=.05,
                         renewal_policy=200, initial_policy=200))
print()

print("SOA Question 6.16: (A) 2408.6")
life = Premiums(interest=dict(d=0.05))
A = life.insurance_equivalence(premium=2143, b=100000)
a = life.annuity_equivalence(premium=2143, b=100000)
p = life.gross_premium(A=A, a=a, benefit=100000, settlement_policy=0,
                       initial_policy=250, initial_premium=.04+.35,
                       renewal_policy=50, renewal_premium=.04+.02)
print(A, a, p)
print()

print("SOA Question 6.20:  (B) 459")
life = Premiums(interest=dict(i=0.04),
                l=lambda x,s: dict(zip([75, 76, 77, 78],
                                   np.cumprod([1,.9,.88,.85]))).get(x+s, 0))
a = life.temporary_annuity(75, t=3)
IA = life.increasing_insurance(75, t=2)
A = life.deferred_insurance(75, u=2, t=1)
print(life.solve(lambda P: P*IA + A*10000 - P*a, target=0, guess=100))
print()

print("Other usage")
life = Premiums(interest=dict(delta=0.06), mu=lambda x,s: 0.04)
print(life.net_premium(0))
```

```
SOA Question 6.29  (B) 20.5
20.480268314431726

SOA Question 6.2: (E) 3604
3604.229940320728

SOA Question 6.16: (A) 2408.6
0.3000139997200056 13.999720005599887 2408.575206281868

SOA Question 6.20:  (B) 459
```

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```
458.83181728297285

Other usage
0.03692697915432344
```

CHAPTER

EIGHT

POLICY VALUES

8.1 Net Policy Value

${}_tV = EPV_t(\text{future benefits}) - EPV_t(\text{future premiums})$

- ${}_0V = E[{}_0L] = 0$ (assumes equivalence principle)
- ${}_nV = E[{}_nL] = 0$ for a n-year term insurance
- ${}_nV = E[{}_nL] =$ endowment benefit for a n-year endowment insurance

Shortcuts for whole life and endowment insurance:

${}_tV = E[{}_tL] = b[1 - \dfrac{\ddot{a}_{x+t}}{\ddot{a}_x}]$ or $b[\dfrac{A_{x+t} - A_x}{1 - A_x}]$ (fully discrete)

${}_tV = E[{}_tL] = b[1 - \dfrac{\overline{a}_{x+t}}{\overline{a}_x}]$ or $b[\dfrac{\overline{A}_{x+t} - \overline{A}_x}{1 - \overline{A}_x}]$ (fully continuous)

8.2 Gross Policy Value

${}_tV^g = E[{}_tL^g] = EPV_t(\text{future benefits}) + EPV_t(\text{future expenses}) - EPV_t(\text{future premiums})$

8.3 Variance of future loss

Whole life and endowment insurance only:

- Net future loss*:

  $Var[{}_tL] = (b + \dfrac{P}{d})^2\ [{}^2A_{x+t:\overline{n-t|}} - (A_{x+t:\overline{n-t|}})^2]$ (discrete)

  $Var[{}_tL] = (b + \dfrac{P}{\delta})^2\ [{}^2\overline{A}_{x+t:\overline{n-t|}} - (\overline{A}_{x+t:\overline{n-t|}})^2]$ (continuous)

- Gross future loss*:

  $Var[{}_tL] = (b + E + \dfrac{G - e_r}{d})^2\ [{}^2A_{x+t:\overline{n-t|}} - (A_{x+t:\overline{n-t|}})^2]$ (discrete)

Shortcuts for variance of net future loss (net premiums under equivalence principle)*:

$$Var[{}_tL] = b^2[\frac{{}^2A_{x+t:\overline{n-t|}} - (A_{x+t:\overline{n-t|}})^2}{(1-A_{x:\overline{n|}})^2}] \text{ (discrete)}$$

$$Var[{}_tL] = b^2[\frac{{}^2\overline{A}_{x+t:\overline{n-t|}} - (\overline{A}_{x+t:\overline{n-t|}})^2}{(1-\overline{A}_{x:\overline{n|}})^2}] \text{ (continuous)}$$

*For whole life insurance, remove the $\overline{n|}$ and $\overline{n-t|}$ notations.

8.4 Expense reserve

$${}_tV^e = {}_tV^g - {}_tV = EPV_t(\text{future expenses}) - EPV_t(\text{future expense loadings})$$

Generally:

- ${}_tV^e < 0$
- ${}_tV > {}_tV^g > 0 > {}_tV^e$

8.5 Present value of loss random variable L

**Note: we have used the terms "expected future loss", "policy value" and "reserves" interchangeably and loosely.

For full discrete whole life or endowment insurance:

- Net future loss

$${}_0L = b\, v^{K_x+1} - P\ddot{a}_{\overline{K_x+1|}} = (b + \frac{P}{d})\, v^{K_x+1} - \frac{P}{d} \text{ (discrete)}$$

$${}_0L = b\, v^{T_x} - P\overline{a}_{\overline{T_x|}} = (b + \frac{P}{\delta})\, v^{T_x} - \frac{P}{\delta} \text{ (continuous)}$$

- Gross future loss

$${}_0L = (b + E + \frac{G - e_r}{d})\, v^{K_x+1} - \frac{G-e_r}{d} + (e_i - e_r)$$

```
"""Policy Values

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"""
from actuarialmath.policyvalues import PolicyValues
print(PolicyValues.help())
```

```
Policy Values
-------------

net_future_loss():
  Assume WL or Endowment Ins for shortcuts since P from equivalence

net_variance_loss():
  Helper for variance of net loss shortcuts of WL or Endowment Ins loss
```

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```
net_policy_variance():
  Shortcuts for variance of future loss for WL or Endow Ins

gross_future_loss():
  Shortcut for WL or Endowment Ins gross future loss

gross_policy_variance():
  Shortcut for gross policy value of WL and Endowment Insurance

gross_policy_value():
  Gross policy values for insurance: t_V = E[L_t]

L_from_t():
  PV of Loss L(t), given T_x (or K_x if discrete)

L_to_t():
  T_x s.t. PV of loss is Z

L_from_prob():
  Percentile of loss PV r.v. L, given probability

L_to_prob():
  Cumulative density of loss PV r.v. L, given percentile value

L_plot():
  Plot loss r.v. L vs T
```

8.6 Examples

```
import matplotlib.pyplot as plt
from actuarialmath.sult import SULT

print("SOA Question 6.24:  (E) 0.30")
life = PolicyValues(interest=dict(delta=0.07))
x, A1 = 0, 0.30   # Policy for first insurance
P = life.premium_equivalence(A=A1, discrete=False)  # Need its premium
policy = life.Policy(premium=P, discrete=False)
def fun(A2):  # Solve for A2, given Var(Loss)
    return life.gross_variance_loss(A1=A1, A2=A2, policy=policy)
A2 = life.solve(fun, target=0.18, guess=0.18)
print()

policy = life.Policy(premium=0.06, discrete=False) # Solve second insurance
variance = life.gross_variance_loss(A1=A1, A2=A2, policy=policy)
print(variance)
print()

print("SOA Question 6.30:  (A) 900")
life = PolicyValues(interest=dict(i=0.04))
policy = life.Policy(premium=2.338, benefit=100, initial_premium=.1,
                        renewal_premium=0.05)
var = life.gross_variance_loss(A1=life.insurance_twin(16.50),
```

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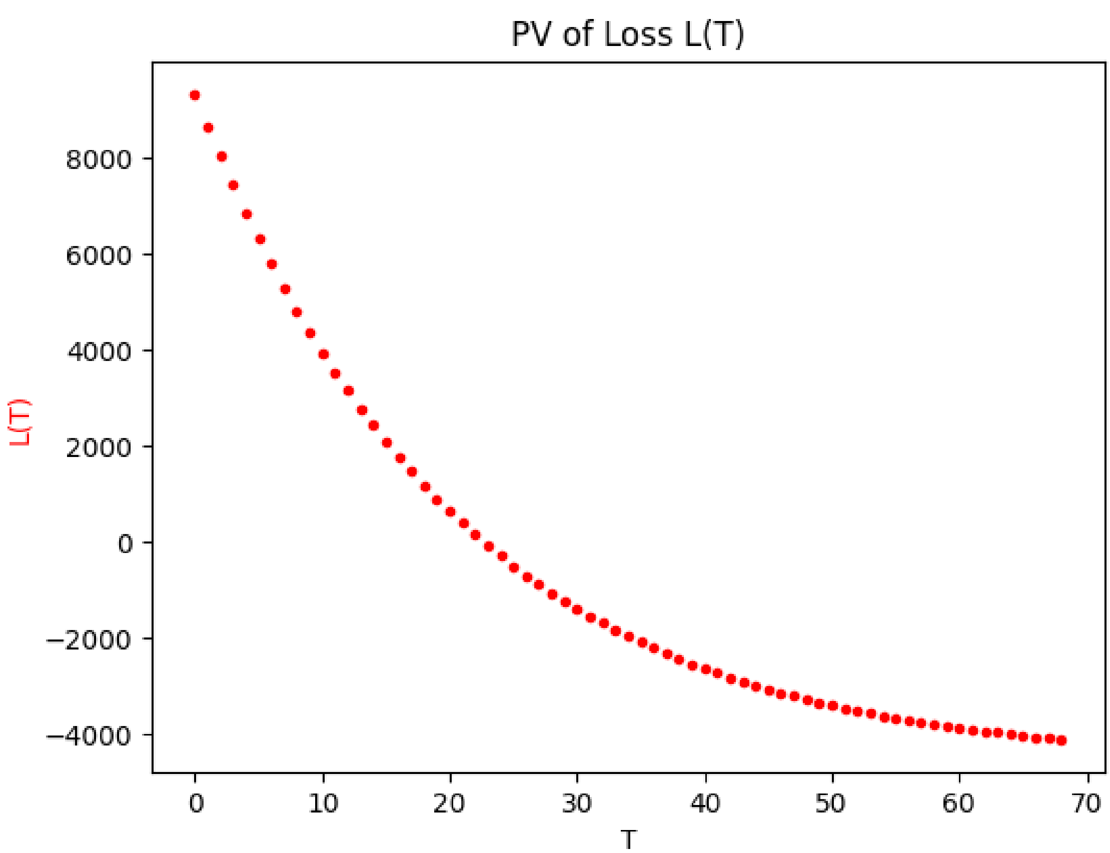
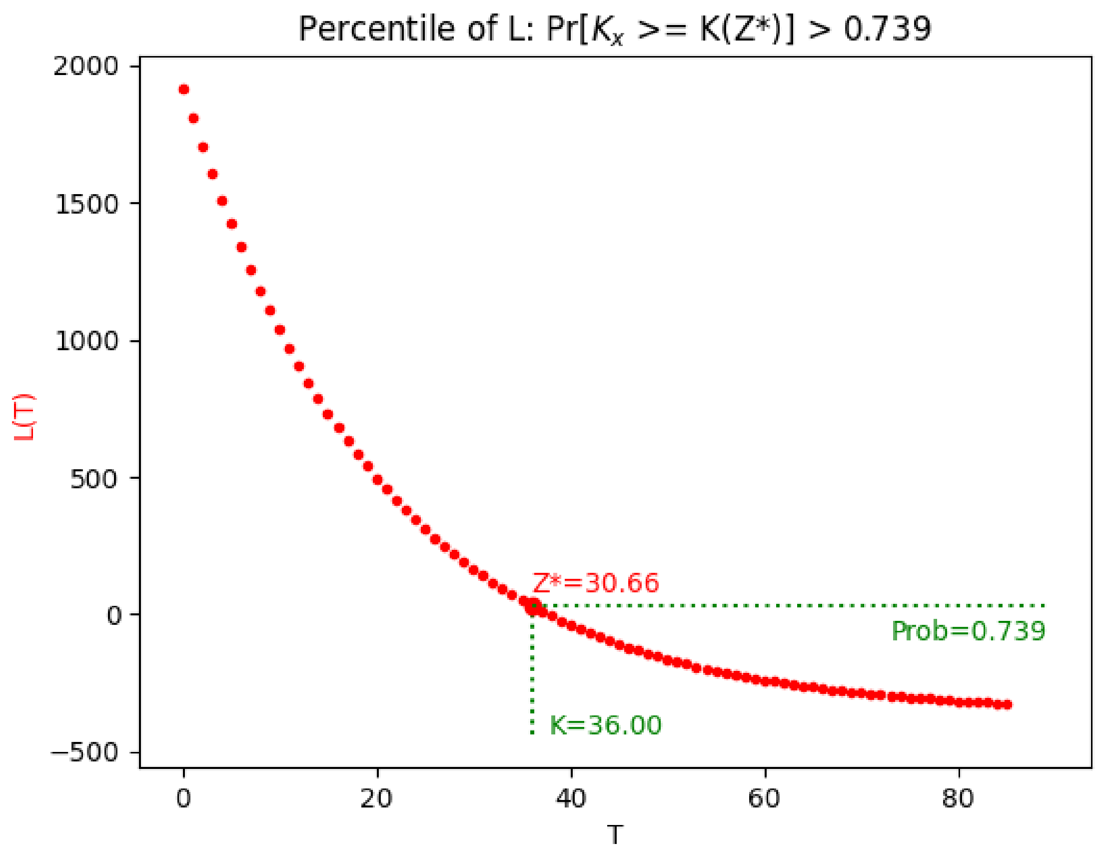
```
                                A2=0.17, policy=policy)
print(var)
print()

print("SOA Question 7.32:  (B) 1.4")
life = PolicyValues(interest=dict(i=0.06))
policy = life.Policy(benefit=1, premium=0.1)
def fun(A2):
    return life.gross_variance_loss(A1=0, A2=A2, policy=policy)
A2 = life.solve(fun, target=0.455, guess=0.455)
policy = life.Policy(benefit=2, premium=0.16)
var = life.gross_variance_loss(A1=0, A2=A2, policy=policy)
print(var)
print()

print("SOA Question 6.12:  (E) 88900")
life = PolicyValues(interest=dict(i=0.06))
a = 12
A = life.insurance_twin(a)
policy = life.Policy(benefit=1000, settlement_policy=20,
                        initial_policy=10, initial_premium=0.75,
                        renewal_policy=2, renewal_premium=0.1)
policy.premium = life.gross_premium(A=A, a=a, **policy.premium_terms)
print(A, policy.premium)
L = life.gross_variance_loss(A1=A, A2=0.14, policy=policy)
print(L)
print()

print("Plot Example -- SOA Question 6.6:  (B) 0.79")
life = SULT()
P = life.net_premium(62, b=10000)
policy = life.Policy(premium=1.03*P, renewal_policy=5,
                        initial_policy=5, initial_premium=0.05, benefit=10000)
L = life.gross_policy_value(62, policy=policy)
var = life.gross_policy_variance(62, policy=policy)
prob = life.portfolio_cdf(mean=L, variance=var, value=40000, N=600)
print(prob, 0.79)
life.L_plot(62, policy=policy)
print()

print("Plot Example -- SOA QUestion 7.6:  (E) -25.4")
life = SULT()
P = life.net_premium(45, b=2000)
policy = life.Policy(benefit=2000, initial_premium=.25, renewal_premium=.05,
                        initial_policy=2*1.5 + 30, renewal_policy=2*.5 + 10)
G = life.gross_premium(a=life.whole_life_annuity(45), **policy.premium_terms)
gross = life.gross_policy_value(45, t=10, policy=policy.set(premium=G))
net = life.net_policy_value(45, t=10, b=2000)
V = gross - net
print(V, -25.4)
T = life.L_to_t(G, policy=policy)
print(G)
life.L_plot(45, T=int(T), policy=policy)

plt.show()
```

```
SOA Question 6.24:  (E) 0.30

0.30419999999999975

SOA Question 6.30:  (A) 900
908.141412994607

SOA Question 7.32:  (B) 1.4
1.3848168384380901

SOA Question 6.12:  (E) 88900
0.3207547169811321 35.38618830746352
88862.59592874818

Plot Example -- SOA Question 6.6:  (B) 0.79
0.7914321142683509 0.79

Plot Example -- SOA QUestion 7.6:  (E) -25.4
-25.44920289521204 -25.4
31.161950196480408
```




Percentile of L: Pr[K_x >= K(Z*)] > 0.739
2000
1500
1000
500
0
−500
L(T)
Z*=30.66
Prob=0.739
K=36.00
0
20
40
60
80
T

CHAPTER

NINE

RESERVES

9.1 Recursion

Gross reserves: $({}_tV^g + G - e)(1+i) = q_{x+t}\,(b+E) + p_{x+t}\;{}_{t+1}V^g$

Net reserves: $({}_tV + P)(1+i) = q_{x+t}\,b + p_{x+t}\;{}_{t+1}V$

Expense reserves: $({}_tV^e + P^e - e)(1+i) = q_{x+t}\,E + p_{x+t}\;{}_{t+1}V^e$

9.2 Interim reserves

$({}_tV + P)(1+i)^r = \;{}_rq_{x+t}\,b\,v^{1-r} + \;{}_rp_{x+t}\;{}_{t+r}V$

${}_{t+r}V\;(1+i)^{1-r} = \;{}_{1-r}q_{x+t+r}\,b + \;{}_{1-r}p_{x+t+r}\;{}_{t+1}V$

9.3 Modified reserves

Full Preliminary Term

- Initial premium: $\alpha = A^{1}_{x:\overline{1|}} = v\,q_x$
- Renewal premium: $\beta = \dfrac{A_{x+1}}{\ddot{a}_{x+1}}$

${}_0V^{FPT} \;=\; {}_1V^{FPT} \;= 0$

${}_tV^{FPT}$ for $(x) \;=\; {}_{t-1}V^{FPT}$ for $(x+1)$

```
"""Recursive, Interim and Modified Reserves

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"""
from actuarialmath.reserves import Reserves
print(Reserves.help())
```

```
Recursive, Interim and Modified Reserves
----------------------------------------

set_reserves():
  None

fill_reserves():
  None

V_plot():
  None

t_V_forward():
  Forward recursion (allows for optional reserve benefit

t_V_backward():
  Backward recursion (allows for optional reserve benefit)

t_V():
  Try to solve time-t Reserve by forward or backward recursion

r_V_forward():
  Forward recursion for interim reserves

r_V_backward():
  Backward recursion for interim reserves

FPT_premium():
  Initial or renewal Full Preliminary Term premiums

FPT_policy_value():
  Compute Full Preliminary Term policy value at time t
```

9.4 Examples

```
import matplotlib.pyplot as plt
from actuarialmath.sult import SULT
from actuarialmath.policyvalues import PolicyValues

print("SOA Question 7.31:  (E) 0.310")
x = 0
life = Reserves().set_reserves(T=3)
print(life._reserves)
G = 368.05
def fun(P):  # solve net premium from expense reserve equation
    return life.t_V(x=x, t=2, premium=G-P, benefit=lambda t: 0,
                    per_policy=5 + .08*G)
P = life.solve(fun, target=-23.64, guess=[.29, .31]) / 1000
print(P)
print()

print("SOA Question 7.13: (A) 180")
life = SULT()
V = life.FPT_policy_value(40, t=10, n=30, endowment=1000, b=1000)
```

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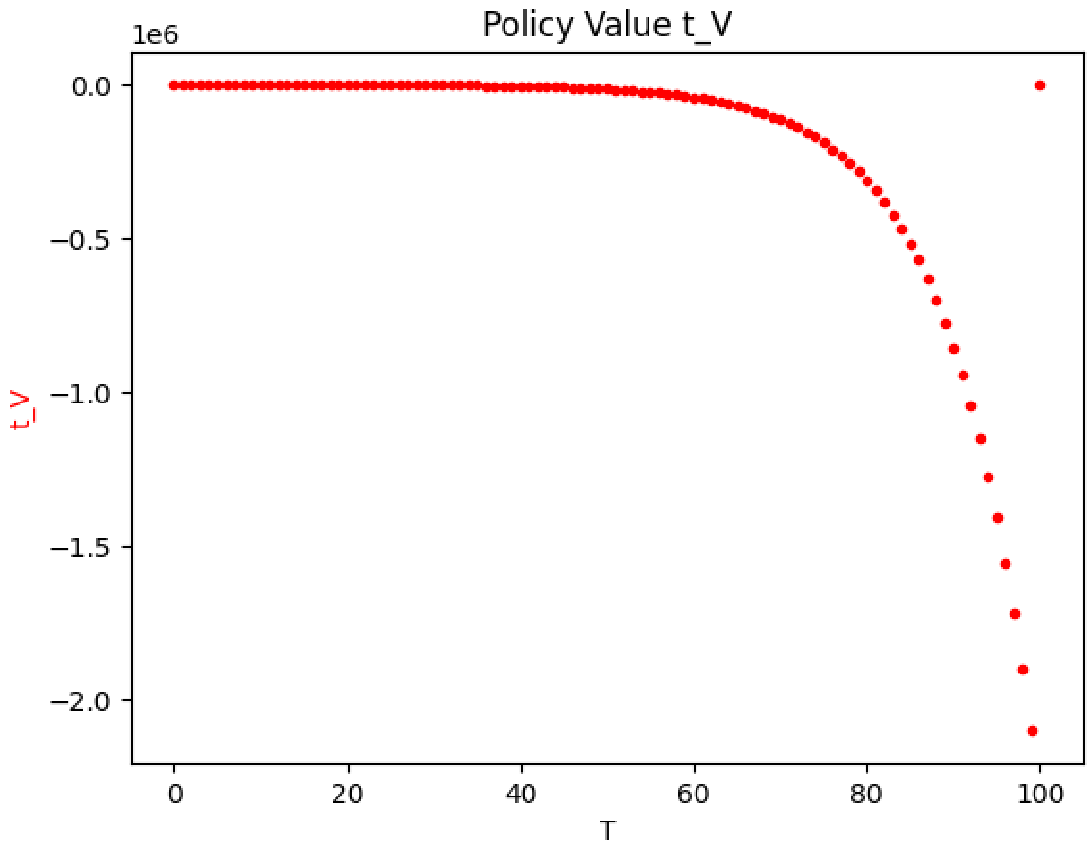
```
print(V)
print()


print("Plot example: TODO from 6.12 -- this needs more work!!!")
life = PolicyValues(interest=dict(i=0.06))
a = 12
A = life.insurance_twin(a)
policy = life.Policy(benefit=1000, settlement_policy=20,
                        initial_policy=10, initial_premium=0.75,
                        renewal_policy=2, renewal_premium=0.1)
policy.premium = life.gross_premium(A=A, a=a, **policy.premium_terms)

life = Reserves(interest=dict(delta=0.06), mu=lambda x,s: 0.04)
life.set_reserves(T=100)
life.fill_reserves(x=0, policy=policy)
life.V_plot()

plt.show()
```

```
SOA Question 7.31:  (E) 0.310
{'V': {0: 0, 3: 0}}
0.309966

SOA Question 7.13: (A) 180
180.1071785904076

Plot example: TODO from 6.12 -- this needs more work!!!
```


Policy Value t_V
1e6
0.0
−0.5
−1.0
−1.5
−2.0
t_V
0
20
40
60
80
100
T

CHAPTER

TEN

MORTALITY LAWS

10.1 Uniform and Beta

Beta (ω, α):

$l_x \sim (\omega - x)^\alpha$

$\mu_x = \dfrac{\alpha}{\omega - x}$

${}_tp_x = (\dfrac{\omega - (x+t)}{\omega - x})^\alpha$

$\overset{\circ}{e}_x = \dfrac{\omega - x}{\alpha + 1}$

Uniform (ω): Beta with $\alpha = 1$

$l_x \sim \omega - x$

$\mu_x = \dfrac{1}{\omega - x}$

${}_tp_x = \dfrac{\omega - (x+t)}{\omega - x}$

$\overset{\circ}{e}_x = \dfrac{\omega - x}{2}$

$\overset{\circ}{e}_{x:\overline{n|}} = \ {}_np_x \ n + \ {}_nq_x \ \dfrac{n}{2}$

${}_nE_x = v^n \dfrac{\omega - (x+n)}{\omega - x}$

$\bar{A}_x = \dfrac{\bar{a}_{\overline{\omega - x|}}}{\omega - x}$

$\bar{A}^{1}_{x:\overline{n|}} = \dfrac{\bar{a}_{\overline{n|}}}{\omega - x}$

10.2 Gompertz and Makeham

Makeham's Law: $c > 1,\ B > 0,\ A \geq -B$

$\mu_x = A + Bc^x$

${}_tp_x = e^{\frac{Bc^x}{\ln c}(c^t-1)-At}$

Gompertz's Law: Makeham's Law with $A = 0$

```
"""Mortality Laws: Uniform, Beta, Gompertz, Makeham

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"""
from actuarialmath.mortalitylaws import MortalityLaws, Uniform, Beta, Makeham,␣
↪Gompertz
print(MortalityLaws.help())
```

```
Shortcuts for Special Mortality Laws
------------------------------------

l_r():
  Fractional age lifes given continuous mortality law: l_[x]+s+r

p_r():
  Fractional age survival given continuous mortality law

q_r():
  Fractional age deferred mortality given continuous mortality law

mu_r():
  Fractional age force of mortality given continuous mortality law

f_r():
  fractional age mortality pdf given continuous mortality law

e_r():
  Fractional age future lifetime given continuous mortality law
```

10.3 Examples

```
print('Beta')
life = Beta(omega=100, alpha=0.5)
print(life.q_x(25, t=1, u=10))      # 0.0072
print(life.e_x(25))                 # 50
print(Beta(omega=60, alpha=1/3).mu_x(35) * 1000)
print()

print('Uniform')
uniform = Uniform(80, interest=dict(delta=0.04))
print(uniform.whole_life_annuity(20))         # 15.53
print(uniform.temporary_annuity(20, t=5))    # 4.35
```

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```
print(Uniform(161).p_x(70, t=1)) # 0.98901
print(Uniform(95).e_x(30, t=40, curtate=False)) # 27.692
print()

uniform = Uniform(omega=80, interest=dict(delta=0.04))
print(uniform.E_x(20, t=5))  # .7505
print(uniform.whole_life_insurance(20, discrete=False))  # .3789
print(uniform.term_insurance(20, t=5, discrete=False))  # .0755
print(uniform.endowment_insurance(20, t=5, discrete=False))  # .8260
print(uniform.deferred_insurance(20, u=5, discrete=False))  # .3033
print()

print('Gompertz/Makeham')
life = Gompertz(B=0.000005, c=1.10)
p = life.p_x(80, t=10)  # 869.4
print(life.portfolio_percentile(N=1000, mean=p, variance=p*(1-p), prob=0.99))

print(Gompertz(B=0.00027, c=1.1).f_x(50, t=10)) # 0.04839
life = Makeham(A=0.00022, B=2.7e-6, c=1.124)
print(life.mu_x(60) * 0.9803)  # 0.00316
```

```
Beta
0.007188905547861446
50.0
13.333333333333332

Uniform
16.03290804858584
4.47503070125663
0.989010989010989
27.692307692307693

0.7505031903214833
0.378867519462745
0.07552885288417432
0.8260320432056576
0.30333866657857067

Gompertz/Makeham
869.3908338193208
0.048389180223511644
0.0031580641631654026
```

CHAPTER

ELEVEN

CONSTANT FORCE

11.1 Constant force of mortality

$_tp_x = e^{-\mu t}$

Future lifetime:

$\mathring{e}_x = \frac{1}{\mu}$

$\mathring{e}_{x:\overline{n|}} = \frac{1}{\mu}(1 - e^{-\mu n})$

Pure endowment:

$_nEx = e^{-(\mu+\delta)n}$

Insurance:

$\bar{A}_x = \frac{\mu}{\mu + \delta}$

$\bar{A}_{x:\overline{t|}} = \frac{\mu}{\mu + \delta}(1 - e^{-\mu t})$

Annuities:

$\bar{a}_x = \frac{1}{\mu + \delta}$

$\bar{a}_{x:\overline{t|}} = \frac{1}{\mu + \delta}(1 - e^{-\mu t})$

```
"""Constant Force mortality law shortcuts

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"""
from actuarialmath.constantforce import ConstantForce
print(ConstantForce.help())
```

```
Constant Force of Mortality: memoryless exponential distribution of deaths
--------------------------------------------------------------------------
```

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```
e_x():
  Expected lifetime E[T_x] is memoryless: does not depend on (x)

E_x():
  Shortcut for APV of whole life: does not depend on age x

whole_life_insurance():
  Shortcut for APV of whole life: does not depend on age x

temporary_annuity():
  Shortcut for temporary life annuity: does not depend on age x

term_insurance():
  Shortcut for APV of term life: does not depend on age x

Z_t():
  Shortcut for T_x (or K_x) given survival probability for insurance

Y_t():
  Shortcut for T_x (or K_x) given survival probability for annuity
```

11.2 Examples

```python
from scipy.stats import norm
import math

print("SOA Question 6.36:  (B) 500")
life = ConstantForce(mu=0.04, interest=dict(delta=0.08))
a = life.temporary_annuity(50, t=20, discrete=False)
A = life.term_insurance(50, t=20, discrete=False)
print(a,A)
def fun(R):
        return life.gross_premium(a=a, A=A, initial_premium=R/4500,
                                  renewal_premium=R/4500, benefit=100000)
R = life.solve(fun, target=4500, guess=[400, 800])
print(R)
print()

print("SOA Question 6.31:  (D) 1330")
life = ConstantForce(mu=0.01, interest=dict(delta=0.05))
A = life.term_insurance(35, t=35) + life.E_x(35, t=35) * 0.51791 # A_35
A = (life.term_insurance(35, t=35, discrete=False)
        + life.E_x(35, t=35) * 0.51791)    # A_35
P = life.premium_equivalence(A=A, b=100000, discrete=False)
print(P)
print()

print("SOA Question 6.27:  (D) 10310")
life = ConstantForce(mu=0.03, interest=dict(delta=0.06))
x = 0
payments = (3 * life.temporary_annuity(x, t=20, discrete=False)
        + life.deferred_annuity(x, u=20, discrete=False))
benefits = (1000000 * life.term_insurance(x, t=20, discrete=False)
```

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```
        + 500000 * life.deferred_insurance(x, u=20, discrete=False))
print(benefits, payments)
print(life.term_insurance(x, t=20), life.deferred_insurance(x, u=20))
P = benefits / payments
print(P)
print()


print("SOA Question 5.4:  (A) 213.7")
life = ConstantForce(mu=0.02, interest=dict(delta=0.01))
P = 10000 / life.certain_life_annuity(40, u=life.e_x(40, curtate=False),
                                        discrete=False)
print()


print("SOA Question 5.1: (A) 0.705")
life = ConstantForce(mu=0.01, interest=dict(delta=0.06))
EY = life.certain_life_annuity(0, u=10, discrete=False)
print(life.p_x(0, t=life.Y_to_t(EY)))  # 0.705
print()


print("Other examples")
life = ConstantForce(mu=0.03, interest=dict(delta=0.04))
print(life.whole_life_annuity(20, discrete=False))  # 14.286
print(life.temporary_annuity(20, t=5, discrete=False))  # 4.219

life = ConstantForce(mu=0.04, interest=dict(delta=0.07))
#print(life.T_p(30, 0.7), 1000*life.certain.Z_t(30))  # 8.9169  535.7, , 0.7 ???

life = ConstantForce(mu=.03, interest=dict(delta=.04))
print(life.E_x(20, t=5))  # .7047
print(life.whole_life_insurance(20, discrete=False))  # .4286
print(life.term_insurance(20, t=5, discrete=False))  # .1266
print(life.endowment_insurance(20, t=5, discrete=False))  # .8313
print(life.deferred_insurance(20, u=5, discrete=False))  # .3020

life1 = ConstantForce(mu=.04, interest=dict(delta=.02))
life2 = ConstantForce(mu=.05, interest=dict(delta=.02))
life3 = ConstantForce(mu=.05, interest=dict(delta=.03))

A1 = life1.term_insurance(0, t=5, discrete=False)
E1 = life1.E_x(0, t=5)
A2 = life2.term_insurance(5, t=7, discrete=False)
E2 = life2.E_x(5, t=7)
A3 = life3.whole_life_insurance(12, discrete=False)
print(A1, E1, A2, E2, A1 + E1 * (A2 + E2 * A3))


life = ConstantForce(mu=.04, interest=dict(delta=.06))
A1 = 10 * life.deferred_insurance(0, u=5, discrete=False)
A2 = 10 * 10 * life.deferred_insurance(0, u=5, moment=2, discrete=False)
E = 100 * A1
V = 100 * (A2 - A1**2)
print(A1, A2, E, V)  # 2.426, 11.233, 242.6, 534.8
print(E + norm.ppf(0.95) * math.sqrt(V)) # 281
```

```
SOA Question 6.36:  (B) 500
7.577350389254893 0.3030940155701957
500.0

SOA Question 6.31:  (D) 1330
1326.5406293909457

SOA Question 6.27:  (D) 10310
305783.51862973545 29.66002470618696
0.26992967028309356 0.053452469414929524
10309.617799001708

SOA Question 5.4:  (A) 213.7

SOA Question 5.1: (A) 0.705
0.7053680433746505

Other examples
14.28571428571429
4.21874157544695
0.7046880897187136
0.42857142857142866
0.1265622472634085
0.831250336982122
0.3020091813080202
0.17278785287885476 0.740818220681718 0.27669543272541713 0.6126263941844162 0.
↪6614218680727285
2.426122638850533 11.233224102930535 242.61226388505327 534.7153044187462
280.64771434478587
```

CHAPTER

TWELVE

LIFE TABLE

12.1 Pure endowment

$$ {}_tE_x = v^t \frac{l_{x+t}}{l_x} $$

$$ {}^2_tE_x = v^{2t} \frac{l_{x+t}}{l_x} = v^t \ {}_tE_x $$

```
"""Life Tables

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"""

from actuarialmath.lifetable import LifeTable
print(LifeTable.help())
```

```
Life Tables
-----------

fill():
  Fill in missing lives and mortality. Does not check consistency

l_x():
  Lookup l_x from life table

d_x():
  Compute from lifetable lifes at x_t divided by lifes at x

p_x():
  t_p_x = lifes beginning year x+t divided lives beginning year x

q_x():
  Deferred mortality: u|t_q_x = (l[x+u] - l[x+u+t]) / l[x]

f_x():
  probability density function of mortality
```

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```
mu_x():
  Compute mu_x from p_x in life table

e_x():
  Expected curtate lifetime from sum of lifes in table

E_x():
  Pure Endowment from life table and interest rate

frame():
  Return life table values in a DataFrame
```

12.2 Examples

```
import math
import matplotlib.pyplot as plt
import numpy as np
import pandas as pd

print("SOA Question 6.53:  (D) 720")
x = 0
life = LifeTable(interest=dict(i=0.08), q={x:.1, x+1:.1, x+2:.1}).fill()
A = life.term_insurance(x, t=3)
P = life.gross_premium(a=1, A=A, benefit=2000, initial_premium=0.35)
print(A, P)
print(life.frame())
print()

print("SOA Question 6.41:  (B) 1417")
x = 0
life = LifeTable(interest=dict(i=0.05), q={x:.01, x+1:.02}).fill()
P = 1416.93
a = 1 + life.E_x(x, t=1) * 1.01
A = (life.deferred_insurance(x, u=0, t=1)
        + 1.01 * life.deferred_insurance(x, u=1, t=1))
print(a, A)
P = 100000 * A / a
print(P)
print(life.frame())
print()


print("SOA Question 3.11:  (B) 0.03")
life = LifeTable(q={50//2: .02, 52//2: .04}, udd=True).fill()
print(life.q_r(50//2, t=2.5/2))
print(life.frame())
print()


print("SOA Question 3.5:  (E) 106")
l = [99999, 88888, 77777, 66666, 55555, 44444, 33333, 22222]
a = LifeTable(l={age:l for age,l in zip(range(60, 68), l)}, udd=True)\
    .q_r(60, u=3.4, t=2.5)
```

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```
b = LifeTable(l={age:l for age,l in zip(range(60, 68), l)}, udd=False)\
    .q_r(60, u=3.4, t=2.5)
print(100000 * (a - b))
print()


print("SOA Question 3.14:  (C) 0.345")
life = LifeTable(l={90: 1000, 93: 825},
                    d={97: 72},
                    p={96: .2},
                    q={95: .4, 97: 1}, udd=True).fill()
print(life.q_r(90, u=93-90, t=95.5-93))
print(life.frame())
print()

print("Other usage examples")
l = [110, 100, 92, 74, 58, 38, 24, 10, 0]
table = LifeTable(l={age:l for age,l in zip(range(79, 88), l)},
                    interest=dict(i=0.06), maxage=87)
print(table.mu_x(80))
#    print(table.temporary_annuity(80, t=4, m=4, due=True)) # 2.7457
#    print(table.whole_life_annuity(80, m=4, due=True, woolhouse=True)) # 3.1778
#    print(table.whole_life_annuity(80, m=4, due=False, woolhouse=True)) # 2.9278
print(table.temporary_annuity(80, t=4)) # 2.7457
print(table.whole_life_annuity(80)) # 3.1778
print('*', table.whole_life_annuity(80, discrete=False)) # 2.9278

l = [100, 90, 70, 50, 40, 20, 0]
table = LifeTable(l={age:l for age,l in zip(range(70, 77), l)},
                    interest=dict(i=0.08), maxage=76)
print(table.A_x(70),
        table.A_x(70, moment=2))  # .75848, .58486
print(table.endowment_insurance(70, t=3)) # .81974
print('*', table.endowment_insurance(70, t=3, discrete=False)) # .81974

print(table.E_x(70, t=3)) # .39692
print('*', table.term_insurance(70, t=3, discrete=False)) # .43953
print('*', table.endowment_insurance(70, t=3, discrete=False)) # .83644
print(table.E_x(70, t=3, moment=2)) # .31503
print('*', table.term_insurance(70, t=3, moment=2, discrete=False)) # .38786
print('*', table.endowment_insurance(70, t=3, moment=2, discrete=False)) # .70294


l = [1000, 990, 975, 955, 925, 890, 840]
table = LifeTable(l={age:l for age,l in zip(range(70, 77), l)},
                    interest=dict(i=0.08), maxage=76)
print(table.increasing_annuity(70, t=4, discrete=True))
print(table.decreasing_annuity(71, t=5, discrete=False))

print('*', table.endowment_insurance(70, t=3, discrete=False)) # .7976

l = [100, 90, 70, 50, 40, 20, 0]
table = LifeTable(l={age:l for age,l in zip(range(70, 77), l)},
                    interest=dict(i=0.08), maxage=76)
print(1e6*table.whole_life_annuity(70, variance=True)) #1743784
```

```
SOA Question 6.53:  (D) 720
0.23405349794238678 720.1646090534978
          l        d    q    p
0  100000.0  10000.0  0.1  0.9
1   90000.0   9000.0  0.1  0.9
2   81000.0   8100.0  0.1  0.9
3   72900.0      NaN  NaN  NaN

SOA Question 6.41:  (B) 1417
1.9522857142857144 0.027662585034013608
1416.9332301924137
          l       d     q     p
0  100000.0  1000.0  0.01  0.99
1   99000.0  1980.0  0.02  0.98
2   97020.0     NaN   NaN   NaN

SOA Question 3.11:  (B) 0.03
0.0298
           l       d     q     p
25  100000.0  2000.0  0.02  0.98
26   98000.0  3920.0  0.04  0.96
27   94080.0     NaN   NaN   NaN

SOA Question 3.5:  (E) 106
106.16575827938624

SOA Question 3.14:  (C) 0.345
0.345
         l      d    q    p
90  1000.0    NaN  NaN  NaN
93   825.0    NaN  NaN  NaN
95   600.0  240.0  0.4  0.6
96   360.0  288.0  0.8  0.2
97    72.0   72.0  1.0  0.0
98     0.0    NaN  NaN  NaN

Other usage examples
0.08338160893905101
3.013501078071164
3.564334685633434
* 3.0554886512477943
0.7584803549199998 0.5848621157624223
0.8197429253670677
* 0.8364390776999064
0.3969161205100847
* 0.4395229571898218
* 0.8364390776999064
0.31508481344155215
* 0.3878582966930021
* 0.7029431101345542
8.373488543413096
6.7673201739834585
* 0.7976061232001961
1744071.8039800397
```

CHAPTER

THIRTEEN

SULT

13.1 Standard ultimate life table

From SOA's "Excel Workbook for FAM-L Tables":

- interest rate $i = 0.05$
- 100000 initial lifes aged 20
- Makeham's Law with $A = 0.00022,\ B = 0.0000027,\ c = 1.124$

13.2 Temporary Annuity

$A^{1}_{x:\overline{t|}} = A_x - {}_tE_x\ A_{x+t} = A_{x:\overline{t|}} - {}_tE_x$

${}^2A^{1}_{x:\overline{t|}} = {}^2A_x - {}^2_tE_x\ {}^2A_{x+t} = {}^2A_x - v^t\ {}_tE_x\ {}^2A_{x+t}$

```
"""Standard Ultimate Life Table

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"""

from actuarialmath.sult import SULT
print(SULT.help())
```

```
Standard Ultimate Life Table
----------------------------

frame():
  Displays SULT table used in FAM-L exam
```

13.3 Examples

```
print("SOA Question 6.52:  (D) 50.80")
sult = SULT()
a = sult.temporary_annuity(45, t=10)
other_cost = 10 * sult.deferred_annuity(45, u=10)
P = sult.gross_premium(a=a, A=0, benefit=0,
                        initial_premium=1.05, renewal_premium=0.05,
                        initial_policy=100 + other_cost, renewal_policy=20)
print(a, P)
print()


print("SOA Question 6.47:  (D) 66400")
sult = SULT()
a = sult.temporary_annuity(70, t=10)
A = sult.deferred_annuity(70, u=10)
P = sult.gross_premium(a=a, A=A, benefit=100000, initial_premium=0.75,
                        renewal_premium=0.05)
print(P)
print()


print("SOA Question 6.43:  (C) 170")
sult = SULT()
a = sult.temporary_annuity(30, t=5)
A = sult.term_insurance(30, t=10)
other_expenses = 4 * sult.deferred_annuity(30, u=5, t=5)
P = sult.gross_premium(a=a, A=A, benefit=200000, initial_premium=0.35,
                        initial_policy=8 + other_expenses, renewal_policy=4,
                        renewal_premium=0.15)
print(P)
print()


print("SOA Question 6.39:  (A) 29")
sult = SULT()
P40 = sult.premium_equivalence(sult.whole_life_insurance(40), b=1000)
P80 = sult.premium_equivalence(sult.whole_life_insurance(80), b=1000)
p40 = sult.p_x(40, t=10)
p80 = sult.p_x(80, t=10)
P = (P40 * p40 + P80 * p80) / (p80 + p40)
print(P)
print()


print("SOA Question 6.37:  (D) 820")
sult = SULT()
benefits = sult.whole_life_insurance(35, b=50000 + 100)
expenses = sult.immediate_annuity(35, b=100)
a = sult.temporary_annuity(35, t=10)
print(benefits, expenses, a)
print((benefits + expenses) / a)
print()
```

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```
print("SOA Question 6.35:  (D) 530")
sult = SULT()
A = sult.whole_life_insurance(35, b=100000)
a = sult.whole_life_annuity(35)
print(sult.gross_premium(a=a, A=A, initial_premium=.19, renewal_premium=.04))
print()


print("SOA Question 5.8: (C) 0.92118")
sult = SULT()
a = sult.certain_life_annuity(55, u=5)
print(sult.p_x(55, t=math.floor(a)))
print()


print("SOA Question 5.3:  (C) 0.6239")
sult = SULT()
t = 10.5
print(t * sult.E_r(40, t=t))
print()


print("SOA Question 4.17:  (A) 1126.7")
sult = SULT()
median = sult.Z_t(48, prob=0.5, discrete=False)
benefit = lambda x,t: 5000 if t < median else 10000
print(sult.A_x(48, benefit=benefit))
print()


print("SOA Question 4.14:  (E) 390000    ")
sult = SULT()
p = sult.p_x(60, t=85-60)
mean = sult.bernoulli(p)
var = sult.bernoulli(p, variance=True)
F = sult.portfolio_percentile(mean=mean, variance=var, prob=.86, N=400)
print(F * 5000 * sult.interest.v_t(85-60))
print()


print("SOA Question 4.5:  (C) 35200")
sult = SULT()
print(100000 * sult.Interest(delta=0.05).v_t(sult.Z_t(45, prob=.95)))
print()


print("SOA Question 3.9:  (E) 3850")
sult = SULT()
p1 = sult.p_x(20, t=25)
p2 = sult.p_x(45, t=25)
mean = sult.bernoulli(p1) * 2000 + sult.bernoulli(p2) * 2000
var = (sult.bernoulli(p1, variance=True) * 2000
        + sult.bernoulli(p2, variance=True) * 2000)
print(sult.portfolio_percentile(mean=mean, variance=var, prob=.99))
print()
```

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```
print("SOA Question 3.8:  (B) 1505")
sult = SULT()
p1 = sult.p_x(35, t=40)
p2 = sult.p_x(45, t=40)
mean = sult.bernoulli(p1) * 1000 + sult.bernoulli(p2) * 1000
var = (sult.bernoulli(p1, variance=True) * 1000
        + sult.bernoulli(p2, variance=True) * 1000)
print(sult.portfolio_percentile(mean=mean, variance=var, prob=.95))
print()


print("SOA Question 3.4:  (B) 815")
sult = SULT()
mean = sult.p_x(25, t=95-25)
var = sult.bernoulli(mean, variance=True)
print(sult.portfolio_percentile(N=4000, mean=mean, variance=var, prob=.1))
print()

print("Other usage examples")
print(sult.temporary_annuity(80, t=10)*20000) # ~130770

E = sult.E_x(60, t=5)
print(E, E*sult.a_x(65, benefit=(lambda x,t: 1000 + .05*t)))

print(sult.whole_life_annuity(60))  # 14.9041
print(sult.whole_life_annuity(60, discrete=False))  # ~13.9041
print(sult.deferred_annuity(60, u=10))  #6.9485
print(sult.temporary_annuity(60, t=10))  # 7.9555
print(sult.temporary_annuity(60, t=15))  # 10.5282
print(sult.temporary_annuity(60, t=10))
print(sult.E_x(60, t=10))  #
print(sult.temporary_annuity(60, t=10, discrete=False))  # ~7.5341
print(sult.certain_life_annuity(60, u=10))  # 15.0563
print(sult.whole_life_annuity(60, variance=True, discrete=False)) # ~10.6182
print(sult.endowment_insurance(60, t=10)) # .62116
print(sult.whole_life_insurance(60, moment=2)) # .10834
print(sult.whole_life_insurance(70, moment=2)) # .21467
print(sult.endowment_insurance(60, t=10, moment=2)) # .38732
print(sult.temporary_annuity(60, t=10, variance=True)) #.6513


print(sult.p_x(70)) # 0.989587
print(math.log(sult.p_x(70, t=2)) / -2)  # 0.011103

A1 = sult.whole_life_insurance(20, discrete=False)
#print(A1, sult.whole_life(20))
A2 = sult.whole_life_insurance(50, discrete=False)
#print(A2, sult.whole_life(50))
A3 = sult.whole_life_insurance(70, discrete=False)
E2 = sult.E_x(20, t=30)
E3 = sult.E_x(20, t=50)
print(A1, E2, A2, E3, A3, 125*A1 + E2*175*A2 - E3*250*A3)  # 5,335

print(sult.whole_life_insurance(50)) # .18931
print(sult.term_insurance(50, t=20)) # .0402
```

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```
print(sult.endowment_insurance(50, t=20))  # ~.31471
print(sult.E_x(50, t=20),
        sult.whole_life_insurance(70),
        sult.deferred_insurance(50, u=20)) # .14911

print(sult.whole_life_insurance(50, discrete=False)) # .19400
print(sult.term_insurance(50, t=20, discrete=False)) # .0412
print(sult.endowment_insurance(50, t=20, discrete=False))  #.38944
print(sult.E_x(50, t=20),
        sult.whole_life_insurance(70, discrete=False),
        sult.deferred_insurance(50, u=20, discrete=False)) # .15281
print()

print("Standard Ultimate Life Table at i=0.05")
sult.frame()
```

```
SOA Question 6.52:  (D) 50.80
8.0750937741422 50.80135534704229

SOA Question 6.47:  (D) 66400
66384.13293704337

SOA Question 6.43:  (C) 170
171.22371939459944

SOA Question 6.39:  (A) 29
29.033866427845496

SOA Question 6.37:  (D) 820
4836.382819496279 1797.2773668474615 8.092602358383987
819.7190338249138

SOA Question 6.35:  (D) 530
534.4072234303344

SOA Question 5.8: (C) 0.92118
0.9211799771029529

SOA Question 5.3:  (C) 0.6239
6.23871918627528

SOA Question 4.17:  (A) 1126.7
1126.774772894844

SOA Question 4.14:  (E) 390000
389322.86778416135

SOA Question 4.5:  (C) 35200
36787.94411714415

SOA Question 3.9:  (E) 3850
3850.144345130047

SOA Question 3.8:  (B) 1505
1504.8328375406456
```

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```
SOA Question 3.4:  (B) 815
815.0943255167722

Other usage examples
135770.41601330126
0.7668687235541889 10395.824343647037
14.904074300627297
14.39854504493635
6.9485261567485095
7.955548143878787
10.52811141378872
7.955548143878787
0.5786434508971754
7.74293075434579
15.05634783239256
10.621710987844606
0.6211643741010102
0.10834081779190481
0.21466683367433795
0.38732012436971175
0.6504506203786906
0.9895866730368528
0.01110329636560811
0.05042882180359127 0.22808348858314714 0.1940084167264656 0.0794272723450692 0.
↪4388106987422976 5.333988893971686
0.18930786030072838
0.04020082061028696
0.3884385332061035
0.34823771259581654 0.4281760254481774 0.14910703969044142
0.1940084167264656
0.041197982733875926
0.38943569532969247
0.34823771259581654 0.4388106987422976 0.15281043399258967

Standard Ultimate Life Table at i=0.05
```

```
          l_x       q_x      a_x      A_x     2A_x  a_x:10   A_x:10   a_x:20  \
20   100000.0  0.000250  19.9664  0.04922  0.00580  8.0991  0.61433  13.0559
21    99975.0  0.000253  19.9197  0.05144  0.00614  8.0990  0.61433  13.0551
22    99949.7  0.000257  19.8707  0.05378  0.00652  8.0988  0.61434  13.0541
23    99924.0  0.000262  19.8193  0.05622  0.00694  8.0986  0.61435  13.0531
24    99897.8  0.000267  19.7655  0.05879  0.00739  8.0983  0.61437  13.0519
..        ...       ...      ...      ...      ...     ...      ...      ...
96    17501.8  0.192887   3.5597  0.83049  0.69991  3.5356  0.83164   3.5597
97    14125.9  0.214030   3.3300  0.84143  0.71708  3.3159  0.84210   3.3300
98    11102.5  0.237134   3.1127  0.85177  0.73356  3.1050  0.85214   3.1127
99     8469.7  0.262294   2.9079  0.86153  0.74930  2.9039  0.86172   2.9079
100    6248.2  0.289584   2.7156  0.87068  0.76427  2.7137  0.87078   2.7156

      A_x:20    5_E_x   10_E_x   20_E_x
20   0.37829  0.78252  0.61224  0.37440
21   0.37833  0.78250  0.61220  0.37429
22   0.37837  0.78248  0.61215  0.37417
23   0.37842  0.78245  0.61210  0.37404
```

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```
24   0.37848  0.78243  0.61205  0.37390
..       ...      ...      ...      ...
96   0.83049  0.19872  0.01330  0.00000
97   0.84143  0.16765  0.00827  0.00000
98   0.85177  0.13850  0.00485  0.00000
99   0.86153  0.11173  0.00266  0.00000
100  0.87068  0.08777  0.00136  0.00000

[81 rows x 12 columns]
```

CHAPTER FOURTEEN

SELECT LIFE TABLE

14.1 Select and ultimate life table

```
"""Select and Ultimate Life Table

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"""
from actuarialmath.selectlife import Select
print(Select.help())
```

```
Implement select and ultimate mortality life table

    Select: when mortality depends on the age when a person is selected
    - newly selected policyholder is in the best health condition possible
    - the selection process wears off

    Ultimate: after several years, selection has no effect on mortality.

--------------------------------------------------

fill():
  Fills in missing mortality values. Does not check for consistency

l_x():
  Returns number of lifes computed from select table

p_x():
  t_p_[x]+s by chain rule: prod(1_p_[x]+s+y) for y in range(t)

q_x():
  t|u_q_[x]+s = [x]+s survives u years, does not survive next t

e_x():
  Returns curtate expected life time computed from select table

A_x():
  Returns insurance value computed from select table
```

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```
a_x():
  Returns annuity value computed from select table
```

14.2 Examples

```
from actuarialmath.sult import SULT

print("SOA Question 3.2:  (D) 14.7")
e_curtate = Select.e_curtate(e=15)
life = Select(l={65: [1000, None,],
                 66: [955, None]},
              e={65: [e_curtate, None]},
              d={65: [40, None,],
                 66: [45, None]}, udd=True).fill()
print(life.e_r(66))
print(life.frame('e'))
print()


print("SOA Question 4.16:  (D) .1116")
q = [.045, .050, .055, .060]
q_ = {50+x: [0.7 * q[x] if x < 4 else None,
             0.8 * q[x+1] if x+1 < 4 else None,
             q[x+2] if x+2 < 4 else None]
      for x in range(4)}
life = Select(q=q_, interest=dict(i=.04)).fill()
print(life.term_insurance(50, t=3))
print()


print("SOA Question 4.13:  (C) 350 ")
life = Select(q={65: [.08, .10, .12, .14],
                 66: [.09, .11, .13, .15],
                 67: [.10, .12, .14, .16],
                 68: [.11, .13, .15, .17],
                 69: [.12, .14, .16, .18]}, interest=dict(i=.04)).fill()
print(life.deferred_insurance(65, t=2, u=2, b=2000))
print()


print("SOA Question 3.13:  (B) 1.6")
life = Select(l={55: [10000, 9493, 8533, 7664],
                 56: [8547, 8028, 6889, 5630],
                 57: [7011, 6443, 5395, 3904],
                 58: [5853, 4846, 3548, 2210]},
              e={57: [None, None, None, 1]}).fill()
print(life.e_r(58, s=2))
print()


print("SOA Question 3.12: (C) 0.055 ")
life = Select(l={60: [10000, 9600, 8640, 7771],
```

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```
                    61: [8654, 8135, 6996, 5737],
                    62: [7119, 6549, 5501, 4016],
                    63: [5760, 4954, 3765, 2410]}, udd=False).fill()
print(life.q_r(60, s=1, t=3.5) - life.q_r(61, s=0, t=3.5))

print()


print("SOA Question 3.7: (b) 16.4")
life = Select(q={50: [.0050, .0063, .0080],
                    51: [.0060, .0073, .0090],
                    52: [.0070, .0083, .0100],
                    53: [.0080, .0093, .0110]}).fill()
print(1000*life.q_r(50, s=0, r=0.4, t=2.5))
print()



print("SOA Question 3.6:  (D) 15.85")
life = Select(q={60: [.09, .11, .13, .15],
                    61: [.1, .12, .14, .16],
                    62: [.11, .13, .15, .17],
                    63: [.12, .14, .16, .18],
                    64: [.13, .15, .17, .19]},
                e={61: [None, None, None, 5.1]}).fill()
print(life.e_x(61))
print()


print("SOA Question 3.3:  (E) 1074")
life = Select(l={50: [99, 96, 93],
                    51: [97, 93, 89],
                    52: [93, 88, 83],
                    53: [90, 84, 78]})
print(10000*life.q_r(51, s=0, r=0.5, t=2.2))

print()


print("SOA Question 3.1:  (B) 117")
life = Select(l={60: [80000, 79000, 77000, 74000],
                    61: [78000, 76000, 73000, 70000],
                    62: [75000, 72000, 69000, 67000],
                    63: [71000, 68000, 66000, 65000]})
print(1000*life.q_r(60, s=0, r=0.75, t=3, u=2))
print()


print("Other usage examples")
life = Select(minage=20, maxage=30, n=3)
life.set_select(column=3, select_age=False, q=SULT()['q']).fill()
print(life._select)
print(life.frame('l'))
print('--------------')
print(life.frame('q'))
print('==============')
```

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```
life = Select(q={21: [0.00120, 0.00150, 0.00170, 0.00180],
                 22: [0.00125, 0.00155, 0.00175, 0.00185],
                 23: [0.00130, 0.00160, 0.00180, 0.00195]}).fill()
print(life.frame('l'))
print('--------------')
print(life.frame('q'))
print('==============')
print(life.p_x(21, 1, 4))  #0.99317
```

```
SOA Question 3.2:  (D) 14.7
14.67801047120419
           0          1
e
65  14.50000  14.104167
66  14.17801  13.879121

SOA Question 4.16:  (D) .1116
0.1115661982248521

SOA Question 4.13:  (C) 350
351.0578236056159

SOA Question 3.13:  (B) 1.6
1.6003382187147688

SOA Question 3.12: (C) 0.055
0.05465655938591829

SOA Question 3.7: (b) 16.4
16.420207214428586

SOA Question 3.6:  (D) 15.85
5.846832

SOA Question 3.3:  (E) 1074
1073.684210526316

SOA Question 3.1:  (B) 117
116.7192429022082

Other usage examples
{'A': {20: {}, 21: {}, 22: {}, 23: {}, 24: {}, 25: {}, 26: {}, 27: {}, 28: {}, 29:
↪{}, 30: {}}, 'a': {20: {}, 21: {}, 22: {}, 23: {}, 24: {}, 25: {}, 26: {}, 27: {}
↪, 28: {}, 29: {}, 30: {}}, 'q': {20: {3: 0.0002620983481647125}, 21: {3: 0.
↪00026732142024017664}, 22: {3: 0.0002731921206805849}, 23: {3: 0.
↪0002797907468233207}, 24: {3: 0.0002872075506188112}, 25: {3: 0.
↪0002955439724024386}, 26: {3: 0.0003049140275073012}, 27: {3: 0.
↪0003154458646110006}, 28: {3: 0.0003272835170716096}, 29: {3: 0.
↪00034058887111311693}, 30: {3: 0.00035554387766526267}}, 'd': {20: {}, 21: {},␣
↪22: {}, 23: {}, 24: {}, 25: {}, 26: {}, 27: {}, 28: {}, 29: {}, 30: {}}, 'l':
↪{20: {0: 100000}, 21: {}, 22: {}, 23: {}, 24: {}, 25: {}, 26: {}, 27: {}, 28: {},
↪ 29: {}, 30: {}}, 'e': {20: {}, 21: {}, 22: {}, 23: {}, 24: {}, 25: {}, 26: {},␣
↪27: {}, 28: {}, 29: {}, 30: {}}}
         0
```

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```
l
20  100000
--------------
           3
q
20  0.000262
21  0.000267
22  0.000273
23  0.000280
24  0.000287
25  0.000296
26  0.000305
27  0.000315
28  0.000327
29  0.000341
30  0.000356
==============
                0             1             2             3
l
21  100000.000000  99880.000000  99730.180000  99560.638694
22   99834.996495  99710.202749  99555.651935  99381.429544
23   99665.273502  99535.708646  99376.451512  99197.573900
--------------
          0        1        2        3
q
21  0.00120  0.00150  0.00170  0.00180
22  0.00125  0.00155  0.00175  0.00185
23  0.00130  0.00160  0.00180  0.00195
==============
0.9931675400449915
```

CHAPTER

FIFTEEN

RECURSION

15.1 Chain rule

$_{t+n}p_x = \ _np_x \cdot \ _tp_{x+n}$

$_{t+n}E_x = \ _nE_x \cdot \ _tE_{x+n}$

15.2 Future lifetime

Complete expectation of life:

$\mathring{e}_x = \mathring{e}_{x:\overline{m|}} + \ _mp_x \ \mathring{e}_{x+m}$

- One-year recursion: $\mathring{e}_x = \mathring{e}_{x:\overline{1|}} + p_x \ \mathring{e}_{x+1}$
- Temporary expectation: $\mathring{e}_{x:\overline{m+n|}} = \mathring{e}_{x:\overline{m|}} + \ _mp_x \ \mathring{e}_{x+m:\overline{n|}}$

Curtate expectation of life:

$e_x = e_{x:\overline{m|}} + \ _mp_x \ e_{x+m}$

- One-year recursion: $e_x = p_x(1 + e_{x+1})$
- Temporary expectation: $e_{x:\overline{m+n|}} = e_{x:\overline{m|}} + \ _mp_x \ e_{x:\overline{n|}}$

15.3 Insurance

$A_x = v\, q_x + v\, p_x\, A_{x+1} \Rightarrow A_{x+1} = \dfrac{A_x - v\, q_x}{v\, p_x}$

$A^1_{x:\overline{t|}} = v\, q_x + v\, p_x\, A^1_{x+1:\overline{t-1|}}$

$A_{x:\overline{0|}} = b$

$A_{x:\overline{1|}} = q_x\, v\, b + p_x\, v\, b = v\, b$

$IA_x = v\, q_x + v\, p_x\, A_{x+1}$

$IA^1_{x:\overline{t|}} = v\, q_x + v\, p_x\, (A_{x+1} + IA^1_{x+1:\overline{t-1|}})$

$$DA^{1}_{x:\overline{t|}} = t\, v\, q_x + v\, p_x\, (DA^{1}_{x+1:\overline{t-1|}})$$

15.4 Annuities

$$\ddot{a}_x = 1 + v\, p_x\, \ddot{a}_{x+1} \Rightarrow \ddot{a}_{x+1} = \frac{\ddot{a}_x - 1}{v\, p_x}$$

$$\ddot{a}_{x:\overline{t|}} = 1 + v\, p_x\, \ddot{a}_{x+1:\overline{t-1|}}$$

```
"""Recursion Formulas and Identity Relationships

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"""
from actuarialmath.recursion import Recursion
print(Recursion.help())
```

```
Recursion and Alternate Formulas
--------------------------------

set_q():
  Set in key-value store

set_p():
  Set in key-value store

set_e():
  Set in key-value store

set_E():
  Set in key-value store

set_A():
  Set in key-value store

set_IA():
  Set in key-value store

set_DA():
  Set in key-value store

set_a():
  Set in key-value store
```

15.5 Examples

```
from actuarialmath.constantforce import ConstantForce

print("SOA Question 6.48:  (A) 3195")
life = Recursion(interest=dict(i=0.06), depth=5)
x = 0
life.set_p(0.95, x=x, t=5)
life.set_q(0.02, x=x+5)
life.set_q(0.03, x=x+6)
life.set_q(0.04, x=x+7)
a = 1 + life.E_x(x, t=5)
A = life.deferred_insurance(x, u=5, t=3)
P = life.gross_premium(A=A, a=a, benefit=100000)
print(P)
print()

print("SOA Question 6.40: (C) 116 ")
# - standard formula discounts/accumulates by too much (i should be smaller)
x = 0
life = Recursion(interest=dict(i=0.06)).set_a(7, x=x+1).set_q(0.05, x=x)
a = life.whole_life_annuity(x)
A = 110 * a / 1000
print(a, A)
life = Recursion(interest=dict(i=0.06)).set_A(A, x=x).set_q(0.05, x=x)
A1 = life.whole_life_insurance(x+1)
P = life.gross_premium(A=A1 / 1.03, a=7) * 1000
print(P)
print()

print("SOA Question 6.17:  (A) -30000")
x = 0
life = ConstantForce(mu=0.1, interest=dict(i=0.08))
A = life.endowment_insurance(x, t=2, b=100000, endowment=30000)
a = life.temporary_annuity(x, t=2)
P = life.gross_premium(a=a, A=A)
print(A, a, P)

life1 = Recursion(interest=dict(i=0.08))\
        .set_q(life.q_x(x, t=1) * 1.5, x=x, t=1)\
        .set_q(life.q_x(x+1, t=1) * 1.5, x=x+1, t=1)
policy = life1.Policy(premium=P * 2, benefit=100000, endowment=30000)
L = life1.gross_policy_value(x, t=0, n=2, policy=policy)
print(L)
print()
```

```
SOA Question 6.48:  (A) 3195
[ Pure Endowment: 5_E_0 ]
    pure endowment 5_E_0 = 5_p_0 * v^5
[ Pure Endowment: 5_E_0 ]
    pure endowment 5_E_0 = 5_p_0 * v^5
[ Term Insurance: A_5^1:3 ]
    forward: A_5 = qv + pvA_6
       forward: A_6 = qv + pvA_7
          endowment insurance - pure endowment = A_7^1:1
    pure endowment 1_E_7 = 1_p_7 * v^1
```

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```
[ Term Insurance: A_5^1:3 ]
    pure endowment 1_E_7 = 1_p_7 * v^1
          endowment insurance - pure endowment = A_7^1:1
       forward: A_6 = qv + pvA_7
    forward: A_5 = qv + pvA_6
3195.1189176587473

SOA Question 6.40: (C) 116
[ Whole Life Annuity: a_0 ]
    forward: a_0 = 1 + E_0 a_1
    pure endowment 1_E_0 = 1_p_0 * v^1
7.2735849056603765 0.8000943396226414
[ Whole Life Insurance: A_1 ]
    backward: A_1 = (A_0/v - q) / p
          backward: A_1 = (A_0/v - q) / p
                backward: A_1 = (A_0/v - q) / p
                backward: A_1 = (A_0/v - q) / p
116.51945397474269

SOA Question 6.17:  (A) -30000
37251.49857703497 1.8378124241073728 20269.478042694187
[ Term Insurance: A_0^1:2 ]
    forward: A_0 = qv + pvA_1
       endowment insurance - pure endowment = A_1^1:1
    pure endowment 1_E_1 = 1_p_1 * v^1
[ Temporary Annuity: a_0:2 ]
    forward: a_0:2 = 1 + E_0 a_1:1
    pure endowment 1_E_0 = 1_p_0 * v^1
       1-year discrete annuity: a_x:1 = 1
[ Pure Endowment: 2_E_0 ]
    chain Rule: 2_E_0 = E_0 * 1_E_1
    pure endowment 1_E_1 = 1_p_1 * v^1
    pure endowment 1_E_0 = 1_p_0 * v^1
-30107.42633581125
```

CHAPTER

SIXTEEN

MTHLY

16.1 1/mthly insurance

$$K_x^{(m)} = \frac{1}{m}\lfloor m T_x \rfloor$$

$$A_x^{(m)} = \sum_{k=0}^{\infty} v^{\frac{k+1}{m}} ~ _{\frac{k}{m}|\frac{1}{m}}q_x$$

$$A_{x:\overline{t|}}^{(m)} = \sum_{k=0}^{mt-1} v^{\frac{k+1}{m}} ~ _{\frac{k}{m}|\frac{1}{m}}q_x$$

16.2 Annuity twin

$$A_x^{(m)} = 1 - d^{(m)}~\ddot{a}_x^{(m)} \iff \ddot{a}_x^{(m)} = \frac{1 - A_x^{(m)}}{d^{(m)}}$$

$$A_{x:\overline{t|}}^{(m)} = 1 - d^{(m)}~\ddot{a}_{x:\overline{t|}}^{(m)} \iff \ddot{a}_{x:\overline{t|}}^{(m)} = \frac{1 - A_{x:\overline{t|}}^{(m)}}{d^{(m)}}$$

16.3 Immediate annuity

$$a_x^{(m)} = \ddot{a}_x^{(m)} - \frac{1}{m}$$

$$a_{x:\overline{t|}}^{(m)} = \ddot{a}_{x:\overline{t|}}^{(m)} - \frac{1}{m}(1 - ~_tE_x)$$

```
"""Mthly-paid insurance and annuities

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"""
from actuarialmath.mthly import Mthly
print(Mthly.help())
```

```
1/Mthly insurance and annuities
-------------------------------

v_m():
  Return discount rate after k mthly periods

p_m():
  Return survival rate over k mthly periods

q_m():
  Return mortality rate over k mthly periods

Z_m():
  Return PV of insurance r.v. Z and probability by mthly period

E_x():
  Return pure endowment factor

A_x():
  Compute insurance factor with mthly benefits

whole_life_insurance():
  Whole life insurance: A_x

term_insurance():
  Term life insurance: A_x:t^1

deferred_insurance():
  Deferred insurance n|_A_x:t^1 = discounted whole life

endowment_insurance():
  Endowment insurance: A_x:t = term insurance + pure endowment

immediate_annuity():
  Immediate mthly annuity

annuity_twin():
  Return annuity twin of mthly insurance

annuity_variance():
  Variance of mthly annuity from mthly insurance moments

whole_life_annuity():
  Whole life mthly annuity: a_x

temporary_annuity():
  Temporary mthly life annuity: a_x:t

deferred_annuity():
  Deferred mthly life annuity n|t_a_x =  n+t_a_x - n_a_x
```

16.4 Examples

```
from actuarialmath.premiums import Premiums
from actuarialmath.lifetable import LifeTable

print("SOA Question 6.4:  (E) 1893.9")
mthly = Mthly(m=12, life=Premiums(interest=dict(i=0.06)))
A1, A2 = 0.4075, 0.2105
mean = mthly.annuity_twin(A1)*15*12
var = mthly.annuity_variance(A1=A1, A2=A2, b=15 * 12)
S = Premiums.portfolio_percentile(mean=mean, variance=var, prob=.9, N=200)
print(S / 200)
print()

print("SOA Question 4.2:  (D) 0.18")
life = LifeTable(q={0: .16, 1: .23}, interest=dict(i_m=.18, m=2),
                    udd=False).fill()
mthly = Mthly(m=2, life=life)
Z = mthly.Z_m(0, t=2, benefit=lambda x,t: 300000 + t*30000*2)
print(Z)
print(Z[Z['Z'] >= 277000].iloc[:, -1].sum())
print()
```

```
SOA Question 6.4:  (E) 1893.9
1893.912859650868

SOA Question 4.2:  (D) 0.18
               Z         p
m
1  275229.357798  0.083485
2  277754.397778  0.076515
3  277986.052822  0.102903
4  276285.832315  0.090297
0.17941813045022975
```

CHAPTER SEVENTEEN

UDD MTHLY

17.1 Annuities

$\alpha(m) = \dfrac{id}{i^{(m)}\ d^{(m)}}$

$\beta(m) = \dfrac{i - i^{(m)}}{i^{(m)}\ d^{(m)}}$

$\ddot{a}_x^{(m)} = \alpha(m)\ \ddot{a}_x - \beta(m)$

$\ddot{a}_{x:\overline{n|}}^{(m)} = \alpha(m)\ \ddot{a}_{x:\overline{n|}} - \beta(m)(1 - \ {}_tE_x)$

${}_{u|}\ddot{a}_x^{(m)} = \alpha(m)\ {}_{u|}\ddot{a}_x - \beta(m)\ {}_uE_x$

17.2 Insurance

Discrete insurance:

Whole life: $A_x^{(m)} = \dfrac{i}{i^{(m)}} A_x$

Temporary: $A_{x:\overline{t|}}^{1\ (m)} = \dfrac{i}{i^{(m)}} A^1_{x:\overline{t|}}$

Endowment: $A_{x:\overline{t|}}^{(m)} = \dfrac{i}{i^{(m)}} A^1_{x:\overline{t|}} + \ {}_tE_x$

Deferred: ${}_{u|}A_x^{(m)} = \ {}_uE_x\ \dfrac{i}{i^{(m)}} A_{x+u}$

Continuous insurance:

Whole life: $\overline{A}_x = \dfrac{i}{\delta} A_x$

Temporary: $\overline{A}^1_{x:\overline{t|}} = \dfrac{i}{\delta} A^1_{x:\overline{t|}}$

Endowment: $\overline{A}_{x:\overline{t|}} = \dfrac{i}{\delta} A^1_{x:\overline{t|}} + \ {}_tE_x$

Deferred: ${}_{u|}\overline{A}_x = \ {}_uE_x\ \dfrac{i}{\delta} A_{x+u}$

Double the force of interest:

$$^2\overline{A}_x = \frac{i^2 - 2i}{2\delta}\ ^2A_x$$

$$^2A_x^{(m)} = \frac{i^2 - 2i}{(i^{(m)})^2 - 2i^{(m)}}\ ^2A_x$$

```python
"""Mthly with UDD assumption

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"""
from actuarialmath.udd import UDD
print(UDD.help())
```

```
UDD 1/Mthly Shortcuts
---------------------

alpha():
  1/Mthly UDD interest rate beta function

beta():
  1/Mthly UDD interest rate alpha function

whole_life_insurance():
  1/Mthly UDD Whole life insurance: A_x

term_insurance():
  1/Mthly UDD Term insurance: A_x:t

deferred_insurance():
  Deferred insurance n|_A_x:t^1 = discounted whole life

whole_life_annuity():
  1/Mthly UDD Whole life annuity: a_x

temporary_annuity():
  1/Mthly UDD Temporary life annuity: a_x:t

deferred_annuity():
  1/Mthly UDD Deferred life annuity n|t_a_x =  n+t_a_x - n_a_x

frame():
  Display 1/mthly UDD interest function values
```

17.3 Examples

```
from actuarialmath.sult import SULT
from actuarialmath.recursion import Recursion

print("SOA Question 7.9:  (A) 38100")
sult = SULT(udd=True)
x, n, t = 45, 20, 10
a = UDD(m=12, life=sult).temporary_annuity(x+10, t=n-10)
print(a)
A = UDD(m=0, life=sult).endowment_insurance(x+10, t=n-10)
print(A)
print(A*100000 - a*12*253)
policy = sult.Policy(premium=253*12, endowment=100000, benefit=100000)
print(sult.gross_future_loss(A=A, a=a, policy=policy))
print()

print("SOA Question 6.49:  (C) 86")
sult = SULT(udd=True)
a = UDD(m=12, life=sult).temporary_annuity(40, t=20)
A = sult.whole_life_insurance(40, discrete=False)
P = sult.gross_premium(a=a, A=A, benefit=100000, initial_policy=200,
                        renewal_premium=0.04, initial_premium=0.04)
print(P/12)
print()

print("SOA Question 6.38:  (B) 11.3")
x, n = 0, 10
life = Recursion(interest=dict(i=0.05))
life.set_A(0.192, x=x, t=n, endowment=1, discrete=False)
life.set_E(0.172, x=x, t=n)
a = life.temporary_annuity(x, t=n, discrete=False)
print(a)

def fun(a):      # solve for discrete annuity, given continuous
    life = Recursion(interest=dict(i=0.05))
    life.set_a(a, x=x, t=n).set_E(0.172, x=x, t=n)
    return UDD(m=0, life=life).temporary_annuity(x, t=n)
a = life.solve(fun, target=a, guess=a)  # discrete annuity
P = life.gross_premium(a=a, A=0.192, benefit=1000)
print(P)
print()

print("SOA Question 6.32:  (C) 550")
x = 0
life = Recursion(interest=dict(i=0.05)).set_a(9.19, x=x)
benefits = UDD(m=0, life=life).whole_life_insurance(x)
payments = UDD(m=12, life=life).whole_life_annuity(x)
print(benefits, payments)
print(life.gross_premium(a=payments, A=benefits, benefit=100000)/12)
print()

print("SOA Question 6.22:  (C) 102")
life = SULT(udd=True)
a = UDD(m=12, life=life).temporary_annuity(45, t=20)
A = UDD(m=0, life=life).whole_life_insurance(45)
```

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```
print(life.gross_premium(A=A, a=a, benefit=100000)/12)
print()

print("Interest Functiona at i=0.05")
print("----------------------------")
print(UDD.frame())
print()
UDD.frame()
```

```
SOA Question 7.9:  (A) 38100
7.831075686716718
0.6187476755196442
38099.62176709247
38099.62176709246

SOA Question 6.49:  (C) 86
85.99177833261696

SOA Question 6.38:  (B) 11.3
[ Temporary Annuity: a_0:10 ]
    Annuity twin: a = (1 - A) / d
16.560714925944584
11.308644185253657

SOA Question 6.32:  (C) 550
0.5763261529803323 8.72530251348809
550.4356936711871

SOA Question 6.22:  (C) 102
102.40668704849178

Interest Functiona at i=0.05
----------------------------
       i(m)     d(m)   i/i(m)   d/d(m)  alpha(m)  beta(m)
1   0.05000  0.04762  1.00000  1.00000   1.00000  0.00000
2   0.04939  0.04820  1.01235  0.98795   1.00015  0.25617
4   0.04909  0.04849  1.01856  0.98196   1.00019  0.38272
12  0.04889  0.04869  1.02271  0.97798   1.00020  0.46651
0   0.04879  0.04879  1.02480  0.97600   1.00020  0.50823
```

	i(m)	d(m)	i/i(m)	d/d(m)	alpha(m)	beta(m)
1	0.05000	0.04762	1.00000	1.00000	1.00000	0.00000
2	0.04939	0.04820	1.01235	0.98795	1.00015	0.25617
4	0.04909	0.04849	1.01856	0.98196	1.00019	0.38272
12	0.04889	0.04869	1.02271	0.97798	1.00020	0.46651
0	0.04879	0.04879	1.02480	0.97600	1.00020	0.50823

CHAPTER

EIGHTEEN

WOOLHOUSE MTHLY

18.1 Annuities

Whole life annuity:

$$\ddot{a}_x^{(m)} = \ddot{a}_x - \frac{m-1}{2m} - \frac{m^2-1}{12m^2}(\mu_x + \delta)$$

Temporary annuity:

$$\ddot{a}_{x:\overline{t|}}^{(m)} = \ddot{a}_x^{(m)} - \ _tE_x\ \ddot{a}_{x+t}^{(m)}$$

- Approximate $\mu_x \approx -\frac{1}{2}(\ln p_{x-1} + \ln p_x)$

```
"""Mthly with Woolhouse approximation

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"""
from actuarialmath.woolhouse import Woolhouse
print(Woolhouse.help())
```

```
Woolhouse 1/Mthly Shortcuts
---------------------------

mu_x():
  Approximate or compute mu_x if not given

insurance_twin():
  Return insurance twin of mthly annuity

whole_life_insurance():
  1/Mthly Woolhouse Whole life insurance: A_x

term_insurance():
  1/Mthly Woolhouse Term insurance: A_x:t

deferred_insurance():
  1/Mthly Woolhouse Deferred insurance = discounted term or whole life
```

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```
whole_life_annuity():
  1/Mthly Woolhouse Whole life annuity: a_x

temporary_annuity():
  1/Mthly Woolhouse Temporary life annuity: a_x

deferred_annuity():
  1/Mthly Woolhouse Temporary life annuity: a_x
```

18.2 Examples

```
from actuarialmath.sult import SULT
from actuarialmath.recursion import Recursion
from actuarialmath.udd import UDD

print("SOA Question 7.7:  (D) 1110")
x = 0
life = Recursion(interest=dict(i=0.05)).set_A(0.4, x=x+10)
a = Woolhouse(m=12, life=life).whole_life_annuity(x+10)
print(a)
policy = life.Policy(premium=0, benefit=10000, renewal_policy=100)
V = life.gross_future_loss(A=0.4, policy=policy.future)
print(V)
policy = life.Policy(premium=30*12, renewal_premium=0.05)
V1 = life.gross_future_loss(a=a, policy=policy.future)
print(V, V1, V+V1)
print()


print("SOA Question 6.25:  (C) 12330")
life = SULT()
woolhouse = Woolhouse(m=12, life=life)
benefits = woolhouse.deferred_annuity(55, u=10, b=1000 * 12)
expenses = life.whole_life_annuity(55, b=300)
payments = life.temporary_annuity(55, t=10)
print(benefits + expenses, payments)
def fun(P):
    return life.gross_future_loss(A=benefits + expenses, a=payments,
                                  policy=life.Policy(premium=P))
P = life.solve(fun, target=-800, guess=[12110, 12550])
print(P)
print()


print("SOA Question 6.15:  (B) 1.002")
life = Recursion(interest=dict(i=0.05)).set_a(3.4611, x=0)
A = life.insurance_twin(3.4611)
udd = UDD(m=4, life=life)
a1 = udd.whole_life_annuity(x=x)
woolhouse = Woolhouse(m=4, life=life)
a2 = woolhouse.whole_life_annuity(x=x)
print(life.gross_premium(a=a1, A=A)/life.gross_premium(a=a2, A=A))
```

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```
print()

print("SOA Question 5.7:  (C) 17376.7")
life = Recursion(interest=dict(i=0.04))
life.set_A(0.188, x=35)
life.set_A(0.498, x=65)
life.set_p(0.883, x=35, t=30)
mthly = Woolhouse(m=2, life=life, three_term=False)
print(mthly.temporary_annuity(35, t=30))
print(1000 * mthly.temporary_annuity(35, t=30))
print()
```

```
SOA Question 7.7:  (D) 1110
12.141666666666666
5260.0
5260.0 -4152.028174603174 1107.9718253968258

SOA Question 6.25:  (C) 12330
98042.52569470297 8.019169307712845
12325.781125438532

SOA Question 6.15:  (B) 1.002
1.0022973504113772

SOA Question 5.7:  (C) 17376.7
[ Pure Endowment: 30_E_35 ]
    pure endowment 30_E_35 = 30_p_35 * v^30
17.37671459632958
[ Pure Endowment: 30_E_35 ]
    pure endowment 30_E_35 = 30_p_35 * v^30
17376.71459632958
```

CHAPTER

NINETEEN

ADJUST MORTALITY

19.1 Extra mortality risk

1. Add constant to force of mortality: $\mu_{x+t} + k \Rightarrow {}_tp_x* = {}_tp_x\ e^{-kt}$
2. Multiply force of mortality by constant: $\mu_{x+t} \cdot k \Rightarrow {}_tp_x* = ({}_tp_x)^k$
3. Mutiply mortality rate by a constant: $q_x \to q_x \cdot k$
4. Age rating – add years to age: $(x) \to (x + k)$

```
"""Adjust Mortality

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"""
from actuarialmath.adjustmortality import Adjust
print(Adjust.help())
```

```
Adjust mortality by extra risk
------------------------------

q_x():
  Add constant to mortality rate or age rating

p_x():
  Adjust force of mortality by adding or multiplying a constant
```

19.2 Examples

```
from actuarialmath.selectlife import Select
from actuarialmath.sult import SULT

print("SOA Question 5.5: (A) 1699.6")
life = SULT()
adjust = Adjust(life=life)
```

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```
q = adjust(extra=0.05, adjust=Adjust.ADD_FORCE)['q']
select = Select(n=1)\
            .set_select(column=0, select_age=True, q=q)\
            .set_select(column=1, select_age=False, a=life['a']).fill()
print(100*select['a'][45][0])
print()

print("SOA Question 4.19:  (B) 59050")
life = SULT()
adjust = Adjust(life=life)
q = adjust(extra=0.8, adjust=Adjust.MULTIPLY_RATE)['q']
select = Select(n=1)\
            .set_select(column=0, select_age=True, q=q)\
            .set_select(column=1, select_age=False, q=life['q']).fill()
print(100000*select.whole_life_insurance(80, s=0))
print()

print("Other usage examples")
life = SULT()
adjust = Adjust(life=life)(extra=0.05, adjust=Adjust.ADD_FORCE)
print(life.p_x(45), adjust.p_x(45))
```

```
SOA Question 5.5: (A) 1699.6
1699.6076593190103

SOA Question 4.19:  (B) 59050
59050.59973285648

Other usage examples
0.9992288829941123 0.9504959153149807
```

CHAPTER TWENTY

FAM-L SOLUTIONS

```python
"""Solutions to SOA FAM-L sample questions

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"""
import math
import pandas as pd
import matplotlib.pyplot as plt
import numpy as np
from typing import Union, Optional
from actuarialmath.life import Life
from actuarialmath.survival import Survival
from actuarialmath.lifetime import Lifetime
from actuarialmath.insurance import Insurance
from actuarialmath.annuity import Annuity
from actuarialmath.premiums import Premiums
from actuarialmath.policyvalues import PolicyValues
from actuarialmath.reserves import Reserves
from actuarialmath.recursion import Recursion
from actuarialmath.lifetable import LifeTable
from actuarialmath.sult import SULT
from actuarialmath.selectlife import Select
from actuarialmath.constantforce import ConstantForce
from actuarialmath.adjustmortality import Adjust
from actuarialmath.mthly import Mthly
from actuarialmath.udd import UDD
from actuarialmath.woolhouse import Woolhouse
```

```python
class SOA:
    """To keep score of solutions correct"""
    def __init__(self, terminate: bool = False):
        self.score = {}
        self.terminate = terminate

    def __call__(self, solution: Union[float, str], answer: Union[float, str],
                 msg: str, rel_tol: float = 0.01):
        """Grade this question, and accumulate score"""
        section, question = str(msg).split('.')
        msg = "SOA Question " + str(msg) + ":"
        if isinstance(solution, str) or isinstance(answer, str):
            correct = (solution == answer)
```

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```
        else:
            correct = math.isclose(solution, answer, rel_tol=rel_tol)
        print(msg, '[', solution, ']', answer, '*' * 10 * (1-correct))
        if section not in self.score:
            self.score[section] = {}
        self.score[section][question] = correct
        if self.terminate:
            assert correct, msg

    def summary(self):
        """Return final score and by section"""
        score = {int(k): [len(v), sum(v.values())] for k, v in self.score.items()}
        out = pd.DataFrame.from_dict(score, orient='index',
                                     columns=['num', 'correct'])
        out.loc[0] = [sum(out['num']), sum(out['correct'])]
        return out.sort_index()

soa = SOA(terminate=False)
```

20.1 Tables

```
print("Interest Functiona at i=0.05")
UDD.frame()
```

```
Interest Functiona at i=0.05
```

```
      i(m)     d(m)    i/i(m)   d/d(m)   alpha(m)  beta(m)
1   0.05000  0.04762  1.00000  1.00000   1.00000  0.00000
2   0.04939  0.04820  1.01235  0.98795   1.00015  0.25617
4   0.04909  0.04849  1.01856  0.98196   1.00019  0.38272
12  0.04889  0.04869  1.02271  0.97798   1.00020  0.46651
0   0.04879  0.04879  1.02480  0.97600   1.00020  0.50823
```

```
print("Values of z for selected values of Pr(Z<=z)")
print(Life.frame().to_string(float_format=lambda x: f"{x:.3f}"))
```

```
Values of z for selected values of Pr(Z<=z)
z         0.842  1.036  1.282  1.645  1.960  2.326  2.576
Pr(Z<=z)  0.800  0.850  0.900  0.950  0.975  0.990  0.995
```

```
print("Standard Ultimate Life Table at i=0.05")
SULT().frame()
```

```
Standard Ultimate Life Table at i=0.05
```

```
          l_x       q_x      a_x      A_x     2A_x   a_x:10   A_x:10   a_x:20  \
20   100000.0  0.000250  19.9664  0.04922  0.00580  8.0991  0.61433  13.0559
21    99975.0  0.000253  19.9197  0.05144  0.00614  8.0990  0.61433  13.0551
```

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```
22    99949.7  0.000257  19.8707  0.05378  0.00652  8.0988  0.61434  13.0541
23    99924.0  0.000262  19.8193  0.05622  0.00694  8.0986  0.61435  13.0531
24    99897.8  0.000267  19.7655  0.05879  0.00739  8.0983  0.61437  13.0519
..        ...       ...      ...      ...      ...     ...      ...      ...
96    17501.8  0.192887   3.5597  0.83049  0.69991  3.5356  0.83164   3.5597
97    14125.9  0.214030   3.3300  0.84143  0.71708  3.3159  0.84210   3.3300
98    11102.5  0.237134   3.1127  0.85177  0.73356  3.1050  0.85214   3.1127
99     8469.7  0.262294   2.9079  0.86153  0.74930  2.9039  0.86172   2.9079
100    6248.2  0.289584   2.7156  0.87068  0.76427  2.7137  0.87078   2.7156

      A_x:20    5_E_x   10_E_x   20_E_x
20   0.37829  0.78252  0.61224  0.37440
21   0.37833  0.78250  0.61220  0.37429
22   0.37837  0.78248  0.61215  0.37417
23   0.37842  0.78245  0.61210  0.37404
24   0.37848  0.78243  0.61205  0.37390
..       ...      ...      ...      ...
96   0.83049  0.19872  0.01330  0.00000
97   0.84143  0.16765  0.00827  0.00000
98   0.85177  0.13850  0.00485  0.00000
99   0.86153  0.11173  0.00266  0.00000
100  0.87068  0.08777  0.00136  0.00000

[81 rows x 12 columns]
```

20.2 2 Survival models

SOA Question 2.1: (B) 2.5

```python
def fun(omega):  # Solve first for omega, given mu_65 = 1/180
    return Lifetime(l=lambda x,s: (1 - (x+s)/omega)**0.25).mu_x(65)
omega = int(Lifetime.solve(fun, target=1/180, guess=(106, 126)))
life = Lifetime(l=lambda x,s: (1 - (x+s)/omega)**0.25, maxage=omega)
soa(2.5, life.e_x(106, curtate=True), 2.1)
```

```
SOA Question 2.1: [ 2.5 ] 2.4786080555423604
```

SOA Question 2.2: (D) 400

```python
p1 = (1. - 0.02) * (1. - 0.01)  # 2_p_x if vaccine given
p2 = (1. - 0.02) * (1. - 0.02)  # 2_p_x if vaccine not given
v = math.sqrt(Life.conditional_variance(p=.2, p1=p1, p2=p2, N=100000))
soa(400, v, 2.2)
```

```
SOA Question 2.2: [ 400 ] 396.5914603215815
```

SOA Question 2.3: (A) 0.0483

```python
B, c = 0.00027, 1.1
life = Survival(S=lambda x,s,t: (math.exp(-B * c**(x+s)
```

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```
                                       * (c**t - 1)/math.log(c))))
soa(0.0483, life.f_x(x=50, t=10), 2.3)
```

```
SOA Question 2.3: [ 0.0483 ] 0.048327399045049846
```

SOA Question 2.4: (E) 8.2

```
life = Lifetime(l=lambda x,s: 0. if (x+s) >= 100 else 1 - ((x+s)**2)/10000.)
soa(8.2, life.e_x(75, t=10, curtate=False), 2.4)
```

```
SOA Question 2.4: [ 8.2 ] 8.20952380952381
```

SOA Question 2.5: (B) 37.1

```
life = Recursion().set_e(25, x=60, curtate=True)
life.set_q(0.2, x=40, t=20).set_q(0.003, x=40)
def fun(e):  # solve e_40 from e_40:20 = e_40 - 20_p_40 e_60
    return life.set_e(e, x=40, curtate=True).e_x(x=40, t=20, curtate=True)
life.set_e(life.solve(fun, target=18, guess=[36, 41]), x=40, curtate=True)
soa(37.1, life.e_x(41, curtate=True), 2.5)
```

```
[ Lifetime: e_41 ]
    backward e_41 = e_41:1 + p_41 e_42
       shortcut 1-year curtate e_40:1
SOA Question 2.5: [ 37.1 ] 37.11434302908726
```

SOA Question 2.6: (C) 13.3

```
life = Survival(l=lambda x,s: (1 - (x+s)/60)**(1/3))
soa(13.3, 1000*life.mu_x(35), 2.6)
```

```
SOA Question 2.6: [ 13.3 ] 13.340451278922776
```

SOA Question 2.7: (B) 0.1477

```
life = Survival(l=lambda x,s:
                    (1-((x+s)/250) if (x+s)<40 else 1-((x+s)/100)**2))
soa(0.1477, life.q_x(30, t=20), 2.7)
```

```
SOA Question 2.7: [ 0.1477 ] 0.1477272727272727
```

SOA Question 2.8: (C) 0.94

```
def fun(p):  # Solve first for mu, given start and end proportions
    mu = -math.log(p)
    male = Lifetime(mu=lambda x,s: 1.5 * mu)
    female = Lifetime(mu=lambda x,s: mu)
    return (75 * female.p_x(0, t=20)) / (25 * male.p_x(0, t=20))
soa(0.94, Lifetime.solve(fun, target=85/15, guess=[0.89, 0.99]), 2.8)
```

```
SOA Question 2.8: [ 0.94 ] 0.9383813306903798
```

20.3 3 Life tables and selection

SOA Question 3.1: (B) 117

```
life = Select(l={60: [80000, 79000, 77000, 74000],
                 61: [78000, 76000, 73000, 70000],
                 62: [75000, 72000, 69000, 67000],
                 63: [71000, 68000, 66000, 65000]})
soa(117, 1000*life.q_r(60, s=0, r=0.75, t=3, u=2), 3.1)
```

```
SOA Question 3.1: [ 117 ] 116.7192429022082
```

SOA Question 3.2: (D) 14.7

```
e_curtate = Select.e_curtate(e=15)
life = Select(l={65: [1000, None,],
                 66: [955, None]},
              e={65: [e_curtate, None]},
              d={65: [40, None,],
                 66: [45, None]}, udd=True).fill()
soa(14.7, life.e_r(66), 3.2)
```

```
SOA Question 3.2: [ 14.7 ] 14.67801047120419
```

SOA Question 3.3: (E) 1074

```
life = Select(l={50: [99, 96, 93],
                 51: [97, 93, 89],
                 52: [93, 88, 83],
                 53: [90, 84, 78]})
soa(1074, 10000*life.q_r(51, s=0, r=0.5, t=2.2), 3.3)
```

```
SOA Question 3.3: [ 1074 ] 1073.684210526316
```

SOA Question 3.4: (B) 815

```
sult = SULT()
mean = sult.p_x(25, t=95-25)
var = sult.bernoulli(mean, variance=True)
p = sult.portfolio_percentile(N=4000, mean=mean, variance=var, prob=0.1)
soa(815, p, 3.4)
```

```
SOA Question 3.4: [ 815 ] 815.0943255167722
```

SOA Question 3.5: (E) 106

```
l = [99999, 88888, 77777, 66666, 55555, 44444, 33333, 22222]
a = LifeTable(l={age:l for age,l in zip(range(60, 68), l)}, udd=True)\
    .q_r(60, u=3.4, t=2.5)
b = LifeTable(l={age:l for age,l in zip(range(60, 68), l)}, udd=False)\
    .q_r(60, u=3.4, t=2.5)
soa(106, 100000 * (a - b), 3.5)
```

```
SOA Question 3.5: [ 106 ] 106.16575827938624
```

SOA Question 3.6: (D) 15.85

```
life = Select(q={60: [.09, .11, .13, .15],
                 61: [.1, .12, .14, .16],
                 62: [.11, .13, .15, .17],
                 63: [.12, .14, .16, .18],
                 64: [.13, .15, .17, .19]},
             e={61: [None, None, None, 5.1]}).fill()
soa(5.85, life.e_x(61), 3.6)
```

```
SOA Question 3.6: [ 5.85 ] 5.846832
```

SOA Question 3.7: (b) 16.4

```
life = Select(q={50: [.0050, .0063, .0080],
                 51: [.0060, .0073, .0090],
                 52: [.0070, .0083, .0100],
                 53: [.0080, .0093, .0110]}).fill()
soa(16.4, 1000*life.q_r(50, s=0, r=0.4, t=2.5), 3.7)
```

```
SOA Question 3.7: [ 16.4 ] 16.420207214428586
```

SOA Question 3.8: (B) 1505

```
sult = SULT()
p1 = sult.p_x(35, t=40)
p2 = sult.p_x(45, t=40)
mean = sult.bernoulli(p1) * 1000 + sult.bernoulli(p2) * 1000
var = (sult.bernoulli(p1, variance=True) * 1000
        + sult.bernoulli(p2, variance=True) * 1000)
soa(1505, sult.portfolio_percentile(mean=mean, variance=var, prob=.95), 3.8)
```

```
SOA Question 3.8: [ 1505 ] 1504.8328375406456
```

SOA Question 3.9: (E) 3850

```
sult = SULT()
p1 = sult.p_x(20, t=25)
p2 = sult.p_x(45, t=25)
mean = sult.bernoulli(p1) * 2000 + sult.bernoulli(p2) * 2000
var = (sult.bernoulli(p1, variance=True) * 2000
        + sult.bernoulli(p2, variance=True) * 2000)
soa(3850, sult.portfolio_percentile(mean=mean, variance=var, prob=.99), 3.9)
```

```
SOA Question 3.9: [ 3850 ] 3850.144345130047
```

SOA Question 3.10: (C) 0.86

```
interest = Life.Interest(v=0.75)
L = 35*interest.annuity(t=4, due=False) + 75*interest.v_t(t=5)
interest = Life.Interest(v=0.5)
R = 15*interest.annuity(t=4, due=False) + 25*interest.v_t(t=5)
soa(0.86, L / (L + R), "3.10")
```

```
SOA Question 3.10: [ 0.86 ] 0.8578442833761983
```

SOA Question 3.11: (B) 0.03

```
life = LifeTable(q={50//2: .02, 52//2: .04}, udd=True).fill()
soa(0.03, life.q_r(50//2, t=2.5/2), 3.11)
```

```
SOA Question 3.11: [ 0.03 ] 0.0298
```

SOA Question 3.12: (C) 0.055

```
life = Select(l={60: [10000, 9600, 8640, 7771],
                 61: [8654, 8135, 6996, 5737],
                 62: [7119, 6549, 5501, 4016],
                 63: [5760, 4954, 3765, 2410]}, udd=False).fill()
soa(0.055, life.q_r(60, s=1, t=3.5)-life.q_r(61, s=0, t=3.5), 3.12)
```

```
SOA Question 3.12: [ 0.055 ] 0.05465655938591829
```

SOA Question 3.13: (B) 1.6

```
life = Select(l={55: [10000, 9493, 8533, 7664],
                 56: [8547, 8028, 6889, 5630],
                 57: [7011, 6443, 5395, 3904],
                 58: [5853, 4846, 3548, 2210]},
              e={57: [None, None, None, 1]}).fill()
soa(1.6, life.e_r(58, s=2), 3.13)
```

```
SOA Question 3.13: [ 1.6 ] 1.6003382187147688
```

SOA Question 3.14: (C) 0.345

```
life = LifeTable(l={90: 1000, 93: 825},
                 d={97: 72},
                 p={96: .2},
                 q={95: .4, 97: 1}, udd=True).fill()
soa(0.345, life.q_r(90, u=93-90, t=95.5 - 93), 3.14)
```

```
SOA Question 3.14: [ 0.345 ] 0.345
```

20.4 4 Insurance benefits

SOA Question 4.1: (A) 0.27212

```
life = Recursion(interest=dict(i=0.03))
life.set_A(0.36987, x=40).set_A(0.62567, x=60)
life.set_E(0.51276, x=40, t=20).set_E(0.17878, x=60, t=20)
Z2 = 0.24954
A = (2 * life.term_insurance(40, t=20)
        + life.deferred_insurance(40, u=20))
soa(0.27212, math.sqrt(life.insurance_variance(A2=Z2, A1=A)), 4.1)
```

```
SOA Question 4.1: [ 0.27212 ] 0.2721117749374753
```

SOA Question 4.2: (D) 0.18

```
life = LifeTable(q={0: .16, 1: .23},
                    interest=dict(i_m=.18, m=2),
                    udd=False).fill()
mthly = Mthly(m=2, life=life)
Z = mthly.Z_m(0, t=2, benefit=lambda x,t: 300000 + t*30000*2)
soa(0.18, Z[Z['Z'] >= 277000].iloc[:, -1].sum(), 4.2)
```

```
SOA Question 4.2: [ 0.18 ] 0.17941813045022975
```

SOA Question 4.3: (D) 0.878 – multi recursion on endowment insurance

```
life = Recursion(interest=dict(i=0.05)).set_q(0.01, x=60)
def fun(q):   # solve for q_61
    return life.set_q(q, x=61).endowment_insurance(60, t=3)
q = life.solve(fun, target=0.86545, guess=0.01)
life.set_q(q, x=61).set_interest(i=0.045)
A = life.endowment_insurance(60, t=3)
soa(0.878, A, "4.3")
```

```
[ Endowment Insurance: A_60:3 ]
    forward: A_60 = qv + pvA_61
       forward: A_61 = qv + pvA_62
SOA Question 4.3: [ 0.878 ] 0.8777667236003878
```

SOA Question 4.4 (A) 0.036

```
life = Insurance(f=lambda *x: 0.025,
                    maxage=40+40,
                    interest=dict(v_t=lambda t: (1 + .2*t)**(-2)))
benefit = lambda x,t: 1 + .2 * t
A1 = life.A_x(40, benefit=benefit, discrete=False)
A2 = life.A_x(40, moment=2, benefit=benefit, discrete=False)
soa(0.036, life.insurance_variance(A2=A2, A1=A1), 4.4)
```

```
SOA Question 4.4: [ 0.036 ] 0.03567680106032681
```

SOA Question 4.5: (C) 35200

```
sult = SULT(interest=dict(delta=0.05))
Z = 100000 * sult.Z_from_prob(45, 0.95)
soa(35200, Z, 4.5)
```

```
SOA Question 4.5: [ 35200 ] 34993.774911115455
```

SOA Question 4.6: (B) 29.85

```
sult = SULT()
life = LifeTable(interest=dict(i=0.05),
                 q={70+k: .95**k * sult.q_x(70+k) for k in range(3)}).fill()
A = life.term_insurance(70, t=3, b=1000)
soa(29.85, A, 4.6)
```

```
SOA Question 4.6: [ 29.85 ] 29.84835110355902
```

SOA Question 4.7: (B) 0.06

```
def fun(i):
    life = Recursion(interest=dict(i=i), verbose=False).set_p(0.57, x=0, t=25)
    return 0.1*life.E_x(0, t=25) - life.E_x(0, t=25, moment=life.VARIANCE)
soa(0.06, Recursion.solve(fun, target=0, guess=[0.058, 0.066]), 4.7)
```

```
SOA Question 4.7: [ 0.06 ] 0.06008023738770262
```

SOA Question 4.8 (C) 191

```
v_t = lambda t: 1.04**(-t) if t < 1 else 1.04**(-1) * 1.05**(-t+1)
life = SULT(interest=dict(v_t=v_t))
soa(191, life.whole_life_insurance(50, b=1000), 4.8)
```

```
SOA Question 4.8: [ 191 ] 191.1281281882354
```

SOA Question 4.9: (D) 0.5

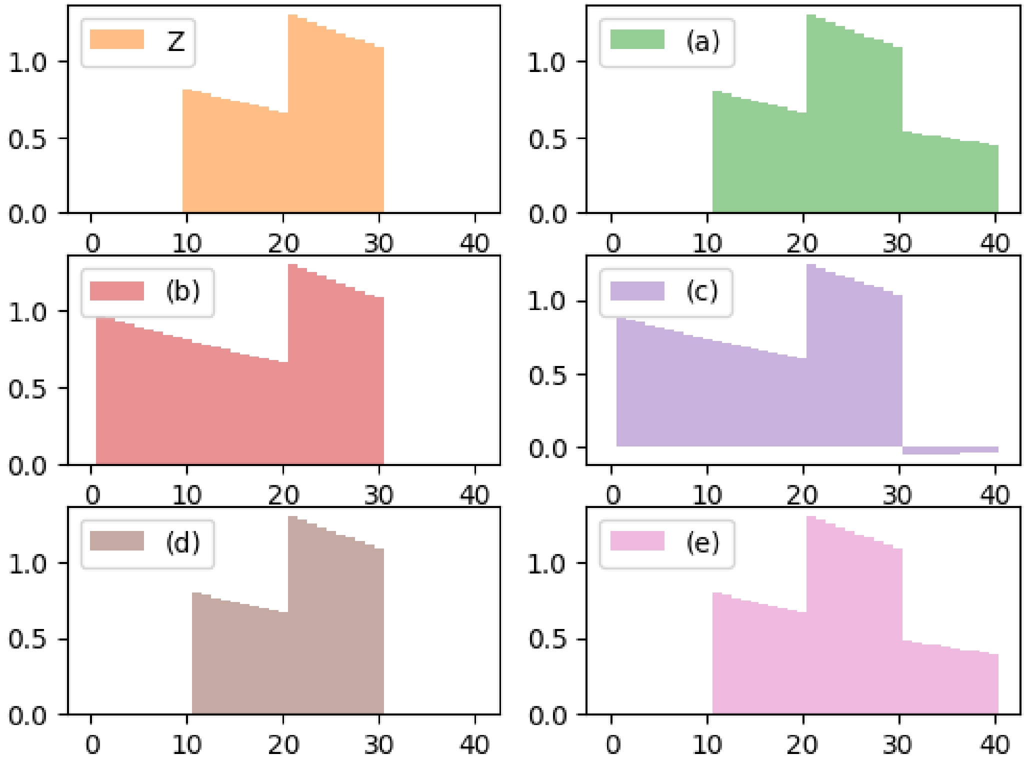
```
life = Recursion().set_A(0.39, x=35, t=15, endowment=1)\
                   .set_A(0.25, x=35, t=15)
E = life.E_x(35, t=15)
life = Recursion().set_A(0.32, x=35)\
                  .set_E(E, x=35, t=15)
def fun(A):
    return life.set_A(A, x=50).term_insurance(35, t=15)
A = life.solve(fun, target=0.25, guess=[0.35, 0.55])
soa(0.5, A, 4.9)
```

```
[ Pure Endowment: 15_E_35 ]
    endowment - term insurance = 15_E_35
SOA Question 4.9: [ 0.5 ] 0.5
```

SOA Question 4.10: (D)

```
life = Insurance(interest=dict(i=0.01), S=lambda x,s,t: 1, maxage=40)
def fun(x, t):
    if 10 <= t <= 20: return life.interest.v_t(t)
    elif 20 < t <= 30: return 2 * life.interest.v_t(t)
    else: return 0
def A(x, t):  # Z_x+k (t-k)
    return life.interest.v_t(t - x) * (t > x)
x = 0
benefits=[lambda x,t: (life.E_x(x, t=10) * A(x+10, t)
                            + life.E_x(x, t=20)* A(x+20, t)
                            - life.E_x(x, t=30) * A(x+30, t)),
            lambda x,t: (A(x, t)
                            + life.E_x(x, t=20) * A(x+20, t)
                            - 2 * life.E_x(x, t=30) * A(x+30, t)),
            lambda x,t: (life.E_x(x, t=10) * A(x, t)
                            + life.E_x(x, t=20) * A(x+20, t)
                            - 2 * life.E_x(x, t=30) * A(x+30, t)),
            lambda x,t: (life.E_x(x, t=10) * A(x+10, t)
                            + life.E_x(x, t=20) * A(x+20, t)
                            - 2 * life.E_x(x, t=30) * A(x+30, t)),
            lambda x,t: (life.E_x(x, t=10)
                            * (A(x+10, t)
                            + life.E_x(x+10, t=10) * A(x+20, t)
                            - life.E_x(x+20, t=10) * A(x+30, t)))]
fig, ax = plt.subplots(3, 2)
ax = ax.ravel()
for i, b in enumerate([fun] + benefits):
    life.Z_plot(0, benefit=b, ax=ax[i], verbose=False, color=f"C{i+1}")
    ax[i].legend(["(" + "abcde"[i-1] + ")" if i else "Z"])
z = [sum(abs(b(0, t) - fun(0, t)) for t in range(40)) for b in benefits]
soa('D', "ABCDE"[np.argmin(z)], '4.10')
```

```
SOA Question 4.10: [ D ] D
```



SOA Question 4.11: (A) 143385

```
A1 = 528/1000    # E[Z1]  term insurance
C1 = 0.209       # E[pure_endowment]
C2 = 0.136       # E[pure_endowment^2]
B1 = A1 + C1     # endowment = term + pure_endowment
def fun(A2):
    B2 = A2 + C2    # double force of interest
    return Insurance.insurance_variance(A2=B2, A1=B1)
A2 = Insurance.solve(fun, target=15000/(1000*1000), guess=[143400, 279300])
soa(143385, Insurance.insurance_variance(A2=A2, A1=A1, b=1000), 4.11)
```

```
SOA Question 4.11: [ 143385 ] 143384.99999999997
```

SOA Question 4.12: (C) 167

```
cov = Life.covariance(a=1.65, b=10.75, ab=0)   # E[Z1 Z2] = 0 nonoverlapping
soa(167, Life.variance(a=2, b=1, var_a=46.75, var_b=50.78, cov_ab=cov), 4.12)
```

```
SOA Question 4.12: [ 167 ] 166.82999999999998
```

SOA Question 4.13: (C) 350

```
life = Select(q={65: [.08, .10, .12, .14],
                 66: [.09, .11, .13, .15],
                 67: [.10, .12, .14, .16],
                 68: [.11, .13, .15, .17],
```

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```
                    69: [.12, .14, .16, .18]}, interest=dict(i=.04)).fill()
soa(350, life.deferred_insurance(65, t=2, u=2, b=2000), 4.13)
```

```
SOA Question 4.13: [ 350 ] 351.0578236056159
```

SOA Question 4.14: (E) 390000

```
sult = SULT()
p = sult.p_x(60, t=85-60)
mean = sult.bernoulli(p)
var = sult.bernoulli(p, variance=True)
F = sult.portfolio_percentile(mean=mean, variance=var, prob=.86, N=400)
soa(390000, F * 5000 * sult.interest.v_t(85-60), 4.14)
```

```
SOA Question 4.14: [ 390000 ] 389322.86778416135
```

SOA Question 4.15 (E) 0.0833

```
life = Insurance(mu=lambda *x: 0.04, interest=dict(delta=0.06))
benefit = lambda x,t: math.exp(0.02*t)
A1 = life.A_x(0, benefit=benefit, discrete=False)
A2 = life.A_x(0, moment=2, benefit=benefit, discrete=False)
soa(0.0833, life.insurance_variance(A2=A2, A1=A1), 4.15)
```

```
SOA Question 4.15: [ 0.0833 ] 0.08334849338238598
```

SOA Question 4.16: (D) 0.11

```
q = [.045, .050, .055, .060]
q_ = {50+x: [0.7 * q[x] if x < len(q) else None,
                0.8 * q[x+1] if x+1 < len(q) else None,
                q[x+2] if x+2 < len(q) else None]
        for x in range(4)}
life = Select(q=q_, interest=dict(i=.04)).fill()
soa(0.1116, life.term_insurance(50, t=3), 4.16)
```

```
SOA Question 4.16: [ 0.1116 ] 0.1115661982248521
```

SOA Question 4.17: (A) 1126.7

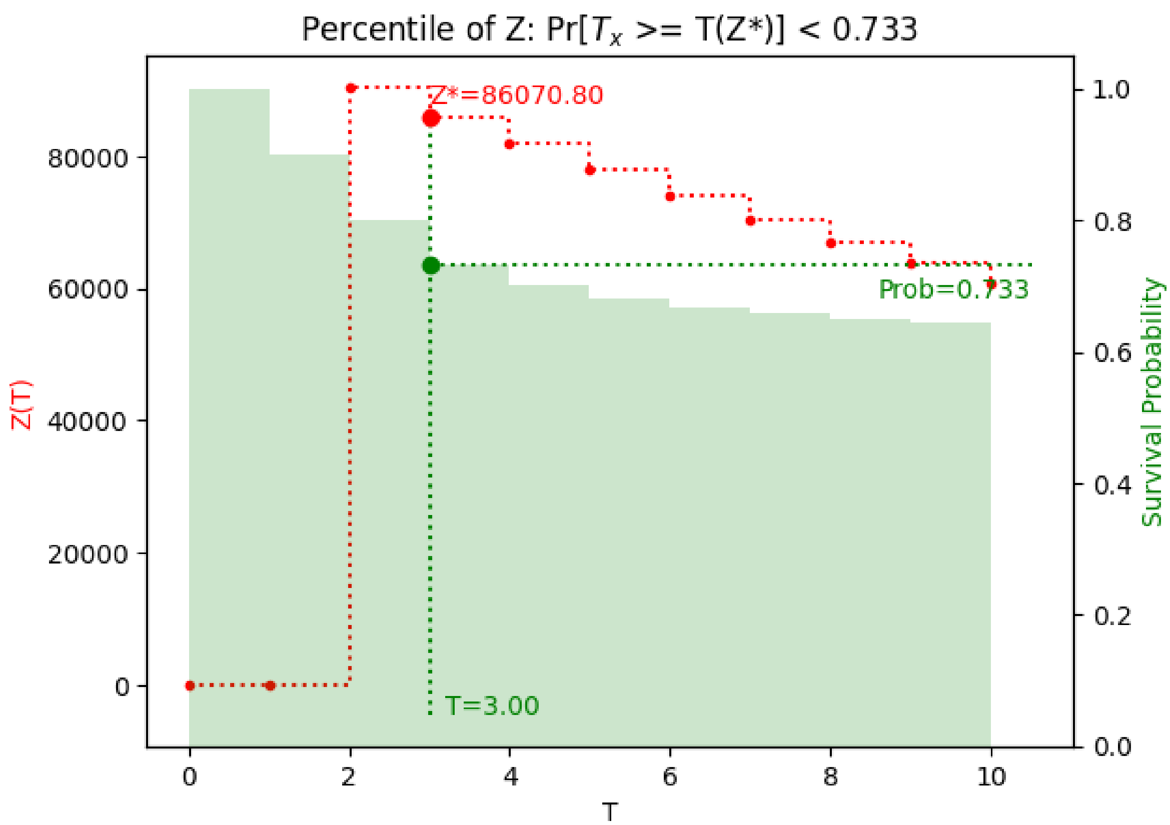
```
sult = SULT()
median = sult.Z_t(48, prob=0.5, discrete=False)
benefit = lambda x,t: 5000 if t < median else 10000
A = sult.A_x(48, benefit=benefit)
soa(1130, A, 4.17)
```

```
SOA Question 4.17: [ 1130 ] 1126.774772894844
```

SOA Question 4.18 (A) 81873

```
life = Insurance(interest=dict(delta=0.05),
                 maxage=10,
                 f=lambda x,s,t: .1 if t < 2 else .4*t**(-2))
benefit = lambda x,t: 0 if t < 2 else 100000
prob = 0.9 - life.q_x(0, t=2)
x, y = life.survival_curve()
T = life.Z_t(0, prob=prob)
life.Z_plot(0, T=T, benefit=benefit, discrete=False, curve=(x,y))
Z = life.Z_from_t(T) * benefit(0, T)
#plt.show()
soa(81873, Z, 4.18)
```

```
SOA Question 4.18: [ 81873 ] 81873.07530779815
```



SOA Question 4.19: (B) 59050

```
life = SULT()
adjust = Adjust(life=life)
q = adjust(extra=0.8, adjust=Adjust.MULTIPLY_RATE)['q']
select = Select(n=1)\
         .set_select(column=0, select_age=True, q=q)\
         .set_select(column=1, select_age=False, q=life['q']).fill()
soa(59050, 100000*select.whole_life_insurance(80, s=0), 4.19)
```

```
SOA Question 4.19: [ 59050 ] 59050.59973285648
```

20.5 5 Annuities

SOA Question 5.1: (A) 0.705

```
life = ConstantForce(mu=0.01, interest=dict(delta=0.06))
EY = life.certain_life_annuity(0, u=10, discrete=False)
a = life.p_x(0, t=life.Y_to_t(EY))
soa(0.705, a, 5.1)  # 0.705
```

```
SOA Question 5.1: [ 0.705 ] 0.7053680433746505
```

SOA Question 5.2: (B) 9.64

```
x, n = 0, 10
life = Recursion(interest=dict(i=0.05))
life.set_A(0.3, x).set_A(0.4, x+n).set_E(0.35, x, t=n)
a = life.immediate_annuity(x, t=n)
soa(9.64, a, 5.2)
```

```
SOA Question 5.2: [ 9.64 ] 9.639999999999999
```

SOA Question 5.3: (C) 6.239

```
sult = SULT()
t = 10.5
soa(6.239, t * sult.E_r(40, t=t), 5.3)
```

```
SOA Question 5.3: [ 6.239 ] 6.23871918627528
```

SOA Question 5.4: (A) 213.7

```
life = ConstantForce(mu=0.02, interest=dict(delta=0.01))
P = 10000 / life.certain_life_annuity(40, u=life.e_x(40, curtate=False),
                                        discrete=False)
soa(213.7, P, 5.4) # 213.7
```

```
SOA Question 5.4: [ 213.7 ] 213.74552118275955
```

SOA Question 5.5: (A) 1699.6

```
life = SULT()
adjust = Adjust(life=life)
q = adjust(extra=0.05, adjust=Adjust.ADD_FORCE)['q']
select = Select(n=1)\
         .set_select(column=0, select_age=True, q=q)\
         .set_select(column=1, select_age=False, a=life['a']).fill()
soa(1700, 100*select['a'][45][0], 5.5)
```

```
SOA Question 5.5: [ 1700 ] 1699.6076593190103
```

SOA Question 5.6: (D) 1200

```
life = Annuity(interest=dict(i=0.05))
var = life.annuity_variance(A2=0.22, A1=0.45)
mean = life.annuity_twin(A=0.45)
soa(1200, life.portfolio_percentile(mean, var, prob=.95, N=100), 5.6)
```

```
SOA Question 5.6: [ 1200 ] 1200.6946732201702
```

SOA Question 5.7: (C)

```
life = Recursion(interest=dict(i=0.04))
life.set_A(0.188, x=35).set_A(0.498, x=65).set_p(0.883, x=35, t=30)
mthly = Woolhouse(m=2, life=life, three_term=False)
soa(17376.7, 1000 * mthly.temporary_annuity(35, t=30), 5.7)
```

```
[ Pure Endowment: 30_E_35 ]
    pure endowment 30_E_35 = 30_p_35 * v^30
SOA Question 5.7: [ 17376.7 ] 17376.71459632958
```

SOA Question 5.8: (C) 0.92118

```
sult = SULT()
a = sult.certain_life_annuity(55, u=5)
soa(0.92118, sult.p_x(55, t=math.floor(a)), 5.8)
```

```
SOA Question 5.8: [ 0.92118 ] 0.9211799771029529
```

SOA Question 5.9: (C) 0.015

```
x, p = 0, 0.9  # set arbitrary p_x = 0.9
life1 = Recursion().set_a(21.854, x=x).set_p(p, x=x)
life2 = Recursion().set_a(22.167, x=x)
def fun(k):
    life2.set_p((1 + k) * p, x=x)
    return life1.whole_life_annuity(x+1) - life2.whole_life_annuity(x+1)
soa(0.015, life2.solve(fun, target=0, guess=[0.005, 0.025]), 5.9)
```

```
SOA Question 5.9: [ 0.015 ] 0.015009110961925157
```

20.6 6 Premium Calculation

SOA Question 6.1: (D) 35.36

```
life = SULT(interest=dict(i=0.03))
soa(35.36, life.net_premium(80, t=2, b=1000, return_premium=True), 6.1)
```

```
SOA Question 6.1: [ 35.36 ] 35.35922286190033
```

SOA Question 6.2: (E) 3604

```
life = Premiums()
A, IA, a = 0.17094, 0.96728, 6.8865
P = life.gross_premium(a=a, A=A, IA=IA, benefit=100000,
                        initial_premium=0.5, renewal_premium=.05,
                        renewal_policy=200, initial_policy=200)
soa(3604, P, 6.2)
```

```
SOA Question 6.2: [ 3604 ] 3604.229940320728
```

SOA Question 6.3: (C) 0.390

```
life = SULT()
P = life.net_premium(45, u=20, annuity=True)
t = life.Y_to_t(life.whole_life_annuity(65))
p = 1 - life.p_x(65, t=math.floor(t) - 1)
soa(0.39, p, 6.3)
```

```
SOA Question 6.3: [ 0.39 ] 0.39039071872030084
```

SOA Question 6.4: (E) 1890

```
mthly = Mthly(m=12, life=Reserves(interest=dict(i=0.06)))
A1, A2 = 0.4075, 0.2105
mean = mthly.annuity_twin(A1)*15*12
var = mthly.annuity_variance(A1=A1, A2=A2, b=15 * 12)
S = Reserves.portfolio_percentile(mean=mean, variance=var, prob=.9, N=200)
soa(1890, S / 200, 6.4)
```

```
SOA Question 6.4: [ 1890 ] 1893.912859650868
```

SOA Question 6.5: (D) 33

```
life = SULT()
P = life.net_premium(30, b=1000)
def fun(k):
    return (life.Y_x(30, t=k) * P
            - life.Z_x(30, t=k) * 1000)
soa(33, min([k for k in range(20, 40) if fun(k) < 0]), 6.5)
```

```
SOA Question 6.5: [ 33 ] 33
```

SOA Question 6.6: (B) 0.79

```
life = SULT()
P = life.net_premium(62, b=10000)
policy = life.Policy(premium=1.03*P, renewal_policy=5,
                        initial_policy=5, initial_premium=0.05, benefit=10000)
L = life.gross_policy_value(62, policy=policy)
var = life.gross_policy_variance(62, policy=policy)
prob = life.portfolio_cdf(mean=L, variance=var, value=40000, N=600)
soa(.79, prob, 6.6)
```

```
SOA Question 6.6: [ 0.79 ] 0.7914321142683509
```

SOA Question 6.7: (C) 2880

```
life=SULT()
a = life.temporary_annuity(40, t=20)
A = life.E_x(40, t=20)
IA = a - life.interest.annuity(t=20) * life.p_x(40, t=20)
soa(2880, life.gross_premium(a=a, A=A, IA=IA, benefit=100000), 6.7)
```

```
SOA Question 6.7: [ 2880 ] 2880.2463991134578
```

SOA Question 6.8: (B) 9.5

```
life = SULT()
initial_cost = (50 + 10 * life.deferred_annuity(60, u=1, t=9)
                + 5 * life.deferred_annuity(60, u=10, t=10))
soa(9.5, life.net_premium(60, initial_cost=initial_cost), 6.8)
```

```
SOA Question 6.8: [ 9.5 ] 9.526003201821927
```

SOA Question 6.9: (D) 647

```
life = SULT()
a = life.temporary_annuity(50, t=10)
A = life.term_insurance(50, t=20)
initial_cost = 25 * life.deferred_annuity(50, u=10, t=10)
P = life.gross_premium(a=a, A=A, benefit=100000,
                       initial_premium=0.42, renewal_premium=0.12,
                       initial_policy=75 + initial_cost, renewal_policy=25)
soa(647, P, 6.9)
```

```
SOA Question 6.9: [ 647 ] 646.8608151974504
```

SOA Question 6.10: (D) 0.91

```
x = 0
life = Recursion(interest=dict(i=0.06)).set_p(0.975, x=x)
a = 152.85/56.05   # solve a_x:3, given net premium and benefit APV

def fun(p):    # solve p_x+2, given a_x:3
    return life.set_p(p, x=x+1).temporary_annuity(x, t=3)
life.set_p(life.solve(fun, target=a, guess=0.975), x=x+1)

def fun(p):    # finally solve p_x+3, given A_x:3
    return life.set_p(p, x=x+2).term_insurance(x=x, t=3, b=1000)
p = life.solve(fun, target=152.85, guess=0.975)
soa(0.91, p, "6.10")
```

```
SOA Question 6.10: [ 0.91 ] 0.90973829505257
```

SOA Question 6.11: (C) 0.041

```
life = Recursion(interest=dict(i=0.04))
life.set_A(0.39788, 51)
life.set_q(0.0048, 50)
A = life.whole_life_insurance(50)
P = life.gross_premium(A=A, a=life.annuity_twin(A=A))
life.set_q(0.048, 50)
A = life.whole_life_insurance(50)
soa(0.041, A - life.annuity_twin(A) * P, 6.11)
```

```
[ Whole Life Insurance: A_50 ]
    forward: A_50 = qv + pvA_51
[ Whole Life Insurance: A_50 ]
    forward: A_50 = qv + pvA_51
SOA Question 6.11: [ 0.041 ] 0.04069206883563675
```

SOA Question 6.12: (E) 88900

```
life = PolicyValues(interest=dict(i=0.06))
a = 12
A = life.insurance_twin(a)
policy = life.Policy(benefit=1000, settlement_policy=20,
                        initial_policy=10, initial_premium=0.75,
                        renewal_policy=2, renewal_premium=0.1)
policy.premium = life.gross_premium(A=A, a=a, **policy.premium_terms)
L = life.gross_variance_loss(A1=A, A2=0.14, policy=policy)
soa(88900, L, 6.12)
```

```
SOA Question 6.12: [ 88900 ] 88862.59592874818
```

SOA Question 6.13: (D) -400

```
life = SULT(interest=dict(i=0.05))
A = life.whole_life_insurance(45)
policy = life.Policy(benefit=10000, initial_premium=.8, renewal_premium=.1)
def fun(P):   # Solve for premium, given Loss(t=0) = 4953
    return life.L_from_t(t=10.5, policy=policy.set(premium=P))
policy.premium = life.solve(fun, target=4953, guess=100)
L = life.gross_policy_value(45, policy=policy)
soa(-400, L, 6.13)
```

```
SOA Question 6.13: [ -400 ] -400.94447599879277
```

SOA Question 6.14 (D) 1150

```
life = SULT(interest=dict(i=0.05))
a = (life.temporary_annuity(40, t=10)
       + 0.5 * life.deferred_annuity(40, u=10, t=10))
A = life.whole_life_insurance(40)
P = life.gross_premium(a=a, A=A, benefit=100000)
soa(1150, P, 6.14)
```

```
SOA Question 6.14: [ 1150 ] 1148.5800555155263
```

SOA Question 6.15: (B) 1.002

```
life = Recursion(interest=dict(i=0.05)).set_a(3.4611, x=0)
A = life.insurance_twin(3.4611)
udd = UDD(m=4, life=life)
a1 = udd.whole_life_annuity(x=x)
woolhouse = Woolhouse(m=4, life=life)
a2 = woolhouse.whole_life_annuity(x=x)
P = life.gross_premium(a=a1, A=A)/life.gross_premium(a=a2, A=A)
soa(1.002, P, 6.15)
```

```
SOA Question 6.15: [ 1.002 ] 1.0022973504113772
```

SOA Question 6.16: (A) 2408.6

```
life = Premiums(interest=dict(d=0.05))
A = life.insurance_equivalence(premium=2143, b=100000)
a = life.annuity_equivalence(premium=2143, b=100000)
p = life.gross_premium(A=A, a=a, benefit=100000, settlement_policy=0,
                       initial_policy=250, initial_premium=0.04 + 0.35,
                       renewal_policy=50, renewal_premium=0.04 + 0.02)
soa(2410, p, 6.16)
```

```
SOA Question 6.16: [ 2410 ] 2408.575206281868
```

SOA Question 6.17: (A) -30000

```
x = 0
life = ConstantForce(mu=0.1, interest=dict(i=0.08))
A = life.endowment_insurance(x, t=2, b=100000, endowment=30000)
a = life.temporary_annuity(x, t=2)
P = life.gross_premium(a=a, A=A)
life1 = Recursion(interest=dict(i=0.08))
life1.set_q(life.q_x(x, t=1) * 1.5, x=x, t=1)
life1.set_q(life.q_x(x+1, t=1) * 1.5, x=x+1, t=1)
policy = life1.Policy(premium=P*2, benefit=100000, endowment=30000)
L = life1.gross_policy_value(x, t=0, n=2, policy=policy)
soa(-30000, L, 6.17)
```

```
[ Term Insurance: A_0^1:2 ]
    forward: A_0 = qv + pvA_1
       endowment insurance - pure endowment = A_1^1:1
    pure endowment 1_E_1 = 1_p_1 * v^1
[ Temporary Annuity: a_0:2 ]
    forward: a_0:2 = 1 + E_0 a_1:1
    pure endowment 1_E_0 = 1_p_0 * v^1
       1-year discrete annuity: a_x:1 = 1
[ Pure Endowment: 2_E_0 ]
    chain Rule: 2_E_0 = E_0 * 1_E_1
    pure endowment 1_E_1 = 1_p_1 * v^1
    pure endowment 1_E_0 = 1_p_0 * v^1
SOA Question 6.17: [ -30000 ] -30107.42633581125
```

SOA Question 6.18: (D) 166400

```
life = SULT(interest=dict(i=0.05))
def fun(P):
    A = (life.term_insurance(40, t=20, b=P)
            + life.deferred_annuity(40, u=20, b=30000))
    return life.gross_premium(a=1, A=A) - P
P = life.solve(fun, target=0, guess=[162000, 168800])
soa(166400, P, 6.18)
```

```
SOA Question 6.18: [ 166400 ] 166362.83871487685
```

SOA Question 6.19: (B) 0.033

```
life = SULT()
policy = life.Policy(initial_policy=.2, renewal_policy=.01)
a = life.whole_life_annuity(50)
A = life.whole_life_insurance(50)
policy.premium = life.gross_premium(A=A, a=a, **policy.premium_terms)
L = life.gross_policy_variance(50, policy=policy)
soa(0.033, L, 6.19)
```

```
SOA Question 6.19: [ 0.033 ] 0.03283273381910885
```

SOA Question 6.20: (B) 459

```
life = LifeTable(interest=dict(i=0.04),
                    p={75: 0.9, 76: 0.88, 77: 0.85}).fill()
a = life.temporary_annuity(75, t=3)
IA = life.increasing_insurance(75, t=2)
A = life.deferred_insurance(75, u=2, t=1)
def fun(P):
    return life.gross_premium(a=a, A=P*IA + A*10000) - P
soa(459, life.solve(fun, target=0, guess=[449, 489]), "6.20")
```

```
SOA Question 6.20: [ 459 ] 458.83181728297353
```

SOA Question 6.21: (C) 100

```
life = Recursion(interest=dict(d=0.04))
life.set_A(0.7, x=75, t=15, endowment=1)
life.set_E(0.11, x=75, t=15)
def fun(P):
    P = float(P)
    return (P * life.temporary_annuity(75, t=15)
            - life.endowment_insurance(75, t=15, b=1000, endowment=15*P))
P = life.solve(fun, target=0, guess=(80, 120))
soa(100, P, 6.21)
```

```
SOA Question 6.21: [ 100 ] 100.85470085470084
```

SOA Question 6.22: (C) 102

```
life=SULT(udd=True)
a = UDD(m=12, life=life).temporary_annuity(45, t=20)
A = UDD(m=0, life=life).whole_life_insurance(45)
P = life.gross_premium(A=A, a=a, benefit=100000) / 12
soa(102, P, 6.22)
```

```
SOA Question 6.22: [ 102 ] 102.40668704849178
```

SOA Question 6.23: (D) 44.7

```
x = 0
life = Recursion().set_a(15.3926, x=x)\
                  .set_a(10.1329, x=x, t=15)\
                  .set_a(14.0145, x=x, t=30)
def fun(P):
    per_policy = 30 + (30 * life.whole_life_annuity(x))
    per_premium = (0.6 + 0.1 * life.temporary_annuity(x, t=15)
                   + 0.1 * life.temporary_annuity(x, t=30))
    a = life.temporary_annuity(x, t=30)
    return (P * a) - (per_policy + per_premium * P)
P = life.solve(fun, target=0, guess=[30.3, 49.5])
soa(44.7, P, 6.23)
```

```
SOA Question 6.23: [ 44.7 ] 44.70806635781144
```

SOA Question 6.24: (E) 0.30

```
life = PolicyValues(interest=dict(delta=0.07))
x, A1 = 0, 0.30   # Policy for first insurance
P = life.premium_equivalence(A=A1, discrete=False)  # Need its premium
policy = life.Policy(premium=P, discrete=False)
def fun(A2):  # Solve for A2, given Var(Loss)
    return life.gross_variance_loss(A1=A1, A2=A2, policy=policy)
A2 = life.solve(fun, target=0.18, guess=0.18)

policy = life.Policy(premium=0.06, discrete=False) # Solve second insurance
variance = life.gross_variance_loss(A1=A1, A2=A2, policy=policy)
soa(0.304, variance, 6.24)
```

```
SOA Question 6.24: [ 0.304 ] 0.30419999999999975
```

SOA Question 6.25: (C) 12330

```
life = SULT()
woolhouse = Woolhouse(m=12, life=life)
benefits = woolhouse.deferred_annuity(55, u=10, b=1000 * 12)
expenses = life.whole_life_annuity(55, b=300)
payments = life.temporary_annuity(55, t=10)
def fun(P):
    return life.gross_future_loss(A=benefits + expenses, a=payments,
                                  policy=life.Policy(premium=P))
P = life.solve(fun, target=-800, guess=[12110, 12550])
soa(12330, P, 6.25)
```

```
SOA Question 6.25: [ 12330 ] 12325.781125438532
```

SOA Question 6.26 (D) 180

```
life = SULT(interest=dict(i=0.05))
def fun(P):
    return P - life.net_premium(90, b=1000, initial_cost=P)
P = life.solve(fun, target=0, guess=[150, 190])
soa(180, P, 6.26)
```

```
SOA Question 6.26: [ 180 ] 180.03164891315885
```

SOA Question 6.27: (D) 10310

```
life = ConstantForce(mu=0.03, interest=dict(delta=0.06))
x = 0
payments = (3 * life.temporary_annuity(x, t=20, discrete=False)
            + life.deferred_annuity(x, u=20, discrete=False))
benefits = (1000000 * life.term_insurance(x, t=20, discrete=False)
            + 500000 * life.deferred_insurance(x, u=20, discrete=False))
P = benefits / payments
soa(10310, P, 6.27)
```

```
SOA Question 6.27: [ 10310 ] 10309.617799001708
```

SOA Question 6.28 (B) 36

```
life = SULT(interest=dict(i=0.05))
a = life.temporary_annuity(40, t=5)
A = life.whole_life_insurance(40)
P = life.gross_premium(a=a, A=A, benefit=1000,
                        initial_policy=10, renewal_premium=.05,
                        renewal_policy=5, initial_premium=.2)
soa(36, P, 6.28)
```

```
SOA Question 6.28: [ 36 ] 35.72634219391481
```

SOA Question 6.29 (B) 20.5

```
life = Premiums(interest=dict(i=0.035))
def fun(a):
    return life.gross_premium(A=life.insurance_twin(a=a), a=a,
                              initial_policy=200, initial_premium=.5,
                              renewal_policy=50, renewal_premium=.1,
                              benefit=100000)
a = life.solve(fun, target=1770, guess=[20, 22])
soa(20.5, a, 6.29)
```

```
SOA Question 6.29: [ 20.5 ] 20.480268314431726
```

SOA Question 6.30: (A) 900

```
life = PolicyValues(interest=dict(i=0.04))
policy = life.Policy(premium=2.338, benefit=100, initial_premium=.1,
                        renewal_premium=0.05)
var = life.gross_variance_loss(A1=life.insurance_twin(16.50),
                                A2=0.17, policy=policy)
soa(900, var, "6.30")
```

```
SOA Question 6.30: [ 900 ] 908.141412994607
```

SOA Question 6.31: (D) 1330

```
life = ConstantForce(mu=0.01, interest=dict(delta=0.05))
A = (life.term_insurance(35, t=35, discrete=False)
        + life.E_x(35, t=35) * 0.51791)     # A_35
P = life.premium_equivalence(A=A, b=100000, discrete=False)
soa(1330, P, 6.31)
```

```
SOA Question 6.31: [ 1330 ] 1326.5406293909457
```

SOA Question 6.32: (C) 550

```
x = 0
life = Recursion(interest=dict(i=0.05)).set_a(9.19, x=x)
benefits = UDD(m=0, life=life).whole_life_insurance(x)
payments = UDD(m=12, life=life).whole_life_annuity(x)
P = life.gross_premium(a=payments, A=benefits, benefit=100000)/12
soa(550, P, 6.32)
```

```
SOA Question 6.32: [ 550 ] 550.4356936711871
```

SOA Question 6.33: (B) 0.13

```
life = Insurance(mu=lambda x,t: 0.02*t, interest=dict(i=0.03))
x = 0
var = life.E_x(x, t=15, moment=life.VARIANCE, endowment=10000)
p = 1- life.portfolio_cdf(mean=0, variance=var, value=50000, N=500)
soa(0.13, p, 6.33, rel_tol=0.02)
```

```
SOA Question 6.33: [ 0.13 ] 0.12828940905648634
```

SOA Question 6.34: (A) 23300

```
life = SULT()
def fun(benefit):
    A = life.whole_life_insurance(61)
    a = life.whole_life_annuity(61)
    return life.gross_premium(A=A, a=a, benefit=benefit,
                                initial_premium=0.15, renewal_premium=0.03)
b = life.solve(fun, target=500, guess=[23300, 23700])
soa(23300, b, 6.34)
```

```
SOA Question 6.34: [ 23300 ] 23294.288659265632
```

SOA Question 6.35: (D) 530

```
sult = SULT()
A = sult.whole_life_insurance(35, b=100000)
a = sult.whole_life_annuity(35)
P = sult.gross_premium(a=a, A=A, initial_premium=.19, renewal_premium=.04)
soa(530, P, 6.35)
```

```
SOA Question 6.35: [ 530 ] 534.4072234303344
```

SOA Question 6.36: (B) 500

```
life = ConstantForce(mu=0.04, interest=dict(delta=0.08))
a = life.temporary_annuity(50, t=20, discrete=False)
A = life.term_insurance(50, t=20, discrete=False)
def fun(R):
    return life.gross_premium(a=a, A=A, initial_premium=R/4500,
                              renewal_premium=R/4500, benefit=100000)
R = life.solve(fun, target=4500, guess=[400, 800])
soa(500, R, 6.36)
```

```
SOA Question 6.36: [ 500 ] 500.0
```

SOA Question 6.37: (D) 820

```
sult = SULT()
benefits = sult.whole_life_insurance(35, b=50000 + 100)
expenses = sult.immediate_annuity(35, b=100)
a = sult.temporary_annuity(35, t=10)
P = (benefits + expenses) / a
soa(820, P, 6.37)
```

```
SOA Question 6.37: [ 820 ] 819.7190338249138
```

SOA Question 6.38: (B) 11.3

```
x, n = 0, 10
life = Recursion(interest=dict(i=0.05))
life.set_A(0.192, x=x, t=n, endowment=1, discrete=False)
life.set_E(0.172, x=x, t=n)
a = life.temporary_annuity(x, t=n, discrete=False)

def fun(a):    # solve for discrete annuity, given continuous
    life = Recursion(interest=dict(i=0.05), verbose=False)
    life.set_a(a, x=x, t=n).set_E(0.172, x=x, t=n)
    return UDD(m=0, life=life).temporary_annuity(x, t=n)
a = life.solve(fun, target=a, guess=a)   # discrete annuity
P = life.gross_premium(a=a, A=0.192, benefit=1000)
soa(11.3, P, 6.38)
```

```
[ Temporary Annuity: a_0:10 ]
    Annuity twin: a = (1 - A) / d
SOA Question 6.38: [ 11.3 ] 11.308644185253657
```

SOA Question 6.39: (A) 29

```
sult = SULT()
P40 = sult.premium_equivalence(sult.whole_life_insurance(40), b=1000)
P80 = sult.premium_equivalence(sult.whole_life_insurance(80), b=1000)
p40 = sult.p_x(40, t=10)
p80 = sult.p_x(80, t=10)
P = (P40 * p40 + P80 * p80) / (p80 + p40)
soa(29, P, 6.39)
```

```
SOA Question 6.39: [ 29 ] 29.033866427845496
```

SOA Question 6.40: (C) 116

```
# - standard formula discounts/accumulates by too much (i should be smaller)
x = 0
life = Recursion(interest=dict(i=0.06)).set_a(7, x=x+1).set_q(0.05, x=x)
a = life.whole_life_annuity(x)
A = 110 * a / 1000
life = Recursion(interest=dict(i=0.06)).set_A(A, x=x).set_q(0.05, x=x)
A1 = life.whole_life_insurance(x+1)
P = life.gross_premium(A=A1 / 1.03, a=7) * 1000
soa(116, P, "6.40")
```

```
[ Whole Life Annuity: a_0 ]
    forward: a_0 = 1 + E_0 a_1
    pure endowment 1_E_0 = 1_p_0 * v^1
[ Whole Life Insurance: A_1 ]
    backward: A_1 = (A_0/v - q) / p
          backward: A_1 = (A_0/v - q) / p
                backward: A_1 = (A_0/v - q) / p
                backward: A_1 = (A_0/v - q) / p
SOA Question 6.40: [ 116 ] 116.51945397474269
```

SOA Question 6.41: (B) 1417

```
x = 0
life = LifeTable(interest=dict(i=0.05), q={x:.01, x+1:.02}).fill()
a = 1 + life.E_x(x, t=1) * 1.01
A = (life.deferred_insurance(x, u=0, t=1)
        + 1.01 * life.deferred_insurance(x, u=1, t=1))
P = 100000 * A / a
soa(1417, P, 6.41)
```

```
SOA Question 6.41: [ 1417 ] 1416.9332301924137
```

SOA Question 6.42: (D) 0.113

```
x = 0
life = ConstantForce(interest=dict(delta=0.06), mu=0.06)
policy = life.Policy(discrete=True, premium=315.8,
                     T=3, endowment=1000, benefit=1000)
L = [life.L_from_t(t, policy=policy) for t in range(3)]    # L(t)
Q = [life.q_x(x, u=u, t=1) for u in range(3)]        # prob(die in year t)
Q[-1] = 1 - sum(Q[:-1])  # follows SOA Question Solution incorrect treat endowment!
p = sum([q for (q, l) in zip (Q, L) if l > 0])
soa(0.113, p, 6.42)
```

```
SOA Question 6.42: [ 0.113 ] 0.11307956328284252
```

SOA Question 6.43: (C) 170

```
sult = SULT()
a = sult.temporary_annuity(30, t=5)
A = sult.term_insurance(30, t=10)
other_expenses = 4 * sult.deferred_annuity(30, u=5, t=5)
P = sult.gross_premium(a=a, A=A, benefit=200000, initial_premium=0.35,
                       initial_policy=8 + other_expenses, renewal_policy=4,
                       renewal_premium=0.15)
soa(170, P, 6.43)
```

```
SOA Question 6.43: [ 170 ] 171.22371939459944
```

SOA Question 6.44: (D) 2.18

```
life = Recursion(interest=dict(i=0.05)).set_IA(0.15, x=50, t=10)
life.set_a(17, x=50).set_a(15, x=60).set_E(0.6, x=50, t=10)
A = life.deferred_insurance(50, u=10)
IA = life.increasing_insurance(50, t=10)
a = life.temporary_annuity(50, t=10)
P = life.gross_premium(a=a, A=A, IA=IA, benefit=100)
soa(2.2, P, 6.44)
```

```
SOA Question 6.44: [ 2.2 ] 2.183803457688809
```

SOA Question 6.45: (E) 690

```
life = SULT(udd=True)
policy = life.Policy(benefit=100000, premium=560, discrete=False)
p = life.L_from_prob(35, prob=0.75, policy=policy)
soa(690, p, 6.45)
```

```
SOA Question 6.45: [ 690 ] 689.2659416264196
```

SOA Question 6.46: (E) 208

```
life = Recursion(interest=dict(i=0.05)).set_IA(0.51213, x=55, t=10)
life.set_a(12.2758, x=55).set_a(7.4575, x=55, t=10)
A = life.deferred_annuity(55, u=10)
IA = life.increasing_insurance(55, t=10)
```

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```
a = life.temporary_annuity(55, t=10)
P = life.gross_premium(a=a, A=A, IA=IA, benefit=300)
soa(208, P, 6.46)
```

```
SOA Question 6.46: [ 208 ] 208.12282139036515
```

SOA Question 6.47: (D) 66400

```
sult = SULT()
a = sult.temporary_annuity(70, t=10)
A = sult.deferred_annuity(70, u=10)
P = sult.gross_premium(a=a, A=A, benefit=100000, initial_premium=0.75,
                       renewal_premium=0.05)
soa(66400, P, 6.47)
```

```
SOA Question 6.47: [ 66400 ] 66384.13293704337
```

SOA Question 6.48: (A) 3195 – example of deep insurance recursion

```
x = 0
life = Recursion(interest=dict(i=0.06), depth=5).set_p(.95, x=x, t=5)
life.set_q(.02, x=x+5).set_q(.03, x=x+6).set_q(.04, x=x+7)
a = 1 + life.E_x(x, t=5)
A = life.deferred_insurance(x, u=5, t=3)
P = life.gross_premium(A=A, a=a, benefit=100000)
soa(3195, P, 6.48)
```

```
[ Pure Endowment: 5_E_0 ]
    pure endowment 5_E_0 = 5_p_0 * v^5
[ Pure Endowment: 5_E_0 ]
    pure endowment 5_E_0 = 5_p_0 * v^5
[ Term Insurance: A_5^1:3 ]
    forward: A_5 = qv + pvA_6
       forward: A_6 = qv + pvA_7
          endowment insurance - pure endowment = A_7^1:1
    pure endowment 1_E_7 = 1_p_7 * v^1
[ Term Insurance: A_5^1:3 ]
    pure endowment 1_E_7 = 1_p_7 * v^1
          endowment insurance - pure endowment = A_7^1:1
       forward: A_6 = qv + pvA_7
    forward: A_5 = qv + pvA_6
SOA Question 6.48: [ 3195 ] 3195.1189176587473
```

SOA Question 6.49: (C) 86

```
sult = SULT(udd=True)
a = UDD(m=12, life=sult).temporary_annuity(40, t=20)
A = sult.whole_life_insurance(40, discrete=False)
P = sult.gross_premium(a=a, A=A, benefit=100000, initial_policy=200,
                       renewal_premium=0.04, initial_premium=0.04) / 12
soa(86, P, 6.49)
```

```
SOA Question 6.49: [ 86 ] 85.99177833261696
```

SOA Question 6.50: (A) -47000

```
life = SULT()
P = life.premium_equivalence(a=life.whole_life_annuity(35), b=1000)
a = life.deferred_annuity(35, u=1, t=1)
A = life.term_insurance(35, t=1, b=1000)
cash = (A - a * P) * 10000 / life.interest.v
soa(-47000, cash, "6.50")
```

```
SOA Question 6.50: [ -47000 ] -46948.2187697819
```

SOA Question 6.51: (D) 34700

```
life = Recursion()
life.set_DA(0.4891, x=62, t=10)
life.set_A(0.0910, x=62, t=10)
life.set_a(12.2758, x=62)
life.set_a(7.4574, x=62, t=10)
IA = life.increasing_insurance(62, t=10)
A = life.deferred_annuity(62, u=10)
a = life.temporary_annuity(62, t=10)
P = life.gross_premium(a=a, A=A, IA=IA, benefit=50000)
soa(34700, P, 6.51)
```

```
[ Increasing Insurance: IA_62:10 ]
    identity IA_62:10: (11)A - DA
SOA Question 6.51: [ 34700 ] 34687.207544453246
```

SOA Question 6.52: (D) 50.80 – hint: set face value benefits to 0

```
sult = SULT()
a = sult.temporary_annuity(45, t=10)
other_cost = 10 * sult.deferred_annuity(45, u=10)
P = sult.gross_premium(a=a, A=0, benefit=0,     # set face value H = 0
                        initial_premium=1.05, renewal_premium=0.05,
                        initial_policy=100 + other_cost, renewal_policy=20)
soa(50.8, P, 6.52)
```

```
SOA Question 6.52: [ 50.8 ] 50.80135534704229
```

SOA Question 6.53: (D) 720

```
x = 0
life = LifeTable(interest=dict(i=0.08), q={x:.1, x+1:.1, x+2:.1}).fill()
A = life.term_insurance(x, t=3)
P = life.gross_premium(a=1, A=A, benefit=2000, initial_premium=0.35)
soa(720, P, 6.53)
```

```
SOA Question 6.53: [ 720 ] 720.1646090534978
```

SOA Question 6.54: (A) 25440

```
life = SULT()
s = math.sqrt(life.net_policy_variance(45, b=200000))
soa(25440, s, 6.54)
```

```
SOA Question 6.54: [ 25440 ] 25441.694847703857
```

20.7 7 Policy Values

SOA Question 7.1: (C) 11150

```
life = SULT()
x, n, t = 40, 20, 10
A = (life.whole_life_insurance(x+t, b=50000)
        + life.deferred_insurance(x+t, u=n-t, b=50000))
a = life.temporary_annuity(x+t, t=n-t, b=875)
L = life.gross_future_loss(A=A, a=a)
soa(11150, L, 7.1)
```

```
SOA Question 7.1: [ 11150 ] 11152.108749338717
```

SOA Question 7.2: (C) 1152

```
x = 0
life = Recursion(interest=dict(i=.1)).set_q(0.15, x=x).set_q(0.165, x=x+1)
life.set_reserves(T=2, endowment=2000)

def fun(P):  # solve P s.t. V is equal backwards and forwards
    policy = dict(t=1, premium=P,
                  benefit=lambda t: 2000, reserve_benefit=True)
    return life.t_V_backward(x, **policy) - life.t_V_forward(x, **policy)
P = life.solve(fun, target=0, guess=[1070, 1230])
soa(1152, P, 7.2)
```

```
SOA Question 7.2: [ 1152 ] 1151.5151515151515
```

SOA Question 7.3: (E) 730

```
x = 0  # x=0 is (90) and interpret every 3 months as t=1 year
life = LifeTable(interest=dict(i=0.08/4),
                 l={0:1000, 1:898, 2:800, 3:706}).fill()
life.set_reserves(T=8, V={3: 753.72})
life.set_reserves(V={2: life.t_V_forward(x=0, t=2, premium=60*0.9,
                                         benefit=lambda t: 1000)})
V = life.t_V_forward(x=0, t=1, premium=0, benefit=lambda t: 1000)
soa(730, V, 7.3)
```

```
SOA Question 7.3: [ 730 ] 729.998398765594
```

SOA Question 7.4: (B) -74 – split benefits into two policies

```
life = SULT()
P = life.gross_premium(a=life.whole_life_annuity(40),
                        A=life.whole_life_insurance(40),
                        initial_policy=100, renewal_policy=10,
                        benefit=1000)
P += life.gross_premium(a=life.whole_life_annuity(40),
                        A=life.deferred_insurance(40, u=11),
                        benefit=4000)   # for deferred portion
policy = life.Policy(benefit=1000, premium=1.02*P,
                        renewal_policy=10, initial_policy=100)
V = life.gross_policy_value(x=40, t=1, policy=policy)
policy = life.Policy(benefit=4000, premium=0)
A = life.deferred_insurance(41, u=10)
V += life.gross_future_loss(A=A, a=0, policy=policy) # for deferred portion
soa(-74, V, 7.4)
```

```
SOA Question 7.4: [ -74 ] -73.942155695248
```

SOA Question 7.5: (E) 1900

```
x = 0
life = Recursion(interest=dict(i=0.03), udd=True).set_q(0.04561, x=x+4)
life.set_reserves(T=3, V={4: 1405.08})
V = life.r_V_backward(x, s=4, r=0.5, benefit=10000, premium=647.46)
soa(1900, V, 7.5)
```

```
SOA Question 7.5: [ 1900 ] 1901.766021537228
```

Answer 7.6: (E) -25.4

```
life = SULT()
P = life.net_premium(45, b=2000)
policy = life.Policy(benefit=2000, initial_premium=.25, renewal_premium=.05,
                        initial_policy=2*1.5 + 30, renewal_policy=2*.5 + 10)
G = life.gross_premium(a=life.whole_life_annuity(45), **policy.premium_terms)
gross = life.gross_policy_value(45, t=10, policy=policy.set(premium=G))
net = life.net_policy_value(45, t=10, b=2000)
V = gross - net

soa(-25.4, V, 7.6)
```

```
SOA Question 7.6: [ -25.4 ] -25.44920289521204
```

SOA Question 7.7: (D) 1110

```
x = 0
life = Recursion(interest=dict(i=0.05)).set_A(0.4, x=x+10)
a = Woolhouse(m=12, life=life).whole_life_annuity(x+10)
policy = life.Policy(premium=0, benefit=10000, renewal_policy=100)
V = life.gross_future_loss(A=0.4, policy=policy.future)
policy = life.Policy(premium=30*12, renewal_premium=0.05)
V += life.gross_future_loss(a=a, policy=policy.future)
soa(1110, V, 7.7)
```

```
SOA Question 7.7: [ 1110 ] 1107.9718253968258
```

SOA Question 7.8: (C) 29.85

```
sult = SULT()
x = 70
q = {x: [sult.q_x(x+k)*(.7 + .1*k) for k in range(3)] + [sult.q_x(x+3)]}
life = Recursion(interest=dict(i=.05)).set_q(sult.q_x(70)*.7, x=x)\
                                        .set_reserves(T=3)
V = life.t_V(x=70, t=1, premium=35.168, benefit=lambda t: 1000)
soa(29.85, V, 7.8)
```

```
SOA Question 7.8: [ 29.85 ] 29.85469179271202
```

SOA Question 7.9: (A) 38100

```
sult = SULT(udd=True)
x, n, t = 45, 20, 10
a = UDD(m=12, life=sult).temporary_annuity(x+10, t=n-10)
A = UDD(m=0, life=sult).endowment_insurance(x+10, t=n-10)
policy = sult.Policy(premium=253*12, endowment=100000, benefit=100000)
V = sult.gross_future_loss(A=A, a=a, policy=policy)
soa(38100, V, 7.9)
```

```
SOA Question 7.9: [ 38100 ] 38099.62176709246
```

SOA Question 7.10: (C) -970

```
life = SULT()
G = 977.6
P = life.net_premium(45, b=100000)
policy = life.Policy(benefit=0, premium=G-P, renewal_policy=.02*G + 50)
V = life.gross_policy_value(45, t=5, policy=policy)
soa(-970, V, "7.10")
```

```
SOA Question 7.10: [ -970 ] -971.8909301877826
```

SOA Question 7.11: (B) 1460

```
life=Recursion(interest=dict(i=0.05)).set_a(13.4205, x=55)
policy=life.Policy(benefit=10000)
def fun(P):
    return life.L_from_t(t=10, policy=policy.set(premium=P))
P = life.solve(fun, target=4450, guess=400)
V = life.gross_policy_value(45, t=10, policy=policy.set(premium=P))
soa(1460, V, 7.11)
```

```
SOA Question 7.11: [ 1460 ] 1459.9818035330218
```

SOA Question 7.12: (E) 4.09

```
benefit = lambda k: 26 - k
x = 44
life = Recursion(interest=dict(i=0.04)).set_q(0.15, x=55)
life.set_reserves(T=25, endowment=1, V={11: 5.})
def fun(P):  # solve for net premium, from final year reserves recursion
    return life.t_V(x=x, t=24, premium=P, benefit=benefit)
P = life.solve(fun, target=0.6, guess=0.5)     # solved net premium
V = life.t_V(x, t=12, premium=P, benefit=benefit)  # recursion formula
soa(4.09, V, 7.12)
```

```
SOA Question 7.12: [ 4.09 ] 4.089411764705883
```

Answer 7.13: (A) 180

```
life = SULT()
V = life.FPT_policy_value(40, t=10, n=30, endowment=1000, b=1000)
soa(180, V, 7.13)
```

```
SOA Question 7.13: [ 180 ] 180.1071785904076
```

SOA Question 7.14: (A) 2200

```
x = 45
life = Recursion(interest=dict(i=0.05)).set_q(0.009, x=50)
life.set_reserves(T=10, V={5: 5500})
def fun(P):  # solve for net premium, from year 6 reserve
    return life.t_V(x=x, t=6, premium=P*0.96 - 50,
                    benefit=lambda t: 100000 + 200)
P = life.solve(fun, target=7100, guess=[2200, 2400])
soa(2200, P, 7.14)
```

```
SOA Question 7.14: [ 2200 ] 2197.8174603174602
```

SOA Question 7.15: (E) 50.91

```
x = 0
life = Recursion(udd=True, interest=dict(i=0.05)).set_q(0.1, x=x+15)
life.set_reserves(T=3, V={16: 49.78})
V = life.r_V_forward(x, s=15, r=0.6, benefit=100)
soa(50.91, V, 7.15)
```

```
SOA Question 7.15: [ 50.91 ] 50.91362826922369
```

SOA Question 7.16: (D) 380

```
life = Select(interest=dict(v=.95), A={86: [683/1000]},
              q={80+k: [.01*(k+1)] for k in range(6)}).fill()
x, t, n = 80, 3, 5
A = life.whole_life_insurance(x+t)
a = life.temporary_annuity(x+t, t=n-t)
V = life.gross_future_loss(A=A, a=a,
                           policy=life.Policy(benefit=1000, premium=130))
soa(380, V, 7.16)
```

```
SOA Question 7.16: [ 380 ] 381.6876905200001
```

SOA Question 7.17: (D) 1.018

```
x = 0
life = Recursion(interest=dict(v=math.sqrt(0.90703)))
life.set_q(0.02067, x=x+10)
life.set_A(0.52536, x=x+11)
life.set_A(0.30783, x=x+11, moment=2)
A1 = life.whole_life_insurance(x+10)
A2 = life.whole_life_insurance(x+10, moment=2)
ratio = (life.insurance_variance(A2=A2, A1=A1)
            / life.insurance_variance(A2=0.30783, A1=0.52536))
soa(1.018, ratio, 7.17)
```

```
[ Whole Life Insurance: A_10 ]
    forward: A_10 = qv + pvA_11
[ Whole Life Insurance: A_10 ]
    forward: A_10 = qv + pvA_11
SOA Question 7.17: [ 1.018 ] 1.0182465434445054
```

SOA Question 7.18: (A) 17.1

```
x = 10
life = Recursion(interest=dict(i=0.04)).set_q(0.009, x=x)
def fun(a):
    return life.set_a(a, x=x).net_policy_value(x, t=1)
a = life.solve(fun, target=0.012, guess=[17.1, 19.1])
soa(17.1, a, 7.18)
```

```
SOA Question 7.18: [ 17.1 ] 17.07941929974385
```

SOA Question 7.19: (D) 720

```
life = SULT()
policy = life.Policy(benefit=100000, initial_policy=300, initial_premium=.5,
                        renewal_premium=.1)
P = life.gross_premium(A=life.whole_life_insurance(40),
                        **policy.premium_terms)
A = life.whole_life_insurance(41)
a = life.immediate_annuity(41)   # after premium and expenses are paid
V = life.gross_future_loss(A=A, a=a, policy=policy.set(premium=P).future)
soa(720, V, 7.19)
```

```
SOA Question 7.19: [ 720 ] 722.7510208759086
```

SOA Question 7.20: (E) -277.23

```
life = SULT()
S = life.FPT_policy_value(35, t=1, b=1000)  # is 0 for FPT at t=0,1
policy = life.Policy(benefit=1000, initial_premium=.3, initial_policy=300,
                        renewal_premium=.04, renewal_policy=30)
P = life.gross_premium(A=life.whole_life_insurance(35),
```

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```
                        **policy.premium_terms)
R = life.gross_policy_value(35, t=1, policy=policy.set(premium=P))
soa(-277.23, R - S, "7.20")
```

```
SOA Question 7.20: [ -277.23 ] -277.19303323929216
```

SOA Question 7.21: (D) 11866

```
life = SULT()
x, t, u = 55, 9, 10
P = life.gross_premium(IA=0.14743, a=life.temporary_annuity(x, t=u),
                       A=life.deferred_annuity(x, u=u), benefit=1000)
policy = life.Policy(initial_policy=life.term_insurance(x+t, t=1, b=10*P),
                       premium=P, benefit=1000)
a = life.temporary_annuity(x+t, t=u-t)
A = life.deferred_annuity(x+t, u=u-t)
V = life.gross_future_loss(A=A, a=a, policy=policy)
soa(11866, V, 7.21)
```

```
SOA Question 7.21: [ 11866 ] 11866.30158100453
```

SOA Question 7.22: (C) 46.24

```
life = PolicyValues(interest=dict(i=0.06))
policy = life.Policy(benefit=8, premium=1.250)
def fun(A2):
    return life.gross_variance_loss(A1=0, A2=A2, policy=policy)
A2 = life.solve(fun, target=20.55, guess=20.55/8**2)
policy = life.Policy(benefit=12, premium=1.875)
var = life.gross_variance_loss(A1=0, A2=A2, policy=policy)
soa(46.2, var, 7.22)
```

```
SOA Question 7.22: [ 46.2 ] 46.2375
```

SOA Question 7.23: (D) 233

```
life = Recursion(interest=dict(i=0.04)).set_p(0.995, x=25)
A = life.term_insurance(25, t=1, b=10000)
def fun(beta):  # value of premiums in first 20 years must be equal
    return beta * 11.087 + (A - beta)
beta = life.solve(fun, target=216 * 11.087, guess=[140, 260])
soa(233, beta, 7.23)
```

```
[ Term Insurance: A_25^1:1 ]
    endowment insurance - pure endowment = A_25^1:1
    pure endowment 1_E_25 = 1_p_25 * v^1
SOA Question 7.23: [ 233 ] 232.64747466274176
```

SOA Question 7.24: (C) 680

```
life = SULT()
P = life.premium_equivalence(A=life.whole_life_insurance(50), b=1000000)
soa(680, 11800 - P, 7.24)
```

```
SOA Question 7.24: [ 680 ] 680.291823645397
```

SOA Question 7.25: (B) 3947.37

```
life = Select(interest=dict(i=.04), A={55: [.23, .24, .25],
                                       56: [.25, .26, .27],
                                       57: [.27, .28, .29],
                                       58: [.20, .30, .31]})
V = life.FPT_policy_value(55, t=3, b=100000)
soa(3950, V, 7.25)
```

```
SOA Question 7.25: [ 3950 ] 3947.3684210526353
```

SOA Question 7.26: (D) 28540 – backward-forward reserve recursion

```
x = 0
life = Recursion(interest=dict(i=.05)).set_p(0.85, x=x).set_p(0.85, x=x+1)
life.set_reserves(T=2, endowment=50000)
benefit = lambda k: k*25000
def fun(P):  # solve P s.t. V is equal backwards and forwards
    policy = dict(t=1, premium=P, benefit=benefit, reserve_benefit=True)
    return life.t_V_backward(x, **policy) - life.t_V_forward(x, **policy)
P = life.solve(fun, target=0, guess=[27650, 28730])
soa(28540, P, 7.26)
```

```
SOA Question 7.26: [ 28540 ] 28542.392566782808
```

SOA Question 7.27: (B) 213

```
x = 0
life = Recursion(interest=dict(i=0.03)).set_q(0.008, x=x)
life.set_reserves(V={0: 0})
def fun(G):  # Solve gross premium from expense reserves equation
    return life.t_V(x=x, t=1, premium=G-187, benefit=lambda t: 0,
                                per_policy=10 + 0.25*G)
G = life.solve(fun, target=-38.70, guess=[200, 252])
soa(213, G, 7.27)
```

```
SOA Question 7.27: [ 213 ] 212.970355987055
```

SOA Question 7.28: (D) 24.3

```
life = SULT()
PW = life.net_premium(65, b=1000)    # 20_V=0 => P+W is net premium for A_65
P = life.net_premium(45, t=20, b=1000)   # => P is net premium for A_45:20
soa(24.3, PW - P, 7.28)
```

```
SOA Question 7.28: [ 24.3 ] 24.334725400123975
```

SOA Question 7.29: (E) 2270

```
x = 0
life = Recursion(interest=dict(i=0.04)).set_a(14.8, x=x)\
                                       .set_a(11.4, x=x+10)
def fun(B):   # Solve for benefit B given net 10_V = 2290
    return life.net_policy_value(x, t=10, b=B)
B = life.solve(fun, target=2290, guess=2290*10)
policy = life.Policy(initial_policy=30, renewal_policy=5, benefit=B)
G = life.gross_premium(a=life.whole_life_annuity(x), **policy.premium_terms)
V = life.gross_policy_value(x, t=10, policy=policy.set(premium=G))
soa(2270, V, 7.29)
```

```
SOA Question 7.29: [ 2270 ] 2270.743243243244
```

SOA Question 7.30: (E) 9035

```
policy = SULT.Policy(premium=0, benefit=10000)  # premiums=0 after t=10
L = SULT().gross_policy_value(35, policy=policy)
V = SULT(interest=dict(i=0)).gross_policy_value(35, policy=policy) # 10000
soa(9035, V-L, "7.30")
```

```
SOA Question 7.30: [ 9035 ] 9034.654127845053
```

SOA Question 7.31: (E) 0.310

```
x = 0
life = Reserves().set_reserves(T=3)
G = 368.05
def fun(P):  # solve net premium from final year expense reserve equation
    return life.t_V(x=x, t=2, premium=G-P, benefit=lambda t: 0,
                              per_policy=5 + .08*G)
P = life.solve(fun, target=-23.64, guess=[.29, .31]) / 1000
soa(0.310, P, 7.31)
```

```
SOA Question 7.31: [ 0.31 ] 0.309966
```

SOA Question 7.32: (B) 1.4

```
life = PolicyValues(interest=dict(i=0.06))
policy = life.Policy(benefit=1, premium=0.1)
def fun(A2):
    return life.gross_variance_loss(A1=0, A2=A2, policy=policy)
A2 = life.solve(fun, target=0.455, guess=0.455)
policy = life.Policy(benefit=2, premium=0.16)
variance = life.gross_variance_loss(A1=0, A2=A2, policy=policy)
soa(1.39, variance, 7.32)
```

```
SOA Question 7.32: [ 1.39 ] 1.3848168384380901
```

```python
from datetime import datetime
print(datetime.now())
soa.summary()
```

```
2022-11-07 06:59:42.501415
```

```
   num  correct
0  136      136
2    8        8
3   14       14
4   19       19
5    9        9
6   54       54
7   32       32
```