# Probability Theory and Mathematical Statistics Notes

## Contents

# 1 Chapter 1: Basic Probability

## 1.1 Set Operations and Logic

- **Complement/Inverse:** $A \cup \overline{A} = S$ and $A\overline{A} = \emptyset$.
- **Circuit Systems:**
  - Series connection corresponds to intersection: $\bigcap A_i$.
  - Parallel connection corresponds to union: $\bigcup A_i$.
- **Difference Formulas:**
  - If $A \subset B$, then $P(B - A) = P(B) - P(A)$.
  - In general, $P(B - A) = P(B) - P(AB)$.

## 1.2 Probability Properties and Inclusion-Exclusion

**Example:** Persons A, B, and C arrive. The probability of each arriving is 0.4. The probability that at least 2 arrive is 0.3. The probability that all arrive is 0.05. Find the probability that at least 1 arrives.

**Solution:** Given:

$$P(A) = P(B) = P(C) = 0.4$$
$$P(AB \cup BC \cup AC) = 0.3$$
$$P(ABC) = 0.05$$

Using the relationship for the union of intersections:

$$P(AB \cup BC \cup AC) = P(AB) + P(BC) + P(AC) - 2P(ABC)$$

$$0.3 = \sum P(AB) - 2(0.05) \implies \sum P(AB) = 0.4$$

To find the probability of at least 1 person arriving ($P(A \cup B \cup C)$):

$$P(A \cup B \cup C) = \sum P(A) - \sum P(AB) + P(ABC)$$

$$P(A \cup B \cup C) = (0.4 + 0.4 + 0.4) - 0.4 + 0.05$$
$$P(A \cup B \cup C) = 1.2 - 0.4 + 0.05 = 0.85$$

## 1.3 Classical Probability (Sampling)

**Example:** There are 8 balls: 3 red and 5 yellow. Draw 2 balls without replacement. What is the probability of drawing 1 red and 1 yellow?

**Solution:** Using Combinations:

$$P = \frac{C_1^3 C_1^5}{C_2^8} = \frac{3 \times 5}{28} = \frac{15}{28}$$

Alternatively, using the Multiplication Rule (considering order then summing):

$$P = 2 \times \frac{3}{8} \times \frac{5}{7} = \frac{15}{28}$$

## 1.4 The Matching Problem

**Example:** There are $n$ people and $n$ numbers. What is the probability that no person matches their number (derangement)?

**Formula:**

$$P = \sum_{k=0}^{n} \frac{(-1)^k}{k!}$$

## 1.5 Conditional Probability and Independence

**Definition:**

$$P(B|A) = \frac{P(AB)}{P(A)}$$

**Total Probability Theorem:**

$$P(A) = \sum P(AB_j) = \sum P(B_j)P(A|B_j)$$

**Bayes' Theorem:**

$$P(B_i|A) = \frac{P(B_i)P(A|B_i)}{\sum_j P(B_j)P(A|B_j)}$$

**Example:** 10 items total, 8 good, 2 defective. Let $A_i$ be the event that the $i$-th draw is good.

- If sampling without replacement: $P(A_2|A_1) = \frac{7}{9}$.
- Note: The joint probability is $\frac{8}{10} \times \frac{7}{9}$, but since condition $A_1$ is assumed, we only look at the second stage.

**Independence/Chain Rule:**

$$P(A_{i_1}...A_{i_k}) = \prod_{j=1}^{k} P(A_{i_j})$$

if mutually independent.

## 1.6 Calculations with Set Operations

**Example:** Given $P(B) = 0.4$, $P(A \cup B) = 0.5$. Find $P(A|\overline{B})$.

**Solution:**

$$P(A|\overline{B}) = \frac{P(A \cap \overline{B})}{P(\overline{B})} = \frac{P(A) - P(AB)}{1 - P(B)}$$

First, find $P(A) - P(AB)$ using the addition rule:

$$P(A \cup B) = P(A) + P(B) - P(AB)$$

$$0.5 = (P(A) - P(AB)) + 0.4$$
$$P(A) - P(AB) = 0.1$$

Substituting back:

$$P(A|\overline{B}) = \frac{0.1}{1 - 0.4} = \frac{0.1}{0.6} = \frac{1}{6}$$

**Note:** If A and B were independent, then $P(\overline{B}|A) = P(\overline{B}) = 0.6$.

## 1.7 Sampling Example (Urn Problem)

**Example:** An urn contains 5 White and 6 Black balls (Total 11). Draw 2 balls.

1. Find the probability of 1 White and 1 Black.
2. Find the probability of at least 1 Black.

**Solution:** 1. One of each:

$$\frac{C_1^5 C_1^6}{C_2^{11}}$$

2. At least one black (1 minus probability of all white):

$$1 - \frac{C_2^5}{C_2^{11}}$$

## 1.8 Two-Urn Transfer Problem (Total Probability & Bayes)

**Example:**

- Urn A: 5 Red, 4 White.
- Urn B: 4 Red, 5 White.
- **Process:** Draw 1 ball from A, put it into B. Then draw 1 ball from B.

**Questions:** 1. What is the probability that the ball drawn from B is White? 2. If the ball drawn from B is White, what is the probability that the ball transferred from A was White?
**Solution:**
1. Let $W_B$ be the event B draws White. Let $W_A$ be transfer White, $R_A$ be transfer Red.

$$P(W_B) = P(R_A)P(W_B|R_A) + P(W_A)P(W_B|W_A)$$

$$P(W_B) = \frac{5}{9} \times \frac{5}{10} + \frac{4}{9} \times \frac{6}{10}$$

(Note: Denominator 10 implies B has 9 balls initially + 1 transferred).
2. Bayes' Theorem: Find $P(W_A|W_B)$.

$$P(W_A|W_B) = \frac{P(W_A \cap W_B)}{P(W_B)} = \frac{\frac{4}{9} \times \frac{6}{10}}{\text{Result from (1)}}$$

# 2 Chapter 2: Random Variables and Distributions

## 2.1 Discrete Random Variables and Common Distributions

1. **0-1 Distribution (Bernoulli Distribution):** $X \sim B(1, p)$ or $X \sim 0 - 1(p)$.

$$P(X = k) = p^k (1 - p)^{1-k}, \quad k = 0, 1$$

2. **Binomial Distribution:** $X \sim B(n, p)$.

$$P(X = k) = C_k^n p^k (1 - p)^{n-k}, \quad k = 0, 1, \ldots, n$$

3. **Poisson Distribution:** $X \sim P(\lambda)$.

$$P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}, \quad k = 0, 1, 2, \ldots, \quad (\lambda > 0)$$

**Poisson Approximation:** When $n$ is large ($> 10$) and $p$ is small ($< 0.1$), the Binomial distribution can be approximated by Poisson with $\lambda = np$:

$$C_k^n p^k (1 - p)^{n-k} \approx \frac{\lambda^k e^{-\lambda}}{k!}$$

4. **Geometric Distribution:** $X \sim G(p)$. Represents the number of trials to get the first success.

$$P(X = k) = p(1 - p)^{k-1}, \quad k = 1, 2, 3, \ldots$$

5. **Pascal Distribution (Negative Binomial):** Waiting for the $r$-th success.

$$P(X = k) = C_{r-1}^{k-1} p^r (1 - p)^{k-r}, \quad k = r, r + 1, \ldots$$

## 2.2 Distribution Functions (CDF) and Probability Density Functions (PDF)

### 2.2.1 Cumulative Distribution Function (CDF)

Defined as $F(x) = P(X \leq x)$.

- Property: $0 \leq F(x) \leq 1$.
- For discrete variables: $F(x) = \sum_{x_k \leq x} P(X = x_k)$.

### 2.2.2 Probability Density Function (PDF)

For continuous variables, $F(x) = \int_{-\infty}^{x} f(t)dt$.

**Example:** Given a PDF:

$$f(x) = \begin{cases} 1/3 & 0 < x < 1 \\ 2/9 & 3 < x < 6 \\ 0 & \text{others} \end{cases}$$

Calculate $F(x)$:

$$F(x) = \begin{cases} 0 & x < 0 \\ \int_0^x \frac{1}{3}dt = \frac{x}{3} & 0 \le x < 1 \\ 1/3 & 1 \le x < 3 \quad \text{(Value holds constant in gap)} \\ 1/3 + \int_3^x \frac{2}{9}dt & 3 \le x < 6 \\ 1 & x \ge 6 \end{cases}$$

## 2.3 Continuous Random Variables and Common Distributions

1. **Uniform Distribution:** $X \sim U(a, b)$.

$$f(x) = \begin{cases} \frac{1}{b-a} & a < x < b \\ 0 & \text{others} \end{cases}$$

2. **Exponential Distribution:** $X \sim E(\lambda)$.

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x > 0 \\ 0 & x \le 0 \end{cases}$$

3. **Normal (Gaussian) Distribution:** $X \sim N(\mu, \sigma^2)$.

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad -\infty < x < +\infty$$

Curve is symmetric about $x = \mu$.

## 2.4 Selected Examples

### 2.4.1 Discrete Probability Calculation

**Example:** Roll a dice twice. Let $X$ be the smaller number of points. Find the distribution law of $X$.
**Solution:** Total outcomes = 36.

- $X = 1$: (1 with 1..6) + (2..6 with 1). Total 11. $P = 11/36$.
- $X = 2$: (2 with 2..6) + (3..6 with 2). Total 9. $P = 9/36$.
- Pattern continues ($2n - 1$ logic decreasing).

Distribution Table:

| $X$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $P$ | 11/36 | 9/36 | 7/36 | 5/36 | 3/36 | 1/36 |

### 2.4.2 From CDF to Probabilities

**Example:** Given $X$ has CDF:

$$F(x) = \begin{cases} 0 & x < 1 \\ \ln x & 1 \le x < e \\ 1 & x \ge e \end{cases}$$

1. Find $f(x)$. 2. Find $P(X < 2)$ and $P(0 < X \le 3)$.

**Solution:** 1. $f(x) = F'(x) = \frac{1}{x}$ for $1 \le x < e$, and 0 otherwise. 2. $P(X < 2) = F(2) = \ln 2$. $P(0 < X \le 3) = F(3) - F(0)$. Note that since max range is $e \approx 2.718$, $F(3) = 1$. $P(0 < X \le 3) = 1 - 0 = 1$. Alternatively via integration: $\int_1^e \frac{1}{x}dx = \ln e - \ln 1 = 1$.

### 2.4.3 Finding Constants in CDF

**Example:** $f(x) = 2e^{-2x}$ for $x \geq 0$. Find $F(x)$.

$$F(x) = \int_0^x 2e^{-2t} dt = -e^{-2t}\Big|_0^x = 1 - e^{-2x}$$

Using boundary conditions $F(+\infty) = 1$, the integration constant resolves.
**Example:** Continuity of CDF.

$$f(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & others \end{cases}$$

$$F(x) = \begin{cases} C_1 & x \leq 0 \\ x + C_2 & 0 < x < 1 \\ C_3 & x > 1 \end{cases}$$

Since $F$ must be a valid CDF:

- $x \leq 0 \implies F(x) = 0 \implies C_1 = 0$.
- $x > 1 \implies F(x) = 1 \implies C_3 = 1$.
- Continuity at $x = 0$: $\lim_{x\to 0^-} 0 = \lim_{x\to 0^+}(0 + C_2) \implies C_2 = 0$.

### 2.4.4 Quadratic Equation Probability

**Example:** $X \sim U(0, 5)$ (implied). What is the probability that the equation $4t^2 + 4Xt + (X + 2) = 0$ has real solutions for $t$?

**Solution:** Discriminant $\Delta \geq 0$:

$$(4X)^2 - 4(4)(X + 2) \geq 0$$
$$16X^2 - 16(X + 2) \geq 0 \implies X^2 - X - 2 \geq 0$$
$$(X - 2)(X + 1) \geq 0$$

Solution implies $X \geq 2$ or $X \leq -1$. Given domain $X \in (0, 5)$: We look for $P(X \geq 2)$. For Uniform distribution $U(0, 5)$, $f(x) = 1/5$.

$$P(X \geq 2) = \frac{5 - 2}{5 - 0} = \frac{3}{5}$$

### 2.4.5 Normal Distribution Symmetry

**Example:** $X \sim N(2, \sigma^2)$. Given $P(2 < X < 4) = 0.3$. Find $P(X < 0)$.

**Solution:** Due to symmetry about $\mu = 2$: $P(0 < X < 2) = P(2 < X < 4) = 0.3$. Total probability is 1, so half probability is 0.5.

$$P(X < 2) = 0.5$$
$$P(X < 0) + P(0 < X < 2) = 0.5$$
$$P(X < 0) + 0.3 = 0.5 \implies P(X < 0) = 0.2$$

## 2.5 Functions of Random Variables

### 2.5.1 Discrete Case

**Example:** $X$ has distribution:

| $X$ | -2 | -1 | 0 | 1 | 3 |
|---|---|---|---|---|---|
| $P$ | 1/5 | 1/6 | 1/5 | 1/15 | 11/30 |

Find distribution of $Y = X^2$. Possible values for $Y$: 0, 1, 4.

- $P(Y = 0) = P(X = 0) = 1/5$.
- $P(Y = 1) = P(X = -1) + P(X = 1) = 1/6 + 1/15 = 5/30 + 2/30 = 7/30$.
- $P(Y = 4) = P(X = -2) = 1/5$.
- $P(Y = 9) = P(X = 3) = 11/30$.

### 2.5.2 Continuous Case

**Example:** $X \sim U(0,2)$. Find the PDF of $Y = X^3$. $f_X(x) = 1/2$ for $0 < x < 2$. Range of $Y$: $(0,8)$.
CDF Method:

$$F_Y(y) = P(Y \leq y) = P(X^3 \leq y) = P(X \leq y^{1/3})$$

For $y < 0$:

$$F_Y(y) = \int_0^{y^{1/3}} f(x)dx = 0$$

For $0 < y < 8$:

$$F_Y(y) = \int_0^{y^{1/3}} \frac{1}{2}dx = \frac{1}{2}y^{1/3}$$

For $8 < y$:

$$F_Y(y) = \int_0^2 \frac{1}{2}dx = 1$$

Differentiating to find PDF:

$$f_Y(y) = F_Y'(y) = \frac{1}{2} \cdot \frac{1}{3}y^{-2/3} = \frac{1}{6\sqrt[3]{y^2}}, \quad 0 < y < 8$$

# 3 Chapter 3: Multidimensional Random Variables

## 3.1 Definitions and Basic Properties

### 3.1.1 Joint Distribution Function (CDF)

The joint cumulative distribution function for random variables $X$ and $Y$ is defined as:

$$F(x,y) = P(X \leq x, Y \leq y)$$

### 3.1.2 Marginal Distribution Functions

$$F_X(x) = F(x, +\infty) = \lim_{y \to \infty} F(x,y)$$

$$F_Y(y) = F(+\infty, y) = \lim_{x \to \infty} F(x,y)$$

### 3.1.3 Conditional Distribution (Discrete Case)

$$P(X = x_i | Y = y_j) = \frac{P(X = x_i, Y = y_j)}{P(Y = y_j)}$$

### 3.1.4 Joint Probability Density Function (PDF)

For continuous variables, the joint PDF $f(x,y)$ satisfies:

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x,y)dxdy = 1$$

### 3.1.5 Marginal PDFs

$$f_X(x) = \int_{-\infty}^{+\infty} f(x,y)dy$$

$$f_Y(y) = \int_{-\infty}^{+\infty} f(x,y)dx$$

### 3.1.6 Conditional PDFs

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}$$

### 3.1.7 Independence

$X$ and $Y$ are independent if and only if:

$$f(x, y) = f_X(x) f_Y(y)$$

(Or for discrete: $P(X = x, Y = y) = P(X = x)P(Y = y)$ for all pairs).

## 3.2 Discrete Example: Determination of Parameters

**Example** Given the joint probability mass function (PMF) of discrete random variables $X$ and $Y$:

$$P(Y \leq 0 \mid X < 2) = 0.5$$

The joint PMF is presented in the table below:

| $X \setminus Y$ | -1 | 0 | 1 |
|---|---|---|---|
| 1 | a | 0.2 | 0.2 |
| 2 | 0.1 | 0.1 | b |

**Solution:** The total probability must sum to 1:

$$P(X = 1, Y = -1) + P(X = 1, Y = 0) + \cdots + P(X = 2, Y = 1) = 1$$

$$a + 0.2 + 0.2 + 0.1 + 0.1 + b = 1 \implies a + b + 0.6 = 1 \implies a + b = 0.4 \quad \text{(Mistake in image: } a + b = 1)$$

Assuming the formula in the image is correct: $a + b = 1$ implies a different set of probabilities must be in the table.

Let's use the given conditional probability $P(Y \leq 0 \mid X < 2) = 0.5$.

- **Event** $B$**:** $X < 2$ $P(B) = P(X = 1) = P(X = 1, Y = -1) + P(X = 1, Y = 0) + P(X = 1, Y = 1) = a + 0.2 + 0.2 = a + 0.4$
- **Event** $A \cap B$**:** $Y \leq 0$ **and** $X < 2$ $P(A \cap B) = P(X = 1, Y \leq 0) = P(X = 1, Y = -1) + P(X = 1, Y = 0) = a + 0.2$

Using the conditional probability formula:

$$P(Y \leq 0 \mid X < 2) = \frac{P(Y \leq 0 \cap X < 2)}{P(X < 2)} = \frac{a + 0.2}{a + 0.4}$$

We are given that this value is 0.5. Thus, we have the equation:

$$\frac{a + 0.2}{a + 0.4} = 0.5$$

Solving for $a$:

$$a + 0.2 = 0.5(a + 0.4)a + 0.2 = 0.5a + 0.20.5a = 0a = 0$$

Since $a = 0$ and the sum of all probabilities must be 1:

$$a + 0.2 + 0.2 + 0.1 + 0.1 + b = 10 + 0.6 + b = 1b = 0.4$$

Therefore, the values are $a = 0$ and $b = 0.4$.

## 3.3 Discrete Example: Balls in Boxes

**Problem:** Place 2 balls randomly into 4 boxes (numbered 1, 2, 3, 4). Let $X$ be the number of balls in box 1. Let $Y$ be the number of balls in box 2.

**1. Joint Distribution Law**

| $X \setminus Y$ | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 1/4 | 4/16 | 1/16 |
| 1 | 4/16 | 2/16 | 0 |
| 2 | 1/16 | 0 | 0 |

**2. Marginal Distributions**

| $X$ | 0 | 1 | 2 |
|---|---|---|---|
| $P$ | $\frac{9}{16}$ | $\frac{6}{16}$ | $\frac{1}{16}$ |

| $Y$ | 0 | 1 | 2 |
|---|---|---|---|
| $P$ | $\frac{9}{16}$ | $\frac{6}{16}$ | $\frac{1}{16}$ |

**3. Conditional Probability**

| $X$ | 0 | 1 | 2 |
|---|---|---|---|
| $P(X \mid Y = 1)$ | $\frac{1/4}{3/8}$ | $\frac{1/8}{3/8}$ | $\frac{0}{3/8}$ |

**4. Independence** Check if $P_{ij} = P_i \cdot P_j$. For example: $P(X = 2) = 1/16$, $P(Y = 2) = 1/16$. $P(X = 2)P(Y = 2) = 1/256$. But $P(X = 2, Y = 2) = 0$. Since $1/256 \neq 0$, $X$ and $Y$ are **not independent**.

## 3.4 Continuous Example: Triangular Region

**Problem:** Given joint PDF:

$$f(x, y) = \begin{cases} kx & 0 \le x \le y \le 1 \\ 0 & \text{others} \end{cases}$$

**1. Find Constant k**

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x, y) dx dy = 1$$

$$\int_0^1 dy \int_0^y kx dx \quad \text{OR} \quad \int_0^1 dx \int_x^1 kx dy$$

Using the second order ($dy$ first):

$$\int_0^1 kx \left( \int_x^1 dy \right) dx = \int_0^1 kx(1 - x) dx = k \int_0^1 (x - x^2) dx$$

$$= k \left[ \frac{1}{2}x^2 - \frac{1}{3}x^3 \right]_0^1 = k \left( \frac{1}{2} - \frac{1}{3} \right) = \frac{k}{6}$$

$$\frac{k}{6} = 1 \implies k = 6$$

**2. Probability Calculation** Find $P(X + Y \le 1)$. Intersection of region $0 \le x \le y$ and $x + y \le 1$. The line $y = 1 - x$ intersects $y = x$ at $x = 1/2$.

$$P = \int_0^{1/2} dx \int_x^{1-x} 6x dy$$

$$= \int_0^{1/2} 6x(1 - x - x) dx = \int_0^{1/2} (6x - 12x^2) dx$$

$$= \left[ 3x^2 - 4x^3 \right]_0^{1/2} = 3(1/4) - 4(1/8) = \frac{3}{4} - \frac{1}{2} = \frac{1}{4}$$

**3. Marginal PDFs**

$$f_X(x) = \int_x^1 6x dy = 6x(1 - x), \quad 0 < x < 1$$

$$f_Y(y) = \int_0^y 6x dx = 6 \left[ \frac{x^2}{2} \right]_0^y = 3y^2, \quad 0 < y < 1$$

**4. Conditional PDF**

$$f_{Y|X}(y|x) = \frac{f(x, y)}{f_X(x)} = \frac{6x}{6x(1 - x)} = \frac{1}{1 - x}, \quad x < y < 1$$

**5. Independence** Since $f(x, y) \neq f_X(x) f_Y(y)$ (or simply because the domain is coupled), they are **not independent**.

### 3.5 Example: Finding the PDF of the Difference of Two Random Variables

**Given** the joint probability density function (PDF) of the random variables $(X, Y)$:

$$f(x,y) = \begin{cases} \frac{1}{2}e^{-y/2}, & 0 < x < 1, y > 0 \\ 0, & \text{otherwise} \end{cases}$$

**Find** the cumulative distribution function (CDF) $F_Z(z)$ and the probability density function (PDF) $f_Z(z)$ of $Z = X - Y$.

1. Finding the Cumulative Distribution Function $F_Z(z)$

The CDF is $F_Z(z) = P\{Z \le z\} = P\{Y \ge X - z\}$. The domain of $f(x,y)$ is $D = \{(x,y) \mid 0 < x < 1, y > 0\}$.

1. Case 1: $z < 0$

$$F_Z(z) = \int_0^1 \left( \int_{x-z}^{\infty} \frac{1}{2} e^{-y/2}\, dy \right) dx$$

$$F_Z(z) = \int_0^1 e^{-(x-z)/2}\, dx = e^{z/2} \left[ -2e^{-x/2} \right]_0^1$$

$$F_Z(z) = 2e^{z/2}(1 - e^{-1/2}), \quad z < 0$$

2. Case 2: $0 \le z < 1$ We use the complement $P\{Y < X - z\}$, where integration is over $z < x < 1$.

$$P\{Y < X - z\} = \int_z^1 \left( \int_0^{x-z} \frac{1}{2} e^{-y/2}\, dy \right) dx$$

$$P\{Y < X - z\} = \int_z^1 \left( 1 - e^{-(x-z)/2} \right) dx = \left[ x + 2e^{z/2}e^{-x/2} \right]_z^1$$

$$P\{Y < X - z\} = 2e^{(z-1)/2} - z - 1$$

Therefore:

$$F_Z(z) = 1 - P\{Y < X - z\} = z + 2 - 2e^{(z-1)/2}, \quad 0 \le z < 1$$

3. Case 3: $z \ge 1$ Since $x - z < 0$ for all $x \in (0,1)$, the condition $Y \ge X - z$ covers the entire domain $D$.

$$F_Z(z) = 1, \quad z \ge 1$$

4. Finding the Probability Density Function $f_Z(z)$ We differentiate $f_Z(z) = F_Z'(z)$:

1. Case 1: $z < 0$

$$f_Z(z) = \frac{d}{dz}\left[ 2e^{z/2}(1 - e^{-1/2}) \right] = (1 - e^{-1/2})e^{z/2}, \quad z < 0$$

2. Case 2: $0 < z < 1$

$$f_Z(z) = \frac{d}{dz}\left[ z + 2 - 2e^{(z-1)/2} \right] = 1 - e^{(z-1)/2}, \quad 0 < z < 1$$

3. Case 3: $z > 1$

$$f_Z(z) = \frac{d}{dz}[1] = 0, \quad z > 1$$

3. Conclusion

The PDF of $Z = X - Y$ is:

$$f_Z(z) = \begin{cases} (1 - e^{-1/2})e^{z/2}, & z < 0 \\ 1 - e^{(z-1)/2}, & 0 \le z < 1 \\ 0, & z \ge 1 \end{cases}$$

# 4 Chapter 4: Numerical Characteristics of Random Variables

## 4.1 Expectation (Mean)

The expectation $E(X)$ represents the probability reward.

- **Discrete Random Variable:**
$$E(X) = \sum x_i p_i$$
- **Continuous Random Variable:**
$$E(X) = \int_{-\infty}^{+\infty} x f(x) dx$$

### 4.1.1 Properties of Expectation

- **Linearity:**
$$E(c_0 + \sum c_i X_i) = c_0 + \sum c_i E(X_i)$$
- **Multiplication (Independence):** If $X_i$ are independent, then:
$$E(\prod X_i) = \prod E(X_i)$$

## 4.2 Variance

Variance $D(X)$ or $Var(X)$ is a measure of the degree of deviation.

$$Var(X) = E[(X - E(X))^2] = E(X^2) - (E(X))^2$$

### 4.2.1 Properties of Variance

If $X$ and $Y$ are independent random variables:

$$Var(aX + bY + c) = a^2 Var(X) + b^2 Var(Y)$$

## 4.3 Covariance and Correlation Coefficient

### 4.3.1 Covariance

$$Cov(X, Y) = E[(X - E(X))(Y - E(Y))] = E(XY) - E(X)E(Y)$$

### 4.3.2 Correlation Coefficient

$$\rho_{XY} = \frac{Cov(X, Y)}{\sqrt{Var(X)Var(Y)}}$$

## 4.4 Expectation and Variance of Common Distributions

| | $0-1$ | $B(n,p)$ | $P(\lambda)$ | $U(a,b)$ | $e(\lambda)$ | $N(\mu,\sigma^2)$ |
|---|---|---|---|---|---|---|
| $E(X)$ | $p$ | $np$ | $\lambda$ | $\frac{a+b}{2}$ | $\frac{1}{\lambda}$ | $\mu$ |
| $D(X)$ | $p(1-p)$ | $np(1-p)$ | $\lambda$ | $\frac{(b-a)^2}{12}$ | $\frac{1}{\lambda^2}$ | $\sigma^2$ |

## 4.5 Examples and Applications

### 4.5.1 Example 1: Variance of a Linear Combination

Given independent variables $X_1 \sim U(0,6)$, $X_2 \sim N(0,4)$, and $X_3 \sim P(3)$. Find $D(X_1 - 2X_2 + 3X_3)$.
**Solution:**

$$D(X_1) = \frac{(6-0)^2}{12} = 3, \quad D(X_2) = 4, \quad D(X_3) = 3$$

$$D(X_1 - 2X_2 + 3X_3) = D(X_1) + (-2)^2 D(X_2) + (3)^2 D(X_3)$$
$$= 3 + 4(4) + 9(3) = 3 + 16 + 27 = 46$$

### 4.5.2 Example 2: Finding $E(X^2)$ for a Binomial Variable

Given $X \sim B(10, 0.4)$. Find $E(X^2)$.
**Solution:**

$$E(X^2) = D(X) + (E(X))^2$$
$$E(X) = 10(0.4) = 4, \quad D(X) = 10(0.4)(0.6) = 2.4$$
$$E(X^2) = 2.4 + (4)^2 = 18.4$$

### 4.5.3 Example 3: Properties of Covariance

(1) $\mathrm{Cov}(X - Y, Y)$ (2) $\rho_{XY}$

$$\begin{array}{ccc|ccc|cccc} & X & & & Y & & & & XY & \\ 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 & 4 \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{7}{12} & \frac{1}{3} & 0 & \frac{1}{12} \end{array}$$

(1)

$$\begin{aligned} \mathrm{Cov}(X - Y, Y) &= \mathrm{Cov}(X, Y) - \mathrm{Cov}(Y, Y) \\ &= E(XY) - E(X)E(Y) - D(Y) \\ &= E(XY) - E(X)E(Y) - \underbrace{(E(Y^2) - E^2(Y))}_{\text{Variance } D(Y)} \end{aligned}$$

(2) 0, Uncorrelated

### 4.5.4 Example 4: Correlation for Joint PDF

Given the joint PDF $f(x, y) = \begin{cases} \frac{3}{4}x^2y & 0 \le x \le 2, 0 \le y \le 1 \\ 0 & \text{others} \end{cases}$. Find $\rho_{XY}$.
**Solution:**
$E(XY) = \int_0^1 dy \int_0^2 xy \left(\frac{3}{4}x^2y\right) dx = 1$
$E(X) = \int_0^1 dy \int_0^2 x \left(\frac{3}{4}x^2y\right) dx = \frac{3}{2}$
$E(Y) = \int_0^2 dx \int_0^1 y \left(\frac{3}{4}x^2y\right) dy = \frac{2}{3}$

$$Cov(X, Y) = E(XY) - E(X)E(Y) = 1 - \left(\frac{3}{2}\right)\left(\frac{2}{3}\right) = 0$$

Since $Cov(X, Y) = 0$, the variables are uncorrelated, thus $\rho_{XY} = 0$.

## 4.6 Chebyshev's Inequality

Chebyshev's inequality provides a bound on the probability deviation from the mean:

$$P(|X - E(X)| < \epsilon) \ge 1 - \frac{D(X)}{\epsilon^2}$$

(The complementary form: $P(|X - E(X)| \ge \epsilon) \le \frac{D(X)}{\epsilon^2}$)

### 4.6.1 Example: Applying Chebyshev's Inequality

Given $E(X) = 100$ and $D(X) = 50$. Find the lower bound for $P\{80 < X < 120\}$.
**Solution:** We seek $P(80 < X < 120) = P(|X - 100| < 20)$. Here, $\epsilon = 20$.

$$P(|X - E(X)| < 20) \ge 1 - \frac{D(X)}{\epsilon^2} = 1 - \frac{50}{20^2}$$
$$= 1 - \frac{50}{400} = 1 - \frac{1}{8} = \frac{7}{8}$$

The lower bound is $\frac{7}{8}$.

# 5 Chapter 5: Limit Theorems

## 5.1 Chebyshev's Inequality

Chebyshev's inequality provides a bound on the probability deviation from the mean:

$$P(|X - E(X)| \geq \epsilon) \leq \frac{Var(X)}{\epsilon^2}$$

or the complementary form:

$$P(|X - E(X)| < \epsilon) \geq 1 - \frac{Var(X)}{\epsilon^2}$$

## 5.2 Laws of Large Numbers (LLN)

### 5.2.1 Khinchine's Law of Large Numbers

Let $X_1, X_2, \ldots, X_n$ be a sequence of independent and identically distributed (i.i.d.) random variables, and $E(X_i) = \mu < \infty$. Then the sample mean converges in probability to the expectation:

$$\frac{1}{n}\sum_{i=1}^{n} X_i \to \mu, \quad \text{as } n \to \infty$$

## 5.3 Central Limit Theorem (CLT)

Let $X_1, X_2, \ldots, X_n$ be a sequence of independent and identically distributed (i.i.d.) random variables, with $E(X_i) = \mu$ and $Var(X_i) = \sigma^2$. If $n$ is large, the distribution of the sum and the sample mean can be approximated by a Normal distribution:

- **Distribution of the Sum:**

$$\sum_{i=1}^{n} X_i \sim N(n\mu, n\sigma^2)$$

- **Distribution of the Sample Mean:**

$$\frac{1}{n}\sum_{i=1}^{n} X_i \sim N(\mu, \frac{\sigma^2}{n})$$

The standardized random variable $Z$ approaches the standard normal distribution:

$$Z = \frac{X - \mu}{\sigma} \sim N(0, 1)$$

## 5.4 Applications of the Central Limit Theorem

### 5.4.1 Example 1: Sum of Exponential Lifetimes

Electronic components follow an Exponential distribution with a mean life of 100 hours. Randomly select 16 independent components. Find the probability that the total lifetime ($\sum X_i$) is greater than 1920 hours.

**Solution:** For an Exponential distribution $X \sim E(\lambda)$, we have $E(X) = 1/\lambda$ and $D(X) = 1/\lambda^2$. Given $E(X) = 100$, so $D(X) = 100^2 = 10000$. By the CLT, the sum of $n = 16$ independent random variables is approximately:

$$\sum_{i=1}^{16} X_i \sim N(n\mu, n\sigma^2) = N(16 \cdot 100, 16 \cdot 100^2) = N(1600, 160000)$$

We want to find $P(\sum X_i > 1920)$.

$$P(\sum X_i > 1920) = 1 - P(\sum X_i \leq 1920)$$

Standardizing the variable:

$$P(\sum X_i \leq 1920) \approx P\left(\frac{\sum X_i - 1600}{\sqrt{160000}} \leq \frac{1920 - 1600}{\sqrt{160000}}\right)$$

$$= P\left(Z \leq \frac{320}{400}\right) = P(Z \leq 0.8)$$

Using the standard normal CDF $\Phi(z)$:

$$P(\sum X_i > 1920) \approx 1 - \Phi(0.8)$$

#### 5.4.2 Example 2: Probability of Defects (Normal Approximation to Binomial)

100 components are produced, and each one has a 0.1 probability of being defective. Find the probability that the number of defective components is less than 15.

**Solution:** Let $X_i$ be 1 if the $i$-th component is defective, and 0 otherwise. $X_i \sim B(1, 0.1)$. The total number of defective components is $S = \sum_{i=1}^{100} X_i$.

$$E(S) = np = 100 \cdot 0.1 = 10$$

$$D(S) = np(1-p) = 100 \cdot 0.1 \cdot 0.9 = 9$$

Since $n = 100$ is large, we can use the CLT approximation: $S \sim N(10, 9)$. We want to find $P(S < 15)$. Standardizing the variable (without continuity correction):

$$P(S < 15) = P\left(\frac{S - 10}{\sqrt{9}} < \frac{15 - 10}{\sqrt{9}}\right)$$

$$= P\left(Z < \frac{5}{3}\right) = \Phi(1.67)$$

The probability is approximately $\Phi(1.67) \approx 0.9525$.

# 6 Chapter 6: Samples and Sampling Distributions

## 6.1 Sample Statistics

Let $X_1, X_2, \ldots, X_n$ be a sample of size $n$ from a population.

### 6.1.1 Definitions of Common Sample Statistics

- **Sample Mean:**
$$\overline{X} = \frac{1}{n} \sum X_i$$
- **Sample Variance:**
$$S^2 = \frac{1}{n-1} \sum (X_i - \overline{X})^2$$
- **Sample $k$-th Moment (Moment about the Origin):**
$$A_k = \frac{1}{n} \sum X_i^k$$
- **Sample Central $k$-th Moment:**
$$B_k = \frac{1}{n} \sum (X_i - \overline{X})^k$$

### 6.1.2 Estimation of Population Characteristics

When the population numerical characteristics are unknown, the following are used as their estimators:

- Sample mean $\overline{X}$ estimates population mean $\mu = E(X)$.
- Sample variance $S^2$ estimates population variance $\sigma^2 = E(X-\mu)^2$.
- Sample $k$-th moment $A_k$ estimates population $k$-th moment $\mu_k = E(X^k)$.
- Sample central $k$-th moment $B_k$ estimates population central $k$-th moment $\nu_k = E(X-\mu)^k$.

## 6.2 Important Sampling Distributions

### 6.2.1 Distribution of the Sample Mean

If $X_1, X_2, \ldots, X_n$ are samples from $N(\mu, \sigma^2)$, then the sample mean follows a Normal distribution:

$$\overline{X} \sim N(\mu, \frac{\sigma^2}{n})$$

### 6.2.2 Chi-Square ($\chi^2$) Distribution

- If $Y_1 \sim \chi^2(m)$ and $Y_2 \sim \chi^2(n)$ are independent, then:

$$Y_1 + Y_2 \sim \chi^2(m+n)$$

### 6.2.3 t-Distribution (Student's t-Distribution)

- If $X \sim N(0,1)$ and $Y \sim \chi^2(n)$ are independent, then the $T$-statistic follows a $t$-distribution with $n$ degrees of freedom, $T \sim t(n)$:

$$T = \frac{X}{\sqrt{Y/n}}$$

- The square of a $t$-distributed variable is an $F$-distributed variable:

$$t^2 \sim F(1,n)$$

### 6.2.4 F-Distribution

- If $X \sim \chi^2(n_1)$ and $Y \sim \chi^2(n_2)$ are independent, then the $F$-statistic follows an $F$-distribution with $(n_1, n_2)$ degrees of freedom, $F \sim F(n_1, n_2)$:

$$F = \frac{X/n_1}{Y/n_2}$$

- The inverse of an $F$-distributed variable is also $F$-distributed:

$$\frac{1}{F} \sim F(n_2, n_1)$$

# 7 Chapter 7: Parameter Estimation

## 7.1 Point Estimation

### 7.1.1 Method of Moments

The method of moments estimates the unknown parameters $\theta_1, \ldots, \theta_k$ by equating the first $k$ population moments to the corresponding sample moments.

1. **Population $l$-th Moment (Moment about the Origin):**

$$\mu_l = E(X^l) = \int_{-\infty}^{+\infty} X^l f(x; \theta_1, \ldots, \theta_k) dx$$

2. **Sample $l$-th Moment (Moment about the Origin):**

$$A_l = \frac{1}{n}\sum_{i=1}^{n} X_i^l$$

3. **Solving for Estimators:** Set $A_l = \mu_l(\theta_1, \dots, \theta_k)$ and solve the system of equations for the parameters $\hat{\theta}_1, \dots, \hat{\theta}_k$.

**Example: Moment Estimator for a Distribution Parameter** Given the PDF:

$$f(x) = \begin{cases} \sqrt{\theta}x^{\sqrt{\theta}-1} & 0 < x < 1 \\ 0 & \text{others} \end{cases}$$

Find the moment estimator $\hat{\theta}$.

**Solution:** Equate the first population moment $E(X)$ to the first sample moment $\overline{X}$:

$$E(X) = \int_0^1 x\left(\sqrt{\theta}x^{\sqrt{\theta}-1}\right)dx = \sqrt{\theta}\int_0^1 x^{\sqrt{\theta}}dx = \frac{\sqrt{\theta}X^{\sqrt{\theta}+1}}{\sqrt{\theta}+1}\Big|_0^1 = \frac{\sqrt{\theta}}{\sqrt{\theta}+1}$$

Setting $E(X) = \overline{X}$:

$$\overline{X} = \frac{\sqrt{\theta}}{\sqrt{\theta}+1}$$

Solving for $\sqrt{\theta}$:

$$\overline{X}(\sqrt{\theta}+1) = \sqrt{\theta} \implies \overline{X}\sqrt{\theta} + \overline{X} = \sqrt{\theta}$$

$$\overline{X} = \sqrt{\theta}(1-\overline{X}) \implies \sqrt{\theta} = \frac{\overline{X}}{1-\overline{X}}$$

The moment estimator is:

$$\hat{\theta} = \left(\frac{\overline{X}}{1-\overline{X}}\right)^2$$

### 7.1.2 Maximum Likelihood Estimation (MLE)

1. **Construct the Likelihood Function $L(\theta)$:**

$$L(\theta) = \prod_{i=1}^{n} f(x_i;\theta) \quad \text{(Continuous Case)}$$

$$L(\theta) = \prod_{i=1}^{n} P(X = x_i;\theta) \quad \text{(Discrete Case)}$$

2. **Take the Log-Likelihood Function $\ln L(\theta)$:**

3. **Calculate the First Derivative:**

$$\frac{d\ln L}{d\theta}$$

4. **Solve the Likelihood Equation:** Set $\frac{d\ln L}{d\theta} = 0$ and solve for $\hat{\theta}$.

## 7.2 Properties of Estimators

### 7.2.1 Unbiasedness and Efficiency

**Example: Comparing Unbiased Estimators** Let $X_1$ and $X_2$ be independent random samples from $N(\mu, 1)$. Consider two estimators for $\mu$:

$$\hat{\mu}_1 = \frac{1}{3}X_1 + \frac{2}{3}X_2 \quad \text{and} \quad \hat{\mu}_2 = \frac{1}{4}X_1 + \frac{3}{4}X_2$$

1. **Check for Unbiasedness:**

$$E(\hat{\mu}_1) = E\left(\frac{1}{3}X_1 + \frac{2}{3}X_2\right) = \frac{1}{3}E(X_1) + \frac{2}{3}E(X_2) = \frac{1}{3}\mu + \frac{2}{3}\mu = \mu$$

$$E(\hat{\mu}_2) = E\left(\frac{1}{4}X_1 + \frac{3}{4}X_2\right) = \frac{1}{4}E(X_1) + \frac{3}{4}E(X_2) = \frac{1}{4}\mu + \frac{3}{4}\mu = \mu$$

Both estimators are unbiased.

2. **Compare Efficiency (Variance):**

$$D(\hat{\mu}_1) = D\left(\frac{1}{3}X_1 + \frac{2}{3}X_2\right) = \left(\frac{1}{3}\right)^2 D(X_1) + \left(\frac{2}{3}\right)^2 D(X_2) = \frac{1}{9}(1) + \frac{4}{9}(1) = \frac{5}{9}$$

$$D(\hat{\mu}_2) = D\left(\frac{1}{4}X_1 + \frac{3}{4}X_2\right) = \left(\frac{1}{4}\right)^2 D(X_1) + \left(\frac{3}{4}\right)^2 D(X_2) = \frac{1}{16}(1) + \frac{9}{16}(1) = \frac{10}{16} = \frac{5}{8}$$

Since $D(\hat{\mu}_1) = 5/9 \approx 0.556$ and $D(\hat{\mu}_2) = 5/8 = 0.625$, we have $D(\hat{\mu}_1) < D(\hat{\mu}_2)$. Therefore, $\hat{\mu}_1$ is a **better** (more efficient) estimator.

## 7.3 Interval Estimation (Confidence Intervals)

For a confidence level of $1-\alpha$, the confidence interval is calculated using the following formulas, typically for a Normal population $X \sim N(\mu, \sigma^2)$.

### 7.3.1 Confidence Interval for the Population Mean $\mu$

- **$\sigma^2$ is Known:** Uses the $Z$ (Standard Normal) distribution.

$$\left(\overline{X} \pm \frac{\sigma}{\sqrt{n}} Z_{\alpha/2}\right)$$

- **$\sigma^2$ is Unknown:** Uses the $t$ (Student's t) distribution with $n-1$ degrees of freedom.

$$\left(\overline{X} \pm \frac{S}{\sqrt{n}} t_{\alpha/2}(n-1)\right)$$

### 7.3.2 Confidence Interval for the Population Variance $\sigma^2$

- **$\mu$ is Known:** Uses the $\chi^2$ (Chi-square) distribution with $n$ degrees of freedom.

$$\left(\frac{\sum_{i=1}^n (X_i-\mu)^2}{\chi^2_{\alpha/2}(n)}, \frac{\sum_{i=1}^n (X_i-\mu)^2}{\chi^2_{1-\alpha/2}(n)}\right)$$

- **$\mu$ is Unknown:** Uses the $\chi^2$ distribution with $n-1$ degrees of freedom, where $(n-1)S^2 = \sum (X_i - \overline{X})^2$.

$$\left(\frac{(n-1)S^2}{\chi^2_{\alpha/2}(n-1)}, \frac{(n-1)S^2}{\chi^2_{1-\alpha/2}(n-1)}\right)$$

**Example: Calculating Confidence Intervals**

$X \sim N(\mu, \sigma^2)$, sample size $n = 6$. Given observed sample variance $S^2$ corresponding to $S = 0.226$ . The confidence level is $1 - \alpha = 95\%$.

1. **Confidence Interval for $\mu$ ($\sigma^2$ unknown):** The calculation uses the $t$-distribution with $n-1 = 5$ degrees of freedom and $t_{0.025}(5)$.

$$\left(\overline{X} \pm \frac{S}{\sqrt{n}} t_{\alpha/2}(n-1)\right) = \left(14.95 \pm \frac{0.26}{\sqrt{6}} t_{0.025}(5)\right)$$

2. **Confidence Interval for $\sigma^2$ ($\mu$ unknown):** The calculation uses the $\chi^2$ distribution with $n-1 = 5$ degrees of freedom.

$$\left(\frac{5S^2}{\chi^2_{\alpha/2}(5)}, \frac{5S^2}{\chi^2_{1-\alpha/2}(5)}\right)$$

# 8 Chapter 8: Hypothesis Testing

## 8.1 Concepts and Structure of Hypothesis Testing

- Assumption $\rightarrow$ Sampling $\rightarrow$ Decision (Small probability of refusal)
- Left-sided test $H_0 : \theta \geq \theta_0$, $H_1 : \theta < \theta_0$
- Right-sided test $H_0 : \theta \leq \theta_0$, $H_1 : \theta > \theta_0$
- Two-sided test $H_0 : \theta = \theta_0$, $H_1 : \theta \neq \theta_0$

Table 1: Decision Table

| **Decision** | $H_0$ **True** | $H_0$ **False** |
|---|---|---|
| Reject $H_0$ | Type I Error ($\alpha$) | Correct Decision |
| Accept $H_0$ | Correct Decision | Type II Error ($\beta$) |

## 8.2 Inference for the Population Mean ($\mu$)

1. **Case 1: Population Variance $\sigma^2$ is Known ($\sigma^2$ known)**

$$Z = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} \sim N(0,1)$$

**Hypothesis Test (Critical Region):**

$$|Z| \geq Z_{\alpha/2}$$

2. **Case 2: Population Variance $\sigma^2$ is Unknown ($\sigma^2$ unknown)**

$$T = \frac{\bar{X} - \mu_0}{S/\sqrt{n}} \sim t(n-1)$$

**Hypothesis Test (Critical Region):**

$$|T| \geq t_{\alpha/2}(n-1)$$

## 8.3 Inference for the Population Variance ($\sigma^2$)

1. **Case 3: Population Mean $\mu$ is Known ($\mu$ known)**

$$W = \frac{\sum_{i=1}^{n}(X_i - \mu)^2}{\sigma_0^2} \sim \chi^2(n)$$

**Hypothesis Test (Confidence Interval Lower/Upper Bounds for $\sigma^2$):**

$$\chi^2_{1-\alpha/2}(n) \leq W \leq \chi^2_{\alpha/2}(n)$$

2. **Case 4: Population Mean $\mu$ is Unknown ($\mu$ unknown)**

$$V = \frac{(n-1)S^2}{\sigma_0^2} \sim \chi^2(n-1)$$

**Hypothesis Test (Confidence Interval Lower/Upper Bounds for $\sigma^2$):**

$$\chi^2_{1-\alpha/2}(n-1) \leq V \leq \chi^2_{\alpha/2}(n-1)$$

## 8.4 Examples

### 8.4.1 Example: Mean $\mu$ Test (Variance Known)

- **Problem:** The weight of salt in each bag $X \sim N(500, 15^2)$ grams. A sample of $n = 25$ bags is taken, with a sample mean $\overline{x} = 511.22$ grams. Is this normal? Use $\alpha = 0.05$.
- **Hypothesis:** $H_0 : \mu = \mu_0 = 500$, $H_1 : \mu \neq \mu_0$.
- **Test Statistic:** $Z = \frac{\overline{x}-\mu_0}{\sigma/\sqrt{n}} \sim N(0, 1)$.
- **Critical Value:** $Z_{\frac{\alpha}{2}} = 1.96$.
- **Rejection Region ($W$):** $W = (-\infty, -1.96) \cup (1.96, +\infty)$.
- **Calculation:**
$$Z = \frac{511.22 - 500}{15/\sqrt{25}} = 3.74$$
- **Decision:** Since $3.74 \in W$, reject $H_0$ (conclude $H_1$). The process is not normal.

### 8.4.2 Example: Variance $\sigma^2$ Test

- **Problem:** Required $\sigma_0^2 = 0.0004$. A sample of $n = 20$ bags has a sample variance $S = 0.003$. Is the standard met? Let $\alpha = 0.05$.
- **Hypothesis:** $H_0 : \sigma^2 = \sigma_0^2 = 0.0004$; $H_1 : \sigma^2 \neq \sigma_0^2$.
- **Test Statistic:** $\chi^2 = \frac{(n-1)S^2}{\sigma_0^2} \sim \chi^2(n-1)$.
- **Rejection Region ($W$):** $W = (0, \chi^2_{1-\frac{\alpha}{2}}(n-1)) \cup (\chi^2_{\frac{\alpha}{2}}(n-1), +\infty)$.
- **Critical Values ($n - 1 = 19, \alpha = 0.05$):**
  - $\chi^2_{0.975}(19) = 8.907$
  - $\chi^2_{0.025}(19) = 32.852$
- **Rejection Region ($W$):** $W = (0, 8.907) \cup (32.852, +\infty)$.
- **Calculation:**
$$\chi^2 = \frac{19 \times 0.003}{0.0004} = 142.5$$
- **Decision:** Since $142.5 \in W$, reject $H_0$ (conclude $H_1$). The standard is not met.