New Practice Problem

$X|\theta \sim \text{Poisson}(\theta)$; $\theta \sim \text{Gamma}(a, b)$ b = rate parameter

$$p(\theta) = \frac{b^a}{\Gamma(a)} \theta^{a-1} e^{-b\theta} \quad I(\theta > 0)$$

a.) Derive $p(\theta | x_{1:n})$, $x_1, \dots, x_n \overset{iid}{\sim} \text{Poisson}(\theta)$

$$p(\theta | x_{1:n}) \propto \boxed{p(x_{1:n} | \theta)}\, p(\theta)$$

$$\propto \left[\prod_{i=1}^{n} \frac{e^{-\theta} \theta^{x_i}}{x_i!} \right] \times \boxed{\frac{b^a}{p(a)}} \theta^{a-1} e^{-b\theta}$$

$$\propto e^{-n\theta} \theta^{\Sigma x_i} \theta^{a-1} e^{-b\theta}$$

$$\propto e^{-\theta(n+b)} \theta^{(\Sigma_i x_i + a - 1)} \propto \theta^{\Sigma x_i + a - 1} e^{-\theta(n+b)}$$

$$\Rightarrow \theta | x_{1:n} \sim \text{Gamma}\left(\sum_{i=1}^{n} x_i + a, \; n+b \right)$$

b.) Assume the squared error loss fn. Provide the Bayes rule (under the model defined above).

sq. error loss $\Rightarrow$ Bayes rule is posterior mean $E[\theta | x_{1:n}]$

$$E[\theta | x_{1:n}] = \frac{\sum_{i=1}^{n} x_i + a}{n+b} \quad \text{since} \quad E[\theta] = \frac{a}{b}.$$

c.) $p(\tilde{x} \mid x_{1:n}) = \int_{\Theta} \underbrace{p(\tilde{x}\mid\theta)} \underbrace{p(\theta \mid x_{1:n})}_{\text{Gamma}\left(\theta \,\middle|\, \boxed{a_n = a + \sum_{i=1}^{n} x_i,\ b_n = n+b}\right)} d\theta$

new observation

$$= \int_0^\infty \left[\frac{\theta^{\tilde{x}} e^{-\theta}}{\tilde{x}!}\right] \times \frac{b_n^{a_n}}{\Gamma(a_n)} \theta^{a_n - 1} e^{-b_n \theta}\, d\theta$$

(CANNOT DROP TERMS AS WE DO w/ POSTERIOR)

$$= \frac{b_n^{a_n}}{\tilde{x}!\,\Gamma(a_n)} \underbrace{\int_0^\infty \underbrace{\theta^{\tilde{x}+a_n-1} e^{-\theta(b_n+1)}\, d\theta}_{\text{kernel Gamma}(\theta \mid \tilde{x}+a_n,\ b_n+1)} \times \underbrace{\frac{(b_n+1)^{\tilde{x}+a_n}}{\Gamma(\tilde{x}+a_n)}}_{\text{normalizing constant}}}_{=1}$$

$$\times \frac{\Gamma(\tilde{x}+a_n)}{(b_n+1)^{\tilde{x}+a_n}}$$ (inverse of the normalizing constant)

$$= \frac{b_n^{a_n}\,\Gamma(\tilde{x}+a_n)}{\tilde{x}!\,\Gamma(a_n)\,(b_n+1)^{\tilde{x}+a_n}}$$

(want to show this is a dist^n)

$$= \frac{\Gamma(\tilde{x}+a_n)}{\tilde{x}!\,\Gamma(a_n)} \times b_n^{a_n} \left(\frac{1}{1+b_n}\right)^{\tilde{x}+a_n}$$

$$= \frac{\Gamma(\tilde{x}+a_n)}{\tilde{x}!\,\Gamma(a_n)} \left(\frac{b_n}{1+b_n}\right)^{a_n} \left(\left(\frac{1}{1+b_n}\right)^{\tilde{x}}\right)$$

$$= \frac{\Gamma(\tilde{x}+a_n)}{\tilde{x}!\,\Gamma(a_n)} \left(\frac{b_n}{1+b_n}\right)^{a_n} \left(1 - \frac{b_n}{1+b_n}\right)^{\tilde{x}}$$

$$\tilde{x} \mid x_{1:n} \sim \text{Negative Bin}\left(a_n,\ \frac{b_n}{b_n+1}\right).$$

☒