Practice Exercise 2023

Exam I Practice Problem

We write $X \sim \text{Poisson}(\theta)$ if X has the Poisson distribution with rate $\theta > 0$, that is, its p.m.f. is

$$p(x|\theta) = \text{Poisson}(x|\theta) = e^{-\theta}\theta^x/x!$$

for $x \in \{0, 1, 2, \dots\}$ (and is 0 otherwise). Suppose $X_1, \dots, X_n \overset{\text{iid}}{\sim} \text{Poisson}(\theta)$ given θ, and your prior is

$$p(\theta) = \text{Gamma}(\theta|a, b) = \frac{b^a}{\Gamma(a)}\theta^{a-1}e^{-b\theta}\mathbb{1}(\theta > 0).$$

1. **Derive** (step by step) $p(\theta \mid x_{1:n})$ where $x_{1:n} = (X_1, \dots, X_n)$.
2. Assume the squared error loss function. **Provide** the Bayes rule (under the model defined above). (You do not have to prove the result).
3. **Derive** (step by step) the posterior predictive distribution of $p(\tilde{x} \mid x_{1:n})$, for a new observation $\tilde{x}$. Identify the distribution of the posterior predictive and its parameters for full credit.

Solutions:

1. $$p(\theta \mid x_{1:n}) = \text{Gamma}(a_n = a + \sum_i x_i, b_n = b + n).$$

2. The Bayes' rule is the posterior mean, which is $\frac{a_n}{b_n}$ under the updated posterior distribution above.

3. $$p(\tilde{x} \mid x_{1:n}) = \frac{\Gamma(\tilde{x} + a_n)}{\Gamma(a_n)\tilde{x}!} \times b_n^{a_n}(\frac{1}{b_n + 1})^{\tilde{x}+a_n} \tag{0.1}$$
$$\implies \tilde{x} \mid x_{1:n} \sim \text{NegativeBinomial}(a_n, \frac{b_n}{b_n + 1}). \tag{0.2}$$