Module 7: Exercise Multinomial Dirichlet Conjugacy

Rebecca C. Steorts

Agenda

- Dirichlet distribution
- The Dirichlet-Multinomial

Dirichlet

A Dirichlet distribution[1] is a distribution of the K-dimensional probability simplex[2]

$$\triangle_K = \{(\pi_1, \ldots, \pi_k) : \pi_k \geq 0, \sum_k \pi_k = 1\}.$$

We say that $(\pi_1, \ldots, \pi_k)$ is Dirichlet distributed:

$$(\pi_1, \ldots, \pi_k) \sim \text{Dir}(\alpha_1, \ldots, \alpha_k)$$

if

$$p(\pi_1, \ldots, \pi_k) = \frac{\Gamma(\sum_k \alpha_k)}{\prod_k \Gamma(\alpha_k)} \prod_{k=1}^{K} \pi_k^{\alpha_k - 1}.$$

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[1] This is the multivariate version of the Beta distribution.

[2] In geometry, a simplex is a generalization of the notion of a triangle or tetrahedron to arbitrary dimensions.
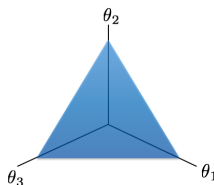
Dirichlet distribution

Let
$$\theta \sim \text{Dirichlet}(\alpha_1, \ldots, \alpha_k)$$

where the probability density function is

$$p(\theta \mid \alpha) \propto \prod_{k=1}^{m} \theta_k^{\alpha_k - 1},$$

where $\sum_k \theta_k = 1, \theta_i \geq 0$ for all i
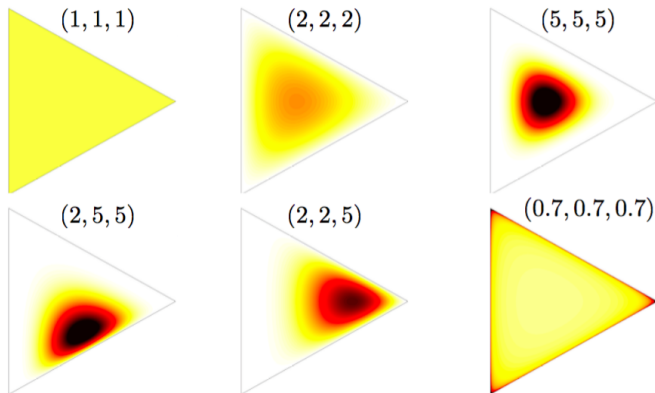
Dirichlet distribution

Figure 1: Far left: We get a uniform prior on the simplex. Moving to the right we get things unimodal. On the bottom, we get distributions that are multimodal at the corners.

Multinomial-Dirichlet

In order to proceed with the lab, we'll need to learn about the Multinomial or Categorical distribution.[3]

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[3]This is the multivariate generalization of of the Binomial distribution.

Multinomial or Categorical distribution

Assume $X = (x_1, x_2, \ldots x_n)$ where $x_i \in \{1, \ldots, m\}$.

1. n: number of observations.

2. m: number of categories or bins.

Assume $\theta = (\theta_1, \ldots, \theta_m) \geq 0$, where $\sum_i \theta_i = 1$.

Assume that

$$X \mid \theta \overset{ind}{\sim} \text{Multinomial}(\theta)$$

or

$$X \mid \theta \overset{ind}{\sim} \text{Categorical}(\theta)$$

$$P(X_i = j \mid \theta) = \theta_j$$

where $j \in \{1, \ldots, m\}$ and $i = 1, \ldots, n$.

Binomial Special Case

Consider $j \in 1, 2$. Let $i = 1, \ldots, n$.

Then $\theta = (\theta_1, \theta_2) \geq 0$, where $\theta_1 + \theta_2 = 1$.

$$X \mid \theta \overset{ind}{\sim} \text{Categorical}(\theta).$$

1. Suppose that j = 1. Then $P(X_i = 1 \mid \theta) = \theta_1$.
2. Suppose that j = 2. Then $P(X_i = 2 \mid \theta) = \theta_2$.

This is equivalent to $X \sim \text{Binomial}(n, \theta_1)$.

Multinomial or Categorical distribution

Consider $j \in 1, 2, 3$. Let $i = 1, \ldots, n$.

Then $\theta = (\theta_1, \theta_2, \theta_3) \geq 0$, where $\theta_1 + \theta_2 + \theta_3 = 1$.

$$X \mid \theta \overset{ind}{\sim} \text{Categorical}(\theta).$$

1. Suppose that $j = 1$. Then $P(X_i = 1 \mid \theta) = \theta_1$.
2. Suppose that $j = 2$. Then $P(X_i = 2 \mid \theta) = \theta_2$.
3. Suppose that $j = 3$. Then $P(X_i = 3 \mid \theta) = \theta_3$.

Example: Extend heights example to three categories of gender, which might be male, female, and other.

Conjugate prior (Dirichlet)

$$\boldsymbol{\theta} \sim \text{Dirichlet}(\boldsymbol{\alpha})$$

Recall the density of the Dirichlet is the following:

$$p(\boldsymbol{\theta} \mid \boldsymbol{\alpha}) \propto \prod_{j=1}^{m} \theta_j^{\alpha_j - 1},$$

where $\sum_j \theta_j = 1, \theta_i \geq 0$ for all i

Likelihood

Define the data as $\boldsymbol{X} = (x_1, \ldots, x_n)$, $x_i \in \{1, \ldots m\}$. Consider

$$p(\boldsymbol{X} \mid \theta) = \prod_{i=1}^{n} P(X_i = x_i \mid \theta) \tag{1}$$

$$= \prod_{i=1}^{n} \theta_{x_i} = \theta_{x_1} \times \theta_{x_2} \times \theta_{x_n} \tag{2}$$

$$= \prod_{i=1}^{n} \prod_{j=1}^{m} \theta_j^{I(x_i=j)} = \prod_{j=1}^{m} \prod_{i=1}^{n} \theta_j^{I(x_i=j)} \tag{3}$$

$$= \prod_{j=1}^{m} \theta_j^{\sum_i I(x_i=j)} \tag{4}$$

$$= \prod_{j=1}^{m} \theta_j^{c_j} \tag{5}$$

where $c = (c_1, \ldots c_m)$, $c_j = \#\{i : x_i = j\}$.

Likelihood, Prior, and Posterior

$$p(\boldsymbol{X} \mid \boldsymbol{\theta}) = \prod_{j=1}^{m} \theta_j^{c_j}$$

$$P(\boldsymbol{\theta}) \propto \prod_{j=1}^{m} \theta_j^{\alpha_j - 1} I(\sum_j \theta_j = 1, \theta_i \geq 0 \forall i)$$

Then

$$P(\boldsymbol{\theta} \mid \boldsymbol{X}) \propto \prod_{j=1}^{m} \theta_j^{c_j} \times \prod_{j=1}^{m} \theta_j^{\alpha_j - 1} I(\sum_j \theta_j = 1, \theta_i \geq 0 \forall i) \qquad (6)$$

$$\propto \prod_{j=1}^{m} \theta_j^{c_j + \alpha_j - 1} I(\sum_j \theta_j = 1, \theta_i \geq 0 \forall i) \qquad (7)$$

This implies

$$\boldsymbol{\theta} \mid \boldsymbol{X} \sim \text{Dirichlet}(\boldsymbol{c} + \boldsymbol{\alpha}).$$

Takeaways

1. Dirichlet is conjugate for Categorical or Multinomial.[4]

2. Useful formula:

$$\prod_i \text{Multinomial}(x_i \mid \theta) \times \text{Dir}(\boldsymbol{\theta} \mid \boldsymbol{\alpha}) \propto \text{Dir}(\boldsymbol{\theta} \mid \boldsymbol{c} + \boldsymbol{\alpha}).$$

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[4]The word Categorical seems to be used in CS and ML. The word Multinomial seems to be used in Statistics and Mathematics.