$$X_1, \dots, X_n \overset{iid}{\sim} \text{Geom}(\theta)$$

$$\theta \sim \text{Beta}(a, b)$$

a.) $p(x \mid \theta) = \theta(1-\theta)^x$, for $x \in 0, 1, 2, \dots$

$$p(x_{1:n} \mid \theta) = \theta^n (1-\theta)^{\sum_{i=1}^n x_i}$$

b.) $p(\theta \mid x_{1:n}) \propto \theta^n (1-\theta)^{\sum x_i} \times \theta^{a-1} (1-\theta)^{b-1}$

$$\propto \theta^{n+a-1} (1-\theta)^{\sum x_i + b - 1}$$

$$\Rightarrow \theta \mid x_{1:n} \sim \text{Beta}(n+a, \textstyle\sum x_i + b)$$

c.) $p(x_{1:n}) = \int p(x_{1:n} \mid \theta)\, p(\theta)\, d\theta$

$$= \int_0^1 \theta^n (1-\theta)^{\sum x_i} \frac{1}{B(a,b)} \theta^{a-1} (1-\theta)^{b-1}\, d\theta$$

$$= \frac{1}{B(a,b)} \int_0^1 \theta^{n+a-1} (1-\theta)^{\sum x_i + b - 1}\, d\theta$$

$$= \frac{1}{B(a,b)} \times B(n+a, \textstyle\sum x_i + b)$$

let $a_n = n+a$
$b_n = \sum x_i + b$

$$= \frac{\Gamma(a_n)\Gamma(b_n)}{\Gamma(a_n + b_n)} \times \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}$$

$$= \frac{\Gamma(n+a)\Gamma(\sum x_i + b)\Gamma(a+b)}{\Gamma(n+a+b+\sum x_i)\Gamma(a)\Gamma(b)}$$

d.) post. predictive

$$p(x_{n+1} | x_{1:n}) = \int_0^1 \underbrace{p(x_{n+1} | \theta)}_{\text{Geomertic}} \underbrace{p(\theta | x_{1:n})}_{\text{updated Beta}} \, d\theta$$

$$= \int_0^1 \theta (1-\theta)^{x_{n+1}} \times \text{Beta}(\theta | a_n, b_n) \, d\theta$$

$$= \int_0^1 \theta (1-\theta)^{x_{n+1}} \frac{1}{B(a_n, b_n)} \theta^{a_n - 1} (1-\theta)^{b_n - 1} \, d\theta$$

$$= \frac{1}{B(a_n, b_n)} \int_0^1 \theta^{a_n - 1 - 1} (1-\theta)^{x_{n+1} + b_n - 1} \, d\theta$$

$a_n = n + \alpha$

$b_n = \sum x_i + b$

$$= \frac{1}{B(a_n, b_n)} \times B(a_n - 1, x_{n+1} + b_n)$$

$$= \frac{B(n + a - 1, x_{n+1} + \sum_{i=1}^{n} x_i + b)}{B(n + a, b + \sum x_i)}.$$