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EXERCISE 2

Consider the configuration of planar vectors shown on the right. Let α be the angle between the unit vectors a and b and let c = 2b<a, b> − a, so that the angle between b and c is also α. We further consider the unit vector d and denote the angle between c and d by β. We finally compute the halfway-vector e = a+d/||a+d|| and denote the angle between b and e by γ. Show that γ = β/2. Note that d may also lie on the other side of c, i.e. between b and c or even between a and b, so you need to distinguish these different cases.

We know that:

- α is the angle between a and b, and between b and c
- c = 2*b<a,b>-a
- d is an unit vector
- β is the angle between c and d
- halfway-vector $\frac{a+d}{||a+d||}$
- γ is the angle between b and e

We need to show that $\gamma = \frac{\beta}{2}$

- We know that:
  - $cos(\beta) = \frac{<d,c>}{||d||||c||}$
  - $cos(\gamma) = \frac{<b,e>}{||b||||e||}$

For d between a and c:

1. We will assume that $\gamma = \frac{\beta}{2}$ is true

   That's means: $cos(\gamma) = cos(\frac{\beta}{2})$ and $2\gamma = \beta$

2.

Math property: $$cos(\frac{angle}{2}) = \sqrt{\frac{1}{2} * (a + cos(angle))}$$

So:

$$cos(\frac{\beta}{2}) = \sqrt{\frac{1}{2} * (a + cos(\beta))} = \sqrt{\frac{1}{2} * (1 + \frac{< d,c >}{||d||||c||})}$$

3. If $\gamma = \frac{\beta}{2}$ is true $\Rightarrow 2\gamma = \beta$ and $cos(2 * \gamma) = cos(\beta)$ are also true

4.

Math property:

$$cos(2 * angle) = cos^2(angle) - sin^2(angle) = 2 * cos^2(angle) - 1$$

So:

$$cos(2 * \gamma) = cos^2(\gamma) - sin^2(\gamma) = 2 * cos^2(\gamma) - 1$$

We assume that: $cos(2 * \gamma) = cos(\beta)$, so:

$$2 * cos^2(\gamma) - 1 = cos(\beta);$$

$$2 * cos^2(\gamma) - 1 = \frac{<d,c>}{||d||||c||};$$

$$2 * cos^2(\gamma) = \frac{<d,c>}{||d||||c||} + 1;$$

$$cos^2(\gamma) = \frac{1}{2} * (\frac{<d,c>}{||d||||c||} + 1);$$

$$cos(\gamma) = \sqrt{\frac{1}{2} * (\frac{<d,c>}{||d||||c||} + 1)};$$

We already show that $cos(\frac{\beta}{2}) = \sqrt{\frac{1}{2} * (1 + \frac{<d,c>}{||d||||c||})}$, so this show that $cos(\gamma) = cos(\frac{\beta}{2})$.
And if $cos(\gamma) = cos(\frac{\beta}{2})$ that means that $\gamma = \frac{\beta}{2}$

For d between b and c:

(We will call the new angle β')

That means that $\beta = -\beta'$

Math property:

$$cos(angle) = cos \quad (-angle)$$

So:

$$cos(\beta) = cos \quad (-\beta) \,;$$

$$cos(\beta) = cos \quad (-\beta') \,;$$

In this case, the value of β don't change, so, we will assume that $\frac{\beta\prime}{2} = \gamma'$

Which the same method than before, we show that $\frac{\beta\prime}{2} = \gamma'$ is true