Exercise 1 [5 points]

Before implementing your first raytracer, let's do some math. You are given two vectors $\mathbf{x} = (\sqrt{2}, 1, 0)^T$, $\mathbf{y} = (1, 1, 1)^T$, and matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 1 \\ -1 & -3 & -3 \end{pmatrix}$$

- Task 1 (2 points): Compute the cosine of the angle between vectors $\mathbf{x}$ and $\mathbf{y}$.

TASK 1:

cosine of the angle between vector x and y

$x = (\sqrt{2}, 1, 0)^T \qquad y = (1, 1, 1)^T$

α = angle between x and y

$\langle x, y \rangle = \|x\| \cdot \|y\| \cos\alpha$;

$$\cos\alpha = \frac{\langle x, y \rangle}{\|x\| \cdot \|y\|} = \frac{\vec{x} \cdot \vec{y}}{\|x\| \cdot \|y\|}$$

$$\vec{x} \cdot \vec{y} = (\sqrt{2}, 1, 0)^T \cdot (1, 1, 1)^T$$
$$= \sqrt{2} \cdot 1 + 1 \cdot 1 + 1 \cdot 0 = \sqrt{2} + 1$$

$$\|x\| = \sqrt{(\sqrt{2})^2 + 1^2 + 0^2} = \sqrt{2+1} = \sqrt{3}$$
$$\|y\| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$$

$$\cos\alpha = \frac{\sqrt{2} + 1}{\sqrt{3} \cdot \sqrt{3}} = \frac{\sqrt{2}+1}{3} = 0{,}804$$

$\hookrightarrow \alpha = 0.6368$ rad.

solution: cosine of the angle between vector x and y is 0,804.

- Task 2 (2 points): Compute vector **z** that is the vector perpendicular to vectors **x** and **y**.

TASK 2:

Compute vector z that is the vector perpendicular to vectors x and y.

if z is perpendicular to x and y, means:

$$\vec{z} \cdot \vec{x} = 0 \quad \text{and} \quad \vec{z} \cdot \vec{y} = 0$$

because, being perpendicular means that the angle between the vectors is 90°, so:

$$\cos(90°) = 0 \;;$$

$$\cos(90°) = \frac{\vec{z} \cdot \vec{x}}{\|z\| \cdot \|x\|} \;;$$

$$0 = \frac{\vec{z} \cdot \vec{x}}{\|z\| \cdot \|x\|} \;;$$

$$0 = \vec{z} \cdot \vec{x} \quad \text{(it's the same for } \vec{z} \cdot \vec{y}\text{)}$$

We just need to simplify:

if $z = (z_1, z_2, z_3) \Rightarrow$

$$\vec{z} \cdot \vec{x} = 0; \quad z_1 \cdot \sqrt{2} + z_2 \cdot 1 + z_3 \cdot 0 = 0$$

$$\vec{z} \cdot \vec{y} = 0; \quad z_1 \cdot 1 + z_2 \cdot 1 + z_3 \cdot 1 = 0$$

$z_1 = \lambda \quad \lambda \in \mathbb{R}$

we clear the ecuation:

$$z_2 = -\sqrt{2}\lambda$$

$$\lambda - \sqrt{2}\lambda + z_3 = 0; \quad z_3 = \sqrt{2}\lambda - \lambda \quad z_3 = (\sqrt{2} - 1)\lambda$$

solution: $z = (\lambda, -\sqrt{2}\lambda, (\sqrt{2}-1)\lambda)$

f.e: $\lambda = 1$

$\hookrightarrow z = (1, -\sqrt{2}, (\sqrt{2}-1))$

- Task 3 (1 point): Compute vector $\mathbf{u}$ defined as $\mathbf{u} = A\mathbf{z}$

TASK 3:

Compute vector u defined as $u = Az$

$$u = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 1 \\ -1 & -3 & -3 \end{pmatrix} (\lambda, -\sqrt{2}\lambda, (\sqrt{2}-1)\lambda)$$

$$= (\lambda, -2\sqrt{2}\lambda, -(\sqrt{2}-1)\lambda)$$

$$= (\lambda, -2\sqrt{2}\lambda, (1-\sqrt{2})\lambda)$$

solution: $u = (\lambda, -2\sqrt{2}\lambda, (1-\sqrt{2})\lambda)$