Università degli Studi di Trento – Dipartimento di Fisica

Computational Physics 1

Recitation class, April 21, 2021

Ordinary differential equations

Indice

1 The age of the Universe

In a simple cosmological model based on the Friedmann equation, the universal scale factor $a(t)$ (that is, the ratio between the dimension of the Universe at time t and its present value) is given by the equation

$$\frac{1}{H_0^2}\left(\frac{da}{dt}\right)^2 = \frac{\Omega_0}{a} + \Omega_\Lambda a^2, \tag{1}$$

where $H_0 = 70.4$ km s^{-1} Mpc^{-1} * is the Hubble constant, while $\Omega_0 = 0.27$ and $\Omega_\Lambda = 0.73$ are adimensional constants proportional to matter density (including dark matter) and dark energy density, respectively. Denoting by $t = 0$ the present time, the initial condition of Eq. (1) is $a(0) = 1$.

Compute:

1. The value $t_{\min}$ for which $a(t_{\min}) = 0$,
2. the time $t_{1/2}$ for which $a(t_{1/2}) = 0.5$,
3. the time t_2 for which $a(t_2) = 2.0$,

and possibly plot $a(t)$ per $t_{\min} < t < 0$.

2 A model for the Sun

The condition of hydrostatic equilibrium for a self gravitating spherical mass distribution is

$$\frac{dp(r)}{dr} = -G\frac{M(r)\ \rho(r)}{r^2}, \tag{2}$$

*1 Mpc = 1 Megaparsec = 3.08×10^{19} km.

where G the universal gravitational constant, $\rho(r)$ the mass density as a function of the distance r from the center, and $M(r)$ is the total mass, that is

$$\begin{aligned} M(r) &= 4\pi \int_0^r \rho(x) x^2 \, dx, &(3)\\ \frac{dM}{dr} &= 4\pi r^2 \rho(r). &(4) \end{aligned}$$

The hydrostatic equilibrium equation needs to be completed with a relation between local pressure and density. A very common model is based on a polytropic equation of state

$$p(r) = K\rho(r)^{(n+1)/n}, \tag{5}$$

where K and n are two constants. Differentiating (2) and using (4) one gets

$$\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dp}{dr}\right) = -4\pi r^2 \rho(r), \tag{6}$$

from which, using (5) as a function of the adimensional variable θ defined by $\rho(r) = \rho_0 \theta(r)^n$, one obtains

$$p(r) = K\rho_0^{(n+1)/n}\theta(r)^{n+1},$$

and, substituting $r = \alpha\xi$ with

$$\alpha^2 = \frac{(n+1)K\rho_0^{1/n-1}}{4\pi G},$$

we finally get the Lane–Emden equation

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2 \frac{d\theta}{d\xi}\right) + \theta^n = 0. \tag{7}$$

It is useful to introduce the variables $\theta(\xi)$ and

$$\eta(\xi) = -\xi^2 \frac{d\theta}{d\xi},$$

obtaining the form

$$\frac{d}{d\xi}\begin{pmatrix} \theta \\ \eta \end{pmatrix} = \begin{pmatrix} -\frac{\eta}{\xi^2} \\ \xi^2 \theta^n \end{pmatrix}.$$

Notice that the boundary conditions for $\xi = 0$ are

$$\begin{aligned} \theta(0) &= 1 \\ \eta(0) &= 0. \end{aligned}$$

From the solution of the Lane–Emden equation, an estimate of the star radius is

$$R = \alpha\xi_0, \tag{8}$$

where ξ_0 is the smallest value for which $\theta(\xi_0) = 0$. Analogously, the star mass is given by

$$M = 4\pi\rho_0 \int_0^R r^2\theta(\xi)^n \, dr = 4\pi\rho_0\alpha^3 \int_0^{\xi_0} \xi^2\theta(\xi)^n \, d\xi \equiv \rho_0\alpha^3 I, \tag{9}$$

where

$$I = 4\pi \int_0^{\xi_0} \xi^2\theta(\xi)^n \, d\xi,$$

is known once Eq. (7) has been solved. In the case of the Sun, one can obtain the central density eliminating α from (8) and (9), obtaining

$$\rho_0 = \frac{\xi_0^3}{I}\frac{M_\odot}{R_\odot^3}.$$

Compute the value of n for which (7) describes the known properties of the Sun, that is

$$\begin{aligned} M_\odot &= 2 \times 10^{30} \text{ kg} \\ R_\odot &= 7 \times 10^{8} \text{ m} \\ \rho_0 &= 1.622 \times 10^{5} \text{ kg} \cdot \text{m}^{-3}. \end{aligned}$$

and estimate the pressure at its center.